On generalized fuzzy ideals of ordered $\mathcal{LA}$-semigroups

Sobre ideales difusos generalizados de $\mathcal{LA}$-semigrupos ordenados

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Abstract. The left, right and intra-regular classes of an ordered semigroup play an essential role in studying the structural properties of the associative structure. In this paper, we have discussed these classes in a non-associative structure called an ordered $\mathcal{LA}$-semigroup. As an application of our results we get characterizations of intra-regular ordered $\mathcal{LA}$-semigroup in terms of different $(\in, \in \vee q_k)$-fuzzy ideals.

Keywords: Ordered $\mathcal{LA}$-semigroup, intra-regular, fuzzy ideal.

Resumen. Las clases izquierda, derecha e intra-regulares de un semigrupo ordenado juegan un rol esencial en el estudio de propiedades estructurales de la estructura asociativa. En este artículo, discutimos estas clases en una estructura no-asociativa llamada un $\mathcal{LA}$-semigrupo ordenado. Como aplicación de nuestros resultados obtenemos caracterizaciones de los $\mathcal{LA}$-semigrupos ordenados intra-regulares en términos de los diferentes ideales $(\in, \in \vee q_k)$-difusos.

Palabras claves: $\mathcal{LA}$-semigrupos ordenados, intra-regular, ideales difusos.

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1. Introduction and Preliminaries

The concept of fuzzy sets was first proposed by Zadeh [20] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It gives us a tool to model the uncertainty present in a phenomenon that does not have sharp boundaries. Many papers on fuzzy

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sets have been published, showing the importance and their applications to set theory, algebra, real analysis, measure theory and topology etc.

Murali [9] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. In [12], the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. The concept of an (α,β)-fuzzy subgroup was first considered by Bhakat and Das in [1] and [2] by using the "belongs to" relation $(\in)$ and "quasi coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. The idea of an (α,β)-fuzzy subgroup is a viable generalization of Rosenfeld's fuzzy subgroup [13]. The concept of an $(\in,\in\vee q)$-fuzzy sub-nearring of a nearring was introduced by Davvaz [3]. Jun et. al. gave the concept of an $(\in,\in\vee q)$-fuzzy ordered semigroup [5]. Moreover, $(\in,\in\vee q_k)$-fuzzy ideals, $(\in,\in\vee q_k)$-fuzzy quasi-ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals of a semigroup are defined in [14]. In [6], Jun and Song initiated the study of an (α,β)-fuzzy interior ideals of a semigroup.

In mathematics, algebraic structures play an important role with wide range of applications in many fields such as theoretical physics, information sciences and many more. This provides enough inspiration to review various concepts and results from the field of abstract algebra in broader frameworks of fuzzy theory.

The concept of a left almost semigroup ($\mathcal{LA}$-semigroup) was first given by M. A. Kazim and M. Naseeruddin [7] in 1972. An $\mathcal{LA}$-semigroup is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. An $\mathcal{LA}$-semigroup with a right identity becomes a commutative semigroup with an identity [10]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$-semigroup was given in [11] as follows: a commutative inverse semigroup $(S,\circ)$ becomes an $\mathcal{LA}$-semigroup $(S,\cdot)$ where $a\cdot b=b\circ a^{-1}$, for all $a,b\in S$. An $\mathcal{LA}$-semigroup S with a left identity becomes a semigroup under the binary operation "$\circ$"defined as follows: for all $x,y\in S$ and for a fixed element $a\in S, x\circ y=(xa)y$ [15]. An $\mathcal{LA}$-semigroup is a generalization of a semigroup [10] and has applications in connection with semigroups as well as with other branches of mathematics.

An $\mathcal{LA}$-semigroup [7] is a groupoid S satisfying the following left invertive law

$$(ab)c=(cb)a, \quad \text{for all } a,b,c\in S. \tag{1}$$

In an $\mathcal{LA}$-semigroup, the medial law [7] holds

$$(ab)(cd)=(ac)(bd), \quad \text{for all } a,b,c,d\in S. \tag{2}$$

If a left identity in an $\mathcal{LA}$-semigroup exists, then it is unique [10]. An $\mathcal{LA}$-semigroup S with a left identity satisfies the following laws

$$(ab)(cd)=(dc)(ba), \quad \text{for all } a,b,c,d\in S. \tag{3}$$

and

$$a(bc)=b(ac), \quad \text{for all } a,b,c\in S. \tag{4}$$

An ordered $\mathcal{LA}$-semigroup (po-$\mathcal{LA}$-semigroup) [8] is a structure $(S, \cdot, \leq)$ in which the following conditions hold:

1. $(S, \cdot)$ is an $\mathcal{LA}$-semigroup.

2. $(S, \leq)$ is a poset.

3. For all a, b and $x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$.

Example 1.1. [19, 18] Consider the $\mathcal{LA}$-semigroup $S = \{a, b, c, d, e\}$ with a left identity d.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Then S becomes an ordered $\mathcal{LA}$-semigroup with the order below.

$$\leq := \{(a, a), (a, b), (c, c), (a, c), (d, d), (a, e), (e, e), (b, b)\}.$$

In this paper S denotes an ordered $\mathcal{LA}$-semigroup.

A fuzzy subset or a fuzzy set f of a non-empty set S is an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is the unit segment of the real line. A fuzzy subset f is a class of objects endowed with membership grades, having the form $f = \{(s, f(s)) \mid s \in S\}$.

