Relativistic Wave Equation for Radiant Electron

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Abstract

In this article a relativistic wave equation for the accelerated radiant electron, independent of Dirac formalism, based on the concept of radiation of classical electrodynamics is proposed. This work describes the solution for the free electron case, the electromagnetic potentials, the invariance under Lorentz transformations, and an application to the atomic model and the conductor media.

Keywords:

Relativistic Wave Equation, Radiant Electron, Accelerated Electron.

1 Introduction

Relativistic wave equations describe the movement of high energies particles that travel at speeds close to light. The first relativistic wave equation that appeared on the scene was formulated by Klein-Gordon (known also as Schoringer′s relativistic equation), which describes the behaviour of zero spin particles [22]. The second equation in this sense, also known as Dirac equation, describes the electron combining quantum mechanics and special relativity [4]. This equation predicts the existence of antielectrons (positrons) and its extension allows the development of quantum electrodynamics (see [24]). However, it does not describe to electron completely omitting some characteristics as the electron radius (denominated also Lorentz radius) and the half-life, despite being concepts of the classical electrodynamics. From this point of view, Larmor and Lienard Wiechert in [2, 3] shown that charged particles (in particular, electrons) when they are accelerated emit radiation. Shortly after, Dirac postulated a quantum theory of the radiation relating the interaction between the electron and the radiation from the phenomenological perspective without taking into account the half-life and the electron radius [6]. The main objective of this work is studying the following relativistic wave equation, taking as starting point the classical theory of the electron [7], defined by

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2\hbar u_e}{3r_e}\boldsymbol{f}_l\cdot(\boldsymbol{S}-\boldsymbol{\beta})\right]\psi_0(\boldsymbol{x},t), \tag{1.1}$$

where $\boldsymbol{S}$ is the associated vector to the spin angular momentum (see [13]), $\boldsymbol{\beta} := \frac{\boldsymbol{v}}{c}$ being $\boldsymbol{v}$ the electron velocity and c the speed of light, $\boldsymbol{f}_l$ stands for the Lorentz force defined as $\boldsymbol{f}_l = e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$ (see [1]), ψ is a spinor associated with states of positive and negative energy with positive and negative parity and $\boldsymbol{a}_0$ is a vector, which is defined in (2.5), and depends

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on the electron radius, half-life and Lorentz force (see [12, 20]). Here, ψ_0 is a spinor obtained by applying the density operator of associated current to energy and parity change $\hat{\jmath}_e$, defined in (2.6), on ψ. This operator, transforms the spinor componentes from ψ to ψ_0 and describes the transition from positive to negative energy, from negative to positive energy and its parity change. Spinors ψ and ψ_0 are defined by the following way:

$$\psi(\boldsymbol{x},t)=\begin{bmatrix}\psi_+^{(+)}(\boldsymbol{x},t)\\ \psi_-^{(+)}(\boldsymbol{x},t)\\ \psi_+^{(-)}(\boldsymbol{x},t)\\ \psi_-^{(-)}(\boldsymbol{x},t)\end{bmatrix} \quad \text{and} \quad \psi_0(\boldsymbol{x},t)=\begin{bmatrix}\psi_-^{(-)}(\boldsymbol{x},t)\\ -\psi_+^{(-)}(\boldsymbol{x},t)\\ -\psi_-^{(+)}(\boldsymbol{x},t)\\ \psi_+^{(+)}(\boldsymbol{x},t)\end{bmatrix},$$

in which the sign of the superscript indicates the state of the energy and the sign of the subscript indicates the parity change. Below we will briefly comment on some results about relativistic wave equations found in the literature during the last years. Walker *et al.* propose, that given the solution of the Weyl equation for the neutrino, more solutions of this equation can be generated by applying a differential linear operator. These operators are known as symmetry operators and can be applied to all relativistic wave equations with spin $s = 1/2$ [17]. In 2001, Niederle and Nikitin [23], formulate new relativistic wave equations for massive particles with arbitary spin, which interact with the external electromagnetic field, based on the wave functions, which are irreducible tensors of the $2n(n = s - 1/2)$. In 2017, Marsch [16] proposes a second-order relativistic wave equation for massive particles loaded with arbitrary spin. In that same year, Simulik [21] proposes in his work, a relativistic wave equation of arbitrary spin in quantum mechanics and field theory, taking as an example $s = 2$.

