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COHOMOLOGY THEORIES IN SYNTHETIC DIFFERENTIAL GEOMETRY

by

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One way of formulating De Rham's theorem 'smoothly in parameters' is to construct the De Rham cohomology groups, and the (duals of the) singular homology groups as sheaves of smooth modules over the space of parameters, and then to assert that these sheaves are canonically isomorphic.

In the last two sections 5 and 6 of this paper we will derive such a version of De Rham's theorem (see p.257), as well as similar isomorphisms of sheaves of smooth modules for some other variants of De Rham's theorem (p. 260, 264). These theorems will follow from more general results asserting the validity of De Rham's theorem in the smooth Grothendieck topos G described e.g. in Moerdijk & Reyes (1983).

The plan of this paper is as follows. In the first two sections, we will give a synthetic description of the De Rham cohomology and the singular homology of an arbitrary smooth space M. In the third section, we prove a synthetic version of De Rham's theorem, and in section 4 we show that results of Moerdijk & Reyes (1983) enable us to interpret this synthetic theorem in the topos G. As a by-product, we will obtain some 'comparison theorems' which essentially tell us that the cohomology of a manifold in G is in a sense the same as its cohomology in sets, i.e. its cohomology as defined in classical differential topology. This interpretation of the results of section 3 in the topos G immediately yields the 'smoothly in parameters' theorems of section 5 and 6 which we mentioned above.

This paper is an extended version of our paper "De Rham's theorem in a smooth topos", (1984).

§1. The De Rham cohomology. In classical differential geometry the De Rham complex of a manifold is built up from differential forms and exterior differentiation. In the context of synthetic differential geometry, these building blocks can be defined for any object M, since all objects are 'smooth spaces'. Thus, to defined these notions, let M be any smooth space. An *infinitesimal* n-*cube on* M is an element of $M^{D^n}\times D^n$, i.e. n+1-tuple $(\gamma,h_1,\ldots,h_n)$.

The *object of intinitesimal* n-chains, $C_n(M)$, is the free R-module generated by the infinitesimal n-cubes on M. So an element of $C_n(M)$ is a formal linear combination

$$\sum_{i=1}^{p} a_i(\gamma_i,h_1^i,\ldots,h_n^i)$$

where $a_i \in R$ and $(\gamma_i,h_1^i,\ldots,h_n^i) \in M^{D^n}\times D^n$.

An n-*form on* M is a map

$$M^{D^n}\times D^n \xrightarrow{\omega} R$$

$$(\gamma,h_1,\ldots,h_n)\mapsto \int_{(\gamma,h_1,\ldots,h_n)}\omega$$

assigning a number (a 'size', like length, area, volume, etc.) to every infinitesimal n-cube, subject to the following conditions:

1. ***homogeneity:*** $\omega(a_i\cdot\gamma,h_1,\ldots,h_n) = a\cdot\omega(\gamma,h_1,\ldots,h_n)$, where $a_i\cdot\gamma:D^n \to M$ is defined by

$$a_i\cdot\gamma(x_1,\ldots,x_i,\ldots,x_n) = \gamma(x_1,\ldots,ax_i,\ldots,x_n),$$

for every $a \in R$ and infinitesimal n-cube $(\gamma,h_1,\ldots,h_n)$.

2. ***alternance:*** $\omega(\sigma\gamma,h_1,\ldots,h_n) = \mathrm{sgn}(\sigma)\cdot\omega(\gamma,h_{\sigma(1)},\ldots,h_{\sigma(n)})$, where σ is any permutation of $\{1,\ldots,n\}$, and $\sigma\gamma$ is γ composed with the co-ordinate permutation induced by σ, i.e.

$$\sigma\gamma(x_1,\ldots,x_n) = \gamma(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \;;$$

$\mathrm{sgn}(\sigma)$ is the signature of σ.

3. ***degeneracy:*** $\omega(\gamma,h_1,\ldots,0,\ldots,h_n) = 0$.

The *object of* n-*forms on* M is denoted by $\Lambda^n(M)$.

Note that by the Kock-Lawvere axiom, $R^D \simeq R\times R$, and the degeneracy condition, each n-form ω on M can be written as

$$\omega(\gamma,h_1,\ldots,h_n) = h_1\cdot\ldots\cdot h_n\cdot\tilde{\omega}(\gamma)$$

for a unique map $\tilde{\omega}:M^{D^n} \to R$. This map $\tilde{\omega}$ satisfies the homogeneity condition $(\tilde{\omega}(a_i\cdot\gamma) = a\cdot\tilde{\omega}(\gamma))$ and is alternating $(\tilde{\omega}(\sigma\gamma) = \mathrm{sgn}(\sigma)\tilde{\omega}(\gamma))$. Thus we obtain a 1-1 correspondence between elements $\omega \in \Lambda^n(M)$ and alternating homogeneous maps $\tilde{\omega}:M^{D^n} \to R$, and we will often identify the two.

If $\omega:M^{D^n}\times D^n \to R$ is an n-form on M, we will write

$$\int_{(-)}\omega:C_n(M) \to R$$

for the unique R-linear map extending ω.

Taking the boundary of an infinitesimal n-cube defines an R-linear *boundary operator*

$$\partial:C_{n+1}(M) \to C_n(M)$$

given by the formula

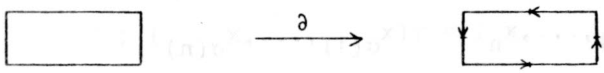
$$\partial(\gamma,h_1,\ldots,h_{n+1}) = \sum_{i=1}^{n} \sum_{\alpha=o,1} (-1)^{i+\alpha} F_{i\alpha}(\gamma,h_1,\ldots,h_n),$$

where $F_{i\alpha}(\gamma,h_1,\ldots,h_{n+1})$ is the infinitesimal n-cube

$$([(x_1,\ldots,x_n) \mapsto \gamma(x_1,\ldots,\alpha\cdot h_i,\ldots,x_n)],h_1,\ldots,\hat{h}_i,\ldots,h_{n+1}).$$

Thus, for example, if $\gamma:D^2 \to R^2$ is the embedding, then

$$\partial(\gamma,h_1,h_2) = (\gamma(-,0),h_1) + (\gamma(h_1,-),h_2) - (\gamma(-,h_2),h_1) - (\gamma(0,-),h_2)$$



We observe that spelling out the definition of ∂ yields that

$$\partial\circ\partial = 0.$$

If we put $C_n(M) = (0)$ for $n < 0$ then we obtain a so-called (*differential*) *complex*. In general, a complex A (of R-modules) is a sequence

$$\ldots\to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \ldots \quad (n \in \mathbb{Z})$$

or

$$\ldots\to A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \to \ldots \quad (n \in \mathbb{Z})$$

of R-module and R-linear maps, such that $\partial_n\partial_{n+1} = 0$, or $d_{n+1}d_n = 0$. (Usually, the subscripts on ∂ and d are omitted). If A and B are complexes, a *map of complexes*, or a *chain map* $f:A \to B$ is a sequence of R-linear maps $f_n:A_n \to B_n$ which preserve the structure of the complex, i.e. commute with the ∂'s, or the d's. (Again, we suppress subscripts on f).

Given this terminology, the construction of the complex $C_\bullet(M) = \{C_n(M)\}$ is (covariantly) functorial in M: a map $M \overset{f}{\to} N$ induces R-linear maps

$$f_*:C_n(M) \to C_n(N)$$

defined on generators by composition, i.e. $f_*(\gamma,h_1,\ldots,h_n) =$

$(f\circ\gamma, h_1, \ldots, h_n)$, and since

$$f_*(\partial(\gamma, h_1, \ldots, h_{n+1})) = \partial(f_*(\gamma, h_1, \ldots, h_{n+1}))$$

this yields a map of complexes.

The boundary operator $C_{n+1}(M) \xrightarrow{\partial} C_n(M)$ enables us to define an R-linear map $\Lambda^n(M) \xrightarrow{d} \Lambda^{n+1}(M)$, called the *exterior differentiation map,* by putting for each n-form $\omega : M^{D^n} \times D^n \to R$,

$$\int_{(\gamma, h_1, \ldots, h_{n+1})} d\omega = \int_{\partial(\gamma, h_1, \ldots, h_{n+1})} \omega.$$

This is well-defined, since as is easily checked, $d\omega : M^{D^{n+1}} \times D^{n+1} \to R$ is indeed homogeneous and alternating, and satisfies the degeneracy condition. Moreover, since $\partial^2 = 0$, we find that $d^2 = 0$. Observe that the defining equation for d is 'Stokes' theorem' for *infinitesimal* n-chains. Below, we will see how to prove the usual form of Stokes' theorem for big n-chains.

Again, the construction of $\Lambda^n(M)$ is (contravariantly) functorial in M: a map $f : M \to N$ induces R-linear maps

$$f^* : \Lambda^n(N) \to \Lambda^n(M)$$

by composition: if ω is an n-form on N and $(\gamma, h_1, \ldots, h_n)$ is an infinitesimal n-chain on M, then

$$f^*(\omega)(\gamma, h_1, \ldots, h_n) = \omega(f\circ\gamma, h_1, \ldots, h_n),$$

and we extend to $\Lambda^n(N)$ by linearity. Thus by definition,

$$\int_{f_*(\gamma, h_1, \ldots, h_n)} \omega = \int_{(\gamma, h_1, \ldots, h_n)} f^*(\omega).$$

The f^* together (for each n) give a chain map $f^* : \Lambda^\cdot(N) \to \Lambda^\cdot(M)$, since

$$d(f^*\omega) = f^*(d\omega).$$

We remark here that if M is R^n (or more generally a manifold in the classical sense) we obtain the usual notions of form and exterior differentiation. This point will be proved in section 4 below, where a comparison is made between the classical approach and the (model theory of the) synthetic aproach.

The *De Rham complex* of R-modules (and R-linear maps) of and arbitrary object M is the sequence

$$\ldots \to \Lambda^{n-1}(M) \xrightarrow{d} \Lambda^{n}(M) \xrightarrow{d} \Lambda^{n+1}(M) \to \ldots$$

where $\Lambda^n(M)$ is defined above for $n \geqslant 0$, and $\Lambda^n(M) = (0)$ for $n < 0$. The *De Rham cohomology* R-modules of M are defined, as in the classical case, by

$$H^n(M) = F^n(M) / E^n(M)$$

where

$$F^n(M) = \mathrm{Ker}(\Lambda^n(M) \xrightarrow{d} \Lambda^{n+1}(M)) \text{ ("the \textit{closed} n-forms")}$$

and

$$E^n(M) = \mathrm{Im}(\Lambda^{n-1}(M) \xrightarrow{d} \Lambda^{n}(M)) \text{ ("the \textit{exact} n-forms").}$$

(Note that $E^n(M) \subset F^n(M)$ since $d^2 = 0$). If $f:M \to N$, then by naturality of d, $f^*:\Lambda^n(N) \to \Lambda^n(M)$ maps closed forms on N to closed forms on M, and exact forms on N to exact ones on M, so we obtain a map $f^* = H^n(f):H^n(N) \to H^n(M)$, making $H^n(-)$ into a contravariant functor.

In the terminology of the De Rham cohomology, the integration axiom of Kock-Reyes (1981) can be stated as

$$H^1([0,1]) = (0)$$

where $I = [0,1] = \{x \in R \mid 0 \leqslant x \leqslant 1\}$ is the unit interval defined by a preorder relation < which is compatible with the ringstructure on R ($0 \leqslant 1$; $x \leqslant y \Rightarrow x+z \leqslant y+z$; and $x < y$, $0 \leqslant t \Rightarrow xt \leqslant yt$) as well as with the infinitesimal structure (x nilpotent $\to 0 \leqslant x \leqslant 0$). Using the integration axiom, we can define integration of a form along a *finite n-cube* $\gamma:I^n \to M$ by the formula

$$\int_\gamma \omega = \int_o^1 \ldots \int_o^1 \tilde{\omega}((h_1,\ldots,h_n) \to \gamma(t_1+h_1,\ldots,t_n+h_n))dt_1\ldots dt_n .$$

Just as for the infinitesimal chains, we define the *object of finite* n-*chains*, $\Gamma_n(M)$, as the free R-module generated by the set of maps $I^n \to M$, and an R-linear boundary operator $\partial:\Gamma_{n+1}(M) \to \Gamma_n(M)$. These definitions are again functorial in an obvious way, and the integral

$$\int:\Gamma_n(M)\times\Lambda_n(M) \to R$$

is R-linear in each variable separately, while moreover again by definition

$$\int_{f_*\gamma} = \int_\gamma f^*(\omega) \qquad (\text{where } f_*\gamma = f\circ\gamma).$$

Less trivial is the extension of Stokes' identity, used to define exterior differentiation, from infinitesimal n-chains to finite n-chains:

PROPOSITION. (Stokes' theorem) *For any* $\gamma \in \Gamma_{n+1}(M)$, $\omega \in \Lambda_n(M)$, $\int_\gamma d\omega = \int_{\partial\gamma} \omega$.

Proof. See Kock-Reyes-Veit (1980), or Kock (1981). ▲

We now check the three 'axioms' for a cohomology theory, namely the homotopy invariance (or Poincaré lemma), the Mayer-Vietoris sequence, and the disjoint-union lemma.

POINCARE LEMMA. *The De Rham cohomology of* R^n *is the same as that of a one-point space* $\{*\}$:

$$H^q(R^n) = H^q(\{*\}) = \begin{cases} R & \text{if } q = 0 \\ (0) & \text{if } q \neq 0. \end{cases}$$

We shall derive this lemma from the following

PROPOSITION. *Let* $F:I\times M \to N$ *be a homotopy from* F_0 *to* F_1. *Then for each* n *there is an* R-*linear map*

$$K = K_n:\Lambda^n(N) \to \Lambda^{n-1}(M)$$

such that for every $\omega \in \Lambda^n(N)$,

$$\int_\tau [F_1^*(\omega) - F_0^*(\omega) - (K_{n+1}d\omega + dK_n\omega)] = 0$$

for all $\tau : I^n \to M$.

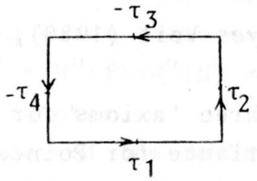
Proof. (For a proof in 'classical lenguage', see appendix 2). Define for $\gamma \in M^{D^{n-1}}$

$$K_n(\omega)(\gamma) = \int_0^1 \omega((h_1, \dots, h_n) \mapsto F_{t+h_1}(\gamma(h_2, \dots, h_n)))dt.$$

It is trivial to check that $K_n(\omega) : M^{D^{n-1}} \to R$ is homogeneous and alternating, so this defines an n-1-form $K_n(\omega) \in \Lambda^{n-1}(M)$. For notational purposes, let us assume that $n = 2$, and take any $\tau : I^2 \to M$. Then

$$\int_\tau dK_2(\omega) = \int_{\partial\tau} K_2\omega = \int_{\tau_1} K_2\omega + \int_{\tau_2} K_2\omega - \int_{\tau_3} K_2\omega - \int_{\tau_4} K_2\omega$$

where $\tau_1 = \tau(-,0)$, $\tau_2 = \tau(1,-)$, $\tau_3 = \tau(-,1)$, $\tau_4 = \tau(0,-)$



Now define a 3-cube $\rho : I^3 \to M$ by

$$\rho(x_1, x_2, x_3) = F_{x_1}(\tau(x_2, x_3)),$$

and compute $\int_\rho d\omega$ in two ways.

