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ALMOST-HOMEOMORPHISMS AND ALMOST-TOPOLOGICAL PROPERTIES

by

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ABSTRACT. A function is said to be an almost-homeomorphism if it is a bijective almost continuous function (see [25]) with an almost continuous inverse. We characterize such functions in several ways and obtain the relationship between almost-homeomorphisms and semi-homeomorphisms (see [8]). We study those properties which are preserved under this class of functions -the almost topological properties - and characterize them as the semi-regular properties (see [3]). We also introduce the concept of an almost topological class and study the relationship between this classes and the topological, semi-topological, and ρ-topological classes.

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Introduction. Many authors, among them Biswas [1], Crossley and Hildebrand [8], Fomin [10], Neubrum [17], and Piotroswki [24], have introduced and studied weak forms of homeomorphisms. In section 1 of this paper, using the definition of an almost-continuous function of Singal [25], we introduce the concept of an almost-homeomorphism. In section 2 we obtain some necessary conditions for a function to be an almost-homeomorphism. In section 3 we obtain the relationsphip between almost-homeomorphisms and semi-homeomorphisms; we introduce the notion of an almost topological property and give some examples of topological properties which are also almost topological properties. We characterize the almost topological properties as the semi-regular properties (see [3]). In section 4 we introduce the concept of an almost topological class and we study its relationship with the concepts of topological class, semi-topological class and, ρ-topological class.

In this work the δ-continuos and δ-open functions of [4] play an important role.

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§0. **Preliminaries**. Our notation is standard. Spaces will always mean topological spaces in which no separation axioms are assumed, unless explicitly stated. The closure (resp. δ-interior) of a subset A of a space (X,τ) will be denoted by $\bar{A}^{\tau}$ (resp. $\mathring{A}^{\tau}$) or simply $\bar{A}$ (resp. $\mathring{A}$). The δ-closure (resp. δ-interior) [4] of A in X will be denoted by $\underline{A}$ (resp. $\underset{o}{A}$). If

τ and σ are two topologies in a given set X, τ is said to be finer than σ if $\sigma \subseteq \tau$, and the relationship is expressed as $\tau \leqslant \sigma$. The empty set will be denoted by $\emptyset$. A subset A of X is called *regularly open,* (resp. *regularly closed*) if it is the interior (resp. closure) of its own closure (resp. interior) [4]. Let (X,τ) be a topological space. By τ^* we denoted the topology (δ-*topology*, prop. 1.1 of [4]) which has as a base the family of regularly open sets of (X,τ). This topology is called the semiregularization of τ, and every element of τ^* is said δ-*open* set [4]. A subset A of X is said to be *semi-open* [13] if there exists an open set U of X such that $U \subseteq A \subseteq \bar{U}$. The family of all semi-open sets of X will be denoted by SO(X).

DEFINITION 0.1, [4],[21]. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be δ-*continuous* if the inverse image of each δ-open subset of Y is a δ-open subset of X, that is, if $\tau^* \leqslant f^{-1}(\sigma^*)$.

DEFINITION 0.2, [4]. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be δ-*open* (resp. δ-*closed*) if the image of each δ-open (resp. δ-closed) subset of X is a δ-open (resp. δ-closed) subset of Y, i.e. if $\sigma^* \leqslant f(\tau^*)$).

LEMMA 0.1. *Let* (X,τ) *be a topological space, then for every subset* A *of* X *we have:* $\underline{A} = \bar{A}^{\tau^*}$ *and* $\underset{\circ}{A} = \mathring{A}^{\tau^*}$. ▲

DEFINITION 0.3, [25]. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *almost-continuous* if the inverse image of every regularly open subset of Y is an open subset of X. Also f is called *almost-open* (resp. *almost-closed*) is the image of every regularly open (resp. regularly-closed) subset of X is an open (resp. closed) subset of Y.

§ 1. Almost-homeomorphisms.

DEFINITION 1.1 A bijective function $f:(X,\tau) \to (Y,\sigma)$ is said to be an *almost-homeomorphism* if f and f^{-1} are almost-continuous functions.