This work is organized as follows. In Section 2 the relativistic wave equation for the radiant electron is building from radiation concept of the classical electrodynamics. In Section 3, it is shown the solution for the free electron and its relation with the density operator $\hat{\jmath}_e$, relativistic wave equations in terms of Lienard-Wiechert retarded potentials are exhibited and Lorentz invariance of the relativistic wave equation using the Lorentz transformation is proved (see [2, 3]). In Section 4 the radiant electron in the atomic model and conductor medium is described. Finally, in Section 5, a brief analysis on the obtained results is made, and a description of the behavior of the electron in a conductor medium, coupling the electromagnetic fields with the Poynting vector.

2 Deduction of the relativistic wave equation for the radiant electron

In this part, it is considered an accelerated and radiant electron defined by means of the relativistic energy in the following system of equations

$$E\psi_+^{(+)} = \left[c|\boldsymbol{p}| + m_e c^2 - \boldsymbol{a}_0\cdot(\boldsymbol{S}-\boldsymbol{\beta})\right]\psi_+^{(-)}, \tag{2.1}$$

$$E\psi_-^{(+)} = -\left[c|\boldsymbol{p}| + m_e c^2 - \boldsymbol{a}_0\cdot(\boldsymbol{S}-\boldsymbol{\beta})\right]\psi_-^{(-)}, \tag{2.2}$$

$$E\psi_-^{(-)} = -\left[c|\boldsymbol{p}| + m_e c^2 - \boldsymbol{a}_0\cdot(\boldsymbol{S}-\boldsymbol{\beta})\right]\psi_-^{(+)}, \tag{2.3}$$

$$E\psi_+^{(-)} = \left[c|\boldsymbol{p}| + m_e c^2 - \boldsymbol{a}_0\cdot(\boldsymbol{S}-\boldsymbol{\beta})\right]\psi_+^{(+)}, \tag{2.4}$$

To follow will be defined the vector $\boldsymbol{a}_0$ in terms of Lorentz force. For that, it is used the following formula of the radiant electron power, relating the Lorentz force and the vector $\boldsymbol{a}_0$:

$$P = \frac{r_e^2}{m_e c\hbar u_e}\boldsymbol{a}_0\cdot\boldsymbol{f}_l,$$

where u_e is a constant characteristic, whose value is obtained by calculating $c^2 t_e/E_0 \approx 2,9 \times 10^{50} fm^2/MeV \cdot s$ and r_e stands for the Lorentz radius, which was obtained at the non-relativistic limit as can be seen in [12, 20]. Here, the energy of rest E_0 corresponds to 0.511 MeV and which was introduced for dimensional reasons. Since it is considered an accelerated electron, the above expression is equated to Larmor's formula [3]

$$\frac{r_e^2}{m_e c\hbar u_e}\boldsymbol{a}_0 \cdot \boldsymbol{f}_l = \frac{2}{3}\frac{r_e}{m_e c}\boldsymbol{f}_l \cdot \boldsymbol{f}_l,$$

Therefore,

$$\boldsymbol{a}_0 = \frac{2\hbar u_e}{3r_e}\boldsymbol{f}_l.$$

and replace in the equations (2.1) - (2.4). The equations (2.1) - (2.4) show a transition from positive and negative energy and of parity change. This transition can be represented by defining an operator that acts on each component of the spinor $\psi(\boldsymbol{x}, t)$, denoted by $\hat{\jmath}_e$, satisfying the following properties:

1. $det(\hat{\jmath}_e) = 1$,
2. $\hat{\jmath}_e \hat{\jmath}_e^\dagger = \mathbb{I}_4$, where $\mathbb{I}_4$ is a identity matrix of size 4.
3. The operator $\hat{\jmath}_e$ associates each component of $\psi(\boldsymbol{x}, t)$ with each component of $\psi_0(\boldsymbol{x}, t)$, respectively.

