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Weighted Composition Operators on Multidimensional Lorentz Spaces and a Glimpse on Multipliers Between Bounded p-variation Spaces

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Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. “Immortality” may be a silly word, but probably a mathematician has the best chance of whatever it may mean.

G. H. Hardy

Acknowledgments

First of all, I want to thank my advisor, Professor René Erlín Castillo, for his guidance and encouragement during my studies. He has always been willing to discuss about mathematics with me, and he has offered me not only his strong academic support but also his friendship, which I greatly appreciate.
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Bogotá, September 2017.

Abstract

In this thesis we study the multidimensional Lorentz spaces via the two-dimensional decreasing rearrangement. In particular, results of interpolation, quasinormability and completeness are stablished, and weights that define a norm are characterized.
The boundedness, compactness and closed range of the weighted composition operator defined on those spaces are also characterized.
Finally, we present the Bounded p-variation spaces, and then we characterize the set of multipliers between them.

Keywords: Decreasing rearrangement, multidimensional rearrangement, multiplication operator, multipliers, composition operator, compact operator, Lorentz spaces, Bounded variation spaces.

Resumen

En esta tesis se estudian los espacios de Lorentz multidimensionales via el reordenamiento decreciente bidimensional. En particular, se establecen resultados de interpolación, cuasi-normabilidad y completitud, y se caracterizan los pesos que definen una norma.
La acotación, compacidad y rango cerrado del operador composición con peso definido en esos espacios también son caracterizados.
Finalmente, se presentan los espacios de p-variación acotada, y se caracteriza el conjunto de multiplicadores entre ellos.

Palabras clave: Reordenamiento decreciente, reordenamiento multidimensional, operador multiplicación, multiplicadores, operador composición, operador compacto, espacios de Lorentz, espacios de variación acotada.

Conventions and notations

$\mathbb{R}, \mathbb{Z}, \mathbb{N}$ stand, respectively, for the field of real numbers, the group of integers and the semigroup of natural numbers.
If $n \in \mathbb{N} \setminus \{0\}$, we denote $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\}$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.
For a Lebesgue measurable set E of $\mathbb{R}^n$, χ_E denotes the characteristic function of the set E and $m_n(E)$ denotes the Lebesgue measure of E. The abbreviation a.e. stands for almost everywhere. In addition, all functions are assumed to be measurable.

Contents

1 Introduction

1.1 Some historical background and basic definitions

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $\lambda \geq 0$, $D_f(\lambda)$, the distribution function of f, is defined as

$$D_f(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right). \tag{1.1}$$

Observe that D_f depends only on the absolute value $|f|$ of the function f and D_f may take the value $+\infty$.
The distribution function D_f provides information about the size of f, but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from (1.1) that D_f is a decreasing function of λ (not strictly necessarily) and continuous from the right. For more details on distribution function see [14, 26, 30].
By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \leq t\}, \quad t \geq 0, \tag{1.2}$$

where we use the convention that $\inf \emptyset = \infty$. f^* is decreasing and right-continuous. Notice that

$$f^*(0) = \inf\{\lambda > 0 : D_f(\lambda) \leq 0\} = \|f\|_\infty,$$

since

$$\|f\|_\infty = \inf\{\alpha \geq 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

Also observe that if D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \leq D_f(t)\} = t.$$

This fact demonstrates that f^* is the inverse function of the distribution function D_f. Let $\mathcal{F}(X, \mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions on X.
Let $(X, \mathcal{A}_0, \mu)$ and $(Y, \mathcal{A}_1, \nu)$ be two measure spaces. Two functions $f \in \mathcal{F}(X, \mathcal{A}_0)$ and $g \in \mathcal{F}(X, \mathcal{A}_1)$ are said to be equimeasurable if they have the same distribution function, that is, if

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) = \nu\left(\{y \in Y : |g(y)| > \lambda\}\right), \quad \text{for all } \lambda \geq 0. \tag{1.3}$$

So then there exists only one right-continuous decreasing function f^* equimeasurable with f. Hence the decreasing rearrangement is unique.
Decreasing rearrangements of functions were introduced by Hardy and Littlewood [33]; the authors attribute their motivation to understanding cricket averages.
One of the most important properties of f^* is that

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} = \left(\int_0^\infty (f^*(t))^p \, dt \right)^{1/p},$$

which is obtained from the fact that f and f^* are equimeasurable. This allows us to study L_p spaces via decreasing reordering. In this way, the Lorentz spaces $\Lambda^p(w)$ are the spaces of all functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^\infty (f^*(t))^p \, w(t) \, dt \right)^{1/p}$$

is finite. Here w is a weight in $\mathbb{R}^+$ and $0 < p < \infty$.
Lorentz spaces were introduced by G. G. Lorentz in [42, 43] as a generalization of classical Lebesgue spaces L_p, and have become a standard tool in mathematical analysis, cf. [4, 14, 16, 19, 20, 25, 23, 24, 30].
The spaces $L^{p,q}$ are defined to be $\Lambda^p(w)$ with $w(t) = \frac{q}{p} t^{q/p-1}$.
In [35], Hunt did a general treatment of the $L^{p,q}$ spaces. Elementary properties, topological properties, interpolation theorems and some applications were studied there. The $L^{p,q}$ spaces play a central role in the study of Banach function spaces. Oftentimes, the methods used to investigate the $L^{p,q}$ spaces are useful for obtaining results for more generalized Banach function spaces. And results for the $L^{p,q}$ spaces often have natural analogues in the more generalized settings.

1.2 Multiplication, Composition and Weighted Composition Operators

1.2.1 Multiplication Operator

If we denote (Ω, Σ, μ) for a σ-finite and complete measure space and $L_0(\Omega)$ is the linear space of all classes of Σ-measurable functions on Ω, then every function $u : \Omega \to \mathbb{R}$ measurable on Ω allows us to define a linear transformation which assigns to every $f \in L_0(\Omega)$, the function $M_u(f) \in L_0(\Omega)$ defined by

$$M_u(f)(t) = u(t) \cdot f(t), t \in \Omega, f \in L_0(\Omega). \tag{1.4}$$

In the case in which normed and complete subspaces of $L_0(\Omega)$ are considered as the domain of M_u, this operator will be called multiplication operator induced by the symbol u. These operators have received considerable attention in the last years, specially in the L_p spaces and play an important role in the study of operators in Hilbert spaces.

Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L_2 space (see [31]).

The basic properties of the multiplication operator on spaces of measurable functions have been studied by many mathematicians. Among them we can name Abrahamese [1] (1978), Halmos [32] (1961), Axler [8] (1982), Takagi [50] (1993), Takagi and Yokouchi [51] (1999), Komal and Gupta [39] (2001), Arora, Datt and Verma [5] (2006), Castillo, León and Trousselot [25] (2009), Douglas [28] (2012), among others. Notably, Castillo, Ramos and Salas in [23] (2014), studied the properties of the multiplication operator M_u in Köthe spaces. The problems studied about the multiplication operator on those spaces are the following:

What are the properties required on the symbol u for the multiplication operator $M_u : X \to Y$, with X and Y Banach subspaces of $L_0(\Omega)$ to be continuous, compact, Fredholm, and have finite or closed range?

It is also of some interest to try to give a formula of the essential norm of M_u in terms of the symbol u.

1.2.2 Composition Operators

Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space and let $T : X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$.

If $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathcal{A}$ with $\mu(A) = 0$, then T is said to be nonsingular. This condition means that the measure $\mu \circ T^{-1}$, defined by $\mu \circ T^{-1}(A) = \mu\left(T^{-1}(A)\right)$ for $A \in \mathcal{A}$ is absolutely continuous with respect to μ (this is usually denoted $\mu \circ T^{-1} \ll \mu$). Then the Radon-Nikodym theorem ensures the existence of a non-negative locally integrable function f_T on X such that

$$\mu \circ T^{-1}(A) = \int_A f_T \, d\mu \quad \text{for } A \in \mathcal{A}.$$

Any measurable nonsingular transformation T induces a linear operator (composition operator) C_T from $\mathfrak{F}(X, \mathcal{A}, \mu)$ into itself defined by

$$C_T(f)(x) = f\left(T(x)\right), x \in X, f \in \mathfrak{F}(X, \mathcal{A}, \mu),$$

where $\mathfrak{F}(X, \mathcal{A}, \mu)$ denotes the linear space of all equivalence classes of $\mathcal{A}$-measurable functions on X, where we identify any two functions that are equal μ-almost everywhere on X.

Here the nonsingularty of T guarantees that the operator C_T is well defined as a mapping of equivalence classes of functions into itself since $f = g$ μ-a.e. implies $C_T(f) = C_T(g)$ μ-a.e.

The first appearance of a composition transformation was in 1871 in a paper of Schrljeder, where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z),$$

for every z, in a suitable domain, if the function T is given. A solution was given by Koenigs in 1884. In 1925, this operators were employed in the Littlewood subordination theory. In early 1931 Koopman used the composition operators to study problems of mathematical physics, specially classical mechanics. In those days, these operators were known as substitutes operators. The systematic study of composition operators was initiated by Nordgren in 1968. After that, the study of composition operators has been extended in several directions by many mathematicians. For more details about composition operators in spaces of measurable functions, see Singh and Kumar [48] (1977), Kumar [40] (1980), Komal and Pathania [38] (1991), Takagi and Yokouchi [51] (1999), Cui, Hudzik, Kumar and Maligranda [27] (2004), Arora, Datt and Verma [6] (2007), among others.
In recent years, R.E. Castillo and other authors have done studies on spaces of functions and operator theory, as is shown in [21, 22, 23], in which they have studied some properties of multiplication and composition operator on Bloch spaces and Köthe spaces. In [24], R.E. Castillo, F. Vallejo and J.C. Ramos-Fernández did a remarkable study of the multiplication and composition operators in Weak L_p spaces. In [19] we studied the composition operator in Orlicz-Lorentz spaces. In [20] we studied the multiplication operator in Orlicz-Lorentz spaces.

1.2.3 Weighted Composition Operator

Now we talk about a more general operator which encapsulates the classical multiplication and composition operators.
Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $T : X \to X$ be a measurable transformation (i.e. $T^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$) and non-singular (i.e. $\mu(T^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ with $\mu(A) = 0$, which means that μT^{-1} is absolutely continuous with respect to μ ($\mu T^{-1} \ll \mu$)) and $\mu : X \to \mathbb{C}$ be a measurable function. The linear transformation $W_{u,T}$ is defined as follows:

$$\begin{aligned} W_{u,T} : \mathcal{F}(X, \mathcal{A}) &\to \mathcal{F}(X, \mathcal{A}) \\ f &\mapsto W_{u,T}(f) = u \circ T \cdot f \circ T, \end{aligned}$$

where

$$\begin{aligned} W_{u,T} : X &\to \mathbb{C} \\ x &\mapsto (W_{u,T}(f))(x) = u(T(x)) \cdot f(T(x)). \end{aligned}$$

If the operator $W_{u,T}$ is bounded and has range in $\Lambda_2^p(w)$, then it is called the weighted composition operator on $\Lambda_2^p(w)$.

The reader may note that this operator generalize the multiplication and composition operators defined previously, as is shown below:

1. If $u = 1$, then $W_{u,T} = W_{1,T} = C_T : f \mapsto f \circ T$ is called the composition operator induced by T.

2. If $T = I_X$, identity on X, then $W_{u,T} = W_{u,I_X} = M_u : f \mapsto u \cdot f$ is called the multiplication operator induced by u.

We will discuss more about this operator in Chapter 3.

1.3 Multidimensional Lorentz Spaces

Since many operations with functions defined on function spaces are iterative, C. J. Neugebauer suggested that it should be possible to obtain multivariate rearrangements by such a process. For simplicity, we are going to reduce the definitions to the two-dimensional case (the definitions for higher dimensions are analogous). Basically, the multidimensional rearrangement can be obtained as an iterative process. More precisely, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function and we take $f_x(y) = f(x, y)$, then the two-dimensional rearrangement of f may be obtained in the following way

$$\tilde{f}(s,t) = \left(f_x^*(\cdot, t)\right)_y^*(s).$$

That is, we first rearrange with respect to y (keeping x fixed) to obtain a function which depends on x and t. Then, this new function is rearranged with respect to x (keeping t fixed) to finally obtain the function $\tilde{f}$. The order in which the reordering takes place is very important, because, in general, we do not get the same function if we first rearrange with respect to x and then with respect to y, we show this in Example 2.1.28. This is a huge difference with respect to the classical one-variable decreasing rearrangement defined in (1.2), which is unique. See [15] for some related work.

In [12] there is another way to obtain the multidimensional rearrangement. There, the authors define the decreasing rearrangement E^* of a set E and use this and the layer cake formula to define the multidimensional rearrangement of a function f as

$$f_2^*(s,t) = \int_0^\infty \chi_{\{|f|>\lambda\}^*}(s,t)\, d\lambda.$$

Although, at first, those definitions look different, it is impressive that they lead to the same result, i.e., $f_2^* = \tilde{f}$.

The two-dimensional Lorentz space $\Lambda_2^p(w)$ is the space of all functions f for which the norm

$$\|f\|_{\Lambda_2^p(w)} := \left(\int_{\mathbb{R}_+^2} (f_2^*(x))^p \, w(x)\, dx \right)^{1/p}$$

is finite. Here w is a nonnegative, locally integrable function on $\mathbb{R}^2_+$, not identically 0.
One of the reasons to study the space $\Lambda_2^p(w)$ is that it is the standard space to consider multidimensional analogs of classical inequalities: Hardy's inequality, Chebyshev's inequality, embeddings for weighted Lorentz spaces, etc. (see [4], [9], [11],[13], [17], [47], [49].)
The aim of this thesis is to study the compactness, boundedness and closed range of the weighted composition operator defined on the space $\Lambda_2^p(w)$.
This thesis is organized as follows. In Chapter 2 we state the basic theory of the Multidimensional Lorentz Spaces. In Chapter 3 we study properties (boundedness, compactness, closed range) of the Weighted Composition Operator defined on the Multidimensional Lorentz Space. Chapter 4 is devoted to the boundedness of the Multiplication Operator defined on Bounded p-variation Spaces.
This thesis is written as a monograph, based on the following papers:

(a) R. E. Castillo and H. C. Chaparro, *Weighted Composition Operator on Two-Dimensional Lorentz Spaces*, Math. Inequal. Appl. **20** (2017), no 3, 773-799.

(b) H. C. Chaparro, *On Multipliers between Bounded Variation Spaces*, Ann. Funct. Anal., to appear.

2 Multidimensional Lorentz spaces

Since one the objectives of this thesis is to study the behavior of certain type of operators acting on multidimensional Lorentz spaces, it is fair enough to devote the present chapter to the study of important properties about those spaces. The theory about multidimensional Lorentz spaces was developed in [9, chapter 5] and also in [12]. However, for the sake of completeness and convenience of the reader, we give here some definitions and results that may be found in the above references. We present here the calculations with great detail, and we also give some new results. Besides, we include some examples and graphics to illustrate some of the concepts.

2.1 Two-dimensional decreasing rearrangement

Definition 2.1.1. We say that a set $D \subset \mathbb{R}^2_+$ is decreasing (and denote that with $D \in \Delta_d$) if the function χ_D is decreasing on each variable.

Example 2.1.2. *If f is decreasing on $(0, b]$, then the set*

$$D = \{(x, y) : y \leq f(x)\}$$

*is a decreasing set (see Figure **2-1**.) Fix y_0 and take $x_1, x_2 \in [0, b)$ with $x_1 \leq x_2$. We have to show that*

$$\chi_D(x_1, y_0) \geq \chi_D(x_2, y_0).$$

Indeed,

$$\chi_D(x_2, y_0) = \begin{cases} 1, & \text{if } (x_2, y_0) \in D \\ 0, & \text{if } (x_2, y_0) \notin D \end{cases} = \begin{cases} 1, & \text{if } y_0 \leq f(x_2) \\ 0, & \text{if } y_0 > f(x_2) \end{cases}$$

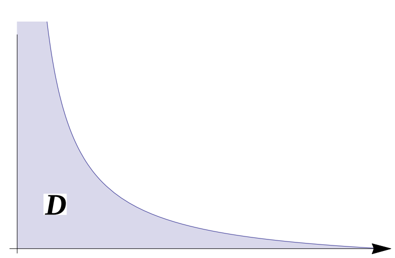

Figure 2-1: The graph of a decreasing set.

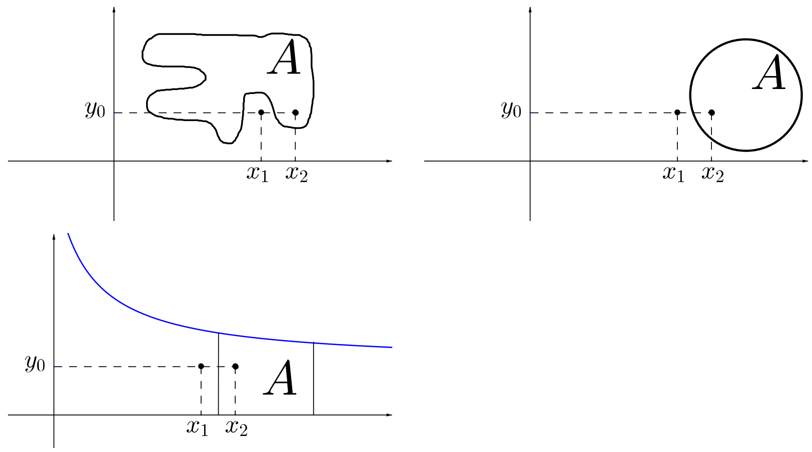

Figure 2-2: The graphs of some non-decreasing sets.

- *Case 1: If $\chi_D(x_2, y_0) = 0$, there's nothing to prove.*
- *Case 2: If $\chi_D(x_2, y_0) = 1$, then $(x_2, y_0) \in D$, it means that $y_0 \leq f(x_2)$. Since f is decreasing, $f(x_2) \leq f(x_1)$. In this way $(x_1, y_0) \in D$ and then $\chi_D(x_1, y_0) = 1$. So $\chi_D(x_2, y_0) \leq \chi_D(x_1, y_0)$.*

We have shown that χ_D is decreasing in the first variable. A similar argument (interchanging x and y) shows that f is decreasing in the second variable.

Example 2.1.3. *The sets shown in Figure **2-2** are not decreasing sets. In all cases $x_1 \leq x_2$ but $\chi_A(x_1, y_0) = 0 < 1 = \chi_A(x_2, y_0)$, so χ_A is not decreasing in the first variable.*

Definition 2.1.4. Let $E \subset \mathbb{R}^2$ and $\varphi_E(x) = m_1(E_x) = m_1(\{y \in \mathbb{R} : (x, y) \in E\})$, $x \in \mathbb{R}$. Let the function φ_E^*, defined by

$$\begin{aligned} \varphi_E^*(s) &= \inf\{\lambda : m_1(\{x \in \mathbb{R} : \varphi_E(x) > \lambda\}) \leq s\} \\ &= \inf\{\lambda : D_{\varphi_E}(\lambda) \leq s\} \quad ((s \geq 0)). \end{aligned}$$

φ_E^* is the usual decreasing rearrangement of φ_E (see [14, p. 39]). Then, the two dimensional decreasing rearrangement of the set E is

$$E^* = \{(s, t) \in \mathbb{R}_+^2 : 0 < t < \varphi_E^*(s)\}.$$

Example 2.1.5. *Let $E = [a, b] \times [c, d]$. We are going to calculate E^*. We have that*

$$\begin{aligned} \varphi_E(x) &= m_1(E_x) \\ &= m_1(\{y \in \mathbb{R} : (x, y) \in E\}) \\ &= m_1(\{y \in \mathbb{R} : (x, y) \in [a, b] \times [c, d]\}) \end{aligned}$$

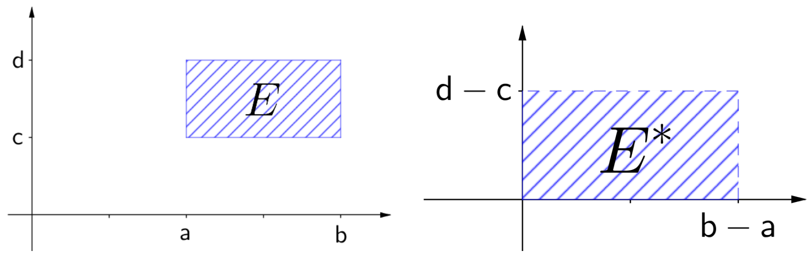

Figure 2-3: The graphs of E and E^* in Example 2.1.5.

$$= (d-c)\chi_{[a,b]}(x).$$

Then

$$\varphi_E^*(s) = (d-c)\chi_{[0,b-a)}(s).$$

So

$$\begin{aligned} E^* &= \{(s,t) \in \mathbb{R}_+^2 : 0 < t < \varphi_E^*(s)\} \\ &= \{(s,t) \in \mathbb{R}_+^2 : 0 < t < (d-c)\chi_{[0,b-a)}(s)\} \\ &= [0, b-a) \times (0, d-c). \end{aligned}$$

Example 2.1.6. *Take*

$$E = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\} = \{(x,y) \in \mathbb{R}^2 : -\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2}\}.$$

Then

$$\begin{aligned} \varphi_E(x) &= m_1\left(\{y \in \mathbb{R} : -\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2}\}\right) \\ &= \begin{cases} \sqrt{(\sqrt{r^2 - x^2} + \sqrt{r^2 - x^2})^2}, & \text{if } -r \le x \le r \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 2\sqrt{r^2 - x^2}, & \text{if } -r \le x \le r \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

After some routine calculation, we see that

$$D_{\varphi_E}(\lambda) = \begin{cases} 2\sqrt{r^2 - \frac{\lambda^2}{4}}, & 0 \le \lambda \le 2r \\ 0, & \lambda \ge 2. \end{cases}$$

and

$$\varphi_E^*(s) = \begin{cases} \sqrt{4r^2 - s^2}, & 0 \le s \le 2r \\ 0, & s \ge 2. \end{cases}$$

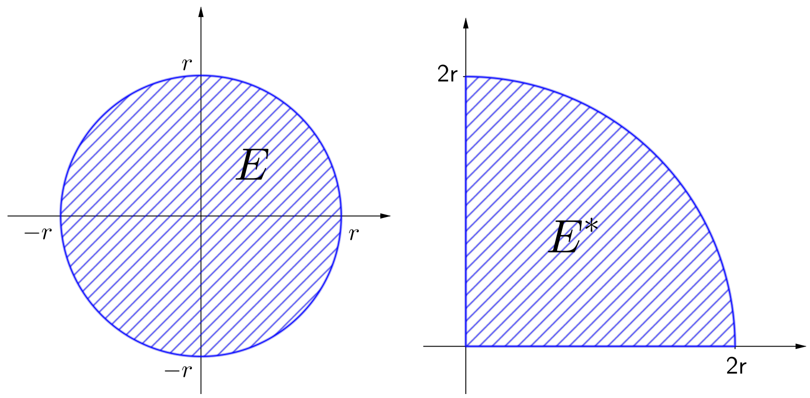

Figure 2-4: The graphs of E and E^* in Example 2.1.6.

This way

$$\begin{aligned} E^* &= \{(s,t) \in \mathbb{R}^2_+ : 0 < t < \varphi^*_E(s)\} \\ &= \{(s,t) \in \mathbb{R}^2_+ : 0 < t < \sqrt{4r^2 - s^2}, 0 < s < 2r\}. \end{aligned}$$

The following definition is inspired by the so-called Layer Cake Formula (see [41, p. 26]), which states that one can recover the value of a function f by integrating the characteristic function of the level sets of f.

Definition 2.1.7 (Layer Cake Formula)**.** The two dimensional decreasing rearrangement f_2^* for a function f on $\mathbb{R}^2$ is given by

$$f_2^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)\, dt, \quad x \in \mathbb{R}^2_+.$$

Example 2.1.8. *Let's calculate f_2^* for $f(x,y) = k\chi_{[a,b]}(x)\chi_{[c,d]}(y)$, $k \in \mathbb{R}$. In this case*

$$\begin{aligned} E &= \{(x,y) \in \mathbb{R}^2 : |f(x,y)| > t\} \\ &= \{(x,y) \in [a,b] \times [c,d] : |k\chi_{[a,b]}(x)\chi_{[c,d]}(y)| > t\}. \end{aligned}$$

Also,

$$\begin{aligned} \varphi_E(x) &= m_1 \left(\{y \in \mathbb{R} : (x,y) \in E\}\right) \\ &= m_1 \left(\{y \in \mathbb{R} : |k\chi_{[a,b]}(x)\chi_{[c,d]}(y)| > t\}\right) \\ &= m_1 \left(\{y \in [c,d] : |k\chi_{[a,b]}(x)| > t\}\right) \\ &= m_1 \left(\{y \in [c,d] : |k| > t\}\right) \chi_{[a,b]}(x) \\ &= (d-c)\chi_{(0,|k|)}(t)\chi_{[a,b]}(x). \end{aligned}$$

Thus,

$$\varphi^*_E(s) = (d-c)\chi_{(0,|k|)}(t)\chi_{[0,b-a)}(s).$$

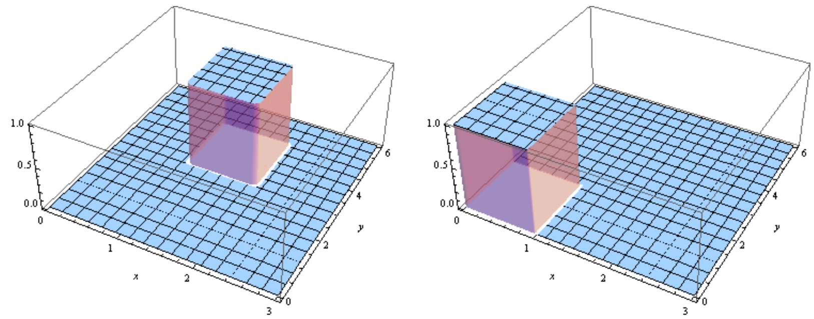

Figure 2-5: The graphs of f and f_2^* in Example 2.1.8.

Then

$$\begin{aligned} f_2^*(s,u) &= \int_0^\infty \chi_{\{|f|>t\}^*}(s,u)\,dt \\ &= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<\varphi_E^*(s)\}}(s,u)\,dt \\ &= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(d-c)\chi_{(0,|k|)}(t)\chi_{[0,b-a)}(s)\}}(s,u)\,dt \\ &= \int_0^{|k|} \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(d-c)\chi_{[0,b-a)}(s)\}}(s,u)\,dt \\ &= \int_0^{|k|} \chi_{[0,d-c)}(u)\chi_{[0,b-a)}(s)\,dt \\ &= |k|\chi_{[0,d-c)}(u)\chi_{[0,b-a)}(s). \end{aligned}$$

So,

$$f_2^*(s,t) = |k|\chi_{[0,b-a)}(s)\chi_{[0,d-c)}(u).$$

Example 2.1.9. *Now, we calculate f_2^* for $f(x,y) = x\chi_{[0,1]}(x)\chi_{[0,1]}(y)$. In this case*

$$\begin{aligned} E &= \{(x,y)\in\mathbb{R}^2 : |f(x,y)|>t\} \\ &= \{(x,y)\in\mathbb{R}^2 : |x\chi_{[0,1]}(x)\chi_{[0,1]}(y)|>t\} \\ &= \{(x,y)\in[0,1]\times[0,1] : x>t\}. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi_E(x) &= m_1\left(\{y\in\mathbb{R} : (x,y)\in E\}\right) \\ &= m_1\left(\{y\in[0,1] : x>t\}\right) \\ &= (1-t)\chi_{[0,1]}(x). \end{aligned}$$

Then

$$\varphi_E^*(s) = (1-t)\chi_{[0,1)}(s).$$

So,

$$
\begin{aligned}
f_2^*(s,u) &= \int_0^\infty \chi_{\{|f|>t\}^*}(s,u)\,dt \\
&= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<\varphi_E^*(s)\}}(s,u)\,dt \\
&= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(1-t)\chi_{[0,1)}(s)\}}(s,u)\,dt.
\end{aligned}
\tag{2.1}
$$

Now,

$$
\begin{aligned}
\chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(1-t)\chi_{[0,1)}(s)\}}(s,u) &= \chi_{[0,1)\times\{u\in\mathbb{R}_+ : 0<u<1-t\}}(s,u) \\
&= \chi_{[0,1)\times(0,1-t)}(s,u).
\end{aligned}
\tag{2.2}
$$

Since

$$\chi_{A\times B}(x,y) = \chi_A(x)\chi_B(y) = \chi_B(y)\chi_A(x) = \chi_{B\times A}(y,x).$$

Returning to (2.2) we obtain

$$\chi_{[0,1)\times(0,1-t)}(s,u) = \chi_{(0,1-t)\times[0,1)}(u,s).$$

Replacing in (2.1),

$$
\begin{aligned}
f_2^*(u,s) &= \int_0^\infty \chi_{(0,1-t)\times[0,1)}(u,s)\,dt \\
&= \int_0^\infty \chi_{(0,1-t)}(u)\chi_{[0,1)}(s)\,dt \\
&= \chi_{[0,1)}(s)\int_0^\infty \chi_{(0,1-t)}(u)\,dt.
\end{aligned}
\tag{2.3}
$$

Since

$$
\begin{aligned}
\chi_{(0,1-t)}(u) &= \begin{cases} 1, & \text{if } 0<u<1-t \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } t<u+t<1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } t<u+t \\ 0, & \text{otherwise} \end{cases} \cdot \begin{cases} 1, & \text{if } u+t<1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } 0<u \\ 0, & \text{otherwise} \end{cases} \cdot \begin{cases} 1, & \text{if } t<1-u \\ 0, & \text{otherwise} \end{cases} \\
&= \chi_{(0,\infty)}(u)\chi_{(0,1-u)}(t).
\end{aligned}
\tag{2.4}
$$

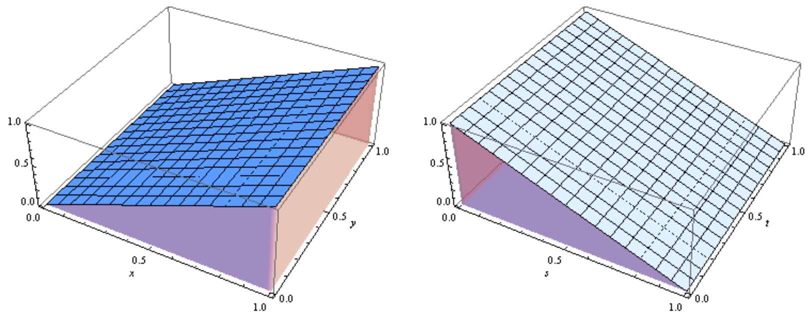

Figure 2-6: The graphs of f and f_2^* in Example 2.1.9.

From (2.3) and (2.4), we deduce that

$$\begin{aligned} f_2^*(u,s) &= \chi_{[0,1)}(s)\int_0^\infty \chi_{(0,\infty)}(u)\chi_{(0,1-u)}(t)\,dt \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)\int_0^\infty \chi_{(0,1-u)}(t)\,dt \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)m[(0,\infty)\cap(0,1-u)] \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)(1-u)\chi_{(0,1)}(u). \end{aligned}$$

This means that

$$f_2^*(u,s) = (1-u)\chi_{(0,1)}(u)\chi_{[0,1)}(s).$$

Just to keep the notation, we write

$$f_2^*(s,t) = (1-s)\chi_{(0,1)}(s)\chi_{[0,1)}(t).$$

Remark 2.1.10. The examples above show that, in general, it is not easy to calcule f_2^*. However, in Theorem 2.1.25 we will show a better way to find it as an iterative rearrangement.

We give now some elementary properties for this new rearrangement definition.

Proposition 2.1.11. *Let $E, F \subset \mathbb{R}^2$. Then,*

a) $m_2(E) = m_2(E^*)$ and $E^* \subset F^*$ if $E \subset F$.

b) $E = E^*$ if and only if E is a decreasing set of $\mathbb{R}_+^2$.

c) $f^* = \chi_{F^*}$ if and only if $f = \chi_E$ and $E^* = F^*$. In particular, $(\chi_E)_2^* = \chi_{E^*}$.

d) If $E \cap F = \emptyset$ then $m_2((E \cup F)^* \setminus E^*) = m_2(F)$.

