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DERIVATION-BOUNDED GROUPS

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ABSTRACT. For some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms. In this paper we present a phenomenon of this type with the classical word problem for finitely presented groups. A presentation of a group is called E_n-*derivation-bounded* (E_n-d.b.), if a function $k \in E_n$ exists which bounds the derivations of the words defining the unit element. For E_n-d.b. presentations a pure combinatorial E_n-algorithm for solving the word problem exists. It is proved that the property of being E_n-d.b. is an invariant of finite presentations, but that the degree of complexity of the pure combinatorial algorithm may be as far as possible from the degree of complexity of the word problem itself.

The complexity of logical theories and of algorithmic problems in algebraic structures has been object of intensive studies during the last years ([Av], [Av-Madl], [Can], [Can-Gat], [Fer-Rac], [Gat], [Madl]). One interesting aspect in the proofs of good lower and upper bounds is the fact that some of these result were achieved not only by using combinatorial methods but also by using algebraic arguments. Even more, for some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms.

In this paper we want to present a phenomenon of this type within the classical word problem for finitely presented groups ([M-K-S]).

Let $\Sigma = \{s_1,\ldots,s_m\}$ be a finite alphabet, $\bar{\Sigma} = \{\bar{s}_1,\ldots,\bar{s}_m\}$ a disjoint copy of Σ ($\bar{s}_i$ is the formal inverse of s_i), $\underline{\Sigma} = \Sigma \cup \bar{\Sigma}$, and $\underline{\Sigma}^*$ the set of words over $\underline{\Sigma}$. For $w = a_1\ldots a_n \in \underline{\Sigma}^*$, $a_i \in \underline{\Sigma}$, let be $w^{-1} = \bar{a}_n\ldots\bar{a}_1$ ($\bar{\bar{s}} = s$), let $n = |w|$ be the length of w, e the empty word, and $L \subset \underline{\Sigma}^*$.

The group G given by the *presentation* $<\Sigma;L>$ can be viewed as the set of equivalence classes of the Thue system

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$$T = (\Sigma; \{w = e | w \in L \cup L^{-1} \cup (s\bar{s}, \bar{s}s: s \in \Sigma\}),$$

where $u \sim v$ if there is a derivation from u to v in T. The set of equivalence classes forms a group with the operations $[u]\cdot[v] = [uv]$ and $[u]^{-1} = [u^{-1}]$, $[e]$ being the unit element.

Σ is the set of *generators*, and L is the set of *defining relators* of this presentation. If Σ is finite, $\langle\Sigma;L\rangle$ is a *finitely generated* (f.g) *presentation* of G, and G is called f.g.. If L is finite, too, then $\langle\Sigma;L\rangle$ is a *finite presentation* of G, and G is *finitely presented* (f.p.).

The *word problem* for the presentation $\langle\Sigma;L\rangle$ of G is the problem of deciding for an arbitrary word $w \in \Sigma^*$ whether w defines the unit element of G or not, i.e. the membership to the set $\{w \in \Sigma^* \mid w \underset{G}{=} e\} = \{w \in \Sigma^* \mid$ there is a derivation from w to e in T$\}$. It is well known that the complexity of the word problem for G is independent of the chosen f.g. presentation for G, and we can speak therefore about the complexity of the word problem *for* G.

We call an algorithm solving the word problem for $\langle\Sigma;L\rangle$ a *natural algorithm* (n.a.) if for $w \underset{G}{=} e$ it produces a derivation $w = w_o \to \ldots \to w_m = e$ in the Thue system T. Of course the length of a produced derivation is a lower bound for the complexity of a n.a..

From each solution of the word problem for $\langle\Sigma;L\rangle$ we can define a n.a. simply by generating all derivation in T for the words w with $w \underset{G}{=} e$, in some ordering.

Some questions concerning the n.a. arise. Does the complexity of any n.a. give information about the complexity of the word problem? Of course, it gives an upper bound, but does it give a lower bound in any way, too? Starting with an algorithm which solves the word problem can we produce a n.a. of the same complexity? Given two presentations of the same group, what is the relation between the complexities of natural algorithms in both presentation?

We introduce the concept of *derivation bounded presentations* to formulate these questions more precisely and also to give the answers. Let K be any complexity class of word functions. We will restrict ourselves to the *Grzegorczyk classes* E_n which are well known ([Weih]). A finite presentation $\langle\Sigma;L\rangle$ is called *K-derivation-bounded* (K-d.b.) if there is a function $k \in K$ such that every word $w \in \Sigma^*$ which defines the unit element of $\langle\Sigma;L\rangle$ can be derived to e in T, within no more than $|k(w)|$ steps.

For a K-d.b. presentation there is always a standard n.a. for solving the word problem. In order to decide for a word $w \in \Sigma^*$ whether $w \underset{G}{=} e$, just produce all possible derivation in T which start with w, of length bounded by $|k(w)|$, and test whether e has been derived. If $K = E_n$ $(n \geq 3)$ this is an E_n-algorithm. In particular the word problem for an E_n-d.b. finite presentation is decidable.

On the other hand if there is a natural E_n-algorithm solving the word problem for <Σ;L> then <Σ;L> is E_n-d.b..

We will prove the following results.

(a) If a f.g. group has an E_n-d.b. finite presentation for some $n \geqslant 1$ then every finite presentation of this group is E_n-d.b.. So the standard n.a. is an E_n-algorithm for all finite presentations of this group, for $n \geqslant 3$.

(b) Every f.g. group G with E_n-decidable word problem ($n \geqslant 3$), and hence any countable group with E_n-decidable word problem ([Ott]), can be embedded into a f.p. group having an E_n-d.b. presentation. This means that a n.a. of the same complexity can effectively be constructed from an algorithm solving the word problem for G, but in general for a larger group only. The restriction of this n.a. solves the word problem for G, but is general it is not a n.a. for G. These two facts give the hope that at least for f.p. E_n-d.b. groups with $n \geqslant 3$ an optimal n.a. exists. But this hope is disappointed by the following fact.

(c) For every $n \geqslant 4$ there is a f.p. E_n-but not E_{n-1}-d.b. group G having an E_3-decidable word problem. So G has no natural E_{n-1}-algorithm for solving the word problem although there is an E_3-algorithm for solving it. Thus the complexity of any n.a. may be as far as possible from the complexity of the word problem. These results show that combinatorial properties of a Thue system are not sufficient to prove good complexity bounds for the word problem. Similar results can be proved for semigroups.

Since there is a f.p. group with E_3-decidable word problem such that none of its finite presentations allows a natural E_3-algorithm, the following question seems to be natural: is there an infinite "easy" presentation of this group for which a natural E_3-algorithm exists?

Of course one could take all relators of the group as defining relators of a presentation, which then trivially is E_0-d.b., since each derivation is of length 1. But such a presentation is not "easy" because the full complexity of the word problem is contained in the defining relators and so in the presentation. Let an *easy* presentation of a group be one for which the set of defining relators is E_1-decidable. Then we have:

(d) Every f.g. group G with E_n-decidable word problem ($n \geqslant 3$) has a f.g. presentation with an E_1-decidable set of defining relators which allows a natural E_n-algorithm for solving the word problem.

Similar questions may be posed for finitely axiomatized (f.a.) theories. Are natural decision algorithms for f.a. theories optimal, or are there easily decidable theories for which the optimal proofs in any finite axiomatization are too long?

1. E_n-DERIVATION-BOUNDED GROUPS.

1.1. DEFINITION. Let $G = \langle\Sigma;L\rangle$ be a group, and let $w \in \Sigma^*$ be such that $w \underset{G}{=} e$.

a) A *derivation from* w is a sequence of words $w = w_o, w_1, \ldots, w_k \equiv e$ from Σ^* such that w_{i+1} is formed by insertion of a word u between any consecutive symbols of w_i, or before w_i, or after w_i, or by deletion of a word u if it forms a block of consecutive symbols of w_i. In both cases u must be a member of $L \cup L^{-1} \cup \{s\bar{s}, \bar{s}s : s \in \Sigma\}$. Here L^{-1} is defined as $\{w^{-1} | w \in L\}$, where $e^{-1} \equiv e$, $(ws)^{-1} \equiv \bar{s}w^{-1}$, $(w\bar{s})^{-1} \equiv sw^{-1}$, and $\equiv$ denotes the identity of the free monoid Σ^*. k is the *length* of this derivation.

b) Let be $n \geqslant 1$, $\langle\Sigma;L\rangle$ is E_n-*derivation-bounded* (E_n-d.b.) if there is a function $k \in E_n(\Sigma)$ satisfying for all $w \in \Sigma^*$: $w \underset{G}{=} e$ implies that there is a derivation from w of length $\leqslant |k(w)|$, where $|\ |$ denotes the length of a word, i.e. the number of letters. Then k is called an E_n-*bound* for $\langle\Sigma;L\rangle$.

Of course a natural algorithm for solving the word problem exists for a finite E_n-d.b. presentation.

1.2. LEMMA. *Let* $n \geqslant 1$, *and* $k \in E_n(\Sigma)$ *be such that* $k(e) \equiv e$. *Then there is a monotonous function* $k_1 \in E_n(\Sigma)$ *satisfying:* $|k_1(u)|+|k_1(v)| \leqslant |k_1(uv)|$ *and* $|k(w)| \leqslant |k_1(w)|$ *for all* $u,v,w \in \Sigma^*$.

Proof. **n = 1**. Let $k \in E_1(\Sigma)$ with $k(e) \equiv e$. Then $\exists c \geqslant 1\ \forall w \geqslant \Sigma^*$ ($|k(w)| \leqslant c|w|$). Define k_1 by $k_1(w) \equiv w^c$, then $k_1 \in E_1(\Sigma)$, k_1 is monotonous, and $|k(w)| \leqslant |k_1(w)|$ for every $w \in \Sigma^*$. Let $u,v \in \Sigma^*$ then $|k_1(u)|+|k_1(v)| = c|u| + c|v| = c|uv| = |k_1(uv)|$. **n $\geqslant$ 2**. Let $k \in E_n(\Sigma)$ with $k(e) \equiv e$. Then there is a monotonous function $k' \in E_n(\Sigma)$ satisfying $|k(w)| \leqslant |k'(w)|$ and $k'(e) \equiv e$. Define $k_1(e) \equiv e$, $k_1(ws) \equiv vk(k_1(w), k'(ws))$, where the function $vk \in E_1(\Sigma)$ denotes the concatenation of two words. Then

$$k_1(s_{i_1}^{\mu_1}..s_{i_r}^{\mu_r}) \equiv \bigodot_{j=1}^{r} k'(s_{i_1}^{\mu_1}..s_{i_j}^{\mu_j})$$

and therefore $|k_1(w)| \leqslant |w|\cdot|k'(w)|$; since k' is monotonous and $n \geqslant 2$, $k_1 \in E_n(\Sigma)$ k_1 is also monotonous, and k_1 a bound for k. Now,

$$|k_1(u)| + |k_1(v)| = \sum_{j=1}^{|u|} |k'(s^j)| + \sum_{j=1}^{|v|} |k'(s^j)|$$

$$\leqslant \sum_{j=1}^{|u|} |k'(s^j)| + \sum_{j=1}^{|v|} |k'(us^j)| = \sum_{j=1}^{|uv|} |k'(s^j)| = |k_1(uv)|.$$

This proves Lemma 1.2.

1.3. REMARK. If k is an E_n-bound it may be assumed that $k(e) \equiv e$. But then because of 1.2 it may be assumed that k is monotonous and satisfies $|k(u)| + |k(v)| \leqslant |k(uv)|$.

Now we give an example of an E_o-d.b. presentation.

1.4. LEMMA. $F = \langle\Sigma;\emptyset\rangle$, *the free group generated by* Σ, *is* E_o-d.b.

Proof. Define $k(w) \equiv w$, then $k \in E_o(\Sigma)$. Now let $w \in \Sigma^*$ such that $w \underset{F}{=} e$. This means $\gamma_f(w) \equiv e$, where γ_f denotes the function calculating the free reduction. But the execution of the free reduction gives a derivation from w of length $\frac{1}{2}|w|$. So k is an E_o-bound for $\langle\Sigma;\emptyset\rangle$.

The following three propositions give technics to construct E_n-d.b. presentations of groups from given E_n-d.b. presentations, such that the groups defined by the given presentations are embedded into the groups defined by the constructed presentations.

1.5. PROPOSITION. *Let* $H_1 = \langle\Sigma_1;L_1\rangle$ *and* $H_2 = \langle\Sigma_2;L_2\rangle$ *be groups such that* $\langle\Sigma_1;L_1\rangle$ *and* $\langle\Sigma_2;L_2\rangle$ *are* E_n-d.b. *for some* $n \geq 2$. *Then*

a) *the presentation* $\langle\Sigma_1 \cup \Sigma_2;L_1,L_2\rangle$ *of* $H_1{*}H_2$ *is* E_n-d.b., *and*

b) *the presentation* $\langle\Sigma_1 \cup \Sigma_2;L_1,L_2,ab\bar{a}\bar{b}\colon a \in \Sigma_1, b \in \Sigma_2\rangle$ *of* $H_1{\times}H_2$ *is* E_n-d.b.

Proof. Without loss of generality it may be assumed that Σ_1 and Σ_2 are disjoint alphabets. Let $k_1 \in E_n(\Sigma_{-1})$ and $k_2 \in E_n(\Sigma_{-2})$ be E_n-bounds for $\langle\Sigma_1;L_1\rangle$ and $\langle\Sigma_2;L_2\rangle$, respectively, and let $w \equiv u_ov_ou_1v_1\ldots u_lv_l$, $u_i \in \Sigma_{-1}^*$, $v_i \in \Sigma_{-1}^*$, where u_i and v_i are the syllables of w.

a) $w = e$ in $H_1{*}H_2$. Then there is an $i \in \{0,\ldots,l\}$ such that $e \neq u_i \underset{H_1}{=} e$ or $e \neq v_i \underset{H_2}{=} e$. So within no more than $|k_1(u_i)|$, respectively $|k_2(v_i)|$, steps w can be derived to a word w' containing less syllables than w. Hence there is a derivation from w of length

$$\mu \leq |k_i \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|,$$

where $\Pi_{\Sigma_i} \in E_1(\Sigma_{-1} \cup \Sigma_{-2})$ denotes the projection onto Σ_{-i}^*.