Using the above properties, it follows that the matrix form of the operator $\hat{\jmath}_e$ is given by:

$$\hat{\jmath}_e \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Thus, it is possible to rewrite (1.1) as

$$E\begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix} = \left[c|\boldsymbol{p}| + m_e c^2 - \frac{2\hbar u_e}{3r_e}\boldsymbol{f}_l \cdot (\boldsymbol{S} - \boldsymbol{\beta})\right] \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix} = \left[c|\boldsymbol{p}| + m_e c^2 - \frac{2\hbar u_e}{3r_e}\boldsymbol{f}_l \cdot (\boldsymbol{S} - \boldsymbol{\beta})\right]\hat{\jmath}_e\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} = c\hat{\boldsymbol{\jmath}}_e \cdot \boldsymbol{p}\psi + \left[mec^2 - \frac{2\hbar u_e}{3r_e}\boldsymbol{f}_l \cdot (\boldsymbol{S} - \boldsymbol{\beta})\right]\hat{\jmath}_e\psi,$$

Similarly, it is possible to rewrite the term $\hat{\jmath}_e\psi$ as ψ_0. Then, (1.1) becomes:

$$i\hbar\frac{\partial \psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2\hbar u_e}{3r_e}(\boldsymbol{f}_l \cdot \boldsymbol{S}) - \frac{2\hbar u_e}{3r_e}(\boldsymbol{f}_l \cdot \boldsymbol{\beta})\right]\hat{\jmath}_e\psi. \tag{2.6}$$

The following bold notation will be used to define the following vectors with repeated components, that is,

$$\boldsymbol{\psi}_0 := (\psi_0, \psi_0, \psi_0) \quad \text{and} \quad \hat{\boldsymbol{j}}_{\boldsymbol{e}} := \left(\hat{j}_e, \hat{j}_e, \hat{j}_e\right).$$

Applying (2.6) in the spinor ψ, it follows that

$$\hat{j}_e \psi(\boldsymbol{x}, t) = \psi_0(x, t), \tag{2.7}$$

where $\boldsymbol{x} := (x, y, z)$. On the other hand, let $\boldsymbol{p}$ be the momentum of electron with components (p_x, p_y, p_z). Then, from (2.7), it holds that

$$\begin{aligned}(p_x, p_y, p_z) \cdot \boldsymbol{\psi}_0 &= p_x\psi_0 + p_y\psi_0 + p_z\psi_0 \\ &= p_x\hat{j}_e\psi + p_y\hat{j}_e\psi + p_z\hat{j}_e\psi \\ &= \hat{j}_e p_x\psi + \hat{j}_e p_y\psi + \hat{j}_e p_z\psi \\ &= \boldsymbol{j_e} \cdot \boldsymbol{p}\psi,\end{aligned}$$

which implies

$$\hat{\boldsymbol{j}}_e \cdot \boldsymbol{p}\psi(\boldsymbol{x}, t) = \boldsymbol{p} \cdot \boldsymbol{\psi}_0.$$

One of the characteristics to take into account the electron in the atomic model are the magnetic and electric dipole moments. From (2.6) two equations were obtained which describe the radiant electron in terms of the electric and magnetic dipole moments $\boldsymbol{\mu} = \frac{e\hbar}{2m_e c}\boldsymbol{S}$ and $\boldsymbol{d} = \frac{f' e\hbar}{m_e c}\boldsymbol{S}$, defined in [13, 18],

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{4}{3}\frac{m_e u_e c}{e r_e}(\boldsymbol{f}_l \cdot \boldsymbol{\mu}) - \frac{2}{3}\frac{\hbar u_e}{r_e}(\boldsymbol{\beta} \cdot \boldsymbol{f}_l)\right]\hat{j}_e\psi,$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2}{3}\frac{m_e c u_e}{e f' r_e}(\boldsymbol{f}_l \cdot \boldsymbol{d}) - \frac{2}{3}\frac{\hbar u_e}{r_e}(\boldsymbol{\beta} \cdot \boldsymbol{f}_l)\right]\hat{j}_e\psi.$$

By separately defining the Lorentz force for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the following four equations were obtained (with charge density defined as $\rho = \frac{e}{V}$ but when evaluating at $V = 1fm^3$, is obtained $\rho = e$):

1. **Magnetic field - Magnetic moment :**

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{4}{3}\frac{m_e c u_e}{e r_e}(\boldsymbol{j} \times \boldsymbol{B} \cdot \boldsymbol{\mu})\right]\hat{j}_e\psi. \tag{2.8}$$

2. **Electric Field - Electric dipole moment :**

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2}{3}\frac{m_e c u_e}{f' r_e}(\boldsymbol{E} \cdot \boldsymbol{d}) - \frac{2}{3}\frac{\hbar u_e}{r_e c}(\boldsymbol{E} \cdot \boldsymbol{j})\right]\hat{j}_e\psi. \tag{2.9}$$