Proof. a) If $E \subset \mathbb{R}^2$, we know that the x-section of E is $E_x = \{y \in \mathbb{R} : (x,y) \in E\}$. A result from measure theory (see [29, Theorem 2.36]) ensures that

$$m_2(E) = \int_{-\infty}^{\infty} m_1(E_x)\,dx = \int_{-\infty}^{\infty} \varphi_E(x)\,dx. \tag{2.5}$$

Besides,

$$\int_{-\infty}^{\infty} \varphi_E(x)\,dx = \int_0^{\infty} \varphi_E^*(t)\,dt. \tag{2.6}$$

If we look at the distribution of φ_E^*, we have that

$$\int_0^{\infty} \varphi_E^*(t)\,dt = \int_0^{\infty} D_{\varphi_E^*}(\lambda)\,d\lambda = \int_0^{\infty} m_1\left(\{s \in (0,\infty) : \varphi_E^*(s) > \lambda\}\right)\,d\lambda. \tag{2.7}$$

Remembering that

$$E^* = \{(s,\lambda) \in \mathbb{R}_+^2 : 0 < \lambda < \varphi_E^*(s)\}.$$

We see that

$$\begin{aligned}(E^*)_\lambda &= \{s \in \mathbb{R}_+ : 0 < \lambda < \varphi_E^*(s)\} \\ &= \{s \in \mathbb{R}_+ : \varphi_E^*(s) > \lambda\}, \quad \text{with } \lambda > 0.\end{aligned}$$

Returning to (2.7),

$$\begin{aligned}\int_0^{\infty} \varphi_E^*(t)\,dt &= \int_0^{\infty} m_1\left(\{s \in (0,\infty) : \varphi_E^*(s) > \lambda\}\right)\,d\lambda \\ &= \int_0^{\infty} m_1\left(E^*\right)_\lambda\,d\lambda \\ &= m_2\left(E^*\right), \text{the same result used in (2.5).}\end{aligned}$$

Which allows us to conclude that

$$m_2\left(E\right) = m_2\left(E^*\right).$$

Let's see that $E \subset F$ implies $E^* \subset F^*$. Indeed,

$$\begin{aligned}E \subset F &\Rightarrow E_x \subset F_x \\ &\Rightarrow m_1\left(E_x\right) \leq m_1\left(F_x\right) \\ &\Rightarrow \varphi_E(x) \leq \varphi_F(x) \\ &\Rightarrow \varphi_E^*(s) \leq \varphi_F^*(s).\end{aligned}$$

Then, if $(s,t) \in E^*$, $0 < t < \varphi_E^*(s) \leq \varphi_F^*(s)$, that is $0 < t < \varphi_F^*(s)$, so $(s,t) \in F^*$.

b) If E is a decreasing set, then there exists $r > 0$ such that

$$E = \{(x,y) \in \mathbb{R}_+^2 : 0 < x < r, 0 < y < \varphi_E(x)\}.$$

Besides, if E is a decreasing set, then φ_E is decreasing too. Let's see that.

If E is a decreasing set, then χ_E is decreasing on each variable. So,

$$x_1 \leq x_2 \Rightarrow \chi_E(x_1,y) \geq \chi_E(x_2,y)$$

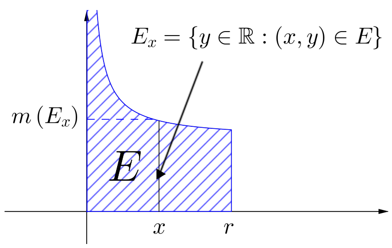

Figure 2-7: The function $\varphi_E(x) = m_1(E_x)$ used in the proof of Proposition 2.1.11 b).

$$\begin{aligned}&\Rightarrow \chi_{E_{x_1}}(y) \geq \chi_{E_{x_2}}(y)\\&\Rightarrow E_{x_1} \supset E_{x_2}\\&\Rightarrow m_1(E_{x_1}) \geq m_1(E_{x_2})\\&\Rightarrow \varphi_E(x_1) \geq \varphi_E(x_2).\end{aligned}$$

Now, since φ is decreasing, then $\varphi_E = \varphi_E^*$, thus

$$E = \{(x,y) \in \mathbb{R}_+^2 : 0 < x < r, 0 < y < \varphi_E(x)\} = \{(x,y) \in \mathbb{R}_+^2 : 0 < y < \varphi_E^*(x)\} = E^*.$$

In the another direction, if $E = E^*$, since E^* is a decreasing set, so is E. Let's show that E^* is indeed a decreasing set.

Fix $s \in (0,\infty)$ and let $t_1, t_2 \in (0,\infty)$ with $t_1 \leq t_2$. We want to show that

$$\chi_{E^*}(s,t_1) \geq \chi_{E^*}(s,t_2).$$

- *Case 1:* If $(s,t_1) \in E^*$, then $\chi_{E^*}(s,t_1) = 1$, and there's nothing to prove.
- *Case 2:* If $(s,t_1) \notin E^*$, remembering that $E^* = \{(s,t) \in \mathbb{R}_+^2 : 0 < t < \varphi_{E^*}(s)\}$, then it holds that $t_1 \geq \varphi_{E^*}(s)$. Since $t_1 \leq t_2$, then $t_2 \geq \varphi_{E^*}(s)$. In this way $(s,t_2) \notin E^*$. So

$$\chi_{E^*}(s,t_1) = 0 \geq 0 = \chi_{E^*}(s,t_2).$$

We have shown that χ_{E^*} is decreasing in the second variable. In a similar way one can prove that χ_{E^*} is decreasing in the second variable. So χ_{E^*} is decreasing in each variable, which means that E^* is decreasing.

c) For $f = \chi_E$, we have

$$\begin{aligned}(\chi_E)_2^*(x) &= \int_0^\infty \chi_{\{x\in\mathbb{R}^2:|\chi_E(x)|>t\}^*}(x)\,dt\\&= \int_0^1 \chi_{\{x\in\mathbb{R}^2:\chi_E(x)>t\}^*}(x)\,dt\\&= \int_0^1 \chi_{E^*}(x)\,dt\end{aligned}$$

$$= \chi_{E^*}(x).$$

Reciprocally, suppose that $f_2^* = \chi_{F^*}$:

- If $x \notin F^*$,
$$f_2^*(x) = \int_0^\infty \chi_{\{x:|f(x)|>t\}^*}(x)\,dt = \chi_{F^*}(x) = 0.$$
Then $\chi_{\{x:|f(x)|>t\}^*}(x) = 0$, so $x \notin \{|f| > t\}^*$. Hence $\{|f| > t\}^* \subset F^*$, for all $t > 0$.
- If $x \in F^*$,
$$f_2^*(x) = \int_0^\infty \chi_{\{x:|f(x)|>t\}^*}(x)\,dt = \chi_{F^*}(x) = 1.$$
Then,
$$\int_0^\infty \chi_{\{x:|f(x)|>t\}^*}(x)\,dt = \int_0^1 \chi_{\{x:|f(x)|>t\}^*}(x)\,dt + \int_1^\infty \chi_{\{x:|f(x)|>t\}^*}(x)\,dt = 1.$$
Which holds only if $x \in \{|f| > t\}^*$ for $0 < t < 1$ and $x \notin \{|f| > t\}^*$ for $t > 1$.

From the above we conclude that $\{|f| > t\}^* = F^*$ if $0 < t < 1$, and $\{|f| > t\}^* = \emptyset$ if $t > 1$. Hence $t < f(x) \leq 1$ (if $f \neq 0$) for all $0 < t < 1$. This implies that $f(x) = 1$ for all $t \in (0,1)$. This way we found a set E ($E = (0,1)$) such that $f = \chi_E$ and $E^* = F^*$.

d) Since $E \subset E \cup F$, from a) we have that $E^* \subset (E \cup F)^*$. In addition, from the definition of E^* one concludes that $m_2(E^*) < \infty$. So,

$$\begin{aligned} m_2\left((E \cup F)^* \setminus E^*\right) &= m_2\left((E \cup F)^*\right) - m_2\left(E^*\right) \\ &= m_2\left(E \cup F\right) - m_2\left(E\right) = m_2\left(E\right) + m_2\left(F\right) - m_2\left(E\right) = m_2\left(F\right). \quad \square \end{aligned}$$

Example 2.1.12. *Let's see an application of part c) of Proposition 2.1.11. In Examples 2.1.5 and 2.1.6 we saw that if $E = [a,b] \times [c,d]$ and $F = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$, then*

$$E^* = [0, b-a) \times (0, d-c) \text{ and } F^* = \{(s,t) \in \mathbb{R}_+^2 : 0 < t < \sqrt{4r^2 - s^2}, 0 < s < 2r\}.$$

Then, for the functions χ_E and χ_F, we have

$$(\chi_E)_2^*(s,t) = \chi_{E^*}(s,t), \quad (\chi_F)_2^*(s,t) = \chi_{F^*}(s,t).$$

*See Figure **2-8**.*

Remark 2.1.13. Looking at Definition 2.1.1 and Proposition 2.1.11, the following questions naturally came up.

1. If $E, F \subset \mathbb{R}^2$ are decreasing sets,
   a) is $E \cup F$ decreasing too?

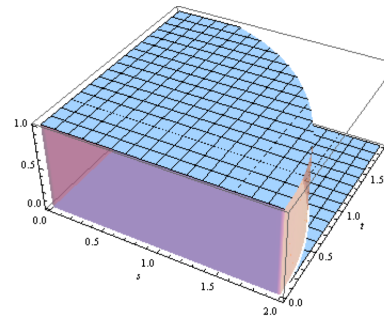

Figure 2-8: $(\chi_E)_2^*$ is χ_{E^*}.

b) is $E \cap F$ decreasing too?

2. If E, F are any subsets of $\mathbb{R}^2$,

a) It is true that $(E \cup F)^* = E^* \cup F^*$?

b) It is true that $(E \cap F)^* = E^* \cap F^*$?

We are going to show that the answer to the first question is affirmative, and the answer to the second question is negative.

Proof of 1. a). Suppose that E and F are decreasing sets. Fix y_0 and take x_1, x_2 with $x_1 \leq x_2$. We want to see that

$$\chi_{E\cup F}(x_1, y_0) \geq \chi_{E\cup F}(x_2, y_0).$$

- Case 1: If $\chi_{E\cup F}(x_2, y_0) = 0$, there's nothing to prove.
- Case 2: If $\chi_{E\cup F}(x_2, y_0) = 1$, then

$$\begin{aligned} 1 = \chi_{E\cup F}(x_2, y_0) &= \chi_E(x_2, y_0) + \chi_F(x_2, y_0) - \chi_{E\cap F}(x_2, y_0) \\ &\leq \chi_E(x_2, y_0) + \chi_F(x_2, y_0) \\ &\leq \chi_E(x_1, y_0) + \chi_F(x_1, y_0), \text{since } E \text{ and } F \text{ are decreasing sets.} \end{aligned}$$

So, $\chi_E(x_1, y_0) = 1$ or $\chi_F(x_1, y_0) = 1$. i.e. $(x_1, y_0) \in E$ or $(x_1, y_0) \in F$, thus $(x_1, y_0) \in E \cup F$ which implies that $\chi_{E\cup F}(x_1, y_0) = 1$, so the condition $\chi_{E\cup F}(x_1, y_0) \geq \chi_{E\cup F}(x_2, y_0)$ is fulfilled.

We showed that $\chi_{E\cup F}$ is decreasing in the first variable. In a total analogous way one can prove that $\chi_{E\cup F}$ is decreasing in the second variable. □

Proof of 1. b). Suppose that E and F are decreasing sets. Fix y_0 and take x_1, x_2 with $x_1 \leq x_2$. We want to see that

$$\chi_{E\cap F}(x_1, y_0) \geq \chi_{E\cap F}(x_2, y_0).$$

- Case 1: If $\chi_{E\cap F}(x_2, y_0) = 0$, there's nothing to prove.
- Case 2: If $\chi_{E\cap F}(x_2, y_0) = 1$, then

$$\begin{aligned}
1 = \chi_{E\cap F}(x_2, y_0) &= \chi_E(x_2, y_0) \cdot \chi_F(x_2, y_0) \\
&\leq \chi_E(x_1, y_0) \cdot \chi_F(x_1, y_0), \text{since } E \text{ and } F \text{ are decreasing sets.} \\
&= \chi_{E\cap F}(x_1, y_0).
\end{aligned}$$

From this we conclude that $\chi_{E\cap F}(x_1, y_0) = 1$, thus $\chi_{E\cap F}(x_1, y_0) \geq \chi_{E\cap F}(x_2, y_0)$.

Again, we omit the proof for the second variable. □

Now we provide some counterexamples for the second question. Take $E = [-1, 1] \times [-1, 1]$ and $F = [-1, 1] \times [1, 3]$. From Example 2.1.5 we know that $E^* = [0, 2) \times (0, 2)$ and $F^* = [0, 2) \times (0, 2)$, then

$$E^* \cup F^* = [0, 2) \times (0, 2).$$

On the other hand, $E \cup F = [-1, 1] \times [-1, 3]$, then

$$(E \cup F)^* = [0, 2) \times (0, 4).$$

So,

$$(E \cup F)^* \neq E^* \cup F^*.$$

Now we show that $(E \cap F)^* \neq E^* \cap F^*$. For this, take $E = [-1, 1] \times [-1, 1]$ and $F = [-1, 1] \times [0, 2]$. Then $E^* = [0, 2) \times (0, 2)$ and $F^* = [0, 2) \times (0, 2)$, so

$$E^* \cap F^* = [0, 2) \times (0, 2).$$

But $E \cap F = [-1, 1] \times [0, 1]$, thus

$$(E \cap F)^* = [0, 2) \times (0, 1).$$

We conclude that $(E \cap F)^* \neq E^* \cap F^*$.

Remark 2.1.14. Although we showed in Remark 2.1.13 that, in general

$$(A \cap B)^* \neq A^* \cap B^*.$$

There exists an inclusion relationship between those sets. Actually, the following is true

$$(A \cap B)^* \subset A^* \cap B^*.$$

We proceed to prove it.
Let $x = (s, t) \in (A \cap B)^*$. Then $x \in \left\{(s, t) \in \mathbb{R}^2_+ : 0 < t < \varphi^*_{A\cap B}(s)\right\}$, where $\varphi_{A\cap B}(x) = m_1\left((A \cap B)_x\right)$. Since $A \cap B \subset A$ and $A \cap B \subset B$, we have

$$(A \cap B)_x \subset A_x \qquad \text{and} \qquad (A \cap B)_x \subset B_x$$

$$\begin{array}{rlcl}
\Rightarrow & m_1\left((A\cap B)_x\right) \le m_1\left(A_x\right) & \text{and} & m_1\left((A\cap B)_x\right) \le m_1\left(B_x\right) \\
\Rightarrow & \varphi_{A\cap B}(x) \le \varphi_A(x) & \text{and} & \varphi_{A\cap B}(x) \le \varphi_B(x) \\
\Rightarrow & \varphi^*_{A\cap B}(s) \le \varphi^*_A(s) & \text{and} & \varphi^*_{A\cap B}(s) \le \varphi^*_B(s).
\end{array}$$

So, if $x = (s,t) \in (A\cap B)^*$,

$$\begin{array}{rlcl}
 & 0 < t < \varphi^*_{A\cap B}(s) \le \varphi^*_A(s) & \text{and} & 0 < t < \varphi^*_{A\cap B}(s) \le \varphi^*_B(s) \\
\Rightarrow & 0 < t < \varphi^*_A(s) & \text{and} & 0 < t < \varphi^*_B(s) \\
\Rightarrow & (s,t) \in A^* & \text{and} & (s,t) \in B^*
\end{array}$$

Hence $x = (s,t) \in A^* \cap B^*$.

Remark 2.1.15. In a similar way one can prove that $A^* \cap B^* \subset (A\cup B)^*$.

The following results give more information about the level sets of f and f_2^*.

Lemma 2.1.16. *If f is a measurable function on $\mathbb{R}^2$ and $t > 0$, then*

$$\{f_2^* > t\} \subset \{|f| > t\}^* \subset \{f_2^* \ge t\}.$$

Proof. By definition,

$$f_2^*(x) > t \Leftrightarrow \int_0^\infty \chi_{\{|f|>s\}^*}(x)\, ds > t, \quad (x = (x_1, x_2)).$$

For $x = (x_1, x_2) \in \{f_2^* > t\}$, let's see that $x \in \{|f| > t\}^*$.
Note that

$$\chi_{\{|f|>s\}^*}(x) = \begin{cases} 1, & \text{if } x \in \{|f| > s\}^* \\ 0, & \text{if } x \notin \{|f| > s\}^*. \end{cases} \tag{2.8}$$

Taking $E = \{|f| > s\} = \{(a,b) : |f(a,b)| > s\}$, it holds that

$$\varphi_E(a) = m_1\left(E_a\right) \overset{\text{notation}}{=} \varphi_s(a).$$

So $\varphi^*_E(r) = \varphi^*_s(r)$. This way

$$\{|f| > s\}^* = \{(r,t) : 0 < t < \varphi^*_s(r)\}. \tag{2.9}$$

Then $x = (x_1, x_2) \in \{|f| > s\}^* \Leftrightarrow 0 < x_2 < \varphi^*_s(x_1)$. Getting back to (2.8),

$$\begin{aligned}
\chi_{\{|f|>s\}^*}(x) &= \begin{cases} 1, & \text{if } \varphi^*_s(x_1) > x_2 \\ 0, & \text{if } \varphi^*_s(x_1) \le x_2 \end{cases} \\
&= \chi_{\{\varphi^*_s(x_1) > x_2\}}(s).
\end{aligned}$$

Hence,

$$f_2^* > t \Leftrightarrow \int_0^\infty \chi_{\{|f|>s\}^*}(x)\, ds > t$$

$$\begin{aligned} &\Leftrightarrow \int_0^\infty \chi_{\{\varphi_s^*(x_1)>x_2\}}(s)\,ds > t \\ &\Leftrightarrow m\left(\{s : \varphi_s^*(x_1) > x_2\}\right) > t. \end{aligned} \tag{2.10}$$

Note that

$$\begin{aligned} s' < s &\Rightarrow \{|f| > s\} \subseteq \{|f| > s'\} \\ &\Rightarrow \{|f| > s\}^* \subseteq \{|f| > s'\}^* \\ &\Rightarrow \{(x_1, x_2) : 0 < x_2 < \varphi_s^*(x_1)\} \subseteq \{(x_1, x_2) : 0 < x_2 < \varphi_{s'}^*(x_1)\} \\ &\Rightarrow \varphi_s^*(x_1) \leq \varphi_{s'}^*(x_1). \end{aligned}$$

This tell us that the set

$$\{s : \varphi_s^*(x_1) > x_2\},$$

is an interval, and it has the form

$$(0, s) \text{ or } (0, s]. \tag{2.11}$$

Let's check this. Take $s' \in (0, s)$, then

$$\begin{aligned} &s < s' \\ &\Rightarrow x_2 < \varphi_s^*(x_1) \leq \varphi_{s'}^*(x_1) \\ &\Rightarrow x_2 < \varphi_{s'}^*(x_1) \\ &\Rightarrow s' \in \{s : \varphi_s^*(x_1) > x_2\} \\ &\Rightarrow (0, s) \subseteq \{s : \varphi_s^*(x_1) > x_2\}. \end{aligned}$$

Now, take $s \in \{s : \varphi_s^*(x_1) > x_2\}$. We have that

$$\begin{aligned} &\varphi_s^*(x_1) > x_2 \\ &\Rightarrow \varphi_{s'}^*(x_1) > \varphi_s^*(x_1) > x_2, \forall\ s' < s \\ &\Rightarrow s' \in (0, s). \end{aligned}$$

Returning to (2.10),

$$\begin{aligned} f_2^*(x) > t &\Leftrightarrow m_1\left(\{s : \varphi_s^*(x_1) > x_2\}\right) > t \\ &\Rightarrow m_1\left((0, s)\right) > t \\ &\Rightarrow s > t \\ &\Rightarrow x_2 < \varphi_s^*(x_1) \leq \varphi_t^*(x_1) \\ &\Rightarrow \varphi_t^*(x_1) > x_2 \\ &\Rightarrow x = (x_1, x_2) \in \{(x_1, x_2) : 0 < x_2 < \varphi_t^*(x_1)\} \\ &\Rightarrow x \in \{|f| > t\}^*. \end{aligned}$$

Reciprocally,

$$\begin{aligned} x=(x_1,x_2)\in\{|f|>t\}^* &\Rightarrow \varphi_t^*(x_1)>x_2, \text{ by using (2.9)}\\ &\Rightarrow m_1\left(\{s:\varphi_s^*(x_1)>x_2\}\right)\geq t, \text{by the same reason given in (2.11)}\\ &\Rightarrow f_2^*(x)\geq t, \text{ using (2.10)}\\ &\Rightarrow x\in\{f_2^*\geq t\}. \end{aligned}$$

□

Lemma 2.1.17. *Let f and g be two measurable functions on $\mathbb{R}^2$ and $t>0$. Then*

$$\chi_{\{|f+g|>t\}^*}(x+y)\leq\chi_{\{|f|>t/2\}^*}(x)+\chi_{\{|g|>t/2\}^*}(y),$$

for $x=(x_1,x_2), y=(y_1,y_2)\in\mathbb{R}^2$.

Proof. Let

$$\begin{aligned} \varphi_{f,t}(a)&=m_1\left(\{b\in\mathbb{R}:|f(a,b)|>t\}\right),\\ \varphi_{g,t}(a)&=m_1\left(\{b\in\mathbb{R}:|g(a,b)|>t\}\right),\\ \varphi_{f+g,t}(a)&=m_1\left(\{b\in\mathbb{R}:|(f+g)(a,b)|>t\}\right). \end{aligned}$$

Let's see that

$$\varphi_{f+g,t}(a)\leq\varphi_{f,t/2}(a)+\varphi_{g,t/2}(a). \tag{2.12}$$

Indeed, it's enough to show that

$$\{b\in\mathbb{R}:|(f+g)(a,b)|>t\}\subseteq\{b\in\mathbb{R}:|f(a,b)|>t/2\}\cup\{b\in\mathbb{R}:|g(a,b)|>t/2\}. \tag{2.13}$$

Let $b\in\mathbb{R}$ such that $b\notin\{b\in\mathbb{R}:|f(a,b)|>t/2\}\cup\{b\in\mathbb{R}:|g(a,b)|>t/2\}$. Then $|f(a,b)|\leq t/2$ and $|g(a,b)|\leq t/2$. So

$$|(f+g)(a,b)|\leq|f(a,b)|+|g(a,b)|\leq t/2+t/2=t.$$

Hence $b\notin\{b\in\mathbb{R}:|(f+g)(a,b)|>t\}$. This justifies (2.13).
Besides, if $x\notin\{|f|>t/2\}^*$, then $\varphi_{f,t/2}^*(x_1)\leq x_2$. Similarly, if $y\notin\{|g|>t/2\}^*$, then $\varphi_{g,t/2}^*(y_1)\leq y_2$. So

$$\begin{aligned} \varphi_{f+g,t}(a)&\leq\left(\varphi_{f,t/2}+\varphi_{g,t/2}\right)(a), \quad \text{using (2.12)}\\ \Rightarrow\varphi_{f+g,t}^*(x_1+y_1)&\leq\left(\varphi_{f,t/2}+\varphi_{g,t/2}\right)^*(x_1+y_1), \quad {}^* \text{ is monotone}\\ &\leq\varphi_{f,t/2}^*(x_1)+\varphi_{g,t/2}^*(y_1), \quad \text{a property of } {}^*\\ &\leq x_2+y_2. \end{aligned}$$

i.e. $x+y\notin\{|f+g|>t\}^*$.
We have shown that if $x\notin\{|f|>t/2\}^*$ and $y\notin\{|g|>t/2\}^*$, then $x+y\notin\{|f+g|>t\}^*$. This is equivalent to

$$\{x+y:|(f+g)(x+y)|>t\}^*\subseteq\{x:|f(x)|>t/2\}^*\cup\{y:|g(y)|>t/2\}^*.$$

Taking characteristic functions, we have that

$$\begin{aligned}\chi_{\{x+y:|(f+g)(x+y)|>t\}^*}(w) &\leq \chi_{\{x:|f(x)|>t/2\}^*\cup\{y:|g(y)|>t/2\}^*}(w)\\ &\leq \chi_{\{x:|f(x)|>t/2\}^*}(w) + \chi_{\{y:|g(y)|>t/2\}^*}(w).\end{aligned}$$

This is

$$\chi_{\{w:|(f+g)(w)|>t\}^*}(x+y) \leq \chi_{\{w:|f(w)|>t/2\}^*}(x) + \chi_{\{w:|g(w)|>t/2\}^*}(y).$$ □

In the next Proposition we show that the two-dimensional rearrangement has similar properties to the classic one. Compare with [14, Proposition 1.7], [26, Theorem 4.5] and [30, Proposition 1.4.5.].

Proposition 2.1.18. *Suppose that f, g and f_n ($n = 1, 2, \ldots$), are measurable functions on $\mathbb{R}^2$ and let $c \in \mathbb{C}$. Then the decreasing two-dimensional rearrangement f_2^* is a non-negative function on $\mathbb{R}^2_+$, decreasing in each variable. Moreover,*

a) $|g| \leq |f|$ *a.e* $\Rightarrow g_2^* \leq f_2^*$;

b) $(cf)_2^* = |c| f_2^*$;

c) If f is decreasing in each variable, then $f_2^* = f$;

d) $(f+g)_2^*(x+y) \leq 2\left(f_2^*(x) + g_2^*(y)\right)$;

e) $|f| \leq \liminf_{n\to\infty} |f_n| \Rightarrow f_2^* \leq \liminf_{n\to\infty} (f_n)_2^*$. *In particular, if $|f_n| \uparrow f$ then $(f_n)_2^* \uparrow f_2^*$;*

f) $(f_2^*(x))^p = (f^p(x))_2^*$, $(0 < p < \infty)$;

g) If f is a symmetric function (i.e. $f(x_1, x_2) = f(x_2, x_1)$), then f_2^* is symmetric.

Proof. The fact that f_2^* is a non-negative function follows from Definition 2.1.7. The fact that f_2^* is decreasing in each variable follows from the fact that if E is a decreasing set, then χ_E is decreasing in each variable. Let's see it.
We already know that $\{|f| > t\}^*$ is a decreasing set (see b) in the proof of Proposition 2.1.11).
Fix $x_1 \in (0, \infty)$ and let $y_1, y_2 \in (0, \infty)$ with $y_1 \leq y_2$. We want to see that

$$\chi_{\{|f|>t\}^*}(x_1, y_1) \geq \chi_{\{|f|>t\}^*}(x_1, y_2).$$

- If $(x_1, y_2) \notin \{|f| > t\}^*$: There's nothing to prove.
- If $(x_1, y_2) \in \{|f| > t\}^*$: Then $\chi_{\{|f|>t\}^*}(x_1, y_2) = 1$. Since $\{|f| > t\}^*$ is a decreasing set, and $y_1 \leq y_2$, it holds that

$$\begin{aligned} & \chi_{\{|f|>t\}^*}(x_1, y_2) \leq \chi_{\{|f|>t\}^*}(x_1, y_1)\\ \Rightarrow \quad & 1 \leq \chi_{\{|f|>t\}^*}(x_1, y_1)\\ \Rightarrow \quad & \chi_{\{|f|>t\}^*}(x_1, y_1) = 1.\end{aligned}$$

In any case, $\chi_{\{|f|>t\}^*}(x_1, y_1) \geq \chi_{\{|f|>t\}^*}(x_1, y_2)$. This proves that $\chi_{\{|f|>t\}^*}$ is decreasing in the second variable. Then, for a fixed x_1 and $y_1 \leq y_2$, we have

$$\begin{aligned} & \chi_{\{|f|>t\}^*}(x_1, y_2) \leq \chi_{\{|f|>t\}^*}(x_1, y_1) \\ \Rightarrow \quad & \int_0^\infty \chi_{\{|f|>t\}^*}(x_1, y_2)\, dt \leq \int_0^\infty \chi_{\{|f|>t\}^*}(x_1, y_1)\, dt \\ \Rightarrow \quad & f_2^*(x_1, y_2) \leq f_2^*(x_1, y_1). \end{aligned}$$

So f_2^* is decreasing in the second variable. The proofs for the first variable are similar.

a)

$$\begin{aligned} |g| \leq |f| \text{ a.e.} \quad \Rightarrow \quad & \{|g| > t\} \subseteq \{|f| > t\} \text{ a.e.} \\ \Rightarrow \quad & \{|g| > t\}^* \subseteq \{|f| > t\}^*, \text{by Proposition 2.1.11 a)} \\ \Rightarrow \quad & \chi_{\{|g|>t\}^*}(x) \leq \chi_{\{|f|>t\}^*}(x) \\ \Rightarrow \quad & \int_0^\infty \chi_{\{|g|>t\}^*}(x)\, dt \leq \int_0^\infty \chi_{\{|f|>t\}^*}(x)\, dt \\ \Rightarrow \quad & g_2^*(x) \leq f_2^*(x). \end{aligned}$$

b) If $c = 0$, the result is trivial. If $c \neq 0$, then

$$(cf)_2^*(x) = \int_0^\infty \chi_{\{|cf|>t\}^*}(x)\, dt = \int_0^\infty \chi_{\left\{|f|>\frac{t}{|c|}\right\}^*}(x)\, dt.$$

Taking $u = \frac{t}{|c|}$, we have $du = \frac{1}{|c|}dt$, i.e. $|c|du = dt$. Hence

$$(cf)_2^*(x) = \int_0^\infty \chi_{\{|f|>u\}^*}(x)|c|\, du = |c| \int_0^\infty \chi_{\{|f|>u\}^*}(x)\, du = |c| f_2^*(x).$$

c) If f is decreasing in each variable, the set $\{|f| > t\}$ is a decreasing set. Note that

$$\{|f| > t\} = \{f > t\} \cup \{f < -t\} = \{f > t\} \cup \{-f > t\}.$$

Let's see that those sets are both decreasing sets. Indeed, fix $x_1 \in (0, \infty)$ and take $y_1, y_2 \in (0, \infty)$ with $y_1 \leq y_2$. We want to show that

$$\chi_{\{f>t\}}(x_1, y_1) \geq \chi_{\{f>t\}}(x_1, y_2).$$

- Case 1: If $(x_1, y_2) \notin \{f > t\}$, there's nothing to prove.
- Case 2: If $(x_1, y_2) \in \{f > t\}$, then $\chi_{\{f>t\}}(x_1, y_2) = 1$. Besides,

$$(x_1, y_2) \in \{f > t\} \Rightarrow f(x_1, y_2) > t.$$

Since f is decreasing in each variable,

$$f(x_1, y_1) \geq f(x_1, y_2) > t \Rightarrow (x_1, y_1) \in \{f > t\}.$$

Hence,

$$\chi_{\{f>t\}}(x_1, y_1) = 1 \geq 1 = \chi_{\{f>t\}}(x_1, y_2).$$

The proof for the first variable is similar. This shows that $\{f > t\}$ is a decreasing set. Now, fix $x_1 \in (0,\infty)$ and take $y_1, y_2 \in (0,\infty)$ with $y_1 \leq y_2$. Let's see that

$$\chi_{\{-f>t\}}(x_1,y_1) \geq \chi_{\{-f>t\}}(x_1,y_2).$$

- Case 1: If $(x_1,y_2) \notin \{-f > t\}$, there's nothing to prove.
- Case 2: If $(x_1,y_2) \in \{-f > t\}$, then $\chi_{\{-f>t\}}(x_1,y_2) = 1$. Besides,

$$(x_1,y_2) \in \{-f > t\} \Rightarrow -f(x_1,y_2) > t.$$

Since f is decreasing in each variable,

$$\begin{aligned} f(x_1,y_1) \geq f(x_1,y_2) \Rightarrow & \quad -f(x_1,y_2) \leq -f(x_1,y_1) \\ \Rightarrow & \quad t < -f(x_1,y_2) \leq -f(x_1,y_1) \\ \Rightarrow & \quad t < -f(x_1,y_1) \\ \Rightarrow & \quad (x_1,y_1) \in \{-f > t\} \\ \Rightarrow & \quad \chi_{\{-f>t\}}(x_1,y_1) = 1. \end{aligned}$$

