Análisis de un esquema de diferencias finitas para la solución numérica de una ecuación de convección difusión fraccionaria

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2014

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Research Thesis
In Partial fulfillment of the Requirements for the Degree of
Magister en Ciencias - Matemática Aplicada

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Mathematics Subject:
Numerical Analysis

Research group:
Cálculo Científico y Modelamiento Matemático

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To my mother and father
Luz Marina Rodriguez P. and
Pedro Elberto Amador O.

Acknowledgments

I would like to thank my family for the support they provided me through my entire life and in particular, I must acknowledge my brother and friend Andres Felipe Amador. A very special thanks to the teachers and friends.
I would to express my gratitude to my supervisor Carlos Daniel Acosta, whose expertise and understanding, help to graduate. I appreciate his vast knowledge and skill in many areas, and his assistance in writing reports and this tesis. I would like to thank Carlos Enrique Mejia for the assistence they provided at all levels of the research project.
In conclusion, I must also acknowledge support by Universidad Nacional de Colombia through the project *Mathematics and Computation,* Hermes code 20305 and COLCIENCIAS through the project *Programa jóvenes investigadores e innovadores 2012 (contrato 566),* Hermes code 16243.

Abstract

The nonlinear time fractional convection diffusion equation (TFCDE) is obtained from a standard nonlinear convection diffusion equation by replacing the first-order time derivative with a fractional derivative (in Caputo sense) of order $\alpha \in (0,1)$. Developing numerical methods for solving fractional partial differential equations is of increasing interest in many areas of Science and Engineering. In this thesis an explicit conservative finite difference scheme for TFCDE is introduced. We find its CFL condition and prove encouraging results regarding stability, namely, monotonicity, the TVD property and several bounds. Illustrative numerical examples are included in order to evaluate potential uses of the new method. Finally, we develop a graphical user interface (GUI) based in tool GUIDE of MATLAB for numerical solution TFCDE.

Keywords: Caputo fractional derivative, finite difference scheme, stability, CFL, TVD, GUI.

Resumen

La ecuación de difusión- convección en el tiempo fraccional no lineal (TFCDE), es obtenido desde una ecuación de difusión - convección no lineal estándar reemplazando la derivada temporal de primer orden, con una derivada fraccional (en el sentido de Caputo) de orden entre 0 y 1. El desarrollo de métodos numéricos que solucionen ecuaciones de este tipo tiene gran interés en muchas áreas de la ciencia y la ingeniera. En esta tesis nosotros introducimos un esquema de diferencias finito conservativo para resolver una TFCDE. Nosotros encontramos su condición CFL y probamos resultados interesantes sobre estabilidad, monotonía, una propiedad TVD y varias cotas. Se desarrollan ejemplos numéricos para evaluar el potencial uso del nuevo método numérico. Finalmente, desarrollamos una interface gráfica (GUI) basados en la herramienta GUIDE de MATLAB para solución numérica de TFCDE.

Palabras clave: Derivada fraccional de Caputo, esquema de diferencia finito, estabilidad, CFL, TVD, GUI

Contents

1 Introduction

In this chapter we start our study of fractional calculus by considering the historical and theoretical development of the various concepts of derivative introduced by Grunwald-letnikove, Riemann-Liouville and Caputo, and then we do a literature review of the analytical and numerical theory of fractional differential equations. We consider a finite difference squeme approximating the Caputo derivative, with which we complete our numerical model in the subsequent chapters. Concepts are illustrated in several examples.
The study of integration and derivation of arbitrary order began in 1695, when L'Hospital inquired Leibnitz about what would happen if one wanted to find a derivative of order 1/2. In 1819, the French mathematician Lacroix dedicated several pages of his book "Integral transforms of generalized functions" to study derivatives of arbitrary order, and examined conditions for obtaining the derivative of order 1/2 of the function $f(x) = x^a$, as an example. Lacroix's work is considered the first attempt to generalize the concept of derivative to arbitrary order. Fourier also explored this field in 1822, but the admissible functions to be derived by his method were very few.
In 1832, Liouville defined the derivative of arbitrary order of a function as a series, confronting the problem of establishing conditions for assuring convergence, and covering some other cases for deriving functions of the type $f(x) = x^a$. Finally, Liouville centered his attention to integrals of arbitrary order, defining them as the inverse operation to taking derivatives of arbitrary order. In 1876, Riemann used the results of Liouville for stating his definition of fractional derivatives, with the problem of requiring an unknown function. Given the importance of Liouville and Riemann's theory, the first definition of fractional integral given by Laurant in 1884 was named after them.
In 1867, Grunwald defined the fractional derivative as the limit of a difference quotient, and defined definite-integral formulas for the q-th derivative. In 1967, Caputo formulated a definition for fractional derivative that allowed physical interpretation to initial conditions of many problems that the Riemann-Liouville derivative does not allow. In 1969, Caputo published his book "Elasticita e dissipazione" in which he used his definition to formulate and solve problems of viscoelasticity and seismology.

1.1 Literature review

The theory of fractional calculus is considered as an old but yet novel topic when related to differential equations, and many authors have studied analytical solutions to fractionary differential equations with diverse senses of derivative. Liu et al. [11] worked on a time fractional advection-dispersion equation using the Fourier-Laplace transform to obtain a solution, while Alibaud [2] studied an equation with fractional derivative on the diffusive term, and defined an entropy formulation for fractional conservation laws in order to prove existence and uniqueness of a solution. El-Shaed [9] investigated existence and multiplicity of positive solution for a nonlinear fractional ordinary differential equation. Mainardi et al. [13] considered two forms of time-fractional diffusion in the senses of Caputo for the time derivative and Riemman for the space derivative, obtaining elementary solutions. Regarding systems of ordinary fractional differential equations, Erturk and Momani [10] used the method of differential transform for obtaning approximated numerical solutions. Nathael and Boris [1] complemented their work of 2007 and studied a fractional Burgers equation. Ferreira [12] established existence of a solution to certain ordinary fractional differential equation. Using the maximun principle for the generalized time-fractional diffusion equation, Luchko [19] proved uniqueness of a solution to such problem. Liang and Zhang [17] established existence of a unique, positive and strictly increasing solution to a nonlinear ordinary fractional differential equation. Zhang and Wei [32] considered a linear systems of degenerate fractional differential equations and studied existence and uniqueness of solutions.

Various fractional differential equations have been solved using numerical methods. Meerschaert and Tadjeran [20] developed an implicit Euler method based on a modified Grunwald aproximation for approximating numerically solutions to a fractional advention dispersion flow equation. Yuste and Acedo [31] considered a fractional diffusion equation and used an explicit finite difference squeme with time going backwards, centered space and the Grundwald-Letnikov discretization of the Riemann-Liouville derivative. Cui [7] used an implicit compact finite difference scheme to solve numerically a fractional diffusion equation, considered in the sense of Riemann-Liouville. Cifani and Jacobsen [6], based on the work in entropy solutions for fractal conservation laws by Alibaud [2], solved a fractional degenerate convection-diffusion equation by means of an explicit numerical scheme, and established covergence to the entropy solution. An implicit scheme to approximate numerically a nonlinear fractional variable diffusion equation is developed by Zhuang et al. [23], in the sense of Riemann-Liouville. In the case of a time fractional advention dispersion equation, Ibrahim and Serife [15] proposed a Crank-Nicholson difference scheme with fractional derivatives in the sense of Riemann-Liouville.

In this thesis, we consider fractional derivatives in the sense of Caputo, in which several authors have worked. Fawang et al [28] proposed an explicit finite difference method for a time fractional diffusion equation, much of the theory to discretize the Caputo derivative is used in this thesis. Lin and Xu [18] considered the time fractional diffusion equation, in which the fractional derivative is discretized as in [28] but using a Galerkin spectral method for the spatial derivative. Chen and Sun [29] and Zhuang et al [22] studied the same problem as in [18] but using a Kansa method and MLS (Moving Least-Squares) method for the spatial derivative, respectively. Mohebbi and Abbaszadeh [21] considered the time fractional advention dispersion equation and used an implicit compact finite difference scheme which discretizes the Caputo fractional derivative in the same way as the previous articles. A variable order in the fractional term is presented by Chen et al. [27], where a finite difference scheme where the fractional derivative depends on the Coimbra variable order fractional operator is proposed.

This thesis is about differential equations in the area of fractional derivatives, this has been motivated by the interesting and novel applications of fractional differential equations to physics, chemistry and engineering [4]. The study of fractional equation with nonlinear terms is reduced and very recent.

1.2 Thesis objectives

Primary objective

Develop an effective numerical method based on a finite difference discretisation technique, for solving convection difusion equations involving fractional derivative in the sense of Caputo.