PROPOSITION 1.1. *If* $f:(X,\tau) \to (Y,\sigma)$ *is a homeomorphism then* f *is an almost-homeomorphism.*

Proof. Since every continuous function is almost continuous ([25], remark 2.1), this is obvious. ▲

Remark 1. The converse of Prop. 1.1 is not true, as the following example shows.

Example 1. Let $X = \{a,b,c\}$, and let $\sigma = \{\emptyset,\{a\},\{b,c\}, \{c\}, \{a,c\},X\}$ and $\tau = \{\emptyset,\{a\},\{b,c\}, X\}$ be two topologies on X. It is obvious that $\sigma \leqslant \tau$. Let $f:(X,\sigma) \to (X,\tau)$ be the identity function on X. It follows that f is not a homeomorphism because f^{-1} is not a continuous function. But f is an almost-homeomorphism.

PROPOSITION 1.2. *Let* $f:(X,\tau) \to (Y,\sigma)$ *be a function. The following conditions are equivalent:*

1) *the function* f *is almost-open,*
2) *the image of each* δ*-open subset of* X *is an open subset of* Y, *i.e.* $\sigma \leqslant f(\tau^*)$,
3) *for every* $A \subseteq X$, $f(\underset{\circ}{A}) \subseteq \overset{\circ}{f}(A)$.

Proof. The implication (1) ⇒ (2) is obvious since any δ-open subset of X is the union of regularly open subsets of X. (2) ⇒ (3) Let $y \in f(\underset{\circ}{A})$. Then there exists $x \in \underset{\circ}{A}$ such that $f(x) = y$. Since $x \in \underset{\circ}{A}$, there exists a neighourhood U of x such that $\mathring{\bar{U}} \subseteq A$, $f(\mathring{\bar{U}}) \subseteq f(A)$. By (2), $f(\mathring{\bar{U}})$ is an open subset of Y such that $y = f(x) \in f(\mathring{\bar{U}})$ and hence $f(x) = y \in f(\mathring{A})$. (3) $\Rightarrow$(1) Let A be a regularly open subset of X, then $A = \underset{\circ}{A}$, hence $f(A) = f(\underset{\circ}{A})$; by (3) we obtain $f(A) \subseteq \mathring{f}(A)$. Since $\mathring{f}(A) \subseteq f(A)$, we obtain also $f(A) = \mathring{f}(A)$ and hence f(A) is open in Y. ▲

The proof of the following proposition is omitted since it is similar to that of proposition 1.2.

PROPOSITION 1.3, *Let* $f:(X,\tau) \rightarrow (Y,\sigma)$ *be a function. Then the following conditions are equivalent:*

1) *the function* f *is almost-closed,*
2) *the image of each* δ*-closed subset of* X *is a closed subset of* Y,
3) *for every* $A \subseteq X$, $\overline{f(A)} \subseteq f(\underline{A})$. ▲

Remark 2. It is obvious that if f is a bijective function then f is almost-open iff f is almost-closed.

COROLLARY 1.3. *A function* $f:(X,\tau) \rightarrow (Y,\sigma)$ *is* δ*-open* (resp. δ*-closed*) *iff for each* $A \subseteq X$, $f(\underset{\circ}{A}) \subseteq \underset{\circ}{f(A)}$ (resp. $\underline{f(A)} \subseteq f(\underline{A})$).

Proof. If f is δ-open, by Prop. 2.4 of [4], $f:(X,\tau^*) \rightarrow (Y,\sigma^*)$ is an open function; then by theorem 11.3 of [9], $f(\mathring{A}^{\tau^*}) \subseteq \mathring{f}(A^{\tau^*})$ for any $A \subseteq X$. Hence by lemma 0.1 $f(\underset{\circ}{A})$ $\underset{\circ}{f}(A)$. In similar way we prove the corollary when f is a closed function.

Remark 3. It is immediate that if f is a bijective function then f is δ-open iff f is δ-closed.

