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Convex sets, convex cones, affine spaces and affine cones

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Dedicated to the memory of Professor Jean Dieudonné

ABSTRACT. We study the relationship between maps and convexity, particularly from the following viewpoint: when does a desired property result from another weaker one? See Propositions 11, 15, 19, 20, 22 and 28.

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1. Introduction

A map between affine spaces is an affine space map iff it is a convex set map. A convex subset X of an affine space Y has an affine space structure inducing on X the convex set structure induced on X by Y iff X is an affine subspace of Y. Such an affine space structure on X is unique, namely equal to that induced on X by Y. Two affine space structures on a set induce the same convex set structure on it iff they are equal (Proposition 11; this proposition becomes false if we replace affine space and affine subspace by convex cone and convex subcone, or by affine cone and affine subcone, respectively). A map

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between affine cones is an affine cone map iff it is a convex cone map. A convex subcone X of an affine cone Y has an affine cone structure inducing on X the convex cone structure induced on X by Y iff X is an affine subcone of Y. Such an affine cone structure induced on X is unique, namely equal to that induced on X by Y. Two affine cone structures on a set induce the same convex cone structure on it iff they are equal (Proposition 15; this proposition becomes false if we replace convex cone and convex subcone by convex set and convex subset, or by affine space and affine subspace, respectively). Proposition 11 implies Proposition 15 (Comment 18, (II)). A quasiorder on a convex cone is compatible with its convex cone structure iff it is compatible with its convex set structure. Let f be a convex set map from a convex cone X onto a convex set Y. There is one and only one convex cone structure on Y such that f is a convex cone map. Such a convex cone structure on Y induces the given convex set structure on Y (Proposition 19). An equivalence relation on an affine space is compatible with its affine space structure iff it is compatible with its convex set structure. Let f be a convex set map from an affine space X onto a convex set Y. There is one and only one affine space structure on Y such that f is an affine space map. Such an affine space structure on Y induces the given convex set structure on Y (Proposition 20; this proposition becomes false if we replace equivalence relation by quasiorder). An equivalence relation on an affine cone is compatible with its affine cone structure iff it is compatible with its convex set structure. Let f be a convex set map from an affine cone X onto a convex set Y. There is one and only one affine cone structure on Y such that f is an affine cone map. Such an affine cone structure on Y induces the given convex set structure on Y (Proposition 22; this proposition becomes false if we replace equivalence relation by quasiorder. It remains true if we replace convex set by convex cone or affine space). Proposition 19 and 20 together imply Proposition 22 (Comment 26). If X is an affine cone, X is a convex cone and an affine space in a compatible way. Conversely, if X is a convex cone and a affine space in a compatible way, there is one and only one affine cone structure on X inducing the given convex cone structure on X. This affine cone structure on X also induces the given affine space structure on X. A convex cone X has an affine cone structure inducing the given convex cone structure on X iff it has a compatible affine space structure. Then such an affine cone and affine space structures on X are unique, and the affine cone structure induces the affine space structure. An affine space X has an affine cone structure inducing the given affine space structure on X iff it has a compatible convex cone structure. Then such an affine cone structure induces the convex cone structure. The injective map which to an affine cone structure on X associates the induced convex cone structure on X is bijective between all affine cone structures on X inducing the given affine space structure on X and all convex cone structures on X that are compatible with the given affine space structure on X (Proposition 28). A key tool in the proofs of Propositions 11, 15 and 28 is the following result. Consider an interval I of $\mathbb{R}$ with nonvoid interior I^o, a convex set X, and convex set maps $f_0, f_1 : I \to X$. There are $\lambda_0, \lambda_1 \in I$, $\lambda_0 \neq \lambda_1$, such that $f_0(\lambda_0) = f_1(\lambda_0)$, $f_0(\lambda_1) = f_1(\lambda_1)$ iff $f_0(\lambda) = f_1(\lambda)$ for all $\lambda \in I^o \cup \{\lambda_0, \lambda_1\}$ (Proposition 8).

2. Notation and terminology

1. Notation. Consider the following systems: $\mathbb{N}$ of all positive integers, $\mathbb{N}^*$ of all strictly positive integers, $\mathbb{R}$ of all real numbers, $\mathbb{R}^*$ of all real numbers different from zero, $\mathbb{R}_+$ of all positive real numbers, $\mathbb{R}_+^*$ of all strictly positive real numbers, the closed interval $\mathbb{I}$ of extremities 0 and 1 in $\mathbb{R}$, and the open interval $\mathbb{J}$ of extremities 0 and 1 in $\mathbb{R}$.

We review here a minimum about convexity. See the bibliography at the end for further aspects.

2. Definition. A *convex set* X is a set in which we are given a *convex combination map* that to every $n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathbb{J}, \lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in X$, associates

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1 \leq i \leq n} \lambda_i x_i \in X,$$

so that the following axioms hold:

Commutativity. If $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}, \lambda_1 + \cdots + \lambda_n = 1, x_1, \ldots, x_n \in X$, and σ is a permutation of $\{1, \ldots, n\}$, then

$$\sum_{1 \leq i \leq n} \lambda_{\sigma(i)} x_{\sigma(i)} = \sum_{1 \leq i \leq n} \lambda_i x_i.$$

Associativity. If $m, n, m_j \in \mathbb{N}^*$ $(j = 1, \ldots, n)$, $\lambda_{ij}, \mu_j \in \mathbb{J}, x_{ij} \in X$ $(i = 1, \ldots, m_j; j = 1, \ldots, n)$,

$$\sum_{1 \leq i \leq m_j} \lambda_{ij} = 1 \quad (j = 1, \ldots, n), \qquad \sum_{1 \leq j \leq n} \mu_j = 1,$$

then

$$\sum_{1 \leq j \leq n} \mu_j \left(\sum_{1 \leq i \leq m_j} \lambda_{ij} x_{ij} \right) = \sum_{\substack{1 \leq i \leq m_j \\ 1 \leq j \leq n}} (\mu_j \lambda_{ij}) x_{ij}.$$

Distributivity. If $n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathbb{J}, \lambda_1 + \cdots + \lambda_n = 1, x \in X$, then $\lambda_1 x + \cdots + \lambda_n x = x$.

3. Definition. A *convex cone* X is a set in which we are given an *addition map* and a *multiplication map*

$$(x_1, x_2) \in X \times X \longmapsto x_1 + x_2 \in X,$$
$$(\lambda, x) \in \mathbb{R}_+^* \times X \longmapsto \lambda x \in X,$$

so that the following axioms hold:

$$x_2 + x_1 = x_1 + x_2, \quad (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3),$$
$$\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2, \quad (\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x,$$
$$\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x, \quad 1x = x,$$

for all $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}_+^*$, $x_1, x_2, x_3, x \in X$. A convex cone is a convex set by the structure inducing map.

4. Definition. An *affine space* X is a set in which we are given an *affine combination map* that to every $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n$, associates

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1 \leq i \leq n} \lambda_i x_i \in X,$$

so that the following axioms hold: *Commutativity, Associativity, Distributivity* for an affine space have the same formulation as in Definition 2 of a convex set provided we replace $\mathbb{J}$ by $\mathbb{R}^*$. An affine space is a convex set by the structure inducing map.

5. Definition. An *affine cone* X is a set in which we are given an *addition map* and a *multiplication map*

$$(x_1, x_2) \in X \times X \longmapsto x_1 + x_2 \in X,$$
$$(\lambda, x) \in \mathbb{R}^* \times X \longmapsto \lambda x \in X,$$

so that the following axioms hold: for an affine cone, they have the same formulations in Definition 3 of a convex cone provided we replace $\mathbb{R}_+^*$ by $\mathbb{R}^*$, with the additional requirement $\lambda_1 + \lambda_2 \in \mathbb{R}^*$ in $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$ besides $\lambda_1, \lambda_2 \in \mathbb{R}^*$. An affine cone is a convex cone and an affine space, hence a convex set, by the structure inducing maps.

6. Definition. The concepts of a *convex subset* of a convex set, and of a *cartesian product of convex sets* are clear. A *convex set map* $f : X \to Y$ between convex sets is defined by requiring its graph $\{(x, f(x)); x \in X\}$ to be a convex subset of $X \times Y$. Similarly for convex cones, affine spaces, and affine cones.

