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KAN FIBRATIONS WHICH ARE HOMOMORPHISMS OF SIMPLICIAL GROUPS

by

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INTRODUCTION.

In [3] C. RUIZ gave a definition for "fibration" on the category $Annb$ of Banach rings which is closely related to a functor

$$GR : Annb \to \Delta^\circ \mathrm{Gr}$$

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which takes its values on the category of the simplicial groups. It was assumed that the fibration f of $Annb$ are the Banach ring homomorphisms such that the homomorphisms $GR(f)$ of simplicial groups, are Kan fibrations. This notion happened to be equivalent (Cf. [3; p. 169]) to the notion of Serre fibration given by KAROUBI and VILLAMAYOR [4].

More generally, let $F : Annb \to Gr$ be a functor of Mayer-Vietoris [5], and let

$$R : Annb \to \Delta^\circ Annb$$

be the functor defined in [3], p. 140. We say that a homomorphism f of $Annb$ is a Kan F-fibration if the morphism of simplicial groups $R \circ (\Delta^\circ F)(f)$ associated to f (where $\Delta^\circ F : \Delta^\circ Annb \to \Delta^\circ Gr$ is the functor that prolongates F dimension by dimension) is a Kan fibration. The foregoing case is obtained by taking $F = Gl$, the linear group.

In [6] it was proved that, in the discrete case, this notion of fibration and the one of F-fibration due to GERSTEN concide [5]. In order to get those equivalences, it was necessary to characterize the Kan fibrations which are homomorphisms of simplicial groups. The results in this paper complete those of QUILLEN [2] and C. RUIZ SALGUERO [3], proposition C. 2.1.2.

We proceed as follows: by reducing the problem to study a Kan homomorphism we work the problem in the case of simplicial sets and we show the equivalence between "Kan's relative property" and the notion of "cohereditary set". The consequences are sumarized in theorem 1.6.1. In §2 these results are applied to simplicial groups and completed with a study on "cohereditary equivalence relations". The main results are given in theorems 2.2.2, 2.3.4, 2.3.5.

1. KAN'S RELATIVE PROPERTY

1.1. *Epimorphisms of simplicial groups.*

1.1.1. LEMMA [1]. *Every simplicial group satisfies Kan's extension condition.*

1.1.2. LEMMA. *If $f: G \to H$ is an epimorphism of simplicial sets in which H satisfies the Kan extension condition, then every box*

$$h_o, \ldots, \hat{h}_k, \ldots, h_{n+1}, \quad h_i \in H_n$$

of elements of H in dimension n, is lifted into a box of elements of G. That is to say, there exist

$$g_o, \ldots, \hat{g}_k, \ldots, g_{n+1}, \quad g_i \in G_n$$

where $d_i g_j = d_{j-1} g_i$ for $i < j$, $i, j \neq k$, and $f(g_i) = h_i$.

Proof: Since we have assumed that H satisfies Kan's condition there exist a "filler" $h \in H_{n+1}$ of the given box : $d_i(h) = h_i$ $(i \neq k)$. On the other hand, since f is surjective in each dimension, there exist $g \in G_{n+1}$ such that $f(g) = h$. The faces $g_i = d_i(g)$, $i \neq k$, provide the desired box.

1.1.3. LEMMA. *Every epimorphism of simplicial groups is a Kan fibration.*

Proof. Let N be the kernel of a given epimorphism $f: G \to H$. Let us take a box $g_o, \ldots, \hat{g}_k, \ldots, g_{n+1}$ in dimension n, with image $h_o = f(g_o), \ldots, h_{n+1} = f(g_{n+1})$, and suppose that this last one is filled with $h \in H_{n+1}: d_i(h) = h_i$, $i \neq k$. We will prove that there is $g \in G_{n+1}$ such that

i) $f(g) = h$
ii) $d_i(g) = g_i \quad i \neq k$.