Define for $s \in \Sigma_1 \cup \Sigma_{-2}$, $U_s(w) = s^{|w|}$, which is an E_1-function. Let $k(w) = vk(k_1 \circ U_{a_1}(w), k_2 \circ U_{b_1}(w))$ for some $a_1 \in \Sigma_1$, $b_1 \in \Sigma_2$. Then $k \in E_n(\Sigma_{-1} \cup \Sigma_{-2})$ with

$$|k(w)| = |k_1 \circ U_{a_1}(w)| + |k_2 \circ U_{b_1}(w)| \geq |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|$$

since $|\Pi_{\Sigma_1}(w)| \leq |w| = |U_{a_1}(w)|$ and $|\Pi_{\Sigma_2}(w)| \leq |w| = |U_{b_1}(w)|$. Hence k is an E_n-bound for $\langle\Sigma_{-1} \cup \Sigma_{-2};L_1,L_2\rangle$.

b) $w = e$ in $H_1{\times}H_2$. Then $w = \Pi_{\Sigma_1}(w)\,\Pi_{\Sigma_2}(w)$ in $H_1{\times}H_2$, $\Pi_{\Sigma_1}(w) \underset{H_1}{=} e$, and $\Pi_{\Sigma_2}(w) \underset{H_2}{=} e$. There is a derivation from $\Pi_{\Sigma_1}(w)$ of length not exceeding $|k_1 \circ \Pi_{\Sigma_1}(w)|$ in $\langle\Sigma_1;L_1\rangle$, and there is a derivation from $\Pi_{\Sigma_2}(w)$ of length not exceeding $|k_1 \circ \Pi_{\Sigma_2}(w)|$ in $\langle\Sigma_2;L_2\rangle$. w can be derived to $\Pi_{\Sigma_1}(w)\Pi_{\Sigma_2}(w)$ by sequences of the form $ba \underset{1}{\to} ab\bar{a}\bar{b}ba \underset{2}{\to} ab$. Therefore $\Pi_{\Sigma_1}(w)\Pi_{\Sigma_2}(w)$ can be derived from w within no more than $3|\Pi_{\Sigma_1}(w)|\cdot|\Pi_{\Sigma_2}(w)|$ steps. Define $VK(w,e) \equiv e$, $VK(w,us) \equiv vk(VK(w,u),w)$ Then

$$VK(w,u) \equiv w^{|u|} \text{ and } VK \in E_2(\Sigma_1 \cup \Sigma_2).$$

Now let $k(w) \equiv vk((VK(w,w))^3, vk(k_1 \circ U_{a_1}(w), k_2 \circ U_{b_1}(w)))$. Since $n \geq 2$, $k \in E_n(\Sigma_1 \cup \Sigma_2)$ and

$|k(w)| \geq 3|w|^2 + |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)| \geq 3|\Pi_{\Sigma_1}(w)| \cdot |\Pi_{\Sigma_2}(w)| + |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|$.

Hence k is an E_n-bound for $\langle \Sigma_1 \cup \Sigma_2; L_1 L_2, ab\bar{a}\bar{b}: a \in \Sigma_1, b \in \Sigma_2 \rangle$.

1.6. PROPOSITION. *Let* $H = \langle \Sigma; L \rangle$ *be* E_n-d.b. *for some* $n \geq 3$.

a) *If* $H^* = \langle H, t; \bar{t}u_i t v_i^{-1}: i = 1, \ldots \ell \rangle$ *is an* HNN-*extension of* H *with rewriting functions* ω_u *for* $\langle u_1, \ldots, u_\ell \rangle_H$ *and* ω_v *for* $\langle v_1, \ldots, v_\ell \rangle_H$ *bounded by polynomials, then the given presentation of* H^* *is* E_n-d.b.

b) *If* $H^* = \langle H, t_1, \ldots, t_k; \bar{t}_i u_{ij} t_i v_{ij}^{-1}: j = 1, \ldots, \ell_i, i = 1, \ldots, k \rangle$ *is an* HNN-*extension of* H *with rewriting functions* ω_{u_i} *for* $\langle u_{i1}, \ldots, u_{i\ell_i} \rangle_H$ *and* ω_{v_i} *for* $\langle v_{i1}, \ldots, v_{i\ell_i} \rangle_H$, $i = 1, \ldots, k$, *bounded by polynomials, then the given presentation of* H^* *is* E_n-d.b. (See [Lyn-Sch] for the definition of HNN-extension).

Proof. As part (b) is nothing else than a finite iteration of part (a) it suffices to prove part (a).

Define $\varphi: U = \langle u_1, \ldots, u_\ell \rangle_H \to V = \langle v_1, \ldots, v_\ell \rangle_H$ as follows: If $w \in \Sigma^* \cap U$, then $w \underset{H}{=} \omega_u(w) \equiv \prod_{j=1}^{\lambda} u_{ij}^{\varepsilon_j}$. Let $\varphi(w) \equiv \prod_{j=1}^{\lambda} v_{ij}^{\varepsilon_j}$. Define $\bar{\varphi}: V \to U$ analogously. Now φ and $\bar{\varphi}$ realize the isomorphisms used for constructing the HNN-extension H^* of H. ω_u and ω_v are bounded by polynomials, and so are φ and $\bar{\varphi}$. Therefore $c \geq 1$ and $d \geq 2$ can be chosen in such a way that for all $w \in \Sigma^*$, $|\omega_u(w)|$, $|\omega_v(w)|$, $|\varphi(w)|$, $|\bar{\varphi}(w)| \leq c|w|^d$ are valid.

Define $f(e) \equiv e$, $f(ws) \equiv f(w)s$, $s \in \Sigma$.

$$f(wt) \equiv \begin{cases} u\varphi(v) & \text{if } f(w) \equiv u\bar{t}v,\ v \in \Sigma^* \cap U \\ f(w)t & \text{otherwise} \end{cases}$$

$$f(w\bar{t}) \equiv \begin{cases} u\bar{\varphi}(v) & \text{if } f(w) \equiv utv,\ v \in \Sigma^* \cap V \\ f(w)\bar{t} & \text{otherwise.} \end{cases}$$

According to [Av-Madl] 3.2, p.94, f is a t-reduction function for H^* satisfying

$$\forall w \in (\Sigma \cup \{t\})^* \quad |f(w)| \leq 2^{2^{cd|w|}}.$$

Let $k_H \in E_n(\Sigma)$ be an E_n-bound for $\langle \Sigma; L \rangle$, and let be $w \in (\Sigma \in \{t\})^*$ such that $w \underset{H^*}{=} e$. Then $f(w) \in \Sigma^*$ and $f(w) \underset{H}{=} e$, $f(w)$ results from w by pinching out $\frac{1}{2}|w|_t$ t-pinches, and subsequently $f(w)$ can be derived to e in $\langle \Sigma; L \rangle$ within no more than $|k_H \circ f(w)|$ steps. Let

$$w \equiv w_o t^{\mu_1} w_1 \ldots t^{\mu_k} w_k, \quad w_o, \ldots, w_k \in \Sigma^*, \quad \mu_1, \ldots, \mu_k \in \{\pm 1\}$$

and

$$t^{\mu_i} w_i t^{\mu_{i+1}}$$

be the lefmost t-pinch contained in w.

$$\mu_i = -1 : w \equiv w_o t^{\mu_1} \ldots w_{i-1} \bar{t} w_i t w_{i+1} \ldots w_k \xrightarrow{(1)}$$

$$w_o \ldots w_{i-1} \bar{t} w_i (\omega_u(w_i))^{-1} \omega_u(w_i) t w_{i+1} \ldots w_k \xrightarrow{(2)}$$

$$w_o..w_{i-1}\bar{t}\omega_u(w_i)tw_{i+1}..w_k \xrightarrow{(3)}$$

$$w_o..t^{\mu_{i-1}}w_{i-1}\ (w_i)w_{i+1}t^{\mu_{i+2}}..w_k \ \equiv:\ w'$$

ad (1), $|\omega_u(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega_u(w_i))^{-1} \underset{H}{=} e$, and so $w_i(\omega_u(w_i))^{-1}$ can be derived to e in $<\Sigma;L>$ within at most $|k_H(w_i(\omega_u(w_i))^{-1}|$ steps.

ad (3), $|\omega_u(w_i)|_u$ = the number of generators $u_1,\ldots,u_\ell$ in $\omega_u(w_i)$. Now (3) can be realized by $|\omega_u(w_i)|_u$ steps of the following kind:

(a) Insertion of $t\bar{t}$.

(b) Insertion of $v_j^{-1}v_j$ by using trivial relators.

(c) Deletion of $\bar{t}u_jtv_j^{-1}$.

Hence within at most

$$m_1 = |\omega_u(w_i)|+|k_H(w_i(\omega_u(w_i))^{-1})|+|\omega_u(w_i)|_u\cdot(2+\max_{j=1,..,\ell}|v_j|)$$

steps the first t-pinch of w can be pinched out.

$$m_1 \leqslant |\omega_u(w_i)|\cdot\{3+\max_{j=1,..,\ell}|v_j|\}+ k_H(w_i(\ _u(w_i))^{-1})| =:\ m_2$$

since

$$|\omega_u(w_i)|_u \leqslant |\omega_u(w_i)|.$$

$$\mu_i = 1 : w \equiv w_o t^{\mu_1}..w_{i-1}tw_i\bar{t}w_{i+1}..w_k \xrightarrow{(1)}$$

$$w_o..w_{i-1}tw_i(\omega_v(w_i))^{-1}\omega_v(w_i)\bar{t}w_{i+1}..w_k \xrightarrow{(2)}$$

$$w_o..w_{i-1}t\omega_v(w_i)\bar{t}w_{i+1}..w_k \xrightarrow{(3)}$$

$$w_o..t^{\mu_{i-1}}w_{i-1}\ \bar{}\ (w_i)w_{i+1}t^{\mu_{i+2}}...w_k \ \equiv:\ w'$$

ad (1), $|\omega_v(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega_v(w_i))^{-1} \underset{H}{=} e$, and so $w_i(\omega_v(w_i))^{-1}$ can be derived to e in $<\Sigma;L>$ within no more than $|k_H(w_i(\omega_v(w_i))^{-1})|$ steps.

ad (3), by $|\omega_v(w_i)|_v$ steps of the following kind (3) can be realized:

(a) Insertion of $\bar{t}u_j^{-1}t\bar{t}u_jt$ by using trivial relators.

(b) Deletion of $v_j\bar{t}u_j^{-1}t$ $(\equiv(\bar{t}u_jtv_j^{-1})^{-1})$ and of $t\bar{t}$.

In this way u_jt is derived from tv_j. Hence within at most

$$m_1^+ = |\omega_v(w_i)|+|k_H(w_i(\omega_v(w_i))^{-1})|+|\omega_v(w_i)|_v\cdot(4+\max_{j=1,..,\ell}|u_j|)$$

steps the first t-pinch of w can be pinched out.

$$m_1^+ \leqslant |\omega_v(w_i)|\cdot\{5+\max_{j=1,\ldots,\ell}|u_j|\}+|k_H(w_i(\omega_i(w_i))^{-1})| =:\ m_2^+$$

since

$$|\omega_v(w_i)|_v \leqslant |\omega_v(w_i)|.$$

Let $A = \max_{j=1,\ldots,\ell} \{|u_j|, |v_j|\}$, and $a \in \Sigma$. Now the first t-pinch of w can be pinched out in at most $c|w|^d \cdot \{5+A\} + |k_H(a^{(c+1)|w|^d})|$ steps. Let w_i' be the word formed by pinching out the first i t-pinches of w.

ASSERTION. *Let* $i \in \{1,2,\ldots,\frac{1}{2}|w|_t\}$. *Then* $|w_i'| \leqslant (c+1)^{d^{2i-1}}|w|^{d^i}$, *and* w_i' *can be derived from* w_{i-1}' *within* m_i' *steps where* m_i' *satisfies*

$$m_i' \leqslant (5+A)\cdot(c+1)^{d^{2i-1}}|w|^{d^i} + |k_H(a^{(c+1)^{d^{2i-1}}|w|^{d^i}})|.$$

Proof. By induction on i.

$i = 1$: $w_1' \equiv w'$, then $|w_1'| = |w| - |w_i| - 2 + |\varphi^\mu(w_i)|$

$$\leqslant |w| + c|w_i|^d \leqslant |w| + c|w|^d \leqslant (c+1)|w|^d \leqslant (c+1)^d|w|^d.$$

$$m_1' \leqslant c|w|^d(5+A) + |k_H(a^{(c+1)|w|^d})| \leqslant (5+A)(c+1)^d|w|^d + k_H(a^{(c+1)^d|w|^d})|.$$

$i \to i+1$: w_{i+1}' is formed from w_i' by pinching out a t-pinch, then

$$|w_{i+1}'| \leqslant |w_i'| + c|w_i'|^d \leqslant (c+1)|w_i'|^d$$
$$< (c+1)\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d = (c+1)^{d^{2i+1}}|w|^{d^{i+1}}$$
$$\leqslant (c+1)^{d^{2i+1}}|w|^{d^{i+1}},$$

and

$$m_{i+1}' < (5+A)c|w'|^d + |k_H(a^{(c+1)|w_i'|^d})|$$
$$< (5+A)c\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d + |k_H(a^{(c+1)\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d})|$$
$$= (5+A)c(c+1)^{d^{2i}}|w|^{d^{i+1}} + |k_H(a^{(c+1)^{d^{2i+1}}|w|^{d^{i+1}}})|$$
$$\leqslant (5+A)(c+1)^{d^{2i+1}}|w|^{d^{i+1}} + |k_H(a^{(c+1)^{d^{2i+1}}|w|^{d^{i+1}}})|$$

Let w^+ be the word formed by pinching out all t-pinches of w. Then $w^+ \equiv w_{\frac{1}{2}|w|_t}'$, and hence

$$|w^+| \leqslant (c+1)^{d^{|w|_t - 1}}|w|^{d^{(\frac{1}{2})|w|_t}} < \{(c+1)|w|\}^{d^{|w|}}$$

The derivation from w to w^+ can be performed within

$$m^+ = \sum_{i=1}^{\frac{1}{2}|w|_t} m_i' \leqslant \sum_{i=1}^{\frac{1}{2}|w|_t} \{(5+A)(c+1)^{d^{2i-1}}|w|^{d^i} + k_H(a^{(c+1)^{d^{2i-1}}|w|^{d^i}})|\}$$
$$\leqslant \tfrac{1}{2}|w|_t\cdot\{5+A)(c+1)^{d^{|w|}}w^{d^{|w|}} + |k_H(a^{((c+1)|w|)^{d^{|w|}}})|\}$$

steps. At last, w^+ is derived to e in $\langle\Sigma;L\rangle$ within at most

$$|k_H(w^+)| \leqslant |k_H(a^{((c+1)|w|)^{d^{|w|}}})| \quad \text{steps.}$$

Hence there is a derivation from w in the given presentation of H^* of length not exceeding

$$m_w = m^+ + |k_H(w^+)| \leqslant |w|\{(5+A)((c+a)|w|)^{d^{|w|}} + |k_H(a^{((c+1)|w|)^{d^{|w|}}})|\}.$$

Define $d_1(w) \equiv a^{d^{|w|}}$, $d_2(w) \equiv VK(w, a^{c+1})$, and

$$d_3(w,e) \equiv a, \quad d_3(w,us) \equiv VK(d_3(w,u), w).$$

Then $d_1 \in E_3(\Sigma \cup \{t\})$, $d_2 \in E_2(\Sigma \cup \{t\})$, $d_3 \in E_3(\Sigma \cup \{t\})$, $d_2(w) \equiv w^{c+1}$, $|d_2(w)| \equiv (c+1)|w|$, and $d_3(w,u) \equiv a^{|w|^{|u|}}$. $d_4(w) \equiv d_3(d_2(w), d_1(w))$ is a function from $E_3(\Sigma \cup \{t\})$ satisfying

$$d_4(w) \equiv a^{((c+1)|w|)^{d^{|w|}}}$$

and $k(w) \equiv VK(vk(VK(d_4(w), a^{5+A}), k_H \circ d_4(w)), w)$ is from $E_n(\Sigma \cup \{t\})$ satisfying:

$$|k(w)| = |w|\{(5+A)((c+1)|w|)^{d^{|w|}} + k_H(a^{((c+1)|w|)^{d^{|w|}}})|\} .$$

Hence k is an E_n-bound for the given presentation of H^*. Thus this presentation is E_n-d.b.

