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ON REGULARLY VARYING FUNCTIONS WITH APPLICATIONS TO FUNCTIONS THEORY

by

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RESUMEN. Se estudia el comportamiento asintótico de funciones positivas no decrecientes de orden ρ, con respecto a funciones de variación regular de índice ρ, es decir funciones ρ-homogéneas en el límite (esto es, $R(\alpha x)/R(x) \to \alpha^{\rho}$ cuando $x \to \infty$). Se dan aplicaciones a la teoría de series enteras de Taylor y Dirichlet.

§1. Introducción. A function $R(x)$ is said to be *regularly varying* at infinity if it is real valued, positive and measurable on $[a,\infty)$ for some $a > 0$ and if for each $\alpha > 0$,

$$\lim_{x\to\infty} \frac{R(\alpha x)}{R(x)} = \alpha^{\rho} \tag{1.1}$$

for some fixed ρ in the interval $(-\infty,\infty)$. The number ρ is called the *index of regular variation* of the function $R(x)$. It is well known (see, e.g. [17]) that if $R(x)$ is a function of regular variation of index ρ then the following hold true:

$$\lim_{x\to\infty}\frac{\log R(x)}{\log x} = \rho \qquad (1.2)$$

For each $\sigma < \rho$,

$$\lim_{x\to\infty}\frac{1}{x^{-\sigma}R(x)}\int_b^x t^{-\sigma}R(t)\frac{dt}{t} = \frac{1}{\rho-\sigma}\;. \qquad (1.3)$$

It is obvious that $R(x)$ is regulary varying if and only if it can be written in the form $R(x) = x^{\rho}L(x)$ where $L(x)$ is *slowly varying*, i.e., for each $\alpha > 0$

$$\lim_{x\to\infty}\frac{L(\alpha x)}{L(x)} = 1. \qquad (1.4)$$

The basic properties of regularly varying functions were first studied by Karamata ([9], [10]) and later, important contributions were made by a number of mathematicians including Bojanić [2], Seneta [16] and others. The aim of the present paper is to study the asymptotic behavior of a positive, non-decreasing function of order ρ ($0 < \rho < \infty$) as defined by Hayman [4] relative to a regularly varying function of index ρ (as defined above). The results thus obtained have been applied to get refinements, sharpenings and generalizations of various known results in function theory. Furthermore, the techniques used by us are, in most of the cases, different from those employed by earlier workers.

To obtain the results in a general setting, we first consider the behavior of these functions with respect to another increasing continuous function $m(x)$.

§2. <u>Results.</u> Let $m(x)$ be a real valued, indefinitely increasing, continuous function of x, defined in the interval (b,∞), where $-\infty \leqslant b < \infty$ and $m(b+) = -\infty$, and let $\phi(x)$ be a positive, non-decreasing function defined on $[a,\infty)$ such that $\phi(x)$ is of order ρ $(0 < \rho < \infty)$ with respect to $m(x)$, i.e.

$$\limsup_{x\to\infty} \frac{\log\phi(x)}{m(x)} = \rho \tag{2.1}$$

Let $R(x)$ be a regularly varying function of index ρ such that $\phi(x)$ is of 'Mean Type' with respect to $R(e^{m(x)})$, that is, $0 < \delta \leqslant \gamma < \infty$, where

$$\lim_{x\to\infty} {}^{\sup}_{\inf} \frac{\phi(x)}{R(e^{m(x)})} = {}^{\gamma}_{\delta} . \tag{2.2}$$

For $\lambda \geqslant 0$, we put

$$\Psi(x) = e^{-\lambda m(x)} \int_a^x \phi(t) e^{\lambda m(t)} dm(t), \tag{2.3}$$

and define

$$\lim_{x\to\infty} {}^{\sup}_{\inf} \frac{\Psi(x)}{R(e^{m(x)})} = {}^{T}_{t} , \tag{2.4}$$

$$\lim_{x\to\infty} {}^{\sup}_{\inf} \frac{\Psi(x)}{\phi(x)} = {}^{C}_{D} . \tag{2.5}$$