Let $x \in G_{n+1}$ be such that $f(x) = b$. Then there exist $e_i \in N_n$ $(i \neq k)$ such that $g_i = e_i \cdot d_i(x)$. The box $e_o, \ldots, \hat{e}_k, \ldots, e_{n+1}$ is filled by an element $e \in N_{n+1}$. We take $g = e \cdot x$ and this concludes the proof of lemma.

The condition of surjectivity can be weakened, and one of our purposes is to exhibit a sequence of equivalent conditions on an homomorphism of simplicial groups which are also equivalent to Kan's condition.

1.2. The decomposition of a morphism and Kan's condition.

1.2.1. We begin by decomposing a given homomorphism $f: G \to H$ in an epimorphism and a monomorphism

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ & {\scriptstyle p_f}\searrow \quad \swarrow{\scriptstyle i_f} & \\ & K & \end{array}$$

where $K = im(f)$. According to Lemma 1.1.3, p_f is a Kan fibration. Therefore, in order to f be a Kan fibration a sufficient condition is that i_f satisfies Kan's condition. More precisely:

PROPOSITION. In order that a homomorphism of simplicial groups $f: G \to H$ to satisfy Kan's condition, a necessary and sufficient condition is that the injection

$$i_f : Im(f) \to H$$

associated to f, satisfies Kan's condition.

We can prove that such condition is a necessary one in a less restrictive way. In order to do so let $g: X \to Y$ be a simplicial map, let $K = Im(g)$, $p_g: X \to K$,

$i_g : K \to Y$ be as before. Then

1.2.2. PROPOSITION. If K and g satisfy Kan's condition so does i_g .

Proof. Let $k_1, \ldots, \hat{k}_q, \ldots, k_{n+1}$, be a box of K whose image by i_g is filled with an element $y \in Y_{n+1}$,

$$d_j(y) = i_g(k_j) \ , \ j \neq q :$$

By lemma 1.1.2., there exist a lifting of the given box $(k_j)_j$ in X. That is to say, there exist a box $x_0, \ldots, \hat{x}_q, \ldots, x_{n+1}$, with $x_i \in X_{n+1}$, such that $p_g(x_j) = k_j$, $j \neq q$.

Since by hypothésis g satisfies Kan's condition, there exist $x \in X_{n+1}$ satisfying the following two properties

- • $g(x) = y$
- •• $d_j(x) = x_j \quad , \quad j \neq q$

The element $k = p_g(x)$ satisfies the two desired properties :

$$i_g(k) = i_g\, p_g(x) = g(x) = y \quad ;$$

$$d_i(k) = d_i\, p_g(x) = p_g\, d_i(x) = p_g(x_i) = k_i \ ,$$

$(i \neq q)$.

1.3. Kan's relative property .

The proposition above lead us to study monomorphisms which are Kan fibrations and to extend the results so obtained to maps wich are not, in general, injective.

1.3.1. DEFINITION. A given simplicial subset X of a simplicial set Y

is said to satisfy *Kan condition relative to* Y if the injection $i : X \to Y$ is a Kan fibration.

This means, of course, that if a box of X is filled in Y then the filler belongs necessarily to X.

There exist a close relation between Kan's relative condition and connectedness. In order to stablish it, let us recall the definition and some properties of "connected components".

1.3.2. Let X be a simplicial set. Let $x,y \in X_0$. We write $x \sim y$ if there exist $z \in X_1$, such that $d_0(z) = x$ and $d_1(z) = y$. The equivalence relation generated by this relation will be denoted again by $\sim$.

Recall that $\pi_0(X)$ is the quotient of X_0 by this relation. By definition, a simplicial set X is said to be connected if $\pi_0(X)$ is a singleton. By the way, it is also true, that if $\{A_\lambda\}$ is a collection of simplicial sub-sets, and $x_0 \in X$ is such that i) A_λ is connected ; ii) $x_0 \in A_\lambda$ for every λ, then $\cup A_\lambda$ is also a connected simplicial set.