7. Definition. A binary relation $R \subset X \times Y$ between convex sets X and Y is said to be *compatible* with their convex set structures when R is a convex subset of $X \times Y$. Similarly for convex cones, affine space, and affine cones.

3. Good conduct of maps vis-à-vis convexity

8. Proposition. *Consider an interval I of $\mathbb{R}$ with nonvoid interior I^o, a convex set X, and convex set maps $f_0(\lambda_0) = f_1(\lambda_0), f_0(\lambda_1) = f_1(\lambda_1)$ iff $f_0(\lambda) = f_1(\lambda)$ for all $\lambda \in I^o \cup \{\lambda_0, \lambda_1\}$.*

Proof. We firstly prove the proposition in the following particular case. Fix $x, x_0, x_1 \in X$. We have

$$(1-\nu)x_0 - \nu[(1-\lambda)x + \lambda x_1] = (1-\nu)x_1 + \nu[(1-\lambda)x + \lambda x_0]$$

for $\lambda, \nu \in I$ if we choose ν so that $1-\nu = \nu\lambda, \nu = 1/(1+\lambda)$. Hence the identity

$$\frac{\lambda}{1+\lambda}x_0 + \frac{1}{1+\lambda}[(1-\lambda)x + \lambda x_1] = \frac{\lambda}{1+\lambda}x_1 + \frac{1}{1+\lambda}[(1-\lambda)x + \lambda x_0]$$

for $\lambda \in I, x, x_0, x_1 \in X$ in any convex set X. Let $I = \mathbb{I}$, and

$$f_0(x) = (1-\lambda)x + \lambda x_0,$$
$$f_1(\lambda) = (1-\lambda)x + \lambda x_1$$

for $x \in I$. We have $f_0(0) = f_1(0) = x$ and $f_0, f_1 : I \to X$ are convex set maps. Then

$$\Lambda = \{x \in I : f_0(\lambda) = f_1(\lambda)\} = \{\lambda \in I : (1-\lambda)x + \lambda x_0 = (1-\lambda)x + \lambda x_1\}$$

is a convex subset of I (an interval) containing 0. Thus $[0, \lambda] \subset \Lambda$ iff $\lambda \in \Lambda$, and $\Lambda = I$ iff $1 \in \Lambda$. Assume that $\lambda \in \Lambda, 0 < \lambda < 1$. Using the above identity, we have

$$\frac{\lambda}{1+\lambda}x_0 + \frac{1}{1+\lambda}[(1-\lambda)x + \lambda x_0] = \frac{\lambda}{1+\lambda}x_1 + \frac{1}{1+\lambda}[(1-\lambda)x + \lambda x_1],$$

$$\frac{1-\lambda}{1+\lambda}x + \frac{2\lambda}{1+\lambda}x_0 = \frac{1-\lambda}{1+\lambda}x + \frac{2\lambda}{1+\lambda}x_1, \quad 0 < \lambda < \frac{2\lambda}{1+\lambda} < 1, \quad \frac{2\lambda}{1+\lambda} \in \Lambda.$$

Define inductively

$$\lambda_n \in \Lambda \text{ by } \lambda_0 = \lambda, \lambda_{n+1} = \frac{2\lambda_n}{1+\lambda_n}, \quad n \in \mathbb{N}.$$

to get a strictly increasing subsequence of Λ. Call $\mu \in]0,1]$ its limit, $\mu = 2\mu/(1+\mu)$, hence $\mu = 1$. This proves that $[0,1[$ is contained in Λ and so

$$(1-\lambda)x + \lambda x_0 = (1-\lambda)x + \lambda x,$$

for all $\lambda \in [0,1[$ iff this equality is true for some $\lambda \in [0,1[$. We secondly prove the proposition in the general case. Let $\lambda_0, \lambda_1 \in I, \lambda_0 < \lambda_1, f_0(\lambda_0) =$

$f_1(\lambda_0), f_0(\lambda_1) = f_1(\lambda_1)$. By convexity, we have $f_0(\lambda) = f_1(\lambda)$ for all $\lambda_0 \leq \lambda \leq \lambda_1$. We claim that this equality is true for all $\lambda \in I^o, \lambda \geq \lambda_1$. There is nothing to prove if λ_1 is the largest extremity of I. Assume then $\lambda_1 \in I^o$. Fix $\mu \in I, \mu > \lambda$. Set $x = f_0(x_0) = f_1(\lambda_0), x_0 = f_0(\mu), x_1 = f_1(\mu)$. We have

$$\lambda_1 = (1-\Theta)\lambda_0 + \Theta\mu \quad \text{with} \quad \Theta = \frac{\lambda_1 - \lambda_0}{\mu - \lambda_0} \in \mathbb{J},$$
$$(1-\Theta)x + \Theta x_0 = (1-\Theta)x + \Theta x_1 \quad \text{because} \quad f_0(\lambda_1) = f_1(\lambda_1),$$
$$(1-\sigma)x + \sigma x_0 = (1-\sigma)x + \sigma x_1 \text{ for all } \sigma \in [0,1[\text{ (by the first part)},$$
$$f_0[(1-\sigma)\lambda_0 + \sigma\mu] = f_1[(1-\sigma)\lambda_0 + \sigma\mu] \text{ for all } \sigma \in [0,1[,$$
$$f_0(\nu) = f_1(\nu) \text{ for all } \nu \in [\lambda_0, \mu[,$$
$$f_0(\lambda) = f_1(\lambda) \text{ for all } \lambda \in I^o, \lambda \geq \lambda_1.$$

If we use the convex set map $\lambda \longmapsto -\lambda$, $\lambda \in \mathbb{R}$, we conclude from the preceding case that also $f_0(\lambda) = f_1(\lambda)$ for all $\lambda \in I^o, \lambda \leq \lambda_0$. ☑

9. Lemma. *Consider the following statements:*

(I) *A map $f : X \to Y$ between affine spaces is an affine space map iff it is a convex set map.*

(I*) *If $f : X \to Y$ is a convex set map between affine spaces, then $f(X)$ is an affine subspace of Y.*

(II) *A convex subset X of an affine space Y has an affine space structure inducing on X the convex set structure induced on X by Y iff X is an affine subspace of Y. Such an affine space structure on X is unique, namely equal to that induced on X by Y.*

(II*) *A convex subset X of an affine space Y has an affine space structure inducing on X the convex set structure induced on X by Y iff X is an affine subspace of Y.*

(III) *Two affine space structures on a set X induce the same convex set structure on X iff they are equal.*

Then (I), (I), (II), (II*) are equivalent, and they imply (III).*

Proof. Clearly (I) $\Longrightarrow$(I*) $\Longrightarrow$ (II*), (I) $\Longrightarrow$ (II) $\Longrightarrow$ (II*), (II) $\Longrightarrow$ (III). Let us prove that (II*) $\Longrightarrow$ (I). Necessity in (I) is clear. We see sufficiency as follows. Assume that $f : X \to Y$ is a convex set map. The graph G of f is a convex subset of $X \times Y$. The bijective convex set map $x \in X \leftrightarrow (x, f(x)) \in G$ transfers the affine space structure on X to an affine space structure on G inducing on G the convex set structure induced on G by $X \times Y$. Apply the necessity part of (II*) to G and $X \times Y$ to conclude that G is an affine subspace of $X \times Y$. Hence $f : X \to Y$ is an affine space map. ☑

10. Comment. (II*) is a tautology when $X = Y$. Even so, (II*) $\Longrightarrow$ (III) may be seen as follows. Only necessity in (III) needs to be proved. Let a set X have two affine space structures inducing the same convex set structures on X. Denote X by X_1 and X_2 when endowed with them. X has the same convex set structure induced by X_1 and X_2 (it does not matter). The diagonal Δ is a convex subset of X^2, hence of the affine space $X_1 \times X_2$. The bijective convex set map $x \in X \leftrightarrow (x, x) \in \Delta$ transfers the affine space structure on X_i $(i = 1, 2)$ to an affine space structure on Δ inducing on Δ the convex set structure induced on Δ by $X_1 \times X_2$. It follows from (II*) and a choice $i = 1$ or $i = 2$ (it does not matter) that Δ is an affine subspace of $X_1 \times X_2$. Hence, if $\lambda \in \mathbb{R}$, $x_1, x_2 \in X$, we have

$$(1-\lambda)(x_1, x_2) + \lambda(x_2, x_2) \in \Delta(X_1 \times X_2),$$
$$((1-\lambda)x_1 + \lambda x_2(X_1), (1-\lambda)x_1 + \lambda x_2(X_2)) \in \Delta,$$
$$(1-\lambda)x_1 + \lambda x_2(X_1) = (1-\lambda)x_1 + \lambda x_2(X_2),$$

where $(X_1 \times X_2), (X_1), (X_2)$ mean affine combinations in the sense of the product $X_1 \times X_2, X_1, X_2$ respectively. Thus $X_1 = X_2$. Since (II) = (II*) $\cap$ (III) and we just saw that (II*) $\Longrightarrow$ (III), then (II*) $\Longrightarrow$ (II). We may also prove (II*) $\Longrightarrow$ (II) by the reasoning used in showing (II*) $\Longrightarrow$ (I) in the proof of Lemma 9, once (I) $\Longrightarrow$ (II).