Now we prove

THEOREM 1. *For the constants as defined above, we have*

$$\frac{\delta}{(\rho+\lambda)\gamma} \leqslant D \leqslant C \leqslant \frac{\gamma}{(\rho+\lambda)\delta} \tag{2.6}$$

$$\delta \leqslant (\rho+\lambda)t \leqslant (\rho+\lambda)T \leqslant \gamma. \tag{2.7}$$

Proof. We shall prove only (2.6), the proof of (2.7) being analogous. For a given $\varepsilon > 0$, there exists x_0 such that for all $x > x_0$,

$$(\delta-\varepsilon)R(e^{m(x)}) < \phi(x) < (\gamma+\varepsilon)R(e^{m(x)}). \tag{2.8}$$

By (2.3), we have

$$\begin{aligned}\Psi(x) &< O(1)+e^{-\lambda m(x)}\int_{x_0}^{x}(\gamma+\varepsilon)e^{\lambda m(t)}R(e^{m(t)})dm(t)\\ &= O(1)+(\gamma+\varepsilon)e^{-\lambda m(x)}\int_{e^{m(x_0)}}^{e^{m(x)}}u^{\lambda}R(u)\frac{du}{u}\\ &\sim O(1)+\frac{(\gamma+\varepsilon)}{\rho+\lambda}R(e^{m(x)}),\end{aligned}$$

the last assertion being a consequence of (1.3). Hence, dividing by $\phi(x)$ and passing to the limit, we get $C \leqslant \frac{\gamma}{(\rho+\lambda)\delta}$. Similarly, use of the left hand inequality of (2.8) gives

$$D \geqslant \frac{\delta}{(\rho+\lambda)\gamma}.$$

This proves Theorem 1.

THEOREM 2. *We have for* $\lambda > 0$,

$$\rho T = \left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \limsup_{x\to\infty} \frac{\phi(x)}{R(e^{m(x)})}\left\{1-\lambda\frac{\Psi(x)}{\phi(x)}\right\}^{-\rho/\lambda} ; \qquad (2.9)$$

and for $\lambda = 0$,

$$e\rho T = \limsup_{x\to\infty} \frac{\phi(x)}{R(e^{m(x)})} \exp\left\{\rho\frac{\Psi(x)}{\phi(x)}\right\} \qquad (2.10)$$

Proof. First let $0 < \lambda < \infty$ and let $\{x_n\}$ be any increasing sequence of positive numbers tending to infinity. It is easy to see that for $x > a$ and $n = 1,2,\ldots$.

$$\lambda\Psi(x)e^{\lambda m(x)} \geqslant \lambda\Psi(x_n)e^{\lambda m(x_n)} + \phi(x_n)\{e^{\lambda m(x)} - e^{\lambda m(x_n)}\},$$

so that

$$\frac{\Psi(x)}{R(e^{m(x)})} \geqslant \frac{\lambda\Psi(x_n)e^{\lambda m(x_n)} + \phi(x_n)\{e^{\lambda m(x)} - e^{\lambda m(x_n)}\}}{\lambda R(e^{m(x)})e^{\lambda m(x)}} \qquad (2.11)$$

Let $\{y_n\}$ be another sequence defined by

$$e^{\lambda m(y_n)} = \left(\frac{\rho+\lambda}{\rho}\right)\left(1-\lambda\frac{\Psi(x_n)}{\phi(x_n)}\right)e^{\lambda m(x_n)}. \qquad (2.12)$$

Since, from (2.6) we have $0 < D \leqslant C < \infty$, it follows that there exist constants A_1 and A_2 such that

$$A_1 e^{m(x_n)} < e^{m(y_n)} < A_2 e^{m(x_n)}$$

for all large n. It follows that $y_n \to \infty$ as $n \to \infty$. Further, by (1.4), if $R(x) = x^\rho L(x)$, then

$$L(e^{m(y_n)}) \sim L(e^{m(x_n)})$$

an $n \to \infty$. Thus (2.11) leads to

$$\frac{\Psi(y_n)}{R(e^{m(y_n)})} \geqslant \frac{1}{\rho}\left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \frac{\phi(x_n)}{R(e^{m(x_n)})}\left\{1-\lambda\frac{\Psi(x_n)}{\phi(x_n)}\right\}^{-\rho/\lambda}.$$

