Scattering equations formalism for QCD and supersymmetric Yang-Mills theories

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Abstract

Scattering amplitudes are one of the most important observables in perturbative quantum field theory, because they allow for the calculation of cross-sections, which are central to collision experiments. In this thesis, we perform a thorough review of some of the modern method for the calculation of tree-level amplitudes, the leading order contributions to the perturbative expansion of scattering amplitudes, focusing on the gauge theories that make up the standard model of particle physics. We will study methods for their calculation that overcome the issues and inefficiencies of Feynman diagrams, focusing on the Cachazo-He-Yuan (CHY) formalism, which provides closed formulas for tree amplitudes in arbitrary dimension as integrals over n-punctured Riemann spheres localized on the solutions to a set of constraints that relate the punctures over the Riemann spheres to the kinematic invariants of the process, known as the scattering equations. We will introduce the CHY formalism for pure Yang-Mills amplitudes, as well as one of its supersymmetric generalizations, the so-called maximally supersymmetric or $\mathcal{N} = 4$ super Yang-Mills theory (SYM). We will introduce the notion of basis amplitudes for Yang-Mills and Quantum Chromodynamics (QCD), which are based on the idea of color decomposition, the separation of color and kinematic degrees of freedom, and see how one can obtain CHY representations for QCD amplitudes. We will use one of these representations, given in terms of basis amplitudes, to derive soft theorems for the CHY integrand of QCD, which is a first step into obtaining constraints on its mathematical structure.

Resumen

Las amplitudes de dispersion representan algunos de los observables más imporantes de la teoría cuántica de campos perturbativa, puesto que estos permiten el cálculo de las secciones eficaces, las cuales son centrales para los experimentos de colisión. En esta tesis presentamos una revisión de algunos de los métodos modernos para el cálculo de amplitudes a nivel árbol, las cuales constituyen la primera contribución a la expansión perturbativa de las amplitudes, haciendo énfasis en el caso de las teorías de gauge a partir de las cuales el modelo estandar de la física de partículas está construido. Estudiaremos métodos que evitan los problemas e ineficiencias de los diagramas de Feynman, enfocándonos en el formalismo Cachazo-He-Yuan (CHY), el cual provee fórmulas cerradas para las amplitudes a nivel árbol en dimensión arbitrary como integrales sobre esferas de Riemann con n punturas, localizadas en la solución de un conjunto de ecuaciones que relacionan dichas punturas con el espacio de invariantes cinemáticos asociados al proces, las cuales son conocidas como las ecuaciones de scattering. Haremos una introducción al formalismo CHY para amplitudes en teorías de Yang-Mills, así como una de sus generalizaciones supersimétricas, conocida como supersimetría maximal o símplemente teoría de super Yang-Mills $\mathcal{N} = 4$. Discutiremos la noción de las amplitudes base en Yang-Mills y Cromodinámica Cuántica (QCD), la cual está basada en la idea de la descomposición de color, la cual representa una forma de separar los grados de libertad cinemáticos de los grados de libertad de color, y veremos distintas alternativas para obtener representations CHY para las amplitudes de QCD. Usaremos una de estas representaciones, la cual estará dada en términos de amplitudes base, para obtener teoremas de emisión infraroja sobre el integrando CHY de la QCD, lo cual representa un primer paso en la obtención de restricciones sobre su estructura matemática.

Contents

Introduction

Currently, at the Large Hadron Collider (LHC) at CERN, our knowledge of fundamental physics is being tested and possibly extended through particle collisions occurring at ever-higher energy scales. This, in turn, produces experimental measurements of higher accuracy, which demands the calculation of more precise theoretical predictions. The key observable in these experiments often is the scattering cross section, which brings information concerning the probability that a given process will occur as function of the energy and momentum of the particles involved. More precisely it amounts, in the so-called corpuscular or probabilistic definition [1], to the probability that a given scattering process happens when there is only one-target particle per unit surface. To obtain a cross-section in the framework of quantum field theory, one needs first to compute a quantity known as the scattering amplitude, which contains the information on the interactions that the colliding particles undergo. We will first provide a definition of the scattering amplitudes in any interacting field theory which, for example, can be specified either by a Lagrangian or a Hamiltonian. In general, working with a Hamiltonian, it will have the structure

$$H = H_0 + H_{int} \tag{1.1}$$

where H_0 denotes the free, non-interacting Hamiltonian and H_{int} contains the interactions. Let the states $|\phi_n\rangle = |p_1...p_n\rangle$ represent the eigenstates of the free Hamiltonian, which can be obtained from the free vacuum $|0\rangle$ by acting upon it with creation operators, obtained from the quantization of the free classical theory. We assume the interactions to be perturbations to the free theory and to be localized to a particular region of space; when these conditions are satisfied, we can define the so-called *in* and *out*, or asymptotic states, which are states with the property that they behave as free states in the far past, $t \to -\infty$ and in the far future, $t \to \infty$ respectively. However, these states do not in general behave as free states when they are close to the region where the interaction occurs. Therefore, an arbitrary in state $|\phi_1\rangle_{in} = |p_1...p_k\rangle_{in}$ has a non-vanishing overlap with an out state $_{out}\langle\phi_2| = {}_{out}\langle p_{k+1}...p_n|$[1], which can be calculated as

$$\begin{aligned} \mathcal{P}(\phi_1 \to \phi_2) &= {}_{out}\langle p_{k+1}...p_n|p_1...p_k\rangle_{in} \\ &= \langle p_{k+1}...p_n|S|p_1...p_k\rangle \end{aligned} \tag{1.2}$$

[1]For simplicity, in writting the in and out states, we omit other possible degrees of freedom or Lorentz indices, such as polarizations or helicities.

where, in the second line, we have written the scalar product of the in and out states in terms of the expectation value of an operator between two free states: the operator S is known as the S-matrix, and for a general field theory can be defined as

$$S = \lim_{\substack{t_1 \to -\infty \\ t_2 \to \infty}} U(t_2, t_1) \tag{1.3}$$

where $U(t_2, t_1)$ is the time-evolution operator in the Dirac interaction picture,

$$U(t_2, t_1) = e^{iH_0 t_2} e^{-iH(t_2 - t_1)} e^{-iH_0 t_1} = \mathcal{T} \exp\left(-i \int_{t_2}^{t_1} dt H_{int}(t)\right) \tag{1.4}$$

Here, $\mathcal{T}$ denotes a time-ordering operator. Since the transition probability $\mathcal{P}(\phi_1 \to \phi_2)$ is calculated as an expectation value between two free states, there can be the case where no interactions happens. Then, we can write the S-matrix as

$$S = 1 + iT \tag{1.5}$$

where T is known as the transfer matrix, and contains all the information on the interaction. Moreover, the matrix elements of S (or T) should reflect the conservation of the total four-momenta. Then, we can define the invariant Feynman amplitude, or simply the scattering amplitude $\mathcal{A}_n$, through the relation

$$\langle p_{k+1} ... p_n | iT | p_1 ... p_k \rangle = (2\pi)^D \delta^D \left(\sum_{i=1}^{k} p_i - \sum_{j=k+1}^{n} p_j \right) i\mathcal{A}_n(\phi_1 \to \phi_2) \tag{1.6}$$

where we have assumed a D dimensional spacetime, with metric

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, ...), \tag{1.7}$$

and all the labels as the momenta, polarizations and helicities are assumed to be contained in ϕ_1 and ϕ_2. It is the objects $\mathcal{A}_n(\phi_1 \to \phi_2)$ which we will study throughout this thesis. In the particular case of $2 \to (n-2)$ particle scattering, one can calculate the differential cross-section from the knowledge of the amplitude as [2]

$$\begin{aligned} d\sigma = & \frac{1}{2E_A E_B |v_A - v_B|} \left(\prod_f \frac{d^{D-1} p_f}{(2\pi)^{D-1}} \frac{1}{2E_f} \right) \times \\ & |A_n(A, B \to \{p_f\})|^2 (2\pi)^D \delta^D (p_A + p_B - \sum_f p_f) \end{aligned} \tag{1.8}$$

which allow us to connect the scattering amplitudes to the observables of a collision process, and which underpin the relevance of the calculation of amplitudes.

Scattering amplitudes enjoy the so-called crossing symmetry, which is the statement that a particle with momentum p in the initial state can be interpreted as an antiparticle with momentum $-p$.Hence, we can consider that all particles are outgoing and simply define the n-point amplitudes

$$A_n = A_n(p_1, ..., p_n) \equiv A_n(p) \tag{1.9}$$

denoting with p the complete set of n on-shell momenta on which the amplitudes depend.

The idea that the interactions contained in H_{int} are perturbations to the free dynamics governed by H_0 means that the scattering amplitudes must be calculated as a perturbative expansion, usually in the coupling constants of the theory, which we assume to be small. Hence, an n-point scattering amplitude can be arranged schematically as

$$A_n(p) = \sum_L A_n^L(p) \tag{1.10}$$

where, if we assume that the theory under consideration has a single coupling constant g, roughly $\frac{A_n^{L+1}}{A_n^L} \sim g$. We refer to A_n^L as the L-loop contribution to the amplitude. In this thesis, we will be interested in the structure of the leading contributions, known as tree-level amplitudes. To a fixed loop order, the amplitudes are also organized by the number of external amplitudes, which is known as the multiplicity. In this study, we will be interested in the properties of tree-level amplitudes with arbitrary multiplicity.

The theoretical framework used for the description of the all known elementary particles and their electromagnetic, weak and strong nuclear interactions is the Standard Model (SM), a gauge theory based on the group $SU(3) \times SU(2) \times U(1)$. However, events at the LHC are dominated by Quantum Chromodynamics (QCD) interactions because the kinematic regimes involved in proton-proton collision imply that LHC works largely as a gluon-gluon collider.The dynamics of gluons and quarks is regulated by interactions with a non-abelian $SU(3)$ gauge quantum field theory. In order to separate the SM signals and possibly new physics, beyond SM, a highly precise prediction of the QCD signals is required. There are large simplifications by computing amplitudes for an arbitrary $SU(N)$ gauge group and evaluate at the end the results at $N = 3$.

In the last 30 years, there has been considerable work on the calculation of QCD perturbative scattering amplitudes. The traditional approach of Feynman diagrams, based on the locality of the interactions in a Lagrangian formulation, is not always efficient specially for processes where the number of diagrams grow quickly with number of external legs and at higher perturbative orders. As an example, consider gluon scattering at tree level. We would have to compute

$$\begin{array}{rl} gggg & \text{4 diagrams} \\ ggggg & \text{25 diagrams} \\ gggggg & \text{220 diagrams} \end{array}$$

to obtain the amplitude. Furthermore, to obtain the cross section, we have to square the sum; in the 5 gluon case, this yields $(25)^2 = 625$ terms. For the 6 gluon case, this is $(220)^2 = 48400$ terms. However, most of these terms cancel and the sum turns out to be a relatively simple result. As an example, the spin averaged sum of squared amplitudes for four gluon scattering in Yang-Mills is given by

$$\frac{1}{4}\sum|\mathcal{A}_4|^2 \propto \left(3 - \frac{st}{u^2} - \frac{su}{t^2} - \frac{tu}{s^2}\right) \tag{1.11}$$

where s, t, u denote the usual Mandelstam variables for a $2 \to 2$ process. This hints at the fact that amplitudes have a simpler, inherent structure than the one suggested by Feynman diagrams. It is one of our goals to show that this is in fact the case.

Doing a procedure known as color decomposition, one can separate the gauge group structure from the kinematic information of an arbitrary $SU(N)$ amplitude, obtaining kinematic objects which are known in the literature as color ordered amplitudes. One of the benefits of this decomposition is that the number of diagrams contributing to an n-gluon tree primitive amplitude is reduced with respect to the original, color-dressed amplitude. In [3], it is shown that the number of diagrams for a given n is

n	3	4	5	6	7 ...
# diagrams	1	3	10	38	154 ...

In general, the number of trivalent graphs[2] that contribute to the n-gluon tree amplitude is counted by the Catalan numbers, C_n which are given by

$$C_n = \prod_{k=2}^{n} \frac{n+k}{n}. \tag{1.12}$$

Evidently, the procedure of summing over Feynman diagrams becomes inefficient beyond low multiplicity. Moreover, as seen for example in the analysis by the CMS collaboration in [4], multileg amplitudes in Yang-Mills theories and QCD are of phenomenological relevance because they provide a large portion of the background against which to compare the data obtained at a particle accelerator, such as the LHC, in order to distinguish missing information from possible new physics, such as supersymmetry.

In this thesis, we review different formalisms that surpass the need for Feynman diagrams and which allow for the calculation of amplitudes based on very thorough analyses of the physical properties of amplitudes, which are imposed by Lorentz invariance and locality (relativity), as well as unitarity (quantum mechanics). Therefore, in a sense, one can arrive at the conclusion that scattering amplitudes provide the prime example of the need to unify special relativity and quantum mechanics in order to describe the interactions of fundamental particles. We focus particularly in the so-called Cachazo-He-Yuan (CHY) formalism, which, in a nutshell, allows one to obtain closed-form formulas for tree amplitudes of arbitrary multiplicity and in any number of spacetime dimensions as complex multivariate contour integrals localized to the solution of a set of algebraic so-called scattering equations. We study the problem of the CHY formula for QCD primitive amplitudes, whose properties still

[2] By trivalent graphs, we mean Feynman diagrams that contain only three-gluon vertices.

represent an open research problem, focusing particularly on its soft limit properties, for which we provide a new, original result concerning the factorization on soft limits of the CHY integrand (to be defined in Chapters 4 and 6) for a particular set of QCD primitive amplitudes with arbitrary number of gluons and up to 2 quark-antiquark pairs.

As a starting step, we do a very general review of the properties of tree-level amplitudes in gauge theories. We introduce the notion of color decomposition [5], and focus on the study of the color ordered amplitudes. In particular, we will specialize to the so-called primitive amplitude, which are a subset of the color-ordered amplitudes which are invariant under a cyclic permutation of the external legs. We will see that color-ordered amplitudes in pure Yang-Mills theories automatically satisfy this property, and that one begins to find color-ordered amplitudes which are not primitive when considering multiquark amplitudes. We then discuss the spinor-helicity formalism in four dimensions, which gives us a set of variables in which amplitudes take simpler expressions than those in terms of the momenta and polarizations, exploiting the fact that the vector representation of the Lorentz group is isomorphic to the bispinor representation. We will see how $SL(2,\mathbb{C})$ invariant contractions of the helicity spinors allow us to write the different kinematic invariant of the scattering process, and introduce an explicit formula for n-point tree level scattering amplitudes with a particular set of of helicities: the so-called maximally helicity violating (MHV) amplitudes[6]. We also analyse the behaviour of gauge theory amplitudes in soft limits, which occur when a gluon is emitted with a momentum very small with respect to the momentum of the other particles in the scattering process, which leads to Weinberg's soft theorem[7], [8]. Afterwards, we review one of the most efficient methods for the analytic computation of tree-amplitudes, known as on-shell recursion. By using Lorentz invariance, we will see that one can completely fix the kinematic dependence of amplitudes with three external particles. Since these vanish for physical, massless momenta, we introduce the notion of complex valued momenta. We will see that this allows one to have non-vanishing three point amplitudes where the on-shell conditions and momentum conservation are still satisfied. Later, we will see how unitarity and locality determine the singularity structure of tree amplitudes to come only in the form of simple poles that occur when a propagator goes on-shell, and that the residues in these poles can be written as the product of lower multiplicity amplitudes. Combining these results will allow us to obtain the Britto-Cachazo-Feng-Witten (BCFW)[9] recursion relations, which express an n-point tree amplitude in gauge theories in terms of lower point amplitudes by introducing a complex deformation of the external momenta and calculating a contour integral whose residues coincide exactly with those of the physical amplitudes at their poles, which is a consequence of Weinberg's polology theorem [10] applied to the deformed amplitude. Finally, we discuss the supersymmetric generalization of gauge theories, focusing on the maximally supersymmetric Yang-Mills theory, or $\mathcal{N}=4$ SYM. We will review the superfield and superamplitude formalism, as well as the supersymmetric Ward identities and discuss the effective supersymmetry of tree-level QCD amplitudes with massless quarks.

Moving on to Chapter 3, we introduce the so-called scattering equations, which are at the heart of the Cachazo-He-Yuan (CHY) formalism, which was mainly developed in the work by Cachazo et al in [11],[12],[13],[14] and [15]. These equations will be at the center of the rest of this thesis, since their importance lies in the fact that they provide an alternative way to define and calculate scattering amplitudes; this occurs because the scattering equations provide a map from momentum space to n-punctured Riemann spheres, in a way such that all the information on the kinematic singularities of amplitudes can be understood through the deformation of these Riemann surfaces, and the knowledge of the behaviour of these singularities can be used to completely determine the tree amplitudes in a variety of theories, such as Yang-Mills theories and Einstein gravity. We will analyse some of the details of this mapping, and provide the various forms of the scattering equations, both for massless and massive particles, as well as their reduction to four dimensions in terms of helicity spinors. We will see that the scattering equations possess the particular property of decoupling in kinematic limits which coincide with the poles of tree-amplitudes. This provides an indication at the fact that amplitudes can be written as formulas localized on their solutions. This observation naturally leads us to the problem of solving the scattering equations; first, we will see that they can be transformed from a set of rational equations to a set of polynomial equations [16], and we will use both forms to obtain some solutions for particular numbers of external particles. We finish the chapter by making a brief review of the approaches to their solution for arbitrary number of external particles.

In Chapter 4, we introduce the CHY formalism for massless particles. First, we will see how to define integrals localized on the solutions to the scattering equations, which gives the CHY representation of a tree-level amplitude. These are closed formulas which give the tree amplitudes of a theory with arbitrary multiplicity. One striking feature of this representation is that it follows from Lorentz invariance and kinematic constraints, and does not require the specification of any particular Lagrangian. This fact might lead to a formulation of quantum field theory where the building blocks are the amplitudes, not the Lagrangians. We will discuss the form of the integrand that gives pure Yang-Mills theory amplitudes and see how it can be related to amplitudes in perturbative quantum gravity and scalar ϕ^3 theory. We proceed by reviewing one of the various methods for the calculation of the integrands that appear in the CHY representation, which is based on interpreting them as multidimensional contour integrals [17][18] and the amplitudes themselves as the global residue of the integrand with respect to the polynomial scattering equations. We compute a couple of examples using the CHY integrals, and provide a list of the different methods which have been introduced in the literature. We close the discussion of CHY representation by introducing the so-called connected formalism. This formalism is based on the ideas of Witten, whom in [19] argued that the tree superamplitudes of $\mathcal{N} = 4$ SYM can be calculated as the scattering amplitudes of a so-called twistor string theory whose target space is a Calabi-Yau supermanifold. These ideas led to the development of the Roiban-Spradlin-Volovich (RSV) formula[20], and we discuss its connection to the CHY formalism in four dimensions.

Color-decomposition can be understood as process in which one expands a tree-level amplitude in gauge theory in terms of basis amplitudes, with the special property that they do not depend on the color degrees of freedom of the theory. Chapter 5 provides a general discussion of this idea and how to reduce this basis to a minimal set of linearly independent primitive amplitudes. First, we see that color decomposition implies the existence of linear relations between primitive amplitudes with different external orderings, known as the $U(1)$ decoupling identity and the Kleiss-Kuijf (KK) relations[21]. The origin of these relations is purely group-theoretical, and does not depend on the kinematics of the process. Afterwards, we introduce the Bern-Carrasco-Johansson relations [22], which are a consequence of a very special property of gauge theory amplitudes, known as color-kinematics duality. Basically, color-kinematics duality is the assertion that one can always write gauge theory amplitudes in way such that their kinematic numerators satisfy Jacobi-like relations, which are dual to the corresponding relations of color factors. Assuming the duality to hold results in further linear relations between primitive amplitudes, that however, are kinematic in nature. We review a proof of a special kind of these relations, known as the fundamental BCJ relations [23], for amplitudes that include quarks and give a formula for the general BCJ identities. Having discussed the various relations between gauge theory primitive amplitudes, we construct the minimal basis of primitive QCD amplitudes in the notation of words and shuffle algebras, which provide a compact way to account for the external orderings of amplitudes. We will see that multiquark amplitudes need other identities, known as no-crossed lines relations, which state that primitive amplitudes with crossed fermion lines vanish. Finally, we briefly study how Yang-Mills theory amplitudes can be used to obtain tree amplitudes in perturbative quantum gravity by the usage of the Kawai-Lewellen-Tye (KLT) relations [24], and discuss how these relations are reflected in the CHY formalism.

Chapter 6 is devoted to the study of the proposals of CHY representations for amplitudes with quarks in QCD, which is still an open problem. Although a prescription for the CHY representation for pure Yang-Mills theory which captures all the property of gluon amplitudes exists, there is, to this date and to the best of the authors' knowledge, no known closed-form formula for the integrand which would allow for the calculation of amplitudes with fermions within the CHY formalism. However, it is possible to prove that, given the existence of a minimal basis of QCD primitive amplitudes, there must exist a pair of integrands whose product, integrated on the support of the scattering equations, allow one to obtain QCD primitive amplitudes. This proof, developed in the work by Weinzierl, De La Cruz and Kniss [25] [26], is performed by formally evaluating the CHY integral, so that any amplitude is written as a linear expansion of integrands evaluated at the inequivalent solutions to the scattering equations. Interpreting this expansion in terms of vectors and matrices, it is shown that the existence of a CHY representation for QCD primitive amplitudes is equivalent to being able to invert the linear expansion to obtain one of the integrands in terms of the basis amplitudes. We will perform this procedure explicitly when the number of quark-antiquark lines, n_q, satisfies $n_q \leq 2$. For arbitrary number of quarks, we will see that it is still possible to invert the matrix equation relating the primitive amplitudes to the CHY integrands, if one assumes that the matrix that gives the general BCJ identities (which is not a square matrix) has full row rank. Afterwards, we perform an analytic calculation of the BCJ matrix for the case of six massless quarks, and show that it does have full rank by calculating the determinant of the square matrix constructed from multiplying the BCJ matrix by its transpose. On the other hand, we discuss the usage of the connected formalism of $\mathcal{N} = 4$ SYM to obtain connected formulas for QCD color-ordered amplitudes with massless quarks[27]. This is possible because, as first discovered in [28], tree-level color-ordered QCD amplitudes with massless quarks can be written as linear combinations of $\mathcal{N} = 4$ SYM gluon-gluino amplitudes, and these in turn can be obtained from the connected formula for the superamplitudes. This procedure provides us with explicit integrands for QCD, and we can regard them as CHY representations for massless QCD in four dimensions. We then give a brief comparison of both approaches, emphasizing on both their virtues and their possible drawbacks.

The last chapter of this thesis treats the subject of soft theorems within the framework of the CHY formalism. Soft factorization theorems in gauge theories are of central importance because they are a consequence of gauge and Lorentz invariance, and their existence is a reflection of these properties on the scattering amplitudes. We will first derive the leading soft limits for scalar ϕ^3 and Yang-Mills theories [13], by using their explicit CHY representations. Afterwards, we discuss in detail the main contribution of this thesis given to the problem of the CHY representation for QCD, which is also an original result of our work: with the knowledge that the CHY integrand for QCD primitive amplitudes with $n_q \leq 2$ can be expanded in terms of the basis primitive amplitudes, we will show that, in the limit when a gluon is emitted with a soft momenta, the integrand also factorizes into the integrand for the original particles without the soft gluon, times an eikonal factor. The existence of such a factorization theorem can be understood as a constraint on the possible form that the CHY integrand for QCD with this different particle contents can take. We also check that one recovers the soft theorem for amplitudes by integrating the soft factor of the integrand over the puncture associated with the soft momenta, which shows the self-consistency of our calculation.

We finish our discussion by drawing our conclusions, and discussing both the possible generalizations of our results and some of the open problems related to the CHY representation.

Tree amplitudes in gauge theories

In this chapter, we introduce methods for the calculation of tree-level amplitudes in gauge theories in four dimensions, as well as some of their physical properties. For a review of some of the topics treated in this chapter, we refer the reader to [29],[30].

We will start discussing the basic concepts of gauge theory amplitudes based on Lie groups, as well as the coupling of gauge bosons to fermions. This will lead us to the concept of color decomposition, a process that allow us to separate the gauge group information from the kinematic degrees of freedom of the amplitudes. We will give examples of this process and discuss the resulting kinematic objects, the so-called primitive amplitudes, on which we will be focused throughout this thesis.

Focusing on the special properties of the Lorentz group in four dimensions we arrive at the spinor-helicity formalism, which are variables that encode more naturally the kinematic dependence of amplitudes than the usual four-momentum variables. We will see how to write the wavefunctions for either gauge bosons or fermions in terms of these spinor products, and how they allow to obtain compact expressions for scattering amplitudes. Further consideration of the properties of gauge theory amplitudes will allow us to obtain the behaviour of a general amplitude in which a gluon is emitted with momenta $P_\mu \to 0$, known as soft limits.

Then, we will see that Lorentz invariance fixes the kinematic dependence of all three-point amplitudes in any theory of massless particles; this will also introduce the need for complex-valued momenta in order for these three-particle amplitudes to be non-vanishing. We will then see how the general properties of the S-matrix determine the analytic structure of the scattering amplitudes by determining their possible singularities: in the case of tree amplitude, these will only be single poles, and their residues at these poles will have the form of products of lower multiplicity amplitudes. The knowledge of this factorization properties will allow us to calculate any tree-level n-point primitive amplitude in gauge theory by the use of the so-called Britto-Cachazo-Feng-Witten (BCFW) recursion relations. This recursion is based on a complex deformation of the external momenta, which then allows to express the amplitude only in terms of its residues. Since these residues are given by the product of lower-point amplitudes as a consequence of the unitarity of the S-matrix, we will be able to calculate tree amplitudes with n external particles by the knowledge of lower multiplicity tree amplitudes. This process can then be iterated, so that fixing the three-point amplitudes provides a seed for the recursion: this means that the physical properties of amplitudes completely fixes their values, independent of the particular form that the Lagrangian describing the theory might have.

Finally, the $\mathcal{N} = 1$ and $\mathcal{N} = 4$ supersymmetric extensions of gauge theories are briefly discussed. The focus will be on the special relations that supersymmetry impose on the scattering amplitudes, the so-called supersymmetry Ward identities, and on the on-shell superspace formalism of $\mathcal{N} = 4$ super Yang-Mills theory, which leads naturally to the concept of superamplitudes.

2.1. Basic concepts of gauge theory amplitudes

Yang-Mills theory are the non-Abelian gauge field theories which describes the self-interaction of gauge bosons. Besides of Lorentz invariance, these theories enjoy a local, continuous group of symmetries, known as gauge transformations. These symmetries are based on semi-simple Lie groups. A Lie algebra is specified through the commutation relations of the group generators, t^a, as

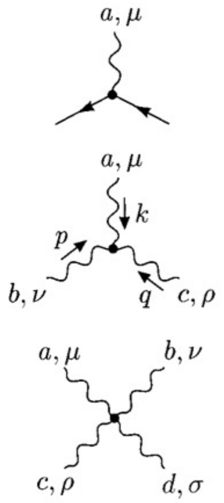
$$[t^a, t^b] = i f^{abc} t^c \tag{2.1}$$

where f^{abc} is a set of numbers known as the structure constants of the gauge group. They can always be taken to be completely antisymmetric in the indices a, b, c. An important property of the structure constants is that they satisfy the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{2.2}$$

For most applications in particle phenomenology, the gauge group is taken to be $SU(N)$ for some integer N, whose self-representations can be given in terms of $N \times N$ unitary matrices with unit determinant. One can also couple the gauge bosons to matter fields, like fermions or scalars, and each of the different states of the theory will transform according to some representation of the gauge group. For example, the external states of a given amplitude in QCD fill out two irreducible representations of the gauge group $SU(N)$; the gauge bosons transform in the adjoint representation, with indices $a, b, c...$ that take values on the set $\{1, 2, ..., N^2-1\}$. The (anti)quarks transform in the (anti)fundamental representation, with indices $\bar{i}, \bar{j}...$ and $i, j...$, respectively, that take values on the set $\{1, 2, ...N\}$. We will be mainly interested in the interactions between gauge bosons and fermions. We will use the terms "gauge boson" and "gluon" interchangeably, and we will refer to fermions transforming in the fundamental representation of the gauge group as quarks. Even though this applies, strictly speaking, to the gauge group $SU(3)$ of the strong interactions in the Standard Model, we will use this terminology throughout this thesis. For completeness, the Feynman rules for QCD are shown in Fig.2.1.

A couple of observations are in order. From a practical point of view, Feynman rules and Feynman diagrams will serve as an illustration of the structure of simple amplitudes to derive some of their properties. However, we will see that there are more efficient methods to perform the calculation of a particular amplitude. Much more important is that the methods we will develop are based on the physical and



$$= \quad ig\gamma^{\mu}t^{a}$$

$$= \quad gf^{abc}\big[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}\big]$$

$$= \quad -ig^{2}\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]$$

Fig. 2.1: Feynman rules for QCD

mathematical properties of scattering amplitudes, such as Lorentz invariance, locality and unitarity. The fact that many simplifications which are obscured by the usual Lagrangian formalism arise by emphasizing the most basic properties an amplitude is expected to satisfy provides us with an indication that, perhaps, quantum field theory can be reformulated in such a way that the space-time properties, such as operator-valued distributions interacting locally (that is, field interactions) can appear as emergent rather than manifest properties.

Following the convention of taking all the particles in an amplitude to be in the final state, instead of talking of the amplitude for the annihilation of a pair of gluons into a quark-antiquark pair, for example, we will talk about the four point amplitude with two gluons and a quark-antiquark pair. This, of course, implies that some particles will have negative energy. However, there is no problem with this technicality, and we can obtain the physical amplitudes by crossing symmetry to recover the initial states. One consequence of this convention is that, using Feynman rules, one usually attaches a polarization vector to an initial state gluon and the complex conjugate to a final state gluon. When talking about polarization vectors, we will simply drop the complex conjugate with the understanding that it refers to a final state gluon.

A basic property of Yang-Mills or QCD amplitudes (to any order in perturbation theory) is gauge invariance, which manifests itself in the form of the so-called Ward identity: suppose we are given an n-point amplitude, in which one of the external particles is a gluon. Then, the amplitude can be written schematically as

$$A_n^{YM} = \epsilon_{\mu}(k)A_n^{\mu} \tag{2.3}$$

where k denotes the four momentum of the gluon we singled out. Gauge invariance is then the statement that, if we replace the polarization vector $\epsilon_{\mu}(k)$ with its corresponding four-momentum, the result should vanish; that is,

$$A_n = \epsilon_\mu(k) A_n^\mu \to k_\mu A_n^\mu = 0 \tag{2.4}$$

A particularly interesting use of gauge invariance is as a consistency check: if one is given a function of momenta and polarizations which is a gauge theory amplitude, it should be possible to verify directly that replacing any polarization vector with its corresponding momenta should yield a vanishing result. This will be one of the consistency checks we will perform when we study the CHY representation of Yang-Mills amplitudes.

2.2. Color ordered amplitudes

Using Feynman rules we can obtain, in principle, any QCD tree-level (or loop) amplitude by constructing all possible diagrams contributing to the scattering process, assigning each line its value through the Feynman rules, and summing those expressions. However, as we have already mentioned, the number of diagrams contributing to a single process, even at tree level, grows quickly with the number of external legs. Moreover, since each diagram is a gauge-dependent quantity built out of off-shell objects, as propagators, calculations become rapidly cumbersome and involve a big amount of unphysical information, whose manifestation is to include a huge number of terms which do not explicitly cancel but that drops out in the final result.

Another issue is that gauge theory amplitudes have many degrees of freedom: not only do they depend on the kinematics, but they also contain information on the gauge group. One way to simplify this problem is to arrange the diagrams contributing to a given matrix element into a smaller subset of quantities which are gauge-invariant. This can be achieved through the process of color decomposition. As a first step, we define a new set of generators T^a, normalized to

$$Tr(T^a T^b) = \delta^{ab} \tag{2.5}$$

which are related to the generators t^a of the previous paragraph by a simple rescaling, $T^a = \sqrt{2} t^a$. With this normalization, the structure constants can be written as

$$f^{abc} = -\frac{i}{\sqrt{2}} Tr\left([T^a, T^b] T^c\right) \tag{2.6}$$

With our choice, the amplitudes will have explicit factors of $1/\sqrt{2}$ when directly applying the Feynman rules. The generators T^a satisfy the completeness relation

$$(T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N} \delta_{i_1}^{\bar{j}_1} \delta_{i_2}^{\bar{j}_2} \tag{2.7}$$

which is nothing more than the statement that the matrices T^a form a basis for $N \times N$ traceless matrices. A proof of this statement can be found in Appendix A. We can illustrate the process of color decomposition using the completeness relation of the generators. As a simple example, consider the four gluon amplitude. The four diagrams contributing to this process are shown in Fig. 2.2.

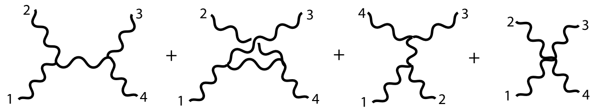


Fig. 2.2: Feynman diagrams contributing to the process $0 \to gggg$. The labels attached to each line indicate the ordering of the external particles.

As is usual in four particle scattering, we label the diagrams with propagators as the s, t and u channel diagrams, according to the Mandelstam invariant appearing in its denominator. In Yang-Mills theory, one can associate a different color factor to each channel, and we can use this to label each diagram. These can be written schematically as

$$\begin{aligned} D_1 &= c_s K_s && (2.8)\\ D_2 &= c_t K_t && (2.9)\\ D_3 &= c_u K_u && (2.10)\\ D_4 &= c_s K'_s + c_t K'_t + c_u K'_u && (2.11) \end{aligned}$$

where K_i, K'_i contain all the kinematic information of the diagram D_i, and the color factors c_s, c_t, c_u are defined as

$$\begin{aligned} c_s &= f^{a_1 a_2 b} f^{b a_3 a_4} && (2.12)\\ c_t &= f^{a_1 a_3 b} f^{b a_4 a_2} && (2.13)\\ c_u &= f^{a_1 a_4 b} f^{b a_2 a_3} && (2.14) \end{aligned}$$

where a_i is the adjoint color index of gluon i. A consequence of the completeness relation (2.7) is that, for any $N \times N$ matrices A and B,

$$Tr(T^a A)Tr(T^a B) = Tr(AB) - \frac{1}{N} Tr(A)Tr(B) \qquad (2.15)$$

which we can use, along with the definition of the structure constants, to write a general expression of the form $f^{abe} f^{cde} = f^{abe} f^{ecd}$ as a sum of generator traces

$$\begin{aligned} f^{abe} f^{ecd} &= Tr([T^a, T^b]T^e)Tr([T^c, T^d]T^e) \\ &= Tr(T^a T^b T^c T^d) - Tr(T^a T^b T^d T^c) - Tr(T^a T^c T^d T^b) + Tr(T^a T^d T^c T^b) \end{aligned} \qquad (2.16)$$

Writing the color factors c_s, c_t, c_u in terms of traces and grouping kinematic terms with the same trace factor, we can write the 4-gluon amplitude as

$$\begin{aligned} \mathcal{A}_4^{YM}(gggg) &= A_4(1234)Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + A_4(1243)Tr(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ &+ A_4(1324)Tr(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) + A_4(1342)Tr(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) \\ &+ A(1423)Tr(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) + A(1432)Tr(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) \\ &= \sum_{\sigma \in S_4/\mathbb{Z}_4} Tr(T^{\sigma(a_1)}T^{\sigma(a_2)}T^{\sigma(a_3)}T^{\sigma(a_4)}) A_4(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \end{aligned} \qquad (2.17)$$

where $S_4/\mathbb{Z}_4$ is the set of non-cyclic permutations of four objects. The quantities $A_4(\sigma)$, for some permutation σ, are known as color-ordered amplitudes. These amplitudes contain all the kinematic information of the full, or color-dressed, amplitude $\mathcal{A}_4$. Now, the color-dressed amplitude is a gauge-invariant object, in the sense of (2.4). Since this is clearly a kinematic statement, this information cannot be carried by the color factor, and the color-ordered amplitude must have some special property that makes the full amplitude gauge-invariant. In fact, if we let $\{\sigma\}$ and $\{\sigma'\}$ denote two permutation of the gluon color indices, the traces satisfy the partial orthogonality [[31]],

$$\sum_{a_i=1}^{N^2-1} Tr(T^{\sigma_1}...T^{\sigma_n}) \left[Tr(T^{\sigma'_1}...T^{\sigma'_n})\right]^* = N^{n-2}(N^2-1)\left(\delta_{\{\sigma\}\{\sigma'\}} + \mathcal{O}(N^{-2})\right) \quad (2.18)$$

where the Kronecker delta is equal to 1 when both permutations are the same, up to cyclic permutations. This orthogonality is sufficient to guarantee that each partial amplitude is a gauge-invariant quantity. In fact, each color-ordered amplitude can be computed from the full amplitude by projection using the orthogonality of the traces.

This so-called trace decomposition can be generalized to color-dressed n-gluon tree level amplitudes as

$$\mathcal{A}_n^{YM} = \sum_{\sigma\in S_n/\mathbb{Z}_n} Tr(T^{\sigma(a_1)}T^{\sigma(a_2)}\cdots T^{\sigma(a_n)})A_n(\sigma(1),\sigma(2),\cdots,\sigma(n)) \quad (2.19)$$

in terms of primitive amplitudes $A_n(\sigma(1),\sigma(2),\cdots,\sigma(n))$. The number of primitive amplitudes appearing in the sum (2.19) is $(n-1)!$, which is nothing but the number of permutations of $(n-1)$ elements. In a sense, color decomposition provides an expansion of a gauge theory amplitude in which the basis elements are the color-ordered amplitudes and the coefficients carry all the gauge group information.

The procedure of color decomposition allows us to put the information on the gauge group aside and focus on the kinematic properties of the amplitudes, which as we will see, can be calculated by a careful analysis of their physical properties. However, primitive amplitudes, in a sense, "remember" that they belong to a theory with gauge symmetry: this is translated into the appearance of different linear relations between them. For example, the partial amplitudes in the expansion (2.19) satisfy a set of group-theoretical linear constraints, known as Kleiss Kuijf (KK) relations [21], which allow to write some partial amplitudes as a linear combinations of other partial amplitudes with different external orderings. These relations make the trace basis of amplitudes overcomplete; a decomposition which explicitly uses the KK-independent amplitudes as a basis, using structure constants as color factors, as well as further ways to reduce the amplitude basis will be discussed further in chapter 5.

One can also consider a gauge theory in which the fermions transform in the adjoint representation (one example of this will be given when we discuss supersymmetric gauge theories). In this case, the color decomposition for an n-point amplitude with $2n_f$ fermions and $n-2n_f$ is still given in terms of traces, as in Eq.(2.19). In this case, there is no fundamental difference between the color structure of amplitudes with fermions with those of pure Yang-Mills theory.

Along the same lines, one can make color decompositions for QCD amplitudes including quarks. However, for an arbitrary number of quarks, the decomposition cannot be given in terms of a single color structure, because the $1/N$ term in the completeness relation do not drop out, as it does for pure gluon amplitudes [1]. For example, if we consider the process $q\overline{q} \to (n-2)g$, the full tree amplitude can written as [5]

$$\mathcal{A}_{n,1} = \sum_{\sigma \in S_{n-2}} (T^{\sigma_1}...T^{\sigma_{n-2}})^{i}_{\overline{j}}\, A\,(q,\sigma,\overline{q}) \tag{2.20}$$

where i is the color index of the quark q and $\overline{j}$ the color index of the antiquark $\overline{q}$. For a higher number of quark-antiquark pairs, there are also color decompositions. These do not, in general, have the same properties that the above primitive amplitudes[2]. The amplitudes appearing in a general color decomposition for QCD are known as partial amplitudes.

To illustrate the difference between primitive and partial amplitudes, we will explicitly work out the color decomposition for the $\mathcal{A}_5(q_1\bar{q}_2q_3\bar{q}_4g)$ amplitude, where the quark q_1 and the antiquark $\bar{q}_2$ have the same flavour, q_3 and $\bar{q}_4$ also have the same flavour, but the flavours of 1 and 2 are different from the flavours of 3 and 4, and g is a gluon. For this example, we will use the standard Feynman diagram expansion. To do this, we will extract the color structure of the Feynman diagrams contributing to the amplitude and leave the kinematic dependence implicit. There are five diagrams at tree level, which are shown in Fig. 2.3.

Consider the first diagram shown in Fig. 2.3. Denoting this diagram by F_1, it will have the general structure

$$F_1 = c_1 D_1 \tag{2.21}$$

where c_1 contains all color factors and D_1 holds the kinematic information. Using the Feynman rules of Fig. 2.1, modified to our convention for the generators and the completeness relation (2.7), we can write

$$\begin{aligned} c_1 &= T^{a_5}_{i_1 j} T^{a}_{j j_2} T^{a}_{i_3 j_4} \\ &= T^{a_5}_{i_1 j_4}\delta_{i_3 j_2} - \frac{1}{N} T^{a_5}_{i_1 j_2}\delta_{i_3 j_4} \end{aligned} \tag{2.22}$$

where a_5 is the color of the external gluon, i_1, i_3 the colors of the quarks q_1, q_3 and j_2, j_4 the colors of the antiquarks $\bar{q}_2, \bar{q}_4$, respectively (a sum over repeated indices is understood in all cases). A similar computation yields the color factors for the remaining diagrams where the gluon is emitted from a quark line. The results are

[1] Which occurs because the photon associated to the $U(1)$ generator has Abelian dynamics

[2] In particular, they are not cyclic invariant because of the second term in (2.7)).

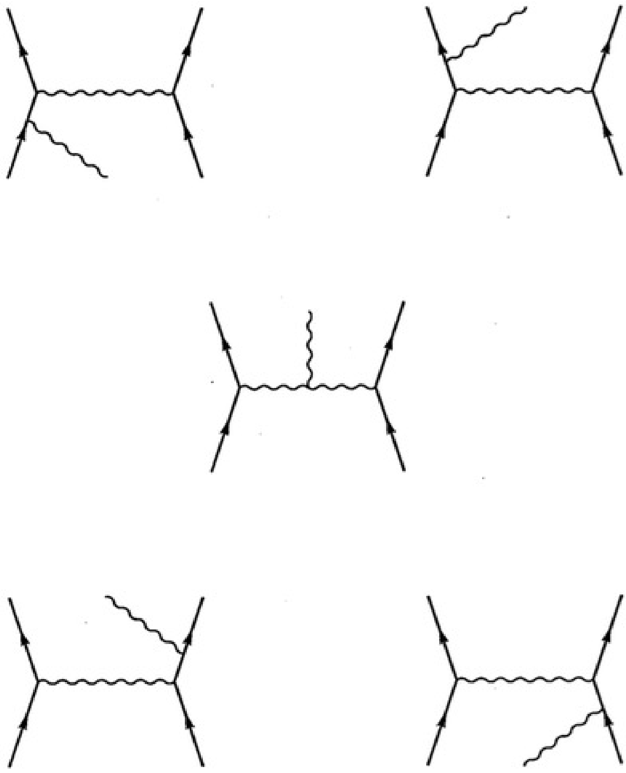


Fig. 2.3: Feynman diagrams for the four quarks plus one gluon amplitude

$$
\begin{aligned}
c_2 &= T^{a_5}_{i_3 j_2}\delta_{i_1 j_4} - \frac{1}{N} T^{a_5}_{i_1 j_2}\delta_{i_3 j_4} \\
c_3 &= T^{a_5}_{i_3 j_2}\delta_{i_1 j_4} - \frac{1}{N} T^{a_5}_{i_3 j_4}\delta_{i_1 j_2} \\
c_4 &= T^{a_5}_{i_1 j_4}\delta_{i_3 j_2} - \frac{1}{N} T^{a_5}_{i_3 j_4}\delta_{i_1 j_2}
\end{aligned}
\tag{2.23}
$$

Now, we consider the center diagram, in which we find a three gluon vertex. Again, we call this diagram $F_5 = c_5 D_5$, where c_5 is its color factor. The calculation is a little more involved in this case, since it includes a contracted structure constant. Using (2.6), we can see that

$$
\begin{aligned}
c_5 &= T^{a}_{i_1 j_2} f^{a b a_5} T^{b}_{i_3 j_4} \\
&= T^{a}_{i_1 j_2} T^{b}_{i_3 j_4} \left[-\frac{i}{\sqrt{2}} Tr \left(T^a T^b T^5 - T^a T^{a_5} T^b \right) \right] \\
&= -\frac{i}{\sqrt{2}} \left(c_{5;1} - c_{5;2} \right)
\end{aligned}
\tag{2.24}
$$

where, after some color algebra,

$$
\begin{aligned}
c_{5;1} &= T^{a}_{i_1 j_2} T^{b}_{i_3 j_4} Tr \left(T^a T^b T^{a_5} \right) \\
&= T^{a_5}_{i_3 j_2}\delta_{i_1 j_4} - \frac{1}{N} T^{a_5}_{i_1 j_2}\delta_{i_3 j_4} - \frac{1}{N} T^{a_5}_{i_3 j_4}\delta_{i_1 j_2}
\end{aligned}
\tag{2.25}
$$

and

$$
\begin{aligned}
c_{5;2} &= T^{a}_{i_1 j_2} T^{b}_{i_3 j_4} Tr \left(T^a T^{a_5} T^b \right) \\
&= T^{a_5}_{i_1 j_4}\delta_{i_3 j_2} - \frac{1}{N} T^{a_5}_{i_1 j_2}\delta_{i_3 j_4} - \frac{1}{N} T^{a_5}_{i_3 j_4}\delta_{i_1 j_2}
\end{aligned}
\tag{2.26}
$$

therefore, when we compute the difference $c_{5;1} - c_{5;2}$, the $1/N$ terms will cancel in pairs, and the color structure for this diagram is

$$
\begin{aligned}
c_5 &= -\frac{i}{\sqrt{2}} \left(T^{a_5}_{i_3 j_2}\delta_{i_1 j_4} - T^{a_5}_{i_1 j_4}\delta_{i_3 j_2} \right) \\
&= \frac{i}{\sqrt{2}} \left(T^{a_5}_{i_1 j_4}\delta_{i_3 j_2} - T^{a_5}_{i_3 j_2}\delta_{i_1 j_4} \right)
\end{aligned}
\tag{2.27}
$$

The structure of all these color factors is the product of a matrix element of the generator times a quark color Kronecker delta. Thus, we can factorize each of these color factors, which will yield an expansion of $\mathcal{A}_5(q_1\bar{q}_2q_3\bar{q}_4g)$ in terms of color factors time kinematic factors; this is the color decomposition we are looking for. As we anticipated, we will obtain two color structures, where one will have an explicit $1/N$ factor. In the case of primitive amplitudes, we only expect one color structure to appear at tree level, which in this case is the one associated to the leading (in $1/N$) term. Moreover, the partial amplitudes obtained by this method are not cyclic invariant, as we will verify by explicit computation in Appendix D. Schematically, we can write the color decomposition as

$$
\mathcal{A}_5(q_1\bar{q}_2q_3\bar{q}_4g) = \sum_i a_i A_i - \frac{1}{N} \sum_i \tilde{a}_i \tilde{A}_i
\tag{2.28}
$$

where we define

$$\begin{aligned} a_1 &= T^{a_5}_{i_1 j_4} \delta_{i_3 j_2} \\ a_2 &= T^{a_5}_{i_3 j_2} \delta_{i_1 j_4} \\ \tilde{a}_1 &= T^{a_5}_{i_1 j_2} \delta_{i_3 j_4} \\ \tilde{a}_2 &= T^{a_5}_{i_3 j_4} \delta_{i_1 j_2} \end{aligned} \tag{2.29}$$

the reason for the additional term, proportional to $1/N$, is because the $U(1)$ generator (related to the $1/N$ term in the completeness relation (2.7)) cannot be ignored, as opposed to the pure gluon case where the structure constants associated to a $U(1)$ gluon are identically zero, and the color factors of gluon amplitudes are built entirely out of structure constants. The partial amplitudes are defined, in terms of the kinematic factors D_i, as

$$\begin{aligned} A_1 &= D_1 + D_4 + D_5 \\ A_2 &= D_2 + D_3 - D_5 \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \tilde{A}_1 &= D_1 + D_2 \\ \tilde{A}_2 &= D_3 + D_4 \end{aligned} \tag{2.31}$$

Each of these amplitudes is separately gauge-invariant. It is interesting to note that the diagram with the three gluon vertex contributes to two distinct color ordered amplitudes. In a purely diagrammatic analysis, it can be seen that it is topologically equivalent to orient the external gluon emitted from the propagator downwards or upwards. This means that, using a reformulation of Feynman diagrams which only take into account diagrams with a particular external ordering (which can be obtained from the so-called color ordered Feynamn rules), this diagrams gives the same contribution to two different color-ordered amplitudes, up to a global sign.

Different ways to work out color decompositions for amplitudes with an arbitrary number of quark-antiquark pairs have been found, for example, using shuffle relations [32] and Dyck words [33]. These decompositions provide an interesting mathematical relation between amplitudes and combinatorics, and we will explore them further in Chapter 5.

2.3. Spinor-Helicity formalism for massless particles

Scattering amplitudes in four dimensions are usually written as functions of Lorentz invariant quantities build out of the momenta and wavefunctions of the external particles, according to a particular set of Feynman rules. Even though Lorentz invariance is manifest in these expressions, the dependence of the amplitudes on the external momenta is very much obscure. However, by choosing an appropriate set of variables, this dependence can be made explicit. With this motivation, we introduce the so-called spinor-helicity variables, which are special to scattering in four dimensions. Interestingly, it will turn out that these variables also have the advantage of producing extremely compact expressions for some special kinds of amplitudes.

The finite representations of the Lorentz group are labelled by two integer or semi-integers (j, j'), and the total spin is $j + j'$. Therefore, the spin 1 representation is labelled by $(\frac{1}{2}, \frac{1}{2})$. But, we also know that the spin $\frac{1}{2}$ representations of the Lorentz group are either the $(\frac{1}{2}, 0)$ or the $(0, \frac{1}{2})$, which correspond to left-handed or right-handed two component spinors, respectively. So, in a sense, it is more natural to represent four vectors p^μ as bispinors $p_{\alpha\dot{\alpha}}$, where the indices $\alpha, \dot{\alpha} = 1, 2$. For the unfamiliar reader, these indices can be raised or lowered with the 2D Levi-Civita tensor, with components

$$\begin{aligned} \varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = 1 \\ \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1 \end{aligned} \tag{2.32}$$

such that, for a left-handed spinor ξ_α and a right-handed spinor $\tilde{\eta}_{\dot{\alpha}}$,

$$\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta, \quad \tilde{\eta}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\tilde{\eta}_{\dot{\beta}} \tag{2.33}$$

The Pauli matrices [3] gives us a map from the vector to the bispinor representation, through the relation

$$p_{\alpha\dot{\alpha}} = p^\mu(\sigma_\mu)_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 - p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 + p_3 \end{pmatrix} \tag{2.34}$$

where we have written the relation in matrix form. It is straightforward to see that $det(p_{\alpha\dot{\alpha}}) = p^2 = m^2$. Hence, all physical information is contained in this new representation for on-shell momentum of the external particles.

An interesting feature of the bispinor representation is that, when the particles are massless, $det(p_{\alpha\dot{\alpha}}) = 0$. A result from linear algebra tells us that any rank 1 2×2 matrix can be written as the outer product of a pair of two component vectors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} \tag{2.35}$$

the two-component vectors λ and $\tilde{\lambda}$ are known as helicity spinors, and we will see that tree level amplitudes of massless particles can be written entirely in terms of $SL(2, \mathbb{C})$ invariant contractions of these spinors. Now, we will see how the two component spinors are embedded in the four component Dirac spinors.

2.3.1. Spinor-helicity for fermions

Consider the Dirac equation in momentum space

$$\not{p}u(p) = mu(p) \tag{2.36}$$

[3] See Appendix B for definitions and conventions.

where $u(p)$ is a four component Dirac spinor. Although we will be concerned with the massless case, there is no technical difficulty in considering massive states in the few following lines. In the chiral representation of the Gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.37}$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, we can construct the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

which will be used in the definition of the chirality projectors (chirality and helicity are the same for massless particles).

An arbitrary four component Dirac spinor can be written in terms of two 2-component spinors, λ_α and $\bar{\eta}_{\dot{\alpha}}$ as

$$u(p) = \begin{pmatrix} \lambda_\alpha \\ \bar{\eta}_{\dot{\alpha}} \end{pmatrix} \tag{2.39}$$

where the two component spinors, known as Weyl spinors, satisfy the equations

$$p_\mu \bar{\sigma}^\mu \lambda - m\eta = 0, \quad p_\mu \sigma^\mu \eta - m\lambda = 0 \tag{2.40}$$

which are known as the Weyl equations. In the massless limit, they decouple and, in fact, their solutions are related to each other by complex conjugation because the momenta is real. For this reason, we write

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{2.41}$$

so that $\tilde{\lambda}_{\dot{\alpha}} = \lambda^\dagger_\alpha$. We can define left handed Dirac spinors,

$$u_L(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \tag{2.42}$$

and right handed Dirac spinors

$$u_R(p) = \begin{pmatrix} 0 \\ \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix} \tag{2.43}$$

as the embedding of the two component Weyl spinors into four component objects. If we define the projectors

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2} \tag{2.44}$$

we can see that

$$\begin{gathered} P_L u_L(p) = u_L(p), \quad P_L u_R(p) = 0 \\ P_R u_L(p) = 0, \quad P_R u_R(p) = u_R(p) \end{gathered} \tag{2.45}$$

that is, the spinors have definite chirality. Since they describe massless particles, they also have a definite helicity.

We also need spinors $v(p)$ to describe antiparticles; however, since they obey the same equation as $u(p)$ in the massless case, we will use the same spinors for those solutions, baring in mind that the helicity of the spinor $v(p)$ is the opposite to the physical helicity of the antiparticle. Thus, $v_L(p)$ describes a right-handed antifermion and $v_R(p)$ a left-handed antifermion. Recalling our convention, we consider all particles involved in the scattering process as final-state particles. If needed, the initial states can be recovered by crossing symmetry. One useful consequence of this convention is that momentum conservation is written as

$$\sum_i p_i = 0 \tag{2.46}$$

It is common in the literature to introduce a Dirac bra-ket notation for the spinors,

$$u_R(p) = |p\rangle, \quad u_L(p) = |p], \quad \bar{u}_R(p) = [p|, \quad \bar{u}_L(p) = \langle p| \tag{2.47}$$

We can form scalar, Lorentz-invariant quantities with products of these spinors. These are the so-called spinor products

$$\langle ij\rangle \equiv \bar{U}_L(p_i)U_R(p_j) = \varepsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta, \quad [ij] \equiv \bar{U}_R(p_i)U_L(p_j) = -\varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}} \tag{2.48}$$

(it can easily be seen that expressions like $\langle ij]$ or $[ij\rangle$ vanish, as well as $\langle ii\rangle$ and $[ii]$, due to the antisymmetry of the Levi-Civita tensor). The usual completeness relation can be written in terms of helicity spinors as

$$\not{p} = |p\rangle[p| + |p]\langle p| \tag{2.49}$$

Various identities satisfied by the spinor products can be found in Appendix B.

2.3.2. Vector boson polarizations

Polarization vectors can be conveniently written in terms of spinors [34, 35, 36]. Let $\epsilon_\mu(p)$ be the polarization vector of a gluon with momentum p. Let q be an arbitrary lightlike four-momentum satisfying $p \cdot q \neq 0$. Then, we can write

$$\epsilon_+^\mu(p) = \frac{1}{\sqrt{2}}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle}, \quad \epsilon_-^\mu(p) = -\frac{1}{\sqrt{2}}\frac{[q|\gamma^\mu|p\rangle}{[qp]} \tag{2.50}$$

where the subscripts $(+,-)$ refer to the helicity $h = \pm 1$ of the associated gluon. Positive helicity gluons are also known as right-handed while negative helicity gluons are also known as left-handed. The momentum q is known as the reference momenta, and it holds the information of gauge-invariance for each leg separately, which means that there can be n independent reference momenta. In fact, let us choose two reference momenta q and p for the same polarization $\epsilon_\mu(k)$. Also, let us suppose, without loss of generality, that it is right-handed. Then, we can see that

$$
\begin{aligned}
\epsilon_+^\mu(k,p) - \epsilon_+^\mu(k,q) &= \frac{1}{\sqrt{2}}\frac{\langle p|\gamma^\mu|k]}{\langle pk\rangle} - \frac{1}{\sqrt{2}}\frac{\langle q|\gamma^\mu|k]}{\langle qk\rangle} \\
&= \frac{1}{\sqrt{2}}\left(\frac{\langle p|\gamma^\mu|k]\langle qk\rangle - \langle q|\gamma^\mu|k]\langle pk\rangle}{\langle pk\rangle\langle qk\rangle}\right) \\
&= \frac{1}{\sqrt{2}}\left(-\frac{\langle p|\gamma^\mu|k]\langle kq\rangle + \langle q|\gamma^\mu|k]\langle kp\rangle}{\langle pk\rangle\langle qk\rangle}\right) \\
&= \frac{1}{\sqrt{2}}\left(-\frac{\langle p|\gamma^\mu \not{k}|q\rangle + \langle q|\gamma^\mu \not{k}|p\rangle}{\langle pk\rangle\langle qk\rangle}\right) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle pk\rangle\langle qk\rangle}\left(\langle q|\not{k}\gamma^\mu|p\rangle + \langle q|\gamma^\mu\not{k}|p\rangle\right) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle pk\rangle\langle qk\rangle}\left(\langle q|(\not{k}\gamma^\mu + \gamma^\mu\not{k})|p\rangle\right) \\
&= \frac{1}{\sqrt{2}}\frac{\langle qp\rangle}{\langle pk\rangle\langle qk\rangle}\cdot 2k^\mu
\end{aligned}
\tag{2.51}
$$

then, if we dot the gluon-stripped amplitude $A_{n\mu}$ with the difference of both polarization vectors, we obtain zero as result, due to gauge-invariance. Thus, any gauge theory amplitude is independent of the choice of reference momenta. This is a very convenient property when doing explicit calculations, because an amplitude can be greatly simplified by choosing an appropriate set of reference momenta.

