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ON THE MULTIPLICATION AND CONVOLUTION OF HOMOGENEOUS DISTRIBUTIONS

by

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Abstract. We investigate several notions of the multiplicative, resp. the convolution product, in the case of both factors being homogeneous distributions. Explicit criteria for the existence of some of these products are given under the assumption that one of the factors is C^∞ outside the origin.

Introduction.

The impossibility of generally defining the multiplicative product in the space of distributions led to a multitude of different concepts (see [15], [21] for comparison). In the last analysis, all these definitions are based on the fact that two distributions can be multiplied if the Fourier transforms, at least of local pieces, are convolvable. This becomes most evident in L. Hörmander's wave front set criterion and its extensions (cf. e. g. [8, Theorem 8.2.10], [14, Cor. 1], [21, Cor. 1.3]).

The purpose of this paper is to state necessary and sufficient conditions for the existence of the multiplicative and the convolution product in the case of both factors being homogeneous distributions. The motivation for specializing on this context stems from the analysis of homogeneous differential operators $P(\partial)$ with constant coefficients, which gives rise to studying the homogeneous distributions $P(ix)^\lambda$ ($\lambda \in \mathbf{C}$) and their Fourier transforms (see [30]).

After having introduced the notation in Section 1, we clarify, in Section 2, the relationship of several definitions of the multiplicative, resp. the convolution product, in the case when both factors are homogeneous distributions. In Section 3, we present an explicit necessary and sufficient criterion for the existence of the multiplicative model-product if at least one of the factors is C^∞ outside the origin. In Section 4, the convolution is investigated under these same assumptions. We confine our attention to this situation, since the multiplication of two arbitrary homogeneous distributions involves the multiplication of their characteristics and is thus equally difficult in description as that of arbitrary, not necessarily homogeneous, distributions.

1. Basic concepts and notation.

We write $\mathbb{N}$ for the set of all natural numbers including zero. The letter n is used for the dimension of the underlying space $\mathbb{R}^n$. The distance from the origin to a generic point $x = (x_1, \ldots, x_n)$ in this space, i. e. $(x_1^2 + \cdots + x_n^2)^{1/2} = |x|$, will usually be denoted by the letter r. α always denotes a multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$; x^α resp. ∂^α denote the corresponding polynomial resp. higher derivative. As usual, the n-dimensional Laplace operator is denoted by Δ_n or simply Δ. Re z denotes the real part of a complex number z.

For the spaces of the theory of distributions, the notations of **[25]** are used. The Fourier transformation $\mathcal{F} : S' \longrightarrow S'$ is also normalized as in [**25**]. $\langle \phi, T \rangle$ stands for the value of a distribution T on a test function ϕ. We denote by $\check{S} = S(-x)$ the reflected distribution with respect to the origin of a distribution S. We write S_{n-1} for the unit sphere in $\mathbb{R}^n$ and denote the dual space of the space of C^∞-functions on S_{n-1} by $\mathcal{D}'(\mathsf{S}_{n-1})$. A distribution S is called homogeneous on $\mathbb{R}^n$ (of degree λ) or simply homogeneous, if $S(cx) = c^\lambda S$ for $c > 0$. Every homogeneous distribution the support of which is not reduced to the origin can be represented as $S = f \cdot r^\lambda$ for some $f \in \mathcal{D}'(\mathsf{S}_{n-1})$. f is called the characteristic of S. Conversely, the distributions of the type $f \cdot r^\lambda$ are homogeneous at least in $\mathbb{R}^n \setminus 0$. A collection of basic statements on homogeneous distributions can be found in [**30**]. For the definitions of convolution and of S'-convolution, the reader is referred to [**3**] and [**29**] and to the bibliography quoted therein.

2. Comparison of some versions of the multiplicative and of the convolution product.

Let Σ denote a set of sequences of test functions belonging to $\mathcal{D}$ resp. S. The *Σ-products* $[S] \cdot [T]$ resp. $[S] \cdot T$ are said to *exist in* $\mathcal{D}'$ or S' iff all the sequences $(\rho_j * S) \cdot (\sigma_j * T)$ resp. $(\rho_j * S) \cdot T$ converge in $\mathcal{D}'$ or S' for $(\rho_j), (\sigma_j) \in \Sigma$. In place of Σ, the following sets appear in the bibliography:

$$\Gamma = \{(\rho_j) : \rho_j \in \mathcal{D}, \text{ a) } \operatorname{supp} \rho_j \to \{0\}, \text{ b) } \forall j \in \mathbb{N} : \int \rho_j(x)\, dx = 1, \text{ c) } \forall j \in \mathbb{N} : \forall x \in \mathbb{R}^n : \rho_j(x) \geq 0\}$$

cf. [**7**], [**20**], [**27**];

$$\Delta = \{(\rho_j) : \rho_j \in \mathcal{D}, \text{ a), b) as above c)' } \exists a_j > 0 : a_j \to 0 \text{ and } \forall j \in \mathbb{N} : \operatorname{supp} \rho_j \subset \{x : |x| \leq a_j\} \text{ and } \forall \alpha \in \mathbb{N}^n : \exists C_\alpha > 0 : \forall j \in \mathbb{N} : a_j^{|\alpha|} \int |\partial^\alpha \rho_j(x)|\, dx < C_\alpha\}$$

cf. [**1**], [**15**];

$$\Delta' = \{(\rho_j) : \rho_j \in \mathcal{D}, \text{ a), b) as above, c)'' } \forall \alpha \in \mathbb{N}^n : \exists C_\alpha > 0 : \forall j \in \mathbb{N} : \int \left| |x|^{|\alpha|} \partial^\alpha \rho_j(x) \right| dx < C_\alpha\}$$

cf. [**26**];

$$\Delta^S = \{(\rho_j) : \rho_j \in \mathcal{S},\ \text{a)'} \lim_{j\to\infty} \rho_j\chi = 0 \text{ in } \mathcal{S} \text{ for } \chi \in \mathcal{O}_M \text{ with } 0 \notin \operatorname{supp}\chi,$$

$$\text{b), c)" as above}\}$$

cf. [26, §3], [15];

$$\Delta_m^S = \{(a_j^n \rho_{i(j)}(a_j x)) : \rho_1, \rho_2 \in \mathcal{S},\ i : \mathbb{N} \longrightarrow \{1,2\},\ a_j \to \infty,$$

$$\int \rho_1(x)\,dx = \int \rho_2(x)\,dx = 1\}$$

cf. [15]; $\Delta_m = \Delta_m^S \cap \mathcal{D}^{\mathbb{N}}$, cf. [15].