On the one hand, by definition of $\int$,

$$\int_\rho d\omega = \int_0^1\int_0^1\int_0^1 d\omega[(h_1, h_2, h_3) \mapsto F_{t_1+h_1}(\tau(t_2+h_2, t_3+h_3))]dt_1dt_2dt_3$$

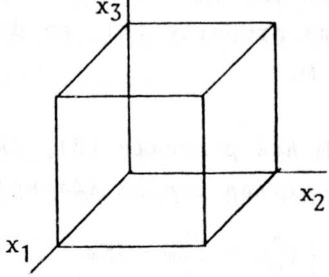
$$= \int_0^1\int_0^1 K_3(d\omega)[(h_1, h_2) \mapsto \tau(t_2+h_2, t_3+h_3)]dt_2dt_3$$

$$= \int_\tau K_3(d\omega).$$

On the other hand, by Stokes' theorem,

$$\int_\rho d\omega = \int_{\partial\rho} \omega = \int_f \omega - \int_{ba} \omega + \int_\ell \omega - \int_r \omega + \int_t \omega - \int_{bo} \omega$$

(where $\partial\rho = (f-ba) + (\ell-r) + (t-bo)$, f refers to the restriction of ρ to the front of the cube below, ba to the back, etc.)



Now $\int_f \omega = \int_{F_1^*\tau} \omega = \int_\tau F_1^*\omega$, and $\int_{ba} \omega = \int_\tau F_0^*\omega$. We claim that

$$\int_\ell \omega = \int_{\tau_4} K_2\omega, \quad \int_r \omega = \int_{\tau_2} K_2\omega, \quad \int_t \omega = \int_{\tau_3} K_2\omega, \quad \int_{bo} \omega = \int_{\tau_1} K\omega.$$

Indeed,

$$\int_{\tau_4} K_2\omega = \int_0^1 K_2(\omega)[h \to \tau(0,t_2+h)]dt_2$$

$$= \int_0^1\int_0^1 \omega[(h_1, h_2) \to F_{t_1+h_1}(\tau(0,t_2+h_2))]\, dt_1 dt_2$$

$$= \int_\ell \omega ,$$

and the other three identities are similar. By putting the derived equalities together, one completes the proof of the proposition. ▲

In the above proposition, it would be more natural to conclude that $F_1^*\omega - F_0^*\omega = Kd\omega + dK\omega$. Unfortunately we do not know whether in general, for $\omega \in \Lambda^n(M)$, $\int_\tau \omega = 0$ for all

$\tau : I^n \to M$ implies that $\omega = 0$. However, this is the case if M has the following extension property.

(E) The canonical map $M^{D_2^n} \overset{r}{\to} M^{D^n}$ induced by the inclusion $D \to D_2 = \{x \in R \mid x^3 = 0\}$ is a retraction (i.e. there is a section i, $r \circ i = 1$).

Every R^n has property (E), and more generally, so do all formal manifolds (in any of the senses proposed). Moreover, if an object M has property (E), so do all exponentials M^N and all retracts of M.

COROLLARY 1. *If* M *has property* (E), *then the conclusion of the above proposition can be strengthened to*

$$F_1^* \omega - F_o^* \omega = Kd\omega + dK\omega$$

Proof. As just claimed, it suffices to show that if M has property (E) then for any $\omega \in \Lambda^n(M)$, $\int_\tau \omega = 0$ for every $\tau : I^n \to M$ implies that $\omega = 0$. We do the case $n = 1$ only. To show that $\omega = 0$, choose an infinitesimal 1-cube $(\gamma, h_o) \in M^D \times D$, and extend γ to a map $\tilde{\gamma} : D_2 \to M$ by an application of property (E). For notation, let $\phi_{h_o} : I \to I$ be the function $t \mapsto h_o t$, and define $f : D \to R$ by $f(t) = \omega[h \to \tilde{\gamma}(t+h)]$. It suffices to show that

$$\int_{(\gamma, h_o)} \omega = \int_{\gamma \circ \phi_{h_o}} \omega .$$

since then by assumption on ω, it follows that $\int_{(\gamma, h_o)} \omega = 0$. Now

$$\begin{aligned}
\int_{\gamma \circ \phi_{h_o}} \omega &= \int_0^1 \omega[h \mapsto \gamma(\phi_{h_o}(t+h))]\,dt && \text{(by definition)} \\
&= \int_0^1 \omega[h \mapsto (h_o \cdot \tilde{\gamma})(h_o t + h)]\,dt \\
&= \int_0^1 h_o \cdot \omega[h \mapsto \tilde{\gamma}(h_o t + h)]\,dt && \text{(by homogeneity)} \\
&= \int_0^1 h_o f(t h_o)\,dt
\end{aligned}$$

$$= \int_0^{h_0} f(t)dt \qquad \text{(substitution)}$$

$$= h_0 \cdot f(0) \qquad \text{(Kock-Lawvere axiom)}$$

$$= \int_{(\gamma,h_0)} \omega \;. \blacktriangle$$

COROLLARY 2. *Assume that* M *has property* (E). *If* $M \overset{f}{\underset{g}{\rightrightarrows}} N$ *are homotopic, then* $H^{\cdot}(f) = H^{\cdot}(g) : H^{\cdot}(N) \to H^{\cdot}(M)$. *In particular, if* M *and* N *are homotopy equivalent, then* $H^{\cdot}(M) \cong H^{\cdot}(N)$ (whence the Poincaré lemma). ▲

Let us now turn to the Mayer-Vietoris sequence. Recall that a partition of unity subordinate to a cover {U,V} of M is a pair of maps $\rho_U, \rho_V : M \to R$ such that for all $x \in M$, $\rho_U(x) + \rho_V(x) = 1$, and moreover, for all $x \in M$

$$x \in U \text{ or } \rho_U(x) = 0, \text{ and } x \in V \text{ or } \rho_V(x) = 0.$$

PROPOSITION. *Assume that* $M = U \cup V$, *where* U *and* V *are étale subjects of* M (*i.e., if* $\phi : D^q \to M$ *and* $\phi(0) \in U$, *then* $\mathrm{im}(\phi) \subset U$; *similarly for* V). *If* {U,V} *has a partition of unity subordinate to it, then the sequence*

$$0 \to \Lambda^q(M) \to \Lambda^q(U) \oplus \Lambda^q(V) \to \Lambda^q(U \cap V) \to 0$$

$$\omega \mapsto (i_U^*(\omega),\ i_V^*(\omega))$$

$$(\mu,\nu) \mapsto i_{U\cap V}^*(\nu) - i_{U\cap V}^*(\mu)$$

is exact (the i-denote the inclusions).

Proof. The fact that the first map is monic follows from the fact that {U,V} is an étale cover.

To show exactness in the middle, let $(\mu,\nu) \in \Lambda^q(U) \oplus \Lambda^q(V)$ be such that $i_{U\cap V}^*(\nu) = i_{U\cap V}^*(\mu)$. Define

$$\omega = \rho_U \cdot \mu + \rho_V \cdot \nu$$

where $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to {U,V}, and

$$(\rho_U\cdot\mu)(\phi) = \begin{cases} \rho_U\phi(0)\cdot\mu(\phi) & \text{if } \phi(0) \in U \\ 0 & \text{if } \rho_U(\phi(0)) = 0 \end{cases}$$

and similarly for $\rho_V\cdot\nu$. (Notice that since U and V are étale, this definition makes sense). Then

$$i_U^*\omega - \mu = i_U^*(\rho_U\cdot\mu) + i_U^*(\rho_V\cdot\mu) - (\rho_{U|U}\cdot\mu + \rho_{V|U}\cdot\mu)$$

But $i_U^*(\rho_U\cdot\mu) = \rho_{U|U}\cdot\mu$, and by definition of a partition of unity and the fact that $i_{U\cap V}^*(\mu) = i_{U\cap V}^*(\nu)$, also $i_U^*(\rho_V\cdot\nu) = \rho_{V|U}\cdot\mu$. Hence $i_U^*\omega = \mu$. Similarly it follows that $i_V^*(\omega) = \nu$, so the sequence is exact in the middle.

To show that the right hand map is epic, we show similarly that any $\omega \in \Lambda^q(U\cap V)$ comes from the pair $(-\rho_V\cdot\omega, \rho_U\cdot\omega)$. ▲

This short exact sequence is called the Mayer-Vietoris sequence, and it induces a long exact sequence, as in the following

COROLLARY. *Under the hypothesis of the preceding proposition, there is a long exact sequence*

$$\ldots \to H^q(M) \to H^q(U) \oplus H^q(V) \to H^q(U\cap V) \to H^{q+1}(M) \to \ldots$$

The linear map $d^*: H^q(U\cap V) \to H^{q+1}(M)$, *the so-called Bockstein homomorphism, may be described by*

$$d^*[\omega] = \begin{cases} [-d(\rho_V\cdot\omega)] & \textit{on } U \\ [d(\rho_U\cdot\omega)] & \textit{on } V. \end{cases}$$

Proof. This is some simple homological algebra. The general situation is that we are given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of complexes (i.e. each level $0 \to A^q \xrightarrow{f} B^q \xrightarrow{g} C^q \to 0$ is exact).

To define $d^*: H^q(C) \to H^{q+1}(A)$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{q-1} \arrow[r, "f"] \arrow[d, "d"] & B^{q-1} \arrow[r, "g"] \arrow[d, "d"] & C^{q-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q} \arrow[r, "f"] \arrow[d, "d"] & B^{q} \arrow[r, "g"] \arrow[d, "d"] & C^{q} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0
\end{tikzcd}$$

and take $[c] \in H^q(C)$, so $dc = 0$. Write $c = g(b)$ for some $b \in B^q$, and observe that $dc = dg(b) = g(db) = 0$, whence $ab = f(a)$ for some $a \in A^{q+1}$. Define $d^*[c]$ to be this $[a]$. This all looks like a horrible application of the axiom of choice (which is not available in the synthetic context), but it is not, and moreover d^* is well-defined on equivalence classes. Both assertions follow from the fact that $d^*[c]$ is the *unique* equivalence class $[a]$ such that for some $b \in B^q$, $f(a) = db$ and $[g(b)] = [c]$. To see this, assume that both a and a' are candidates, i.e.

$$f(a) = db, \quad [g(b)] = [c], \quad \text{some } b$$

$$f(a') = db', \quad [g(b')] = [c], \text{ some } b'.$$

Then since $[g(b)] = [g(b')]$, $g(b-b') = dc_o$ for some c_o. g is epi, so $c_o = g(b_o)$ for some b_o. But then $g(b-b'-db_o) = dc_o - dc_o = 0$, hence $b-b'-db_o = f(a_o)$ for some a_o.

Thus $f(da_o) = db-db'-d^2b_o = f(a)-f(a')$, and since f is mono, $da_o = a-a'$, i.e. $[a] = [a']$. Linearity of d^* is now obvious. It remains to show that the long sequence is exact, which is easy and can safely be left to the reader. ▲

As a final remark, we note the following proposition, the proof of which is obvious.

PROPOSITION. *If* $M = \bigcup_\alpha M_\alpha$ *is a disjoint union, then* $H^q(M) \simeq \pi_\alpha M^q(M_\alpha)$. ▲

§2. Singular homology.

Let M be a smooth space. A *singular* q-*simplex* of M is a map $\Delta_q \xrightarrow{\sigma} M$, where $\Delta_q (q \geqslant 0)$ is the standard q-simplex

$$[e_o,\dots,e_q] = \{(x_o,\dots,x_q) \in R^{q+1} \mid 0 \leqslant x_i \leqslant 1 \text{ and } \textstyle\sum x_i = 1\}$$

($\{e_o,\dots,e_q\}$ denotes the standard base of R^{q+1}). We let $S_q(M)$ be the free R-module generated by the singular q-simplices; the elements of $S_q(M)$ are called *singular* q-*chains*. There is an R-linear *boundary operator*

$$\partial = \partial_q : S_q(M) \to S_{q-1}(M)$$

defined on generators $\Delta_q \xrightarrow{\sigma} M$ by

$$\partial_q(\sigma) = \sum_{j=o}^{2} (-1)^j \sigma \circ \varepsilon_q^j ,$$

where $\varepsilon_q^j : \Delta_{q-1} \to \Delta_q$ is the j-th face of Δ_q, i.e. $(\varepsilon_q^j(x_o,\dots,x_{q-1}) = (x_o,\dots,x_{j-1},0,x_j,\dots,x_q)$. Since $\partial \circ \partial = 0$ (as is easily checked), this defines a complex $S_\bullet(M)$ if we agree that $S_q(M) = (0)$ for $q < 0$. Note that this definition of $S_\bullet(M)$ is functorial in M: a map $M \xrightarrow{f} N$ induces R-linear maps $f_* : S_q(M) \to S_q(N)$ for each q, defined on generators by composition, i.e. $f_*(\sigma) = f \circ \sigma$, and this yields a chain map $f_* : S_\bullet(M) \to S_\bullet(M)$ because $\partial(f_*\sigma) = f_*(\partial\sigma)$.

As usual, we define submodules $B_q(M) = \mathrm{Im}(\partial_{q+1})$ ('*boundaries*') and $Z_q(M) = \mathrm{Ker}(\partial_q)$ ('*cycles*') of $S_q(M)$, and since $\partial^2 = 0$, $B_q(M) \subset Z_q(M)$ so we can define the q-dimensional *singular homology* R-module of M by

$$H_q(M;R) = Z_q(M)/B_q(M).$$

Clearly, $H_q(-;R)$ is a covariant functor.

We proceed now as in the case of the De Rham cohomology by proving the three key properties, viz. the Poincaré lemma or homotopy invariance, the existence of the (longexact) Mayer-Vietoris sequence, and the disjoint union lemma.