3. **Magnetic field - Electric dipole moment :**

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2}{3}\frac{m_e c u_e}{e f' r_e}(\boldsymbol{j} \times \boldsymbol{B} \cdot \boldsymbol{d})\right]\hat{j}_e\psi. \tag{2.10}$$

4. **Electric field - Magnetic moment :**

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p} \cdot \boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{4}{3}\frac{m_e c u_e}{r_e}(\boldsymbol{E} \cdot \boldsymbol{\mu}) - \frac{2}{3}\frac{\hbar u_e}{r_e c}(\boldsymbol{E} \cdot \boldsymbol{j})\right]\hat{j}_e\psi. \tag{2.11}$$

These are the relativistic wave equations that describe the radiant electron in the electric and magnetic cases coupled with their respective electric dipole and magnetic moments, being f' the analogous factor to the Lande gyromagnetic factor g.

3 Free electron, Electromagnetic Potentials and Invariance under Lorentz Transformations

In this section, we analyse the solution for a free electron, taking the force of Lorentz $\boldsymbol{f}_l = \boldsymbol{0}$. Then, we defined the relativistic wave equations for an electron in the presence of the Lienard - Wiechert retarded potentials, using the minimal substitutions for momentum $c\boldsymbol{p} \longrightarrow c\boldsymbol{p} - e\boldsymbol{A}$, and energy $E \longrightarrow E - e\phi$, and finally we will show their invariance under Lorentz transformations. For a free electron (that is, absence of electromagnetic fields), the equation (2.6) is defined as

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + m_e c^2 \hat{j}_e \psi \tag{3.1}$$

d - Wiechertand, in a similar way as was stated in [11], it is proposed to consider solutions of the following form:

$$\psi(x) = \begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix} e^{\pm ip\cdot x}. \tag{3.2}$$

Applying $\hat{j}_e$ in (3.2), the following relations were obtained for each spinor component (the complete derivation can be seen in Appendix A);

$$\hat{j}_e\psi_+^{(+)} = -\psi_+^{(-)\dagger}, \tag{3.3}$$

$$\hat{j}_e\psi_-^{(+)} = \psi_-^{(-)\dagger}, \tag{3.4}$$

$$\hat{j}_e\psi_-^{(-)} = -\psi_-^{(+)\dagger}, \tag{3.5}$$

$$\hat{j}_e\psi_+^{(-)} = \psi_+^{(+)\dagger}. \tag{3.6}$$

According to the above, we can say that equations (3.3) - (3.6) describe the transition from positive to negative energy states and from negative to positive without parity change. In the previous section, the motion equations for a radiant electron in the presence of electromagnetic fields were considered without coupling the of vectorial and scalar potentials, respectively. Let $\boldsymbol{A}$ be a vector potential and ϕ a scalar potential in the same way as it was done [5, 13, 14]. Using the following modifications

$$c\boldsymbol{p} \longrightarrow c\boldsymbol{p} - e\boldsymbol{A} \quad \text{and} \quad E \longrightarrow E - e\phi,$$

and from (2.6), it holds

$$\begin{aligned}\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)\psi =& c\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\cdot\boldsymbol{\psi}_0 \\ &+ \left[m_e c^2 - \frac{2\hbar u_e}{3r_e}(\boldsymbol{f}_l(\boldsymbol{A},\phi)\cdot\boldsymbol{S}) - \frac{2\hbar u_e}{3r_e\rho c}(\boldsymbol{f}_l(\boldsymbol{A},\phi)\cdot\boldsymbol{J})\right]\hat{j}_e\psi.\end{aligned}$$

This last implies that the electromagnetic fields in the Lorentz force must be expressed in terms of potentials $\boldsymbol{A}$ and ϕ (see [14]). As an accelerated electron is being described, Barut in [13] considered from the point of view of relativistic kinematics an electron that moves in a proper time τ in a world line. This consideration created a delayed electromagnetic field produced at a time τ_1 from Lienard - Wiechert potentials. Therefore, the wave equations in the Lienard -