1.7. PROPOSITION. *The* $H = \langle\Sigma;L\rangle$ *be* E_n-b.d. *for some* $n \geq 2$. *If* $H^* = \langle H,t;tu_itu_i^{-1} : i = 1,\dots,\ell\rangle$ *is an* HNN-*extension of* H *with the identity as isomorphism and with a rewriting function* $\omega \in E_n(\Sigma)$ *for* $\langle u_1,\dots,u_\ell\rangle_H$, *then the given presentation of* H^* *is* E_n-d.b.

Proof. Define $f(e) \equiv e$, $f(ws) \equiv f(w)s$, $s \in \Sigma$,

$$f(wt^\mu) \equiv \begin{cases} uv & \text{if } f(w) \equiv ut^{-\mu}v, \quad v \in \Sigma^* \cap \langle u_1,\dots,u_\ell\rangle_H \\ f(w)t^\mu & \text{otherwise} \end{cases}$$

f is a t-reduction fuction for H^* satisfying $|f(w)| \leq |w|$. Let $w \in (\Sigma \cup \{t\})^*$ with $w =_{H^*} e$. Then $f(w) \in \Sigma^*$ and $f(w) =_H e$. Therefore w can be derived to e by first pinching out all the t-pinches of w and thereafter deriving the resulting word to e in $\langle\Sigma;L\rangle$. $\omega(e) \equiv e$ may be assumed. Then according to Lemma 1.2 there is a monotonous function $\omega_2 \in E_n(\Sigma)$ satisfying $|\omega(w)| \leq |\omega_2(w)|$ and $|\omega_2(u)|+|\omega_2(v)| \leq |\omega_2(uv)|$ for every $w,u,v \in \Sigma^*$.

Let $k_H \in E_n(\Sigma)$ be an E_n-bound for $\langle\Sigma;L\rangle$, and let $w \in w_o t^{\mu_1}..t^{\mu_r}w_r$, $w_o,..,w_r \in \Sigma^*$, $\mu_1,\dots,\mu_r \in \{\pm1\}$, with $w =_{H^*} e$ contain the t-pinch $t^{\mu_i}w_it^{\mu_{i+1}}$. This t-pinch can be pinched out by the following sequence of operations:

$$w \equiv w_o t^{\mu_1}w_1..w_{i-1}t^{\mu_i}w_it^{\mu_{i+1}}w_{i+1}..w_r \xrightarrow{(1)}$$

$$w_o..w_{i-1}t^{\mu_i}w_i(\omega(w_i))^{-1}\omega(w_i)t^{\mu_{i+1}}w_{i+1}..w_r \xrightarrow{(2)}$$

$$w_o..w_{i-1}t^{\mu_i}\omega(w_i)t^{\mu_{i+1}}w_{i+1}..w_r \xrightarrow{(3)}$$

$$w_o..t^{\mu_{i-1}}w_{i-1}\omega(w_i)w_{i+1}t^{\mu_{i+2}}..w_r \xrightarrow{(4)}$$

$$w_o..t^{\mu_{i-1}}w_{i-1}w_i(\omega(w_i))^{-1}\omega(w_i)w_{i+1}t^{\mu_{i+2}}..w_r \xrightarrow{(5)}$$

$$w_o..t^{\mu_{i-1}}w_{i-1}w_iw_{i+1}t^{\mu_{i+2}}..w_r \equiv : w' .$$

ad (1), $|\omega(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega(w_i))^{-1} \underset{H}{=} e$, and hence $w_i(\omega(w_i))^{-1}$ can be derived to e in $\langle\Sigma;L\rangle$ within at most $|k_H(w_i(\omega(w_i))^{-1})|$ steps.

ad (3), $|\omega(w_i)|_u$ steps of the following form:

$\mu_i = -1$: (a) Insertion of $tu_j^{-1}u_j\bar{t}$ by using trivial relators.
(b) Deletion of $\bar{t}u_j tu_j^{-1}$

In this way $u_j\bar{t}$ is derived from $\bar{t}u_j$.

$\mu_i = 1$: (a) Insertion of $\bar{t}u_j^{-1}t\bar{t}u_j t$ by using trivial relators
(b) Deletion of $u_j\bar{t}u_j^{-1}t$ $(\equiv(\bar{t}u_j tu_j^{-1})^{-1})$ and of $t\bar{t}$:

$$tu_j \to tu_j\bar{t}u_j^{-1}t\bar{t}u_j t \to t\bar{t}u_j t \to u_j t.$$

ad (4), $w_i(\omega(w_i))^{-1}$ can be derived from e by inverting the derivation of (2).

ad (5), $|\omega(w_i)|$ trivial relators are deleted.

Hence the t-pinch of w can be pinched out within

$$m' \leqslant |\omega(w_i)|+|k_H(w_i(\omega(w_i))^{-1})|+|\omega(w_i)|_u\cdot(4+\max_{j=1,\dots,\ell}|u_j|)$$
$$+ |k_H(w_i(\omega(w_i))^{-1}|+|\omega(w_i)| \leqslant |\omega(w_i)|\cdot(6+\max_{j=1,\dots,\ell}|u_j|)+2|k_H(w_i(\omega(w_i))^{-1})|$$

steps, since $|\omega(w_i)|_u \leqslant |\omega(w_i)|$. Let $A = \max_{j=1,\dots,\ell}|u_j|$. Then

$$m' \leqslant |\omega(w_i)|(6+A)+2|k_H(w_i(\ (w_i))^{-1})| \leqslant 2|k_H(w_i\omega_2(w_i))|+(6+A)|\omega_2(w_i)|,$$

since $|\omega(w_i)| = |\omega(w_i)^{-1}| \leqslant |\omega_2(w_i)|$, and k_H being monotonous

$$\leqslant 2|k_H(a^{|w|+|\omega_2(a^{|w|})|})+(6+A)|\omega_2(a^{|w|})\ ,$$

since $|w_i| \leqslant |w|$, and k_H and ω_2 being monotonous.

$\frac{1}{2}|w|_t$ t-pinches must be pinched out. Of course $|w'| \leqslant |w|$. Hence w can be derived to f(w) in the given presentation of H^* within m^* steps where m^* satisfies:

$$m^* \leqslant \tfrac{1}{2}|w|_t\cdot(2|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+ (6+A)|\omega_2(a^{|w|})|\}$$
$$\leqslant |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}.$$

f(w) is derived to e in $\langle\Sigma;L\rangle$ within at most $\tilde{m} \leqslant |k_H\circ f(w)| \leqslant |k_H(a^{|w|})|$ steps, as $|f(w)| \leqslant |w|$ and k_H being monotonous. Hence w can be derived to e in the given presentation of H^* within m steps where m satisfies:

$$m = m^*+\tilde{m} \leqslant |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}+|k_H(a^{|w|})|.$$

Define

$$k(w) \equiv vk(VK(vk(k_H\circ vk(U_a(w),U_a\circ\omega_2\circ U_a(w)),\ VK(\omega_2\circ U_a(w),a^{3+A})),w),k_H\circ U_a(w)).$$

Then $k \in E_n(\Sigma \cup \{\underline{t}\})$ and k satisfies:

$$|k(w)| = |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}+ k_H(a^{|w|})|.$$

Therefore k is an E_n-bound for the given presentation of H^*, which is E_n-d.b. herewith.

2. AN EMBEDDING INTO DERIVATION-BOUNDED GROUPS.

The proposition in Sec. 1 give examples of embeddings of d.b. groups into d.b. groups. But now the question arises whether a group possessing no E_n-d.b. presentation can be cambedded into a E_n-d.b. group. The answer to this question is given by the next theorem and its corollary.

2.1. THEOREM. *Let* $G = <\Sigma;L>$ *be f.g. with* $WP_G \in E_n(\Sigma)$, *i.e. the word problem for the given presentation* $<\Sigma;L>$ *of* G *is* E_n*-decidable, for some* $n \geq 3$. *Then there is a finite* E_n-d.b. *presentation* $<\Delta;M>$ *of a group* H *such that* G *can be embedded in* H.

Proof. Starting with $<\Sigma;L>$ we construct $<\Delta;M>$ in a few number of steps. Let $\hat{L} = \{w \in \underline{\Sigma}^* | w \underset{G}{=} e\}$, and $\hat{G} = <\Sigma;\hat{L}>$. Then $\hat{G}$ is f.g., $WP_{\hat{G}} \in E_n(\underline{\Sigma})$, $\hat{G} \cong G$, via the identity mapping, and for each word $w \in \underline{\Sigma}^*$ with $w \underset{\hat{G}}{=} e$ there is a derivation of length 1 in $<\Sigma;L>$, because $w = e$ in $\hat{G}$ implies $w \in \hat{L}$.

Let $\tilde{\Sigma} = \{\tilde{s} | s \in \Sigma\}$ be a copy of Σ satisfying $\tilde{\Sigma} \cap \underline{\Sigma} = \emptyset$, $\Sigma_o = \Sigma \cup \tilde{\Sigma}$, and let $\varphi: \underline{\Sigma}^* \to \Sigma_o^*$ be defined by $\varphi(s) \equiv s$, $\varphi(\bar{s}) \equiv \tilde{s}$. Let $L_o = \{w \in \Sigma_o^* | \exists u \in \hat{L}: \varphi(u) \equiv w\}$ and $G_o = <\Sigma_o;L_o>$, then G_o is f.g., $WP_{G_o} \in E_n(\underline{\Sigma}_o)$, $G_o \cong G$ via φ, the defining relators of G_o do contain only positive letters, and for every word $w \in \underline{\Sigma}_o^*$ with $w \underset{G_o}{=} e$ there is a derivation of length $\leq 2|w|+1$ in $<\Sigma_o;L_o>$, because at first all letters of the form $\bar{s}$ $(s \in \Sigma)$ contained in w must be substituted by $\tilde{s}$ by means of the derivation $\bar{s} \to \bar{s}s\tilde{s} \to \tilde{s}$, then all the letters of the form $\bar{\tilde{s}}$ $(\tilde{s} \in \tilde{\Sigma})$ contained in w must be substituted by s by means of the derivation $\bar{\tilde{s}} \to \bar{\tilde{s}}\tilde{s}s \to s$, as $s\tilde{s}, \tilde{s}s \in L_o$, and at last the produced word $w' \in \Sigma_o^*$ can be deleted in one step.

L_o is an E_n-decidable subset of Σ_o^*. Hence there is a Turing Machine $T = (\Sigma_o, Q_T, q_o, \beta)$, where Q_T is a finite set of states, $q_o \in Q_T$ is the initial state of T, and β is the transition function of T, and a function $g \in E_n(\Sigma_o)$ such that T computes the characteristic function of the set L_o and g is a time bound for T.

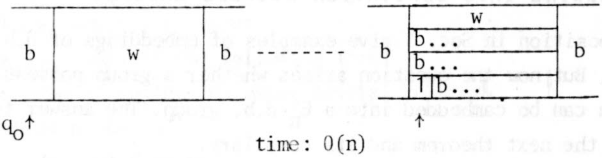
Now T can be modified to get a Turing Machine $\tilde{T} = (\tilde{\Sigma}_o, Q_{\tilde{T}}, q_o, \tilde{\beta})$, where $\tilde{\Sigma}_o$ is a finite alphabet including Σ_o, satisfying the following two conditions:

(1) There is a special state $q_a \in Q_{\tilde{T}}$ called the *accepting state* such that starting at $q_o w$, $\tilde{T}$ eventually reaches the state q_a if and only if $w \in L_o$.

(2) There is a function $k_T \in E_n \in (\tilde{\Sigma}_o \cup Q_{\tilde{T}})$ satisfying for all $u, v \in \Sigma_o^*$, $q_j \in Q_{\tilde{T}}$: starting at the configuration uq_jv, $\tilde{T}$ halts within $|k_T(ug_jv)|$ steps if $\tilde{T}$ reaches the accepting states q_a afeter all.

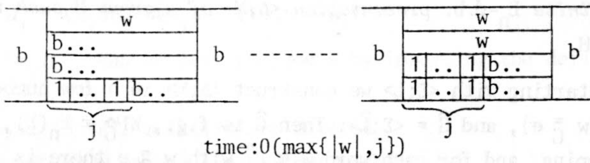
Especially it is E_n-decidable whether starting at uq_jv, $\tilde{T}$ eventually reaches the state q_a. For that $\tilde{T}$ works as follows:

Start:



The tape is divided into four tracks. The input is copied onto track N° 1. Below the leftmost letter of the copied input a "1" is printed onto track N° 4.

Loop:



Track N° 1 is copied onto track N° 2, and track N° 4 is copied onto track N° 3. If a letter $a \in \tilde{\Sigma}_o - \Sigma_o$ is contained in w, or if a letter $a \neq 1$ is contained in the inscription of track N° 4, $\tilde{T}$ halts at the state q_-, a nonaccepting state. Otherwise $\tilde{T}$ simulates T starting at $q_o w$ on its track N° 2. Ahead of each step of this simulation a "1" is erased from track N° 3. If T halts and accepts, then $\tilde{T}$ halts at state q_a. If T halts without accepting, then $\tilde{T}$ halts at state q_-. If the whole inscription of track N° 3 is erased before reaching the end of the computation of T, then $\tilde{T}$ breaks off the simulation of T, cleans track N° 2, adds a "1" to the inscription of track N° 4, and starts the loop again. For carrying out this computation, $\tilde{T}$ needs two additional tracks as scratch paper to note the direction of the beginning of the inscription of track N° 1, and with it the beginning of the inscriptions of tracks N° 3 and N° 4, and the direction in which the actual cell of track N° 2 is situated in relation to the position of the head of $\tilde{T}$.