11. Proposition.

(I) *A map $f : X \to Y$ between affine spaces is an affine space map iff it is a convex set map.*

(II) *A convex subset X of an affine space Y has an affine space structure inducing on X the convex set structure induced on X by Y iff X is an affine subspace of Y. Such an affine space structure on X is unique, namely equal to that induced on X by Y.*

(III) *Two affine space structures on a set X induce the same convex set structure on X iff they are equal.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume $f : X \to Y$ to be a convex set map. Introduce

$$x_0 \in X, \quad x_1 \in X, \quad y_0 = f(x_0) \in Y, \quad y_1 = f(x_1) \in Y,$$
$$x_\lambda = (1-\lambda)x_0 + \lambda x_1 \in X, \quad y_\lambda = (1-\lambda)y_0 + \lambda y_1 \in Y,$$

where $\lambda \in \mathbb{R}$. We want to prove

$$f(x_\lambda) = y_\lambda. \tag{1}$$

This is true for $\lambda \in \mathbb{I}$, because f is a convex set map. Assume $\lambda > 1$. If $\mu, \nu \in \mathbb{R}$, we have

$$(1-\mu)x_0 + \mu[(1-\nu)x_0 + \nu x_1] = (1-\mu\nu)x_0 + \mu\nu x_1$$

provided $\mu\nu \neq 1$. It results that we have in X, likewise in Y by a similar computation

$$(1-\mu)x_0 + \mu x_\nu = x_{\mu\nu}, \tag{2}$$

$$(1-\mu)y_0 + \mu y_\nu = y_{\mu\nu}, \tag{3}$$

provided $\mu\nu \neq 1$. Assume $\nu > 1, 0 < 1/\nu < 1$. We have

$$f(x_{\mu\nu}) = y_{\mu\nu}, \tag{4}$$

provided $0 \leq \mu \leq 1/\nu$, because $0 \leq \mu\nu \leq 1$, $\mu\nu \in \mathbb{I}$ and f is a convex set map. Thus

$$f[(1-\mu)x_0 + \mu x_\nu] = (1-\mu)y_0 + \mu y_\nu, \tag{5}$$

if $0 \leq \mu < 1/\nu$, by (2), (3) and (4), because then $\mu\nu \neq 1$. Besides that,

$$f[(1-\mu)x_0 + \mu x_\nu] = (1-\mu)y_0 + \mu f(x_\nu), \tag{6}$$

for $\mu \in \mathbb{I}$, because f is a convex set map. So

$$(1-\mu)y_0 + \mu f(x_\nu) = (1-\mu)y_0 + \mu y_\nu, \tag{7}$$

for $0 \leq \mu < 1/\nu$, by (5) and (6). Proposition 8 implies that (7) holds for $0 \leq \mu < 1$. Hence

$$f[(1-\mu)x_0 + \mu x_\nu] = (1-\mu)y_0 + \mu y_\nu, \tag{8}$$

for $0 \leq \mu < 1$, by (7) because f is a convex set map. Therefore (4) holds provided $0 \leq \mu < 1, \mu\nu \neq 1$, because of (2), (3) and (8), where $\nu > 1$. Actually (4) is true also if $\mu\nu = 1$. Fix $\nu \in \mathbb{J}$. Given any $\lambda > 1$, define $\nu = \lambda/\mu$. Then $\lambda = \mu\nu, \mu\nu \neq 1, \nu > 1$ since $0 < \mu < 1 < \lambda$. Therefore (4) shows that (1) holds for every $\lambda > 1$, as wanted. Assume $\lambda < 0$. By using the parameter change $\lambda \in \mathbb{R} \mapsto (1-\lambda) \in \mathbb{R}$, we reduce ourselves to the preceding case with x_0 and x_1 interchanged. Thus (I) is proved.

(II) follows obviously from (I), and (III) results clearly from (ii). ☑

12. Comment. It is easily seen that Proposition 11 becomes false if we replace affine space and affine subspace by convex cone and convex subcone, or by affine cone and affine subcone, respectively.

13. Lemma. *Consider the following statements:*

(I) *A map $f : X \to Y$ between affine cones is an affine cone map iff it is a convex cone map.*

(I*) *If $f : X \to Y$ is a convex cone map between affine cones, then $f(X)$ is an affine subcone of Y.*

(II) *A convex subcone X of an affine cone Y has an affine cone structure inducing on X the convex cone structure induced on X by Y iff X is an affine subcone of Y. Such an affine cone structure on X is unique, namely equal to that induced on X by Y.*

(II*) *A convex subcone X of an affine cone Y has an affine cone structure inducing on X the convex cone structure induced on X by Y iff X is an affine subcone of Y.*

(III) *Two affine cone structures on a set X induce the same convex cone structure on X iff they are equal.*

Then (I), (I), (II), (II*) are equivalent, and they imply (III).*

Proof. Analogous to that of Lemma 9. ☑

14. Comment. We have (II*) $\implies$ (III). Since (II) $\implies$ (II*) $\cap$ (III), then (II*) $\implies$ (II). We may also prove (II*) $\implies$ (II) by the reasoning used in showing (II*) $\implies$ (i) in the proof of Lemma 13, once (i) $\implies$ (II). All this is analogous to Comment 10.

15. Proposition.

(I) *A map $f : X \to Y$ between affine cones is an affine cone map iff it is a convex cone map.*

(II) *A convex subcone X of an affine cone Y has an affine cone structure inducing on X the convex cone structure induced on X by Y iff X is an affine subcone of Y. Such an affine cone structure on X is unique, namely equal to that induced on X by Y.*

(III) *Two affine cone structures on a set X induce the same convex cone structure on X iff they are equal.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume $f : X \to Y$ to be a convex cone map. All we need to prove is

$$f(\lambda x) = \lambda f(x) \text{ for } \lambda \in \mathbb{R},\ \lambda < 0,\ x \in X.$$

It is enough to prove it for $\lambda = -1$, that is

$$f(-x) = -f(x). \tag{1}$$

Now, we have

$$(1-\mu)f(x) + \mu[2f(-x)] = f[(1-\mu)x] + f(-2\mu x) = f[(1-3\mu)x] \tag{2}$$

for $\mu \in \mathbb{R}$, $0 < \mu < 1$, $\mu \neq 1/3$, and

$$(1 - \mu)f(x) + \mu[-2f(x)] = (1 - 3\mu)f(x) \tag{3}$$

for $\mu \in \mathbb{R}$, $\mu \neq 1/3$. We then have

$$(1 - \mu)f(x) + \mu[2f(-x)] = (1 - \mu)f(x) + \mu[-2f(x)] \tag{4}$$

for $\mu \in \mathbb{R}$, $0 < \mu < 1/3$, since $1 - 3\mu > 0$ in (1) and (3). Proposition 8 implies that (4) holds for $0 \leq \mu < 1$. Choose $\mu = 2/3$ in (4) to conclude

$$\begin{aligned}(1/3)f(x) + (4/3)f(-x) &= (1/3)f(x) + (-4/3)f(x)\\ f[(1/3)x + (4/3)(-x)] &= -f(x),\end{aligned}$$

proving (1), as wanted. Thus (I) is proved.

(II) follows obviously from (I), and (III) results clearly from (II). ☑

16. Comment. It is easily seen that Proposition 15 becomes false if we replace convex cone and convex subcone by convex set and convex subset, or by affine space and affine subspace, respectively.