For the notation as well as for a comparison of the resulting notions of a product, we refer to [15]. In particular, there it is shown that, fortunately, many of them are equivalent. Some of these equivalences were stated already in [26] and [27]. Especially, the equivalence of the Δ_m– and of the Δ–products relies on the characterization of the value of a distribution at a point given by S. Łojasiewicz (see [18]). The remaining set of generally inequivalent notions is reduced further if S and T are homogeneous.

PROPOSITION 1 AND DEFINITION. *Let S and T be homogeneous distributions. Then all the product definitions $[S]\cdot T$, $[S]\cdot[T]$, $[T]\cdot S$ in $\mathcal{D}'$ and in $\mathcal{S}'$ with respect to the classes Δ, Δ', Δ^S, Δ_m^S, Δ_m coincide.*

If one or, equivalently, all these products exist, then they are called the *model-product* or, for short, the *m-product.*

PROOF: By symmetry, it is sufficient to consider the product types $[S]\cdot T$ and $[S]\cdot[T]$ only and to disregard $[T]\cdot S$. Then, evidently, the existence of the Δ_m-product $[S]\cdot T$ in $\mathcal{D}'$ is the most general notion, i. e., it is implied by the existence of any other product. Conversely, the Δ^S-product $[S]\cdot[T]$ in $\mathcal{S}'$ is the most restricted notion. Let us suppose $[S]\cdot T$ to exist in $\mathcal{D}'$ with respect to Δ_m. Then by [26, Prop. 4 and 5], the Δ'-product $[S]\cdot[T]$ in $\mathcal{D}'$ exists as well. In a first step, we shall show that this Δ'-product is well defined in $\mathcal{S}'$ too.

By the weak-star completeness of $\mathcal{S}'$, it is sufficient to prove the convergence of the sequences $\langle\phi, (S*\rho_j)(T*\sigma_j)\rangle$ for $(\rho_j), (\sigma_j) \in \Delta'$ and $\phi \in \mathcal{S}$. An easy argument gives:

$$\phi = \sum_{k=1}^{\infty} \phi_k, \text{ where } \phi_k \in \mathcal{D}, \operatorname{supp}\phi_k \subset \{x : k-1 \le |x| \le k+1\}$$

$$\text{and } \forall i \in \mathbb{N} : \lim_{k\to\infty} k^i \phi_k(kx) = 0 \text{ in } \mathcal{D}.$$

If λ and μ are the homogeneity degrees of S and T respectively, then we have:

$$\langle\phi, (S*\rho_j)(T*\sigma_j)\rangle = \sum_{k=1}^{\infty} k^{\lambda+\mu+n} \langle\phi_k(kx), (S*\rho_{j,k})(T*\sigma_{j,k})\rangle,$$

where $\rho_{j,k} := k^n\rho_j(kx)$ and $\sigma_{j,k} := k^n\sigma_j(kx)$. Let us arrange the indices j,k in such a way that $(j,k) < (j',k')$ if $j+k < j'+k'$ and denote by $(j(i),k(i))$ the i-th index. The resulting sequences $\tilde{\rho}_i := \rho_{j(i),k(i)}$ and $\tilde{\sigma}_i$ belong to Δ' too. Hence $(S*\tilde{\rho}_i)\cdot(T*\tilde{\sigma}_i)$ converges, by assumption, to $[S]\cdot[T]$ in $\mathcal{D}'$ and thus uniformly on the bounded subset $\{k^{\lambda+\mu+n+2}\ \phi_k(kx) : k \in \mathbb{N}\setminus 0\}$ of $\mathcal{D}$. This implies the convergence of the sum in question.

Next we have to consider sequences $(\rho_j), (\sigma_j) \in \Delta^S$. If $\phi \in \mathcal{D}$, $\phi(x) = 1$ in a neighbourhood of 0, then we can choose $M_j \in \mathbb{N}$ with $M_j \to \infty$ such that $\lim_{j\to\infty} (1-\phi(M_j x))\rho_j = 0$ in $\mathcal{S}$. This is a consequence of the condition a)' in the definition of Δ^S. Since the convolution constitutes a partially continuous mapping from $\mathcal{S} \times \mathcal{S}'$ to $\mathcal{O}_M$ (see [25, p. 248]), we obtain

$$\lim_{j\to\infty} (\rho_j * S - \phi(M_j x)\rho_j * S) = 0 \text{ in } \mathcal{O}_M .$$

Furthermore, $\sigma_j * T$ is a convergent sequence and hence bounded in $\mathcal{S}'$. Consequently, the hypocontinuity of the multiplication mapping from $\mathcal{O}_M \times \mathcal{S}'$ to $\mathcal{S}'$ (see [25, Th. X, p. 246]) yields:

$$\lim_{j\to\infty} (\rho_j * S) \cdot (\sigma_j * T) = \lim_{i\to\infty} (\phi(M_j x)\rho_j * S) \cdot (\sigma_j * T) \quad \text{in } \mathcal{S}'.$$

Applying the same procedure to σ_j we have reduced the problem to proving the convergence of $U_j := (\phi(M_j x)\rho_j * S)\cdot(\phi(M_j x)\sigma_j * T)$ in $\mathcal{S}'$. Evidently, $c_j := \int \phi(M_j x)\rho_j(x)\,dx$ converges to 1. Moreover, an application of the Leibniz's product rule shows that $\phi(M_j x)\rho_j/c_j$ belongs to Δ', and similarly for σ. Hence the convergence of U_j follows by assumption and the proof is complete. ∎

REMARKS. 1) The first part of the proof, which exploits the homogeneity of S and T, bears a certain resemblance to the proof of Satz 8 in [29], which states the equivalence of the concepts of convolution and of $\mathcal{S}'$-convolution in the context of homogeneous distributions. Similarly to the example given thereupon, it is possible to find nonhomogeneous $S, T \in \mathcal{S}'$ such that $[S]\cdot T$ exists in $\mathcal{D}'$ but not in $\mathcal{S}'$. For instance, $S = \delta_x \otimes 1_y$, $T = e^y/(x^2 + e^{2y}) \in \mathcal{S}'(\mathbb{R}^2)$ will do. An example in one dimension is given in [16].

2) The second part of the proof, which makes no use of the homogeneity of S and T, employs a similar method as in [26, §3] and [22, p. 357, 358]. Let us mention, however, that Shiraishi treats the Δ^S-product in $\mathcal{D}'$ and not in $\mathcal{S}'$.