Now the following is satisfied: for $w \in L_o$ starting at $q_o w$, T halts and accepts. Hence starting at q_o,w $\tilde{T}$ reaches the state q_a. On the other hand if starting at $q_o w$, $\tilde{T}$ reaches the state q_a, then $w \in \Sigma_o^*$, and T halts and accepts starting at $q_o w$, i.e. $w \in L_o$.

With the input $w \in \Sigma_o^*$, T does not carry out more than $|g(w)|$ steps, $\tilde{T}$ simulates T step by step. Each step of this simulation takes $\tilde{T}$ at most $0(|g(w)|)$ steps, for $\tilde{T}$ must erase a "1" from track N° 3. Altogether, $\tilde{T}$ simulates $\sum_{i=1}^{|g(w)|} i = 0(|g(w)|^2)$ steps of T.

Hence $\tilde{T}$ needs $0(|g(w)|^3)$ steps to carry through the simulation of T with input w. If $\tilde{T}$ is started at an arbitrary configuration $uq_j v$, it simulates T starting at a configuration depending on $uq_j v$ on track N° 2 for as many steps as the inscription of track N° 3 tells. This takes no more than $0(|uv|^2)$ steps. Afterwards $\tilde{T}$ simulates T, starting at a well defined initial configuration $q_o w$, where w is the inscription of track N° 1 of the configuration uq_jv, if $w \in \Sigma_o^*$. Otherwise $\tilde{T}$ halts at state q_-. As $n \geqslant 3$ there is a function $k_{\tilde{T}} \in E_n(\tilde{\Sigma}_o \cup Q_{\tilde{T}})$ satisfying condition (2).

According to [Av-Madl], p.89, a semigroup $\Delta_{\tilde{T}} = (S \cup Q; \pi)$ where $S = \tilde{\Sigma}_o \cup \{h\}$, $Q = Q_{\tilde{T}} \cup \{q\}$, and

$$\pi = \{F_i q_{i_1} G_i = H_i q_{i_2} K_i \mid q_{i_1}, q_{i_2} \in Q,\ F_i, G_i, H_i, K_i \in S^*, \quad i = 1,\dots,N\}$$

can be constructed from $\tilde{T}$, satisfying:

(3) $\forall w \in \tilde{\Sigma}_o^* \ (hg_o wh \underset{\Delta_{\tilde{T}}}{=} q \Longleftrightarrow w \in L_o)$.

(4) If $uq_jv = q$, then there is a derivation from uq_jv to q in $\Delta_{\tilde{T}}$ of length not exceeding $2|k_T(uq_jv)| + |uq_jv|$, because it may be assumed that $k_{\tilde{T}}$ is non-decreasing ([Weih]).

Let $u, v \in S^*$ with $uq_jv \underset{\Delta_T}{=} q$. Then $uq_jv \equiv q$, or $u \equiv hu'$, $v \equiv v'h$, $q_j \neq q$, and starting at $u'q_jv'$, $\tilde{T}$ reaches the accepting state q_a. But for doing so, $\tilde{T}$ does not need more than $|k_T(u'q_jv')|$ steps. Hence $uq_jv \equiv hu'q_jv'h$ can be derived to $h\tilde{u}q_a\tilde{v}h$ in $\Delta_{\tilde{T}}$ within at most $|k_{\tilde{T}}(u'q_jv')|$ steps. Of course $|\tilde{u}\tilde{v}| < |u'v'| + |k_{\tilde{T}}(u'q_jv')|$, since $\tilde{T}$ can increase the length of its tape inscription by at most one per step. It takes $\Delta_{\tilde{T}}$ $|\tilde{u}\tilde{v}|$ steps to derive hq_ah from $h\tilde{u}q_a\tilde{v}h$ by erasing $\tilde{u}\tilde{v}$; hq_ah can be derived to q within one step. Hence $\Delta_{\tilde{T}}$ can derive q from uq_jv within at most

$$2|k_{\tilde{T}}(uq_jv)| + |u'v'| + 1 \leqslant 2|k_{\tilde{T}}(uq_jv)| + |uq_jv| \quad \text{steps.}$$

Define $k_\Delta(w) \equiv vk(vk(k_{\tilde{T}}(w), k_{\tilde{T}}(w)), w)$. Then $k_\Delta \in E_n(S \cup Q)$, and k_Δ bounds the derivation of words $w \in (S \cup Q)^*$ with $w \underset{\Delta_{\tilde{T}}}{=} q$ to q in $\Delta_{\tilde{T}}$.

Now a Britton tower of groups is constructed:

$D_o = \langle x; \emptyset\rangle$

$D_1 = \langle x, S; \bar{s}xs = x^2 (s \in S)\rangle$

$D_2 = \langle D_1, Q; \emptyset\rangle$

$D_3 = \langle D_2, r; \bar{r}_i \bar{F}_i q_{i_1} G_i r_i = \bar{H}_i q_{i_2} k_i, \bar{r}_i s x r_i = s\bar{x} (s \in S,\ i = 1,\dots,N)\rangle$ where $\overline{(s_{i_1}..s_{i_m})} \equiv \bar{s}_{i_1}..\bar{s}_{i_m}$

$D_4 = \langle D_3, t; \bar{t}xt = x, \bar{t}rt = r (r \in R)\rangle$

$D_5 = \langle D_4, k; \bar{k}xk = x, \bar{k}rk = r (r \in R), \bar{k}\bar{q}tqk = \bar{q}tq\rangle$

$\cong \langle D_5, t_o, k_o; t_o = (\bar{h}q_o)^{-1} t(\bar{h}g_o), k_o = hk\bar{h}\rangle$

$\cong \langle D_3, t_o, k_o; (\bar{h}q_o\bar{t}_o\bar{q}_o h)a(\bar{h}q_o t_o \bar{q}_o h) = a,\ (\bar{h}\bar{k}_o h)a(\bar{h}k_o h) = a (a \ \{x\} \cup R),$
$(\bar{h}\bar{k}_o h)\bar{g}\bar{h}q_o t_o \bar{q}_o h g(\bar{h}k_o h) = \bar{q}\bar{h}q_o t_o \bar{q}_o hq\rangle =: \langle S_6; M_6\rangle = D_6$

$\Sigma_o' = \{s' \mid s \in \Sigma_o\}$, $': \Sigma_o^* \to \Sigma_o'^*$ is defined by $(s^\mu)' \equiv s'^\mu$,
$(\tilde{s}^\mu)' \equiv \tilde{s}'^\mu$, $L_o' = \{w \in \Sigma_o'^* \mid \exists u \in L_o : u' \equiv w\}$, and $G' = \langle \Sigma_o'; L_o'\rangle$.

Then $G' = G_o$ via $'$.

$H_o \doteq D_6 \times G' = \langle S_6 \cup \Sigma_o'; M_6, L_o', as' = s'a (a \in S_6, s' \in \Sigma_o')\rangle$

$H_1 = \langle H_o, d; \bar{d}ss'd = s, \bar{d}\bar{k}_o s k_o d = \bar{k}_o s k_o (s \in \Sigma_o), \bar{d}t_o d = t_o, \bar{d}\bar{k}_o t_o k_o d = \bar{k}_o t_o k_o\rangle$

$H_2 = \langle H_1, z; \bar{z}sz = s, \bar{z}\bar{k}_o s k_o z = \bar{k}_o s k_o (s \in \Sigma_o), \bar{z}t_o z = t_o d, \bar{z}\bar{k}_o t_o k_o z = \bar{k}_o t_o k_o d\rangle$

$\Delta = S_6 \cup \Sigma_o' \cup \{d, z\}$.

Let M be the set of defining relators of the given presentation of H_2 and $M = \tilde{M} - (L_o' - \{s'\tilde{s}', \tilde{s}'s' \mid s' \in \Sigma'\})$ where $\Sigma' = (\Sigma)' \subseteq \Sigma_o'$

REMARK. $\forall s \in \Sigma(s\bar{s}, \bar{s}s \in \hat{L}) \Rightarrow \forall s \in \Sigma(s\tilde{s}, \tilde{s}s \in L_o) \Rightarrow \forall s' \in \Sigma'(s'\tilde{s}', \tilde{s}'s' \in L_o')$

Let $H = \langle\Delta; M\rangle$. In [Av-Madl] Satz 1.1, p.184, Avenhaus and Madlener prove: H is f.p., $WP_H \in E_n(\Delta)$, and G embeds in H. It remains to show that $\langle\Sigma; M\rangle$ is E_n-d.b.

According to [Ott]§15, pp.156-173, the following assertions are valid:

D_o is E_o-d.b.

D_1, D_2, D_3, and D_4 are E_3-d.b.

D_5 is E_n-d.b.

For proving these assertions propositions 1.4 until 1.7 are used. At the last part one has to construct a rewriting function $\omega \in E_n(\{\underline{x,t}\} \cup \underline{S} \cup \underline{Q} \cup \underline{R})$, for $\langle x, \bar{q}tq, R\rangle_{D_4}$. After that, proposition 1.7 can be applied. Analogously there is an E_n-rewriting function for $\langle hx\bar{h}, hr\bar{h}, hq\bar{h}q_o t_o \bar{q}_o hq\bar{h}\rangle$ in $D_4' = \langle D_3, t_o; (\bar{h}q_o t_o \bar{q}_o h)\cdot a(\bar{h}q_o t_o \bar{q}_o h) = a \ (a \in \{x\} \cup R)\rangle$ where D_4' is E_3-d.b. just like D_4. Hence D_6 is E_n-d.b., too.

(5) $\langle\Sigma; M\rangle$ is E_n-d.b.

Proof. a) Let $w' \in \tilde{L}_o' = L_o' - \{s'\tilde{s}', \tilde{s}'s' \mid s' \in \Sigma'\}$, and $w \equiv (w')^{('-1)} \in L_o \subseteq \Sigma_o^*$. Then $hq_o wh \underset{\Delta_T}{=} q$, and hence $k_o(w^{-1}t_o w) \underset{D_6}{=} (w^{-1}t_o w)k_o$, due to [Av-Madl], p.185. $w' \in \tilde{L}_o' \subseteq L_o' \Rightarrow w' \underset{H_2}{=} e \Rightarrow w' \underset{H}{=} e$, since $H \cong H_2$ via the identity. Now w' can be derived to e in $\langle\Delta; M\rangle$ as follows:

$$w' \xrightarrow{(1)} w^{-1}\bar{t}_o ww^{-1}t_o ww'd\bar{d} \xrightarrow{(2)} w^{-1}\bar{t}_o ww^{-1}t_o dw\bar{d} \xrightarrow{(3)}$$

$$w^{-1}\bar{t}_o w\bar{z}w^{-1}t_o wz\bar{d} \xrightarrow{(4)} w^{-1}\bar{t}_o w\bar{z}\bar{k}_o w^{-1}t_o wk_o z\bar{d} \xrightarrow{(5)}$$

$$w^{-1}\bar{t}_o w\bar{k}_o w^{-1}t_o wk_o d\bar{d} \xrightarrow{(6)} w^{-1}\bar{t}_o ww^{-1}t_o wd\bar{d} \xrightarrow{(7)} e.$$

ad (1), $2|w|+2$ trivial relators are inserted.

ad (2), by using the commutation relators of H_o w and w' can be mixed within at most $3|w|^2$ steps:

$$ww' \to s_{i_1}s_{i_1}'s_{i_2}s_{i_2}' .. s_{i_\lambda}s_{i_\lambda}'.$$

After that:

$$s_{i_1}s_{i_1}' .. s_{i_\lambda}s_{i_\lambda}'d \to d\bar{d}s_{i_1}s_{i_1}'d\bar{d}s_{i_2}s_{i_2}'d\bar{d} .. s_{i_\lambda}s_{i_\lambda}'d.$$

(Insertion of $\lambda = |w|$ trivial relators)

$$\to ds_{i_1}s_{i_2} .. s_{i_\lambda} = dw$$

(Insertion of $\bar{s}_{i_j}s_{i_j}$ and deletion of $\bar{d}s_{i_j}s_{i_j}'d\bar{s}_{i_j}$).

Taken altogether this derivation doesn't need more than $3|w|^2+3|w|$ steps.

ad (3), $w^{-1}t_o dw \equiv \bar{s}_{i_\lambda}..\bar{s}_{i_1}t_o ds_{i_1}..s_{i_\lambda}$

$$\to (\bar{z}\bar{s}_{i_\lambda}z)(\bar{z}s_{i_\lambda}z\bar{s}_{i_\lambda})..(\bar{z}\bar{s}_{i_1}z)(\bar{z}s_{i_1}z\bar{s}_{i_1})(t_o d\bar{z}\bar{t}_o z)(\bar{z}t_o z)$$
$$(s_{i_1}\bar{z}\bar{s}_{i_1}z)(\bar{z}s_{i_1}z)..(s_{i_\lambda}\bar{z}\bar{s}_{i_\lambda}z)(\bar{z}s_{i_\lambda}z)$$

(Insertion of $6|w|+3$ trivial relators),

$$\to \bar{z}\bar{s}_{i_\lambda}z..\bar{z}\bar{s}_{i_1}z\bar{z}t_o z\bar{z}s_{i_1}z..\bar{z}s_{i_\lambda}z$$

(Deletion of $2|w|+1$ relators of the form $\bar{z}s_{i_j}z\bar{s}_{i_j}, s_{i_j}\bar{z}\bar{s}_{i_j}z, t_o d\bar{z}\bar{t}_o z$),

$$\to zw^{-1}t_o wz$$

(Deletion of $2|w|$ trivial relators).

Altogether (3) needs at most $|10|w$ +4 steps.

ad (4), $w^{-1}t_o w \to (w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)(\bar{k}_o w^{-1}t_o wk_o)$

(Insertion of $2|w|+3$ trivial relators).

