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Embeddings on Spaces of Generalized Bounded Variation

Inmersiones en espacios generalizados de variación acotada

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Abstract. In this paper we show the validity of some embedding results on the space of (ϕ, α)-bounded variation, which is a generalization of the space of Riesz p-variation.

Key words and phrases. (ϕ, α)-Bounded variation, Riesz p-variation, Embedding.

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Resumen. En este trabajo se muestra la validez de algunos resultados de inmersión en el espacio de variación (ϕ, α)-acotada, que es una generalización del espacio de Riesz de variación p-acotada.

Palabras y frases clave. Variación (ϕ, α)-acotada, variación p-acotada, inmersión.

1. Introduction

Two centuries ago, around 1880, C. Jordan (see [5]) introduced the notion of a function of bounded variation and established the relation between those functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such as F. Riesz, N. Wiener, R. E. Love, H. Ursell, L. C. Young, W. Orlicz, J. Musielak, L. Tonelli, L. Cesari, R. Caccioppoli, E. de Giorgi, O. Oleinik, E. Conway, J. Smoller, A. Vol'pert, S. Hudjaev, L. Ambrosio, G. Dal Maso, among many others. It is noteworthy to mention that many of these generalizations where motivated by problems in such areas as calculus of variations, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation in mathematical physics see, e.g., the monograph [7]. We just want to point out the recent generalization on bounded variation in the framework of variable spaces [2].

In his 1910 paper F. Riesz (see [6]) defined the concept of bounded p-variation ($1 \leqslant p < \infty$) and proved that, for $1 < p < \infty$, this class coincides with the class of functions f, absolutely continuous with derivative $f' \in L_p[a,b]$. Moreover the p-variation of a function f on $[a,b]$ is given by

$$\mathsf{V}_p(f,[a,b]) = \mathsf{V}_p(f) = \left\|f'\right\|^p_{L^p[a,b]}.$$

Although we will not make much references, for that see [1] and the references therein, we want to stress that there is a vast literature on the topic of bounded variation.

In [4] the first and third named authors of the present paper generalized the concept of bounded p-variation introducing a strictly increasing continuous function $\alpha : [a,b] \to \mathbb{R}$ and considering the bounded p-variation with respect to α. This new concept was called (p,α)-bounded variation and denoted by $\mathsf{BV}_{(p,\alpha)}[a,b]$. Recently the authors went a step further and generalized the (p,α)-variation to (ϕ,α)-variation and gave a characterization of this newly introduced spaces, see [3].

In this paper, we continue previous work [3, 4] in the study of (ϕ,α)-variation space and give some embedding results in it.

2. Preliminaries

In this section, we gather definitions, notations and results that will be used throughout the paper. Let α be any strictly increasing, continuous function defined on $[a,b]$.

Definition 2.1. A function $f : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous with respect to* α if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\left\{(a_j,b_j)\right\}_{j=1}^n$ are disjoint open subintervals of $[a,b]$, then

$$\sum_{j=1}^n \left|\alpha(b_j) - \alpha(a_j)\right| < \delta \quad \text{implies} \quad \sum_{j=1}^n \left|f(b_j) - f(a_j)\right| < \varepsilon.$$

Thus, the collection α-$\mathsf{AC}[a,b]$ of all α-absolutely continuous functions on $[a,b]$ is a function space and an algebra of functions.

Definition 2.2. A function $f : [a,b] \to \mathbb{R}$ is said to be α–*Lipschitz* if there exists a constant $M > 0$ such that

$$\big|f(x)-f(y)\big| \leqslant M\big|\alpha(x)-\alpha(y)\big|,$$

for all $x, y \in [a,b]$, $x \neq y$.

By α–Lip we will denote the space of functions which are α–Lipschitz. If $f \in \alpha$–Lip we define

$$\begin{aligned}\mathsf{Lip}_\alpha(f) &= \inf\big\{M>0 : \big|f(x)-f(y)\big| \leq M\big|\alpha(x)-\alpha(y)\big|,\ x\neq y \in [a,b]\big\}\\ &= \sup\left\{\frac{\big|f(x)-f(y)\big|}{\big|\alpha(x)-\alpha(y)\big|} : x \neq y \in [a,b]\right\}.\end{aligned}$$

The space α–$\mathsf{Lip}[a,b]$ equipped with the norm

$$\big\|f\big\|_{\mathsf{Lip}_\alpha} = \big|f(a)\big| + \mathsf{Lip}_\alpha(f) = \big\|f\big\|_{\alpha\text{–}\mathsf{Lip}[a,b]}$$

is a Banach space.