The following results are direct generalizations of well-known facts on homeomorphisms.

THEOREM 1.1 *Let* $(X,\tau) \to (Y,\sigma)$ *be a bijective function. Then the following conditions are equivalent:*

1) *the function* f *is an almost-homeomorphism,*
2) *the function* f *is almost-continuous and almost-open* (resp. *almost-closed*),
3) *the function* f *is* δ*-continuos and* δ*-open,* (resp. δ*-closed*),
4) *the function* f *and* f^{-1} *are* δ*-continuous functions,*
5) *the subset* $f(A)$ *is regularly open* (resp. *closed*) *in* Y *if and only if* A *is regularly open in* X (resp. *regularly closed*),
6) $f(\tau^*) = \sigma^*$, (resp. $f(\tau^*_c) = \sigma^*_c$),
7) *the subset* $f^{-1}(B)$ *is regularly open in* X *if and only if* B *is regularly open in* Y (resp. *regularly closed*). ▲

THEOREM 1.2 *Let* $f:(X,\tau) \to (Y,\sigma)$ *be a bijective function. Then the following conditions are equivalent:*

1) *the function* f *is an almost-homeomorphism,*
2) *for every regularly open subset* A *of* X *and for every regularly open subset* $f(B)$ *of* Y *we have* $f(\bar{A}) = \overline{f(A)}$ *and* $f(\bar{B}) = \overline{f(B)}$,
3) *for every regularly closed subset* A *of* X *and for every regularly closed subset* $f(B)$ *of* Y *we have* $f(\mathring{A}) = \mathring{f}(A)$ *and* $f(\mathring{B}) = \mathring{f}(B)$,
4) *for every subset* A *of* X *we have* $f(\underline{A}) = \underline{f(A)}$,
5) *for every subset* B *of* X *we have* $f(\underset{\circ}{B}) = \underset{\circ}{f}(B)$.

Proof. We prove only the implications (1) $\Rightarrow$(2), (3) $\Rightarrow$(1) and (4) $\Rightarrow$(5). The implications (2) $\Rightarrow$(3), (1) $\Rightarrow$ (3), (1) $\Leftrightarrow$(4) and (5) $\Rightarrow$(4) are straightforward and are left to the reader. We prove first (1) $\Rightarrow$ (2): let $A \subseteq X$ be a regularly open subset of X. Then f(A) and $f(\bar{A})$ are open and closed subsets of Y, respectively (by Th. 1.1 (2)), hence $\overline{f(A)} \subseteq f(\bar{A})$. Since $A = f^{-1}f(A)$ and $\overline{f(A)}$ is a regularly closed subset of Y, $f^{-1}(\overline{f(A)})$ is a closed subset of X and hence $\bar{A} \subseteq f^{-1}(\overline{f(A)})$ and $f(\bar{A}) \subseteq \overline{f(A)}$. We have then $f(\bar{A}) = \overline{f(A)}$. Let f(B) be a regularly open subset of Y. Then by theorem 1.1 $f^{-1}f(B) = B$ is a regular open subset of X and by the proof above we have $f(\bar{B}) = \overline{f(B)}$.

(3) $\Rightarrow$ (1): by (2) of Prop. 1.2, we prove that f is an almost-continuous and almost-open function. Then, let $f(B) \subseteq Y$ be a regularly open subset of Y. If $C = f^{-1}(\overline{f(B)})$, then $f(C) = \overline{f(B)}$ is a regularly closed subset of Y, we apply (3) to f(C) obtaining $f(\mathring{C}) = \mathring{\overbrace{f(C)}} = \mathring{\overline{f(B)}}$, then $f(\mathring{C}) = f(B)$; hence we have $B = \mathring{C}$, then B is open and so f is almost-continuous.