Let $k_6 \in E_n(S_6)$ be an E_n-bound for $\langle S_6;M_6\rangle$. Then $w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o$ can be derived to e in $\langle S_6;M_6\rangle$ within at most $|k_6(w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)| = |k_6(x^{4|w|+4})|$ steps. Hence;

$$(w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)(\bar{k}_o w^{-1}t_o wk_o) \to \bar{k}_o w^{-1}t_o wk_o \text{ in } \langle\Delta;M\rangle$$

within at most $|k_6(x^{4|w|+4})|$ steps, and (4) can be carried out within not more than $|k_6(x^{4|w|+4})|+2|w|+3$ steps.

ad (5), $\bar{z}\bar{k}_o w^{-1}t_o wk_o z \equiv \bar{z}\bar{k}_o\bar{s}_{i_\lambda}..\bar{s}_{i_1}t_o s_{i_1}..s_{i_\lambda}k_o z$

$$\to (\bar{k}_o\bar{s}_{i_\lambda}k_o)(\bar{k}_o s_{i_\lambda}k_o\bar{z}\bar{k}_o\bar{s}_{i_\lambda}k_o z)(\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o)(\bar{k}_o s_{i_{\lambda-1}}k_o\bar{z}\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o z)..$$
$$(\bar{z}\bar{k}_o t_o k_o zd\bar{k}_o\bar{t}_o k_o)(\bar{k}_o t_o k_o d)(\bar{z}\bar{k}_o s_{i_1}k_o z\bar{k}_o\bar{s}_{i_1}k_o)$$
$$(\bar{k}_o s_{i_1}k_o)(\bar{z}\bar{k}_o s_{i_2}k_o z\bar{k}_o\bar{s}_{i_2}k_o)..(\bar{z}\bar{k}_o s_{i_\lambda}k_o z\bar{k}_o\bar{s}_{i_\lambda}k_o)(\bar{k}_o s_{i_\lambda}k_o)\bar{z}\bar{k}_o k_o z$$

(Insertion of $10|w|+6$ trivial relators),

$$\to \bar{k}_o\bar{s}_{i_\lambda}k_o\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o..\bar{k}_o\bar{s}_{i_1}k_o\bar{k}_o t_o k_o d\bar{k}_o s_{i_1}k_o..\bar{k}_o s_{i_\lambda}k_o\bar{z}\bar{k}_o k_o z$$

(Deletion of $2|w|+1$ relators of the form $\bar{z}\bar{k}_o s_{i_j}k_o z\bar{k}_o\bar{s}_{i_j}k_o, \bar{k}_o s_{i_j}k_o\bar{z}\bar{k}_o\bar{s}_{i_j}k_o z$ $(s_{i_j} \in \Sigma_o)$, $\bar{z}\bar{k}_o t_o k_o zd\bar{k}_o\bar{t}_o k_o$),

$$\to \bar{k}_o w^{-1}t_o k_o d\bar{k}_o s_{i_1}k_o..\bar{k}_o s_{i_\lambda}k_o.$$

(Deletion of $|w|+2$ trivial relators),

$$\to \bar{k}_o w^{-1}t_o k_o d(\bar{k}_o s_{i_1}k_o\bar{d}\bar{k}_o\bar{s}_{i_1}k_o d)(\bar{d}\bar{k}_o s_{i_1}k_o d)..(\bar{k}_o s_{i_\lambda}k_o\bar{d}\bar{k}_o\bar{s}_{i_\lambda}k_o d)(\bar{d}\bar{k}_o s_{i_\lambda}k_o d).$$

(Insertion of $5|w|$ trivial relators),

$$\to \bar{k}_o w^{-1}t_o k_o d\bar{d}\bar{k}_o s_{i_1}k_o d..\bar{d}\bar{k}_o s_{i_\lambda}k_o d$$

(Deletion of $|w|$ relators of the form $\bar{k}_o s_{i_j}k_o\bar{d}\bar{k}_o\bar{s}_{i_j}k_o d$ $(s_{i_j} \in \Sigma_o)$).

$$\to \bar{k}_o w^{-1}t_o wk_o d.$$

(Deletion of $2|w|$ trivial relators).

Hence (5) can be carried out within $21|w|+9$ steps.

ad (6), $\bar{k}_o w^{-1} t_o w k_o \rightarrow (\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w)(w^{-1} t_o w)$

(Insertion of $2|w|+1$ trivial relators).

Hence $\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w$ can be derived to e in $<S_6;M_6>$ within $|k_6(x^{4|w|+4})|$, steps, and so $(\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w)(w^{-1} t_o w) \rightarrow w^{-1} t_o w$ in $<\Delta;M>$ within at most $|k_6(x^{4|w|+4})|$ steps. Hence (6) doesn't need more than $|k_6(x^{4|w|+4})|+2|w|+1$ steps altogether.

ad (7), $2|w|+2$ trivial relators are deleted.

Taken altogether, there is a derivation from w' in $<\Delta;M>$ of length not exceeding $2|k_6(x^{4|w|+4})|+3|w|^2+42|w|+21$. Define

$$k'(w) \equiv vk(vk(k_6 U_x(w^8), k_6 U_x(w^8)), VK(VK(w,w), x^{66})).$$

Then $k' \in E_n(\underline{\Delta})$, k' is nondecreasing, for all $u,v \in \underline{\Delta}^*(|k'(u)|+|k'(v)| \leqslant |k'(uv)|)$ and for every $w' \in L_o'$ there is a derivation from w' in $<\Delta;M>$ of length bounded by $|k'(w')|$.

b) Let $w \in \underline{\Delta}^*$ with $|w|_z = 0$ and $w \overline{\overline{\scriptstyle H}}\, e$, and so $w \overline{\overline{\scriptstyle H}}\, e$. According to the proof of Proposition 1.6 (a), w can be derived to e in H_1 in the following way:

$$w \xrightarrow{(1)} w' \xrightarrow[\text{in } H_o]{(2)} \pi_{\underline{S}_6}(w')\pi_{\underline{\Sigma}_o'}(w') \xrightarrow[\text{in } D_6]{(3)} \pi_{\underline{\Sigma}_o'}(w') \xrightarrow[\text{in } G']{(4)} e$$

(d-pinches are pinched out in H_1, in step (1))

This derivation can be simulated in $<\Delta;M>$:

ad (1), d-pinches are pinched out in the following way:

$$d^{-\mu} u d^{\mu} \rightarrow \bar{d}^{\mu} u(\omega_\mu(u))^{-1} \omega_\mu(u) d^{\mu}$$

(Insertion of $|\omega_\mu(u)|$ trivial relators),

$$\rightarrow \bar{d}^{\mu} u_1 u_2 \omega_\mu(u) d^{\mu}$$

(Within $3(|u|+|\omega_\mu(u)|)^2$ steps $u(\omega_\mu(u))^{-1}$ can be transformed into $u_1 u_2$ where $u_1 \in \underline{S}_6^*$ and $u_2 \in \underline{\Sigma}_o'^*$),

$$\rightarrow \bar{d}^{\mu} u_2 \omega_\mu(u) d^{\mu}$$

$(u(\omega_\mu(u))^{-1} \overline{\overline{\scriptstyle H}}\, e$, and so $u_1 \overline{\overline{\scriptstyle D_6}}\, e$ and $u_2 \overline{\overline{\scriptstyle G'}}\, e$. But then u_1 can be derived to e in $<S_6;M_6>$ within at most $|k_6(u_1)|$ steps),

$$\rightarrow \bar{d}^{\mu} \omega_\mu(u) d^{\mu}$$

(In u_2, $\bar{s}'$ is substituted by s', and $\bar{\bar{s}}'$ is substituted by s': $\bar{s}' \rightarrow \bar{s}'s'\tilde{s}' \rightarrow \tilde{s}'$, and $\bar{\bar{s}} \rightarrow s'\tilde{s}'\bar{\bar{s}}' \rightarrow s'$. Let $\tilde{u}_2$ be the result of these substitutions. Then $\tilde{u}_2$ can be derived from u_2 within at most $2|u_2|$ steps. Since $e \,\overline{\overline{\scriptstyle G'}}\, u_2 \,\overline{\overline{\scriptstyle G'}}\, \tilde{u}_2$, $\tilde{u}_2 \in L_o'$, and because of (a), $\tilde{u}_2$ can be derived to e in $<\Delta;M>$ within no more than $|k'(\tilde{u}_2)|$ steps),

$$\rightarrow \varphi^{\mu}(u)$$

($7|\omega_\mu(u)|$ steps of the form: Insertion of trivial relators, deletion of trivial relators, and deletion of a d-relator).

Let $A_1 = \langle ss', \bar{k}_o s k_o (s \in \Sigma_o), t_o, \bar{k}_o t_o k_o \rangle_{H_o}$, $B_1 = \langle s, \bar{k}_o s k_o (s \in \Sigma_o), t_o, k_o t_o k_o \rangle_{H_o}$, and φ and $\bar{\varphi}$ denote function realizing the isomorphisms $A_1 \to B_1$ and $B_1 \to A_1$, respectively. According to [Av-Madl] Lemma 1.4, p.187, there are constants $c \geq 1$ and $d \geq 2$ sastisfying $|\omega_{A_1}(w)|$, $|\omega_{B_1}(w)|$, $|\varphi(w)|$, $|\bar{\varphi}(w)| \leq c|w|^d$. Hence for pinching out the d-pinch $\bar{d}^\mu u d^\mu$ one doesn't need more than

$$8c|u|^d + 3(c+1)^2|u|^{2d} + |k_6(x^{(c+1)|u|^d})| + 2(c+1)|u|^d + |k'(x^{(c+1)|u|^d})|$$
$$\leq 13(c+1)^2|u|^{2d} + k_6(x^{(c+1)|u|^d})| + |k'(x^{(c+1)|u|^d})|$$

steps in $\langle\Delta;M\rangle$. Let w_i' be the word formed from w by pinching out i d-pinches. Then by the proof of Prop. 1.6 (a),

$$|w_i'| \leq (c+1)^{d^{2i-1}}|w|^{d^i}.$$

Therefore every d-pinch $\bar{d}^\mu u d^\mu$ pinched out at (1) is bounded by

$$|u| \leq (c+1)^{d^{|w|}}|w|^{d^{|w|}}$$

Hence there is a function $k_1' \in E_n(N)$ bounding the number of steps needed for carrying out (1), since $n \geq 3$. Of course w' satisfies $|w'| \leq ((c+1)|w|)^{d^{|w|}}$.

ad (2), by using the commutation relators of H_o and some trivial relators, w' can be transformed into $\pi_{S_6}(w')\pi_{\Sigma_o}'(w')$, within at most $3|w'|^2$ steps. So this transformation can be bounded by a function $k_2' \in E_n(N)$.

ad (3), there is a derivation from $\pi_{S6}(w')$ in $\langle S_6;M_6\rangle$ consisting of no more than $|k_6 \circ \pi_{S6}(w')| \leq |k_6(x^{|w'|})|$ steps, and so there is a function $k_3' \in E_n(N)$ bounding this derivation.

ad (4), within at most $2|\pi_{\Sigma_o}'(w')|$ steps each $\bar{s}'$ and each $\tilde{s}'$ contained in $\pi_{\Sigma_o}'(w')$ can be substituted by $\tilde{s}'$ or s', respectively. In this way $\pi_{\Sigma_o}'(w')$ is transformed into a word $\tilde{w} \in L_o'$ which can be derived to e in $\langle\Delta;M\rangle$ within at most $|k'(x^{|\tilde{w}|})| \leq |k'(x^{|w'|})|$ steps because of (a). Hence (4) is bounded by a function $k_4 \in E_n(N)$, too.

So there is a function $\tilde{k} \in E_n(N)$ bounding the derivations from w to e in $\langle\Delta;M\rangle$ for all $w \in \Delta^*$ satisfying $|w|_z = 0$ and $w \underset{H}{=} e$.

c) Let $w \in \Delta^*$ with $|w|_z > 0$ and $w \underset{H}{=} e$, and so $w \underset{H_2}{=} e$. According to the proof of Prop. 1.6 (a), w can be derived to e in H_2 as follows:

$$w \xrightarrow{(1)} w' \xrightarrow[\text{in } H_1]{(2)} e$$

(z-pinches are pinched out in H_2, in steps (1)).
This derivation can be simulated in $\langle\Delta;M\rangle$:

ad (1), z-pinches are pinched out in the following way

$$\bar{z}^\mu u z^\mu \to \bar{z}^\mu u(\omega_\mu(u))^{-1}\omega_\mu(u) z^\mu$$

(Insertation of $|\omega_\mu(u)|$ trivial relators),

$$\to \bar{z}^{\mu}\omega_{\mu}(u)z^{\mu}$$

$(u(\omega_{\mu}(u))^{-1} \underset{H_1}{=} e$ and $|u(\omega_{\mu}(u))^{-1}|_z = 0$. Hence $u(\omega_{\mu}(u))^{-1}$ can be derived to e in $<\Delta;M>$ within at most $\tilde{k}(|u|+|\omega_{\mu}(u)|)$ steps because of (b)),

$$\to \varphi^{\mu}(u)$$

$(8|\omega_{\mu}(u)|$ steps of the form: insertion of trivial relators, deletion of a z-relator, and deletion of trivial relators). Let

$A_2 = <s, \bar{k}_o s k_o (s \in \Sigma_o), t_o, \bar{k}_o t_o k_o>_{H_1}$, $B_2 = <s, \bar{k}_o s k_o (s \in \Sigma_o), t_o d, \bar{k}_o t_o k_o d>_{H_1}$,

and φ and $\bar{\varphi}$ denote functions realizing the isomorphisms $A_2 \to B_2$ and $B_2 \to A_2$, respectively. Because of [Av-Madl] Lemma 1.5, p.187, there are constants $\alpha,\beta \geq 2$ satisfying:

$$|\omega_{A_2}(w)|,\ |\omega_{B_2}(w)|,\ |\varphi(w)|,\ |\bar{\varphi}(w)| \leq \alpha|w|^{\beta}.$$

Hence for pinching out the z-pinch $z^{\mu}uz^{\mu}$ one only needs $\alpha|u|^{\beta}+\tilde{k}((\alpha+1)|u|^{\beta}) + 8\alpha|u|^{\beta} = 9\alpha|u|^{\beta}+\tilde{k}((\alpha+1)|u|^{\beta})$ steps.

Let w_i' denote the word formed from w by pinching out i z-pinches. By the proof of Prop. 1.6 (1), $|w_i'| \leq (\alpha+1)^{\beta^{2i-1}}|w|^{\beta^i}$. Hence any z-pinch $\bar{z}^{\mu}uz^{\mu}$ pinched out at (1) satisfies $|u| \leq ((\alpha+1)|w|)^{\beta^{|w|}}$, and therefore the number of steps necessary to realize (1) can be bounded by a function $k_1'' \in E_n(N)$. Furthermore $|w'| \leq ((\alpha+1)|w|)^{\beta^{|w|}}$.

ad (2), $|w'|_z = 0$ and $e \underset{H}{=} w \underset{H}{=} w'$. Hence, because of (b), w' can be derived to e in $<\Delta;M>$ within at most $\tilde{k}(|w'|)$ steps and so, there is a function $k_H' \in E_n(\Delta)$ bounding the derivations of all words $w \in \Delta^*$ with $w \underset{H}{=} e$ in $<\Delta;M>$. Therefore $<\Delta;M>$ is E_n-d.b.

2.2. COROLLARY. *Every countable group* G *having an* E_n*-decidable word problem for some* $n \geq 3$ *can be embedded into a f.p. group* H *possessing a finite* E_n*-d.b. presentation.*

Proof. Every countable group G having an E_n-decidable word problem for some $n \geq 2$ can be embedded into a f.g. group G_1 having an E_n-decidable word problem too ([Ott] Thm. 12.1, p.117).