Set $x \in S$ and $A_x = \{(y, z) \in S \times S \mid x \leq yz\}$.

Assume that S is an ordered $\mathcal{LA}$-semigroup and let $F(S)$ denote the set of all fuzzy subsets of S. Define the operation $\circ$ and the relation $\subseteq$ in the expression $(F(S), \circ, \subseteq)$. Then $(F(S), \circ, \subseteq)$ is an ordered $\mathcal{LA}$-semigroup [8].

For $\emptyset \neq A \subseteq S$, we define

$$(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}.$$

If $A = \{a\}$, then we usually write $(a]$.

A non-empty subset A of an S is called a left (right) ideal of S if

1. $SA \subseteq A$ $(AS \subseteq A)$.

2. If $a \in A$ and $b \in S$ are such that $b \leq a$, then $b \in A$.

Equivalently, $(SA] \subseteq A((AS] \subseteq A)$ if S has a left (or right) identity.

A non-empty subset A of an S is called an interior ideal of S if

1. $(SA)S \subseteq A$.

2. If $a \in A$ and $b \in S$ are such that $b \leq a$, then $b \in A$.

Equivalently, $((SA)S] \subseteq A$ if S has a left (or right) identity.

A subset A of S is called a two-sided ideal of S if it is both a left and a right ideal of S.

For $\emptyset \neq A \subseteq S$ and $k \in [0,1)$, the k-characteristic function $(C_A)_k$ is defined by

$$(C_A)_k = \begin{cases} \dfrac{1-k}{2} & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let f and g be any two fuzzy subsets of S. We define the product $f \circ_k g$ by

$$(f \circ_k g)(x) = \begin{cases} \bigvee\limits_{(y,z)\in A_x} \left\{ f(y) \wedge g(z) \wedge \dfrac{1-k}{2} \right\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset \end{cases}, \text{ where } k \in [0,1).$$

For $k \in [0,1)$, the symbols $f \cap_k g$ and $f \cup_k g$ mean the following fuzzy subsets of S:

$$(f \cap_k g)(x) = f(x) \wedge g(x) \wedge \frac{1-k}{2}, \text{ for all } x \in S.$$
$$(f \cup_k g)(x) = f(x) \vee g(x) \vee \frac{1-k}{2}, \text{ for all } x \in S.$$

For $k \in [0,1)$, the order relation $\subseteq_k$ between any two fuzzy subsets f and g of S is defined by

$$f \subseteq_k g \text{ if and only if } f(x) \leq g(x) \wedge \frac{1-k}{2}, \text{ for all } x \in S.$$

2. Basic definitions and results

In what follows, $k \in [0,1)$ and $t, r \in (0,1]$ unless otherwise specified.

A fuzzy subset f of S of the form

$$f(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy subset f in a set S, Pu and Liu [12] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t \in qf$) if $f(x) \geq t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee qf$ (resp. $x_t \in \wedge qf$) means that $x_t \in f$ or $x_t \in qf$ (resp. $x_t \in f$ and $x_t \in qf$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. Generalizing the concept of $x_t q f$, Jun [4] defined $x_t q_k f$ if $f(x) + t + k > 1$ and $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

We have given the concept of $(\in, \in \vee q_k)$-fuzzy ideals in an ordered $\mathcal{LA}$-semigroup [18]. We have developed a very basic theory for an ordered $\mathcal{LA}$-semigroup in terms of $(\in, \in \vee q_k)$-fuzzy ideals and shown that $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, $(\in, \in \vee q_k)$-fuzzy interior ideals and $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideals need not to be coincide in an ordered $\mathcal{LA}$-semigroup. But on the other hand, we have proved that all these $(\in, \in \vee q_k)$-fuzzy ideals to be coincident in a left regular class of an ordered $\mathcal{LA}$-semigroup. We have connected an ideal theory with an $(\in, \in \vee q_k)$-fuzzy ideal theory by using the notions of duo and $(\in, \in \vee q_k)$-fuzzy duo.

In this paper, we have extended the work carried out in [18] and characterized an intra-regular ordered $\mathcal{LA}$-semigroup by using the properties of $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals and $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals.

Definition 2.1. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$*-fuzzy left (right) ideal of* S if

(i) For all $x, y \in S,\ x \leq y, y_t \in f \Rightarrow x_t \in \vee q_k f$.

(ii) For all $x, y \in S,\ y_t \in f \Rightarrow (xy)_t \in \vee q_k f \quad (y_t \in f \Rightarrow (yx)_t \in \vee q_k f)$.

Theorem 2.2. *[18] If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if and only if*

(i) $x \leq y \Rightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f(xy) \geq f(y) \wedge \frac{1-k}{2} \quad \big(f(xy) \geq f(x) \wedge \frac{1-k}{2}\big)$, *for all* $x, y \in S$.

Definition 2.3. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$*-fuzzy $\mathcal{LA}$-subsemigroup* of S if for all x, $y \in S$, $x_t \in f$ and $y_r \in f \Rightarrow (xy)_{t \wedge r} \in \vee q_k f$.

Theorem 2.4. *[18] If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S if and only if $f(xy) \geq f(x) \wedge f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.*

Definition 2.5. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$*-fuzzy generalized bi-ideal* of S if

(i) For all $x, y \in S,\ x \leq y,\ y_t \in f \Rightarrow x_t \in \vee q_k f$.