Specific objectives

1. Develop a numerical method for solving evolutionary problems composed of fractional-order differential equations with terms of diffusion and convection.
2. Make the stability analysis and convergence of the numerical method.
3. Expose numerical experiments to test the theory used.
4. Compare the efficiency of the numerical method in relation to others.

1.3 Preliminary concepts

In this section we define the most common definitions of fractional derivatives.

Grunwald-Letnikov derivative

The Grunwald-Letnikov derivative is a direct generalization of the following formula of difference quotient that holds for derivatives of integer order

$$D^n f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh), \quad n \in \mathbb{N}^+. \tag{1-1}$$

where $\binom{n}{k}$ is the usual notation for the binomial coefficients.

Definition 1.3.1. *Let $\alpha \in \mathbb{R}^+$ and $a < t$. The Grunwald-Letnikov derivative of order α of a function f, in case it exists, is defined as*

$$ {}_a^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{[\frac{t-a}{h}]} (-1)^k \binom{\alpha}{k} f(t-kh) $$

where $[\cdot]$ is the floor function. Let us evaluate the limit

$$ {}_a^{GL}D_t^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau \tag{1-2} $$

where $\Gamma(\cdot)$ is the function gamma. The formula (1-2) has been obtained under the assumption that the derivates $f^{(k)}(t)$ are continuous in the closed interval $[a,t]$ and the m is an integer number satisfying the condition $m > p-1$.

Riemann-Liouville derivatives

Before studying fractional derivatives in the sense of Riemann-Liouville, we introduce the concept of fractional integrals generalizing the Cauchy formula that reduces the calculation of the n-fold primative of a function $f(t)$ to a single integral of convolution type

$$ {}_aJ_t^n f(t) = \frac{1}{(n+1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \quad n \in \mathbb{N}, \text{ where } t > a. \tag{1-3} $$

Extend to any positive real value by using the Gamma function, we obtain

Definition 1.3.2. *Let $[a,b] \subset \mathbb{R}$ and $f \in L^1(a,b)$. The Riemann-Liouville integral of order $\alpha \in \mathbb{R}^+$ is defined as*

$$ {}_aJ_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{1-4} $$

For the operator D^n (derivative of order n) and the operator J^n defined in (1-3), $D^nJ^n = Id$ and $J^nD^n \neq Id$, so D^n is a left-inverse but not a right-inverse of J^n. This fact motivates the definition of fractional derivative in the Riemann-Liouville sense.

Definition 1.3.3. *Let $[a,b] \subset \mathbb{R}$ and $f \in L^1(a,b)$. The Riemann-Liouville derivative of order $\alpha \in \mathbb{R}^+$ of the function f is defined as*

$$ {}_a^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad m-1 < \alpha < m, \text{ where } t \in (a,b). \tag{1-5} $$

Caputo derivative

We define the Caputo derivative operator by ${}^{C}D^{\alpha} = J^{m-\alpha}D^{m}f(t)$ where $m-1<\alpha\leq m$, and $J^{m-\alpha}$ as in (1-4).

Definition 1.3.4. *Let $\alpha \in \mathbb{R}^{+}$ and $m = [a]+1$. Then, for $f:[a,b]\to\mathbb{R}$ such that f is m times differentiable on (a,b) and $D^{m}f \in L^{1}(a,b)$, the Caputo derivative of order α is defined by*

$$(1\text{-}6)\qquad {}_{a}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, \quad m-1<\alpha<m, \text{ where } t\in(a,b).$$

The following example extracted from [14], gives an exact formula for the Caputo derivative for a special class of functions.

Example 1.3.1. *The Caputo derivative of the power function satisfies*

$$(1\text{-}7)\qquad {}_{0}^{C}D_{t}^{\alpha}t^{p} = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, & m-1<\alpha<m,\ p>m-1,\ p\in\mathbb{R} \\ 0, & m-1<\alpha<m,\ p>m-1,\ p\in\mathbb{N} \end{cases}$$

Comparison between Riemann-Liouville and Caputo derivatives

The Riemann-Liouville derivative is in disadvantage with respect to the Caputo derivative. The latter allows us to work with physical initial conditions expressed in terms of derivatives. Another difference between the Riemann-Liouville and Caputo definitions is that the Caputo derivative of a constant is 0, whereas when a is finite, the Riemann-Liouville derivative of a constant is not equal to 0. More generaly,

$${}^{RL}D^{\alpha}f(t) = D^{m}J^{m-\alpha}f(t) \neq J^{m-\alpha}D^{m}f(t) =^{C} D^{\alpha}f(t).$$

On the other hand, as $\alpha \to (m-1)^{+}$,

$$\begin{aligned} &{}_{0}^{RL}D_{t}^{\alpha}f(t) \to D^{m}Jf(t) = D^{m-1}f(t) \\ &{}_{0}^{C}D_{t}^{\alpha}f(t) \to JD^{m}f(t) = D^{m-1}f(t) - D^{m-1}f(0^{+}). \end{aligned}$$

The Laplace transform for the Riemann-Liouville derivatives requires knowing the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and its integer derivatives of order $1,2,\ldots,m-1$. For the Caputo derivatives, it requires knowing the (bounded) initial values of the function and its integer derivatives of order $k=1,2,\ldots,m-1$, in analogy with the case when $\alpha=m$. Simplifying notation, let us denote ${}_{0}^{C}D_{t}^{\alpha}f$ by f_{t}^{α} from now onwards. For further developments on fractional calculus see [24], [25], [8] and [30].

1.4 Approximation method for the Caputo fractional derivative

When $0 < \alpha < 1$, the Caputo fractional derivative of order α can be approximated using the following scheme (see [28]). Let $0 < \alpha < 1$, $N \in \mathbb{N}$ be a positive integer, $f : [0, T] \to \mathbb{R}$ be a function, and $\Delta t = \frac{T}{N}$. The grid is composed by the points $(x, t^n) = (x, n\Delta t)$, where $n = 0, 1, \ldots N$.

$$\begin{aligned} f_t^{\alpha}(x, t^n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t^n} \left(\frac{\partial f(x,\tau)}{\partial \tau} \right) \frac{d\tau}{(t^n - \tau)^{\alpha}} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{k\Delta t}^{(k+1)\Delta t} \left(\frac{\partial f(x,\tau)}{\partial \tau} \right) \frac{d\tau}{(t^n - \tau)^{\alpha}}. \end{aligned}$$

Approximating the time derivative with a value τ_k between t^k and t^{k+1}. This term is approximated by a forward difference

$$f_t^{\alpha}(x, t^n)(x, t^n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{f(x, t^{k+1}) - f(x, t^k)}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \frac{d\tau}{(t - \tau)^{\alpha}} + r_{\Delta t}^{k+1} \tag{1-8}$$

Where $r_{\Delta t}^{k+1}$ is the truncation error. We can obtain $|r_{\Delta t}^{k+1}| \leq C_f (\Delta t)^{2-\alpha}$ (see [18]). By solving the integral on the right side, the equation (1-8) can be rewritten as

$$\begin{aligned} f_t^{\alpha}(x, t^n) =& \frac{1}{\Delta t^{\alpha} \Gamma(2-\alpha)} \sum_{k=0}^{n-1} [f(x, t^{n-k}) - f(x, t^{n-1-k})][k^{1-\alpha} - (k-1)^{1-\alpha}] \\ & + O((\Delta t)^{2-\alpha}). \end{aligned} \tag{1-9}$$

For functions of one variable,

$$\begin{aligned} f_t^{\alpha}(t^n) =& \frac{1}{\Delta t^{\alpha} \Gamma(2-\alpha)} \sum_{k=0}^{n-1} [f(t^{n-k}) - f(t^{n-1-k})][k^{1-\alpha} - (k-1)^{1-\alpha}] \\ & + O((\Delta t)^{2-\alpha}), \end{aligned} \tag{1-10}$$

so

$$f_t^{\alpha}(t^n) \approx \frac{1}{\Delta t^{\alpha} \Gamma(2-\alpha)} \left(f(t^n) b_0 - \sum_{k=1}^{n-1} f(t^{n-k})(b_{k-1} - b_k) + f(t^0) b_{n-1} \right), \tag{1-11}$$

where $b_k = k^{1-\alpha} - (k-1)^{1-\alpha}$. The constants b_k satisfy the following properties:

$$1 = b_0 > b_1 > b_2 > \cdots \to 0$$

$$c_k = b_{k-1} - b_k, \quad \sum_{k=1}^{n} c_k = 1 + n^{1-\alpha} - (n+1)^{1-\alpha}$$

$$\sum_{k=1}^{\infty} c_k = 1, \quad 1 > 2 - 2^{1-\alpha} = c_1 > c_2 > \cdots \to 0. \tag{1-12}$$

These properties where studied in [28], and they play an essential role for the stability study developed in [5].