3. F.P. E_n-DERIVATION BOUNDED GROUPS AND THE WORD PROBLEM.

For finite E_n-d.b. presentations of groups there is a standard natural algorithm for solving the word problem. But of what degree of complexity is this algorithm, and how is this degree of complexity related to the selected finite presentation?

3.1. THEOREM. *Let* $H = <\Sigma;L>$ *be f.p. and* E_n*-d.b. for some* $n \geq 3$. *Then the standard natural algorithm for* $<\Sigma;L>$, *as it is described in the introduction,* *is an* E_n*-algorithm. In particular the word problem for* $<\Sigma;L>$ *is* E_n*-decidable.*

Proof. Let $\Sigma = \{s_1,..,s_m\}$, $L = \{w_1,..,w_\ell\} \subseteq \Sigma^*$, and $k \in E_n(\Sigma)$ be an E_n-bound for $<\Sigma;L>$. Without loss of generality $m \geqslant 3$ may be assumed, for otherwise auxiliary generators and defining relators can be added.

If $w \in \Sigma^*$ with $w \underset{H}{=} e$, then there is a derivation from w in $<\Sigma;L>$ of length not exceeding $|k(w)|$. During each step of this derivation a word $u \in Rel = L \cup L^{-1} \cup \{s\bar{s}, \bar{s}s \mid s \in \Sigma\}$ is inserted or deleted. L contains ℓ, and Σ contains m elements only. Hence there are only $2(\ell+m)$ possible choices for u. Define λ as the length of the longest possible word u. Then every word v found in that bounded derivation from w satisfies $|v| \leqslant |w|+\lceil\frac{\lambda}{2}\rceil\cdot k(w)|$, where $\lceil\mu\rceil$ denotes the least natural number greater than or equal to μ, because in order to derive a word of greater length from w more than $\frac{1}{2}|k(w)|$ steps are necessary, but then in order to derive this word to e more than $\frac{1}{2}|k(w)|$ steps are needed, again contradicting the fact that the derivation from w is bounded by $|k(w)|$. Define

$$\mu_w = |w|+\lceil\tfrac{\lambda}{2}\rceil\cdot|k(w)|.$$

A step of a derivation can be encoded as a triple (i_1,i_2,i_3) of natural numbers such that $i_1 \in \{0,1\}$, $i_2 \in \{1,2,\ldots,2(\ell+m)\}$, and $i_3 \in \{0,1,2,..,\mu_w\}$. Here $i_1 = 0$ stands for "insertion", $i_1 = 1$ for "deletion" of the relator with the number i_2 at the position described by i_3. Hence there are $\nu_w = 2\cdot 2\cdot(\ell+m)\cdot(\mu_w+1)$ different steps which can be chosen in a derivation of w. Therefore there are not more than $(\nu_m)^{|k(w)|}$ possible derivations from w of length $|k(w)|$. In order to decide $w \underset{H}{=} e$, it is sufficient to apply these derivations one after another to w, and to test whether one of these derivations produces e. Define $f_1(e) \equiv e$, $f_2(e) \equiv s_1$, $f_1(ws) \equiv f_2(w)$, $f_2(ws) \equiv vk(f_1(w),s_1)$ then $f_1,f_2 \in E_1(\Sigma)$, satisfying

$$f_1(w) \equiv s_1^{\lceil\frac{\lfloor w\rfloor}{2}\rceil}, \quad f_2(w) \equiv s_1^{\lceil\frac{\lfloor w\rfloor+1}{2}\rceil}$$

Let $ML(w) \equiv vk(U_{s_1}(w),VK(U_{s_1}\circ k(w),f_1(s_1^\lambda)))$ where $\lambda = \max_{u\in Rel}|u|$. Then $ML \in E_n(\Sigma)$ and

$$ML(w) \equiv s_1^{|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|} \equiv s_1^{\mu_w}$$

Each step in a derivation is described by a triple

$(i_1,i_2,i_3) \in \{0,1\}\times\{1,2,..,2(\ell+m)\}\times\{0,1,..,\mu_w\}$, and so it can be encoded as a word over Σ, namely as

$$s_1^{i_1+1}\ s_2^{i_2}\ s_3^{i_3+1}$$

which is a word of length not exceeding $2+2(\ell+m)+\mu_w+1 = 2(\ell+m)+3+|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|$. Hence a derivation of w can be described by a word of length at most

$$(2(\ell+m)+3+|w|+\lceil\tfrac{\lambda}{2}\rceil\cdot|k(w)|)\cdot|k(w)|.$$

Let

$$LDA(w) \equiv VK(vk(s_1^{2(\ell+m)+3},ML(w)),k(w))$$

then $LDA \in E_n(\Sigma)$ satisfying

$$LDA(w) \equiv s_1^{(2(\ell+m)+3+|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|)\cdot|k(w)|}.$$

In order to decide whether $w \underset{H}{=} e$ is valid or not one only has to check whether there is a word u of length at most $|LDA(w)|$ describing a derivation from w to e in $\langle\Sigma;L\rangle$. Now a Turing Machine M will be defined to test for a pair $(w,u) \in (\Sigma^*)^2$ whether u is the description of a derivation from w, by trying yo apply u to w. In an initial part of u is the description of a derivation from w to e, then M will halt with its output tape being empty, but if u doesn't meet this condition, then M will print the letter "s_1" and halt.

Let M have two input tapes, one output tape, and four auxiliary tapes.

1) w is the inscription of the first input tape, and u is the inscription of the second one.

2) w is copied onto the first auxiliary tape, while u is copied onto the second one. This can be done within $2|w|+2|u|+3$ steps. i.e. *amount of time* (*A.t.*) $= 2|w|+2|u|+3$.

3) The elements of the set Rel are printed onto the third auxiliary tape separated by a "b", respectively.

A.t. $\leqslant 2\,(\lambda+1)\cdot 2\cdot(\ell+m) \leqslant 8\lambda(\ell+m)$.

I1:

..b	w	b..

↑

I2:

..b	u	b..

↑

A1:

..b	w	b..

↑

A2:

..b	u	b..

↑

A3:

..b	w_1	b	w_1^{-1}	b	w_2	b..b	w_1^{-1}	b	$s_1\bar{s}_1$	b..b	$\bar{s}_m s_m$	b..

↑

4) If u starts with a letter $s \neq s_1$, then outputs s_1 and halts. *A.t.* = 3.

If u starts with s_1, then mind "insertion". *A.t.* = 3.

If u starts with s_1^2, then mind "deletion". *A.t.* = 4.

If u starts with s_1^i for an $i > 2$, then outputs s_1 and halts. *A.t.* = 4.

A2:

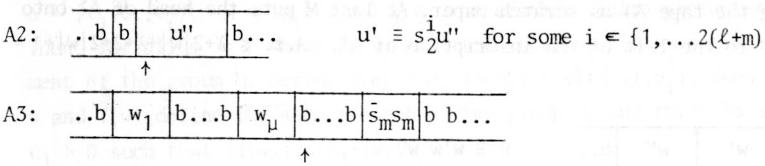
..b	b	u'	b..

↑

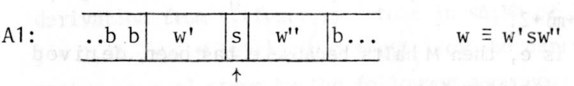
$u \equiv s_1^i u'$ for some $i \in \{1,2\}$.

If u' starts with a letter $s \neq s_2$, then outputs s_1 and halts. *A.t.* = 2.

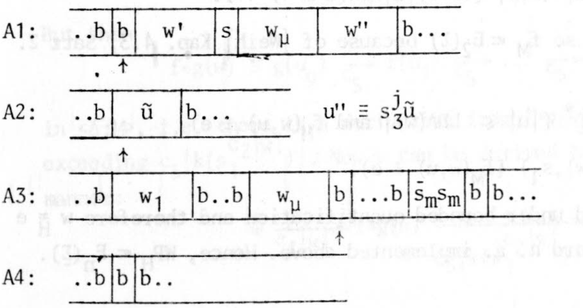
If u' starts with s_2^i, then for i-1 times M puts the head of its third auxiliary tape onto the next symbol "b" to the right of the actual position of the head. After that this head performs one step to the right. *A. t.* $\leqslant i(\lambda+1)+1$.
If M reads a "b" on its third auxiliary tape, then output s_1 and halts. *A. t.* = 2.
Otherwise, the head of A3 is pointing to the relator which shall be inserted or deleted from w.



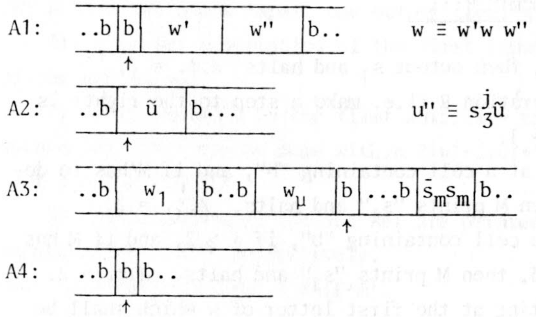
If u'' starts with a letter s ≠ s_3, then output s_1 and halts *A.t.* = 2.
If u'' starts with s_3^j, then the operation R (i.e. make a step to the right) is executed on A1, j-1 times. *A.t.* = j.
If the head of A1 is now pointing at a cell containing "b", and if M has to delete the relator marked on A3, then M prints "s_1" and halts. *A.t.* = 2.
If the head of A1 is pointing at a cell containing "b", if j $\geqslant$ 2, and if M has to insert the relator marked on A3, then M prints "s_1" and halts. *A.t.* = 2.
Otherwise, the head of A1 is pointing at the first letter of w which shall be erased or behind which the indicated relator shall be inserted.



5) ***Insertion:*** The indicated relator is copied from A3 onto A4, subsequently w'' is appended at the rigth end of this copy, and at last w'' is erased from A1. *A.t.* $\leqslant \lambda+|w|+1$.
If j = 1, then the inscription of A4 is copied onto A1, in the course of which it is erased from A4. Otherwise the inscription of A4 is appended to the inscription of A1 (w's), at which it is erased from A4. The head of A1 is put onto the first "b" to the left of the inscription of A1.
A.t. $\leqslant |w|+2(|w|+\lambda+1)+|w|+\lambda+1 = 4|w|+3\lambda+3$.



Deletion. The indicated relator is compared to the subword of w, beginning at the position the head of A1 is pointing at. By doing so, the subword of w is erased. If this subword of w and the indicated relator do nos coincide, then M prints "s_1" and halts. Otherwise an initial part or an internal segment of w has been erased. In the first case the head of A1 performs one step to the left, in the second case M appends the remained end of w to the remained initial part by using the tape A4 as scratch paper. At last M puts the head of A1 onto the first "b" to the left of the inscription of A1. *A.t.* $\leqslant \lambda+2|w|+\lambda+2+2|w|+1 = 4|w|+2\lambda+3$.



6) The head of tape A3 returns to the left.
A.t. $\leqslant (\lambda+1)\cdot 2\cdot(1+m)+2 \leqslant 4\lambda(1+m)+2$.
If the inscription of tape A1 is e, then M halts because e has been derived from w. Otherwise M continues with step (4).
A.t. = 2.
Of course M eventually halts satisfying $f_M(w,u) \equiv e$ iff an initial part of u is describing a derivation from w. Altogether M has the following amount of time.

$$T_M(w,u) \leqslant 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot\{4+|u|(\lambda+1)+1+2+|u|+2+ 5(|w|+\lambda|u|)+4\lambda+4+4\lambda(\ell+m)+2+2\}.$$

(In the course of the computation w may grow, but it cannot become larger than $|w| + |u|$)

$$= 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot\{5|w|+(6\lambda+2)|u|+4\lambda(\ell+m+1)+17\}.$$

But λ,ℓ,m are constants, and so $f_M \in E_2(\Sigma)$ because of [Weih] Kap. 4.3, Satz 2. Now we have:

$$w \underset{H}{=} e \quad \text{iff} \quad \exists u \in \Sigma^* \; (|u| \leqslant \text{LDA}(w)| \text{ and } f_M(w,u) \equiv e)$$

$$\text{iff} \quad \exists u \leqslant vk(\text{LDA}(w),s_1) \; (f_M(w,u) \equiv e).$$

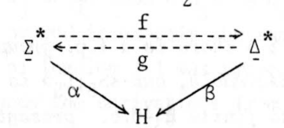
But as $n \geqslant 3$, $E_n(\Sigma)$ is closed under bounded quantification and therefore $w \underset{H}{=} e$ is E_n-decidable by the standard n. a. implemented above. Hence, $WP_H \in E_n(\Sigma)$.

Next we prove that E_n-derivation boundedness is an invariant of finite presentations.

3.2. THEOREM. *Let* $H = \langle\Sigma;L\rangle$ *be* f.p. *and* E_n-d.b. *for some* $n \geq 1$. *Then every finite presentation for* H *is* E_n-d.b., *too*.

Proof. Let be Σ, L, and k as in the proof of Theorem 3.1, and let $\langle\Delta;M\rangle$, $\Delta = \{t_1,..,t_r\}$, $M = \{u_1,..,u_s\} \subseteq \underline{\Delta}^*$, be another finite presentation for H. Then, for all $s_i \in \Sigma$ there is $v_i \in \underline{\Delta}^+$ such that s_i and v_i define the same element of the group H. Define $f(e) \equiv e$, $f(ws_i^{\mu}) \equiv vk(f(w),v_i^{\mu})$. Then for all $w \in \underline{\Sigma}^*$, w and f(w) define the same element of the group H, and there is a constant $c_1 > 0$ such that $|f(w)| \leq c_1 \cdot |w|$.

$\forall t_j \in \Delta\, \exists x_j \in \underline{\Sigma}^+$, t_j and x_j define the same element of H. Moreover, $g(e) \equiv e$, $g(wt_j^{\mu}) \equiv vk(g(w),x_j^{\mu})$. Then for all $w \in \underline{\Delta}^*$, w and g(w) define the same element of H, and there is a constant $c_2 > 0$ such that $|g(w)| \leq c_2 \cdot |w|$.



Then $\beta(w) \underset{H}{=} \alpha\circ g(w) \underset{H}{=} \beta\circ f\circ g(w)$, and so $w \underset{H}{=} f\circ g(w)$. Also $|f\circ g(w)| \leq c_1|g(w)| \leq c_1 \cdot c_2 \cdot |w|$. Especially $t_j^{\mu}(f\circ g(t_j^{\mu}))^{-1} \underset{H}{=} e$. Hence for each $t_j^{\mu} \in \underline{\Delta}$ there is a derivation from $t_j^{\mu}(f\circ g(t_j))^{-1}$ to e in $\langle\Delta;M\rangle$ of length $\ell_{j,\mu}$. If $c_3 = \max\{\ell_{j,\mu} \mid j = ,\ldots,r, \mu \in \{\pm 1\}\}$, then $f\circ g(t_j)$ can be derived from t_j^{μ} in $\langle\Delta;M\rangle$ within at most $c_4 = c_3+1$ steps by the following sequence:

$$t_j^{\mu} \xrightarrow[c_3]{} t_j^{\mu}t_j^{\mu}(f\circ g(\bar{t}_j^{\mu}))^{-1} \xrightarrow[T]{} (f\circ g(\bar{t}_j^{\mu}))^{-1} \equiv f\circ g(t_j^{\mu}).$$

Hence every word $w \in \underline{\Delta}^*$ can be derived to $f\circ g(w)$ within $c_4|w|$ steps.