Moreover, using the definitions (2.50), we can show that, for example, the right-handed polarization vectors satisfy the usual transversality condition

$$
p_\mu \epsilon_R^\mu(p) = \frac{1}{\sqrt{2}}\frac{\langle q|p_\mu\gamma^\mu|p]}{\langle qp\rangle} = \frac{1}{\sqrt{2}}\frac{\langle q|\not{p}|p]}{\langle qp\rangle} = 0 \tag{2.52}
$$

due to the Dirac equation, $\not{p}|p] = 0$. Also, they are properly normalized,

$$
|\epsilon_R^\mu(p)|^2 = \frac{1}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle}\frac{\langle p|\gamma^\mu|q]}{[pq]} = \frac{1}{2}\frac{2\langle qp\rangle[qp]}{\langle qp\rangle[pq]} = -1 \tag{2.53}
$$

Armed with the helicity spinors and the polarization vectors written in terms of them, we can try to calculate any amplitude of massless particles as a function of spinor products. Since both the spinors and polarization vectors are defined as quantities with definite helicity, it is natural to consider amplitudes where the external particles have an specific helicity, the so-called helicity amplitudes. Then, generally, an n-point amplitude is written as

$$
A_n = A_n(\{\lambda_i, \tilde{\lambda}_i, h_i\}) \tag{2.54}
$$

that is, as a function of the list of spinors and helicities of the external particles. Helicity amplitudes in gauge theory may take strikingly simple forms for some special helicity configurations. These are known as MHV amplitudes, and will be our next topic of discussion.

2.3.3. MHV classification

Consider an n-point gluon tree amplitude. Typically, it will be a function of Lorentz invariants of the form

$$p_i \cdot p_j, \quad p_i \cdot \epsilon_j, \quad \epsilon_i \cdot \epsilon_j \tag{2.55}$$

and we want to rewrite them in terms of spinor variables. First, using the Fierz identity (B.16), it is easy to see that, for a pair of gluons with momenta p_i and p_j with respective reference momenta q_i and q_j,

$$\begin{array}{rcl} \epsilon_+(p_i) \cdot \epsilon_+(p_j) & \propto & \langle q_i q_j \rangle \\ \epsilon_-(p_i) \cdot \epsilon_-(p_j) & \propto & [q_i q_j] \\ \epsilon_+(p_i) \cdot \epsilon_-(p_j) & \propto & \langle q_i p_j \rangle [q_j p_i] \end{array} \tag{2.56}$$

Now, looking at the Feynman rules in Fig. 2.1, we can see that an arbitrary n-gluon tree amplitude takes the schematic form

$$A_n^{YM} \backsim \sum \frac{\prod(\epsilon_i \cdot \epsilon_j)(\epsilon_i \cdot p_j)(p_i \cdot p_j)}{\prod P^2} \tag{2.57}$$

where the sum runs over the different diagrams contributing to the process, the product goes over all possible contractions appearing in the diagram, and P denotes the momentum flowing through each propagator of the diagram. Now, consider for a moment tree diagrams with only cubic vertices. The three gluon vertex has mass dimension $+1$. Then, if we start "sewing" cubic vertices to construct an arbitrary n-point diagram, each time we need to add a new vertex and a new propagator to the diagram; this means that both the number of vertices and propagators in a given diagram grow linearly with n. In general, there will be $(n-2)$ vertices and $(n-3)$ propagators. Since propagators have mass dimension -2, then

$$[A_n] \sim \frac{(mass)^{n-2}}{(mass^2)^{n-3}} \sim (mass)^{4-n} \tag{2.58}$$

which is the mass dimension that an amplitude with n external states in an arbitrary four-dimensional field theory should have in order to give the cross section the dimensions of area. However, it will be important that the mass dimension of the numerator cannot exceed $(n-2)$ for an n-point amplitude.

Now, suppose we wanted to compute an helicity amplitude with either all positive or all negative helicities. Recalling that we can choose each reference momenta to have an arbitrary value, we could make the choice $q_i = q$ for all $i = 1, ..., n$. Then,

$$\epsilon_+(p_i) \cdot \epsilon_+(p_j) = \epsilon_-(p_i) \cdot \epsilon_-(p_j) = 0 \tag{2.59}$$

Since each external gluons contributes with one polarization vector to the amplitude, each diagram is the product of a tensor in Lorentz indices times the n polarization vectors of the process. Thus, in order to obtain a non-zero result for either helicity configuration, every polarization vector should appear contracted as $\epsilon_i \cdot k_j$, where k is a some linear combination of external momenta. However, we have seen that the mass dimension of the numerator of a Yang-Mills amplitude is at most $(n-2)$. This means that we can, at most, contract $(n-2)$ of the external polarization vectors with momentum factors coming from the cubic vertices. This implies that each diagrams must contain at least one product of polarization vectors. Hence,

$$\begin{array}{rcl} A_n^{YM}(1^+2^+....n^+) & = & 0 \\ A_n^{YM}(1^-2^-....n^-) & = & 0 \end{array} \tag{2.60}$$

Similarly, assume that we wanted to calculate the amplitudes $A_n^{YM}(1^-2^+....n^+)$ or $A_n^{YM}(1^+2^-....n^-)$. Then, if we take the reference momenta $q_i = p_1$ for $i \neq 1$, the equal-helicity polarization vector products would all vanish, and $\epsilon_{1+}\cdot\epsilon_{i-} = \epsilon_{1-}\cdot\epsilon_{i+} = 0$. Therefore, we find again that all possible products of polarization vectors vanish, and from the same argument as before, we obtain

$$\begin{aligned} A_n^{YM}(1^-2^+....n^+) &= 0 \\ A_n^{YM}(1^-2^+....n^+) &= 0 \end{aligned} \tag{2.61}$$

The first non-vanishing helicity amplitudes are those where two particles have positive helicity and the rest have negative helicity, and vice-versa. These amplitudes are known as Maximally Helicity Violating (MHV) and anti-MHV amplitudes, respectively. These were first proposed by Parke and Taylor [6], who guessed their general structure after performing specific calculations with up to six external gluons, and were proven afterwards with the of-shell recursive relations of Berends and Giele [37] and are given, for n gluon scattering, by

$$A_n(1^-...i^+...j^+...n^-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle (n-1)n\rangle\langle n1\rangle} \tag{2.62}$$

and

$$A_n(1^+...i^-...j^-...n^+) = (-1)^n\frac{[ij]^4}{[12][23]...[(n-1)n][n1]} \tag{2.63}$$

where we have omitted the powers of the gauge coupling that appear as prefactors. The denomination of "maximally helicity violating" comes from thinking of the n-point amplitude as a $2 \to (n-2)$ process; in this scenario, it is the non-vanishing amplitude where the difference between the total incoming and outcoming helicity takes its maximum value. MHV amplitudes have the remarkable property that they do not have any multiparticle pole, which renders them remarkably simple. Sometimes, MHV amplitudes are also referred to as holomorphic and anti-MHV amplitudes as antiholomorphic [4], because of the fact that the former only depend on angle spinor products and the latter on square spinor products.

Usually, "flipping" more positive helicities into negative ones yield more complicated expressions. In particular, gluon amplitudes with $k = 3$ negative helicity and $(n-3)$ positive helicity gluons are known as Next-to-MHV, or NMHV amplitudes. In general, an n-gluon amplitude with $K+2$ negative helicity and $(n-K-2)$ positive helicity gluons is known as $N^K MHV$ amplitude.

Full QCD amplitudes with quarks satisfy similar relations. As an example, consider an amplitude with one $q\bar{q}$ pair and $n-2$ external gluons. Since helicity is a conserved quantity along any fermion line, the quark and antiquark must have opposite helicites. In this case, MHV and anti-MHV amplitudes occur when all gluon helicites are the same, except for one. These are given by

$$A(q^-2^+...i^-...(n-1)^+\bar{q}^+) = ig^{n-2}\frac{\langle 1i\rangle^3\langle ni\rangle}{\langle 12\rangle\langle 23\rangle...\langle (n-1)n\rangle\langle n1\rangle} \tag{2.64}$$

[4]They are also known as googly, in the context of twistor theory

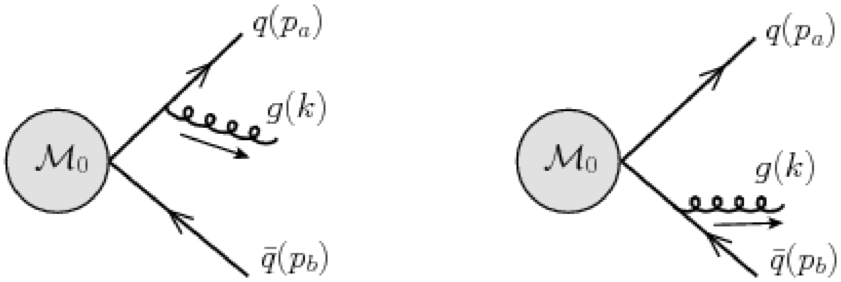


Fig. 2.4: Emission of a soft gluon from a quark line

and

$$A(q^{-}2^{+}...i^{-}...(n-1)^{+}\bar{q}^{+}) = ig^{n-2}\frac{[1i][ni]^3}{[12][23]...[(n-1)n][n1]} \tag{2.65}$$

These formulas can be obtained from the pure gluon MHV formulas by using the supersymmetric Ward identities, which we introduce at the end of this chapter.

2.3.4. Soft limits

There are a special kind of singularities in which the amplitude factorizes when approaching a pole, which are known as soft limits. These occur when the momenta of some external massless particles is very small compared to the momenta of the remaining particles [5]. In this limit, the amplitudes exhibit a universal factorization (in the sense that the factorization is independent of whom the soft particles couples to). In particular, we focus on the single soft limit where only one external particle has a soft momentum that can be taken as $k^\mu = \delta K^\mu$, where K^μ is some constant, finite momentum and $\delta \to 0$ parametrizes the soft limit. This can occur, for example, when a photon or gluon is emitted with very small momenta. Seeing that we are interested in gauge theories, we will consider the case of a color ordered amplitude where the soft gluon is emitted from a quark line. This is illustrated in 2.4.

Let A_n denote the amplitude without the soft gluon, and consider how this amplitude is modified when the gluon is inserted in the line with a quark of momentum p_a and antiquark of momentum p_b. To simplify the calculation, we assume the quarks to be massless. We can write the amplitude A_n as

$$A_n = \bar{u}(p_a)A'_n(p_a, p_b)v(p_b) \tag{2.66}$$

that is, we define A'_n as the n-point amplitude stripped off the fermion wave functions. The resulting amplitude from the insertion of the gluon on the quark line can be written as

$$A_{n+1} = -ig\bar{u}(p_a)\left[\gamma^\mu\epsilon_\mu(k)\tfrac{i(\not{p}_a+\not{k})}{(p_a+k)^2}A'_n(p_a+k,p_b) + \tfrac{i(\not{p}_b-\not{k})}{(p_b-k)^2}A_n`(p_a,p_b+k)\right]v(p_b) \tag{2.67}$$

Now, since the gluon has a soft momenta $k = \delta K$ with $\delta \to 0$, we can make the following approximations

[5]These are usually given the name of hard particles.

$$
\begin{aligned}
&A'_n(p_a+k,p_b) \approx A'_n(p_a,p_b+k) \approx A'_n(p_a,p_b)\\
&\not{p}_a + \not{k} \approx \not{p}_a\\
&\not{p}_b + \not{k} \approx \not{p}_b
\end{aligned}
\tag{2.68}
$$

hence, the numerators are simplified by using the anticommutation relation of the Gamma matrices, from which it follows that

$$
\begin{aligned}
\not{p}_b \gamma^\mu \epsilon_\mu v(p_b) &= 2p_b \cdot \epsilon v(p_b)\\
\bar{u}(p_a)\gamma^\mu \epsilon_\mu \not{p}_a &= \bar{u}(p_a) 2p_a \cdot \epsilon
\end{aligned}
\tag{2.69}
$$

and the squares in the denominators can be simplified to

$$
\begin{aligned}
(p_a+k)^2 &= 2p_a \cdot k\\
(p_b-k)^2 &= -2p_b \cdot k
\end{aligned}
\tag{2.70}
$$

Putting everything together, we find that, in the soft limit, the amplitude factorizes

$$
A_{n+1} \approx \frac{1}{\delta} g S(a,k,b) A_n \tag{2.71}
$$

where $S(a,k,b)$ is known as the eikonal or Weinberg soft factor, and is given by

$$
S(a,k,b) = \frac{p_a \cdot \epsilon}{p_a \cdot k} - \frac{p_b \cdot \epsilon}{p_b \cdot k} \tag{2.72}
$$

We note that the soft factor does not depend on the spin of the particles from which the gluon is emitted, only on their momenta. This reflects the fact that low energy gluon (or photon) emission must be understood as a classical phenomena. Moreover, the soft factor enjoys a couple of properties, the first of which is the antisymmetry under the exchange of the momenta labelled by a and b, that is

$$
S(b,k,a) = -S(a,k,b) \tag{2.73}
$$

and what could be called a Schouten identity,

$$
S(a,k,b) + S(b,k,c) = S(a,k,c) \tag{2.74}
$$

These identities will play a crucial role in chapter 7, where we study the soft behaviour of the CHY integrand for QCD and show that it enjoys similar factorization properties to those of the amplitudes in the soft limit.

Soft theorems also have strong physical consequences: in the case we have derived, they imply the existence of a conserved charge associated to the coupling of the spin-1 massless particle to the other particles (for example, Weinberg's theorem for soft photon emission implies the conservation of electric charge). Similarly, gravitons, understood as spin-2 massless particles, also enjoy soft factorization theorems, and their existence implies that these particles must couple in a unique, universal way to every kind of matter field. Moreover, they also imply that there are no interacting theories of massless particles with spin greater than 2 [8].

Having discussed the singularity structure of amplitudes on general grounds and specialized on the case of gauge theories, we are in a position to use the analytic properties of amplitudes we have derived to develop a framework in which to calculate tree amplitudes recursively in the number of external legs, which is the subject of the following section.

2.4. On-shell recursion relations

The fact that we can write explicit expressions for arbitrary multiplicity amplitudes is astonishing from the point of view of Feynman diagrams: for arbitrary n, we would need to compute the sum of about $\mathcal{O}(n!)$ diagrams, each of which is not a gauge invariant quantity defined in terms of the wavefunctions of the external particles, like the polarization vectors fo gluons or Dirac spinors for quarks. Then, by some miracle, we arrive to extremely simple formulas like (2.62). Summing an arbitrarily high number of diagrams and obtaining such a simple result, however, does not seem like an easy problem at all. This is because, as we have mentioned, Feynman diagrams do not know too much about the physical properties of amplitudes. In this spirit, we introduce a scheme that will allow us to construct amplitudes only from on-shell, gauge invariant blocks, which generally fits into what we will call on-shell recursion relations. As a first step, we will construct the building blocks upon which to start the recursion, which will be the lowest multiplicity tree amplitudes we can possibly calculate: three-point amplitudes.

2.4.1. Three-point amplitudes and complex momenta

Consider the scattering process of three massless particles with momenta p_1, p_2 and p_3. The amplitude for this process can be a function of the three kinematic invariants s_{12}, s_{23} and s_{31} (the invariant s_{123} is identically zero due to momentum conservation). Given the amount of variables it can depend, this amplitude ought to be very simple. In fact, let us use momentum conservation to manipulate, for example,

$$\begin{aligned} s_{12} &= (p_1 + p_2)^2 \\ &= p_3^2 \\ &= 0 \end{aligned} \tag{2.75}$$

from the on-shell conditions and the fact that the particles are massless. Similarly, one can show that $s_{23} = s_{31} = 0$. Furthermore, as proved in Appendix B,

$$s_{ij} = \langle ij\rangle[ji] \tag{2.76}$$

for an arbitrary, massless four momenta p_i and p_j, and that $[ji] = \langle ij\rangle^*$. Hence,

$$s_{ij} = |\langle ij\rangle|^2 = |[ij]|^2 \tag{2.77}$$

so, whenever s_{ij} vanishes, so do the spinor products. We have exhausted the possible Lorentz invariants we can build out of the momenta to construct the amplitude, which seems to imply that there are no non-vanishing three-point amplitudes of massless particles. This is of particular interest in Yang-Mills theory, where the external states are exactly massless (as opposed to massless limits of fermions, which occur only on high energy regimes). If we set $n = 3$ in the Parke-Taylor formula (2.62), we would obtain

$$A_3^{YM}(1^-2^-3^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{2.78}$$

From this expression, we can say that $A_3^{YM} = 0$, roughly because there are more powers of the momenta in the numerator than the denominator, and all spinor products tend to zero. However, this expression is not entirely useless. In fact, the complex conjugate relation $[ji] = \langle ij\rangle^*$ is only valid with Lorentzian metric when the momenta p_i and p_j have real components. If we allow for the momenta to have complex components, the angle and square bracket products are independent, but the relation (2.76) is still valid. Hence, we have two possible kinematic configurations

- $\langle ij\rangle \neq 0$ and $[ij] = 0$, or
- $\langle ij\rangle = 0$ and $[ij] \neq 0$

then, the amplitude (2.78) is a perfectly valid, non-vanishing result for the three-point amplitude satisfying momentum conservation and the on-shell conditions, provided that the momenta are complex and all square brackets vanish. The most general form that a three-point amplitude in an arbitrary field theory can take is [38]

$$A_3 = A_3^H(\langle 12\rangle, \langle 23\rangle, \langle 31\rangle) + A_3^A([12], [23], [31]) \tag{2.79}$$

where the superscripts H and A refer to "holomorphic" and "antiholomorphic", respectively. For each of the two kinematic configurations, one should find that either the holomorphic or antiholomorphic term vanishes. As we will see shortly, Lorentz covariance of the amplitudes will allow us to determine the general form of A_3^H and A_3^A, up to multiplicative constants. Using the physical constraint that the complete three-point amplitude must vanish when the momenta is taken to be real, we will be able to determine, given the helicities of the the external particles, which part vanishes on each kinematic configuration.

2.4.2. Little group scaling

For massless particles, the bispinor representation can be realized through the completeness relation

$$\not{p} = |p\rangle[p| + |p]\langle p| \tag{2.80}$$

The physically meaningful quantity is the momentum, not the spinors. Hence, we can see that there is a redundancy in the bispinor decomposition of the momenta: (2.80) is invariant under the transformation $|p\rangle \to t|p\rangle$, $|p] \to t^{-1}|p]$. For real momenta, t is restricted to be a complex phase[6]; for complex momenta, t might be an arbitrary complex number. The origin of this redundancy can be traced to the little group, the subset of Lorentz transformations which leave the momentum of an on-shell particle invariant. For a massless particle whose four momentum can be

[6] In this case, the transformation of, for example, the angle spinor, implies the transformation of the square spinor through complex conjugation.

parametrized by its energy as $p^\mu = (E, 0, 0, E)$, the little group is the group $SO(2)$ of orthogonal transformations in the $x-y$ plane. Since $SO(2)$ is isomorphic to $U(1)$, the states of massless particles only pick up a phase under little group transformations. These transformations act on the one-particle states which define the irreducible representations of the Poincaré group. Assuming that amplitudes inherit the transformation properties of these one-particle states under little group transformations will allow us to obtain the values of three-point amplitudes in a variety of theories with massless particles.This means that, in order to find how the amplitudes scale under little group transformations, we assume that the only objects with non-trivial transformations are the external wavefunctions. Then, by definition of the scaling transformation, the fermion wavefunctions, which are the spinors, scale as t^{-2h}, where $h = \pm\frac{1}{2}$ denotes the helicity. Also, by inspection of 2.50, the polarization vectors also scale as t^{-2h}, where $h = \pm 1$ is the helicity of the vector boson. In general, an n-point amplitude of massless particles scales under little group transformation of the spinors for particle i, $|i\rangle \to |i'\rangle = t_i|i\rangle$, $|i] \to |i'] = t_i^{-1}|i]$ as

$$A_n\left(1, 2, ..., i', ..., n\right) = t_i^{-2h} A_n(1, 2, ..., i, ...n) \tag{2.81}$$

This scaling property will suffice to determine any three-point amplitude of massless particles, subject to the condition of complex momenta. This scaling property can be recast in terms of the action of the helicity operator,

$$\left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} - \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}\right) A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = -2h_i A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) \tag{2.82}$$

where h_i is the helicity of the i-th particle. Since the holomorphic term only depends on λ and the antiholomorphic part only depends on $\tilde{\lambda}$, this relation can be recast as the pair of equations

$$\left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + 2h_i\right) A_3^H = 0 \tag{2.83}$$

and

$$\left(\tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} - 2h_i\right) A_3^A = 0 \tag{2.84}$$

To find the solution to, say, (2.83), consider the special kinematics in which $[ij] = 0$. As an ansatz, the more general form that the amplitude can take is

$$A_3^H(1^{h_1}, 2^{h_2}, 3^{h_3}) = k_H \langle 12\rangle^{c_1} \langle 23\rangle^{c_2} \langle 31\rangle^{c_3} \tag{2.85}$$

where k_H is a scalar function of the momenta. This function can be fixed in terms of the coupling appearing in the interaction vertex of the three particles that form the external states of A_3, as we will argue later due to dimensional analysis. Hence, this procedure completely fixes the structure of the three-point amplitudes to all loop orders, up to quantum corrections to the coupling constants.

To determine the coefficients c_1, c_2 and c_3, we use our ansatz and Eq. (2.83). Consider, for example, the action of the helicity operator for particle 1. The derivative with respect to the spinor λ_1 will pick up a factor of $(c_1 + c_3)$, coming from the spinor products $\langle 12\rangle$ and $\langle 31\rangle$ in which this spinor appears. This yields the condition

$$-2h_1 = c_1 + c_3 \tag{2.86}$$

Similarly, by using the remaining equations, one obtain the equations

$$-2h_2 = c_1 + c_2, \quad -2h_3 = c_2 + c_3 \tag{2.87}$$

Then, we have obtained a simple linear system which can be solved for the unknown exponents c_1, c_2, c_3, which can be solved in terms of the helicities to yield

$$A_3^H(1^{h_1}, 2^{h_2}, 3^{h_3}) = k_H \langle 12\rangle^{h_3-h_1-h_2} \langle 23\rangle^{h_1-h_2-h_3} \langle 31\rangle^{h_2-h_1-h_3} \tag{2.88}$$

By making the substitution $\langle ij\rangle \to [ij]$ and $h_i \to -h_i$, we immediately obtain the antiholomorphic solution,

$$A_3^A(1^{h_1}, 2^{h_2}, 3^{h_3}) = k_A [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_1+h_3-h_2} \tag{2.89}$$

which implies that the exact three-particle amplitude is given by

$$A_3(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = k_H \langle 12\rangle^{d_3} \langle 23\rangle^{d_1} \langle 31\rangle^{d_2} + k_A [12]^{-d_3} [23]^{-d_1} [31]^{-d_2} \tag{2.90}$$

where $d_1 = h_1 - h_2 - h_3$, $d_2 = h_2 - h_1 - h_3$ and $d_3 = h_3 - h_1 - h_2$. To obtain the correct physical behaviour in the limit where the momenta are taken to be real, we must require that A_3 vanishes when both angle and square spinor products go to zero simultaneously. It is easy to see that the sum of the powers of the terms in the holomorphic part is $p_H = -h_1 - h_2 - h_3$, and $p_A = -p_H$ in the antiholomorphic part. Then, if $p_H > 0$, we must take $k_H = 0$ in order to avoid an inconsistent, divergent result, while $p_H < 0$ means we must take $k_A = 0$. The case $p_H = 0$ is special, since neither the holomorphic nor the antiholomorphic part can be discarded on these grounds. However, since we will not find amplitudes with such helicity configurations, we will simply omit such particular cases.

Now, we consider a few specific examples. If we take $h_1 = h_2 = -1$ and $h_3 = 1$, we should obtain the helicity amplitude for three spin one massless bosons. Note that $p_H = -1$; hence, we will obtain an holomorphic configuration. Inserting these values in (2.88) we obtain

$$\begin{aligned} A_3(g_1^-, g_2^-, g_3^+) &= k_H \langle 12\rangle^3 \langle 23\rangle^{-1} \langle 31\rangle^{-1} \\ &= k_H \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \\ &= k_H \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \end{aligned} \tag{2.91}$$

which is nothing more than the three-gluon MHV amplitude. Two comments are in order here. First, recall that, in four dimensions, an n-point amplitude must have mass dimension $(4 - n)$. Now, since the spinor products satisfy $\langle ij\rangle \sim \sqrt{s_{ij}}$, each spinor product has dimensions of mass. Therefore, we can see that the total mass dimensions of the three point amplitude in Eq.(2.91) is $[k_H] + 1$, where $[k_H]$ is the mass dimension of the constant k_H. Thus, in order for the amplitude to have the correct mass dimension, we must have $[k_H] = 0$. Since the only dimensionless parameter in Yang-Mills theory is the gauge coupling g, we conclude that k_H should be proportional to g at all orders in perturbation theory. This is a very strong statement about the general structure of the three-point amplitudes.

Also, note that the amplitude in Eq. (2.91) is completely antisymmetric under the exchange of two boson labels. However, observables involving bosons must be completely symmetric under the exchange of any pair of particle labels. This problem is solved in Yang-Mills theory if we assume that this three-particle amplitude has gone through a process of color decomposition, and we set the full amplitude to be $A_3(g_1^-, g_2^-, g_3^+)$ times a structure constant f_{abc}. However, we obtained this result only through Lorentz covariance, so there is no a priori information on the gauge group. What we have obtained is a very powerful result: Lorentz covariance implies that there cannot be an interacting theory of less than three massless particles of spin 1 (this result actually generalizes to odd spin), and that the coupling constant of these massless bosons must be completely antisymmetric in its indices.

Similarly, with $h_1 = -1/2$, $h_2 = -1$ and $h_3 = 1/2$, which corresponds to a quark-antiquark-gluon amplitude (the quarks must have opposite helicity) we obtain

$$\begin{aligned} A_3(q_1^-, g_2^-, \bar{q}_3^+) &= k_H \langle 12\rangle^2 \langle 23\rangle^0 \langle 31\rangle^{-1} \\ &= k_H \frac{\langle 12\rangle^2}{\langle 31\rangle} \\ &= k_H \frac{\langle 12\rangle^3 \langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \end{aligned} \tag{2.92}$$

which again reproduces the MHV formula with a quark-antiquark pair. Finally, before pointing out the subtleties of this approach, consider the case $h_1 = h_2 = -2$, $h_3 = 2$, which would be the MHV analog of the three graviton amplitude[7]. Denoting a graviton by G, we see that the net effect in comparison to the gluon case is to multiply every helicity by two. Hence,

$$A_3(G_1^-, G_2^-, G_3^+) = k_H \frac{\langle 12\rangle^8}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2} = (A_3(g_1^-, g_2^-, g_3^+))^2 \tag{2.93}$$

Apart from the fixing of the constant k_H, this is an example of the profound (and celebrated) statement *gravity* = (*gauge theory*)^2. In chapter 5, we will discuss two particular ways in which this statement is realized: the KLT relations, which have their origin in string theory and relate closed and open string amplitudes, and the double copy formulation obtained from the BCJ relations, which have their origin in the so-called color-kinematics duality. However, this does not take anything away from the surprising fact that the on-shell three point correlation functions of Einstein gravity are exactly equal to the square of the Yang-Mills case, for this is not at all obvious from either the Lagrangians or the Feynman rules of either theories.

[7] The wavefunctions of a graviton are tensor products of gauge boson helicity vectors, $\epsilon_\pm^{\mu\nu} = \epsilon_\pm^\mu \epsilon_\pm^\nu$

2.4.3. Unitarity and locality

We have used Lorentz invariance to determine the form of all possible three-point amplitudes that can be calculate in a relativistic field theory. This procedure provides an example of the constraints imposed by the physical properties of amplitudes in their kinematic structure. Now, we will consider the consequences of the unitarity of the S-matrix and the locality of interactions. This reasoning will lead us to conclude that the only singularities that a tree amplitude can possess are simple poles that occur when a propagator goes on-shell, and that the residue at each pole will be given in terms of products of lower point amplitudes, whose multiplicity is specified by the structure of the singular propagator under consideration.

As we mention in the introduction, the S-matrix is defined as the operator which gives the transition probability from one free state to another. Then, since probability must be conserved, the S-matrix has to be an unitary operator

$$S^{\dagger}S = 1 \tag{2.94}$$

where 1 is the identity operator in the space where we define the S-matrix. We can obtain constrains on the scattering amplitudes by looking at the conditions that the unitarity of the S-matrix imposes on the transfer matrix T, whose definition we recall

$$S = 1 + iT \tag{2.95}$$

hence, we find that the matrix T must satisfy the condition

$$-i(T - T^{\dagger}) = T^{\dagger}T \tag{2.96}$$

Now, consider two free states $\langle\phi_1|$ and $|\phi_2\rangle$ such that the total number of external particles is n. Then, we can calculate the matrix element of Eq.(2.96) between these states to obtain a statement involving amplitudes, namely

$$-i(\mathcal{A}_n(\phi_1 \to \phi_2) - \mathcal{A}_n^*(\phi_2 \to \phi_1)) = \langle\phi_1|T^{\dagger}T|\phi_2\rangle \tag{2.97}$$

we assume that, generically, the amplitudes depend on the momenta p_i of the external particles. Inserting a complete set of free states, we obtain

$$\begin{aligned}&-i(\mathcal{A}_n(\phi_1 \to \phi_2) - \mathcal{A}_n^*(\phi_2 \to \phi_1))\\ &= \sum_m \left(\prod_{i=1}^{m}\int \frac{d^4q_i}{(2\pi)^4}\delta^{(+)}(q_i^2)\right) A^*(\phi_1 \to \{q_i\})A(\phi_2 \to \{q_i\})\end{aligned} \tag{2.98}$$

where the sum goes over all possible sets of m final on-shell particles into which the states ϕ_1 and ϕ_2 can scatter into, and

$$\delta^{(+)}(q_i^2) = \theta(E_{q_i})\delta(q_i^2) \tag{2.99}$$

where $\theta(x)$ denotes the Heaviside step function. An important special case is when $|\phi_1\rangle = |\phi_2\rangle = |p_1p_2\rangle$ are both the same two particle state. Then, one can use the different kinematic integrations to reconstruct a cross-section of the form Eq. (1.8) on the right hand side of Eq. (2.98). On the left hand side, we obtain the imaginary part of the amplitude for the state $|p_1p_2\rangle$ to scatter onto itself. Then, we obtain the so-called optical theorem

$$\mathrm{Im}\, A_4(p_1p_2 \to p_1p_2) = 2E_{cm}p_{cm}\sigma_{tot}(p_1p_2 \to \ anything) \tag{2.100}$$

where E_{cm} and p_{cm} are the center of mass energy and momenta of the process. Here, the appearance of an imaginary part can be understood from the fact that we can assume evaluate amplitudes on complex momenta, as we have done in the previous section.

Now, due to crossing symmetry, we can move all the states into the out-state $\langle\phi_1|$. Then, we find the condition

$$\mathrm{Im}(\mathcal{A}_n) = \sum_m \left(\prod_{i=1}^m \int \frac{d^4q_i}{(2\pi)^4}\delta^{(+)}(q_i^2)\right)\mathcal{A}_{n_L+m}\mathcal{A}_{n_R+m} \tag{2.101}$$

where n_L,n_R count the number of particles in the in- and out- states, and are subject to the constraint $n_L + n_R = n$. This result allows for the calculation of the discontinuity of the n-point amplitude when the momentum invariant $P^2 = (\sum_L p_i)^2$ goes on-shell, where p_i label the momentum of the external particles and the sum over L means to sum only over the momenta that was originally in the state $|\phi_2\rangle$, when regarding the amplitude as a complex function of the momenta. Since we have a product of amplitudes in the right-hand side of Eq.(2.101), it relates amplitudes of different multiplicity and loop order. In particular, the discontinuities of one-loop amplitudes are determined by products of tree amplitudes, which means that, understanding tree amplitudes as classical quantities and loop amplitudes as quantum effects, unitarity implies that the structure of the quantum theory is determined by its classical counterpart.

Now, in order to derive the factorization properties of amplitudes near a multiparticle pole, we will use a result known as the polology theorem and the LSZ reduction formula. Its proofs can be found in the textbooks by Weinberg [10] and Schwartz [39].

Consider a theory of an interacting scalar field $\psi(x)$. Then, the most general observables we can compute in this theory are the momentum space correlation function

$$G_n(p_1, ..., p_n) = \int \prod_{j=1}^n d^4x_j e^{ip_j\cdot x_j}\langle\Omega|\mathcal{T}\{\psi(x_1)...\psi(x_n)\}|\Omega\rangle \tag{2.102}$$

where $|\Omega\rangle$ is the vacuum of the interacting theory. The momenta on which the correlation function depends on may be off-shell (but they do satisfy momentum conservation). The polology theorem states that, if there is a sum of the momenta $P^\mu = p_1^\mu...p_r^\mu$ for some $1 < r < n$ and if there is a one particle state $|\Psi\rangle$ of mass m such that $\langle\Psi|\psi(x_1)...\psi(x_r)|\Omega\rangle \neq 0$, then G_n will have a pole at $P^2 = m^2$ and the residue at this pole is given by the product of lower-point correlation functions. That is,

$$G_n(p_1,...,p_n) = (2\pi)^4\delta^4(\sum p)\frac{i}{P^2-m^2+i0}G_\Psi^{1,r}(G_\Psi^{r+1,n})^\dagger + \text{non-singular} \quad (2.103)$$

where "non-singular" refers to the remaining contributions, which are finite when P^μ goes on-shell, and $G_\Psi^{1,r}$ are the lower point correlation functions,

$$G_\Psi^{1,r} = \int\prod_{j=2}^{r} d^4x_j e^{ip_j\cdot x_j}\langle\Omega|\mathcal{T}\{\psi(0)\psi(x_2)...\psi(x_r)\}|\Omega\rangle \quad (2.104)$$

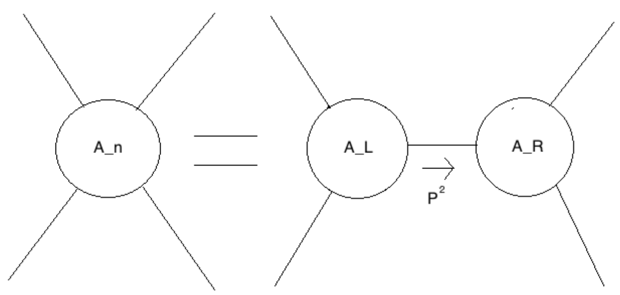
On the other hand, the LSZ reduction formula allow us to obtain amplitudes from correlation functions as

$$A_n(p_1,...,p_n) = i^n\int\prod_{k=1}^{n} d^4x_k e^{ip_k\cdot x_k}(\partial_k^2+m^2)\langle\Omega|\mathcal{T}\{\psi(x_1)...\psi(x_n)\}|\Omega\rangle \quad (2.105)$$

where we have written the formula for the case of scalar fields. Then, mixing these results, we find that near a multiparticle pole an n-point amplitude factorizes as (specializing to the case of gluons, which are massless),

$$A_n = A_{n_L+1}\frac{1}{P^2}A_{n_R+1} + \ finite \quad (2.106)$$

where by finite we mean terms that are not singular when $P^2 \to 0$. This can be represented pictorically as



This property of factorization is the manifestation of the unitary nature of the S-matrix and the locality of field interactions on the scattering amplitudes.

2.5. BCFW recursion formula

Now, we are in a position to derive the Britto-Cachazo-Feng-Witten (BCFW) recursion formula [9]. These are relations which construct n-point tree amplitudes as the sum of products of lower point amplitudes. Hence, finding all the amplitudes in a given theory for a fixed number of external particles, one can calculate recursively, in the number of external legs, a higher point amplitude. The idea is to perform a deformation of the on-shell momenta of the external particles parameterized by a complex number. Then, following our arguments on the analytic structure of tree amplitudes, we will be able to construct the contour integral of a rational function related to the amplitude whose residues will sum into the amplitude we desire. The poles of this function will include all poles of the amplitude and, because of unitarity, these poles will be simple and the residue at each pole will be the product of two lower point amplitudes.

Our first task is to see how to actually make the complex shifts of the momenta. To do this, we must ask ourselves: what constraints are put on the momenta of a scattering process? For massless scattering, we know that the external momenta must be on-shell, that is, $p_i^2 = 0$, and that momentum conservation must be satisfied. Hence, we should consider shifts that preserve these conditions. It turns out that the simplest way to do this is by shifting two momenta, say p_1 and p_n in the following manner

$$\hat{p}_1 = p_1 - zv, \quad \hat{p}_n = p_n + zv \tag{2.107}$$

where z is a complex number and v is a light-like four vector. Since $\hat{p}_1 + \hat{p}_n = p_1 + p_n$, momentum conservation is preserved for an arbitrary light-like v. However, imposing the on-shell condition $\hat{p}_1^2 = 0$ implies that v must also satisfy

$$v \cdot p_1 = 0 \tag{2.108}$$

and an equivalent relation with p_n. It turns out that there are two solutions to the constraints $v^2 = 0$ and $v \cdot p_1 = v \cdot p_n = 0$. Instead of trying to solve these equations, let us study these shifts from the perspective of the helicity spinors. Again, we choose p_1 and p_n, but let us think that we shift the spinors, as

$$\begin{aligned} \hat{\lambda}_1 &= \lambda_1 - z\lambda_n, \quad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \\ \hat{\lambda}_n &= \lambda_n, \quad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + z\tilde{\lambda} \end{aligned} \tag{2.109}$$

which can be translated to Dirac spinor notation as

$$\begin{aligned} |\hat{1}\rangle &= |1\rangle - z|n\rangle, \quad |\hat{1}] = |1] \\ |\hat{n}\rangle &= |n\rangle, \quad |\hat{n}] = |n] + z|1] \end{aligned} \tag{2.110}$$

Now, to obtain the momentum shifts associated with this spinor shifts, we use the relation

$$p^\mu = \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}} \;\; = \frac{1}{2}\langle p|\gamma^\mu|p] \tag{2.111}$$

to obtain

$$\begin{aligned} \hat{p}_1^\mu &= p_1^\mu - \frac{1}{2}z\langle n|\gamma^\mu|1] \\ \hat{p}_n^\mu &= p_n^\mu + \frac{1}{2}z\langle n|\gamma^\mu|1] \end{aligned} \tag{2.112}$$

and we identify $v^\mu = \frac{1}{2}z\langle n|\gamma^\mu|1]$. The fact that $v \cdot p_1 = v \cdot p_n = 0$ follows from the Dirac equation, and $v^2 = 0$ is a consequence of the Fierz identity.

Now, of what use is this complex shift? To answer this, consider a tree amplitude $A_n(p_1, \{p_i\}, p_n)$, where we denote by $\{p_i\}$ the remaining momenta on which the amplitude depends on. If we shift the momenta according to (2.112), we obtain a rational function of z which we will call $A_n(z)$. Setting $z = 0$ restores the physical values of the momenta, which implies that the amplitude we want to calculate is given by $A_n(0)$. So, let C denote the circle at infinity in the extended complex plane, and consider the integral

$$I = \oint_C \frac{dz}{2\pi i} \frac{A_n(z)}{z} \tag{2.113}$$

Using the residue theorem, we can see that

$$\oint_C \frac{dz}{2\pi i} \frac{A_n(z)}{z} = A_n(0) + \sum_{z_i} Res\left(\frac{A_n(z)}{z}, z_i\right) \tag{2.114}$$

where z_i are the poles of $A_n(z)$, and $A_n(0)$ is simply the pole at $z = 0$. If $A_n(z) \to 0$ as $|z| \to \infty$, the integral vanishes because there is no pole at infinity. Assuming this does happen, we find that

$$A_n(0) = -\sum_{z_i} Res\left(\frac{A_n(z)}{z}, z_i\right) \tag{2.115}$$

But, as we have argued, $A_n(0)$ is the physical amplitude we actually want to calculate. Thus, we need to find the value of the residues of $A_n(z)$. We will now focus on the case of gauge theory partial amplitudes.

As we have discussed, an amplitude develops a pole when a propagator goes on-shell. In the particular case of partial amplitudes, due to the fact that the external lines are ordered according to some permutation of $\{1, 2, ..., n\}$, these can only take the form

$$\frac{1}{(p_i + p_{i+1} + ... + p_j)^2} \tag{2.116}$$

that is, each propagator can only carry a sum of adjacent momenta. Moreover, these poles are simple. Hence, since the propagator which produce singularities in z necessarily has to carry either $\hat{p}_1(z)$ or $\hat{p}_n(z)$, these can only take the form

$$\begin{aligned} \frac{1}{\hat{P}_i(z)^2} &= \frac{1}{(\hat{p}_1(z) + p_2 + ... + p_{i-1})^2} = \frac{1}{(p_i + p_{i+1} + ... + \hat{p}_n(z))^2} \\ &= \frac{1}{P_i^2 - z\langle n|P_i|1]} \end{aligned} \tag{2.117}$$

where $P_i = p_1 + ... + p_{i-1}$ and $\langle n|P_i|1] = \lambda_{n\alpha} P_i^{\alpha\dot{\alpha}} \tilde{\lambda}_{1\dot{\alpha}}$, we see that $A_n(z)$ can only develop poles at the locations

$$z = z_i \equiv \frac{P_i^2}{\langle n|P_i|1]}, \quad i = 3, ..., n-1 \tag{2.118}$$

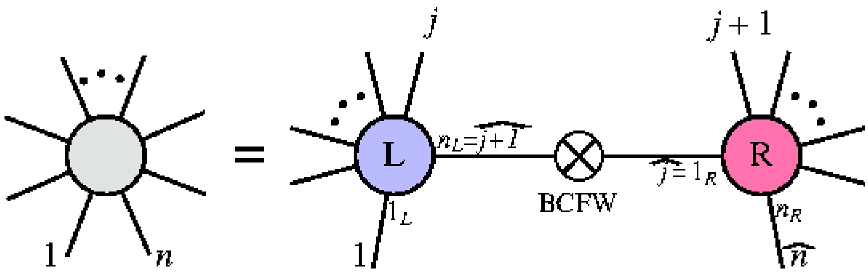


Fig. 2.5: Diagrammatic representation of the BCFW recursion

Now, consider the graphical representation in Fig. 2.5. In a given tree level color-ordered diagram, a propagator separates two diagrammatic structures. The one at the left is related to all the particles that appear before particle i in the cyclic ordering and the one at the right is related to all the particles appearing at the right in the cyclic ordering. If one isolates, for example, the structure to the left, it is not hard to convince oneself that the sum of all the "left structures" obtained when separating all diagrams contributing to the partial amplitude by the propagator i, one obtains all the possible diagrams that contribute to the $i-1$ particle partial amplitude whose on-shell states correspond to all particles at the left of i. Similarly, the same happens with the structure to the right. Therefore, this graphical picture implies that, near to a particular pole z_i, the amplitude factorizes as

$$A_n(z \sim z_i) \sim \tfrac{1}{z-z_i}\tfrac{-1}{\langle n|P_i|1]}\textstyle\sum_s A_L(\hat{1}(z_i), 2, ..., i-1, -\hat{P}^s(z_i))A_R(\hat{P}^{\bar{s}}(z_i), i, ..., n-1, \hat{n}(z_i)) \tag{2.119}$$

where the sum over s is over the helicities of the propagating particle between the left and right subamplitude, and $\bar{s} = -s$. For example, in QCD, the singular propagator may be associated either to a gluon or quark propagator going on-shell[8], and one must sum over these possible assignments. Also, one must sum over the helicities of the propagating particles. This is an example of the general factorization property we found by studying the implications of the unitarity of the S-matrix, resulting in Eq. (2.106). So, in pure Yang-Mills theory, it is a sum over $s = \pm 1$.

Now, if we consider our integrand $A_n(z)/z$ and recall the value of the poles z_i, we can see that (without writing the explicit argument of the amplitudes)

$$\begin{aligned}
\lim_{z\to z_i}\frac{A_n(z)}{z} &= \frac{1}{z-z_i}\frac{-1}{z_i\langle n|P_i|1]}\sum_s A_L^s(z_i)A_R^{\bar{s}}(z_i)\\
&= -\frac{1}{z-z_i}\sum_s A_L^s(z_i)\frac{1}{P_i^2}A_R^{\bar{s}}(z_i)
\end{aligned} \tag{2.120}$$

which implies that

$$Res\left(\frac{A_n(z)}{z}, z_i\right) = -\sum_s A_L^s(z_i)\frac{1}{P_i^2}A_R^{\bar{s}}(z_i) \tag{2.121}$$

With this, we finally obtain the BCFW recursion relation

[8]Quark propagators will appear only if there are external quarks in the process as well

$$A_n = \sum_{i=2}^{n-1} \sum_s A_L^s(z_i) \frac{1}{P_i^2} A_R^{\bar{s}}(z_i) \tag{2.122}$$

It should be emphasized that the $1/P^2$ term, although it has the algebraic form of a propagator, is evaluated at on-shell kinematics, and does not carry any off-shell degrees of freedom associated to the propagation of virtual particles. Basically, this formula implies that any tree level amplitude can be constructed from the knowledge of the 3-point amplitudes, which we already calculated based on Lorentz covariance and locality. In the context of Yang-Mills theory, this seems to imply that only the three gluon vertex is needed to calculate the scattering amplitudes of the theory. Although in a sense this is true, the amplitudes must be gauge invariant, and the four gluon vertex is needed, in order to preserve gauge invariance of the Yang-Mills Lagrangian. However, from the point of view of the on-shell amplitudes, the three gluon vertex captures all the information needed to construct the tree S-matrix of the theory. In Appendix E, we apply the BCFW relations to prove the Parke-Taylor formula.

By this point, we have seen how to organize the different degrees of freedom of scattering amplitudes in four dimensions, focusing on the amplitudes of theories with a gauge group. First, color decomposition allow us to separate the gauge group information from the actual dynamics of the theories. As a second step, we introduce the spinor-helicity variables to parametrize the momentum dependence of the amplitudes, and show how these allow us to obtain highly compact expressions for amplitudes that, in principle, are very complicated to calculate (such as the MHV amplitudes at high multiplicity). Finally, we have shown how Lorentz invariance determines all possible three-point amplitudes and that the unitarity and locality of the S-matrix, allow us to construct tree amplitudes of arbitrary multiplicity by using three-point amplitudes as building blocks in a recursive approach. Now, we will go a step further into another aspect of gauge theory amplitudes, which will be the final subject of this chapter: their supersymmetric extensions.

2.6. Supersymmetric gauge theories

The Coleman-Mandula theorem [40] puts severe restrictions on the possible symmetries of any relativistic, interacting field theory: it states that the Lie group symmetries must be a direct product between the Poincaré group and an internal symmetry group, such as isospin. This implies, in other words, that the only conserved quantities that can appear with Lorentz indices are the Poincaré generators, P^μ for translations and $M^{\mu\nu}$ for rotations and boosts.

However, the theorem does not rule out symmetry transformations whose conserved charges carry spinor indices; these are the so-called supersymmetry transformations. One particular property of a supersymmetry transformation is that it transforms bosonic fields (scalars or vector bosons) into fermionic fields and vice versa. Furthermore, the infinitesimal parameter of a supersymmetry transformation is a Grassmann or anticommuting variable.

The supersymmetry generators, $Q_\alpha, Q^\dagger_{\dot\beta}$, are defined to satisfy

$$\begin{aligned}\{Q_\alpha, Q^\dagger_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2P_{\alpha\dot\beta} \\ \{Q_\alpha, Q_\beta\} &= 0 \\ \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} &= 0\end{aligned} \tag{2.123}$$

where P_μ is the momentum operator, and $\{,\}$ denotes an anticommutator. One can increase the number of supersymmetry generators by considering an internal symmetry group (which we will always take to be $SU(\mathcal{N})$) and use the set of generators

$$Q^A_\alpha, (Q^B_{\dot\beta})^\dagger, \quad A, B = 1, 2, ..., \mathcal{N} \tag{2.124}$$

which satisfy

$$\{Q^A_\alpha, (Q^B_{\dot\beta})^\dagger\} = 2\delta^{AB} P_{\alpha\dot\beta} \tag{2.125}$$

this is known as extended supersymmetry. In four dimensions, the maximum value of $\mathcal{N}$ that we can consider in order to avoid states of helicity higher than 1 is $\mathcal{N} = 4$ [41]. We will discuss this maximally supersymmetric Yang-Mills theory, or simply $\mathcal{N} = 4$ super Yang-Mills, at the end of this section

Since supersymmetry mixes the bosonic and fermionic degrees of freedom of the field theory under consideration, one necessary (but not sufficient) condition for a theory to be supersymmetric is that the number of bosonic degrees of freedom equals the fermionic degrees of freedom. These requirement introduces the concept of a supermultiplet, which is a multiplet of bosonic and fermionic states that are connected through supersymmetry transformations. Following [42], in order to construct these multiples, we first consider the case of a massive state $|s\rangle$. In the rest frame, where

$$P_\mu = (m, 0, 0, 0) \tag{2.126}$$

the non-trivial anticommutation relations 2.123 read

$$\{Q_\alpha, Q^\dagger_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta} \tag{2.127}$$

Now, if we assume that $Q_\alpha|s\rangle = 0$, we can easily construct the representation of the supersymmetry algebra by acting on the state $|a\rangle$ as

$$|a\rangle, \quad Q^\dagger_1|a\rangle, \quad Q^\dagger_2|a\rangle, \quad Q^\dagger_1 Q^\dagger_2|a\rangle \tag{2.128}$$

that is, in the same way as the representations of two anticommuting annihilation operators. If we assume that $|a\rangle$ represents a spin-j particle, the states $Q^\dagger_\beta|a\rangle$ have spin $j \pm \frac{1}{2}$ if $j \neq 0$ while for $j = 0$ they must have spin $\frac{1}{2}$, Finally, the state $Q^\dagger_1 Q^\dagger_2|a\rangle$ has the same spin as $|a\rangle$, since it transforms as a singlet under right-handed rotations. This procedure then yields $(2j - \frac{1}{2} + 1$ and $2j + \frac{1}{2} + 1$ fermionic (Weyl) states corresponding to the $2(2j + 1)$ bosonic states, for $j \neq 0$, and a set of two scalar plus two Weyl fermion states for $j = 0$. In particular, this implies that, in order to construct a consistent supersymmetric theory which includes scalar fields, they must be complex.

For massless states, we take the momentum to be in the frame

$$P_\mu = (E, 0, 0, E) \tag{2.129}$$

where E is the total energy of the particle. Then,

$$\{Q_\alpha, Q^\dagger_\beta\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.130}$$

which implies that Q_2 and $Q^\dagger_2$ vanish for any massless representation. The only possible states we can construct are then

$$|b\rangle, \quad Q^\dagger_1|b\rangle \tag{2.131}$$

and, if $|b\rangle$ has helicity λ then $Q^\dagger_1|b\rangle$ has helicity $\lambda + \frac{1}{2}$. In order to preserve CPT invariance, one must introduce a conjugate multiplet with helicities $\{-\lambda, -\lambda - \frac{1}{2}\}$. A particular example of a massless multiplet, which will be important for us, is the vector multiplet with helicites

$$\lambda = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\} \tag{2.132}$$

On the other hand, we are interested in gauge theories. If we want to construct a gauge theory with supersymmetry, the complete field content of the theory must transform in the same representation of the gauge group, which, due to the construction of the gauge field, will always be the adjoint representation. Hence, QCD cannot be supersymmetric, since quarks transform in the fundamental representation of the gauge group. We will first study the simplest supersymmetric gauge theory, $\mathcal{N} = 1$ Super Yang-Mills (SYM) theory.

2.6.1. $\mathcal{N} = 1$ SYM

In order for us to obtain a supersymmetric gauge theory, there must be both bosonic and fermionic degrees of freedom, and all of these must transform in the adjoint representation of the gauge group. Hence, consider a gauge group $G = SU(N)$. The $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in four dimensions can be defined by the Lagrangian

$$\mathcal{L}_{\mathcal{N}=1} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + i\lambda^{a\dagger} \bar{\sigma}^\mu D_\mu \lambda^a \tag{2.133}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the field strength, λ^a is a massless Weyl spinor transforming in the adjoint representation and the covariant derivative acts on λ as $D_\mu \lambda^a = \partial_\mu \lambda^a + g f^{abc} A^b_\mu \lambda^c$.

The spectrum of this theory is composed of the vector multiplet, Eq. (2.132). The particle associated to this fermionic field is generically known as a Gaugino. We will refer to them as gluinos, which are the superpartners of the gluons.

From this point of view, it is necessary that λ satisfies the Weyl condition; that is,

$$\lambda = \pm\gamma_5 \lambda \tag{2.134}$$

this is because the vector field A_μ carries only the two transversal, physical degrees of freedom, while a general spinor in four dimensions has four degrees of freedom. The Weyl condition reduces the degrees of freedom of the spinor field by a factor of two, and hence we have the same number of bosonic and fermionic degrees of freedom, which are required for supersymmetry. In four dimensions, one could also impose the Majorana condition

$$\lambda = C\bar{\lambda}^T \tag{2.135}$$

where C is the charge conjugation operator, which also halves the number of fermionic degrees of freedom. However, Weyl spinors are defined to be chirality eigenstates, and since they must be massless for the theory to be supersymmetric, they naturally describe helicity eigenstates, which makes them particularly well suited for the study of helicity amplitudes.

Supersymmetry Ward identities

Upon canonical quantization, the free fields can be written as plane wave expansions, in which the coefficients are creation and annihilation operators. Since supersymmetry transforms bosonic fields into fermionic fields and vice-verse, it is natural to expect that the supersymmetry generators $Q, Q^\dagger = \tilde{Q}$ have a non-trivial action on the ladder operators. In fact, if we let $a_\pm(p)$ be the operator that annihilates a ± 1 helicity gluon with momentum p and $b_\pm(p)$ be the operator that annihilates a $\pm\frac{1}{2}$ helicity gluino, using the explicit form of the supersymmetry transformation, one obtains

$$\begin{aligned}
&[\tilde{Q}, a_+(p_i)]_\pm = 0, && [Q, a_+(p_i)]_\pm = [i|b_+(p_i) \\
&[\tilde{Q}, b_+(p_i)]_\pm = |i\rangle a_+(p_i), && [Q, b_+(p_i)] = 0 \\
&[\tilde{Q}, b_-(p_i)]_\pm = 0, && [Q, b_-(p_i)]_\pm = [i|a_-(p_i) \\
&[\tilde{Q}, a_-(p_i)]_\pm = |i\rangle b_-(p_i), && [Q, a_-(p_i)] = 0
\end{aligned} \tag{2.136}$$

where $[\cdot,\cdot]_\pm$ denotes a graded commutator, which is an anticommutator when both arguments are fermionic, and is a commutator otherwise, and $|i\rangle$, $[i|$ denote helicity spinors. From this, we can see that $\tilde{Q}$ increases the helicity by $+\frac{1}{2}$ and Q decreases the helicity by $-\frac{1}{2}$.

We can define an n-point amplitude as the vacuum expectation value of n annihilation operators acting on the out bra vacuum state $\langle 0|$, with the S-matrix operator $\hat{S}$ acting on the in ket vacuum state $|0\rangle$,

$$A_n = \langle 0|\mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\hat{S}|0\rangle \tag{2.137}$$

where $\mathcal{O}_i(p_i)$ denotes the annihilation operator associated to a particle of momentum p_i. If we assume that the vacuum is supersymmetric, i.e. $Q|0\rangle = \tilde{Q}|0\rangle = 0$ and that the supersymmetry generators commute with the S-matrix, We find that

$$\begin{aligned}
0 &= \langle 0|[\tilde{Q}, \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\hat{S}]_{pm}|0\rangle \\
&= \sum_{i=1}^{n}(-1)^{\sum_{j<i}|\mathcal{O}_j|}\langle 0|\mathcal{O}_1(p_1)...[\tilde{Q}, \mathcal{O}_i(p_i)]_\pm...\mathcal{O}_n(p_n)\hat{S}|0\rangle
\end{aligned} \tag{2.138}$$

where $|\mathcal{O}_j$ is 0 when it is associated to a bosonic operator and -1 when it is associated to a fermionic operator. Interpreting each of the terms in the sum as an amplitude, we obtain linear relations, valid to all orders in perturbation theory, between amplitudes with different external states that are linked by supersymmetry transformations. These relations are known as supersymmetric Ward identities. For example, letting $\mathcal{O}_i(p_i) = a_+(p_i)$ for $i \neq 1$ and $\mathcal{O}_1(p_1) = b_+(p_1)$, we can use the commutation relations (2.136) to obtain

$$\begin{aligned} 0 &= \langle 0|[\tilde{Q}, b_+(p_1)a_+(p_2)...a_+(p_n)\hat{S}]_\pm|0\rangle \\ &= \langle 0| \left(|1\rangle a_+(p_1)...a_+(p_n)\hat{S} + b_+(p_1)[\tilde{Q}, b_+(p_1)a_+(p_2)...a_+(p_n)\hat{S}]_\pm\right)|0\rangle \\ &= |1\rangle\langle 0|a_+(p_1)...a_+(p_n)\hat{S}|0\rangle \\ &= |1\rangle A_n(g_1^+...g_n^+) \end{aligned} \tag{2.139}$$

from which we can see that the all-plus gluon amplitude vanishes. Since the SUSY Ward identities are independent from perturbation theory, we can see that in $\mathcal{N}=1$ SYM, the all-plus gluon amplitudes vanishes to all orders in perturbation theory. From our analysis of the polarization vectors in spinor-helicity variables, we expected this to happen at tree-level; since the coupling of gluons to gluinos is cubic, there cannot be internal gluinos propagating on any diagram at tree-level. Hence, this implies that gluon tree amplitudes are equal in pure Yang-Mills theory, QCD and $\mathcal{N} = 1$ SUSY. In fact, gluons can only have cubic or quartic couplings with scalars, which means that tree amplitudes whose external states are only gluons are equal in any Yang-Mills theory with the same gauge group, with any degree of supersymmetry. However, the cancellations that occur due to supersymmetry are far more powerful than the simple relations found at tree-level in Yang-Mills. Similarly, taking $\mathcal{O}_i(p_i) = a_+(p_i)$ for $i \neq 1, 2$, $\mathcal{O}_1(p_1) = a_-(p_1)$ and $\mathcal{O}_2(p_2) = b_+(p_2)$, we find

$$\begin{aligned} 0 &= \langle 0|[\tilde{Q}, a_-(p_1)b_+(p_2)a_+(p_3)...a_+(p_n)\hat{S}]_\pm|0\rangle \\ &= |1\rangle A_n(\lambda_1^-\lambda_2^+g_3^+...g_n^+) + \langle 0|a_-(p_1)[\tilde{Q}, b_+(p_2)a_+(p_3)...a_+(p_n)\hat{S}]_\pm|0\rangle \\ &= |1\rangle A_n(\lambda_1^-\lambda_2^+g_3^+...g_n^+) + |2\rangle A_n(g_1^-g_2^+...g_n^+) \end{aligned} \tag{2.140}$$

from which, "dotting" the spinors $\langle 1|$ or $\langle 2|$, we obtain, respectively

$$A_n(\lambda_1^-\lambda_2^+g_3^+...g_n^+) = A_n(g_1^-g_2^+...g_n^+) = 0 \tag{2.141}$$

which, again, is one of the cancellations we found from gluon amplitudes at tree level. Moreover, since gluons have only cubic couplings with both quarks and gluinos, the only difference between QCD gluon-quark and $\mathcal{N} = 1$ SUSY gluon-gluino amplitudes, at tree-level, is in their color factors: the $gq\bar{q}$ vertex carries a generator matrix T^a_{ij}, while the $g\lambda\bar{\lambda}$ vertex carries a structure constant f^{abc}. Hence, after color decomposition, not only are gluon amplitudes in both QCD and $\mathcal{N} = 1$ SUSY equal at tree-level, but gluon-quark and gluon-gluino amplitudes are equal as well. Hence, one can consider that massless QCD is effectively supersymmetric at tree-level.