Taking limits as $n \to \infty$, we get

$$\rho T \geqslant \rho \limsup_{x\to\infty} \frac{\Psi(y_n)}{R(e^{m(y_n)})}$$

$$\geqslant \left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \limsup_{n\to\infty} \frac{\phi(x_n)}{R(e^{m(x_n)})}\left\{1-\lambda\frac{\Psi(x_n)}{\phi(x_n)}\right\}^{-\rho/\lambda}.$$

Since the above inequality holds for any increasing sequence $\{x_n\}$ tending to infinity, we get

$$\rho T \geqslant \left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \limsup_{x\to\infty} \frac{\phi(x)}{R(e^{m(x)})}\left\{1-\lambda\frac{\Psi(x)}{\phi(x)}\right\}^{-\rho/\lambda} \tag{2.13}$$

To obtain the reverse inequality, we use the fact that $(1-pu)p^u \leqslant (1+u)^{-(1+u)}$ for p, u satisfying $0 \leqslant p,\ u < \infty$ (p and u not simultaneously zero). Substituting $u = \lambda/\rho$ and $p = \lambda\frac{\Psi(x)}{\phi(x)}$, we get

$$\rho\frac{\Psi(x)}{R(e^{m(x)})} \leqslant \left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \frac{\phi(x)}{R(e^{m(x)})}\left\{1-\lambda\frac{\Psi(x)}{\phi(x)}\right\}^{-\rho/\lambda}. \tag{2.14}$$

Proceeding to limits, this leads to

$$\rho T \leqslant \left(\frac{\rho}{\rho+\lambda}\right)^{(\rho+\lambda)/\lambda} \limsup_{x\to\infty} \frac{\phi(x)}{R(e^{m(x)})}\left\{1-\lambda\frac{\Psi(x)}{\phi(x)}\right\}^{-\rho/\lambda}. \tag{2.15}$$

Combining (2.13) and (2.15), we get (2.9). The proof of (2.10) is similar (or else it can be obtained from (2.9) by letting $\delta \to 0$).

We now obtain a chain of inequalities between the constants defined earlier. Later on, using these inequalities, we obtain relations that throw intrinsic light or the relative asymptotic behavior of $\phi(x)$, $\Psi(x)$ and $R(x)$.

THEOREM 3. (i) *For* $\lambda > 0$ *we have*

$$\frac{\delta}{(\rho+\lambda)\gamma} \leqslant \frac{t}{\gamma} \leqslant D \leqslant \frac{1}{\lambda}\Big[1-\frac{\rho}{\rho+\lambda}\Big\{\frac{\gamma}{(\rho+\lambda)T}\Big\}^{\lambda/\rho}\Big] \leqslant \frac{T}{\gamma}$$

$$\leqslant \frac{1}{\rho+\lambda} \leqslant \frac{t}{\delta} \leqslant C \leqslant \frac{1}{\lambda}\Big[1-\frac{\rho}{\rho+\lambda}\Big\{\frac{\delta}{(\rho+\lambda)T}\Big\}^{\lambda/\rho}\Big] \leqslant \frac{T}{\delta} \leqslant \frac{\gamma}{(\rho+\lambda)\delta}. \tag{2.16}$$

(ii) *For* $\lambda = 0$, *we have*

$$\frac{\delta}{\rho\gamma} \leqslant \frac{t}{\gamma} \leqslant D \leqslant \frac{1}{\rho}[1+\log(\rho T/\gamma)] \leqslant \frac{T}{\gamma} \leqslant \frac{1}{\rho} \leqslant \frac{t}{\delta}$$

$$\leqslant C \leqslant \frac{1}{\rho}[1+\log(\rho T/\delta)] \leqslant \frac{T}{\delta} \leqslant \frac{\gamma}{\rho\delta}. \tag{2.17}$$