PROPOSITION. *Let* $F:I\times M \to N$ *be a homotopy from* F_o *to* F_1. *Then for each* q *there is an* R-*linear map*

$$P = P_q : S_q(M) \to S_{q+1}(N)$$

such that for every $\sigma:\Delta_q \to M$,

$$F_{1*}(\sigma) - F_{o*}(\sigma) = \partial P_q(\sigma) + P_{q-1}(\partial\sigma).$$

Proof. We will define a triangulation P_q of $I\times\Delta_q$, i.e. a sum $P_q \in S_q(I\times\Delta_q)$ of maps $\Delta_{q+1} \to I\times\Delta_q$, and then for $\sigma:\Delta_q \to M$ we let $P_q(\sigma) \in S_{q+1}(N)$ be the composition (= sum of compositions)

$$\Delta_{q+1} \xrightarrow{P_q} I\times\Delta_q \xrightarrow{1\times\sigma} I\times M \xrightarrow{F} N.$$

Each of the maps involved in the definition of P_q will be affine, and it is useful to introduce some notation. Recall that a singular q-simplex σ on a convex subset $M \subset R^n$ is called affine if there are points $m_o,\dots,m_q \in M$ such that

$$\sigma(x_o,\dots,x_q) = x_o m_o + \dots + x_q m_q;$$

such an affine simplex is denoted by $[m_o,\dots,m_q]$. So $\partial[m_o,\dots,m_q] = \sum_{k=o}^{q}(-1)^k[m_o,\dots,\hat{m}_k,\dots,m_q]$. If $S = [m_o,\dots,m_q]$ is an affine q-simplex, the k-th q-1-simplex occurring in ∂S will be denoted by $S(\hat{k})$, i.e.

$$S(\hat{k}) = [m_o,\dots,\hat{m}_k,\dots,m_q].$$

In $I\times\Delta_q$, we distinguish the points $e_j^\alpha = (\alpha, e_j)$ for $\alpha = 0,1$, $j = 0,\dots,q$. Let us write S_j for the affine q+1-simplex $[e_j^1,\dots,e_q^1,e_o^o,\dots,e_j^o]$ on $I\times\Delta_q$ $(j = 0,\dots,q)$. We now define P_q by

$$P_q = \sum_{j=o}^{q} - S_j, \quad \text{is q is even}$$

$$P_q = \sum_{j=o}^{q} (-1)^j S_j, \text{ if q is odd.}$$

Let us verify that indeed

$$\partial P_q(\sigma) - P_{q-1}(\partial\sigma) = F_1{*}(\sigma) - F_o{*}(\sigma)$$

for every $\sigma:\Delta_q \to M$. From the definition of $P_q(\sigma)$ given above it is clear that it sufficies to consider the 'generic' case where $\sigma = [e_o,\ldots,e_q] = \mathrm{id}:\Delta_q \to \Delta_q$, and $F:I\times M \to N$ is the identity $I\times\Delta_q \to I\times\Delta_q$. Thus, we verify that for each q,

$$\partial P_q([e_o,\ldots,e_q]) - \sum_{k=o}^{q}(-1)^k P_{q-1}([e,\ldots,\hat{e}_k,\ldots,e_q])$$
$$= [e_o^1,\ldots,e_q^1] - [e_o^o,\ldots,e_q^o].$$

Indeed, if q is even,

$$\partial P_q([e_o,\ldots,e_q]) = -\sum_{j=o}^{q}\sum_{k=o}^{q+1}(-1)^k S_j(\hat{k})$$

while

$$P_{q-1}(\partial[e_o,\ldots,e_q]) = \sum_{k=o}^{q}(-1)^k P_{q-1}([e_o,\ldots,\hat{e}_k,\ldots,e_q])$$
$$= \sum_{k=o}^{q}(-1)^k \sum_{\substack{j=o\\ k\neq k}}^{q}(-1)^{p_j^k}[e_j^1,\ldots,\hat{e}_k^1,\ldots,e_o^o,\ldots\hat{e}_k^o,\ldots,e_j^o]$$

(where $p_j^k = j$ if $j < k$, $p_j^k = j-1$ if $j > k$. Note that only one of e_k^1, e_k^o is omitted, depending on whether $j < k$ or $k < j$);

$$= \sum_{j=o}^{q}\Big[\sum_{k=o}^{j-1}(-1)^k(-1)^{j-1}S_j(\widehat{q-j+k+1}) + \sum_{k=j+1}^{q}(-1)^k(-1)^j\, S_j(\widehat{k-j})\Big]$$
$$= \sum_{j=o}^{q}\Big[\sum_{\ell=q-j+1}^{q}(-1)^{\ell-q+j-1}(-1)^{j-1}S_j(\hat{\ell}) + \sum_{\ell=1}^{q-j}(-1)^{\ell+j}(-1)^j S_j(\hat{\ell})\Big]$$
$$= \sum_{j=o}^{q}\Big[\sum_{\ell=1}^{q}(-1)^{\ell}S_j(\hat{\ell})\Big],\quad \text{since q is even.}$$

Hence $$\partial P_q + P_{q-1}\partial = -\sum_{j=o}^{q}\sum_{k=o}^{q+1}(-1)^k S_j(\hat{k}) + \sum_{j=o}^{q}\sum_{k=1}^{q}(-1)^k S_j(\hat{k})$$
$$= \sum_{j=o}^{q}[S_j(\widehat{q+1}) - S_j(\hat{0})] \quad (\text{q es even})$$

$$= [e_o^1,\dots,e_q^1] = [e_o^o,\dots,e_q^o],$$

since everthing else cancels because of $S_j(\widehat{q+1}) = S_{j-1}(\hat{0})$. And if q is odd,

$$P_q([e_o,\dots,e_q]) = \sum_{j=o}^{q} (-1)^j \sum_{k=o}^{q+1} (-1)^k S_j(\hat{k}),$$

while

$$P_{q-1}(\partial[e_o,\dots,e_q]) = \sum_{k=o}^{q} (-1)^k P_{q-1}([e_o,\dots,\hat{e}_k,\dots,e_q])$$

$$= \sum_{k=o}^{q} (-1)^k \sum_{\substack{j=o\\ j\neq k}}^{q} [e_j^1,\dots,\hat{e}_k^1,\dots,e_q^1,e_o^o,\dots,\hat{e}_k^o,\dots,e_j^o]$$

$$= \sum_{j=o}^{q} [\sum_{k=o}^{j-1} (-1)^{k+1} S_j(\widehat{q-j+k+1}) + \sum_{k=j+1}^{q} (-1)^{k+1} S_j(\widehat{k-j})]$$

$$= \sum_{j=o}^{q} [\sum_{\ell=q-j+1}^{q} (-1)^{\ell-q+j} S_j(\hat{\ell}) + \sum_{=1}^{q-j} (-1)^{\ell+j+1} S_j(\hat{\ell})]$$

$$= \sum_{j=o}^{q} \sum_{\ell=1}^{q} (-1)^{\ell+j+1} S_j(\hat{\ell}), \text{ since q is odd,}$$

and from this is immediately follows as in the case where q is even that $\partial P_q([e_o,\dots,e_q]) - P_{q-1}(\partial[e_o,\dots,e_q]) = [e_o^1,\dots,e_q^1] - [e_o^o,\dots,e_q^o]$. This completes the proof. ▲

COROLLARY 1. (Homotopy invariance) *If* $M \underset{g}{\overset{f}{\rightrightarrows}} N$ *are homotopic maps, then* $H_\bullet(f;R) = H_\bullet(g;R): H_\bullet(M;R) \to H_\bullet(N;R)$. *In particular, if* M *and* N *are homotopy equivalent, then* $H_\bullet(M;R) \simeq H_\bullet(N;R)$. ▲

COROLLARY 2. (Poincaré Lemma) *Let* $M \subset R^n$ *be convex and inhabited. Then*

$$H_q(M;R) = \begin{cases} R & \text{if } q = 0 \\ (0) & \text{if } q > 0 \end{cases}$$

Proof. If M is a single point, this is clear; and if M is arbitrary convex, inhabited, it is contractible, hence by corollary 1 it has the same singular homology as a single point. ▲

We now turn to the exactness of the Mayer-Vietoris sequence. Things are considerably more difficult here than in the case of the De Rham cohomology. Let $M = U \cup V$, and let $S_q^{\{U,V\}}(M)$ be the submodule of $S_q(M)$ generated by $S_q(U) \cup S_q(V)$. Then from the short exact sequence

$$0 \to S_\cdot(U \cap V) \to S_\cdot(U) \oplus S_\cdot(V) \to S_\cdot^{\{U,V\}}(M) \to 0$$

we obtain (as usual) a long exact sequence which is just like the one of Mayer-Vietoris but for the fact that the homology $H_\cdot^{\{U,V\}}(M;R)$ of the complex $S_\cdot^{\{U,V\}}(M)$ appears instead of $H_\cdot(M;R)$. What is the connection between the two? To answer this question, we shall from now on assume that

1. R is Archimedean
2. Δ_q is compact, for each $q \geqslant 0$
3. Every finite cover of Δ_q has a (finite) open refinement (each $q \geqslant 0$).

(As for the consistency of these assumptions relative to SDG, see section 4 below).

PROPOSITION. *The canonical map* $H_\cdot^{\{U,V\}}(M;R) \to H_\cdot(M;R)$ *induced by the inclusion* $S_\cdot^{\{U,V\}}(M) \longrightarrow S_\cdot(M)$ *is an isomorphism.*

Proof. We apply assumptions 1.-3. to a special chain map

$$sd^M: S_\cdot(M) \to S_\cdot(M),$$

viz the *barycentric subdivision*. sd^M is natural in M, and hence completely determined by the chains $(sd^{\Delta q})_q(\mathrm{id}) \in S_q(\Delta_q)$, which are defined as follows. Slightly more general, we define for each affine complex $\Delta_q \xrightarrow{\sigma} M$ into a convex $M \subset R$ a chain $sd_q^M(\sigma) \in S_q(M)$ by induction on q:

$$sd_o^M([m_o]) = [m_o]$$

$$sd_q^M([m_o,\ldots,m_q]) = (-1)^q\,[sd(\partial[m_o,\ldots,m_q]),b],$$

where $b = \sum_{j=0}^{q} \frac{1}{q+1} m_j$ is the barycenter of $[m_0, \ldots, m_q]$, and the outer brackets $[\ ]$ are interpreted as: if $\tau = \sum_{i=0}^{n} a_i [n_0^i, \ldots, n_{q-1}^i]$ is a chain of affine q-1-simplices, then $[\tau, b]$ is the chain $\sum_{i=0}^{n} a_i [n_0^i, \ldots, n_{q-1}^i, b]$ of affine q-simplices.

Thus in particular, we have defined $(sd^{\Delta_q})_q(id) \in S_q(\Delta_q)$ for each $q \geq 0$, and as just said this determines $sd_q^M(\sigma)$ for every q-simplex $\Delta_q \xrightarrow{\sigma} M$ by

$$sd_q^M(\sigma) = \sigma_*((sd^{\Delta_q})_q(id)).$$

(Note that in case σ happens to be affine, this definition of $sd_q^M(\sigma)$ coincides with the one already given). One easily checks that each $sd^M : S_\bullet(M) \to S_\bullet(M)$ is a chain map. The proof is now completed by noting the properties of sd stated in the following three lemmas.

LEMMA 1. *Every singular simplex in* $(sd^{\Delta_q})_q{}^m(id)$ *has diameter* $\leq (q/q+1)^m \mathrm{diam}(\Delta_q)$.

Proof. trivial induction on q.

LEMMA 2. *Let* $M = U \cup V$. *For each singular* q-*simplex* $\sigma : \Delta_q \to M$ *there is an* $m \geq 0$ *such that every simplex in* $sd^m(\sigma)$ (*where* $sd = (sd^M)_q$) *factors through either* U *or* V, *i.e.* $sd^m(\sigma) \in S_q^{\{U,V\}}(M)$.

Proof. Since $\Delta_q = \sigma^{-1}(U) \cup \sigma^{-1}(V)$, we also have (by assumption 3 on Δ_q) that $\Delta_q = \mathrm{Int}\,\sigma^{-1}(U) \cup \mathrm{Int}\,\sigma^{-1}(V)$. From compactness of Δ_q (assumption 2) we obtain a Lebesgue number $\lambda > 0$ for this cover. Since R is Archimedean (assumption 1), there is an $m \geq 0$ such that $(q/q+1)^m \mathrm{diam}(\Delta_q) < \lambda$. Then every simplex in $sd^m(id)$ factors through $\mathrm{Int}(\sigma^{-1}(U))$ or through $\mathrm{Int}(\sigma^{-1}(V))$, and this implies that every simplex in $sd^m(\sigma)$ factors through U or V.

LEMMA 3. *For every* M *there are* R-*linear maps*

$$R_q = R_q^M : S_q(M) \to S_{q+1}(M)$$

(*natural in* M) *such that for every* $\sigma \in S_q(M)$,

$$sd_q^M(\sigma) - \sigma = \partial R_q(\sigma) + R_{q-1}(\partial\sigma).$$

Consequently, the map $H_q(M;R) \to H_q(M;R)$ *induced by the chain map* sd^M *is the identity.*

Proof. As in the definition of sd_q^M, because of naturality in M all of R_q^M is determined by fixing $R_q^{\Delta_q}(id)$. This will be done by induction on q: For $q = 0$, there is only one choice $\Delta_1 \to \Delta_o$ for $R_o^{\Delta_o}(id)$. And if $R_{q-1}^{\Delta_{q-1}}$ is defined as is required by the lemma, consider $sd_q^{\Delta_q}(id) - id \in S_q(\Delta_q)$: Since

$$\begin{aligned}
&\partial(sd_q^{\Delta_q}(id) - id - R_{q-1}^{\Delta_{q-1}}(\partial(id)))\\
&= sd_{q-1}^{\Delta_{q-1}}(\partial(id)) - \partial(id) - \partial R_{q-1}^{\Delta_{q-1}}(\partial id)\\
&= R_{q-2}^{\Delta_{q-2}}(\partial\partial id) \quad \text{(by induction hypothesis)}\\
&= 0\,,
\end{aligned}$$

it follows from the contractibility of Δ_q that there exists a $\sigma \in S_{q+1}(\Delta_q)$ such that

$$\partial\sigma = sd_q^{\Delta_q}(id) - id - R_q^{\Delta_{q-1}}(\partial id)$$

(by the Poincaré lemma). Thus for $R_q^{\Delta_q}$ we can take this σ. (The reader may suspect that in order to obtain $R_q^{\Delta_q}(\Delta_q)$ as a function of q we have to apply the axiom of dependent choices (on q), which is not available in the synthetic context. But this is not so, since the Poincaré lemma does not merely yield the existence of a σ as above: by applying the proposition preceding the Poincaré lemma (p.237) to a fixed contraction of Δ_q we obtain an explicit description of σ!) Putting these three lemmas together, we complete the proof of the proposition. ▲

COROLLARY. (Mayer-Vietoris sequence). *Assume that* $M = U \cup V$. *Then there is a long exact sequence*

$$\cdots \to H_{q+1}(M) \xrightarrow{\delta_*} H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(M) \xrightarrow{\delta_*} \cdots$$

deduced from the short exact sequence

$$0 \to S_\bullet(U \cap V) \to S_\bullet(U) \oplus S_\bullet(V) \to S_\bullet^{\{U,V\}}(M) \to 0. \quad \blacktriangle$$

As a final property of singular homology that we need, we have

PROPOSITION. *If* $M = \coprod_\alpha M_\alpha$ *is a disjoint union of a family* $(M_\alpha)_\alpha$ *indexed by a decidable set* $\{\alpha\}$, *then*

$$H_\bullet(M;R) = \oplus_\alpha H_\bullet(M_\alpha;R)$$

where $\oplus_\alpha$ *denotes the coproduct of the family* $\{H_\bullet(M_\alpha;R)\}_\alpha$

Proof. This follows immediately from the fact that Δ_q is indecomposable (because of the integration axiom) (i.e. if $\Delta_q = A \cup B$, A, B disjoint, then $\Delta_q = A$ or $\Delta_q = B$), and thus any map $\Delta_q \overset{\sigma}{\to} M$ factors through some M_α. ▲

§3. A synthetic version of the Rham's theorem.