DEFINITION. Let X a simplicial set, $x \in X_0$. We call the connected component of x in X and denote it by $C(X,[x])$, the union of all connected simplicial sub-sets of X containing x. It follows from the foregoing remark that the connected component $C(X,[x])$ is the largest connected simplicial sub-set which contains x.

1.4. Cohereditary Sets.

1.4.1. DEFINITION. A simplicial sub-set A of a simplicial set X is said to be *cohereditary* if it satisfies the following condition :

(C) In order that an element $x \in X_n$ belong to A_n it is necessary and sufficient that there exist a face

$$\omega^* : X_n \to X_m \quad , \quad \omega : [m] \to [n] \quad ,$$

such that $\omega^*(x) \in A_m$.

1.4.2. **Condition (C) is equivalent to the following one :**

(C') In order that an element $x \in X_n$ $(n \geq 0)$ belongs to A_n a necessary and sufficient condition is that there exist a face

$$d_i : X_n \to X_{n-1}$$

for some i, $0 \leq i \leq n$, such that

$$d_i(x) \in A_{n-1} .$$

Let us prove that *(C') => (C)*. Let $x \in X_n$. Assume that there exist a face $\omega^* : X_n \to X_m$ such that $\omega^*(x) \in A_n$. According to McLane's descomposition of ω^*, we have

$$\omega^* = s_{j_1} \circ \ldots \circ s_{j_p} \circ d_{i_1} \circ \ldots \circ d_{i_q} \circ$$

with $n + q - p = m$. Therefore

$$d_{j_p} \circ \ldots \circ d_{j_1} \circ \omega^*(x) = d_{i_1} \circ \ldots \circ d_{i_q}(x) \in A_{n-p} .$$

Applying condition *(C')* we get

$$d_{i_2} \circ \ldots \circ d_{i_q}(x) \in A_{n-p+1} .$$

Applying *(C')* successively we get

$$x \in A_{n-p+q} = A_m .$$

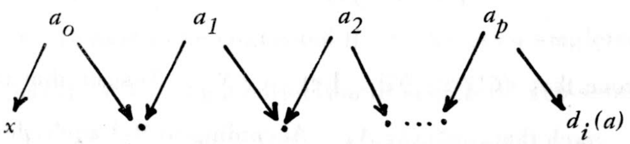
1.4-3. It is clear that unions, as well as intersections of cohereditary sub-sets, are cohereditary. As we will see in the next numeral, the set $C(X,[x])$ is the smallest cohereditary sub-set of X containing x. Moreover every cohereditary subset of X is a union of subsets of this kind.

On $X' = \coprod_{n \geq o} X_n$, the subsets $A' = \bigcup_{n \geq o} A_n$, where $A = \{A_n\}$ is a cohereditary simplicial subset of X, form a topology on X, to wich we will refer as the cohereditary topology of X.

1.5. *Cohereditary sets and connectedness.*

1.5.1. PROPOSITION. *$C(X,[x])$ is a cohereditary subset of X.*

Proof. 1) Let $a \in X_1$, and assume that $d_i(a) \sim x$. Then there exist a chain



where the arrows are either d_o or d_1.

It is clear that a simple process of induction reduces the proof of this part to show that if an element $a \in X_1$, $d_o(a) = x$ (or $d_1(a) = x$), then a belongs to the connected component of x in X, In order to prove it let L be the smallest simplicial subset of X containing $C(X,[x])$ and a. Let's prove that $\pi_o(L)$ is a singleton, which implies that $C(X,[x]) = L$ and then $a \in C(X,[x])$

In order to prove that L is connected recall that in dimension n, L is the union of $C(X,[x])_n$ and the set formed by the $\omega^*(a)$'s where

$\omega^*: X_1 \to X_n$ is a face. Therefore, for $n = 0$, in the McLane's descomposition of ω^*

$$\omega^* = s_{j_1} \circ \dots \circ s_{j_p} \circ d_{i_1} \circ \dots \circ d_{i_q}$$

there cannot be any degeneracies and moreover there can be only one face: d_i. This implies that

$$L_o = C(X, [x]) \cup \{ d_o(a), d_1(a) \} ;$$

since it has been assumed that the faces of a are equal to x, then certainly $\pi_o(L)$ is a singleton.