2.6.2. $\mathcal{N} = 4$ SYM

A possible way towards obtaining other supersymmetric gauge theories is to consider the generalization of the Lagrangian 2.133 to space-time dimensions $D \neq 4$. For general D, the number of degrees of freedom of the massless vector field is $(D-2)$, while a Dirac spinor in D dimensions has $2^{\frac{D}{2}}$ degrees of freedom. Hence, different supersymmetric theories can be constructed in different number of dimensions by reducing the fermionic degrees of freedom with either Majorana or Weyl conditions or by including scalar field to increase the bosonic degrees of freedom. One possible case is to take $D = 10$. Here, both the Majorana and Weyl conditions can be applied to the spinor field, and one obtains exactly eight bosonic and fermionic degrees of freedom. This yields a supersymmetric theory in ten dimensions, with action

$$S = \int d^{10}x \left(-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + i\lambda^{a\dagger} \bar{\sigma}^\mu D_\mu \lambda^a \right) \tag{2.142}$$

In [41] [43] , it is shown that performing a dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ gauge theory to four dimensions, one obtains what is known as maximally supersymmetric Yang-Mills theory, or simply $\mathcal{N} = 4$ super Yang-Mills; it is known as maximal because it has the greatest number of possible supersymmetry generators that don't require the addition of particles with spin greater than one. Denoting with Γ^I the ten-dimensional representation of the Clifford algebra, the action of $\mathcal{N} = 4$ SUSY can be written as

$$S = \int d^4x \Big[-\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \tfrac{1}{2}(D_\mu \phi^a_I)^2 + \tfrac{i}{2} \bar{\psi}^a \not{D} \psi^a + \tfrac{g}{2} \bar{\psi}^a \Gamma^I [\phi_I, \psi]^a + \tfrac{g^2}{4}([\phi_I, \phi_J]^a)^2 \Big] \tag{2.143}$$

where all fields transform in the adjoint representation of the gauge group, ψ^a are Majorana-Weyl spinors and ϕ^a_I is a set of real scalar fields with an $SO(6)$ interal symmetry index $I = 1, ..., 6$ that allows to obtain a Lorentz invariant coupling of the extra components of the Clifford algebra that remain after performing the dimensional reduction. These six scalars can also be arranged in the antisymmetric two-index representation of $SU(4)$, $\varphi^{AB} = -\varphi^{BA}$ subject to the self-duality condition $\bar{\varphi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\varphi^{CD}$. This simplifies the description of the theory, because the $\mathcal{N} = 4$ supersymmetry generators transform in the fundamental representation of $SU(4)$.

On-shell superspace, superfields and superamplitudes

The spectrum of $\mathcal{N} = 4$ SUSY can be organized in terms of the representation of $SU(4)$ under which they transform. Specifically, we have a positive helicity gluon g^+ transforming in the trivial representation and a negative helicity gluon g^- transforming in the totally antisymmetric four index representation; the positive helicity gluinos λ^A that transform in the $\mathbf{4}$ representation and the negative helicity gluinos $\lambda^{ABC} \sim \bar{\lambda}_D$ transforming $\bar{\mathbf{4}}$ representation, and six scalars S^{AB} transforming in the totally antisymmetric 2-index representation. These states can all be organized in what is known as a superfield. In order to do this, one introduces the Grassmann variables η_A which transform in the fundamental representation of $SU(4)$. The superfield is defined as

$$\Omega = g^+ + \eta_A\lambda^A - \frac{1}{2!}\eta_A\eta_B S^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C\lambda^{ABC} + \eta_1\eta_2\eta_3\eta_4 g^- \tag{2.144}$$

in the previous equation, one must think of each state, say g^+ or S^{AB} as representing the creation or annihilation operator associated to each external particle. Hence, what we are doing essentially is to introduce the Grassmann variables η_A in order to describe the states of the theory in terms of $SU(4)$ invariants. Moreover, assigning helicity $+\frac{1}{2}$ to all the Grassmann variables, we can see that the superfield have uniform helicity $+1$. The signs in each of the term of the superfield are chosen so that the differential operators $1, \partial^A, \partial^A\partial^B, \partial^A\partial^B\partial^C, \partial^1\partial^2\partial^3\partial^4$, acting on the superfield, extract the correct component with the corresponding number of indices after setting $\eta_A = 0$.

The supersymmetry generators (2.125) can be realized in terms of spinors and Grassmann variables as

$$q^{Aa} = [p|^a \frac{\partial}{\partial\eta_A}, \quad q_A^{\dagger\dot{a}} = |p\rangle^{\dot{a}}\eta_A \tag{2.145}$$

These realization of the supersymmetry generators, along with the superfield description of the states, is known as the on-shell superspace of $\mathcal{N} = 4$ SUSY.

The most important use of superfields is that the allow us to define objects known as superamplitudes. If we let $\Omega_i = \Omega(p_i)$ be a superfield (or superwavefunction) associated to the $i - th$ external particle, we can define a superamplitude as

$$\mathcal{A}_n^S[\Omega_1, ..., \Omega_n] \equiv \langle 0|\Omega_1, ..., \Omega_n S|0\rangle \tag{2.146}$$

The object $\mathcal{A}_n^S$ is then a function of the on-shell momenta p_i of the external particles and the Grassmann variables associated to each superfield, η_{iA}. In particular, the superamplitude will be a polynomial in the Grassmann variables, and the coefficients in the expansion of the superamplitude in terms of the Grassmannians are the amplitudes whose external states come from the spectrum of $\mathcal{N} = 4$ SUSY; these are also known as component amplitudes, and correspond to the amplitudes we have been calculating and whose properties we have studied so far. Moreover, in order to have external states that combine to a $SU(4)$ singlet, the Grassmann polynomial whose coefficient is an $N^K MHV$ amplitude must be of degree $4(K+2)$.

Component amplitudes can be obtained from the superamplitudes using derivative operators; for example, the MHV gluon amplitude can be calculated from the superamplitude as

$$A_n[g_1^+, ..., g_i^-, ..., g_j^-, ...g_n^+] = \left(\prod_{A=1}^{4} \frac{\partial}{\partial\eta_{iA}}\right)\left(\prod_{B=1}^{4} \frac{\partial}{\partial\eta_{jB}}\right) \mathcal{A}_n^S[\Omega_1, ..., \Omega_n]_{\eta_{kc}=0} \tag{2.147}$$

The fact that the theory is supersymmetric is reflected on the superamplitudes by their vanishing under the action of the supercharges

$$
\begin{aligned}
Q^A &= \sum_{i=1}^{n} q_i^A = \sum_{i=1}^{n} [i| \frac{\partial}{\partial \eta_A} \\
\tilde{Q}_A &= \sum_{i=1}^{n} q_{iA}^{\dagger} = \sum_{i=1}^{n} |i\rangle \eta_{iA}
\end{aligned}
\tag{2.148}
$$

Similarly to the way in which the Lorentz invariance of regular amplitudes is implemented through a momentum conservation delta function, the vanishing of the superamplitudes under the action of the supercharges can be implemented by defining a supermomentum conservation delta function

$$
\begin{aligned}
\delta^8(\tilde{Q}) &= \frac{1}{2^4} \prod_{A=1}^{4} \tilde{Q}_{A\dot{a}} \tilde{Q}_A^{\dot{a}} \\
&= \frac{1}{2^4} \prod_{A=1}^{4} \sum_{i,j=1}^{n} \langle ij \rangle \eta_{iA} \eta_{jA}
\end{aligned}
\tag{2.149}
$$

Then, since this delta function already has degree two in the Grassmann variables, the term of the superamplitude which corresponds to the $N^K MHV$ sector can be written as

$$
\mathcal{A}_n^{S\, N^K MHV} = \delta^8(\tilde{Q}) P_n^{(4K)} \tag{2.150}
$$

where $P_n^{(4K)}$ is a degree $4K$ polynomial in η_{iA}. For example, the superamplitude associated to the MHV amplitudes is given by

$$
\mathcal{A}_n^{S\, MHV}(1, ..., n) = \frac{\delta^8(\tilde{Q})}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle} \tag{2.151}
$$

Finding the superamplitude would allow one to calculate any amplitude in the theory by calculating derivatives with respect to the Grassmann variables. In Chapter 4, we will introduce the so-called connected formalism, which allows for the calculation of the superamplitude in terms of integrals localized on the solutions to rational equations, which happen to be exactly the four dimensional version of the scattering equations.

The scattering equations

Let Φ_n be the momentum configuration space of n on-shell massless particles,

$$\Phi_n = \{(p_1, p_2, ..., p_n) \in (\mathbb{C}M)^n | \sum_i p_i = 0,\, p_i^2 = 0,\, \forall i \in \{1, 2, ...n\}\} \tag{3.1}$$

where $\mathbb{C}M$ is the space of complex-valued, D-dimensional momenta. Using the elements of this configuration space, we can construct the Mandelstam invariants $S_{i_1...i_r} = (p_{i_1} + ... + p_{i_r})^2$, which are the variables that the Lorentz-invariant scattering amplitudes will depend on.

In the previous chapter, we have studied various properties of scattering amplitudes in four dimensional spacetime. We found that, for massless particles, we can define the spinor-helicity variables to encode all the kinematic dependence of the amplitudes. This is particular to four dimensions, because the little group of massless particles, $SO(2)$, is an Abelian group and it is straightforward to define quantities with definite transformation properties under the action of the little group (which amounts to knowing how the states change under Lorentz transformations). Furthermore, singularities of the amplitudes, which give the information on their analytic structure, are captured in the spinor products.

Recall the Parke-Taylor formula, Eq. (2.62),

$$A_n(1^- ... i^+ ... j^+ ... n^-) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle (n-1)n \rangle \langle n1 \rangle} \tag{3.2}$$

and consider the following parametrization of the spinors, $\lambda_{i\alpha} = (z_i, 1)$, which is always possible due to little group scaling. In these special variables, the spinor products take a particularly simple form

$$\langle ij \rangle = z_i - z_j \equiv z_{ij} \tag{3.3}$$

The momenta associated to these spinors is then written as

$$k_i^\mu = \frac{1}{2}(1 + |z_i|^2, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - |z_i|^2) \tag{3.4}$$

which is easily seen to satisfy the on-shell condition $k_i^2 = 0$. Choosing the reference spinors to have a similar form, that is, $\lambda_{q\alpha} = (z_q, 1)$, the polarization vectors can be written similarly, for example

$$\epsilon^{\mu}_{i,+} = \frac{1}{2}(1 + z_q\bar{z}_i, z_q + \bar{z}_i, -i(z_q - \bar{z}_i), 1 - z_q\bar{z}_i) \tag{3.5}$$

In these variables, the Parke-Taylor formula takes a particularly simple form

$$A_n(1^- ... i^+ ... j^+ ... n^-) = \frac{z_{ij}^4}{z_{12}z_{23}...z_{n1}} \tag{3.6}$$

Then, we can think of the amplitude as a function of several complex variables on $\mathbb{C}^n$. In terms of the z variables, the statement of momentum conservation becomes

$$\sum_i z_i = 0 \tag{3.7}$$

that means that the amplitude is defined on the hyperplane of $\mathbb{C}^n$ defined by Eq.(3.7). Moreover, from the denominator structure of Eq.(3.5), we can see that the poles of the amplitude occur when $z_k \to z_m$ for $k \neq m$, and the associated residue would be the Parke-Taylor amplitude for $(n-1)$ particles, defined in $\mathbb{C}^{n-1}$, along the deformed hyperplane

$$2z_m + \sum_{i \neq k,m} z_i = 0 \tag{3.8}$$

This is a very simple characterization of the singularities that the amplitudes possesses. However, MHV amplitudes have the particular property that they only have two-particle poles: that is, their only singularities occur when a kinematic invariant s_{ij} goes on-shell. However, more complicated amplitudes can have multi-particle poles, associated to the invariant $s_{i_1...i_k}$ going on-shell. Characterizing the singularities of an amplitude is difficult in terms of the Mandelstam invariants; however, we have seen that the variables z_i provide a simple way in which we can describe the singularity structure of scattering amplitudes. The natural question that arises is if it is possible to find a set of variables that enjoy similar properties to those that the z_i have in four dimensions, but in arbitrary spacetime dimensions, and how they are related to the space of kinematic invariants. In this chapter, we introduce the scattering equations, which provide a map from Φ_n to the moduli space of n-punctured Riemann spheres and show that they encapsulate all the physical properties of amplitudes in terms of the punctures over the Riemann sphere, through the study of their singularities.

3.1. Mapping of momentum space to the Riemann sphere

Consider a Riemann sphere, the set defined by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A puncture is a specific point that is taken out of a set. When we talk about n-punctured Riemann spheres, we refer to the set $\hat{\mathbb{C}}/\{z_1, ..., z_n\}$ for some specific choice of the points $z_1, ..., z_n$. Thus, a punctured sphere is defined by the position of its punctures. In order to find a map from the momentum configuration space to the Riemann sphere, we must find a function that takes momentum invariants into the punctures. One simple way to this is given by Cauchy's theorem,

$$p_i^\mu = \frac{1}{2\pi i}\oint_{|z-z_i|<\delta} dz\, \omega^\mu \tag{3.9}$$

where $i = 1, 2, ...n$, labels the external momenta. Here, δ defines a contour on $\hat{\mathbb{C}}$ that surrounds the puncture z_i and,

$$\omega^\mu = \sum_{i=1}^{n} \frac{p_i^\mu}{z - z_i} \tag{3.10}$$

The map provided by Eq. (3.9) would not work if it had a pole at infinity, because its residue at $z = z_i$ would not be p_i. However, by deforming the contour, one can see that the residue at infinity is simply the sum of all the residues at the n punctures, which is equal to the sum of the external momenta. This sum, of course, vanishes due to momentum conservation. On the other hand, each external momenta must satisfy the on-shell condition $p_i^2 = 0$; this allows one to obtain a constraint on ω^μ. We can calculate the contraction

$$\begin{aligned} F &= \omega^\mu \omega_\mu \\ &= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{p_i \cdot p_j}{(z - z_i)(z - z_j)} \end{aligned} \tag{3.11}$$

Then, since $F(z)$ is proportional to p_i^2, it must vanish everywhere; in order for this to happen, all its residues must vanish, and we can see that F has both simple and double poles. The residue at a double pole $z = z_i$ is simply p_i^2, which is zero because of the on-shell conditions. The absence of simple poles yields a non-trivial constraint,

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\oint_{|z-z_i|\leq\delta} dz F(z) \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{p_i \cdot p_j}{z_i - z_j} \end{aligned} \tag{3.12}$$

which must hold for $1 \leq i \leq n$. Hence, we define

$$f_i(z, p) = \sum_{j\neq i} \frac{s_{ij}}{z_{ij}} \tag{3.13}$$

where $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$ and $z_{ij} = z_i - z_j$. For a fixed set of momenta, that is, for a fixed n-tuple $p \in \phi_n$, the scattering equations are the system of equations which comes from the constraint of having vanishing residue at the simple poles, and are defined by

$$f_i(z, p) = 0,\ 1 \leq i \leq n \tag{3.14}$$

The scattering equations were first introduced in [44] by Fairlie, while looking for an alternative way to calculate amplitudes in the so-called dual models (what would after some time become string theory) in the Veneziano model such that it would be free of tachyons. Afterwards, they appeared in the work of Gross and Mende [45], [46] as the solutions to the saddle-point integrals that govern high-energy string scattering. The system (3.14) was given the name of the *scattering equations* by Cachazo, He and Yuan [11], who found them to generalize the RSV formula constraints to arbitrary spacetime dimension, and their properties were subsequently studied in the series of papers [12], [13], [14] by the same authors.

The scattering equations form the backbone of the so-called CHY representation of tree-level amplitudes, which is our main object of study. The idea of the CHY formalism is to write closed formulas for amplitudes in arbitrary dimension, localized to the solutions of the scattering equations. This is because, as we will show, the scattering equations contain all the information on the analytic structure of the scattering amplitudes and have factorization properties that mirror, in a sense, those of the amplitudes. However, we must understand in what sense the equations "factorize". We will also explore some its properties as well as its solutions. But before, let us see how can be generalized to the case of massive momenta. Also, we will see the form the scattering equations take in four dimensions, in terms of helicity spinors.

Generalizations to massive particles

Now, assume that the on-shell momenta of the external particles may be massive, that is, $p_i^2 = m_i^2$. In this case, the Mandelstam invariants s_{ij} do not take the simple form of the product $2p_i \cdot p_j$, which have implicitly used throughout our previous treatment of the scattering equations. This means that the numerator of each term in the scattering function $f_i(z,p)$ must be modified, in order to account for the fact that the momenta may not be massless. As first proposed in [47], one possible generalization of the scattering equations is defined by setting

$$f_i^{mass}(z,p) = \sum_{j \neq i} \frac{2p_i \cdot p_j + \Delta_{ij}}{z_{ij}} = 0 \tag{3.15}$$

where the constants Δ_{ij} are symmetric with respect to its indices, $\Delta_{ij} = \Delta_{ji}$ and satisfy the constraints

$$\sum_{j \neq i} \Delta_{ij} = 2m_i^2 \tag{3.16}$$

There is no unique way to choose the constants Δ_{ij}. One particular election of these constants, which we will use when discussing the CHY representation of QCD amplitudes, is to take

$$\Delta_{q_i \bar{q}_i} = m_{q_i}^2 \tag{3.17}$$

and the remaining constants equal to zero. Here q_i denotes the label of a quark with mass m_{q_i} and $\bar{q}_i$ the label of its corresponding antiquark.

As long as the constants Δ_{ij} are chosen to satisfy the constraints of symmetry and Eq.(3.16), these will enjoy the same properties of the massless scattering equations, which we derive in the next few paragraphs.

Scattering equations in 4D

The scattering equations have a natural reduction to four dimensions, which was first studied in [48] and then exploited in [49] to obtain a four dimensional version of the CHY representation for Yang-Mills theory.

Since we are working in four dimensions, it is natural to solve the equations in terms of helicity spinors. Now, recall that we have derived the scattering equations, from the condition that the square of the polynomial $P(z)$ vanishes (which amounts to demanding that we map the whole of $\mathbb{CP}^1$ to the null cone in momentum space). In four dimensions, this is equivalent to the existence of spinors $\lambda(z)$, $\tilde{\lambda}(z)$, such that

$$P^{\alpha\dot{\alpha}}(z) = \lambda^{\alpha}(z)\tilde{\lambda}^{\dot{\alpha}}(z) \tag{3.18}$$

This means that scattering equations in four dimensions can be written as a set of equations determining the punctures in terms of the spinors associated to the external momenta. This can be achieved in two different ways, related to each other by a linear transformation. The first one is a set of polynomial equations of degree $d = 1, ..., n-3$:

$$\begin{aligned} &\sum_{i=1}^{n} t_i \sigma_i^m \tilde{\lambda}_i^{\dot{\alpha}} = 0, \quad m = 0, 1, ..., d, \\ &\lambda_i^{\alpha} - t_i \sum_{m=0}^{d} \rho_m^{\alpha} \sigma_i^m = 0, \quad i = 1, ..., n \end{aligned} \tag{3.19}$$

in the variables σ_i and t_i for $i = 1, 2, ..., n$ and ρ_m^{α} for $m = 0, ..., d$. These equations decompose into $(n-3)$ sectors labelled by d, which are in a one-to-one correspondence with the helicity sector $K = d-1$ of amplitudes in the $N^K MHV$ classification.

The second form of the equations involves a rational set of equations, in which one divides the n particles in the process into two sets of K and $n-K$ particles, respectively. This form of the scattering equations reads

$$\begin{aligned} &\tilde{\lambda}_I^{\dot{\alpha}} - \sum_{i=K+1}^{n} \frac{t_i t_I \tilde{\lambda}_i^{\dot{\alpha}}}{\sigma_I - \sigma_i} = 0, \quad I = 1, .., K \\ &\lambda_i^{\alpha} - \sum_{I=1}^{K} \frac{t_i t_I \lambda_I^{\alpha}}{\sigma_I - \sigma_i} = 0, \quad i = K+1, .., n \end{aligned} \tag{3.20}$$

These forms of the scattering equations can be used to write closed formulas for the superamplitudes of $\mathcal{N} = 4$ SUSY, through the Roiban-Spradlin-Volovich (RSV) formula, which was derived from the twistor string formalism of Witten [19] [20]. Since the constraints Eqs. (3.19),(3.20) are equivalent to the scattering equations, the RSV formula provides a CHY representation for superamplitudes in four spacetime dimensions. We will introduce this formalism in Chapter 4.

3.2. Properties of the scattering equations

The scattering equations (3.14) are invariant under Möbius or $SL(2,\mathbb{C})$ transformations; that is, under transformations of the form

$$z_i \to \zeta_i = \frac{az_i + b}{cz_i + d}, \quad \forall i \tag{3.21}$$

where $ad-bc = 1$ in the sense that, if $z = (z_1, ..., z_n)$ is a solution to the scattering equations, then $\zeta = (\zeta_1, ..., \zeta_n)$ will also be a solution. To show this, we note that

$$\zeta_i - \zeta_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)} \tag{3.22}$$

where the condition $ad - bc = 1$ has been used. This implies

$$\begin{aligned} f_i(\zeta, p) &= \sum_{j\neq i} \frac{(cz_i + d)(cz_j + d)s_{ij}}{z_{ij}} \\ &= (cz_i + d)\left[c\sum_{j\neq i}\frac{z_j s_{ij}}{z_{ij}} + df_i(z,p)\right] \\ &= c(cz_i + d)\sum_{j\neq i}\frac{[-z_{ij} + z_i]s_{ij}}{z_{ij}} \\ &= c(cz_i + d)\left[-\sum_{j\neq i} s_{ij} + z_i f_i(z,p)\right] \\ &= c(cz_i + d)\left(-2p_i \cdot \sum_{j\neq i} p_j\right) \\ &= c(cz_i + d)(-2p_i^2) \qquad\qquad = 0 \end{aligned} \tag{3.23}$$

where we have used $f_i(z,p) = 0$, conservation of momentum and the fact that, for massless particles, $s_{ij} = 2p_i \cdot p_j$. This means that, out of the n equations, only $(n-3)$ are actually independent. This is reflected on the additional conditions [1],

$$\sum_i f_i(z,p) = 0, \quad \sum_i z_i f_i(z,p) = 0, \quad \sum_i z_i^2 f_i(z,p) = 0 \tag{3.24}$$

which can be easily proved algebraically. The first one

$$\sum_i f_i(z,p) = 0 \tag{3.25}$$

is a simple consequence of the fact that $z_{ji} = -z_{ij}$. On the other hand, we can see that

[1]Which can be seen to be direct consequences of $SL(2,\mathbb{C})$ invariance, by considering an infinitesimal Möbius transformation

$$\begin{aligned}
\sum_i z_i f_i(z,p) &= \sum_{ij} \frac{z_i s_{ij}}{z_{ij}} \\
&= \sum_{ij} \frac{1}{2} \frac{(z_i + z_i) s_{ij}}{z_{ij}} \\
&= \frac{1}{2} \sum_{ij} \left(\frac{z_i s_{ij}}{z_{ij}} + \frac{z_j s_{ji}}{z_{ji}} \right) \\
&= \frac{1}{2} \sum_{ij} \frac{z_{ij} s_{ij}}{z_{ij}} \\
&= \frac{1}{2} \sum_{ij} s_{ij} \\
&= \frac{1}{2} \sum_i 2p_i \cdot \left(\sum_{j \neq i} p_j \right) \\
&= -\sum_i p_i^2 = 0
\end{aligned} \tag{3.26}$$

where, in the third line, we have exchanged the indices i, j and in the fourth line we have used the symmetry of the kinematic invariants, $s_{ij} = s_{ji}$, and the relation $z_{ji} = -z_{ij}$. Finally, we have used momentum conservation and the on-shell conditions on the momenta. The remaining relation with z_i^2 is obtained in analogous fashion. These three conditions imply that we have more equations than variables to solve for and, in practice, will allow us to fix the values of three of the punctures.

We can show that the massive scattering equations, Eq. (3.15) are also $SL(2,\mathbb{C})$ invariant, provided the condition Eq. (3.16) holds. Under the transformation Eq. (3.22), $f_i^{mass}(z,p)$ transforms as

$$f_i^{mass}(\zeta,p) = (cz_i + d)^2 f_i^{mass}(z,p) - c(cz_i + d) \sum_{j \neq i} (2p_i \cdot p_j + \Delta_{ij}) \tag{3.27}$$

hence, if the equations $f_i^{mass} = 0$ are $SL(2,\mathbb{C})$ invariant, the second term must vanish. Since

$$\sum_{j \neq i} 2p_i \cdot p_j = -2p_i^2 = -2m_i^2 \tag{3.28}$$

we find that the equations are indeed invariant if Eq. (3.16) is satisfied.

It is clear from the fact that the scattering equations are invariant under a continuous set of transformations that they have an infinite number of solutions. However, a very large amount of them are related to each other by a Möbius transformation and, in fact, there is a finite number of solutions which are not related by $SL(2,\mathbb{C})$ transformations; these are known as inequivalent solutions. In the following paragraphs, we will study the behaviour of the scattering equations on different factorization channels. We will first focus on soft limits and use them to determine the number of inequivalent solutions for a given n and then consider how the system factorizes on more general singularities.

3.2.1. Singular kinematic behaviour of the scattering equations

As we have discussed in Chapter 2, an amplitude of massless particles develops a singularity when a kinematic invariant $s_{i_1 \ldots i_k}$ vanishes. When approaching this singularity, the amplitude factorizes into smaller amplitudes due to its analytic structure. This implies that, if the scattering equations serve as building blocks upon which one can construct scattering amplitudes, they must have special properties in these singular kinematic configurations which mirrors in some sense those of the amplitudes. We have mentioned that the scattering equations factorize when approaching a pole of the amplitude, but, what do we understand as factorization of the system of equations? As we will see, the physical factorization of amplitudes is reflected on the scattering equations by decoupling them, in the sense that the complete set of equations separates into independent subsets of equations for the punctures associated to the different subset of particles that are compose the lower-point amplitudes that appear in the residues of the n-point amplitude

As a first step towards showing that this is indeed the case, which will also be of particular interest to us throughout this thesis, let us consider one special singular configuration: the limit when one of the external particles is emitted with a soft momenta, by which we mean that $|p_n| << |p_i|$, where we take the soft particle to have the label n and $i = 1, ..., n-1$. The fact that the scattering equations decouple in the soft limit was first discovered in [12].

Soft limit of the scattering equations

Suppose that, in some n particle scattering event, the momentum of the n-th particle obeys $p_n \to 0$. One way to parametrize this soft limit is to write $p_n = \epsilon P$, for some arbitrary D-momentum P and consider the limit $\epsilon \to 0$. In this limit, we find

$$f_i(z,p) = \sum_{j \neq i,n} \frac{s_{ij}}{z_{ij}} + \epsilon \frac{s_{in}}{z_{in}} = 0 \tag{3.29}$$

for $i \neq n$. Then, the leading order (in ϵ) term of each of these equations is nothing more but the associated scattering equation which describes a process with $(n-1)$ external particles. On the other hand, the remaining equation is

$$f_n(z,p) = \epsilon \sum_{j \neq n} \frac{s_{nj}}{z_{nj}} = 0 \tag{3.30}$$

which is solved by setting either $\epsilon = 0$ or the remaining term equal to zero. The interesting point is that, for $\epsilon \to 0$, the system $f_i = 0$ for $i \neq n$ becomes independent of z_n. Therefore, the equation $f_n = 0$ decouples from the original system and can be converted into a polynomial equation for z_n, where the coefficients of this polynomial depend on the momenta and the remaining punctures z_i, $i = 1, ..., n-1$. Rationalizing and using momentum conservation, one can show that this polynomial equation is of degree $(n-3)$. Hence, there are $(n-3)$ complex roots for z_n. Hence, in this singular kinematic configuration, the scattering equations exhibit properties akin to those of the amplitude: as anticipated, they "factorize", in the sense that the information on the momenta of the hard particles decouples from the information on the kinematics of the soft emitted particle. This is an indication that the scattering equations should have similar properties to those of the amplitudes in general factorization channels. Before showing that this is in fact true, let us consider how the soft limit behaviour of the scattering equations allow us to deduce its number of independent solutions.

As we have argued, once we take the soft limit, the equation associated to the soft momenta decouples from the rest of the equations, and becomes a polynomial equation of degree $(n-3)$. This means that, if we let N_n be the number of solutions to the original system and N_{n-1} the number of solutions to the system of $(n-1)$ scattering equations, we see that the total number of solutions in the soft limit is given by

$$N_n = (n-3) \times N_{n-1} \tag{3.31}$$

This defines a recursion relation for the number of solutions N_n, which is easily solved in terms of N_4, the number of solutions to the scattering equations with four external particles, as

$$N_n = (n-3)! \times N_4 \tag{3.32}$$

Moreover, since the scattering equations obey the additional constraints Eq.(3.24), there is only one independent equation for $n = 4$ and, as we will show explicitly when we derive some particular solutions to the scattering equations, this equation has only one independent solution. This implies that, for an arbitrary number of external particles n, the scattering equations have $(n-3)!$ independent solutions. Also, as shown in [12], the factorization in soft limits provide an algorithm to find approximate solutions to the scattering equations.

General factorization channels

Now, we consider a general multiparticle singularity. Without loss of generality, we consider the factorization channel defined by the invariant

$$s_{1,..,n_L} = (p_1 + ... + p_{n_L})^2 \equiv s_L \tag{3.33}$$

where $2 \leq n_L \leq n-2$. This defines a partition of the set $\{1, 2, ..., n\}$ into two sets, namely $L = \{1, 2, ..., n_L\}$ and $R = \{n_R, ..., n\}$ where $n_R = n - n_L$. We want to study the behaviour of the scattering equations when $s_L \to 0$. We follow the discussion in [50].

In order to describe the solutions to the scattering equations in this factorization channel, the punctures are also split into two sets,

$$\begin{aligned} z_i &= \frac{\chi}{u_i}, \quad i \in L \\ z_i &= \frac{v_i}{\chi}, \quad i \in R \end{aligned} \tag{3.34}$$

Thus, we introduce a new variable χ and solve for the u_i, v_i and χ. Since $SL(2,\mathbb{C})$ invariance allows us to fix three of the solutions, the introduction of this new variable allows us to fix one further value, which we may take to be two of the u's and two of the v's. This means that we will take χ as a variable whose value is determined by solving the scattering equations.

Now, recall our discussion of the Parke-Taylor amplitude at the beginning of the chapter. Introducing the parametrization of the spinors $\lambda_i = (z_i, 1)$, we argued that all singularities of the amplitudes occurred when two or more of the z variables where equal to each other. Hence, if the scattering equations are to provide a mapping from momentum space to the Riemann space that correctly maps the kinematic singularities of the amplitudes, we expect that there will be solutions where $\chi \to 0$, so that every puncture associated to the particles whose Mandelstam invariant goes on-shell is equal. To check that this is indeed the case, we will assume that χ is a small number and study the behaviour of the scattering equations as an expansion in χ. To this end, we consider the scattering equations with $i \in R$. In general, these will be given by

$$\begin{aligned} f_{i\in R} &= \sum_{j\neq i} \frac{s_{ij}}{z_{ij}} \\ &= \sum_{j\in L} \frac{s_{ij}}{\frac{v_i}{\chi} - \frac{\chi}{u_i}} + \sum_{\substack{j\in R\\ j\neq i}} \frac{s_{ij}}{\frac{v_i}{\chi} - \frac{v_j}{\chi}} \\ &= \chi \sum_{\substack{j\in R\\ j\neq i}} \frac{s_{ij}}{v_i - v_j} + \sum_{j\in L} \frac{s_{ij}}{\frac{v_i}{\chi} - \frac{\chi}{u_i}} \end{aligned} \tag{3.35}$$

Now, rewriting the denominator of the second term, we can expand around $\chi = 0$

$$\begin{aligned} \frac{1}{\frac{v_i}{\chi} - \frac{\chi}{u_i}} &= \frac{\chi}{v_i} \frac{1}{1 - \frac{\chi^2}{u_i v_i}} \\ &= \frac{\chi}{v_i}\left(1 + \frac{\chi^2}{u_i v_i} + ...\right) \\ &= \frac{\chi}{v_i} + \frac{\chi^3}{u_i v_i^2} + ... \end{aligned} \tag{3.36}$$

hence, to subleading order in χ, setting $f_{i\in R} = 0$, we obtain the equation

$$\chi \sum_{\substack{j\in R\\ j\neq i}} \frac{s_{ij}}{v_i - v_j} + \frac{\chi}{v_i} \sum_{j\in L} s_{ij} + \chi^3 \sum_{j\in L} \frac{s_{ij}}{v_i^2 u_i} = 0 \tag{3.37}$$

From this equation we can see that there is, indeed, a solution for $\chi = 0$. Beyond the leading order in χ, one can also have solutions were $\chi \neq 0$; these correspond to the fact that the factorization of scattering amplitudes when an arbitrary propagator goes on-shell is a phenomenon that only occurs at leading order, that is, when we say that amplitudes factorize along a kinematic singularity, we mean that the residue associated with the pole at, say, $s_L = 0$ is a product of lower point amplitudes. However, we should not expect this to happen at higher orders in the Laurent expansion.

Now, we must check that the original equations do decouple, as in the case of the soft limits, into two sets, each associated with the particles in L and R, respectively. To do this, we first consider Eq. (3.37) at leading order in χ, that is

$$\begin{aligned} &\chi \sum_{\substack{j\in R\\ j\neq i}} \frac{s_{ij}}{v_i - v_j} + \frac{\chi}{v_i} \sum_{j\in L} s_{ij} = 0 \\ &\chi \left(2\frac{p_i \cdot p_L}{v_i} + \sum_{\substack{j\in R\\ j\neq i}} \frac{s_{ij}}{v_i - v_j} \right) = 0 \end{aligned} \tag{3.38}$$

where we defined $p_L = \sum_{j\in L} p_j$. If we associate the puncture $v_L = 0$ to the momentum p_L, we find that the term in brackets is exactly the system of scattering equations for the punctures associated to the particles in R plus an additional particle, which we can expect to find from our discussion of unitarity and the optical theorem: when we approach a singularity, the amplitude factorizes into two lower-point amplitudes which are connected by a "propagator", and each of the product amplitudes sees that propagator as an external, on-shell particle. In analogous fashion, one can show that the remaining scattering equations decouple at leading order, which results into two independent sets of equations, each of them describing a lower-point amplitude. This is the essential property which suggests that the scattering equations can be used to describe scattering amplitudes in any number of dimensions.

3.3. Polynomial form of the scattering equations

The scattering equations (3.14) are a deceptively difficult system of equations. The first natural step to perform if we wanted to solve them would be to convert them into polynomial equations for the punctures. This procedure can be realized in a general form. as shown by Dolan and Goddard [16], who found a system of equations equivalent to (3.14) which is known as the polynomial form of the scattering equations. To state these polynomial equations, define $I = \{1, 2, ..., n\}$ and consider a subset $S \subseteq I$. Then, if we let

$$P_S = \sum_{i\in S} p_i, \;\; z_S = \prod_{i\in S} z_i \tag{3.39}$$

where for completeness we let $P_\emptyset = 0$ and $z_\emptyset = 1$ for the empty set $\emptyset$. For $1 \leq m \leq n$, we can define the polynomials

$$h_m(z,p) = \frac{1}{m!} \sum_{S\subset I, |s|=m} P_S^2 z_S \tag{3.40}$$

That is, we take all subsets of I with m elements, and then sum over the product of their corresponding P_S^2 and z_S. From this definition, it becomes evident that each polynomial h_m is a homogeneous function of degree m. Moreover, each polynomial is linear in each different puncture; that is,

$$\frac{\partial^2 h_m}{\partial z_j^2} = 0, \quad \forall m, j = 1, ..., n \tag{3.41}$$

Furthermore, it is easy to see that the polynomials h_1, h_{n-1} and h_n are identically zero, using momentum conservation and the on-shell conditions. For $2 \leq m \leq n-2$, the system of the scattering equations is equivalent to solving

$$h_m(z, p) = 0 \tag{3.42}$$

where, as before, we solve for the puncture z given a configuration of momenta specified by p. This form of the equations also allows to determine the number of inequivalent solutions to the scattering equations due to Bézout's theorem, which states that the number of solutions to a system of polynomial equations is bounded by the product of its degrees. From the definition of the polynomials $h_m(z, p)$, one can see that this bound is $(n-3)!$. Moreover, since the punctures z_i are different for non-singular kinematics, the equations exactly saturate the bound of Bézout's theorem, confirming our analysis using the soft limit.

3.3.1. Overview of the equivalence

It is an interesting exercise to see how to derive (3.42) from (3.14). To start with, define

$$g_m(z, p) = \sum_{i=1}^{n} z_i^{m+1} f_i(z, p) \tag{3.43}$$

For $m \geq 2$, the function g_m is a polynomial of degree m in z_i. Note that, for the particular values $m = -1, 0, 1$, g_m vanishes identically due to (3.24). Now, using a similar strategy to the one we used to prove these identities, we can write

$$\begin{aligned} g_m(z, p) &= \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{p_i \cdot p_j z_i^{m+1}}{z_i - z_j} \\ &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} p_i \cdot p_j \frac{z_i^{m+1} - z_j^{m+1}}{z_i - z_j} \\ &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} p_i \cdot p_j \sum_{r=1}^{m} z_i^r z_j^{m-r} \end{aligned} \tag{3.44}$$

Now, if we let $-1 \leq m \leq n-2$ and define the $n \times n$ matrix with entries $Z_{mi} = z_i^{m+1}$, we can write (3.43) as the matrix equation $g_m = Z_{mi} f_i$. Furthermore, since Z_{mi} is nothing else but a Vandermonde matrix[2], it has the determinant

$$detZ = \prod_{i<j}^{n} (z_i - z_j) \tag{3.45}$$

[2] A Vandermonde matrix is an $n \times n$ matrix whose rows are the values of a geometric progression

which is non-singular since $z_i \neq z_j$ for $i \neq j$. Hence, imposing the scattering equations $f_i(z,p) = 0$ imply that $g_m(z,p) = 0$ for all values of m, which then yields a polynomial system of equations equivalent to (3.14). Now, let $m = 2$. We can see that, for example,

$$\begin{aligned}
g_2(z,p) &= \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i \cdot p_j z_i^2 + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i \cdot p_j z_i z_j \\
&= \sum_{j\neq i} p_j \cdot \left(\sum_i p_i z_i^2\right) + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i \cdot p_j z_i z_j \\
&= -\sum_i p_i^2 z_i^2 + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i \cdot p_j z_i z_j \\
&= \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i \cdot p_j z_i z_j \\
&= \frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} (p_i + p_j)^2 z_i z_j \\
&= h_2(z,p)
\end{aligned} \tag{3.46}$$

where the first term in the sum vanishes due to the on-shell conditions and we obtain the squares of the momenta using momentum conservation to evaluate the sum over j. This construction can be generalized to arbitrary $2 \leq m \leq n-2$, and, as shown in [16], yield the relations

$$h_m(z,p) = 2\sum_{r=1}^{m}(-1)^r g_r(z,p)\Sigma_{m-r} \tag{3.47}$$

where

$$\Sigma_r = \sum_{\substack{S\in I \\ |S|=r}} z_S \tag{3.48}$$

that is, Σ_r is the product of all the z's whose indices are in the subset S of I. Then, since the scattering equations $f_i = 0$ are equivalent to $g_r = 0$, then they are also equivalent to the condition $h_m = 0$ due to Eq. (3.47), which are just Eq.(3.42).

3.4. Particular solutions

Now, we turn to the problem of solving the scattering equations, in either the standard (3.14) or polynomial (3.42) form. This is important because, as we have mentioned, the CHY representation will give us formulas for amplitudes in terms of integrals localized into the solutions to the scattering equations, which means that, in practice, we will define some quantities which depend on the punctures and, in order to obtain the amplitude, we will need to evaluate those functions on the solutions. We will consider only the simplest of cases, which are with $n = 4$ and $n = 5$ external particles, and we will see how both forms yield the same solutions. We do not consider the case $n = 3$ because, in this case, the $SL(2, \mathbb{C})$ constraints imply that the system can be solved by any arbitrary values of z_1, z_2 and z_3 as long as $z_1 \neq z_2 \neq z_3$. For higher multiplicity, this means that we will fix the values of three of the punctures and solve for the remaining punctures. In all cases, we will fix $z_1 = 1$, $z_{n-1} = 0$ and $z_n = \infty$.

3.4.1. Solution for n=4

The scattering equations (3.14) for $n = 4$ external, massless particles are given by

$$f_i(z,p) = \sum_{j\neq i}^{4} \frac{s_{ij}}{z_{ij}} \tag{3.49}$$

Now, consider the particular set of equations obtained when we fix $z_1 = 1$, $z_3 = 0$ and $z_4 = \infty$. With this choice for the fixed punctures, it is clear that f_4 vanishes identically for any finite value of z_1, z_2 and z_3. On the other hand, the remaining equations are

$$\begin{aligned} f_1(z,p) &= \sum_{j=2,3} \frac{s_{1j}}{z_1 - z_j} \\ &= \frac{s_{12}}{1 - z_2} - s_{13} = 0, \end{aligned} \tag{3.50}$$

$$\begin{aligned} f_2(z,p) &= \sum_{j=1,3} \frac{s_{2j}}{z_2 - z_j} \\ &= \frac{s_{12}}{z_2 - 1} + \frac{s_{23}}{z_2} = 0 \end{aligned} \tag{3.51}$$

and, the last one,

$$\begin{aligned} f_3(z,p) &= \sum_{j=1,2} \frac{s_{3j}}{z_3 - z_j} \\ &= \frac{s_{31}}{z_3 - z_1} + \frac{s_{32}}{z_3 - z_2} \\ &= -s_{13} - \frac{s_{23}}{z_2} = 0 \end{aligned} \tag{3.52}$$

which has the solution $z_2 = -\frac{s_{23}}{s_{13}}$. This value can also be seen to be a solution to the other two equations, become an equation for z_2, using momentum conservation and the on-shell conditions.

We will now obtain this solution using the polynomial form of the scattering equations, 3.42. In this case, there is only one polynomial equation, and the first step is to write down the polynomial $h_2(z,p)$ explicitly. In order to do this, we first identify all the subsets of $\{1, 2, 3, 4\}$ which contain two elements. The number of this subsets is

$$N_2 = \frac{4!}{2!(4-2)!} = 6 \tag{3.53}$$

which is just the binomial coefficient. These are given by $\{1,2\}$, $\{1,3\}$,$\{1,4\}$, $\{2,3\}$,$\{2,4\}$ and $\{3,4\}$. Then,

$$h_2(z,p) = s_{12}z_1z_2 + s_{13}z_1z_3 + s_{14}z_1z_4 + s_{23}z_2z_3 + s_{24}z_2z_4 + s_{34}z_3z_4 \tag{3.54}$$

Now, we first fix the values $z_1 = 1$ and $z_3 = 0$. These yields the partially fixed polynomial

$$h_2(z,p) = s_{12}z_2 + s_{14}z_4 + s_{24}z_2z_4 \tag{3.55}$$

we can set this polynomial equal to zero, and solve for z_2 to obtain

$$z_2 = -\frac{s_{14}z_4}{s_{12} + s_{24}z_4} \tag{3.56}$$

From this, we see that, for every value we take of z_4 we obtain a different value of z_2. However, this is not a problem because we expect the scattering equations to have an infinite set of solutions, related to each other through $SL(2,\mathbb{C})$ transformations. In particular, if we let $z_4 = \infty$, we obtain

$$z_2 = -\frac{s_{14}}{s_{24}} = -\frac{s_{23}}{s_{13}} \tag{3.57}$$

where we used momentum conservation in the last equality. This is nothing but the same solution we found when using the original form of the scattering equations.

3.4.2. Solution for n=5

Now, we will solve the scattering equations for five external particles using the polynomial form of the scattering equations.

We need to determine the two polynomials h_2 and h_3, which after being set to zero, will be the equations to solve. Hence, we must find the number of distinct subsets with two and three elements of the set $\{1,2,3,4,5\}$. Simple combinatorics show that there are 10 of each kind of subsets. The two-element subsets are

$$\begin{matrix} \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\ \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \end{matrix} \tag{3.58}$$

while the subsets with three elements are given by

$$\begin{matrix} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,3,4\} & \{1,3,5\} \\ \{1,4,5\} & \{2,3,4\} & \{2,3,5\} & \{2,4,5\} & \{3,4,5\} \end{matrix} \tag{3.59}$$

Then, using the definition 3.40, we can write

$$\begin{aligned} h_2 =& s_{12}z_1z_2 + s_{13}z_1z_3 + s_{14}z_1z_4 + s_{15}z_1z_5 + s_{23}z_2z_3 \\ &+ s_{24}z_2z_4 + s_{25}z_2z_5 + s_{34}z_3z_4 + s_{35}z_3z_5 + s_{45}z_4z_5 \end{aligned} \tag{3.60}$$

For h_3, we would find terms of the form $s_{123}z_1z_2z_3$, involving three-particle invariants. However, using momentum conservation we can see that, for example,

$$\begin{aligned} s_{123} &= (p_1 + p_2 + p_3)^2 \\ &= (p_4 + p_5)^2 \\ &= s_{45} \end{aligned} \tag{3.61}$$

which yields a term $z_1 z_2 z_3 s_{45}$. Then, h_3 is a polynomial with terms of the form $z_i z_j z_k s_{lm}$, where, if we think of the labels as a set, $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. Then,

$$\begin{aligned} h_3 =& s_{12} z_3 z_4 z_5 + s_{13} z_2 z_4 z_5 + s_{14} z_2 z_3 z_5 + s_{15} z_2 z_3 z_4 + s_{23} z_1 z_4 z_5 \\ &+ s_{24} z_1 z_3 z_5 + s_{25} z_1 z_3 z_4 + s_{34} z_1 z_2 z_5 + s_{35} z_1 z_2 z_4 + s_{45} z_1 z_2 z_3 \end{aligned} \tag{3.62}$$

Now, we use our values for the fixed punctures $z_1 = 1$, $z_4 = 0$ and $z_5 = \infty$. In order to implement the value $z_5 = \infty$ in a consistent manner, we note that the equations $h_m = 0$ for $m = 2, 3$ are equivalent to the equations $\frac{h_m}{z_5} = 0$ for any non-zero value of z_5. Hence, we divide each equation by z_5 and take the limit $z_5 \to \infty$ in the resulting equations. These results in the system of equations

$$\begin{aligned} h'_2 &= s_{25} z_2 + s_{35} z_3 + s_{13} = 0 \\ h'_3 &= s_{34} z_2 + s_{24} z_3 + s_{14} z_2 z_3 = 0 \end{aligned} \tag{3.63}$$

Solving for z_3 in the equation $h'_2 = 0$, we find

$$z_3 = -\frac{s_{13} + s_{25} z_2}{s_{35}} \tag{3.64}$$

which, upon substitution in $h'_3 = 0$ yields the quadratic equation for z_2

$$s_{14} s_{25} z_2^2 + (s_{13} s_{14} + s_{24} s_{25} - s_{34} s_{35}) z_2 + s_{13} s_{24} = 0 \tag{3.65}$$

Which, of course, has two different solutions, which we simply call z_2^+ and z_2^-, and these in turn determine two values z_3^+ and z_3^- for the puncture z_3. Then, the two independent solutions to the scattering equations that we expect to find are given by the sets

$$\begin{aligned} z_1 = 1,\ z_2 = z_2^+,\ z_3 = z_3^+,\ z_4 = 0,\ z_5 = \infty \\ z_1 = 1,\ z_2 = z_2^-,\ z_3 = z_3^-,\ z_4 = 0,\ z_5 = \infty \end{aligned} \tag{3.66}$$

which is what we expect from the general counting of solutions. Note that, technically, although we look for equations to solve for each individual puncture, what we call an independent solution to the scattering equations is the complete n-tuple $(z_1^i, z_2^i, ..., z_n^i)$ and that there might be some punctures equal to each other in different solution n-tuples.

We finish this chapter with some comments on how to approach the solution of the scattering equations in general scenarios. In [51], it is shown that in $D = 4$ spacetime dimensions, out of the $(n-3)!$ solutions, one can always write two of them for any multiplicity as cross-ratios of spinor products and that, for $n = 6$, the remaining $(6-3)! - 2 = 4$ solutions can be written in terms of algebraic functions which depend on the Mandelstam invariants. In the series of related papers [52], [53], [54] solutions are derived in special kinematic scenarios; afterwards, in [55] elimination theory is used to obtain the $(n-3)!$ degree polynomial equation for each individual puncture starting from the polynomial form (3.14). The polynomial equation is also worked out in [56] and is shown to be identified with the hyperdeterminant of a multidimensional matrix, and use that construction to give another proof of the fact that the scattering equations have $(n-3)!$ solutions for a given number n of external particles.

In this chapter, we have introduced the scattering equations and showed that they provide a consistent mapping from momentum space to the Riemann sphere with the property of characterizing the singularity structure of scattering amplitudes. Now, we need to relate these mapping to the amplitudes themselves, and that is the subject of the following chapter.

The CHY representation of tree-level amplitudes

In this chapter, we introduce the Cachazo, He, Yuan (CHY) formalism [12][13][14] for tree level scattering amplitudes. We will first define what it means for an amplitude to have a CHY representation and the different elements that compose the CHY formula for the amplitudes of a particular theory. Following that, we will introduce the objects that enter into the CHY integrand for pure Yang-Mills theory, and show how the ingredients entering the formula are related to amplitudes in scalar ϕ^3 theory and Einstein gravity. As a next step, we briefly discuss one way in which the CHY integrals can be calculated and illustrate it with some basic examples. Finally, we will discuss the analogue of the CHY representation in four dimensions for $\mathcal{N}=4$ superamplitudes, the so-called connected formalism.

4.1. CHY formula

Let $\mathcal{A}_n^0$ denote an arbitrary tree amplitude with n external particles. In general, this amplitude will be a function of the momenta of the external particles, as well as their wavefunctions. Also, in the case of gluon scattering, it may also depend on the cyclic ordering of the particles, if we consider color-ordered amplitudes. For this reason, we write the arguments of $\mathcal{A}_n^0$ implicitly as $\mathbf{x}$. A theory is said to have a CHY representation if we can write

$$\mathcal{A}_n^0(\mathbf{x}) = i \oint_C d\Omega_{CHY} \mathcal{I}(z, \mathbf{x}) \tag{4.1}$$

that is, if we can write the amplitude in terms of an universal integration measure $d\Omega_{CHY}$ and a theory-dependent integrand $\mathcal{I}(z, \mathbf{x})$. The contour C is defined so that it encloses all the inequivalent solutions to the scattering equations. In general, the integrand will be of the form

$$\mathcal{I}(z, \mathbf{x}) = \mathcal{I}_L(z, \mathbf{x}) \mathcal{I}_R(z, \mathbf{x}) \tag{4.2}$$

and we refer to $\mathcal{I}_L$, $\mathcal{I}_R$ as half-integrands. Although such a separation seems arbitrary, we will see through various examples that the CHY integrands usually split into two factors, in such a way that the various degrees of freedom of the amplitudes factorize. For example, a Yang-Mills primitive amplitude is a function of the cyclic ordering of the external particles, w, and the polarizations of each external gluon, denoted by ϵ. As we will see, the CHY integrand for Yang-Mills can be written as the product of two functions, one of which only depends on the cyclic ordering and the other which depends only on the polarizations [15]. In fact, the factorization of the dependence on the cyclic orderings and the external polarizations will be one of the properties we demand of the CHY representation for multiquark QCD amplitudes.

4.1.1. Definition of the measure

In the previous chapter, we constructed the scattering equations as a map from momentum space to an n-punctured Riemann sphere and showed that the kinematic singularities of the scattering amplitudes are in correspondence with the singular behaviour of the scattering equations. Hence, if we want to associate to each Riemann sphere a scattering amplitude, the integral in Eq. (4.2) should be localized to the solutions of the scattering equations. However, we know that the solutions to the scattering equations are $SL(2,\mathbb{C})$ invariant. This means that we must perform a Fadeev-Popov procedure to fix this gauge redundancies. A natural candidate for the measure is then given by

$$d\Omega_{CHY} = \frac{dz_1...dz_n}{d\omega} \prod{}' \delta(f_a(z,p)) \tag{4.3}$$

where

$$\prod{}' \delta(f_a(z,p)) = (-1)^{i+j+k} z_{ij} z_{jk} z_{ki} \prod_{a \neq i,j,k} \delta(f_a(z,p)) \tag{4.4}$$

and

$$d\omega = (-1)^{p+q+r} \frac{dz_p dz_q dz_r}{z_{pq} z_{qr} z_{rp}} \tag{4.5}$$

is the invariant $SL(2,\mathbb{C})$ measure. The gauge fixing procedure can be understood as follows: in general, there are n punctures for a process with n external particles. Therefore, the natural way to localize a quantity which depends on the n punctures to the solutions of the scattering equations would be to integrate over $d^n z = dz_1...dz_n$ and weight the integral with the product of n delta functions $\delta(f_i)$. However, the $SL(2,\mathbb{C})$ invariance of the scattering equations allow us to fix three of the puncture to any arbitrary, different values. To compensate for this fact and in order to obtain a finite result after the integration, we must choose a gauge orbit for the integral, and this is realized by dividing over the invariant volume of $SL(2,\mathbb{C})$, which is given by the integral of $d\omega$. Hence, we simply define the CHY measure to include a factor of $1/d\omega$, which cancels the gauge redundancy of the integral.

The phases $(-1)^{i+j+k}$ and $(-1)^{p+q+r}$ guarantee that the integral is independent of the choice of i, j, k, p, q, r. In practice, we will always take two sets such that $\{i, j, k\} = \{p, q, r\}$.

Since the integral is completely localized, we can formally evaluate it as a sum of the integrand, divided by an appropriate Jacobian, calculated at each of the independent solution to the scattering equations. This Jacobian is obtained by using the transformation properties of the delta functions, and to define it we first introduce the $n \times n$ matrix

$$(\Phi_n)_{ab} := \begin{cases} \frac{s_{ab}}{z_{ab}^2} & \text{if } a \neq b \\ -\sum_{\substack{c=1 \\ c\neq a}} (\Phi_n)_{ac} & \text{if } a = b \end{cases} \tag{4.6}$$

The Jacobian is defined as

$$J_n(z,p) = \frac{1}{det'(\Phi_n)} = (-1)^{i+j+k+q+p+r} \frac{z_{ij} z_{jk} z_{ki} z_{pq} z_{qr} z_{rp}}{det[\Phi_n]^{ijk}_{pqr}} \tag{4.7}$$

Here, we denote by $[\Phi_n]^{ijk}_{pqr}$ the minor of Φ_n obtained by deleting the rows i, j, k and the columns p, q, r. With this, we can write

$$\mathcal{A}^0_n = \sum_{(j)} J(z^{(j)}, p) \mathcal{I}(z^{(j)}, \mathbf{x}) \tag{4.8}$$

where the sum runs over $(j) = 1, 2, ..., (n-3)!$ and $z^{(j)}$ is the (j)-th inequivalent solution to the scattering equations. We will find this form of the CHY representation more useful when we consider the construction of the integrand for QCD primitive amplitudes. The CHY representation of scattering amplitudes, as localized integrals on the solutions to the scattering equations, was first introduced in [14].

Determining what is the CHY representation of the tree amplitudes of a particular field theory amounts to finding a prescription for the integrand $\mathcal{I}(z, \mathbf{x})$. This non-trivial task is simplified by the mathematical and physical assumptions one can make depending on the theory under consideration. One very general statement about the integrand can be made on the basis of the $SL(2, \mathbb{C})$ invariance that the integral must inherit from the scattering equations. So, as a first step, we will attack this problem.

4.1.2. General transformation properties of the integrand

Although the solutions to the scattering equations are $SL(2, \mathbb{C})$ invariant, the functions $f_i(z, p)$ themselves change under Möbius transformations. To see this, consider the transformation

$$z \to \zeta = \frac{az + b}{cz + d} \tag{4.9}$$

then, using Eq. (3.22), it is easy to see that, in either the massive or massless case,

$$f_i(\zeta, p) = (cz_i + d)^2 f_i(z, p) \tag{4.10}$$

which means that, using the properties of the Delta function,

$$\delta(f_i(\zeta, p)) = \frac{1}{(cz_i + d)^2} \delta(f_i(z, p)) \tag{4.11}$$

On the other hand, since

$$\frac{d\zeta}{dz} = \frac{1}{(cz+d)^2} \tag{4.12}$$

we can see that

$$d^n\zeta = \prod_{i=1}^{n} \frac{1}{(cz_i+d)^2} d^n z \tag{4.13}$$

On the other hand, the gauge fixing term $d\omega$ is Möbius invariant by construction. Hence, the CHY measure is not invariant under Möbius transformations; instead, it picks up an overall z-dependent factor

$$d\Omega'_{CHY} = \left(\prod_{i=1}^{n} \frac{1}{(cz_i+d)^4} \right) d\Omega_{CHY} \tag{4.14}$$

What we want, however, is to have a completely $SL(2,\mathbb{C})$ invariant object under the integral, due to the invariance of the solutions to the scattering equations, which are the values to which we are localizing the integral. Hence, independent of the theory, we require that the integrand $\mathcal{I}(z,\mathbf{x})$ will transform under Möbius transformations as

$$\mathcal{I}(\zeta,\mathbf{x}) = \left(\prod_{i=1}^{n} (cz_i+d)^4 \right) \mathcal{I}(z,\mathbf{x}) \tag{4.15}$$

This Möbius covariance property of the integrand then becomes one of the consistency checks that any integrand candidate must satisfy in order to have a consistent CHY representation. Thus, by construction, the scattering equations limit the possible integrands which can be used in order to define the tree level S-matrix of a particular theory.