We also introduce the space α–$\mathsf{Lip}^0[a,b]$ as

$$\alpha\text{–}\mathsf{Lip}^0[a,b] = \big\{f \in \alpha\text{–}\mathsf{Lip}[a,b] : f(a)=0\big\},$$

in this case we sometimes use $\big|f\big|_{\alpha\text{–}\mathsf{Lip}[a,b]}$ for $\mathsf{Lip}_\alpha(f)$.

The following proposition is not hard to prove

Proposition 2.3. *Let $f, g \in \alpha$–$\mathsf{Lip}[a,b]$, then*

i) $\mathsf{Lip}_\alpha(cf) = |c|\mathsf{Lip}_\alpha(f)$,

ii) $\mathsf{Lip}_\alpha(f+g) \leqslant \mathsf{Lip}_\alpha(f) + \mathsf{Lip}_\alpha(g)$,

iii) $\mathsf{Lip}_\alpha(f) = 0$ if and only if $f = const$,

iv) $f \in \alpha$-$\mathsf{Lip}[a,b]$ if and only if $f - f(a) \in \alpha$-$\mathsf{Lip}^0[a,b]$.

Definition 2.4. Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing function with $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$. Then such a function is know as a *ϕ-function.*

2.1. Functions of (ϕ, α)-Bounded Variation

Definition 2.5. Let f be a real–valued function on $[a,b]$ and ϕ be a ϕ-function. Let $\Pi = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ be a partition of $[a,b]$. We Consider

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi) = \sum_{j=1}^{n} \phi\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)$$

and

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(f;[a,b]\big) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = \sup_{\Pi} \sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi),$$

where the supremum is taken over all partitions Π of $[a,b]$. $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ is called the *Riesz* (ϕ,α)*–variation* of f on $[a,b]$. If $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty$, we say that f is a function of *Riesz* (ϕ,α)*-bounded variation*. The set of all these functions is denoted by

$$\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b] = \big\{f : [a,b] \to \mathbb{R} \mid \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty\big\}.$$

Note, if we set $\phi(t) = t^p (1 \leqslant p < \infty)$ we get back the concept of (p,α)-bounded variation defined in [4]. In the case $\phi = \alpha =$ *identity function*, then we get back the classical concept of bounded variation, denoted by $\mathsf{BV}[a,b]$.

Definition 2.6. Let ϕ be a convex ϕ-function. Then

$$\big\{f : [a,b] \to \mathbb{R} \mid \exists\lambda > 0 \text{ such that } \lambda f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\big\} = \big\{f : [a,b] \to \mathbb{R} \mid \exists\lambda > 0 \text{ such that } \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) < +\infty\big\}$$

is called the *vector space of* (ϕ,α)*-bounded variation function in the sense of Riesz* and we denote it by $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

Definition 2.7. Let ϕ be a convex ϕ-function. Then

$$\mathsf{RBV}^{0}_{(\phi,\alpha)}[a,b] = \big\{f : [a,b] \to \mathbb{R} \mid f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b] \text{ and } f(a) = 0\big\}$$

is the *vector space of Riesz* (ϕ,α)*-variation which vanishes at* a. Let us now define the Minkowski functional $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)} : \mathsf{RBV}^{0}_{(\phi,\alpha)}[a,b] \to \mathbb{R}^+$ by the formula $|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\big\{\varepsilon > 0 \mid \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \leqslant 1\big\}$.

Definition 2.8. Let ϕ be a convex ϕ-function. We define a *Luxemburg type norm* by

$$\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)} : \mathsf{RBV}_{(\phi,\alpha)}[a,b] \to \mathbb{R}$$

with $f \mapsto |f(a)| + |f - f(a)|^{\mathsf{R}}_{(\phi,\alpha)} = |f(a)| + \inf\left\{\varepsilon > 0 \;\middle|\; \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\}$.