Wiechert delayed potentials are

$$\left(i\hbar\frac{\partial}{\partial t_{ret}} - \frac{e^2}{4\pi R}\frac{1}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}|_{\tau=\tau_0}\right)\psi = c\left(\boldsymbol{p}_R - \frac{e^2}{4\pi cR}\frac{\boldsymbol{\beta}}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}|_{\tau=\tau_0}\right)\cdot\boldsymbol{\psi}_0$$
$$+\left[m_ec^2 - \frac{4m_ecu_e}{er_e}\left\{\frac{e^2}{4\pi R^2}\frac{1}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}[(1-\beta^2)(\hat{\boldsymbol{n}}-\boldsymbol{\beta}) - \frac{\boldsymbol{R}}{c}\times[(\hat{\boldsymbol{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]]\cdot\boldsymbol{\mu}\right\}\right]\hat{j}_e\psi$$
$$-\frac{2\hbar u_e}{r_e}\left\{\frac{e^2}{4\pi R^2}\frac{1}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}[(1-\beta^2)(\hat{\boldsymbol{n}}-\boldsymbol{\beta}) - \frac{\boldsymbol{R}}{c}\times[(\hat{\boldsymbol{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]]\cdot\boldsymbol{\beta}\right\}\hat{j}_e\psi, \quad (3.7)$$

$$\left(i\hbar\frac{\partial}{\partial t_{ret}} - \frac{e^2}{4\pi R}\frac{1}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}|_{\tau=\tau_1}\right)\psi = c\left(\boldsymbol{p}_R - \frac{e^2}{4\pi cR}\frac{\boldsymbol{\beta}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}|_{\tau=\tau_1}\right)\cdot\boldsymbol{\psi}_0$$
$$+\left[m_ec^2 + \frac{4m_ecu_e}{er_e}\left\{\boldsymbol{j}_{ret}\times\frac{e}{4\pi Rc}\frac{1}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}\dot{\boldsymbol{\beta}}\cdot\boldsymbol{\mu}\right\}\right]\hat{j}_e\psi, \quad (3.8)$$

where $t_{ret} = t - \frac{|x^0 - y^0(\tau_0)|}{c}, \boldsymbol{R} = \boldsymbol{x} - \boldsymbol{y}(\tau_0), \hat{\boldsymbol{n}} = \frac{\boldsymbol{R}}{|\boldsymbol{R}|}$, $R := |\boldsymbol{R}| = ct - ct_{ret}$ and $\boldsymbol{j}_{ret}$ the retarded current density. The function $\psi = \psi(R, t_{ret})$ satisfying (3.7) and (3.8) is called the *retarded wave function.*

On the other hand, it will be show that (2.6) is invariant under Lorentz transformations. For that, it is used that $\hbar = c = 1$ and

$$i\partial_\mu\psi^\mu + \left(\frac{2u_e}{3r_e}f_\mu S^\mu - m_e + \frac{2u_e}{3r_e}f_\mu v^\mu\right)\psi_0 = 0, \quad (3.9)$$

where $\psi^\mu = (\psi, \boldsymbol{\psi}_0)$. Initially, it is considered an electron within a reference system Σ' with x' -coordinates and described by the wave function ψ'. Then, moving from Σ' to another reference system Σ with x -coordinates, the following transformations take place:

$$\psi^{\mu\prime} = \Lambda^\mu_\alpha\psi^\alpha, \quad \partial'_\mu = \Lambda^\beta_\mu\partial_\beta, \quad v'^\mu = \Lambda^\mu_\beta v^\beta, \quad (3.10)$$
$$f'_\mu = \Lambda^\alpha_\mu f_\alpha, \quad S'^\mu = \Lambda^\mu_\beta S^\beta, \quad \psi'_0(x') = \psi_0(\Lambda x). \quad (3.11)$$

Replacing the transformations (3.10)-(3.11) in (3.9), it is obtained

$$i\partial'_\mu\psi'^\mu + \left(\frac{2u_e}{3r_e}f'_\mu S'^\mu - m_e + \frac{2u_e}{3r_e}f'_\mu v'^\mu\right)\psi_0(\Lambda x) = 0,$$
$$i\Lambda^\mu_\alpha\Lambda^\beta_\mu\partial_\beta\psi^\alpha + \left(\frac{2u_e}{3r_e}\Lambda^\alpha_\mu f_\alpha\Lambda^\beta_\mu S^\beta - m_e + \frac{2u_e}{3r_e}\Lambda^\alpha_\mu f_\alpha\Lambda^\mu_\beta v^\beta\right)\psi_0(\Lambda x) = 0,$$
$$i\Lambda^\mu_\alpha\Lambda^\beta_\mu\partial_\beta\psi^\alpha + \left(\frac{2u_e}{3r_e}\Lambda^\alpha_\mu\Lambda^\mu_\beta f_\alpha S^\beta - m_e + \frac{2u_e}{3r_e}\Lambda^\alpha_\mu\Lambda^\mu_\beta f_\alpha v^\beta\right)\psi_0(\Lambda x) = 0.$$