Proof. (i) From (2.9), we have

$$\begin{cases} T \geqslant \frac{\delta}{\rho+\lambda}\{\frac{\rho+\lambda}{\rho}(1-\lambda C)\}^{-\rho/\lambda} \\ T \geqslant \frac{\gamma}{\rho+\lambda}\{\frac{\rho\ \lambda}{\rho}(1-\lambda D)\}^{-\rho/\lambda}. \end{cases} \tag{2.18}$$

The above inequalities, in view of the obvious relation $t \leqslant \min(\delta C, \gamma D)$, lead to

$$\begin{cases} \frac{t}{\gamma} \leqslant D \leqslant \frac{1}{\lambda}\Big[1-\frac{\rho}{\rho+\lambda}\Big\{\frac{\gamma}{(\rho+\lambda)T}\Big\}^{\lambda/\rho}\Big] \leqslant \frac{T}{\gamma} \\ \frac{t}{\delta} \leqslant C \leqslant \frac{1}{\lambda}\Big[1-\frac{\rho}{\rho+\lambda}\ \frac{\delta}{(\rho+\lambda)T}\Big\}^{\lambda/\rho}\Big] \leqslant \frac{T}{\delta}\ . \end{cases} \tag{2.19}$$

(2.19) combined with (2.6) and (2.7) yields (2.16). The inequalities (2.17) follow in a similar manner on using the particular cases of (2.6) and (2.7) for $\lambda = 0$. This proves Theorem 3.

Next we have

THEOREM 4. (i) *for* $\lambda \geqslant 0$, *we have*

$$\frac{e^{-m(\omega)}}{\rho+\lambda} \leqslant \lim_{x\to\infty}{}^{\sup}_{\inf} \frac{\Psi(x)}{\phi(x)} \leqslant \frac{e^{m(\omega)}}{\rho+\lambda} \; ; \tag{2.20}$$

where, for $\lambda > 0$, ω *is the root lying in the interval* $[m^{-1}(0),\infty)$ *of the equation*

$$(\rho+\lambda)T\frac{e^{\lambda m(x)/\rho}-1}{\lambda} = te^{(\rho+\lambda)m(x)/\rho}-T \; ; \tag{2.21}$$

and for $\lambda = 0$, ω *is the root lying in the interval* $[m^{-1}(0),\infty)$ *of the equation*

$$Tm(x) = te^{m(x)}-T \tag{2.22}$$

The inequalities in (2.20) *are sharp.*

(ii) *If* $\Psi(x) \simeq TR(e^{m(x)})$, *then* $C = D = \frac{1}{\rho+\lambda}$ *and* $\phi(x) \simeq (\rho+\lambda)TR(e^{m(x)})$.

(iii) *If* $\phi(x) \simeq \gamma R(e^{m(x)})$, *then* $C = D = \frac{1}{\rho+\lambda}$ *and* $\Psi(x) \simeq \gamma(\rho+\lambda)^{-1}R(e^{m(x)})$.

Proof. (i) For $\lambda > 0$, we consider the function

$$P(x) = te^{(\rho+\lambda)m(x)/\rho}-\frac{\delta}{\lambda}e^{(\rho+\lambda)m(x)/\rho}-e^{m(x)} \; .$$

It is easy to check that for all x, we have

$$P(x) \leq \frac{\delta}{\rho+\lambda}\left[\frac{\rho\delta}{(\rho+\lambda)(\delta+\lambda t)}\right]^{\rho/\lambda} .$$

The above result, on combination with (2.16), gives

$$P(x) \leq T. \qquad (2.23)$$

If ω is the root of the equation (2.21) lying in the interval $[m^{-1}(0),\infty)$, then (2.23) for $x = \omega$ gives

$$\frac{T}{\delta} \leq \frac{e^{m(\omega)}}{\rho+\lambda}$$

Hence, from (2.16), we have

$$C \leq \frac{e^{m(\omega)}}{\rho+\lambda} .$$

To prove the left hand inequality of (2.20), we consider the function

$$Q(x) = \frac{\gamma t}{T}e^{(\rho+\lambda)m(x)/\rho} - \frac{t(\rho+\lambda)^2}{\lambda}\left[e^{(\rho+\lambda)m(x)/\rho} - e^{m(x)}\right].$$