Let $A \subseteq X$ be a regularly open subset of X, then $\bar{A}$ is a regularly closed subset of X. We apply (3) to $\bar{A}$ and obtain $f(\mathring{\bar{A}}) = \mathring{\overbrace{f(\bar{A})}}$; hence $f(A) = \mathring{\overbrace{f(\bar{A})}}$. Therefore, f(A) is an open subset of Y. This shows that f is almost-open function.

(4) $\Rightarrow$ (5): we apply (4) to $A = X-B$, with $B \subseteq X$, and obtain $f(\underline{X-B}) = \underline{f(X-B)}$, which gives:

$$Y-f(\underline{X-B}) = Y-\underline{f(X-B)} = Y-(\underline{f(X)-f(B)}) = Y-(\underline{Y-f(B)}) = f(\underset{\circ}{B}).$$

On the other hand we have:

$$Y-f(\underline{X-B}) = f(X)-f(\underline{X-B}) = f(X-(\underline{X-B})) = f(\underset{\circ}{B}),$$

hence, $f(\underset{\circ}{B}) = \underset{\circ}{f}(B)$. ▲

The following proposition if an immediate consequence of Theorem 1.1.

PROPOSITION 1.4 *A function* $f:(X,\tau) \to (Y,\sigma)$ *is an almost-homeomorphism iff* $f:(X,\tau^*) \to (Y,\sigma^*)$ *is a homeomorphism.* Δ

§2. Properties of almost-homeomorphisms.

DEFINITION 2.1, [26]. A space (X,τ) is said to be *nearly-compact* (resp. *almost-compact*) if every open cover admits of a finite subfamily, the interior of the closure (resp.closure) of whose members cover the space.

DEFINITION 2.2, [31]. A T_2 space (X,τ) is said to be T_2-*closed* if (X,τ) is closed in every T_2 extension of it.

DEFINITION 2.3, [5]. A space (X,τ) is said to be *weakly-compact* if every *regular cover* [5] (an open cover $\{A_i\}_{i\in J}$ such that every A_i contanined a regular closed C_i veryfing $\bigcup_{i\in J} \overset{o}{C}_i = X$) admits of a finite subfamily the closure of whose members cover the space.

PROPOSITION 2.1 *Let* $f:(X,\tau) \to (Y,\sigma)$ *be an almost continuous function. If* (X,τ) *is almost-compact and* Y *is a* T_2 *space, then* f *is an almost-closed function.*

Proof. Let A be a regularly closed subset of X. By theorem 2.6 of [25], $f/_A:A \to Y$ is an almost continuous function. Since A is almost-compact, $f(A)$ is T_2-closed subset of Y (Th.3.3 of [26]). Since Y is a T_2 space, $f(A)$ is a closed subset of Y. ▲

Combining Prop. 2.1 with Th. 1.1 we obtain the following corollary:

COROLLARY 2.1 *If* $f:(X,\tau) \rightarrow (Y,\sigma)$ *is a bijective almost-continuos function from a* T_2*-closed space onto* T_2*-space, then* f *is an almost-homeomorphism.* ▲

PROPOSITION 2.2 *Let* $f:(X,\tau) \rightarrow (Y,\sigma)$ *be an almost-continuous function. If every regularly closed subset of* X *is weakly-compact and* Y *is a* $T_{2-\frac{1}{2}}$ *space (or a completely Hausdorff [30] space), then* f *is an almost-closed function.*

Proof. Let A be a regularly closed subset of X. By theorem 2.6 of [25], $f/_A : A \rightarrow Y$ is an almost continuous function. By hypothesis, A is weakly-compact space and hence by theorem 3.4 of [6] f(A) is weakly-compact space in Y. Since Y is a $T_{2-\frac{1}{2}}$ space, by theorem 2 of [7] f(A) a closed subset of Y. ▲

An immediate consequence of this fact and theorem 1.1 is the next result.

COROLLARY 2.2 *If* $f:(X,\tau) \rightarrow (Y,\sigma)$ *is a bijective almost continuos function* X *where every regularly closed subset is weakly-compact, into a* $T_{2-\frac{1}{2}}$ *space* Y*, then it is an almost-homeomorphism.* ▲

§3. Almost-homeomorphisms, semi-homeomorphisms, Almost-topological properties and semiregular properties. We recall some definition used in the following.