(ii) For all $x,\ y, z \in S,\ x_t \in f$ and $z_r \in f \Rightarrow ((xy)z)_{t \wedge r} \in \vee q_k f$.

Theorem 2.6. *[18] If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if*

(i) $x \leq y \Rightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Definition 2.7. If an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S is also an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S, then f is called an $(\in, \in \vee q_k)$-*fuzzy bi-ideal* of S.

Theorem 2.8. *[18] An $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if*

(i) $x \leq y \Rightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Let us consider the ordered $\mathcal{LA}$-semigroup in Example 1.1. If we define a fuzzy subset $f : S \longrightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.8 & \text{for } x = a, \\ 0.5 & \text{for } x = b, \\ 0.1 & \text{for } x = c, \\ 0.3 & \text{for } x = d, \\ 0.6 & \text{for } x = e. \end{cases}$$

Then f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, but it is not an $(\in, \in \vee q_k)$-fuzzy right ideal of S, because

$$f(bd) \ngeq f(b) \wedge \frac{1-k}{2} \text{ for all } k \in [0, 0.8]. \tag{5}$$

On the other hand it is easy to see that every $(\in, \in \vee q_k)$-fuzzy right ideal of S with a left identity is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

If we define a fuzzy subset $f : S \longrightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.4 & \text{for } x = a, \\ 0.4 & \text{for } x = b, \\ 0.4 & \text{for } x = c, \\ 0.2 & \text{for } x = d, \\ 0.5 & \text{for } x = e. \end{cases}$$

Then it is easy to see that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S but it is not an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S which can be seen from the following:

$$f(ae) \ngeq f(e) \wedge \frac{1-k}{2} \, (f(ea) \ngeq f(e) \wedge \frac{1-k}{2}) \text{ for all } k \in [0, 0.2].$$

On the other hand it is easy to see that every $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is an $(\in, \in \vee q_k)$-fuzzy interior (bi-and quasi) ideal of S. It follows that all $(\in, \in \vee q_k)$-fuzzy ideals need not to be coincident in S even if S has a left identity.

Again let us consider an ordered $\mathcal{LA}$-semigroup $S = \{a, b, c\}$ with left identity b in the following multiplication table.

.	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

$$\leq := \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}.$$

Define a fuzzy subset $g : S \longrightarrow [0,1]$ as follows:

$$g(x) = \begin{cases} 0.8 & \text{for } x = a, \\ 0.6 & \text{for } x = b, \\ 0.5 & \text{for } x = c. \end{cases}$$

One can easily check that S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Lemma 2.9. *[18] Let A be a non-empty subset of S, then A is a left (right, two-sided) ideal of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.*

Lemma 2.10. *[18] For any non-empty subsets A and B of S, $C_A \circ_k C_B = (C_{(AB]})_k$ and $C_A \cap_k C_B = (C_{A\cap B})_k$.*

Definition 2.11. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-*fuzzy semiprime* if for all $x \in S$, $x_t^2 \in f \Rightarrow x_t \in \vee q_k f$.

Lemma 2.12. *[18] Every right (left, two-sided) ideal of S is semiprime if and only if their characteristic function is $(\in, \in \vee q_k)$-fuzzy semiprime.*

Corollary 2.13. *Every right (left, two-sided) ideal of S is semiprime if every fuzzy right (left, two-sided) ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.*

The converse of Corollary 2.13 is not true in general [18].

Definition 2.14. An element a of S is called a *left regular element* of S if there exists any $x \in S$ such that $a \leq xa^2$ and S is called left regular if every element of S is left regular.

Definition 2.15. An element a of S is called an *intra-regular element of S* if there exist any $x, y \in S$ such that $a \leq (xa^2)y$ and S is called intra-regular if every element of S is intra-regular.

In [17], we have shown that a left regular and an intra-regular classes of an ordered $\mathcal{LA}$-semigroup S coincide in S with left identity.

Theorem 2.16. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) Every $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (ii)$ Assume that f is a $(\in, \in \vee q_k)$-fuzzy two-sided ideal of intra-regular S with left identity and let $a \in S$, then there exist $b, c \in S$ such that $a \leq (ba^2)c$. Therefore

$$f(a) \geq f((ba^2)c) = f((ba^2)(ec)) = f((ce)(a^2b)) = f(a^2((ce)b)) \geq f(a^2) \wedge \frac{1-k}{2}.$$

Thus f is $(\in, \in \vee q_k)$-fuzzy semiprime.

$(ii) \Rightarrow (i)$ Let every $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime. Since $(a^2S]$ is a two-sided ideal of S with left identity [8], therefore by using Corollary 2.13, $(a^2S]$ is semiprime. Clearly $a^2 \in (a^2S]$ [8], therefore $a \in (a^2S]$. Thus

$$\begin{aligned} a &\in (a^2S] = ((aa)S] = ((Sa)a] \subseteq ((Sa)(a^2S)] = (((a^2S)a)S] \\ &= (((aS)a^2)S] = ((Sa^2)(aS)] \subseteq ((Sa^2)S], \end{aligned}$$

which shows that S is intra-regular. □

Corollary 2.17. *Every $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of an intra-regular S with left identity is an $(\in, \in \vee q_k)$-fuzzy semiprime.*

Lemma 2.18. *[16] Let A and B be the subsets of S. Then the following statements are true.*

(i) $A \subseteq (A]$.