Example 1.4.1. *Suppose $0 < \alpha < 1$, following (1-7) the Caputo derivative is given by*

$$^{C}D^{\alpha}t^2 = \frac{\Gamma(2+1)}{\Gamma(2-\alpha+1)} t^{2-\alpha} \tag{1-13}$$

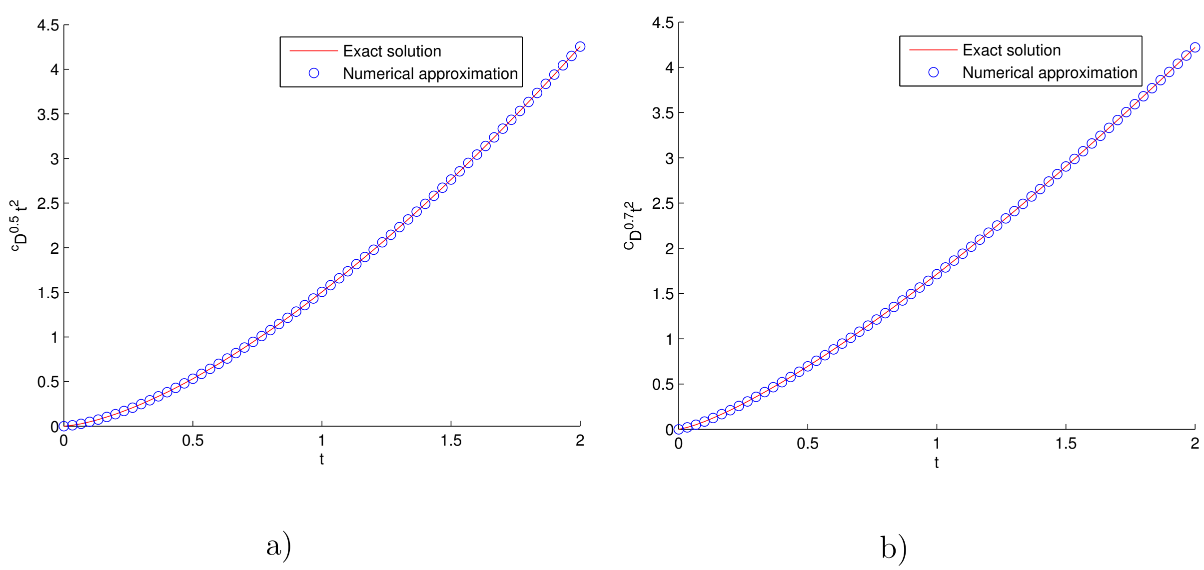
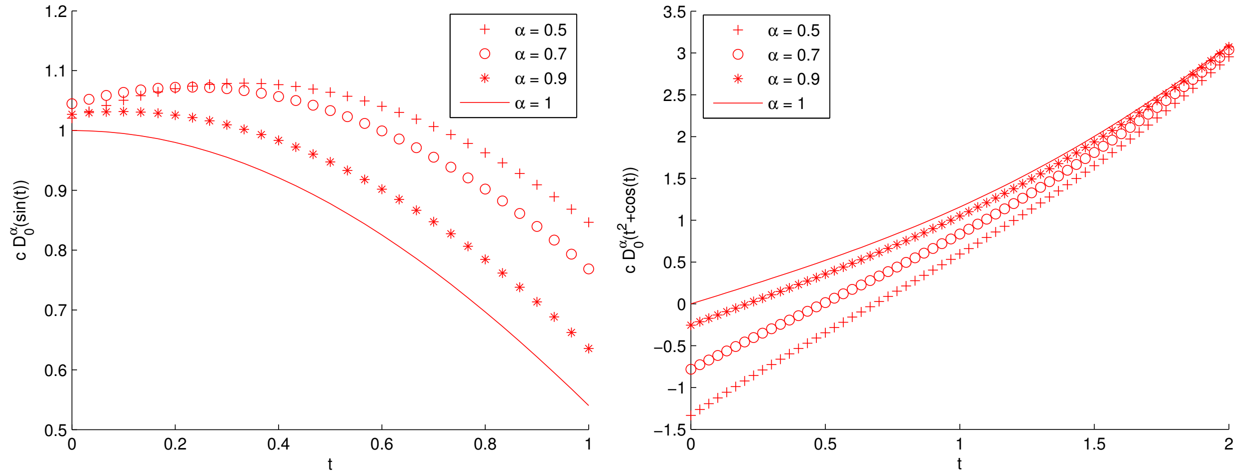


Figure **1-1**: a) Comparison of the numerical approximation (1-10) and exact solution (1-13) for $f(t) = t^2$ using with $\Delta t = 1/100$ and $\alpha = 0.5$. b) Comparison of the numerical approximation (1-10) and exact solution (1-13) for $f(t) = t^2$ using with $\Delta t = 1/100$ and $\alpha = 0.7$.

Example 1.4.2. *Using (1-10) will approach its derivative for the follows functions:*

1. *We now consider $f(t) = \sin(t)$ where $t \in [0, 1]$.*
2. *We now consider $f(t) = t^2 + \cos(t)$ where $t \in [0, 2]$.*



a) b)

Figure **1-2**: a) Comparison of the numerical approximation for $f(t) = \sin(t)$ using (1-10) with $\Delta t = 1/50$. b) Comparison of the numerical approximation for $f(t) = t^2 + \cos(t)$ using (1-10) with $\Delta t = 1/50$

Concluding remarks

Fractional equations is a topic of increasing interest, and many authors have worked intensively on it. It is very important for us to work on numerical methods for time fractional equations with derivatives in the sense of Caputo. Much of what had been done concerns proving stability and convergence of numerical approximations to derivatives, but very few of these numerical models treated nonlinear terms.

Numerical examples about approximations to Caputo derivative show good performance of the method developed.

2 Finite difference scheme for a TFCDE

In this chapter, we present the work developed in [5] on finite difference scheme for a time fractional nonlinear convection diffusion equation. Using (1-10) and finite difference scheme we develop a numerical scheme for which we present a stability analysis based on a CFL condition. The proofs develped in this chapter are based on an inductive step, because the memory scheme requires it. The chapter concludes with several examples, for each of these examples we compare the exact solution with the numerical solution, finding the converge order and relative order in ∞-norm for several non integer values (time derivative) and space step.

2.1 Mathematical model

We consider the following Cauchy problem of the form

$$u_t^\alpha + cu_x = A(u)_{xx}, \quad 0 < \alpha < 1, \quad (x,t) \in \Pi_T := \mathbb{R} \times (0,T), \quad T > 0 \tag{2-1}$$

with initial condition given by

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}.$$

Here c is a positive constant, the integrated diffusion coefficient $A(u)$ is defined by

$$A(u) = \int_0^u a(s)\mathrm{d}s, \quad a(u) \geq 0, \quad a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \tag{2-2}$$

and u_t^α denotes Caputo fractional derivative of order α defined by (1-6)

$$u_t^\alpha(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t-\xi)^\alpha} d\xi.$$

The diffusion function $a(s)$ is allowed to vanish on intervals of positive length and thus, in principle, (2-1) might be a strongly degenerate parabolic equation.

2.2 Numerical scheme

We begin our discussion of a finite difference scheme for equation (2-1) by defining a grid of points in the (x,t) strip. Let Δx be a positive real number, N be a positive integer and let us define $\Delta t = \frac{T}{N}$. The grid will be the points $(x_j, t^n) = (j\Delta x, n\Delta t)$ for all $j \in \mathbb{Z}$ and $n = 0, 1, ..., N$.
Following (1-9), the Caputo fractional derivative at time t^{n+1} can be approximated by

$$u_t^\alpha\left(x, t^{n+1}\right) = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} b_k \left[u\left(x, t^{n-k+1}\right) - u\left(x, t^{n-k}\right)\right] + O\left((\Delta t)^{2-\alpha}\right),$$

for $n = 0, 1, ..., N-1$ and weights $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ for $k = 0, 1, ..., n$.
The partial derivatives with respect to x are approximated in a straightforward way by

$$\frac{\partial u(x_j,t)}{\partial x} = \frac{u(x_j,t) - (x_{j-1},t)}{\Delta x} + O(\Delta x) \tag{2-3}$$

and

$$\frac{\partial^2 A(u(x_j,t))}{\partial x^2} = \frac{A(u(x_{j+1},t)) - 2A(u(x_j,t)) + A(u(x_{j-1},t))}{(\Delta x)^2} + O\left((\Delta x)^2\right) \tag{2-4}$$

Let us denote by v_j^n the numerical approximation of $u(x_j, t^n)$. The numerical method for the solution of (2-1) is obtained from the previous approximations and is given by the explicit finite difference scheme