17. Lemma.

(I) *A map $f : X \to Y$ between affine cones is an affine cone map iff it is a convex cone map and an affine space map.*

(II) *A convex subcone and affine subspace X of an affine cone Y has an affine cone structure inducing on X the convex cone structure and affine space structure induced on X by Y iff X is an affine subcone of Y. Such an affine cone structure on X is unique, namely equal to that induced on X by Y.*

(III) *Two affine cone structures on a set X induce the same convex cone structure and the same affine space structure on X iff they are equal.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume $f : X \to Y$ to be a convex cone map and an affine space map. All we need to prove is

$$f(\lambda x) = \lambda f(x) \quad \text{for} \quad \lambda \in \mathbb{R}, \quad \lambda < 0, \quad x \in X.$$

It is enough to prove it for $\lambda = -1$, that is

$$f(-x) = -f(x) \text{ for } x \in X. \tag{1}$$

Now, choose $\alpha_1, \alpha_2 \in \mathbb{R}^*_+, \mu \in \mathbb{R}, \mu \neq 0, 1$ so that $(1-\mu)\alpha_1 + \mu\alpha_2 = -1$. Notice that $(1 - \mu)\alpha_1, \mu\alpha_2, (1 - \mu)\alpha_1 + \mu\alpha_2 \in \mathbb{R}^*$, therefore we have

$$\begin{aligned}-1 &= (1 - \mu)\alpha_1 + \mu\alpha_2,\\ -x &= (1 - \mu)\alpha_1 x + \mu\alpha_2 x,\\ f(-x) &= (1 - \mu)f(\alpha_1 x) + \mu f(\alpha_2 x)\\ &= (1 - \mu)[\alpha_1 f(x)] + \mu[\alpha_2 f(x)]\\ &= [(1 - \mu)\alpha_1 + \mu\alpha_2]f(x) = -f(x),\end{aligned}$$

as wanted in (1). Thus (I) is proved. (II) follows obviously from (I), and (III) results clearly from (II). ☑

18. Comment. (I) Lemma 17 is a weakened version of Proposition 15, with a simpler proof. (II) Proposition 11 implies Proposition 15 via Lemma 17. Let us examine sufficiency in the cases (I) of Propositions 11 and 15. Assume $f : X \to Y$ to be a convex cone map between affine cones. Then f is a convex set map. By Proposition 11, (I), f is an affine space map. Lemma 17, (I), implies that f is an affine cone map, thus proving Proposition 15, (I). Likewise for (II) and (III) in both propositions.

19. Proposition.

(I) *A quasiorder R on a convex cone X is compatible with its convex cone structure iff it is compatible with its convex set structure.*

(II) *Let $f : X \to Y$ be a convex set map from a convex cone X onto a convex set Y. There is one and only one convex cone structure on Y such that f is a convex cone map. Such a convex cone structure on Y induces the given convex set structure on Y.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume R compatible with the convex set structure of X. Fix $\lambda \in \mathbb{R}_+^*, x_0, x_1 \in X, x_0 R x_1$. We claim that

$$(\lambda x_0) R (\lambda x_0). \tag{1}$$

To prove it, define

$$u(\mu) = (1-\mu)(\lambda x_0) + \mu(\lambda x_1) \in X \text{ for } \mu \in \mathbb{I}. \tag{2}$$

Choose any $\alpha, \beta \in \mathbb{I}, 0 < \beta - \alpha < \inf(1, 1/\lambda)$. We cannot have $\alpha = 0, \beta = 1$ at the same time. Define $\Theta = (\beta - \alpha)\lambda \in \mathbb{J}$. Introduce

$$w = \frac{(1-\beta)(\lambda x_0) + \alpha(\lambda x_1)}{1 - \Theta} \in X, \tag{3}$$

which is a convex cone combination of x_0 and x_1 since we cannot have $1 - \beta = 0, \alpha = 0$ at the same time, and $1 - \Theta \neq 0$. Then (2) and (3) give

$$u(\alpha) = (1-\Theta)w + \Theta x_0, \tag{4}$$

$$u(\alpha) = (1-\Theta)w + \Theta x_1. \tag{5}$$

Therefore, (4) and (5) imply $u(\alpha) R u(\beta)$, by $x_0 R x_1$ and compatibility of R with the convex set structure of X (to visualize, draw a picture in a plane, by plotting $x_0, x_1, \lambda x_0, \lambda x_1, w, u(\alpha), u(\beta)$). Next, choose $n \in \mathbb{N}^*, n > \sup(1, \lambda)$.

Define $\gamma_i = i/n \in \mathbb{I}$ for $i = 0, \ldots, n$. We have $0 < \gamma_i - \gamma_{i-1} = 1/n < \inf(1, 1/\lambda)$ for $i = 1, \ldots, n$. Hence $u(\gamma_{i-1}) R u(\gamma_i)$ for $i = 1, \ldots, n$, by the preceding case. Transitivity of R implies $u(\gamma_0) R u(\gamma_n)$, that is $u(0) R u(1)$, hence (1) as wanted. Finally, fix $t_0, t_1, x_0, x_1 \in X$ to $R x_0, t_1 R x_1$. We claim that

$$(t_0 + t_1) R (x_0 + x_1). \tag{6}$$

To prove it, notice that

$$[(t_0 + t_1)/2] R (x_0 + x_1)/2]$$

by compatibility of R with the convex set structure of X. Use the preceding result and multiply both sides by 2 to get (6). Thus (I) is proved. (II) Existence in the first assertion of (II) is easily seen to be equivalent to sufficiency in (I) when R is an equivalence relation. Uniqueness is obvious. The second assertion of (II) is clear. ☑

20. Proposition.

(I) *An equivalence relation R on an affine space X is compatible with its affine space structure if it is compatible with its convex set structure.*

(II) *Let $f : X \to Y$ be a convex set map from an affine space X onto a convex set Y. There is one and only one affine space structure on Y such that f is an affine space map. Such an affine space structure on Y induces the given convex set structure on Y.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume R compatible with the convex set structure of X. It is enough to prove the following. Fix $\lambda \in \mathbb{R}, t, x_0, x_1 \in X, x_0 R x_1$. We claim that

$$[(1-\lambda)t + \lambda x_0] R [(1-\lambda)t + \lambda x_1]. \tag{1}$$

This is true for $\lambda \in \mathbb{I}$, because R is compatible with the convex set structure of X. To prove the claim (1), we turn to the cases $\lambda > 1$ and $\lambda < 0$; hence $\lambda \neq 0, 1$. Define

$$u_0(\lambda) = (1-\lambda)t + \lambda x_0 \in X, \tag{2}$$

$$u_1(\lambda) = (1-\lambda)t + \lambda x_1 \in X, \tag{3}$$

so that our claim (1) becomes

$$u_0(\lambda) R u_1(\lambda). \tag{4}$$

Consider

$$u(\mu) = (1-\mu) u_0(\lambda) + \mu u_1(\lambda) \in X, \tag{5}$$

for $\mu \in \mathbb{R}$. Choose any $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \beta \neq 0$, so that $\Theta = (\beta - \alpha)\lambda \in \mathbb{J}$, hence $\Theta \neq 1$. Introduce

$$(6) \qquad w = \frac{(1-\lambda)t + (1-\beta)\lambda x_0 + \alpha\lambda x_1}{1-\Theta} \in X,$$

which is an affine combination of t, x_0 and x_1 since $(1-\lambda) + (1-\beta)\lambda + \alpha\lambda = 1 - \Theta \neq 0$. Then (2), (3), (5) and (6) imply

$$(7) \qquad u(\alpha) = (1-\Theta)w + \Theta x_0 \text{ because } \lambda \neq 0, 1, \alpha \neq 1,$$

$$(8) \qquad u(\beta) = (1-\Theta)w + \Theta x_1 \text{ because } \lambda \neq 0, 1, \beta \neq 0.$$