Hence,

$$\chi_{\{-f>t\}}(x_1,y_1) = 1 \geq 1 = \chi_{\{-f>t\}}(x_1,y_2).$$

The proof for the first variable is similar. This shows that $\{-f > t\}$ is a decreasing set. It follows from Remark 2.1.13 that the set $\{|f| > t\}$ is a decreasing set. Then, it follows from Proposition 2.1.11 b) that

$$\begin{aligned} & \{|f| > t\} = \{|f| > t\}^* \\ \Rightarrow \quad & \chi_{\{|f|>t\}}(x) = \chi_{\{|f|>t\}^*}(x) \\ \Rightarrow \quad & \int_0^\infty \chi_{\{|f|>t\}}(x)\,dt = \int_0^\infty \chi_{\{|f|>t\}^*}(x)\,dt \\ \Rightarrow \quad & f(x) = f_2^*(x). \end{aligned}$$

d)

$$\begin{aligned} (f+g)_2^*(x+y) &= \int_0^\infty \chi_{\{|f+g|>t\}^*}(x+y)\,dt \\ &\leq \int_0^\infty \left[\chi_{\{|f|>t/2\}^*}(x) + \chi_{\{|g|>t/2\}^*}(y)\right] dt, \quad \text{using Lemma 2.1.17} \\ &= \int_0^\infty \chi_{\{|f|>t/2\}^*}(x)\,dt + \int_0^\infty \chi_{\{|g|>t/2\}^*}(y)\,dt \\ &= 2\int_0^\infty \chi_{\{|f|>u\}^*}(x)\,du + 2\int_0^\infty \chi_{\{|g|>u\}^*}(y)\,du \\ &= 2\left[f_2^*(x) + g_2^*(y)\right]. \end{aligned}$$

e) Let
$$E^t := \{(x,y) : |f(x,y)| > t\} \quad \text{and} \quad E_n^t := \{(x,y) : |f_n(x,y)| > t\}.$$
Take $f_x(y) = f(x,y)$ and
$$\varphi_{f,t}(x) = m_1(\{y : |f(x,y)| > t\}) = m_1(\{y : |f_x(y)| > t\}) = D_{f_x}(t),$$
where D_{f_x} is the distribution function of f_x. Then
$$\begin{aligned}
|f| \le \liminf_{n\to\infty} |f_n| \Rightarrow \quad & |f_x| \le \liminf_{n\to\infty} |(f_n)_x| \quad \text{a.e.} \\
\Rightarrow \quad & D_{f_x} \le \liminf_{n\to\infty} D_{(f_n)} \\
\Rightarrow \quad & \varphi_{f,t} \le \liminf_{n\to\infty} \varphi_{f_n} \quad \text{a.e. } \forall\, t > 0 \\
\Rightarrow \quad & \varphi^*_{f,t} \le \liminf_{n\to\infty} \varphi^*_{f_n} \quad \text{a.e. } \forall\, t > 0 \\
\Rightarrow \quad & \chi_{(E^t)^*} \le \liminf_{n\to\infty} \chi_{(E_n^t)^*} \\
\Rightarrow \quad & \chi_{\{|f|>t\}^*} \le \liminf_{n\to\infty} \chi_{\{|f_n|>t\}^*} \\
\Rightarrow \quad & \int_0^\infty \chi_{\{|f|>t\}^*}(x)\, dt \le \int_0^\infty \liminf_{n\to\infty} \chi_{\{|f_n|>t\}^*}(x)\, dt \\
& \le \liminf_{n\to\infty} \int_0^\infty \chi_{\{|f_n|>t\}^*}(x)\, dt \\
& = \liminf_{n\to\infty} (f_n)_2^*.
\end{aligned}$$
Where we used Fatou's lemma in the last inequality. Hence, $f_2^* \le \liminf_{n\to\infty} (f_n)_2^*$. The second part is an immediate consequence of the first.

f) We have
$$\begin{aligned}
(f^p)_2^*(x) &= \int_0^\infty \chi_{\{|f^p|>t\}^*}(x)\, dt \\
&= \int_0^\infty \chi_{\{|f|>t^{1/p}\}^*}(x)\, dt, \quad u = t^{1/p} \Rightarrow u^p = t \Rightarrow pu^{p-1}du = dt, \\
&= \int_0^\infty \chi_{\{|f|>u\}^*}(x) pu^{p-1}\, du \\
&= p\int_0^\infty \chi_{\{|f|>t\}^*}(x) t^{p-1}\, dt.
\end{aligned}$$
In view of Lemma 2.1.16, we have
$$\begin{aligned}
& \{|f| > t\}^* \supseteq \{f_2^* > t\} \\
\Rightarrow \quad & \chi_{\{|f|>t\}^*} \ge \chi_{\{f_2^*>t\}}.
\end{aligned}$$

So

$$\begin{aligned}(f^p)_2^*(x) &= p\int_0^\infty \chi_{\{|f|>t\}^*}(x)t^{p-1}\,dt \\ &\geq p\int_0^\infty \chi_{\{f_2^*>t\}}(x)t^{p-1}\,dt \\ &= p\int_0^{f_2^*(x)} t^{p-1}\,dt \\ &= (f_2^*(x))^p\,.\end{aligned}$$

Then

$$(f_2^*(x))^p \leq (f^p)_2^*(x). \tag{2.14}$$

In the other hand, if we take $0 < r < 1$ and $t \geq 0$,

$$rt \leq t \Rightarrow \{f_2^* \geq t\} \subseteq \{f_2^* \geq rt\}\,.$$

Using Lemma 2.1.16,

$$\begin{aligned} &\{|f| > t\}^* \subseteq \{f_2^* \geq t\} \subseteq \{f_2^* \geq rt\} \\ \Rightarrow\quad &\{|f| > t\}^* \subseteq \{f_2^* \geq rt\} \\ \Rightarrow\quad &\chi_{\{|f|>t\}^*} \leq \chi_{\{f_2^* \geq rt\}}.\end{aligned}$$

Therefore,

$$\begin{aligned}(f^p)_2^*(x) &= p\int_0^\infty \chi_{\{|f|>t\}^*}(x)t^{p-1}\,dt \\ &\leq p\int_0^\infty \chi_{\{f_2^*\geq rt\}^*}(x)t^{p-1}\,dt \\ &= p\int_0^{f_2^*(x)/r} t^{p-1}\,dt \\ &= \left(\frac{f_2^*(x)}{r}\right)^p.\end{aligned}$$

Since the last inequality is valid for all $0 < r < 1$, we obtain

$$(f^p)_2^*(x) \leq (f_2^*(x))^p\,. \tag{2.15}$$

From (2.14) and (2.15) we conclude that

$$(f^p)_2^*(x) = (f_2^*(x))^p\,.$$

g) Let's call $x = (x_1, x_2)$, $u = (x_2, x_1)$. Since f is symmetric, $f(x) = f(u)$. Then

$$\{x : |f(x)| > t\} = \{u : |f(u)| > t\}$$

$$\begin{aligned}
&\Rightarrow && \{x : |f(x)| > t\}^* = \{u : |f(u)| > t\}^* \\
&\Rightarrow && \chi_{\{x:|f(x)|>t\}^*}(x) = \chi_{\{u:|f(u)|>t\}^*}(u) \\
&\Rightarrow && \int_0^\infty \chi_{\{x:|f(x)|>t\}^*}(x)\, dt = \int_0^\infty \chi_{\{u:|f(u)|>t\}^*}(u)\, dt \\
&\Rightarrow && f_2^*(x) = f_2^*(u) \\
&\Rightarrow && f_2^*(x_1, x_2) = f_2^*(x_2, x_1).
\end{aligned}$$

i.e. f_2^* is symmetric. □

The next proposition will be very useful in order to prove our main results, since it allows to consider the special and easy case of simple functions. The result is similar to [29, Theorem 2.10].

Proposition 2.1.19. *If f is a measurable function on $\mathbb{R}^2$, then there exists a sequence $(s_n)_n$ of measurable functions such that*

a) $0 \leq (s_1)_2^* \leq \cdots \leq (s_n)_2^* \leq f_2^*$,

b) $(s_n)_2^* \to f_2^*$ *when* $n \to \infty$ *a.e.*

Proof. The existence of the sequence is standard. Also,

$$0 \leq |s_1| \leq |s_2| \leq \cdots \leq |s_n| \leq |f| \underset{\text{Proposition 2.1.18 a)}}{\Rightarrow} 0 \leq (s_1)_2^* \leq (s_2)_2^* \leq \cdots \leq (s_n)_2^* \leq f_2^*.$$

And

$$|s_n| \uparrow |f| \underset{\text{Proposition 2.1.18 e)}}{\Rightarrow} (s_n)_2^* \uparrow f_2^*.$$

An observation. If $s(x) = \sum_{j=1}^n a_j \chi_{E_j}$ with $a_1 > a_2 > \cdots > a_n > 0$ and $E_i \cap E_j = \emptyset$, then

$$s_2^*(x) = \sum_{j=1}^n a_j \chi_{F_j^* \setminus F_{j-1}^*}(x), \tag{2.16}$$

where $F_j = \bigcup_{k=1}^j E_k$ and $F_0 = \emptyset$. Observe that from Proposition 2.1.11 d) we have

$$m_2\left(F_j^* \setminus F_{j-1}^*\right) = m_2\left(\left(E_j \cup \bigcup_{k=1}^{j-1} E_k\right)^* \setminus \left(\bigcup_{k=1}^{j-1} E_k\right)^*\right) = m_2\left(E_j\right).$$ □

In the next corollary, we show some properties relating the two-dimensional rearrangement with the classical one. We show that this new rearrangement is finer and gives more information than the other.

Corollary 2.1.20. *Let f and g be two measurable functions on $\mathbb{R}^2$.*

a) If $f_2^* = g_2^*$, then $f^* = g^*$, and, in general, the reciprocal isn't true.

b) $(f_2^*)^* = f^*$.

Proof. a) Note that if $f_2^* = g_2^*$, then

$$\int_0^\infty \chi_{\{|f|>t\}^*}(x)\,dt = \int_0^\infty \chi_{\{|g|>t\}^*}(x)\,dt.$$

Therefore $\{|f|>t\}^* = \{|g|>t\}^*$. Using Proposition 2.1.11 a), we obtain $|\{|f|>t\}| = |\{|g|>t\}|$. Then

$$D_f = m_2\left(\{x \in \mathbb{R}^2 : |f(x)| > t\}\right) = m_2\left(\{x \in \mathbb{R}^2 : |g(x)| > t\}\right) = D_g(t).$$

Hence

$$f^*(s) = \inf\{t > 0 : D_f(t) \le s\} = \inf\{t > 0 : D_g(t) \le s\} = g^*(s).$$

i.e. $f^* = g^*$.

To see that the reciprocal is not true, consider the decreasing sets $A = (0,1) \times (0,2)$, $B = (0,2) \times (0,1)$ and the functions $f = \chi_A$ and $g = \chi_B$. We have

$$\begin{aligned} D_f(\lambda) &= m(\{x \in \mathbb{R}^2 : |f(x)| > \lambda\}) \\ &= m_2\left(\{x \in \mathbb{R}^2 : \chi_{(0,1)\times(0,2)}(x) > \lambda\}\right) = \begin{cases} 0, & \text{if } \lambda \ge 1 \\ 2, & \text{if } 0 \le \lambda < 1. \end{cases} \end{aligned}$$

Therefore,

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \le t\} = \begin{cases} 0, & \text{if } t \ge 2 \\ 1, & \text{if } 0 < t < 2 \end{cases} = \chi_{(0,2)}(t).$$

Similarly,

$$\begin{aligned} D_g(\lambda) &= m_2\left(\{x \in \mathbb{R}^2 : |g(x)| > \lambda\}\right) \\ &= m_2\left(\{x \in \mathbb{R}^2 : \chi_{(0,2)\times(0,1)}(x) > \lambda\}\right) = \begin{cases} 0, & \text{if } \lambda \ge 1 \\ 2, & \text{if } 0 \le \lambda < 1. \end{cases} \end{aligned}$$

Then $g^*(t) = \chi_{(0,2)}(t)$. Hence $f^* = g^*$. However,

$$f_2^*(x) = \left(\chi_{(0,1)\times(0,2)}\right)^*(x) \underset{Proposition 2.1.11c)}{=} \chi_{[(0,1)\times(0,2)]^*}(x) \underset{Proposition 2.1.11b)}{=} \chi_{(0,1)\times(0,2)}(x).$$

In a similar way,

$$g_2^*(x) = \left(\chi_{(0,2)\times(0,1)}\right)^*(x) \underset{Proposition 2.1.11c)}{=} \chi_{[(0,2)\times(0,1)]^*}(x) \underset{Proposition 2.1.11b)}{=} \chi_{(0,2)\times(0,1)}(x).$$

So,

$$f_2^*(x) = \chi_{(0,1)\times(0,2)}(x) \neq \chi_{(0,2)\times(0,1)}(x) = g_2^*(x).$$

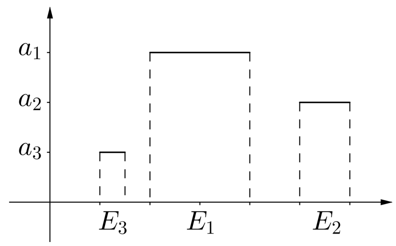

Figure 2-9: The simple function f in the proof of Corollary 2.1.20.

Remark 2.1.21. The above counterexample can be generalized by taking $A = (0, a) \times (0, b)$, $B = (0, b) \times (0, a)$, $f = \chi_A$, $g = \chi_B$ with $a \neq b$.

b) Remember that a simple function can be written as

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where the E_j are mutually disjoint sets, with finite measure, and $a_1 > a_2 > \cdots > a_n > 0$. Another way to write s is

$$s(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x),$$

where the b_k are positive and each F_k have finite measure, and they form an increasing sequence $F_1 \subset F_2 \subset \cdots \subset F_n$.

Comparing the two expressions, we see that

$$b_k = a_k - a_{k-1}, \quad F_k = \bigcup_{j=1}^{k} E_j \quad (k = 1, \dots, n).$$

In this case (see [14, p. 40]), the rearrangement of s is given by

$$f^* = \sum_{k=1}^{n} b_k \chi_{[0, m_2(F_k)]}.$$

We proceed to prove that $f_2^* = f^*$.

- Case f is simple: In this case

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x), \quad \text{with} \quad b_k = a_k - a_{k-1}, \quad F_k = \bigcup_{j=1}^{k} E_j \quad (k = 1, \dots, n).$$

(See Figure **2-9**). Then

$$f_2^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)\, dt$$

$$= \int_{a_2}^{a_1} \chi_{E_1^*}(x)\,dt + \int_{a_3}^{a_2} \chi_{(E_1 \cup E_2)^*}(x)\,dt + \cdots + \int_{0}^{a_n} \chi_{(E_1 \cup \cdots \cup E_n)^*}(x)\,dt$$
$$= \chi_{E_1^*}(x)[a_1 - a_2] + \chi_{(E_1 \cup E_2)^*}(x)[a_2 - a_3] + \cdots + \chi_{(E_1 \cup \cdots \cup E_n)^*}(x)[a_n]$$
$$= \sum_{k=1}^{n} b_k \chi_{F_k^*}(x).$$

Obviously $b_k > 0$ and F_k^* is an increasing sequence of sets. Hence

$$\begin{aligned}(f_2^*)^* &= \left(\sum_{k=1}^{n} b_k \chi_{F_k^*}(x) \right)^* \\ &= \sum_{k=1}^{n} b_k \chi_{[0, m_2(F_k^*)]}, \quad \text{see [14, p. 40]} \\ &= \sum_{k=1}^{n} b_k \chi_{[0, m_2(F_k)]}, \quad \text{by Proposition 2.1.11 a)} \\ &= f^*, \quad \text{again, by [14, p. 40].}\end{aligned}$$

We can also calculate $(f_2^*)^*$ using the canonical representation of f as follows. Taking $f = \sum_{j=1}^{n} a_j \chi_{E_j}$, where $a_1 > \cdots > a_n > 0$ and $E_i \cap E_j = \emptyset$ if $i \neq j$, we have $f_2^* = \sum_{j=1}^{n} a_j \chi_{F_j^* \smallsetminus F_{j-1}^*}$ where $F_j = \bigcup_{k=1}^{j} E_k$ and $F_0 = \emptyset$. Note that f_2^* is a simple function given in canonical form since $\left\{ \left(F_j^* \smallsetminus F_{j-1}^* \right) \right\}_{j=1}^{n}$ is a sequence of mutually disjoint sets. Therefore, setting

$$m_k := \sum_{i=1}^{k} m_2\left(F_j^* \smallsetminus F_{j-1}^*\right) \underset{Proposition 2.1.11d)}{=} \sum_{i=1}^{k} m_2\left(E_j\right),$$

we have

$$\begin{aligned}(f_2^*)^* &= \left(\sum_{j=1}^{n} a_j \chi_{F_j^* \smallsetminus F_{j-1}^*} \right)^* \\ &= \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j]}, \quad \text{see [14] page 38} \\ &= f^*.\end{aligned}$$

- Case f is a measurable function: From Proposition 2.1.19 we know that there exists a sequence $(s_n)_n$ of simple functions such that

$$\begin{aligned} & (s_n)_2^* \uparrow f_2^* \\ \Rightarrow \quad & ((s_n)_2^*)^* \uparrow (f_2^*)^* \quad (\text{A property of } ^*) \\ \Rightarrow \quad & s_n^* \uparrow (f_2^*)^*.\end{aligned}$$

Since also $s_n \uparrow f \Rightarrow s_n^* \uparrow f^*$, by the uniqueness of the limit we conclude that $(f_2^*)^* = f^*$.

Observe that using b) one can give an alternative proof of a). □

2.1.1 Hardy-Littlewood inequality for the two-dimensional decreasing rearrangement

For measurable functions f, g defined on $\mathbb{R}^2$, the Hardy-Littlewood inequality (see [14, Theorem 2.2]) states that

$$\int_{\mathbb{R}^2} |f(x)g(x)|\,dx \le \int_0^\infty f^*(s)g^*(s)\,ds.$$

We will show that the two-dimensional rearrangement allows us to obtain a better estimate. In order to prove it, we state the following lemma.

Lemma 2.1.22. *Let g a simple non-negative function on $\mathbb{R}^2$ and let E an arbitrary set of $\mathbb{R}^2$. Then*

$$\int_E g(x)\,dx \le \int_{E^*} g_2^*(x)\,dx.$$

Proof. Let

$$g(x) = \sum_{j=1}^n a_j \chi_{E_j}(x),$$

where $a_1 > \cdots > a_n > 0$, $a_{n+1} = 0$, and $E_j \subset \mathbb{R}^2$ are of finite measure such that $E_i \cap E_j = \emptyset$ if $i \neq j$. Another representation of g is

$$g(x) = \sum_{j=1}^n b_j \chi_{F_j}(x),$$

where $b_j > 0$, $b_j = a_j - a_{j-1}$, and $F_j = \bigcup_{i=1}^j E_i$. Then

$$\begin{aligned} g_2^*(x) &= \int_0^\infty \chi_{\{|g|>t\}^*}(x)\,dt \\ &= \int_0^\infty \chi_{\left\{\sum_{j=1}^n a_j\chi_{E_j}>t\right\}^*}(x)\,dt \\ &= \int_{a_2}^{a_1} \chi_{E_1^*}(x)\,dt + \int_{a_3}^{a_2} \chi_{(E_1\cup E_2)^*}(x)\,dt + \cdots + \int_0^{a_n} \chi_{(E_1\cup\cdots\cup E_n)^*}(x)\,dt \\ &= (a_1 - a_2)\,\chi_{E_1^*}(x) + (a_2 - a_3)\,\chi_{(E_1\cup E_2)^*}(x) + \cdots + a_n\chi_{(E_1\cup\cdots\cup E_n)^*}(x) \\ &= \sum_{j=1}^n b_j \chi_{F_j^*}(x). \end{aligned} \tag{2.17}$$

Since $(F_j \cap E)^* \subset F_j^* \cap E^*$ (see Remark 2.1.14), we have

$$\begin{aligned}\int_E g(x)\,dx &= \int_E \sum_{j=1}^n b_j \chi_{F_j}(x)\,dx \\ &= \sum_{j=1}^n b_j \left(\int_E \chi_{F_j}(x)\,dx\right) \\ &= \sum_{j=1}^n b_j m_2\left(F_j \cap E\right) \\ &= \sum_{j=1}^n b_j m_2\left((F_j \cap E)^*\right) \\ &= \sum_{j=1}^n b_j \left(\int_{(F_j\cap E)^*} dx\right) \\ &\le \sum_{j=1}^n b_j \left(\int_{F_j^*\cap E^*} dx\right) \\ &= \sum_{j=1}^n b_j \left(\int_{E^*} \chi_{F_j^*}(x)\,dx\right) \\ &= \int_{E^*} \sum_{j=1}^n b_j \chi_{F_j^*}(x)\,dx \\ &= \int_{E^*} g_2^*(x)\,dx. \end{aligned}$$ □

Theorem 2.1.23. *If f and g are measurable functions on $\mathbb{R}^2$, then*

$$\int_{\mathbb{R}^2} |f(x)g(x)|\,dx \le \int_{\mathbb{R}^2_+} f_2^*(x) g_2^*(x)\,dx \le \int_0^\infty f^*(t)g^*(t)\,dt.$$

Proof. Since $f_2^* = |f|_2^*$ and $f^* = |f|^*$, it's enough to prove the theorem only for non-negative f and g.

In view of Proposition 2.1.18 e), and the monotone convergence theorem, there's no loss of generality in assuming that f and g are simple functions. Let

$$f(x) = \sum_{j=1}^n b_j \chi_{F_j}(x),$$

where $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset \mathbb{R}^2$ are sets of finite measure, and $b_j > 0$. Then, by Lemma 2.1.22, we have

$$\int_{\mathbb{R}^2} f(x)g(x)\,dx = \int_{\mathbb{R}^2} \left(\sum_{j=1}^n b_j \chi_{F_j}(x)\right) g(x)\,dx$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(\int_{\mathbb{R}^2} b_j \chi_{F_j}(x) g(x) \right) dx \\
&= \sum_{j=1}^{n} b_j \int_{F_j} g(x)\, dx \\
&\le \sum_{j=1}^{n} b_j \int_{F_j^*} g_2^*(x)\, dx, \quad \text{by Lemma 2.1.22} \\
&= \sum_{j=1}^{n} \left(\int_{\mathbb{R}^2_+} b_j \chi_{F_j^*}(x) g_2^*(x)\, dx \right) \\
&= \int_{\mathbb{R}^2_+} f_2^*(x) g_2^*(x)\, dx.
\end{aligned}
$$

For the second inequality, take

$$F(x) = f_2^*(x) \quad \text{and} \quad G(x) = g_2^*(x) \quad (x \in \mathbb{R}^2_+).$$

By the Hardy-Littlewood inequality,

$$\int_{\mathbb{R}^2_+} F(x)G(x)\, dx \le \int_0^\infty F^*(t)G^*(t)\, dt.$$

And from Corollary 2.1.20 b)

$$F^* = (f_2^*)^* = f^* \quad \text{and} \quad G^* = (g_2^*)^* = g^*.$$

Therefore,

$$\int_{\mathbb{R}^2_+} f_2^*(x) g_2^*(x)\, dx \le \int_0^\infty f^*(t) g^*(t)\, dt. \qquad \square$$

Corollary 2.1.24. *If f is a non-negative measurable function on $\mathbb{R}^2$, and D is a decreasing set, then*

$$\sup_{E^*=D} \int_E f(x)\, dx \le \int_D f_2^*(x)\, dx \le \int_0^{m_2(D)} f^*(t)\, dt,$$

both inequalities may be strict for some f and some D.

Proof. From Theorem 2.1.23 we know that

$$\int_{\mathbb{R}^2} |f(x)g(x)|\, dx \le \int_{\mathbb{R}^2_+} f_2^*(x) g_2^*(x)\, dx \le \int_0^\infty f^*(t) g^*(t)\, dt.$$

Taking $g = \chi_D$, we obtain

$$\int_{\mathbb{R}^2} |f(x)| \chi_D(x)\, dx \quad \le \quad \int_{\mathbb{R}^2_+} f_2^*(x) \left(\chi_D\right)_2^*(x)\, dx \quad \le \quad \int_0^\infty f^*(t) \left(\chi_D\right)^*(t)\, dt$$

$$\Rightarrow \quad \int_D |f(x)|\,dx \quad \le \quad \int_{\mathbb{R}^2_+} f_2^*(x)\chi_{D^*}(x)\,dx \quad \le \quad \int_0^\infty f^*(t)\chi_{[0,m_2(D))}(t)\,dt$$

$$\Rightarrow \quad \int_D |f(x)|\,dx \quad \le \quad \int_{D^*} f_2^*(x)\,dx \quad \le \quad \int_0^{m_2(D)} f^*(t)\,dt.$$

Since D is decreasing, $D^* = D$. Hence

$$\int_D |f(x)|\,dx \le \int_D f_2^*(x)\,dx \le \int_0^{m_2(D)} f^*(t)\,dt.$$

Note that $\int_D f_2^*$ is an upper bound for the set

$$\left\{ \int_D |f(x)|\,dx : D \text{ is decreasing} \right\}.$$

Then

$$\sup_{\{E:E^*=D\}} \int_E |f(x)|\,dx \le \int_D f_2^*(x)\,dx.$$

Let's see that the inequality

$$\sup_{E^*=D} \int_E f(x)\,dx \le \int_D f_2^*(x)\,dx,$$

may be strict. For this, consider the sets $A = (3,4) \times (0,1)$, $B = (4,6) \times (0,2)$, $D = (0,1) \times (0,2)$, and the function $f(x) = 2\chi_A(x) + \chi_B(x)$. On the one hand, if E is any set with $E^* = D$,

$$\begin{aligned}
\int_E f(x)\,dx &= \int_E \left(2\chi_A(x) + \chi_B(x)\right)\,dx \\
&= 2\int_E \chi_A(x)\,dx + \int_E \chi_B(x)\,dx \\
&= 2m(A \cap E) + m(B \cap E).
\end{aligned}$$

- Case 1: If $A \cap E = \emptyset$,

$$\int_E f(x)\,dx = m_2\left(B \cap E\right) \le m_2\left(E\right) = m_2\left(E^*\right) = m_2\left(D\right) = 2.$$

- Case 2: If $B \cap E = \emptyset$,

$$\int_E f(x)\,dx = 2m_2\left(A \cap E\right) \le 2m_2\left(A\right) = 2.$$

- Case 3: If $A \cap E \neq \emptyset$, since $E^* = D = (0,1) \times (0,2)$, we have $E = (a,b) \times (c,d)$ with $3 < a < 4$, $0 < c < 1$ (or $0 < d < 1$), $b - a = 1$ and $d - c = 2$.

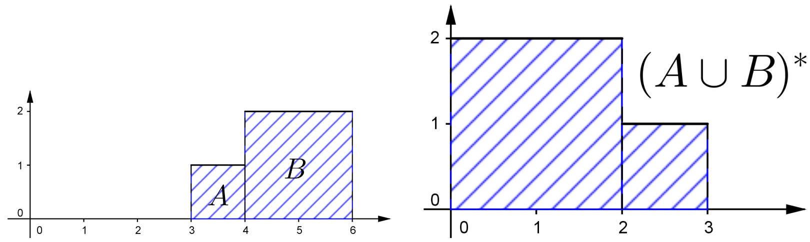

Figure 2-10: The graphs of the sets $A \cup B$ and $(A \cup B)^*$ in the proof of Corollary 2.1.24.

Note that $A \cap E = (a, 4) \times (c, 1)$, $m_2(A \cap E) = (4 - a)(1 - c)$. Taking $F = (a, 4) \times (1, 2 - c)$, we have $(A \cap E) \cap F = \emptyset$ and $m_2(F) = (4 - a)(1 - c) = m_2(A \cap E)$. Hence,

$$\begin{aligned} & (A \cap E) \cup F \subseteq (a, 4) \times (c, d) \\ \Rightarrow \quad & m_2((A \cap E) \cup F) \leq (4 - a)(d - c) \\ \Rightarrow \quad & m_2(A \cap E) + m_2(F) \leq (4 - a)(d - c) \\ \Rightarrow \quad & 2m_2(A \cap E) \leq (4 - a)(d - c). \end{aligned}$$

In a similar way,

$$B \cap E \subseteq (4, b) \times (c, d) \Rightarrow m_2(B \cap E) \leq (b - 4)(d - c).$$

Therefore,

$$\begin{aligned} 2m_2(A \cap E) + m_2(B \cap E) &\leq (4 - a)(d - c) + (b - 4)(d - c) \\ &= (d - c)(4 - a + b - 4) \\ &= m_2(E) \\ &= 2. \end{aligned}$$

In any case, $\int_E f(x)\,dx \leq 2$. In the other hand, we calculate $\int_D f_2^*(x)\,dx$.
Since $f(x) = 2\chi_A(x) + \chi_B(x)$, using the observation given in the proof of Proposition 2.1.19, we have that $f_2^* = 2\chi_{A^*}(x) + \chi_{(A\cup B)^* \smallsetminus A^*}(x)$, where $A = (3, 4) \times (0, 1)$ and $B = (4, 6) \times (0, 2)$. We know that $A^* = (0, 1) \times (0, 1)$. Let's calculate $(A \cup B)^*$.

$$\varphi_{A\cup B}(x) = m_1((A \cup B)_x) = \chi_{(3,4)}(x) + 2\chi_{(4,6)}(x),$$

then

$$\varphi^*_{A\cup B}(x) = 2\chi_{(0,2)}(x) + \chi_{(2,3)}(x),$$

so

$$\begin{aligned} (A \cup B)^* &= \left\{(s, t) \in \mathbb{R}^2_+ : 0 < t < \varphi^*_{A\cup B}(s)\right\} \\ &= \left\{(s, t) \in \mathbb{R}^2_+ : 0 < t < 2\chi_{(0,2)}(s) + \chi_{(2,3)}(s)\right\} \end{aligned}$$

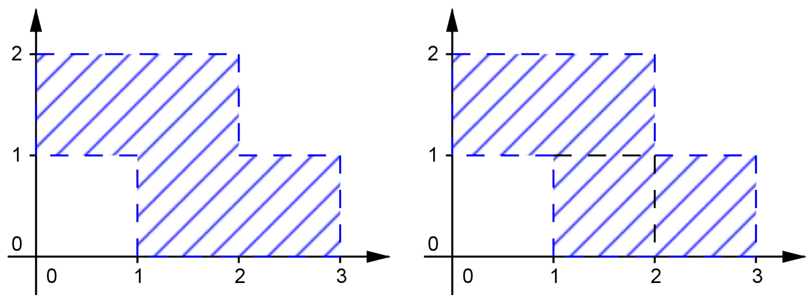

Figure 2-11: Two sets which are equal a.e.

$$= (0,2)\times(0,2)\cup(2,3)\times(0,1) \quad \text{(See Figure } \mathbf{2\text{-}10}).$$

From the above, we conclude that

$$\begin{aligned} f_2^* &= 2\chi_{A^*}(x) + \chi_{(A\cup B)^*\setminus A^*}(x) \\ &= 2\chi_{(0,1)\times(0,1)}(x) + \chi_{[(0,2)\times(0,2)\cup(2,3)\times(0,1)]\setminus(0,1)\times(0,1)}(x). \end{aligned}$$

Hence,

$$\begin{aligned} \int_D f_2^* &= \int_0^1 \left(\int_0^2 f_2^*(x,y)\,dy\right)dx \\ &= \int_0^1 \left[\int_0^2 \left(2\chi_{(0,1)\times(0,1)}(x,y) + \chi_{[(0,2)\times(0,2)\cup(2,3)\times(0,1)]\setminus(0,1)\times(0,1)}(x,y)\right)dy\right]dx. \end{aligned}$$

Note that

$$[(0,2)\times(0,2)\cup(2,3)\times(0,1)]\setminus(0,1)\times(0,1) = \underbrace{(0,2)\times(1,2)\cup(1,3)\times(0,1)}_{\text{disjoint}} \quad \text{a.e.}$$

(See Figure **2-11**). Then

$$\begin{aligned} \int_D f_2^* &= \int_0^1 \left[\int_0^2 \left(2\chi_{(0,1)\times(0,1)}(x,y) + \chi_{[(0,2)\times(0,2)\cup(2,3)\times(0,1)]\setminus(0,1)\times(0,1)}(x,y)\right)dy\right]dx \\ &= \int_0^1 \left[\int_0^2 \left(2\chi_{(0,1)\times(0,1)}(x,y) + \chi_{(0,2)\times(1,2)\cup(1,3)\times(0,1)}(x,y)\right)dy\right]dx \\ &= \int_0^1 \left[\int_0^2 \left(2\chi_{(0,1)\times(0,1)}(x,y) + \chi_{(0,2)\times(1,2)}(x,y) + \chi_{(1,3)\times(0,1)}(x,y)\right)dy\right]dx \\ &= \int_0^1 \left[\int_0^2 \left(2\chi_{(0,1)}(x)\chi_{(0,1)}(y) + \chi_{(0,2)}(x)\chi_{(1,2)}(y) + \chi_{(1,3)}(x)\chi_{(0,1)}(y)\right)dy\right]dx \\ &= \int_0^1 \left[2\chi_{(0,1)}(x) + \chi_{(0,2)}(x) + \chi_{(1,3)}(x)\right]dx \\ &= 3. \end{aligned}$$

So, we conclude that

$$\int_E f(x)\,dx \le 2 < 3 = \int_D f_2^*(x)\,dx.$$

For the second inequality, consider $D_\varepsilon = (0,\varepsilon)\times(0,1/\varepsilon)$, and the same function f. We have $f = 2\chi_A + \chi_B$ with $A = (3,4)\times(0,1)$, $m_2(A) = 1$; $B = (4,6)\times(0,2)$, $m_2(B) = 4$. Then

$$\begin{aligned} f^* &= 2\chi_{[0,m_2(A))} + \chi_{[m_2(A),m_2(A)+m_2(B))} \\ &= 2\chi_{[0,1)} + \chi_{[1,5)}. \end{aligned}$$

Since $m_2(D_\varepsilon) = \varepsilon\cdot\frac{1}{\varepsilon} = 1$, we have

$$\int_0^{m_2(D_\varepsilon)} f^*(t)\,dt = \int_0^1 \big(2\chi_{[0,1)}(t) + \chi_{[1,5)}(t)\big)\,dt = 2, \quad \forall\,\varepsilon > 0.$$

But

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{D_\varepsilon} f_2^*(x)\,dx \\ &= \lim_{\varepsilon\to 0}\int_0^\varepsilon \left[\int_0^{1/\varepsilon} \big(2\chi_{(0,1)}(x)\chi_{(0,1)}(y) + \chi_{(0,2)}(x)\chi_{(1,2)}(y) + \chi_{(1,3)}(x)\chi_{(0,1)}(y)\big)\,dy\right]dx \\ &= \lim_{\varepsilon\to 0} 2m_1((0,1)\cap(0,\varepsilon))\cdot m_1((0,1)\cap(0,1/\varepsilon)) \\ &\quad + m_1((0,2)\cap(0,\varepsilon))\cdot m_1((1,2)\cap(0,1/\varepsilon)) + m_1((1,3)\cap(0,\varepsilon))\cdot m_1((0,1)\cap(0,1/\varepsilon)) \\ &= 0. \end{aligned}$$

□

2.1.2 Two-dimensional decreasing rearrangement as an iterated rearrangement

The definition of the two-dimensional rearrangement is based in a geometrical approach: we obtain the rearrangement of the function by summing the rearrangement of its level sets (layer-cake formula). The following theorem shows that this is equivalent to an iterative process, in which one rearranges separately with respect to each variable (see [15] for some related work).