DEFINITION 3.1, [8]. A function $f:X \to Y$ is said to be *irresolute* if, for any semi-open [13] subset S of Y, $f^{-1}(S)$ is semi-open in X.

DEFINITION 3.2, [8]. A function $f:X \to Y$ is said to be *pre-semi-open* if for any semi-open subset S of X, f(S) is semi-open in Y.

DEFINITION 3.3, [8]. Two topological spaces X and Y are said to be *semi-homeomorphic* if there exists a bijective function f such that f is irresolute and pre-semi-open. Such an f is called a *semi-homeomorphism.*

THEOREM 3.1 *Homeomorphisms are semi-homeomorphisms, and semi-homeomorphisms are almost-homeomorphisms.*

Proof. It is shown in theorem 1.9 of [8] that homeomorphism implies semi-homeomorphism. It is shown in prop. 4.2 of [14] that semi-homeomorphism implies almost-homeomorphism. ▲

REMARK 4. It follows from example 1.2 of [8] and example page 251 of [14] that the converses of the implications in theorem 3.1 are not true in general.

Utilizing Th. 1.1 (4) and Prop. 3.2 of [4] we have the following proposition:

PROPOSITION 3.1 *Almost-homeomorphism is an equivalence relation between topological spaces.*

DEFINITION 3.4 A topological property which is preserved under almost-homeomorphisms is said to be an *almost-topological* property.

The following results are direct consequence of respective definitions.

PROPOSITION 3.2 *If* P *is an almost-topological property, then* P *is a semi-topological* [8] *property and hence* P *is a topological property.* ▲

PROPOSITION 3.3 *If* P *is a topological property but it is not a semi-topological property, then* P *is not an almost-topological property.* ▲

It follows from [8] that T_o, T_1, T_3, T_4, T_5, regularity, complete normality, normality, first countability. second countability, compactness, lindeloffness, metrizability, and local connectedness are not semi-topological properties. By prop. 3.3 they are not almost-topological properties either.

We give now some examples of almost-topological properties.

a) *Separation properties*

PROPOSITION 3.4 *Let* (X,τ) *be a* T_2 (resp. $T_{2-\frac{1}{2}}$) *topological space. If* $f:(X,\tau) \to (Y,\sigma)$ *is an almost-open bijection, then* (Y,σ) *is* T_2 (resp. $T_{2-\frac{1}{2}}$).

Proof. We prove only the $T_{2-\frac{1}{2}}$ case, since the proof of the T_2 case is analogous. Let $y_1 \neq y_2 \in Y$. Then there exist $x_1 \neq x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since X is T_2, there exist two open sets U and V such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Then also $\mathring{\bar{U}} \cap \mathring{\bar{V}} = \emptyset$. By hypothesis f is an almost-open function, then $f(\mathring{\bar{U}})$ and $f(\mathring{\bar{V}})$ are subsets of ψ such that $y_1 \in f(\mathring{\bar{U}})$, $y_2 \in f(\mathring{\bar{V}})$ and $f(\mathring{\bar{U}}) \cap f(\mathring{\bar{V}}) = \emptyset$. ▲

An immediate consequence of this fact and theorem 1.1 is the next result.

COROLLARY 3.4 T_2 (resp. $T_{2-\frac{1}{2}}$) *is an almost-topological property.*

LEMMA 3.1, [11]. *Let* (Y,σ) *be a regular space. If* $f:(X,\tau) \rightarrow (Y,\sigma)$ *is continuous then* $f:(X,\tau^*) \rightarrow (Y,\sigma)$ *is continuous.* ▲