For every $u \in Rel$, $f(u) \underset{H}{=} e$, and therefore there is a derivation from f(u) to e in $\langle\Delta;M\rangle$ of length ℓ'_u. If $c_5 = \max\{\ell'_u \mid u \in Rel\}$, then f(u) can be derived to e in $\langle\Delta;M\rangle$ within no more than c_5 steps. Let $w \in \underline{\Delta}^*$ with $w \underset{H}{=} e$, then g(w) $\underset{H}{=}$ e, too. Hence there is a derivation from g(w) to e in $\langle\Sigma;L\rangle$ of length not exceeding $|k\circ g(w)| \leq |k(s_1^{c_2|w|})|$:

$$g(w) \equiv u_o \to u_1 \to .. \to e.$$

But then

$$f\circ g(w) \equiv g(u_o) \xrightarrow[c_5]{} f(u_1) \xrightarrow[c_5]{} \ldots \xrightarrow[c_5]{} f(e) \equiv e$$

in $\langle\Delta;M\rangle$, i.e. there is a derivation from $f\circ g(w)$ to e in $\langle\Delta;M\rangle$ of length not exceeding $c_5|k(s_1^{c_2|w|})|$. Now w can be derived to e in $\langle\Delta;M\rangle$ in the following manner:

$$w \xrightarrow[c_4|w|]{} f\circ g(w) \xrightarrow[c_5|k(s_1^{c_2|w|})|]{} e.$$

Of course there is an E_n-function bounding this derivation. Hence <Σ;M> is E_n-d.b.

The last theorem shows that the property of being E_n-d.b. does not depend on the chosen finite presentation. It merely depends on the group. Hence a f.p. group is called E_n-d.b. if one, and therewith each, of its finite presentations is E_n-d.b. A conclusion of the proof of the last theorem is the fact that even every f.g. presentation of a f.p. E_n-d.b. group is E_n-d.b. But of course each f.p. E_n-d.b. group has a f.g. E_o-d.b. presentation, i.e. $<\Sigma;\{w \in \Sigma^* | w \underset{G}{=} e\}>$ for example. Therefore the property of being E_n-d.b. does depend on the chosen f.g. presentation of a group.

It remains to answer the question wheter for f.p. E_n-d.b. groups with $n \geq 3$ an optimal n.a. exists. The following theorem gives an answer in the negative sense.

3.3. THEOREM. *For every* $n \geq 4$ *there is a* f.p. *group* $G_5 = <S_5;L_5>$ *such that the word problem for* G_5 *is* E_3*-decidable, but* $<S_5;L_5>$ *is only* E_n*-, but not* E_{n-1}-d.b. *Especially there is no finite* E_3-d.b. *presentation for* G_5.

Proof. Let $n \geq 4$. The f.p. group G_5 will now be constructed in the same manner as the group D_5 has been constructed in the proof of Theorem 2.1. Only the underlying Turing Machine will be modified. Let $S' = \{s_1,s_2,s_3\}$ and $L = S'^*$, and let $T = (S',Q_T,q_o,\beta)$ be a single tape machine acting as follows. For every $w \in S'^*$, starting at $q_o w$, T computes $A_n(w,w)$ where $A_n \in E_n(S')$ denotes the n-th Ackermann function over S' ([Weih]). After that T enters the accepting state q_a and halts. For carrying out this computation T has to execute more than $|A_n(w,w)|$ steps. On the other hand, T can be chosen in such a way that there exists a function $g \in E_n(S')$ which bounds the time, i.e. the number of steps T needs for its computation ([Weih] Kap.4.4, Satz 1).

Now T can be modified to get $\tilde{T} = (\tilde{S},Q_{\tilde{T}},q_o,\tilde{\beta})$, where $\tilde{S}$ is a finite alphabet containing S' such that there is a function $k_{\tilde{T}} \in E_n(\tilde{S} \cup Q_{\tilde{T}})$ satisfying.

$\forall u,v \in S^* \forall q_j \in Q_{\tilde{T}}$, starting at the configuration uq_jv, $\tilde{T}$ halts in the accepting state q_a within at most $|k_{\tilde{T}}(uq_jv)|$ steps.

This modification is done in the same way as the one used in the proof of theorem 2.1, with the only exception that the non-accepting state q_- is omitted, i.e. instead of entering q_-, $\tilde{T}$ enters the accepting state q_a. Since for every $w \in S'^*$, starting at $q_o w$, T halts in the state q_a, $\tilde{T}$ also halts in the state q_a, starting at any configuration uq_jv. The execution time of T is bounded by the function $g \in E_n(S')$. Hence there is a function $k_{\tilde{T}} \in E_n(\tilde{S} \cup Q_{\tilde{T}})$ satisfying the condition formulated above. Of course, starting at $q_o w$, $\tilde{T}$ has to carry out more than $|A_n(w,w)|$ steps for avery $w \in S'^*$, too.

CLAIM. *Let* $S = \tilde{S} \cup \{h\}$, $Q = Q_T \cup \{q\}$ *and* $\Delta = (S \cup Q;\pi)$, *where* $\pi = \{F_i q_{i_1} G_1 = H_i q_{i_2} K_i \mid q_{i_1}, q_{i_2} \in Q, F_i, G_i, H_i, K_i \in S^*, i = 1,\dots,N\}$ *is the semigroup constructed from* T *according to* [Av-Madl], *p.89. Then the following three conditions are satisfied:*

(1) $\forall u,v \in S^* \forall q_j \in Q\ (uq_jv \underset{\Delta}{=} q \iff uq_jv \equiv q$ or $u \equiv hu'$, $v \equiv v'h$, with $u',v' \in \tilde{S}^*$ and $q_j \neq q)$.

(2) $\forall w \in S^* \forall q_j \in Q\ (uq_jv \underset{\Delta}{=} q \rightarrow \exists$ derivation from uq_jv to q in Δ, of length not exceeding $2|k_{\tilde{T}}(uq_jv)|+|uq_jv|)$.

(3) $\forall w \in S'^*$ $(hq_owh \underset{\Delta}{=} q$, but there is no derivation from hq_owh to g in Δ of length $\leqslant |A_n(w,w)|)$.

Proof.

ad (1) "$\Rightarrow$". Let $uq_jv \underset{\Delta}{=} q$, but $uq_jv \not\equiv q$. Then $q_j \neq q$, $u \equiv hu'$ and $v \equiv v'h$ for some $u',v' \in \tilde{S}^*$.

"$\Leftarrow$". Let $u',v' \in \tilde{S}^*$, $q_j \in Q_{\tilde{T}}$. Then $u'q_jv' \underset{\tilde{T}}{\overset{*}{\rightarrow}} q_a$, and so $hu'q_jv'h \underset{\Delta}{=} hq_ah \underset{\Delta}{=} q$.

ad (2) This can be proved in exactly the same way as the corresponding statement in the proof of Theorem 2.1 was proved. Hence there is a function $k_\Delta \in E_n(S \cup Q)$ which bounds the derivations from $w \in (S \cup Q)^*$ to q in Δ if $w \underset{\Delta}{=} q$.

ad (3) Δ simulates $\tilde{T}$, step by step. But starting at q_ow, $\tilde{T}$ has to execute more than $|A_n(w,w)|$ steps before reaching q_a. Therefore Δ has a carry out more than $|A_n(w,w)|$ steps to reach q, too, when started at hq_owh.

Now a Britton tower of groups is constructed:

$G_o = \langle x,\emptyset\rangle$, $S_o = \{x\}$,

$G_1 = \langle G_o,S;\bar{s}xs = x^2 (s \in S)\rangle$, $S_1 = S_o \cup S$,

$G_2 = \langle G_1,Q;\emptyset\rangle$, $S_2 = S_1 \cup Q$,

$G_3 = \langle G_2,R;\bar{r}_i\bar{F}_iq_{i_1}G_ir_i = \bar{H}_iq_{i_2}k_i, \bar{r}_isxr_i = s\bar{x} (s \in S, 1 \leqslant i \leqslant N)\rangle$, $S_3 = S_2 \cup R$,

$G_4 = \langle G_3,t;\bar{t}xt = x, \bar{t}rt = r (r \in R^2)\rangle$, $S_4 = S_3 \cup \{t\}$,

$G_5 = \langle G_4,k;\bar{k}ak = a (a \in \{x,\bar{q}tq\} \cup R)\rangle$, $S_5 = S_4 \cup \{k\}$, $R_x = R \cup \{x\}$.

Of course $G_o,G_1,\dots,G_5$ are f.p. Furthermore they satisfy ([Av-Madl]):

(α) For $i = 1,\dots,4$, G_i is an HNN-extension of G_{i-1}, there is a reduction function $f_i \in E_3(\underline{S}_i)$ for G_i, and the word problem for G_i is E_3-decidable.

(β) There is a function $g \in E_3(\underline{S}_3)$ satisfying:

- $\forall w \in \underline{S}_3^*$ $(g(w) \underset{G_3}{=} uw$ for some $u \subset \underline{R}_x^*)$.
- If $w \in \underline{S}_3^*$ is R-reduced, there is no $u \in \underline{R}_x^*$ such that there is a R-pinch in $ug(w)$ just on the border $u - g(w)$.
- If $w \in \underline{S}_3^*$ is R-reduced, and if $g(w) \equiv ur_i^{\mu}v$ where $u \in \underline{S}_2^*$, $v \in \underline{S}_3^*$, then w has the form $u'r_i^{\mu}v$ for some $u' \in \underline{S}_3^*$.

(γ) Over $\underline{S}_3^*$ define the predicate: $\tilde{P}(u) \iff \exists w_1,w_2 \in \underline{R}_x^* (w_1uw_2 \underset{G_3}{=} q)$.

- If $u \in \underline{S}_3^*$ is reduced and $v \equiv g((g(u))^{-1})^{-1}$, then:
  $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$.
- If $v \in \underline{S}_2^*$ is reduced and v' is the result of deleting all x and $\bar{x}$ symbols of v, then:

$\tilde{P}(v)$ iff $\exists X,Y \in S^*$, $q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $\tilde{P}(Xq_jY)$).
- $\forall X,Y \in S^*$, $q_j \in Q$ ($\tilde{P}(\bar{X}q_jT)$ iff $Xq_jY \underset{\Delta}{=} q$).

Assertion. $\tilde{P} \in E_3(\underline{S}_3)$.

Proof. Let $u' \in \underline{S}_3^*$. Then $u \equiv f_3(u')$ satisfies $u' \underset{\bar{G}_3}{=} u$, and so $\tilde{P}(u')$ iff $\tilde{P}(u)$. Let $v \equiv g((g(u))^{-1})^{-1}$. Then because of (γ), $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$, since u is reduced. Let $\tilde{v} \equiv f_2(v)$, and $v' \equiv \pi_{\underline{S} \cup Q}(\tilde{v})$. If $v \in \underline{S}_2^*$, then the following is true because of (γ):

$$\tilde{P}(v) \text{ iff } \exists X,Y \in S^*, q_j \in Q \ (v' \equiv \bar{X}q_jY \text{ and } \tilde{P}(\bar{X}q_jY)).$$

Altogether we have thus:
$\tilde{P}(u')$ iff $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$
iff $v \in \underline{S}_2^*$ and $\exists X,Y \in S^*, q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $\tilde{P}(\bar{X}q_jY)$)
iff $v \in \underline{S}_2^*$ and $\exists X,Y \in S^*, q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $Xq_jY \underset{\Delta}{=} q$).
But $u, v, \tilde{v}$, and v', and therewith also Xq_jY, are E_3-computable from u'. $Xq_jY \underset{\Delta}{=} q$ is E_1-decidable because of (1). Hence $\tilde{P} \in E_3(\underline{S}_3)$.

Now let $u \in \underline{S}_4^*$ be such that $f_4(u) \equiv u_o t^{\mu_1} u_1 .. t^{\mu_m} u_m$, $u_i \in \underline{S}_3^*$, $\mu_i \in \{\pm 1\}$. According to the proof of [Av-Mad] Lemma 4.9, p.102, the following assertion is satisfied:

$$u \in \langle x, \bar{q}tq, R\rangle_{G_4} \text{ iff } u_o u_1 .. u_m \in \langle x, R\rangle_{G_3} \text{ and } \bigwedge_{i=0}^{m-1} \tilde{P}((u_o u_1 .. u_i)^{-1}).$$

But $\langle x,R\rangle_{G_3}$ is E_3-decidable because of the proof of [Av-Madl] Lemma 4.6, p.100. Hence $\langle x,\bar{q}tq,R\rangle_{G_4}$ is E_3-decidable and so G_5 is an E_3-admissible HNN-extension of G_4. Hence $WP_{G_5} \in E_3(\underline{S}_5)$.

According to [Ott] §15, pp.156-173, the presentation $\langle S_5;L_5\rangle$ of G_5 is E_n-d.b.