2) Suppose that we have proved that for each $a \in X_n$, for which there exist a face $d_i(a) \in C(X, [x])_{n-1}$ it holds that $a \in C(X, [x])_n$. And let us prove the property for $n+1$: let $b \in X_{n+1}$, be such that $d_i(b) \in C(X, [x])_n$, for some i. We will prove first, that for every j, $d_j(b) \in C(X, [x])_n$. In order to do this, and according to the induction hypothesis, it is enough to prove that, for some k, $d_k(d_j(b)) \in C(X,[x])_{n-1}$. For $i < j$ we take $k = i-1$. For $j \geq i$ we take $k = i$. Therefore every face

$$d_{i_p} \circ \dots \circ d_{i_o}(b) \in C(X, [x]).$$

With a similar process to that in part 1), let L be the smallest simplicial subset containing $C(X, [x])$ and b. We have

$$L_o = C(X, [x])_o \cup \{ \omega^*(b) \mid \omega^* = d_{i_o} \circ \dots \circ d_{i_n} \} .$$

We get then, that $L_o = C(X, [x])_o$. Since $L_1 \supset C(X, [x])_1$ we have the following diagram

$$\begin{array}{ccc} L_1 & \supseteq & C_1 \\ \downdownarrows & & \downdownarrows \\ L_o & = & C_o \\ \downarrow & & \downarrow \\ \pi_o(L) & \twoheadleftarrow & \pi_o(C) \end{array}$$

where $C = C(X, [x])$. From this we get that $\pi_o(L) = \pi_o(C)$ is a singleton. Since L is connected and contains $C(X, [x])$ we have that $L = C(X, [x])$ and therefore that $b \in C(X, [x])$.

1.5.2. PROPOSITION. *Let A be a simplicial subset of X and $x \in A$. If A is cohereditary, then $C(X, [x]) \subseteq A$.*

Proof. By induction on n, let us prove that $C(X, [x])_n \subseteq A_n$. For $n = 0$ it is enought to notice that if a chain

$$\begin{array}{ccccccccccc} & x_o & & & x_1 & & & & x_q & & \\ \swarrow & & \searrow & \swarrow & & \searrow & & \swarrow & & \searrow & \\ a_o & & & a_1 & & & \cdots & & & & a_p \end{array}$$

(where the dimensions of the x_i's and also those of the a_j's can be different) has some element in A, then all of them are in A. Assume, then, the result for n. Let $a \in C(X, [x])_{n+1}$. Then $d_o(a) \in C(X, [x])_n \subseteq A_n$. Since A is cohereditary $a \in A_{n+1}$.

1.5.3. COROLLARY. *The connected components of X are the smallest* *coheredi tary subsets of X. Moreover : in order that $A \subseteq X$ to be cohereditary it is necessary and sufficient that A is a union of connected components of X.*

Proof. Let us prove the last assertion. According to 1.5.2. if A is cohereditary we have

$$\bigcup_{x \in A_o} C(X, [x]) \subseteq A .$$

On the other hand, for each $a \in A$, $y = d_{i_1} \circ \dots \circ d_{i_p}(a) \in A_o$ and $y \in C(X,[y])$. Since $C(X,[y])$ is cohereditary, then $a \in C(X,[y])$. From it we get $A = \bigcup_{x \in A_o} C(X,[x])$.

1.6. Kan's relative condition and cohereditary sets.

1.6.1. THEOREM. Let X be a simplicial set and $A \subseteq X$. Then the following conditions are equivalent:

i) A satisfies Kan's condition relative to X.

ii) A is cohereditary in X.

iii) A is a union of connected components of X.

iv) the set $A' = \coprod_{n \geq o} A_n$ is open for the cohereditary topology of X (Cf. 1.4.3.).

v) the complementary of A in X is a simplicial subset of X.