Taking into account the product property of Lorentz matrices $\Lambda^\alpha_\mu\Lambda^\mu_\beta = \delta^\alpha_\beta, \Lambda^\beta_\mu\Lambda^\mu_\alpha = \delta^\beta_\alpha$, the invariance under Lorentz transformations was actually demonstrated by using the following formalism:

$$i\delta^\beta_\alpha\partial_\beta\psi^\alpha + \left(\frac{2u_e}{3r_e}\delta^\alpha_\beta f_\alpha S^\beta - m_e + \frac{2u_e}{3r_e}\delta^\alpha_\beta f_\alpha v^\beta\right)\psi_0(\Lambda x) = 0,$$
$$i\partial_\alpha\psi^\alpha + \left(\frac{2u_e}{3r_e}f_\beta S^\beta - m_e + \frac{2u_e}{3r_e}f_\beta v^\beta\right)\psi_0(\Lambda x) = 0.$$

4 Energy levels of the accelerated electron in the atomic model

In this section the extension to the atomic model and an application to the conductor media was considered. The coupling of the electric field with the magnetic moment in terms of the quantum number associated with the angular orbital momentum l and the Bohr magnetron μ_B, was defined. The eigenvalues for the operator $\hat{j}_e$ were calculated using the coupling with the operator associated with the orbital angular moment $\boldsymbol{L}$, with the objective to obtain the energy of the electron for the degenerated states, denoted by $E_{nlm;j}$. The case of a relativistic accelerated electron in an atom rotating around the nucleus was considered, defining the coupling of the electric field $\boldsymbol{E}$ and its magnetic momentum $\boldsymbol{\mu}$ of (2.11), defined by

$$\boldsymbol{E}\cdot\boldsymbol{\mu} = -\frac{4m_e c u_e Z k_C e}{3r_e r^2}\mu_B\sqrt{l(l+1)}.$$

Here, μ_B is the Bohr magneton defined in [13, 18], k_C is Coulomb constant and Z is atomic number. The coupling of the electric field and current density $\boldsymbol{E}\cdot\boldsymbol{j}$ (for electrons travelling at a velocity $v \approx c$), is defined as

$$\boldsymbol{E}\cdot\boldsymbol{j} = \frac{2\hbar u_e Z k_C e^2}{3r^2 r_e}.$$

From above and (2.11), it is getting the following relativistic wave equation for the accelerated electron in the atomic model

$$i\hbar\frac{\partial\psi}{\partial t} = c\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p}\psi + \left[m_e c^2 + \frac{4}{3}\frac{e^2 m_e c u_e Z k_C}{r^2 r_e}\mu_B\sqrt{l(l+1)} + \frac{2}{3}\frac{Z k_C \hbar u_e e^2}{r_e r^2}\right]\hat{j}_e\psi. \tag{4.1}$$

Doing β as

$$\beta = \frac{4}{3}\frac{e^2 m_e c u_e Z k_C}{r_e}\mu_B\sqrt{l(l+1)} + \frac{2}{3}\frac{Z k_C \hbar u_e e^2}{r_e},$$

(4.1) is expressed by

$$i\hbar\frac{\partial\psi}{\partial t} = c\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p}\psi + \left[m_e c^2 + \frac{\beta}{r^2}\right]\hat{j}_e\psi. \tag{4.2}$$

Factor $\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p}$ was analysed using the same procedure presented in [4] using the following equation

$$(\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p})(\hat{\boldsymbol{j}}_e\cdot\boldsymbol{r}) = \boldsymbol{r}\cdot\boldsymbol{p} + i\hat{\boldsymbol{j}}_e\cdot\boldsymbol{L}. \tag{4.3}$$

Replacing $\hat{\boldsymbol{j}}_e\cdot\boldsymbol{r} = r\hat{j}_e$ in (4.3), it holds

$$\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p} = \frac{\hat{j}_e}{r}(\boldsymbol{r}\cdot\boldsymbol{p} + i\hat{\boldsymbol{j}}_e\cdot\boldsymbol{L}). \tag{4.4}$$