Again, it follows that for all x

$$Q(x) \leq (\rho+\lambda)t\left[\frac{\rho T(\rho+\lambda)}{(\rho+\lambda)^2-\lambda\gamma}\right]^{\rho/\lambda} .$$

Using (2.16), this easily leads to

$$Q(x) \leq \gamma$$

wchich, for $x = \omega$ gives

$$\frac{e^{-m(\omega)}}{\rho+\lambda} \leq \frac{t}{\gamma} \leq D.$$

This proves (2.20). When $\lambda = 0$, the inequalities (2.20) can be obtained in an analogous fashion by using (2.17). To see that the inequalities in

(2.20) are sharp, it is enough to take $\phi(x) = e^{\rho m(x)}$.

(ii) If $\Psi(x) = TR(e^{m(x)})$, then from (2.4) we have $T = t$ and so by (2.21) or (2.22), the root ω satisfies $e^{m(\omega)} = 1$. Hence, from (2.20), we have $C = D = \frac{1}{\rho+\lambda}$. Also (2.16) or (2.17) implies that $D = \frac{T}{\gamma}$ and $C = \frac{T}{\delta}$, i.e. $\gamma = \delta = (\rho+\lambda)T$. Thus $\phi(x) \simeq (\rho+\lambda)TR(e^{m(x)})$.

(iii) When $\phi(x) \simeq \gamma R(e^{m(x)})$, we have $\gamma = \delta$. From (2.6) we have $C = D = \frac{1}{\rho+\lambda}$ and (2.7) implies that $T = t = \frac{\gamma}{\rho+\lambda}$, i.e. $\Psi(x) \simeq \gamma(\rho+\lambda)^{-1}R(e^{m(x)})$ as $x \to \infty$.

This completes the proof of Theorem 4.

§3. <u>Applications</u>. In this section, we give a few applications of the above results to function theory.

(i) ENTIRE TAYLOR SERIES. Let f, defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z = re^{i\theta}$, be an entire function. If we set

$$M(r) = \max_{|z|=r} |f(z)|,$$

$$\mu(r) = \max_{n\geqslant 0}\{|a_n|r^n\},$$

$$\nu(r) = \max\{n:\mu(r) = |a_n|r^n\};$$

then $M(r)$, $\mu(r)$ and $\nu(r)$ are called respectively the maximum modulos, the maximum term and the rank of $f(z)$ for $|z| = r$. Similarly the geometric mean

$G(r)$ and the weighted geometric mean $g_k(r)$ of $f(z)$ are defined as

$$G(r) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta\right),$$

$$g_k(r) = \exp\left(\frac{k+1}{2\pi}\int_0^{r} \log G(x)x^k dx\right), \quad -1 < k < \infty$$

(a) It is well known (see, e.g., [1]) that if f has at least one zero then $\log G(r)$ is an indefinitely increasing function of r. If f is of order ρ, i.e.

$$\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r} = \rho ,$$

set

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log G(r)}{r^\rho} = \begin{matrix}\gamma\\ \delta\end{matrix} ,$$

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log g_k(r)}{r^\rho} = \begin{matrix}A\\ B\end{matrix} .$$

Kamthan and Jain [8] showed that

$$\frac{\delta(k+1)}{\rho+k+1} \leqslant B \leqslant \delta\left\{1-\frac{\rho}{\rho+k+1}\left(\frac{\delta}{\gamma}\right)^{(k+1)/\rho}\right\} \tag{3.1}$$

$$\gamma\left\{\frac{\rho}{(\rho+k+1)-(k+1)(\delta/\gamma)}\right\}^{\rho/(k+1)} \leqslant \frac{(\rho+k+1)A}{k+1} \leqslant \gamma \tag{3.2}$$