Proof. Let us see that i) => ii). Let $x \in X_n$ be such that $d_i(x) \in A$. If we have shown that $d_i(x) \in A$ for every i, then the $d_i(x)$, $i \neq k$, form a box in A for which $x \in X$ is a filler. Since A satisfies Kan's condition relative to X, then $x \in A$.

In order to prove that " if there exist j such that $d_j(x) \in A$, then $d_i d_j(x) \in A$ for every i ", we use induction on the dimension of x. In fact, if $dim(x) = 1$ it is clear that when $x_j = d_j(x) \in A_o$ these elements alone form a box in A, which is filled in X, and this filler x is, by hypothesis, in A.

Now let $dim(x) = n+1$, $d_j(x) \in A$. Consider, without loosing generality, $i < j$. The relation $d_i d_j(x) = d_{j-1} d_i(x)$ implies that the element $y = d_i(x)$, whose dimension equals n, has the face d_{j-1} in A. All of its faces are in A, by the induction hypothesis. Those faces conform a box because A satisfies Kan's condition relative to X.

It is obvious that ii) => i)

Proposition 1.5.3. states the equivalence ii) and iii).

By definition iii) => ii) (Cf. 1.4.3.). Conversely, let A be a simplicial subset of X. $A' = \coprod_{n \geq o} A_n$. Assume that A' is open for the cohereditary topology of X'. Then there exist a cohereditary simplicial subset B of X, such that

$$B' = \coprod_{n \geq o} B_n = A' \quad ;$$

it is clear now that $B_n = A_n$, $n \geq 0$ and then A is cohereditary.

ii) <=> i) is also evident, since in order that $B = \bigcup_{n \geq o} B_n$ $(B_n \subseteq X_n)$ to be a simplicial subset of X it is necessary and sufficient that B to be stable relatively to the faces; i.e.,

$$x \in B_m \Rightarrow \omega^*(x) \in B_n, \ \omega : [n] \to [m] .$$

These facts been established, let A be a simplicial subset of X. Let

$B_n = X_n - A_n$. Then in order that $B = \bigcup B_n$ to be a simplicial subset a necessary and sufficient condition is that for every $\omega : [n] \to [m]$ and $x \in X_m$,

$$x \notin A_m \Rightarrow \omega^*(x) \notin A_n$$

which is equivalent to : A is cohereditary .

1.6.2. *Simplicial Topologies.* Let $\mathcal{O} = \{ A^\lambda \}$ be a collection of simplicial subsets of a simplicial set X . We say that $\mathcal{O}$ is a simplicial topology on X if the unions as well as finite intersections (both given dimension by dimension) of elements of $\mathcal{O}$ are elements of $\mathcal{O}$ and if the empty set (dimension by dimension) and X itself belong to $\mathcal{O}$. These topologies can be compared in the way used for Top . If $A \in \mathcal{O}$, the graded set $\{B_n\}_{n \geq o}$ $B_n = X_n - A_n$ is said to be closed for the topology $\mathcal{O}$. It is convenient to notice that B is not in general a simplicial subset of X . According to the previous theorem, part v), we have

COROLLARY. The cohereditary simplicial topology on X is the finest of the topologies on X which satisfy the condition : " B is closed $\Rightarrow$ B is a simplicial subset of X ".

1.6.3. *COROLLARY. Let $f : X \to Y$ be a simplicial map. Assume that in the descomposition of f ,*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p_f}\searrow \quad \swarrow{\scriptstyle i_f} & \\ & f(X) & \end{array}$$

p_f and $f(X)$ satisfy Kan's condition. Then the following conditions are equivalent :

a) f is a Kan fibration.

b) i_f is a Kan fibration.

c) $f(X) = Im(f)$ satisfies Kan's condition relatively to Y.

d) $f(X)$ is cohereditary in Y.

e) $f(X)$ is a union of connected components of Y.