PROPOSITION 2 AND DEFINITION. *Let S and T be homogeneous distributions. Then all the Γ-products $[S]\cdot T$, $[S]\cdot[T]$, $[T]\cdot S$ in $\mathcal{D}'$ and in $\mathcal{S}'$ are equivalent.*

If one and hence all these products exist, then we shall call them the *product in the sense of Mikusiński, Hirata and Ogata* or the *MHO-product*.

PROOF. The equivalence of $[S]\cdot T$ and $[S]\cdot[T]$ in $\mathcal{D}'$ with respect to Γ is proven in [27]. The existence of these products in $\mathcal{S}'$, provided they are defined in $\mathcal{D}'$, is shown along the lines of the first part of the proof to Proposition 1. ∎

REMARK. A comparison of the Δ- and the Γ-product with some other methods of defining the product of distributions is carried out in [21].

Finally, we shall consider the convolution product and its relation to the multiplicative product by means of the exchange formula. Following [**15**], we speak of the existence of the *Σ-convolutions* $[S]*T$ resp. $[S]*[T]$ *in* $\mathcal{D}'$ or $\mathcal{S}'$ iff all the sequences $(\rho_j S)*T$ resp. $(\rho_j S)*(\sigma_j T)$ converge in $\mathcal{D}'$ or $\mathcal{S}'$. Herein Σ denotes some set of unit sequences and $(\rho_j),(\sigma_j)\in\Sigma$. If the existence of the limits is required in $\mathcal{D}'$ only, then we obtain the usual definition of convolution (see Section 1) when taking for Σ the set

$$E=\{(\rho_j):\rho_j\in\mathcal{D},\quad \text{a) } \rho_j\to 1 \text{ in } \mathcal{E},\quad \text{b) } \{\rho_j\}\subset\mathcal{D}_{L^\infty} \text{ bounded}\}$$

cf. [**3**, Theorem (1.3)], [**15**, Theorem 3]. For $S,T\in\mathcal{S}'$, we obtain the concept of $\mathcal{S}'$-convolvability if we demand the limits to exist in $\mathcal{S}'$. The larger class $E^S=\{(\rho_j):\rho_j\in\mathcal{S},\text{ a), b) as above}\}$ does not yield anything new in this case (see [**15**, Cor. 1, p. 89]). Together with Satz 8 of [**29**], this furnishes the motivation for the following definition.

DEFINITION. Let S,T be homogeneous distributions. The *product in the sense of the Fourier transformation* or *$\mathcal{F}$-product* is said to exist iff $\mathcal{F}S$ and $\mathcal{F}T$ are convolvable or, what amounts to the same, are $\mathcal{S}'$-convolvable.

In order to clarify the relationship between these three notions of a product, let us recall the exchange theorem in the form given by A. Kamiński. In analogy to Δ_m^S and Δ_m, he introduces the sets

$$E_m^S=\{(\rho_{i(j)}(a_j x)):\rho_1,\rho_2\in\mathcal{S},\ i:\mathbb{N}\longrightarrow\{1,2\},\ a_j>0,\ a_j\to 0,\ \rho_1(0)=\rho_2(0)=1\}$$

and $E_m=E_m^S\cap\mathcal{D}^{\mathbb{N}}$. According to Theorems 4 and 5 of [**15**], the multiplicative Δ_m^S-products $[S]\cdot T$ resp. $[S]\cdot[T]$ in $\mathcal{S}'$ are equivalent to the E_m^S-convolution-products $[\mathcal{F}S]*\mathcal{F}T$ resp. $[\mathcal{F}S]*[\mathcal{F}T]$ through the Fourier exchange formula. To be complete, let us state the following proposition.

PROPOSITION 3. *Let S,T be homogeneous distributions. Then the various convolution products $[S]*T$, $[S]*[T]$, $[T]*S$ in $\mathcal{D}'$ and in $\mathcal{S}'$ with respect to E_m and E_m^S are all equivalent and, in turn, are equivalent to (the inverse Fourier transform of) the m-product of $\mathcal{F}S$ and $\mathcal{F}T$.*

PROOF. The equivalence of the E_m^S-products $[S]*T$ resp. $[S]*[T]$ in $\mathcal{S}'$ to the m-product is a consequence of the exchange theorem of A. Kamiński mentioned above. So it remains to prove that the existence of $[S]*T$ in $\mathcal{D}'$ with respect to E_m implies that one in $\mathcal{S}'$ with respect to E_m^S. In contrast to the proof of Proposition 1, we shall use here a different method, which will allow to prove this in one step only. Obviously,

$$h_\epsilon(\phi,\rho):=\langle\phi,(\rho(\epsilon x)S)*T\rangle=\langle\phi(x+y)\rho(\epsilon x),S_x\otimes T_y\rangle=\langle\rho(\epsilon x),S\cdot(\phi*\check{T})\rangle$$

are separately continuous bilinear functionals of $\phi,\rho\in\mathcal{S}$ for $\epsilon>0$. Hence by the kernel theorem, there exist $U_\epsilon\in\mathcal{S}'(\mathbb{R}^{2n})$ such that $h_\epsilon(\phi,\rho)=\langle\phi\otimes\rho,U_\epsilon\rangle$. If λ resp. μ are the degrees of homogeneity of S resp. T, then U_ϵ is homogeneous of degree $\lambda+\mu$. By assumption, for $\phi,\rho\in\mathcal{D}$, $\langle\phi\otimes\rho,U_\epsilon\rangle$ converges to $\langle\phi\otimes\rho,(S*T)\otimes\delta\rangle$, where

$S * T$ has to be understood as the E_m-product $[S] * T$ in $\mathcal{D}'$. If interpreting U_ϵ as a family of continuous linear mappings from $\mathcal{D}$ to $\mathcal{D}'$, the Banach-Steinhaus theorem (cf. e. g. [24, Cor. 1, p. 69]) asserts that U_ϵ uniformly converges on products $B_1 \times B_2$ of bounded sets $B_1, B_2 \subset \mathcal{D}$. (Note that $\mathcal{D}$ is a Montel space.) Making again use of Schwartz's kernel theorem (see e. g. [6]) we infer that $U_\epsilon \to (S * T) \otimes \delta$ in $\mathcal{D}'(\mathbb{R}^{2n})$. To complete the proof, given $\psi \in \mathcal{S}(\mathbb{R}^{2n})$, we use a decomposition $\psi = \sum_{k=1}^{\infty} \psi_k$ as in the proof of Proposition 1 and obtain

$$\langle \psi, U_\epsilon \rangle = \sum_{k=1}^{\infty} k^{\lambda+\mu+2n} \langle \psi_k(kx), U_\epsilon \rangle \to \langle \psi, (S * T) \otimes \delta \rangle,$$

since $\{k^{\lambda+\mu+2n+2} \psi_k(kx) : k \in \mathbb{N} \setminus 0\}$ is a bounded set in $\mathcal{D}(\mathbb{R}^{2n})$. ∎

From the inclusion $E_m^S \subset E^S$, we deduce that the existence of the $\mathcal{F}$-product implies that of the m-product. But even more is true. By [7], the $\mathcal{S}'$-convolvability of two tempered distributions ensures the validity of the exchange formula with the MHO-product used. Furthermore, on account of Proposition 3.3 of [21], the MHO-product is more special than the m-product. Symbolically: "$\mathcal{F}$" $\Longrightarrow$ "MHO" $\Longrightarrow$ "m". The reverse implications are not valid.