4.1.3. CHY measure using the polynomial form of the scattering equations

The CHY representation provides an extremely compact expression for tree level scattering amplitudes; the fact that one can write a closed formula for amplitudes of arbitrary multiplicity is striking by itself, from the point of view of Feynman diagrams. However, in order to obtain this compact formulas, one must introduce the auxiliary variables z_i and solve the constraints imposed by the scattering equations in order to determine the values that the punctures take as functions of the Mandelstam invariants. As we have mentioned in the previous chapter, this is a highly non-trivial task. This can be simplified by noting that the delta functions in Eq. (4.3) can also be understood as denominators, that is, as contour integrals on $\mathbb{C}^n$ over a contour C which encloses the simultaneous solutions to the scattering equations. This can be visualized in the case of $n=4$ external particles, were the gauge fixing of the CHY integrals implies that there is only an integration over one puncture; say, z_2. Using Eq. (4.3), for a general integrand $\mathcal{I}_4(z,\mathbf{x})$, we can write the four particle amplitude related to this integrand as

$$A_4^I = \oint_C dz_2 \mathcal{I}_4^{GF}(z,\mathbf{x}) \delta\left(\sum_{i\neq 2} \frac{s_{2i}}{z_2 - z_i} \right) \tag{4.16}$$

where $\mathcal{I}_4^{GF}(z, \mathbf{x})$ denotes the integrand obtained by including the factors that arise from the gauge fixing procedure. Now, recall that, in a real variable, we define the Dirac delta function as the distribution which, for any well-behaved functions g, satisfies

$$g(x) = \int_{-\infty}^{\infty} dx' g(x')\delta(x - x') \tag{4.17}$$

How does one obtain a similar definition for a complex variable? For an analytic function $y(z)$, Cauchy's theorem states that

$$\frac{1}{2\pi i}\oint_C dz' \frac{y(z')}{z' - z} = y(z) \tag{4.18}$$

where C is a contour that encloses the point $z' = z$. Hence, for any analytic function, we can define the complex delta function as [1]

$$\delta(z' - z) = \frac{1}{2\pi i}\frac{1}{z' - z} \tag{4.19}$$

Applying this definition to the $(n - 3)$ integrations of the CHY representation, we rewrite the measure as

$$d\Omega_{CHY} = \frac{1}{(2\pi i)^{n-3}}\frac{d^n z}{d\omega}\prod{}' \frac{1}{f_a(z, p)} \tag{4.20}$$

and the CHY integrals are actually multidimensional complex contour integrals, under the conditions that the integrands are meromorphic functions on the solutions to the scattering equations. Then, we will write the amplitudes as

$$\mathcal{A}_n^0(\mathbf{x}) = i\frac{(-1)^{i+j+k}}{(2\pi i)^{n-3}}\oint_{\mathcal{C}} \frac{d^n z}{d\omega}\frac{z_{ij}z_{jk}z_{ki}}{\prod' f_a(z, p)}\mathcal{I}(z, \mathbf{x}) \tag{4.21}$$

where $\mathcal{C}$ is a contour in $\mathbb{C}^n$ that encloses the simultaneous solutions to the scattering equations, $f_i(z, p) = 0$. Now, since the scattering equations are equivalent to the polynomial equations Eq. (3.40) , there should be a contour integral equivalent to Eq.(4.21) where the denominators are written in terms of the polynomials $h_m(z, p)$ given in Eq. (3.40). Using Eqs.(3.43) and (3.47), it is possible to do this by calculating the Jacobian of the transformations from f_i to h_m. The result, which was proven in [16], can be written as

$$\mathcal{A}_n^0(\mathbf{x}) = i\frac{(-1)^n}{(2\pi)^{n-3}}\oint_{\mathcal{C}} \frac{\prod_{i<j} z_{ij}}{\prod_{m=2}^{n-2} h_m(z, p)}\mathcal{I}(z, \mathbf{x}) \tag{4.22}$$

We will use this form in Appendix F, when we use the CHY representation to calculate a 5-point example in scalar ϕ^3 theory. Now, we will introduce the CHY integrand which allows to calculate n-point tree amplitudes in pure Yang-Mills theory, and see how the recipes of the integrand allow one to define the integrands for both scalar ϕ^3 theory and Einstein gravity.

[1] note that this definition means that, unlike the real case, the delta function is antisymmetric in the exchange of its arguments.

4.2. CHY integrand for Yang-Mills theory

We have mentioned that the CHY integrand $\mathcal{I}(z, \mathbf{x})$ usually takes the form of a product, $\mathcal{I}(z, \mathbf{x}) = \mathcal{I}_L(z, \mathbf{x})\mathcal{I}_R(z, \mathbf{x})$. These half-integrands are defined so that the information on some of the various degrees of freedom of a given amplitude are related only to one of the half-integrands. Primitive amplitudes in Yang-Mills theory have essentially three types of degrees of freedom: the cyclic ordering of the external particles, the momenta and the polarizations. The CHY integrand for Yang-Mills separates the information on the cyclic ordering from the information on that from the polarizations. We will be able to write the tree-level primitive amplitudes in Yang-Mills theories as

$$A_n^{YM}(w, p, \epsilon) = \frac{(-1)^{i+j+k}}{(2\pi)^{n-3}} i \oint_{\mathcal{C}} \frac{d^n z}{d\omega} \frac{z_{ij} z_{jk} z_{ki}}{\prod_{a \neq i,j,k} f_a(z, p)} C_n(w, z) E_n(z, p, \epsilon) \tag{4.23}$$

Now, we define the two half-integrands in Eq.(4.23). On one hand, the first half-integrand of Yang-Mills theory is the so-called Parke-Taylor factor

$$C^n(w, z) = \frac{1}{z_{l_1 l_2} \ldots z_{l_n l_1}} \tag{4.24}$$

where $w = l_1 \ldots l_n$ is a word of length n. Essentially, a word is a set of labels that specifies the external ordering of a primitive or color-decomposed amplitude. We review the formalism of words and shuffle algebras to organize this dependence of amplitudes in Appendix G. These receive the name of Parke-Taylor factor because of their similarity to the denominator of the MHV gluon amplitude when written as in 3.6.

On the other hand, the information on the polarizations is encoded in what we will call the polarization function, $E_n(z, p, \epsilon)$, where it is understood that it depends on the complete set of polarization vectors ϵ_i, $i = 1, 2, ..., n$. From the structure of Feynman diagrams, we know that an arbitrary Yang-Mills amplitude is a multilinear function of the external polarization vectors; hence, we should expect that, if the polarization function does encode all the information on the polarization vectors, it must be a multilinear function of these vectors. Moreover, it should also hold the information on gauge invariance, since we already know that each color-ordered amplitude is a gauge-invariant object and the Parke-Taylor factor has no dependence on the polarizations. In order to define the polarization function for a given number of external particles n, we introduce the $(2n) \times (2n)$ matrix

$$\Psi_n = \begin{pmatrix} A_n & -C_n^T \\ C_n & B_n \end{pmatrix} \tag{4.25}$$

which is constructed in terms of three $n \times n$ matrices A_n,B_n and C_n, with entries

$$A_{ab} = \begin{cases} \frac{2p_a \cdot p_b}{z_{ab}} & a \neq b, \\ 0 & a = b, \end{cases} \tag{4.26}$$

$$B_{ab} = \begin{cases} \frac{2\epsilon_a \cdot \epsilon_b}{z_{ab}} & a \neq b, \\ 0 & a = b, \end{cases} \tag{4.27}$$

and, finally,

$$C_{ab} = \begin{cases} \frac{2\epsilon_a \cdot p_b}{z_{ab}} & a \neq b, \\ -\sum\limits_{j=1, j\neq a}^{n} \frac{2\epsilon_a \cdot p_j}{z_{aj}} & a = b. \end{cases} \tag{4.28}$$

There are two obvious properties Ψ_n which can be seen from its definition: it is antisymmetric and is always even-dimensional. An important feature of any matrix with these two properties (with dimension $(2n) \times (2n)$) is that its determinant can be written as the square of a polynomial of degree n in its matrix entries with integer coefficients, which only depends on the size of the matrix. This polynomial is known as the Pfaffian. For an arbitrary, $(2n) \times (2n)$ antisymmetric matrix M with entries m_{ij}, one can define the Pfaffian as

$$\text{Pf}\ (M) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \prod_{i=1}^{n} m_{\sigma(2i-1)\sigma(2i)} \tag{4.29}$$

where, we encounter a sum over permutations of $2n$ elements and $sgn(\sigma)$ equals 1 if the permutation is even and -1 if it is odd. For example, if we have a 4×4 antisymmetric matrix M with entries

$$M = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \tag{4.30}$$

its Pfaffian is given by

$$\text{Pf}\ (M) = af - be + cd \tag{4.31}$$

which can be easily seen to square to the determinant of M. One important property of the Pfaffian is that it can be expanded in terms of Pfaffians of minors, in similar fashion to the determinant, as

$$\text{Pf}\ (M) = \sum_{j \neq i} (-1)^{i+j+1+\theta(i-j)} m_{ij} \text{Pf}\ (M_{ij}) \tag{4.32}$$

Where $\theta(x)$ denotes the Theta function, and M_{ij} is the minor obtained by removing rows and columns i, j of the matrix M.

Now, we can see that the $n \times (2n)$ matrix $(A, -C^T)$ has two null vectors, $\vec{n}_1 = (1, ..., 1)$ and $\vec{n}_2 = (z_1, z_2, ..., z_n)$. To see this, note that either of the products $\vec{n}_k(A, -C^T)$ is a $1 \times (2n)$ vector whose first n entries vanish due to the scattering equations, in the case of $\vec{n}_1$, and because of the constraints imposed on the scattering equations by $SL(2, \mathbb{C})$ invariance of the scattering equations in the case of $\vec{n}_2$. On the other hand, to see that the remaining n entries vanish, we can see that, in the case of $\vec{n}_1$, each entry is the sum of a column of C, which vanishes due to the diagonal element C_{aa}. On the other hand from its definition, for a fixed value $1 \leq a \leq n$, we can see that

$$
\begin{aligned}
\sum_{b=1}^{n} z_b C_{ab} &= 2\sum_{\substack{b=1\\ b\neq a}} \frac{z_b \epsilon_a \cdot p_b}{z_a - z_b} - 2z_a \sum_{\substack{b=1\\ b\neq a}} \frac{\epsilon_a \cdot p_b}{z_a - z_b} \\
&= -2\epsilon_a \cdot \left(\sum_{\substack{b=1\\ b\neq a}} p_b \right) + 2z_a \sum_{\substack{b=1\\ b\neq a}} \frac{\epsilon_a \cdot p_b}{z_a - z_b} - 2z_a \sum_{\substack{b=1\\ b\neq a}} \frac{\epsilon_a \cdot p_b}{z_a - z_b} \\
&= -2\epsilon_a \cdot p_a = 0
\end{aligned}
\tag{4.33}
$$

where we used momentum conservation and the transversality of polarization vectors. This means that two rows of Ψ_n are linearly dependent, and the Pfaffian of Ψ_n vanishes. However, if we denote the minor of Ψ_n obtained by removing both rows and columns i, j for $1 \leq i \leq j \leq n$ by $(\Psi_n)^{ij}_{ij}$, we obtain a matrix with non-vanishing Pfaffian. With this, we define

$$
E_n(z, p, \epsilon) = \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}' \ \Psi_n \tag{4.34}
$$

where we define the shorthand notation $\mathrm{Pf}' \ \Psi_n = \mathrm{Pf} \ (\Psi_n)^{ij}_{ij}$. Gauge invariance is easily proven by noting that, if we take $\epsilon_k \to p_k$ for an arbitrary k, the Pfaffian vanishes because the columns k and $k+n$ of Ψ_n become identical after noting that

$$
C_{kk} = -\sum_{b\neq k} \frac{2\epsilon_k \cdot p_b}{z_{kb}} \to -\sum_{b\neq k} \frac{2p_k \cdot p_b}{z_{kb}} \tag{4.35}
$$

The formula Eq.(4.23) was first proposed by Cachazo, He and Yuan in [13] and later proven to yield the correct Yang-Mills tree amplitudes by Dolan and Goddard in [57] by showing that it satisfies the same BCFW recursion relations as Yang-Mills tree amplitudes. This integral representation for Yang-Mills amplitudes was also obtained afterwards by Mason and Skinner [58] as the worldsheet integral of the n-point correlation function of an ambitwistor superstring theory, and by Bjerrum-Bohr et al in [59] as the field theory limit of open string correlators.

4.2.1. Integrands for scalar ϕ^3 theory and gravity

The Parke-Taylor factor Eq. (4.24) and the polarization function Eq. (4.29) can also be used to define the CHY representation of scalar ϕ^3 theory and Einstein gravity.

We can define a scalar theory with two gauge groups $U(N_1)$ and $U(N_2)$ by the lagrangian

$$
\mathcal{L} = \partial^\mu \phi^{aa'} \partial_\mu \phi_{aa'} + \frac{1}{3!} f_1^{abc} f_2^{a'b'c'} \phi_{aa'} \phi_{bb'} \phi_{cc'} \tag{4.36}
$$

where f_1^{abc} and $f_2^{a'b'c'}$ are the structure constants of $U(N_1)$ and $U(N_2)$, respectively. This is known as a biadjoint scalar theory. The amplitudes of the theory can be color decomposed in the same fashion as Yang-Mills tree amplitudes, but in this case it can be done with respect to both gauge groups, yielding partial amplitudes which depend on two external orderings; denoting such an amplitude by $m_n(w_1|w_2)$, where $w_1 = l_1 l_2 ... l_n$, $w_2 = l'_1 l'_2 ... l'_n$, its CHY representation can be written as

$$m_n(w_1|w_2) = \frac{(-1)^{i+j+k}}{(2\pi)^{n-3}} i \oint_{\mathcal{C}} \frac{d^n z}{d\omega} \frac{z_{ij} z_{jk} z_{ki}}{\prod_{a \neq i,j,k} f_a(z,p)} C_n(w_1, z) C_n(w_2, z) \tag{4.37}$$

The complete amplitude m_n^{BA} is then obtained by multiplying by the traces $\mathrm{Tr}(T^{l_1} T^{l_2} ... T^{l_n})$ and $\mathrm{Tr}(T^{l'_1} T^{l'_2} ... T^{l'_n})$ and then summing over all possible permutations. Performing this procedure and setting all traces equal to 1 yield the amplitude of ordinary $\lambda\phi^3$ theory.

On the other hand, tree-level graviton amplitudes M_n^E can be computed from the formula

$$M_n^E = \frac{(-1)^{i+j+k}}{(2\pi)^{n-3}} i \oint_{\mathcal{C}} \frac{d^n z}{d\omega} \frac{z_{ij} z_{jk} z_{ki}}{\prod_{a \neq i,j,k} f_a(z,p)} E_n^2(z,p,\epsilon) \tag{4.38}$$

which means that the integrand will be proportional to a determinant. In Chapter 5, we will see that the similarity of the integrands is no coincidence, and that this formula for graviton amplitudes can be obtained from "squaring" the Yang-Mills formula and applying the so-called Kawai-Lewellen-Tye (KLT) orthogonality of the Parke-Taylor factors, which is the manifestation of the field theory limit of the KLT relations [24] between gravity and gauge theory amplitudes in the CHY formalism.

It is interesting to note that CHY representations for other theories, such as Dirac-Born-Infeld (DBI), Einstein-Yang-Mills (EYM) Yang-Mills scalar and the $U(N)$ non-linear sigma model have been obtained in [15] and [50], by taking the integrands above and applying different manipulations, such as dimensional reduction. One of the main features found with these representations is that the KLT relations are not exclusive to gravity and Yang-Mills amplitudes, but rather, that one can find general KLT relations between pairs of theories whenever one of them has a structure of color-ordering. We will see how this happens when we study the KLT relations from the CHY perspective in Chapter 5.

4.3. Calculation of CHY integrals

Now, we will see how to calculate amplitudes by using the CHY representation. First, we present two elementary examples: the three gluon amplitude and the four scalar amplitude in the biadjoint scalar theory. Three and four point amplitudes in the CHY representation have the virtue of not needing any mathematical tools beyond algebra and basic complex analysis. Afterwards, we will introduce one approach to calculating a general n-point amplitude using the CHY representation, which is based on the interpretation of the CHY integrals as multidimensional residues developed in [17] by Søgaard and Zhang and further refined in [18] by Bosma, Søgaard and Zhang.

Three-gluon amplitude

As a first example, we will use Eq. (4.23) to calculate the three-gluon amplitude. Explicitly,

$$A_3^{YM}(w,p,\epsilon) = (-1)^{i+j+k} i \oint_{\mathcal{C}} \frac{d^3 z}{d\omega} \frac{z_{ij} z_{jk} z_{ki}}{\prod_{a\neq i,j,k} f_a(z,p)} C_3(w,z) E_3(z,p,\epsilon) \tag{4.39}$$

We will take the cyclic order $w = 123$. As we mentioned in the previous chapter, the solution to scattering equations for $n = 3$ is completely fixed by $SL(2,\mathbb{C})$ invariance and can be fixed to any three arbitrary values. Moreover, in Eq. (4.39), the only possible choice for the indices i, j, k is a permutation of $1, 2, 3$. Hence, without loss of generality, we take $i = 1$, $j = 2$ and $k = 3$. Then,

$$\begin{aligned} A_3^{YM}(w,p,\epsilon) &= i \frac{(z_{12} z_{23} z_{31})^2}{z_{12} z_{23} z_{31}} E_3(z,p,\epsilon) \\ &= i(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) E_3(z,p,\epsilon) \end{aligned} \tag{4.40}$$

where we have replaced the explicit expression of the Parke-Taylor factor Eq.(4.24) for $n = 3$ and used the form of the $SL(2,\mathbb{C})$ invariant measure $d\omega$, Eq. (4.5) with $p = 1$, $q = 2$ and $r = 3$. Therefore, all we need to do is to compute the polarization function for $n = 3$, using Eq.(4.34). We choose to calculate the minor with $i = 2$ and $j = 3$ to calculate the reduced Pfaffian $\mathrm{Pf}'\ \Psi_3$, obtaining the 4×4 matrix

$$(\Psi_3)^{23}_{23} = \begin{pmatrix} 0 & \left(\frac{2\epsilon_1\cdot p_2}{z_1-z_2} - \frac{2\epsilon_1\cdot p_3}{z_3-z_1}\right) & \frac{2\epsilon_2\cdot p_1}{z_1-z_2} & -\frac{2\epsilon_3\cdot p_1}{z_3-z_1} \\ -\left(\frac{2\epsilon_1\cdot p_2}{z_1-z_2} - \frac{2\epsilon_1\cdot p_3}{z_3-z_1}\right) & 0 & \frac{2\epsilon_1\cdot \epsilon_2}{z_1-z_2} & -\frac{2\epsilon_1\cdot \epsilon_3}{z_3-z_1} \\ -\frac{2\epsilon_2\cdot p_1}{z_1-z_2} & -\frac{2\epsilon_1\cdot \epsilon_2}{z_1-z_2} & 0 & \frac{2\epsilon_2\cdot \epsilon_3}{z_2-z_3} \\ \frac{2\epsilon_3\cdot p_1}{z_3-z_1} & \frac{2\epsilon_1\cdot \epsilon_3}{z_3-z_1} & -\frac{2\epsilon_2\cdot \epsilon_3}{z_2-z_3} & 0 \end{pmatrix} \tag{4.41}$$

Now, we can use the formula Eq. (4.31) for the Pfaffian of a 4×4 matrix, to obtain

$$\mathrm{Pf}\ (\Psi_3)^{23}_{23} = \left(\frac{2\epsilon_1\cdot p_2}{z_1-z_2} - \frac{2\epsilon_1\cdot p_3}{z_3-z_1}\right)\frac{2\epsilon_2\cdot\epsilon_3}{z_2-z_3} + \frac{4(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_1)}{(z_1-z_2)(z_3-z_1)} - \frac{4(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)}{(z_1-z_2)(z_3-z_1)} \tag{4.42}$$

This expression can be simplified by noting that

$$\begin{aligned} \frac{2\epsilon_1\cdot p_2}{z_1-z_2} - \frac{2\epsilon_1\cdot p_3}{z_3-z_1} &= \frac{\epsilon_1\cdot p_2(z_3-z_1) - \epsilon_1\cdot p_3(z_1-z_2)}{(z_1-z_2)(z_3-z_1)} \\ &= \frac{-z_1\epsilon_1\cdot(p_2+p_3) + \epsilon_1\cdot p_2 z_3 + \epsilon_1\cdot p_3 z_2}{(z_1-z_2)(z_3-z_1)} \\ &= \frac{\epsilon_1\cdot p_2(z_3-z_2)}{(z_1-z_2)(z_3-z_1)} \end{aligned} \tag{4.43}$$

where the term proportional to z_1 in the numerator is identically zero because $\epsilon_1\cdot(p_2+p_3) = -\epsilon_1\cdot p_1 = 0$ using momentum conservation and the transversality of the polarization vectors, and we write $\epsilon_1\cdot p_3 = -\epsilon_1\cdot p_2$. Similarly, writing $\epsilon_2\cdot p_1 = -\epsilon_2\cdot p_3$ in the numerator of the second term, we obtain

$$\text{Pf}\ (\Psi_3)^{23}_{23} = -\frac{4[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot p_2)+(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_3)]}{(z_1-z_2)(z_3-z_1)} \tag{4.44}$$

Hence, including the prefactor $-(z_2-z_3)^{-1}$, we obtain the polarization function

$$E_3(z,p,\epsilon) = \frac{4[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot p_2)+(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_3)]}{(z_1-z_2)(z_2-z_3)(z_3-z_1)} \tag{4.45}$$

Multiplying by the Parke-Taylor factor, the z dependence cancels, as expected. This reflects the fact that, in kinematics where the three-point amplitudes are non-vanishing (for example, in complex momenta), the three-point amplitudes have vanishing residues in all their possible singularities, and we obtain the three-point Yang-Mills amplitudes

$$A_3^{YM}(w,p,\epsilon) = 4i[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot p_2)+(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_3)] \tag{4.46}$$

which can be seen to be the correct result by using the Feynman rule for the three-gluon vertex in Yang-Mills theory.

Four scalar amplitude in biadjoint theory

For simplicity, we will calculate the four scalar amplitude for the same external ordering $w_1 = w_2 = 1234$ for the two Parke-Taylor factors that will appear in the integrand of 4.37, with $n=4$. Then, using the polynomial form of the scattering equations, our task is to calculate the integral

$$m_4(1234|1234) = i\oint_C \frac{1}{2\pi i}\frac{d^4z}{d\omega}\frac{\prod_{i<j}^4 z_{ij}}{h_2(z,p)}\frac{1}{(z_{12}z_{23}z_{34}z_{41})^2} \tag{4.47}$$

where we have used the fact that the only non-trivial polynomials h_m occur for $2\le m\le n-2$, and with $n=4$ the only possible value that m can take is 2. Now, we first need to write down the polynomial $h_2(z,p)$ explicitly. In order to do this, we first identify all the subsets of $\{1,2,3,4\}$ which contain two elements. The number of this subsets can be found using the binomial coefficient

$$N_2 = \frac{4!}{2!(4-2)!} = 6 \tag{4.48}$$

The subsets are easy to find, and are $\{1,2\}$, $\{1,3\}$,$\{1,4\}$, $\{2,3\}$,$\{2,4\}$ and $\{3,4\}$. Then,

$$h_2(z,p) = s_{12}z_1z_2 + s_{13}z_1z_3 + s_{14}z_1z_4 + s_{23}z_2z_3 + s_{24}z_2z_4 + s_{34}z_3z_4 \tag{4.49}$$

Notice that, since the particles are assumed to be massless, four point kinematics implies that

$$\begin{aligned} &s_{12}=s_{34}, \quad s_{13}=s_{24}, \quad s_{14}=s_{23} \\ &0 = p_i^2 = (p_j+p_k+p_l)^2 = s_{jk}+s_{jl}+s_{kl} \end{aligned} \tag{4.50}$$

where, in the second equation, the indices i, j, k, l take any value out of $1, 2, 3, 4$ in such a way that $i \neq j \neq k \neq l$. For example,

$$s_{12} + s_{14} + s_{24} = 0 \tag{4.51}$$

which is equivalent to the relation $s+t+u = 0$ of the usual Mandelstam variables for massless particles, only written in a different notation. Now, we can easily write out the product

$$\prod_{i<j}^{4} z_{ij} = z_{12}z_{13}z_{14}z_{23}z_{24}z_{34} \tag{4.52}$$

With these equations, we have the explicit form of our integrand. Now, let us take

$$d\omega = \frac{dz_1 dz_3 dz_4}{z_{13}z_{34}z_{41}} \tag{4.53}$$

(notice that the $(-1)^{i+j+k}$ factor equals one in this case). Then, prior to fixing the values of the three "free" punctures, we put all the factors of the integrand together to obtain

$$\begin{aligned} m_4(1234|1234) &= -i \oint_C \frac{dz_2}{2\pi i} \frac{z_{12}z_{13}^2 z_{14}^2 z_{23}z_{24}z_{34}^2}{z_{12}^2 z_{23}^2 z_{34}^2 z_{41}^2} \frac{1}{h_2} \\ &= i \oint_C \frac{dz_2}{2\pi i} \frac{z_{13}^2 z_{42}}{z_{12}z_{23}} \frac{1}{h_2} \end{aligned} \tag{4.54}$$

where we have used the fact that $z_{ij} = -z_{ji}$. Now, we perform the gauge-fixing $z_1 = 1, z_3 = 0, z_4 = \infty$; the first two conditions are easily applied, and we are left with

$$m_4(1234|1234) = i \oint_C \frac{1}{z_2(1-z_2)} \lim_{z_4 \to \infty} \frac{z_{42}}{h_2} \tag{4.55}$$

where h_2 is implicitly evaluated at $z_1 = 1, z_3 = 0$. Now, we can easily calculate the limit

$$\begin{aligned} \lim_{z_4 \to \infty} \frac{z_{42}}{h_2} &= \lim_{z_4 \to \infty} \frac{z_4 - z_2}{z_4(s_{14} + s_{24}z_2) + s_{12}z_2} \\ &= \frac{1}{s_{14} + s_{24}z_2} \end{aligned} \tag{4.56}$$

thus, we must solve the contour integral

$$m_4(1234|1234) = i \oint_C \frac{dz_2}{2\pi i} \frac{1}{z_2(1-z_2)} \frac{1}{s_{14} + z_2 s_{24}} \tag{4.57}$$

Since the integrand vanishes for $z_2 \to \infty$, there is no residue at infinity. Morever, although the integrand has two singularities (at $z_4 = 1$ and $s_{13} + z_4 z_{34} = 0$), only the second one contributes because, for general kinematics, $z_i \neq z_j$ for $i \neq j$ given a particular solution to the scattering equations. Note that, so far, we have not needed to solve the scattering equations, even though we already found the solution in Chapter 3. This is because we are implicitly solving them when finding the poles of the polynomials h_m. In this case, we find a pole at

$$z_2 = -\frac{s_{14}}{s_{24}} \tag{4.58}$$

The evaluation of the integral is a simple application of the residue theorem, and yields

$$\begin{aligned} m_4(1234|1234) &= -i\frac{s_{24}}{s_{12}s_{14}} \\ &= i\left(\frac{1}{s_{12}} + \frac{1}{s_{24}}\right) \end{aligned} \tag{4.59}$$

Beyond four-points, the integrals we have to perform in the CHY formalism take the form of multidimensional contour integrals. Therefore, our task is to introduce the concepts that allow for the calculation of such integrals.

4.3.1. Multidimensional residues

Consider a function $p = (p_1, ..., p_n)$, defined in $\mathbb{C}^n$. Each of the components is an scalar function that takes a point in $\mathbb{C}^n$ into the complex plane. Assume that the system of equations $p_i(z) = 0$ has a finite number of solutions $z^{(j)}$. Furthermore, let $f(z)$ denote a function in $\mathbb{C}^n$ which is non-singular at all the solutions $z^{(j)}$. The local residue of f with respect to the divisors p_i at $z^{(j)}$ is defined as the integral

$$Res_{(p_1,...,p_n)}(f, z^{(j)}) = \frac{1}{(2\pi i)^n}\oint_{T_\delta} \frac{f(z)dz_1 \wedge ... \wedge dz_n}{p_1(z)...p_n(z)} \tag{4.60}$$

where $\wedge$ stands for the wedge product (that is, we understand the measure as a differential form in $\mathbb{C}^n$), the integration contour is taken to be an n-torus,

$$T_\delta = \{z \in \mathbb{C}^n | \, |p_i(z)| = \delta\} \tag{4.61}$$

where δ is a small number such that T_δ encloses the point $z^{(j)}$. The contour is oriented according to

$$d\,arg\,p_1 \wedge d\,arg\,p_2 \wedge ... \wedge d\,arg\,p_n \geq 0 \tag{4.62}$$

The global residue of f with respect to the divisors $p_1, ..., p_n$ is defined as the sum of all its local residues,

$$Res_{(p_1,...,p_n)}(f) = \sum_j Res_{(p_1,...,p_n)}(f, z^{(j)}) \tag{4.63}$$

In practice, one can encounter three situations. The first one is when the divisors are each functions of only one complex variable, i.e. $p_i(z) = p_i(z_i)$. Then, each local residue is said to be factorizable and can be calculated as the product of one-dimensional contour integrals,

$$Res_{(p_1,...,p_n)}(f, z^{(j)}) = \left(\prod_{i=1}^{n} \oint_{|p_i(z_i)|=\delta} \frac{dz_i}{2\pi i}\frac{1}{p_i(z_i)}\right) f(z^{(j)}) \tag{4.64}$$

where $z^{(j)}$ is the vector in $\mathbb{C}^n$ whose components are given by the solutions to the n independent equations $f_i(z_i^{(j)}) = 0$.

In the general case where each divisor is a function of several complex variables, one can calculate the Jacobian

$$J(z) = \mathrm{Det}\left(\frac{\partial(p_1, ..., p_n)}{\partial(z_1, ..., z_n)}\right) \tag{4.65}$$

and evaluate it at the point $z = z^{(j)}$. Then, if $J(z^{(j)}) \neq 0$, we say the residue is non-degenerate and can be evaluated by performing the change of variables

$$x_i = p_i(z), \quad i = 1, ..., n \tag{4.66}$$

which then yields

$$Res_{(p_1,...,p_n)}(f, z^{(j)}) = \frac{1}{(2\pi i)^n} \oint_{T_\delta} \frac{f(z)dz_1 \wedge ... \wedge dz_n}{p_1(z)...p_n(z)} = \frac{f(z^{(j)})}{J(z^{(j)})} \tag{4.67}$$

On the other hand, if $J(z^{(j)}) = 0$, we say that the local residue is degenerate, and algebraic geometry methods are required for their calculation.

From this discussion, we see that the CHY representation provides a concise mathematical formulations of tree level amplitudes: an n-point amplitude is the global residue of a holomorphic function with respect to the functions $f_i(z, p)$, $1 \leq i \leq n$, or with respect to the polynomial functions $h_m(z, p)$, $2 \leq m \leq n-2$. If the solutions to the scattering equations are known, we can apply Eq. (4.67) to calculate each local residue and then sum all the local residues to obtain the amplitude. We will apply this method in Appendix F to calculate the 5-point amplitude in ϕ^3 scalar theory.

However, as we have discussed, the solution to the scattering equations for $n \geq 6$ is in general very difficult to obtain. Moreover, since the amplitudes are given by the global residue, one can focus on the structure of the complete sum instead of the structure of each individual residue. Taking advantage of the fact that the divisors of the CHY integrand can be written as the polynomials h_m, the authors of [17], [18] developed an approach based on algebraic geometry which allows for the computation of the global residue without calculating the local residues individually. For this discussion, we will need some concepts of abstract algebra reviewed in Appendix H.

The idea is as follows: consider a set of n polynomials $p_i(z)$ in the ring $R = \mathbb{C}[z_1, ..., z_n]$, and assume that the ideal $I = \langle p_1, p_2, ..., p_n \rangle$ is zero-dimensional, that is, that the set of solutions to the equations $p_i = 0$ for $i = 1, 2, ..., n$ contains only a finite number of points. This means that the quotient ring R/I is a finite dimensional vector space with coefficients in $\mathbb{C}$. Hence, for any function $f(z)$ which is non-singular in the points $z^{(j)}$ where the polynomials vanishes, one defines the global residue as in Eq. (4.63) as the sum of all local residues, which will simply be denoted by Res(f). We will refer to f generically as the numerator.

Due to a theorem from algebraic geometry[2], one can define a symmetric, non degenerate inner product on the quotient ring R/I as

[2]Whose proof can be found, for example, in [60]

$$\langle f_1, f_2\rangle = \mathrm{Res}(f_1 f_2) \tag{4.68}$$

Now, since R/I is finite dimensional, all of its elements can be expanded in a basis $\{e_i\}$. From the existence of an inner product in R/I, one deduces the existence of a dual basis Δ_i that satisfies

$$\langle e_i, \Delta_j\rangle = \delta_{ij} \tag{4.69}$$

Hence, we can expand the numerator f in the basis $\{e_i\}$

$$f(z) = \sum_i c_i e_i \tag{4.70}$$

Also, we can expand the identity 1 in the dual basis,

$$1 = \sum_j u_i \Delta_i \tag{4.71}$$

This allows to write the global residue as

$$\begin{aligned}\mathrm{Res}(f) &= \mathrm{Res}(f \times 1)\\ &= \langle f, 1\rangle\\ &= \sum_{i,j} c_i u_j \langle e_i, \Delta_j\rangle\\ &= \sum_{i,j} c_i u_j \delta_{ij}\\ &= \sum_i c_i u_i\end{aligned} \tag{4.72}$$

In particular, this means that, if one of the elements in the dual basis is a constant, say Δ_r, the expansion of the identity in the dual basis is simply $1 = \frac{\Delta_r}{\Delta_r}$, which implies

$$\mathrm{Res}(f) = \frac{c_r}{\Delta_r} \tag{4.73}$$

The usefulness of this approach, as discussed in [17], is that the dual basis always has a constant element for the ideal generated by the polynomials h_m. The basis $\{e_i\}$ can be calculated as Gröbner basis (which we define in Appendix H) and the dual basis is then obtained by the Bezoutian matrix method. In [18], this method was further refined by showing that the polynomials h_m form what is known as Macauly H-basis. A review of the full algorithm to calculate CHY integrals based on these H-basis is provided in [61]. This approach is further explored and applied to loop amplitudes in [62].

We finish this section by mentioning some of the other approaches to the calculation of CHY integrals that have been developed. In [63], Cachazo and Gomez introduce rules for the calculation of a particular class of integrals based on graph theory, Hamiltonian cycles and the KLT relations; later, Baadsgaard et all provide the so-called "integration rules" for CHY integrands with simple poles in [64], based on combinatorics, and further generalize this rules to CHY loop integrands in [65].

Also, in [66], Cardona et al. introduce the so-called cross-ratio identities, which allow to reduce integrands with higher order poles as sum of integrals with simple poles, which can be evaluated by the integration rules.

4.4. Connected formalism

To finish this chapter, we introduce the so-called connected prescription or connected formalism for tree-level superamplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory.

As we discuss in Appendix C, when scattering amplitudes are transformed from momentum to twistor space, they are supported on algebraic curves, which is very similar to the scattering amplitudes obtained in the perturbative expansion in string theory, where L-loop order contribution to the n-point amplitude can be written as an integral over a genus L Riemann surface. For example, tachyon tree amplitudes in closed string theory are given by the closed formula

$$\mathcal{A}_n^{closed} \sim \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C})} \int \prod_{i=1}^{n} d^2 z_i \mathrm{exp}\left(\frac{\alpha'}{2}\sum_{j,l} p_j \cdot p_l \log|z_j - z_l|\right) \tag{4.74}$$

which, for n = 4, gives the famous Virasoro-Shapiro amplitude. In this expression, α' is the inverse string tension. The integration in 4.74 is performed over n insertion points of the Riemann sphere, and the exponential is obtained from the string path integral as the expactation value of a product of vertex operators at the insertion points $z_1, z_2, ..., z_n$. However, the integral is not localized, but rather performed over an ordered region $|z_1 < |z_2| < ... < |z_n|$. The fact that gauge theory amplitudes are supported on algebraic curves in twistor space led Witten [19] to propose that the scattering amplitudes in $\mathcal{N} = 4$ SUSY could be computed form the D-instanton expansion of the so-called topological B model, whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$.

4.4.1. RSV formula

Originally, Witten conjectured that different instanton expansions (for example, the completely connected expansion with a degree D instanton or the completely disconnected expansion with D degree 1 instantons) should give different contributions to the $\mathcal{N} = 4$ scattering amplitudes at each fixed loop order. However, each of these expansions happened to yield the complete tree level S-matrix, thus providing different prescriptions for the calculation of amplitudes in Yang-Mills theory. For example, the completely disconnected instantons yield the so called CSW rules, [67], where amplitudes are computed using a set of diagrams where each vertex corresponds to an MHV amplitude. In [20] Roiban, Spradlin and Volovich introduce the so-called connected prescription, which is the formula arising from Witten's twistor string theory when using the completely connected D-instanton. In the original notation, the RSV formula can be written as

$$\mathcal{A}_n^{\mathcal{N}=4} = i(2\pi)^4 \sum_{d=1}^{n-3} \int dM_{n,d} \prod_{i=1}^{n} \delta^2(\lambda_i^\alpha - \xi_i P_i^\alpha) \textstyle\prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} t_i \sigma_i^k \tilde{\lambda}_i^{\dot\alpha}\right) \delta^4\left(\sum_{i=1}^{n} t_i \sigma_i^k \eta_{iA}\right) \tag{4.75}$$

where P_i^α, $i = 1, 2$ are degree d polynomials

$$P_i^\alpha = \sum_{k=0}^{d} \rho_k^\alpha \sigma_i^k \tag{4.76}$$

and the measure is given by

$$dM_{n,d} = \frac{d^{2d+2}\rho d^n\sigma d^n t}{\mathrm{Vol}(GL(2)} \prod_{i=1}^{n} \frac{1}{t_i(\sigma_i - \sigma_{i+1}} \tag{4.77}$$

Where we identify $\sigma_{n+1} \equiv \sigma_1$. We can see some similarities with the CHY formula, Eq. (4.1). First, we have to include a factor of $\mathrm{Vol}(GL(2))$ that cancels a gauge redundancy of the integral; in this case, the one can fix four of the integration variables to arbitrary values, just like we fix three of the z integration variables of the scattering equations. Moreover, we see the appearance of the Parke-Taylor-like factor

$$\frac{1}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)...(\sigma_n - \sigma_1)} \tag{4.78}$$

Also, as pointed out in the original RSV paper, the integral is fully localized to the solutions of a set of polynomial equations. To note this, we can see that the number of integration variables, after substracting the four integration variables fixed by the $GL(2)$ symmetry is $2n + 2d - 2$ for a given d. On the other hand, each term in the sum has $2n + 2d + 2$ delta functions. However, the delta functions set

$$\sum_{i=1}^{n} t_i \sigma_i^k \tilde{\lambda}_i^{\dot{\alpha}} = 0 \tag{4.79}$$

For every $1 \leq k \leq d$. Hence, any linear combination of them should also be zero on the support of the delta functions; this yields

$$\begin{aligned} 0 &= \sum_{k=1}^{d} \rho_k^\alpha \left(\sum_{i=1}^{n} t_i \sigma_i^k \tilde{\lambda}_i^{\dot{\alpha}} \right) \\ &= \sum_{i=1}^{n} t_i \tilde{\lambda}_i^{\dot{\alpha}} \left(\sum_{k=1}^{d} \rho_k^\alpha \sigma_i^k \right) \\ &= \sum_{i=1}^{n} t_i P_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \\ &= \sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \\ &= \sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}} \end{aligned} \tag{4.80}$$

where, in the second to last step, we use another set of delta functions to transform $\xi_i P_i^\alpha$ into λ_i^α. What we just obtained is the statement of momentum conservation; this means that, by introducing a Jacobian, one can pull out a momentum conservation delta function explicitly. In four dimensions, this reduces the number of delta functions by 4, yielding a total of $2n + 2d - 2$ delta functions, which is exactly the same as the number of integration variables. Hence, the integral is completely localized to the solutions of the equations

$$\begin{aligned} &\sum_{i=1}^{n} t_i \sigma_i^k \tilde{\lambda}_i^{\dot{\alpha}} = 0, \quad k = 0, ..., d, \ \dot{\alpha} = 1, 2 \\ &\lambda_i^{\alpha} - t_i \sum_{k=0}^{d} \rho_k^{\alpha} \sigma_i^k = 0, \quad i = 1, ..., n, \ \alpha = 1, 2 \end{aligned} \tag{4.81}$$

which are just the four dimensional form of the scattering equations, Eq.(3.19). Indeed, the first appearance of the scattering equations in the work of Cachazo et al. in [11] came from trying to generalize the constraints of the RSV formula to arbitrary space-time dimension. After the introduction of the scattering equations, it was shown by Mason and Skinner [58] that both the CHY formula and the RSV formula can be derived from ambitwistor string theory in ten and four dimensions, respectively, and later generalized to arbitrary number of supersymmetry in four dimensions by Geyer, Lipstein and Mason in [68].

With this, we finish our first approach to the CHY representation of tree level amplitudes. We have shown it allows to write closed formulas for amplitudes in a variety of theories, independent of the spacetime dimension. However, one problem which we did not touch upon is that the set of theories which have a known CHY integrand seem to include only bosons. This is not a problem in the connected formalism, since one can extract component amplitudes with an arbitrary number of gluinos from the complete superamplitude; however, not only gluinos are fundamentally different from quarks, since they transform in different representations of the gauge groups, but are also massless if supersymmetry is unbroken. In chapter 6, we attack this problem from two different point of views: on one hand, we will use the approach of [26] to show that, under suitable assumptions, there must exist a CHY representation for QCD primitive amplitudes with an arbitrary number of massive quarks with the property that the half-integrands separate the information on the polarizations and the external orderings in a way that mirrors that of the pure Yang-Mills case. On the other, we will use a rewritten version of the connected formula and the results of [28] to write down QCD amplitudes with massless quarks as linear combinations of gluon-gluino amplitudes in $\mathcal{N} = 4$ SUSY, which yields the four dimensional integrands of the CHY representation for amplitudes with quarks.

Basis QCD amplitudes

Recall that, in Chapter 2, we discussed that an arbitrary Yang-Mills n-point tree amplitude $\mathcal{A}_n^{YM}$ can be color decomposed according to Eq.(2.19) as

$$\mathcal{A}_n^{YM} = \sum_{w \in S_n/Z_n} \mathrm{Tr}(T^{(a_{l_1})} T^{\sigma(a_{l_2})} \cdots T^{\sigma(a_{l_n})}) A_n(l_1 l_2 ... l_n) \tag{5.1}$$

Where the objects A_n are what we have called primitive amplitudes, and we have rewritten the permutations in terms of words. We can think of color decomposition as a process where we expand the complete, color-dressed amplitude into a basis of amplitudes where the coefficients are purely group-theoretical factors. Moreover, as we discuss in Appendix G, we can understand color-ordered amplitudes as linear operators that act on the vector space of words, that is, of external orderings. Hence, it would be both interesting and useful to determine a minimal set of linearly independent color-ordered (more precisely, of primitive) amplitudes, which would form a basis for this vector space. This immediately poses a question: what is the size of the basis? Note that in Eq. (2.19) we only sum over non-cyclic permutations; this is because the trace of a product of matrices is cyclic, which means that primitive amplitudes are invariant under a cyclic permutation of its external indices[1]. Naively, one would be led to think that there are $(n-1)!$ primitive amplitudes, corresponding to all the non-cyclic permutations of the external legs. However, we will see that these amplitudes are not all independent, and thus the size of the basis can be reduced.

In this Chapter, we will consider the different linear relations that exist between primitive amplitudes, and how they allow one to reduce the size of the amplitude basis for a given number of external particles. We will use the CHY representation of Yang-Mills amplitudes to prove different relations between primitive amplitudes.

Recall that the CHY integrand for Yang-Mills primitive amplitudes splits into two parts: the Parke-Taylor factor $C_n(w)$ and the polarization function $E_n(z, p, \epsilon)$. Since the polarization function does not depend on the external ordering, all the properties related to the cyclic ordering should be reflected on the Parke-Taylor factor

[1]This becomes obvious from the point of view of Feynman diagrams: a cyclic permutation corresponds to a rotation of the diagram, which does not change the structure of the amplitude.

$$C_n(w) = \frac{1}{z_{l_1 l_2} \ldots z_{l_n l_1}} \tag{5.2}$$

where we use a generic word of length n, given by $w = l_1 l_2 \ldots l_n$. We will prove that the Parke-Taylor factors satisfy the so-called $U(1)$-decoupling and reflection identities, as well Kleiss-Kuijf (KK) relations, which are purely group-theoretical in nature. This will suffice to prove that the Yang-Mills primitive amplitudes also satisfy these relations, since the Parke-Taylor factor carry all the information on the external orderings. We will then see how the KK relations allow one to reduce the number of basis amplitudes to $(n-2)!$ in the case of gluons.

Afterwards, we will introduce the Bern-Carrasco-Johansson (BCJ) relations, which are a set of linear relations between primitive amplitudes that also depend on the kinematics, and underpin the so-called Color-Kinematics duality. In a sense, the BCJ relations arise from constructing the amplitudes out of quantities where the kinematics have algebraic properties that mirror those of the color factors. We will discuss the different types of BCJ relations, and briefly comment on their relation to gravity tree amplitudes. The BCJ relation will allow us to reduce the size of the amplitude basis to $(n-3)!$.

Following this discussion we will construct the amplitude basis for QCD amplitudes, by introducing the no-crossed fermion line relations and the so-called Dyck words. The no-crossed fermion line relations will be seen to be related to the KK relations, and the consequence of this will be to reduce the size of the amplitude basis by a factor of $2(n_q-1)/(n_q)!$ when the amplitude has n_q quark-antiquark pairs. These will form the backbone of the construction of the CHY integrand for QCD primitive amplitudes which we will work out in Chapter 6.

We will end this chapter by introducing the Kawai-Lewellen-Tye (KLT) relations, which allow one to obtain gravitational amplitudes as the sum of certain "squares" of Yang-Mills amplitudes. We will see how this are obtained in the CHY formalism and how their origin is linked to the existence of a color decomposition. This will allow us to see the KLT relations in a much more general context, in which one can obtain objects that may behave as amplitudes from the knowledge of the amplitudes of theories that have any gauge degrees of freedom.

5.1. Consequences of trace decomposition

5.1.1. Cyclic invariance

As we have mentioned, the fact that the trace of a product of matrices is cyclic in its arguments imply that the primitive amplitudes in the expansion Eq. (2.19) are also invariant under a cyclic permutation of their labels. If we consider a word $w = l_1 l_2 \ldots l_n$, this condition is

$$A_n(l_1 l_2 \ldots l_n) = A_n(l_2 \ldots l_n l_1) \tag{5.3}$$

This is known as cyclic invariance. This is a property only of primitive amplitudes, in the case of quarks: in general, the partial amplitude that arise in the $\frac{1}{N}$ terms of the $SU(N)$ completeness relations do not satisfy this. Since cyclic invariance implies that all amplitudes related by a cyclic permutation of particle labels are equal, this reduces the basis from $n!$ to $(n-1)!$, and we can use it to fix the position of one of the external legs, for example, to set $l_1 = 1$. Then, we rewrite

$$\mathcal{A}_n^{YM} = \sum_{S_{n-1}} \mathrm{Tr}(T^{(a_1)} T^{\sigma(a_{l_2})} \cdots T^{\sigma(a_{l_n})}) A_n(1 l_2 ... l_n) \tag{5.4}$$

where we sum over all permutations of the $(n-1)$ labels $l_2, l_3, ..., l_n$. Now, let us state and prove the reflection and $U(1)$ decoupling identities, using the properties of the Parke-Taylor factors.

5.1.2. Reflection and U(1) decoupling identities

Through this section, we use the word $w = l_1 l_2 ... l_n$ to specify the external ordering of the primitive amplitudes. First, consider the reflection identity, which states that, given a primitive Yang-Mills amplitude,

$$A_n(w) = (-1)^n A_n(w^T) \tag{5.5}$$

where $w^T = l_n ... l_2 l_1$ is the word with the order reversed. Then, what we need to show is

$$C_n(w, z) = (-1)^n C_n(w^T, z) \tag{5.6}$$

which is trivial, since each factor in the denominator of the Parke-Taylor factor satisfies $z_{ab} = z_a - z_b = -(z_b - z_a) = -z_{ba}$. Then, to obtain $C_n(w^T)$ from $C_n(w)$, we should simply flip the two labels of each factor in the denominator. Since there are n such terms in an n-point Parke-Taylor factor, we pick up an overall sign of $(-1)^n$.

Now, to state the $U(1)$ decoupling (or photon decoupling) identity, we recall that, for two words $u = k_1 ... k_j$, $v = k_{j+1} ... k_r$,

$$u \sqcup\!\sqcup\, v = \sum_{shuffles\ \sigma} k_{\sigma(1)} ... k_{\sigma(r)} \tag{5.7}$$

where the sum over all permutations of the set $k_1 ... k_r$ that preserves the relative order of the letters that make up the words u and w. When we write an amplitude (or a Parke-Taylor factor) whose argument is a shuffle product, what we mean is

$$A_r(u \sqcup\!\sqcup\, v) = \sum_{shuffles\ \sigma} A_r(k_{\sigma(1)} ... k_{\sigma(r)}) \tag{5.8}$$

That is, every time a shuffle product appears in the argument of an amplitude we assume that there is an implicit sum over the terms appearing in it.

Let $w_{n-2} = l_2 ... l_{n-1}$ be a subword, such that $w = l_1 w_{n-2} l_n$. The $U(1)$ decoupling identity states that

$$C_n(l_1(w_{n-2} \sqcup\!\sqcup\, l_n)) = 0 \tag{5.9}$$

We will prove that this is a simple algebraic identity in terms of the Parke-Taylor factor, but directly in terms of amplitudes it is a consequence of the fact that we can couple an additional $U(1)$ photon to an $SU(N)$ gauge theory in order to have a theory with the gauge group $U(N)$; since the $U(1)$ photon has Abelian dynamics, all the structure constants involving such a photon vanish and that is reflected on the partial amplitudes as the $U(1)$ decoupling identity.

In order to prove this identity, let $w^j = l_1 l_2 ... l_j l_n l_{j+1} ... l_{n-1}$ be the word where l_n is between the letters l_j and l_{j+1} for $j = 1, 2, ..., n-1$. Then, we can rewrite

$$C_n(l_1(w_{n-2} \sqcup l_n)) = \sum_{j=1}^{n-2} C_n(w^j) + C_n(l_1 l_2 ... l_n) \tag{5.10}$$

then, to show that the $U(1)$ decoupling identity holds, we prove the equivalent statement

$$\sum_{j=1}^{n-2} C_n(w^j) = -C_n(l_1 l_2 ... l_n) \tag{5.11}$$

To do this, we can see that any word w^j can be rewritten as

$$\begin{aligned} C_n(w^j) &= \frac{1}{z_{l_1 l_2} ... z_{l_j l_n} z_{l_n l_{j+1}} ... z_{l_{n-1} l_1}} \frac{z_{l_j l_{j+1}}}{z_{l_j l_{j+1}}} \\ &= C_{n-1}(l_1 ... l_{n-1}) \frac{z_{l_j l_{j+1}}}{z_{l_j l_n} z_{l_n l_{j+1}}} \\ &= C_{n-1}(l_1 ... l_{n-1}) Z(l_j, l_n, l_{j+1}) \end{aligned} \tag{5.12}$$

where we define

$$Z(a, b, c) = \frac{z_{ac}}{z_{ab} z_{bc}} = \frac{1}{z_{ab}} + \frac{1}{z_{bc}} \tag{5.13}$$

where the second line is obtained by adding and subtracting z_c in the denominator. This allows us to show that

$$Z(a, b, c) + Z(c, b, d) = \frac{1}{z_{ab}} + \frac{1}{z_{bc}} + \frac{1}{z_{cb}} + \frac{1}{z_{bd}} = Z(a, b, d) \tag{5.14}$$

due to the antisymmetry of z_{ab}. Using this relation iteratively, one finds that the cross-ratios $Z(a, b, c)$ satisfy the Eikonal identity,

$$\sum_{j=a}^{c-1} Z(j, b, j+1) = Z(a, b, c) \tag{5.15}$$

Hence, we can see that

$$\begin{aligned} \sum_{j=1}^{n-2} C_n(w^j) &= \sum_{j=1}^{n-2} C_{n-1}(l_1 ... l_{n-1}) Z(l_j, l_n, l_{j+1}) \\ &= C_{n-1}(l_1 ... l_{n-1}) \sum_{j=1}^{n-2} Z(l_j, l_n, l_{j+1}) \\ &= C_{n-1}(l_1 ... l_{n-1}) Z(l_1, l_n, n-1) \\ &= -C_n(l_1 ... l_n) \end{aligned} \tag{5.16}$$

which is what we wanted to prove. Note that the $U(1)$ identity is independent of the values of the variables z; it just relies on the cyclic structure of the Parke-Taylor factors. This confirms that it does not depend on the kinematic configuration of the amplitude, only on the fact that it can be color decomposed.

5.2. Kleiss-Kuijf relations

Now, we will prove what can be seen as generalization of the $U(1)$ decoupling identities, the so-called Kleiss-Kuijf relations [21]. Afterwards, we will see how they apply to the case with quarks.

The amplitudes independent under the KK relations naturally span a basis in which one can expand the color-dressed amplitude, in which the color factors are not traces of gauge group generators, but rather products of structure constants contracted in a suitable way. This basis forms what is known as the Del Duca-Dixon-Maltoni (DDM) decomposition [69].

5.2.1. Derivation of the KK relations

In this section, we derive the KK relations for pure Yang-Mills theory, using the CHY representation. Our argument closely follows [70].

Let $w_1 = l_{\alpha_1}...l_{\alpha_{n-m-2}}$ and $w_2 = l_{\beta_1}...l_{\beta_m}$ be two subwords of the word $w = l_1...l_n$, subject to the conditions

$$\{l_1\} \cup \{l_{\alpha_1}...l_{\alpha_{n-m-2}}\} \cup \{l_{\beta_1}...l_{\beta_m}\} \cup \{l_n\} = \{l_1, l_2, ..., l_n\} \tag{5.17}$$

Then, we can state the KK relations as

$$C_n(l_1 w_1 l_n w_2) = (-1)^m C_n(l_1(w_1 \sqcup\!\sqcup\, w_2^T) l_n) \tag{5.18}$$

We will prove the KK relations by induction, both on m and on n. As a short-hand notation, let $k = n - m - 2$ be the length of the subword w_1. For $m = 1$ and for arbitrary m, the KK relations are simply the $U(1)$ decoupling identity we just have proven, which are true for arbitrary n. Now, assume as the inductive hypothesis that the relations are valid for all multiplicities equal to or less than $(n+1)$ and for some $m > 1$. Then, we can see that

$$\begin{aligned} C_{n+1}(l_1 w_1 l_n w_2 l_{\beta_{m+1}}) &= C_n(l_1 w_1 l_n w_2) Z(l_{\beta_m}, l_{\beta_{m+1}}, l_1) \\ &= (-1)^m C_n(l_1(w_1 \sqcup\!\sqcup\, w_2^T) l_n) Z(l_{\beta_m}, l_{\beta_{m+1}}, l_1) \end{aligned} \tag{5.19}$$

where we have used the inductive hypothesis. Now, we rewrite the sum over shuffles in a more explicit way. In order to this, define $\not{w}_2 = l_{\beta_1}...l_{\beta_{m-1}}$ as the word w_2 without its last letter; for $s = 1, ..., k$, let $w_s = l_{\alpha_1}...l_{\alpha_s}$ be a subword of w_1 of length s, and w_s^C be the word such that $w_s w_s^C = w_1$, which we will call the complement of w_s. Finally, we let

$$\tilde{C}_n(l_1...l_n) = \frac{1}{z_{l_1 l_2}...z_{l_{n-1} l_n}} = z_{l_n l_1} C_n(l_1...l_n) \tag{5.20}$$

With these definitions, we can write

$$C_n(l_1(w_1 \sqcup\!\sqcup w_2^T)l_n) = \sum_{s=1}^{k} \frac{\tilde{C}_{s+2}(l_1, w_s, l_{\beta_m})\tilde{C}_{n-s-3}(l_{\beta_m}(w_s^C \sqcup\!\sqcup \psi_2^T)l_n)}{z_{l_n l_1}} \tag{5.21}$$

Furthermore, we can reconstruct a Parke-Taylor factor out of $\tilde{C}_{s+2}$ by multiplying with the cross-ratio,

$$\tilde{C}_{s+2}(l_1, w_s, l_{\beta_m})Z(l_{\beta_m}, l_{\beta_{m+1}}, l_1) = C_{s+3}(l_1 w_s l_{\beta_m} l_{\beta_{m+1}})z_{l_{\beta_m} l_1} \tag{5.22}$$

With this, we can see that

$$\begin{aligned} C_{n+1}(l_1 w_1 l_n w_2 l_{\beta_{m+1}}) &= (-1)^m \sum_{s=1}^{k} C_{s+3}(l_1, w_s, l_{\beta_m}, l_{\beta_{m+1}})\tilde{C}_{n-s-3}(l_{\beta_m}(w_s^C \sqcup\!\sqcup \psi_2^T)l_n)\frac{z_{l_{\beta_m} l_1}}{z_{l_n l_1}} \\ &= (-1)^{m+1} \sum_{s=1}^{k} C_{s+3}(l_1(w_s \sqcup\!\sqcup' l_{\beta_{m+1}})l_{\beta_m})\tilde{C}_{n-s-3}(l_{\beta_m}(w_s^C \sqcup\!\sqcup \psi_2^T)l_n)\frac{z_{l_{\beta_m} l_1}}{z_{l_n l_1}} \\ &= (-1)^{m+1} \sum_{s=1}^{k} C_{s+3}(l_1(w_s \sqcup\!\sqcup' l_{\beta_{m+1}})l_{\beta_m})z_{l_{\beta_m} l_1}\tilde{C}_{n-s-2}(l_{\beta_m}(w_s^C \sqcup\!\sqcup \psi_2^T)l_1) \\ &= (-1)^{m+1} \sum_{s=1}^{k} \tilde{C}_{s+3}(l_1(w_s \sqcup\!\sqcup' l_{\beta_{m+1}})l_{\beta_m})\tilde{C}_{n-s-2}(l_{\beta_m}(w_s^C \sqcup\!\sqcup \psi_2^T)l_1) \\ &= (-1)^{m+1} C_{n+1}(l_1(w_1 \sqcup\!\sqcup (w_2 l_{\beta_{m+1}})^T)l_n \end{aligned} \tag{5.23}$$

where, in the second line, we use the $U(1)$ relation to move the letter l_{β_m} to the right by introducing the sum, and in the other we regroup terms to arrive at the $(n+1)$-points Parke-Taylor factor.