2. COHEREDITARY SIMPLICIAL GROUPS

2.1. *Kan's relative condition and cohereditary notion on $\Delta^\circ Gr$.*

2.1.1. We will say that a simplicial sub-group G of a simplicial group H satisfies Kan's condition relatively to H (respectively, is cohereditary in H) if the underlying simplicial subset of G satisfies Kan's condition relatively to (respectively, is cohereditary in) the underlying simplicial subset of H.

2.1.2. Caracterization of cohereditary subgroups.

PROPOSITION. Let G be a simplicial subgroup of H. In order that G to be cohereditary in H a necessary and sufficient condition is that the equality $d_o(x) = 1$ (in H) implies $x \in G$. Or equivalently, that for every $n \in \mathbb{N}$, $n > 0$

$$Ker(d_o : H_n \to H_{n-1}) \subseteq G_n .$$

Proof. Let us see that the condition implies property *(C')* of *1.4.2.* . Let $h \in H_n$ be such that for some $i \geq 0$, $d_i(h) \in G_{n-1}$. We will prove that $h \in G$, in two steps :

Step 1. Let us see that $ker(d_i) \subseteq G$. It is clear that $Ker(d_o) \subseteq G$. Suppose that for every j, $1 \leq j < i$, $Ker(d_j) \subseteq G$ and let us prove that $Ker(d_i) \subseteq G$.

Let $h \in Ker(d_i)$. Let $x = (s_{i-1}(d_{i-1}(h))^{-1})\, h$. Then $d_{i-1}(x) = 1$. Therefore $x \in G$, by the induction hypothesis. So we have that $d_i(x) \in G$. But $d_i(x) = (d_{i-1}(h))^{-1} d_i(h) = d_{i-1}(h)^{-1}$, then $d_{i-1}(h) \in G$ and thus $h = (s_{i-1}\, d_{i-1}(h)).x \in G$.

Step 2. We will prove that if $d_i(h) \in G$ then $h \in G$. Again we use induction on i. For $i=0$, let $h \in H_n$ such that $d_o(h) \in G$. Then the element $x = (s_o(d_o(h)^{-1})).h$ satisfies $d_o(x) = 1$. By hypothesis, $x \in G$. Therefore $h = s_o d_o(h).x \in G$. Assume now, for every j, $1 \leq j < i$, that "$d_j(x) \in G_o \Rightarrow x \in G$", where $x \in H$. And let us prove that "$d_i(x) \in G \Rightarrow x \in G$". Let $h \in H_{n+1}$, $d_i(h) \in G_n$. Then the element $x = (s_{i-1}(d_{i-1}(h))^{-1}).h$ is such that $d_{i-1}(x) = 1$. Therefore by step 1, $x \in G$ and then $d_{i-1}(h) = (d_i(x)^{-1}).d_i(h) \in G$. Induction guarantees that $h \in G$.

2.1.3. In view of the equivalence established in 1.6.1. between Kan's relative condition and cohereditary subsets we can claim :

COROLLARY. Let G be a simplicial sub-group of H. In order that G to satisfy Kan's condition relative to H, a necessary and sufficient condition is that for every $n \geq 1$,

$$Ker(d_o : H_n \rightarrow H_{n-1}) \subseteq G_n .$$

Or equivalently, if for every $n \geq 0$, *and every* $h \in H_n$, *and* $d_o \ldots d_o(h) = 1$ *in* H, *then* $h \in G$.