EXAMPLE. Let S be homogeneous of degree λ. a) The $\mathcal{F}$-product of δ and S exists $\Longleftrightarrow \mathcal{F}S \in \mathcal{D}'_{L^1}$. In particular, this implies $S \in \mathcal{C}(\mathbb{R}^n)$ and $(\operatorname{Re}\lambda > 0$ or $S \in \mathbb{C})$.

b) The MHO-product of δ and S exists $\Longleftrightarrow S \in \mathbb{C}$ or $(\operatorname{Re}\lambda > 0$ and $S \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$. To verify this, notice that the existence of the Γ-product $[T] \cdot S$ is, by [27], equivalent to the condition: $\forall \phi \in \mathcal{D} : (\phi T) * \check{S}$ is an L^∞-function near 0 and continuous in 0.

c) The m-product of δ and S exists $\Longleftrightarrow S \in \mathbb{C}$ or $\operatorname{Re}\lambda > 0$. In fact, the existence of this m-product means that S has a value in 0 (see e. g. [15, Lemma, p. 91 and its proof]). This, in turn, implies that $S(\epsilon x) = \epsilon^\lambda S$ converges to a constant.

REMARK. There exist homogeneous distributions S which are continuous functions and the Fourier transform $\mathcal{F}S$ of which does not belong to $\mathcal{D}'_{L^1}$. Clearly, it is impossible to encounter examples in $\mathbb{R}^1$. In $\mathbb{R}^2$ we may write $S = f(\varphi) \cdot r^\lambda$ and develop f in a Fourier series: $f(\varphi) = \sum_{m \in \mathbb{Z}} a_m e^{im\varphi}$. Then

$$\mathcal{F}S = f_1(\varphi) \cdot r^{-2-\lambda} \text{ where } f_1(\varphi) = \sum_{m \in \mathbb{Z}} b_m e^{im\varphi}, \; b_m = a_m \frac{i^{|m|}}{\pi^{1+\lambda}} \cdot \frac{\Gamma(1 + \frac{\lambda - |m|}{2})}{\Gamma(\frac{-\lambda - |m|}{2})},$$

at least if $\lambda \notin \mathbb{N}$. If $S = \check{S}$ and $\lambda = 1$, then this reduces to $f_1(\varphi) = -(2\pi)^{-2}(1 + \partial_\varphi^2) f(\varphi + \pi/2)$. Due to the two-fold differentiation in the last formula, it is possible to find $f \in \mathcal{C}(\mathbb{S}_1)$ such that $f_1 \cdot r^{-3} \notin \mathcal{D}'_{L^1}(\mathbb{R}^2)$. Since this example is not essential for the following, we shall not go into the last detail.

3. On the m-product with a homogeneous distribution which is diferentiable outside the origin.

We state a necessary and sufficient condition for the existence of the m-product of two homogeneous distributions one of which is C^∞ off the origin.

PROPOSITION 4. *Let S, T be homogeneous distributions of the degree λ and μ respectively. Suppose that S is C^∞ outside the origin. Then the m-product of S and T exists if and only if $\partial^\alpha(\mathcal{F}S)\cdot\mathcal{F}\check{T} = 0$ in $\mathbb{R}^n \setminus 0$ for $|\alpha| \le -Re(\lambda+\mu)-n$.*

REMARK. By the lemma in [29, p. 481], $\mathcal{F}S$ is C^∞ in $\mathbb{R}^n \setminus 0$, and thus the product in the condition above is defined in the sense of the multiplication mapping from $\mathcal{E}\times\mathcal{D}'$ to $\mathcal{D}'$ on $\mathbb{R}^n\setminus 0$.

PROOF. a) Let us assume first that the m-product of S and T exists. Then, by the exchange theorem, $S_1 := \mathcal{F}S$ and $T_1 := \mathcal{F}T$ are E_m-convolvable. These are homogeneous distributions of the degree $-\lambda-n$ and $-\mu-n$ respectively. Therefore, if $\phi,\rho\in\mathcal{D}$, $0\notin \operatorname{supp}\rho$, we have:

$$0 = \lim_{\epsilon\to 0}\langle\phi, (\rho(\epsilon x)S_1) * T_1\rangle = \lim_{\epsilon\to 0}\epsilon^{\lambda+\mu}\langle\phi\left(\frac{x}{\epsilon}\right), (\rho S_1) * T_1\rangle.$$

Since $U := (\rho S_1) * T_1 \in \mathcal{D} * \mathcal{D}' \subset \mathcal{E}$ and $\phi\in\mathcal{D}\subset\mathcal{E}'$, we can apply the method of moment asymptotic expansion (cf. [4]), which yields, for $l\in\mathbb{N}$,

$$\langle\phi(\tfrac{x}{\epsilon}), U\rangle = \sum_{|\alpha|=0}^{l}\frac{\epsilon^{|\alpha|+n}}{\alpha!}\langle\phi, x^\alpha\rangle(\partial^\alpha U)(0) + o(\epsilon^{l+n}).$$

The existence of the above limit for all $\phi\in\mathcal{D}$ furnishes:

$\forall\alpha$ with $|\alpha|\le -\mathrm{Re}(\lambda+\mu)-n$:

$$0 = (\partial^\alpha U)(0) = \langle\partial^\alpha(\rho S_1), \check{T}_1\rangle = (-1)^{|\alpha|}\langle\rho, S_1\cdot\partial^\alpha\check{T}_1\rangle.$$