The following results were shown in [3]

Lemma 2.9. *Let* ϕ *be a convex* ϕ*-function. Let* $f \in \mathsf{RBV}^{0}_{(\phi,\alpha)}[a,b]$. *Then*

i) $|f|^{\mathsf{R}}_{(\phi,\alpha)} \neq 0 \Longrightarrow \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big) \leqslant 1$;

ii) $|f|^{\mathsf{R}}_{(\phi,\alpha)} < k \Longleftrightarrow \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(f/k\big) \leqslant 1,\ k > 0$;

iii) $0 \leqslant |f|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant 1 \Longrightarrow \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leqslant |f|^{\mathsf{R}}_{(\phi,\alpha)}$;

Lemma 2.10. *Let* ϕ *be a* ϕ*-function, then* $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *if and only if* $f - f(a) \in \mathsf{RBV}^{0}_{(\phi,\alpha)}[a,b]$.

3. Embedding Results

Definition 3.1. Let ϕ be a convex ϕ-function. If $\lim_{n\to\infty} \frac{\phi(x)}{x} = +\infty$, then we say that ϕ *satisfies the* ∞_1*-condition.*

Using the ∞_1-condition we obtain several relations among some spaces.

Theorem 3.2. *Let ϕ be a convex ϕ-function which satisfy the ∞_1-condition. Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, then f is absolutely continuous with respect to α on $[a,b]$, that is, $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \alpha\text{-}\mathsf{AC}[a,b]$.*

Proof. Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Given $\varepsilon > 0$, let us consider (a_j, b_j), $j = 1, 2, \ldots, n$ a finite collection of disjoint subintervals contained in $[a,b]$. Let $m > 0$ such that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \frac{m\varepsilon}{2}$. Since ϕ satisfy the ∞_1-condition, there exists $x_0 \in (0,\infty)$ such that $\phi(x) \geqslant mx$ for $x \geqslant x_0$. Next, let us consider the following set

$$E = \left\{ j \in \{1,2,\ldots,n\} \;\middle|\; \frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)} \geqslant x_0 \right\}.$$

If $j \in E$, then

$$x_0 \leqslant \frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)}.$$

Since ϕ satisfy the ∞_1-condition we have

$$m\frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)} \leqslant \phi\left(\frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)}\right)$$

and thus

$$\left|f(b_j) - f(a_j)\right| \leqslant \frac{1}{m}\phi\left(\frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)}\right)\big(\alpha(b_j) - \alpha(a_j)\big).$$

From this last inequality we obtain

$$\begin{aligned}
\sum_{j=1}^{m} \left|f(b_j) - f(a_j)\right| &= \sum_{j\in E} \left|f(b_j) - f(a_j)\right| + \sum_{j\notin E} \left|f(b_j) - f(a_j)\right| \\
&\leqslant \frac{1}{m}\sum_{j\in E} \phi\left(\frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)}\right)\big(\alpha(b_j) - \alpha(a_j)\big) + x_0 \sum_{j\notin E}\big(\alpha(b_j) - \alpha(a_j)\big) \\
&\leqslant \frac{1}{m}\sum_{j=1}^{n} \phi\left(\frac{\left|f(b_j) - f(a_j)\right|}{\alpha(b_j) - \alpha(a_j)}\right)\big(\alpha(b_j) - \alpha(a_j)\big) + x_0 \sum_{j=1}^{n}\big(\alpha(b_j) - \alpha(a_j)\big) \\
&< \frac{1}{m}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) + x_0 \sum_{j=1}^{n}\big(\alpha(b_j) - \alpha(a_j)\big).
\end{aligned}$$

Now, choose $0 < \delta < \varepsilon/(2x_0)$. Thus, if $\sum_{j=1}^{n} \big(\alpha(b_j) - \alpha(a_j)\big) < \delta$, then

$$\sum_{j=1}^{n} \big|f(b_j) - f(a_j)\big| < \frac{\varepsilon}{2} + x_0\delta < \varepsilon.$$

Finally, collecting all of this information we conclude that, given $\varepsilon > 0$ there exists $\delta > 0$ such that for all finite family of disjoint subintervals $\big\{(a_j, b_j) \mid j = 1, 2, \dots, n\big\}$ of $[a, b]$ such that $\sum_{j=1}^{n} \big(\alpha(b_j) - \alpha(a_j)\big) < \delta$, then $\sum_{j=1}^{n} \big|f(b_j) - f(a_j)\big| < \varepsilon$, which means that $f \in \alpha\text{-}\mathsf{AC}[a, b]$. ☑