Suppose firstly $\lambda > 1$. Then $\Theta \in \mathbb{J}$ amounts to $0 < \beta - \alpha < 1/\lambda$. Therefore, (7) and (8) imply $u(\alpha)Ru(\beta)$, by x_0Rx_1 and compatibility of R with the convex set structure of X, if $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \beta \neq 0, 0 < \beta - \alpha < 1/\lambda$ (to visualize, draw a picture in a plane, by plotting $t, x_0, x_1, u_0(\lambda), u_1(\lambda), w, u(\alpha), u(\beta)$). Next, choose $n \in \mathbb{N}^*, n > \lambda$. Define $\gamma_i = i/n \in \mathbb{I}$ for $i = 0, \ldots, n$. We have $0 < \gamma_i - \gamma_{i-1} = 1/n < 1/\lambda, \gamma_{i-1} \neq 1, \gamma_i \neq 0$, for $i = 1, \ldots, n$. Hence $u(\gamma_{i-1})Ru(\gamma_i)$ for $i = 1, \ldots n$ by the preceding case. Transitivity of R implies $u(\gamma_0)Ru(\gamma_n)$, that is $u(0)Ru(1)$, hence (4) as claimed. Suppose secondly $\lambda < 0$. This case is treated similarly, except for some changes (it does not result from the case $\lambda > 1$ by the usual parameter change $\lambda \in \mathbb{R} \mapsto 1 - \lambda \in \mathbb{R}$). Then $\Theta \in \mathbb{J}$ amounts to $0 < \alpha - \beta < 1/(-\lambda)$. Therefore, (7) and (8) imply $u(\alpha)Ru(\beta)$, by x_0Rx_1 and compatibility of R with the convex set structure of X, if $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \beta \neq 0, 0 < \alpha - \beta < 1/(-\lambda)$ (to visualize, draw a picture in a plane, by plotting $t, x_0, x_1, u_0(\lambda), u_1(\lambda), w, u(\alpha), u(\beta)$). Let, more generally, $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \beta \neq 0, \beta < \alpha$. There are $n \in \mathbb{N}^*, \gamma_0 = \beta < \gamma_1 < \cdots < \gamma_{n-1} < \gamma_n = \alpha$ such that $\gamma_i \neq 1, \gamma_{i-1} \neq 0, 0 < \gamma_i - \gamma_{i-1} < 1/(-\lambda)$ for $i = 1, \ldots, n$. Hence $u(\gamma_i)Ru(\gamma_{i-1})$ for $i = 1, \ldots, n$, by the preceding case. Transitivity of R implies $u(\gamma_n)Ru(\gamma_0)$, hence $u(\alpha)Ru(\beta)$ if $\alpha, \beta \in \mathbb{R}, \quad \alpha \neq 1, \quad \beta \neq 0, \quad \beta < \alpha$. Fix $\rho, \sigma \in \mathbb{R}$ so that $\rho < 0 < 1 < \sigma$. If we apply the conclusion we just get to the choices $\alpha = 0, \beta = \rho$, and $\alpha = \sigma, \beta = \rho$, and $\alpha = \sigma, \beta = 1$, we have

$$\begin{aligned} &u(0)Ru(\rho) \quad \text{since} \quad 0 \neq 1, \quad \rho \neq 0, \quad \rho < 0, \\ &u(\sigma)Ru(\rho) \quad \text{since} \quad \sigma \neq 1, \quad \rho \neq 0, \quad \rho < \sigma, \\ &u(\sigma)Ru(1) \quad \text{since} \quad \sigma \neq 1, \quad 1 \neq 0, \quad \rho < \sigma. \end{aligned}$$

By symmetry and transitivity of R, we have $u(0)Ru(1)$, hence (4) as claimed. Thus (I) is proved. (II) Existence in the first assertion of (II) is easily seen to be equivalent to sufficiency in (I). Uniqueness is obvious. The second assertion of (II) is clear. ☑

21. Comment. Although Proposition 19, (I) is true for a quasiorder, hence an equivalence relation, it is easily seen that Proposition 20, (I) is false for a quasiorder.

22. Proposition.

(I) *An equivalence relation R on an affine cone X is compatible with its affine cone structure iff it is compatible with its convex set structure.*

(II) *Let $f : X \to Y$ be a convex set map from an affine cone X onto a convex set Y. There is one and only one affine cone structure on Y such that f is an affine cone map. Such an affine cone structure induces the given convex set structure on Y.*

Proof. (I) Necessity is clear. Let us prove sufficiency. Assume R compatible with the convex set structure of X. By Proposition 19, (I), we see that R is compatible with the convex cone structure of X. All we need to prove is

$$(\lambda x_0)R(\lambda x_1) \text{ for } \lambda \in \mathbb{R}, \lambda < 0, x_0, x_1 \in X, x_0 R x_1.$$

It is enough to prove it for $\lambda = -1$, that is

(1) $$(-x_0)R(-x_1) \text{ for } x_0, x_1 \in R, x_0 R x_1.$$

To this goal, define

(2) $$u(\mu) = (1-\mu)(-x_0) + \mu(-x_1) \text{ for } \mu \in \mathbb{R}.$$

Choose any $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \beta \neq 0$ so that $\Theta = \alpha - \beta \in \mathbb{J}$. We cannot have $\alpha = 0, \beta = 1$ at the same time. Introduce

$$w = \frac{(1-\beta)(-x_0) + \alpha(-x_1)}{1-\Theta} \in X,$$

which is an affine cone combination of $-x_0$ and $-x_1$ since we cannot have $1-\beta = 0, \alpha = 0$ at the same time, and $1 - \Theta \neq 0$. Then (2) and (3) give

(4) $$u(\alpha) = (1-\Theta)w + \Theta x_0 \text{ because } \alpha \neq 1,$$

(5) $$u(\beta) = (1-\Theta)w + \Theta x_0 \text{ because } \beta \neq 0.$$

Therefore, (4) and (5) imply $u(\alpha)Ru(\beta)$, by $x_0 R x_1$ and compatibility of R with the convex set structure of X (to visualize, draw a picture in a plane by plotting $x_0, x_1, -x_0, -x_1, w, u(\alpha), u(\beta)$). Fix $\rho, \sigma, \tau \in \mathbb{R}$ so that $\rho < 0 < \sigma < 1 < \tau, -\rho < 1, \sigma - \rho < 1, \tau - \sigma < 1$, hence $\tau - 1 < 1$. If we apply the conclusion we just got to the choice $\alpha = 0, \beta = \rho$, and $\alpha = \sigma, \beta = \rho$, and $\alpha = \tau, \beta = \sigma$, and $\alpha = \tau, \beta = 1$, we have

$$u(0)Ru(\rho) \text{ since } 0 \neq 1, \quad \rho \neq 0, \quad 0 < \rho - 0 < 1,$$

$$u(\sigma)Ru(\rho) \text{ since } \sigma \neq 1, \quad \rho \neq 0, \quad 0 < \sigma - \rho < 1,$$

$$u(\tau)Ru(\sigma) \text{ since } \tau \neq 1, \quad \sigma \neq 0, \quad 0 < \tau - \sigma < 1,$$

$$u(\tau)Ru(1) \text{ since } \tau \neq 1, \quad 1 \neq 0, \quad 0 < \tau - 1 < 1.$$

By symmetry and transitivity of R, we have $u(0)Ru(1)$, hence (1) as claimed. Thus (I) is proved. (II) Existence in the first assertion of (II) is easily seen to be equivalent to sufficiency in (I). Uniqueness is obvious. The second assertion in (II) is clear. ☑

23. Comment. Although Proposition 19, (I) is true for a quasiorder, hence an equivalence relation, it is easily seen that Proposition 22, (I) is false for a quasiorder.