(ii) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(iii) $(A](B] \subseteq (AB]$.

(iv) $(A] = ((A]]$.

(vi) $((A](B]] = (AB]$.

Lemma 2.19. *[8] In intra-regular S with left identity, $f \circ S = f$ and $S \circ f = f$ holds for every fuzzy two-sided ideal f of S.*

Note that $S = (S^2]$ if S has a left identity or if S is intra-regular.

Theorem 2.20. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $(AB] = (BA]$, where both A and B are two-sided ideals of S such that they are semiprime.

(iii) $f \circ_k g = g \circ_k f$, where both f and g are $(\in, \in \vee q_k)$-fuzzy two-sided ideals of S such that they are $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f and g be any $(\in, \in \vee q_k)$-fuzzy two-sided ideals of intra-regular S, then by using Lemma 2.19, we have

$$\begin{aligned} f \circ_k g &= (f \circ g) \wedge \frac{1-k}{2} = ((S \circ f) \circ g) \wedge \frac{1-k}{2} = ((g \circ f) \circ S) \wedge \frac{1-k}{2} \\ &= (g \circ f) \wedge \frac{1-k}{2} = g \circ_k f. \end{aligned}$$

Therefore $f \circ_k g = g \circ_k f$. Now by Corollary 2.17, f and g are $(\in, \in \vee q_k)$-fuzzy semiprime.

$(iii) \Rightarrow (ii)$ Let A and B be any two-sided ideals of S, then by Lemma 2.9, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy two-sided ideals of S. Let $x \in (AB]$, then by Lemma 2.10, we have

$$\frac{1-k}{2} = (C_{(AB]})_k(x) = (C_A \circ_k C_B)(x) = (C_B \circ_k C_A)(x) = (C_{(BA]})_k(x).$$

This implies that $x \in (BA]$ and therefore $(AB] = (BA]$. Now by Corollary 2.13, A and B are semiprime.

$(ii) \Rightarrow (i)$ is simple. □

Theorem 2.21. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $f \circ_k (h \circ_k g) = g \circ_k (h \circ_k f)$, for any $(\in, \in \vee q_k)$-fuzzy two-sided ideals f and g of S such that they are $(\in, \in \vee q_k)$-fuzzy semiprime and any h that is any $(\in, \in \vee q_k)$-fuzzy right ideal of S.

Proof. $(i) \Rightarrow (ii)$ Let f and g be any $(\in, \in \vee q_k)$-fuzzy two-sided ideals of S and let h be any $(\in, \in \vee q_k)$-fuzzy right ideal of S with left identity. By Corollary 2, they are $(\in, \in \vee q_k)$-fuzzy semiprime. Then by using Lemma 2.19, we have

$$\begin{aligned} f \circ_k (h \circ_k g) = h \circ_k (f \circ_k g) &= (h \circ (f \circ g)) \wedge \frac{1-k}{2} = (h \circ ((f \circ g) \circ S)) \wedge \frac{1-k}{2} \\ &= (h \circ ((S \circ g) \circ f)) \wedge \frac{1-k}{2} = ((S \circ g) \circ (h \circ f)) \wedge \frac{1-k}{2} \\ &= g \circ_k (h \circ_k f). \end{aligned}$$

Thus $f \circ_k (h \circ_k g) = g \circ_k (h \circ_k f)$.

$(ii) \Rightarrow (i)$ By using Lemma 2.19 and given assumption, we have

$$\begin{aligned} f \circ_k g &= (f \circ g) \wedge \frac{1-k}{2} = (f \circ (S \circ g)) \wedge \frac{1-k}{2} = (g \circ (S \circ f)) \wedge \frac{1-k}{2} \\ &= g \circ f \wedge \frac{1-k}{2} = g \circ_k f \end{aligned}$$

for any f and g $(\in, \in \vee q_k)$-fuzzy two-sided ideals that are $(\in, \in \vee q_k)$-fuzzy semiprime. Now by using Theorem 2.20, S is intra-regular. □

Definition 2.22. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-*fuzzy idempotent* if $f^2 = f$.

Lemma 2.23. *[18] The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) Every $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is $(\in, \in \vee q_k)$-fuzzy idempotent.

Lemma 2.24. *[18] Let f be an $(\in, \in \vee q_k)$-fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S if and only if f satisfies the following conditions.*

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, for all x and y in S.

(ii) $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Let f be an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S, define the fuzzy subset f_k as follows: $f_k = f \wedge \frac{1-k}{2}$.