To make the KK relations a little more clear, let us consider a simple example with $n = 6$. Consider the word $w = 132645$. We want to use the KK relations to expand the Parke-Taylor factor $C_6(w)$ in terms of six-point Parke-Taylor factor with external orderings of the form $1w_4 6$. In order to do this, we can see that the subwords w_1 and w_2 that enter the definition of the KK relations are given by

$$w_1 = 32, \quad w_2 = 45 \tag{5.24}$$

therefore, we can calculate the shuffle product

$$w_1 \sqcup\!\sqcup w_2 = 3245 + 3425 + 3452 + 4352 + 4325 + 4532 \tag{5.25}$$

moreover, since w_2 is composed of two letters, the overall sign in the KK relations is $(-1)^2 = 1$. Hence,

$$C_6(132645) = C_6(132456) + C_6(134256) + C_6(134526) + C_6(143526) + C_6(143256) + C_6(145326) \tag{5.26}$$

which can be checked to hold directly from the definition of the Parke-Taylor factors.

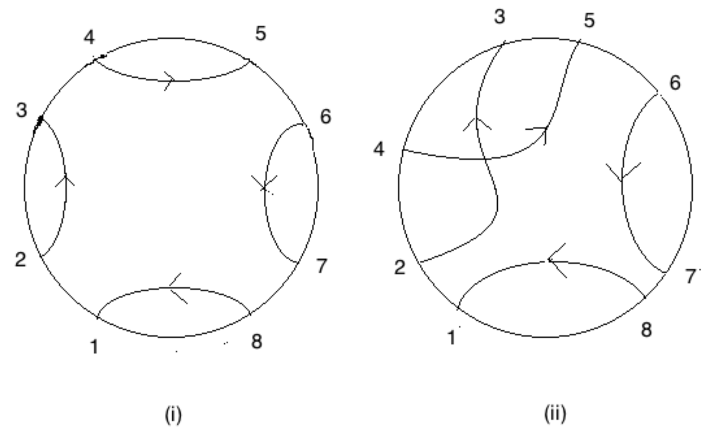


Fig. 5.1: (i) graph representing the quark amplitude with ordering (12345678); (ii) graph with crossed fermion lines

In terms of constructing an amplitude basis, we can use the KK relations to fix the position of a second leg, say, $l_n = n$. Then, in the case of Yang-Mills theory, we arrive at a basis with elements of the form $A_n(1w_{n-2}n)$, where w_{n-2} is a word of length $(n-2)$ whose letters are a permutation of the labels $\{2, 3, ..., n-1\}$. The size of this basis is $(n-2)!$.

In the case of quarks, the KK relations also hold when one considers primitive amplitudes. However, these amplitudes satisfy additional constraints which can be understood as KK relations. To illustrate them, consider the case of pure quark amplitudes; that is, an $n = 2n_q$ point amplitude, where n_q is the number of quark-antiquark pairs. As discussed by Melia in [71], one can consider a basis of amplitudes of the form $A_n(1w_{n-2}n)$, where the label n indicates an antiquark and 1 its corresponding quark. These amplitudes can be described graphically in the following way: draw a circle to represent the plane. Then, write the quark labels clockwise around the circle, according to their cyclic ordering, and connect each quark with its corresponding antiquark. This is illustrated in Fig.(5.1).

These graphs not only are useful to clear the structure of the color-ordered Feynman diagrams that contribute to the primitive amplitude under consideration, but also allow one to see that, having fixed the labels to specify each quark and antiquark, there are some primitive amplitudes (in this case, $A_8(12354678)$) which vanish, because the corresponding pairs cannot be connected along the circle in a planar way. This means that there are less than $(n-2)!$ independent, non-vanishing primitive amplitudes; moreover, some of them can be seen to satisfy linear relations. For example, due to the antisymmetry of the color-ordered Feynman rules, one can see that

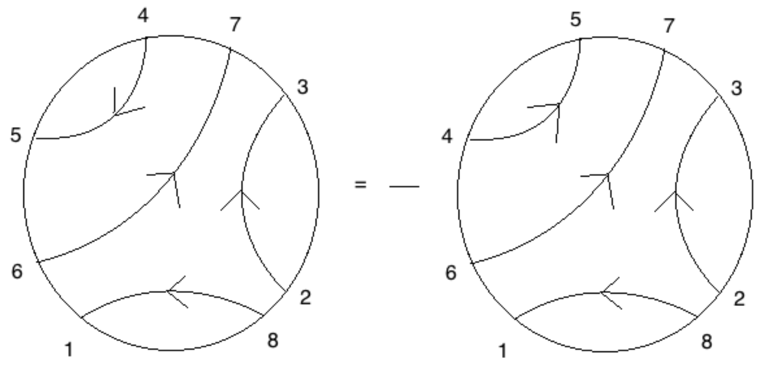


Fig. 5.2: Linear relation between primitive amplitudes with different orderings

$$A_8(16547328) = -A_8(15647328) \tag{5.27}$$

which is represented graphically in Fig.(5.2). As pointed out by Melia, this rather trivial relation can be seen to be a KK relation where we take the words $w_1 = 64732$ and $w_2 = 5$, and where the remaining amplitudes appearing in the sum of Eq. (5.18) vanish because they involve crossed fermion lines. A systematic study of these relations reveals that these quark KK relations reduce the number of basis amplitudes from $(n-2)!$ to $(n-2)!/(n_q)!$. This fact wil play a major role when we count the number of basis amplitudes for a generic QCD amplitude. However, in order to do this, we first need to discuss the BCJ relations, which are the subject of the next section.

5.3. BCJ relations: color-kinematics duality

Now, we introduce the Bern-Carrasco-Johansson (BCJ) relations, at tree level. These are linear relations involving primitive amplitudes with different external orderings, independent under the KK relations, that involve coefficients which are rational functions of the momentum invariants $s_{i_1 \ldots i_k}$. These relations were first conjectured by Bern, Carrasco and Johansson in [22] for the case of pure Yang-Mills amplitudes, based on the idea that the one could find a set of kinematic numerators for the gauge theory Feynman diagrams that satisfied an algebra analogous to that of the color factors. They provided non-trivial examples for fixed number of particles and provided a possible generalization to arbitrary number of external legs, which was later proven by taking the field theory limit of open string amplitudes in [72][73], and later in [74], [75] and [76] by applying the BCFW recursion relations both in Yang-Mills theory and $\mathcal{N} = 4$ SYM. They were later conjectured to generalize to QCD amplitudes with arbitrary number of massive or massless quarks in [77] by Johansson and Ochirov. Afterwards, proofs of the so-called fundamental BCJ relations for QCD amplitudes and that they allow to obtain a minimal basis for QCD primitive amplitudes were provided in [23] and [33], respectively.

We will first introduce these relations by considering the simple example of the four gluon tree-amplitude. Then, we will discuss the fundamental BCJ relations and review the proof given in [23] for QCD primitive amplitudes. Afterwards, we introduce the general BCJ relations, which will allow us to write the minimal basis of QCD amplitudes. Finally, we will illustrate how the BCJ relations can be derived in $\mathcal{N} = 4$ SYM from the connected formalism.

5.3.1. Introduction: four gluon amplitude

When discussing the color decomposition of pure Yang-Mills theory amplitudes, we found that the color structure of the four gluon amplitude can be arranged in terms of the color factors $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$, $c_t = f^{a_1 a_3 b} f^{b a_4 a_2}$ and $c_u = f^{a_1 a_4 b} f^{b a_2 a_3}$, that each of these factor appears in the s-channel, t-channel, and u-channel diagrams, respectively, and that the four gluon vertex contains a sum of all these factors, which allows this vertex term to be distributed into the diagrams with only three gluon indices. This decomposition can be done similarly for any tree level n-gluon amplitude, in such a way that it can be written in terms of diagrams with three-gluon vertices only [78]. Then, we can write the color-dressed amplitude as

$$\mathcal{A}_n = \sum_{i \in trivalent} \frac{c_i n_i}{\prod_a p_a^2} \tag{5.28}$$

The denominator is given as the product of all propagators appearing in a given diagram, and the numerators n_i are purely kinematic functions, which can, however, be in general not local. In particular, the four gluon amplitude will have the structure

$$\mathcal{A}_4(4g) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \tag{5.29}$$

where the four gluon vertex has been split into the different channels through their corresponding color factors. Since the color factors satisfy the Jacobi identity

$$c_s + c_t + c_u = 0 \tag{5.30}$$

There is not a unique way to rearrange the four gluon vertex into the other diagrams, which implies that the numerators n_i are not uniquely defined. Furthermore, this implies a new symmetry of the amplitude: if we redefine $n_s \to n_s + sf(p, \epsilon)$, $n_t \to tf(p, \epsilon)$, $n_u \to uf(p, \epsilon)$ for some arbitrary function f, the modified amplitudes acquires a term proportional to $c_s + c_t + c_u$, which vanishes due to the Jacobi identity. This is known as a generalized gauge transformation.

The *color-kinematics* duality is the statement that one can always find a representation for $\mathcal{A}_n$ such that, if there are three color factors c_i, c_j, c_k embedded in the amplitude that satisfy the Jacobi identity $c_i + c_j + c_k = 0$, their corresponding numerators also satisfy the same algebraic relation, namely $n_i + n_j + n_k = 0$.

Cyclic invariance and the Kleiss-Kuijf relations imply that there are only $(n-2)!$ independent primitive amplitudes for a given gluon amplitude $\mathcal{A}_n$. Since the expansion of the color-dressed amplitudes in primitive amplitudes sets the coefficients to be color factors, there will also be only $(n-2)!$ independent color factors, in this case, under the Jacobi identities. If we assume color-kinematics duality to hold, then there will also be only $(n-2)!$ independent numerator factors n_i. Then, we can write, for some ordering w_i,

$$A_n(w_i) = \sum_{j=1}^{(n-2)!} \Theta_{ij}\hat{n}_j \tag{5.31}$$

where the primitive amplitudes $A_n(w_i)$ are all linearly independent, as well as all numerators $\hat{n}_i$. The matrix Θ_{ij} was introduced by [79], and is known as the propagator matrix. Consider the case $n = 4$ and take the independent partial amplitudes to be $A_4(1,2,3,4)$ and $A_4(1,3,2,4)$. From the color-ordered Feynman rules, one can see that

$$A_4(1,2,3,4) = -\frac{n_s}{s} + \frac{n_u}{u}, \quad A_4(1,3,2,4) = -\frac{n_u}{u} + \frac{n_t}{t} \tag{5.32}$$

Now, assuming that color-kinematics holds, $n_t = -n_s - n_u$. Taking the independent numerators to be $\hat{n}_1 = n_s$, $\hat{n}_2 = n_u$, we find

$$\Theta_{ij} = \begin{pmatrix} -\frac{1}{s} & \frac{1}{u} \\ -\frac{1}{t} & -\frac{1}{u} - \frac{1}{t} \end{pmatrix} \tag{5.33}$$

If the matrix Θ_{ij} was invertible, color-kinematics duality would be trivial since one could always find the numerators in terms of color-ordered amplitudes and use the Jacobi identities to generate whatever numerators are missing. However, this is not the case; Θ_{ij} does not have full rank, and is therefore not invertible. Using Eq.(5.31), we can write

$$\hat{n}_1 = -sA_4(1,2,3,4) + \frac{s}{u}\hat{n}_2 \tag{5.34}$$

and then

$$A_4(1,3,2,4) = -\frac{\hat{n}_1}{t} - \frac{\hat{n}_2}{t} - \frac{\hat{n}_2}{u} = \frac{s}{t}A_4(1,2,3,4) - \left(\frac{s}{tu} + \frac{1}{t} + \frac{1}{u}\right)\hat{n}_2 \tag{5.35}$$

Now, since $s + t + u = 0$ for massless particles, the second term above vanishes and we obtain the identity

$$A_4(1,3,2,4) = \frac{s}{t}A_4(1,2,3,4) \tag{5.36}$$

This is an example of the BCJ relations.

An interesting consequence of color-kinematics duality are the so-called double copy relations. These relations allow to find gravitational amplitudes from their Yang-Mills counterparts. At tree-level, suppose that numerators n_i have been found, which satisfy the color-kinematics duality. Then, if in the expansion Eq.(5.28) we replace each color factor by its corresponding kinematic factor, the quantity

$$\mathcal{A}_n^{grav} = \sum_{i \in trivalent} \frac{n_i^2}{\prod_a p_a^2} \tag{5.37}$$

gives the tree n-point graviton amplitude in pure Einstein gravity. One could go further, and expand 5.28 in terms of numerators $\tilde{n}_i$ which do not satisfy color-kinematics. Then, if one replaces $c_i \to n_i$, where n_i is a set of numerators which do satisfy the duality, the resulting expansion, with numerators of the form $n_i \tilde{n}_i$ will also yield expressions for gravitational amplitudes in theories beyond pure Einstein gravity. As an example, let us consider the the 4 graviton amplitude. In terms of the kinematic numerators n_s, n_t and n_u, this amplitude would be given by

$$A_4^{grav} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{(n_s + n_t)^2}{u} \tag{5.38}$$

assuming color-kinematics duality to hold. Hence, doing the same identification of (n_s, n_u) with $(\hat{n}_1, \hat{n}_2)$, we can set $\hat{n}_2 = 2$, and obtain

$$A_4^{grav} = -\frac{su}{t}(A_4(1234))^2 = -uA_4(1234)A_4(1324) \tag{5.39}$$

which is an example of the KLT relations we will discuss later in this chapter. Therefore, since the KLT relations are known to hold, this verifies the validity of the double copy relations for four point amplitudes.

5.3.2. Fundamental BCJ relations

The simplest BCJ relations, known as the *Fundamental BCJ relations*, can be written in the form

$$\sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} 2p_2 \cdot p_j \right) A_n(1, 3, ..., i, 2, i+1, ...n) = 0 \tag{5.40}$$

or, equivalently, using momentum conservation on the sum over j, as

$$\sum_{i=2}^{n-1} \left(\sum_{j=1}^{i} 2p_2 \cdot p_j \right) A_n(1, 3, ..., i, 2, i+1, ...n) = 0 \tag{5.41}$$

These relations were shown to hold, for gluon primitive amplitudes, in [72]-[74], For QCD primitive amplitudes, where particle 2 is a gluon, they were later proven in [23], and we will review their argument, which is based on the BCFW recursion relations.

Recall that in Chapter 2, we introduced the complex momentum shifts by shifting two of the momentum variables in such a way that momentum conservation and the on-shell conditions were still satisfied. However, one can consider shifting more than two momenta; this will, in general, improve the large-z behaviour of the shifted amplitudes. To prove the fundamental BCJ relations, Weinzierl et al consider a three particle shift, that is, they deform the momenta $p_1 \to \hat{p}_1(z)$, $p_2 \to \hat{p}_2(z)$ and $p_n \to \hat{p}_n(z)$, in such a way that

$$\begin{aligned}\hat{p}_1(z) + \hat{p}_2(z) + \hat{p}_n(z) &= p_1 + p_2 + p_n \\ \hat{p}_i^2(z) = p_i^2, &\quad i = 1, 2, n\end{aligned} \tag{5.42}$$

which guarantees that both momentum conservation and the on-shell conditions hold for the shifted momenta. Then, they show that, if particle 2 is a gluon and particles 1 and n are either two gluons or a quark-antiquark pair, there exists a shift such that the boundary term in the BCFW recursion relations vanishes. We will not construct such a shift, and just take as a given that it does exist. Define the quantity

$$I_n(z) = \sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} 2\hat{p}_2 \cdot \hat{p}_j \right) A_n(\hat{1}, 3, ..., i, \hat{2}_g, i+1, ..., n+1, \hat{n}) \tag{5.43}$$

where we write 2_g to emphasize that particle 2 is a gluon and $\hat{p}_j$ is a shifted momenta whenever $j = 1, n$ and the unshifted, physical momenta otherwise. Hence, in order to prove the fundamental BCJ relations, we need to show that

$$I_n(0) = \frac{1}{2\pi i} \oint_C \frac{dz}{z} I_n(z) = 0 \tag{5.44}$$

where the contour C encloses all the poles $I_n(z)$ plus the pole at $z = 0$. Then, since the shift of the momenta p_1, p_2 and p_n is defined so that $I_n(z) \to 0$ when $|z| \to \infty$, we can see that

$$I_n(0) = -\sum_{\alpha} \text{Res} \left(\frac{I_n(z)}{z} \right)_{z_\alpha} \tag{5.45}$$

where z_α are the poles of $I_n(z)$. Now, we proceed by induction on n. We take the first step with $n = 3$. In this case, we do not perform any contour integral and simply evaluate at $z = 0$ to obtain

$$I_3(0) = 2p_2 \cdot p_3 A_3(1, 2_g, 3) \tag{5.46}$$

Three particle kinematics imply that $p_1 + p_2 + p_3 = 0$. Moreover, $p_1^2 = p_3^2 = m^2$, because these particles either belong to the same quark line or are both gluons, and $p_2^2 = 0$. Hence, for generic values of the momenta the amplitude $A_3(1, 2_g, 3)$ will be non-vanishing (in the all-gluon or massless case we assume the momenta to be complex). Thus, since

$$2p_2 \cdot p_3 = (p_2 + p_3)^2 - m^2 = p_1^2 - m^2 = 0 \tag{5.47}$$

we conclude that $I_3(0) = 0$, which means that the fundamental BCJ relations hold for $n = 3$.

Now, we perform the inductive step by assuming that $I_n(0) = 0$ for all $j < n$, and prove it for n.

Recalling from Eq.(2.121) that

$$\text{Res} \left(\frac{A_n(z)}{z} \right)_{z_\alpha} = -\sum_s A_L^s(z_\alpha) \frac{1}{P_i^2} A_R^{\bar{s}}(z_\alpha) \tag{5.48}$$

due to the factorization properties imposed by unitarity, we can see that

$$\text{Res}\left(\frac{I_n(z)}{z}\right)_{z_\alpha} = -\sum_{i=2}^{n-2}\left(\sum_{j=i+1}^{n} 2\hat{p}_2\cdot\hat{p}_j\right)\sum_s A_{k+1}(\hat{1},\hat{2}_g,...,k,\hat{P})\frac{i}{P^2}A_{n-k+1}(-\hat{P},...,n-1,\hat{n}) \quad (5.49)$$

with the understanding that the hatted quantities are evaluated at $z = z_\alpha$. That is, the residue of $I_n(z)$ at a particular pole z_α is the sum of the residues of different shifted amplitudes, and the sum over s is over the helicities of the internal propagating particle that connects the left and right subamplitudes. As in [23], we introduce the short-hand notation

$$A_n(\hat{1},\hat{2}_g,...,k,\hat{P}|-\hat{P},k+1,...,n-1,\hat{n}) = \sum_s A_{k+1}(\hat{1},\hat{2}_g,...,k,\hat{P})\frac{i}{P^2}A_{n-k+1}(-\hat{P},...,n-1,\hat{n}) \quad (5.50)$$

This means that the sum over residues is replaced by a sum over the possible insertion points of particle k, and we can write

$$\begin{aligned} I_n(0) &= \sum_{i=2}^{n}\sum_{j=i+1}^{n}\sum_k (2\hat{p}_2\cdot\hat{p}_j)A_n(\hat{1},\hat{2}_g,...,k,\hat{P}|-\hat{P},k+1,...,n-1,\hat{n}) \\ &= R_1 + R_2 + R_n \end{aligned} \quad (5.51)$$

where we split the sum over k according to the three possible the shifted legs can enter the subamplitudes: the first one will be the one on which the shifted leg $\hat{1}$ is on the left subamplitude, while the other two shifted legs are on the right subamplitude; the second one will have the leg $\hat{2}_g$ on the left subamplitude, while the legs $\hat{1}$ and $\hat{n}$ will be on the right subamplitude, and the final one will have legs $\hat{1}$ and $\hat{2}_g$ in the left subamplitude. Similarly to the two particle shifts, one cannot have a configuration where all shifted legs belong to either the left or the right subamplitude, because their z dependence would cancel in order to preserve momentum conservation. Then, we find that

$$R_1 = \sum_{i=2}^{n}\sum_{j=i+1}^{n}(2\hat{p}_2\cdot\hat{p}_j)\sum_{k=3}^{i} A_n(\hat{1},3,...,k,\hat{P}|-\hat{P},k+1,...,i,\hat{2}_g,i+1,...,n-1,\hat{n}), \quad (5.52)$$

$$R_2 = \sum_{i=2}^{n}\sum_{j=i+1}^{n}(2\hat{p}_2\cdot\hat{p}_j)\sum_{k=2}^{i}\sum_{\substack{l=1\\(k,l)\neq(i,i)}}^{n-1} A_n(k+1,...,i,\hat{2}_g,i+1,..,l,\hat{P}|-\hat{P},l+1,...,\hat{n},\hat{1},...,k) \quad (5.53)$$

and, finally

$$R_n = \sum_{i=2}^{n}\sum_{j=i+1}^{n}(2\hat{p}_2\cdot\hat{p}_j)\sum_{k=i}^{n-2} A_n(\hat{1},3,...i,\hat{2}_g,i+1,...,k,\hat{P}|-\hat{P},k+1,...,n-1,\hat{n}) \quad (5.54)$$

Now, we proceed to show that each of these terms vanishes. In the case of R_1 and R_n, one simply must exchange the order of the sums over i and k; for R_1, this is

$$\sum_{i=2}^{n-1}\sum_{k=3}^{i} f(i,k) = \sum_{k=3}^{n-1}\sum_{i=k}^{n-1} f(i,k) \quad (5.55)$$

with which we can write

$$R_1 = \sum_{k=3}^{n-1} \left[\sum_{i=k}^{n-1} \left(\sum_{j=i+1}^{n} 2\hat{p}_2 \cdot \hat{p}_j \right) A_n(\hat{1}, 3, ..., k, \hat{P}| - \hat{P}, k+1, ..., i, \hat{2}_g, i+1, ..., n-1, \hat{n}) \right] \tag{5.56}$$

Now, if we use Eq. (5.50) to rewrite the residue $A_n(\hat{1}, 3, ..., k, \hat{P}| - \hat{P}, k+1, ..., i, \hat{2}_g, i+1, ..., n-1, \hat{n})$ in terms of the subamplitudes, we find that nested within the sum over k is a term of the form

$$\sum_{i=k}^{n-1} \left(\sum_{j=i+1}^{n} 2\hat{p}_2 \cdot \hat{p}_j \right) A_{n-k+2}(-\hat{P}, k+1, ..., i, \hat{2}_g, i+1, ..., n-1, \hat{n}) = 0 \tag{5.57}$$

which vanishes because it is just the BCJ fundamental relation for $(n-k+2)$ particles, and for $k > 3$ we have $n - k + 2 < n$. This is enough to guarantee that $R_1 = 0$. The case of R_n is similar: once the sums over i and k are exchanged as

$$\sum_{i=2}^{n-1} \sum_{k=i}^{n-2} f(i,k) = \sum_{k=2}^{n-2} \sum_{i=2}^{k} f(i,k) \tag{5.58}$$

the BCJ fundamental relations for $k-2$ particles appear on the left subamplitude, after using momentum conservation on the invariants $2\hat{p}_2 \cdot \hat{p}_j$, and vanish because of the induction hypothesis. To complete the proof, we need to show that R_2 vanishes; in order to this, we exchange the order of the sums in order to rewrite

$$\begin{aligned} R_2 &= \sum_{k=2}^{n-2} \sum_{l=k+1}^{n-1} \sum_{i=k}^{l} \left(\sum_{j=i+1}^{n} 2\hat{p}_2 \cdot \hat{p}_j \right) A_n(k+1, ..., i, \hat{2}_g, i+1, ..., l, \hat{P}| - \hat{P}, l+1, ..., \hat{n}, \hat{1}, 3, ..., k) \\ &= \sum_{k,l,i} \left(\left(\sum_{j=i+1}^{l} + \sum_{j=l+1}^{n} \right) 2\hat{p}_2 \cdot \hat{p}_j \right) A_n(k+1, ..., i, \hat{2}_g, i+1, ..., l, \hat{P}| - \hat{P}, l+1, ..., \hat{n}, \hat{1}, 3, ..., k) \end{aligned} \tag{5.59}$$

Then, R_2 splits into two different terms. In the first one, where the sum over j goes from $i+1$ to l, we can reconstruct the BCJ fundamental relations with the left subamplitudes $A_{l-k+2}(k+1, ..., i, \hat{2}_g, i+1, ..., l, \hat{P})$ and these vanish due to the induction hypothesis, since $l - k + 2 < n$. On the other term, we can freely exchange the sums over i and j, since the sum over j no longer depends on i, and we find terms of the form

$$\sum_{i=k}^{l-1} A_{l-k+2}(k+1, ..., i, \hat{2}_g, i+1, ..., l, \hat{P}) = 0, \tag{5.60}$$

and this linear combination vanishes due the the $U(1)$ decoupling identity Eq. (5.9). Hence, we have shown that R_2 is also zero, and thus

$$I_n(0) = 0 \tag{5.61}$$

which proves the fundamental BCJ relations. As we have already mentioned, the proof that the three-particle shift that makes the recursion relations valid exists is provided in the paper [23], based on four dimensional kinematics and helicity spinors, suitably constructed for massive particles. Given that the only role of dimensionality is in providing the spinor variables to construct the shifts, we will assume in the following that there exist analogous shifts in arbitrary dimensions so that the fundamental BCJ relations are satisfied in spacetime dimension $D \neq 4$.

5.4. Construction of the basis for QCD primitive amplitudes

Armed with the different relations among primitive amplitudes we have developed so far, we will construct a minimal basis of QCD primitive amplitudes. In order to do this consistently for amplitudes with an arbitrary number of quark-antiquark pairs, we will need a few more identities, which we have already mentioned in our discussion of the KK relations, the so-called "no-crossed fermion line" relations. Also, we will introduce the concept of generalized Dyck words, which allow us to organize appropriately the labels of the quarks. This will lead us to define an standard orientation of fermion lines. Using these new properties of multiquark amplitudes and the relations developed above, we will construct the QCD basis in terms of the different external orderings that the basis amplitudes depend on.

To clarify notation, we recall from Appendix G that a generic QCD primitive amplitude with n external particles can be split according to

$$n = n_g + 2n_q \tag{5.62}$$

where n_g is the number of gluons and n_q the number of quark-antiquark lines. The set of all possible external orderings of the primitive amplitudes is given in terms of words

$$W_0 = \{w = l_1 l_2 ... l_n | l_i \in A, l_i \neq l_j \, if \, i \neq j\} \tag{5.63}$$

where A, the set of all particle labels, is what we call the alphabet.

5.4.1. Fermion lines and Dyck words

In our treatment of the KK relations, we found that quark amplitudes could be visualized by using a class of circle graphs, around which the quark lines were cyclically ordered. We saw that not all of the quark amplitudes could be represented with a circle graph in which the quark lines were all connected in a planar way, and that this implied that quark primitive amplitudes with crossed fermion lines vanish. In a general QCD primitive amplitude with an arbitrary number of quark-antiquark pairs and gluons, the appearance of a crossed fermion line would imply that there is a vertex that joins two quarks of different flavour, which is not possible. Therefore, QCD primitive amplitudes satisfy the so-called "no crossed fermion line" relations,

$$A_n(...q_i, q_j, ..., \bar{q}_i, \bar{q}_j, ...) = A_n(..., q_i, \bar{q}_j, ..., \bar{q}_i, q_j, ...) = 0 \tag{5.64}$$

Primitive amplitudes with no crossed fermion lines are non-vanishing, and their external orderings may de described by generalized Dyck words [71] [80]. In order to define them, let us consider an alphabet, C, with two elements

$$C = \{(,)\} \tag{5.65}$$

A Dyck word is defined to be any word $w = l_1...l_k$ where $l_i \in C$, $i = 1, 2, ..., k$ such that no bracket is unmatched. For example, the word $w_1 = ((()))$ is a Dyck word, while $w_2 = (()(()$ is not. We can generalize this idea [26] by defining opening brackets $(_i$ and closing brackets $)_i$ for $i = 1, 2, ..., n_q$, with which we will form an alphabet

$$D = \{(_1, (_2, ...(_{n_q},)_1,)_2, ...,)_{n_q}\} \tag{5.66}$$

Then, a generalized Dyck word is a word $w_D = l_1...l_m$ where $l_i \in D$ for $i = 1, 2, ..., m$) such that each opening bracket $(_i$ is matched by its corresponding closing bracket $)_i$. For example, the word $w_1 = (_i(_j)_j)_i$ is a generalized Dyck word while $w_2 = (_i(_j)_i)_j$ is not. We will consider generalized Dyck words in which each opening and closing bracket occurs exactly once and have length $2n_q$. The number of such words is

$$N_{dyck} = \frac{(2n_q)!}{(n_q + 1)!} \tag{5.67}$$

We associate the quark q_i with the opening bracket $\rightarrow (_i$ and the antiquark $\bar{q}_i$ with the closing bracket $\rightarrow)_i$. Define a projection operator P that acts on words $w \in W_0$ such that $P(w) = P(l_1)P(l_2)...P(l_n)$, where

$$P(q_i) = (_i, \quad P(\bar{q}_i) =)_i, \quad P(g_j) = e \tag{5.68}$$

were e stands for the empty word. The action of this operator allow us to introduce the concept of orientation of the fermion lines, and to define an standard orientation for these lines. We say that a primitive amplitude $A_n(w)$ is in the standard orientation if the word $P(w)$ is a generalized Dyck word.Then, the set

$$Dyck_{n_q} = \{w \in W_0 | P(w) \ is \, a \, generalized \, Dyck \, word\} \tag{5.69}$$

Contains all possible words in which the fermion labels are in the standard orientation. It is obvious that some of the non-vanishing primitive amplitudes with quarks are not in the standard orientation. However, it is always possible to bring an amplitude to the standard orientation using cyclic invariance, the KK relations and the fact that amplitudes with crossed fermion lines vanish. We will outline how this is realized, and refer the reader to the detailed procedure given in [26].

The idea is to consider the class of amplitudes

$$A_n(w_{k-1}q_i w_k \bar{q}_j w_{k+1} q_j u_k \bar{q}_i u_{k-1}) \tag{5.70}$$

where w_{k-1},w_k, w_{k+1}, u_{k-1} and u_k are subwords of the word defining the external ordering. The subscript k denotes the so-called level of the fermion line, which is the number of fermion lines that separate the k-the fermion line from the line $n1$ in a circle graph. For example, in graph (i) of Fig.(5.1) the line 32 is of level one, the line 54 is of level two, and so on. With this notation, we are assuming that the line $q_i - \bar{q}_i$ is already in the standard orientation and is of level $(k - 1)$. Furthermore, the fermion lines in w_{k-1} and u_{k-1} are already in the standard orientation. However, the fermion line $q_j - \bar{q}_j$, which is of level k, is not in the standard orientation. Then, if we let

$$w_k = l_{i_1}...l_{i_r}, \quad u_k = l_{j_1}...l_{j_s} \tag{5.71}$$

be two words which may contain fermion lines of arbitrary level $L \geq k$. Also, assume that w_{k+1} may contain fermion lines of level $l \geq k+1$. Then, we can bring the line $q_j - \bar{q}_j$ into the standard orientation by expressing the amplitude in Eq. (5.70) as a linear combination of amplitudes with the standard orientation of the line $q_j - \bar{q}_j$ as

$$\begin{aligned} &A_n(w_{k-1}q_i w_k \bar{q}_j w_{k+1} q_j u_k \bar{q}_i u_{k-1}) \\ &= (-1)^{\gamma+1} \sum_{a=0}^{r} \sum_{b=0}^{s} A_n(w_{k-1} q_i l_{i_1}...l_{i_a} q_j w'_{k+1} \bar{q}_j l_{j_b+1}...l_s \bar{q}_i u_{k-1}) \end{aligned} \tag{5.72}$$

where t is the length of the word w_{k+1}, and

$$w'_{k+1} = (l_{i_a}...l_{i_r}) \sqcup\!\sqcup\, w^T_{k+1} \sqcup\!\sqcup\, (l_{j_1}...l_{j_b}) \tag{5.73}$$

We refer to Eq.(5.72) as the fermion orientation relation. Recursively applying this relation, an amplitude with no crossed fermion lines but with an arbitrary number of amplitudes with the wrong orientation of the fermion lines can be written in terms of amplitudes where all fermion lines are in the standard orientation. It is worthwhile to note that some of the amplitudes appearing in the sum may vanish due to the appearance of crossed fermion lines.

5.4.2. Definition of a basis for QCD primitive amplitudes

All the relations among amplitudes described above allows us to write a given primitive amplitude in terms of basis of amplitudes. Since primitive amplitudes with $n_q \leq 1$ satisfy cyclic invariance, KK and the full BCJ relations, the size of this basis is $(n-3)!$. For amplitudes with $n_q \geq 2$, the size of the basis is reduced by a factor $2(n_q-1)/(n_q)!$. We summarize this as

$$N_b = \begin{cases} (n-3)! & \text{if } n_q \leq 1 \\ (n-3)!\frac{2(n_q-1)}{(n_q)!} & \text{if } n_q \geq 2 \end{cases} \tag{5.74}$$

Now, let $N_{per} = n!$ be the number of permutations of particle labels. Since the factor $2(n_q-1)/(n_q)!$ is always less than or equal to 1, the number of independent primitive amplitudes for $n_q \geq 2$ is always smaller or equal to $(n-3!)$. This will play a crucial role when we try to obtain a CHY representation for QCD amplitudes.

Now, we can define the set of words corresponding to the n particle amplitude basis. For $n_q = 0$, we set

$$B_n = \{w \in W_0 | l_1 = g_1,\, l_{n-1} = g_{n-1},\, l_n = g_n\} \tag{5.75}$$

For $n_q = 1$,

$$B_n = \{w \in W_0 | l_1 = q_1,\, l_{n-1} = g_{n-2},\, l_n = \bar{q}_1\} \tag{5.76}$$

and, for $n_q \geq 2$,

$$B_n = \{w \in W_0 | l_1 = q_1,\, l_{n-1} \in \{\bar{q}_2, ..., \bar{q}_{n_q}\},\, l_n = \bar{q}_1\} \tag{5.77}$$

To construct this basis, we use cyclic invariance, the KK and BCJ relations, the no-crossed fermion lines relation and we bring all quark-antiquark pairs into the standard order. First, we use cyclic invariance to fix leg 1 to be g_1 for the pure gluonic case and q_1 for primitive amplitudes with $n_q \neq 0$. We define the set $W_1 \subset W_0$ as

$$W_1 = \begin{cases} \{w \in W_0 | l_1 = g_1\} & \text{if } n_q = 0 \\ \{w \in W_0 | l_1 = q_1\} & \text{if } n_q \neq 0 \end{cases} \tag{5.78}$$

The number of words in this set is $(n-1)!$. Then, using the KK relations, we can fix leg n to be g_n for $n_q = 0$ and to be $\bar{q}_1$ for $n_q \neq 0$. We then define the subset $W_2 \subset W_1$ as

$$W_2 = \begin{cases} \{w \in W_1 | l_n = g_n\} & \text{if } n_q = 0 \\ \{w \in W_1 | l_n = \bar{q}_1\} & \text{if } n_q \neq 0 \end{cases} \tag{5.79}$$

Now, the number of words in W_2 is $(n-2)!$ and correspond to all words with the first and last letters fixed. We then set to zero any amplitude with crossed fermion lines, and put every other one in the standard orientation. Again, define a subset $W_3 \subset W_2$ as

$$W_3 = \begin{cases} W_2 & \text{if } n_q \leq 1 \\ \{w \in W_2 | w \in Dyck_{n_q}\} & \text{if } n_q \geq 2 \end{cases} \tag{5.80}$$

Finally, we use the fundamental BCJ relations to fix leg $n-1$ to be a g_{n-1} in the pure gluonic case, g_{n-2} for $n_q = 1$ and an antiquark $\bar{q} \in \{\bar{q}_2, ..., \bar{q}_{n_q}\}$ for the case $n_q \geq 2$; this is done by removing any gluon from that position and, since the amplitudes are in the standard orientation, that position will necessarily be occupied by an antiquark. In general, we define the basis B_n as the set

$$B_n = \begin{cases} \{l_1 l_2 ... l_n \in W_3 | l_{n-1} = g_{n-1}, & n_q = 0 \\ \{l_1 l_2 ... l_n \in W_3 | l_{n-1} = g_{n-2}, & n_q = 1 \\ \{l_1 l_2 ... l_n \in W_3 | l_{n-1} \in \{\bar{q}_2, ..., \bar{q}_{n_q}\}, & n_q \geq 2 \end{cases} \tag{5.81}$$

Then, the set B_n contains all possible orderings of the particle labels, such that the amplitudes with these orderings form a basis for the primitive amplitudes of a given n particle QCD scattering process. Note that $B_n \subseteq W_3 \subseteq W_2 \subseteq W_1 \subseteq W_0$.

5.4.3. General BCJ identities

Now, we can state the general BCJ identities by relating the Parke-Taylor factors whose external orderings are in W_3 to those whose external orderings are in W_2 when $n_q = 0$. That is, we expand the $(n-2)!$, KK independent Parke-Taylor factors in terms of the $(n-3)!$ BCJ independent ones. We note that, although cyclic invariance and the KK relations are valid for arbitrary Parke-Taylor factors, these will satisfy the BCJ relations only when evaluated on the solutions to the scattering equations, and the primitive amplitudes will of course satisfy them.

Consider the alphabet

$$A_o = \{1, 2, ..., n\} \tag{5.82}$$

this is simply the set of n-particle labels, were we omit the information on flavour. Furthermore, let $w_k = l_1...l_k$ be a subword of length k of a word $w \in W_0$, where we understand W_0 as defined by the alphabet A_o. For this subword, define

$$s(w_k) = \sum_{\sigma in S_k} l_{\sigma(1)}...l_{\sigma(k)} \tag{5.83}$$

Now, assume that $w = 1w_{n-2}n \in W_2$ can be written in terms of subwords $w_1 = l_1...l_j$ and $w_2 = l_{j+1}...l_{n-3}$ in such a way that $w = 1w_1(n-1)w_2n$. Then, the general BCJ relations can be written as

$$C_(w, z^{(j)}) = \sum_{w'} F_{ww'} C(w', z^{(j)}) \tag{5.84}$$

where the sum is over the subset of B whose elements appear in the sum $1(w_1 \sqcup s(w_2))(n-1)n$. For fixed w, we define $F_{ww'} = 0$ if w' does not appear in the sum; otherwise, if we write $w' = 1(\sigma)(n-1)n$ for some permutation σ of $23...(n-2)$, the matrix elements are given by [26]

$$F_{ww'} = \prod_{k=j+1}^{n-3} \frac{\mathcal{F}(1(\sigma)(n-1)|l_k)}{\tilde{s}_{n,l_k,...,l_3}} \tag{5.85}$$

where the denominators are linear combinations of momentum invariants

$$\tilde{s}_{m_1...m_k} = \sum_{i<j} (2p_{m_i} \cdot p_{m_j} + 2\Delta_{m_i m_j}) \tag{5.86}$$

with Δ_{ij} defined by Eq.(3.17). For a permutation $\rho = 1\sigma(n-1)$, the function $\mathcal{F}(\rho|l_k)$ can be written as

$$\mathcal{F}(\rho|l_k) = \mathcal{F}_1(\rho|l_k) + \mathcal{F}_2(\rho|l_k) \tag{5.87}$$

where

$$\mathcal{F}_1(\rho|l_k) = \begin{cases} \sum_{r=1}^{t_{l_k}-1} \mathcal{G}(l_k, \rho_r) & \text{if } t_{l_k} < t_{l_k+1} \\ -\sum_{r=t_{l_k}+1}^{n-1} \mathcal{G}(l_k, \rho_r) & \text{if } t_{l_k} > t_{l_k+1} \end{cases} \tag{5.88}$$

and

$$\mathcal{F}_2(\rho|l_k) = \begin{cases} \hat{s}_{n,l_k,...,l_3} & \text{if } t_{l_{k-1}} < t_{l_k} < t_{l_k+1} \\ -\hat{s}_{n,l_k,...,l_3} & \text{if } t_{l_{k-1}} > t_{l_k} > t_{l_k+1} \end{cases} \tag{5.89}$$

here, we use t_a to denote the position of letter a in the string of letters ρ, except for $t_{l_{n-2}}$ and t_{l_j}, which are always fixed to the positions

$$t_{l_{n-2}} = t_{l_{n-4}}, \quad t_{l_j} = n \tag{5.90}$$

Finally, the function $\mathcal{G}(l_k, \rho_r)$ is defined by

$$\mathcal{G}(l_k, \rho_r) = \begin{cases} 2p_{l_k} \cdot p_{\rho_r} + 2\Delta_{l_k \rho_r} & \text{if } \rho_r = 1, (n-1) \\ 2p_{l_k} \cdot p_{\rho_r} + 2\Delta_{l_k \rho_r} & \text{if } \rho_r = l_t \text{ and } t < k \\ 0 \text{ in other cases} \end{cases} \tag{5.91}$$

The BCJ relations for primitive amplitudes are then written as

$$A_n(w) = \sum_{w'} F_{ww'} A_n(w') \tag{5.92}$$

with the same conditions as the BCJ relations for the Parke-Taylor factors. We will explicitly construct the matrix $F_{ww'}$ when the alphabet consists of three massless quark-antiquark pairs of different flavour in Chapter 6, where we will see that this matrix plays a crucial role in defining whether one can construct a CHY representation for QCD primitive amplitudes.

5.5. KLT relations

To finish this chapter, we introduce the so-called Kawai-Lewellen-Tye (KLT) relations, first found by the authors in [24], as a set of identities between closed and open string amplitudes. In this context, they arise from the fact that closed string tree amplitudes are calculated as integrals over the moduli space of the Riemann sphere, while open string tree amplitudes are calculated as integrals over its boundary. Based on this observation, Kawai, Lewellen and Tye managed to rewrite the closed string integrals as a sum over products of two open string integrals, weighted by kinematic factors. Taking the field theory limit, one finds relations between graviton and gluon scattering amplitudes. In this section, we will start by reviewing the kinematic factors appearing in the field theory limit of the KLT relations, the so-called momentum kernel, and state the KLT relations. Following this, we will see how the Parke-Taylor factors, evaluated at the solutions to the scattering equations, allow one to define an inner product under which Parke-Taylor factors evaluated at two different, inequivalent solutions to the scattering equations are orthogonal, and we will see how this orthogonality is related to the existence of KLT relations (beyond those between gravity and gauge theory amplitudes).

5.5.1. KLT relations and the momentum kernel

Consider the set of primitive Yang-Mills amplitudes $A_n(w, p, \epsilon)$ such that $w \in B$, where B denotes the basis defined in Eq.(5.81). Gravitons are spin-2 massless particles, and their polarization states can be described by two index Lorentz tensors constructed as the product of two gluon polarization vectors, $\epsilon_{\pm}^{\mu\nu} = \epsilon_{\pm}^{\mu}\epsilon_{\pm}^{\nu}$ [2]. Then, a logical idea would be to construct graviton amplitudes out of products of two Yang-Mills amplitudes. This is the content of the KLT relations; if we let $M_n^E(p, \epsilon)$ denote the n-graviton tree amplitude, we can write it as

[2]Note that we only consider products where both polarization vectors have the same helicities. The product of polarization vectors with different helicities do not describe gravitons, but rather the dilaton (a scalar) or an antisymmetric tensor. We will not be concerned with such fields throughout this work.

$$M_n^E(p,\epsilon) = \sum_{w_1,w_2\in B_n} A_n(w_1,p,\epsilon)\tilde{S}_n[w_1|\bar{w}_2]A_n(w_2,p,\epsilon) \tag{5.93}$$

where, for a given word $w = l_1...l_{n-1}l_n$, we the $\bar{w} = l_1...l_nl_{n-1}$, and the bilinear function $\tilde{S}_n[w|\bar{v}]$ is the so-called momentum kernel

$$\tilde{S}_n[w|\bar{v}] = (-1)^n \prod_{i=2}^{n-2}\left(\hat{s}_{\ell_1\ell_i} + \sum_{j=2}^{i-1}\theta_{\bar{v}}(\ell_j,\ell_i)(\hat{s}_{\ell_i\ell_j})\right) \tag{5.94}$$

where $\theta_{\bar{v}}(\ell_j,\ell_i)$ is 1 if the letter ℓ_j comes before ℓ_i in the word $\bar{v}$ and is zero otherwise, and we have introduced the notation

$$\hat{s}_{ij} = 2p_i\cdot p_j \tag{5.95}$$

This terminology was first introduced in [81], where the KLT relations were written in a form akin to Eq. (5.93) and some its properties were derived. A generalization of the momentum kernel to include massive quarks was introduced in [26], and be written as

$$S_n[w|\bar{v}] = (-1)\prod_{i=2}^{n-2}\left(\hat{s}_{\ell_1\ell_i} + \Delta_{\ell_1\ell_i} + \sum_{j=2}^{i-1}\theta_{\bar{v}}(\ell_j,\ell_i)(\hat{s}_{\ell_i\ell_j} + \Delta_{\ell_i\ell_j})\right) \tag{5.96}$$

which we will need in order to construct the CHY representation for QCD primitive amplitudes, and to derive the soft limit of the integrand. Now, let us work out the relation of the momentum kernel with the Parke-Taylor factors $C_n(w)$.

5.5.2. KLT orthogonality of Parke-Taylor factors

In [12], Cachazo et al noted that the momentum kernel allowed the definition of an inner product, under which suitable Parke-Taylor factors are orthogonal. To define this inner product, consider two words $w_1,w_2\in B_n$. A general Parke-Taylor factor is a function of an external ordering w and the punctures z. Noting by $z^{(i)}$ the inequivalent solutions to the scattering equations, with $i = 1,2,...,(n-3)!$, we can construct

$$(i,j) = \sum_{w_1,w_2\in B_n} C_n(w_1,z^{(i)})\tilde{S}_n[w_1|\bar{w}_2]C_n(w_2,z^{(j)}) \tag{5.97}$$

Then, KLT orthogonality is the claim that the Parke-Taylor factors, evaluated at different solutions to the scattering equations, are orthogonal with respect to the inner product (i,j); that is,

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2})}} = \delta_{ij} \tag{5.98}$$

The proof of this statement, which is a bit technical, can be found in [12]. We also note that

$$(i,i) = \frac{1}{J(z^{(i)},p)} \tag{5.99}$$

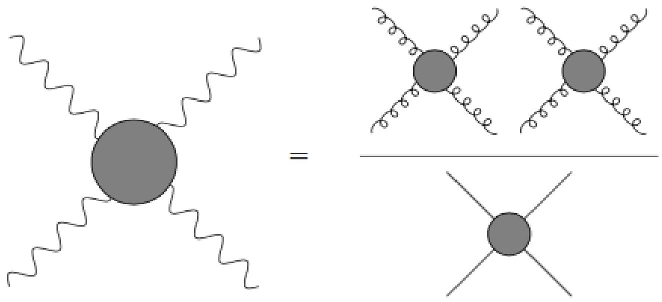


Fig. 5.3: Four graviton amplitude as the product of two four gluon amplitudes over a scalar amplitude. Image taken from [61].

where $J(z,p)$ is the Jacobian of the scattering equations, defined in (4.7). One consequence of KLT orthogonality is related to the biadjoint scalar amplitudes defined in Eq. (4.37), which, as we recall, depend on two external cyclic orderings u and v. Rewriting them as a sum over the inequivalent solutions to the scattering equations, we obtain

$$m_n(u|v) = \sum_{(j)} J(z^{(j)},p) C_n(u, z^{(j)}) C_n(v, z^{(j)}) \tag{5.100}$$

Now, assume that $u, v \in B_n$, and define the $(n-3)! \times (n-3)!$ matrix $m^n_{u\bar{v}}$ whose entries are given by the biadjoint amplitudes $m_n(u|\bar{v})$. Then, if we denote by $S^n_{u\bar{v}}$ the $(n-3)! \times (n-3)!$ matrix whose entries are given by the momentum kernel $S_n[u|\bar{v}]$, we have the relation

$$\tilde{S}^n_{u\bar{v}} m^n_{\bar{v},w} = \delta_{uw} \tag{5.101}$$

that is, the momentum kernel forms a matrix which is the inverse of the matrix formed by the biadjoint scalar amplitudes with different external orderings.We can prove this statement by defining the $(n-3)! \times (n-3)!$ matrices

$$A_{(i)w} = \sqrt{J(z^{(i)})} C_n(z^{(i)}, w), \quad D_{(i)\bar{w}} = \sqrt{J(z^{(i)})} C_n(z^{(i)}, \bar{w}) \tag{5.102}$$

where the index i ranges over the inequivalent solutions to the scattering equations and the indices w and $\bar{w}$ take values on the basis B, with the understanding that elements of the form $\bar{w}$ have the last two letters exchanged with respect to the ordering induced by B_n. Then, we KLT orthogonality can be written as

$$A \tilde{S}^n D^T = I_{sol} \tag{5.103}$$

where I_{sol} is the identity in the space of solutions to the scattering equations. Now, both matrices, A and B, are invertible, because they are constructed from a set of Parke-Taylor factors that are independent under cyclic invariance, KK and BCJ relations, and Jacobians are non-vanishing for general kinematics. Then, multiplying by $(D^T)^-1$ on the right and D^T on the left, KLT orthogonality is equivalent to

$$D^T A\tilde{S}^n = I_{perm} \tag{5.104}$$

where I_{perm} is the identity on the space of the different permutations of the words $w \in B$. Moreover, we can see that

$$(D^T A)_{\bar{w}v} = \sum_{(j)} J_n(z^{(j)}) C_n(\bar{w}, z^{(j)}) C_n(v, z^{(j)}) \tag{5.105}$$

which is nothing but the biadjoint scalar amplitudes. With this, we obtain a refined interpretation of the KLT relations between gravitons and Yang-Mills: a graviton amplitude is the sum of products of two Yang-Mills amplitudes, divided over an scalar amplitude with cubic interactions. This is depicted in Fig. 5.3.

5.5.3. KLT relations in the CHY formalism

To finish this chapter, let us see how KLT orthogonality allow us to obtain the CHY representation for gravity, Eq. (4.38) from the CHY representation of Yang-Mills theory. We will develop first a general argument valid for any two field theories with a gauge group, and then consider the particular case of Yang-Mills.

To start, consider two general field theories whose tree amplitudes can be color decomposed with respect to some gauge group G. Their CHY integrand will then have the form

$$\mathcal{I}_n^i(z, w, \mathbf{y}) = C_n(w, z) I_{Rn}^i(z, \mathbf{y}) \tag{5.106}$$

where $i = 1, 2$ denotes that the right integrands belong to possibly different theories, and $\mathbf{y}$ denotes the variables upon which this integrands can depend other than the punctures. In order for $\mathcal{I}_n^i$ to be CHY integrands, we must impose the condition

$$I_{Rn}^i(\zeta, \mathbf{y}) \to \left(\prod_{i=1}^n (cz_i + d) \right)^2 I_{Rn}^i(z, \mathbf{y}) \tag{5.107}$$

for an $SL(2, \mathbb{C})$ transformation $\zeta = (az + b)/(cz + d)$. The n-point amplitudes of this theories are then written as

$$A_n^i(w) = \sum_{(j)} J(z^{(j)}, p) \mathcal{I}_n^i(z, w, \mathbf{y}) \tag{5.108}$$

Now, omitting the explicit dependence on $\mathbf{y}$ and the momenta p, consider the combination

$$
\begin{aligned}
A_n^3 &= \sum_{w,v\in B_n} A_n^1(w)\tilde{S}_n[w|\bar{v}]A_n^2(\bar{v}) \\
&= \sum_{(j),(k)} J(z^{(j)})J(z^{(k)})I_{Rn}^1(z^{(j)})I_{Rn}^2(z^{(k)}) \sum_{w,v\in B_n} C_n^1(w,z^{(j)})\tilde{S}_n[w|\bar{v}]C_n^2(\bar{v},z^{(k)}) \\
&= \sum_{(j),(k)} \frac{\delta_{jk}}{J(z^{(k)})} J(z^{(j)})J(z^{(k)})I_{Rn}^1(z^{(j)})I_{Rn}^2(z^{(k)}) \\
&= \sum_{(j)} J(z^{(j)})I_{Rn}^1(z^{(j)})I_{Rn}^2(z^{(j)})
\end{aligned}
\tag{5.109}
$$

Now, we can see that the combination $I_{Rn}^1 I_{Rn}^2$ transforms properly as a CHY integrand under a Möbius transformation. Therefore, we can assume that the quantity A_n^3 yields the n-point tree amplitudes of a third theory, whose CHY representation can be constructed out of those of the other two theories. This means that, whenever we have two sets of amplitudes belonging to theories with gauge groups, we can construct amplitudes of new theories by calculating the linear combination of the amplitudes of the two known theories weighted by the momentum kernel. This construction immediately yields the CHY integrand of the new theory as the product of the half-integrands of the original theories that are separated from the Parke-Taylor factors. In particular, recall that the integrand for a primitive Yang-Mills theory amplitude can be written as

$$
\mathcal{I}_n^{YM}(z,w,p,\epsilon) = C_n(w,z)E_n(z,p,\epsilon) \tag{5.110}
$$

where the polarization function is defined as the reduced Pfaffian in Eq.(4.34. The KLT combination of Yang-Mills primitive amplitudes is nothing but the tree-level graviton scattering amplitudes. Thus, we conclude that the CHY integrand for gravity is given by

$$
\mathcal{I}_n^{GR}(z,p,\epsilon) = E_n^2(z,p,\epsilon) \tag{5.111}
$$

in agreement with Eq.(4.38) in Chapter 4.

CHY representation for fermions

In this chapter, we will introduce two of the possible schemes that allow us to define a CHY representation for amplitudes with fermions.

The first scheme exploits the fact that QCD primitive amplitudes have an external cyclic ordering, and as we have seen, multiple linear relations between primitive amplitudes allow us to obtain a minimal set of independent primitive amplitudes, which we have called the basis. We saw that the number of basis amplitudes depended on the specific particle content of the amplitudes under consideration, and that the maximum number of these independent amplitudes for n external partons was $(n-3)!$. In particular, the CHY integrand for QCD primitive amplitudes we construct will be a direct generalization of the Yang-Mills integrand,

$$A_n^{QCD}(w,p,\epsilon) = i\sum_{(j)} J_n(z^{(j)})\hat{C}_n(w,z^{(j)})\hat{E}_n(z,p,\epsilon) \tag{6.1}$$

in the sense that the function $\hat{C}_n$ will carry all the information on the external orderings and $\hat{E}_n(z,p,\epsilon)$ will only depend on the external polarizations, which we denote generically by ϵ (of course, this also includes spinor wavefunctions for the quarks). These functions will be referred to as the generalized Parke-Taylor factor and generalized polarization function, respectively. Rewriting the CHY representation as a matrix equation between the basis amplitudes and the generalized polarization function evaluated at different inequivalent solutions to the scattering equations, we will see that it is always possible to invert the equation and write the integrand $\hat{E}_n(z,p,\epsilon)$ in terms of the basis amplitudes, assuming that the matrix $F_{ww'}$ defined by Eq.(5.85) has full row rank. We will verify this conjecture for the case of the six quark amplitude analytically. One obvious drawback of this approach is that the CHY integrand will be given in terms of the amplitudes themselves; therefore, what we accomplish with this is to obtain a proof that the CHY representation always exists for QCD primitive amplitudes in arbitrary dimension.

The second way to obtain the CHY representation for QCD amplitudes will be based on the construction of a connected formalism prescription for amplitudes with massless quarks. Using the relation between $\mathcal{N}=4$ SYM and QCD amplitudes with up to four massless quark-antiquark pairs obtained in [28] (which were later shown to also hold for arbitrary, massless QCD primitive amplitudes in [80]), one uses the connected formulas for the superamplitudes of $\mathcal{N}=4$ SYM to write down the equivalent formulas for the case of QCD. As we have discussed, the scattering equations in four spacetime dimensions, when written in terms of spinors, are exactly the same constraints of the connected prescription formulas. Therefore, we can interpret these connected formulas as the CHY representation of QCD amplitudes with fermions. However, in this case, we only obtain results which are valid in $D = 4$, and that only hold for massless fermions.

It should be noted that CHY integrands for amplitudes with fermions can also be obtained from string theory, as shown in [59]. We will not work out this in detail, but we offer a brief outline of the idea. The n-gluon open string amplitude in the Ramond-Neveu-Schwarz (RNS) formalism for the superstring can be written as an integral over the expectation value a product of n vertex operators on the disk as

$$\mathcal{A}_n^{string} = \frac{1}{\alpha' g_o^2} \langle cU^{(-1)}(z_1) cU^{(-1)}(z_{n-1}) cU^{(0)}(z_n) \int \prod_{i=2}^{n-2} dz_i U^{(0)}(z_2) ... U^{(0)}(z_n) \rangle \quad (6.2)$$

Here, α' is the inverse string tension and g_o is the open string coupling. The subscript on the vertex operators U denote the ghost picture in which they are defined, and the integral is to be performed over the ordered region $z_{n-2} > z_{n-3} > ... > z_2$, once the values of z_1, z_{n-1}, z_n are fixed to arbitrary values, which is the analogue of the $SL(2, \mathbb{C})$ gauge fixing of the scattering equations. Evaluating the expectation value by performing the corresponding Grassmann integrations, the integrand of Eq. (6.2) becomes an expansion in α', whose leading order is the Pfaffian of the CHY formalism for Yang-Mills theory. The subleading terms then become proportional to the scattering equations, and thus, introducing by hand the product of delta functions in the CHY measure Eq. (4.3), one obtains a new dual model whose low energy limit (which corresponds to $\alpha' \to 0$) precisely coincides with the CHY formulas for tree amplitudes. Performing this procedure for the vertex operators associated to superstring amplitudes with fermions, it is possible to obtain the CHY representation by this prescription after calculating the correlation functions of the appropriate vertex operators and taking the leading term in the limit $\alpha' \to 0$. We refer the interested reader to the original paper for details.

6.1. Construction of the CHY integrand for QCD

In this section, we will review the construction of the CHY integrand for QCD primitive amplitudes developed in [26].