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} b_k \left[v_j^{n-k+1} - v_j^{n-k}\right] + c\frac{v_j^n - v_{j-1}^n}{\Delta x} = \frac{A\left(v_{j+1}^n\right) - 2A\left(v_j^n\right) + A\left(v_{j-1}^n\right)}{(\Delta x)^2}.$$

Let $\lambda = \Gamma(2-\alpha)(\Delta t)^\alpha / \Delta x$, $\mu = \lambda/\Delta x$ and $A_j^n = A\left(v_j^n\right)$. If $n = 0$, the numerical scheme can be written

$$v_j^1 = v_j^0 - c\lambda\left(v_j^0 - v_{j-1}^0\right) + \mu\left(A_{j+1}^0 - 2A_j^0 + A_{j-1}^0\right). \tag{2-5}$$

Likewise, if $n \geq 1$, the numerical scheme becomes

$$v_j^{n+1} = v_j^n - c\lambda\left(v_j^n - v_{j-1}^n\right) + \mu\left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right) - \sum_{k=1}^{n} b_k \left[v_j^{n-k+1} - v_j^{n-k}\right]. \tag{2-6}$$

An alternative way to write scheme (2-6) is

$$\begin{aligned} v_j^{n+1} &= v_j^n - c\lambda\left(v_j^n - v_{j-1}^n\right) + \mu\left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right) \\ &\quad - b_1 v_j^n + \sum_{k=1}^{n-1} d_k v_j^{n-k} + b_n v_j^0, \end{aligned} \tag{2-7}$$

where $d_k = b_k - b_{k+1}$ for $k = 1, 2, \cdots, n-1$.
Sometimes it is appropriate to consider the method in sequence form. Let $v^n = \left(v_j^n\right)_{j\in\mathbb{Z}}$. Method (2-5)-(2-6) is represented by an expression of the form

$$v^{n+1} = \mathcal{H}\left(v^n, v^{n-1}, \cdots, v^0; j\right) \tag{2-8}$$

where the right hand side in (2-8) corresponds to the right hand side in (2-5) or (2-6), depending on the value of n.
The first feature of scheme (2-5)-(2-6) is that it allows a conservative form, which guarantees that the numerical method does not converge to non-solutions.

Lemma 2.2.1. *Scheme (2-5)-(2-6) is conservative, that is, it admits a conservation form. More precisely,*

$$v_j^{n+1} = v_j^n - \lambda\left(\psi_j^n - \psi_{j-1}^n\right), \tag{2-9}$$

where

$$\psi_j^0 = cv_j^0 - \frac{1}{\Delta x}\left(A_{j+1}^0 - A_j^0\right), \quad \textit{for } n = 0$$

$$\psi_j^n = cv_j^n - \frac{1}{\Delta x}\left(A_{j+1}^n - A_j^n\right) - \sum_{k=1}^{n} b_k \psi_j^{n-k}, \quad \textit{for } n \geq 1$$

Proof. The case $n = 0$ follows from (2-5). Suppose it is possible to achieve the conservation form (2-9) for $k = 0, 1, \cdots, n-1$, that is

$$v_j^{k+1} = v_j^k - \lambda\left(\psi_j^k - \psi_{j-1}^k\right).$$

For $k = n$,

$$\begin{aligned}
v_j^{n+1} &= v_j^n - c\lambda\left(v_j^n - v_{j-1}^n\right) + \mu\left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right) \\
&\quad - \sum_{k=1}^{n} b_k\left(v_j^{n-k+1} - v_j^{n-k}\right) \\
&= v_j^n - \lambda\left\{c\left(v_j^n - v_{j-1}^n\right) - \frac{1}{\Delta x}\left[\left(A_{j+1}^n - A_j^n\right) - \left(A_j^n - A_{j-1}^n\right)\right]\right\} \\
&\quad + \lambda\sum_{k=1}^{n} b_k\left(\psi_j^{n-k} - \psi_{j-1}^{n-k}\right).
\end{aligned}$$

□

We end this section by clarifying that convergence issues are not addressed here although they are important. Since nonlinear equations may have several weak solutions, an entropy condition is usually required to identify the *physically correct* solution. These ideas, along with the notion of nonlinear stability, are treated by many authors. For an initial boundary value problem of a strongly degenerate parabolic equation in which the time derivative is not fractional we recommend [26].
Next section deals with conditional stability and other properties of scheme (2-5)-(2-6).

2.3 Stability analysis

Explicit schemes require certain restrictions on the discretization parameters in order for the method to be useful. We begin by introducing the CFL condition for scheme (2-5)-(2-6), which is

$$c\lambda + 2\mu \|a\|_\infty \leq 2 - 2^{1-\alpha}. \tag{2-10}$$

The equation (2-10) implies the following estimates:

$$\begin{aligned} 0 &\leq 1 - b_1 - c\lambda - 2\mu \|a\|_\infty \\ 0 &\leq 1 - c\lambda - 2\mu \|a\|_\infty \end{aligned} \tag{2-11}$$

Provided the CFL condition is satisfied, two important properties of the method are derived.

2.3.1 Monotonicity property

Let v_j^n and u_j^n be two discrete functions to which method (2-8) can be applied. The numerical method (2-8) is called a *monotone* method in the following sense if

$$v_j^0 \leq u_j^0 \quad \text{for all } j \Longrightarrow v_j^n \leq u_j^n \quad \text{for all } j \text{ and all } n$$

Theorem 2.3.1. *If the CFL condition (2-10) holds, then method (2-8) is monotone.*

Proof. Suposse $u_j^0 \leq v_j^0$ for all $j \in \mathbb{Z}$. For all n, we denote $A_j^n = A\left(v_j^n\right)$ and $\bar{A}_j^n = A(u_j^n)$. For $n = 1$ monotonicity is proved as follows:

$$\begin{aligned} v_j^1 - u_j^1 &= \left(v_j^0 - u_j^0\right) - c\lambda\left(\left(v_j^0 - u_j^0\right) - \left(v_{j-1}^0 - u_{j-1}^0\right)\right) \\ &\quad + \mu\left(\left(A_{j+1}^0 - \bar{A}_{j+1}^0\right) - 2\left(A_j^0 - \bar{A}_j^0\right) + \left(A_{j-1}^0 - \bar{A}_{j-1}^0\right)\right) \\ &= \int_{u_j^0}^{v_j^0} \left(1 - c\lambda - 2\mu a(u)\right) du + \mu \int_{u_{j+1}^0}^{v_{j+1}^0} a(u) du + \mu \int_{u_{j-1}^0}^{v_{j-1}^0} a(u) du \\ &\geq 0 \end{aligned}$$

The condition (2-11) allows nonnegativity of the first of the three integrals. Now, suppose $u_j^k \leq v_j^k$ for $k = 0, 1, \cdots, n$ and all $j \in \mathbb{Z}$. Thus

$$\begin{aligned}
v_j^{n+1} - u_j^{n+1} &= \int_{u_j^n}^{v_j^n} (1 - c\lambda - 2\mu a(u))\, du + \mu \int_{u_{j+1}^n}^{v_{j+1}^n} a(u) du + \mu \int_{u_{j-1}^0}^{v_{j-1}^n} a(u) du \\
&- b_1 \int_{u_j^n}^{v_j^n} du + \sum_{k=1}^{n-1} d_k \left(v_j^{n-k} - u_j^{n-k}\right) + b_n \left(v_j^0 - u_j^0\right) \\
&= \int_{u_j^n}^{v_j^n} (1 - b_1 - c\lambda - 2\mu a(u))\, du + \mu \int_{u_{j+1}^n}^{v_{j+1}^n} a(u) du + \mu \int_{u_{j-1}^0}^{v_{j-1}^n} a(u) du \\
&+ \sum_{k=1}^{n-1} d_k \left(v_j^{n-k} - u_j^{n-k}\right) + b_n \left(v_j^0 - u_j^0\right) \\
&\geq 0
\end{aligned}$$

Where we have taken into consideration that $1 - b_1 = 2 - 2^{1-\alpha}$ and the CFL condition.

□

2.3.2 Stability bounds

The next theorem establishes two stability bounds in the $\infty-$norm and the $1-$norm respectively and it includes a total variation diminishing property of importance in case a convergence analysis is sought.