In section 1 we have seen how the integration axiom allows us to define for any q-form ω on M the integral $\int_\gamma \omega$ along an n-chain $\gamma: I^q \to M$. From this, we can define the integral

$$\int_\sigma \omega$$

for a simplicial q-chain $\sigma: \Delta_q \to M$ in any of the standard ways. Let us quickly describe one version (which seems notationally not too involved) in more detail. For this, we temporarily replace the standard simplices $\Delta_q = [e_o, \ldots, e_q]$ by their isomorphic copies (also called Δ_q)

$$\Delta_q = \{(x_1, \ldots, x_q) \in R^q \mid 0 \leqslant x_q \leqslant \ldots \leqslant x_1 \leqslant 1\}.$$

Observe that the faces of this Δ_q are the maps $\varepsilon^i: \Delta_{q-1} \to \Delta_q$,

$\varepsilon^0(x_1,\ldots,x_{q-1}) = (x_1,\ldots,x_{q-1},0)$, $\varepsilon^i(x_1,\ldots,x_{q-1}) = (x_1,\ldots x_i,x_i,x_{i+1},\ldots,x_{q-1})$ $(1 \leq i \leq q-1)$, and $\varepsilon^q(x_1,\ldots,x_{q-1}) = (1,x_1,\ldots,x_{q-1})$. There is an obvious (orientation preserving) projection

$$\pi_q : I^q \to \Delta_q,\ (x_1,\ldots,x_q) \to (x_1,x_1x_2,\ldots,x_1\ldots x_q),$$

by use of which we can define the above integral $\int_\sigma \omega$ as

$$\int_\sigma \omega = \int_{\sigma\circ\pi_q} \omega\ .$$

Writing out the boundary $\partial\pi_q$ as a sum of maps $I^{q-1} \to \Delta_q$ immediately gives that $\partial\pi_q = (\partial\Delta_q)\circ\pi_{q-1}$ modulo some degenerate chains $I^{q-1} \to \Delta_q$ (these are affine chains whose image has a dimension $<q-1$, so the integral over any $q-1$-form vanishes). Consequently, we obtain Stokes' theorem for simplices if ω is any $q-1$-form on M and $\gamma:\Delta_q \to M$ is a simplicial q-chain, then

$$\int_{\partial\gamma} \omega = \int_{\partial\Delta_q} \gamma^*(\omega) = \int_{\partial\Delta_q\circ\pi_{q-1}} \gamma^*(\omega) \qquad \text{(by definition)}$$

$$= \int_{\partial\pi_q} \gamma^*(\omega) = \int_{\pi_q} \gamma^*(d\omega) \qquad \text{(by cubical Stokes')}$$

$$= \int_\gamma d\omega \qquad \text{(again by definition)}.$$

Having defined $\int_\sigma \omega$ for $\omega \in \Lambda^q(M)$ and generators $\sigma \in S_q(M)$, we extend this to a map

$$\Lambda^q(M)\times S_q(M) \xrightarrow{\int} R,\quad (\omega,\sigma) \to \int_\sigma \omega$$

which is R-linear in both ω and σ separately. Clearly, this integration is natural in M, in the sense that

$$\int_{f_*(\sigma)} \omega = \int_\sigma f^*(\omega).$$

Because of (the simplicial form of) Stokes' theorem the restriction of the integral to

$$F^q(M)\times Z_q(M) \xrightarrow{\int} R\ ,$$

sends exact forms as well as boundaries to 0, and thus we may pass to quotients to obtain an R-linear map

$$H^q(M) \xrightarrow{I} H_q(M;R)^* = \mathrm{Hom}_R(H_q(M;R),R)$$
$$[\omega] \mapsto ([\sigma] \mapsto \int_\sigma \omega).$$

To formulate De Rham's theorem, let

$$\mathcal{R} = \{M \mid H^q(M) \xrightarrow{I} H_q(M;R)^* \text{ is an isomorphism for every } q\}.$$

If we assume as in section 2 that R is Archimedean, each Δ_q is compact and indecomposable, and every finite cover of Δ_q has an open refinement, then we obtain the following synthetic version of *De Rham's theorem*.

THEOREM. *The class $\mathcal{R}$ has the following closure properties:*

(1) *$\mathcal{R}$ contains R^n for each $n \geq 0$ (and also infinitesimal spaces such as D, D_∞, Δ, etc.)*

(2) *Let $\{U,V\}$ be an étale cover of M having a partition of unity subordinated to it. If U, V, and $U \cap V$ belong to $\mathcal{R}$ then so does M.*

(3) *If $M = \coprod_\alpha M_\alpha$ is a disjoint union indexed by a decidable set $\{\alpha\}$, and each M_α belongs to $\mathcal{R}$, then so does M.*

(4) *If X is a retract of an object M in $\mathcal{R}$, then $X \in \mathcal{R}$.*

Proof. (1) follows from the two Poincaré lemmas, for De Rham cohomology and singular homology, (3) follows from the two disjoint union lemmas. (4) is almost trivial: Let $X \underset{i}{\overset{r}{\rightleftarrows}} M$ be given such that $ri = \mathrm{id}_X$, and consider the diagram

$$\begin{tikzcd}
H^q(M) \arrow[r, shift right, "H^q(i)"'] \arrow[d, "I_M"'] & H^q(X) \arrow[l, shift right, "H^q(r)"'] \arrow[d, "I_X"'] \\
H_q(M;R)^* \arrow[r, shift right, "H_q(i)^*"'] & H_q(X;R)^* \arrow[l, shift right, "H_q(r)^*"'] \arrow[u, dashed, "J"']
\end{tikzcd}$$

An inverse $J:H_q(M;R)^* \to H^q(X)$ is given by the composite $H^q(r)\circ I_M \circ H_q(i)^*$, since both squares commute and by functoriality

$$H^q(i)\circ H^q(r) = \mathrm{id},\ H_q(i)^*\circ H_q(r)^* = \mathrm{id}.$$

(2) is only slightly more involved: it follows from the 5-lemma applied to the diagram obtained from the long Mayer-Vietoris sequence for De Rham cohomology, and its dual for singular homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U)\oplus H^q(V) \arrow[r] \arrow[d] & H^q(U\cap V) \arrow[r, "d^*"] \arrow[d] & H^{q+1}(M) \arrow[r] \arrow[d] & H^{q+1}(U)\oplus H^{q+1}(V) \arrow[r] \arrow[d] & H^{q+1}(U\cap V) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & H_q(U)^*\oplus H_q(V)^* \arrow[r] & H_q(U\cap V)^* \arrow[r, "\delta^*"'] & H_{q+1}(M)^* \arrow[r] & H_{q+1}(U)^*\oplus H_{q+1}(V)^* \arrow[r] & H_{q+1}(U\cap V)^* \arrow[r] & {}
\end{tikzcd}$$

Indeed, this diagram is commutative#: the only nontrivial square is the one involving the Bockstein homomorphisms d^* and δ^*, and for this case we have the following.

LEMMA. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of complexes $\ldots\to A^q \xrightarrow{q} A^{q+1}\to\ldots$ etc., and let $0 \to C \xrightarrow{\beta} B \xrightarrow{\alpha} A \to 0$ be an exact sequence of complexes $\ldots\to A_{q+1} \xrightarrow{\partial} A_q \to\ldots$. Let $\phi^q_{AA}:A^q \to A^*_q$, ϕ^q_{BB}, ϕ^q_{CC} be R-linear pairings such that both*

$$\text{(1)}\quad \begin{tikzcd}
A^q \arrow[r, "f^q"] \arrow[d, "\phi^q_{AA}"'] & B^q \arrow[d, "\phi^q_{BB}"] \\
A^*_q \arrow[r, "\alpha^*_q"] & B^*_q
\end{tikzcd}$$

, and similarly for ϕ_{BB} and ϕ_{CC},

and

$$\text{(2)}\quad \begin{tikzcd}
A_q \arrow[r] \arrow[d, "\phi^q_{AA}"'] & A^{q+1} \arrow[d, "\phi^{q+1}_{AA}"] \\
A^*_q \arrow[r, "\partial^*"] & A^*_{q+1}
\end{tikzcd}$$

, and similarly for ϕ_{BB} and ϕ_{CC},

commute, then the diagram.

Provided that the dual of the Long Mayer-Vietoris sequence is exact.

$$\begin{tikzcd}
H^q(C) \arrow[r, "d^*"] \arrow[d, "\phi^q_{C\mathcal{B}}"'] & H^{q+1}(A) \arrow[d, "\phi^{q+1}_{A\mathcal{A}}"] \\
H_q(\mathcal{C})^* \arrow[r, "\delta^*"] & H_{q+1}(\mathcal{A})^*
\end{tikzcd} \tag{3}$$

also commutes, where δ^* *is the dual of* δ^*.

Proof. Recall the definition of the Bockstein maps $d^*: H^q(C) \to H^{q+1}(A)$ and $\delta_*: H_{q+1}(\mathcal{A}) \to H_q(\mathcal{C})$: given $[c] \in H^q(C)$, we find $b \in B^q$, $a \in A^{q+1}$ such that $f(a) = db$ and $g(b) = c$ and put $d^*[c] = [a]$, while given $[\underline{a}] \in H_{q+1}(\mathcal{A})$ we find $\underline{b} \in \mathcal{B}_{q+1}$ and $\underline{c} \in \mathcal{C}_q$ such that $\alpha(\underline{b}) = \underline{a}$ and $\partial\underline{b} = \beta(\underline{c})$, and put $\delta_*[\underline{a}] = [\underline{c}]$.

Now we compute (in the computation, we only use commutativity of 2 for B, $\mathcal{B}$, but the others are used to define 3 überhaupt): let $[c] \in H^q(C)$, with a and b as above, and $[\underline{a}] \in H^{q+1}(\mathcal{A})$, with $\underline{b}$ and $\underline{c}$ as above. Then

$$\begin{aligned}
\phi^{q+1}_{A\mathcal{A}}(d^*[c])([\underline{a}]) &= [\phi^{q+1}_{A\mathcal{A}}(a)(\alpha(\underline{b}))] \\
&= [\phi^{q+1}_{B\mathcal{B}}(f(a))(\underline{b})] && \text{(by 1)} \\
&= [\phi^{q+1}_{B\mathcal{B}}(db)(\underline{b})] \\
&= [\phi^{q}_{B\mathcal{B}}(b)(\partial\underline{b})] && \text{(by 2)} \\
&= [\phi^{q}_{B\mathcal{B}}(b)(\beta\underline{c})] \\
&= [\phi^{q}_{C\mathcal{C}}(gb)(\underline{c})] && \text{(by 1)} \\
&= [\phi^{q}_{C\mathcal{C}}([c])(\delta_*[\underline{a}])].
\end{aligned}$$

§4. De Rham's theorem with parameters and the comparison theorems.

In this section we obtain a version of De Rham's theorem with parameters, simply by interpreting the result of section 3 in the topos G introduced by Dubuc (1981) as a model for SDG.

We recall (cf. Reyes (1982), Moerdijk & Reyes (1983)) that we have a diagram

$$\mathcal{M} \hookrightarrow \mathcal{G} \underset{B}{\overset{\Delta}{\leftleftarrows}}\!\!\!\xrightarrow{\Gamma} \text{Sets},$$

where $\mathcal{M}$ is the category of manifolds (with a countable basis, i.e. embeddable into some $\mathbb{R}^n$), and $\mathcal{G}$ is Dubuc's topos of sheaves over the site $\mathbb{G}$ of finitely generated germ-determined C^∞-rings. The embedding s factors through $\mathbb{G}$, is full and faithful, and preserves transversal pullbacks as well as open covers; Γ is the global sections functor, Δ the constant functor, and B is the right adjoint of Γ,

$$\Delta \dashv \Gamma \dashv B.$$

$\mathcal{G}$ is a model for SDG; for example, the Koch-Lawvere axiom is valid (Koch (1981)), and so is the integration axiom (Que & Reyes (1982)).

As promised in section 1, we will begin by showing that our notion of form does not differ from the usual one whenever the two make sense. Indeed,

PROPOSITION. *For any manifold* $M \in \mathcal{M}$, Γ *maps a morphism* $\Lambda^q(s(M)) \overset{d}{\to} \Lambda^{q+1}(s(M))$ *in* $\mathcal{G}$ *to the map* $\Lambda^q(M) \overset{d}{\to} \Lambda^{q+1}(M)$, *where the first denotes the interpretation of the synthetic definition of form and exterior differentiation in* $\mathcal{G}$, *while the second denotes the usual vector space of forms and exterior differentiation map from classical differential geometry. Moreover, if* $M \overset{f}{\to} N$ *in* $\mathcal{M}$, *then similarly* Γ *maps* $s(f)^*: \Lambda^q(sN) \to \Lambda^q(sM)$ *in* $\mathcal{G}$ *to the usual pullback map* $f^*:\Lambda^q(N) \to \Lambda^q(M)$.

Proof. The global sections of $\Lambda^q(sM)$ are the maps $s(M)^{D^q} \to R$ in $\mathcal{G}$ such that in $\mathcal{G}$ it holds that they are homogeneous and alternating. But $s(M)^{D^q}$ is just the q-th iterate of the tangent bundle, $s(M)^{D^q} \simeq s(T^q(M))$, while $R = s(\mathbb{R})$, so these are the maps $T^q(M) \to \mathbb{R}$ in $\mathcal{M}$ which are homogeneous and alternating in $\mathcal{G}$. Classically, on the other hand, $\Lambda^q(M)$ is defined as the set of maps $T(M)\times_M \ldots \times_M T(M) \to \mathbb{R}$ (q-fold fibered product) all of whose fibers $T_x(M)\times\ldots\times T_x(M) \to \mathbb{R}$ are alternating and $\mathbb{R}$-linear in each variable separately. Thus, to show $\Gamma(\Lambda^q(s(M))) \simeq \Lambda^q(M)$ it suffices to prove *synthetically* (hence in G) that alternating homogeneous maps $M^{D^q} \to R$ are in 1-1 correspondence with alternating maps $M^D \times_M \ldots \times_M M^D \to R$ which are (pointwise) R-linear in each coordinate separately (where M is a manifold, so the fibers $(M^D)_x = T_x(M)$ have a vector space structure). In fact by local parametrization it suffices to consider the case $M = R^n$, and for ease of notation we will take $q = 2$. Suppose given a map $(R^n)^{D^2} \xrightarrow{\omega} R$ which is homogeneous and alternating. By the Kock-Lawvere axiom, each $f: D^2 \to R^n$ is given as

$$f(x,y) = \underline{a} + x\cdot\underline{b} + y\cdot\underline{c} + xy\cdot\underline{d}$$

for unique vectors $\underline{a},\underline{b},\underline{c},\underline{d} \in R^n$. To show that ω is determined by its restriction to $(R^n)^D \times_R (R^n)^D$ (consisting of such f with $\underline{d} = \underline{0}$) we show that $\omega(f)$ does not depend on $\underline{d}$. Indeed, writing $\omega_{\underline{a}}$ for the restriction of ω to the fiber over $\underline{a}$, $\omega_a(\underline{b},\underline{c},\underline{d})$ for $\omega(f)$, we get for all $\underline{b},\underline{c},\underline{d} \in R^n$

$$\omega_{\underline{a}}(\underline{b},\underline{c},\underline{d}) = -\omega_{\underline{a}}(\underline{c},\underline{b},\underline{d}) \qquad \text{(alternating)}$$

and hence

$$\omega_{\underline{a}}(\underline{0},\underline{0},\underline{d}) = 0$$

Moreover, for fixed $\underline{c}$ and $\underline{b}$ respectively, $\omega_{\underline{a}}(\underline{b},-,-)$ and $\omega_{\underline{a}}(-,\underline{c},-)$ are R-linear maps $(R^n)^2 \to R$ (R-linearity follows from homogeneity, cf. Kock (1981), p.51), so

$$\begin{aligned}\omega_{\underline{a}}(\underline{b},\underline{c},\underline{d}) - \omega_{\underline{a}}(\underline{b},\underline{c},\underline{0}) &= \omega_{\underline{a}}(\underline{b},\underline{0},\underline{d}) \\ &= \omega_{\underline{a}}(\underline{b},\underline{0},\underline{0}) + \omega_{\underline{a}}(\underline{0},\underline{0},\underline{d}) \\ &= 0 + 0 = 0.\end{aligned}$$

The fact that Γ preserves exterior differentiation is now immediate from the fact that both $\Gamma(d)$ and the 'classical' d satisfy Stokes' theorem (since Γ trivially preserves the boundary operator ∂).