Since we will work with massive quarks, we will need the generalization of the scattering equations to massive particles, given in Eq. (3.15). Also, since the form of the CHY representation as a sum over the inequivalent solutions will prove to be more useful for our purposes, we will need to generalize the Jacobian, Eq.(4.7), to account for the modification of the scattering equations. The only modification need is in the matrix (Φ_n), which will be given by

$$(\Phi_n)_{ab} := \begin{cases} \frac{2p_a \cdot p_b + 2\Delta_{ab}}{z_{ab}^2} & \text{if } a \neq b \\ -\sum_{\substack{c=1 \\ c \neq a}} (\Phi_n)_{ac} & \text{if } a = b \end{cases} \quad (6.3)$$

in terms of which, as before, the Jacobian is

$$J_n(z,p) = \frac{1}{det'(\Phi_n)} = (-1)^{i+j+k+q+p+r} \frac{z_{ij}z_{jk}z_{ki}z_{pq}z_{qr}z_{rp}}{det[\Phi_n]^{ijk}_{pqr}} \tag{6.4}$$

Now, consider an n-point QCD primitive amplitude $A_n^{QCD} = A_n$ such that $n = n_g + 2n_q$, with n_g the number of gluons and n_q the number of quark-antiquark pairs. We would like to show that A_n can be written as a sum over the inequivalent solutions to the scattering equations as

$$A_n(w,p,\epsilon) = i \sum_{(j)} J_n(z^{(j)}) \hat{C}_n(w,z^{(j)}) \hat{E}_n(z^{(j)},p,\epsilon) \tag{6.5}$$

where the function $\hat{C}_n(w,z^{(j)})$ carries all the information on the cyclic ordering and the function $\hat{E}_n(z,p,\epsilon)$ contains all the dependence on the polarizations of the external particles. In order to respect the $SL(2,\mathbb{C})$ invariance of the scattering equations, we demand that, under a Möbius transformation

$$z \to \zeta = \frac{az+b}{cz+d}, \quad ad - bc = 1 \tag{6.6}$$

the half-integrands $\hat{C}_n(w,z^{(j)})$ and $\hat{E}_n(z,p,\epsilon)$ transform as

$$\begin{aligned} \hat{C}_n(w,\zeta) &= \left(\prod_{i=1}^{n}(cz_i+d)^2\right)\hat{C}_n(w,z) \\ \hat{E}_n(\zeta,p,\epsilon) &= \left(\prod_{i=1}^{n}(cz_i+d)^2\right)\hat{E}_n(z,p,\epsilon) \end{aligned} \tag{6.7}$$

Note that, however, there is no unique way to construct such two integrands, because a rescaling $\hat{C} \to t\hat{C}$, $\hat{E} \to \frac{1}{t}\hat{E}$ leaves the product $\hat{C}\hat{E}$ invariant. More generally, we can multiply $\hat{C}$ by any function of cross-ratios of the punctures, which is $SL(2,\mathbb{C})$ invariant, and divide $\hat{E}$ by this same function. Such a modification not only leaves the product $\hat{C}\hat{E}$ invariant, but also leaves the transformation properties of the half-integrands unaltered. Therefore, this implies that there is a freedom in the definition of either the generalized Parke-Taylor factor $\hat{C}$ or the generalized polarization function $\hat{E}$. To fix this freedom, we will constrain the structure of the generalized Parke-Taylor factors: for $n_q \leq 1$, we will assume that $\hat{C}_n(w) = C_n(w)$ for all n, that is, for the pure Yang-Mills case and the QCD amplitudes with one quark line, we define the generalized Parke-Taylor factor as the standard Parke-Taylor factors given in Eq.(4.24). For amplitudes with $n_q \geq 2$, we also demand that $\hat{C}_n(w) = C_n(w)$, but only for the subset of words with the standard orientations of fermion lines, which were described in Chapter 5. Since these subset of words exactly define the amplitude basis for $n_q \geq 2$, we will impose cyclic invariance, the KK and fundamental BCJ relations, as well as the the fermion orientation relations to express all the other generalized Parke-Taylor factors as linear combinations of standard Parke-Taylor factors. In a nutshell, this means that we will take the generalized Parke-Taylor factors to satisfy the different linear relations that primitive amplitudes must satisfy. Since the dependence on the external orderings will be encoded completely in the generalized Parke-Taylor factors, this will guarantee that the primitive amplitudes satisfy them by construction. Let us describe how to do this.

For $w \in W_3$, we have simply defined

$$\hat{C}_n(w, z) = C_n(w, z) \tag{6.8}$$

since, for $n_q \geq 2$, the words in W_3 are exactly those where the fermion lines are all in the standard orientations. Now, for $w \in W_2/W_3$, we first define

$$\hat{C}_n(w) = 0 \tag{6.9}$$

for words with crossed fermion lines; this guarantees that the amplitudes with crossed fermion lines vanish. For words with no crossed fermion lines but with that are in the wrong orientation, we expand the generalized Parke-Taylor factors as

$$\begin{aligned} &\hat{C}_n(w_{k-1} q_i w_k \bar{q}_j w_{k+1} q_j u_k \bar{q}_i u_{k-1}) \\ &= (-1)^{\gamma+1} \sum_{a=0}^{r} \sum_{b=0}^{s} \hat{C}_n(w_{k-1} q_i l_{i_1} ... l_{i_a} q_j w'_{k+1} \bar{q}_j l_{j_b+1} ... l_s \bar{q}_i u_{k-1}) \end{aligned} \tag{6.10}$$

with the conventions used in Eq. (5.72). This means that, after applying Eq. (6.10) recursively enough times, we will be able to expand the generalized Parke-Taylor factors with the wrong orientation of the fermion lines in terms of standard Parke-Taylor factors with the standard orientation of the fermion lines.

For $w \in W_1/W_2$, the word w has the general form

$$w = l_1 w_1 l_n w_2 \tag{6.11}$$

for two subwords w_1 and w_2. Letting $|w_2|$ denote the length of w_2, we demand that they can be expanded according to the KK relations,

$$\hat{C}_n(l_1 w_1 l_n w_2, z) = (-1)^{|w_2|} \hat{C}_n(l_1 (w_1 \sqcup\!\sqcup\, w_2^T) l_n, z) \tag{6.12}$$

which gives the generalized Parke-Taylor factors with external ordering $w \in W_1/W_2$ in terms of generalized Parke-Taylor factors with legs l_1 and l_2 fixed. Finally, for words $w \in W_0/W_1$, which are the words that do not have the letter l_1 fixed at position one, such as $w = w'_1 l_1 w'_2$ for some subwords w'_1, w'_2, we demand cyclic invariance

$$\hat{C}_n(w'_1 l_1 w'_2, z) = \hat{C}_n(1 w'_1 w'_2, z) \tag{6.13}$$

hence, ultimately, we define the generalized Parke-Taylor factor as a linear combination of standard Parke-Taylor factors whose orderings belong to the amplitude basis.

In order to define the generalized polarization function $\hat{E}_n$, we rewrite Eq. (6.5) as a matrix equation relating an $N_b = (n-3)!$ dimensional vector of basis primitive amplitudes, indexed by the word w that specifies its external ordering,
$A_w = A_n(w)$
and a $N_s = (n-3)!$ vector of generalized polarization functions, where each entry is evaluated at the j-th solution to the scattering equations,

$$\hat{E}_j = \hat{E}_n(z^{(j)}, p, \epsilon) \tag{6.14}$$

Then, the CHY formula Eq. (6.5) for the primitive amplitude $A_n(w)$ can be written as

$$A_w = i\hat{M}_{wj}\hat{E}_j \tag{6.15}$$

where a sum over j is understood and $\hat{M}$ is the $N_b \times N_s$ matrix, with entries

$$\hat{M}_{wj} = J_n(z^{(j)}, p)\hat{C}_n(w, z^{(j)}) \tag{6.16}$$

This definition can also we done, of course, for $w \notin B_n$. However, we will see that it is sufficient to consider only the primitive amplitudes in the basis B in order to define $\hat{E}_n$. The benefit of using this notation is that we can understand the CHY representation for QCD primitive amplitudes (or, in fact, for any theory with color-ordered amplitudes) as that of finding a basis in which we can expand the primitive amplitudes, such that the information on the external orderings is separated from the information on the spin, and which localizes to the solutions to the scattering equations. Then, given the definition of the generalized Parke-Taylor amplitude, a CHY representation for QCD primitive amplitudes will exist if Eq. (6.15) can be inverted to obtain the generalized polarization function $\hat{E}$. However, recall that

$$N_b = \begin{cases} (n-3)! & \text{if } n_q \leq 1 \\ (n-3)!\frac{2(n_q-1)}{(n_q)!} & \text{if } n_q \geq 2 \end{cases} \tag{6.17}$$

hence, the matrix $\hat{M}$ is an square matrix for $n_q = 0, 1, 2$ [1], but is in general a rectangular matrix with less rows than columns for $n_q > 2$. Let us treat these cases separately.

To simplify the discussion, we will work with the alphabet A_o, given in Eq. (5.82), which we remind is given by

$$A_0 = \{1, 2, ..., n\} \tag{6.18}$$

that is, we omit the information on the flavour of the particles. With this terminology, the set W_2 is simply given by

$$W_2 = \{w \in W_0 | l_1 = 1, l_n = n\} \tag{6.19}$$

Moreover, since the size of the basis B for $n_q = 0, 1, 2$ is $(n-3)!$, we define the set

$$B_{n_q \leq 2} = \{w \in W_0 | l_1 = 1, l_{n-1} = n-1, l_n = n\} \tag{6.20}$$

which provides a basis for amplitudes with $n_q \leq 2$. Furthermore, since in all these cases the words $w \in B_{n_q \leq 2}$ have the standard orientation of the fermion lines, the matrix in Eq.(6.16) simplifies, because all of the generalized Parke-Taylor factors in it become standard Parke-Taylor factors, and we introduce the notation for this simplified matrix

[1] for $n_q = 2$, the factor $\frac{2(n_q-1)}{(n_q)!}$ is equal to 1

$$M_{wj} = J_n(z^{(j)}, p)C_n(w, z^{(j)}) \tag{6.21}$$

which is a square matrix. Furthermore, for these values of n_q, we can define the matrix $N_s \times N_b$ matrix (which is also square),

$$N_{jw} = \sum_{v \in B_{n_q \leq 2}} S_n[w|\bar{v}]C_n(\bar{v}, z^{(j)}) \tag{6.22}$$

where $S_n[w|\bar{v}]$ is the massive momentum kernel, given in Eq.(5.96). Now, since both the momentum kernel and the Jacobian are modified in the same way by the inclusion of massive particles, we can see that KLT orthogonality holds when we replace $\tilde{S}_n$ by S_n and the Jacobian by its massive form. Therefore, we can conclude that N_{jw} defines an inverse to M_{wj},

$$\begin{aligned} N_{iw}M_{wj} &= \sum_{w,v \in B_{n_q \leq 2}} J_n(z^{(j)})C_n(w, z^{(j)})S_n[w|\bar{v}]C_n(\bar{v}, z^{(i)}) \\ &= \delta_{ij} \end{aligned} \tag{6.23}$$

Moreover, note that the j-th row of N_{jw} depends only on the j-th inequivalent solution $z^{(j)}$ to the scattering equations, and is independent of the other solutions $z^{(k)}$ for $k \neq j$. Hence, given a primitive amplitude $A_n(w) = A_w$ with $w \in B_{n_q \leq 2}$ which can be expanded in terms of the generalized polarization functions

$$A_w = iM_{wj}\hat{E}_j \tag{6.24}$$

multiplying by N_{iw} from the left, we find

$$\hat{E}_j = -iN_{jw}A_w \tag{6.25}$$

or, rewriting $\hat{E}_j$ and A_w in terms of its arguments,

$$\hat{E}_n(z^{(j)}, p, \epsilon) = -i \sum_{w \in B_{n_q \leq 2}} S_n[w|\bar{v}]C_n(\bar{v}, z^{(j)})A_n(w, p, \epsilon) \tag{6.26}$$

hence, we have shown that, for at least two quark-antiquark pairs, which may either be massive or massless and for an arbitrary number of gluons, there must exist a CHY representation in the form of Eq.(6.5). Note that, as we anticipated, this does not provide a closed expression which allow us to compute the amplitudes, since we are expanding the generalized polarization function in term of the basis amplitudes. Nonetheless, the expansion in Eq.(6.26) will allow us to derive the leading soft limit of the generalized polarization function by using the known soft limit of the primitive amplitudes, which is the main result of this thesis and will be worked out in detail in Chapter 7.

What happens when $n_q > 2$? In this case, the number of basis amplitudes is smaller than the number of inequivalent solutions to the scattering equations, $N_b < N_s$. Hence, the matrix $\hat{M}$ is not a square matrix. However, since we have restricted $w \in B$, the generalized Parke-Taylor factors also become standard Parke-Taylor factors as in the $n_q \leq 2$ case,

$$\hat{M}_{wj} = J_n(z^{(j)}, p)C_n(w, z^{(j)}) \tag{6.27}$$

In this case, to solve Eq.(6.15) for the generalized polarization function in terms of the primitive basis amplitudes, we must find a right inverse to $\hat{M}$. Before we proceed, let us introduce some basic notions of pseudo-inverse matrices, which generalize the notion matrix inversion to non-square matrices. We follow partially the treatment in [82], where pseudo-inverses have also been studied in the context of loop BCFW relations. For an extensive treatise of the theory of generalized inverses, we refer the reader to [83].

Let A be an $m \times n$ matrix, and let x and b denote two n dimensional vectors. Then, consider the problem of solving the following linear system

$$Ax = b \tag{6.28}$$

where we want to solve for x. If A were a square, non-singular matrix, the solution would be easily obtained in terms of the inverse A^{-1} as $x = A^{-1}b$. For a non-square matrix, there is no simple notion of matrix inversion (which means that the system is either under- or over-determined). In this case, we define the generalized inverse G as the $n \times m$ matrix that satisfies

$$AGA^T = A \tag{6.29}$$

which allows to obtain a solution to the system $Ax = b$. However, in order to provide a solution to the system, G must also satisfy the condition

$$AGb = b \tag{6.30}$$

and the general solution to the linear system is given in terms of an arbitrary vector y as

$$x = Gb + (I + GA)y \tag{6.31}$$

where I is the $n \times n$ identity matrix.

With this conditions, it can be seen that, given a non-square matrix A, a right-inverse is defined by

$$A_R^{-1} = A^T(AA^T)^{-1} \tag{6.32}$$

which exists only if the square matrix AA^T is invertible. Moreover, the generalized inverse G can be seen to be non-unique for, if we define

$$G' = G + (I - GA)Y + W(I - AG) \tag{6.33}$$

for arbitrary matrices Y and W of size $n \times n$ and $m \times m$, respectively, and where I denotes the corresponding identity matrices, then

$$AG'A = AGA = A \tag{6.34}$$

which allows one to obtain a family of generalized inverses. Hence, when trying to obtain a right inverse for $\hat{M}_{wj}$, we will make the simplest choice, given by the right inverse defined in Eq.(6.32). This will only be possible if

$$\text{rank}\hat{M}_{wj} = N_b \tag{6.35}$$

that is, if the matrix $\hat{M}_{wj}$ has full row rank. Having full row rank means that the N_b vectors formed from the rows of $\hat{M}$ are linearly independent. It should be noted that the right inverse is not unique [2], and we will be interested in finding a right inverse with the property that the entries on its j-th row is independent on the k-th inequivalent solution to the scattering equations for $k \neq j$. In order to do this, we expand the Parke-Taylor factors in Eq. (6.27) in terms of the Parke-Taylor factors with $w' \in B_{n_q \leq 2}$ using the general BCJ relations,

$$C_n(w, z^{(j)}) = F_{ww'} C_n(w', z^{(j)}) \tag{6.36}$$

where the elements of $F_{ww'}$ have been defined in Eq.(5.85). What is important to us is that the coefficients $F_{ww'}$ are purely kinematic, and do not depend on the solutions to the scattering equations. We then have

$$\hat{M}_{wj} = F_{ww'} M_{w'j} \tag{6.37}$$

From our previous discussion, we know that the matrix $M_{w'j}$ is square and invertible. Hence, the condition on the rank of $\hat{M}$ becomes a condition on the rank of F,

$$\text{rank} F_{ww'} = N_b \tag{6.38}$$

which is a purely kinematic statement, since F does not depend on the solutions to the scattering equations. There is no proof of this statement to the best of my knowledge, and the equivalence to a weaker statement is provided in Appendix C of [26]. In this thesis, we will explicitly calculate the elements of $F_{ww'}$ for the non-trivial case of six massless quarks, and we will show that, in this case, the square matrix FF^T has a non-vanishing determinant, which is equivalent to $F_{ww'}$ having full rank.

Now, assuming that F does have full rank, we take the right inverse to be

$$G = F^T (FF^T)^{-1} \tag{6.39}$$

and then the right inverse of $\hat{M}$ can be written as

$$\hat{N}_{jw} = N_{jw'} G_{w'w} \tag{6.40}$$

and then, multiplying Eq. (6.15) from the left and writting out the explicit elements of $N_{jw'}$, we obtain the generalized polarization function

$$\hat{E}_n(z, p, \epsilon) = -i \sum_{u,v \in B_{n_q \leq 2}} \sum_{w \in B_n} S_n[u|\bar{v}] G_{uw} C(\bar{v}, z) A_n(w, p, \epsilon) \tag{6.41}$$

Note that we could have constructed a right inverse to $\hat{M}$ directly, as

$$\hat{M}^T (\hat{M}\hat{M}^T)^{-1} \tag{6.42}$$

however, such an inverse could possibly mix Parke-Taylor factors evaluated at different, inequivalent solutions to the scattering equations in a given row j. The specific construction of the right inverse $\hat{N}_{jw}$ has the property that its j-th row is depends only on the j-th solution $z^{(j)}$ to the scattering equations.

[2]In contrast to the case of square matrices, were the inverse A^{-1} to a given matrix A is always unique

6.1.1. F matrix for six massless quarks

Now, we consider the explicit calculation of the matrix elements $F_{ww'}$ for the case of the six quark amplitude. If we denote the quarks and their corresponding antiquarks according to

$$q_1 = 1, \quad q_2 = 2, \quad q_3 = 3 \quad \bar{q}_3 = 4, \quad \bar{q}_2 = 5, \quad \bar{q}_1 = 6. \tag{6.43}$$

We can construct the basis

$$B = \{123456, 125346, 132546, 134256\}. \tag{6.44}$$

We also define the set

$$B_{n_q \leq 2} = \{123456, 124356, 132456, 134256, 142356, 143256\} \tag{6.45}$$

which would correspond to the basis for the six gluon amplitude. The matrix $\mathbb{F}_{6q}$, whose elements are $F_{ww'}$, will be of 4×6 size and it can be regarded as the linear operator relating the multiquark amplitude with the purely gluonic one. In order to determine the non-trivial elements (that is, those different from one or zero), one fixes the word w and using the linear expansion appearing in the BCJ relations, Eq. (5.84), one can find the words w' for which the element $F_{ww'}$ is non-trivial. Since the letters $1, 5, 6$ are fixed in the elements of $B_{n_q \leq 2}$ and the letters $1, 6$ are fixed in the elements of B, it is sufficient to consider the words in B which are not in $B_{n_q \leq 2}$. The amplitudes with those external orderings are the only ones that will have a non-trivial linear expansion in terms of the amplitudes whose external orderings are in $B_{n_q \leq 2}$. As an example of the calculation of the elements, we consider the word $w = 125346$. To determine the words $w' \in B_{n_q \leq 2}$ that contribute to the linear expansion of w, we consider two subwords of w,

$$w_1 = 2, \quad w_2 = 34 \tag{6.46}$$

these are chosen in such a way that $w = 1w_1 5 w_2 6 \notin B_{n_q \leq 2}$. Then, we calculate

$$s(w_2) = 34 + 43 \tag{6.47}$$

Finally, we need to calculate the shuffle product

$$w_1 \sqcup\!\sqcup\, s(w_2) = 234 + 243 + 324 + 342 + 324 + 432 \tag{6.48}$$

If we let σ denote an arbitrary term appearing on the sum in 6.48, we construct the word $w' = 1\sigma 56$, and we can calculate the element $F_{ww'}$ for the fixed w and each of those w'. In this particular case, we need to calculate six matrix elements. For example, we can calculate the matrix element for $w = 125346$ and $w' = 123456$. Using the definition of $F_{ww'}$ in Eq. (5.85), we can see that, in this case,

$$F_{ww'} = \frac{\mathcal{F}(1\sigma 5|3)}{\hat{s}_{6,\ell_2,\ell_3}} \frac{\mathcal{F}(1\sigma 5|4)}{\tilde{s}_{6,\ell_3}} \tag{6.49}$$

where $\sigma = 234$. Then, we form the string $\rho = 12345$, and we need to calculate the factors $\mathcal{F}(\rho|3)$ and $\mathcal{F}(\rho|4)$, defined in Eq. (5.87). For a general factor $\mathcal{F}(\rho|\ell_k)$, we recall that one needs to determine the position of ℓ_k in the string $\rho = 1\sigma(n-1)$, t_{ℓ_k}, and compare it with the positions of ℓ_{k+1} and ℓ_{k-1}; furthermore, if we let ℓ_j be the last letter appearing in w_1, which in our case is the letter 2, the position of said letter is always fixed to $t_{\ell_j} = n$, the number of particles in the scattering process. Finally, for $k = n-4$, we always have the condition $t_{\ell_{n-4}} = t_{\ell_{n-2}}$.

For the factor $\mathcal{F}(\rho|3)$ in our example, since $\ell_2 = 3$ and $\ell_3 = 4$, $t_{\ell_2} < t_{\ell_3}$. Furthermore, since $\ell_1 = 2$, from the general condition $t_{\ell_j} = n$, we find $t_2 = 6$ so that $\mathcal{F}(\rho|3) = 2(p_1 + p_2) \cdot p_3$. From the general condition $t_{\ell_{n-2}} = t_{\ell_{n-4}}$, we find that $t_{\ell_3} > t_{\ell_4}$ therefore $\mathcal{F}(\rho|4) = -2p_4 \cdot p_5$. This implies

$$F_{ww'} = -\frac{4(p_1 + p_2) \cdot p_3 (p_4 \cdot p_5)}{\tilde{s}_{6,3,4}\tilde{s}_{6,4}}. \tag{6.50}$$

The calculation of the rest of matrix elements proceeds similarly. That is, once w is fixed, one constructs the sum in (6.48) and calculates the element $F_{ww'}$ for each of those two words. In the case $w = w'$, $F_{ww'} = 1$ and the rest of elements are zero. With this in mind, we can arrange the matrix $\mathbb{F}_{6q}$, and find the square matrix

$$\mathbb{F}_{6q}\mathbb{F}_{6q}^T = \begin{bmatrix} 1 & 0 & 0 & D \\ 0 & 1 & A & E \\ 0 & A & A^2 + B^2 + C^2 & AE + BF + CG \\ D & E & AE + BF + CG & D^2 + E^2 + F^2 + G^2 + H^2 + I^2 \end{bmatrix}.$$

The non-zero elements are rational functions of various kinematic invariants and masses, and are given by

$$\begin{aligned} A &= -\frac{2[(p_1 + p_3) \cdot p_4 + m_3^2]}{\tilde{s}_{6,4}}, \quad B = \frac{2[p_4 \cdot (p_1 + p_2 + p_3) + m_3^2]}{\tilde{s}_{4,6}}, \quad C = \frac{2p_1 \cdot p_4}{\tilde{s}_{4,6}} \\ E &= -\frac{4\left((p_2 + p_5) \cdot p_4\right)(p_1 \cdot p_3)}{\tilde{s}_{6,3,4}\tilde{s}_{6,4}}, \quad F = -\frac{4(p_1 \cdot p_3)(p_4 \cdot p_5)}{\tilde{s}_{6,3,4}\tilde{s}_{6,4}} \\ G &= -\frac{2p_1 \cdot p_4 \left(2p_3 \cdot (p_2 + p_5) + \tilde{s}_{6,4,3}\right)}{\tilde{s}_{6,4,3}\tilde{s}_{6,4}} \\ H &= -\frac{2[(p_1 + p_2) \cdot p_4][2p_3 \cdot p_5 + \tilde{s}_{6,4,3}]}{\tilde{s}_{6,4,3}\tilde{s}_{6,4}}, \quad I = -\frac{-2p_1 \cdot p_4 \left(2p_3 \cdot p_5 + \tilde{s}_{6,4,3}\right)}{\tilde{s}_{6,4,3}\tilde{s}_{6,4}} \end{aligned} \tag{6.51}$$

and D is given by the expression on (6.50). For general kinematics, this matrix has a non-vanishing determinant. Thus, by explicitly computing the matrix elements and the determinant, we have confirmed the claim [26] that the CHY representation exists for the six quark primitive amplitude in QCD [84].

6.2. Connected formalism prescription for QCD amplitudes

A different route to obtaining CHY/Connected formulas for QCD color-ordered amplitudes with massless quarks, which is the subject we review in this chapter, was provided in [27]. The idea is to exploit the fact that, as first detailed in [28] for amplitudes with up to four quark lines and then shown to hold for an arbitrary number of quark lines in [85], all color-ordered amplitudes in QCD can be written as linear combinations of gluon-gluino amplitudes in $\mathcal{N}=4$ SYM. Therefore, one can use the connected formula , Eq.(4.75), which contains all the information on the superamplitudes of $\mathcal{N}=4$ SYM; in particular, by suitable Grassmann differentiation (or integration), one can extract the component amplitudes corresponding to gluon-gluino amplitudes. We will see, through a couple of examples, that combining the different connected formulas for gluon-gluino amplitudes gives simple forms for the gluon-quark QCD amplitudes. As a first step, we will see how to combine the $\mathcal{N}=4$ component amplitudes to obtain QCD amplitudes. Since connected formulas are explicitly four-dimensional, we can also make use of helicity amplitudes. However, besides the fact that this approach cannot be applied for massive fermions, it is not valid in arbitrary dimension, which is one of the main advantages of the CHY formalism.

The fundamental difference between QCD and $\mathcal{N}=4$ SYM amplitudes lies in the fact that, while quarks transform in the fundamental representation of the gauge group and there can be an arbitrary number of quark flavours, gluinos transform in the adjoint representation and come only in four flavours which arise from the $SU(4)$ R-symmetry of the theory. Moreover, the spectrum of $\mathcal{N}=4$ contains six $SU(4)$ antisymmetric scalars S_{AB}, which give rise to flavour changing interactions for the gluinos via the interaction term $SAB\psi^A\psi^B$. The first problem is solved by color-decomposition, because once one separates the color information to obtain the color-ordered amplitude, the information on the gauge group (and hence, the representation under which the external states transform) is irrelevant, and then it is possible to obtain the full quark-gluon amplitude by using gluon-gluino partial amplitudes.

However, the non-trivial problem one has to solve in order to express QCD amplitudes in terms of $\mathcal{N}=4$ SYM amplitudes is to avoid the exchange of internal scalars, which of course are not present in QCD. Also, it is not obvious that the flavours of all the quark lines can be taken to be different from each other. In order to this, one first notes that, similarly to the pure gluon sector, $\mathcal{N}=1$ SYM is a closed subsector of $\mathcal{N}=4$ SYM. By closed sector we mean that, if out of the four gluino flavours, we single out one of them (say, ψ^1), then amplitudes whose external states are only formed by gluons g and gluinos ψ^1 are the same in $\mathcal{N}=4$ SYM and in $\mathcal{N}=1$ SYM, where the spectrum is simply a gluon and a gluino. This happens because the scalars in $\mathcal{N}=4$ SYM only couple different flavoured scalars; hence, the only internal particles that can be exchanged to produce $\psi^1\bar{\psi}^1$ pairs are gluons, and these are exactly the same interactions that appear in $\mathcal{N}=1$ SYM. This is similar to the argument of why pure gluon tree amplitudes are the same in all gauge theories: since both the scalars and fermions couple only to the gluon by pairs, there can be no internally propagating scalar or fermion which produces a pair of gluons; if an internal fermion generates a gluon, it must be accompanied by a corresponding antifermion and similarly in the case of scalars. The consequence of this is that QCD color-ordered amplitudes with an arbitrary number of massless quark lines with the same flavour are identical to the corresponding amplitude, where the gluino ψ^1 replaces the quark, in $\mathcal{N}=4$ SYM. If we want to calculate amplitudes with a higher number of flavours, we start to face the problem of the scalar exchange that couple

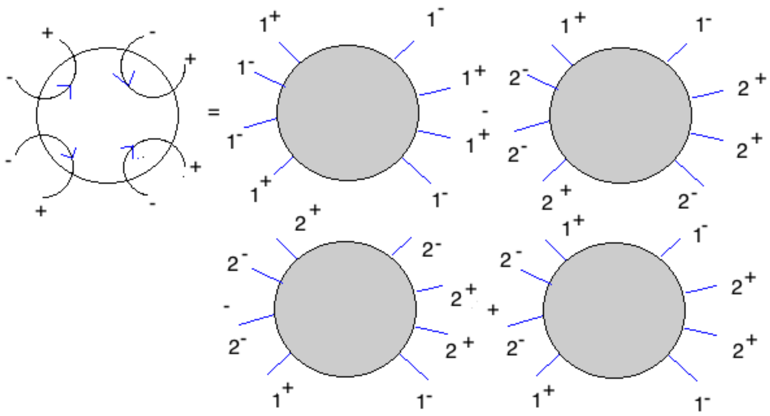


Fig. 6.1: Color-ordered QCD amplitude with four quark-lines (left hand side) as a linear combination of $\mathcal{N} = 4$ SYM gluon-gluino amplitudes. The number indicates the flavour and $\pm$ the helicity.

different flavoured gluinos. In order to avoid the exchange of scalars, one needs to take in account the following facts about the interaction vertices of $\mathcal{N} = 4$ SYM:

1. The scalars S_{AB} cannot couple equal-flavoured gluinos due to their antisymmetry in the $SU(4)$ indices A, B,

2. fermion helicity is conserved in gauge theories. Therefore, a Yukawa interaction coupling the scalar S_{AB} with two gluinos $\psi^{+,A}$ and $\psi^{-,B}$, as well as gluon-gluino-gluino vertex where both gluinos carry the same helicity vanish (in the convention where all particles are outgoing),

3. a gluon-gluino-gluino vertex with different flavour gluinos vanish because gluons do not change flavour

In particular, the fact that gluons cannot change gluino flavours mean that one can focus on amplitudes with only external gluinos. In order to avoid this exchange, we first look at the position of the gluinos in the color ordering. If two adjacent gluinos have the same helicity, then we choose them to have the same flavour; that is, configurations of the form $(..., A^{\pm}, B^{\pm}, ...)$ for $A \neq B$ are forbidden, where we note each gluino simply by its flavour. Moreover, it is necessary to also forbid configurations of the form $(..., A^{\pm}, C^{\pm}, C^{\mp}, B^{\pm}, ...)$ because the line $(C^{\pm}, C^{\mp})$ can be produced from an internal gluon and connect the line $(A^{\pm}, B^{\pm})$. Similarly, if the two adjacent gluinos have opposite helicity, they are either chosen to have the same flavour, or they are given a different flavour which matches the quark fermion flow. This can be illustrated with graphs similar to the ones we used in Chapter 5 to describe multiquark primitive amplitudes. As an example, consider the graphs given in Fig.(6.1). Algebraically, this relation means

$$
\begin{aligned}
&A_8^{QCD}(\bar{q}_1^-, q_1^+, \bar{q}_2^-, q_2^+, q_3^+, \bar{q}_3^-, q_4^+, \bar{q}_4^-) = A_8^{\mathcal{N}=4}(1^-, 1^+, 1^-, 1^+, 1^+, 1^-, 1^+, 1^-) \\
&- A_8^{\mathcal{N}=4}(2^-, 1^+, 1^-, 2^+, 2^+, 2^-, 2^+, 2^-) - A_8^{\mathcal{N}=4}(2^-, 2^+, 2^-, 2^+, 2^+, 1^-, 1^+, 2^-) \\
&+ A_8^{\mathcal{N}=4}(2^-, 1^+, 1^-, 2^+, 2^+, 1^-, 1^+, 2^-)
\end{aligned} \tag{6.52}
$$

In order to write down similar formulas for more than four quark flavours, one must use flavour recursions, as derived in [80], which allow to write multiflavour color-ordered (and also, primitive) amplitudes in terms of amplitudes with only one flavour, which can be directly extracted from the $\mathcal{N} = 1$ subsector of $\mathcal{N} = 4$ SYM.

Now, to write connected formulas for the color-ordered QCD amplitudes from the connected formulation of $\mathcal{N} = 4$ SYM, we fill find convenient to rewrite the RSV formula, Eq.(4.75), using the rational form of the 4D scattering equations, given in Eq. (3.20). In order to do this, we first introduce the short hand notation

$$
(ab) = \frac{\sigma_a - \sigma_b}{t_a t_b} \tag{6.53}
$$

then, as shown in [68], the superamplitude in the sector with k negative helicity particles and be written as

$$
\begin{aligned}
\mathcal{A}_{n,k}^{\mathcal{N}=4} &= \int \frac{\prod_{a=1}^n d^2\sigma_a}{\text{vol}\, GL(2,\mathbb{C})} \prod_{I\in -} \delta^2\left(\tilde{\lambda}_I - \sum_{i\in +} \frac{\tilde{\lambda}_i}{(Ii)}\right) \prod_{i\in +} \delta^2\left(\lambda_i - \sum_{I\in -} \frac{\lambda_I}{(iI)}\right) \\
&= \prod_{I\in -} \delta^{0|4}\left(\eta_I - \sum_{i\in +} \frac{\eta_i}{(Ii)}\right) \frac{1}{(12)(23)...(n1)}
\end{aligned} \tag{6.54}
$$

where $+$ and $-$ denote two sets into which we split the particles, such that there are k particles in $-$ and the remaining $n-k$ particles in $+$, and we denote the elements of $-$,$+$ with I and i, respectively.

As discussed in Chapter 2, component amplitudes can be obtained from the superamplitude by calculating derivatives of the Grassmann variables η, according to the superfield prescription of Eq.(2.144). Grassmann differentiation can of course be also performed as Grassmann integration, and it is in this way which it proves more easy to perform the calculations.

For example, assume we want the component amplitude with $(n-2)$ gluons and one pair of gluinos $\{\bar{\psi}_{I,A}, \psi_i^A\}$, with $I \in -$ and $i \in +$. This means that we take negative helicity gluinos to be in the set $-$ and positive helicity gluinos to be in the set $+$. Ignoring the contribution from the gluons, we need to perform the Grassmann integrals over [3] $(d^3\eta_I)_A d\eta_i^A$. This leaves only one delta function,

$$
\delta^{0|4}\left(\eta_I - \frac{\eta_i}{(Ii)}\right) = \frac{1}{(Ii)}\delta^{0|4}(\eta_I - \eta_i) \tag{6.55}
$$

[3] Recall that the positive helicity gluinos are associated to a single Grassmann variable η^A, while the negative helicity gluinos are associated to the product of three Grassmann variables in the definition of the superfield.

using the properties of the fermionic delta functions. Hence, the overall effect of the integration over the Grassmann variables of a gluino-antigluino pair is to introduce a Jacobian factor $\mathcal{J} = \frac{1}{(Ii)}$. This generalizes to m gluino pairs, which we can label by $(I_1, i_1)...(I_m, i_m)$, and yields the formula

$$A_{n,k}^{g;\psi\bar{\psi}} = \int \frac{\prod_{a=1}^{n} d^2\sigma_a}{\text{vol}\, GL(2,\mathbb{C})} \frac{\det \Xi}{(12)(23)...(n1)} \prod_{I\in -} \delta^2\left(\tilde{\lambda}_I - \sum_{i\in +} \frac{\tilde{\lambda}_i}{(Ii)}\right) \prod_{i\in +} \delta^2\left(\lambda_i - \sum_{I\in -} \frac{\lambda_I}{(iI)}\right) \tag{6.56}$$

where Ξ is the $m \times m$ matrix with entries, for $r, s = 1, 2, ...m$,

$$\Xi_{rs} = \frac{\delta^{A_{I_r} A_{i_s}}}{(I_r i_s)} \tag{6.57}$$

where A_{I_r}, A_{i_s} refer to the flavour of the gluinos. Naturally, to avoid an ambiguity with respect to the overall sign of the determinant, one must choose in which order to label the rows and columns of Ξ; these are arranged such that row r corresponds to I_r and column s to i_s.

From our discussion of the relation between $\mathcal{N} = 4$ SYM and QCD color-ordered amplitudes, we know that the component amplitudes with only one flavour are exactly equal to the amplitudes in single-flavour QCD. Hence, from Eq.(6.56), we obtain the connected formula for single-flavour, massless QCD by setting $\delta^{A_i A_i} = 1$ in all the entries of Ξ.

Now, consider the case where there are two quark lines with distinct flavours, which we denote by $\{a_1, a_2\}$ and $\{b_1, b_2\}$. These two lines can be ordered in two possible ways: $(a_1^-, b_1^-, b_2^+, a_2^+)$ and $(a_1^-, a_2^+, b_1^-, b_2^+)$. As before, the position of the gluons can be ignored.

In the first case, note that the vertices connect the different flavour gluinos a_1^-, b_1^- and a_2^+, b_2^+. Therefore, in order to avoid the propagation of an internal scalar S^{AB}, we identify the QCD color-ordered amplitude with the $\mathcal{N} = 4$ SYM amplitude with $A = B$, which guarantees that the diagrams with scalar exchange vanish. This corresponds to taking the entries of Ξ for two gluinos of equal flavour, which results in the Jacobian

$$\mathcal{J}(a_1^-, b_1^-, b_2^+, a_2^+) = \begin{vmatrix} \frac{1}{(a_1 a_2)} & \frac{1}{(a_1 b_2)} \\ \frac{1}{(b_1 a_2)} & \frac{1}{b_1 b_2)} \end{vmatrix} \tag{6.58}$$

In the second case, the vertices already connect the equal flavour gluinos. In this case, to avoid contributions with the form of amplitudes with equal flavours (which would arise from gluon exchange), one takes two different flavours, $A \neq B$, which cancel the off-diagonal elements of Ξ and the Jacobian simplifies

$$\mathcal{J}(a_1^-, a_2^+, b_1^-, b_2^+) = \frac{1}{(a_1 a_2)} \frac{1}{b_1 b_2)} \tag{6.59}$$

In general, the connected formula obtained from this procedure for an arbitrary number of quark-antiquark pairs has the form

$$A_n^{g;q\bar{q}} = \int \frac{\prod_{a=1}^n d^2\sigma_a}{\text{vol}\, GL(2,\mathbb{C})} \frac{\mathcal{J}(\{\sigma_{q,\bar{q}}\})}{(12)(23)...(n1)} \prod_{I\in -} \delta^2\left(\tilde{\lambda}_I - \sum_{i\in +} \frac{\tilde{\lambda}_i}{(Ii)}\right) \prod_{i\in +} \delta^2\left(\lambda_i - \sum_{I\in -} \frac{\lambda_I}{(iI)}\right) \tag{6.60}$$

where the Jacobian J is obtained from the matrix Ξ, by setting to zero different elements of the one-flavour matrix with entries $\Xi_{rs} = 1/(I_r i_s)$, depending on the ordering of the external quarks. This construction is given explicitly for up to four quark-antiquark pairs in [27].

Let us do some final remarks on this two alternative approaches. On one hand, the method based on inverting the CHY representation to obtain the polarization function in terms of the primitive amplitudes has the advantage of satisfying, by construction, the various linear relations between primitive amplitudes with different external orderings in arbitrary spacetime dimension. Also, it allows one to obtain mathematical properties of the integrand by using the known properties of the amplitudes (for example, the subject of Chapter 7 will be to show that the generalized polarization function for $n_q \leq 2$ factorizes on soft limits to leading order, in a way similar to the primitive amplitudes). However, as we already mentioned, it has the obvious problem of depending on the amplitudes, making it useless for the calculation of the amplitudes themselves.

On the other hand, the connected formalism provides us with explicit formulas for the integrands with fermions. However, although cyclic invariance and the KK relations are manifest from the definition in terms of the Parke-Taylor like factor $1/(12)...(n1)$, the BCJ relations are not so obvious, and would require a case-by-case check, because there is no explicit expression for the fermion Jacobian $\mathcal{J}$ with m quark-antiquark pairs, even though it can be calculated in principle for any number of quark-antiquark lines. Moreover, there is no clear way to generalize the result to massive quarks, because the spectrum of $\mathcal{N} = 4$ SYM is massless, and the introduction of masses breaks the supersymmetry of the theory, which makes the whole superfield and superamplitude construction fall apart. Moreover, the formula is only valid in four dimensions, and its generalization to arbitrary dimension is complicated due to the fact of choosing to express the fermion Jacobian in terms of spinors and helicity amplitudes, which of course exist only in four dimensions[4]. It would be interesting to see how these two approaches can be connected to each other, and if the existence of the four dimensional connected formulas provide hints into the proof that $F_{ww'}$ has full row rank.

[4]In $D > 4$ dimensions, the little group of a massless particle is $SO(D-2)$, which is of course non-Abelian, and thus the simplicity of the $SO(2)$ little group in four dimensions is lost.

Soft limits in the CHY formalism

In this final chapter, we study the factorization of amplitudes in the soft limit using the CHY representation and present an original result, not given still in the literature, concerning the soft behaviour of the generalized polarization function for $n_q \leq 2$.

We will show that using the prescription for the integrands of Yang-Mills theory, Eq.(4.23) and for the biadjoint scalar theory, Eq.(4.37) and the decoupling of the scattering equations discussed in Chapter 3, it is possible to derive the leading soft factors when a gluon or a massless scalar is emitted with momenta $p_s \to 0$, respectively.

Finally, we will use the universal soft limit for gluon emission derived in Chapter 2 and the expansion of the generalized polarization function in terms of primitive amplitudes for $n_q \leq 2$ given in Eq. (6.26) to show that the polarization function also exhibits a factorization property in the single soft limit, which will be the main result of this thesis. We will check the self-consistency of our result by showing that the soft factor of the generalized polarization function integrates over the puncture associated to the soft momenta to yield the eikonal factor for the amplitudes. We close this chapter with a discussion of the difficulties that arise when one tries to perform a similar procedure in order to derive a factorization theorem for the generalized polarization function with an arbitrary number of quarks.

The fact that the CHY representation allow one to obtain explicit factorization theorems in certain kinematic limits for arbitrary multiplicity and in any spacetime dimension has been widely exploited in the literature, of which we quote a few results (but by no means intend to by extensive). Subleading soft theorems for both Yang-Mills and gravity [1] were obtained in [86], by performing a similar analysis to ours; later on, the result was extended to the sub-subleading order for gravity amplitudes in [87]. Double soft theorems, which occur when two massless particles are emitted with momenta $p_1 = \epsilon P_1$, $p_2 = \epsilon P_2$ with $\epsilon \to 0$ (that is, they are controlled by the same scale) were derived in [88] for various effective theories such as Dirac-Born-Infeld (DBI) and the Non-linear Sigma model (NLSM), as well as Yang-Mills scalar. The double soft graviton theorem was derived afterwards in [89]. Also, collinear li-

[1] The subleading terms for soft theorems in QCD were worked out from the point of view of on-shell recursion relations in [91].

mits were considered in [90], where agreement was found with the leading collinear factorization when 2 out of n external gluons become collinear, and it was shown that, at the subleading order, the amplitude can be written as the convolution of $(n-1)$-point gluon amplitudes and a so-called universal collinear kernel.

As a reminder to the reader and in order to clear the notation of this chapter, we denote by $s_{ij} = (p_i + p_j)^2$, which in the case of massless particles is equal to $2p_i \cdot p_j$. However, this is no longer true for massive particles, and we will use the notation $\hat{s}_{ij} = 2p_i \cdot p_j$ with the understanding that, in this relation, the momenta p_i and p_j may be massive.

7.1. Soft limits in the CHY formalism

Let us establish the general notation we will need for the derivation of the soft theorems.

Recall that the CHY representation can be written as the multidimensional contour integral

$$\mathcal{A}_n^0(\mathbf{x}) = i\frac{(-1)^{i+j+k}}{(2\pi i)^{n-3}} \oint_C \frac{d^n z}{d\omega} \frac{z_{ij} z_{jk} z_{ki}}{\prod' f_a(z,p)} \mathcal{I}_n(z,\mathbf{x}) \tag{7.1}$$

where the contour C encloses the $(n-3)!$ inequivalent solutions to the scattering equations. Now, suppose that one of the particles of the scattering process, say the one with momentum p_n, is massless. Hence, it makes sense to consider the limit when this momentum is very small, which as in Chapter 3, we parametrize as $p_n = \epsilon P_n$, with $\epsilon \to 0$. In this limit, the (massless) scattering equations

$$f_i(z,p) = \sum_{j \neq i} \frac{s_{ij}}{z_{ij}} = 0 \tag{7.2}$$

decouple, in the sense that, for $i \neq n$, we can write

$$f_i(z,p) = \sum_{j \neq i,n} \frac{s_{ij}}{z_{ij}} + \epsilon \frac{s_{in}}{z_{in}} = 0 \tag{7.3}$$

and, for $i = n$,

$$f_n(z,p) = \epsilon \sum_{j \neq n} \frac{s_{nj}}{z_{nj}} = 0 \tag{7.4}$$

which, as we discussed, means that if we keep only the leading order terms in ϵ, in the soft limit the equations $f_i(z,p) = 0$ for $i \neq n$ become a system of scattering equations for the $(n-1)$ particles were the soft particle is omitted, and the equation $f_n(z,p) = 0$ becomes an order $(n-3)$ polynomial equation for z_n, whose coefficients depend on the remaining hard punctures and the momenta. This means that, by performing a contour deformation, if we can separate the dependence of the integrand $\mathcal{I}_n(z,\mathbf{x})$ on the puncture z_n associated to the soft momenta, we can integrate over this puncture individually as a simple contour integral in one complex variable to obtain the leading contribution to the soft factor associated with the emission of the particle with momenta p_n. We will first see how this occurs in the biadjoint scalar ϕ^3 theory, and afterwards we will consider the case of Yang-Mills theory.

7.1.1. Leading soft limit in ϕ^3

The integrand for biadjoint scalar theory is given by the product of two standard Parke-Taylor factors with possibly different external orderings,

$$\mathcal{I}_n^{\phi^3}(z, w_1, w_2) = C_n(w_1, z)C_n(w_2, z) \tag{7.5}$$

In general,

$$\begin{aligned} w_1 &= l_1^1 ... l_{s-1}^1 l_s^1 l_{s+1}^1 ... l_n^1 \\ w_2 &= l_1^2 ... l_{t-1}^2 l_t^2 l_{t+1}^2 ... l_n^2 \end{aligned} \tag{7.6}$$

Suppose that particle n is in arbitrary positions given by the letters l_s^1 in the word w_1 and l_t^2 in the word w_2. If we assume that the letters l_{s-1}^1 and l_{s+1}^1 are associated to particles i and j respectively, with $1 \leq i, j \leq n$, and similarly that the letters l_{t-1}^2 and l_{t+1}^2 are associated to the particles k and m, with $1 \leq k, m \leq n$, we can rewrite the n particle Parke-Taylor factors in the integrand $\mathcal{I}_n^{\phi^3}(z, w_1, w_2)$ in terms of Parke-Taylor factors for $(n-1)$ particles, in analogous fashion to how we did when deriving the KK relations,

$$\mathcal{I}_n^{\phi^3}(z, w_1, w_2) = Z(i, n, j)Z(k, n, m)C_{n-1}(w_1', z)C_{n-1}(w_2', z) \tag{7.7}$$

where

$$\begin{aligned} w_1' &= l_1^1 ... l_{s-1}^1 l_{s+1}^1 ... l_n^1 \\ w_2' &= l_1^2 ... l_{t-1}^2 l_{t+1}^2 ... l_n^2 \end{aligned} \tag{7.8}$$

and $Z(a, b, c)$ is defined by Eq. (5.13). Note that the $(n-1)$ particle Parke-Taylor factors are independent of the puncture z_n. Hence, in the soft limit $p_n \to 0$, choosing not to gauge fix any of the punctures i, j, k, m, n, the CHY integral for the n-point scalar amplitude can be written as

$$m_n(w_1|w_2) = \oint_C d\Omega_{CHY}^{(n-1)} C_{n-1}(w_1', z)C_{n-1}(w_2', z) \left(\tfrac{1}{2\pi i} \oint_{f_n=0} \tfrac{dz_n}{f_n} Z(i, n, j)Z(k, n, m) \right) \tag{7.9}$$

where by $d\Omega_{CHY}^{(n-1)}$ we denote the CHY measure associated to $(n-1)$ particles. Now, omitting the explicit integration contour, let us introduce a special notation for the one-dimensional contour integral,

$$I_{ijkm}^n \equiv \frac{1}{2\pi i} \oint \frac{dz_n}{f_n} Z(i, n, j)Z(k, n, m) \tag{7.10}$$

We want to evaluate this integral for generic values of the indices i, j, k, m and n. As a first step, let us rewrite the denominator $1/f_n$ explicitly as

$$\frac{1}{f_n} = \frac{\prod_{a \neq n} z_{na}}{\sum_{a \neq n} s_{na} \prod_{b \neq a, n} z_{nb}} \tag{7.11}$$

which can be easily obtained by multiplying $\frac{1}{f_n}$ by the factor $(\prod_{a\neq n} z_{na})/(\prod_{a\neq n} z_{na})$ and noting that the denominators appearing in each of the terms of the sum defining f_n, which are of the form s_{nb}/z_{nb}, cancel exactly against the term z_{nb} in the product $\prod_{a\neq n} z_{na})$. Hence, we rewrite

$$I^n_{ijkm} = \frac{1}{2\pi}\oint dz_n \frac{\prod_{a\neq n} z_{na}}{\sum_{a\neq n} s_{na} \prod_{b\neq a,n} z_{nb}} Z(i,n,j)Z(k,n,m) \tag{7.12}$$

The contour of integration has to enclose the $(n-3)$ solutions to the equation $f_n = 0$. Then, as a first step, we notice that there is no residue at infinity, because there are $(n-1)$ power of z_n in the numerator and $(n-2)+2 = n$ in the denominator, which come from multiplying the denominators of $Z(i,n,j)Z(k,n,m)$ with the factor that remains from $1/f_n$. Moreover, using the residue theorem, it is possible to see that the only non-vanishing residues come from the denominators in $Z(i,n,j)Z(k,n,m)$. Hence, these factors determine completely the structure of the integral.

Before performing the actual evaluation, let us consider some symmetry constraints on the integral I^n_{ijkm}. First, by definition, it is obvious that, when $i = j$ or $k = m$, we have

$$I^n_{iikm} = I^n_{ijkk} = 0 \tag{7.13}$$

because of the numerators in

$$Z(a,b,c) = \frac{z_{ac}}{z_{ab}z_{bc}} \tag{7.14}$$

Moreover, since $Z(a,b,c) = -Z(c,b,a)$, the integral is antisymmetric under the exchange of its first two indices keeping the third and fourth fixed

$$I^n_{jikm} = -I^n_{ijkm} \tag{7.15}$$

and is also antisymmetric under the exchange of the third and fourth indices, keeping the first two fixed

$$I^n_{ijmk} = -I^n_{ijkm} \tag{7.16}$$

From these two properties, we can see that under a simultaneous exchange of both pairs of particle labels, the integral is symmetric

$$I^n_{jimk} = I^n_{ijkm} \tag{7.17}$$

Moreover, all of the following combinations are vanishing

$$I^n_{iiii} = I^n_{iiij} = I^n_{iikk} = 0 \tag{7.18}$$

Hence, the only non-vanishing integrals are given by the combinations $ijij$ for $i \neq j$, and $ijim$ for $i \neq j \neq m$. All other non-vanishing integrals are related to these cases by symmetry or antisymmetry under exchange of the different indices.

Now, let us evaluate the integrals of the type $ijij$ and $ijim$. First, let us write out the integral explicitly

$$I^n_{ijij} = \frac{1}{2\pi i}\oint dz_n \frac{\prod_{a\neq n} z_{na}}{\sum_{a\neq n} s_{na}\prod_{b\neq a,n} z_{nb}} \frac{z^2_{ij}}{z^2_{in}z^2_{nj}} \tag{7.19}$$

Notice that the product in the numerator contains one power each of z_{ni} and z_{nj}. This cancels the squares in the denominator, leaving only two single poles. Furthermore, when evaluating the residue at each pole (which are either $z_n = z_i$ or $z_n = z_j$), a generic term in the sum

$$\sum_{a\neq n} s_{na} \prod_{b\neq a,n} z_{nb} \tag{7.20}$$

will contain either a factor of z_{ii} or z_{jj}, which are of course zero. The only non-vanishing term when calculating the residue, say, at z_i, is then

$$s_{ni}\prod_{b\neq i} z_{ib} \tag{7.21}$$

therefore, using the residue theorem, the integral is obtained as the sum of the two residues at $z_n = z_i$ and $z_n = z_j$, that is

$$\begin{aligned} I^n_{ijij} &= \frac{1}{2\pi i}\oint dz_n \frac{\prod_{a\neq n} z_{na}}{\sum_{a\neq n} s_{na}\prod_{b\neq a,n} z_{nb}} \frac{z^2_{ij}}{z^2_{in}z^2_{nj}} \\ &= \frac{1}{2\pi i} z^2_{ij}\oint dz_n \frac{\prod_{a\neq n,i,j} z_{na}}{\sum_{a\neq n} s_{na}\prod_{b\neq a,n} z_{nb}} \frac{1}{z_{ni}z_{nj}} \\ &= z^2_{ij}\left[\frac{1}{z_{ij}}\frac{\prod_{a\neq i,j} z_{ia}}{s_{ni}\prod_{b\neq i} z_{ib}} + \frac{1}{z_{ji}}\frac{\prod_{a\neq i,j} z_{ja}}{s_{nj}\prod_{b\neq j} z_{jb}}\right] \\ &= z^2_{ij}\left[\frac{1}{z^2_{ij}}\frac{1}{s_{ni}} + \frac{1}{z^2_{ji}}\frac{1}{s_{nj}}\right] \\ &= \frac{1}{s_{ni}} + \frac{1}{s_{nj}} \end{aligned} \tag{7.22}$$

Hence, the integral is independent of the hard punctures, which is what we need in order to obtain a soft theorem. A similar calculation yields

$$I^n_{ijim} = \frac{1}{s_{ni}} \tag{7.23}$$

These results can be compactly written in terms of Kronecker delta functions as

$$I^n_{ijkm} = \frac{1}{s_{ni}}\left(\delta_{ik} - \delta_{im}\right) - \frac{1}{s_{nj}}\left(\delta_{jk} - \delta_{jm}\right) \tag{7.24}$$

which is a simple consequence of the two integrals we calculated explicitly and the different symmetries under the exchange of indices we have discussed. This means that, in the soft limit when the momenta of a scalar $p_n \to 0$, the biadjoint amplitude factorizes as

$$m_n(w_1|w_2) \approx I^n_{ijkm} m_{n-1}(w'_1|w'_2) \tag{7.25}$$

which depends on the position of the scalar n in the external orderings w_1 and w_2. A comment is in order: at first glance, it seems like there is a mismatch in the dimensionality of the quantities appearing in Eq. (7.25). However, this occurs because we have omitted throughout our calculations the coupling constant of the theory, λ. To see that reintroducing this coupling constant restores the dimensional consistency of this equation, consider the case $D=4$. In this scenario, the coupling constant has mass dimension $+1$, and an n-point tree amplitude goes like λ^{n-2}. Hence, when taking the soft limit, one must attach a factor of λ to the soft factor I^n_{ijkm} in order to reconstruct the λ^{n-3} factor in the definition of the $(n-1)$-point amplitude $m_{n-1}(w_1'|w_2')$. Thus, the rescaled soft factor $\lambda \times I^n$ has mass dimension -1, which then makes Eq. (7.25) consistent. It is worth noting that this soft factor for biadjoint scalars with cubic interaction has also been found by using a so-called transmutation operator applied to the gluon soft factor in [92].

This result not only provides a leading soft theorem for biadjoint ϕ^3 theory, but as we will also see, it will allow us to prove the self-consistency of the soft factorization of QCD primitive amplitudes.

7.1.2. Leading soft limit for Yang-Mills theory

Now, we use the reduced Pfaffian, Eq. (4.34), as first worked out in [13], to obtain the leading soft theorem for Yang-Mills theory. Note that we have already derived the Weinberg soft factor (also known as eikonal factor) from a Feynman diagram analysis in Chapter two. This provides an alternative derivation of the eikonal factor, which was also one of the first hints at the fact that the Pfaffian provides a correct prescription for the evaluation of gauge theory tree amplitudes from the CHY representation.

We recall that the n-point gluon primitive amplitude with external ordering w can be written as the CHY integral

$$A_n^{YM}(w,p,\epsilon) = \frac{(-1)^{i+j+k}}{(2\pi)^{n-3}} i \oint_{\mathcal{C}} \frac{d^n z}{d\omega} \frac{z_{ij}z_{jk}z_{ki}}{\prod_{a\neq i,j,k} f_a(z,p)} C_n(w,z) E_n(z,p,\epsilon) \tag{7.26}$$

where E_n is the reduced Pfaffian,

$$E_n(z,p,\epsilon) = \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}'\ \Psi_n \tag{7.27}$$

of the matrix Ψ_n, which we defined in Eq.(4.25). For simplicity, we will derive the soft limit for the canonical ordering $w = 12...(n-1)n$, and assume that gluon n has a soft momenta $p_n \to 0$. Then, we choose to compute $\mathrm{Pf}'\ \Psi_n = \mathrm{Pf}\ (\Psi_n)^{ij}_{ij}$ eliminating columns and rows i,j such that $i,j \neq n$. As we discussed in Chapter 4, the Pfaffian satisfies an expansion in terms of Pfaffians of minors, similarly to the determinant, as in Eq.(4.32). The leading term, in the soft limit, for the Pfaffian of n gluons is given by

$$\mathrm{Pf}\ (\Psi_n)^{ij}_{ij} \approx C_{nn} \mathrm{Pf}(\Psi_{n-1})^{ijn(2n)}_{ijn(2n)} \tag{7.28}$$

where C_{nn} is the diagonal element of the C matrix in the definition of Ψ_n, which is given by

$$C_{nn} = -\sum_{a\neq n} \frac{2\epsilon_n \cdot p_a}{z_{na}} \tag{7.29}$$

Hence, separating the Parke-Taylor factor in analogous fashion to the procedure we performed to obtain the soft theorem for the scalars, we find that

$$\begin{aligned} A_n^{YM}(12...(n-1)n) &\approx \left(\frac{1}{2\pi i} \oint_{f_n=0} \frac{dz_n}{f_n} \sum_{a\neq n} \frac{2\epsilon_n \cdot p_a}{z_{na}} \frac{z_{n-1,1}}{z_{n-1,n} z_{n1}} \right) A_{n-1}^{YM}(12...(n-1)) \\ &= S(n-1,n,1) A_{n-1}^{YM}(12...(n-1)) \end{aligned} \tag{7.30}$$

where

$$S(n-1,n,1) = \frac{1}{2\pi i} \oint_{f_n=0} \frac{dz_n}{f_n} \sum_{a\neq n} \frac{2\epsilon_n \cdot p_a}{z_{na}} \frac{z_{n-1,1}}{z_{n-1,n} z_{n1}} \tag{7.31}$$

Similarly to the case of the scalars, after writing the explicit expression for f_n, we can multiply and divide by $\prod_{a\neq n} z_{na}$ to rewrite the integral as

$$S(1,n,n-1) = \frac{1}{2\pi i} \oint_{f_n=0} dz_n \frac{\sum_{a\neq n} 2\epsilon_n \cdot p_a \prod_{b\neq a,n} z_{nb}}{\sum_{a\neq n} s_{na} \prod_{b\neq a,n} z_{nb}} \frac{z_{n-1,1}}{z_{n-1,n} z_{n1}} \tag{7.32}$$

Now, this integral has the same pole structure of Eq. (7.12). Thus, to evaluate it, we simply compute the residues at $z_n = z_{n-1}$ and $z_n = z_1$. By a similar argument to the one we used when evaluating I^n_{ijkm}, only one term of the sums in the numerator and denominator contribute to each residue, and the product of the remaining punctures cancels out, yielding

$$S(n-1,n,1) = -\frac{\epsilon_n \cdot p_{n-1}}{p_n \cdot p_{n-1}} + \frac{\epsilon_n \cdot p_1}{p_n \cdot p_1} \tag{7.33}$$

where the minus in the first term comes from writing the denominator $z_{n-1,n} = -z_{n,n-1}$ when calculating the residue at $z_n = z_{n-1}$. This result agrees with the one derived in Eq.(2.72).