Theorem 2.3.2. *If the CFL condition (2-10) is satisfied then the following inequalities hold:*

$$\begin{gathered}
\|v^n\|_\infty \leq \left\|v^0\right\|_\infty, n = 1, 2, \cdots, N \\
\|v^n\|_1 \leq \left\|v^0\right\|_1, n = 1, 2, \cdots, N \\
\sum_j \left|v_{j+1}^{n+1} - v_j^{n+1}\right| \leq \sum_j \left|v_{j+1}^n - v_j^n\right|, n = 1, 2, \cdots, N
\end{gathered}$$

Proof. First observe that

$$\begin{aligned}
v_j^1 &= v_j^0 - c\lambda \left(v_j^0 - v_{j-1}^0\right) + \mu \left(A_{j+1}^0 - 2A_j^0 + A_{j-1}^0\right) \\
&= (1 - c\lambda)\, v_j^0 + c\lambda v_{j-1}^0 + \mu \left(\left(A_{j+1}^0 - A_j^0\right) - \left(A_j^0 - A_{j-1}^0\right)\right) \\
&= (1 - c\lambda)\, v_j^0 + c\lambda v_{j-1}^0 + \mu \left(a(\zeta_{j+1/2}^0) \left(v_{j+1}^0 - v_j^0\right) - a(\zeta_{j-1/2}^0) \left(v_j^0 - v_{j-1}^0\right)\right) \\
&= \left(1 - c\lambda - \mu a(\zeta_{j+1/2}^0) - \mu a(\zeta_{j-1/2}^0)\right) v_j^0 \\
&+ c\lambda v_{j-1}^0 + \mu a(\zeta_{j+1/2}^0) v_{j+1}^0 + \mu a(\zeta_{j-1/2}^0) v_{j-1}^0
\end{aligned}$$

for some values $\zeta_{j\pm1/2}^0$ between $v_{j\pm1}^0$ and v_j^0 respectively. Taking into account (2-11) implies the following stimates

$$1 - c\lambda - \mu a(\zeta^0_{j+1/2}) - \mu a(\zeta^0_{j-1/2}) \geq 1 - c\lambda - 2\mu \left\| a \right\|_\infty \geq 0.$$

Then

$$\begin{aligned}\left|v^1_j\right| \leq \left(1 - c\lambda - \mu a(\zeta^0_{j+1/2}) - \mu a(\zeta^0_{j-1/2})\right)\left|v^0_j\right| + c\lambda\left|v^0_{j-1}\right| \\ + \mu a(\zeta^0_{j+1/2})\left|v^0_{j+1}\right| + \mu a(\zeta^0_{j-1/2})\left|v^0_{j-1}\right| \leq \left\|v^0\right\|_\infty.\end{aligned}$$

Also,

$$\begin{aligned}\sum_j \left|v^1_j\right| &\leq \sum_j \left(1 - c\lambda - \mu a(\zeta^0_{j+1/2}) - \mu a(\zeta^0_{j-1/2})\right)\left|v^0_j\right| + \sum_j c\lambda\left|v^0_{j-1}\right| \\ &+ \mu\sum_j a(\zeta^0_{j+1/2})\left|v^0_{j+1}\right| + \mu\sum_j a(\zeta^0_{j-1/2})\left|v^0_{j-1}\right| \\ &\leq \sum_j \left|v^0_j\right| - c\lambda\sum_j\left(\left|v^0_j\right| - \left|v^0_{j-1}\right|\right) - \mu\sum_j\left(a(\zeta^0_{j+1/2})\left|v^0_j\right| - a(\zeta^0_{j-1/2})\left|v^0_{j-1}\right|\right) \\ &- \mu\sum_j\left(a(\zeta^0_{j-1/2})\left|v^0_j\right| - a(\zeta^0_{j+1/2})\left|v^0_{j+1}\right|\right) \\ &\leq \sum_j \left|v^0_j\right|.\end{aligned}$$

Similarly, we get

$$\sum_j \left|v^1_j - v^1_{j-1}\right| \leq \sum_j \left|v^0_j - v^0_{j-1}\right|.$$

To conclude the proof, we proceed by induction. Suppose the following inequalities are satisfied:

$$\begin{aligned}\left\|v^k\right\|_\infty \leq \left\|v^0\right\|_\infty, k = 1, 2, \cdots, n-1 < N \\ \left\|v^k\right\|_1 \leq \left\|v^0\right\|_1, k = 1, 2, \cdots, n-1 < N \\ \sum_j \left|v^k_{j+1} - v^k_j\right| \leq \sum_j \left|v^{k-1}_{j+1} - v^{k-1}_j\right|, k = 1, 2, \cdots, n-1 < N\end{aligned}$$

Thus for $k = n$, we have

$$\begin{aligned}v^{n+1}_j &= v^n_j - c\lambda\left(v^n_j - v^n_{j-1}\right) + \mu\left(A^n_{j+1} - 2A^n_j + A^n_{j-1}\right) \\ &- b_1 v^n_j + \sum_{k=1}^{n-1} d_k v^{n-k}_j + b_n v^0_j \\ &= \left(1 - b_1 - c\lambda - \mu a(\zeta^n_{j+1/2}) - \mu a(\zeta^n_{j-1/2})\right) v^n_j \\ &+ c\lambda v^n_{j-1} + \mu a(\zeta^n_{j+1/2}) v^n_{j+1} + \mu a(\zeta^n_{j-1/2}) v^n_{j-1} \\ &+ \sum_{k=1}^{n-1} d_k v^{n-k}_j + b_n v^0_j.\end{aligned}$$

By the CFL condition (2-10), we obtain

$$\begin{aligned}\left|v_j^{n+1}\right| \leq & \left(1-b_1-c\lambda-\mu a(\zeta_{j+1/2}^n)-\mu a(\zeta_{j-1/2}^n)\right)\left|v_j^n\right| \\ & + c\lambda\left|v_{j-1}^n\right| + \mu a(\zeta_{j+1/2}^n)\left|v_{j+1}^n\right| + \mu a(\zeta_{j-1/2}^n)\left|v_{j-1}^n\right| \\ & + \sum_{k=1}^{n-1} d_k\left|v_j^{n-k}\right| + b_n\left|v_j^0\right| \\ \leq & \left(1-b_1+\sum_{k=1}^{n-1} d_k + b_n\right)\left\|v^0\right\|_\infty = \left\|v^0\right\|_\infty .\end{aligned}$$

since $\sum_{k=1}^{n-1} d_k = b_1 - b_n$. Similarly, we obtain the other inequalities. □

In the next section we present numerical examples. We work a linear time fractional diffusion equation and two nonlinear time fractional convection diffusion equation.

2.4 Numerical experiments

In this section we present the numerical results of the method (2-5) on several test problems. For each example we make a comparison between exact solution and numerical solution for several values of α.

Example 2.4.1. *This experiment is a linear time fractional diffusion equation with constant diffusion $a(u) = \bar{a} = 0.001$ for all u. The right hand side term $f(x,t)$ is chosen in such a way that the equation has a unique polynomial solution. The problem is the following:*

(2-12) $$u_t^\alpha = 0.001 u_{xx} + f(x,t),\ 0<\alpha<1,\ x\in[0,1],\ 0<t\leq 1,$$

The exact solution is given by

$$u(x,t) = 10x^2(1-x)(t+1)^2$$

Table **2-1**: Numerical results for example 2.4.1

	$\alpha = 1/2$		$\alpha = 2/3$		$\alpha = 3/4$	
Δx	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$
1/16	0.413849	–	0.40372	–	0.39757	–
1/32	0.084805	2.2868	0.173122	1.22156	0.16875	1.2363
1/64	0.0062903	3.7529	0.029929	2.53217	0.046724	1.8526
1/128	0.0003967	3.987	0.0038678	2.952	0.0076661	2.6076

For this problem the CFL condition (2-10) becomes

$$2\Gamma(2-\alpha)\bar{a}\frac{(\Delta t)^\alpha}{(\Delta x)^2} \leq 2 - 2^{1-\alpha}$$

and indicates that Δt behaves like $O\left((\Delta x)^{\frac{2}{\alpha}}\right)$.
Table **2-1** shows results for three different values of α and suggests that the order of accuracy is about $2/\alpha$ for Δx as the main discretization parameter. This is consistent with the theory, because monotone numerical methods are at most first order accurate (see [16], Theorem 15.6.)