The case of f^* is obvious. ▲

Thus, the classical representation theorem that every form on M is locally of the form $\Sigma f(\underline{x})dx_{i_1}\wedge\ldots\wedge dx_{i_n}$ holds in G for all objects of the form s(M). (In fact this can also be shown directly by a synthetic argument).

To pave the way for some results to be formulated in section 5, we remark here that a similar analysis yields that if X is a locally closed subspace of some $\mathbb{R}^n$, regarded as an object of $\mathbb{G}$ (cf. Moerdijk & Reyes (1983)), the sections of the sheaf $\Lambda^q(s(M))$ over X correspond to the usual q-forms on M which are smoothly varying in X, i.e. q-forms on X×M which are locally of the form

$$\sum_{i_1<\ldots<i_q} f_{i_1\ldots i_q}(x,\underline{m})dm_{i_1}\wedge\ldots\wedge dm_{i_q}$$

(where f is smooth), while the X-component of d, $d_X:\Lambda^q(sM)(X) \to \Lambda^{q+1}(sM)(X)$ comes from pointwise (for points of X) applying the usual $d:\Lambda^q(M) \to \Lambda^{q+1}(M)$.

In order to interpret the result of section 3 in G, let us check that the assumptions made there hold in G:

LEMMA 1. *The following hold in* G

1) R *is Archimedean*
2) *each* Δ_q *is compact*
3) $\Delta_q = A \cup B \Rightarrow \Delta_q = \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

Proof. (1) was proved in Moerdijk & Reyes (1983)). (2) and (3) were also proved there, but for the case with Δ_q replaced by $I = [0,1] \subset R$. The same proofs, however, apply to any object of the form s(M) (M compact for (2)), in particular to Δ_q. ▲

LEMMA 2. 1) *If in* M, $U \subset M$ *is open, then in* G *the inclusion* $s(U) \to s(M)$ *is étale.*

2) *If* $\{\rho_U,\rho_V\}$ *is a partition of unity subordinate to an open cover* $\{U,V\}$ *of* M *in* M, *then the same holds in* G *for* $\{s(\rho_U),s(\rho_V)\}$ *with respect to the étale cover* $\{s(U),s(V)\}$ *of* s(M).

3) *For every* M *in* M, $s(M)^{D_2^q} \to s(M)^{D^q}$ *is an epimorphism in* G.

Proof: easy and omitted. ▲

THEOREM. (De Rham's theorem 'with parameters'). *For any* $M \in M$, *the canonical map*

$$H^q(sM) \xrightarrow{I} H_q(sM;R)^*$$
$$[\omega] \mapsto ([\gamma] \mapsto \int_\gamma \omega)$$

is an isomorphism in G.

Proof. We have to show that $s(M) \in R$, where R is as in the formulation of De Rham's theorem given in section 3.

Let $\mathcal{O} = \{U \in M \text{ open} \mid s(U) \in R\}$. M has a basis of sets diffeomorphic to some $\mathbb{R}^k$, hence $\mathcal{O}$ contains a basis for M by 1) of De Rham's theorem (section 3). Also by the same theorem, $\mathcal{O}$ is closed under finite unions# and disjoint countable unions (since in G, the natural number object has decidable equality). But then $\mathcal{O}$ contains all the open subsets of M, and in particular M itself, for if U is any open subset of M, we may write $U = \bigcup_{n=o}^{\infty} V_n$ with each V_n relatively compact. Now construct by induction an open cover $\{W_n\}$ of U such that each W_n is a finite union of relatively compact basic open sets (sets diffeomorphic to some R^k), hence $W_n \in \mathcal{O}$, such that

$$\bar{V}_k \subset \bigcup_{n=o}^{k} W_n \subset \bigcup_{n=o}^{k+1} \bar{W}_n \subset \bigcup_{n=o}^{k+1} W_n ,$$

and $W_{n+2} \cap W_k = \phi$ for each $k \leqslant n$. Then $U = \bigcup_n W_n = \bigcup_{n \text{ odd}} W_n \cup \bigcup_{n \text{ even}} W_n \in \mathcal{O}$. ▲

COROLLARY. (Classical De Rham) *For any manifold* $M \in M$, *the canonical map*

$$H^q(M) \to H_q(M;\mathbb{R})^* = \mathrm{Hom}_{\mathbb{R}}(H_q(M;\mathbb{R}),\mathbb{R})$$
$$[\omega] \mapsto ([\gamma] \mapsto \int_\gamma \omega)$$

is an isomprphism.

Proof. We have already observed that the global sections functor Γ 'preserves' the notions of form and exterior derivative (cf. the proposition above), and also, trivially,

---

See note at the end of the paper.

Γ preserves the notion of q-simplex and boundary of such. Thus by exactness of Γ (Γ has both adjoints) Γ also preserves $H^q(M)$ and $H_q(M;R)^*$, i.e. $\Gamma(H^q(sM) \simeq H^q(M)$, $\Gamma(\mathrm{Hom}_R(H_q(M;R),R) \simeq \mathrm{Hom}_{\mathbb{R}}(H_q(M;\mathbb{R}),\mathbb{R})$. So the corollary follows by applying Γ to the preceding theorem. ▲

A similar argument allows us to conclude

THEOREM. (First Comparison) *For any* $M \in M$ *and any set* X, $H_q(M;\mathbb{R}) \simeq \mathrm{Free}_{\mathbb{R}}(X)$ *iff* $H_q(s(M) \simeq \mathrm{Free}_R(\Delta X)$ *in G.*

Proof. Γ preserves free module, so $\Leftarrow$ follows by applying Γ to the exact sequence

$$0 \to Z_q(sM) \to B_q(sM) \to \mathrm{Free}_R(\Delta X) \to 0.$$

For $\Rightarrow$, we need two lemmas:

LEMMA 1. *If* $F_1 \overset{\alpha}{\to} F_2$ *is a homomorphism in G of free R-modules with constant bases, then* $\mathrm{Im}(\alpha)$, $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Cok}(\alpha)$ *are also free with constant bases.*

LEMMA 2. *If F is a free module in G with constant basis, then every epimorphism* $M \to F$ *of R-modules in G splits.*

As for the proof of lemma 2, let $F = \mathrm{Free}_R(\Delta X)$. By applying Γ we obtain a split diagram of vector spaces over $\mathbb{R}$ in Sets,

$$\Gamma(M) \underset{s}{\overset{\Gamma(a)}{\rightleftarrows}} \mathrm{Free}_{\mathbb{R}}(X).$$

But we have canonical bijections

$$\frac{\frac{\frac{\mathrm{Free}_{\mathbb{R}}(X) \xrightarrow{s} \Gamma(M) \quad \text{in } \mathrm{Mod}_{\mathbb{R}}(\mathrm{Sets})}{X \xrightarrow{s} \Gamma(M) \quad \text{in Sets}}}{\Delta X \xrightarrow{s} M \quad \text{in } G}}{\mathrm{Free}_R(\Delta X) \longrightarrow M \quad \text{in } \mathrm{Mod}_R(G)}$$

and clearly $\alpha\circ s = \mathrm{id}$. The proof of lemma 1 is similar.

To complete the proof of the theorem, we now apply an induction argument on open subsets of M, just as in the proof of De Rham's theorem with parameters on p. 251 above, using the long Mayer-Vietoris sequence and the coproduct lemma from section 2, with $\Delta(\mathbb{N})$ as the index set. For example, as the induction step for finite unions we need to conclude from the Mayer-Vietoris sequence:

$$\to H_q(U\cap V) \to H_q(U) \oplus U_q(V) \to H_q(M) \to H_{q-1}(U\cap V) \to H_{q-1}(U) \oplus U_{q-1}(V) \to \dots$$

that $H_q(M)$ is free with constant basis if $H.(U)$, $H.(V)$ and $H.(U\cap V)$ are. But more generally, if

$$\dots \to F_1 \to F_2 \to A \to F_3 \to F_4 \to \dots$$

is an exact sequence of R-modules with the F's being free on a constant basis, then so is A: just apply lemmas 1 and 2 to the diagram

$$\begin{tikzcd} F_1 \arrow[r] & F_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & F_3 \arrow[r] & F_4 \\ & F'_2 \arrow[u] & F'_3 \arrow[ur] & & \end{tikzcd}$$

Observe that, as a consequence of the classical De Rham theorem, $H^q(M)$ is always of the form $(\mathrm{Free}_{\mathbb{R}}(X))^* \simeq \mathbb{R}^X$.

COROLLARY. (Second Comparison Theorem) *For any manifold* $M \in \mathcal{M}$ *and any set* X, $H^q(M) \simeq \mathbb{R}^X$ *iff* $H^q(s(M)) \simeq R^{\Delta X}$ *in* $\mathcal{G}$.

Proof. Just notice that $(\mathrm{Free}_R(\Delta X))^* \simeq R^{\Delta X}$, and combine the first comparison theorem with the version of De Rham's theorem proved on p. 251. ▲

§5. Applications.

On hearing the expression 'De Rham's theorem with parameters' the classically minded reader probably has in mind something quite different from our theorem of section 4. Presumably, he is thinking of construing the De Rham cohomology groups, as well as the (duals of the) singular homology groups, as sheaves of smooth modules over the space of parameters, and then asserting that these sheaves are canonically isomorphic. In this section, we derive such a theorem from the main result of section 4. For unexplained notations, the reader is referred to Godement (1958).

Let the manifold $X \in M$ be our space of parameters, and let $\mathbb{R}_\infty$ be the sheaf on X of smooth real-valued functions, i.e. $\mathbb{R}_\infty(U) = C^\infty(U,\mathbb{R})$ for each open $U \subset X$, with obvious restrictions. Starting from this ringed space $(X,\mathbb{R}_\infty)$, we shall construct, for each $M \in M$, several $\mathbb{R}_\infty$-Modules on X.

First of all, there is the sheaf $\mathcal{A}^q(M)$ on X of (smooth) q-forms on M depending (smoothly) on parameters from X:

$\mathcal{A}^q(M)(U)$ = the set of q-forms on $U\times M$ which locally are of the form $\sum_{i_1<\ldots<i_q} f_{i_1\ldots i_q}(u,\underline{m})\,dm_{i_1}\wedge\ldots\wedge dm_{i_q}$

(with all the functions $f_{i_1\ldots i_q}$ smooth). Clearly, $\mathcal{A}^q(M)$ is indeed a sheaf on X, with obvious restrictions. Furthermore, exterior differentiation (with respect to the $\underline{m}$-variables only) defines natural transformations

$$\mathcal{A}^q(M) \xrightarrow{d^{q+1}} \mathcal{A}^{q+1}(M), \quad d^{q+1} = \{d_U^{q+1}\}_U$$

for each q, thus giving rise to a sheaf complex.

We now wish to form the sheaf cohomology of this sheaf complex $\mathcal{A}^\bullet(M)$. So let us define, for each open $U \subseteq X$,

$$\mathcal{F}^q(M)(U) = \mathrm{Ker}(d_U^{q+1})$$

$$\mathcal{E}^q(M)(U) = \mathrm{Im}(d_U^q)$$

$$H^q(M)(U) = F^q(M)(U)/E^q(M)(U).$$

Fortunately, to define the sheaf cohomology we do not have to pass to the associated sheaves of E or H, since

PROPOSITION. *$F^q(M)$, $E^q(M)$, and $H^q(M)$ are sheaves on X, and carry a natural $\mathbb{R}_\infty$-Module structure.*

Proof. $\mathcal{A}^q(M)$ is a sheaf for each q, and it has an obvious $\mathbb{R}$-Module structure. This structure is inherited by $F^q(M)$, $E^q(M)$ and $H^q(M)$, so we only need to show that these are sheaves. For $F^q(M)$, this is obvious from the fact that $\mathcal{A}^q(M)$ is a sheaf.

And E^q is a sheaf, essentially because a form which is locally exact is globally so by the existence of partitions of unity. More explicitly, if $\{U_\alpha\}_\alpha$ is an open cover of U and we are given a compatible family $\{\omega_\alpha\}$, $\omega_\alpha \in \mathcal{A}^q(M)(U_\alpha)$, such that each ω_α is of the form $d\lambda_\alpha$ for some $\lambda_\alpha \in \mathcal{A}^{q-1}(M)(U_\alpha)$, then if $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, we may put

$$\omega = \textstyle\sum_\alpha \rho_\alpha \cdot \omega_\alpha \in \mathcal{A}^q(M)(U)$$

$$\lambda = \textstyle\sum_\alpha \rho_\alpha \cdot \lambda_\alpha \in \mathcal{A}^{q-1}(M)(U)$$

and it trivially follows that $d\lambda = \omega$.

Finally, to show that $H^q(M)$ is a sheaf, choose a compatible family $[\omega_\alpha] \in H^q(M)(U_\alpha)$ for a cover $\{U_\alpha\}$ of U, i.e. for some $\lambda_{\alpha\beta} \in \mathcal{A}^{q-1}(M)(U_\alpha \cap U_\beta)$

$$\omega_\alpha|U_\alpha \cap U_\beta - \omega_\beta|U_\alpha \cap U_\beta = d\lambda_{\alpha\beta}.$$

Again take a partition of unity $\{\rho_\alpha\}$ as above, and let $\omega = \sum_\alpha \rho_\alpha \cdot \omega_\alpha$. We complete the proof by showing that for each β,

$$[\omega]_{|U_\beta} = [\omega_\beta] \quad \text{in } H^q(M)(U_\beta);$$

Indeed, since $E^q(M)$ has been shown to be a sheaf, it sufficies to check that $\omega_{\alpha|U_\alpha \cap U_\beta} - \omega_{\beta|U_\alpha \cap U_\beta}$ is exact on each element of the cover $\{U_\alpha \cap U_\beta\}_\alpha$ of U_β. But

$$\begin{aligned}
\omega_\alpha|U_\beta \cap U_\alpha - \omega_\beta|U_\alpha \cap U_\beta &= \textstyle\sum_\alpha \rho_\alpha \cdot \omega_\alpha|U_\alpha \cap U_\beta - \sum_\alpha \rho_\alpha \cdot \omega_\beta|U_\alpha \cap U_\beta \\
&= \textstyle\sum_\alpha \rho_\alpha \cdot (\omega_\alpha|U_\alpha \cap U_\beta - \omega_\beta|U_\alpha \cap U_\beta) \\
&= \textstyle\sum_\alpha \rho_\alpha d\lambda_{\alpha\beta} \\
&= d\textstyle\sum_\alpha \rho_\alpha \lambda_{\alpha\beta}. \quad \blacktriangle
\end{aligned}$$

We now define the singular homology $\mathbb{R}_\infty$-Modules, starting from the sheaf $S_q(M)$ of (smooth) simplicial q-chains which vary smoothly along the parameterspace X; $S_q(M)$ is defined to be the associated sheaf of the presheaf which assigns to an open $U \subseteq X$ the free $\mathbb{R}_\infty(U)$-module generated by the C_∞-maps $U\times\Delta_q \to M$. Thus, elements of $S_q(M)$ locally look like formal expressions of the form

$$\sum_{i=1}^{n} a_i(u)\sigma_i(u,t)$$

with both $a_i: U \to \mathbb{R}$ and $\sigma_i: U\times\Delta_q \to M$ smooth.