Now let $w \in S'^*$, then $q_o w \underset{T}{\overset{*}{\to}} ..q_a..$; so $q_o w \underset{T}{\overset{*}{\to}} ..q_a..$, and therefore $hq_o wh \underset{\Delta}{=} q$. $\bar{k}\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whk \underset{\bar{G}_5}{=} \bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ according to [Rot] Lemma 12.13, p.229. Therefore, there is a derivation from $\bar{k}\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whk\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ to e in $\langle S_5;L_5\rangle$. During this derivation $\bar{k}$ and k must be eliminated by using relators of the form $\bar{k}aka^{-1}$ ($a \in \{x,\bar{q}tq\} \cup R$). But for that, $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ must be rewritten into a word $u \in (\underline{\{x,\bar{q}tq\} \cup R})^*$. Let $u \equiv u_o \bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_1}qu_1$, $u \in \underline{R}_x$, $\mu_i \in \{\pm 1\}$ be such that

$$\bar{h}w^{-1}\bar{g}_o ht\bar{h}q_o wh \underset{\bar{G}_4}{=} u \equiv u_o\bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_1}qu_1.$$

$\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ is t-reduced in G_4. Hence there is an $i \in \{1,\dots,\ell\}$ such that $u \equiv u_o\bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_\ell}u_\ell \underset{\bar{G}_4}{=} u_o..u_{i-1}\bar{q}tqu_iu_{i+1}..u_\ell \underset{\bar{G}_4}{=} \gamma_f(u_o..u_{i-1})\bar{q}tq\gamma_f(u_i..u_\ell)$, where γ_f denotes the free reduction. Then $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh \underset{\bar{G}_4}{=} u \underset{\bar{G}_4}{=} v_1\bar{q}tqv_2$ with $v_1 \equiv \gamma_f(u_o..u_{i-1})$ and $v_2 \equiv \gamma_f(u_i..u_\ell)$. So, $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whv_2^{-1}\bar{q}\bar{t}\bar{q}v_1^{-1} \underset{\bar{G}_4}{=} e$. Hence there is a $v_3 \in \underline{R}_x^*$ freely reduced with $\bar{h}q_o whv_2^{-1}\bar{q} \underset{\bar{G}_3}{=} v_3$. But $v_3^{-1}\bar{h}q_o whv_2^{-1} \underset{\bar{G}_3}{=} q$ with $v_3^{-1}, v_2^{-1} \in \underline{R}_x^*$ freely reduced. So $|v_3^{-1}|_R = |v_2^{-1}|_R$. According to the proof of [Rot] Lemma 12.18, p.304, $\pi_R(v_2^{-1})$ describes a derivation from hq_owh to q in Δ. Because of (3) such a derivation contains more than $|A_n(w,w)|$ steps. This means $|v_2^{-1}|_R > |A_n(w,w)|$, and therefore $|A_n(w,w)| \leq |v_2^{-1}|_R \leq |v_2^{-1}| \leq |u_1..u_1| \leq |u| - 3$. Therefore, a word of length $2|w|+7$, namely $\bar{h}w^{-1}\bar{q}_oht\bar{h}q_owh$, is substituted by a word of length $> |A_n(w,w)|+3$, namely u.

Let $\alpha = \max\{|y|: y \in L_5 \cup L_5^{-1} \cup \{s\bar{s},\bar{s}s | s \in S_5\}\}$. Then in order to construct a word of length $> |A_n(w,w)|+4$ from a word of length $2|w|+7$, at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps are necessary. Hence a derivation from $\bar{h}w^{-1}\bar{q}_oht\bar{h}q_owh$ to a word $u \in (\{x,\bar{q}tq\} \cup R)^*$ needs at leas $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps. Therefore every derivation from $\bar{k}\bar{h}w^{-1}\bar{g}_oht\bar{h}q_owhk\bar{h}w^{-1}\bar{q}_ot\bar{h}q_owh$ to e in $<S_5;L_5>$ needs at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps, i.e. in order to derive a word of length $3|w|+16$ to e in $<S_5;L_5>$ at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps are necessary.

Hence $<S_5;L_5>$ is not E_{n-1}-d.b., which proves Theorem 3.3.

3.4. COROLLARY. *For every* $n \geq 4$ *there is a f.p. group having an* E_3*-decidable word problem such that each finite presentation of this group is* E_n*-, but not* E_{n-1}-d.b.

Proof. Theorem 3.3 and Theorem 3.2.

3.5. COROLLARY. *For every* $4 \leq m < n$ *there is a f.p. group such that the word problem for this group is* E_m*-, but not* E_{m-1}*-decidable, and each finite presentation of this group is* E_n*-, but not* E_{n-1}-d.b.

Proof. Let $G_1 = <\Sigma_1;L_1>$ be f.p. having an E_3-decidable word problem and being E_n-, but not E_{n-1}-d.b. (3.3). Let $H = <\Delta;M>$ be f.g. having an E_m-, but not E_{m-1}-decidable word problem. Then there is a group $G_2 = <\Sigma_2;L_2>$ which is f.p. and E_m-d.b. s.t. $H \hookrightarrow G_2$ (2.1). According to 3.1, G_2 has an E_m-decidable word problem. The word problem of G_2 is not E_{m-1}-decidable since the word problem of H is not either. Hence G_2 is not E_{m-1}-d.b. because of 3.1. Let $G = G_1 * G_2 = <\Sigma_1 \cup \Sigma_2;L_1,L_2>$. Then G is f.p., the word problem for G is E_m-, but not E_{m-1}-decidable, and the given presentation of G, and therewith each finite presentation of G, is E_n-, but not E_{n-1}-d.b. (1.5 a)).

This last corollary shows that even for f.p. groups the complexity of a n.a. for solving the word problem can be of an arbitrarily higher degree than the complexity of the word problem itself.

3.6. REMARK. According to a remark in [Av-Madl], p.93, the word problem of the group G_5 constructed in the proof ot Theorem 3.3 is even E_2-decidable, since the special word problem of the underlying semigroup Δ is E_1-decidable because of (1), p.155. Hence for every $n \geq 3$ there is a f.p. group having an E_2-decidable word problem and being E_n-, but not E_{n-1}-d.b.

4. NATURAL E_n-ALGORITHMS FOR E_n-DECIDABLE GROUPS.

For f.p. groups the property of E_n-derivation-boundedness leads to a natural E_n-algorithm for solving the word problem of the group. If a presentation has infinitely many relators we have infinitely many possibilities of inserting a relator in each step of a derivation, but only a finite number of deletions of a defining relator are possible, since only subwords are deleted. For non-f.p. groups a stronger concept of derivation-boundedness is therefore needed which guarantees the existence of a natural algorithm of the same complexity. There are several different possible definitions of d.b. group presentations for non-f.p. groups. We choose the following one, in which the allowed derivations are restricted.

4.1. DEFINITION. Let $G = \langle\Sigma;L\rangle$ f.g. The presentation $\langle\Sigma;L\rangle$ is *strongly* E_n-*derivation bounded* (s. E_n-d.b.) if there is a function $k \in E_n(\Sigma)$ such that for any $w \underset{G}{=} e$ in Σ^*, there is a derivation $w \equiv w_0 \to w_1 \to \ldots \to w_\ell \equiv e$ in $\langle\Sigma;L\rangle$ such that (i) $\ell \leq |k(w)|$, (ii) only trivial relators are inserted. Such a derivation is called a *strongly* E_n-*bounded derivation.*

4.2. OBSERVATION. a) Let $G = \langle\Sigma;L\rangle$ f.p. Then for all $n \geq 1$, $\langle\Sigma;L\rangle$ is s.-E_n-d.b. iff $\langle\Sigma;L\rangle$ is E_n-d.b. (The insertion of a relator u can be simulated by the insertion of uu^{-1} by using trivial relators and the delection of u^{-1}. So the length of the derivation is at most increased by the factor $\mu = (\max\{|u|: u \in L\}+1)$.

b) Let $n,p \geq 0$, and $g := \max\{n,p,3\}$. If $G = \langle\Sigma;L\rangle$ is s.E_n-d.b. with $L \subseteq \Sigma^+$, E_p-decidable, then there is a natural algorithm $x \in E_q(\Sigma)$ for the word problem of $\langle\Sigma;L\rangle$, i.e.

$$x(w) \equiv \begin{cases} (w_0,w_1,\ldots,w_\ell) & \text{if } w \underset{G}{=} e \text{, and } w \equiv w_0 \to w_1 \to \ldots \to w_\ell \equiv e \text{ is a strongly} \\ & E_n\text{-bounded derivation from } w \text{ to } e \text{ in } \langle\Sigma;L\rangle. \\ \# & \text{if } w \underset{G}{\neq} e. \end{cases}$$

The proof of this fact is similar to the proof of Theorem 3.1. The only difference is that only strongly E_n-bounded derivations are considered.

c) The property of being strongly E_n-d.b. is dependent on the chosen presentation of the group. Let $n \geq 2$, $\Delta = \{a_i | i \geq 1\}$, and

$$G = \langle\Delta; a_1^i (i \geq 1), a_i a_1^{A_{n+1}(i,i)} (i \geq 2)\rangle,$$

where A_{n+1} is the n+1st Ackermann-function ([Rit]Def.1.1, p.1028).

For all $w \in \underline{\Delta}^*$, $w \underset{G}{=} e$, i.e. $G \cong \langle e\rangle$, and so $WP_G \in E_1(\underline{\Delta})$. Let $F = \langle b,c;\emptyset\rangle$ and $K = F*G \cong F$. Then $WP_K \in E_1(\underline{\Delta} \cup \underline{\{b,c\}})$. Finally let

$$H = \langle K,t;\bar{t}b^n cb^n cb^n cb^n t = b^n cb^n a_n cb^n cb^n :\ n \geq 1\rangle$$

$$\cong \langle b,c,t;\ (\bar{b}\bar{c}\bar{b}\bar{t}bcbcbctb\bar{c}\bar{b}\bar{c})^i (i \geq 1), \bar{b}^i\bar{c}\bar{b}^i\bar{t}b^i cb^i cb^i cb^i t\bar{b}^i\bar{c}\bar{b}^i\bar{c}(\bar{b}\bar{c}\bar{b}\bar{t}bcbcbctb\bar{c}\bar{b}\bar{c})^{A_{n+1}(i,i)} :\ i \geq 2\rangle$$

$$=: \langle\Sigma;L_{n+1}\rangle$$

$$\cong \langle\Sigma;\bar{t}b^i cb^i cb^i cb^i t\bar{b}^i\bar{c}\bar{b}^i\bar{c}\bar{b}^i\bar{c}\bar{b}^i :\ i \geq 1\rangle =: \langle\Sigma;L'\rangle.$$

Then $\langle\Sigma;L'\rangle$ is s.E_2-d.b. and $\langle\Sigma;L_{n+1}\rangle$ is s.E_{n+1}-d.b. but not s.E_n-d.b.

Since there are f.p. groups with E_3-decidable word problem for which no finite presentation allows a natural E_3-algorithm (the group $G_5 = \langle S_5;L_5\rangle$ in 3.3 has this property), we ask whether there is an infinite strongly E_3-d.b. presentation for this group, and further on whether this is the case for all E_n-decidable f.g. groups.

For the group G_5 we get that the presentation

$$\langle S_5;L_5,\bar{k}\bar{t}Y^{-1}\bar{q}_j\bar{X}^{-1}ht^{\varepsilon}\bar{h}Xq_jYhk\bar{h}Y^{-1}\bar{q}_j\bar{X}^{-1}h\bar{t}^{\varepsilon}\bar{h}Xq_jYh:\ \varepsilon \in \{\pm 1\}, X,Y \in \tilde{S}^*,\ q_j \in Q-\{q\}\rangle.$$

has an E_1-decidable set of defining relators, and that it is in fact s.E_3-d.b. So a natural E_3-algorithm exists for this special presentation. We want to prove that such easy presentation can be constructed for all E_n-decidable f.g. groups ($n \geq 3$). Therefore we need the following technical lemma, which is proved by standard methods.

4.3. LEMMA. *Let Σ with $|\Sigma| > 1$, $t \in \Sigma$, be a finite alphabet, and $\emptyset \neq L \subseteq \Sigma^*$ be E_n-decidable for some $n \geq 3$. Then there is a function $g \in E_1(\Sigma)$ such that*

(a) $g(\{t^i \mid i \geq 0\}) = L$.

(b) *There exists a function $k \in E_n(\Sigma)$ satisfying:*
$\forall w \in \Sigma^* (w \in L \rightarrow \exists i \leq |k(w)|:\ g(t^i) \equiv w)$,

i.e. L is enumerated by an E_1-function g such that for each word w an index can be calculated by an E_n-function.

4.4. THEOREM. *Let $G = \langle\Sigma;L\rangle$ be f.g. with E_n-decidable word problem for some $n \geq 3$, and let $t \notin \Sigma$. Then G has a non-finite presentation $\langle\Sigma,t;L_g\rangle$ such that*

(1) $L_g \subseteq (\underline{\Sigma} \cup \{t\})^*$ *is E_1-decidable.*

(2) $\langle\Sigma,t;L_g\rangle$ *is strongly E_n-d.b.*

Proof. Let $\tilde{L} := \{w \in \Sigma^+ \mid w \underset{G}{=} e\}$. $\tilde{L}$ is E_n-decidable in $\underline{\Sigma}^*$, and so $\tilde{L}$ is E_n-decidable in $(\underline{\Sigma} \cup \{\underline{t}\})^*$. Because of Lemma 4.3 there is a function $g \in E_1(\underline{\Sigma} \cup \{\underline{t}\})$ such that $g(\{t^i \mid i \geq 0\}) = \tilde{L}$ and there exists a function $k \in E_n(\underline{\Sigma} \cup \{\underline{t}\})$ satisfying:

$$\forall w \in (\underline{\Sigma} \cup \{\underline{t}\})^* (w \in \tilde{L} \rightarrow \exists i \leq |k(w)| (g(t^i) \equiv w)).$$

Let $L_g = \{t, t^i g(t^i) : i \geqslant 0\}$. Then

$$<\Sigma,t;L_g> = <\Sigma,t;t,t^i g(t^i) : i \geqslant 0> \cong <\Sigma;g(t^i) : i \geqslant 0> = <\Sigma;\tilde{L}>$$
$$\cong <\Sigma;L> = G,$$

and so $<\Sigma,t;L_g>$ is a f.g. presentation of G.

a) ***Claim.*** *L_g is E_1-decidable in $(\underline{\Sigma} \cup \{\underline{t}\})^*$.* We have $w \in L_g$ iff $w \equiv t$ or $w \equiv t^i v$ with $v \in \underline{\Sigma}^+$ and $v \equiv g(t^i)$.

b) ***Claim.*** *$<\Sigma,t;L_g>$ is strongly E_n-d.b.* Let $w \underset{G}{=} e$. Then we have the following derivation, where $w' \in \underline{\Sigma}^*$: $w \underset{1}{\to} w' \underset{2}{\to} t^i w' \underset{3}{\to} e$.

ad 1, all t^ε which appear in w are deleted. This takes $|w|_t \leqslant |w|$ steps, and $w' \equiv \pi_{\Sigma}(w)$ satisfies $|w'| \leqslant |w|$ and $w' \underset{G}{=} e$.

ad 2, if $w' \equiv e$ then we are ready. Let $w' \not\equiv e$. Then $w' \in \tilde{L}$ and because of (b) there is an $i \leqslant |k(w')|$ with $g(t^i) \equiv w'$. Insertion of i trivial relators $\bar{t}t$ and deletion of i relators $\bar{t}$ result in $t^i w'$. Here $2i \leqslant 2|k(w')|$ steps are sufficient.

ad 3, $t^i w' \equiv t^i g(t^i) \in L_g$, and so $t^i w'$ can be deleted within one step. Thus we have a derivation of w to e in $<\Sigma,t;L_g>$ of length $m \leqslant |w|+2|k(w')|+1$ in which only trivial relators are inserted. Hence the presentation $<\Sigma,t;L_g>$ is s.E_n-d.b.

4.5. COROLLARY. *Let $G = <\Sigma;L>$ be f.g. with E_n-decidable word problem for some $n \geqslant 3$. Then there exists a f.g. presentation for G with an E_1-decidable set of defining relators such that the word problem for this presentation can be solved by a natural E_n-algorithm.*

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