Theorem 2.25. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $f_k = (S \circ_k f)^2$, where f is any $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

Proof. $(i) \Rightarrow (ii)$ Let f be any $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S with left identity, then it is easy to see that $S \circ_k f$ is also an $(\in, \in \vee q_k)$- fuzzy two-sided ideal of S. Now by using Lemma 2.23, $S \circ_k f$ is $(\in, \in \vee q_k)$-fuzzy idempotent and therefore, we have

$$(S \circ_k f)^2 = (S \circ f)^2 \wedge \frac{1-k}{2} = (S \circ f) \wedge \frac{1-k}{2} \subseteq f \wedge \frac{1-k}{2} = f_k.$$

Now let $a \in S$. Since S is an intra-regular, there exist $b, c \in S$ such that $a \leq (ba^2)c$. Therefore

$$\begin{aligned} a &\leq (b(aa))c = (a(ba))c \leq (((ba^2)c)(ba))(ec) = (ce)((ba)((ba^2)c)) \\ &= (ba)((ce)((ba^2)c)) = (ba)(((ce)(b(aa)))c) = (ba)(((ce)(a(ba)))c) \\ &= (ba)((a((ce)(ba)))c) = (ba)((c((ce)(ba)))a) \\ &= (ba)(pa), \quad \text{where} \quad p = (c((ce)(ba))). \end{aligned}$$

Thus $(ba, pa) \in A_a$, whence $A_a \neq \emptyset$ and

$$
\begin{aligned}
(S \circ_k f)^2(a) &= \bigvee_{(x,y)\in A_a} \{(S \circ_k f)(x) \wedge (S \circ_k f)(y)\} \\
&\geq (S \circ f)(ba) \wedge (S \circ f)(pa) \wedge \frac{1-k}{2} \\
&= \bigvee_{(z,u)\in A_{ba}} \{S(z) \wedge S(u)\} \wedge \bigvee_{(v,w)\in A_{pa}} \{S(v) \wedge S(w)\} \wedge \frac{1-k}{2} \\
&\geq S(b) \wedge f(a) \wedge S(p) \wedge f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}
$$

Thus we get the required $f_k = (S \circ_k f)^2$.

$(ii) \Rightarrow (i)$ Suppose that $f_k = (S \circ_k f)^2$ holds for any $(\in, \in \vee q_k)$-fuzzy two-sided ideal f of S, then by using Lemma 2.24 and given assumption, we have

$$
\begin{aligned}
f_k = (S \circ_k f)^2 = (S \circ f)^2 \wedge \frac{1-k}{2} \subseteq f^2 \wedge \frac{1-k}{2} = (f \circ f) \wedge \frac{1-k}{2} \\
\subseteq (S \circ f) \wedge \frac{1-k}{2} \subseteq f \wedge \frac{1-k}{2} = f_k,
\end{aligned}
$$

which shows that $f_k = f \circ_k f$, therefore by using Lemma 2.23, S is intra-regular. □

Lemma 2.26. *[17] Let A be a subset of an intra-regular S with left identity, then A is a left ideal of S if and only if A is a right ideal of S.*

Theorem 2.27. *The following conditions are equivalent for S with left identity.*

(i) *S is intra-regular.*

(ii) *$R \cap L = (RL]$, where R is any right ideal and L is any left ideal of S such that R is semiprime.*

(iii) *$f \cap_k g = f \circ_k g$, where f is any $(\in, \in \vee q_k)$-fuzzy right ideal and g is any $(\in, \in \vee q_k)$-fuzzy left ideal of S such that f is $(\in, \in \vee q_k)$-fuzzy semiprime.*

Proof. $(i) \Rightarrow (iii)$ Let f be any $(\in, \in \vee q_k)$-fuzzy right ideal and g be any $(\in, \in \vee q_k)$-fuzzy left ideal of S with left identity. Now for $a \in S$ there exist $b, c \in S$ such that $a \leq (ba^2)c$. Therefore

$$
\begin{aligned}
a \leq (b(aa))c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c)))a = (c((ba^2)(bc)))a \\
= (c((cb)(a^2b)))a = (c(a^2((cb)b)))a = (a^2(c((cb)b)))a = pa.
\end{aligned}
$$

where $p = a^2(c((cb)b))$. Thus $(p, a) \in A_a$, whence $A_a \neq \emptyset$ and

$$\begin{aligned}(f \circ_k g)(a) &= (f \circ g)(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{(x,y)\in A_a} \{f(x) \wedge g(y)\} \wedge \frac{1-k}{2} \\ &\geq f(p) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \cap_k g)(a),\end{aligned}$$

which implies that $f \circ_k g \supseteq f \cap_k g$ and by using Lemma 2.24, $f \circ_k g \subseteq f \cap_k g$, therefore $f \cap_k g = f \circ_k g$.

(iii) $\Rightarrow$ *(ii)* Let R be any right ideal and L be any left ideal of S, then by Lemma 2.9, $(C_R)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left ideals of S respectively. By [8], it is obvious that $(RL] \subseteq R \cap L$. Now let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 2.9 and given assumption, we have

$$(C_{(RL]})_k(a) = (C_R \circ_k C_L)(a) = (C_R \cap_k C_L)(a) = C_R(a) \wedge_k C_L(a) = \frac{1-k}{2},$$

which implies that $a \in (RL]$ and therefore $R \cap L = (RL]$. Thus by using Corollary 2.13, R is semiprime.

(ii) $\Rightarrow$ *(i)* By [8], $(Sa]$ and $(a^2S]$ are left and right ideals of S with left identity such that $a \in (Sa]$ and $a^2 \in (a^2S]$. Since by assumption, $(a^2S]$ is semiprime, therefore $a \in (a^2S]$. Now by using Lemma 2.18, we have

$$\begin{aligned} a &\in (a^2S] \cap (Sa] = ((a^2S](Sa]] \subseteq ((a^2S)(Sa)] = ((aS)(Sa^2)] = (((Sa^2)S)a] \\ &= (((Sa^2)(eS))a] \subseteq (((Sa^2)(SS))a] = (((SS)(a^2S))a] = ((a^2((SS)S))a] \\ &\subseteq ((a^2S)S] = ((SS)(aa)] = ((aa)(SS)] \subseteq ((aa)S] = ((Sa)a] \subseteq ((Sa)(a^2S]] \\ &= ((Sa)(a^2S)] = (((a^2S)a)S] = (((aS)a^2)S] \subseteq ((Sa^2)S],\end{aligned}$$

which shows that S is intra-regular. □

By using Lemma 2.26, we have the following corollary.