PROPOSITION 3.5 *Being completely Hausdorff is an almost-topological property.*

Proof. Let (X,τ) be a completely Hausdorff space and let $f:(X,\tau) \rightarrow (Y,\sigma)$ be an almost-homeomorphism. We show that (Y,σ) is a completely Hausdorff space. Let $y_1 \neq y_2 \in Y$. Then there exist $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By hypothesis there exists a Urysohn function $g:(X,\tau) \rightarrow [0,1]$, [30]. Next, we show that $g \circ f^{-1}:(Y,\sigma) \rightarrow [0,1]$ is a continuous function. For any open subset A of $[0,1]$, $(g \circ f^{-1})(A) = f(g^{-1}(A))$. Since $[0,1]$ is a regular space, by lemma 3.1, $g^{-1}(A) \in \tau^*$. By theorem 1.1, f is almost-open and so by Proposition 1.2, $f(g^{-1}(A))$ is an open subset of Y. Therefore, $g \circ f^{-1}$ is continuous. Moreover, since f is bijective, $g \circ f^{-1}$ is a Urysohon function. This shows that (Y,σ) is completely Hausdorff. ▲

DEFINITION 3.4 [27]. A space (X,τ) is said to be *almost regular* if for any regularly closed set A and any $x \notin A$ there exist disjoint open sets U and V such that $A \subseteq U$ and $x \in V$.

DEFINITION 3.5 [28]. A space (X,τ) is said to be *almost completely regular* if for any regularly closed set A and each point $x \in X-A$ there is a continuous function $f:(X,\tau) \to [0,1]$ such that $f(x) = 1$ and $f(A) = 0$.

PROPOSITION 3.6 *Almost regularity (almost-complete regularity) is an almost-topological property.*

Proof. This follows from remark 2, theorem 1.1, and theorem 1 of [20] (prop. 1.4 and Cor. 2.1 of [29]). ▲

b) *Connectedness properties.*

DEFINITION 3.6, [30]. A space (X,τ) is said to be *hyperconnected* if every nonempty open set of X is dense in X.

PROPOSITION 3.7 *Hyperconnectedness is an almost-topological property.*

Proof. Let (X,τ) be a hyperconnected space and let $f:(X,\tau) \to (Y,\sigma)$ be an almost-homeomorphism. We show that (Y,τ) is hyperconnected. Let $A \subseteq Y$ be an open set of Y. Then $A \subseteq \mathring{\bar{A}}$ and $f^{-1}(\mathring{\bar{A}})$ is a regularly open set of X by theorem 1.1 (7). By hypothesis, $\overline{f^{-1}(\mathring{\bar{A}})} = X$ and so $f^{-1}(\mathring{\bar{A}}) = X$. We have $\mathring{\bar{A}} = Y$ and hence $\bar{A} = Y$. This shows that (Y,τ) is hyperconnected. ▲

DEFINITION 3.7 [16]. A space (X,τ) is said to be *almost-locally connected at a point* $p \in X$ if each regularly open neighbourhood of p contains a connected open neighbourhood of p. (X,τ) is said to be *almost-locally conected* if it is almost-locally connected at each of its points.

Utilizing Th. 4 of [15], Cor. 4.7 of [23] and Th.1.1, we have the following proposition.

PROPOSITION 3.8 *Connectedness (almost-local connectedness) is an almost-topological property.* ▲

c) *Covering properties.*

PROPOSITION 3.9. *Near compactness, almost compactness (weak compactness) are almost-topological properties.*

Proof. For near compactness, it is shown in Theorem 3.1 of [26] that the almost continuous and almost-open image of a nearly-compact spaces is nearly-compact, then by theorem 1.1 (2) we have the proposition. For almost compactness, this is a consequence of theorem 3.3 of [26] (theorem 3.4 of [6], and theorem 1.1 (2).) ▲

DEFINITION 3.10 [7]. A $T_{2-\frac{1}{2}}$ space (X,τ) is said to be $T_{2-\frac{1}{2}}$*-closed* if (X,τ) is closed in every $T_{2-\frac{1}{2}}$ extension of it.

The following proposition is an consequence of Prop. 3.4, Prop. 3.9, and Th. 2 of [7].