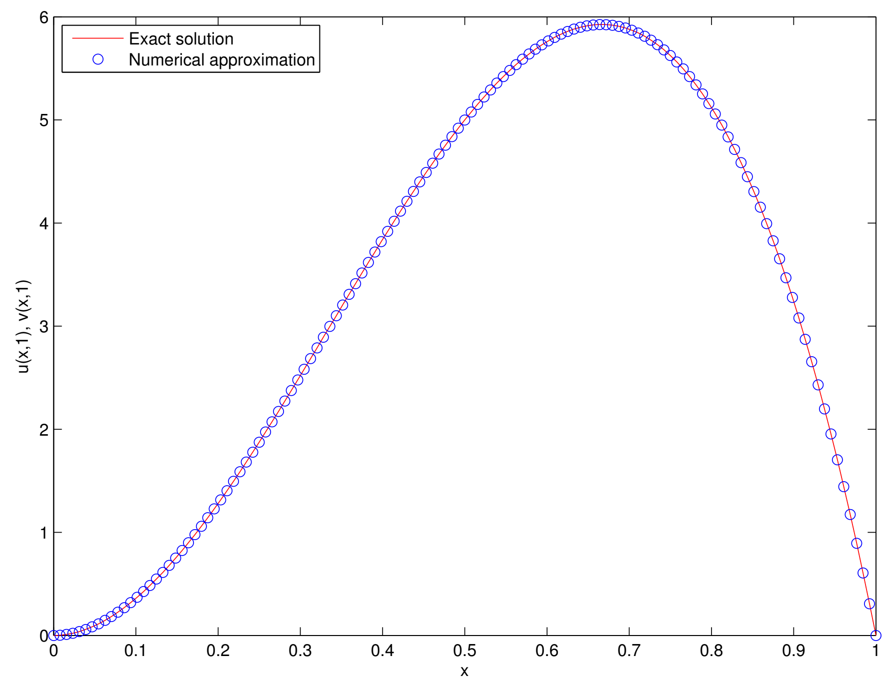


Figure **2-1**: Comparison of the exact solution and numerical solution for example 2.4.1 with $\alpha = 2/3$ and $dx = 1/128$.

Example 2.4.2. *This is a nonlinear time fractional convection diffusion equation and, as before, the right hand side function $f(x,t)$ is chosen so that the equation has a unique polynomial solution.*

$$u_t^\alpha + cu_x = A(u)_{xx} + f(x,t), \quad 0 < \alpha < 1,\ x \in [0,2],\ 0 < t \leq 1 \tag{2-13}$$

$$c = 1,\ A(u) = 4\varepsilon u^2\left(\frac{1}{2} - \frac{u}{3}\right),\ \varepsilon = 0.001 \tag{2-14}$$

Table **2-2**: Numerical results for example 2.4.2

	$\alpha = 1/2$		$\alpha = 2/3$		$\alpha = 3/4$	
Δx	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$
2/16	0.078763	–	0.098771	–	0.114912	–
2/32	0.03808	1.0484	0.045521	1.1175	0.054143	1.0856
2/64	0.018676	1.0278	0.020768	1.1321	0.024813	1.1256
2/128	0.009246	1.0142	0.009582	1.1159	0.01127	1.1386

The exact solution is given by

$$u(x,t) = t^2 x(2-x)$$

Table **2-2** summarizes our results for three different values of α and several discretization parameters. It suggests that the order of accuracy is about 1 for Δx as the main discretization parameter.

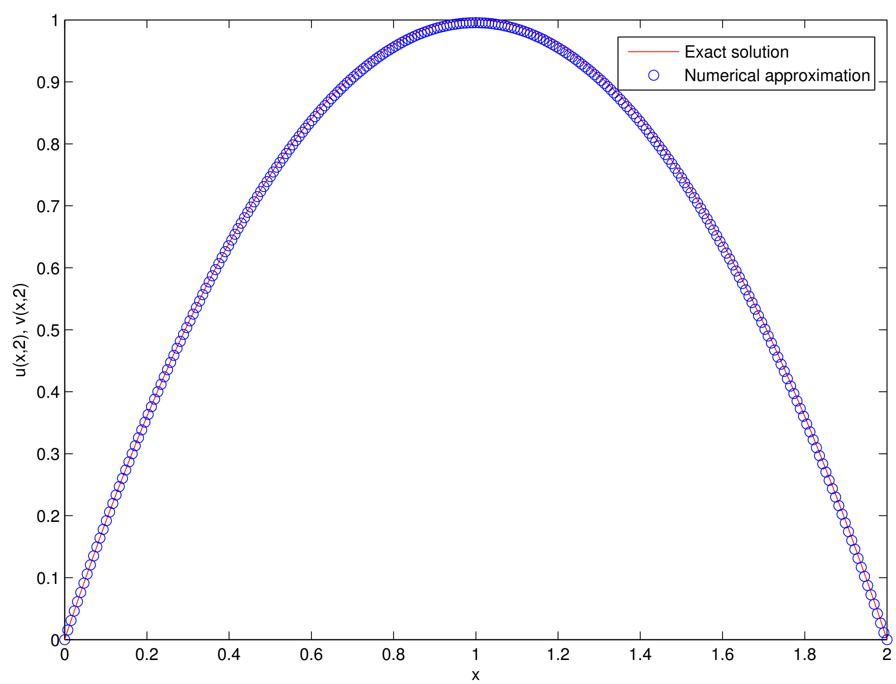


Figure **2-2**: Comparison of the exact solution and numerical solution for example 2.4.2 with $\alpha = 2/3$ and $dx = 1/128$.

Table **2-3**: Numerical results for example 2.4.3

	$\alpha = 1/2$		$\alpha = 2/3$		$\alpha = 3/4$	
Δx	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$	$L_\infty - err$	$Order$
1/16	0.27452	–	0.266241	–	0.26303	–
1/32	0.13907	0.9811	0.135045	0.9792	0.133823	0.9749
1/64	0.070001	0.9903	0.067851	0.9929	0.067228	0.9931
1/128	0.035133	0.9945	0.03395	0.9989	0.03357	1.0018

Example 2.4.3. *Once again, this is a nonlinear time fractional convection diffusion equation and, as before, the right hand side function $f(x,t)$ is chosen so that the equation has a unique closed form solution.*

$$u_t^\alpha + cu_x = A(u)_{xx} + f(x,t), \quad 0 < \alpha < 1,\ x \in [0,1]\,,\ 0 < t \le 1 \tag{2-15}$$

$$c = 1,\ A(u) = \varepsilon \frac{u^{n+1}}{n+1},\ \varepsilon = 0.001,\ n = 2 \tag{2-16}$$

The exact solution is

$$u(x,t) = t^2 \sin(2\pi x)$$

The numerical results are presented in table **2-3** which includes experiments for three different values of α and several discretization parameters. As in the previous example, the table suggests that the order of accuracy is about 1 for Δx as the main discretization parameter. In the tables (**2-1**), (**2-2**) and (**2-3**) the relative order in ∞-norm is calculated by the following formula:

$$\frac{\|u(x,T) - v(x,T)\|_\infty}{\|u(x,T)\|_\infty}. \tag{2-17}$$

Also the convergence order is calculated by the following formula:

$$\text{Convergence order} = \log_{\frac{\Delta x_1}{\Delta x_2}} \frac{e_1}{e_2} \tag{2-18}$$

Concluding remarks

The numerical method has important properties of stability, stability bounds for the numerical solution in the ∞-norm and 1-norm, also a TVD (total variation diminishing) property.

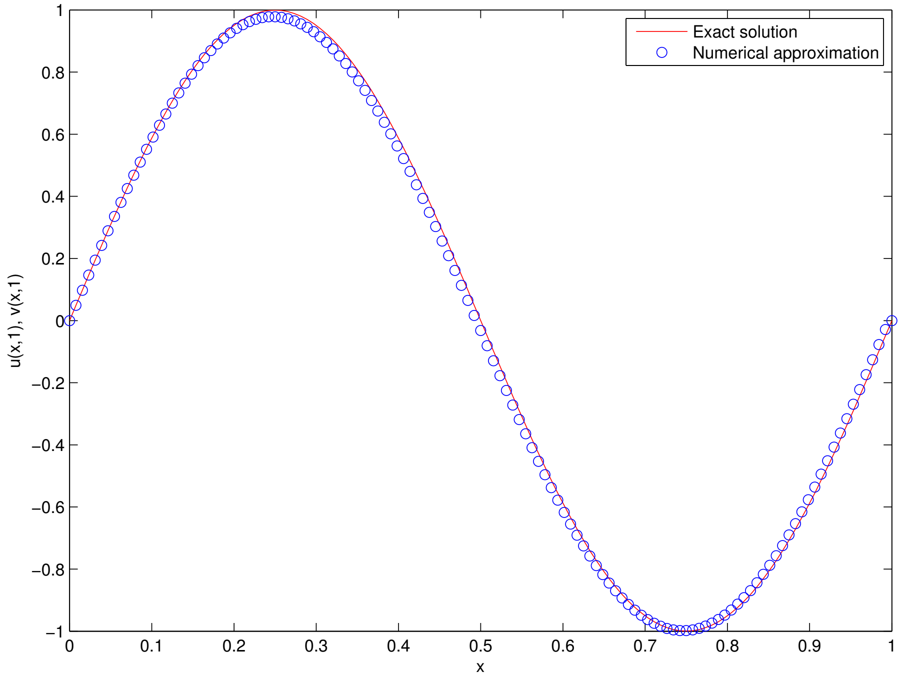


Figure **2-3**: Comparison of the exact solution and numerical solution for example 2.4.3 with $\alpha = 2/3$ and $dx = 1/128$.