The notation used in the proof is as follows: given a function $f(x,y)$ defined on $\mathbb{R}^2$, we write $R_t(x) = (f_x)^{*y}(t)$, where $f_x(y) = f(x,y)$ and $t > 0$ (i.e. R_t is the usual rearrangement of the function f_x with respect to the variable y). In a similar way, we write $\tilde f(s,t) = (R_t)^{*x}(s)$, $s,t > 0$. It's easy to show that, in general, we don't get the same function if we first rearrange with respect to x and the respect to y, as is shown in Example 2.1.28.

Theorem 2.1.25. *If f is a measurable function on $\mathbb{R}^2$, then*

$$f_2^*(s,t) = \tilde f(s,t), \quad \forall\; s,t > 0.$$

Proof. Using Proposition 2.1.19, it's enough to consider the case in which f is a simple function. So, let $f(x,y)=\sum_{j=1}^{n} a_j \chi_{E_j}(x,y)$ with $a_1>a_2>\cdots>a_n>0$, $E_j\cap E_k=\emptyset$ if $j\neq k$. Take $F_k=\bigcup_{j=1}^{k} E_j$, $F_0=\emptyset$, then

$$f_2^*(s,t)=\sum_{j=1}^{n} a_j \chi_{F_j^*\smallsetminus F_{j-1}^*}(s,t).$$

Remember that $\varphi_E(x)=m_1(E_x)=m_1(\{y\in\mathbb{R}:(x,y)\in E\})$ and $E^*=\{(s,t):0<t<\varphi_E^*(s)\}$. Hence,

$$\chi_{E^*}(s,t)=\chi_{\left\{0<t<\varphi_E^*(s)\right\}}(s,t)=\chi_{\left(0,\varphi_E^*(s)\right)}(t)\underset{\text{A property of }^*}{=}\chi_{\left(0,D_{\varphi_E}(t)\right)}(s).$$

So,

$$\begin{aligned}\chi_{F_j^*\smallsetminus F_{j-1}^*}(s,t)&=\chi_{F_j^*}(s,t)-\chi_{F_{j-1}^*}(s,t),\quad \text{since } F_{j-1}^*\subseteq F_j^*\\ &=\chi_{\left(0,D_{\varphi_{F_j}}(t)\right)}(s)-\chi_{\left(0,D_{\varphi_{F_{j-1}}}(t)\right)}(s)\\ &=\chi_{\left[D_{\varphi_{F_{j-1}}}(t),D_{\varphi_{F_j}}(t)\right)}(s).\end{aligned}$$

So we get

$$f_2^*(s,t)=\sum_{j=1}^{n} a_j \chi_{F_j^*\smallsetminus F_{j-1}^*}(s,t)=\sum_{j=1}^{n} a_j \chi_{\left[D_{\varphi_{F_{j-1}}}(t),D_{\varphi_{F_j}}(t)\right)}(s). \tag{2.18}$$

On the other hand, since $\left(\chi_E(x,y)\right)_x=\chi_{E_x}(y)$, then

$$f(x,y)=\sum_{j=1}^{n} a_j\chi_{E_j}(x,y)\Rightarrow f_x(y)=\left(\sum_{j=1}^{n} a_j\chi_{E_j}(x,y)\right)_x=\sum_{j=1}^{n} a_j\chi_{(E_j)_x}(y).$$

Thus

$$\begin{aligned}R_t(x)&=\left(f_x\right)^{*y}(t)\\ &=\sum_{j=1}^{n} a_j\chi_{\left[m_1\left((F_{j-1})_x\right),m_1\left((F_j)_x\right)\right)}(t)\\ &=\sum_{j=1}^{n} a_j\chi_{\left[\varphi_{F_{j-1}}(x),\varphi_{F_j}(x)\right)}(t)\\ &=\sum_{j=1}^{n} a_j\chi_{H_j(t)}(x),\end{aligned}$$

where $H_j(t)=\left\{y:\varphi_{F_{j-1}}(y)\le t<\varphi_{F_j}(y)\right\}$. Hence,

$$\tilde{f}(s,t)=\left(R_t\right)^{*x}(s)=\sum_{j=1}^{n} a_j\chi_{\left[m_1(G_{j-1}(t)),m_1(G_j(t))\right)}, \tag{2.19}$$

where $G_j(t) = \bigcup_{k=1}^{j} H_k(t)$, $G_0 = \emptyset$. Thus, looking at (2.18) and (2.19), it's enough to show that

$$m_1\left(G_j(t)\right) = D_{\varphi_{F_j}}(t).$$

But, indeed

$$\begin{aligned} m_1\left(G_j(t)\right) &= m_1\left(\bigcup_{k=1}^{j} H_k(t)\right) \\ &= \sum_{k=1}^{j} m_1\left(H_k(t)\right) \\ &= \sum_{k=1}^{j} m_1\left(\left\{y : \varphi_{F_{k-1}}(y) \leq t < \varphi_{F_k}(y)\right\}\right) \\ &= m_1\left(\bigcup_{k=1}^{j} \left\{y : \varphi_{F_{k-1}}(y) \leq t < \varphi_{F_k}(y)\right\}\right) \\ &= m_1\left(\left\{y : t < \varphi_{F_j}(y)\right\}\right) \\ &= D_{\varphi_{F_j}}(t), \end{aligned}$$

and this completes the proof. □

As an immediate consequence of the previous theorem, we show how to obtain the two-dimensional rearrangement of a special type of functions.

Corollary 2.1.26. *If g and h are two measurable functions on $\mathbb{R}$, and $f(x,y) = g(x)h(y)$, then $f_2^*(s,t) = g^*(s)h^*(t)$.*

Proof. Let's calculate $\tilde{f}(s,t)$. We know that

$$f_x(y) = f(x,y) = g(x)h(y) \quad \text{For a fixed } x.$$

Then

$$R_t(x) = \left(f_x\right)^{*y}(t) = (g \cdot h)^{*y}(t) = |g(x)|h^*(t).$$

Here we used the following property: $(cf)^* = |c|f^*$. Hence,

$$\tilde{f}(s,t) = \left(R_t\right)^{*x}(s) = \left(|g|h^*\right)^{*x}(s) = |h^*(t)|\left(|g|\right)^{*x}(s) = h^*(t)(g)^{*x}(s) = h^*(t) \cdot g^*(s).$$

Where we used the property appointed above, and also $|f|^* = f^*$. In conclusion,

$$\tilde{f}(s,t) = h^*(t)g^*(s) = f_2^*(s,t).$$ □

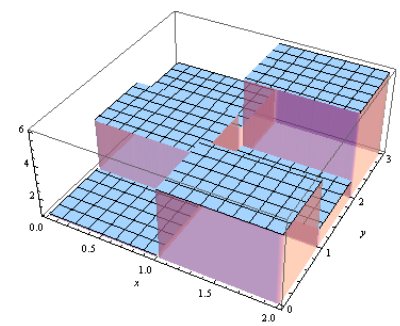

Figure 2-12: The graph of the simple function f used in Example 2.1.28.

Example 2.1.27. *It is shown in [30, Example 1.4.4] that, for the function* $g(x) = 1 - e^{-|x|^2}$, *one has* $g^*(s) = 1$ *for all* $s \geq 0$. *As a consequence of Corollary 2.1.26, we infer that, for the function*

$$f(x,y) = 1 + e^{-|x|^2-|y|^2} - e^{-|x|^2} - e^{-|y|^2} = \left(1 - e^{-|x|^2}\right)\left(1 - e^{-|y|^2}\right).$$

Its two-dimensional decreasing rearrangement is

$$f_2^*(s,t) = 1.$$

Example 2.1.28. *The following example shows us that the order in which the rearrangement takes place is fundamental. That is to say, in general we get different functions if the order to calculate the rearrangement is changed.*
Consider the function $f(x,y) = \sum_{i,j}^{2,3} C(i,j)\chi_{E(i,j)}(x,y)$, *where* $E(i,j) = [i-1,i) \times [j-1,j)$ *and* $C(1,1) = 1$, $C(1,2) = 4$, $C(1,3) = 3$, $C(2,1) = 5$, $C(2,2) = 2$ *and* $C(2,3) = 6$. *That is,*

$$\begin{aligned} f(x,y) = \chi_{[0,1)\times[0,1)}(x,y) + 4\chi_{[0,1)\times[1,2)}(x,y) + 3\chi_{[0,1)\times[2,3)}(x,y) \\ + 5\chi_{[1,2)\times[0,1)}(x,y) + 2\chi_{[1,2)\times[0,1)}(x,y) + 6\chi_{[1,2)\times[2,3)}(x,y). \end{aligned}$$

Let's calculate $f_2^*(s,t) = \tilde{f}(s,t)$. *For that purpose, we are going to write* f *in the following way*

$$\begin{aligned} f(x,y) = \chi_{[0,1)}(x)\left[4\chi_{[1,2)}(y) + 3\chi_{[2,3)}(y) + \chi_{[0,1)}(y)\right] \\ + \chi_{[1,2)}(x)\left[6\chi_{[2,3)}(y) + 5\chi_{[0,1)}(y) + 2\chi_{[1,2)}(y)\right]. \end{aligned}$$

We have

$$R_t(x) = (f_x)^{*y}(t) = \begin{cases} 4\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t) + \chi_{[2,3)}(t), & \text{if } x \in [0,1) \\ 6\chi_{[0,1)}(t) + 5\chi_{[1,2)}(t) + 2\chi_{[2,3)}(t), & \text{if } x \in [1,2). \end{cases}$$

So

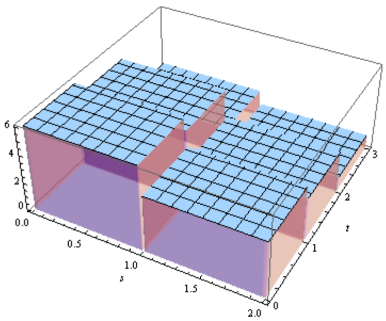

Figure 2-13: The graph of f_2^* in Example 2.1.28.

$$R_t(x) = \left[4\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t) + \chi_{[2,3)}(t)\right]\chi_{[0,1)}(x) + \left[6\chi_{[0,1)}(t) + 5\chi_{[1,2)}(t) + 2\chi_{[2,3)}(t)\right]\chi_{[1,2)}(x).$$

And then

$$R_t(x) = \begin{cases} 6\chi_{[1,2)}(x) + 4\chi_{[0,1)}(x), & \text{if } t \in [0,1) \\ 5\chi_{[1,2)}(x) + 3\chi_{[0,1)}(x), & \text{if } t \in [1,2) \\ 2\chi_{[1,2)}(x) + \chi_{[0,1)}(x), & \text{if } t \in [2,3). \end{cases}$$

Now, $\tilde{f}(s,t) = (R_t)^{*x}(s)$, *therefore*

$$(R_t)^{*x}(s) = \begin{cases} 6\chi_{[0,1)}(s) + 4\chi_{[1,2)}(s), & \text{if } t \in [0,1) \\ 5\chi_{[0,1)}(s) + 3\chi_{[1,2)}(s), & \text{if } t \in [1,2) \\ 2\chi_{[0,1)}(s) + \chi_{[1,2)}(s), & \text{if } t \in [2,3). \end{cases} = \tilde{f}(s,t) = f_2^*(s,t).$$

Now, we calculate the iterated rearrangement but in the reverse order. We will use the following notation

$$f_y(x) = f(x,y), \quad G_t(y) = (f_y)^{*x}(t) \quad \text{and} \quad \hat{f}(s,t) = (G_t)^{*y}(s).$$

We write f *as*

$$f(x,y) = \chi_{[0,1)}(y)\left[5\chi_{[1,2)}(x) + \chi_{[0,1)}(y)\right] + \chi_{[1,2)}(y)\left[4\chi_{[0,1)}(x) + 2\chi_{[1,2)}(x)\right] + \chi_{[2,3)}(y)\left[6\chi_{[1,2)}(x) + 3\chi_{[0,1)}(x)\right].$$

So

$$G_t(y) = (f_y)^{*x}(t) = \begin{cases} 5\chi_{[0,1)}(t) + \chi_{[1,2)}(t), & \text{if } y \in [0,1) \\ 4\chi_{[0,1)}(t) + 2\chi_{[1,2)}(t), & \text{if } y \in [1,2) \\ 6\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t), & \text{if } y \in [2,3). \end{cases}$$

We express G_t *as*

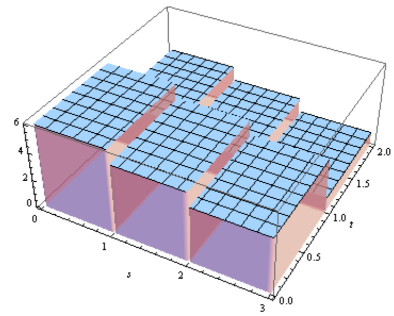

Figure 2-14: The graph of $\hat{f}$ in Example 2.1.28.

$$G_t(y) = \left[5\chi_{[0,1)}(t) + \chi_{[1,2)}(t)\right]\chi_{[0,1)}(y) + \left[4\chi_{[0,1)}(t) + 2\chi_{[1,2)}(t)\right]\chi_{[1,2)}(y) + \left[6\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t)\right]\chi_{[2,3)}(y).$$

Thus

$$G_t(y) = \begin{cases} 6\chi_{[2,3)}(y) + 5\chi_{[0,1)}(y) + 4\chi_{[1,2)}(y), & \text{if } t \in [0,1) \\ 3\chi_{[2,3)}(y) + 2\chi_{[1,2)}(y) + \chi_{[0,1)}(y), & \text{if } t \in [1,2). \end{cases}$$

Since $\hat{f}(s,t) = (G_t)^{*y}(s)$, *we have*

$$(G_t)^{*y}(s) = \begin{cases} 6\chi_{[0,1)}(s) + 5\chi_{[1,2)}(s) + 4\chi_{[2,3)}(s), & \text{if } t \in [0,1) \\ 3\chi_{[0,1)}(s) + 2\chi_{[1,2)}(s) + \chi_{[2,3)}(s), & \text{if } t \in [1,2) \end{cases} = \hat{f}(s,t).$$

Looking at f_2^* *and* $\hat{f}$, *we see that*

$$f_2^*(s,t) \neq \hat{f}(s,t).$$

Another application of Theorem 2.1.25 is that the inequality proved in Proposition 2.1.18 d), can be improved to obtain the classical subadditivity condition $(f+g)_2^*(x+y) \leq f_2^*(x)+g_2^*(y)$.

Corollary 2.1.29. *If f and g are measurable functions on $\mathbb{R}^2$, then*

$$(f+g)_2^*(x+y) \leq f_2^*(x) + g_2^*(y).$$

Proof. With the notation of Proposition 2.1.18, we know that $f_2^* = \tilde{f}$. In this way, we have that

$$\begin{aligned} f_2^*(s,t) &= (R_t)^{*x}(s) \quad \text{where} \quad R_t(x) = (f_x)^{*y}(t) \\ g_2^*(s,t) &= (U_t)^{*x}(s) \quad \text{where} \quad U_t(x) = (g_x)^{*y}(t) \\ (f+g)_2^*(s,t) &= (V_t)^{*x}(s) \quad \text{where} \quad V_t(x) = ([f+g]_x)^{*y}(t). \end{aligned}$$

We know that $[f+g]_x = f_x + g_x$, then

$$\begin{aligned} V_{t_1+t_2}(x) &= ([f+g]_x)^{*y}(t_1+t_2) \\ &= (f_x+g_x)^{*y}(t_1+t_2) \\ &\leq (f_x)^{*y}(t_1) + (g_x)^{*y}(t_2) \\ &= R_{t_1}(x) + U_{t_2}(x). \end{aligned} \tag{2.20}$$

The above inequality is valid since

$$(z+w)^*(t_1+t_2) \leq z^*(t_1) + w^*(t_2).$$

From (2.20) we have that $V_{t_1+t_2} \leq R_{t_1} + U_{t_2}$. Since * is monotone, we have

$$\begin{aligned}(V_{t_1+t_2})^{*x}(s_1+s_2) &\leq (R_{t_1}+U_{t_2})^{*x}(s_1+s_2) \\ &\leq (R_{t_1})^{*x}(s_1) + (U_{t_2})^{*x}(s_2).\end{aligned}$$

That is

$$\left(\widetilde{f+g}\right)(s_1+s_2, t_1+t_2) \leq \tilde{f}(s_1,t_1) + \tilde{g}(s_2,t_2).$$

Taking $x=(s_1,t_1)$ and $y=(s_2,t_2)$, we obtain

$$(f+g)_2^*(x+y) \leq f_2^*(x) + g_2^*(y).$$

□

2.2 The multidimensional Lorentz spaces $\Lambda_2^p(w)$

Now we'll use the two-dimensional decreasing rearrangement to define a Lorentz-type space. Remember the definition of the classical Lorentz space: If v is a weight on $\mathbb{R}^+$ and $0<p<\infty$,

$$\Lambda^p(v) = \left\{ f : \mathbb{R}^n \to \mathbb{C} : \|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p\, v(t)\, dt \right)^{1/p} < \infty \right\}.$$

Definition 2.2.1. We say that a measurable function f on $\mathbb{R}^2$ belongs to the (multidimensional) Lorentz space $\Lambda_2^p(w)$, if the norm $\|f\|_{\Lambda_2^p(w)}$, given by

$$\|f\|_{\Lambda_2^p(w)} := \left(\int_{\mathbb{R}_+^2} (f_2^*(x))^p\, w(x)\, dx \right)^{1/p}, \tag{2.21}$$

is finite. Here w is a non-negative function, locally integrable over $\mathbb{R}_+^2$, not identically 0.

The next result gives an alternative description of the norm $L^p_{\mathbb{R}^2}$ in terms of the two dimensional decreasing rearrangement, i.e., the spaces defined above are a natural generalization of the Lebesgue spaces.

Theorem 2.2.2. *If $0<p<\infty$, then $\Lambda_2^p(1) = L^p_{\mathbb{R}^2}$.*

Proof.

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x)|^p\, dx &= \int_{\mathbb{R}^2} \left(\int_0^{|f(x)|^p} dt \right) dx \\ &= \int_{\mathbb{R}^2} \left(\int_0^\infty \chi_{[0,|f(x)|^p)}(t)\, dt \right) dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left(\int_{\mathbb{R}^2} \chi_{\{x:|f(x)|^p>t\}}(x)\, dx \right) dt \\
&= \int_0^\infty \left(\int_{\{x:|f(x)|^p>t\}} dx \right) dt \\
&= \int_0^\infty m_2 \left(\{x : |f(x)|^p > t\} \right) dt \\
&= \int_0^\infty m_2 \left(\{x : |f(x)|^p > t\}^* \right) dt \\
&= \int_0^\infty \left(\int_{\{x:|f(x)|^p>t\}^*} dx \right) dt \\
&= \int_{\mathbb{R}^2_+} \left(\int_0^\infty \chi_{\{x:|f(x)|^p>t\}^*}(x)\, dt \right) dx \\
&= \int_{\mathbb{R}^2_+} (f^p)^*_2 (x)\, dx \\
&= \int_{\mathbb{R}^2_+} (f^*_2)^p (x)\, dx.
\end{aligned}
$$

Note the use of Fubini's theorem in the third and eighth equality, and the use of Proposition 2.1.18 f) in the last equality. □

2.2.1 The spaces $\Lambda_2^p(w)$ and the rearrangement invariant spaces

It is worth to compare the spaces $\Lambda_2^p(w)$ with the classical rearrangement invariant spaces (see [14]). The following results show that this two types of spaces only agree in very particular cases.

Proposition 2.2.3. *If $\|\cdot\|_{\Lambda_2^p(w)}$ is a rearrangement invariant norm, then w is constant, and so $\Lambda_2^p(w) = L^p_{\mathbb{R}^2}$.*

Proof. Fix $(x, y) \in \mathbb{R}^2_+$, $0 < \varepsilon < \min\{x, y\}$, and define the sets

$$R = (0, x) \times (0, y), P_\varepsilon = (x - \varepsilon, x) \times (y - \varepsilon, y), Q_\varepsilon = (x, x + \varepsilon) \times (0, \varepsilon), A_\varepsilon = (R \setminus P_\varepsilon) \cup Q_\varepsilon.$$

(See Figure **2-15**.) Note that

$$
\begin{aligned}
m_2 (A_\varepsilon) = m_2 ((R \setminus P_\varepsilon) \cup Q_\varepsilon) &= m_2 (R \setminus P_\varepsilon) + m_2 (Q_\varepsilon) \\
&= m_2 (R) - m_2 (P_\varepsilon) + m_2 (Q_\varepsilon) = m_2 (R) - \varepsilon^2 + \varepsilon^2 = m_2 (R).
\end{aligned}
$$

This implies that χ_{A_ε} and χ_R are equimeasurable. Let's see the distribution functions of each one.

$$D_{\chi_{A_\varepsilon}}(\lambda) = m_2 \left(\left\{ x \in \mathbb{R}^2 : |\chi_{A_\varepsilon}(x)| > \lambda \right\} \right) = \begin{cases} m_2 (A_\varepsilon), & \text{if } 0 < \lambda < 1 \\ 0, & \text{if } \lambda \geq 1 \end{cases}$$

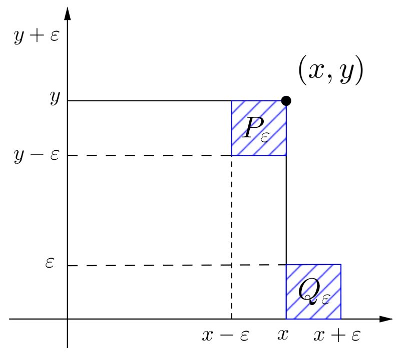

Figure 2-15: The sets P_ε and Q_ε in the proof of Proposition 2.2.3.

$$D_{\chi_R}(\lambda) = m_2\left(\left\{x \in \mathbb{R}^2 : |\chi_R(x)| > \lambda\right\}\right) = \begin{cases} m_2(R), & \text{if } 0 < \lambda < 1 \\ 0, & \text{if } \lambda \geq 1. \end{cases}$$

Then $D_{\chi_{A_\varepsilon}} = D_{\chi_R}$. Since $\|\cdot\|_{\Lambda_2^p(w)}$ is a rearrangement invariant norm and χ_{A_ε} and χ_R are equimeasurable, it holds that

$$\|\chi_R\|_{\Lambda_2^p(w)} = \|\chi_{A_\varepsilon}\|_{\Lambda_2^p(w)}.$$

But

$$\begin{aligned}
\|\chi_R\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2} \left[(\chi_R)_2^*(x)\right]^p w(x)\, dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2} \left[\chi_{R^*}(x)\right]^p w(x)\, dx\right)^{1/p}, \quad \text{by Proposition 2.1.11 c)} \\
&= \left(\int_{R^*} w(x)\, dx\right)^{1/p} \\
&= \left(\int_{R} w(x)\, dx\right)^{1/p}, \quad \text{since } R^* = R.
\end{aligned}$$

And also

$$\begin{aligned}
\|\chi_{A_\varepsilon}\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2} \left[(\chi_{A_\varepsilon})_2^*(x)\right]^p w(x)\, dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2} \left[\chi_{A_\varepsilon^*}(x)\right]^p w(x)\, dx\right)^{1/p}, \quad \text{by Proposition 2.1.11 c)} \\
&= \left(\int_{A_\varepsilon^*} w(x)\, dx\right)^{1/p} \\
&= \left(\int_{A_\varepsilon} w(x)\, dx\right)^{1/p}, \quad \text{since } A_\varepsilon^* = A_\varepsilon.
\end{aligned}$$

Then

$$\begin{aligned}
& \left(\int_R w(x)\,dx\right)^{1/p} = \left(\int_{A_\varepsilon} w(x)\,dx\right)^{1/p} \\
\Rightarrow \quad & \int_R w(x)\,dx = \int_{A_\varepsilon} w(x)\,dx \qquad (2.22)\\
\Rightarrow \quad & \int_R w(x)\,dx = \int_{(R\setminus P_\varepsilon)\cup Q_\varepsilon} w(x)\,dx \\
\Rightarrow \quad & \int_R w(x)\,dx = \int_{R\setminus P_\varepsilon} w(x)\,dx + \int_{Q_\varepsilon} w(x)\,dx \\
\Rightarrow \quad & \int_R w(x)\,dx = \int_R w(x)\,dx - \int_{P_\varepsilon} w(x)\,dx + \int_{Q_\varepsilon} w(x)\,dx \\
\Rightarrow \quad & \int_{P_\varepsilon} w(x)\,dx = \int_{Q_\varepsilon} w(x)\,dx.
\end{aligned}$$

Now, since $m_2(P_\varepsilon) = \varepsilon^2 = m_2(Q_\varepsilon)$, we have

$$\frac{1}{m_2(P_\varepsilon)} \int_{P_\varepsilon} w(x)\,dx = \frac{1}{m_2(Q_\varepsilon)} \int_{Q_\varepsilon} w(x)\,dx.$$

Noting that $P_\varepsilon \xrightarrow[\varepsilon\to 0]{} (x,y)$ and $Q_\varepsilon \xrightarrow[\varepsilon\to 0]{} (x,0)$, we invoke the Lebesgue's differentiation theorem. Hence

$$\begin{aligned}
& \lim_{\varepsilon\to 0} \frac{1}{m_2(P_\varepsilon)} \int_{P_\varepsilon} w(x)\,dx = \lim_{\varepsilon\to 0} \frac{1}{m_2(Q_\varepsilon)} \int_{Q_\varepsilon} w(x)\,dx \\
\Rightarrow \quad & w(x,y) = w(x,0). \qquad (2.23)
\end{aligned}$$

Using a symmetric argument, interchanging x and y, it follows that

$$w(x,y) = w(0,y). \qquad (2.24)$$

From (2.23) and (2.24),

$$w(x,0) = w(x,y) = w(0,y).$$

Since $w(x,y) = w(x,0)$, in particular for $x = 0$ it holds that $w(0,y) = w(0,0)$. Then

$$w(x,y) = w(0,0),$$

i.e. w is constant. $\square$

In a similar way, one can prove the following.

Proposition 2.2.4. *There exists a weight v on $\mathbb{R}^2$ such that $\Lambda_2^p(w) = L^p_{\mathbb{R}^2}(v)$ if and only if $\Lambda_2^p(w) = L^p_{\mathbb{R}^2}$.*

Proof. We will use the same notation as in Proposition 2.2.3.
($\Rightarrow$) Suppose that there exists a weight v on $\mathbb{R}^2$ such that $\Lambda_2^p(w) = L^p_{\mathbb{R}^2}(v)$. We will show that v is constant. In order to do that, let's consider the functions χ_R and χ_{A_ε}. In Proposition 2.2.3 we saw that

$$\|\chi_R\|_{\Lambda_2^p(w)} = \left(\int_R w(x)\,dx\right)^{1/p} < \infty \quad \text{Since } w \text{ is locally integrable.}$$
$$\|\chi_{A_\varepsilon}\|_{\Lambda_2^p(w)} = \left(\int_{A_\varepsilon} w(x)\,dx\right)^{1/p} < \infty \quad \text{Since } w \text{ is locally integrable.}$$

Then $\chi_R, \chi_{A_\varepsilon} \in \Lambda_2^p(w)$. Since we are under the hypothesis of that $\Lambda_2^p(w) = L^p_{\mathbb{R}^2}(v)$, then $\chi_R, \chi_{A_\varepsilon} \in \Lambda_2^p(w) = L^p_{\mathbb{R}^2}(v)$. Also, $m_2(R) = m_2(A_\varepsilon)$ implies the equimeasurability of χ_R and χ_{A_ε}. Since $\|\cdot\|_{L^p_{\mathbb{R}^2}(v)}$ is rearrangement invariant, we have

$$\begin{aligned} \|\chi_R\|^p_{L^p_{\mathbb{R}^2}(v)} &= \|\chi_{A_\varepsilon}\|^p_{L^p_{\mathbb{R}^2}(v)} \\ \Rightarrow \int_{\mathbb{R}^2} [\chi_R(x)]^p\, v(x)\,dx &= \int_{\mathbb{R}^2} [\chi_{A_\varepsilon}(x)]^p\, v(x)\,dx \\ \Rightarrow \int_R v(x)\,dx &= \int_{A_\varepsilon} v(x)\,dx \\ \Rightarrow \int_{P_\varepsilon} v(x)\,dx &= \int_{Q_\varepsilon} v(x)\,dx, \quad \text{see (2.22).} \end{aligned}$$

Using the same argument as in the proof of Proposition 2.2.3, it follows that v is constant, hence $L^p_{\mathbb{R}^2} = L^p_{\mathbb{R}^2}(v) = \Lambda_2^p(w)$.
($\Leftarrow$) Take $v \equiv 1$. □

2.2.2 $\Lambda_2^p(w)$: embeddings, quasinormability and completeness

Embeddings

One can show in an easy way that the embedding results for the spaces $\Lambda_2^p(w)$ are equivalent to the embeddings for the cone of decreasing functions on $L^p_{\mathbb{R}^2_+}$. Those results have been completely characterized in all cases (see [11] and [13]). So, we state without proof the following proposition.