PROPOSITION 3.10 T_2-*closedness* (resp. $T_{2-1/2}$-*closedness*) *is an almost-topological property.* ▲

DEFINITION 3.11, [3]. A topological property P is called *semiregular* provided that (X,τ) has property P if and only if (X,τ^*) has property P.

THEOREM 3.2 *A topological property* P *is an almost-topological property if and only if it is semiregular.*

Proof. Let (X,τ) be a topological space with an almost-topological property P. Since the identity function $i:(X,\tau) \to (X,\tau^*)$ is an almost-homeomorphism, (X,τ^*) has P. So if (X,τ^*) has P, (X,τ) has P, because the identity function $i^{-1}:(X,\tau^*) \to (X,\tau)$ is also an almost-homeomorphism. Conversely, we prove that if P is an semiregular property then P is an almost-topological property. Let (X,τ) be a topological space with a semiregular property P and $f:(X,\tau) \to (Y,\sigma)$ an almost-homeomorphism. Then by prop.1.4, $f:(X,\tau^*) \to (Y,\sigma^*)$ is an homeomorphism and hence (Y,σ^*) has P. Since P is semiregular, (Y,σ) has P and hence P is an almost-topological property. ▲

By theorem 3.2 and results of [12],[22] we have some almost-topological properties.

DEFINITION 3.12, [18],[2]. A space (X,τ) is said to be *weakly locally connected* if each component of (X,τ) is open.

PROPOSITION 3.11, *Extremally disconnectedness* (resp. *weakly local connectedness*) *is an almost-topological property.* ▲

DEFINITION 3.13, [3]. A topological space (X,τ) is said to be S-*closed* (*locally* S-*closed* [22]) if every semi-open [13] cover has a finite subfamily whose closure coves the space (if each point of X has an open neighbourhood which is an S-closed subspace of X).

PROPOSITION 3.13, S-*closedness and locally* S-*closedness are almost-topological properties.* ▲

§4. Almost-topological classes.

If X is a set of points, let S(X) denote the collection of all topological spaces which have X as their set of points.

DEFINITION 4.1, [3]. Let (X,τ) and (X,σ) be two elements of S(X), then (X,τ) and (X,σ) are said to be ρ-*equivalent* if $\tau^* = \sigma^*$.

THEOREM 4.1, ρ-*equivalence is an equivalence relation on the collection* S(X). ▲

Thus, the collection S(X) of topological spaces is partitioned into equivalence classes. Let $\rho[X,\tau^*]$ denote the equivalence class of topological spaces with the same semi-regularization τ^* as (X,τ).

DEFINITION 4.2 The equivalence classes of S(X) under the relation of ρ-equivalence will be called the ρ-*topological classes* of X.

We omit the easy proof of the following results.

THEOREM 4.2 *If* $f:(X,\tau) \to (Y,\sigma)$ *is a* δ*-continuous* (resp. δ*-open*) *function and if* $(X,\tau_1) \in \rho[X,\tau^*]$ *and* $(Y,\sigma_1) \in \rho[Y,\sigma^*]$, *then* $f:(X,\tau_1) \to (Y,\sigma_1)$ *is* δ*-continuous* (resp. δ*-open*). ▲

THEOREM 4.3 *If* $f(X,\tau) \to (Y,\sigma)$ *is an almost-continuous (almost-open) function, and if* $(Y,\sigma_1) \in \rho[Y,\sigma^*]$ $((X,\tau_1) \in \rho[X,\tau^*])$, *then* $f:(X,\tau) \to (Y,\sigma_1)$ $(f:(X,\tau_1) \to (Y,\sigma))$ *is an almost-continuous (almost-open) function.* ▲

THEOREM 4.4 *If* $f:(X,\tau) \to (Y,\sigma)$ *is an almost-homeomorphism and if* $(X,\tau_1) \in \rho[X,\tau^*]$ *and* $(Y,\sigma_1) \in \rho[Y,\sigma^*]$, *then* $f:(X,\tau_1) \to (Y,\sigma_1)$ *is an almost-homeomorphism.* ▲

As was shown in [2], p.103, $\rho[X,\tau^*]$ contains a maximal topological space, denoted by (X,τ_o), in the sense that the topology induced on X by relation "$\leqslant$" is finer than the topology on any other space in $\rho[X,\tau^*]$. However, we have the following result.