Since there exists an operator K defined by $\hbar K = \hat{\boldsymbol{j}}_e\cdot\boldsymbol{L} + \hbar$ (see [5]), and denoting

$$p_r := \frac{1}{r}(\boldsymbol{r}\cdot\boldsymbol{p} - i\hbar) = \frac{\boldsymbol{r}\cdot\boldsymbol{p}}{r} - \frac{i\hbar}{r},$$

from (4.4), it follows that

$$\hat{\boldsymbol{j}}_e\cdot\boldsymbol{p} = \hat{j}_e p_r + \frac{i\hbar K\hat{j}_e}{r}.$$

Replacing the above equality in (4.2), it is obtained the relativistic wave equation for the radiant electron in the atomic model

$$i\hbar\frac{\partial\psi}{\partial t} = c j_e p_r\psi + \frac{i\hbar c K j_e}{r}\psi + \left[m_e c^2 + \frac{\beta}{r^2}\right]\hat{j}_e\psi, \tag{4.5}$$

Moreover, the eigen-values of operator K can be obtained from $\hbar K = \hat{j}_e \cdot \boldsymbol{L} + \hbar$ (see Appendix C), which are defined by

$$k = \pm\sqrt{\left(j+\frac{1}{2}\right)^2 + \frac{1}{2} \pm \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}}}.$$

Now, in order to determine the relativistic energy of the electron associated with the degenerated states, the equation (4.5) is taken at each spinor component ψ

$$E\psi_+^{(+)} = \left(-i\hbar c\frac{\partial}{\partial r} + \frac{i\hbar ck}{r} + m_e c^2 + \frac{\beta}{r^2}\right)\psi_-^{(-)}, \tag{4.6}$$

$$E\psi_-^{(+)} = -\left(-i\hbar c\frac{\partial}{\partial r} + \frac{i\hbar ck}{r} + m_e c^2 + \frac{\beta}{r^2}\right)\psi_+^{(-)}, \tag{4.7}$$

$$E\psi_+^{(-)} = -\left(-i\hbar c\frac{\partial}{\partial r} + \frac{i\hbar ck}{r} + m_e c^2 + \frac{\beta}{r^2}\right)\psi_-^{(+)}, \tag{4.8}$$

$$E\psi_-^{(-)} = \left(-i\hbar c\frac{\partial}{\partial r} + \frac{i\hbar ck}{r} + m_e c^2 + \frac{\beta}{r^2}\right)\psi_+^{(+)}. \tag{4.9}$$

Thus, the following solutions are proposed for each component of the spinor:

$$\psi_+^{(+)} = i\sum_{m=0}^{\infty} a_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0}, \tag{4.10}$$

$$\psi_-^{(-)} = -i\sum_{m=0}^{\infty} b_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0}, \tag{4.11}$$

$$\psi_+^{(-)} = i\sum_{m=0}^{\infty} c_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0}, \tag{4.12}$$

$$\psi_-^{(+)} = -i\sum_{m=0}^{\infty} d_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0}, \tag{4.13}$$

where r_0 is Bohr radius [18], Then, the final energy expression is (see Appendix C)

$$E \equiv E_{nlm;j} = \pm\left[m_e c^2 + \frac{(m_e c)^2\alpha^2}{\hbar^2 n^4 (m \pm k)^2}\left(\frac{4}{3}\frac{e^2 m_e c u_e Z k_C}{r_e}\mu_B\sqrt{l(l+1)} + \frac{2}{3}\frac{Z k_C \hbar u_e e^2}{r_e}\right)\right]. \tag{4.14}$$

Considering the particular case of the initial state $n = 1, l = 0, m = 0, k = \sqrt{3/2}$, E is approximately 10^{46} eV and it was obtained with respect to the energy of the initial state. For the radiant electron case of an atom in a conductor medium with electrical conductivity σ , the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields coupled with the magnetic momentum $\boldsymbol{\mu}$ associated were given as

$$\boldsymbol{E}(r) = \frac{k_C Z e}{r^3}\hat{\boldsymbol{r}}, \tag{4.15}$$

$$\boldsymbol{B}(r) = \frac{Ze}{m_e c^2 r^3}\boldsymbol{L}, \tag{4.16}$$

$$\boldsymbol{\mu} = -(\boldsymbol{\mu})_L\sqrt{l(l+1)}. \tag{4.17}$$

Replacing (4.15) - (4.17) in the $\boldsymbol{j} \times \boldsymbol{B} \cdot \boldsymbol{\mu}$ factor of (2.9), it is obtained

$$|\boldsymbol{j} \times \boldsymbol{B} \cdot \boldsymbol{\mu}| = |\sigma\boldsymbol{E} \times \boldsymbol{B}||\boldsymbol{\mu}| = \frac{\sigma k_C Z^2 e^2 \mu_L m\hbar}{r^5 m_e c^2}\sqrt{l(l+1)}.$$