Proposition 2.2.5. *Let $0 < p_1, p_2 < \infty$ and w_1, w_2 be two weights on $\mathbb{R}^2_+$.*

a) If $p_1 \leq p_2$, then $\Lambda_2^{p_1}(w_1) \subset \Lambda_2^{p_2}(w_2)$, if and only if

$$\sup_{D \in \Delta_d} \frac{w_2(D)^{1/p_2}}{w_1(D)^{1/p_1}} < \infty.$$

b) If $p_1 > p_2$*, then* $\Lambda_2^{p_1}(w_1) \subset \Lambda_2^{p_2}(w_2)$*, if and only if*

$$\sup_{0\le h\downarrow} \int_0^\infty w_1(D_{h,t})^{-r/p_1} d\left(-w_2(D_{h,t})^{r/p_2}\right) < \infty,$$

where $D_{h,t} = \left\{x \in \mathbb{R}_+^2 : h(x) > t\right\}$*, and* $1/r = 1/p_2 - 1/p_1$*.*

Quasinormability and completeness

In the case of classical Lorentz spaces, it was proved in [17] that the quasinormability of the space is equivalent to a doubling condition on the weight (the Δ_2 condition).
We will show that a similar result is valid for the two dimensional rearrangement.
First, note that the spaces $\Lambda_2^p(w)$, $0 < p < \infty$, have the following (quasi)norm properties.

Proposition 2.2.6. *If* $f \in \Lambda_2^p(w)$*, then*

a)

$$\|f\|_{\Lambda_2^p(w)} = 0 \Leftrightarrow f = 0 \quad a.e. \tag{2.25}$$

b)

$$\|cf\|_{\Lambda_2^p(w)} = |c| \, \|f\|_{\Lambda_2^p(w)} . \tag{2.26}$$

Proof. a)

$$\begin{aligned}(\Leftarrow) f = 0 \quad \text{a.e.} &\Rightarrow f_x(y) = f(x,y) = 0 \quad \text{a.e.}\\ &\Rightarrow R_t(x) = (f_x)^{*y}(t) = 0\\ &\Rightarrow \tilde{f}(s,t) = (R_t)^{*x}(s) = 0\\ &\Rightarrow f_2^* = 0.\end{aligned}$$

$$\begin{aligned}(\Rightarrow) \|f\|_{\Lambda_2^p(w)} = 0 &\Rightarrow \left(\int_{\mathbb{R}_+^2} [f_2^*(x)]^p \, w(x) \, dx\right)^{1/p} = 0\\ &\Rightarrow \int_{\mathbb{R}_+^2} [f_2^*(x)]^p \, w(x) \, dx = 0\\ &\Rightarrow [f_2^*(x)]^p \, w(x) = 0\\ &\Rightarrow [f_2^*(x)]^p = 0\\ &\Rightarrow f_2^*(x) = 0\\ &\Rightarrow \int_0^\infty \chi_{\{|f|>t\}^*} \, dt = 0\\ &\Rightarrow \chi_{\{|f|>t\}^*} = 0 \quad \text{a.e.}\\ &\Rightarrow m(\{|f| > t\}^*) = 0 \quad \forall t\\ &\Rightarrow m(\{|f| > t\}) = 0 \quad \forall t\\ &\Rightarrow f(x) = 0 \quad \text{a.e.}\end{aligned}$$

b)

$$\begin{aligned}
\|cf\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2} [(cf)_2^*(x)]^p \, w(x) \, dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2} [|c| f_2^*(x)]^p \, w(x) \, dx\right)^{1/p}, \quad \text{see Proposition 2.1.18 b)} \\
&= |c| \left(\int_{\mathbb{R}_+^2} [f_2^*(x)]^p \, w(x) \, dx\right)^{1/p} \\
&= |c| \, \|f\|_{\Lambda_2^p(w)} .
\end{aligned}$$

□

Due to Proposition 2.2.6, if we check that the triangular inequality (quasi-triangular) is valid, then we conclude that $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm (quasi-norm). The following theorem gives us an equivalent condition for the quasinormability of $\Lambda_2^p(w)$. Compare with [17, Corollary 2.2].

Theorem 2.2.7. *Let $0 < p < \infty$. Then, $\|\cdot\|_{\Lambda_2^p(w)}$ is a quasinorm if and only if there exists a constant $C > 0$ such that*

$$\int_D w(2x) \, dx \leq C \int_D w(x) \, dx, \tag{2.27}$$

for all the decreasing subsets $D \subset \mathbb{R}_+^2$. Besides, with this quasinorm, $\Lambda_2^p(w)$ is a complete quasinormed space.

Proof. ($\Leftarrow$) We will use Proposition 2.1.18 d), and also [11, Theorem 2.2 d)] with $p = q$.

$$\begin{aligned}
\|f+g\|_{\Lambda_2^p(w)}^p &= \int_{\mathbb{R}_+^2} [(f+g)_2^*(x)]^p \, w(x) \, dx \\
&\leq C \int_{\mathbb{R}_+^2} [f_2^*(x/2) + g_2^*(x/2)]^p \, w(x) \, dx, \quad \text{by Proposition 2.1.18 d)} \\
&\leq C \left(\int_{\mathbb{R}_+^2} [f_2^*(x/2)]^p \, w(x) \, dx + \int_{\mathbb{R}_+^2} [g_2^*(x/2)]^p \, w(x) \, dx\right) \\
&\leq C \left(\int_{\mathbb{R}_+^2} [f_2^*(x)]^p \, w(2x) \, dx + \int_{\mathbb{R}_+^2} [g_2^*(x)]^p \, w(2x) \, dx\right) \\
&\leq C \left(\int_{\mathbb{R}_+^2} [f_2^*(x)]^p \, w(x) \, dx + \int_{\mathbb{R}_+^2} [g_2^*(x)]^p \, w(x) \, dx\right) \\
&= C \left(\|f\|_{\Lambda_2^p(w)}^p + \|g\|_{\Lambda_2^p(w)}^p\right).
\end{aligned}$$

Where we used [11, Theorem 2.2 d)] in the last inequality. It follows that $\|f+g\|_{\Lambda_2^p(w)} \leq C\left(\|f\|_{\Lambda_2^p(w)} + \|g\|_{\Lambda_2^p(w)}\right)$.

($\Rightarrow$) Let D and D_1 be two subsets of $\mathbb{R}^2_+$ with $D \cap D_1 = \emptyset$ and $D^* = D_1^*$, and such that if D^* has the representation

$$D^* = \{(x_1, x_2) : 0 < x_1 < r, 0 < x_2 < \phi(x_1); r > 0\},$$

(with $\phi \downarrow$), then

$$(D \cup D_1)^* = \{(x_1, x_2) : 0 < x_1 < 2r, 0 < x_2 < \phi(x_1/2); r > 0\},$$

(this is easily achieved by taking D_1 as a translation of the form $D_1 = D + (N, 0)$, where N is big enough). If $\|\cdot\|_{\Lambda_2^p(w)}$ is a quasinorm, then

$$\|f + g\|_{\Lambda_2^p(w)} \leq C \left(\|f\|_{\Lambda_2^p(w)} + \|g\|_{\Lambda_2^p(w)} \right), \tag{2.28}$$

and taking $f = \chi_D$ and $g = \chi_{D_1}$, we obtain

$$\begin{aligned}
\|f + g\|^p_{\Lambda_2^p(w)} &= \int_{\mathbb{R}^2_+} [(f+g)_2^*(x)]^p \, w(x) \, dx \\
&= \int_{\mathbb{R}^2_+} [(\chi_D + \chi_{D_1})_2^* (x)]^p \, w(x) \, dx \\
&= \int_{\mathbb{R}^2_+} [(\chi_{D \cup D_1})_2^* (x)]^p \, w(x) \, dx \\
&= \int_{\mathbb{R}^2_+} \left[\chi_{(D \cup D_1)^*}(x)\right]^p w(x) \, dx, \quad \text{by Proposition 2.1.11 c)} \\
&= \int_{(D \cup D_1)^*} w(x) \, dx.
\end{aligned}$$

In a similar way, $\|f\|^p_{\Lambda_2^p(w)} = \int_{D^*} w(x)\, dx = \|g\|^p_{\Lambda_2^p(w)}$, then from (2.28) it follows that

$$\int_{(D \cup D_1)^*} w(x) \, dx \leq C \int_{D^*} w(x) \, dx \tag{2.29}$$

Let's denote $E := (D \cup D_1)^*$, and

$$E_1 = \{(x_1, x_2) : 0 < x_1 < 2r, \phi(2x_1) < x_2 < 2\phi(x_1/2); r > 0\}.$$

Obviously $E \cup E_1 = 2D^*$. Since $E_1^* = E = E^*$, we can use equation (2.29) with $D = E$ and $D_1 = E_1$, and then we obtain

$$\begin{aligned}
\int_{2D^*} w(x) \, dx &= \int_{E \cup E_1} w(x) \, dx \\
&= \int_{(E \cup E_1)^*} w(x) \, dx, \quad \text{since } E_1 \cup E = 2D^* \text{ is decreasing}
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_{E^*} w(x)\,dx \\
&= C \int_{(D\cup D_1)^*} w(x)\,dx, \quad \text{since } E^* = ((D\cup D_1)^*)^* = (D\cup D_1)^* \\
&\leq C \int_{D^*} w(x)\,dx, \quad \text{by using (2.29).}
\end{aligned}$$

So, we proved that

$$\int_{2D^*} w(x)\,dx \leq C \int_{D^*} w(x)\,dx,$$

which is equivalent to (2.27).

In order to show that $\Lambda_2^p(w)$ is complete, we have to show that if $(f_k)_k$ is a Cauchy sequence, then there exists a function $f \in \Lambda_2^p(w)$ such that $\|f_j - f\|_{\Lambda_2^p(w)} \to 0$ when $j \to \infty$. Since $\|\cdot\|^p_{\Lambda_2^p(w)}$ is a quasinorm and $(f_k)_k$ is a Cauchy sequence, there exists a constant $C > 0$ such that $\|f_j\|^p_{\Lambda_2^p(w)} \leq C < \infty$ for all $j \in \mathbb{N}$.
Besides, since $(f_j - f_k)_2^*$ is decreasing in each variable, for a fixed $x \in \mathbb{R}^2_+$, if we take $Q_x = \left\{y \in \mathbb{R}^2_+ : 0 < y_k \leq x_k, k = 1,2\right\}$, then

$$\begin{aligned}
& \min_{Q_x} \left[(f_j - f_k)_2^*\right]^p \leq \left[(f_j - f_k)_2^*(y)\right]^p \quad \forall\, y \in Q_x \\
\Rightarrow\quad & \left[(f_j - f_k)_2^*(x)\right]^p \leq \left[(f_j - f_k)_2^*(y)\right]^p \quad \forall\, y \in Q_x \\
\Rightarrow\quad & \left[(f_j - f_k)_2^*(x)\right]^p w(y) \leq \left[(f_j - f_k)_2^*(y)\right]^p w(y) \\
\Rightarrow\quad & \int_{Q_x} \left[(f_j - f_k)_2^*(x)\right]^p w(y)\,dy \leq \int_{Q_x} \left[(f_j - f_k)_2^*(y)\right]^p w(y)\,dy \\
\Rightarrow\quad & \left[(f_j - f_k)_2^*(x)\right]^p \int_{Q_x} w(y)\,dy \leq \int_{\mathbb{R}^2_+} \left[(f_j - f_k)_2^*(y)\right]^p w(y)\,dy
\end{aligned}$$

This way,

$$\left[(f_j - f_k)_2^*(x)\right]^p \int_{Q_x} w(y)\,dy \leq \|f_j - f_k\|^p_{\Lambda_2^p(w)}.$$

Then

$$\begin{aligned}
& (f_j - f_k)_2^* \to 0 \quad \text{a.e.} \\
\Rightarrow\quad & D_{(f_j - f_k)_2^*} \to 0 \quad \text{a.e.} \\
\Rightarrow\quad & D_{(f_j - f_k)} \to 0 \quad \text{a.e.}
\end{aligned}$$

i.e. $(f_k)_k$ is Cauchy in measure. Then there exists a subsequence $\left(f_{k_j}\right)$ which converges pointwise, let's say to a function f which is measurable. By using Proposition 2.1.18 e) and Fatou's lemma, we conclude that $f \in \Lambda_2^p(w)$. Let's see

$$f = \lim_{j\to\infty} f_{k_j}$$

$$\begin{aligned}
\Rightarrow \quad & f_2^* = \lim_{j\to\infty} \left(f_{k_j}\right)_2^* \quad \text{(Proposition 2.1.18 e))} \\
\Rightarrow \quad & \int_{\mathbb{R}^2_+} [f_2^*(x)]^p \, w(x)\, dx \le \liminf_{j\to\infty} \int_{\mathbb{R}^2_+} \left[\left(f_{k_j}\right)_2^*(x)\right]^p w(x)\, dx \quad \text{(Fatou's lemma)} \\
& \le C < \infty \\
\Rightarrow \quad & f \in \Lambda_2^p(w).
\end{aligned}$$

Besides,

$$\lim_{j\to\infty} \left|f_{k_j}(x) - f_i(x)\right| = |f(x) - f_i(x)|, \quad x \in \mathbb{R}^2.$$

Using Fatou's lemma and the fact that $(f_k)_k$ is a Cauchy sequence, we finally obtain

$$\begin{aligned}
\|f - f_i\|_{\Lambda_2^p(w)} &= \left\|f - f_{k_j} + f_{k_j} - f_i\right\|_{\Lambda_2^p(w)} \\
&\le C\left(\left\|f - f_{k_j}\right\|_{\Lambda_2^p(w)} + \left\|f_{k_j} - f_i\right\|_{\Lambda_2^p(w)}\right) \\
&= C\left(\left\|f - f_{k_j}\right\|_{\Lambda_2^p(w)} + \left\|f_i - f_{k_j}\right\|_{\Lambda_2^p(w)}\right) \xrightarrow[i,j\to\infty]{} 0.
\end{aligned}$$

Note that inequality above holds since $\|\cdot\|_{\Lambda_2^p(w)}$ is a quasinorm. □

In order to $\Lambda_2^p(w)$ be a Banach space, we show in the next theorem a necessary condition on the index p.

Theorem 2.2.8. *If $\Lambda_2^p(w)$ is a Banach space, then $p \ge 1$.*

Proof. Since $\Lambda_2^p(w)$ is a Banach space, there exists a norm $\|\cdot\|$ on $\Lambda_2^p(w)$ such that

$$\|g\|_{\Lambda_2^p(w)} \approx \|g\|.$$

So, there exists a constant $C > 1$ such that

$$\frac{1}{C}\|g\|_{\Lambda_2^p(w)} \le \|g\| \le C\|g\|_{\Lambda_2^p(w)}.$$

Then

$$\|g\|_{\Lambda_2^p(w)} \le C\|g\| \le C^2\|g\|_{\Lambda_2^p(w)}.$$

Taking $g = \sum_{k=1}^N f_k$, we obtain

$$\left\|\sum_{k=1}^N f_k\right\|_{\Lambda_2^p(w)} \le C\left\|\sum_{k=1}^N f_k\right\| \le C^2\left\|\sum_{k=1}^N f_k\right\|_{\Lambda_2^p(w)} \le C^2\sum_{k=1}^N \|f_k\|_{\Lambda_2^p(w)}.$$

For all $N \in \mathbb{N}$. Suppose that $0 < p < 1$ and take a decreasing sequence of domains

$$A_{k+1} \subset A_k \subset \cdots \subset \mathbb{R}^2,$$

such that $\int_{A_k^*} w(x)\,dx = 2^{-kp}$. If $f_k = 2^k \chi_{A_k}$, then

$$\begin{aligned}
\|f_k\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2} [(f_k)_2^*(x)]^p \, w(x)\,dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2} \left[\left(2^k \chi_{A_k}\right)_2^*(x) \right]^p w(x)\,dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2} 2^{kp} \chi_{A_k^*}^p w(x)\,dx \right)^{1/p}, \text{Proposition 2.1.11 c) and Proposition 2.1.18 b)} \\
&= \left(2^{kp} \int_{A_k^*} w(x)\,dx \right)^{1/p} \\
&= \left(2^{kp} \cdot 2^{-kp}\right)^{1/p} = 1.
\end{aligned}$$

But for a fixed N, we have

$$\frac{1}{N} \left\| \sum_{k=1}^{N} f_k \right\|_{\Lambda_2^p(w)} \le \tilde{C} < \infty. \tag{2.30}$$

In the other hand, since $\left(\sum_{k=1}^N 2^k \chi_{A_k}\right)_2^* = \sum_{k=1}^N 2^k \chi_{A_k^*}$ (by (2.17)), and $A_{k+1}^* \subset A_k^* \subset \cdots \subset \mathbb{R}_+^2$, we have (taking $A_{N+1} = \emptyset$)

$$\begin{aligned}
\frac{1}{N} \left\| \sum_{k=1}^{N} f_k \right\|_{\Lambda_2^p(w)} &= \frac{1}{N} \left\| \sum_{k=1}^{N} 2^k \chi_{A_k} \right\|_{\Lambda_2^p(w)} \\
&= \frac{1}{N} \left(\int_{\mathbb{R}_+^2} \left[\left(\sum_{k=1}^{N} 2^k \chi_{A_k} \right)_2^* (x) \right]^p w(x)\,dx \right)^{1/p} \\
&= \frac{1}{N} \left(\int_{\mathbb{R}_+^2} \underbrace{\left[\sum_{k=1}^{N} 2^k \chi_{A_k^*}(x) \right]^p}_{(*)} w(x)\,dx \right)^{1/p}
\end{aligned} \tag{2.31}$$

The expression $(*)$ is equivalent to $\sum_{k=1}^N \left[\left(\sum_{j=1}^k 2^j \right) \chi_{A_k^* \setminus A_{k+1}^*} \right](x)$, let's see this.

$$\sum_{k=1}^{N} 2^k \chi_{A_k^*}(x) = 2^1 \chi_{A_1^*}(x) + 2^2 \chi_{A_2^*}(x) + 2^3 \chi_{A_3^*}(x) + \cdots + 2^N \chi_{A_N^*}(x).$$

Besides

$$\sum_{k=1}^{N}\left[\left(\sum_{j=1}^{k}2^j\right)\chi_{A_k^*\setminus A_{k+1}^*}\right](x)$$
$$= \left(2^1\right)\chi_{A_1^*\setminus A_2^*}(x) + \left(2^1+2^2\right)\chi_{A_2^*\setminus A_3^*}(x) + \left(2^1+2^2+2^3\right)\chi_{A_3^*\setminus A_4^*}(x)$$
$$+\cdots+\left(2^1+2^2+2^3+\cdots 2^N\right)\chi_{A_N^*\setminus A_{N+1}^*}(x). \quad (2.32)$$

Since $A_{n+1}\subset A_n$, we have $A_{n+1}^*\subset A_n^*$, and then

$$\chi_{A_{n+1}^*\setminus A_n^*} = \chi_{A_{n+1}^*} - \chi_{A_n^*}.$$

Going back to (2.32),

$$\sum_{k=1}^{N}\left[\left(\sum_{j=1}^{k}2^j\right)\chi_{A_k^*\setminus A_{k+1}^*}\right](x) = \left(2^1\right)\left(\chi_{A_1^*}(x)-\chi_{A_2^*}(x)\right) + \left(2^1+2^2\right)\left(\chi_{A_2^*}(x)-\chi_{A_3^*}(x)\right)$$
$$+\left(2^1+2^2+2^3\right)\left(\chi_{A_3^*}(x)-\chi_{A_4^*}(x)\right)$$
$$+\cdots+\left(2^1+2^2+2^3+\cdots+2^N\right)\left(\chi_{A_N^*}(x) - \underbrace{\chi_{A_{N+1}^*}}_{A_{N+1}=\emptyset}(x)\right)$$
$$= 2^1\chi_{A_1^*}(x) - 2^1\chi_{A_2^*}(x) + 2^1\chi_{A_2^*}(x) - 2^1\chi_{A_3^*}(x) + 2^2\chi_{A_2^*}(x) - 2^2\chi_{A_3^*}(x)$$
$$+ 2^1\chi_{A_3^*}(x) - 2^1\chi_{A_4^*}(x) + 2^2\chi_{A_3^*}(x) - 2^2\chi_{A_4^*}(x) + 2^3\chi_{A_3^*}(x) - 2^3\chi_{A_4^*}(x)$$
$$+\cdots+2^1\chi_{A_N^*}(x) + 2^2\chi_{A_N^*}(x) + \cdots + 2^N\chi_{A_N^*}(x)$$
$$= 2^1\chi_{A_1^*}(x) + 2^2\chi_{A_2^*}(x) + \cdots + 2^N\chi_{A_N^*}(x) = \sum_{k=1}^{N}2^k\chi_{A_k^*}(x).$$

Returning to (2.31),

$$\frac{1}{N}\left\|\sum_{k=1}^{N}f_k\right\|_{\Lambda_2^p(w)} = \frac{1}{N}\left(\int_{\mathbb{R}_+^2}\left(\sum_{k=1}^{N}\left[\left(\sum_{j=1}^{k}2^j\right)\chi_{A_k^*\setminus A_{k+1}^*}\right]\right)^p(x)w(x)\,dx\right)^{1/p}$$
$$= \frac{1}{N}\left(\int_{\mathbb{R}_+^2}\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)\chi_{A_k^*\setminus A_{k+1}^*}(x)w(x)\,dx\right)^{1/p}$$
$$= \frac{1}{N}\left(\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)\int_{\mathbb{R}_+^2}\chi_{A_k^*\setminus A_{k+1}^*}(x)w(x)\,dx\right)^{1/p}$$
$$= \frac{1}{N}\left(\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)\int_{\mathbb{R}_+^2}\left[\chi_{A_k^*}(x)-\chi_{A_{k+1}^*}(x)\right]w(x)\,dx\right)^{1/p}$$
$$= \frac{1}{N}\left(\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)\left(\int_{A_k^*}w(x)\,dx - \int_{A_{k+1}^*}w(x)\,dx\right)\right)^{1/p}$$

$$= \frac{1}{N}\left(\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)\left(2^{-kp}-2^{-(k+1)p}\right)\right)^{1/p}$$
$$= \frac{1}{N}\left(\left(\sum_{k=1}^{N}\left[\sum_{j=1}^{k}2^j\right]^p\right)2^{-kp}\left(1-2^{-p}\right)\right)^{1/p}$$
$$= \frac{C}{N}\left(\sum_{k=1}^{N}\left[\frac{\sum_{j=1}^{k}2^j}{2^k}\right]^p\right)^{1/p}, \quad \text{taking } C=\left(1-2^{-p}\right)^{1/p}. \tag{2.33}$$

Note that

$$\begin{aligned}\sum_{k=1}^{N}\left[\frac{\sum_{j=1}^{k}2^j}{2^k}\right]^p &= \left(\frac{2^1}{2^1}\right)^p+\left(\frac{2^1+2^2}{2^2}\right)^p+\left(\frac{2^1+2^2+2^3}{2^3}\right)^p+\cdots+\\ &\quad\left(\frac{2^1+2^2+2^3+\cdots+2^N}{2^N}\right)^p\\ &= 1+\left(1+2^{-1}\right)^p+\left(1+2^{-1}+2^{-2}\right)^p+\cdots+\\ &\quad\left(1+2^{-1}+2^{-2}+\cdots+2^{1-N}\right)^p\\ &\geq \left(1+2^{-1}\right)^p+\left(1+2^{-2}\right)^p+\cdots+\left(1+2^{-N}\right)^p\\ &\geq \left(1-2^{-1}\right)^p+\left(1-2^{-2}\right)^p+\cdots+\left(1-2^{-N}\right)^p\\ &= \sum_{k=1}^{N}\left(1-2^{-k}\right)^p.\end{aligned} \tag{2.34}$$

Now,

$$\begin{aligned} & 2^k \geq 2 \quad \forall\, k\geq 1\\ \Rightarrow\quad & 2^{-1}\geq 2^{-k}\\ \Rightarrow\quad & 1\geq 2^{-k}+2^{-1}\\ \Rightarrow\quad & 1-2^{-k}\geq 2^{-1}\\ \Rightarrow\quad & \left(1-2^{-k}\right)^p\geq\left(2^{-1}\right)^p=2^{-p}.\end{aligned}$$

Therefore

$$\sum_{k=1}^{N}\left(1-2^{-k}\right)^p\geq\sum_{k=1}^{N}2^{-p}. \tag{2.35}$$

So, going back to (2.33), we obtain

$$\begin{aligned}\frac{1}{N}\left\|\sum_{k=1}^{N}f_k\right\|_{\Lambda_2^p(w)} &= \frac{C}{N}\left(\sum_{k=1}^{N}\left[\frac{\sum_{j=1}^{k}2^j}{2^k}\right]^p\right)^{1/p}\\ &\geq \frac{C}{N}\left(\sum_{k=1}^{N}\left(1-2^{-k}\right)^p\right)^{1/p}, \quad \text{by (2.34)}\end{aligned}$$

$$\geq \frac{C}{N}\left(\sum_{k=1}^{N} 2^{-p}\right)^{1/p}, \quad \text{by } (2.35)$$
$$= C\frac{N^{1/p}}{N} \to \infty \quad \text{when } N \to \infty.$$

Which is a contradiction in view of (2.30). Then, must be $p \geq 1$. □

Finally, we present an important result: the characterization of the weights w for which $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm.

Theorem 2.2.9. *Let $1 \leq p < \infty$ and w a weight on $\mathbb{R}_+^2$. Then, the following assertions are equivalent:*

a) $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm.

b) For all $A, B \subset \mathbb{R}^2$,

$$w\left((A \cap B)^*\right) + w\left((A \cup B)^*\right) \leq w\left(A^*\right) + w\left(B^*\right).$$

c) There exists a decreasing weight v on $\mathbb{R}_+$ such that

$$w(s,t) = v(t), \quad s, t > 0.$$

Proof. *a)* ⇒ *b)*: If $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm, take $A, B \subset \mathbb{R}^2$, $\delta > 0$ and let's define the functions

$$f(x) = \begin{cases} 1+\delta, & \text{if } x \in A \\ 1, & \text{if } x \in (A \cup B) \smallsetminus A\,; \\ 0, & \text{otherwise.} \end{cases} \quad g(x) = \begin{cases} 1+\delta, & \text{if } x \in B \\ 1, & \text{if } x \in (A \cup B) \smallsetminus B \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$f(x) = (1+\delta)\chi_A(x) + \chi_{(A\cup B)\smallsetminus A}(x); \quad g(x) = (1+\delta)\chi_B(x) + \chi_{(A\cup B)\smallsetminus B}(x).$$

Then, according to (2.16) in Proposition 2.1.19:

$$f_2^*(x) = (1+\delta)\chi_{A^*}(x) + \chi_{[(A\cup B\smallsetminus A)\cup A]^*\smallsetminus A^*}(x) = (1+\delta)\chi_{A^*}(x) + \chi_{(A\cup B)^*\smallsetminus A^*}(x).$$

And also $g_2^*(x) = (1+\delta)\chi_{B^*}(x) + \chi_{(A\cup B)^*\smallsetminus B^*}(x)$.
In a similar way, since $(A \cup B) \smallsetminus A = B \smallsetminus A$ and $(A \cup B) \smallsetminus B = A \smallsetminus B$, we have

$$\begin{aligned}(f+g)(x) = f(x) + g(x) &= (1+\delta)\chi_A(x) + \chi_{(A\cup B)\smallsetminus A}(x) + (1+\delta)\chi_B(x) + \chi_{(A\cup B)\smallsetminus B}(x) \\ &= (1+\delta)\left(\chi_A(x) + \chi_B(x)\right) + \left(\chi_{(A\cup B)\smallsetminus A}(x) + \chi_{(A\cup B)\smallsetminus B}(x)\right) \\ &= (1+\delta)\left(\chi_A(x) + \chi_B(x)\right) + \left(\chi_{B\smallsetminus A}(x) + \chi_{A\smallsetminus B}(x)\right)\end{aligned}$$

$$
\begin{aligned}
&= (1+\delta)\left(\chi_A(x)+\chi_B(x)-\chi_{A\cap B}(x)+\chi_{A\cap B}(x)\right)+(\chi_{B\smallsetminus A}(x) \\
&\quad + \chi_{A\smallsetminus B}(x)) \\
&= (1+\delta)\left(\chi_{A\cup B}(x)+\chi_{A\cap B}(x)\right)+\chi_{\underbrace{(A\smallsetminus B)\cup(B\smallsetminus A)}_{\text{symmetric difference}}}(x) \\
&= (1+\delta)\left(\chi_{A\cup B}(x)+\chi_{A\cap B}(x)\right)+\chi_{(A\cup B)\smallsetminus(A\cap B)}(x) \\
&= (1+\delta)\left(\chi_{A\cup B}(x)+\chi_{A\cap B}(x)\right)+\chi_{A\cup B}(x)-\chi_{A\cap B}(x) \\
&= (2+\delta)\chi_{A\cup B}(x)+\delta\chi_{A\cap B}(x) \\
&= (2+\delta)\left[\chi_{A\smallsetminus B}(x)+\chi_{A\cap B}(x)+\chi_{B\smallsetminus A}(x)\right]+\delta\chi_{A\cap B}(x) \\
&= (2+\delta)\left[\chi_{A\smallsetminus B}(x)+\chi_{B\smallsetminus A}(x)\right]+(2+2\delta)\chi_{A\cap B}(x) \\
&= (2+\delta)\chi_{(A\smallsetminus B)\cup(B\smallsetminus A)}(x)+(2+2\delta)\chi_{A\cap B}(x) \\
&= (2+2\delta)\chi_{A\cap B}(x)+(2+\delta)\chi_{(A\smallsetminus B)\cup(B\smallsetminus A)}(x).
\end{aligned}
$$

Once again, using (2.16) in Proposition 2.1.19,

$$
\begin{aligned}
(f+g)_2^*(x) &= (2+2\delta)\chi_{(A\cap B)^*}(x)+(2+\delta)\chi_{\{(A\cap B)\cup[(A\smallsetminus B)\cup(B\smallsetminus A)]\}^*\smallsetminus(A\cap B)^*}(x) \\
&= (2+2\delta)\chi_{(A\cap B)^*}(x)+(2+\delta)\chi_{(A\cup B)^*\smallsetminus(A\cap B)^*}(x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|f+g\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2}\left[(f+g)_2^*(x)\right]^p w(x)\,dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2}\left[(2+2\delta)\chi_{(A\cap B)^*}(x)+(2+\delta)\chi_{(A\cup B)^*\smallsetminus(A\cap B)^*}(x)\right]^p w(x)\,dx\right)^{1/p} \\
&= \left(\int_{(A\cap B)^*}(2+2\delta)^p w(x)\,dx+\int_{(A\cup B)^*\smallsetminus(A\cap B)^*}(2+\delta)^p w(x)\,dx\right)^{1/p} \\
&= \left((2+2\delta)^p w\left((A\cap B)^*\right)+(2+\delta)^p w\left((A\cup B)^*\smallsetminus(A\cap B)^*\right)\right)^{1/p}. \qquad (2.36)
\end{aligned}
$$

Besides

$$
\begin{aligned}
\|f\|_{\Lambda_2^p(w)}+\|g\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2}\left[f_2^*(x)\right]^p w(x)\,dx\right)^{1/p}+\left(\int_{\mathbb{R}_+^2}\left[g_2^*(x)\right]^p w(x)\,dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}_+^2}\left[(1+\delta)\chi_{A^*}(x)+\chi_{(A\cup B)^*\smallsetminus A^*}(x)\right]^p w(x)\,dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}_+^2}\left[(1+\delta)\chi_{B^*}(x)+\chi_{(A\cup B)^*\smallsetminus B^*}(x)\right]^p w(x)\,dx\right)^{1/p} \\
&= \left(\int_{A^*}(1+\delta)^p w(x)\,dx+\int_{(A\cup B)^*\smallsetminus A^*}w(x)\,dx\right)^{1/p}
\end{aligned}
$$

$$+\left(\int_{B^*}(1+\delta)^p w(x)\,dx+\int_{(A\cup B)^*\setminus B^*} w(x)\,dx\right)^{1/p}$$
$$=\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\setminus A^*\right)\right]^{1/p}$$
$$+\left[(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\setminus B^*\right)\right]^{1/p}.$$

From the triangular inequality we have

$$\|f+g\|_{\Lambda_2^p(w)}\leq\|f\|_{\Lambda_2^p(w)}+\|g\|_{\Lambda_2^p(w)},$$

i.e.

$$\left((2+2\delta)^p w\left((A\cap B)^*\right)+(2+\delta)^p w\left((A\cup B)^*\setminus(A\cap B)^*\right)\right)^{1/p}$$
$$\leq\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\setminus A^*\right)\right]^{1/p}+\left[(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\setminus B^*\right)\right]^{1/p}.$$

Using the inequality

$$|a|^{1/p}+|b|^{1/p}\leq 2^{1-1/p}\left(|a|+|b|\right)^{1/p}\quad a,b\in\mathbb{R},p\geq 1. \tag{2.37}$$

(Which is proved in the appendix, see Proposition 5.3.1), we have

$$\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\setminus A^*\right)\right]^{1/p}+\left[(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\setminus B^*\right)\right]^{1/p}$$
$$\leq 2^{1-1/p}\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\setminus A^*\right)+(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\setminus B^*\right)\right]^{1/p}. \tag{2.38}$$

Raising (2.36) and (2.38) to the p-th power, we obtain

$$(2+2\delta)^p w\left((A\cap B)^*\right)+(2+\delta)^p w\left((A\cup B)^*\setminus(A\cap B)^*\right)$$
$$\leq 2^{p-1}\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\setminus A^*\right)+(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\setminus B^*\right)\right]$$

Note that if $C\subset D$, then

$$w(D\setminus C)=\int_{D\setminus C}w=\int_D w-\int_C w=w(D)-w(C).$$

With this, we may write the last inequality as

$$(2+2\delta)^p w\left((A\cap B)^*\right)+(2+\delta)^p\left[w\left((A\cup B)^*\right)-w\left((A\cap B)^*\right)\right]$$
$$\leq 2^{p-1}\left[(1+\delta)^p w\left(A^*\right)+w\left((A\cup B)^*\right)-w\left(A^*\right)+(1+\delta)^p w\left(B^*\right)+w\left((A\cup B)^*\right)-w\left(B^*\right)\right].$$

So

$$\left[(2+2\delta)^p-(2+\delta)^p\right]w\left((A\cap B)^*\right)+(2+\delta)^p w\left((A\cup B)^*\right)$$
$$\leq 2^{p-1}\left[(1+\delta)^p-1\right]\left[w\left(A^*\right)+w\left(B^*\right)\right]+2^p w\left((A\cup B)^*\right).$$

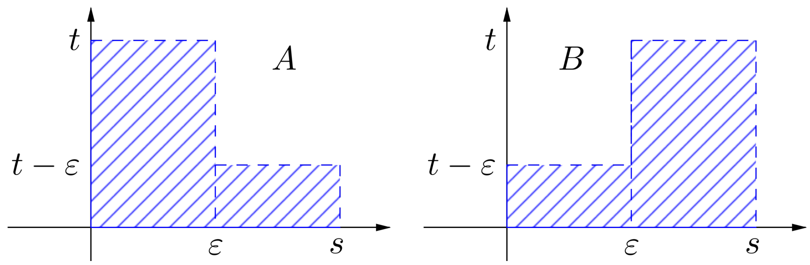

Figure 2-16: The sets A and B in the proof of $b) \Rightarrow c)$ in Theorem 2.2.9.