This is equivalent to the condition stated in the proposition. Indeed, an easy proof by induction shows that:

$$\forall\alpha \text{ with } |\alpha|\le l : h\cdot\partial^\alpha R = 0 \iff \forall\alpha,\beta \text{ with } |\alpha|+|\beta|\le l : \partial^\alpha h\cdot\partial^\beta R = 0$$
$$\iff \forall\alpha \text{ with } |\alpha|\le l : \partial^\alpha h\cdot R = 0,$$

if $G\subset\mathbb{R}^n$ is an open domain, $h\in C^\infty(G)$, $R\in\mathcal{D}'(G)$, $l\in\mathbb{N}$.

b) In order to prove the sufficiency of the condition stated in the proposition, we begin by estimating $\phi * \check{S}_1$ for $\phi\in\mathcal{D}$. Assume that $x\neq 0$, $y\in\mathbb{R}^n$, $|y| < r := |x|$, $l\in\mathbb{N}$. Then

$$\check{S}_1(x-y) = r^{-\lambda-n}\,\check{S}_1\left(\frac{x-y}{r}\right)$$
$$= r^{-\lambda-n}\left[\sum_{|\alpha|\le l}\frac{1}{\alpha!}\left(-\frac{y}{r}\right)^\alpha(\partial^\alpha\check{S}_1)\left(\frac{x}{r}\right) + O\left(\left(\frac{|y|}{r}\right)^{l+1}\right)\right]$$
$$= \sum_{|\alpha|\le l}\frac{1}{\alpha!}(-y)^\alpha(\partial^\alpha\check{S}_1)(x) + O\left(r^{-\mathrm{Re}\,\lambda-n-l-1}\right),$$

if y remains bounded and r converges to ∞. Taking into account analogous estimates for the derivatives of $\check{S}_1$ we obtain for

$$h_l := \phi * \check{S}_1 - \sum_{|\alpha|\leq l} \frac{1}{\alpha!} \langle \phi, (-x)^\alpha \rangle \, \partial^\alpha \check{S}_1,$$

which is C^∞ outside the origin:

$$\forall \alpha : \exists C_{l,\alpha} : \forall x \text{ with } |x| \geq 1 : |\partial^\alpha h_l(x)| \leq C_{l,\alpha} \, r^{-\mathrm{Re}\,\lambda - n - |\alpha| - l - 1} \quad .$$

Let us define l by 0 if $\mathrm{Re}(\lambda+\mu)+n>0$ and else by the integral part of $-\mathrm{Re}(\lambda+\mu)-n$ and, furthermore, choose $\psi \in \mathcal{E}$ with $\psi(x)=0$ for $|x|\leq 1$ and $\psi(x)=1$ for $|x|\geq 2$. It is clearly sufficient to establish the E_m-convolvability of S_1 and ψT_1. This means that

$$\forall \phi, \rho \in \mathcal{D} \text{ with } \rho(0)=1 : \lim_{\epsilon\to 0} \langle \phi * \check{S}_1, (\rho(\epsilon x)\psi T_1)\rangle$$

exists and is independent of the choice of ρ. Assuming the validity of the condition given in the proposition we have:

$$\langle \phi, S_1 * (\rho(\epsilon x)\psi T_1)\rangle = \langle \phi * \check{S}_1, \rho(\epsilon x)\psi T_1\rangle = \langle h_l \psi, \rho(\epsilon x) T_1\rangle.$$

The C^∞-function $g := h_l\psi$ satisfies the following estimates:

$$\forall \alpha : \exists C_\alpha : \forall x \in \mathrm{R}^n : |\partial^\alpha g(x)| \leq C_\alpha r^{-\mathrm{Re}\,\lambda - n - |\alpha| - l - 1} \, .$$

Now consider $T := f(\omega)\cdot r^\mu$ with $f \in \mathcal{D}'(\mathrm{S}_{n-1})$. If $m \in \mathrm{N}$ is chosen sufficiently large, then $\tilde{T} := f(\omega)\cdot r^{\mu-2m}$ belongs to $\mathcal{D}'_{L^1}$ and is homogeneous at least in $\mathrm{R}^n\backslash 0$. Therefore, the distribution $U := (2\pi i)^{-2m}\,\mathcal{F}\tilde{T}$ is a continuous function, and $\Delta^m U = \mathcal{F}T = T_1$, and

$$\exists C>0 : \forall x \in \mathrm{R}^n : |U(x)| \leq C\, r^{-\mathrm{Re}\,\mu - n + 2m} \, \log(2+r).$$

Hence it remains to prove the convergence of

$$\langle g, \rho(\epsilon x) T_1\rangle = \int U(x)\Delta^m\left(g\rho(\epsilon x)\right) dx.$$

Using Leibniz's product rule we cbtain:

$$\Delta^m\left(g\rho(\epsilon x)\right) = \sum_{i=0}^{2m} \epsilon^i g_i(x,\epsilon),$$

wherein the functions $g_i(x,\epsilon)$ satisfy:

$$\forall i : \exists C_i : \forall x \in \mathrm{R}^n : \forall \epsilon > 0 : |g_i(x,\epsilon)| < C_i\, r^{-\mathrm{Re}\,\lambda - n - 2m + i - l - 1} \quad \text{and}$$

$$\exists M>0 : \forall i : \forall 0<\epsilon<\frac{M}{2} : \operatorname{supp} g_i(x,\epsilon) \subset \{x \in \mathrm{R}^n : 1 \leq |x| \leq \frac{M}{\epsilon}\}.$$

From this we deduce that the terms of the sum corresponding to $i > 0$ converge, if integrated, to 0. Indeed:

$$\forall i > 0 : \exists C_i' : \forall \epsilon \text{ with } 0 < \epsilon < \frac{M}{2} : \left| \int U(x) \epsilon^i g_i(x,\epsilon)\, dx \right| \leq$$

$$\leq C_i' \int_1^{M/\epsilon} r^{-\operatorname{Re}\mu - n + 2m} \log r \; \epsilon^i \, r^{-\operatorname{Re}\lambda - n - 2m + i - l - 1} \; r^{n-1}\, dr$$

$$\leq C_i' M^i \int_1^{M/\epsilon} r^{-\operatorname{Re}(\lambda+\mu) - n - l - 2} \log r\, dr,$$

which converges to 0 as $\epsilon \to 0$ because of $-\operatorname{Re}(\lambda + \mu) - n - l - 1 < 0$. Applying Lebesgue's dominated convergence theorem to the remaining term in the sum, i. e., to $g_0(x,\epsilon) = \rho(\epsilon x)\Delta^m g$, we conclude that:

$$\lim_{\epsilon \to 0} \langle \phi, S_1 * (\rho(\epsilon x)\psi T_1) \rangle = \int U(x) \Delta^m g(x)\, dx.$$