Theorem 3.3. *Let ϕ be a convex ϕ-function which satisfy the ∞_1-condition. Let $f \in \alpha\text{-}\mathsf{Lip}^0[a, b]$, then $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a, b]$ and there exists $m > 0$ such that $|f|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant m|f|_{\alpha\text{-}\mathsf{Lip}[a,b]}$ with*

$$m \geqslant \frac{1}{\phi^{-1}\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)}$$

and thus

$$\alpha\text{-}\mathsf{Lip}^0[a, b] \hookrightarrow \mathsf{RBV}^0_{(\phi,\alpha)}[a, b].$$

***Proof*.** Let $f \in \alpha\text{-}\mathsf{Lip}^0[a, b]$, then

$$\frac{\big|f(x) - f(y)\big|}{\alpha(x) - \alpha(y)} \leqslant |f|_{\alpha\text{-}\mathsf{Lip}[a,b]}$$

for all $x, y \in [a, b]$ and $x \neq y$.

Let $\Pi = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ be a partition of $[a, b]$. Since $\lim_{x\to 0} \phi(x) = 0$, there exists $m > 0$ such that

$$\phi\bigg(\frac{1}{m}\bigg) \leqslant \frac{1}{\alpha(b) - \alpha(a)},$$

that is, $m \cdot \phi^{-1}\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big) \geqslant 1$. Now let us consider

$$\sum_{j=1}^{n} \phi\left(\frac{\big|f(x_j) - f(x_{j-1})\big|}{m|f|_{\alpha\text{-}\mathsf{Lip}[a,b]}\big(\alpha(x_j) - \alpha(x_{j-1})\big)}\right)\big(\alpha(x_j) - \alpha(x_{j-1})\big) =: I.$$

Since ϕ is an increasing function, we have

$$I \leqslant \sum_{j=1}^{n} \phi\bigg(\frac{1}{m}\bigg)\big(\alpha(x_j) - \alpha(x_{j-1})\big) \leqslant \phi\bigg(\frac{1}{m}\bigg)\big(\alpha(b) - \alpha(a)\big) \leqslant 1$$

for all partitions Π of $[a,b]$. Then $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{m|f|_{\alpha\text{-}\mathsf{Lip}[a,b]}}\right) \leqslant 1$. By Lemma 2.9 *ii)* we obtain

$$|f|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant m|f|_{\alpha\text{-}\mathsf{Lip}[a,b]}. \tag{1}$$

☑

Corollary 3.4. *Let ϕ be a convex ϕ-function which satisfy the ∞_1-condition, then*

$$\alpha\text{-}\mathsf{Lip}[a,b] \hookrightarrow \mathsf{RBV}_{(\phi,\alpha)}[a,b]$$

and there exists $m > 0$ such that

$$\|f\|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant \max\{1,m\}\|f\|_{\alpha\text{-}\mathsf{Lip}[a,b]}$$

with $m \cdot \phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right) \geqslant 1$.

Proof. Let $f \in \alpha\text{-}\mathsf{Lip}[a,b]$. Then $f - f(a) \in \alpha\text{-}\mathsf{Lip}^0[a,b]$ (see Proposition 2.3) and by Lemma 3.3 we have that $f - f(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Moreover, by (1) there exists $m > 0$ such that

$$\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant m\big|f - f(a)\big|_{\alpha\text{-}\mathsf{Lip}[a,b]}.$$

Observe that

$$\begin{aligned}
\big|f - f(a)\big|_{\alpha\text{-}\mathsf{Lip}[a,b]} &= \mathsf{Lip}_\alpha\big(f - f(a)\big)\\
&= \sup_{x\neq y} \frac{\big|\big(f - f(a)\big)(x) - \big(f - f(a)\big)(y)\big|}{\big|\alpha(x) - \alpha(y)\big|}0\\
&= \sup_{x\neq y} \frac{\big|f(x) - f(y)\big|}{\big|\alpha(x) - \alpha(y)\big|}\\
&= \mathsf{Lip}_\alpha(f).
\end{aligned}$$