By taking $m(r) = \log r$, $\phi(r) = \log G(r)$, $\lambda = k+1$, $\Psi(r) = \frac{\log g_k(r)}{k+1}$, and $R(r) = r^\rho$, it is easy to check that our Theorem 3 leads to a sharpening of the above inequalities. Further, Theorem 2 yields the following exact relation between the growths of $G(r)$ and $g_k(r)$ not obtained in [8], viz;

$$\limsup_{r\to\infty}\frac{\log g_k(r)}{r\rho} =$$

$$(k+1)\left(\frac{\rho}{\rho+k+1}\right)^{(\rho+k+1)/(k+1)} \limsup_{r\to\infty}\frac{\log G(r)}{r^{\rho}}\left\{1-\frac{\log g_k(r)}{\log G(r)}\right\}^{-\rho/(k+1)} \tag{3.3}$$

Again, if we take $m(r) = \log r$, $\phi(r) = n(r)$, the number of zeros of $f(z)$ in the disc $\{z:|z| \leqslant r\}$, and $R(r) = r^{\rho}L(r)$, we get Theorem 5 of Jain [6] from (ii) and (iii) of Theorem 4. Similarly, with the above substitutions, while Theorem 4 (ii) and (iii) give rise to Theorem 2 of [8], Theorem 4 (i) leads to a new relation in this direction. It may be noted that our techniques in all the above theorems are different and simpler than those employed by Kamthan and Jain [8]. Furthermore, if $\rho(r)$ denotes the proximate order [11] of f, then it is known that $r^{\rho(r)}$ is a function of regular variation of index ρ. Thus, if instead of r^{ρ} we choose $R(r) = r^{\rho(r)}$ then all the above results stand sharpened in a generalized form.

(b) Results analogous to those mentioned in (a) have been obtained by Shah ([18], [19]), Gopalakrishna [3], S.K.Singh [21], etc. for the maximum term and rank of the entire function f. However, if we make the substitutions $m(r) = \log r$, $\lambda = 0$, $\phi(r) = \nu(r)$, $\Psi(r) = \log\mu(r)$ and $R(r) = r^{\rho}L(r)$

where $L(x)$ is slowly increasing, then it is easily seen that all the results in [18], [21] etc., are sharpened and generalized by our theorems. Mention must be made of the following result given by Theorem 4 (i) which yields an exact bound not given by analogous result of Shah [19]:

If ω *denotes the root, lying in the interval* $[1,\infty)$ *of the equation* $T\log x = xt - T$, *then*

$$\frac{1}{\rho\omega} \leqslant \lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log\mu(r)}{\nu(r)} \leqslant \frac{\omega}{\rho} \tag{3.4}$$

where, as usual,

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log\mu(r)}{r^{\rho}L(r)} = \frac{T}{t}.$$

(c) For entire functions of slow growth, the concept of logarithmic order ρ^* is used [20]. For this case also, analogous study of the properties mentioned in (a) and (b) has been made by Srivastava [22], Jain and Chugh [7] etc. However, it is a simple matter to verify that if we choose $m(r) = \log\log r$, $R(r) = (\log r)^{\rho^*(r)}$, $\rho^*(r)$ being logarithmic proximate order [22], $\phi(r) = \nu(r)\log r$ for $\lambda = 0$ and $\phi(r) = \log G(r)$ for $\lambda > 0$ and $\Psi(r)$ accordingly, then the results of the above mentioned authors are sharpened and generalized by our theorems.

(ii) ENTIRE DIRICHLET SERIES. Consider the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{3.5}$$

where $s = \sigma + it$, $0 < \lambda_n < \lambda_{n+1}, \ldots, \lambda_n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{3.6}$$

It is well known (see, e.g. |15|) that if the series (3.5) converges absolutely in the s-plane, then F(s) represents an entire funcion. We set

$$M(\sigma) = \sup_{-\infty<t<\infty} |f(\sigma+it)| ;$$

$$\mu(\sigma) = \max_{n\geq 1}\{|a_n| e^{\sigma\lambda_n}\} ,$$

$$\lambda_{\nu(\sigma)} = \max\{\lambda_n : \mu(\sigma) = |a_n| e^{\sigma\lambda_n}\} .$$