COROLLARY 4.4 *if* $f:(X,\tau) \to (Y,\sigma)$ *is an almost-homeomorphism, then* $f:(X,\tau_o) \to (Y,\sigma_o)$ *is an almost-homeomorphism.* ▲

THEOREM 4.5 *If* (X,τ) *and* (X,σ) *are two* ρ*-equivalent spaces, then they are almost-homeomorphic.* ▲

Remark 5. Since an almost-homeomorphism is an equivalence relation (Prop. 3.1), the collection of all topological spaces is partitioned into equivalence classes. Let $A[X,\tau]$ denote the equivalence class of all topological spaces almost-homeomorphic to (X,τ). Let $H[X,\tau]$ and $S[X,\tau]$ denote respectively the equivalence classes of all topological spaces homeomorphic to (X,τ), and those semi-homeomorphic to (X,τ).

DEFINITION 4.3 The equivalence class of (X,τ) under the relation of almost-homeomorphisms will be called the *almost-topological class of* X.

PROPOSITION 4.1 *Let* P *be a topological property. Then* P *is an almost-topological property if and only if for some topological space* (X,τ) *with* P *we have that every* $(Y,\sigma) \in A[X,\tau]$ *has* P. ▲

Remark 6. In [8], $[X,SO(X,\tau)]$ denote the equivalence class of topological spaces with the same collection of semi-open sets as (X,τ), i.e. the semi-topological class of X.

THEOREM 4.6 *Let* (Y,σ) *be a topological space of* $A[X,\tau]$ *then we have:*

$$[Y,SO(Y,\sigma)] \subseteq \rho[Y,\sigma^*] \subseteq A[X,\tau].$$

Proof. We prove that $[Y,SO(Y,\sigma)] \subseteq \rho[Y,\sigma^*]$. Let σ_1 and σ_2 two topologies on Y such that $SO(Y,\sigma_1) = SO(Y,\sigma_2)$ i.e. (Y,σ_1) and (Y,σ_2) are elements of $[Y,SO(Y,\sigma)]$. By $\sigma_{1\alpha}$ and $\sigma_{2\alpha}$ we denote the α-topology on Y [19] associate to σ_1 and σ_2 respectively. From prop. 1 of [19] we have $\sigma_{1\alpha} = \sigma_{2\alpha}$ then by prop. 2.4 of [14] $\sigma_1^* = \sigma_2^*$. Hence (Y,σ_1) and

(Y,σ_2) are ρ-equivalent.

We prove that $\rho[Y,\sigma^*] \subseteq A[X,\tau]$. Let $(Y,\omega) \in \rho[Y,\sigma^*]$. Then by theorem 4.5 there exists an almost-homeomorphism $i:(Y,\omega) \to (Y,\sigma)$. If $f:(X,\tau) \to (Y,\sigma)$ is the almost-homeomorphism by hypothesis, then $f^{-1} \circ i$ is the almost-homeomorphism from (Y,ω) onto (X,τ) so $(Y,\omega) \in A[X,\tau]$. ▲

COROLLARY 4.6 *Let* (X,τ) *be a topological space, we have:*

1) $H[X,\tau] \subseteq S[X,\tau] \subseteq A[X,\tau]$,

2) $A[X,\tau] = \bigcup_{(Y,\sigma)\in A[X,\tau]} \rho[Y,\tau^*]$.

Proof. 1) Obvious.

2) By theorem 4.6 we obtain $A[X,\tau] \supset \bigcup_{(Y,\sigma)\in A[X,\tau]} \rho[Y,\tau^*]$. The proof of the inverse inclusion is similar to that of theorem 4.6. ▲

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