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A comment about estimable functions in linear models with non estimable constraints

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ABSTRACT.
In the Searle's (1987) book, Linear Models for Unbalanced Data, a characterization of the estimable functions in linear models with non estimable constraints is presented. In this *informal* paper, I indicate another characterization of these functions which was developed by Magnus and Neudecker (1988). The aim of the article is to provide a caution signal to users of linear models theory.

Key words: Estimable functions; Linear models; Non estimable constraints.

1. A controversy

On the academic second semester of 1992, I was teaching a graduate course in linear models using Searle's (1987) book. On page 308 of this book, I found the following characterization of the estimable functions in the linear model with non estimable restrictions:

Let us assume we have the linear model

$$\mathbf{Y} = X\beta + \mathbf{e}$$

with the (consistent) non estimable constraint $R\beta = \mathbf{r}$. Then an estimable function under this model is given by

$$\mathbf{q}'\beta + \lambda'(R\beta - \mathbf{r}) \text{ for } \lambda' \text{ such that } \lambda'\mathbf{r} = 0; \text{ any } \lambda' \text{ if } \mathbf{r} = \mathbf{0}, \qquad (1)$$

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where $\mathbf{q}'\beta$ is an estimable function in the unconstrained model. One deduces immediately from (1) that, for all λ, these estimable functions are the same of the unconstrained model because of $R\beta - \mathbf{r} = \mathbf{0}$.

At the beginning of 1993, I sent à letter to Professor Searle in which I indicated with all deference a possible problem whit this characterization. Gently, Prof. Searle answered to me the following argument: the point is right, i.e., one must delete $R\beta - \mathbf{r}$ from expression (1) and set only $R\beta$ in its place. However, the dependence of λ on $\mathbf{r}$ must be maintained.

I consider that this relationship between λ and $\mathbf{r}$ can be misuntertood. Following Magnus and Neudecker (1988, pág. 268) (MN), a parametric function $W\beta$ is estimable if, and only if, $\mathcal{M}(W') \subseteq \mathcal{M}(X' : R')$, where in general $\mathcal{M}(A)$ denotes the column space of the matrix A. This fact implies that $W\beta$ is estimable if, and only if, $W = W_1X + Q_2R$ for some matrices Q_1 and Q_2 (compatible for the indicated products). Consequently,

$$W\beta = (Q_1X)\beta + Q_2R\beta,$$

where $(Q_1X)\beta$ is an estimable function in the unconstrained model. In particular, if W is a row vector, then Q_1 and Q_2 are row vectors, too.

At this point, one can note that Q_2 has not any relationship with $\mathbf{r}$. It depends only on Q_1; at the bottom line, on W!. This fact is formally supported by MN's rigorous treatment of the topic.

Although less critic, we must also prevent the use of Henderson's (1984) characterization of estimable functions, in the restricted linear model whit non estimable constraints.

2. An example

As in Searle's (1987, pág. 244) book, we consider a 1-way-classification experiment with three cells, with the following number of observations per cell: $n_1 = 2$, $n_2 = 2$, and $n_3 = 3$. Suppose that the parameter vector is given by $\beta = (\mu, \beta_1, \beta_2, \beta_3)'$ and that we have the constraint $\beta_1 + \beta_2 + \beta_3 = 0$.

We can observe that the restriction is not estimable because of

$$R = (0, 1, 1, 1) \neq Q_1X \text{ for all } Q_1 \in \mathbb{R}^7,$$

where X is the design matrix. We address the question: is $\mu = (1, 0, 0, 0)\beta$ an estimable function?. It is easy to see that under the unconstrained model the answer is *no* . However, in the restricted model and using Magnus and Neudecker characterization we obtain

$$(1, 0, 0, 0) = (1/3, 0, 0, 1/3, 0, 1/3, 0)X + (-1/3)R,$$

which means that in this model μ is estimable. It is worth noticing that $Q_2 = -1/3$ does not depend on $\mathbf{r} = 0!$. This value is obtained using only $W = (1,0,0,0), X$, and R.

If we use Searle's characterization, any value for Q_2 can be used because of $\mathbf{r} = 0$, in particular $Q_2 = 0$. In this case, we obtain that μ is estimable in the unconstrained model, too. But this is a contradiction.

References

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