This last expression is apparently independent of ρ and hence the proof is complete. ∎

REMARK. If the m-product of the two homogeneous distributions $S = f \cdot r^\lambda$, $T = g \cdot r^\mu$, $f \in C^\infty(\mathsf{S}_{n-1})$, $g \in \mathcal{D}'(\mathsf{S}_{n-1})$, exists, then this product is given by $(f \cdot g) \cdot r^{\lambda+\mu}$. The second case covered by the Proposition 4 amounts to $S = P(\partial)\delta$, P a homogeneous polynomial of degree $k = -\lambda - n$, T as above. Under the assumption that the m-product of S and T exists, we have $S \cdot T = 0$ if $k - \mu \notin \mathbb{N}$ and and else

$$S \cdot T = \sum_{|\alpha| = k - \mu} \frac{1}{\alpha!} [(\partial^\alpha P)(-\partial) T]\, \partial^\alpha \delta.$$

(If the m-product exists, then the distributions $(\partial^\alpha P)(-\partial)T$ are bound to be constants.) All this can be deduced from the first part of the proof if using therein $\rho \in \mathcal{D}$ with $\rho(0) = 1$.

The situation becomes particularly simple if both S and T are C^∞ outside the origin.

COROLLARY. *Let S, T be homogeneous distributions of the degree λ and μ respectively and suppose that S and T are C^∞ outside the origin. Then the following assertions are equivalent:*

(1) *The m-product of S, T exists;*
(2) *The MHO-product of S, T exists;*
(3) *The $\mathcal{F}$-product of S, T exists;*
(4) *$Re(\lambda + \mu) + n > 0$ or $\mathcal{F}S \cdot \mathcal{F}\check{T} = 0$ in $\mathbb{R}^n \setminus 0$.*

PROOF. The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are generally true. (1) $\Longrightarrow$ (4) follows from Proposition 4. (4) $\Longrightarrow$ (3) is a consequence of [29, Satz 10]. ∎

REMARKS. 1) A thorough investigation of the one-dimensional case is given in [**19**, Theorems 4, 5]. It is based on a definition of the product due to W. Ambrose, which, under the assumptions of the Corollary, coincides as well with the m-product.

2) The equivalences stated in the above Corollary are far from being true, if one of the distributions S,T fails to possess a C^∞-characteristic. This already results from the example in Section 2.

4. On the convolution of two homogeneous distributions one of which has a C^∞-characteristic.

In order to analyze the $\mathcal{F}$-product of two homogeneous distributions, we must study the convolvability of their Fourier transforms. Under various assumptions, sufficient conditions for the existence of the convolution product of homogeneous distributions have been stated in [**11**, p. 189, Corollary], [**29**, Section 6] and [**23**, Prop. 2]. And, of course, the necessary and sufficient condition of the Corollary in Section 3 in the case of both factors having C^∞-characteristics has already been affirmed in [**29**, Satz 10]. In the sequel, we shall allow that one of the factors in the convolution product has an arbitrary characteristic.

DEFINITIONS. 1) The norms p_m $(m \in \mathbb{N})$ on $\mathcal{D}(\mathbb{R}^n)$ are defined by

$$p_m : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R} : \phi \longmapsto \max\{|\partial^\alpha \phi(x)| : x \in \mathbb{R}^n, |\alpha| \le m\}.$$

2) If $T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$, then its order in $\mathcal{D}'_{L^1}$ is defined by

$$\operatorname{ord} T := \min\{m \in \mathbb{N} : \exists C > 0 : \forall \phi \in \mathcal{D}(\mathbb{R}^n) : |\langle \phi, T\rangle| \le C\, p_m(\phi)\}.$$

REMARK. In [**10**, Chapter 4, §5], it is shown that a distribution is integrable, i. e., it belongs to $\mathcal{D}'_{L^1}$, iff it is continuous with respect to one of the norms p_m. In this section, we shall identify in some places $f \in \mathcal{D}'(\mathbb{S}_{n-1})$ with the distribution $\tilde{f} \in \mathcal{E}'(\mathbb{R}^n)$ which satisfies $\langle \phi(x), \tilde{f}\rangle = \langle \phi(\omega), f\rangle$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$. It is in this way, that ord f is defined.

PROPOSITION 5. *Let S,T be homogeneous distributions of the degree λ and μ respectively. Suppose that S is C^∞ outside the origin and that $T = f(\omega)\cdot r^\mu$ with $\operatorname{ord} f \le m$. Choose $\psi \in \mathcal{E}$ with $\psi(x) = 0$ for $|x| \le 1$ and $\psi(x) = 1$ for $|x| \ge 2$. Then S and T are convolvable if and only if*

$$i)\ (\psi\partial^\alpha S)\check{T} = 0 \text{ for } |\alpha| \le Re(\lambda+\mu)+n;$$

$$\text{and } ii)\ (\psi\partial^\alpha S)\check{T} \in \mathcal{D}'_{L^1} \text{ for } Re(\lambda+\mu)+n < |\alpha| \le Re(\lambda+\mu)+n+m.$$

PROOF. We proceed in a similar manner as in the part b) of the proof to Proposition 4. For $\phi \in \mathcal{D}$, $l \in \mathbb{N}$ define the distribution h_l by

$$h_l := \phi * S - \sum_{|\alpha|\le l} \frac{1}{\alpha!}\langle \phi, (-x)^\alpha\rangle \partial^\alpha S.$$

It fulfills the following estimates:

$$\forall\alpha : \exists C_\alpha : \forall x \text{ with } |x| \geq 1 : |\partial^\alpha h_l(x)| \leq C_{l,\alpha}\, r^{\operatorname{Re}\lambda - |\alpha| - l - 1} .$$

Let us choose $l \geq \operatorname{Re}(\lambda+\mu)+n+m$. Then $(\psi h_l)\check{T} \in \mathcal{D}'_{L^1}$. In fact, when $\chi \in \mathcal{D}$ we have:

$$\begin{aligned} |\langle \chi, (\psi h_l)\check{T}\rangle| &\leq \int_1^\infty |\langle (\chi h_l \psi)(t\omega), k(\omega)\rangle|\, t^{\operatorname{Re}\mu+n-1}\, dt \\ &\leq C \int_1^\infty \max\{|\partial_x^\alpha (\chi h_l \psi)(tx)| : |\alpha| \leq m, |x| = 1\}\, t^{\operatorname{Re}\mu+n-1}\, dt \\ &\leq C'\, p_m(\chi) \int_1^\infty t^{\operatorname{Re}(\lambda+\mu)+n+m-l-2}\, dt = C''\, p_m(\chi), \end{aligned}$$