Then there exists $m > 0$ such that

$$\begin{aligned}
\big\|f\big\|^{\mathsf{R}}_{(\phi,\alpha)} = \big|f(a)\big| + \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} &\leqslant \big|f(a)\big| + m\big|f - f(a)\big|_{\alpha\text{-}\mathsf{Lip}[a,b]}\\
&= \big|f(a)\big| + m\mathsf{Lip}_\alpha(f) \leqslant \max\{1,m\}\big\|f\big\|_{\alpha\text{-}\mathsf{Lip}[a,b]}.
\end{aligned}$$ ☑

Theorem 3.5. *Let ϕ be a convex ϕ-function which satisfy the ∞_1-condition. Let $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Then $f \in \mathsf{BV}^0[a,b]$ and*

$$\mathsf{V}\big(f;[a,b]\big) \leqslant \left(\alpha(b) - \alpha(a) + \frac{1}{\phi(1)}\right)\big|f\big|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Therefore

$$\mathsf{RBV}^0_{(\phi,\alpha)}[a,b] \hookrightarrow \mathsf{BV}^0[a,b].$$

where $\mathsf{BV}^0[a,b]$ *is the* $\mathsf{BV}[a,b]$ *space of vanishing functions at the point* a*.*

Proof. From Theorem 3.2 we know that f is of bounded variation. If $\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)} = 0$ then $f = 0$ and thus $\mathsf{V}(f) = 0$; then the inequality holds trivially.

If $\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)} \neq 0$, let $\Pi = \left\{a = x_0 < x_1 < \cdots < x_n = b\right\}$ be a partition of $[a,b]$. Let us consider in the proof of Theorem 3.2, $f/\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)}$ in place of f. Then we obtain

$$\begin{aligned}\sum_{j=1}^{n} \frac{\left|f(x_j)-f(x_{j-1})\right|}{\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)}} &\leqslant \big(\alpha(b)-\alpha(a)\big) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right)\\ &\leqslant \big(\alpha(b)-\alpha(a)\big) + \frac{1}{\phi(1)}\end{aligned}$$

where the last inequality follows from Lemma 2.9 *i)*.

Thus

$$\sum_{j=1}^{n} \left|f(x_j)-f(x_{j-1})\right| \leqslant \left(\alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right)\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Since this last inequality holds for any partition Π of $[a,b]$, then

$$V\big(f;[a,b]\big) \leqslant \left(\alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right)\left|f\right|^{\mathsf{R}}_{(\phi,\alpha)}.$$ ☑

Corollary 3.6. *Let* ϕ *be a convex* ϕ*-function which satisfy the* ∞_1*-condition. Let* $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$*, then* $f \in \mathsf{BV}[a,b]$ *and*

$$\left\|f\right\|_{\mathsf{BV}[a,b]} \leqslant \max\left\{1, \alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right\}\left\|f\right\|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Then

$$\mathsf{RBV}_{(\phi,\alpha)}[a,b] \hookrightarrow \mathsf{BV}[a,b].$$

Proof. Taking $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, then, from Lemma 2.10, we have that the function $f - f(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ and by Theorem 3.5 that $f - f(a) \in \mathsf{BV}^0_{(\phi,\alpha)}[a,b]$. Moreover

$$\mathsf{V}\big(f-f(a);[a,b]\big) \leqslant \left(\alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right)\left|f-f(a)\right|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Since $\mathsf{V}\big(f-f(a);[a,b]\big)=\mathsf{V}\big(f;[a,b]\big)$, then

$$\begin{aligned}\big\|f\big\|_{\mathsf{BV}[a,b]}&=\big|f(a)\big|+\mathsf{V}\big(f;[a,b]\big)\\&\leqslant\big|f(a)\big|+\left(\alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right)\big|f-f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\\&\leqslant\max\left\{1,\alpha(b)-\alpha(a)+\frac{1}{\phi(1)}\right\}\big\|f\big\|^{\mathsf{R}}_{(\phi,\alpha)}.\end{aligned}$$