Then

$$[(2+2\delta)^p - (2+\delta)^p]\, w\left((A\cap B)^*\right) + [(2+\delta)^p - 2^p]\, w\left((A\cup B)^*\right) \\ \leq 2^{p-1}\left[(1+\delta)^p - 1\right]\left[w\left(A^*\right) + w\left(B^*\right)\right].$$

Dividing both sides by $2^{p-1}\left[(1+\delta)^p - 1\right]$, we obtain

$$\left[\frac{(2+2\delta)^p - (2+\delta)^p}{2^{p-1}\left[(1+\delta)^p - 1\right]}\right] w\left((A\cap B)^*\right) + \left[\frac{(2+\delta)^p - 2^p}{2^{p-1}\left[(1+\delta)^p - 1\right]}\right] w\left((A\cup B)^*\right) \\ \leq w\left(A^*\right) + w\left(B^*\right). \quad (2.39)$$

Now let's see what happens with the terms containing δ on the left side of (2.39) when $\delta \to 0$,

$$\lim_{\delta\to 0} \frac{(2+2\delta)^p - (2+\delta)^p}{2^{p-1}\left[(1+\delta)^p - 1\right]} \underset{\text{L'Hôpital}}{=} \lim_{\delta\to 0} \frac{p(2+2\delta)^{p-1}\cdot 2 - p(2+\delta)^{p-1}}{2^{p-1}p(1+\delta)^{p-1}} \\ = \frac{p\cdot 2^{p-1}\cdot 2 - p\cdot 2^{p-1}}{p\cdot 2^{p-1}} = \frac{p\cdot 2^{p-1}(2-1)}{p\cdot 2^{p-1}} = 1.$$

$$\lim_{\delta\to 0} \frac{(2+\delta)^p - 2^p}{2^{p-1}\left[(1+\delta)^p - 1\right]} \underset{\text{L'Hôpital}}{=} \lim_{\delta\to 0} \frac{p(2+\delta)^{p-1}}{2^{p-1}p(1+\delta)^{p-1}} = \frac{2^{p-1}}{2^{p-1}} = 1.$$

So, taking the limit when $\delta \to 0$, (2.39) becomes

$$w\left((A\cap B)^*\right) + w\left((A\cup B)^*\right) \leq w\left(A^*\right) + w\left(B^*\right),$$

which is $b)$. So $a)$ implies $b)$.
$b) \Rightarrow c)$: Suppose now that $b)$ holds. Fix $s, t > 0$ and consider, for $\varepsilon > 0$, the sets

$$A = (0,\varepsilon)\times(0,t) \cup (\varepsilon,s)\times(0,t-\varepsilon); \quad B = (0,\varepsilon)\times(0,t-\varepsilon)\cup(\varepsilon,s)\times(0,t).$$

(See Figure **2-16**). Then, we have that $A^* = A$ (since A is decreasing). Let's calculate B^*.

$$\varphi_B(x) = m_1\left(B_x\right) = m_1\left(\{y\in\mathbb{R} : (x,y)\in B\}\right) = \begin{cases} t-\varepsilon, & \text{if } 0 < x < \varepsilon \\ t, & \text{if } \varepsilon < x < s \end{cases}$$

$$= (t-\varepsilon)\chi_{(0,\varepsilon)}(x) + t\chi_{(\varepsilon,s)}(x) = t\chi_{(\varepsilon,s)}(x) + (t-\varepsilon)\chi_{(0,\varepsilon)}(x).$$

Then

$$\varphi_B^*(z) = t\chi_{[0,s-\varepsilon)}(z) + (t-\varepsilon)\chi_{[s-\varepsilon,s)}(z).$$

So

$$B^* = \{(z,t) : 0 < t < \varphi_B^*(z)\} = (0,s-\varepsilon)\times(0,t)\cup(s-\varepsilon,s)\times(0,t-\varepsilon) \quad \text{a.e.}$$

Also, we have

$$A\cap B = (0,\varepsilon)\times(0,t-\varepsilon)\cup(\varepsilon,s)\times(0,t-\varepsilon) \quad (A\cap B\in\Delta_d).$$

Then

$$(A\cap B)^* = (0,\varepsilon)\times(0,t-\varepsilon)\cup(\varepsilon,s)\times(0,t-\varepsilon) = (0,s)\times(0,t-\varepsilon) \quad \text{a.e.}$$

And

$$A\cup B = (0,\varepsilon)\times(0,t)\cup(\varepsilon,s)\times(0,t) \underset{\text{a.e.}}{=} (0,s)\times(0,t)\in\Delta_d.$$

So

$$(A\cup B)^* = (0,s)\times(0,t).$$

To summarize, for the sets

$$A = (0,\varepsilon)\times(0,t)\cup(\varepsilon,s)\times(0,t-\varepsilon); \quad B = (0,\varepsilon)\times(0,t-\varepsilon)\cup(\varepsilon,s)\times(0,t).$$

We have

$$\begin{aligned} A^* &= A, \\ B^* &= (0,s-\varepsilon)\times(0,t)\cup(s-\varepsilon,s)\times(0,t-\varepsilon), \\ (A\cap B)^* &= (0,s)\times(0,t-\varepsilon), \\ (A\cup B)^* &= (0,s)\times(0,t). \end{aligned}$$

Note that

$$\begin{aligned} (A\cup B)^*\setminus B^* &= (0,s)\times(0,t)\setminus[(0,s-\varepsilon)\times(0,t)\cup(s-\varepsilon,s)\times(0,t-\varepsilon)] \\ &= (s-\varepsilon,s)\times(t-\varepsilon,t). \end{aligned}$$

Then

$$\begin{aligned} w\left((s-\varepsilon,s)\times(t-\varepsilon,t)\right) &= w\left((A\cup B)^*\setminus B^*\right) \\ &= w\left((A\cup B)^*\right) - w\left(B^*\right) \\ &\le w\left(A^*\right) - w\left((A\cap B)^*\right) \quad \text{(Using } b)) \\ &= w\left(A^*\setminus(A\cap B)^*\right) \\ &= w\left((0,\varepsilon)\times(t-\varepsilon,t)\right). \end{aligned}$$

i.e.

$$\int_{(s-\varepsilon,s)\times(t-\varepsilon,t)} w(x)\,dx \le \int_{(0,\varepsilon)\times(t-\varepsilon,t)} w(x)\,dx.$$

Dividing by ε^2,

$$\frac{1}{\varepsilon^2}\int_{(s-\varepsilon,s)\times(t-\varepsilon,t)} w(x)\,dx \le \frac{1}{\varepsilon^2}\int_{(0,\varepsilon)\times(t-\varepsilon,t)} w(x)\,dx.$$

Letting $\varepsilon \to 0$ and using the Lebesgue's Differentiation Theorem, we obtain

$$w(s,t) \le w(0,t). \tag{2.40}$$

Now, consider the sets

$$A = (0,s)\times(0,t); \quad B = (0,\varepsilon)\times(\varepsilon,t+\varepsilon)\cup(\varepsilon,s-\varepsilon)\times(0,t)\cup(s-\varepsilon,s)\times(0,t-\varepsilon).$$

We have

$$\begin{aligned}
A^* &= A,\\
B^* &= (0,s-\varepsilon)\times(0,t)\cup(s-\varepsilon,s)\times(0,t-\varepsilon),\\
(A\cap B)^* &= (0,s-2\varepsilon)\times(0,t)\cup(s-2\varepsilon,s)\times(0,t-\varepsilon),\\
(A\cup B)^* &= (0,\varepsilon)\times(0,t+\varepsilon)\cup(\varepsilon,s)\times(0,t).
\end{aligned}$$

Note that

$$\begin{aligned}
(A\cup B)^* \setminus A^* &= [(0,\varepsilon)\times(0,t+\varepsilon)\cup(\varepsilon,s)\times(0,t)] \setminus [(0,s)\times(0,t)]\\
&= (0,\varepsilon)\times(t,t+\varepsilon).
\end{aligned}$$

Then

$$\begin{aligned}
w\left((0,\varepsilon)\times(t,t+\varepsilon)\right) &= w\left((A\cup B)^*\setminus A^*\right)\\
&= w\left((A\cup B)^*\right) - w\left(A^*\right)\\
&\le w\left(B^*\right) - w\left((A\cap B)^*\right) \quad \text{(Using } b))\\
&= w\left(B^*\setminus (A\cap B)^*\right)\\
&= w\left((s-2\varepsilon,s-\varepsilon)\times(t-\varepsilon,t)\right).
\end{aligned}$$

i.e.

$$\int_{(0,\varepsilon)\times(t,t+\varepsilon)} w(x)\,dx \le \int_{(s-2\varepsilon,s-\varepsilon)\times(t-\varepsilon,t)} w(x)\,dx.$$

Dividing by ε^2,

$$\frac{1}{\varepsilon^2}\int_{(0,\varepsilon)\times(t,t+\varepsilon)} w(x)\,dx \le \frac{1}{\varepsilon^2}\int_{(s-2\varepsilon,s-\varepsilon)\times(t-\varepsilon,t)} w(x)\,dx.$$

Letting $\varepsilon \to 0$ and using the Lebesgue's Differentiation Theorem, we obtain

$$w(0,t) \le w(s,t). \tag{2.41}$$

From (2.40) and (2.41) it follows that

$$w(s,t) = w(0,t) = v(t).$$

Finally, we will show that v is decreasing, i.e.

$$v(b) = w(0,b) \le w(0,a) = v(a) \quad \text{if } 0 < a \le b.$$

For $\varepsilon > 0$ small, take

$$A = (0,\varepsilon) \times (0,a); \quad B = (0,\varepsilon) \times (\varepsilon, b).$$

We have

$$\begin{aligned} A^* &= A, \\ B^* &= (0,\varepsilon) \times (0, b-\varepsilon), \\ (A \cap B)^* &= (0,\varepsilon) \times (0, a-\varepsilon), \\ (A \cup B)^* &= (0,\varepsilon) \times (0,b). \end{aligned}$$

Thanks to $b)$ we obtain

$$\begin{aligned} w\left((0,\varepsilon) \times (b-\varepsilon, b)\right) &= w\left((A \cup B)^* \smallsetminus B^*\right) \\ &\le w\left(A^*\right) - w\left((A \cap B)^*\right) \\ &= w\left(A^* \smallsetminus (A \cap B)^*\right) \\ &= w\left((0,\varepsilon) \times (a-\varepsilon, a)\right). \end{aligned}$$

i.e.

$$\int_{(0,\varepsilon)\times(b-\varepsilon,b)} w(x)\,dx \le \int_{(0,\varepsilon)\times(a-\varepsilon,a)} w(x)\,dx.$$

Dividing by ε^2, letting $\varepsilon \to 0$ and using the Lebesgue's Differentiation Theorem, we obtain

$$w(0,b) \le w(0,a),$$

So

$$v(b) \le v(a).$$

$c) \Rightarrow a)$: From Theorem 2.1.25 we know that

$$f_2^*(s,t) = \left(f_x^{*y}(t)\right)^{*x}(s).$$

From the fact that $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm if v is decreasing (see [43]) and the Minkowski inequality, we obtain

$$
\begin{aligned}
\|f+g\|_{\Lambda_2^p(w)} &= \left(\int_{\mathbb{R}_+^2} [(f+g)_2^*(s,t)]^p \, w(s,t)\, ds\; dt\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_0^\infty \left[\left((f_x+g_x)^{*y}(t)\right)^{*x}(s)\right]^p \, ds\right) v(t)\, dt\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_{\mathbb{R}} [(f_x+g_x)^{*y}(t)]^p \, dx\right) v(t)\, dt\right)^{1/p}, \quad \text{since } \int_0^\infty f^* = \int_{\mathbb{R}} f \\
&= \left(\int_{\mathbb{R}} \left(\int_0^\infty [(f_x+g_x)^{*y}(t)]^p \, v(t)\, dt\right) dx\right)^{1/p}, \quad \text{Fubini's Theorem.}
\end{aligned}
$$

From [43] we know that $\|f_x+g_x\|^p_{\Lambda_2^p(w)} \le \left(\|f_x\|_{\Lambda_2^p(w)} + \|g_x\|_{\Lambda_2^p(w)}\right)^p$, then

$$
\begin{aligned}
&\le \left(\int_{\mathbb{R}} \left[\left(\int_0^\infty [(f_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p} + \left(\int_0^\infty [(g_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right]^p dx\right)^{1/p} \\
&= \left\|\left(\int_0^\infty [(f_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p} + \left(\int_0^\infty [(g_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right\|_{L^p(dx)} \\
&\le \left\|\left(\int_0^\infty [(f_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right\|_{L^p(dx)} + \left\|\left(\int_0^\infty [(g_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right\|_{L^p(dx)} \\
&= \left(\int_{\mathbb{R}} \left[\left(\int_0^\infty [(f_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right]^p dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}} \left[\left(\int_0^\infty [(g_x)^{*y}(t)]^p \, v(t)\, dt\right)^{1/p}\right]^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}} \left(\int_0^\infty [(f_x)^{*y}(t)]^p \, v(t)\, dt\right) dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}} \left(\int_0^\infty [(g_x)^{*y}(t)]^p \, v(t)\, dt\right) dx\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_{\mathbb{R}} [(f_x)^{*y}(t)]^p \, v(t)\, dx\right) dt\right)^{1/p} \\
&\quad + \left(\int_0^\infty \left(\int_{\mathbb{R}} [(g_x)^{*y}(t)]^p \, v(t)\, dx\right) dt\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_0^\infty \left[[(f_x)^{*y}(t)]^p\right]^{*x}(s) v(t)\, ds\right) dt\right)^{1/p} \\
&\quad + \left(\int_0^\infty \left(\int_0^\infty \left[[(g_x)^{*y}(t)]^p\right]^{*x}(s) v(t)\, ds\right) dt\right)^{1/p}, \text{again, } \int_0^\infty f^* = \int_{\mathbb{R}} f
\end{aligned}
$$

$$= \left(\int_0^\infty \left(\int_0^\infty \left[\left[(f_x)^{*y}(t)\right]^{*x}(s)\right]^p v(t)\,ds\right) dt\right)^{1/p}$$
$$+ \left(\int_0^\infty \left(\int_0^\infty \left[\left[(g_x)^{*y}(t)\right]^{*x}(s)\right]^p v(t)\,ds\right) dt\right)^{1/p}, \quad \text{since } (|h|^p)^* = (h^*)^p$$
$$= \left(\int_{\mathbb{R}^2_+} [f_2^*(s,t)]^p\, v(t)\, ds\ dt\right)^{1/p} + \left(\int_{\mathbb{R}^2_+} [g_2^*(s,t)]^p\, v(t)\, ds\ dt\right)^{1/p}$$
, by using Theorem 2.1.25
$$= \left(\int_{\mathbb{R}^2_+} [f_2^*(s,t)]^p\, w(s,t)\, ds\ dt\right)^{1/p} + \left(\int_{\mathbb{R}^2_+} [g_2^*(s,t)]^p\, w(s,t)\, ds\ dt\right)^{1/p}$$
, by hypothesis
$$= \|f\|_{\Lambda_2^p(w)} + \|g\|_{\Lambda_2^p(w)}.$$

Thus, in view of (2.25) and (2.26), we see that $\|\cdot\|_{\Lambda_2^p(w)}$ is a norm, and this completes the proof. □

Remark 2.2.10. Observe that the equivalences proved in Theorem 2.2.9 in particular say that

$$\Lambda_2^p(w) = L^p\left(\Lambda^p(v, dy), dx\right),$$

which is a mixed norm space.

3 Weighted Composition Operator on $\Lambda_2^p(w)$

In this chapter the boundedness, compactness and closed range of the Weighted Composition Operator on the space $\Lambda_2^p(w)$ are characterized.

Definition 3.0.11. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $T : X \to X$ be a measurable transformation (i.e. $T^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$) and non-singular (i.e. $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathcal{A}$ with $\mu(A) = 0$, which means that μT^{-1} is absolutely continuous with respect to μ ($\mu T^{-1} \ll \mu$)) and $\mu : X \to \mathbb{C}$ be a measurable function. The linear transformation $W_{u,T}$ is defined as follows:

$$\begin{aligned} W_{u,T} : \mathcal{F}(X, \mathcal{A}) &\to \mathcal{F}(X, \mathcal{A}) \\ f &\mapsto W_{u,T}(f) = u \circ T \cdot f \circ T, \end{aligned}$$

where

$$\begin{aligned} W_{u,T} : X &\to \mathbb{C} \\ x &\mapsto \left(W_{u,T}(f)\right)(x) = u(T(x)) \cdot f(T(x)). \end{aligned}$$

If the operator $W_{u,T}$ is bounded and has range in $\Lambda_2^p(w)$, then it is called the Weighted Composition Operator on $\Lambda_2^p(w)$.

Remark 3.0.12.

1. If $u = 1$, then $W_{u,T} = W_{1,T} = C_T : f \mapsto f \circ T$ is called the composition operator induced by T.

2. If $T = I_X$, identity on X, then $W_{u,T} = W_{u,I_X} = M_u : f \mapsto u \cdot f$ is called the multiplication operator induced by u.

3. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $T : X \to X$ be a measurable and non-singular transformation and $u : X \to \mathbb{C}$ be a measurable function, then T and u induce a Weighted Composition Operator that is well defined on $\mathcal{F}(X, \mathcal{A})$.

   Indeed, remember that $\mathcal{F}(X, \mathcal{A})$ is a set of functions classes where two functions belong to a same class if they are equal almost everywhere with respect to μ. That is to say,

$$f \cong g \Leftrightarrow \mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Let $f, g \in \mathcal{F}(X, \mathcal{A})$ such that $f \cong g$. Then

$$\begin{aligned} &\mu(\{x \in X : f(x) \neq g(x)\}) = 0 \\ \Rightarrow \quad &\mu(\{x \in X : (uf)(x) \neq (ug)(x)\}) = 0 \end{aligned}$$

(We are assuming that $u \neq 0$, otherwise $M_u = M_0 = 0$, which is not of interest). Now,

$$\begin{aligned} &x_0 \in \{x \in X : u(T(x))f(T(x)) \neq u(T(x))g(T(x))\} \\ \Leftrightarrow \quad &u(T(x_0))f(T(x_0)) \neq u(T(x_0))g(T(x_0)) \\ \Leftrightarrow \quad &u(y_0)f(y_0) \neq u(y_0)g(y_0), y_0 = T(x_0) \\ \Leftrightarrow \quad &y_0 = T_{x_0} \in \{x \in X : u(x)f(x) \neq u(x)g(x)\} \\ \Leftrightarrow \quad &x_0 \in T^{-1}(\{x \in X : (uf)(x) \neq (ug)(x)\}). \end{aligned}$$

Therefore

$$\begin{aligned} &\mu(\{x \in X : u(T(x))f(T(x)) \neq u(T(x))g(T(x))\}) \\ &\qquad = \mu\left(T^{-1}(\{x \in X : (uf)(x) \neq (ug)(x)\})\right) = 0, \end{aligned}$$

because of the non-singularity of T. So,

$$\begin{aligned} (W_{u,T}f)(x) &= u(T(x))f(T(x)) \\ &= u(T(x))g(T(x)) \\ &= (W_{u,T}g)(x) \quad \mu-\text{a.e.} \end{aligned}$$

i.e., $W_{u,T}$ is well defined on the classes of $\mathcal{F}(X, \mathcal{A})$.

From now on, $(X, \mathcal{A}, \mu) = (\mathbb{R}^2, \mathcal{B}, m_2)$, which is a σ-finite measure space.

3.1 Boundedness

The boundedness of $W_{u,T}$ is characterized in the following result.

Theorem 3.1.1. *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-singular measurable transformation. Also, suppose that there exists a constant $b \geq 1$ such that $m_1(T_x^{-1}(E)) \leq b m_1(E_x)$ for all $E \subset \mathbb{R}^2$. Then*

$$W_{u,T} : f \mapsto W_{u,T}f = W_{u,T}(f) = u \circ T \cdot f \circ T$$

is bounded on $\Lambda_2^p(w)$ if $u \in L_\infty(\mathbb{R}^2)$. Moreover,

$$\|W_{u,T}\| \leq b^{1/p}\|u\|_\infty.$$

Besides, if $T_x^{-1}(F) \supset F_x$ for all $F \subset \mathbb{R}^2$, then

$$\|W_{u,T}\| = b^{1/p}\|u\|_\infty.$$

Proof. We are going to use iterated rearrangement, since Theorem 2.1.25 ensures that $h_2^* = \tilde{h}$, where $\tilde{h} = ((h_x)^{*y})^{*x}$ and $h_x(y) = h(x,y)$ (sometimes h_x is called the x-section of h).
For $(x,y) \in \mathbb{R}^2$, we know that

$$\begin{aligned}(W_{u,T}f)_x &= (u \circ T \cdot f \circ T)_x(y) \\ &= (u \circ T \cdot f \circ T)(x,y) \\ &= (u \circ T)(x,y) \cdot (f \circ T)(x,y) \\ &= u(T(x,y)) \cdot f(T(x,y)) \\ &= u(T_x(y)) \cdot f(T_x(y)) \\ &= (u \circ T_x)(y) \cdot (f \circ T_x)(y).\end{aligned}$$

So,

$$\begin{aligned}D_{(W_{u,T}f)_x}(\lambda) &= m_1\left(\{y \in \mathbb{R} : |(u \circ T \cdot f \circ T)_x(y)| > \lambda\}\right) \\ &= m_1\left(\{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\}\right) \\ &= m_1\left(\left\{(x,y) \in \mathbb{R}^2 : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\right\}_x\right).\end{aligned}$$

In the other hand, since

$$\begin{aligned}& y_0 \in \left\{(x,y) \in \mathbb{R}^2 : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\right\}_x \\ \Leftrightarrow\ & \{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\} \\ \Leftrightarrow\ & |u(T_x(y_0)) f(T_x(y_0))| > \lambda \\ \Leftrightarrow\ & |u(z_0) f(z_0)| > \lambda,\ z_0 = T_x(y_0) \\ \Leftrightarrow\ & z_0 = T_x(y_0) \in \left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\} \\ \Leftrightarrow\ & y_0 \in T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right) \\ \Leftrightarrow\ & y_0 \in \left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right)\right)_x.\end{aligned}$$

Then

$$\begin{aligned}\{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\} &= T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right) \\ &= \left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right)\right)_x.\end{aligned}$$

Hence

$$\begin{aligned}D_{(W_{u,T}f)_x} &= m_1\left(\{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\}\right) \\ &= m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right)\right) \\ &= m_1\left(\left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y) f(x,y)| > \lambda\right\}\right)\right)_x\right). \qquad (3.1)\end{aligned}$$

Next, since $|u(x,y)| \leq \|u\|_\infty\ \forall\ (x,y) \in \mathbb{R}^2$, it holds that, in particular,

$$(x_0, y_0) \in \left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}^{\complement}$$

$$\begin{aligned}
&\Rightarrow \|u\|_\infty |f(x_0,y_0)| \leq \lambda \\
&\Rightarrow |u(x_0,y_0)||f(x_0,y_0)| \leq \lambda \\
&\Rightarrow (x_0,y_0) \in \left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}^{\complement}.
\end{aligned}$$

Therefore

$$\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\} \subset \left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\},$$

then

$$T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}\right) \subset T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right),$$

and so

$$\begin{aligned}
&m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}\right)\right) \\
&\qquad \leq m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right).
\end{aligned}$$

Thus,

$$D_{\left(W_{u,T}f\right)_x}(\lambda) \leq m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right). \tag{3.2}$$

Let $E = \{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\} \subset \mathbb{R}^2$. By hypothesis we know that

$$m_1\left(T_x^{-1}(E)\right) \leq b m_1\left(E_x\right).$$

Going back to (3.2),

$$\begin{aligned}
D_{\left(W_{u,T}f\right)_x}(\lambda) &\leq m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right) \\
&\leq b m_1\left(\{y \in \mathbb{R} : \|u\|_\infty |f(x,y)| > \lambda\}\right) \\
&= b D_{\|u\|_\infty f_x}(\lambda).
\end{aligned}$$

Now, for any $t \geq 0$,

$$\begin{aligned}
&D_{\left(W_{u,T}f\right)_x}(\lambda) \leq b D_{\|u\|_\infty f_x}(\lambda) \\
\Leftrightarrow \quad &\left\{\lambda > 0 : D_{\|u\|_\infty f_x}(\lambda) \leq \frac{t}{b}\right\} \subset \left\{\lambda > 0 : D_{\left(W_{u,T}f\right)_x}(\lambda) \leq t\right\}.
\end{aligned}$$

Which implies that

$$\begin{aligned}
\left(W_{u,T}f\right)_x^*(t) &= \inf\left\{\lambda > 0 : D_{\left(W_{u,T}f\right)_x}(\lambda) \leq t\right\} \\
&\leq \inf\left\{\lambda > 0 : D_{\|u\|_\infty f_x}(\lambda) \leq \frac{t}{b}\right\} \\
&= \inf\left\{\lambda > 0 : m_1\left(\{y \in \mathbb{R} : \|u\|_\infty |f_x(y)| > \lambda\}\right) \leq \frac{t}{b}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \inf\left\{\lambda > 0 : m_1\left(\left\{y \in \mathbb{R} : |f_x(y)| > \frac{\lambda}{\|u\|_\infty}\right\}\right) \le \frac{t}{b}\right\}, r = \frac{\lambda}{\|u\|_\infty} \\
&= \inf\left\{r\|u\|_\infty > 0 : m_1\left(\{y \in \mathbb{R} : |f_x(y)| > r\}\right) \le \frac{t}{b}\right\} \\
&= \|u\|_\infty \inf\left\{r > 0 : m_1\left(\{y \in \mathbb{R} : |f_x(y)| > r\}\right) \le \frac{t}{b}\right\} \\
&= \|u\|_\infty \inf\left\{r > 0 : D_{f_x}(r) \le \frac{t}{b}\right\} \\
&= \|u\|_\infty \left(f_x\right)^* \left(\frac{t}{b}\right).
\end{aligned}
$$

Now, using Theorem 2.1.25, we rearrange with respect to x to obtain

$$
\left(W_{u,T}f\right)_2^*(s,t) \le \|u\|_\infty f_2^*\left(s, \frac{t}{b}\right),
$$

then

$$
\|W_{u,T}f\|_{\Lambda_2^p(w)} \le b^{1/p}\|u\|_\infty \|f\|_{\Lambda_2^p(w)},
$$

from where

$$
\|W_{u,T}f\| \le b^{1/p}\|u\|_\infty. \tag{3.3}
$$

Now, let us see under what conditions $\|W_{u,T}f\| = b^{1/p}\|u\|_\infty$.
Let $B_\varepsilon = \left\{x \in \mathbb{R}^2 : |u(T(x))| \ge b^{1/p}\|u\|_\infty - \varepsilon\right\}$ (note that $m_2\left(B_\varepsilon\right) > 0$). Then,

$$
|u(T(x,y))\chi_{B_\varepsilon}(T(x,y))| \ge \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\chi_{B_\varepsilon}(T(x,y)) \tag{3.4}
$$

On the other hand, for a fixed x,

$$
\chi_{B_\varepsilon}(T(x,y)) = \chi_{B_\varepsilon}(T_x(y)) = \begin{cases} 1, & \text{if } T_x(y) \in B_\varepsilon \\ 0, & \text{if } T_x(y) \notin B_\varepsilon \end{cases} = \begin{cases} 1, & \text{if } y \in T_x^{-1}\left(B_\varepsilon\right) \\ 0, & \text{if } y \notin T_x^{-1}\left(B_\varepsilon\right) \end{cases} = \chi_{T_x^{-1}(B_\varepsilon)}(y).
$$

And

$$
\chi_{B_\varepsilon}(x,y) = \left(\chi_{B_\varepsilon}\right)_x(y) = \begin{cases} 1, & \text{if } (x,y) \in B_\varepsilon \\ 0, & \text{if } (x,y) \notin B_\varepsilon \end{cases} = \begin{cases} 1, & \text{if } y \in \left(B_\varepsilon\right)_x \\ 0, & \text{if } y \notin \left(B_\varepsilon\right)_x \end{cases} = \chi_{(B_\varepsilon)_x}(y).
$$

By hypothesis, $T_x^{-1}\left(B_\varepsilon\right) \supset \left(B_\varepsilon\right)_x$, then

$$
\chi_{B_\varepsilon}\left(T(x,y)\right) = \chi_{T_x^{-1}(B_\varepsilon)}(y) \ge \chi_{(B_\varepsilon)_x}(y) = \chi_{B_\varepsilon}(x,y).
$$

Going back to (3.4), we have

$$
\begin{aligned}
& |u(T(x,y))\chi_{B_\varepsilon}(T(x,y))| \ge \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\chi_{B_\varepsilon}(x,y) \\
\Rightarrow \quad & |u(T_x(y))\chi_{B_\varepsilon}(T_x(y))| \ge \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\left(\chi_{B_\varepsilon}\right)_x(y)
\end{aligned}
$$

$$\Rightarrow \qquad \left|\left(W_{u,T}\chi_{B_\varepsilon}\right)_x(y)\right| \geq \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\left(\chi_{B_\varepsilon}\right)_x(y)$$