24. Comment. Proposition 22 clearly remains true if we replace convex set by convex cone or affine space.

25. Lemma. *An equivalence relation R on an affine cone X is compatible with its affine cone structure iff it is compatible with the convex cone structure and the affine space structure induced on X by the affine cone structure of X.*

Proof. Necessity is clear. Let us prove sufficiency. Assume R compatible with the convex cone structure and the affine space structure induced on X by the affine cone structure of X. All we need is to prove

$$(\lambda x_0)R(\lambda x_1) \text{ for } \lambda \in \mathbb{R}, \quad \lambda < 0, \quad x_0, x_1 \in X, \quad x_0 R x_1.$$

It is enough to prove it for $\lambda = -1$, that is

(1) $$(-x_0)R(-x_1) \text{ for } x_0, x_1 \in X, x_0 R x_1.$$

Now, choose $\alpha_1, \alpha_2 \in \mathbb{R}_+^*, \mu \in \mathbb{R}, \mu \neq 0, 1$ so that $(1-\mu)\alpha_1 + \mu\alpha_2 = -1$. Notice that $(1-\mu)\alpha_1, \mu\alpha_2, (1-\mu)\alpha_1 + \mu\alpha_2 \in \mathbb{R}^*$, therefore we have

$$\begin{aligned}
&x_0 R x_1,\\
&(\alpha_1 x_0)R(\alpha_1 x_1), (\alpha_2 x_0)R(\alpha_2 x_1),\\
&[(1-\mu)(\alpha_1 x_0) + \mu(\alpha_2 x_0)]R[(1-\mu)(\alpha_1 x_1) + \mu(\alpha_2 x_1)],\\
&\{[(1-\mu)\alpha_1 + \mu\alpha_2]x_0\}R\{[(1-\mu)\alpha_1 + \mu\alpha_2]x_1\},
\end{aligned}$$

proving (1). ☑

26. Comment. (I) We can prove Proposition 22, (I) by using Propositions 19, (I) and 20, (I) as follows. Necessity is clear. Let us prove sufficiency. Assume R compatible with the convex set structure of X. By Proposition 19, (I), we see that R is compatible with the convex cone structure of X. By Proposition 20, (I), we see that R is compatible with the affine space structure of X. Apply sufficiency in Lemma 25 to conclude that R is compatible with the affine cone structure of X. (II) We could prove Proposition 22, (II) by using Propositions 19, (II) and 20 (II) as follows. By Proposition 19, (II), there is a convex cone structure on Y such that f is a convex cone map. By Proposition 20, (II), there is affine space structure on Y such that f is an affine space map. This convex cone structure on Y and this affine space structure on Y induce the same convex set structure on Y, namely the convex set structure given on Y. By Proposition 28 below, there is an affine cone structure on Y inducing that convex cone structure on Y and that affine space structure on Y. Since f is a convex cone map and an affine space map, as said precedingly, Lemma 17, (I) shows that f is an affine cone map. At this point, we could also use Proposition 15, (I).

4. Compatible convex cones and affine spaces are affine cones and viceversa

27. Definition. A convex cone structure and an affine space structure on a set X are said to be *compatible* when they induce the same convex set structure on X.

28. Proposition.

(I) *An affine cone X is a convex cone and an affine space in a compatible way. Conversely, if X is a convex cone and an affine space in a compatible way, there is one and only one affine cone structure on X inducing the given convex cone structure on X. This affine cone structure also induces on X the given affine space structure on X.*

(II) *A convex cone X has an affine cone structure inducing the given convex cone structure on X iff it has a compatible affine space structure. Then, such an affine cone structure and an affine space structure on X are unique, and the affine cone structure induces the affine space structure.*

(III) *An affine space X has an affine cone structure inducing the given affine space structure on X iff it has a compatible convex cone structure. Then, such an affine cone structure induces that convex cone structure. The injective map that to an affine cone structure on X associates the induced convex cone structure on X is bijective between all affine cone structures on X inducing the given affine space structure on X and all convex cone structures on X that are compatible with the given affine space structure on X.*

Proof. (I) The first statement of (I) is trivial. Let us prove its remaining parts. Assume that X is a convex cone and an affine space in a compatible way. We shall split the proof into eight claims.

Claim 1. If $n \in \mathbb{N}^*, \alpha_1, \dots, \alpha_n \in \mathbb{R}_+^*, \lambda_1, \dots, \lambda_n \in \mathbb{R}, \lambda_1 + \dots + \lambda_n = 1, \lambda_1\alpha_1 + \dots + \lambda_n\alpha_n > 0, x \in X$, then

$$\lambda_1(\alpha_1 x) + \dots + \lambda_n(\alpha_n x) = (\lambda_1\alpha_1 + \dots + \lambda_n\alpha_n)x, \tag{1}$$

where we have an affine combination of convex cone products in the left hand side, and a convex cone product in the right hand side. It is enough to prove

(1) for $n = 2$ (it being a triviality for $n = 1$), since the cases $n = 2$ and $n \geq 2$ together imply the case $n + 1$. The claim for $n = 2$ is that, if $x_1, \alpha_2 \in \mathbb{R}_+^*$, $\lambda \in \mathbb{R}$, $(1-\lambda)\alpha_1 + \lambda\alpha_2 > 0$, $x \in X$, then

$$(2) \qquad (1-\lambda)(\alpha_1 x) + \lambda(\alpha_2 x) = [(1-\lambda)\alpha_1 + \lambda\alpha_2]x.$$

If $\alpha_1 = \alpha_2$, then (2) is trivially true. By compatibility, (2) is true for $\lambda \in \mathbb{I}$. We shall consider the cases $\lambda > 1$ and $\lambda < 0$ separately. Define

$$(3) \qquad u(\lambda) = (1-\lambda)(\alpha_1 x) + \lambda(\alpha_2 x) \text{ for } \lambda \in \mathbb{R}.$$

Firstly, assume $1 < \lambda < \rho$, where

$$\rho = \frac{\alpha_1}{\alpha_1 - \alpha_2} > 1 \text{ if } \alpha_1 > \alpha_2, \quad \rho = +\infty \text{ if } \alpha_1 \leq \alpha_2.$$

Notice that, if $\lambda \geq 0$, then

$$(1-\lambda)\alpha_1 + \lambda\alpha_2 > \text{ iff } \lambda < \rho.$$

Fix $\mu \in \mathbb{R}, 1 < \mu < \rho$. We have

$$\begin{aligned}(1-\theta)(\alpha_1 x) + \theta u(\mu) &= (1-\theta)(\alpha_1 x) + \theta[(1-\mu)(\alpha_1 x) + \mu(\alpha_2 x)] \\ &= (1-\theta\mu)(\alpha_1 x) + (\theta\mu)(\alpha_2 x),\end{aligned}$$

for $\theta \in \mathbb{R}$, provided $\theta \neq 1/\mu$ (notice $\mu \neq 0$). Therefore

$$(4) \qquad (1-\theta)(\alpha_1 x) + \theta u(\mu) = [(1-\theta\mu)\alpha_1 + (\theta\mu)\alpha_2]x,$$

by compatibility, provided $0 \leq \theta\mu \leq 1, \theta \neq 1/\mu$, hence provided $0 \leq \theta < 1/\mu$ since then also $\theta \neq 1/\mu$ (notice $0 < 1/\mu < 1$). Define

$$\begin{aligned}&\alpha = (1-\mu)\alpha_1 + \mu\alpha_2 > 0 \quad (\text{because } 0 \leq \mu < \rho), \\ &v = \alpha x \in X.\end{aligned}$$

We have, for $0 \leq \theta \leq 1$,

$$\begin{aligned}(1-\theta)(\alpha_1 x) + \theta v &= (1-\theta)(\alpha_1 x) + \theta(\alpha x) \\ &= [(1-\theta)\alpha_1 + \theta\alpha]x \quad (\text{by compatibility}).\end{aligned}$$

Therefore, for $0 \leq \theta \leq 1$,

$$(5) \qquad (1-\theta)(\alpha_1 x) + \theta v = [(1-\theta\mu)\alpha_1 + (\theta\mu)\alpha_2]x.$$

Hence, in view of (4) and (5),

$$(6) \qquad (1-\theta)(\alpha_1 x) + \theta u(\mu) = (1-\theta)(\alpha_1 x) + \theta v,$$

provided $0 \leq \theta < 1/\mu$. By Proposition 8, we see that (6) remains true for $0 \leq \theta < 1$. Thus, (4) remains true for $0 \leq \theta < 1$, by comparison between (5) and (6). Now, given any $\lambda \in \mathbb{R}, 1 < \lambda < \rho$, fix $\mu \in \mathbb{R}, 1 < \mu < \rho$. If $\theta = \lambda/\mu \in \mathbb{J}$, we have

$$(7) \qquad u(\lambda) = (1-\theta)(\alpha_1 x) + \theta u(\mu)$$

because

$$\begin{aligned}(1-\theta)(\alpha_1 x) + \theta u(\mu) &= (1-\theta)(\alpha_1 x) + \theta[(1-\mu)(\alpha_1 x) + \mu(\alpha_2 x)] \quad (\text{by } (3)) \\ &= (1-\theta\mu)(\alpha_1 x) + (\theta\mu)(\alpha_2 x) \quad (\text{by } \theta\mu = \lambda \neq 1) \\ &= (1-\lambda)(\alpha_1 x) + \lambda(\alpha_2 x) = u(\lambda) \quad (\text{by } (9)).\end{aligned}$$

By the preceding argument and the validity of (4) for $0 \leq \theta < 1$, we see that (3), (4) and (7) imply (2) because $\theta\mu = \lambda$, as wanted. Secondly, in the case $\lambda < 0$, by using the parameter change $\lambda \in \mathbb{R} \mapsto 1 - \lambda \in \mathbb{R}$, we reduce ourselves to the preceding case with α_1 and α_2 interchanged. Thus Claim 1 is proved.