C, C', C'' being constants independent of χ. Hence the convolvability of S and T, i. e., the validity of $\forall\phi \in \mathcal{D} : (\phi * S)\check{T} \in \mathcal{D}'_{L^1}$, is equivalent to the following condition: $\forall\alpha$ with $|\alpha| \leq l : (\psi\partial^\alpha S)\check{T} \in \mathcal{D}'_{L^1}$. Finally, note that a homogeneous distribution U of the degree ν which is not proportional to δ can belong to $\mathcal{D}'_{L^1}$ only if $\operatorname{Re}\nu < -n$. In fact, $\mathcal{F}U$ has to be continuous and is (at least adjointly) homogeneous of the degree $-n-\nu$. ∎

REMARK. According to Proposition 5, the vital point in what considers the convolvability of two homogeneous distributions (one of which has a differentiable characteristic) is the problem of how to decide whether or not a homogeneous distribution belongs to $\mathcal{D}'_{L^1}$. Before tackling this problem we shall solve, as a digression, the analogous question with respect to $\mathcal{D}'_{L^2}$. Since $\mathcal{D}'_{L^1} \subset \mathcal{D}'_{L^2}$, this will provide at least a necessary condition.

PROPOSITION 6. *Let $f \in \mathcal{D}'(\mathsf{S}_{n-1})$, $\lambda \in \mathbb{C}$, $T := f \cdot r^\lambda$. Then $T \in \mathcal{D}'_{L^2}$ if and only if $\operatorname{Re}\lambda < -n/2$ and f belongs to the Sobolev space $H^{\operatorname{Re}\lambda+n/2}(\mathsf{S}_{n-1})$.*

PROOF. a) Let us suppose first that T is homogeneous in $\mathbb{R}^n$. Choose an orthonormal basis

$$\{P_j^J(\omega) : j \in \mathbb{N}, J \in \mathbb{N}^n, |J| = j, j_n \in \{0,1\}\} \subset L^2(\mathsf{S}_{n-1})$$

of spherical harmonics. Herein the functions $P_j^J(\omega)$ are restrictions to S_{n-1} of harmonic polynomials in $\mathbb{R}^n$ of the degree j. If expanding f in a Fourier series with respect to $P_j^J(\omega)$, we can express thereby the Fourier transform of T:

$$\begin{aligned} f = \sum a_j^J P_j^J(\omega), a_j^J \in \mathbb{C} \Longrightarrow \mathcal{F}T &= \sum a_j^J \mathcal{F}\left(P_j^J(x) \cdot r^{\lambda-j}\right) \\ &= \pi^{-\lambda-n/2} \sum a_j^J (-i)^j \frac{\Gamma\left(\frac{n+\lambda+j}{2}\right)}{\Gamma\left(\frac{j-\lambda}{2}\right)} P_j^J(\omega) \cdot r^{-\lambda-n} =: f_1(\omega) \cdot r^{-\lambda-n}, \end{aligned}$$

cf. [2], [9], [12]. $T \in \mathcal{D}'_{L^2}$ iff $\exists m \in \mathbb{N} : (1+r^2)^{-m}\mathcal{F}T \in L^2$. In the case of homogeneous distributions, this amounts to requiring that $\mathcal{F}T \in L^2_{\text{loc}}$. This is fulfilled iff $\operatorname{Re}(-\lambda-$

$n) > -n/2$ and $f_1 \in L^2(S_{n-1})$. From the asymptotic expansion of the Γ-function (cf. [5, Formula 8.328.2]), we obtain:

$$f_1 \in L^2(S_{n-1}) \Longleftrightarrow \sum |a_j^J|^2 \left| \frac{\Gamma\left(\frac{n+\lambda+j}{2}\right)}{\Gamma\left(\frac{j-\lambda}{2}\right)} \right|^2 < \infty \Longleftrightarrow \sum |a_j^J|^2 j^{2\mathrm{Re}\,\lambda+n} < \infty.$$

The $P_j^J(\omega)$ are the eigenfunctions of the Laplace-Beltrami operator on S_{n-1} corresponding to the eigenvalues $\lambda_j = j(j+n-2)$ (see e. g. [28, §31]). Therefore, we can use the characterization of the spaces H^s given in [17, p. 42, which contains a typographical error]:

$$H^s(S_{n-1}) = \{f_1(\omega) = \sum a_j^J P_j^J(\omega) : \sum |a_j^J|^2 j^{2s} < \infty\}.$$

This completes the proof in the case of T being homogeneous in the whole space $\mathbb{R}^n$.

b) T can fail to be homogeneous in $\mathbb{R}^n$ if $\lambda = -n-j,\ j \in \mathbb{N}$. This occurs if the meromorphic distribution-valued function $z \longmapsto f \cdot r^z$ has a simple pole in λ, i. e., iff

$$\operatorname*{Res}_{z=\lambda} f \cdot r^z = \frac{(-1)^j}{j!} \langle (\omega \cdot \nabla)^j, f(\omega) \rangle\, \delta \neq 0$$

(cf. [30, §3]). From this formula, we infer that

$$\exists f' \in \mathcal{D}(S_{n-1}) : \operatorname*{Res}_{z=\lambda} f \cdot r^z = \operatorname*{Res}_{z=\lambda} f' \cdot r^z.$$

Since $f' \cdot r^\lambda \in \mathcal{D}'_{L^2} \Longleftrightarrow \mathrm{Re}\,\lambda < -n/2$, it is now easy to reduce the proof in this case to part a). ∎

REMARK. As has been pointed out by J. Voigt, this method of proof also shows that

$$f \cdot r^\lambda \in H^s(\mathbb{R}^n) \Longleftrightarrow s < 0,\ s - \frac{n}{2} < \mathrm{Re}\,\lambda < -\frac{n}{2} \text{ and } f \in H^{\mathrm{Re}\,\lambda + n/2}(S_{n-1}).$$

In order to characterize the integrability of homogeneous distributions, it will be convenient to introduce a notation.