The numerical results were obtained through routines implemented in programming language MATLAB. The exact solution in the examples 2.4.2 and 2.4.3 were developed solving for the source term (term $f(x,t)$ in (2-14) and(2-16)) with respect to the exact solution.
Comparison between numerical solution and exact solution in the examples 2.4.2 and 2.4.3 (nonlinear models) the convergence order is approximately 1, while for the simplest models (2.4.1) the order the convergence is much is much greater.
The numerical scheme solves several equations, convection and diffusion equations, convection-diffusion equations and the principal model with nonlinear diffusion, for each of these classes of equations with time fractional derivative or integer derivative. Obviously each of the equations has its own CFL condition but for all models this condition restricts the time step to very small values when $\alpha \to 0$, also for the nonlinear equations this constant is smaller in compared to others.

3 Graphical user interface

Based on the visualization tool GUIDE of MATLAB [3], we designed a friendly graphical user interface (GUI) for the numerical solution of the TFCDE, that implements the numerical method developed in Chapter 2 findind the CFL condition for the problem that is inserted. The input can be any of the models developed before or any equation with the same structure as (2.1). Then, the objective of this chapter is to explain all the features of the interface, and develop concrete examples of its use. Our aim was to develop a software application for solving fractional equations.

3.1 Technical specifications

Our interface is composed by nine push buttons, eleven edit-text boxes, one pop-up menu, one static-text box and one figure box. Figures **3-1** and **3-2** show the interface's appearance in spanish and in english, correspondingly.

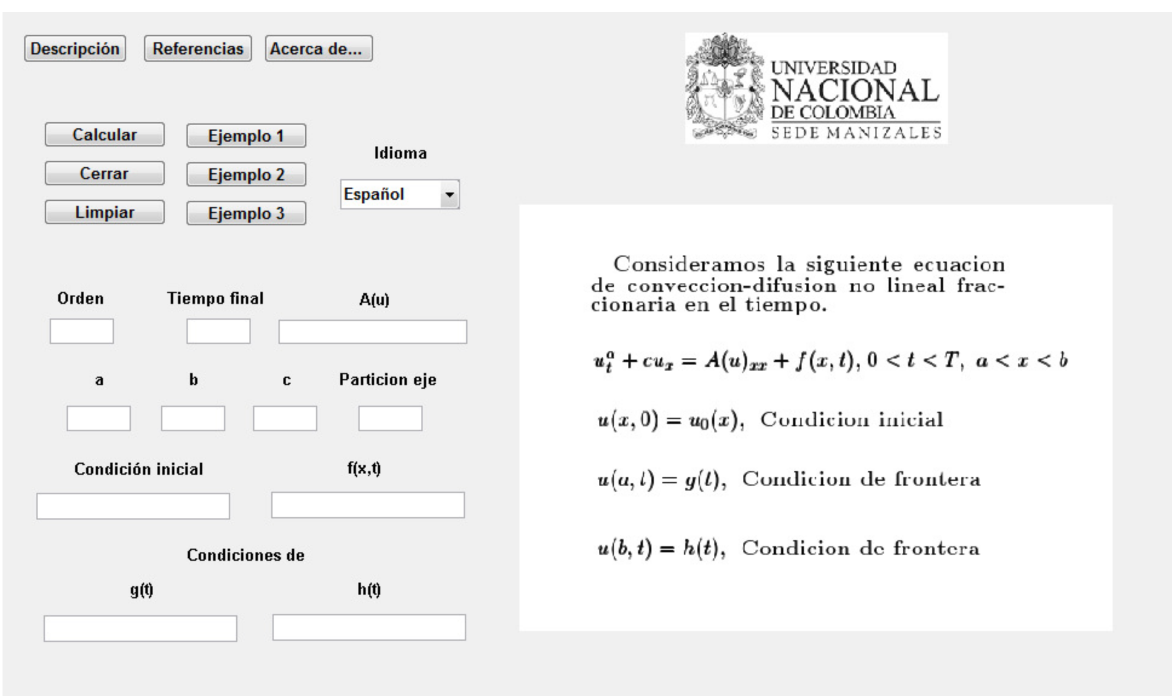


Figure **3-1**: Complete GUI

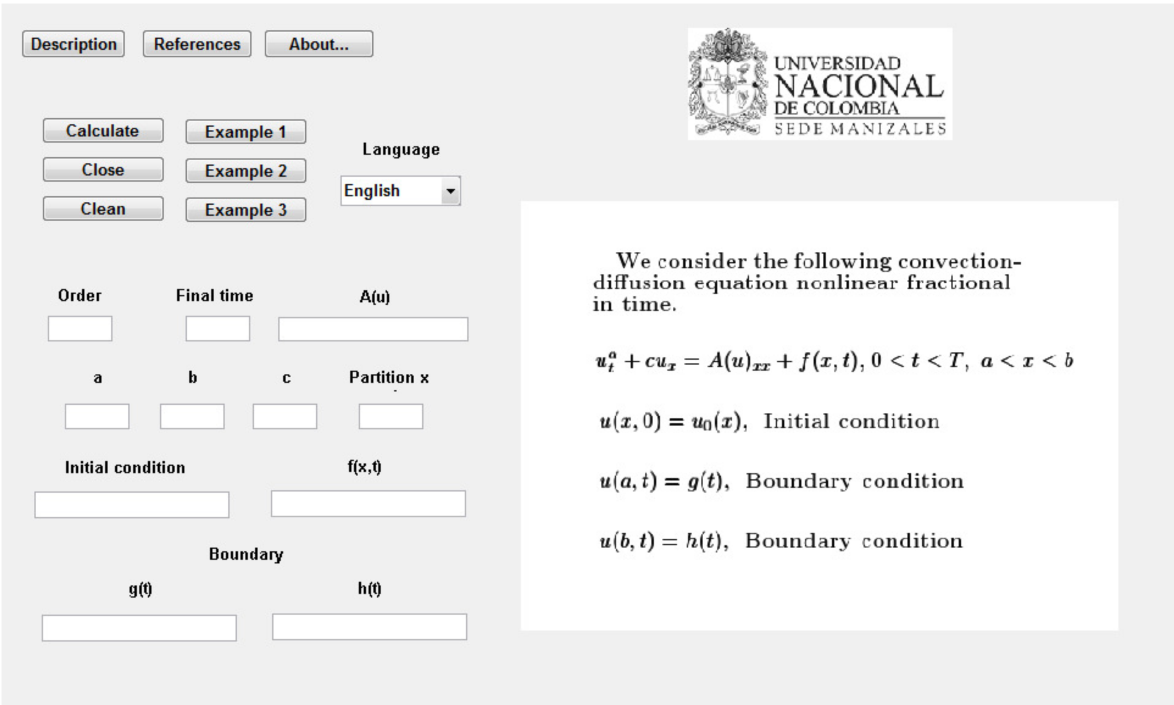


Figure **3-2**: Complete GUI

3.1.1 Edit-text boxes

The edit-text boxes contain the following data:

1. Order: order of the fractional derivative, that is a real number between 0 and 1.
2. c: value of the constant of convection,that must be a positive real value.
3. Final time: a positive real value.
4. $f(x,t)$: source term, that is a function of x and t.
5. $A(u)$: diffusion term, that is a function of u.
6. a: lower limit value of the variable x.
7. b: upper limit value of the variable x.
8. Partition x: number of partitions in the spatial axis, that must be a positive integer value.
9. Initial condition: the function of x to which the solution to the problem is equal at $t = 0$.

10. $g(t)$: boundary condition associated to the lower limit value of the variable x (option a), that is a function of t.

11. $h(t)$: boundary condition associated to the upper limit value of the variable x (option b), that is a function of t.

3.1.2 Push buttons and the pop-up menu

The push buttons execute the following tasks:

1. Calculate: computes the numerical solution.

2. Close: closes the guide.

3. Clean: empties the text-edit boxes and display the problem statement.

4. Examples 1: fills out the edit-text boxes with the data of a particular example. Similarly for buttons Example 2 and Example 3.

5. Description: presents the problem description in the figure box.

6. References: displays a list of references.

7. About...: provides information about we, the authors.

There is only one pop-up menu, that allows the user to choose Spanish or English as the interface language.