With this, we have shown that the CHY formalism provides a very versatile scheme in which to derive factorization theorems of amplitudes in soft limits. In both of the examples we have worked out, we have seen that one of the main features which allow us to obtain the soft theorems is the fact that the integrand themselves factorize in the soft limit into lower point integrands, and that the remaining factors integrate over the soft puncture to yield a function which does not depend on the punctures associated to the hard particles. Therefore, we may ask ourselves: is the soft factorization a general property of CHY integrands? Although we are not able to answer this question in its full generality, we will see that the integrand for primitive QCD amplitudes with $n_q \leq 2$ does have this property by using the leading soft theorem for the primitive amplitudes to derive the soft factor associated to the integrand $\hat{E}_n$. We will also verify that this is self-consistent by integrating the soft factor over the soft puncture, which will yield the correct soft factor for the primitive amplitude.

7.2. Soft limit of the QCD integrand

In this final section, we will derive the soft factorization theorem for the integrand $\hat{E}_n$ in Eq.(6.26), which describes QCD primitive amplitudes with an arbitrary number of gluons and up to $n_q = 2$ massive or massless quark lines. In these cases, we can write the amplitude basis simply as

$$B_n = \{w \in W | l_1 = 1, \quad l_{n-1} = n-1, \quad l_n = n\} \tag{7.34}$$

where, for $n_q = 0$, we identify $g_1 = 1$, $g_{n-1} = n-1$ and $g_n = n$; for $n_q = 1$, we set $q_1 = 1$, $g_{n-2} = n-1$, $\bar{q}_1 = n$ and for $n_q = 2$, we set $q_1 = 1$, $\bar{q}_2 = n-1$ and $\bar{q}_1 = n$.

As a first step, we rearrange the sum in Eq.(6.26) in a convenient fashion. To do this, we let the elements $w \in B_n$ be labelled explicitly as w_i, where $i = 1, 2, ..., (n-3)!$. Then, we can rewrite the sums over B_n as

$$\begin{aligned} \hat{E}_n(z, p, \epsilon) &= -i \sum_{i,j=1}^{(n-3)!} S_n[w_i|\bar{w}_j] C_n(\bar{w}_j, z) A_n(w_i, p, \epsilon) \\ &= -i \sum_{i=1}^{(n-3)!} e_i A_n(w_i, p, \epsilon) \end{aligned} \tag{7.35}$$

where we have chosen to abbreviate one of the sums to put the polarization function as an expansion in terms of primitive amplitudes weighted by some coefficients, which are given by

$$e_i = \sum_{j=1}^{(n-3)!} S_n[w_i|\bar{w}_j] C_n(\bar{w}_j, z) \tag{7.36}$$

7.2.1. Projection of n particle basis into $(n-1)$ particle basis

When we obtained the soft limits for scalar ϕ^3 and Yang-Mills theories, we remarked that one of the properties that allowed us to obtain the soft theorems was that the integrands factorized into integrand soft factors, times $(n-1)$ particle integrands. In order for this to be possible for the QCD integrand, Eq.(7.35), we would like to find a way in which to express the sums over the $(n-3)!$, independent n-point basis amplitudes, as sums over the $(n-4)!$, independent $(n-1)$-point basis amplitudes were the soft gluon has been stripped off. In order to do this, first note that for a fixed word $w_i = 1 l_2^{(i)} l_3^{(i)} ... l_{n-2}^{(i)} (n-1) n \in B_n$, there is some $k \in \{2, ..., n-2\}$ such that $l_k^{(i)} = n-2$, where we take the label $(n-2)$ to describe a gluon, which will be the one we take the soft limit upon.

We need a procedure to convert the sums over the set B_n into sums over the smaller set B_{n-1}, whose elements are all the independent external orderings of $(n-1)$ particle amplitudes obtained from those of B_n by the removal of the soft gluon. To this end, let us define a projection operator

$$\begin{aligned} T : B_n &\to B_{n-1} \\ w_i &\mapsto T(w_i) \end{aligned} \tag{7.37}$$

whose action on w_i is to remove the letter corresponding to the label $(n-2)$. Explicitly

$$\begin{aligned} T(w_i) &= T(1l_2^{(i)}...l_k^{(i)}...l_{n-2}^{(i)}(n-1)n) \\ &= 1l_2^{(i)}...l_{k-1}^{(i)}l_{k+1}^{(i)}...l_{n-2}^{(i)}(n-1)n \\ &= 1l_2^{(i')}...l_{k-1}^{(i')}l_k^{(i')}...l_{n-3}^{(i')}(n-1)n \\ &\equiv w'_{i'} \end{aligned} \tag{7.38}$$

where i' is a new index taking values on $\{1, 2, ..., (n-4)!\}$. Then, the notation $l_t^{(i')}$ indicates which letter occupies the position t in the word $w'_{i'}$.

The function T is not injective, because the gluon $(n-2)$ can be anywhere between positions 2 and $(n-2)$ for an arbitrary external ordering. To illustrate this point, consider the case $n = 6$ and let $w_1 = 123456$, $w_2 = 124356$ and $w_3 = 142356$. We assume the label $n-2 = 4$ refers to a gluon. Then, we can see that

$$T(w_1) = T(w_2) = T(w_3) = 12356 \equiv w'_1 \tag{7.39}$$

Furthermore, we can see that for $n = 6$, there are no more elements in the set B_6 whose image under T is w'_1, and that these elements of B_6 differ from each other only by the position of letter 4. Since the set B_6 has $(6-3)! = 3! = 6$ elements, it is helpful for the general construction to see what happens with all the elements of the set. We have already enumerated three of its elements; the other three are given by $w_4 = 132456$, $w_5 = 134256$, $w_6 = 143256$. We can see that

$$T(w_4) = T(w_5) = T(w_6) = 13256 \equiv w'_2 \tag{7.40}$$

The words w'_1 and w'_2 can be seen to form a basis for five-particle amplitudes, since the set B_5 has $(5-3!) = 2$ independent elements with the first, fourth and fifth letters with fixed values in each word of the set, which in this case are set to the values $1, 5, 6$ that make reference to the labels of the particles of the six-particle amplitudes. Furthermore, we can write the set B_6 as

$$B_6 = S_1 \cup S_2 \tag{7.41}$$

where

$$S_1 = \{w_1, w_2, w_3\} \qquad S_2 = \{w_4, w_5, w_6\}, \qquad S_1 \cap S_2 = \emptyset \tag{7.42}$$

Moreover, we can see that

$$T(S_1) = \{w'_1\}, \;\; T(S_2) = \{w'_2\} \tag{7.43}$$

This means that the projection operator T defines a partition of the set B_6. This construction can be generalized in a straightforward fashion to the case of arbitrary n. That is, we can construct a partition of the basis B_n

$$B_n = \bigcup_{i'=1}^{(n-4)!} S_{i'} \tag{7.44}$$

such that

$$T(S_{i'}) = \{w'_{i'}\} \tag{7.45}$$

for every $i' = 1, 2, ..., (n-4)!$. Using these decomposition, we can write the polarization function as

$$\hat{E}_n = -i \sum_{i',j'=1}^{(n-4)!} \sum_{w_a \in S_{i'}} \sum_{\bar{w}_b \in \bar{S}_{j'}} S_n[w_a|\bar{w}_b] C_n(\bar{w}_b, z) A_n(w_a, p, \epsilon) \tag{7.46}$$

the indices take values $a, b = 1, 2, ..., (n-3)$ and, for $w_a \in S_{i'}$, $\bar{w}_b \in \bar{S}_{j'}$, we have

$$w_a = 1 l_2^{(a)} ... l_k^{(a)} ... l_{n-2}^{(a)} (n-1) n, \quad \bar{w}_b = 1 l_2^{(b)} ... l_p^{(b)} ... l_{n-2}^{(b)} n (n-1) \tag{7.47}$$

where, as before, the letters $(n-1)$ and n have their ordered reversed in words with a bar, and we assume that $l_k^{(a)} = l_p^{(b)} = n-2$ for some $k, p \in \{2, ..., n-2\}$. This simply means that the gluon whose position in the canonical external ordering $12...n$ is specified by $(n-2)$ must be in some position in the words w_a and w_b. Note that, in general, $k \neq p$. Using the shuffle product, we can see that each set $S_{i'}$ contains all the words appearing in the sum

$$w_a = w'_{i'} \sqcup\!\sqcup \, (n-2) \tag{7.48}$$

An important detail in what follows is that the value of a fixes the value of k in the word w_a and the value of b fixes the value of p in the word w_b. That is, specifying a external ordering with the word $w \in B_n$ determines the position of each of the particles in that given ordering. In particular, it specifies the position of the letter $(n-2)$.

7.2.2. Soft behaviour of $\hat{E}_n$

Before we continue, it is useful to see how each of the quantities defining the integrand in Eq.(7.35) behaves in soft limits.

To start with, we use the fact that n-particle Parke-Taylor factors can always be decomposed as the product of an $(n-1)$-particle PT factor, times a function $Z(a, b, c)$. In our case of interest, we write

$$C_n(\bar{w}_b) = Z(l_{p-1}^{(b)}, n-2, l_{p+1}^{(b)}) C_{n-1}(\bar{w}'_{j'}) \tag{7.49}$$

An important property of $Z(a, b, c)$, which we have already discussed in Chapter 5, is that $Z(a, b, c)$ satisfies the eikonal identity

$$\sum_{i=a}^{b} Z(i, x, i+1) = Z(a, x, b) \tag{7.50}$$

On the other hand, if we let $p_{n-2} \to 0$, each primitive amplitude will factorize, at leading order in p_{n-2}, as

$$A_n(w_a) \approx S(l_{k-1}^{(a)}, n-2, l_{k+1}^{(a)}) A_{n-1}(w'_{i'}) \tag{7.51}$$

where $S(a, b, c)$ is the Weinberg soft factor derived in Eq.(2.72). This soft factor also satisfies the eikonal identity

$$\sum_{i=a}^{b} S(i,x,i+1) = S(a,x,b) \tag{7.52}$$

Finally, we need the leading soft behaviour of the momentum kernel. Since the momentum kernel is a polynomial in the kinematic invariants and squared masses, this leading behaviour is given by the linear term in the soft momentum p_{n-2}. In order to find this term, let us parametrize $p_{n-2} \to xp_{n-2}$, where x is a real parameter, and we consider the limit $x \to 0$ of the momentum kernel. Then, noting that $\Delta_{ij} = 0$ whenever i or j are gluon indices, we can expand the momentum kernel as

$$\begin{aligned} S_n[w_a|\bar{w}_b] = (-1)^n &\left(\hat{s}_{1l_2^{(a)}} + \Delta_{1l_2^{(a)}}\right)\left(\hat{s}_{1l_3^{(a)}} + \Delta_{1l_3^{(a)}} + \theta_{\bar{w}_b}(l_2^{(a)}, l_3^{(a)})(\hat{s}_{l_2^{(a)}l_3^{(a)}} + \Delta_{l_2^{(a)}l_3^{(a)}})\right) \\ \times \ldots \times &\left[x\left(\hat{s}_{1(n-2)} + \theta_{\bar{w}_b}(l_2^{(a)}, n-2)\hat{s}_{l_2^{(a)}(n-2)} + \ldots + \theta_{\bar{w}_b}(l_{k-1}^{(a)}, n-2)\hat{s}_{l_{k-1}^{(a)}(n-2)}\right)\right] \\ \times \ldots \times &\left(\hat{s}_{1l_{k+1}^{(a)}} + \Delta_{1l_{k+1}^{(a)}} + \ldots + x\theta_{\bar{w}_b}(n-2, l_{k+1}^{(a)})s_{(n-2)l_{k+1}^{(a)}}\right) \\ \ldots \times &\left(\hat{s}_{1l_{n-2}^{(a)}} + \ldots + x\theta_{\bar{w}_b}(n-2, l_{n-2}^{(a)})\hat{s}_{(n-2)l_{n-2}^{(a)}} + \ldots + \theta_{\bar{w}_b}(l_{n-3}^{(a)}, l_{n-2}^{(a)})(\hat{s}_{l_{n-3}^{(a)}l_{n-2}^{(a)}} + \Delta_{l_{n-3}^{(a)}l_{n-2}^{(a)}})\right) \end{aligned} \tag{7.53}$$

From this we can see that the momentum kernel always is, at least, linear in x. Then, to obtain the leading term in p_{n-2}, we set to zero every term with powers of x greater than one, and set $x = 1$. With this, we obtain

$$S_n[w_a|\bar{w}_b] \approx -\left(\hat{s}_{1(n-2)} + \sum_{r=2}^{k-1} \theta_{\bar{w}_b}(l_r^{(a)}, n-2)\hat{s}_{l_r^{(a)}(n-2)}\right) S_{n-1}[w'_{i'}|\bar{w}'_{j'}] \tag{7.54}$$

Putting everything together, the soft limit of the polarization function can be written as

$$\hat{E}_n(z,p,\epsilon) \approx \sum_{i',j'=1}^{(n-4)!} f^n_{i'j'} S_{n-1}[w'_{i'}|\bar{w}'_{j'}] C_{n-1}(\bar{w}'_{j'}, z) A_{n-1}(w'_{i'}) \tag{7.55}$$

where we define the coefficients

$$f^n_{i'j'} = -\sum_{w_a \in S_{i'}} \sum_{\bar{w}_b \in S_{j'}} \left(\hat{s}_{1(n-2)} + \textstyle\sum_{r=2}^{k-1} \theta_{\bar{w}_b}(l_r^{(a)}, n-2)\hat{s}_{l_r^{(a)}(n-2)}\right) Z(l_{p-1}^{(b)}, n-2, l_{p+1}^{(b)}) S(l_{k-1}^{(a)}, n-2, l_{k+1}^{(a)}) \tag{7.56}$$

The next step is to perform the sums over a and b. If we can show that this sum is independent on the election of i' and j', we will be able to put this factor outside the sums and obtain a soft theorem for the generalized polarization function $\hat{E}_n$. Before tackling the general case, let us see the simplest non-trivial example: the six point amplitudes.

7.2.3. Example: six-point amplitudes

When $n = 6$, the possible ways to distribute the particles are either $n_g = 6, n_q = 0$; $n_g = 4, n_q = 1$ or $n_g = 2, n_q = 2$. Since there is essentially no difference between the different cases, we will ignore the specific particle content. Then, we consider the alphabet

$$\mathbb{A}_6 = \{1, 2, 3, 4, 5, 6\} \tag{7.57}$$

With which we can form the six particle basis

$$B_6 = \{w_1, w_2, w_3, w_4, w_5, w_6\} \tag{7.58}$$

where we take

$$\begin{aligned} w_1 &= 123456 \\ w_2 &= 124356 \\ w_3 &= 142356 \\ w_4 &= 132456 \\ w_5 &= 134256 \\ w_6 &= 143256 \end{aligned} \tag{7.59}$$

Now, let T be the projection operator that removes the letter 4 from an arbitrary $w_i \in B_6$. Then, following our general arguments, we can construct the subsets

$$S_1 = \{w_1, w_2, w_3\}, \qquad S_2 = \{w_4, w_5, w_6\} \tag{7.60}$$

which satisfy

$$T(S_1) = \{w_1'\}, \qquad T(S_2) = \{w_2'\} \tag{7.61}$$

where $w_1' = 12356$, $w_2' = 13256$, and we can define the five particle basis as

$$B_5 = T(S_1) \cup T(S_2) = \{w_1', w_2'\} \tag{7.62}$$

Moreover, since $n = 6$, $(n-3)! = 3! = 6$; hence, we can write the six particle generalized polarization function as

$$\begin{aligned} \hat{E}_6 &= -i \sum_{i,j=1}^{6} S_6[w_i|\bar{w}_j] C_6(\bar{w}_j, z) A_6(w_i) \\ &= -i \sum_{i',j'=1}^{2} \sum_{w_a \in S_i'} \sum_{\bar{w}_b \in \bar{S}_j'} S_6[w_i|\bar{w}_j] C_6(\bar{w}_j, z) A_6(w_i) \\ &\approx -i \sum_{i'j'=1}^{2} f^6_{i'j'} S_5[w'_{i'}|\bar{w}'_{j'}] C_5(\bar{w}'_{j'}) A_5(w'_{i'}) \end{aligned} \tag{7.63}$$

where, in the last line, we have taken the approximate soft expression. The coefficients are defined as in the general expression, Eq.(7.56),

$$\begin{aligned} f^6_{i'j'} &= -\sum_{w_a \in S_{i'}} \sum_{\bar{w}_b \in S_{j'}} \left(\hat{s}_{14} + \sum_{r=2}^{k-1} \theta_{\bar{w}_b}(l_r^{(a)}, 4) \hat{s}_{l_r^{(a)}4} \right) Z(l_{p-1}^{(b)}, 4, l_{p+1}^{(b)}) S(l_{k-1}^{(a)}, 4, l_{k+1}^{(a)}) \\ &= -\sum_{w_a \in S_{i'}} \alpha_a S(l_{k-1}^{(a)}, 4, l_{k+1}^{(a)}) \end{aligned} \tag{7.64}$$

where we define

$$\begin{aligned}
\alpha_a &= \sum_{\bar{w}_b \in S_{j'}} Z(l^{(b)}_{p-1}, 4, l^{(b)}_{p+1}) \left(\hat{s}_{14} + \sum_{r=2}^{k-1} \theta_{\bar{W}_b}(l^{(a)}_r, 4) \hat{s}_{l^{(a)}_r 4} \right) \\
&= \sum_{\bar{w}_b \in S_{j'}} \sum_{r=1}^{k-1} Z(l^{(b)}_{p-1}, 4, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, 4) \hat{s}_{l^{(a)}_r 4} \\
&= \sum_{r=1}^{k-1} \hat{s}_{l^{(a)}_r 4} \sum_{\bar{w}_b \in S_{j'}} Z(l^{(b)}_{p-1}, 4, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, 4)
\end{aligned} \tag{7.65}$$

Now, we will evaluate the sum over b in Eq.(7.65) by considering the explicit forms of $\bar{S}'_j$, evaluating the sum over each set and showing that we obtain the same result. On one hand, consider the set

$$\bar{S}_1 = \{\bar{w}_1, \bar{w}_2, \bar{w}_3\} \tag{7.66}$$

where $\bar{w}_1 = 123465$, $\bar{w}_2 = 124365$, $\bar{w}_3 = 142365$. Then, if we denote by $l^{(b)}_p$ the letter 4 in the word $\bar{w}_b$, as before, we can see that

$$\begin{aligned}
l^{(1)}_p = l^{(1)}_4 &\to l^{(1)}_{p-1} = 3,\ l^{(1)}_{p+1} = 6 \\
l^{(2)}_p = l^{(2)}_3 &\to l^{(2)}_{p-1} = 2,\ l^{(2)}_{p+1} = 3 \\
l^{(3)}_p = l^{(3)}_2 &\to l^{(3)}_{p-1} = 1,\ l^{(3)}_{p+1} = 2
\end{aligned} \tag{7.67}$$

and we can expand

$$\begin{aligned}
&\sum_{\bar{w}_b \in \bar{S}_1} Z(l^{(b)}_{p-1}, 4, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, 4) \\
&= Z(3,4,6)\theta_{\bar{w}_1}(l^{(a)}_r, 4) + Z(2,4,3)\theta_{\bar{w}_2}(l^{(a)}_r, 4) + Z(1,4,2)\theta_{\bar{w}_3}(l^{(a)}_r, 4)
\end{aligned} \tag{7.68}$$

Now, for any value of a, the sum over r in Eq.(7.65) has as upper limit $k-1$, where k is the position of the letter associated to the soft gluon in the word w_a; then, the letter $l^{(a)}_r$ can only take the values $1, 2, 3$, because those are the only letters that can appear before 4 in an arbitrary word $w \in B_6$. Hence, if we substitute any of the values $l^{(a)}_r = 1, 2, 3$ in 7.68, we find

$$\sum_{\bar{w}_b \in S_{j'}} Z(l^{(b)}_{p-1}, 4, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, 4) = Z(l^{(a)}_r, 4, 6) \tag{7.69}$$

An equivalent argument follows if we choose the subset

$$\bar{S}_2 = \{\bar{w}_4, \bar{w}_5, \bar{w}_6\} \tag{7.70}$$

with $\bar{w}_4 = 132465$, $\bar{w}_5 = 134265$ and $\bar{w}_6 = 143265$. In this case,

$$\begin{aligned}
l^{(4)}_p = l^{(4)}_4 &\to l^{(4)}_{p-1} = 2,\ l^{(4)}_{p+1} = 6 \\
l^{(5)}_p = l^{(5)}_3 &\to l^{(5)}_{p-1} = 3,\ l^{(5)}_{p+1} = 2 \\
l^{(6)}_p = l^{(6)}_2 &\to l^{(6)}_{p-1} = 1,\ l^{(6)}_{p+1} = 3
\end{aligned} \tag{7.71}$$

this implies that

$$\begin{aligned}&\sum_{\bar{w}_b\in\bar{S}_2} Z(l^{(b)}_{p-1},4,l^{(b)}_{p+1})\theta_{\bar{w}_b}(l^{(a)}_r,4)\\&=Z(2,4,6)\theta_{\bar{w}_4}(l^{(a)}_r,4)+Z(3,4,2)\theta_{\bar{w}_5}(l^{(a)}_r,4)+Z(1,4,3)\theta_{\bar{w}_6}(l^{(a)}_r,4)\end{aligned}\tag{7.72}$$

and, upon substitution of any of the values $l^{(a)}_r = 1,2,3$, we obtain Eq.(7.69) again. This shows that the sum over b is indeed independent of the specific subset $\bar{S}_{j'}$. Hence,

$$\alpha_a=\sum_{r=1}^{k-1}\hat{s}_{l^a_r4}Z(l^{(a)}_r,4,6)\tag{7.73}$$

Now, we evaluate

$$\begin{aligned}f^6_{i'j'}&=-\sum_{w_a\in S_{i'}}\alpha_a S(l^{(a)}_{k-1},4,l^{(a)}_{k+1})\\&=-\sum_{w_a\in S_{i'}}\sum_{r=1}^{k-1}\hat{s}_{l^{(a)}_r4}Z(l^{(a)}_r,4,6)S(l^{(a)}_{k-1},4,l^{(a)}_{k+1})\end{aligned}\tag{7.74}$$

for a particular subset $S_{i'}\subset B_6$. There are two possibilities,

$$S_1=\{w_1,w_2,w_3\},\ S_2=\{w_4,w_5,w_6\}\tag{7.75}$$

where, as in the case of the sum over b, $w_1 = 123456$, $w_2 = 124356$, $w_3 = 142356$, $w_4 = 132456$, $w_5 = 134256$, $w_6 = 143256$. First, we consider the subset S_1. In this case,

$$\begin{aligned}l^{(1)}_k&=l^{(1)}_4\rightarrow l^{(1)}_{k-1}=3,\ l^{(1)}_{k+1}=5\\l^{(2)}_k&=l^{(2)}_3\rightarrow l^{(2)}_{k-1}=2,\ l^{(2)}_{k+1}=3\\l^{(3)}_k&=l^{(3)}_2\rightarrow l^{(3)}_{k-1}=1,\ l^{(3)}_{k+1}=2\end{aligned}\tag{7.76}$$

Hence, expanding the sum over w_a with this subset, we find

$$\begin{aligned}f^6_{i'j'}&=\sum_{r=1}^{3}s_{l^1_r4}Z(l^{(a)}_r,4,6)S(3,4,5)+\sum_{r=1}^{2}\hat{s}_{l^2_r4}Z(l^{(a)}_r,4,6)S(2,4,3)\\&+\sum_{r=1}^{1}\hat{s}_{l^3_r4}Z(l^{(a)}_r,4,6)S(1,4,2)\\&=S(3,4,5)\left(\hat{s}_{14}Z(1,4,6)+\hat{s}_{24}Z(2,4,6)+\hat{s}_{34}Z(3,4,6)\right)\\&+S(2,4,3)\left(\hat{s}_{14}Z(1,4,6)+\hat{s}_{24}Z(2,4,6)\right)+S(1,4,2)\hat{s}_{14}Z(1,4,6)\\&=\hat{s}_{14}Z(1,4,6)\left(S(1,4,2)+S(2,4,3)+S(3,4,5)\right)\\&+\hat{s}_{24}Z(2,4,6)\left(S(2,4,3)+S(3,4,5)\right)+\hat{s}_{34}Z(3,4,6)S(3,4,5)\\&=\sum_{i=1}^{3}\hat{s}_{i4}Z(i,4,6)S(i,4,5)=\sum_{r\neq4}\hat{s}_{i4}Z(i,4,6)S(i,4,5)\end{aligned}\tag{7.77}$$

where we have used the eikonal identity and added the terms $\hat{s}_{45}Z(5,4,6)S(5,4,5)$, $\hat{s}_{46}Z(6,4,6)S(6,4,5)$, both of which are equal to zero, in order to write $f^{6}_{i'j'}$ in a permutation invariant form. Performing the sum over S_2 goes along the same lines and yields the same result, showing explicitly that the sum over w_a is independent of the subset $S_{i'}$.

7.2.4. Factorization of the integrand for $n_q \leq 2$

Now, we perform the sums over a and b in Eq.(7.56) for arbitrary n. As in the six particle case, we first write

$$f^{n}_{i'j'} = \sum_{w_a \in S_{i'}} \alpha_a S(l^{(a)}_{k-1}, n-2, l^{(a)}_{k+1}) \tag{7.78}$$

That is, we define the sum over b as a coefficient of the sum over a. Now, we want to evaluate the sum

$$\begin{aligned}
\alpha_a &= \sum_{\bar{w}_b \in S_{j'}} Z(l^{(b)}_{p-1}, n-2, l^{(b)}_{p+1}) \left(\hat{s}_{1(n-2)} + \sum_{r=2}^{k-1} \theta_{\bar{W}_b}(l^{(a)}_r, n-2) \hat{s}_{l^{(a)}_r (n-2)} \right) \\
&= \sum_{\bar{w}_b \in S_{j'}} \sum_{r=1}^{k-1} Z(l^{(b)}_{p-1}, n-2, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, n-2) \hat{s}_{l^{(a)}_r (n-2)} \\
&= \sum_{r=1}^{k-1} \hat{s}_{l^{(a)}_r (n-2)} \sum_{\bar{w}_b \in S_{j'}} Z(l^{(b)}_{p-1}, n-2, l^{(b)}_{p+1}) \theta_{\bar{w}_b}(l^{(a)}_r, n-2)
\end{aligned} \tag{7.79}$$

where we have used the facts that $l^{(a)}_1 = 1$ for all values of a and that $\theta_{\bar{w}_b}(1, n-2) = 1$ for any word, since the letter 1 is fixed to be at the first position for an arbitrary $w \in B_n$ (this implies that it is also fixed to be in that position for a word in any subset of B_n). Finally, we have exchanged the order of the sums over r and b. With this, we are in position to perform the sum over b.

The key observation in performing this sum is that, for fixed values of r and a, there is some $q \in \{2, 3, ..., p-1, p+1, ..., n-2\}$ such that $l^{(a)}_r = l^{(b)}_q$ for every value of b. Now, recalling that the value of b fixes the value of p (that is, by choosing a value of b, we are taking a particular element $\bar{w}_b \in \bar{S}_{j'}$, and we can identify the value of p for this element by looking at the position of the label $(n-2)$ in $\bar{w}_b$) and noting that in an analogous fashion, the value of b determines the value of q, only the terms with $q < p$ are non-vanishing. Furthermore, the sum over r has the label $(k-1)$, the position of the last letter before $(n-2)$ in the word w_a, as its upper limit. Therefore, $l^{(a)}_r \neq (n-1), n$, because of the way we have constructed the basis set B_n. This implies

$$\sum_{b \in \bar{S}_{j'}} \theta_{\bar{w}_b}(l^{(a)}_r, n-2) Z(l^{(b)}_{p-1}, n-2.l^{(b)}_{p+1}) = \sum_{p=q+1}^{n-2} Z(l^{(b)}_{p-1}, n-2, l^{(b)}_{p+1}) \tag{7.80}$$

The new sum can be easily performed by using the eikonal identity for $Z(a, b, c)$. Hence, the sum collapses to a single term,

$$\begin{aligned}\sum_{p=q+1}^{n-2} Z(l_{p-1}^{(b)}, n-2, l_{p+1}^{(b)}) &= Z(l_q^{(b)}, n-2, l_{n-1}^{(b)}) \\ &= Z(l_r^{(a)}, n-2, n)\end{aligned} \tag{7.81}$$

where we have used the fact that $l_{n-1}^{(b)} = n$ for every word $\bar{w}_b \in \bar{S}_{j'}$ and our definition of $l_q^{(b)}$. We have obtained, then, the partial result

$$\alpha_a = \sum_{r=1}^{k-1} Z(l_r^{(a)}, n-2, n)\hat{s}_{l_r^{(a)}(n-2)} \tag{7.82}$$

One important feature of this sum is that it is independent of the particular subset $\bar{S}_{j'}$. This is exactly what we need in order to obtain a soft factorization theorem for the generalized polarization function. Inserting this result into $f_{i'j'}^{n}$,

$$f_{i'j'}^{n} = -\sum_{w_a \in S_{i'}} \sum_{r=1}^{k-1} Z(l_r^{(a)}, n-2, n)\hat{s}_{l_r^{(a)}(n-2)} S(l_{k-1}^{(a)}, n-2, l_{k+1}^{(a)}) \tag{7.83}$$

To perform these final sums, it is helpful to order the elements $w_a \in S_{i'}$ by the position of the letter $(n-2)$ in w_a. That is,

$$S_{i'} = \{w_1, w_2, w_3, ..., w_{n-3}\} \tag{7.84}$$

where

$$\begin{aligned} w_1 &= 1(n-2)l_3^{(1)}l_4^{(1)}...l_{n-2}^{(1)}(n-1)n \\ w_2 &= 1l_2^{(2)}(n-2)l_4^{(2)}l_5^{(2)}...l_{n-2}^{(2)}(n-1)n \\ w_3 &= 1l_2^{(3)}l_3^{(3)}(n-2)l_5^{(3)}...l_{n-2}^{(3)}(n-1)n \\ &\cdot \\ &\cdot \\ &\cdot \\ w_{n-3} &= 1l_2^{(n-3)}l_3^{(n-3)}...l_{n-3}^{(n-3)}(n-2)(n-1)n \end{aligned} \tag{7.85}$$

This particular enumeration of the elements of $S_{i'}$ has the advantage that, for any given value of a, we can immediately determine the value of k (which gives the position of $(n-2)$ in the word w_a) as $k = a+1$. Now, let

$$T(S_{i'}) = \{w'_{i'}\} = \{1\ell_2^{(i')}\ell_3^{(i')}...\ell_{n-3}^{(i')}(n-1)n\} \tag{7.86}$$

be the image of $S_{i'}$ under T, which as we have discussed, is a single element. For an arbitrary $w_a \in S_{i'}$, we can identify the letters $l_m^{(a)}$, $m = 2, ..., n-2$ with the letters $l_t^{(i')}$, $t = 2, ..., n-3$ in the following manner: first, we identify the position of $(n-2)$ which, as we have mentioned, will be at position $a+1$ in the word w_a with this particular ordering of the elements in $S_{i'}$. Then, for $m < a+1$,

$$l_m^{(a)} = l_m^{(i')} \tag{7.87}$$

and, for $m > a+1$,

$$l_m^{(a)} = l_{m-1}^{(i')} \tag{7.88}$$

Using this, we can replace all letters $l_r^{(a)} \neq (n-2)$ with the letters of $w'_{i'}$. Then, we can explicitly expand the double sum in Eq.(A.7) as

$$\begin{aligned} -f^n_{i'j'} = & \hat{s}_{l_1^{(i')}(n-2)} Z(l_1^{(i')}, n-2, n) \sum_{i=1}^{n-3} S(l_i^{(i')}, n-2, l_{i+1}^{(i')}) \\ & + \hat{s}_{l_2^{(i')}(n-2)} Z(l_2^{(i')}, n-2, n) \sum_{i=2}^{n-3} S(l_i^{(i')}, n-2, l_{i+1}^{(i')}) \\ & ... + \hat{s}_{l_{n-3}^{(i')}(n-2)} Z(l_{n-3}^{(i')}, n-2, n) \sum_{i=n-3}^{n-3} S(l_i^{(i')}, n-2, l_{i+1}^{(i')}) \end{aligned} \tag{7.89}$$

The number of times each eikonal factor appears is a consequence of the fact that, in the double sum Eq.(7.83), each element w_a determines the value of k and this, in turn, defines how many terms appear on the second sum. We choose to factor out the kinematic invariants and the z dependent terms. Using the eikonal identity for the Weinberg soft factors and the fact that $l_{n-2}^{(i')} = n-1$, allows for direct evaluation of each of the sums appearing in Eq.(7.89), giving the result

$$f^n_{i'j'} = -\sum_{i=1}^{n-3} \hat{s}_{l_i^{(i')}(n-2)} Z(l_i^{(i')}, n-2, n) S(\ell_i^{(i')}, n-2, n-1) \tag{7.90}$$

Moreover, since the string of letters $\ell_2^{(i')}\ell_3^{(i')}...\ell_{n-3}^{(i')}$ is just a permutation of the the string $23...(n-3)$, we can see that

$$\{l_2^{(i')}, l_3^{(i')}, ..., l_{n-3}^{(i')}\} = \{2, 3, ..., n-3\} \tag{7.91}$$

independent of the particular value of i' (which amounts to the election of a particular permutation of $2, 3, ..., (n-3)$). This implies

$$\begin{aligned} f^n_{i'j'} &= -\sum_{i=1}^{n-3} \hat{s}_{i(n-2)} Z(i, n-2, n) S(i, n-2, n-1) \\ &= -\sum_{i \neq n-2} \hat{s}_{i(n-2)} Z(i, n-2, n) S(i, n-2, n-1) \end{aligned} \tag{7.92}$$

where we have used the fact that $Z(n, n-2, n) = 0$ and $S(n-1, n-2, n-1) = 0$ to add the two terms needed to write the soft factor in a permutation invariant form. We have found that the coefficients $f_{i'j'}$ are independent of i', j'; then, we can assert that the generalized polarization function factorizes, in the soft limit $p_{n-2} \to 0$, as

$$\hat{E}_n \approx F_n \hat{E}_{n-1} \tag{7.93}$$

where F_n is given by

$$F_n = -\sum_{i \neq n-2} \hat{s}_{i(n-2)} Z(i, n-2, n) S(i, n-2, n-1) \tag{7.94}$$

We would like to emphasize that this is a result still missing in the literature. Not only we have shown that the QCD integrand for $n_q \leq 2$ shows similar soft factorization properties to that of the pure Yang-Mills case, but we also provide an operational technique to relate primitive amplitude basis with different number of particles, and a way to characterize this relation in terms of partitions.

7.2.5. Integration of the soft factor

Finally, as a consistency check of the calculation, we perform the integral over the soft puncture and see if we recover the eikonal factor associated with the external ordering specified by the word w. For simplicity, we take $w \in B_n$, since we can use cyclic invariance, the KK and BCJ relations in order to obtain the case of an arbitrary ordering by computing suitable linear combinations of basis amplitudes.

Concretely, consider a primitive amplitude $A_n(w, p, \epsilon)$ with some external ordering $w = l_1 l_2 ... l_n$, and assume that, for some $m \in \{1, n\}$, $m = n-2$ (note that the external ordering has no restriction on the position of the soft leg). Furthermore, let $w' = T(w)$. Also, for simplicity of notation, we let $l_{m-1} = q$ and $l_{m+1} = t$ for some $q, t \in \{1, 2, ..., n\}/n-2$. Then, we can write

$$C_n(w, z) = C_{n-1}(w', z) Z(q, n-2, t) \tag{7.95}$$

Then, using the soft limit of the scattering equations, we can rewrite the CHY integral for the primitive amplitude $A_n(w)$ as

$$\begin{aligned} A_n(w) &= A_n(w, p, \epsilon) \\ &\approx \frac{i(-1)^{n-1}}{(2\pi i)^{n-4}} \oint_{\mathcal{C}} \frac{d^{n-1}z}{d\omega} \prod_{a \neq n-2}{}' \frac{1}{f_a(z, p)} \hat{C}_{n-1}(w', z) \hat{E}_{n-1}(z, p, \epsilon) a_n(z) \end{aligned} \tag{7.96}$$

where we define the integral

$$\begin{aligned} a_n(z) &= -\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz_{n-2}}{f_{n-2}} Z(q, n-2, t) F_n \\ &= \frac{1}{2\pi i} \sum_{j \neq n-2} \hat{s}_{j(n-2)} S(j, n-2, n-1) \oint_{\mathcal{C}} \frac{dz_{n-2}}{f_{n-2}} Z(q, n-2, t) Z(j, n-2, n) \end{aligned} \tag{7.97}$$

which, in terms of the integral I^n_{ijkm} defined in Eq. (7.12), we can formally evaluate

$$a_n(z) = \sum_{j \neq n-2} s_{j(n-2)} S(j, n-2, n-1) I^{n-2}_{qtin} \tag{7.98}$$

Therefore, upon substitution of the general result for I^a_{ijkl} into $a_n(z)$, we can see that it is independent of the punctures z_a, $a \neq n-2$. Moreover, the appearance of the Kronecker deltas pick the terms in the sum such that the invariants $\hat{s}_{ij}$ appearing in the numerator and denominator cancel out, and yields the result

$$a_n(z) = S(q, n-2, t) \tag{7.99}$$

by use of the eikonal identity. This is the soft factor associated to the external ordering given by the word w. Hence, we have shown not only that the polarization function factorizes in the soft limit at leading order in the soft momenta, but also that it is consistent with the factorization properties of the primitive amplitudes.

Now, let us give a few closing remarks on the factorization of the generalized polarization function. For the general scenario of a primitive amplitude with $n_q \geq 2$, the main difference with regards to the $n_q \leq 2$ case is the BCJ matrix $F_{ww'}$ because, in order to find a soft factorization theorem, we need to understand the soft behaviour of all the factors that enter the expansion of $\hat{E}_n$. However, besides from the fact that there are no explicit expressions for the elements of the pseudoinverse $G_{w'w}$ [2], it is also not clear how the matrix F itself behaves in soft limits. Since each of its elements are rational functions of the momenta with equal power of the kinematic invariants in both the numerator and denominator, one has three possibilities for the value that a particular element $F_{ww'}$ takes on the soft limit $p_n \to 0$,

$$\begin{aligned} F_{ww'} &\propto \frac{s_{na}}{s_{nb}} \sim 1 \\ F_{ww'} &\propto \frac{s_{na}}{s_{ij}} \sim 0 \\ F_{ww'} &\propto \frac{s_{ij}}{s_{nb}} \sim \infty \end{aligned} \tag{7.100}$$

for some momenta p_a, p_b, p_i and p_j different from p_n, and in the first line, we understand ~ 1 as meaning that the element $F_{ww'}$ becomes independent of the soft momentum, but may depend on some of the hard momenta. There is also a fourth possibility, which is that the element $F_{ww'}$ is independent of p_n. Although we have a explicit definition, there is no simple way to determine which of these cases is satisfied given arbitrary words w and w' such that $F_{ww'}$ is non-trivial. Understanding this behaviour would give hints at the soft limit of the elements of the pseudoinverse $G_{w'w}$, and may allow for the derivation of a soft factorization theorem for the generalized polarization function for an arbitrary QCD primitive amplitude.

[2] which would be of course equivalent to proving the conjecture on the rank of $F_{ww'}$

Conclusions and outlooks

In this thesis, we have shown that, by focusing on the physical and mathematical properties of scattering amplitudes, it is possible to calculate them without resorting to a Lagrangian formulation nor the Feynman diagram expansion. Through the use of spinor-helicity variables and BCFW recursion, we have seen that it is possible to obtain compact formulas for amplitudes with arbitrary number of external particles, which is outright impossible from the point of view of Feynman diagrams.

We have also seen that extending the idea of studying the singularity structure of amplitudes to arbitrary dimensions provides us with the scattering equations, which have allowed us to write closed formulas for the tree amplitudes of Yang-Mills theories and gravity in terms of localized integrals over a set of complex variables. The fact that this is possible is by itself remarkable, since it provides complete solutions to classical Yang-Mills theories and general relativity.

Another interesting conclusion is that the procedure of color decomposition and the existence of a minimal amplitude basis provides the backbone for the existence of a CHY representation for QCD amplitudes. This has allowed us to prove that the CHY integrands has factorization properties parallel to those of scattering amplitudes. Now, let us give a few comments on our results and some directions in which one could extend them.

Even though we have proved that there exists a CHY representation for primitive QCD amplitudes in arbitrary dimension with up to two quark-antiquark pairs, the integrand obtained has the obvious shortcoming of depending on the amplitudes. Therefore, it cannot be used for either an explicit calculation or to derive properties of the amplitudes, since the desired property would be needed as an input. However, it is possible to obtain properties of the integrand itself by using different properties of the scattering amplitudes. These properties, like the factorization on soft limits, put further constraints on the integrand beyond those of $SL(2,\mathbb{C})$ covariance and the independence on the external ordering. Deriving further constraints, such as factorization on collinear limits, could allow to determine the explicit expression for the integrand $\hat{E}_n$ explicitly without an expansion in terms of amplitudes, if not for general n, for a particular number of external particles. This is a direction which we consider requires further exploration.

Also, as we mentioned at the end of Chapter 7, it should be possible to derive a soft theorem for the generalized polarization function for an arbitrary number of quarks, provided the soft behaviour of the BCJ matrix $F_{ww'}$ is understood. We expect that studying the soft behaviour of the BCJ matrix not only allows to determine a soft theorem for the multiquark generalized polarization function, but also that it sheds some light on the existence of its pseudoinverse.

Another interesting perspective is provided by the results in [93], where it is shown that the Pfaffian of Yang-Mills can be written as an expansion over KK independent Parke-Taylor factors,

$$\mathrm{Pf}'\Psi_n = \sum_{w \in W_2} C(1wn) n_{1wn} \tag{7.101}$$

where n_{1wn} are numerators that satisfy color-kinematics duality. Our derivation of the generalized polarization function for QCD primitive amplitudes involved an expansion over the BCJ independent Parke-Taylor factors, which is of course a smaller basis than the one provided above. However, one cannot help but see their similarity, and it would be interesting to see if both results are connected and if the kinematic, color-dual numerators of QCD provide valid CHY integrands for QCD primitive amplitudes.

Finally, we would like to remark that there are many other open problems within the CHY formalism. For example, the generalization to loop amplitudes has been studied, at one-loop order, in [94] and [95] from an ambitwistor string theory perspective, and rules for their integration presented in [65]; in [96], loop amplitudes are obtained from the forward limit of higher dimensional tree amplitudes. More recently in [97] a generalization of the scattering equations to elliptic curves is considered and this so-called elliptic scattering equations are applied in [98], by the same authors, to construct one-loop integrands in scalar ϕ^3 theory. This approach was generalized to the two-loop order in [99]. One-loop propagators from the scattering equations are studied [100],[101], also in scalar ϕ^3 theory. A CHY representation of Yang-Mills one-loop amplitudes which manifestly satisfies the BCJ relations has been worked out in [102]. The two-loop case has been considered in [103], again from the ambitwistor string formalism, but no field theory argument has been provided.

Also, recent work by Weinzierl et al [104] and Arkani-Hamed et al [105] has shown that the different CHY half-integrands of Yang-Mills theory can be written as differential forms -given the name of scattering forms- which only have logarithmic singularities, and whose residues factorize into products of lower-point scattering forms. Moreover, as proved by Mizera [106], the scattering equations themselves define a one-form η, and the scattering amplitudes can be calculated as the intersection numbers of the scattering forms associated to each half-integrand, twisted by the one-form η. This provides a nice geometrical interpretation of scattering amplitudes and the CHY representation, and it may be a fruitful research line, both in the direction of extending this picture to different theories as well as loop level extensions to Yang-Mills, scalar ϕ^3 theory and gravity.

Furthermore, given that the tree-level case is not completely understood, it is obvious that the loop case for amplitudes with fermions is a problem still not worked out in the literature. Along with the other topics we have discussed, we can see that there is still much to be understood about the CHY formalism and its deeper connections to quantum field theory that, perhaps, will reveal hidden structures beyond the usual Lagrangian formalism that allow us to obtain more profound information about the fundamental interactions of nature.

Proof of SU(N) completeness relation

We want to show that the $SU(N)$ hermitean generators T^a satisfy the identity

$$(T^a)_{i_1}^{\bar{j}_1}(T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} - \frac{1}{N}\delta_{i_1}^{\bar{j}_1}\delta_{i_2}^{\bar{j}_2} \tag{A.1}$$

In order to do this, consider an $N \times N$ matrix M. Since the matrices T^a, for $a = 1, 2, ..., N^2 - 1$ are the generators of the Lie algebra of $SU(N)$, we can expand any $N \times N$ matrix in terms of these generators plus the identity matrix, which we shall note as I. Thus, we can write

$$M = c_1 I + c_b T^b \tag{A.2}$$

Now, since the generators are traceless and $Tr(I) = N$, we immediately find that

$$c_1 = \frac{1}{N}Tr(M) \tag{A.3}$$

Furthermore, left-multiplying A.2 by T^a and using $Tr(T^aT^b) = \delta^{ab}$, we find the coefficient c_a

$$c_a = Tr(MT^a) \tag{A.4}$$

Therefore, inserting the coefficients in A.2 and writting the equation in terms of the matrix elements of M, (a sum over all repeated indices is understood) we obtain

$$\begin{aligned} M_{i_1}^{\bar{j}_1} &= Tr(MT^a)(T^a)_{i_1}^{\bar{j}_1} + \frac{1}{N}Tr(M)\delta_{i_1}^{\bar{j}_1} \\ &= M_{\bar{j}_2}^{i_2}(T^a)_{i_2}^{\bar{j}_2}(T^a)_{i_1}^{\bar{j}_1} + \frac{1}{N}\delta_{i_1}^{\bar{j}_1}\delta_{i_2}^{\bar{j}_2}M_{\bar{j}_2}^{i_2} \end{aligned} \tag{A.5}$$

Where we also wrote down the traces in terms of matrix elements. Now, rewritting the right-hand side as

$$M_{i_1}^{\bar{j}_1} = \delta_{i_2}^{\bar{j}_1}\delta_{i_1}^{\bar{j}_2}M_{\bar{j}_2}^{i_2} \tag{A.6}$$

we find that

$$
\begin{aligned}
M^{i_2}_{\bar{j}_2}(T^a)^{\bar{j}_2}_{i_2}(T^a)^{\bar{j}_1}_{i_1} &= \delta^{\bar{j}_1}_{i_2}\delta^{\bar{j}_2}_{i_1}M^{i_2}_{\bar{j}_2} - \frac{1}{N}\delta^{\bar{j}_1}_{i_1}\delta^{\bar{j}_2}_{i_2}M^{i_2}_{\bar{j}_2} \\
&= \left(\delta^{\bar{j}_1}_{i_2}\delta^{\bar{j}_2}_{i_1} - \frac{1}{N}\delta^{\bar{j}_1}_{i_1}\delta^{\bar{j}_2}_{i_2}\right)M^{i_2}_{\bar{j}_2}
\end{aligned}
\tag{A.7}
$$

Equating the coefficients in the sum of both sides of A.7 we obtain A.1.

Spinor-helicity conventions and identities

In this Appendix, we list our conventions for spinor indices and prove some identities useful for manipulating expressions involving spinor products. Our metric convention in 4-dimensional Minkowski spacetime is $g_{\mu\nu} = diag(1, -1, -1, -1)$. The antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ are take the values

$$\begin{aligned} \epsilon^{12} &= \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1 \\ \epsilon^{21} &= \epsilon^{\dot{2}\dot{1}} = \epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -1 \end{aligned} \tag{B.1}$$

Spinor indices are raised and lowered using the Levi-Civita tensors as

$$\eta^{\alpha} = \epsilon^{\alpha\beta}\eta_{\beta}, \quad \tilde{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\eta}_{\dot{\beta}} \tag{B.2}$$

With the usual definitions of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.3}$$

and denoting the three-dimensional vector of Pauli matrices by σ, we define the four-dimensional Sigma matrices

$$\begin{aligned} &(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = (\mathbb{I}, -\sigma), \ (\sigma^{\mu})_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\beta}\beta} = (\mathbb{I}, \sigma) \\ &(\bar{\sigma}_{\mu})^{\dot{\alpha}\alpha} = (\mathbb{I}, \sigma), \ (\sigma_{\mu})_{\alpha\dot{\alpha}} = (\mathbb{I}, -\sigma) \end{aligned} \tag{B.4}$$

The generators of the Lorentz group in the spinor representation can be written in terms of Sigma matrices as

$$\begin{aligned} (\sigma^{\mu\nu})_{\alpha}{}^{\beta} &= \frac{1}{4}\left((\sigma^{\mu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta} - (\sigma^{\nu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\right) \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}\left((\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\sigma^{\nu})_{\alpha\dot{\beta}} - (\bar{\sigma}^{\nu})^{\dot{\alpha}\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\right) \end{aligned} \tag{B.5}$$

An arbitrary four-momentum p^{μ} can be projected into the bispinor representation as

$$\begin{aligned} p_{\alpha\dot{\alpha}} &= p^{\mu}(\sigma_{\mu})_{\alpha\dot{\alpha}} \\ p^{\alpha\dot{\alpha}} &= p_{\mu}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} \end{aligned} \tag{B.6}$$

Since the Sigma matrices and the Levi-Civita tensors are related through

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\nu\dot{\beta}\beta}g_{\mu\nu} = 2\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta} \tag{B.7}$$

and various other relations can be obtained by lowering the various spinor indices. In particular, this allows to obtain the four-momentum from its bispinor representation as

$$p^{\mu} = \lambda^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} = \frac{1}{2}\langle p|\gamma^{\mu}|p] \tag{B.8}$$

due to the various forms of contracting the spinors with the different Sigma matrices, this representation is not unique. The bispinor representation, in the massless case, is realized in terms of two-component spinors

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}} \tag{B.9}$$

where, for real momenta in $(1, -1, -1, -1)$ signature, $(\lambda_{\alpha})^* = \tilde{\lambda}_{\dot{\alpha}}$. The helicity spinors can be identified with the solutions to the massless Dirac equation in four dimensions as

$$\begin{aligned} &u_+(p) = v_-(p) = \begin{pmatrix} \lambda_{\alpha} \\ 0 \end{pmatrix} = |p\rangle,\ u_-(p) = v_+(p) = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} = |p] \\ &\bar{u}_+(p) = \bar{v}_-(p) = (0, \tilde{\lambda}_{\dot{\alpha}}) = [p|,\ \bar{u}_-(p) = \bar{v}_+(p) = (\lambda^{\alpha}, 0) = \langle p| \end{aligned} \tag{B.10}$$

and the spinor products are given by

$$\langle ij\rangle = \lambda_i^{\alpha}\lambda_{j\alpha},\ [ij] = \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_i^{\dot{\alpha}} \tag{B.11}$$

From their definition and the relation $\tilde{\lambda}_{\dot{\alpha}} = \lambda^{\dagger}_{\alpha}$, we can see that

$$\langle pq\rangle = [qp]^* \tag{B.12}$$

Moreover, the spinor products can be related to the Lorentz-invariant product of their momenta through

$$\begin{aligned} \langle ij\rangle[ji] &= \varepsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta}(-\varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_j^{\dot{\alpha}}\tilde{\lambda}_i^{\dot{\beta}}) \\ &= \varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\beta}}\lambda_j^{\beta}\tilde{\lambda}_j^{\dot{\alpha}} \\ &= \varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}p_i^{\alpha\dot{\beta}}p_j^{\beta\dot{\alpha}} \\ &= p^{\alpha}_{i\dot{\alpha}}p^{\dot{\alpha}}_{j\alpha} \\ &= 2p_i\cdot p_j \end{aligned} \tag{B.13}$$

Using both of the above equations, we can see that $|\langle ij\rangle|^2 = |[ij]|^2 = 2p_i\cdot p_j$. Thus, we can think of spinor products as (complex) square roots of the kinematic invariants $s_{ij} = (p_i + p_j)^2 = 2p_i\cdot p_j$ (the last equality is valid only for massless particles.) Furthermore, another consequence of the antisymmetry of the Levi-Civita tensor is that

$$\langle ij\rangle = -\langle ji\rangle,\quad [ij] = -[ji] \tag{B.14}$$

and, using the properties of the Gamma matrices,

$$\langle i|\gamma^\mu|j] = [j|\gamma^\mu|i\rangle \tag{B.15}$$

Furthermore, using the relation $\bar{\sigma}^{\mu\alpha\dot{\alpha}}\bar{\sigma}^{\nu\beta\dot{\beta}}g_{\mu\nu} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$ we can prove the so-called Fierz identity

$$\begin{aligned}
[i|\gamma^\mu|j\rangle[k|\gamma_\mu|l\rangle &= \tilde{\lambda}_{i\dot{\alpha}}\lambda_{j\alpha}\tilde{\lambda}_{k\dot{\beta}}\lambda_{l\beta}\bar{\sigma}^{\mu\alpha\dot{\alpha}}\bar{\sigma}^{\nu\beta\dot{\beta}}g_{\mu\nu} \\
&= 2\tilde{\lambda}_{i\dot{\alpha}}\lambda_{j\alpha}\tilde{\lambda}_{k\dot{\beta}}\lambda_{l\beta}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \\
&= 2\epsilon^{\alpha\beta}\lambda_{j\alpha}\lambda_{l\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_{k\dot{\beta}} \\
&= 2\langle jl\rangle[ki] \\
&= 2[ik]\langle lj\rangle
\end{aligned} \tag{B.16}$$

One final property of spinors is a simple consequence of the fact that each spinor only has two independent components. Consider, for example, the spinors $|i\rangle, |j\rangle, |k\rangle$. Since these are two-component objects embedded in four dimensional vectors, only two of them are linearly independent. Then, we can write

$$|k\rangle = a\,|i\rangle + b\,|j\rangle \tag{B.17}$$

where a, b are complex numbers. Using $\langle ii\rangle = 0$ and the equivalent relation for j, it can be seen that

$$a = \frac{\langle jk\rangle}{\langle ji\rangle}, \quad b = \frac{\langle ik\rangle}{\langle ij\rangle} \tag{B.18}$$

thus, if we dot the expression with another spinor with momentum l, we obtain

$$\begin{aligned}
\langle lk\rangle &= \frac{\langle jk\rangle}{\langle ji\rangle}\langle li\rangle + \frac{\langle ik\rangle}{\langle ij\rangle}\langle lj\rangle \\
&= \frac{\langle jk\rangle}{\langle ji\rangle}\langle li\rangle - \frac{\langle ik\rangle}{\langle ji\rangle}\langle lj\rangle
\end{aligned} \tag{B.19}$$

which, upon multiplying both sides by $\langle ji\rangle$, yields

$$\langle lk\rangle\langle ji\rangle + \langle ik\rangle\langle lj\rangle + \langle kj\rangle\langle li\rangle = 0 \tag{B.20}$$

which is known the Schouten identity. An equivalent expression is obtained if one replaces all angle brackets with square brackets.

Conformal symmetry of MHV amplitudes

We will review the basic properties of conformal transformations. Our first aim is then to show that the trace of the energy-momentum tensor in Yang-Mills theory vanishes. Afterwards, we will see how the conformal symmetry of Yang-Mills manifests itself in the MHV amplitudes, and how the action of the conformal generators on the amplitudes is more naturally realized in twistor space. We partially follow [107] in the discussion of the conformal group.

Consider a flat, d-dimensional spacetime with metric $g_{\mu\nu} = diag(1, -1, ..., -1)$. A local transformation of the coordinates $x \to x'$ is said to be a conformal transformation if it preserves the angles between two given lines in spacetime. In terms of the metric, this condition can be stated as

$$g_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu} \tag{C.1}$$

where $\Lambda(x)$ is a scalar function, assumed to be positive for every value of x, and is known as the scale factor. It is worthwhile to note that, when $\Lambda = 1$, the conformality condition becomes the usual definition of a Lorentz transformation. Moreover, since the conformal transformations are themselves a group (obviously called the conformal group), this implies that the Lorentz group is a subgroup of the conformal group. In fact, the complete Poincaré group is a subgroup of the conformal group, since translations by constant vectors also preserve the angle between any two given lines.