Next we use Theorem 2.1.25 in order to calculate the two-dimensional rearrangement in an iterated way, so

$$\begin{aligned} & \left(W_{u,T}\chi_{B_\varepsilon}\right)_x^{*y}(t) \geq \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\left(\chi_{B_\varepsilon}\right)_x^{*y}(t) \\ \Rightarrow \quad & \left(W_{u,T}\chi_{B_\varepsilon}\right)_2^*(s,t) \geq \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\left(\chi_{B_\varepsilon}\right)_2^*(s,t) \\ \Rightarrow \quad & \|W_{u,T}\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)} \geq \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\|\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)} \\ \Rightarrow \quad & \frac{\|W_{u,T}\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)}}{\|\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)}} \geq b^{1/p}\|u\|_\infty - \varepsilon \\ \Rightarrow \quad & \|W_{u,T}\| \geq b^{1/p}\|u\|_\infty - \varepsilon. \end{aligned}$$

Since the above inequality is valid for all $\varepsilon > 0$, then

$$\|W_{u,T}\| \geq b^{1/p}\|u\|_\infty. \tag{3.5}$$

Combining (3.3) and (3.5), it follows that

$$\|W_{u,T}\| = b^{1/p}\|u\|_\infty. \qquad \square$$

Theorem 3.1.2. *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation such that $T(A_\varepsilon) \subset A_\varepsilon$ for all $\varepsilon > 0$, where $A_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |u(x,y)| > \varepsilon\}$. If $W_{u,T}$ is bounded on $\Lambda_2^p(w)$, then $u \in L_\infty(\mathbb{R}^2)$.*

Proof. Suppose that $W_{u,T}$ is bounded on $\Lambda_2^p(w)$ and $u \notin L_\infty(\mathbb{R}^2)$. Let

$$A_n = \left\{(x,y) \in \mathbb{R}^2 : |u(x,y)| > n\right\}.$$

Since $u \notin L_\infty(\mathbb{R}^2)$, we have that $m_2(A_n) > 0 \ \forall\ n \in \mathbb{N}$.
Besides, $T(A_n) \subset A_n$ implies $A_n \subset T^{-1}(A_n)$, then $(A_n)_x \subset \left(T^{-1}(A_n)\right)_x$ and therefore $\chi_{(A_n)_x}(y) \leq \chi_{(T^{-1}(A_n))_x}(y)$ for all $y \in \mathbb{R}$. Let $\lambda > 0$ and $y \in (A_n)_x$. We have

$$\left|\chi_{(A_n)_x}(y)\right| > \lambda \Rightarrow \left|\chi_{(T^{-1}(A_n))_x}(y)\right| > \lambda,$$

Also,

$$y \in (A_n)_x \Rightarrow (x,y) \in A_n \Rightarrow T(x,y) \in A_n \Rightarrow |u(T(x,y))| > n \Rightarrow |u\left(T_x(y)\right)| > n,$$

then we have

$$\left|u\left(T_x(y)\right)\chi_{(T^{-1}(A_n))_x}(y)\right| > n\lambda,$$

i.e.

$$\left\{y \in \mathbb{R} : \left|\chi_{(A_n)_x}(y)\right| > \lambda\right\} \subset \left\{y \in \mathbb{R} : \left|u\left(T_x(y)\right)\chi_{(T^{-1}(A_n))_x}(y)\right| > n\lambda\right\}$$

$$\Rightarrow m_1\left(\left\{y\in\mathbb{R}:\left|\chi_{(A_n)_x}(y)\right|>\lambda\right\}\right)\le m_1\left(\left\{y\in\mathbb{R}:\left|u\left(T_x(y)\right)\chi_{(T^{-1}(A_n))_x}(y)\right|>n\lambda\right\}\right) \quad (3.6)$$

Since

$$\chi_{(T^{-1}(A_n))_x}(y)=\begin{cases}1, & \text{if } y\in\left(T^{-1}\left(A_n\right)\right)_x\\ 0, & \text{if } y\notin\left(T^{-1}\left(A_n\right)\right)_x\end{cases}=\begin{cases}1, & \text{if } y\in T_x^{-1}\left(A_n\right)\\ 0, & \text{if } y\notin T_x^{-1}\left(A_n\right)\end{cases}=\begin{cases}1, & \text{if } T_x(y)\in A_n\\ 0, & \text{if } T_x(y)\in A_n\end{cases}$$
$$=\chi_{A_n}\left(T_x(y)\right)=\left(\chi_{A_n}\circ T_x\right)(y).$$

Returning to (3.6),

$$m_1\left(\left\{y\in\mathbb{R}:\left|\chi_{(A_n)_x}(y)\right|>\lambda\right\}\right)\le m_1\left(\left\{y\in\mathbb{R}:\left|u\left(T_x(y)\right)\chi_{A_n}\left(T_x(y)\right)\right|>n\lambda\right\}\right),$$

i.e.

$$m_1\left(\left\{y\in\mathbb{R}:\left|\chi_{(A_n)_x}(y)\right|>\lambda\right\}\right)\le m_1\left(\left\{y\in\mathbb{R}:\left|\left(W_{u,T_x}\chi_{A_n}\right)(y)\right|>n\lambda\right\}\right).$$

Therefore,

$$\begin{aligned}
& D_{\chi_{(A_n)_x}}(\lambda)\le D_{W_{u,T_x}\chi_{A_n}}(n\lambda)\\
\Rightarrow\quad & D_{\chi_{(A_n)_x}}(\lambda)\le D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda)\\
\Rightarrow\quad & \left\{\lambda>0:D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda)\le t\right\}\subset\left\{\lambda>0:D_{\chi_{(A_n)_x}}(\lambda)\le t\right\}\\
\Rightarrow\quad & \inf\left\{\lambda>0:D_{\chi_{(A_n)_x}}(\lambda)\le t\right\}\le\inf\left\{\lambda>0:D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda)\le t\right\}\\
\Rightarrow\quad & \left[\chi_{(A_n)_x}\right]^*(t)\le\left[\frac{1}{n}W_{u,T_x}\chi_{A_n}\right]^*(t)\\
\Rightarrow\quad & n\left[\chi_{(A_n)_x}\right]^*(t)\le\left[W_{u,T_x}\chi_{A_n}\right]^*(t).
\end{aligned} \quad (3.7)$$

Note that

$$\chi_{(A_n)_x}(y)=\begin{cases}1, & \text{if } y\in\left(A_n\right)_x\\ 0, & \text{if } y\notin\left(A_n\right)_x\end{cases}=\begin{cases}1, & \text{if } (x,y)\in A_n\\ 0, & \text{if } (x,y)\notin A_n\end{cases}=\chi_{A_n}(x,y)=\left(\chi_{A_n}\right)_x(y).$$

Also

$$\left(W_{u,T_x}\chi_{A_n}\right)(y)=u\left(T_x(y)\right)\chi_{A_n}\left(T_x(y)\right)=u\left(T(x,y)\right)\chi_{A_n}\left(T(x,y)\right)$$
$$=\left(W_{u,T}\chi_{A_n}\right)(x,y)=\left(W_{u,T}\chi_{A_n}\right)_x(y).$$

Returning to (3.7), we obtain

$$n\left[\left(\chi_{A_n}\right)_x\right]^*(t)\le\left[\left(W_{u,T}\chi_{A_n}\right)_x\right]^*(t).$$

By Theorem 2.1.25, we rearrange with respect to x to obtain

$$n\left(\chi_{A_n}\right)_2^*(s,t)\le\left(W_{u,T}\chi_{A_n}\right)_2^*(s,t),$$

which implies

$$n \left\|\chi_{A_n}\right\|_{\Lambda_2^p(w)} \leq \left\|W_{u,T}\chi_{A_n}\right\|_{\Lambda_2^p(w)}.$$

We conclude that given $n \in \mathbb{N}$, there exists $\chi_{A_n} \in \Lambda_2^p(w)$ such that

$$\left\|W_{u,T}\chi_{A_n}\right\|_{\Lambda_2^p(w)} > n \left\|\chi_{A_n}\right\|_{\Lambda_2^p(w)}.$$

Hence $W_{u,T}$ is not bounded, which contradicts the hypothesis of the theorem, so $u \in L_\infty(\mathbb{R}^2)$. □

The next result follows as a consequence from the last two theorems.

Theorem 3.1.3. *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-singular measurable transformation that satisfies the following conditions:*

1. *There exists a constant $b \geq 1$ such that*

$$m_1\left(T_x^{-1}(E)\right) \leq b m_1\left(E_x\right), \text{for all } E \subset \mathbb{R}^2.$$

2. *$T(A_\varepsilon) \subset A_\varepsilon$ for all $\varepsilon > 0$, with $A_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |u(x,y)| > \varepsilon\}$.*

Then $W_{u,T}$ is bounded on $\Lambda_2^p(w)$ if and only if $u \in L_\infty(\mathbb{R}^2)$.

3.2 Compactness

In this section compactness of the Weighted Composition Operator on the space $\Lambda_2^p(w)$ is characterized.

Theorem 3.2.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b \geq 1$ and $\delta > 0$ such that $\delta m_1(E_x) \leq m_1(T_x^{-1}(E)) \leq b m_1(E_x)$ for all $E \subset \mathbb{R}^2$. If $f \in \Lambda_2^p(w)$, then*

$$\alpha \left\|M_u f\right\|_{\Lambda_2^p(w)} \leq \left\|W_{u,T} f\right\|_{\Lambda_2^p(w)} \leq \beta \left\|M_u f\right\|_{\Lambda_2^p(w)},$$

with $\alpha = \delta^{1/p}$, $\beta = b^{1/p}$ $\forall$ $f \in \Lambda_2^p(w)$.

Proof. Let $E = \{(x,y) \in \mathbb{R}^2 : |(uf)(x,y)| > \lambda\}$ and let $t > 0$. By equation (3.1) and the inequality (3.3), we have

$$\begin{aligned}
\left[(W_{u,T}f)_x\right]^*(t) &= \inf\{\lambda > 0 : m_1(\{y \in \mathbb{R} : |u(T_x(y)) f(T_x(y))| > \lambda\}) \leq t\} \\
&= \inf\left\{\lambda > 0 : m_1\left(\left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}\right)\right)_x\right) \leq t\right\}, \text{ by (3.1)} \\
&= \inf\left\{\lambda > 0 : m_1\left(\left(T^{-1}(E)\right)_x\right) \leq t\right\} \\
&\leq \inf\left\{\lambda > 0 : b m_1(E_x) \leq t\right\}, \text{ by (3.3)} \\
&= \inf\left\{\lambda > 0 : b m_1\left(\left\{(x,y) \in \mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \leq t\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \inf\left\{\lambda > 0 : m_1\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \le \frac{t}{b}\right\}\\
&= \inf\left\{\lambda > 0 : D_{(uf)_x}(\lambda) \le \frac{t}{b}\right\}\\
&= [(uf)_x]^*\left(\frac{t}{b}\right)\\
&= [(M_u f)_x]^*\left(\frac{t}{b}\right).
\end{aligned}
$$

Rearranging with respect to x, we obtain

$$
(W_{u,T}f)_2^*(s,t) \le (M_u f)_2^*\left(s,\frac{t}{b}\right).
$$

Hence, if $1 < p < \infty$, it holds that

$$
\|W_{u,T}f\|_{\Lambda_2^p(w)} \le b^{1/p}\|M_u f\|_{\Lambda_2^p(w)} = \beta\|M_u f\|_{\Lambda_2^p(w)}. \tag{3.8}
$$

On the other hand, we know that

$$
D_{\left(W_{u,T}f\right)_x}(\lambda) = m_1\left(\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}\right)\right)_x\right),
$$

and

$$
\left[\left(W_{u,T}f\right)_x\right]^*(t) = \inf\left\{\lambda > 0 : m_1\left(\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}\right)\right)_x\right) \le t\right\}.
$$

Let $S = \{(x,y)\in\mathbb{R}^2 : u(x,y)\neq 0\}$. From the hypothesis we know that for all $F\in\mathcal{B}, F\subset S$

$$
m_1\left(T_x^{-1}(F)\right) \ge \delta m_1(F_x).
$$

Now, let $G = \{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\}$ (note that $G\subset S$), then

$$
\left\{\lambda > 0 : m_1\left(T_x^{-1}(G)\right) \le t\right\} \subset \{\lambda > 0 : \delta m_1(G_x) \le t\},
$$

and it holds that

$$
\begin{aligned}
\left[\left(W_{u,T}f\right)_x\right]^*(t) &= \inf\left\{\lambda > 0 : m_1\left(T_x^{-1}\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}\right)\right) \le t\right\}\\
&= \inf\left\{\lambda > 0 : m_1\left(T_x^{-1}(G)\right) \le t\right\}\\
&\ge \inf\{\lambda > 0 : \delta m_1(G_x) \le t\}\\
&= \inf\left\{\lambda > 0 : \delta m_1\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \le t\right\}\\
&= \inf\left\{\lambda > 0 : m_1\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \le \frac{t}{\delta}\right\}\\
&= \inf\left\{\lambda > 0 : D_{(uf)_x} \le \frac{t}{\delta}\right\}
\end{aligned}
$$

$$= [(uf)_x]^* \left(\frac{t}{\delta}\right)$$
$$= [(M_u f)_x]^* \left(\frac{t}{\delta}\right).$$

i.e.,

$$\left[(W_{u,T} f)_x\right]^* (t) \geq [(M_u f)_x]^* \left(\frac{t}{\delta}\right),$$

Now, we invoke Theorem 2.1.25, and rearrange with respect to x, then

$$(W_{u,T} f)_2^* (s,t) \geq M_u f_2^* \left(s, \frac{t}{\delta}\right),$$

from where

$$\|W_{u,T} f\|_{\Lambda_2^p(w)} \geq \delta^{1/p} \|M_u f\|_{\Lambda_2^p(w)}. \tag{3.9}$$

Looking at inequalities (3.8) and (3.9), it follows that for any $f \in \Lambda_2^p(w)$,

$$\alpha \|M_u f\|_{\Lambda_2^p(w)} \leq \|W_{u,T} f\|_{\Lambda_2^p(w)} \leq \beta \|M_u f\|_{\Lambda_2^p(w)}. \tag{3.10}$$

□

Lemma 3.2.2. *Let M_u be a compact operator. For $\varepsilon > 0$, we define*

$$A_\varepsilon(u) = \left\{x \in \mathbb{R}^2 : |u(x,y)| > \varepsilon\right\} \quad \textit{and} \quad L^w_{A_\varepsilon(u)} = \left\{f\chi_{A_\varepsilon(u)} : f \in \Lambda_2^p(w)\right\}.$$

Then $L^w_{A_\varepsilon(u)}$ is an invariant closed subspace of $\Lambda_2^p(w)$ under M_u. Moreover, $M_u|_{L^w_{A_\varepsilon(u)}}$ is a compact operator.

Proof. Suppose that $h, k \in L^w_{A_\varepsilon(u)}$, then $h = f\chi_{A_\varepsilon(u)}$ and $k = g\chi_{A_\varepsilon(u)}$ with $f, g \in \Lambda_2^p(w)$. For scalars α, β, it holds that

$$\alpha h + \beta k = \alpha \left(f\chi_{A_\varepsilon(u)}\right) + \beta \left(g\chi_{A_\varepsilon(u)}\right) = (\alpha f)\chi_{A_\varepsilon(u)} + (\beta g)\chi_{A_\varepsilon(u)} = (\alpha f + \beta g)\chi_{A_\varepsilon(u)},$$

where $(\alpha f + \beta g) \in \Lambda_2^p(w)$. Hence $\alpha h + \beta k \in L^w_{A_\varepsilon(u)}$ and then $L^w_{A_\varepsilon(u)}$ is a vector subspace of $\Lambda_2^p(w)$.

Besides, if $h \in L^w_{A_\varepsilon(u)}$ with $h = f\chi_{A_\varepsilon(u)}$, $f \in \Lambda_2^p(w)$, then

$$M_u h = uh = u\left(f\chi_{A_\varepsilon(u)}\right) = (uf)\chi_{A_\varepsilon(u)},$$

where $uf \in \Lambda_2^p(w)$. Therefore $M_u h \in L^w_{A_\varepsilon(u)}$, which implies that $L^w_{A_\varepsilon(u)}$ is an invariant subspace of $\Lambda_2^p(w)$ under the operator M_u.

Then, by Theorem 5.1.1, $M_u|_{L^w_{A_\varepsilon(u)}}$ is a compact operator. □

Theorem 3.2.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b \geq 1$ and $\delta > 0$ such that $\delta m_1(E_x) \leq m_1(T_x^{-1}(E)) \leq b m_1(E_x)$ for all $E \subset \mathbb{R}^2$. Suppose that $u : \mathbb{R}^2 \to \mathbb{C}$ is a measurable function such that $W_{u,T}$ is bounded on $\Lambda_2^p(w)$. The following assertions are equivalent:*

i) $W_{u,T}$ *is compact.*

ii) M_u *is compact.*

iii) $L^w_{A_\varepsilon(u)}$ *has finite dimension for* $\varepsilon > 0$.

Proof. $i) \Leftrightarrow ii)$ This follows from inequalities (3.10) of Theorem 3.2.1, and Theorem 5.1.3.
$ii) \Rightarrow iii)$ Let $\varepsilon > 0$, then

$$\begin{aligned} |u(x,y)| \geq \varepsilon \quad &\Rightarrow \quad \left|uf\chi_{A_\varepsilon(u)}(x,y)\right| \geq \left|\varepsilon f\chi_{A_\varepsilon(u)}(x,y)\right|, (x,y) \in \mathbb{R}^2 \\ &\Rightarrow \quad \varepsilon \left(f\chi_{A_\varepsilon(u)}\right)^*_2(s,t) \leq \left(uf\chi_{A_\varepsilon(u)}\right)^*_2(s,t) = \left(M_u f\chi_{A_\varepsilon(u)}\right)^*_2(s,t). \end{aligned}$$

Hence, for $1 < p < \infty$,

$$\left\|M_u f\chi_{A_\varepsilon(u)}\right\|_{\Lambda^p_2(w)} \geq \varepsilon \left\|f\chi_{A_\varepsilon(u)}\right\|_{\Lambda^p_2(w)}.$$

Which implies that $M_u|_{L^w_{A_\varepsilon(u)}}$ is bounded below, therefore is invertible. Hence, since $M_u|_{L^w_{A_\varepsilon(u)}}$ is compact, by Theorem 5.1.2, the dimension of $L^w_{A_\varepsilon(u)}$ is finite.
$iii) \Rightarrow ii)$ Suppose that $L^w_{A_\varepsilon(u)}$ has finite dimension for each $\varepsilon > 0$, in particular, for all $n \in \mathbb{N}$, $L^w_{A_{\frac{1}{n}}(u)}$ has finite dimension.
Define, for each $n \in \mathbb{N}$, $u_n : \mathbb{R}^2 \to \mathbb{C}$ as follows

$$u_n(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in A_{\frac{1}{n}}(u) \\ 0 & \text{if } (x,y) \notin A_{\frac{1}{n}}(u). \end{cases}$$

Then, for each $f \in \Lambda^p_2(w)$, it holds that

$$\begin{aligned} |(u_n - u)f| = |u_n - u||f| \leq \frac{1}{n} f &\Rightarrow ((u_n - u)f)^*_2(s,t) \leq \frac{1}{n} f^*_2(s,t) \\ &\Rightarrow \|(u_n - u)f\|_{\Lambda^p_2(w)} \leq \frac{1}{n} \|f\|_{\Lambda^p_2(w)}. \end{aligned}$$

Therefore

$$\begin{aligned} \|M_{u_n} - M_u\| &= \sup_{\substack{f \in \Lambda^p_2(w) \\ \|f\|_{\Lambda^p_2(w)} = 1}} \|M_{u_n} f - M_u f\|_{\Lambda^p_2(w)} \\ &= \sup_{\substack{f \in \Lambda^p_2(w) \\ \|f\|_{\Lambda^p_2(w)} = 1}} \|M_{u_n - u} f\|_{\Lambda^p_2(w)} \\ &= \sup_{\substack{f \in \Lambda^p_2(w) \\ \|f\|_{\Lambda^p_2(w)} = 1}} \|(u_n - u) f\|_{\Lambda^p_2(w)} \\ &\leq \frac{1}{n} \|f\|_{\Lambda^p_2(w)} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence M_{u_n} converges to M_u uniformly. Next, since $L^w_{A_\varepsilon(u)}$ has finite dimension, we have that M_{u_n} is a finite range operator. By Theorem 5.1.4 M_{u_n} is a compact operator, so Theorem 5.1.5 implies that M_u is a compact operator. □

3.3 Closed Range

In this section we study conditions under which the Weighted Composition Operator on the space $\Lambda_2^p(w)$ has closed range.

Definition 3.3.1. Let $T : X \to Y$ be a linear operator between normed spaces. T it is said to be bounded below if

$$\exists\ m > 0 \text{ such that } m\|x\| \leq \|Tx\|, \forall\ x \in X.$$

Let us see when the operator $W_{u,T}$ is 1-1.

Theorem 3.3.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : \Lambda_2^p(w) \to \Lambda_2^p(w)$ is 1-1 if and only if $u \circ T \neq 0$ and T is surjective.*

Proof. ($\Rightarrow$)

a) Suppose that T is not surjective. Then there exist $F \subset \mathbb{R}^2 \setminus T(\mathbb{R}^2)$ such that $m_2(F) < \infty$ and therefore $0 \neq \chi_F \in \Lambda_2^p(w)$. Now,

$$W_{u,T}(\chi_F)(x) = u(T(x)) \cdot \chi_F(T(x));$$

since $\chi_F(T(x)) = 0$ (because $T(x) \notin F$), we obtain

$$\begin{aligned} & W_{u,T}(\chi_F)(x) = 0, \quad \forall\ x \in \mathbb{R}^2 \\ \Rightarrow \qquad & W_{u,T}(\chi_F) = 0, \text{with } \chi_F \neq 0. \end{aligned}$$

So $\ker(W_{u,T}) \neq \{0\}$ and then $W_{u,T}$ is not 1-1. In conclusion T is surjective.

b) Suppose that $u \circ T = 0$. Let

$$E = \left\{x \in \mathbb{R}^2 : (u \circ T)(x) = 0\right\}, \quad \text{with } m_2(E) > 0.$$

Then there exists $A \subset \mathbb{R}^2$ such that $T^{-1}(A) \subset E$ and $0 < m_2(A) < \infty$ (since $0 < m_2(T^{-1}(A)) < m_2(E)$, by being m_2 non-atomic, and $m_2(A) > 0$, by being T non-singular), so $\chi_A \in \Lambda_2^p(w)$.

Consider

$$\begin{aligned} W_{u,T}(\chi_A)(x) &= (u \circ T)(x) \cdot (\chi_A \circ T)(x) \\ &= u(T(x)) \cdot \chi_A(T(x)) \\ &= u(T(x)) \cdot \chi_{T^{-1}(A)}(x). \end{aligned}$$

- If $T(x) \notin A$, then $\chi_A(T(x)) = 0$, therefore

$$W_{u,T}(\chi_A)(x) = 0.$$

- If $T(x) \in A$, then $x \in T^{-1}(A) \subset E$, so $(u \circ T)(x) = 0$, therefore

$$W_{u,T}\left(\chi_A\right)(x) = 0.$$

Thus,

$$W_{u,T}\left(\chi_A\right)(x) = 0, \quad \forall\, x \in \mathbb{R}^2,$$

Hence $0 \neq \chi_A \in \ker\left(W_{u,T}\right) \neq \{0\}$ and so $W_{u,T}$ is not $1-1$. In conclusion, $u \circ T = 0$ m_2-a.e.

($\Leftarrow$) Let $y \in \mathbb{R}^2$, since T is onto, there exists $x \in \mathbb{R}^2$ such that $Tx = y$. So,

$$\begin{aligned} W_{u,T}f = W_{u,T}g, \text{with } f,g \in \Lambda_2^p(w) &\Rightarrow \left(W_{u,T}f\right)(x) = \left(W_{u,T}g\right)(x), x \in \mathbb{R}^2 \\ &\Rightarrow (u \circ T)(x)f(Tx) = (u \circ T)(x)g(Tx) \\ &\Rightarrow f(Tx) = g(Tx), \text{since } u \circ T \neq 0 \Rightarrow (u \circ T)(x) \neq 0 \\ &\Rightarrow f(y) = g(y), \forall\, y \in \mathbb{R}^2 \\ &\Rightarrow f = g. \end{aligned}$$

Thus, $W_{u,T}$ is $1-1$. □

In the following results, we will denote $S = \operatorname{supp}(u) = \{x \in \mathbb{R}^2 : u(x) \neq 0\}$, the support of u.

Corollary 3.3.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : L_S^w \to L_S^w$ is 1-1, where $L_S^w = \{f\chi_S : f \in \Lambda_2^p(w)\}$.*

Proof. Consider $W_{u,T}\left(\bar{f}\right) = 0$, where $\bar{f} = f\chi_S \in L_S^w$. Then

$$\begin{aligned} 0 = W_{u,T}(\bar{f}) &= u\left(T(x)\right)\bar{f}\left(T(x)\right) \\ &= u\left(T(x)\right)f\left(T(x)\right)\chi_S\left(T(x)\right) \\ &\Rightarrow u\left(T(x)\right)f\left(T(x)\right) = 0 \\ &\Rightarrow f\left(T(x)\right) = 0, \forall\, T(x) \in S, \text{since } T(x) \in S \Leftrightarrow u\left(T(x)\right) \neq 0 \\ &\Rightarrow f\left(T(x)\right)\chi_S\left(T(x)\right) = 0 \\ &\Rightarrow \left(f\chi_S\right)\left(T(x)\right) = 0, \forall\, T(x) \in S \\ &\Rightarrow \bar{f} = 0. \end{aligned}$$

□

Corollary 3.3.4. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : L_S^w \to L_S^w$ is bounded below if and only if $W_{u,T}$ has closed range.*

Proof. It follows from Theorem 5.1.6, since $W_{u,T}$ is $1-1$ on L_S^w. □

Theorem 3.3.5. *Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b \geq 1$ and $\delta > 0$ such that $\delta m_1(E_x) \leq m_1(T_x^{-1}(E)) \leq b m_1(E_x)$ for all $E \subset \mathbb{R}^2$. Suppose that $u:\mathbb{R}^2 \to \mathbb{C}$ is a measurable function. If $W_{u,T}$ is bounded on L_S^w, then $W_{u,T}$ is bounded below on $L_S(w)$ if and only if there exist $r > 0$ such that $|u(x)| \geq r$ a.e. in S.*

Proof. ($\Rightarrow$) Suppose that $W_{u,T}$ is bounded below. There exists $m > 0$ such that

$$\|W_{u,T}f\|_{\Lambda_2^p(w)} \geq m\|f\|_{\Lambda_2^p(w)}, f \in L_S^w. \tag{3.11}$$

Let $r > 0$ such that $r < m$. Let $R = \{x \in S : |u(x)| < r\}$ and suppose that $0 < m_2(E) < \infty$. Then, $\chi_E \in L_S^w$. Now,

$$\begin{aligned} & |u\chi_E(x)| \leq |r\chi_E(x)|, \ \forall\ x \in \mathbb{R}^2 \\ \Rightarrow\quad & (u\chi_E)_2^*(s,t) \leq (r\chi_E)_2^*(s,t) \\ \Rightarrow\quad & \|u\chi_E\|_{\Lambda_2^p(w)} \leq \|r\chi_E\|_{\Lambda_2^p(w)} = r\|\chi_E\|_{\Lambda_2^p(w)}. \end{aligned}$$

So,

$$\begin{aligned} \|W_{u,T}\chi_E\|_{\Lambda_2^p(w)} &\leq \|M_u\chi_E\|_{\Lambda_2^p(w)}, \quad \text{by (3.10)} \\ &= \|u\chi_E\|_{\Lambda_2^p(w)} \\ &\leq r\|\chi_E\|_{\Lambda_2^p(w)} \\ &< m\|\chi_E\|_{\Lambda_2^p(w)}, \end{aligned}$$

which contradicts (3.11). Hence $m_2(E) = 0$, that is to say, there exists $r > 0$ such that $|u(x)| \geq r$ a.e. in S.
($\Leftarrow$) Let $r > 0$ such that $|u(x) \geq r|$ a.e. in S, then

$$|(uf\chi_S)(x)| \geq |(rf\chi_S)(x)|, \ \forall\ f \in \Lambda_2^p(w).$$

From where

$$\|uf\chi_S\|_{\Lambda_2^p(w)} \geq \|rf\chi_S\|_{\Lambda_2^p(w)} = r\|f\chi_S\|_{\Lambda_2^p(w)}.$$

Thus, by inequality (3.10) in Theorem 3.2.1, we obtain

$$\begin{aligned} \|W_{u,T}f\chi_S\|_{\Lambda_2^p(w)} &\geq \delta^{1/p}\|M_uf\chi_S\|_{\Lambda_2^p(w)} \\ &= \delta^{1/p}\|uf\chi_S\|_{\Lambda_2^p(w)} \\ &\geq \delta^{1/p}r\|f\chi_S\|_{\Lambda_2^p(w)}, \end{aligned}$$

which means that $W_{u,T}$ is bounded below. □

4 Multipliers Between BV_p spaces

We saw in Section 1.2 that the multiplication operator M_u is a special and important case of the weighted composition operator $W_{u,T}$. This chapter is devoted to study the boundedness of the operator

$$M_u : \mathrm{BV}_p \to \mathrm{BV}_q,$$

where BV_p stands for the bounded p-variation space, also known as Wiener type variation space (see Definition 4.1.1) for the cases $1 \leq q < p$ and $1 \leq p \leq q$.
Although BV_p-spaces ($1 \leq p < \infty$) are complete normed linear spaces, they are not Köthe spaces. As a consequence of this, BV_p-spaces behave differently compared with Lebesgue spaces, Lorentz spaces (classic and multidimensional), etc. For example, for functions belonging to BV_p, equality almost everywhere does not imply equality in norm. Also, BV_p-spaces lack of lattice property. This makes the study of the boundedness of M_u interesting because we have to develop new techniques in order to attack the problem.

4.1 Introduction

Let E and F be spaces of real (or complex) valued functions defined on a set X. A real (or complex) valued function g defined on X is called a *multiplier* from E to F if the pointwise multiplication fg belongs to F for every $f \in E$. The set of all multipliers from E to F is denoted as $M(E \to F)$. When E and F are normed spaces, then it is natural to consider the operator $M_g : E \to F$ defined as

$$M_g(f) = fg.$$

The operator M_g is called a *multiplication operator* induced by g, and the function g is usually called the *symbol* of the multiplication operator.
It is then of interest to characterize the set $M(E \to F)$ as well as some properties of M_g (such as boundedness, compactness, closed range, etc.) in terms of conditions on the symbol g. For example, Takagi and Yokouchi [51] characterized the set $M(L_p \to L_q)$, where L_p stands for the usual Lebesgue space. Nakai [45] studied the set of multipliers between Lorentz spaces, Castillo and Chaparro studied the multiplication operator defined on Orlicz-Lorentz spaces [20] and also on multidimensional Lorentz spaces [18]. The reader may find more information about this topics in [46].

In order to introduce the bounded variation spaces, we recall that a partition P of $[0,1]$ is a finite set $P = \{t_0, t_1, \ldots, t_m\}$ such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1.$$

Definition 4.1.1. For a function $f : [0,1] \to \mathbb{R}$, we say that f has bounded p-variation if

$$\mathrm{Var}_p(f) = \sup_P \left(\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p \right)^{1/p} < \infty,$$

where the supremum is taken over all partitions P of $[0,1]$. The set of all functions $f : [0,1] \to \mathbb{R}$ with bounded p-variation will be denoted as $\mathrm{BV}_p([0,1])$.