Observe that since every (open) subspace of M is paracompact, the process of passing from the given presheaf (which is separated) to its associated sheaf coincides with the process of closing off under partitions of unity. Thus, for example, if $\{U_\alpha\}$ covers U and for each α we are given formal expressions $\sum_{i=1}^{n_\alpha} a_i^\alpha(u)\sigma_i^\alpha(u,t)$ as elements of the presheaf over U_α, then $S_q(M)(U)$ contains an element which we may denote by

$$\sum_\alpha \sum_{i=1}^{n_\alpha} \rho_\alpha(u)\cdot a_i(u)\cdot\sigma_i^\alpha(u,t)$$

for a partition of unity $\{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$.

At the presheaf level there is an obvious natural transformation induced by composition with the boundary chain $\partial: \Delta_{q-1} \to \Delta_q$, and this yields a sheaf complex

$$\to S_{q+1}(M) \xrightarrow{\partial_{q+1}} S_q(M) \xrightarrow{\partial_q} S_q(M) \to \cdots \qquad \partial_q = \{(\partial_q)_U\}_U\,.$$

To define the singular homology sheaves, we define presheaves $Z_q(M)$, $B_q(M)$ and $H_q(M)$ by

$$Z_q(M)(U) = \mathrm{Ker}(\partial_q)_U$$

$$B_q(M)(U) = \mathrm{Im}(\partial_{q+1})_U$$

$$H_q(M)(U) = Z_q(M)(U)/B_q(M)(U).$$

By the remark on closure under partitions of unity that we just made, we can almost literally copy the proof of the preceding propositions to show that

PROPOSITION. $Z_q(M)$, $B_q(M)$ *and* $H_q(M)$ *are sheaves on* X, *and carry a natural* $\mathbb{R}_\infty$*-Module structure.*

Now we are ready to formulate the more conventional form of De Rham's theorem hinted at in the beginning of this section:

THEOREM. (De Rham's theorem with parameters) *The canonical* $\mathbb{R}_\infty$*-linear map*

$$H^q(M) \xrightarrow{\ I\ } H_q(M)^*$$

of $\mathbb{R}_\infty$*-Modules on the ringed space* $(X,\mathbb{R}_\infty)$ *given by the components*

$$I_U : H^q(M)(U) \to H_q(M)^*(U), \qquad I_U([\omega])([\gamma]) = \int_\gamma \omega$$

is an isomorphism of sheaves. (Here $(-)^*$ denotes the dual in the category of $\mathbb{R}_\infty$-modules. So $H_q(M)^*(U)$ is the set of natural transformations from $H_q(M)|_U$ to $\mathbb{R}_\infty|_U$).

Proof. Restrict everything in the version of De Rham with parameters proved in section 4 for the topos of sheaves on the site $\mathbb{G}$ to the subcategory of $\mathbb{G}$ consisting of open subspaces of X, and read off the different notions involved. ▲

COROLLARY. *Let* ω *be a smooth* X*-form on* M, *i.e.*

$\omega \in \mathcal{A}^q(M)(X)$. *If for each parameter value* $x \in X$, *the form* $\omega(x,-) \in \Lambda^q(M)$ *is exact, then there is an* X-*form* $\alpha \in \mathcal{A}^{q-1}(M)(X)$ *such that* $\omega = d\alpha$.

Proof. The previous theorem tells us that

$$0 \to E^q(M) \to F^q(M) \to H_q(M)^* \to 0$$

is exact in Sh(X). Since $\int_\gamma \omega = 0$ for all $\gamma \in Z_q(M)(X)$, ω is locally in $E^q(M)$, i.e. $\omega \in E^q(M)(X)$ since this is a sheaf. ▲

Recently, this corollary was independently proved in Glass (1983). Both Glass and I we were unaware of the existence of an earlier proof using the method of carapaces, which was pointed out to us by W. van Est (cf. van Est (1958)).

As another corollary, we derive that the De Rham cohomology $\mathbb{R}_\infty$-modules are *vectorbundles*, provided we ensure that their dimension is finite:

COROLLARY. *Let* $T \to M$ *be a retract of a manifold* $M \in \mathcal{M}$ *of* ***finite*** *homology* ***type*** *(i.e.* $H_q(M)$ *is finite dimensional for each* $q \geq 0$. *Then for each* $X \in \mathcal{M}$, *the* $\mathbb{R}_\infty$-*Module* $H^q(T) \in$ Sh(X) ***is locally free****, i.e. there is an open cover* $\{U_\alpha\}$ *of* X *such that for each* α *there is an* $\mathbb{R}_\infty{}_{|U_\alpha}$-*linear isomorphism of sheaves*

$$\phi_\alpha : H^q(T)_{|U_\alpha} \xrightarrow{\sim} \mathbb{R}_\infty^{n_\alpha}{}_{|U_\alpha}, \quad \text{some } n_\alpha \in \mathbb{N}.$$

Proof. $H^q(T)$ is a retract of $H^q(M)$, which is free and of finite type, by the comparison theorems. Since $\mathbb{R}_\infty$ is local, the result now follows from Swan's theorem (see for example Reyes (1978)). ▲

§6. Some remarks on other cohomologies.

Now that we have established the validity of De Rham's theorem for the topos $\mathcal{G}$ (section 4) and (consequently) for smooth $\mathbb{R}_\infty$-Modules ever a space of parameters (section 5), it is natural to ask whether De Rham's theorem holds in G for other cohomologies. We will briefly consider two examples of this question: the case of Čech cohomology, and the case of singular cohomology.

We quickly recall the classical version of De Rham's theorem for Čech cohomology: Let $M \in \mathcal{M}$ be a manifold, and let $\mathcal{U} = \{U_\alpha\}$ be a *good* cover of M, that is, an open cover such that all monempty finite intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ are diffeomorphic to some R^n. Assume that the indexset $\{\alpha\}$ is linearly ordered. The Čech complex (with coefficients in $\mathbb{R}$) is the complex

$$C^o(\mathcal{U},\mathbb{R}) \overset{\delta}{\to} C^1(\mathcal{U},\mathbb{R}) \overset{\delta}{\to} C^2(\mathcal{U},\mathbb{R}) \to \ldots ,$$

where $C^n(\mathcal{U},\mathbb{R})$ is the vector space $\prod_{\alpha_o<\ldots<\alpha_n} F^o(U_{\alpha_o\ldots\alpha_n},\mathbb{R})$ over $\mathbb{R}$ ($F^o(U_{\alpha_o\ldots\alpha_n},\mathbb{R})$ denotes the vector space of locally constant functions $U_{\alpha_o} \cap \ldots \cap U_{\alpha_n} \to \mathbb{R}$), and the boundary operator $\delta: C^n(\mathcal{U},\mathbb{R}) \to C^{n+1}(\mathcal{U},\mathbb{R})$ is defined as follows: if $f = \{f_{\alpha_o\ldots\alpha_n}\} \in C^n(\mathcal{U},\mathbb{R})$, then

$$(\delta f)_{\alpha_o\ldots\alpha_{n+1}} = \sum_{i=o}^{n+1} (-1)^i f_{\alpha_o\ldots\hat{\alpha}_i\ldots\alpha_{n+1}} .$$

The cohomology of this complex is called the Čech cohomology of the good cover $\mathcal{U}$, and is denoted by $H^\bullet(\mathcal{U},\mathbb{R})$.

De Rham's theorem for Čech-cohomology says that in that situation there is a canonical isomorphism

$$H^\bullet(M) \xrightarrow{\sim} H^\bullet(\mathcal{U},\mathbb{R}). \qquad (*)$$

Consequently, $H^\bullet(\mathcal{U},\mathbb{R})$ does not depend on the good cover $\mathcal{U}$. Another immediate corollary is that since compact manifolds have finite good covers, the De Rham cohomology of such is finite dimensional. The proof of the existence of the isomorphism (*) given by A. Weil (cf. Weil (1952)) is completely constructive and explicit, and hence is valid in the synthetic context: (cf. Appendix 1). Consequently, since the embedding $s: \mathcal{M} \to G$ preserves the ingredients of Weil's proof (notably goof open covers, and partitions of unity; preservation of the latter is proved as proposition 5.9 of Moerdijk & Reyes (1983)), we obtain the analogs of our theorems from sections 4 and 5:

THEOREM. *For any* $M \in \mathcal{M}$ *and any good cover* $\mathcal{U}$ *of* M, *the canonical map*

$$H^q(sM) \to H^q(s(\mathcal{U}),\mathbb{R})$$

is an isomorphism in the topos $\mathcal{G}$. ▲

COROLLARY. (De Rham's theorem with parameters, for Čech cohomology). *Let* $\mathcal{U}$ *and* M *be as above, and let* $X \in \mathcal{M}$ *be a space of smooth parameters. Then the canonical homomorphism of* $\mathbb{R}_\infty$*-Modules*

$$H^q(M) \to H^q(\mathcal{U},\mathbb{R}_\infty)$$

over the ringed space $(X,\mathbb{R}_\infty)$ *is an isomorphism.* ▲

Here $H^q(\mathcal{U},\mathbb{R}_\infty)$ is the cohomology of the complex $C^q(\mathcal{U},\mathbb{R}_\infty)$ of sheaves on X, $C^q(\mathcal{U},\mathbb{R})$ being the sheafproduct $\prod_{\alpha_0<\ldots<\alpha_q} F^0(U_{\alpha_0\ldots\alpha_q},\mathbb{R}_\infty)$ of the $\mathbb{R}_\infty$-Modules $F^0(U_{\alpha_0\ldots\alpha_q},\mathbb{R})$, defined by setting for open $W \subset X$:

$$F^0(U_{\alpha_0\ldots\alpha_q},\mathbb{R}_\infty)(W) = \text{smooth functions } f(x,u): W\times U_{\alpha_0\ldots\alpha_q} \to \mathbb{R}$$
$$\text{which locally do not depend on } u$$

(i.e. there are covers $\{W_\xi\}$ of W and $\{U_\eta\}$ of $U_{\alpha_0\ldots\alpha_q}$ such that each $f(x,u)\mid W_\xi\times U_\infty$ does not depend on u).

Notice that if $\mathcal{U}$ is finite, $H^q(\mathcal{U},\mathbb{R}_\infty)$ is a vector bundle, thus giving us the last corollary of the previous section.

Turning to singular cohomology, we have to admit that we do not know whether De Rham's theorem holds synthetically (or in $\mathcal{G}$), at least, when we interpret singular cohomology as the cohomology of the complex

$$\dots \to \mathrm{Hom}_R(S_q(M),R) \xrightarrow{\partial^*} \mathrm{Hom}_R(S_{q+1}(M),R) \to \dots$$

which is the dual of the complex $\dots \to S_{q+1}(M) \to S_q(M) \to \dots$ of section 2. In this case, some form of the axiom of choice seems to be needed to establish the result. The problem here is that the dual of the short exact sequence

$$0 \to S_q(U\cap V) \to S_q(U) \oplus S_q(V) \to S_q^{\{U,V\}}(M) \to 0$$

of section 2, which is

$$0 \to R^{M^{\Delta_q}} \to R^{U^{\Delta_q}} \times R^{V^{\Delta_q}} \to R^{(U\cap V)^{\Delta_q}} \to 0,$$

is not necessarily exact: an arbitrary function $(U\cap V)^{\Delta_q} \to R$, cannot in general be extended to a function $U^{\Delta q} + V^{\Delta q} \to R$, so the sequence is not epic on the right.

A way of circumventing this problem in the topos G is to replace the sheaf $s(M)^{\Delta q} \in G$ by the constant sheaf $\Delta(M^{\Delta q})$ (recall that Δ:Sets $\to G$ is the constant functor). Thus, let $S_{\Delta,q}(sM)$ be the free R-module in G generated by $\Delta(M^{\Delta q})$. $S_{\Delta,q}(sM)$ has a constant basis, so (from lemmas 1,2 of section 4) we get a *split* exact sequence in G,

$$0 \to S_{\Delta,q}(s(U\cap V)) \to S_{\Delta,q}(sU) \oplus S_{\Delta,q}(sV) \to S_{\Delta,q}^{\{U,V\}}(sM) \to 0$$

and therefore its dual in G,

$$0 \to S_{\Delta,q}^{\{U,V\}}(sM)^* \to (S_{\Delta,q}(sU) \oplus S_{\Delta,q}(sV))^* \to S_{\Delta,q}(s(U\cap V))^* \to 0$$

is exact as well. Consequently, if we let $H^q_\Delta(sM)$ denote the cohomology of the complex

$$\dots \to S_{\Delta,q}(sM) \xrightarrow{\partial^*} S_{\Delta,q+1}(sM) \to \dots$$

we obtain a long Mayer-Vietoris sequence.

LEMMA. *Let* $M = U \cup V$ *in* M *as before. Then in* G *there is a long exact sequence*

$$\dots \to H^{q+1}_\Delta(sM) \to H^q_\Delta(sM) \to H^q_\Delta(sU) \oplus H^q_\Delta(sV) \to H^q_\Delta(sU\cap V)) \to \dots$$

Proof. As before we need to show that the restriction to $S^{\{U,V\}}_{\Delta,q}(sM)$ instead of $S_{\Delta,q}(sM)$ in the complex still gives the same cohomology. For this, we only need to observe that the proof for singular homology by barycentric subdivision dualizes, since if K is a homotopy between chain maps, then trivially so is its dual. (Recall that if $f,g:A \to B$ are chain maps, say with $A^q \overset{d}{\to} A^{q+1}$, a homotopy $K:f \simeq g$ is a sequence of maps $A^{q+1} \xrightarrow{K^q} B^q$ such that $f^q - g^q = K^q d + dK^{q-1}$. ▲

Now we obtain exactly as before,

THEOREM. *Let* $M \in \mathcal{M}$ *be a manifold. Then the canonical* R*-linear map*

$$H^q(sM) \to H^q_\Delta(sM)$$

$$[\omega] \longmapsto ([\gamma] \longmapsto \int_\gamma \omega)$$

is an isomorphim. ▲

The schizophrenic character of the isomorphism is apparent: we integrate *internal* (*variable*, in $\mathcal{G}$) forms ω over *external* (constant, from Sets) chains γ. This was reflected in the proof: the splitting in the lemma above comes from Sets, and similarly the homotopy equivalence $S^{\{U,V\}}_{\Delta,q}(M)^* \to S_{\Delta,q}(M)^*$ was brought into $\mathcal{G}$ by dualizing the usual constant homotopy equivalence $S^{\{U,V\}}_{\Delta,q} \to S_{\Delta,q}$ coming 'from outside', from Sets.