In addition to the usual translations, rotations and boosts, the conformal group introduces two additional kind of transformations: dilatations and the so-called special conformal transformations (SCT), which are given by

$$\begin{array}{rcl} x'^{\mu} & = & \alpha x^{\mu} \quad (Dilatations) \\ x'^{\mu} & = & \frac{x^{\mu} - x^2 a^{\mu}}{1 - 2a\cdot x + a^2 x^2} \quad (SCT) \end{array} \tag{C.2}$$

where α is a constant and a_{μ} an arbitrary, constant vector. These transformations are generated, in configuration space, by the operators

$$D = -ix^{\mu}\partial_{\mu} \tag{C.3}$$

for dilatations, and

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \tag{C.4}$$

for special conformal transformations. Along with the usual generators of the Poincaré group

$$P_\mu = -i\partial_\mu, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{C.5}$$

the set of 15 generators $\{M_{\mu\nu}, P_\mu, K_\mu, D\}$ satisfy the conformal algebra

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(g_{\mu\nu} D - M_{\mu\nu}) \\
[K_\rho, M_{\mu\nu}] &= i(g_{\rho\mu} K_\nu - g_{\rho\nu} K_\mu) \\
[P_\rho, M_{\mu\nu}] &= i(g_{\rho\mu} P_\nu - g_{\rho\nu} P_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\nu\rho} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho})
\end{aligned} \tag{C.6}$$

in a general spacetime dimension $d \geq 3$ with Lorentzian metric $g_{\mu\nu} = diag(1, -1, ..., -1)$, the conformal group is identified with $SO(1, d+1)$. The case $d = 2$ is very special, because there the number of conformal generators is infinite. However, we will be concerned with the finite case; in particular, with $d = 4$. As we have mentioned, a particular feature of a conformal field theory is that the trace of the energy-momentum tensor vanishes, $T^\mu_\mu = 0$. Let us check this condition for Yang-Mills theory.

Recall that the Yang-Mills Lagrangian is given by

$$\mathcal{L}_{YM} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \tag{C.7}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{C.8}$$

is the gauge field strength, g is the gauge coupling and f_{abc} are the structure constants of the gauge group. To compute the energy-momentum tensor, we will assume that the metric $g_{\mu\nu}$ is a dynamical variable, and use the definition

$$\delta_g S[A_\mu, g_{\mu\nu}] = \frac{1}{2}\int d^4x \sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu} \tag{C.9}$$

and then set back $g_{\mu\nu}$ to be the Minkowski metric. Unlike the energy-momentum tensor obtained from Noether's theorem, this procedure yields a symmetric, gauge-invariant tensor, which is usually obtained by applying the Belifante prescription for symmetrizing the Noether energy-momentum tensor. First, we write the action as

$$S_{YM} = -\frac{1}{4}\int d^4x \sqrt{-g} g^{\alpha\rho} g^{\beta\sigma} F^a_{\alpha\beta} F_{\rho\sigma} \tag{C.10}$$

and then, using the relations $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}$, $\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}$, a short calculation shows that

$$
\begin{aligned}
\delta_g S_{YM} &= -\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^a_{\rho\sigma}\left(\tfrac{1}{2}\sqrt{-g}g^{\alpha\rho}g^{\beta\sigma}g^{\mu\nu}\delta g_{\mu\nu} - \sqrt{-g}g^{\alpha\rho}g^{\beta\mu}g^{\sigma\nu}\delta g_{\mu\nu} - \sqrt{-g}g^{\beta\sigma}g^{\alpha\mu}g^{\rho\nu}\delta g_{\mu\nu}\right) \\
&= -\frac{1}{4}\int d^4x\sqrt{-g}\delta g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu}F^{a\alpha\beta}F^a_{\alpha\beta} - g^{\alpha\rho}g^{\beta\mu}g^{\sigma\nu}F^a_{\alpha\beta}F^a_{\rho\sigma} - g^{\beta\sigma}g^{\alpha\mu}g^{\rho\nu}F^a_{\alpha\beta}F^a_{\rho\sigma}\right) \\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{\mu\nu}\left(F^{a\mu\sigma}F^{a\nu}_{\sigma} - \frac{1}{4}g^{\mu\nu}F^{a\alpha\beta}F^a_{\alpha\beta}\right)
\end{aligned} \tag{C.11}
$$

where, to obtain the term that is not proportional to the metric, we use the antisymmetry of the field-strength, $F^a_{\nu\mu} = -F^a_{\mu\nu}$. We can immediately identify

$$
T^{\mu\nu}_{YM} = F^{a\mu\sigma}F^{a\nu}_{\sigma} - \frac{1}{4}g^{\mu\nu}F^{a\alpha\beta}F^a_{\alpha\beta} \tag{C.12}
$$

Now, taking $g_{\mu\nu}$ to be the Minkowski metric, we know that $g_{\mu\nu}g^{\mu\nu} = 4$. This implies that $T^{\mu}_{\mu} = g_{\mu\nu}T^{\mu\nu} = 0$. This is enough to show that Yang-Mills theory is conformally invariant as a classical field theory, which implies the conformal invariance of gauge theory tree-level amplitudes. Quantum effects induce the so-called trace anomaly, which imply that the trace of the energy-momentum tensor becomes proportional to the Beta function of the running coupling constant g. The fact that the Beta function of Yang-Mills theory is non-vanishing starting at one-loop breaks the conformal symmetry of the theory.

Our next goal is to see how the global symmetries of Yang-Mills theory manifest themselves on the MHV amplitudes, and try to interpret them. We have already seen from the Lagrangian point of view that the theory is, at least classically, conformal invariant. Now, let us see how these statement is translated to the tree amplitudes of the theory.

The fact that an amplitude has a particular symmetry is realized as an operator statement. That is, if $\mathcal{O}(p)$ is the generator of some particular symmetry transformation, the relation to be satisfied is

$$
\mathcal{O}(p)\mathbf{A}_n = 0 \tag{C.13}
$$

where

$$
\mathbf{A}_n = \delta^4\left(\sum_i p_i\right)A_n(\{p_i, \epsilon_i\}) \tag{C.14}
$$

is the amplitude with the momentum-conserving delta function restored. Since in four dimensions we consider amplitude as functions of the spinor variables, we let

$$
\mathbf{A}_n = \delta^4\left(\sum_i \lambda_i\tilde{\lambda}_i\right)A_n(\{\lambda_i, \tilde{\lambda}_i\}) \tag{C.15}
$$

and, in order to study the action of the conformal generators, we write them in terms of spinor variables. [1]. This is realized in two steps: first, one performs a Fourier

[1] We will use the conventions for spinor indices of Appendix B.

transform from configuration to momentum space, in order to write the generators (C.3)(C.4)(C.5) acting on momentum variables, and then one projects their Lorentz components into spinor components using the appropriate combinations of Sigma matrices. For example, consider the generators of the Lorentz group

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{C.16}$$

In momentum space, this operator takes the form

$$\hat{M}_{\mu\nu} = i\left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\right) \tag{C.17}$$

Now, there are two ways to project the Lorentz generator into spinor variables, namely $m_{\alpha\beta} = \hat{M}^{\mu\nu}(\sigma_{\mu\nu})_{\alpha\beta}$ and $\bar{m}_{\dot{\alpha}\dot{\beta}} = \hat{M}^{\mu\nu}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}$, which correspond to the self-dual and anti-self-dual part of the operator $\hat{M}_{\mu\nu}$. Here, $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are the generators of the Lorentz group in the spinor representation, and are given by

$$\begin{aligned}
(\sigma_{\mu\nu})_\alpha{}^\beta &= \frac{1}{4}\left((\sigma_\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\nu)^{\dot{\alpha}\beta} - (\sigma_\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}\right) \\
(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}\left((\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}(\sigma_\nu)_{\alpha\dot{\beta}} - (\bar{\sigma}_\nu)^{\dot{\alpha}\alpha}(\sigma_\mu)_{\alpha\dot{\beta}}\right)
\end{aligned} \tag{C.18}$$

Moreover, the map from the vector to the bispinor representation can be inverted to yield

$$p^\mu = \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}} \tag{C.19}$$

which then allows to use the chain rule to write

$$\partial_\alpha \equiv \frac{\partial}{\partial \lambda^\alpha} = \frac{1}{2}\sigma^\mu_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}\frac{\partial}{\partial p^\mu} \tag{C.20}$$

hence, we can contract the vector indices of the Lorentz generators with those of the sigma matrices to obtain, for example,

$$(\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} p_\mu \frac{\partial}{\partial p^\nu} = 2\lambda^\beta \partial_\alpha \tag{C.21}$$

Hence, an arbitrary operator which depends on momenta and its derivatives can be transformed to spinor variables. In particular, the conformal generators for a single particle take the form

$$\begin{aligned}
p^{\alpha\dot{\alpha}} &= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, & k_{\alpha\dot{\alpha}} &= \partial_\alpha \partial_{\dot{\alpha}} \\
m_{\alpha\beta} &= i\lambda_{(\alpha}\partial_{\beta)} := \frac{i}{2}\left(\lambda_\alpha \partial_\beta + \lambda_\beta \partial_\alpha\right), & \bar{m}_{\dot{\alpha}\dot{\beta}} &= i\tilde{\lambda}_{(\dot{\alpha}}\partial_{\dot{\beta})} \\
d &= \frac{1}{2}\lambda^\alpha \partial_\alpha + \frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}} + 1
\end{aligned} \tag{C.22}$$

With this representation of the conformal generators, we can readily show that the MHV amplitudes are conformally invariant.

First, the fact that the momentum operator annihilates the amplitude is valid in the distributional sense of

$$x\delta(x) = 0 \tag{C.23}$$

due to the appearance of the momentum-conserving delta function in C.15. Also, they are symmetric under Lorentz transformations generated by m and $\bar{m}$ because, for example,

$$\begin{aligned} m_{\alpha\beta}\langle jk\rangle &= \sum_{i=1}^{n} \lambda_{i(\alpha}\partial_{i\beta)}\lambda_j^{\gamma}\lambda_{k\gamma} \\ &= \frac{1}{2}\sum_{i=1}^{n}\left[\lambda_{i\alpha}\left(\delta_{ij}\delta_{\beta}{}^{\gamma}\lambda_{k\gamma} + \delta_{ik}\epsilon_{\beta\gamma}\lambda_j^{\gamma}\right) + (\alpha \leftrightarrow \beta)\right] \\ &= \frac{1}{2}\left[\lambda_{j\alpha}\lambda_{k\beta} - \lambda_{k\alpha}\lambda_{j\beta} + (\alpha \leftrightarrow \beta)\right] = 0 \end{aligned} \tag{C.24}$$

where we used the fact that the epsilon tensor is antisymmetric. Similarly, the Lorentz generators annihilate all other spinor products. Hence, since the MHV amplitudes are functions of the spinor products, we conclude that the Lorentz generators also annihilate the full amplitudes.

We only need to show that the amplitudes are annihilated by the dilatation and special conformal generators d and $k_{\alpha\dot{\alpha}}$. On one hand, it is easy to see that, if we write $\tilde{d} = d - n$, that is, the differential part of the dilatation operator, then

$$\begin{aligned} \tilde{d}\lambda_i^{\alpha} &= \frac{1}{2}\lambda_i^{\alpha} \\ \tilde{d}\langle ij\rangle &= \langle ij\rangle \end{aligned} \tag{C.25}$$

then, the dilatation operator measures the mass dimension of the object it acts upon, and sums a constant times the object. Hence,

$$d\mathbf{A}_n = \left([\mathbf{A}_n] + n\right)\mathbf{A}_n \tag{C.26}$$

Recalling that the delta function has units of the inverse of its argument, $[\delta^4(p)] = -4$. Since the MHV amplitude has four spinor products in the numerator and n in the denominator, its mass dimension (as it should be) is $4 - n$. Hence,

$$d\mathbf{A}_n = (-4 + 4 - n + n)\mathbf{A}_n = 0 \tag{C.27}$$

Finally, we study the action of the special conformal generator. Note that, since the MHV amplitude only depends on spinor products formed with undotted spinors, the only action of the derivative with respect to dotted spinors appearing on $k_{\alpha\dot{\alpha}}$ is on the delta function. Thus, if we let P denote the total momentum of the scattering process,

$$\begin{aligned}
k_{\alpha\dot{\alpha}}\mathbf{A}_n &= \sum_{i=1}^{n} \partial_{i\alpha}\partial_{i\dot{\alpha}}\left(\delta^4(P)A_n\right) \\
&= \sum_{i=1}^{n} \partial_{i\alpha}\left(\frac{\partial P^{\beta\dot{\beta}}}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}\left(\frac{\partial}{\partial P^{\beta\dot{\beta}}}\delta^4(P)\right)A_n\right) \\
&= \sum_{i=1}^{n} \delta_{\dot{\alpha}}{}^{\dot{\beta}}\partial_{i\alpha}\left(\lambda_i^{\beta}\left(\frac{\partial}{\partial P^{\beta\dot{\beta}}}\delta^4(P)\right)A_n\right) \\
&= \left[\left(n\frac{\partial}{\partial P^{\alpha\dot{\alpha}}} + P^{\beta\dot{\beta}}\frac{\partial}{\partial P^{\beta\dot{\alpha}}}\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\right)\delta^4(P)\right]A_n \\
&+ \left(\frac{\partial}{\partial P^{\beta\dot{\alpha}}}\delta^4(P)\right)\sum_{i=1}^{n}\lambda_i^{\beta}\partial_{i\alpha}A_n
\end{aligned} \tag{C.28}$$

Now, lowering the β index, we may split the operator on the last line into a symmetric and an antisymmetric part

$$\sum_{i=1}^{n}\lambda_{i\beta}\partial_{i\alpha} = \sum_{i=1}^{n}\lambda_{i(\beta}\partial_{i\alpha)} + \epsilon_{\alpha\beta}\sum_{i=1}^{n}\lambda_i^{\gamma}\partial_{i\gamma} \tag{C.29}$$

We recognize the symmetric part as the Lorentz generator, $m_{\beta\alpha}$, which we already know that annihilates the amplitude. Then, recognizing that the part proportional to the Levi-Civita tensor is just the non-vanishing part of the dilatation operator acting on the MHV amplitude A_n, we find that, after rising again the β index,

$$\sum_{i=1}^{n}\lambda_i^{\beta}\partial_{\alpha}A_n = \delta_{\alpha}{}^{\beta}(4-n)A_n \tag{C.30}$$

then, putting everything together,

$$k_{\alpha\dot{\alpha}}\mathbf{A}_n = \left[\left(4\frac{\partial}{\partial P^{\alpha\dot{\alpha}}} + P^{\beta\dot{\beta}}\frac{\partial}{\partial P^{\beta\dot{\alpha}}}\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\right)\delta^4(P)\right]A_n \tag{C.31}$$

Now, let $F(P)$ be an arbitrary test function. Then, we can see that

$$\begin{aligned}
&\int d^4P\, F(P)P^{\beta\dot{\beta}}\frac{\partial}{\partial P^{\beta\dot{\alpha}}}\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\delta^4(P) \\
&= -\int d^4P\left\{\frac{\partial}{\partial P^{\beta\dot{\alpha}}}\left[F(P)P^{\beta\dot{\beta}}\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\delta^4(P)\right] - \frac{\partial}{\partial P^{\beta\dot{\alpha}}}\left(F(p)P^{\beta\dot{\beta}}\right)\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\delta^4(P)\right\} \\
&= -\int d^4P\left[2\delta_{\dot{\alpha}}{}^{\dot{\beta}}F(P) + P^{\beta\dot{\beta}}\frac{\partial F}{\partial P^{\beta\dot{\alpha}}}\right]\frac{\partial}{\partial P^{\alpha\dot{\beta}}}\delta^4(P) \\
&= \int d^4P\left[2\frac{\partial F}{\partial P^{\alpha\dot{\alpha}}} + \frac{\partial}{\partial P^{\alpha\dot{\beta}}}\left(P^{\beta\dot{\beta}}\frac{\partial}{\partial F P^{\beta\dot{\alpha}}}\right)\right]\delta^4(P) \\
&= \int d^4P\left[4\frac{\partial F}{\partial P^{\alpha\dot{\alpha}}} + P^{\beta\dot{\beta}}\frac{\partial^2 F}{\partial P^{\alpha\dot{\beta}}P^{\beta\dot{\alpha}}}\right]\delta^4(P) \\
&= -\int d^4P\, F(P)\left[4\frac{\partial}{\partial P^{\alpha\dot{\alpha}}}\delta^4(P)\right]
\end{aligned} \tag{C.32}$$

where, to obtain the last line, we used $x\delta(x) = 0$ and integrate by parts to pass the derivatives to the delta function. Hence, on a distributional sense, we have the equality

$$P^{\beta\dot\beta}\frac{\partial}{\partial P^{\beta\dot\alpha}}\frac{\partial}{\partial P^{\alpha\dot\beta}}\delta^4(P) = -4\frac{\partial}{\partial P^{\alpha\dot\alpha}}\delta^4(P) \tag{C.33}$$

which implies that $k_{\alpha\dot\alpha}$ annihilates the MHV amplitude. This completes the proof that MHV amplitudes are conformal invariant at tree level.

The representation of the conformal group (C.22) in terms of spinor variables is highly asymmetric. Momentum is realized as a multiplicative operator, while the Lorentz generators are first-order differential operators and the special conformal transformations are second-order differential operators. The natural thing would be to look for a set of variables in which the generators are written in a more uniform manner. In order to do this, consider the following transformation

$$\begin{aligned} \tilde\lambda_{\dot\alpha} &\to i\frac{\partial}{\partial \mu^{\dot\alpha}} \\ \frac{\partial}{\partial\tilde\lambda^{\dot\alpha}} &\to i\mu_{\dot\alpha} \end{aligned} \tag{C.34}$$

In these variables, all the conformal generators become first-order differential operators; their one particle action is given by

$$\begin{aligned} p_{\alpha\dot\alpha} &= i\lambda_\alpha\frac{\partial}{\partial\mu^{\dot\alpha}}, \qquad\qquad k_{\alpha\dot\alpha} = i\mu_{\dot\alpha}\frac{\partial}{\partial\lambda^\alpha} \\ m_{\alpha\beta} &= \frac{i}{2}\left(\lambda_\alpha\frac{\partial}{\partial\lambda^\beta} + \lambda_\beta\frac{\partial}{\partial\lambda^\alpha}\right) \\ \bar m_{\dot\alpha\dot\beta} &= \frac{i}{2}\left(\mu_{\dot\alpha}\frac{\partial}{\partial\mu^{\dot\beta}} + \mu_{\dot\alpha}\frac{\partial}{\partial\mu^{\dot\beta}}\right) \\ d &= \frac{i}{2}\left(\lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} - \mu^{\dot\alpha}\frac{\partial}{\partial\mu^{\dot\alpha}}\right) \end{aligned} \tag{C.35}$$

This is a particularly uniform representation of the conformal generators; in particular, the dilatation operator has become homogeneous, which is a direct consequence of commuting $\mu^{\dot\alpha}$ with the derivative operator. In four dimensions, the conformal group can be identified with $SU(2,2)$, the special unitary group of 2×2 matrices acting on two-dimensional hermitian forms. Its complexification, $SL(4,\mathbb{C})$, has a four-dimensional representation acting on

$$Z^I = (\lambda^\alpha, \mu^{\dot\alpha}) \tag{C.36}$$

the objects Z^I are known as twistors; they were first introduced by Penrose [108] as an alternative geometric description of Minkowski spacetime that made conformal symmetry manifest. Depending on the signature of spacetime, the twistors are either real or complex. For example, in $(+,+,-,-)$ signature, the conformal group is $SL(4,\mathbb{R})$ and it is consistent to take λ and μ as real, two-component spinors. In particular, to obtain an amplitude in twistor space from the corresponding momentum-space amplitude in this signature, one must simply perform a Fourier transform

$$\tilde{A}(\lambda_i, \mu_i) = \int \prod_{j=1}^{n} d^2\tilde{\lambda}_j \exp\left(i[\mu_j \tilde{\lambda}_j]\right) A(\lambda_i, \tilde{\lambda}_i) \tag{C.37}$$

for Lorentzian signature $(+,-,-,-)$, the spinors are complex and the Fourier transform should be replaced by a contour integral. However, the definition of the contour is not unique, and the formalism of $\bar{\partial}$-cohomology is needed to define the twistor transform. Nonetheless, we will perform manipulations of amplitudes from momentum to twistor space as if they were in $(+,+,-,-)$ signature.

Note that choosing to transform $\tilde{\lambda}$ instead of λ is completely arbitrary, and one could also perform a twistor transform which leaves the antiholomorphic spinors untouched, and transforms the holomorphic spinors. The consequence of making such a change is that, for example, there is no longer a simple relation between the MHV and anti-MHV amplitudes; in momentum space, one can obtain the anti-MHV amplitudes as parity conjugates of the MHV ones, which amounts to replacing all angle brackets with square brackets, up to a phase [2]. This symmetry is broken down by the transformation to twistor space. As we will see in a moment, MHV amplitudes will have a particularly simple structure, but this will not be the case for the anti-MHV ones.

When the external states of an amplitude undergo a little group transformation, they pick up a phase. This fact is realized, in spinor variables, as the relation

$$\left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} - \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}\right) A_n(\lambda_i, \tilde{\lambda}_i, h_i) = -2h_i A_n(\lambda_i, \tilde{\lambda}_i, h_i) \tag{C.38}$$

for an arbitrary leg i, and h_i denotes its helicity. In terms of twistor variables, one obtains

$$\left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \mu_i^{\dot{\alpha}} \frac{\partial}{\partial \mu_i^{\dot{\alpha}}}\right) \tilde{A}_n(\lambda_i, \tilde{\lambda}_i, h_i) = -(2h_i + 2)\tilde{A}_n(\lambda_i, \mu_i, h_i) \tag{C.39}$$

The operator on the left-hand side of the last relation can be seen to be equal to $Z^I \frac{\partial}{\partial Z^I}$, which has an identical form to the dilatation operator of the conformal group (although acting on rather different variables). Therefore, we can identify it as the generator of the scalings $Z^I \to tZ^I$ for some complex, non-zero constant t. This implies that the amplitudes are homogeneous functions of the twistor variables of each particle, Z_i^I, of degree $-(2h_i + 2)$. We can identify sets of twistor variables differing by overall scalings, $Z^I \sim tZ^I$ and drop the single point $Z^I = 0$. This identification defines an equivalence relation, and the space where the twistors that amplitudes depend on are defined is actually projective twistor space, $\mathbb{CP}^3$ or $\mathbb{CR}^3$, depending on whether the twistors are real or complex.

Now, let us write the MHV amplitude with the momentum conserving delta function as

$$\mathbf{A}_n^{MHV} = (2\pi)^4 \delta^4(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}) f(\lambda_i) \tag{C.40}$$

[2] Actually, a factor of $(-1)^n$ for an n-point amplitude.

where f corresponds to the Parke-Taylor formula (2.62). The important fact in the following argument is that it only depends on the holomorphic spinors λ_i. We want to perform the twistor transform of the MHV amplitude. Since the only dependence on the antiholomorphic spinors is through the delta function, we write

$$\mathbf{A}_n^{MHV} = \int d^4x \exp(ix_{\beta\dot{\beta}} \sum_i \lambda_i^\beta \tilde{\lambda}_i^{\dot{\beta}}) f(\lambda_i) \tag{C.41}$$

with this representation, the Fourier transform to twistor space can be easily performed,

$$\begin{aligned}
\tilde{A}_n^{MHV}(\lambda_i, \mu_i) &= \int \prod_{j=1}^n d^2\tilde{\lambda}_j \exp\left(i\mu_{j\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}}\right) \int d^4x \exp\left(ix_{\alpha\dot{\alpha}} \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\right) f(\lambda_i) \\
&= \int d^4x \int \prod_{j=1}^n d^2\tilde{\lambda}_j \exp\left[i\tilde{\lambda}_j^{\dot{\alpha}}(\mu_{j\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda_j^\alpha)\right] f(\lambda_i) \\
&= \int d^4x \prod_{i=1}^n \delta^2(\mu_{i\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda_i^\alpha) f(\lambda_i)
\end{aligned} \tag{C.42}$$

How do we interpret this result? For every point $x_{\alpha\dot{\alpha}}$ in spacetime, the delta functions enforce the equations

$$\mu_{i\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda_i^\alpha = 0, \quad \dot{\alpha} = 1,2 \tag{C.43}$$

which define a degree 1 algebraic curve in $\mathbb{RP}^3$ [3]. In $(+,+,-,-)$ signature, this curve is most easily described as straight line; if we take $\lambda_1 \neq 0$, it can be described by the set of parametric equations

$$x = \frac{\lambda_2}{\lambda_1},\ y = \frac{\mu_1}{\lambda_1},\ z = \frac{\mu_2}{\lambda_1} \tag{C.44}$$

hence, MHV amplitudes in twistor space are supported on points which are collinear in $\mathbb{R}^3$. This observation motivated Witten [19] to conjecture that an n gluon scattering amplitude in twistor space is nonzero only if the twistor coordinates Z_i^I of each external particle are supported on an algebraic curve of degree d, which satisfies

$$d = q - 1 + l \tag{C.45}$$

where q is the number of positive helicity gluons and l the number of loops. Furthermore, this curve may have genus $g \geq 0$, but must be bounded by

$$g \leq l \tag{C.46}$$

For example, at tree level, there can only be genus zero curves. Since lines are defined as genus zero curves, our example indeed satisfies these assumptions. Further

[3] An algebraic curve, $\mathcal{C}$ in $\mathbb{RP}^3$ is a curve defined as the set of zeroes of a polynomial equation with real coefficients in the twistor coordinates Z^I. One way to define an algebraic curve is to set two polynomials of degrees d_1, d_2 say $P_1(Z^I)$ and $P_2(Z^I)$, equal to zero. The degree of such an intersection is then $d = d_1 d_2$.

exploration of this conjecture led to the connected formalism for tree-level super-amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory (SYM), which we discuss in Chapter 4.

$A_5(q_1\bar{q}_1q_2\bar{q}_2g)$ helicity amplitude

We will perform the calculation of the color-ordered amplitude $A_5(q_1\bar{q}_1q_2\bar{q}_2g)$, as defined by the color decomposition of 2.30. Attaching all factors of $-ig/\sqrt{2}$ to the primitive amplitudes instead of the color factors and using the notation $U(p_i) = U(i)$ and $\epsilon_\mu(p_i) = \epsilon_\mu(i)$, using the shorthand notations $u(p_i) = u(i)$, etc. We can write A_1 as

$$
\begin{aligned}
A_1 &= D_1 + D_4 + D_5 \\
&= \bar{u}(3)\left(-i\frac{g}{\sqrt{2}}\gamma^\mu\right)v(4)\left(\frac{-ig_{\mu\nu}}{s_{34}}\right)\bar{u}_1\left(-i\frac{g}{\sqrt{2}}\gamma^\alpha\right)\epsilon_\alpha(5)\left(\frac{i(\not{p}_1+\not{p}_5)}{s_{15}}\right)\left(-i\frac{g}{\sqrt{2}\gamma^\nu}\right)v(2) \\
&- \bar{u}_1\left(-i\frac{g}{\sqrt{2}}\gamma^\mu\right)v(2)\left(\frac{-ig_{\mu\nu}}{s_{12}}\right)\bar{u}_3\left(-i\frac{g}{\sqrt{2}}\gamma^\nu\right)\left(\frac{i(\not{k}_4+\not{k}_5)}{s_{45}}\right)\left(-i\frac{g}{\sqrt{2}}\gamma^\alpha\right)\epsilon_\alpha(5)v(4) \\
&+ \bar{u}(1)\left(-i\frac{g}{\sqrt{2}}\gamma^\mu\right)v(2)\left(\frac{-ig_{\mu\rho}}{s_{12}}\right)\left(\frac{ig}{\sqrt{2}}\right)[g^{\rho\alpha}(p_5-p_1-p_2)^\sigma + g^{\alpha\sigma}(p_3+p_4-p_5)^\rho \\
&+ g^{\sigma\rho}(p_1+p_2-p_3-p_4)^\alpha]\epsilon_\alpha(5)\left(-i\frac{g_{\sigma\nu}}{s_{34}}\right)\bar{u}(3)\left(-i\frac{g}{\sqrt{2}}\gamma^\nu\right)v(4) \\
&= i\left(\frac{g}{\sqrt{2}}\right)^3\left\{\frac{1}{s_{15}s_{34}}\bar{u}(3)\gamma^\mu v(4)\bar{u}(1)\not{\epsilon}(5)\left(\not{p}_1+\not{p}_5\right)\gamma_\mu v(2)\right. \\
&- \frac{1}{s_{12}s_{45}}\bar{u}(1)\gamma^\mu v(2)\bar{u}(3)\gamma_\mu\left(\not{p}_4+\not{p}_5\right)\not{\epsilon}(5)v(4) + \frac{2}{s_{12}s_{34}}\Big[\bar{u}(1)\not{\epsilon}(5)v(2)\bar{u}(3)\not{p}_5 v(4) \\
&\left. -\bar{u}(1)\not{k}_5 v(2)\bar{u}(3)\not{\epsilon}(5)v(4) + \bar{u}(1)\gamma^\rho v(2)\bar{u}(3)\gamma_\rho v(4)\,(p_1+p_2)\cdot\epsilon(5)]\right\} \\
&= i\left(\frac{g}{\sqrt{2}}\right)^3(T_1+T_2+T_3)
\end{aligned}
\tag{D.1}
$$

where we have contracted all possible Lorentz indices and T_1, T_2, T_3 denote each term in the sum in brackets. Now, we will consider spinors of definite helicity. In particular, let us consider $A_1(1^+2^-3^+4^-5_g^+)$. Then, in Dirac spinor notation,

$$
\begin{aligned}
\bar{u}(1) &= [1|, \quad v(2) = |2\rangle \\
\bar{u}(3) &= [3|, \quad v(4) = |4\rangle
\end{aligned}
\tag{D.2}
$$

and, for some reference momenta q,

$$
\epsilon_\mu(5) = \frac{1}{\sqrt{2}}\frac{\langle q|\gamma_\mu|5]}{\langle q5\rangle}
\tag{D.3}
$$

which implies, due to the Fierz identity, that

$$
\begin{aligned}
\not\epsilon(5) &= \frac{1}{\sqrt{2}} \frac{\langle q|\gamma^\mu|5]\gamma_\mu}{\langle q5\rangle} \\
&= \frac{\sqrt{2}}{\langle q5\rangle} \left(|q\rangle[5| + |5]\langle q| \right)
\end{aligned}
\tag{D.4}
$$

Then, if we take the reference momentum to be $q = p_1$ and using the anticommutation relation of the Gamma matrices, we can see that

$$
\begin{aligned}
\not\epsilon(5)\left(\not p_1 + \not p_5\right) &= \left(2p_1 \cdot \epsilon(5) - \not p_1 \not\epsilon(5)\right) + \not\epsilon(5)\not p_5 \\
&= -\not p_1 \not\epsilon(5) + \not\epsilon(5)\not p_5
\end{aligned}
\tag{D.5}
$$

Furthermore, the Dirac equation implies $\bar{u}(1)\not p_1 = 0$. Then, the first term contributing to the amplitude simplifies to

$$
\begin{aligned}
T_1 &= \frac{1}{s_{15}s_{34}}[3|\gamma^\mu|4\rangle[1|\left(\frac{\sqrt{2}}{\langle 15\rangle}\left(|1\rangle[5| + |5]\langle 1|\right)\right)\not p_5 \gamma_\mu|2\rangle \\
&= \frac{1}{s_{15}s_{34}}\frac{\sqrt{2}}{\langle 15\rangle}[3|\gamma^\mu|4\rangle\left([15]\langle 1|\not p_5\gamma_\mu|2\rangle\right) \\
&= \frac{1}{s_{15}s_{34}}\frac{\sqrt{2}}{\langle 15\rangle}[15][3|\gamma^\mu|4\rangle\langle 15\rangle[5|\gamma_\mu|2\rangle \\
&= \frac{\sqrt{2}}{s_{15}s_{34}}[15]\langle 4|\gamma^\mu|3]\langle 2|\gamma_\mu|5] \\
&= \frac{2\sqrt{2}}{\langle 15\rangle[51]\langle 34\rangle[43]}[15]\langle 42\rangle[53] \\
&= 2\sqrt{2}\frac{\langle 42\rangle[53]}{\langle 51\rangle\langle 34\rangle[43]}
\end{aligned}
\tag{D.6}
$$

Now, we consider the numerator of T_2, that is

$$
\begin{aligned}
&[1|\gamma^\mu|2\rangle[3|\gamma_\mu(\not p_4 + \not p_5)\not\epsilon(5)|4\rangle \\
&= [1|\gamma^\mu|2\rangle[3|\gamma_\mu\left(|4\rangle[4| + |4]\langle 4| + |5\rangle[5| + |5]\langle 5|\right)\frac{\sqrt{2}}{\langle 15\rangle}\left(|1\rangle[5| + |5]\langle 1|\right)|4\rangle \\
&= \frac{\sqrt{2}}{\langle 15\rangle}[1|\gamma^\mu|2\rangle\left([3|\gamma_\mu|4\rangle[4| + [3|\gamma_\mu|5\rangle[5|\right)\left(|5]\langle 14\rangle\right) \\
&= \sqrt{2}\frac{\langle 14\rangle}{\langle 15\rangle}[1|\gamma^\mu|2\rangle[45][3|\gamma_\mu|4\rangle \\
&= 2\sqrt{2}\frac{\langle 14\rangle}{\langle 15\rangle}[45]\langle 24\rangle[31]
\end{aligned}
\tag{D.7}
$$

hence, after writing the propagators in terms of spinor products,

$$
T_2 = 2\sqrt{2}\frac{\langle 14\rangle\langle 24\rangle[31]}{\langle 12\rangle[21]\langle 54\rangle\langle 15\rangle}
\tag{D.8}
$$

Finally, we have to calculate the third term, corresponding to the three-gluon vertex, in terms of spinors products. First, we consider

$$
\begin{aligned}
\bar{u}(1)\not{\epsilon}(5)v(2)\bar{u}(3)\not{p}_5 v(4) &= [1|\left(\frac{\sqrt{2}}{\langle 15\rangle}\left(\,|1\rangle[5| + |5]\langle 1|\,\right)\right)|2\rangle[3|\not{p}_5|4\rangle \\
&= \frac{\sqrt{2}}{\langle 15\rangle}[15]\langle 12\rangle[35]\langle 54\rangle \equiv T_{3;1}
\end{aligned}
\tag{D.9}
$$

and

$$
\begin{aligned}
\bar{u}(1)\not{k}_5 v(2)\bar{u}(3)\not{\epsilon}(5)v(4) &= [1|\not{p}_5|2\rangle[3|\left(\frac{\sqrt{2}}{\langle 15\rangle}\left(\,|1\rangle[5| + |5]\langle 1|\,\right)\right)|4\rangle \\
&= \frac{\sqrt{2}}{\langle 15\rangle}[15]\langle 52\rangle[35]\langle 14\rangle \equiv T_{3;2}
\end{aligned}
\tag{D.10}
$$

We can calculate the difference between these two terms and use the Schouten identity to obtain

$$
\begin{aligned}
T_{3;1} - T_{3;2} &= \sqrt{2}\frac{[15]}{\langle 15\rangle}[35]\left(\langle 23\rangle\langle 54\rangle - \langle 52\rangle\langle 14\rangle\right) \\
&= \sqrt{2}\frac{[15]}{\langle 15\rangle}[35]\left(-\langle 15\rangle\langle 42\rangle\right) \\
&= \sqrt{2}[15][35]\langle 24\rangle
\end{aligned}
\tag{D.11}
$$

The final term we have to simplify is

$$
\begin{aligned}
\bar{u}(1)\gamma^\rho v(2)\bar{u}(3)\gamma_\rho v(4)\,(p_1 + p_2)\cdot\epsilon(5) &= [1|\gamma^\rho|2\rangle[3|\gamma_\rho|4\rangle\frac{1}{\sqrt{2}}\frac{\langle 1|\not{p}_2|5]}{\langle 15\rangle} \\
&= \frac{1}{\sqrt{2}}\langle 2|\gamma^\rho|1]\langle 4|\gamma_\rho|3]\frac{\langle 12\rangle[25]}{\langle 15\rangle} \\
&= \sqrt{2}\frac{[31][25]\langle 24\rangle\langle 12\rangle}{\langle 15\rangle} \equiv T_{3;3}
\end{aligned}
\tag{D.12}
$$

With this, the third term in the amplitude can be written in terms of spinor products as

$$
\begin{aligned}
T_3 &= T_{3;1} - T_{3;2} + T_{3;3} \\
&= \frac{2\sqrt{2}}{\langle 15\rangle}\frac{1}{\langle 12\rangle[21]}\frac{1}{\langle 34\rangle[43]}\left(\langle 24\rangle\langle 15\rangle[15][35] + \langle 24\rangle\langle 12\rangle[25][31]\right)
\end{aligned}
\tag{D.13}
$$

Therefore, the partial amplitude is given by

$$
\begin{aligned}
A_1(1^+2^-3^+4^-5_g^+) = ig^3&\frac{\langle 24\rangle}{\langle 15\rangle\langle 54\rangle\langle 12\rangle[21]\langle 34\rangle[43]}\left\{[53]\langle 12\rangle[21]\langle 54\rangle + [35]\langle 34\rangle[43]\langle 54\rangle\right. \\
&\left.+\langle 15\rangle[51][53]\langle 54\rangle + \langle 24\rangle[32]\langle 34\rangle[43] + [31]\langle 12\rangle[25]\langle 54\rangle\right\} \\
= ig^3&\frac{\langle 14\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 15\rangle\langle 54\rangle}
\end{aligned}
\tag{D.14}
$$

where the last equality follows after using momentum conservation to eliminate the momentum p_5 from the spinor products. Other helicity configurations may be calculated in the same fashion, starting from (D.1), replacing the corresponding spinors for the helicities under consideration and simplifying their products. The remarkable simplicity of the result shows the utility of spinor products. The calculation of the complete set of helicity amplitudes associated to this process using string theory methods was performed by Kosower in [109].

Derivation of the Parke-Taylor formula

In this appendix, we provide a proof of the MHV formula for gluon amplitudes using the BCFW recursion relations. We will proceed by induction on the number of external legs. Since we have already proven the case $n = 3$ using little group scaling, we will take it as the starting step, and as inductive hypothesis assume that the formula is valid for all $j < n$ for some $n \in \mathbb{N}$, and prove that it holds for this value of n. Without loss of generality, we choose to perform the following BCFW shift

$$\begin{aligned} |n\rangle &\to |n\rangle - z|1\rangle \\ |1] &\to |1] + z|n] \end{aligned} \tag{E.1}$$

With this shift, the BCFW recursion can be expanded to take the form

$$\begin{aligned} &A_n(1^-, 2^+, ..., i^-, ..., n^+) = \sum_{s=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1}, 2, ..., -\hat{P}_{1k}^{-s})\frac{i}{P_{1k}^2}A_{n-k+1}(\hat{P}_{1k}^s, k+1, ..., \hat{n}) \\ &= \sum_{s=\pm}[A_3(\hat{1}^-, 2^+, -\hat{P}_{12}^{-s})\frac{1}{P_{12}^2}A_{n-1}(\hat{P}_{12}^s, 3^+, ..., i^-, ..., \hat{n}^+) + ...+ \\ &+ A_{n-1}(\hat{1}^-, 2^+, ..., \hat{i}^-, ..., -\hat{P}_{1(n-2)}^{-s})\frac{i}{P_{1(n-2)}^2}A_3(\hat{P}_{1(n-2)}^s, (n-1)^+, \hat{n}^+)] \\ &= A_3(\hat{1}^-, 2^+, -\hat{P}_{12}^+)\frac{1}{P_{12}^2}A_{n-1}(\hat{P}_{12}^-, 3^+, ..., i^-, ..., \hat{n}^+)+ \\ &+ +A_{n-1}(\hat{1}^-, 2^+, ..., \hat{i}^-, ..., -\hat{P}_{1(n-2)}^+)\frac{i}{P_{1(n-2)}^2}A_3(\hat{P}_{1(n-2)}^-, (n-1)^+, \hat{n}^+) \end{aligned} \tag{E.2}$$

where, in going from the second to the third line, we see that all other contributions vanish because they are either all-plus gluon amplitudes or have only one negative helicity gluon. Choosing complex kinematics to satisfy

$$[12] = 0, \quad \langle 12\rangle \neq 0 \tag{E.3}$$

we see that the three point amplitude $A_3(\hat{1}^-, 2^+, -\hat{P}_{12}^+)$ vanishes, and we only have to compute one contribution. Using the three-point anti-MHV formula and the inductive hypothesis, we can write

$$
\begin{aligned}
&A_n(1^-, 2^+, ..., i^-, ..., n^+) = \\
&= \left(-i \frac{\langle \hat{1} i \rangle^4}{\langle \hat{1} 2 \rangle ... \langle (n-2) \hat{P}_{1(n-2)} \rangle \langle \hat{P}_{1(n-2)} \hat{1} \rangle} \right) \frac{i}{P^2_{1(n-2)}} \times \\
&\times \left(-i \frac{[(n-1)\hat{n}]^4}{[\hat{P}_{1(n-2)}(n-1)][(n-1)\hat{n}][\hat{n}\hat{P}_{1(n-2)}]} \right) \\
&= \frac{-i}{P^2_{1(n-2)}} \frac{\langle \hat{1} i \rangle^4 [(n-1)\hat{n}]^4}{\langle \hat{1} 2 \rangle ... \langle (n-2) | \hat{P}_{1(n-2)} | \hat{n}] [(n-1)\hat{n}] \langle \hat{1} | \hat{P}_{1(n-2)} | (n-1)]} \\
&= i \frac{\langle 1 i \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}
\end{aligned} \tag{E.4}
$$

where we used momentum conservation in the last step and replaced the explicit expression of $P^2_{1(n-2)} = (p_1 + ... + p_{n-2})^2 = (p_{n-1} + p_n)^2 = s_{(n-1)n} = \langle (n-1) n \rangle [n(n-1)]$. This completes the induction step, thus proving the Parke-Taylor formula for the MHV amplitudes.

5-point ϕ^3 amplitude from the CHY representation

In this appendix, we will use the method of local residues to calculate the 5-point amplitude in ϕ_3 theory. Similarly to the 4-point case, we compute the amplitude for two equal external orderings $\sigma = \bar{\sigma} = 12345$. To avoid clutter, we will not include this as an argument of the amplitude, with the understanding that we have chosen such an ordering.

In order to use 4.22, we need to determine the two polynomials h_2 and h_3 in the denominator of the integrand. Hence, we must find the number of distinct subsets with two and three elements of the set $\{1, 2, 3, 4, 5\}$. Simple combinatorics show that there are 10 of each kind of subsets. The two-element subsets are

$$\begin{array}{ccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\ \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \end{array} \tag{F.1}$$

while the subsets with three elements are given by

$$\begin{array}{ccccc} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,3,4\} & \{1,3,5\} \\ \{1,4,5\} & \{2,3,4\} & \{2,3,5\} & \{2,4,5\} & \{3,4,5\} \end{array} \tag{F.2}$$

Then, using the definition 3.40, we can write

$$\begin{aligned} h_2 =& s_{12}z_1z_2 + s_{13}z_1z_3 + s_{14}z_1z_4 + s_{15}z_1z_5 + s_{23}z_2z_3 \\ &+ s_{24}z_2z_4 + s_{25}z_2z_5 + s_{34}z_3z_4 + s_{35}z_3z_5 + s_{45}z_4z_5 \end{aligned} \tag{F.3}$$

For h_3, we would find terms of the form $s_{123}z_1z_2z_3$, involving three-particle invariants. However, using momentum conservation we can see that, for example,

$$\begin{aligned} s_{123} &= (p_1 + p_2 + p_3)^2 \\ &= (p_4 + p_5)^2 \\ &= s_{45} \end{aligned} \tag{F.4}$$

which yields a term $z_1z_2z_3s_{45}$. Then, h_3 is a polynomial with terms of the form $z_iz_jz_ks_{lm}$, where, if we think of the labels as a set, $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. Then,

$$\begin{aligned} h_3 =& s_{12}z_3z_4z_5 + s_{13}z_2z_4z_5 + s_{14}z_2z_3z_5 + s_{15}z_2z_3z_4 + s_{23}z_1z_4z_5 \\ &+ s_{24}z_1z_3z_5 + s_{25}z_1z_3z_4 + s_{34}z_1z_2z_5 + s_{35}z_1z_2z_4 + s_{45}z_1z_2z_3 \end{aligned} \tag{F.5}$$

Now, using 4.22, the integral we want to calculate is

$$m_5 = -i \oint_C \frac{1}{(2\pi i)^2} \frac{d^5 z}{d\omega} \frac{\prod_{i<j}^5 z_{ij}}{h_2 h_3} \frac{1}{(z_{12}z_{23}z_{34}z_{45}z_{51})^2} \tag{F.6}$$

For this calculation, we choose to gauge fix z_1, z_2 and z_3. Hence

$$d\omega = \frac{dz_1 dz_2 dz_3}{z_{12}z_{23}z_{31}} \tag{F.7}$$

and the product in the numerator is

$$\prod_{i<j}^5 z_{ij} = z_{12}z_{13}z_{14}z_{15}z_{23}z_{24}z_{25}z_{34}z_{35}z_{45} \tag{F.8}$$

Substituting into the integral and cancelling terms, we obtain

$$m_5 = -i \oint_C \frac{dz_4 dz_5}{(2\pi i)^2} \frac{1}{h_2 h_3} \frac{z_{31}^2 z_{14}z_{24}z_{25}z_{35}}{z_{34}z_{45}z_{51}} \tag{F.9}$$

We set $z_1 = 1, z_2 = 0, z_3 = \infty$. Then, e isolate the terms with z_3 and calculate

$$\lim_{z_3 \to \infty} \frac{z_{31}^2 z_{35}}{z_{34} h_2 h_3} = \frac{1}{h_2' h_3'} \tag{F.10}$$

where a prime denotes differentiation with respect to z_3. After setting $z_1 = 1$ and $z_2 = 0$, we obtain

$$h_2' = s_{13} + s_{34}z_4 + s_{35}z_5 \tag{F.11}$$

and

$$h_3' = s_{25}z_4 + s_{24}z_5 + s_{12}z_4z_5 \tag{F.12}$$

On one hand. On the other, we need the part of the integrand which does not involve z_3. This is

$$\frac{z_{14}z_{24}z_{25}}{z_{45}z_{51}} = -\frac{z_4 z_5 (1 - z_4)}{(z_4 - z_5)(1 - z_5)} \tag{F.13}$$

Therefore, renaming $z_4 = x$ and $z_5 = y$, we obtain the integral

$$m_5 = i \oint_C \frac{dxdy}{(2\pi i)^2} \frac{xy(1-x)}{(x-y)(1-y)} \frac{1}{s_{13} + s_{34}x + s_{35}y} \frac{1}{s_{12}xy + s_{25}x + s_{24}y} \tag{F.14}$$

The above integral is a simple example of a multidimensional contour integral; in this case, the contour C (a curve on $\mathbb{C}^2$) encloses the simultaneous solutions to $h_2' = h_3' = 0$. To perform the integral, we first solve the equations $h_2' = h_3' = 0$ explicitly. For example, we can eliminate y in $h_3' = 0$ by using $h_2' = 0$, obtaining

$$y = -\frac{s_{13} + s_{34}x}{s_{35}} \tag{F.15}$$

then, substituting y into $h'_3 = 0$ results in the quadratic equation

$$s_{12}s_{34}x^2 + (s_{12}s_{13} + s_{24}s_{34} - s_{25}s_{35})x + s_{13}s_{24} = 0 \tag{F.16}$$

which yields two solutions x_1, x_2 for x, which are related to a pair of solutions y_1, y_2 for y. This is analogous to our discussion in Chapter 2. Then, the two solutions (x_1, y_1) and (x_2, y_2) are the ones we need to calculate our integral. Then, if we identify

$$f(x, y) = \frac{xy(1-x)}{(x-y)(1-y)} \tag{F.17}$$

the Jacobian we are after is given by changing variables to $x' = h'_2(x, y)$, $y' = h'_3(x, y)$. Since h'_2 is linear in x, y and h'_3 is quadratic, the Jacobian is easily calculated, and yields

$$J(x, y) = s_{34}(s_{24} + s_{12}x) - s_{35}(s_{25} + s_{12}y) \tag{F.18}$$

With this, the amplitude is simply given by

$$\begin{aligned}
m_5 &= i\left(\frac{f(x_1, y_1)}{J(x_1, y_1)} + \frac{f(x_2, y_2)}{J(x_2, y_2)}\right) \\
&= i\frac{s_{24} + s_{25}}{s_{12}((s_{34} + s_{35})(s_{24} + s_{25}) - s_{12}s_{13})} \\
&+ i\frac{s_{24}}{(s_{12} + s_{25})((s_{12} + s_{25})(s_{13} + s_{35}) - s_{24}s_{34})} \\
&- i\frac{s_{25}}{s_{12}^2 s_{34} + s_{12}s_{25}s_{34}} \\
&= i\left(\frac{1}{s_{12}s_{34}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{51}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}}\right)
\end{aligned} \tag{F.19}$$

where the last line is obtained by rewritting the Mandelstam invariants in the numerators using the various kinematic identities obtained from momentum conservation.

Words and shuffle algebras

In his appendix, we introduce the notation of words and shuffle algebras, which will allow us to organize amplitudes that possess an external cyclic ordering due to some color decomposition. In particular, consider QCD amplitudes with n external particles, such that

$$n = 2n_q + n_g \tag{G.1}$$

where n_q is the number of quark-antiquark pairs and n_g the number of gluons. We assume that the quarks have different flavours. The quark labels are $q_1, ..., q_{n_q}$, the antiquark labels are $\bar{q}_1, ..., \bar{q}_{n_q}$, and the gluon labels are $g_1, ..., g_{n_g}$. We define the set

$$A = \{q_1, q_2, ..., q_{n_q}; \bar{q}_1, \bar{q}_2, ...\bar{q}_{n_q}; g_1, g_2, ..., g_{n_g}\} \tag{G.2}$$

which we call an alphabet, and each of its element we call letters. We define a word, w, as an ordered sequence of letters

$$w = l_1 l_2 ....l_n \tag{G.3}$$

Words are a convenient way to group the particle labels on which amplitudes depend on. In particular, we are interested in word made out of n letters, such that no letter is repeated; those words correspond to all possible orderings of the particle labels, which means, all possible orderings of the external particles in a primitive or partial amplitude. We define the set of such words as

$$W_0 = \{w = l_1 l_2 ...l_n | l_i \in A, l_i \neq l_j \, if \, i \neq j\} \tag{G.4}$$

We can define a product of words, called the shuffle product, under which the words form an algebra. Let $w_1 = l_1 l_2 ...l_k$, $w_2 = l_{k+1} l_{k+2} ...l_r$. Then, the shuffle product is defined as

$$w_1 \sqcup\!\sqcup \, w_2 = \sum_{shuffles \, \sigma} l_{\sigma(1)} ...l_{\sigma(r)} \tag{G.5}$$

where the set of "shuffles" is the set of permutations that preserve the relative order of $l_1...l_k$ and $l_{k+1}...l_r$. As an example, consider the words $w_1 = l_1 l_2$, $w_2 = l_3 l_4$. The possible shuffles are $\{l_1 l_2 l_3 l_4\}$, $\{l_1 l_3 l_2 l_4\}$, $\{l_3 l_1 l_2 l_4\}$, $\{l_3 l_4 l_1 l_2\}$, $\{l_1 l_3 l_4 l_2\}$, $\{l_3 l_1 l_4 l_2\}$, and then

$$w_1 \sqcup\!\sqcup\, w_2 = l_1l_2l_3l_4 + l_1l_3l_2l_4 + l_3l_1l_2l_4 + l_3l_4l_1l_2 + l_1l_3l_4l_2 + l_3l_1l_4l_2 \tag{G.6}$$

The shuffle product is both associative and commutative. If we denote the empty word by e, that is, the word with no letters, it will be an identity for the shuffle product. Since words $w \in W_0$ are simply a way to organize the ordering of the external particle labels of a given primitive amplitude A_n, we can interpret amplitudes as functions on the vector space of words,

$$A_n(l_1...l_n) = A_n(w) \tag{G.7}$$

such that these are linear operators,

$$A_n(w_1 + w_2) = A_n(w_1) + A_n(w_2) \tag{G.8}$$

We take G.8 as the definition of a sum of words; a sum of external particle orderings is understood as the sum of primitive amplitudes with those orderings. We will see that words also are a natural language in which to express the different linear relations among primitive amplitudes with distinct orderings.

Rings, ideals and Gröbner bases

In this appendix, we introduce some notions of abstract algebra and algebraic geometry that provide the basis for the calculation of the global residues needed in the CHY formalism. We follow the review [110], where methods of computational algebraic geometry are treated in the context of multiloop integrand reduction.

Let R be an Abelian group under the operation $+$. We say that the triple $(R, +, \times)$ is a ring if $\times$ satisfies

- $(a \times b) \times c = a \times (b \times c), \quad \forall a, b, c \in R,$
- There exists an element $1 \in R$ such that $1 \times a = a \times 1 = a \quad \forall a \in R,$
- $a \times (b + c) = a \times b + a \times c$ and $(b + c) \times a = b \times a + c \times a \; \forall a, b, c \in R.$

Furthemore, a triple $(\mathbb{F}, +, \times)$ is a field if it satisfies the ring axioms and is also an Abelian group under $\times$.

One particular type of ring is the polynomial ring $R = \mathbb{F}[z_1, ..., z_n]$, which is the collection of all polynomials in n variables with coefficients in the field $\mathbb{F}$. Common examples of fields where we one considers polynomial rings are the rational numbers $\mathbb{Q}$ or the complex numbers $\mathbb{C}$.

Now, we can define an ideal I as the subset of the polynomial ring $R = \mathbb{F}[z_1, ..., z_n]$ that satifies

- $0 \in I$, where 0 is the identity of the $+$ operation on the field F,
- For any pair of polynomials $f_1, f_2 \in I$, $(f_1 + f_2) \in I$,
- If $f \in I$ then $-f \in I$,
- $\forall f \in I$ and $\forall h \in R$, $h \times f = hf \in I$

Another definition of an ideal can be given as follows: if we consider a set of polynomials $S = \{f_1, ..., f_k\} \subset R$, the ideal generated by S is defined as

$$I = \langle f_1, ..., f_k \rangle \equiv \{f | f = \sum_{i=1}^{k} h_i f_i, \; h_i \in R\} \tag{H.1}$$

In general terms, the generating set of an ideal is not unique, but can always be chosen to be finite, a result which is known as the Lasker-Noether theorem.

Now, consider an ideal $I \subset R$. For two polynomials $f, g \in R$, define the equivalence relation

$$f \sim g \ iff \ f - g \in I \tag{H.2}$$

Then, for each $f \in R$, we define the equivalence class $[f]$ as the set of all polynomials $g \in R$ such that $g \sim f$. This allows us to define the quotient ring

$$R/I = \{[f] | f \in R\} \tag{H.3}$$

An standard result on ring theory is that the multiplication on the ring R/I is given by $[f_1][f_2] = [f_1 f_2]$.

Let $\mathbb{K}$ be a field and suppose $\mathbb{F} \subset \mathbb{K}$. The n-dimensional $\mathbb{K}$-affine space $A^n_{\mathbb{K}}$ is the set of all n-tuples composed of elements of $\mathbb{K}$. Given $S \subset \mathbb{F}[z_1, ..., z_n]$, we define its algebraic set over $\mathbb{K}$ as

$$Z_{\mathbb{K}}(S) \equiv \{p \in A^n_{\mathbb{K}} | f(p) = 0 \ \forall f \in S\} \tag{H.4}$$

That is, the algebraic set of a subset S of the polynomial ring $\mathbb{F}[z_1, ..., z_n]$ over a field $\mathbb{K}$ is the set of simultaneous zeroes of the elements of S over the field in question. This definition allows us to formulate one of the most important theorems in algebraic geometry, known as the weak nullstellensatz: if $I \subset \mathbb{F}[z_1, ..., z_n]$ is an ideal and $\mathbb{K}$ is an algebraically closed field (which means that at least one root of the polynomial equation $P(x) = 0$ for $P \in \mathbb{K}[x]$ is an element of $\mathbb{K}$), such that $\mathbb{F} \subset \mathbb{K}$, then $Z_{\mathbb{K}}(I) = \emptyset$ implies that $I = \langle I \rangle = \mathbb{F}[z_1, ..., z_n]$.

The weak nulstellensatz is generalized to the so-called Hilbert nullstelensatz. Let $\mathbb{F}$ is an algebraically closed field, $R = \mathbb{F}[z_1, ..., z_n]$ and let $I \subset R$ be an ideal. Then, if $f \in R$ is a polynomial such that $f(p) = 0$ for every point $p \in Z_{\mathbb{F}}(I)$, there exists a positive integer k sucha that $f^k \in I$. Put simply, the Hilbert nullstelensatz provides a way to characterize all polynomials vanishing on the algebraic set of I over $\mathbb{F}$.

One special kind of generating set of an ideal is the so-called Gröbner basis. To define them, we first need to introduce the concepts of monomial orderings.

For any polynomial f, define $LT(f)$ to be the highest degree monomial in f along with its corresponding coefficient. Now, let M be the set of all monomials with coefficient 1 in the ring $R = \mathbb{F}[z_1, ..., z_n]$. A monomial ordering $\prec$ in a ring R is an ordering on M such that

- $\prec$ is a total ordering; that is, if $m_1, m_2 \in M$, then either $m_1 \prec m_2$ or $m_2 \prec m_1$, unless $m_1 = m_2$,
- Given $w \in M$, if $u \prec v$ then $uw \prec vw$,
- $1 \prec vu$ if $u \in M$ and u is not a constant.

There is no unique way to construct a monomial ordering. One possible one is to take $1 \prec z_n \prec z_{n-1} \prec ... \prec z_1$. Then, given two monomials

$$g_1 = z_1^{\alpha_1}...z_n^{\alpha_n}, \quad g_2 = z_1^{\beta_1}...z_n^{\beta_n} \tag{H.5}$$

we can define the following monomial orderings:

- Lexicographic ordering: $g_1 \prec g_2$ if $\alpha_i < \beta_i$ for some $i \in \{1, 2, ..., n\}$ and $\alpha_j = \beta_j$ for $j \in \{1, ..., i-1\}$,
- Degree lexicographic: $g_1 \prec g_2$ if $\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i$. In the case both sums are equal, the "tie" is decided by applying lexicographic ordering,
- Degree reversed lexicographic: $g_1 \prec g_2$ if $\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i$. If the sums are equal, one compares α_n and β_n. If $\alpha_n < \beta_n$ then $g_2 \prec g_1$. If $\alpha_n = \beta_n$, repeat the process with $\alpha_{n-1}, \beta_{n-1}$, and so on.

Having defined the different monomial orderings, we can define a Gröbner basis of an ideal $I \subset \mathbb{F}[z_1, ..., z_n]$ with a given monomial ordering as the set $G(I) = \{g_1, ..., g_m\}$ such that $I = \langle G \rangle$ and with the property that, for each $f \in I$, there exists $g_i \in G(I)$ such that the leading term of g_i divides the leading term of f, i.e. $LT(g_i)|LT(f)$. Gröbner bases allow to solve non-linear polynomial system of equations, and can be understood as a generalization of Gaussian elimination. Also, the allow the computation of polynomial greatest common divisors in multiple variables, which is the reason of their usefulness in the context of integrand reduction.

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