☑

Theorem 3.7. *Let ϕ_1 and ϕ_2 be two convex ϕ-functions such that there exist $t_0>0$ and $c>0$ such that $\phi_2(t)\leqslant\phi_1(ct)$ for $t\geqslant t_0$. Then*

$$\big|f\big|^{\mathsf{R}}_{(\phi_2,\alpha)}\leqslant K\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)},\qquad f\in\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$$

with $K\geqslant c\big(1+\big(\alpha(b)-\alpha(a)\big)\phi_2(t_0)\big)$, that is,

$$\mathsf{RBV}^0_{(\phi_1,\alpha)}[a,b]\hookrightarrow\mathsf{RBV}^0_{(\phi_2,\alpha)}[a,b].$$

Proof. Let $f\in\mathsf{RBV}^0_{(\phi_1,\alpha)}[a,b]$. If $\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}=0$ then the result is trivial. Let $\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\neq 0$. Let $M>1+\big(\alpha(b)-\alpha(a)\big)\phi_2(t_0)$ and $\Pi=\big\{a=x_0<x_1<\cdots<x_n=b\big\}$ be a partition of $[a,b]$. Let us define

$$E=\left\{j\in\{1,\dots,n\}\;\middle|\;\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\geqslant t_0\right\}.$$

Consider

$$\sum_{j=1}^{n}\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{Mc\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)=:I.$$

Since ϕ_2 is convex, $\phi_2(0)=0$ and $1/M<1$ we have

$$\begin{aligned}I&\leqslant\frac{1}{M}\sum_{j=1}^{n}\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\&\leqslant\frac{1}{M}\sum_{j\in E}\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)+\\&\qquad\frac{1}{M}\sum_{j\notin E}\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\end{aligned}$$

if $j \in E$ then $\big(\big|f(x_j)-f(x_{j-1})\big|\big)/\big(c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)\big) \geqslant t_0$ and

$$\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right) \leqslant \phi_1\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right).$$

If $j \notin E$, then $\big(\big|f(x_j)-f(x_{j-1})\big|\big)/\big(c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)\big) < t_0$. Since ϕ_2 is increasing we obtain

$$\phi_2\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{c\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right) \leqslant \phi_2(t_0)$$

and thus

$$\begin{aligned}
I &\leqslant \frac{1}{M}\left[\sum_{j\in E}\phi_1\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big) + \right.\\
&\qquad\qquad\left. \sum_{j\notin E}\phi_2(t_0)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\right]\\
&\leqslant \frac{1}{M}\left[\sum_{j=1}^{n}\phi_1\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}\big(\alpha(x_j)-\alpha(x_{j-1})\big)}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big) + \right.\\
&\qquad\qquad\left. \sum_{j=1}^{n}\phi_2(t_0)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\right]\\
&\leqslant \frac{1}{M}\left[\mathsf{V}^{\mathsf{R}}_{(\phi_1,\alpha)}\left(\frac{f}{\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}}\right) + \phi_2(t_0)\big(\alpha(b)-\alpha(a)\big)\right]\\
&\leqslant \frac{1}{M}\big[1+\phi_2(t_0)\big(\alpha(b)-\alpha(a)\big)\big]\\
&< 1
\end{aligned}$$

where the last inequality comes from the definition of M and the penultimate from Lemma 2.9 *ii)*. Thus $\sigma^{\mathsf{R}}_{(\phi_2,\alpha)}\left(\frac{f}{Mc|f|^{\mathsf{R}}_{(\phi_1,\alpha)}}\right) \leqslant 1$ for all partitions Π of $[a,b]$, and then we get $\mathsf{V}^{\mathsf{R}}_{(\phi_2,\alpha)}\left(\frac{f}{Mc|f|^{\mathsf{R}}_{(\phi_1,\alpha)}}\right) \leqslant 1$ and, one more time, by Lemma 2.9 *ii)* we have

$$\big|f\big|^{\mathsf{R}}_{(\phi_2,\alpha)} \leqslant Mc\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}.$$

Let $K = Mc$; hence

$$\big|f\big|^{\mathsf{R}}_{(\phi_2,\alpha)} \leqslant K\big|f\big|^{\mathsf{R}}_{(\phi_1,\alpha)}.$$ ☑