Just as before we can restrict this isomorphism of sheaves in $\mathcal{G}$ to the category Sh(X) for $X \in \mathcal{M}$, to obtain a result with a more classical appearance, by defining a 'hibrid' cohomology sheaf $H^q_\Delta(M)$ on X carrying an $\mathbb{R}_\infty$-Module structure. First, we define a sheaf $S_{\Delta,q}(M)$ on X whose sections are locally of the form

$$\sum_{i=1}^{n} a_i(u)\sigma_i(t)$$

where $a_i : U \to \mathbb{R}$ and $\sigma_i(t) : U\times\Delta_q \to \mathbb{R}$ are smooth maps. (Just as in the definition of $S_q(M)$ given in section 5, but now with the additional requirement that $\sigma_i(u,t)$ *locally does not depend on* u). Alternatively $S_{\Delta,q}(M)$ is the associated sheaf of the presheaf

$$U \mapsto \text{free } \mathbb{R}_\infty(U) \text{ module generated by } C^\infty(\Delta_q, M).$$

This gives a sheaf complex, of which we can take the dual (in the category of $\mathbb{R}$ -Module over X)

$$\ldots \to S_{\Delta,q}(M)^* \xrightarrow{\partial_q^*} S_{\Delta,q+1}(M)^* \to \ldots .$$

As before, we then show that to obtain the cohomology of this sheaf complex we may define sheaves (not just presheaves, by a partition of unity argument) $Z^q(M)$, $B^q(M)$ and $H^q(M)$ by setting for open $U \subset X$

$$Z^q_\Delta(M)(U) = \mathrm{Ker}(\partial_q^*)_U$$

$$B^q_\Delta(M)(U) = \mathrm{Im}(\partial_{q-1}^*)_U$$

$$H^q_\Delta(M)(U) = Z^q_\Delta(M)(U)/B^q_\Delta(M)(U) .$$

If we unravel the definitions, it turns out that we obtain a result familiar in classical differential geometry (cf. van Est (1958)): for elements $\sigma \in S_{\Delta,q}(M)^*(T)$, T an open subspace of X, we have

$$\begin{array}{ll}
T \xrightarrow{\sigma} S_{\Delta,q}(M)^* & \text{in } G \\
\hline\hline
S_{\Delta,q}(M) \to R^T & \text{in } \mathrm{Mod}_R(G) \\
\hline\hline
\Delta(M^{\Delta q}) \to R^T & \text{in } G \\
\hline\hline
M^{\Delta q} \to \Gamma(R^T) & \text{in Sets} \\
\hline\hline
M^{\Delta q} \to C_\infty(T,\mathbb{R}) & \text{in Sets} \\
\hline\hline
S^\Delta_q(M) \to C^\infty(T,\mathbb{R}) & \text{in } \mathrm{Mod}_{\mathbb{R}}(\text{Sets}) .
\end{array}$$

That is, a T-element of $\mathrm{Hom}_R(S_{\Delta,q}(M),R) \in G$, or equivalently a section of $S_{\Delta,q}(M)^*$ over T, is precisely an $\mathbb{R}^T$-valued singular cochain on M in the sense of van Est (1958). Furthermore, a T-element of $\Lambda^q(M) \in G$ is precisely a differential T-form on M of degree q in van Est's sense, i.e. an element of $\mathcal{A}^q(M)(T)$. Thus we have:

COROLLARY. *Let* $M \in \mathcal{M}$ *be a manifold, and* $X \in \mathcal{M}$ *be the space of parameters. Then the canonical homomorphism*

$$H^q_\Delta(M) \to H^q_\Delta(M)$$

$$[\omega] \mapsto ([\gamma] \mapsto \int_\gamma \omega)$$

of $\mathbb{R}_\infty$*-Modules over the ringed space* $(X, \mathbb{R}_\infty)$ *is an isomorphism.* ▲

And hence by taking the section over T of this isomorphism,

COROLLARY. (van Est (1958)) *The integration* I *is a homomorphism of the complex* Ω *of* T*-forms into the complex* Σ *of* $\mathbb{R}^T$*-valued singular co-chains on* M. *Furthermore,*

$$I^*: H(\Omega) \to H(\Sigma)$$

is an isomorphism. ▲

Note that, as van Est points out in his paper, we can deduce the corollary of section 5, p. 237, form this simpler result.

As a final remark, we note that we could have developed a '*continuous*' singular homology of manifolds, completely parallel to the '*smooth*' singular homology of section 2. Every manifold M lives in G not only as the smooth space s(M) but also as the continuous space c(M),

$$c(M)(\bar{A}) = \mathrm{Cts}(\gamma\bar{A}, M).$$

Using the same arguments as for the earlier comparison theorems, we may derive another comparison theorem.

THEOREM. *For any manifold* $M \in \mathcal{M}$ *and any set* S, $H_q(M,\mathbb{R}) \simeq \mathrm{Free}_{\mathbb{R}}(S)$ *in Sets iff* $H_q(M,\mathbb{R}) \simeq \mathrm{Free}_{\mathbb{R}}(\Delta S)$ *in* G (on the righthand side, $\mathbb{R}$ denotes the Dedekind reals in G, i.e. the objet $c(\mathbb{R})$, Moerdijk & Reyes (1983)). ▲

On applying ρ^*, which preserves the singular homology groups by the general arguments of Moerdijk & Reyes (1983) . we obtain:

THEOREM. (De Rham's theorem in G, for continuous homology). *For any manifold* $M \in \mathcal{M}$, *and any set* S,

$$H^q(sM,\mathbb{R}) = R^{\Delta S} \quad \textit{in } G \textit{ iff } H_q(cM,\mathbb{R})^* = \mathbb{R}^{\Delta S} \quad \textit{in } G. \ ▲$$

(Note that in the definition of $H_q(cM,\mathbb{R})$, the notion of '*continuous* simplex' $\Delta_q \to c(M)$ does *not* occur. We take *all* simplices, just as with $H_q(s(M),R)$, and by definition of $c(M)$ these are automatically the continuous ones).

Reinterpreting this in $Sh(X)$, X a manifold, we obtain a result saying that the "De Rham cohomology *smooth* in X-parameters" agrees with the "singular homology *continuous* in X-parameters". This is a version of De Rham's theorem 'in parameters' which is closest to what seems to be *De Rham's original theorem*, saying that in *Sets* (X is one point),

$$H^q(M,\mathbb{R}) \simeq H_q^{cts}(M,\mathbb{R})^* .$$

Having returned at our starting point, there is nothing left to say.

APPENDIX 1. Weil's version of De Rham's theorem.

Let M be a smooth space. A *good cover* of M is a cover $\{U_\alpha\}_\alpha$ such that all the finite nonempty intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ are isomorphic to some R^n. Fix one such cover $\mathcal{U}$ and assume that the index set $\{\alpha\}$ is linearly ordered (in the synthetic/intuitionistic sense). What follows will be a synthetic argument. Thus, intuitively, every object has a smooth structure and every function is smooth, so we do not need to assume that M is a manifold. Neither does $\mathcal{U}$ necessarily have to consist of '*open*' subsets of M in some sense, but we do need one assumption on $\mathcal{U}$, namely that there is a partition of unity subordinate to it (or to a refinement of $\mathcal{U}$). In particular, we assume that $\mathcal{U}$ is *pointfinite* (not necessarily neighbourhood finite, since we work synthetically) or at least that $\mathcal{U}$ has a pointfinite refinement. Thus, if $f_\alpha : U_\alpha \to V$ are maps into some R-module V, and $\{\rho_\alpha\}$ is a partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}$, then $\Sigma_\alpha \rho_\alpha \cdot f_\alpha$ makes senses as a function $M \to V$.

The De Rham cohomology $H^\bullet(M)$ of M was defined in section 1, and the Čech cohomology $\check{H}^\bullet(\mathcal{U},R)$ in section 6 (classically, but it is obvious how to define the synthetic analogue). Weil's idea for proving that $\check{H}^\bullet(\mathcal{U},R) \simeq H^\bullet(M)$ is to embed both the De Rham complex $\{\Lambda^n(M)\}$ and the Čech complex $\check{C}^n(\mathcal{U},R)$ into a bigger complex (denoted $\ldots \to L^n \to L^{n+1} \to \ldots$ below) and show that both cohomologies are isomorphic to this bigger third cohomology.

Let $U_{\alpha_0 \ldots \alpha_n} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ for each sequence $\alpha_0 < \ldots < \alpha_n$ of indices, and let $\partial_i = \partial_i^n : U_{\alpha_0 \ldots \alpha_n} \to U_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_n}$ be the inclusion . Then we have a diagram

$$\Lambda^q(M) \to \prod_{\alpha_0} \Lambda^q(U_{\alpha_0}) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{\alpha_0<\alpha_1} \Lambda^q(U_{\alpha_0\alpha_1}) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \prod_{\alpha_0<\alpha_1<\alpha_2} \Lambda^q(U_{\alpha_0\alpha_1\alpha_2}) \cdots$$

where the first map is the obvious restriction of forms, and

$\delta_i : \prod_{\alpha_0<\dots<\alpha_n} \Lambda^q(U_{\alpha_0\dots\alpha_n}) \to \prod_{\alpha_0<\dots<\alpha_{n+1}} \Lambda^q(U_{\alpha_0\dots\alpha_{n+1}})$ comes from pulling back a form along ∂_i^n,

$$\delta_i = \partial_i^{n*} : \Lambda^q(U_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{n+1}}) \to \Lambda^q(U_{\alpha_0\dots\alpha_{n+1}}).$$

From this, we obtain a complex

$$\Lambda^q(M) \to \prod_{\alpha_0} \Lambda^q(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} \Lambda^q(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1<\alpha_2} \Lambda^q(U_{\alpha_0\alpha_1\alpha_2}) \to \cdots$$

by defining $\delta = \delta^n : \prod_{\alpha_0<\dots<\alpha_n} \Lambda^q(U_{\alpha_0\dots\alpha_n}) \to \prod_{\alpha_0<\dots<\alpha_{n+1}} \Lambda^q(U_{\alpha_0\dots\alpha_{n+1}})$ as the alternating sum $\sum_{i=0}^{n+1} (-1)^i \delta_i$. Thus, δ maps a sequence $\omega = \{\omega_{\alpha_0\dots\alpha_n}\}$ of forms to the sequence $\delta\omega = \{(\delta\omega)_{\alpha_0\dots\alpha_{n+1}}\}$, where

$$(\delta\omega)_{\alpha_0\dots\alpha_{n+1}} = \sum_{i=0}^{n+1} (-1)^i \delta_i(\omega_{\alpha_0\dots\alpha_n}).$$

Indeed, precisely as in the case of the boundary operator of the singular homology complex, $S_\bullet(M)$ we can show that $\delta^2 = 0$. So we could form its cohomology, but this is not of much use, since

LEMMA. *every sequence*

$$0 \to \Lambda^q(M) \to \prod_{\alpha_0} \Lambda^q(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} \Lambda^q(U_{\alpha_0\alpha_1}) \to \cdots$$

is exact.

Proof. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\mathcal{U}$, and define $K_n : \prod_{\alpha_0<\dots<\alpha_{n+1}} \Lambda^p(U_{\alpha_0\dots\alpha_{n+1}}) \to \prod_{\alpha_0<\dots<\alpha_n} \Lambda^p(U_{\alpha_0\dots}U_{\alpha_n})$ by putting for $\omega = \{\omega_{\alpha_0\dots\alpha_{n+1}}\}$,

$$K_n(\omega)_{\alpha_0\dots\alpha_n} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\dots\alpha_n},$$

where $\omega_{\alpha\alpha_0\dots\alpha_n}$ is interpreted according to the following convention: if $\beta_0\dots\beta_n$ is a sequence of indices (not necessarily increasing and possibly with repetitions, and σ is a permutation of $\{0,\ldots,n\}$ then $\omega_{\beta_0\ldots\beta_n} = \mathrm{sgn}(\sigma)\cdot$ $\cdot\omega_{\beta_{\sigma(0)}\ldots\beta_{\sigma(n)}}$. (So $\omega_{\ldots\alpha\ldots\alpha\ldots} = 0$). Then an easy calculation shows that

$$\delta K_n(\omega) + K_{n+1}(\delta\omega) = \omega,$$

whence the lemma. ▲

Now consider the diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 & 0 \to & \Lambda^1(M) & \xrightarrow{r} & \prod_{\alpha_0} \Lambda^1(U_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} \Lambda^1(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \ldots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
(*) & 0 \to & \Lambda^0(M) & \xrightarrow{r} & \prod_{\alpha_0} \Lambda^0(U_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} \Lambda^0(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \ldots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & 0 \to & F^0(M) & \longrightarrow & C^0(\mathcal{U},R) & \longrightarrow & C^1(\mathcal{U},R) \longrightarrow \ldots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}$$

By the lemma, all the rows except the first are exact, and by the Poincaré lemma, so are all the columns except the first ($\mathcal{U}$ is a good cover). Let us write

$$K^{p,q} = \prod_{\alpha_0<\ldots<\alpha_p} \Lambda^q(U_{\alpha_0\ldots\alpha_p}) ,$$

$\{K^{p,q}\}_{p\geq 0, q\geq 0}$ has the structure of a 'double complex': we have maps $\delta: K^{p,q} \to K^{p+1,q}$ and $d: K^{p,q} \to K^{p,q+1}$ such that $\delta^2 = 0 = d^2$ and $\delta d = d\delta$. From such a double complex we can construct an ordinary complex by summing up the codiagonals: let

$$L^n = \bigoplus_{p+q=n} K^{p,q} , \quad n \geq 0$$

where $\oplus$ denotes the direct sum of R-modules. Then the L^n form a complex with boundary operator

$$D:L^n \to L^{n+1}$$

defined as follows: $D:\oplus_{p+q=n} K^{p,q} \to \oplus_{p+q=n+1} K^{p,q}$ is determined by its components $D_p: K^{p,q} \to \oplus_{p,q=n+1} K^{p,q}$ $(p = 0,\ldots,n)$ which are given by

$$D_p = \delta + (-1)^p d \qquad \begin{array}{ccc} K^{pq+1} & \hookrightarrow & L^{n+1} \\ \uparrow & & \uparrow \\ K^{p,q} & \xrightarrow{\delta} & K^{p+1,q} \end{array}$$

After a quick look it will be clear that $D^2 = 0$.

Let us write $H^n_L(M)$ for the cohomology of this complex, i.e.

$$H^n_L(M) = \mathrm{Ker}(L^n \xrightarrow{D} L^{n+1})/\mathrm{Im}(L^{n-1} \xrightarrow{D} L^n).$$

THEOREM. $H^{\bullet}_L(M)$ *is isomorphic to both the De Rham cohomology* $H^{\bullet}(M)$ *and the Čech cohomology* $H^{\bullet}(\mathcal{U},\mathbb{R})$.