Corollary 2.28. *The following conditions are equivalent for S with left identity.*

(i) *S is intra-regular.*

(ii) *$R = (R^2]$, where R is any right ideal of S such that R is semiprime.*

(iii) *$f_k = f \circ_k f$, where f is any $(\in, \in \vee q_k)$-fuzzy right ideal of S such that f is $(\in, \in \vee q_k)$-fuzzy semiprime.*

Lemma 2.29. *Let B be a non-empty subset of S, then B is a bi-ideal of S if and only if C_B is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof. It is simple. □

Theorem 2.30. *The following conditions are equivalent for S with left identity.*

(i) *S is intra-regular.*

(ii) *$f \cap_k g \subseteq g \circ_k f$, where both f and g are any $(\in, \in \vee q_k)$-fuzzy bi-ideals of S.*

(iii) *Every $(\in, \in \vee q_k)$-fuzzy bi-ideal of S is $(\in, \in \vee q_k)$-fuzzy idempotent.*

(iv) *Every bi-ideal of S is idempotent.*

Proof. $(i) \Rightarrow (ii)$ Let f and g be both $(\in, \in \vee q_k)$-fuzzy bi-ideals of intra-regular S with left identity. Then for every $a \in S$, there exist $b, c \in S$ such that $a \leq (ba^2)c$.

$$\begin{aligned} a &\leq (ba^2)c = (b(aa))c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c)))a \\ &= (c(b((b(aa))c)))a = (c(b((a(ba))c)))a = (c(a((ba))(bc)))a \\ &= ((a(ba))(c(bc)))a = (((c(bc))(ba))a)a \leq (((c(bc))(b((ba^2)c)))a)a \\ &= (((c(bc))((ba^2)(bc)))a)a = (((ba^2)((c(bc))(bc)))a)a \\ &= ((((bc)(c(bc)))(a^2b))a)a = ((a^2(((bc)(c(bc)))b))a)a \\ &= (((aa)(((bc)(c(bc)))b))a)a = (((b((bc)(c(bc))))(aa))a)a \\ &= ((a((b((bc)(c(bc))))a))a)a = (pa)a \end{aligned}$$

where $p = a((b((bc)(c(bc))))a)$. Thus $(pa, a) \in A_a$, whence $A_a \neq \emptyset$ and

$$\begin{aligned} (g \circ_k f)(a) &= (g \circ f)(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{(x,y) \in A_a} \{g(x) \wedge f(y)\} \wedge \frac{1-k}{2} \\ &\geq g(p) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq g(a) \wedge f(a) \wedge \frac{1-k}{2} = (g \cap_k f)(a). \end{aligned}$$

This implies

$$g \circ_k f \supseteq g \cap_k f = f \cap_k g \Rightarrow f \cap_k g \subseteq g \circ_k f.$$

$(ii) \Rightarrow (iii)$ Since f is any $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, therefore

$$f \cap_k f \subseteq f \circ_k f \subseteq f_k \Rightarrow f_k = f \circ_k f.$$

This implies that f is idempotent.

$(iii) \Rightarrow (iv)$ Let B be a bi-ideal of S such that $b \in B$, then by using Lemma 2.29, $(C_B)_k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Now by using Lemma 2.18, we have

$$\frac{1-k}{2} = (C_B)_k(b) = (C_B \circ_k C_B)(b) = (C_{(B^2]})_k(b),$$

which implies that $b \in (B^2]$, therefore $B \subseteq (B^2]$. Since $(B^2] \subseteq B$ is obvious, then $B = (B^2]$.

$(iv) \Rightarrow (i)$ Clearly $(Sa]$ is a bi-ideal of S with left identity, therefore by using given assumption and Lemma 2.18, we have

$$\begin{aligned} a \in (Sa] &= ((Sa](Sa]] = ((Sa)(Sa)] = ((aS)(aS)] = ((aa)(SS)] \\ &= ((ea^2)(SS)] \subseteq ((Sa^2)S]. \end{aligned}$$

Therefore S is intra-regular. □

Corollary 2.31. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $f_k \subseteq f \circ_k f$, for all $(\in, \in \vee q_k)$-fuzzy bi-ideal f of S.

(iii) Every $(\in, \in \vee q_k)$-fuzzy bi-ideal f of S is $(\in, \in \vee q_k)$-fuzzy idempotent.

(iv) Every bi-ideal B of S is idempotent.

Theorem 2.32. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $A \cap B \subseteq (BA]$, where both A and B are any left ideals of S.

(iii) $f \cap_k g \subseteq g \circ_k f$, where both f and g are any $(\in, \in \vee q_k)$-fuzzy left ideals of S.