2.2. *Cohereditary relations.*

2.2.1. *DEFINITION.* A simplicial equivalence relations on a simplicial set X is said to be *cohereditary* if for every pair of elements $x,y \in X_n$ and for every $\omega : [n] \rightarrow [m]$,

$$\omega^*(x) \sim \omega^*(y) \Rightarrow x \sim y .$$

2.2.2. *THEOREM. The following conditions are equivalent :*

a) *R is a cohereditary relation on X.*

b) *The simplicial subset $R \subset X \times X$ (defining the relation) is cohereditary in $X \times X$.*

c) *The canonic injection $R \to X \times X$ is a Kan fibration.*

d) *If*

$$x_o, \ldots, x_k, \ldots, x_{n+1}$$

$$z_o, \ldots, z_k, \ldots, z_{n+1}$$

(where $x_i, z_i \in X_n$) are arbitrary boxes in X such that $x_i \sim z_i$, $i \neq k$, then the fillers x (of the first one) and z (of the second one), whenever they exist, are equivalent : $x \sim z$.

e) *In the quotient simplicial set $Y = X/R$ every face is an injection (and thus a bijection). That is to say Y is a simplicial set of the kind $K(A, 0)$, where A is a set :*

$$K(A, 0)_n = A \quad , \quad n \geq 0$$

with the faces equal to id_A.

f) X/R is a minimal simplicial set (MOORE) such that

$$\pi_o(X/R) = X_o / R_o$$

and

$$\pi_n(X/R) = 0 \quad , \quad n \geq 1 \ .$$

Proof. The equivalences a) <=> b) <=> c) (1.6.1.) and e) <=> d) are evident .

Let us prove that a) => e) . Suposse that $d_i[x] = d_i[y]$. This means that $[d_i(x)] = [d_i(y)]$ or equivalently $d_i(x) \sim d_i(y)$. Since R is cohereditary $x \sim y$. So $[x] = [y]$. Conversely c) => a) because if $\omega^*(x) \sim \omega^*(y)$, $x, y \in X$, then $\omega^*[x] = [\omega^*(x)] = [\omega^*(y)] = \omega^*[y]$. Since the faces in X/R are assumed to be injections. Then $[x] = [y]$ and thus $x \sim y$. Finally, the equivalence e) <=> f) is a very well known fact (Cf. [7]) .

2.2.3. Example : Let A a simplicial subset of X . In order that A to be cohereditary in X , a sufficient condition is that the relation defined by A in X (where the cosets on X are of two kinds : the points of $X_n - A_n$ and the coset A_n) is cohereditary. We recall that the condition is not necessary one (Cf. Proposition 2.3.1) .

2.3. *Fundamental Theorem.*

2.3.1. *Subgroups and cohereditary relations.*

PROPOSITION. Let G be a simplicial subgroup of a simplicial group H. In order that G to be cohereditary in H a necessary and sufficient condition is that the equivalence relation given by " $x \sim y \iff y^{-1}x \in G$ " is cohereditary .

2.3.2. *COROLLARY. Let G be a simplicial subgroup of H. Assume that the inclusion $i : G \to H$ is a Kan fibration. Then : 1) the complementary of G in H is a simplicial subset of H ; 2) The homogeneous quotient set H/G is isomorfic to the space of Eilemberg-McLane, $K(A, 0)$ where $A = H_o / G_o$. Moreover each one of the conditions 1), 2) is sufficient in order that G to satisfy Kan's condition relative to H .*

2.3.3. *PROPOSITION. If G satisfies Kan's condition relatively to H, then the inclusion map induces an isomorphism*

$$\pi_n(G) \cong \pi_n(H), \quad \text{for} \quad n \geq 1,$$

and furthermore $\pi_o(G) \subseteq \pi_o(H)$. *Also the homogeneous quotient set* $\pi_o(H)/\pi_o(G)$ *is precisely* H_o/G_o. *Moreover (if G is cohereditary in H)* G_o *is a normal subgroup of* H_o, *if and only if* $\pi_o(G)$ *is a normal subgroup of* $\pi_o(H)$.