Bounded variation spaces were introduced by Camille Jordan [36] in 1881. Since then, the concept of bounded variation has been generalized in many ways. The one we discuss here was introduced by Wiener [52] in 1924.
There are some special features that distinguish the $\mathrm{BV}_p([0,1])$ spaces from other spaces such as Lebesgue spaces L_p (and its generalizations: Lorentz spaces, Orlicz spaces, etc.). For example, for functions f and g in L_p, if $f = g$ almost everywhere, then their L_p-norms are the same. This is not true for functions in $\mathrm{BV}_p([0,1])$. Even if $f, g \in \mathrm{BV}_p([0,1])$ differ only on one single point, their norms can be very different. So, in the context of $\mathrm{BV}_p([0,1])$, $f = g$ means $f(t) = g(t)$ for all $t \in [0,1]$.
Another important difference between $\mathrm{BV}_p([0,1])$ and L_p spaces, is the lack of the so called *lattice property*: for f, g in L_p, if $|f| \le |g|$ almost everywhere, then $\|f\|_{L_p} \le \|g\|_{L_p}$. This property does not hold in $\mathrm{BV}_p([0,1])$, as is easily shown by defining on $[0,1]$ the functions

$$f(t) = \begin{cases} 0, & \text{if } t \neq 1/2 \\ 1, & \text{if } t = 1/2 \end{cases} \quad \text{and} \quad g(t) = 1.$$

For more details about bounded variation spaces and different types of variations, see [3].
There has been relatively little study of multipliers and multiplication operators on bounded variation spaces. One of the few examples we can cite is [7], where the authors obtained results about the multiplication operator $M_u : \mathrm{BV}_1([0,1]) \to \mathrm{BV}_1([0,1])$.
In this chapter, we characterize completely the set

$$M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1])).$$

In order to describe our answer, it is convenient to divide an argument into two cases:

$$\text{CASE I: } 1 \le q < p, \quad \text{CASE II: } 1 \le p \le q.$$

In Section 4.2 we give some auxiliary results and definitions. In Section 4.3 we state some theorems regarding the characterization described above.

4.2 Auxiliary Results

We state here some auxiliary results that will be useful later.
Let us denote by $B([0,1])$ the set of all bounded functions $f : [0,1] \to \mathbb{R}$ with the norm

$$\|f\|_\infty := \sup_{0\le t\le 1} |f(t)|.$$

It is a well-known fact that $\mathrm{BV}_p([0,1])$ is a subspace of $B([0,1])$ (see [3, p. 85]). Moreover, if we set

$$\|f\|_{\mathrm{BV}_p} := \|f\|_\infty + \mathrm{Var}_p(f).$$

Then $\left(\mathrm{BV}_p([0,1]), \|\cdot\|_{\mathrm{BV}_p}\right)$ becomes a Banach space. With this norm, $\mathrm{BV}_p([0,1])$ is a normalized Banach algebra, i.e.

$$\|fg\|_{\mathrm{BV}_p} \le \|f\|_{\mathrm{BV}_p} \|g\|_{\mathrm{BV}_p}. \tag{4.1}$$

(See [37] for a proof of the above inequality). Besides, since the inequality

$$\mathrm{Var}_p(f)^{1/p} \le \mathrm{Var}_q(f)^{1/q}, \quad 1 \le q \le p < \infty, \tag{4.2}$$

holds, one concludes that

$$\mathrm{BV}_q([0,1]) \subset \mathrm{BV}_p([0,1]), \quad 1 \le q \le p < \infty.$$

In the next lemma we show that the above inclusion is strict. This fact will be useful later.

Lemma 4.2.1. *Given any strictly increasing sequence $\{t_j\}_{j\in\mathbb{N}} \subset [0,1]$, there exists a function f such that*

1. $f \in \mathrm{BV}_p([0,1])$ *but* $f \notin \mathrm{BV}_q([0,1])$, *if* $1 \le q < p$.
2. $\sup_{t\in(t_j,t_{j+1})} f(t) = f(t_j)$.
3. $\inf_{t\in(t_j,t_{j+1})} f(t) = 0$.

Proof. (1) For some $\theta > 0$, consider the zigzag function Z_θ, defined on $[0,1]$ as

$$Z_\theta(t) = \begin{cases} 0 & \text{if } t < t_0 \text{ or } t = (t_j + t_{j+1})/2,\ j = 0,1,2,\dots \\ \dfrac{1}{2^{1/p}(j+1)^\theta} & \text{if } t = t_j,\ j = 0,1,2,\dots \\ \text{linear} & \text{otherwise.} \end{cases} \tag{4.3}$$

It follows that

$$\mathrm{Var}_p(Z_\theta)^p = \sum_{j=1}^{\infty} \frac{1}{j^{p\theta}} \quad (1 \le p < \infty).$$

This means that Z_θ belongs to $\mathrm{BV}_p([0,1])$ only if $p > 1/\theta$. In particular, for $1 \le q < p$, $Z_{1/q}(t) \in \mathrm{BV}_p([0,1])$ and $Z_{1/q}(t) \notin \mathrm{BV}_q([0,1])$ (see [3, p. 89] for a similar discussion).
It is clear that $Z_{1/q}$ also satisfies conditions (2) and (3). □

For any function $f : [0,1] \to \mathbb{R}$ and any set $E \subseteq [0,1]$, we call

$$\mathrm{osc}_E(f) = \sup_{t\in E} f(t) - \inf_{t\in E} f(t).$$

the *oscillation* of f on E. For $1 \le p < \infty$, we define

$$v_p(f) = \sup \left(\sum_{k=1}^{m} \mathrm{osc}_{I_k}(f)^p \right)^{1/p},$$

where the supremum is taken over all collections $\{I_k\}$ of disjoint intervals contained in $[0,1]$. For the proof of the Theorem 4.3.1, which is the main result of this chapter, it will be convenient to use $v_p(f)$ instead of $\mathrm{Var}_p(f)$. We show that they are the same in the following lemma.

Lemma 4.2.2. *For any function $f \in \mathrm{BV}_p([0,1])$,*

$$\mathrm{Var}_p(f) = v_p(f).$$

Proof. Given any partition $P = \{0 = t_0, t_1, \ldots, t_m = 1\}$ of $[0,1]$, we construct a sequence of disjoint intervals

$$I_1 = (t_0, t_1), I_2 = (t_1, t_2), \ldots, I_m = (t_{m-1}, t_m).$$

It is clear that

$$|f(t_j) - f(t_{j-1})| \le \sup_{t\in I_j} f(t) - \inf_{t\in I_j} f(t) = \mathrm{osc}_{I_j}(f), \quad j = 1, 2, \ldots, m.$$

Then

$$\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p \le \sum_{j=1}^{m} \mathrm{osc}_{I_j}(f)^p.$$

From which one concludes that

$$\mathrm{Var}_p(f) \le v_p(f). \tag{4.4}$$

Now we will obtain the reverse inequality. Fix a sequence $X_1, X_2, \ldots, X_m$ of disjoint subintervals of $[0,1]$. Then, for any $\varepsilon > 0$, there exist $x_j \in X_j$ and $x_{j-1} \in X_j$ $(j = 1, \ldots, m)$ such that

$$f(x_j) > \sup_{x\in X_j} f(x) - \varepsilon \quad \text{and} \quad f(x_{j-1}) < \inf_{x\in X_j} f(x) + \varepsilon.$$

Therefore

$$|f(x_j) - f(x_{j-1})| \ge f(x_j) - f(x_{j-1}) > \sup_{x\in X_j} f(x) - \inf_{x\in X_j} f(x) - 2\varepsilon.$$

And then

$$\mathrm{Var}_p(f)^p \ge \sum_{j=1}^{m} |f(x_j) - f(x_{j-1})|^p > \sum_{j=1}^{m} \left(\mathrm{osc}_{X_j}(f) - 2\varepsilon\right)^p.$$

From the above inequality, a standard argument shows that

$$v_p(f) \le \mathrm{Var}_p(f). \tag{4.5}$$

Combining (4.4) and (4.5) we obtain the desired result. ☐

4.3 Multipliers from $\mathrm{BV}_p([0,1])$ to $\mathrm{BV}_q([0,1])$

In order to do the study about multipliers between $\mathrm{BV}_p([0,1])$ and $\mathrm{BV}_q([0,1])$ spaces, we separate it in cases.

CASE I: $\mathbf{1 \leq q < p}$.

Lemma 4.2.1 and the inequality

$$\mathrm{Var}_p(f) \leq \mathrm{Var}_q(f) \quad (1 \leq q < p),$$

shows us that, for $1 \leq q < p$, $\mathrm{BV}_q([0,1])$ is a proper subset of $\mathrm{BV}_p([0,1])$. If we take a function u belonging to $\mathrm{BV}_p([0,1]) \setminus \mathrm{BV}_q([0,1])$, then we cannot induce a multiplier from $\mathrm{BV}_p([0,1])$ into $\mathrm{BV}_q([0,1])$. For the constant function $f(t) = 1 \in \mathrm{BV}_p([0,1])$,

$$M_u(f) = u \cdot f = u \cdot 1 = u \notin \mathrm{BV}_q([0,1]).$$

Because of this, it is natural to restrict ourselves only to symbols u such that $u \in \mathrm{BV}_q([0,1])$. For any subset $A \subset [0,1]$, we denote by $\#(A)$ the counting measure on A, i.e.

$$\#(A) = \begin{cases} \text{number of elements in } A, & \text{if } A \text{ is a finite set} \\ \infty, & \text{if } A \text{ is an infinite set.} \end{cases}$$

Moreover, for a function $u : [0,1] \to \mathbb{R}$, we define

$$\varphi_u(r) = \#\left(\{t \in [0,1] : |u(t)| \geq r\}\right).$$

In the next theorem, we shall see that the function φ_u allows us to characterize the set $M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$.

Theorem 4.3.1. *Suppose $1 \leq q < p$, and let u be a function in $\mathrm{BV}_q([0,1])$. Then $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ if and only if $\varphi_u(r) < \infty$ for all $r > 0$.*

Proof. Fix $u \in \mathrm{BV}_p([0,1])$. Assume that $\varphi_u(r) < \infty$ for all $r > 0$ and take arbitrary $f \in \mathrm{BV}_q([0,1])$. We shall prove that $uf \in \mathrm{BV}_p([0,1])$.
There is no loss of generality in assuming that both u and f are positive. Otherwise, we decompose u and f as

$$u = u^+ - u^-, \quad f = f^+ - f^-,$$

where the superscripts $^+$ and $^-$ stand for the positive and negative parts of the functions, i.e.

$$f^+(t) = \max\{f(t), 0\}, \quad f^-(t) = \max\{-f(t), 0\}.$$

Note that

$$uf = u^+f^+ - u^+f^- - u^-f^+ + u^+f^-,$$

and

$$\mathrm{Var}_p(uf)^{1/p} \leq \mathrm{Var}_p(u^+f^+)^{1/p} + \mathrm{Var}_p(u^+f^-)^{1/p} + \mathrm{Var}_p(u^-f^+)^{1/p} + \mathrm{Var}_p(u^+f^-)^{1/p},$$

Then, it is sufficient to estimate each term separately.
We prove first the converse. If $\varphi(r) < \infty$ for all $r > 0$, then for any interval $I_k \subset [0,1]$ there exists $t_k \in I_k$ such that $u(t_k) = 0$. Since u and f are positive functions, it follows that

$$\inf_{t \in I_k} (uf)(t) = 0 \quad \text{and} \quad \inf_{t \in I_k} u(t) = 0. \tag{4.6}$$

We know also that

$$\sup_{t \in I_k} (uf)(t) \leq \|f\|_\infty \sup_{t \in I_k} u(t). \tag{4.7}$$

From (4.6) and (4.7) we conclude that

$$\mathrm{osc}_{I_k}(uf) \leq \|f\|_\infty \, \mathrm{osc}_{I_k}(u).$$

And then, adding over disjoint intervals I_k,

$$\sum_{k=1}^{m} \mathrm{osc}_{I_k}(uf)^q \leq \|f\|_\infty^q \sum_{k=1}^{m} \mathrm{osc}_{I_k}(u)^q.$$

Therefore

$$\mathrm{Var}_q(uf) \leq \|f\|_\infty \, \mathrm{Var}_q(u) \leq \mathrm{Var}_p(f) \, \mathrm{Var}_q(u).$$

In order to prove the direct implication, by means of a contradiction assume that there exists a number $r_0 > 0$ such that $\varphi(r_0) = \infty$. Then we can find an increasing sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0,1]$ such that $u(t_n) \geq r_0$. Recall the function $Z_{1/q}$ defined in the proof of Lemma 4.2.1. We know that $v_p(Z_{1/q}) < \infty$ and $v_q(Z_{1/q}) = \infty$. For this function, it is true that

$$\inf_{(t_j, t_{j+1})} Z_{1/q}(t) = 0, \quad \text{and also} \quad \inf_{(t_j, t_{j+1})} u(t) \cdot Z_{1/q}(t) = 0,$$

then

$$\begin{aligned}
\mathrm{osc}_{(t_j,t_{j+1})}\left(u \cdot Z_{1/q}\right) &= \sup_{(t_j,t_{j+1})} \left(u \cdot Z_{1/q}\right)(t) - \inf_{(t_j,t_{j+1})} (u \cdot Z_{1/q})(t) \\
&= \sup_{(t_j,t_{j+1})} \left(u \cdot Z_{1/q}\right)(t) \\
&\geq u(t_j) \cdot Z_{1/q}(t_j) \\
&\geq r_0 \cdot Z_{1/q}(t_j) \\
&= r_0 \cdot \sup_{(t_j,t_{j+1})} Z_{1/q}(t) \\
&= r_0 \cdot \left(\sup_{(t_j,t_{j+1})} Z_{1/q}(t) - \inf_{(t_j,t_{j+1})} Z_{1/q}(t) \right)
\end{aligned}$$

$$= r_0 \cdot \mathrm{osc}_{(t_j,t_{j+1})}(Z_{1/q}).$$

From this one concludes that

$$\begin{aligned}\mathrm{Var}_q(u \cdot Z_{1/q})^q &\geq r_0^q \,\mathrm{Var}_q(Z_{1/q})^q \\ &= r_0^q \sum_{k=1}^{\infty} \frac{1}{k} \\ &= \infty.\end{aligned}$$

□

CASE II: $\mathbf{1 \leq p \leq q}$

The study of $M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ for the case $1 \leq p \leq q$ is quite easy. For the sake of completeness, we will do it here.
The following result relies on the fact that $\mathrm{BV}_q([0,1])$ is a normalized Banach algebra, and also on the fact that, for $1 \leq p \leq q$, we have the continuous embedding $\mathrm{BV}_p([0,1]) \hookrightarrow \mathrm{BV}_q([0,1])$.

Theorem 4.3.2. *Suppose* $1 \leq p \leq q$. *Then* $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ *if and only if* $u \in \mathrm{BV}_q([0,1])$. *In this case,* M_u*, the multiplication operator induced by* u*, is a bounded linear operator from* $\mathrm{BV}_p([0,1])$ *into* $\mathrm{BV}_q([0,1])$*, and its norm is given by* $\|M_u\| = \|u\|_{\mathrm{BV}_q}$.

Proof. If $u \in \mathrm{BV}_q([0,1])$, then from (4.1) and (4.2) we get

$$\|uf\|_{\mathrm{BV}_q} \leq \|u\|_{\mathrm{BV}_q} \|f\|_{\mathrm{BV}_q} \leq \|u\|_{\mathrm{BV}_q} \|f\|_{\mathrm{BV}_p} < \infty. \tag{4.8}$$

Then $uf \in \mathrm{BV}_q([0,1])$.
Conversely, if $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$, then, since the constant function $h(t) = 1$ belongs to $\mathrm{BV}_p([0,1])$, we have

$$M_u h(x) = u(x) \cdot h(x) = u(x) \cdot 1 = u(x), \tag{4.9}$$

so $u \in \mathrm{BV}_q([0,1])$.
Finally, from (4.8) and (4.9) we conclude that $\|M_u\| = \|u\|_{\mathrm{BV}_q}$. □

Remark 4.3.3. The results we have obtained in this chapter can be performed, with some modifications, for functions of several variables. For some information about bounded variation in this setting, the reader is refer to [3, p. 91], [10] and [44] .

5 Appendix

5.1 Some results from functional analysis

Below are listed some theorems from functional analysis that we used often in Chapter 3. The interested reader is invited to check [2, 34].

Theorem 5.1.1. *The restriction of a compact operator to an invariant closed subspace is compact.*

Theorem 5.1.2. *Let T be a compact and invertible operator. If T^{-1} is bounded on a normed space X, then* $\dim(X) < \infty$.

Theorem 5.1.3. *Let X and Y be Banach spaces, S and T bounded linear operators form X to Y. If there exists $\alpha > 0$ such that $\|S\| \leq \alpha\|T\|$ and T is compact for all $x \in X$, then S is compact.*

Theorem 5.1.4. *Let X and Y be normed spaces, $T : X \to Y$ be a linear operator. If T is bounded and has finite range, then T is a compact operator.*

Theorem 5.1.5. *Let $\{T_n\}$ be a sequence of linear compact operators from a normed space X into a Banach space Y. If $\{T_n\}$ converges uniformly, i.e., if $\|T_n - T\| \to 0$, then the limit operator T is compact.*

Theorem 5.1.6. *Let $T : X \to X$ be a linear bounded operator between Banach spaces. The following assertions are equivalent.*

a) T is bounded below.

b) T is 1-1 and $T(X)$, the range of T, is closed in Y.

c) There exists the bounded inverse of T, i.e., there exists $T^{-1} : T(X) \to X$ and it is bounded.

Observe that a) and b) are equivalent in any normed spaces without being T bounded.

5.2 A result about sections

While I was doing some calculations for Chapter 3, a question about x-sections and multiplication by an scalar came up. I didn't use this result in this thesis but I want to include it anyway.
Remember that if $E \subset \mathbb{R}^2$, then $E_x = \{y : (x, y) \in E\}$.

Proposition 5.2.1. *If λ is an scalar, then*

$$\lambda E_x \subset (\lambda E)_x \, .$$

Proof. Note that

$$\lambda E_x = \{\lambda y : y \in E_x\} = \{\lambda y : (x, y) \in E\} \, , \text{ and } (\lambda E)_x = \{y : (x, y) \in \lambda E\} \, .$$

$$\begin{aligned}
\text{Let } z \in \lambda E_x \quad &\Rightarrow \quad z = \lambda y, y \in E_x \\
&\Rightarrow \quad z = \lambda y, (x, y) \in E \\
&\Rightarrow \quad z = \lambda y, \lambda(x, y) \in \lambda E \\
&\Rightarrow \quad z = \lambda y, (\lambda x, \lambda y) \in \lambda E; \lambda x = \bar{x}, \lambda y = \bar{y} \\
&\Rightarrow \quad z = \bar{y}, (\bar{x}, \bar{y}) \in \lambda E \\
&\Rightarrow \quad z \in (\lambda E)_x \, . \quad \square
\end{aligned}$$

Remark 5.2.2. Inclusion in Proposition 5.2.1 may be strict. Consider the following example. Take $E = \bar{B}(0, 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$, and take $\lambda = 2$, $x = 1/2$. After some calculations, we see that

$$2E = \bar{B}(0, 2) \Rightarrow (2E)_{1/2} = \left[-\sqrt{7}, \sqrt{7}\right] , \text{ and } 2E_{1/2} = 2\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] = \left[-\sqrt{3}, \sqrt{3}\right] .$$

So $2E_{1/2} \subsetneq (2E)_{1/2}$.

5.3 A useful inequality

In the following proposition, we will prove the inequality (2.37) that was used in Theorem 2.2.9.

Proposition 5.3.1. *Let $a, b \in \mathbb{R}$ with $a \neq 0$. If $p \geq 1$, then*

$$|a|^{1/p} + |b|^{1/p} \leq 2^{1-1/p} \left(|a| + |b|\right)^{1/p} .$$

Proof. Consider the function

$$f(t) = 2^{1-\frac{1}{p}}(1+t)^{\frac{1}{p}} - 1 - t^{\frac{1}{p}}, t \geq 0.$$

We know that

$$\begin{aligned} f'(t) &= 2^{1-\frac{1}{p}}\frac{1}{p}(1+t)^{\frac{1}{p}-1} - \frac{1}{p}t^{\frac{1}{p}-1} \\ &= \frac{1}{p}\left[2^{1-\frac{1}{p}}(1+t)^{\frac{1}{p}-1} - t^{\frac{1}{p}-1}\right] \\ &= \frac{1}{p}\left[\frac{2^{1-\frac{1}{p}}}{(1+t)^{1-\frac{1}{p}}} - \frac{1}{t^{1-\frac{1}{p}}}\right] \\ &= \frac{1}{p}\left[\frac{2^{1-\frac{1}{p}}t^{1-\frac{1}{p}} - (1+t)^{1-\frac{1}{p}}}{(1+t)^{1-\frac{1}{p}}t^{1-\frac{1}{p}}}\right]. \end{aligned}$$

A critical point is $t = 0$. Another critical point is

$$\begin{aligned} & 2^{1-\frac{1}{p}}t^{1-\frac{1}{p}} - (1+t)^{1-\frac{1}{p}} = 0 \\ \Rightarrow \quad & 2^{1-\frac{1}{p}}t^{1-\frac{1}{p}} = (1+t)^{1-\frac{1}{p}} \\ \Rightarrow \quad & 2^{1-\frac{1}{p}} = \left(\frac{1+t}{t}\right)^{1-\frac{1}{p}} \\ \Rightarrow \quad & 2 = \frac{1+t}{t} \\ \Rightarrow \quad & 2 = 1 + \frac{1}{t} \\ \Rightarrow \quad & t = 1. \end{aligned}$$

Derivating one more time,

$$f''(t) = \frac{1}{p}\left[2^{1-\frac{1}{p}}\left(\frac{1}{p}-1\right)(1+t)^{\frac{1}{p}-2} - \left(\frac{1}{p}-1\right)t^{\frac{1}{p}-2}\right].$$

Then

$$\begin{aligned} f''(1) &= \frac{1}{p}\left[2^{1-\frac{1}{p}}\left(\frac{1}{p}-1\right)2^{\frac{1}{p}-2} - \left(\frac{1}{p}-1\right)\right] \\ &= \frac{1}{p}\left[2^{-1}\left(\frac{1}{p}-1\right) - \left(\frac{1}{p}-1\right)\right] \\ &= \frac{1}{p}\left(\frac{1}{p}-1\right)\left(2^{-1}-1\right) = \frac{1}{p}\left(\frac{1}{p}-1\right)\left(-\frac{1}{2}\right) \\ &= \frac{1}{p}\left(1-\frac{1}{p}\right)\left(\frac{1}{2}\right) > 0 \text{ since } p > 1. \end{aligned}$$

Since $f(0) = 2^{1-1/p} - 1 > 0$ and $f(1) = 0$, we see that f has a global minimum at $t = 1$. Therefore

$$\begin{aligned} & f(t) \geq f(1) \ \forall t \in [0, \infty) \\ \Rightarrow \quad & 2^{1-1/p}(1+t)^{1/p} - 1 - t^{1/p} \geq 0 \\ \Rightarrow \quad & 2^{1-1/p}(1+t)^{1/p} \geq 1 + t^{1/p}. \end{aligned}$$

By taking $t = \frac{|b|}{|a|}$,

$$\begin{aligned} & 2^{1-1/p}\left(1 + \frac{|b|}{|a|}\right)^{1/p} \geq 1 + \left(\frac{|b|}{|a|}\right)^{1/p} \\ \Rightarrow \quad & 2^{1-1/p}\left(\frac{|a| + |b|}{|a|}\right)^{1/p} \geq \frac{|a|^{1/p} + |b|^{1/p}}{|a|^{1/p}} \\ \Rightarrow \quad & \frac{2^{1-1/p}\left(|a| + |b|\right)^{1/p}}{|a|^{1/p}} \geq \frac{|a|^{1/p} + |b|^{1/p}}{|a|^{1/p}} \\ \Rightarrow \quad & 2^{1-1/p}\left(|a| + |b|\right)^{1/p} \geq |a|^{1/p} + |b|^{1/p}. \quad \square \end{aligned}$$

5.4 Chebyshev's type inequality

A Chebyshev's type inequality holds in the spaces $\Lambda_2^p(w)$. The result reads as follows.

Proposition 5.4.1 (Chebyshev's type inequality). *Let $f \in \Lambda_2^p(w)$. Then, for any real number $t > 0$, we have*

$$w\left(E^*\right) \leq \frac{\|f\|^p_{\Lambda_2^p(w)}}{t^p},$$

where $E = \{x \in \mathbb{R}^2 : |f(x)| > t\}$ and $w : \mathbb{R}^2 \to \mathbb{R}$ is a weight such that $w\left(E^*\right) = \int_{E^*} w(x)dx$.

Proof. For any $z \in \mathbb{R}^2$ we have

$$\begin{aligned} & t\chi_E(z) \leq |f(z)| \\ \Rightarrow \quad & (t\chi_E)_2^* \leq |f|_2^* \\ \Rightarrow \quad & t\chi_{E^*}(x) \leq f_2^*(x) \\ \Rightarrow \quad & t^p\chi_{E^*}(x) \leq [f_2^*(x)]^p \\ \Rightarrow \quad & t^p \int_{\mathbb{R}^2_+} \chi_{E^*}(x)w(x)\,dx \leq \int_{\mathbb{R}^2_+} [f_2^*(x)]^p\, w(x)\,dx \\ \Rightarrow \quad & t^p \int_{E^*} w(x)\,dx \leq \int_{\mathbb{R}^2_+} [f_2^*(x)]^p\, w(x)\,dx \\ \Rightarrow \quad & t^p w\left(E^*\right) \leq \|f\|^p_{\Lambda_2^p(w)} \\ \Rightarrow \quad & w\left(E^*\right) \leq \frac{\|f\|^p_{\Lambda_2^p(w)}}{t^p}. \quad \square \end{aligned}$$

Bibliography

[1] M. B. Abrahamse, *Multiplication operators*, Hilbert space operators (Proc. Conf., Calif. State Univ., Long Beach, Calif., 1977), Lecture Notes in Math., vol. 693, Springer, Berlin, 1978, pp. 17–36.

[2] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002.

[3] J. Appell, J. Banaś, and N. Merentes, *Bounded variation and around*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 17, De Gruyter, Berlin, 2014.

[4] M. A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 727–735.

[5] S.C. Arora, G. Datt, and S. Verma, *Multiplication operators on Lorentz spaces*, Indian J. Math. **48** (2006), no. 3, 317–329.

[6] ______, *Composition operators on Lorentz spaces*, Bull. Austral. Math. Soc. **76** (2007), no. 2, 205–214.

[7] F. Astudillo-Villalba and J. C. Ramos-Fernández, *Multiplication operators on the space of functions of bounded variation*, Demonstr. Math. **50** (2017), no. 1, 105–115.

[8] A. Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. **336** (1982), 26–44.

[9] S. Barza, *Weighted multidimensional integral inequalities and applications*, Ph.D. thesis, Luleå University of Technology, 1999.

[10] S. Barza and M. Lind, *A new variational characterization of Sobolev spaces*, J. Geom. Anal. **25** (2015), no. 4, 2185–2195.

[11] S. Barza, L. E. Persson, and J. Soria, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr. **210** (2000), 43–58.

[12] ______, *Multidimensional rearrangement and Lorentz spaces*, Acta Math. Hungar. **104** (2004), no. 3, 203–224.

[13] S. Barza, L. E. Persson, and V. Stepanov, *On weighted multidimensional embeddings for monotone functions*, Math. Scand. **88** (2001), no. 2, 303–319.

[14] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.

[15] A. P. Blozinski, *Multivariate rearrangements and Banach function spaces with mixed norms*, Trans. Amer. Math. Soc. **263** (1981), no. 1, 149–167.

[16] M. Carro, L. Pick, J. Soria, and V. D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), no. 3, 397–428.

[17] M. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. **112** (1993), no. 2, 480–494.

[18] R. E. Castillo and H. C. Chaparro, *Weighted composition operator on two-dimensional Lorentz spaces*, Math. Inequal. Appl. **20** (2017), no. 3, 773–799.

[19] R. E. Castillo, H. C. Chaparro, and J. C. Ramos Fernández, *Orlicz-Lorentz spaces and their composition operators*, Proyecciones **34** (2015), no. 1, 85–105.

[20] ______, *Orlicz-Lorentz spaces and their multiplication operators*, Hacet. J. Math. Stat. **44** (2015), no. 5, 991–1009.

[21] R. E. Castillo, D. D. Clahane, J. Farías López, and J. C. Ramos Fernández, *Composition operators from logarithmic Bloch spaces to weighted Bloch spaces*, Appl. Math. Comput. **219** (2013), no. 12, 6692–6706.

[22] R. E. Castillo, J. C. Ramos Fernández, and E. M. Rojas, *A new essential norm estimate of composition operators from weighted Bloch space into μ-Bloch spaces*, J. Funct. Spaces Appl. (2013), Art. ID 817278, 5.

[23] R. E. Castillo, J. C. Ramos Fernández, and M. Salas-Brown, *Properties of Multiplication Operators on Köthe spaces*, arXiv preprint arXiv:1411.1018 (2014).

[24] R. E. Castillo, F. A. Vallejo Narváez, and J. C. Ramos Fernández, *Multiplication and composition operators on weak L_p spaces*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 3, 927–973.

[25] R.E. Castillo, R. León, and E. Trousselot, *Multiplication operator on $L_{(p,q)}$ spaces*, PanAmer. Math. J. **19** (2009), no. 1, 37–44.

[26] R.E. Castillo and H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, Springer International Publishing, New York, 2016.

[27] Y. Cui, H. Hudzik, R. Kumar, and L. Maligranda, *Composition operators in Orlicz spaces*, J. Aust. Math. Soc. **76** (2004), no. 2, 189–206.

[28] R. G. Douglas, *Banach algebra techniques in operator theory*, second ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[29] G. B. Folland, *Real analysis: modern techniques and their applications*, second ed., Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1999.

[30] L. Grafakos, *Classical Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008.

[31] P. R. Halmos, *What does the spectral theorem say?*, Amer. Math. Monthly **70** (1963), 241–247.

[32] ———, *A Hilbert space problem book*, second ed., Graduate Texts in Mathematics, vol. 19, Springer-Verlag, New York-Berlin, 1982.

[33] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54** (1930), no. 1, 81–116.

[34] H. Heuser, *Functional analysis*, John Wiley & Sons, Ltd., Chichester, 1982.

[35] R. A. Hunt, *On* $L(p, q)$ *spaces*, Enseignement Math. (2) **12** (1966), 249–276.

[36] C. Jordan, *Sur la série de fourier*, CR Acad. Sci. Paris **92** (1881), no. 5, 228–230.

[37] R. Kantrowitz, *Submultiplicativity of norms for spaces of generalized BV-functions*, Real Anal. Exchange **36** (2010/11), no. 1, 169–175.

[38] B. S. Komal and R. S. Pathania, *Composition operators on a space of operators*, Indian J. Math. **33** (1991), no. 1, 11–17.

[39] B.S. Komal and S. Gupta, *Multiplication operators between Orlicz spaces*, Integral Equations Operator Theory **41** (2001), no. 3, 324–330.

[40] A. Kumar, *Fredholm composition operators*, Proc. Amer. Math. Soc. **79** (1980), no. 2, 233–236.

[41] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[42] G.G. Lorentz, *Some new functional spaces*, Ann. of Math. (2) **51** (1950), 37–55.

[43] ———, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.

[44] J. Malý, *Absolutely continuous functions of several variables*, J. Math. Anal. Appl. **231** (1999), no. 2, 492–508.

[45] E. Nakai, *Pointwise multipliers on the Lorentz spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. **45** (1996), no. 1, 1–7.

[46] J. C. Ramos-Fernández, *Some properties of multiplication operators acting on Banach spaces of measurable functions*, Bol. Mat. **23** (2016), no. 2, 221–237.

[47] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), no. 2, 145–158.

[48] R.K. Singh and A. Kumar, *Multiplication operators and composition operators with closed ranges*, Bull. Austral. Math. Soc. **16** (1977), no. 2, 247–252.

[49] V. D. Stepanov, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338** (1993), no. 1, 173–186.

[50] H. Takagi, *Fredholm weighted composition operators*, Integral Equations Operator Theory **16** (1993), no. 2, 267–276.

[51] H. Takagi and K. Yokouchi, *Multiplication and composition operators between two L^p-spaces*, Function spaces (Edwardsville, IL, 1998), Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 321–338.

[52] N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Stud. App. Math. **3** (1924), no. 2, 72–94.