Proof. $(i) \Rightarrow (iii)$ Let f and g be both $(\in, \in \vee q_k)$-fuzzy left ideals of an intra-regular S with left identity. Now for any $a \in S$, there exist $b, c \in S$ such that $a \leq (ba^2)c$, then

$$a \leq (ba^2)c = (b(aa))c = (a(ba))c = (c(ba))a.$$

Thus $(c(ba), a) \in A_a$, whence $A_a \neq \emptyset$ and

$$\begin{aligned} (g \circ_k f)(a) &= (g \circ f)(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{(x,y) \in A_a} \{g(x) \wedge f(y)\} \wedge \frac{1-k}{2} \\ &\geq g(p) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq g(a) \wedge f(a) \wedge \frac{1-k}{2} = (g \cap_k f)(a), \end{aligned}$$

which implies that $g \circ_k f \supseteq f \cap_k g$.

$(iii) \Rightarrow (ii)$ Let A and B be any left ideals of S, then by Lemma 2.9, $(C_A)_k$ and $(C_B)_k$ are any $(\in, \in \vee q_k)$-fuzzy left ideals of S. Let $x \in A \cap B$, then by using Lemma 2.10, we have

$$1 = (C_{A\cap B})_k(x) = (C_A \cap_k C_B)(x) \leq (C_B \circ_k C_A)(x) = (C_{(BA]})_k(x),$$

which implies that $a \in (BA]$ and therefore $A \cap B \subseteq (BA]$.

$(ii) \Rightarrow (i)$ Since $(Sa]$ is a left ideal of S with left identity [8] such that $a \in (Sa]$, therefore by using given assumption and Lemma 2.18, we have

$$\begin{aligned} a &\in (Sa] \cap (Sa] \subseteq ((Sa](Sa]] = ((Sa)(Sa)] = ((aS)(aS)] \\ &= ((aa)(SS)] = ((ea^2)(SS)] \subseteq ((Sa^2)S], \end{aligned}$$

hence S is intra-regular. □

Theorem 2.33. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) Every two-sided ideal of S is semiprime.

(iii) Every bi-ideal of S is semiprime.

(iv) Every $(\in, \in \vee q_k)$-fuzzy bi-ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

(v) Every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

(vi) $f_k(a) = f_k(a^2)$, for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f of S and for all $a \in S$.

(vii) $f_k(a) = f_k(a^2)$, for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f of S and for all $a \in S$.

Proof. $(i) \Rightarrow (vii)$ Let S be intra-regular and f be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S with left identity. Let $a \in S$, then there exist $b, c \in S$ such that $a \leq (ba^2)c$, therefore

$$\begin{aligned} a &\leq (ba^2)c = (b(aa))c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c))a = (b(c((ba^2)c)))a \\ &= (b((ba^2)c^2))a = ((ba^2)(bc^2))a = (b^2(a^2c^2))a = (a^2(b^2c^2))a = (a(b^2c^2))a^2 \\ &\leq (((ba^2)c)(b^2c^2))a^2 = ((c^2c)(b^2(ba^2)))a^2 = ((c^2b^2)(c(ba^2)))a^2 \\ &\leq ((c^2b^2)((uv)(ba^2)))a^2 = ((c^2b^2)((a^2v)(bu)))a^2 = ((c^2b^2)((a^2b)(vu)))a^2 \\ &= ((((c^2b^2)(vu))b)(aa)))a^2 = ((ab)(a((c^2b^2)(vu))))a^2 \\ &= ((aa)(b((b^2c^2)(vu))))a^2 = (a^2((b((b^2c^2)(vu))))a^2, \end{aligned}$$

thus, we have

$$f_k(a) \geq f_k((a^2((b((b^2c^2)(vu))))a^2) \geq f(a^2) \wedge f(a^2) \wedge \frac{1-k}{2} = f(a^2) \wedge \frac{1-k}{2} = f_k(a^2),$$

and

$$\begin{aligned} a^2 = aa &\leq ((ba^2)c)((ba^2)c) = ((ba^2)(ba^2))(cc) = ((bb)(a^2a^2))c^2 = (b^2(a^2)^2)c^2 \\ &= (b^2(a^2)^2)c^2 = (b^2(a^2a^2))c^2 = (a^2(b^2a^2))c^2 = (c^2(b^2a^2))a^2 \\ &= (c^2(b^2a^2))(aa) = (a(b^2a^2))(ac^2) = (aa)((b^2a^2)c^2) \\ &= (((b^2a^2)c^2)a)a = (((c^2a^2)b^2)a)a = (((c^2(aa))b^2)a)a \\ &= (((a(c^2a))b^2)a)a = ((ab^2)(a(c^2a)))a = (a^2(b^2(c^2a)))a \\ &= (a^2(c^2(b^2a))a = ((aa)(c^2(b^2a)))a \\ &= (((b^2a)c^2)(aa))a = (a(((b^2a)c)a))a, \end{aligned}$$

therefore $f_k(a^2) \geq f_k(a(((b^2a)c)a))a \geq f(a) \wedge f(a) \wedge \frac{1-k}{2} = f_k(a)$. Hence $f_k(a^2) = f_k(a)$.