Corollary 3.8. *Let ϕ_1 and ϕ_2 be two convex ϕ-functions which satisfy the ∞_1-condition such that there exists $t_0 > 0$ and $c > 0$ such that $\phi_2(t) \leqslant \phi_1(ct)$ for $t \geqslant t_0$. Then*

$$\|f\|^{\mathsf{R}}_{(\phi_2,\alpha)} \leqslant \max\{1, K\}\|f\|^{\mathsf{R}}_{(\phi_1,\alpha)},$$

$f \in \mathsf{RBV}_{(\phi_1,\alpha)}[a,b]$ *with* $K > c\big(1 + (\alpha(b) - \alpha(a))\phi_2(t_0)\big)$ *that is*

$$\mathsf{RBV}_{(\phi_1,\alpha)}[a,b] \hookrightarrow \mathsf{RBV}_{(\phi_2,\alpha)}[a,b].$$

Proof. Let $f \in \mathsf{RBV}_{(\phi_1,\alpha)}[a,b]$, then $f - f(a) \in \mathsf{RBV}^0_{(\phi_1,\alpha)}[a,b]$ and $f - f(a) \in \mathsf{RBV}^0_{(\phi_2,\alpha)}[a,b]$ by Theorem 3.7

$$\big|f - f(a)\big|^{\mathsf{R}}_{(\phi_2,\alpha)} \leqslant K\big|f - f(a)\big|^{\mathsf{R}}_{(\phi_1,\alpha)}$$

and thus

$$\begin{aligned}\|f\|^{\mathsf{R}}_{(\phi_2,\alpha)} &= \big|f(a)\big| + \big|f - f(a)\big|^{\mathsf{R}}_{(\phi_2,\alpha)}\\ &\leqslant \big|f(a)\big| + K\big|f - f(a)\big|^{\mathsf{R}}_{(\phi_1,\alpha)} \leqslant \max\{1, K\}\|f\|^{\mathsf{R}}_{(\phi_1,\alpha)}\end{aligned}$$ ☑

Corollary 3.9. *Let ϕ be a convex ϕ-function which satisfy the ∞_1-condition. If there exists $t_0 > 0$ such that $t^p \leqslant \phi(t)$ for $t \geqslant t_0$, $p \geqslant 1$, then*

$$\mathsf{RBV}_{(\phi,\alpha)}[a,b] \hookrightarrow \mathsf{RBV}_{(p,\alpha)}[a,b].$$

***Remark* 3.10.** In the particular case $\phi(t) = t^p$ $(1 \leqslant p < \infty)$ we have $\|f\|^{\mathsf{R}}_{(\phi,\alpha)} = \|f\|^{\mathsf{R}}_{(p,\alpha)}$. Indeed

$$\begin{aligned}\inf\left\{\varepsilon > 0 \;\middle|\; \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\} &= \inf\left\{\varepsilon > 0 \;\middle|\; \mathsf{V}^{\mathsf{R}}_{(p,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\}\\ &= \inf\left\{\varepsilon > 0 \;\middle|\; \frac{1}{\varepsilon^p}\mathsf{V}^{\mathsf{R}}_{(p,\alpha)}(f) \leqslant 1\right\}\\ &= \inf\left\{\varepsilon > 0 \;\middle|\; \left[\mathsf{V}^{\mathsf{R}}_{(p,\alpha)}(f)\right]^{\frac{1}{p}} \leqslant \varepsilon\right\}\\ &= \left[\mathsf{V}^{\mathsf{R}}_{(p,\alpha)}(f)\right]^{\frac{1}{p}}.\end{aligned}$$

Thus $\big|f(a)\big| + \inf\left\{\varepsilon > 0 \;\middle|\; \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\} = \big|f(a)\big| + \left[\mathsf{V}^{\mathsf{R}}_{(p,\alpha)}(f)\right]^{\frac{1}{p}}$. Therefore

$$\|f\|^{\mathsf{R}}_{(\phi,\alpha)} = \|f\|^{\mathsf{R}}_{(p,\alpha)}.$$

Corollary 3.11. *Let $1 \leqslant q \leqslant p$, then*

$$\mathsf{RBV}_{(p,\alpha)}[a,b] \hookrightarrow \mathsf{RBV}_{(q,\alpha)}[a,b].$$

Proof. Indeed, let us take $\phi_1(t) = t^p$ and $\phi_2(t) = t^q$, next we may apply Theorem 3.7. ☑

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