3.2 A practical experiment

Let us consider the following equation, based on example 2.4.3

$$u_t^\alpha + a u_x = bA(u)_{xx} + f(x,t), \quad 0 < \alpha < 1,\ x \in [0,1],\ 0 < t \leq 1$$
$$a = 1,\ b = 1,\ A(u) = \varepsilon \frac{u^{n+1}}{n+1},\ \varepsilon = 0.01,\ n = 2,$$

where

$$\begin{aligned} f(x,t) =& \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 2\pi t^2 \cos(2\pi x) - \\ & (2\pi)^2 \varepsilon t^{2n+2} (n \cos(2\pi x)^2 \sin(2\pi x)^{n-1} - \sin(2\pi x)^{n+1}), \end{aligned}$$

boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and initial condition

$$u(x,0) = 0.$$

The exact solution to this problem is

$$u(x,t) = t^2 \sin(2\pi x)$$

In order to simulate this example with $\alpha = 0.7$, we would have to fill the blanks of he interface as illustrated in Figure **3-3**

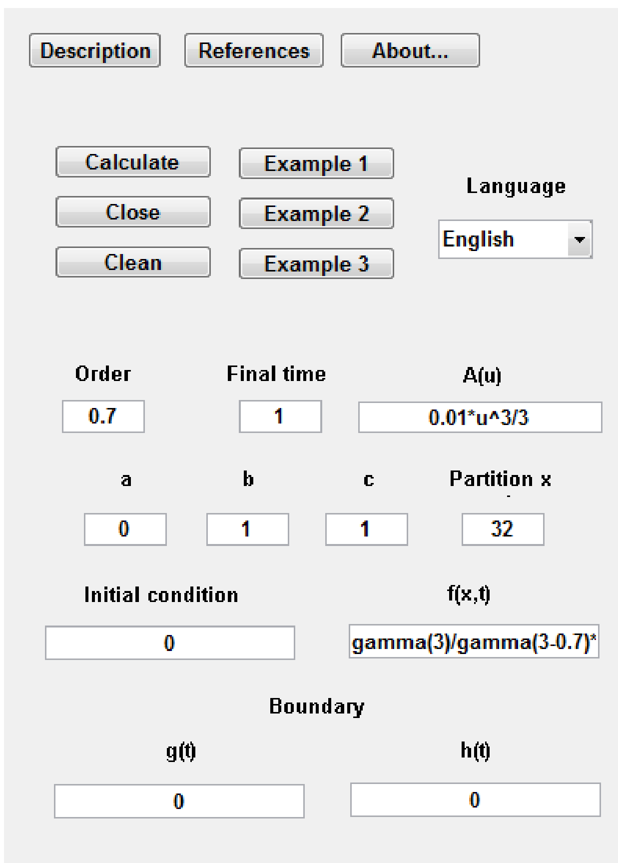


Figure **3-3**: Text-edit boxes with date for example 2.4.3

After clicking on the push botton "Calculate", the figure box shows a graph of the numerical solution as illustrated in Figure **3-4**

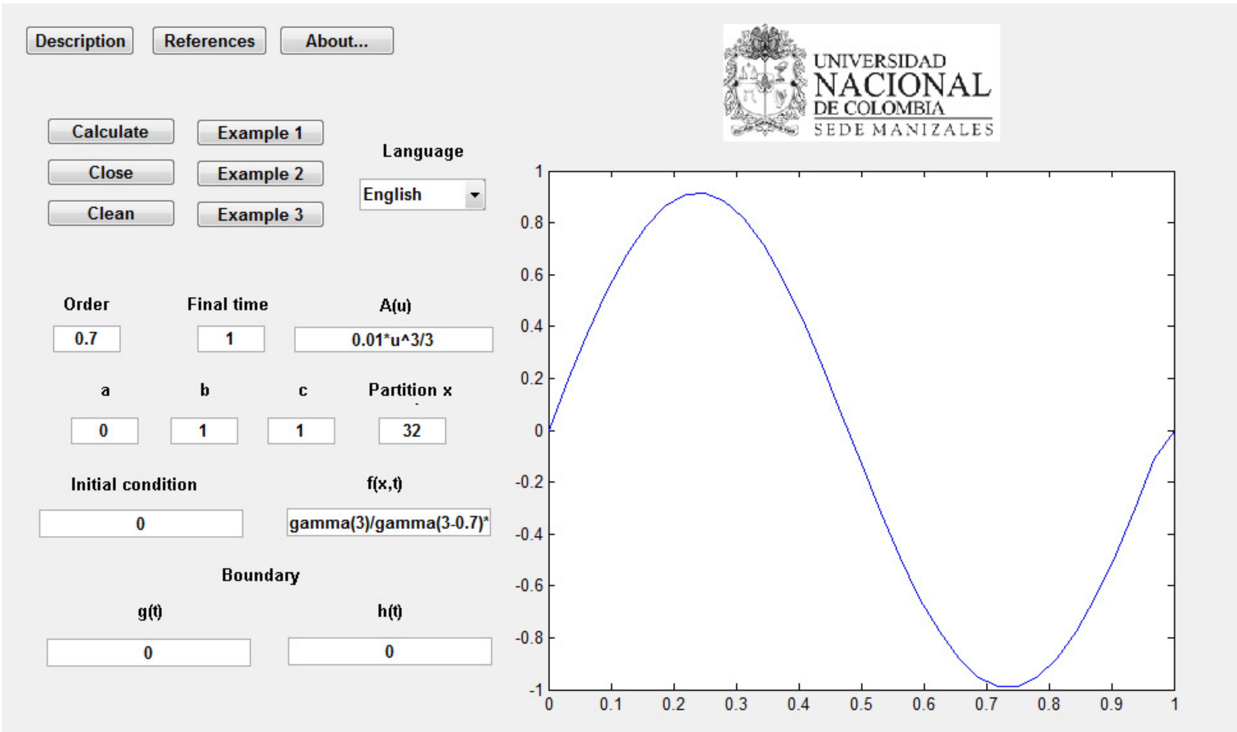


Figure **3-4**: Numerical solution

Concluding remark

We developed a GUI that consists on nine push buttons, eleven edit-text boxes, one pop-up menu, one static-text box and one figure box, which allows to approximate solutions to solve fractional equations and graph such approximations. There are three examples implemented the user accesses by clicking on the push bottons “Example 1”, “Example 2” and “Example 3”. The user can choose the language the options are displayed from spanish or english.
In order to use this interface properly, the user must have basic knowledge in programming in MATLAB, writting functions as inputs appropriately, analyzing graphs and using the solution in the vectorial form, and finally, he/she must also know the conditions studied in Chapter 2 under which a problem has a solution.

4 Conclusions

In this thesis a new method for finding the numerical solution of (2-1) has been presented. It is an explicit conservative finite difference scheme that under a CFL condition satisfies standard stability estimates. Based on an induction argument and the CFL condition, we proved stability bounds (∞-norm and 1-norm) and the TVD property.
Numerical examples were developed in the programming language MATLAB. When programming the numerical method, we found computational difficulties, since it occupies much memory. The numerical scheme solves diffusion and convection equations, convection-diffusion equations, and nonlinear convection-diffusion equations, involving time derivative of fractional order, including order 1. For the nonlinear models, the CFL condition can generate small time step when $\alpha \to 0$, while for values near to 1 the CFL condition generates a reasonable time step. This consequence is meaningful since fractional models with derivative near to 1 are more interesting. In Examples 2.4.2 and 2.4.3 (nonlinear case) the order of convergence is approximately 1, which is good for a nonlinear model.
In Chapter 3, the numerical method studied in this thesis is implemented in a GUI that allows users to interact through edit-text boxes and push buttons. The GUI can be applied to solve diffusion equations, convection-diffusion equations and combinations of both of them with time fractional derivative. The GUI has diverse options such as choosing the language of displying among spanish or english, and show the graph of the solution to the problem in the figure box. The user needs to have basic knowledge in MATLAB in order to use the GUI in the correct way. There are three pre-defined examples the GUI has, in which a user can observe the correct syntax of the program.
With respect to the thesis objectives, we developed an efficient numerical strategy for fractional order differential equation with terms of diffusion (also nonlinear diffusion term) and convection. We also performed stability analysis for the numerical methods used, and studied convergence order in some numerical examples. Convergence analysis for a general problem would be part of a future work. Numerical methods for the nonlinear equations with time fractional derivative are scarce, making comparing the method to others difficult.

Contributions

Oral Presentation: Pedro Alejandro Amador, Carlos Daniel Acosta. "Stability and convergence of finite difference schemes for the time-fractional convection and nonlinear diffusion equations". International Conference on Applied Mathematics and Informatics 2013 (ICAMI

2013). From 24 to 29 November.
This thesis contributes to the area of numerical analysis, more specifically Chapter 2 is part of an article that is in review process [5].
Chapter 3 consists of a graphical user interface in MATLAB, which is the technological contribution of this work.

Directions for future research

It would be interesting to study generalizations of equation (2-1) to equations with a non-linear convection term, and develop a numerical method with similar properties.
If an entropy solution of (2-1) were defined, studying convergence of the method would be very important as well.

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