Proof. Using the exactness of the rows (except the bottom one) of the diagram (*) we will show that the maps $r:\Lambda^n(M) \to \prod_{\alpha_0} \Lambda^n(U_{\alpha_0})$ induce isomorphisms $H^n(M) \overset{r}{\to} H^n_L(M)$. But the definition of the complex $\{L^n(M)\}$ is symmetric in p and q, so by 'reflecting (*) in the diagonal' a completely similar argument will yield that the maps

$$i:C^n(\mathcal{U},\mathbb{R}) \hookrightarrow \prod_{\alpha_0<\ldots<\alpha_n} \Lambda^0(U_{\alpha_0\ldots\alpha_n})$$

induce isomorphisms $H^n(\mathcal{U},\mathbb{R}) \overset{i}{\to} H^n_L(M)$.

So to prove the first isomorphism, define a chainmap $r:A^{\bullet}(M) \to L^{\bullet}$ by

$$r^n:\Lambda^n(M) \to L^n, \quad \omega \longmapsto \{\omega|U_{\alpha_0}\} \in K^{o,n} \subset L^n.$$

(r is indeed a chainmap, since the restriction of D to $K^{o,n}$ is just $\delta+d$, so $D(r^n\omega) = \delta(r^n\omega)+dr^n(\omega) = \{d\omega|U_{\alpha_0}\} = r^{n+1}d\omega$, because $\delta r^n(\omega) = 0$ by exactness of the rows).

Thus r induces a map $r:H^n(M) \to H^n_L(M)$ at the level of cohomology. We claim that at this level, r is an isomorphism.

r *is surjective*: take $\emptyset \in L^n$, say $\emptyset = \sum_{p+q=n} \emptyset_{p,q}$ with $\emptyset_{p,q} \in K^{p,q}$, such that $D\emptyset = 0$. We have to show that $\emptyset$ differs by a boundary DX from some $\emptyset' = \sum_{p+q=n} \emptyset'_{p,q}$ with $\emptyset_{p,q} = 0$ for all p,q except $p = 0$, $q = n$. We do this in n steps, using the induction step which reduces a $\emptyset \in L^n$ with $\emptyset_{p,q} = 0$ for $p = k+1,\dots,n$ to a $\emptyset'$ with $\emptyset'_{p,q} = 0$ for $p = k,\dots,n$. Indeed, since for $\emptyset = \sum_{p,q=n} \emptyset_{p,q}$,

$$D\emptyset = \sum_{p+q=n} \delta\emptyset_{p,q} + (-1)^p d\emptyset_{p,q}$$

it follows that $\delta\emptyset_{n,o} = 0$ (in $K^{n+1,o}$), $d\emptyset_{o,n} = 0$ (in $K^{o,n+1}$) and $\delta\emptyset_{u,v+1} + (-1)^{u+1} d\phi_{u+1,v} = 0$ (in $K^{u+1,v+1}$, for $u+v+1 = n$). So if $\emptyset_{p,q} = 0$ for $p > k$ then $\delta\emptyset_{k,n-k} = 0$. Hence by exactness of the rows, $\emptyset_{k,n-k} = \delta\psi$ for some $\psi \in K^{k-1,n-k}$. Let $\emptyset' = \emptyset - D\psi$. Then $\emptyset'_{p,q} = 0$ for $p \geqslant k$.

r *is injective:* Take $\omega \in F^n(M)$ such that $r(\omega) = D\emptyset$ for some $\emptyset \in L^{n-1}$. As shown above, there is a $\psi \in K^{o,n-1} \subset L^{n-1}$ such that $[\emptyset] = [\psi]$ in $H_L^{n-1}(M)$, so $r\omega = D\emptyset = D\psi$. But ψ is a sequence $\{\psi_\alpha\}$ of $n-1$-forms on U_α such that $\omega_{|U_\alpha} = d\psi_\alpha$, and moreover $\delta\psi = 0$ (since $r\omega = D\psi$), so by exactness of rows there is a global form λ with $\psi_\alpha = \lambda_{|U_\alpha}$ for each α, and we conclude that $d\lambda = \omega$, i.e. $[\omega] = 0$ in $H^n(M)$. ▲

*

APPENDIX 2. A classical proof of the homotopy invariance of De Rham cohomology.

By 'translating' the synthetic argument given in section 1, we give a purely classical proof of the homotopy invariance, which seems to be more direct than the proofs given in the standard texts.

Let M be a (smooth) manifold, and let $\Lambda^p(M)$ denote the (real) vector space of smooth p-forms on M. So an element $\omega \in \Lambda^p(M)$ is a map

$$T(M)\times_M\cdots\times_M T(M) \xrightarrow{\omega} R \qquad \text{(p-fold fibered product)}$$

satisfying the usual conditions. $\Lambda^0(M)$ is the set of smooth maps $M \to R$, and we put $\Lambda^p(M)$ = the zero vector space, for $p < 0$. Exterior differentiation gives a complex

$$\cdots \to \Lambda^{p-1}(M) \xrightarrow{d^{p-1}} \Lambda^p(M) \xrightarrow{d^p} \Lambda^{p+1}(M) \to \cdots$$

and the p^{th} De Rham cohomology space of M is the vector space

$$H^p(M) = \operatorname{Ker}(d^p)/\operatorname{Im}(d^{p-1})$$

of 'closed p-forms modulo exact p-forms'. We write $H^{\bullet}(M)$ for the sequence $\{H^p(M)\}$ of vector spaces.

A smooth map $M \xrightarrow{f} N$ of manifolds induces a linear map $f^* = (f^*)^p : \Lambda^p(N) \to \Lambda^p(M)$ (by composing with the obvious map

$$T(M)\times_M\cdots\times_M T(M) \xrightarrow{df\times_M\cdots{}_M\times df} T(N)\times_N\cdots\times_N T(N)),$$

which commutes with exterior differentiation d. So we get a map $H^p(f):H^p(N) \to H^p(M)$ for each p, i.e. a sequence of maps $H^{\bullet}(f):H^{\bullet}(N) \to H^{\bullet}(M)$. The following well-known theorem, usually refered to as the homotopy invariance of De Rham cohomology, or as the Poincaré lemma, says that $H^{\bullet}(f)$ only depends on the homotopy class of f:

THEOREM. *If* f *and* $g:M \to N$ *are homotopic maps, then* $H^{\bullet}(f) = H^{\bullet}(g)$. *It has an immediate consequence: if* M *and* N *are homotopy equivalent, then* $H^{\bullet}(M) = H^{\bullet}(N)$.

The theorem is proved by showing that if $F:M\times I \to N$ is a (smooth) homotopy form $f = F_0$ to $g = F_1$, we can find for every closed p-form ω on N a p-1-form λ on M such that $d\lambda = F_1^*(\omega)-F_0^*(\omega)$. As usual, this immediately follows from the existence of a chain-homotopy K from F_0^* to F_1^*, i.e. a sequence of linear maps $K^p:\Lambda^p(N) \to \Lambda^{p-1}(M)$ such that for all p, all $\omega \in \Lambda^p(N)$,

$$F_1^*(\omega)-F_o^*(\omega) = d^{p-1}K^p\omega+K^{p+1}(d^p\omega) \tag{1}$$

Such a map K is defined as follows. For a p-form $\omega:TN\times_N\ldots\times_N TN \to R$ on N, $K\omega = K^p\omega$ will be a map $TM\times_M\ldots\times_M\times TM\to R$ (p-1-fold fibered product). Now choose $(x,v_1,\ldots,v_{p-1}) \in TM\times_M\ldots\times_M TM$, $v_i \in T_x(M)$, and let

$$g_{x,\underline{v}}:I \to TN\times_N\ldots\times_N TN$$

be the map

$$g_{x,\underline{v}}(t) = (F^x(t),\ (dF^x)_t(1),\ (dF_t)_x(v_1),\ldots,(dF_t)_x(v_{p-1}))$$

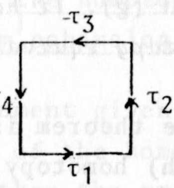
(Here $F^x:I \to N$ is the map $F^x(t) = F(x,t)$). Indeed, the right-hand side is an element of $TN\times_N\ldots\times_N TN$ (p times): $y = F^x(t) \in N$, and $(dF^x)_t$ is a linear map $T_t(I) \to T_y(N)$, i.e. $R \to T_y(N)$, which corresponds to a vector $(dF^x)_t(1)\in T_y(N)$; also $F_t:M \to N$ defines a linear map $(dF_t)_x:T_x(M) \to T_y(N)$, so $(dF_t)_x(v_i)\in T_y(N)$. Now put

$$K\omega(x,v) = \int_o^1 \omega(g_{x,\underline{v}}(t))dt.$$

For fixed x, $K\omega(x,-)$ is alternating and separately linear in $\underline{v}$, so $K\omega$ defines a p-1-form on N, and from the explicit definition we have given it is clear that $K\omega$ is smooth, i.e. $K\omega \in \Lambda^{p-1}(M)$.

We will now verify that (1) holds. For notational convenience, we assume that p = 2. Let $\tau:I^2 \to M$ be any 2-chain on M, and write

$$\int_\tau dK\omega = \int_{\partial\tau} K\omega = \int_{\tau_1} K\omega + \int_{\tau_2} K\omega - \int_{\tau_3} K\omega - \int_{\tau_4} K\omega \tag{2}$$



(where $\tau_1 = \tau(-,0)$, $\tau_2 = \tau(1,-)$, $\tau_3 = \tau(-,1)$, $\tau_4 = \tau(0,-)$). We now define a 3-chain $\rho:I^3 \to M$ by

$$\rho(x_1,x_2,x_3) = F_{x_1}(\tau(x_2,x_3)),$$

and compute $\int_\rho d\omega$ in two ways. On the one hand, by definition (writing $\underline{t} = (t_1, t_2, t_3)$, and $J_{\underline{t}}(\rho) = \left(\frac{\partial\rho}{\partial t_1}(t), \frac{\partial\rho}{\partial t_2}(t), \frac{\partial\rho}{\partial t_2}(t)\right)$, considered as an element of $T_{\rho(\underline{t})}(N)^3$)

$$\int_\rho d\omega = \int_0^1\int_0^1\int_0^1 [\underline{t} \mapsto d\omega(\rho(\underline{t}),\ J_{\underline{t}}(\rho))]\ dt_1 dt_2 dt_3 \tag{3}$$

$$= \int_0^1\int_0^1 [(t_2,t_3) \mapsto \int_0^1 \{t_1 \mapsto d\omega)(g_{\tau(t_2,t_3),J(t_2,t_3)}(t_1))\}dt_1\ ]dt_2 dt_3$$

$$= \int_0^1\int_0^1 [(t_2,t_3) \mapsto K(d\omega)(\tau(t_2,t_3), J_{(t_2,t_3)}(\tau))]dt_2 dt_3$$

$$= \int_\tau K(d\omega).$$

On the other hand, by Stokes theorem,

$$\int_\rho d\omega = \int_{\partial\rho} \omega = \int_f \omega - \int_{ba} \omega - \int_\ell \omega - \int_r \omega + \int_t \omega - \int_{bo} \omega \tag{4}$$

(where $\partial\rho = (f-ba)+(\ell-r)+(t-bo)$, f refers to the restriction of σ to the front of the cube in the picture p.199, ba to the back, etc.). Now clearly,

$$\int_f \omega = \int_{F_o\tau} \omega = \int_\tau F_1^*(\omega), \text{ and } \int_{ba} \omega = \int_\tau F_o^*(\omega). \tag{5}$$

We claim that also

$$\int_\ell \omega = \int_{\tau_4} K\omega,\ \int_r \omega = \int_{\tau_2} K\omega,\ \int_t \omega = \int_{\tau_3} K\omega,\ \int_{bo} \omega = \int_{\tau_1} K\omega\ . \tag{6}$$

Note that from (2)-(6) we get that

$$\int_\tau K(d\omega) = \int_{\partial\rho} \omega = \int_\tau F^*(\omega) - F_o^*(\omega) - \int_{\partial\tau} K\omega\ ,$$

or $\int_\tau (K(d\omega)+dK\omega) = \int_\tau F_1^*(\omega) - F_o^*(\omega)$; and since τ is arbitrary, $F_1^*(\omega) - F_o^*(\omega) = Kd\omega - dK\omega$. So to complete the proof, we only need to verify (6). We will do the first equality, the others are, of course, analogous.

ℓ = "left part of $\partial\rho$": $I^2 \to N$ is the 2-chain $(s,t) \overset{\ell}{\mapsto} F_s(\tau(0,t))$, so by definition

$$\int_\ell \omega = \int_0^1\int_0^1 [(s,t) \mapsto \omega(F_s(\tau(0,t)), \frac{\partial \ell}{\partial s}(s,t), \frac{\partial \ell}{\partial t}(s,t))]\,ds\,dt.$$

But by the chain rule,

$$\frac{\partial \ell}{\partial t}(s,t) := (d\ell)_{(s,t)}(0,1) = (dF_s)_{\tau(o,t)} d\tau_{(0,t)}(0,1).$$

Put $x = \tau(0,t)$, $v = (d\tau)_{(0,t)}(0,1) = \frac{\partial \tau}{\partial t}(0,t) = (d\tau_4)_t(1)$; then

$$\int_{\tau_4} \lambda = \int_0^1 [t \mapsto K\omega(\tau_4(t), (d\tau_4)_t(1))]\,dt$$

$$= \int_0^1 [t \mapsto \int_0^1 \{s \mapsto \omega(g_{x,v}(s))\}ds]\,dt.$$

So from the definition of $g_{x,v}$, it is clear that $\int_\ell \omega = \int_{\tau_4} \lambda$.

*

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NOTE (pág.251)

In G, the long exact Mayer-Vietoris sequence for homology consists of free R-modules on constant bases, as can be shown by induction (using lemmas 1 and 2 of page 252). Hence its dual is also exact. Indeed, a sequence $F_1 \to F_2 \to F_3$ is exact if and only if $0 \to F_1' \to F_2 \to F_3' \to 0$ is exact, where $F_1 \twoheadrightarrow F_1' \to F_2 \twoheadrightarrow F_3' \to F_3$. But by Lemma 1, of F_1, F_2, F_3 are free on constant bases, so are F_1' and F_3'. By Lemma 2, the exactness of the latter sequence is equivalent to $F_3 \simeq F_2' \oplus F_3'$. Obviously, it then follows that $F_3^* \simeq F_2'^* \oplus F_3'^*$, so it suffices to show that the epi-mono factorization is preserved bu dualization, more precisely, that the dual of an epi is a mono (which is clear), and that the dual of a mono is an epi. So let $F_1 \xrightarrow{\mu} F_2$ in G. Where $F_i = \mathrm{Free}_{\mathbb{R}}(\Delta X_i)$. Then in Sets there is a linear map $\lambda\colon \Gamma F_2 \twoheadrightarrow \Gamma F_1$, i.e. $\lambda\colon \mathrm{Free}_{\mathbb{R}}(X_2) \to \mathrm{Free}_{\mathbb{R}}(X_1)$, such that $\lambda \circ \Gamma\mu = \mathrm{id}$, and this map can be lifted to G, i.e. there is an R-linear map $F_2 \overset{\mu}{\twoheadrightarrow} F_1$ with $\nu\mu = \mathrm{id}$ (so $F_2^* \xrightarrow{\mu^*} F_1^*$ is epic). More generally, if F is a free R-module in G, then an $\mathbb{R}$-linear map $\phi\colon \Gamma F_1 \to \Gamma M$ can be lifted uniquely to an R-linear map $\Phi\colon F \to M$ with $\Gamma\Phi = \phi$, as follows immediately from $\Delta \dashv \Gamma$.

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