$(vii) \Rightarrow (vi)$, $(vii) \Rightarrow (v)$ and $(i) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (iii)$ Let $(B]$ be a bi-ideal of S, then by Lemma 2.29, $C_{(B]}$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Let $a^2 \in (B]$, then $C_{(B]}(a) \geq C_{(B]}(a^2)$, therefore $C_{(B]}(a^2) = 1 \leq C_{(B]}(a)$, this implies $C_{(B]}(a) = 1$. Thus $a \in (B]$ and therefore $(B]$ is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Since $(Sa^2]$ is a two-sided ideal of S with left identity containing a^2 [8], thus by using given assumption and Lemma 2.18, we have

$$a \in (Sa^2] = ((SS)a^2] = ((a^2S)S] = (((aa)(SS))S] = (((SS)(aa))S] = ((Sa^2)S].$$

Therefore S is intra-regular. □

Theorem 2.34. *The following conditions are equivalent for S with left identity.*

(i) *S is intra-regular.*

(ii) *$L \cap B \subseteq ((LB]L]$, where L is a left ideal and B is a bi-ideal of S.*

(iii) *$f \cap g \subseteq (f \circ_k g) \circ_k f$, where f is an $(\in, \in \vee q_k)$-fuzzy left ideal and g is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in, \in \vee q_k)$-fuzzy left ideal and g be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of intra-regular S with left identity. Then for any $a \in S$, there exist $b, c \in S$ such that $a \leq (ba^2)c$. Thus

$$\begin{aligned} a &\leq (ba^2)c = (a(ba))c = (c(ba))a \leq (c(b((ba^2)c))a = (b(c((a(ba)c)))a \\ &= (b(a(ba))c^2)a = ((a(ba))(bc^2))a = (((bc^2)(ba))a)a, \end{aligned}$$

which implies that $(((bc^2)(ba))a, a) \in A_a$, whence $A_a \neq \emptyset$ and

$$\begin{aligned}((f \circ_k g) \circ_k f)(a) &= \bigvee_{(x,y)\in A_a} \{(f \circ g)(x) \wedge f(y)\} \wedge \frac{1-k}{2} \\ &\geq f((bc^2)(ba)) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq f(ba) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \cap_k g)(a).\end{aligned}$$

Hence $((f \circ_k g) \circ_k f) \supseteq f \cap_k g$.

$(iii) \Rightarrow (ii)$ Let L be a left ideal and B be a bi-ideal of S. Let $x \in L \cap B$, then

$$\begin{aligned}1 = C_{L\cap B}(x) &= (C_L \cap C_B)(x) \leq ((C_L \circ C_B) \circ C_L)(x) \\ &= ((C_{(LB]}) \circ C_L)(x) = (C_{((LB]L]})(x),\end{aligned}$$

which implies $(C_{((LB]L]})(a) = 1$. Thus $a \in ((LB]L]$ and therefore $L \cap B \subseteq ((LB]L]$.

$(ii) \Rightarrow (i)$ Since $(Sa]$ is both left and bi-ideal of S containing a, therefore by using given assumption and Lemma 2.18, we have

$$a \in (Sa] \cap (Sa] = (((Sa](Sa]](Sa]] = (((Sa)(Sa)](Sa]] = (((SS)(aa)](Sa]] \subseteq ((Sa^2)S].$$

Therefore S is intra-regular. □

Theorem 2.35. *The following conditions are equivalent for S with left identity.*

(i) S is intra-regular.

(ii) $f \cap_k g \cap_k h \subseteq (f \circ_k g) \circ_k (f \circ_k h)$, where f is an $(\in, \in \vee q_k)$-fuzzy left ideal, h is an $(\in, \in \vee q_k)$-fuzzy right ideal and g is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Proof. $(i) \Rightarrow (ii)$ Let f, g and h be any $(\in, \in \vee q_k)$-fuzzy left, right and bi-ideals of an intra-regular S with left identity respectively. Now for any $a \in S$, there exist $b, c \in S$ such that $a \leq (ba^2)c$, therefore

$$\begin{aligned}a \leq (ba^2)c &= (a(ba))c = (c(ba))a \leq (c(b((ba^2)c))a = (c(b(a(ba))c))a \\ &= (c((a(ba))(bc)))a = ((a(ba))(c(bc)))a = (((c(bc))(ba))a)a \\ &\leq (((c(bc))(ba))a)((ba^2)c) = (((c(bc))(ba))a)((b(aa))c) \\ &= (((c(bc))(ba))a)((a(ba))c) = (((c(bc))(ba))a)((c(ba))a),\end{aligned}$$

which implies that $(((c(bc))(ba))a, (c(ba))a) \in A_a$, whence $A_a \neq \emptyset$ and

$$\begin{aligned}
&((f \circ_k g) \circ_k (f \circ_k h))(a) \\
&= \bigvee_{(x,y)\in A_a} \{(f \circ_k g)(x) \wedge (f \circ_k h)(y)\} \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge f(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} = (f \cap_k g \cap_k h)(a),
\end{aligned}$$

which shows that $f \cap_k g \cap_k h \subseteq (f \circ_k g) \circ_k (f \circ_k h)$.

$(ii) \Rightarrow (i)$ Since S is an $(\in, \in \vee q_k)$-fuzzy right ideal of itself, therefore

$$f \cap_k g = f \cap_k g \cap_k S \subseteq (f \circ_k g) \circ_k (f \circ_k S) \subseteq (f \circ_k g) \circ_k f.$$

Thus $f \cap_k g \subseteq (f \circ_k g) \circ_k f$. Hence by using Theorem 2.34, S is intra-regular. □

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