Corollary to Claim 1. In the notation of Claim 1, if also $\lambda \in \mathbb{R}_+^*$, we have

$$\lambda[\lambda_1(\alpha_1 x) + \cdots + \lambda_n(\alpha_n x)] = \lambda_1(\lambda\alpha_1 x) + \cdots + \lambda_n(\lambda\alpha_n x).$$

Claim 2. We have

$$\begin{aligned}[(1-\lambda)(\alpha_1 x_1) + \lambda(\alpha_2 x_1)] + [(1-\lambda)(\alpha_1 x_2) + \lambda(\alpha_2 x_2)] \\ = (1-\lambda)[\alpha_1(x_1 + x_2)] + \lambda[\alpha_2(x_1 + x_2)],\end{aligned}$$

for $\lambda \in \mathbb{R}, \alpha_1, \alpha_2 \in \mathbb{R}_+^*, x_1, x_2 \in X$. Indeed, if $\mu \in \mathbb{R}$, we have

$$\begin{aligned}(1-\mu)&[(1-\lambda)(\alpha_1 x_1) + \lambda(\alpha_2 x_1)] + \mu[(1-\lambda)(\alpha_1 x_2) + \lambda(\alpha_2 x_2)] \\ &= (1-\mu)(1-\lambda)(\alpha_1 x_1) + (1-\mu)\lambda(\alpha_2 x_1) + \mu(1-\lambda)(\alpha_1 x_2) + \mu\lambda(\alpha_2 x_2) \\ &= (1-\lambda)[(1-\mu)(\alpha_1 x_1) + \mu(\alpha_1 x_2)] + \lambda[(1-\mu)(\alpha_2 x_1) + \mu(\alpha_2 x_2)],\end{aligned}$$

by affine space rules. Choose $\mu = 1/2$. By compatibility, if we multiply by 2 and use Corollary to Claim 1, we get Claim 2.

Claim 3. We have

$$\lambda x + [(1-\mu)(\alpha_1 x) + \mu(\alpha_2 x)] = [\lambda + (1-\mu)\alpha_1 + \mu\alpha_2]x$$

for $\lambda, \alpha_1, \alpha_2 \in \mathbb{R}_+^*, \mu \in \mathbb{R}, x \in X$, provided $\lambda + (1-\mu)\alpha_1 + \mu\alpha_2 > 0$. Indeed, if $\nu \in \mathbb{R}$, we have

$$\begin{aligned}(1-\nu)&(\lambda x) + \nu[(1-\mu)(\alpha_1 x) + \mu(\alpha_2 x)] \\ &= (1-\nu)(\lambda x) + \nu(1-\mu)(\alpha_1 x) + \nu\mu(\alpha_2 x) \\ &= [(1-\nu)\lambda + \nu[(1-\mu)\alpha_1 + \nu\mu\alpha_2]x,\end{aligned}$$

by affine space rules and Claim 1, provided $(1-\nu)\lambda + \nu(1-\mu)\alpha_1 + \nu\mu\alpha_2 > 0$, which is the case if we choose $\nu = 1/2$. By compatibility, if we multiply by 2, we get Claim 3.

Claim 4. We shall construct and affine cone structure on X enjoying the properties asserted in the second and third parts of (I). Addition and multiplication

$$\begin{aligned}(x_1, x_2) \in X \times X &\longmapsto x_1 + x_2 \in X, \\ (\lambda, x) \in \mathbb{R}^* \times X &\longmapsto \lambda x \in X,\end{aligned}$$

for that affine cone structure have to be addition for the given convex cone structure, and an extension of multiplication for the given convex cone structure, respectively. All we need to do is to define $\lambda x \in X$ for $\lambda \in \mathbb{R}^*, \lambda < 0, x \in X$, so as to satisfy the affine cone axioms together with addition and multiplication for the given convex cone structure, as well as to be in accordance with the assertions in the second and third parts of (I). By way of motivation, let us reason as follows. Fix $\alpha_1, \alpha_2 \in \mathbb{R}_+^*, \mu \in \mathbb{R}$, so that

$$(1) \qquad (1-\mu)\alpha_1 + \alpha_2 = 1.$$

Then, we have

$$(2) \qquad \lambda x = (1-\mu)(\lambda\alpha_1 x) + \mu(\lambda\alpha_2 x) \in X,$$

for $\lambda \in \mathbb{R}_+^*, x \in X$, by (1) and Claim 1. It hints at the idea of defining $\lambda x \in X$, where $\lambda \in \mathbb{R}, \lambda < 0, x \in X$, as follows. Fix $\alpha_1, \alpha_2 \in \mathbb{R}_+^*, \mu \in \mathbb{R}$, so that

$$(3) \qquad (1-\mu)\alpha_1 + \mu\alpha_1 = -1.$$

Then, define

$$(4) \qquad (-\lambda)x = (1-\mu)(\lambda\alpha_1 x) + \mu(\lambda\alpha_2 x) \in X,$$

for $\lambda \in \mathbb{R}_+^*, x \in X$. If we do that, as we shall, the first step would be to prove that the value of $(-\lambda)x$ in (4) depends only on λ and x, not on the choice of α_1, α_2 and μ satisfying 3. We have a proof of that single valuedness of $(-\lambda)x$ (using Proposition 8); it is less time consuming not to give it, but rather get it as a byproduct if we proceed as follows. Fix α_1, α_2 and μ satisfying (3), and define $(-\lambda)x \in X$ by (4), even at the risk that it might depend on α_1, α_2 and μ. We then prove that such a definition together with addition and multiplication for the given convex cone satisfy all affine cone axioms. Finally, we check that such an affine cone structure on X enjoys the properties asserted in the second and third parts of (I). Uniqueness in the second part of (I) implies that this affine cone structure on X does not depend on α_1, α_2 and μ.

Claim 5. The first axiom to check is

$$(-\lambda)(x_1 + x_2) = (-\lambda)x_1 + (-\lambda)x_2$$

for $\lambda \in \mathbb{R}_+^*, x_1, x_2 \in X$. By definition and Claim 2, we have

$$\begin{aligned}
&(-\lambda)x_1 = (1-\mu)(\lambda\alpha_1 x_1) + \mu(\lambda\alpha_2 x_1),\\
&(-\lambda)x_2 = (1-\mu)(\lambda\alpha_1 x_2) + \mu(\lambda\alpha_2 x_2),\\
&(-\lambda)x_1 + (-\lambda)x_2 = (1-\mu)[\lambda\alpha_1(x_1+x_2)] + \mu[\lambda\alpha_2(x_1+x_2)] =\\
&(-\lambda)(x_1+x_2),
\end{aligned}$$

as wanted in Claim 5.