F(s) is said to be of Ritt-order ρ if

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \rho \tag{3.7}$$

(a) For $0 < \rho < \infty$, and $L(x)$ a slowly increasing function, define

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma} L(e^{\sigma})} = \begin{matrix}T\\ t\end{matrix} ;$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma} L(e^{\sigma})} = \begin{matrix}\gamma\\ \delta\end{matrix} .$$

Rahman [13] showed that if the constants involved are non-zero, finite, then T = t if and only if γ = δ and that if x = κ is that root of the equation $eTx = e^{x}t - eT$ which lies in the interval $(1,\infty)$ then

$$\frac{1}{\rho e^{\kappa}} < \lim_{\sigma\to\infty}{}^{\sup}_{\inf} \frac{\log M(\sigma)}{\lambda_{\nu(\sigma)}} < \frac{e^{\kappa}}{\rho} \qquad (3.8)$$

However, if we take $m(x) = x$, $\phi(x) = \lambda_{\nu(x)}$, $\lambda = 0$, $\Psi(x) = \log\mu(x)$, and $R(e^{x}) = e^{\rho x}L(e^{x})$, then Theorem 4 (ii) and (iii) lead to a simple proof of the first result of Rahman, whereas Theorem 4 (i) yields the following sharp result which is a considerable refinement over (3.8).

If ω *denotes the root, lying in the interval* $[0,\infty)$, *of the equation* $Tx = te^{x} - T$, *then*

$$\frac{1}{\rho e^{\omega}} \leq \lim_{\sigma\to\infty}{}^{\sup}_{\inf} \frac{\log M(\sigma)}{\lambda_{\nu(\sigma)}} \leq \frac{e^{\omega}}{\rho} \qquad (3.9)$$

Further, Theorem 2, with the above substitutions, gives the following exact relation between the growths of $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$:

$$e\rho \limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{e^{\rho\sigma}L(e^{\sigma})}$$

$$= \limsup_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma}L(e^{\sigma})} \exp\left(\rho \frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}}\right) \qquad (3.10)$$

(b) The geometric means and weighted geometric means of an entire Dirichlet series $F(s)$ are defined as

$$G(\sigma) = \exp\left\{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \log|F(\sigma+it)|\,dt\right\}$$

$$g_k(\sigma) = \exp\{\frac{k}{e^{k\sigma}}\int_0^{\sigma} e^{kx}\log G(x)dx\}$$

Results analogous to those in (ii) (a) hace also been obtained, involving $G(\sigma)$ and $g_k(\sigma)$, by Srivastava, Agrawal and Kumar [25]. If we take $m(x) = x$, $\phi(x) = \log G(x)$, $\lambda = k$, $\Psi(x) = \log g_k(x)/k$, and $R(x) = e^{\rho x}L(e^x)$ where $L(x)$ is slowly increasing, then our Theorem 3 gives inequalities sharper than those of Theorems 4 and 5 of [25]. Similarly, Theorem 4 (ii) and (iii) yields a simple alternative proof of Theorem 6 of [25]. However, our Theorems 1 and 4 (i) lead to new results not obtained by these authors.

(c) For entire Dirichlet series of zero order also, the concept of logarithmic Ritt order is used [14]. In this case, one of the authors [23] has made a parallel study connecting $\mu(\sigma)$, $\lambda_{\nu(\sigma)}$, etc. However, it is easily seen that his Theorems 1 to 3 of [23] can be obtained from our Theorems 3 and 4 by making appropriate substitutions. Further, our Theorems 1 and 4 (i) lead to new results not obtained in [23]. Analoguous relations for geometric means can also be obtained from our results.

(iii) SUBHARMONIC AND MEROMORPHIC FUNCTIONS. The concepts of order, mean values, etc. have also been introduced for subharmonic functions (see e.g. [5], [12], [24]). Since our results above have been obtained in a general setting, they can be applied in this case also. Similar remarks apply to meromorphic functions.

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