DEFINITION. Let $f \in \mathcal{D}'(S_{n-1})$ and denote by $\tilde{f}$ the corresponding distribution in $\mathcal{E}'(\mathbb{R}^n)$. For $m \in \mathbb{N}$, $m \geq \operatorname{ord} f$, $\epsilon > 0$, we define

$$q_m(f, \epsilon) := \sup\{|\langle \phi, \tilde{f} \rangle| : \phi \in \mathcal{D}(\mathbb{R}^n),\ p_m(\phi(\epsilon x)) \leq 1\}.$$

REMARK. Since $p_m(\phi(\epsilon x))$ is a monotonically increasing function of ϵ, $q_m(f,\epsilon)$ is monotonically decreasing. If $f = d\nu$ is a measure on the sphere, then $q_0(f,\epsilon) = \int_{S_{n-1}} d|\nu|$ is independent of ϵ and, for $m \in \mathbb{N}$, we have $\lim_{\epsilon \to 0} q_m(f,\epsilon) = q_0$. As it results from the next proposition, the growth of $q(f,\epsilon)$ as $\epsilon \to 0$ is decisive for the integrability of $f \cdot r^\lambda$.

PROPOSITION 7. *Let* $f \in \mathcal{D}'(\mathsf{S}_{n-1})$, $\lambda \in \mathbb{C}$. *Then* $f \cdot r^\lambda \in \mathcal{D}'_{L^1}$ *if and only if the integral* $\int_0^1 \epsilon^{-\mathrm{Re}\,\lambda-n-1}\, q_m(f,\epsilon)\, d\epsilon$ *converges for some* $m \geq \operatorname{ord} f$.

PROOF. a) Suppose first the above integral to be convergent. If $\phi \in \mathcal{D}$, $\phi(x) = 0$ for $|x| \leq 1$, then

$$|\langle \phi, f \cdot r^\lambda \rangle| = \left| \int_1^\infty t^{\lambda+n-1} \langle \phi(t\omega), f \rangle\, dt \right|$$

$$\leq \int_0^1 \epsilon^{-\mathrm{Re}\,\lambda-n-1}\, \left|\langle \phi(\frac{\omega}{\epsilon}), f\rangle\right| d\epsilon \leq p_m(\phi) \int_0^1 \epsilon^{-\mathrm{Re}\,\lambda-n-1}\, q_m(f,\epsilon)\, d\epsilon.$$

Thus $f \cdot r^\lambda \in \mathcal{D}'_{L^1}$.

b) Now assume that $f \cdot r^\lambda \in \mathcal{D}'_{L^1}$ or, what is the same, that

$$\exists m \in \mathbb{N} : \exists C > 0 : \forall \phi \in \mathcal{D} : |\langle \phi, f \cdot r^\lambda \rangle| \leq C\, p_m(\phi).$$

Since $r^{\mathrm{Re}\,\lambda-\lambda}$ lies in $\mathcal{D}_{L^\infty}$ outside the origin and since $\mathcal{D}'_{L^1}$ is a module over this algebra, we can assume without restriction that λ is real. For $j \in \mathbb{N}$, choose $\psi_j \in \mathcal{D}$ such that

$$\operatorname{supp} \psi_j \subset \{x : \frac{1}{2} < |x| < 2\},\ p_m\left(\psi_j(\frac{x}{j})\right) \leq 1,\ \langle \psi_j, f\rangle \geq \frac{1}{2}\, q_m(f, \frac{1}{j}).$$

Let $0 \neq \chi \in \mathcal{D}(\mathbb{R}^1)$ with support in the interval $(0,1)$ with $\chi(t) \geq 0$ for all t and let us define

$$\phi_N(x) := \sum_{j=1}^N \chi(|x| - j)\, \psi_j(\frac{x}{|x|}) \in \mathcal{D}(\mathbb{R}^n).$$

Then $p_m(\phi_N) < C$ with a constant C independent of N, and hence the following expressions are bounded above as $N \to \infty$:

$$|\langle \phi_N, f \cdot r^\lambda \rangle| = \sum_{j=1}^N \int_1^\infty t^{\lambda+n-1}\, \chi(t-j)\, \langle \psi_j(\omega), f\rangle\, dt \geq C' \int_1^N t^{\lambda+n-1}\, q_m(f, \frac{1}{t})\, dt.$$

This proves the "only if"–part of the proposition and thus the proof is complete. ∎

EXAMPLE. Let $f_\nu(\omega) := (\omega_1^2 + \cdots + \omega_{n-1}^2)^{\nu/2} \in L^1(\mathsf{S}_{n-1})$ for $\mathrm{Re}\,\nu > 1-n$ and extend this to a definition of $f_\nu \in \mathcal{D}'(\mathsf{S}_{n-1})$ for every complex ν by analytic continuation and by taking the finite part at the poles $\nu = 1-n-j$, $j \in \mathbb{N}$. If $S = g(\omega) \cdot r^\lambda$ has a C^∞-characteristic g, then we can apply Proposition 5 to examine the convolvability of S and $T := f_\nu \cdot r^\mu$. If $\mathrm{Re}\,\nu > 1-n$, then it is clearly equivalent to the condition $\mathrm{Re}(\lambda+\mu) < -n$. So let us assume that $\mathrm{Re}\,\nu \leq 1-n$. Then $\operatorname{ord} f_\nu = 2-n+[-\mathrm{Re}\,\nu] \geq 1$, and $q_m(f_\nu, \epsilon)$ grows like $\epsilon^{\mathrm{Re}\,\nu+n-1}$ as $\epsilon \to 0$ for $m \geq \operatorname{ord} f_\nu$, since f_ν is homogeneous of the degree ν in the local coordinate system $\omega_1, \ldots, \omega_{n-1}$ near $x_0 = (0, \ldots, 1)$. To ensure the convolvability of S and T, by the condition i) of Proposition 5, we must

require at least that $\mathrm{Re}(\lambda+\mu) < -n$. Now suppose that $\partial^\alpha S \cdot \check{T} = g_\alpha(\omega)\cdot r^{\lambda+\mu-|\alpha|}$ for $0 \le |\alpha| \le \mathrm{Re}(\lambda+\mu)+n+m$ and $|x|>0$. Then $q_m(g_\alpha,\epsilon)/\max\{1,\epsilon^{\mathrm{Re}\,\nu+n-1+j(\alpha)}\}$ converges to a constant as $\epsilon \to 0$ if $\partial^\alpha S$ has a zero of the order $j(\alpha)$ in $-x_0$. The condition ii) of Proposition 5 combined with the characterization of Proposition 7 furnishes then: $\mathrm{Re}(\lambda+\mu-\nu)-|\alpha|-j(\alpha)+1<0$. Putting all this together we obtain that $g\cdot r^\lambda$ and $f_\nu \cdot r^\mu$ are convolvable if and only if $\mathrm{Re}(\lambda+\mu) < -n$, and g has a zero of an order larger than $\mathrm{Re}(\lambda+\mu-\nu)+1$ in $\pm x_0$.

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