Proof. It is enough to consider the homotopy sequence for the Kan fibration $H \to H/G = K(A,0)$ with fiber and group G. Let us show that if $\pi_o(G)$ is normal in $\pi_o(H)$ then G_o is normal in H_o. Let $h \in H_o$ and $g \in G_o$. Then $[h g h^{-1}] = [h][g][h]^{-1} \in \pi_o(G) \subseteq \pi_o(H)$. Then, there exist $g' \in G_o$ such that $[g'] = [h g h^{-1}]$ in $\pi_o(G)$). There exist $h \in H_1$ such that $d_o(k) = g'$, $d_1(k) = h g h^{-1}$. Since G is cohereditary in H, $k \in G$ and then $h g h^{-1} \in G$.

2.3.4. Recall that a simplicial map $K(A,0) \to Y$ is completly determined by a function $f_o : A \to Y_o$. This means that

$$Hom(K(A,0), Y) \approx Hom(A, Y_o);$$

in particular, if Y_o is a singleton, as it is the case when $Y = W(G)$ (the classifying set of MacLane for G) then $Hom(K(A,0), Y)$ is a singleton.

As a consequence there exist one, and only one principal fibering with group G on $K(A,0)$ which corresponds to the unique map

$$K(A,0) \to \overline{W}(G).$$

Then the principal fibering

$$G \to H \to K(A, 0),$$

with group G, is trivial, and then

$$H \approx K(A, 0) \times G$$

as simplicial sets. In particular,

$$\pi_0(H) \approx A \times \pi_0(G), \quad A = H_0 / G_0 .$$

That is to say, from the set theoretical point of view the subgroup G differs from H only by a set $K(A, 0)$ of Eilenberg-MacLane.

2.3.5. In this paper we have obtained the following informations.

THEOREM. If $f : K \to L$ is a homomorphism of simplicial groups. Then the following conditions are equivalent:

a) f is a Kan fibration ;

b) for every i and for every n,

$$Ker\,(d_i : L_n \to L_{n-1}) \subseteq Im(f) ;$$

c) for every n,

$$Ker\,(d_0 : L_n \to L_{n-1}) \subseteq Im(f) .$$

d) In the decomposition of f

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ & \searrow^{p_f} & \downarrow{\scriptstyle i_f} \\ & & Im(f) \end{array}$$

by an epimorphism and a monomorphism, i_f *is a Kan fibration (or equivalently* $Im(f)$ *satisfies Kan's condition relativelly to* L *).*

e) $Im(f)$ *is cohereditary in* L *(or equivalently the equivalence relation defined in* L *by its subgroup* $Im(f)$ *is a cohereditary one).*

f) *The homogeneous quotient simplicial set* $L/Im(f)$ *is minimal and of the kind* $K(A, 0)$ *where* A *is a set* (Cf. [2 : p. 38 , prop. 1]).

Notice that in such a case $A = L_o / Im(f_o)$, *and that* $L/Im(f)$ *satisfies Kan's condition* (Cf. 2.3.2. (2)).

g) $Im(f)$ *is a union of connected components of* L .

h) *For every* $l \in L$ $(\dim l = n)$ *if* $d_o \ldots d_o(l) = 1$ *in* L_o, *then* $l \in Im(f)$. *On the other hand, when* $f : K \to L$ *is a Kan fibration, then*

1) *The simplicial sets* L *and* $Im(f) \times K(V, 0)$ *are isomorphic* $(V = L_o / Im f_o)$. *In particular,*

$$\pi_n(L) = \pi_n(Im(f)), \quad n \geq 1 ;$$

2) *The injection* $Im(f) \to L$ *induces a monomorphism* $\pi_o(Im f) \to \pi_o(L)$, *and we get an isomorphism*

$$\pi_o(L) / \pi_o(Im f) \approx L_o / Im(f_o) ;$$

3) *if* $f_o : K_o \to L_o$ *is surjective then* f *is an epimorphism.*

4) *if* L *is connected then every homomorphism* $f : K \to L$ *satisfying Kan's condition is an epimorphism.*

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