Claim 6. The second axiom to check is

$$(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$$

for $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{R}^*, x \in X$. That is clear if $\lambda_1 > 0, \lambda_2 > 0$. Let $\lambda_1 < 0, \lambda_2 < 0$, hence $\lambda_1 = -\mu_1, \lambda_2 = -\mu_2$, where $\mu_1 > 0, \mu_2 > 0$. We have, by definition and Claim 2,

$$\begin{aligned}
(-\mu_1)x &= (1-\mu)[\alpha_1(\mu_1 x)] + \mu[\alpha_2(\mu_1 x)],\\
(-\mu_2)x &= (1-\mu)[\alpha_1(\mu_2 x)] + \mu[\alpha_2(\mu_2 x)],\\
(-\mu_1)x + (-\mu_2)x &= (1-\mu)[\alpha_1(\mu_1+\mu_2)x] + \mu[\alpha_2(\mu_1+\mu_2)x]\\
[-(\mu_1+\mu_2)]x &= (-\mu_1-\mu_2)x,
\end{aligned}$$

as wanted. Finally, let $\lambda_1 > 0, \lambda_2 < 0$ (the case $\lambda_1 < 0, \lambda_2 > 0$ is identical, by symmetry), hence $\lambda_1 = \mu_1, \lambda_2 = -\mu_2$, where $\mu_1 > 0, \mu_2 > 0, \mu_1 \neq \mu_2$ (notice $\lambda_1 + \lambda_2 \neq 0$). Assume firstly $\mu_1 > \mu_2$. We have, by definition and Claim 3,

$$\begin{aligned}
\mu_1 x + (-\mu_2)x &= \mu_1 x + [(1-\mu)(\mu_2\alpha_1 x) + \mu(\mu_2\alpha_2 x)]\\
&= [\mu_1 + (1-\mu)(\mu_2\alpha_1) + \mu(\mu_2\alpha_2)]x\\
&= (\mu_1 - \mu_2)x,
\end{aligned}$$

as wanted, because $\mu_1 + (1-\mu)(\mu_2\alpha_1) + \mu(\mu_2\alpha_2) = \mu_1 - \mu_2 > 0$. Assume secondly $\mu_2 > \mu_1$. Then

$$(\mu_2 - \mu_1)x = \mu_2 x + (-\mu_1)x$$

by the first case. If we multiply both sides by -1, use

$$-(x_1+x_2)=(-x_1)+(-x_2),\quad (-1)(\lambda x)=(-\lambda)x,$$

by Claim 5 and Claim 7 below, respectively, we get

$$(\mu_1-\mu_2)x=(-\mu_2)x+\mu_1 x,$$

as wanted. Thus Claim 6 was checked.

Claim 7. The third axiom to check is

$$\lambda_1(\lambda_2 x)=(\lambda_1\lambda_2)x$$

for $\lambda_1,\lambda_2\in\mathbb{R}^*$, $x\in X$. That is clear, if $\lambda_1>0,\lambda_2>0$. Let $\lambda_1>0,\lambda_2<0$, hence $\lambda_1=\mu_1,\lambda_2=-\mu_2$, where $\mu_1>0,\mu_2>0$. By definition and Corollary to Claim 1, we have

$$\begin{aligned}(-\mu_1\mu_2)x&=(1-\mu)(\mu_1\mu_2\alpha_1 x)+\mu(\mu_1\mu_2\alpha_2 x)\\&=\mu_1[(1-\mu)(\mu_2\alpha_2 x)+\mu(\mu_2\alpha_2 x)]\\&=\mu_1[(-\mu_2)x],\end{aligned}$$

as wanted. Let $\lambda_1<0,\lambda_2>0$, hence $\lambda_1=-\mu_1,\lambda_2=\mu_2$, where $\mu_1>0,\mu_2>0$. We have, by definition

$$\begin{aligned}(-\mu_1\mu_2)x&=(1-\mu)(\mu_1\mu_2\alpha_1 x)+\mu(\mu_1\mu_2\alpha_2 x)\\&=(1-\mu)[\mu_1\alpha_1(\mu_2 x)]+\mu[\mu_1\alpha_2(\mu_2 x)]\\&=(-\mu_1)(\mu_2 x),\end{aligned}$$

as wanted. Finally, let $\lambda_1<0,\lambda_2<0$, hence $\lambda_1=-\mu_1,\lambda_2=-\mu_2$, where $\mu_1>0,\mu_2>0$. By definition, affine space rules, Corollary to Claim 1, and Claim 1, we have

$$\begin{aligned}(-\mu_1)[(-\mu_2)x]&=(1-\mu)\{\mu_1\alpha_1[(-\mu_2)x]\}+\mu\{\mu_1\alpha_2[(-\mu_2)x]\}\\&=(1-\mu)\{\mu_1\alpha_1[(1-\mu)(\mu_2\alpha_1 x)\\&\quad+\mu(\mu_2\alpha_2 x)]\}+\mu\{\mu_1\alpha_2[(1-\mu)(\mu_2\alpha_1 x)+\mu(\mu_2\alpha_2 x)]\}\\&=(1-\mu)^2(\mu_1\mu_2\alpha_1^2 x)+2(1-\mu)\mu(\mu_1\mu_2\alpha_1\alpha_2 x)+\mu^2(\mu_1\mu_2\alpha_2^2 x)\\&=(\mu_1\mu_2)x,\end{aligned}$$

as wanted, because

$$\begin{aligned}&(1-\mu)^2+2(1-\mu)\mu+\mu^2=[(1-\mu)+\mu]^2=1,\\&(1-\mu)^2(\mu_1\mu_2\alpha_1^2)+2(1-\mu)\mu(\mu_1\mu_2\alpha_1\alpha_2)+\mu^2(\mu_1\mu_2\alpha_2^2)\\&=[(1-\mu)\alpha_1+\mu\alpha_2]^2\mu_1\mu_2=\mu_1\mu_2>0.\end{aligned}$$

Thus Claim 7 was checked.

Claim 8. The affine cone structure we constructed on X obviously induces on X the given convex cone structure. By proposition 15, (III), there is at most one affine cone structure on X inducing the given convex cone structure. Hence, the affine cone structure on X we constructed is that unique one. As such, it does not depend on the choice of λ_1, λ_2 and μ satisfying (3) of Claim 4. The convex cone structure given on X is induced by the affine cone structure we constructed on X. It follows that the convex set structure induced on X by that convex cone structure is also induced by that affine cone structure, hence by the affine space structure induced on X by that affine cone structure. This convex cone structure is also induced by the affine space structure given on X. By Proposition 11, (III), these two affine space structures on X coincide. Hence, the affine cone structure we constructed on X induces on X the affine space structure given on X. We have completed the proof of (I). We now turn to (II). Necessity in the first part of (II) is clear. Sufficiency follows from (I). Uniqueness in the case of an affine cone results from Proposition 15, (III). Uniqueness in the case of an affine space results from Proposition 11, (III), because the convex cone structure given on X induces a convex set structure on X which determines uniquely the compatible affine space structure. The final statement in (II) follows from (I). We now turn to (III). Necessity is clear in the first part of (III). Sufficiency in it and the second part of (III) follow from (I). The map considered in the third part of (III) is injective by Proposition 15, (III). The fact that it is bijective between the indicated sets is seen as follows. It is clearly into the set of values that was stated, by using the convex set structure induced on X by the given affine space structure. It is onto that set by (I). ☑

29. Comment. (I) Consider a diagram formed by the set A, B, C, D, and maps $a : A \to B, b : B \to D, c : A \to c, d : C \to D$, that commute in the sense that $ba = dc$. Then

$$a(A) \subset b^{-1}[d(C)], \quad c(A) \subset d^{-1}[b(B)].$$

If $x \in A, u = a(x) \in B, v = c(x) \in C$, then $b(u) = d(v)$. We say that the diagram is *exact* when, conversely, $u \in B, v \in C, b(u) = d(v)$, then $u = a(x), v = c(x)$ for some $x \in A$. It follows that

$$a(A) = b^{-1}[d(C)], \quad c(A) = d^{-1}[b(B)].$$

Conversely, if

$$a(A) = b^{-1}[d(C)]$$

and d is injective, the diagram is exact. (II) Consider next a set X and the diagram formed by the sets $AC(X)$ of all affine cone structures on X, $CC(X)$ of all convex cone structures on X, $AS(X)$ of all affine space structures on

$X, CS(X)$ of all convex set structures on X, and the structure inducing maps $a : AC(X) \to C(X), b : CC(X) \to CS(X), c : AC(X) \to AS(X), d : AS(X) \to CS(X)$, that commute in the sense that $ba = dc$. By Propositions 11, (III) and 15, (III) respectively, d and a are injective. Proposition 28 means that this diagram is exact. We have

$$a[AC(X)] = b^{-1}\{d[AS(X)]\},$$
$$c[AC(X)] = d^{-1}\{b[CC(X)]\}.$$

The first equality means that the set of all convex cone structures on X that are induced by affine cone structures on X coincides with the set of all convex cone structures on X that have compatible affine space structures on X. The second equality means that the set of all affine space structures on X that are induced by affine cone structures on X coincides with the set of all affine space structures on X that have compatible convex cone structures on X.

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