Here, it is used the Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{E}$ defined by Jackson [14] for perpendicular fields. Therefore, (2.9) can be rewrite as

$$i\hbar \frac{\partial \psi}{\partial t} = c j_e p_r \psi + \frac{i\hbar c K j_e}{r} \psi + \left[m_e c^2 - \frac{4}{3} \frac{u_e}{\rho r_e} \frac{e \sigma k_C Z^2 \hbar^2 \alpha \mu_L m}{r^5} \sqrt{l(l+1)} \right] \hat{j}_e \psi,$$

where ρ is the charge density of the conductor medium y α is the fine structure constant. Thus, the relativistic energy of the radiant electron in a conductor medium is:

$$E_{nlm;j} = \pm \left(m_e c^2 + \frac{4}{3} \frac{u_e}{\rho r_e} \frac{e \sigma k_C Z^2 \hbar^2 \alpha \mu_L m}{(r_0 (m \pm k))^5} \sqrt{l(l+1)} \right). \tag{4.18}$$

5 Final comments and possible applications

This work describes in detail an accelerated electron with the data found in classical electrodynamics, such as the Lorentz radius[1], half life and energy at rest. In Section 2, a relativistic wave equation for the radiant electron is formulated, in which a vector denoted by $\boldsymbol{a}_0$ is introduced in function of Lorentz force $\boldsymbol{f}_l$, which will be referred as the quantum potential vector for accelerated electrons. This vector was defined using Larmor's work [3] in order to describe in detail the behavior of quantum accelerated electrons in the presence of electromagnetic fields by means of equations (2.8) - (2.11). Here it is used a new operator $\hat{j}_e$ defined from the solutions for a free electron whose physical interpretation corresponds to the Density of electric current associated with the energy and the electron parity. On the other hand, from equations (2.9) and (2.10) there is a constant f' with unknown value, which could be determined from the electrical dipole moment (see [13]). In 2013, through the experiment with Torio monoxide molecules ThO led by *ACME collaboration*, an approximate value of the electric dipole moment was obtained, taking into account that the electrons of the experiment travel relativistically. In the same section, the equations for a radiant electron with the electromagnetic potentials were given. However, the potentials of Lienard-Wiechert were taken assuming that an electron travels in a world line, in an advanced proper time τ_1 and retarded τ_0, to obtain the equations (3.7) - (3.8). It is important to mention that electromagnetic fields are not the advanced and delayed fields that Jefimenko denoted in his work (see [9]). One application of equations (3.7) - (3.8), especially (3.8), is the description of the experiment made in 2015 on the emission of radio waves by an electron [10].

In Section 4, the case of a radiant electron is considered in the atomic model, taking equation (2.11) as a starting point. Modifying the coupling of $\boldsymbol{E} \cdot \boldsymbol{\mu}$ in terms of Bohr magneton gives an expression of the relativistic energy of the accelerated electron in the energetic levels associated with degenerate states described in equation (4.14). Using this formula, the energy was calculated for the ground level (i.e., for $n = 1, l = 0$ and $m = 0$) of a hydrogen atom, and it was found that its value exceeds the energy of the ground state of the same by a factor of 10^{46} without to be obtained experimentally. Some parameters such as the fine-structure constant in second order, the constant u_e, the electrostatic constant k_C and the electron radius were taken into consideration here. From the above, an interesting topic of research would be to obtain the energy of the ground state experimentally.

A particular case of research of relevant importance are the equations below that describe

[1] although this was determined without taking into account quantum effects

the behavior of accelerated electrons of an atom in a conductive medium

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{4}{3}\frac{m_e c u_e}{\rho r_e}(\sigma\boldsymbol{E}\times\boldsymbol{B}\cdot\boldsymbol{\mu})\right]\hat{\jmath}_e\psi,$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2}{3}\frac{m_e c u_e}{f' r_e}(\boldsymbol{E}\cdot\boldsymbol{d}) - \frac{2}{3}\frac{\hbar u_e}{r_e c}(\sigma|\boldsymbol{E}|^2)\right]\hat{\jmath}_e\psi,$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{2}{3}\frac{m_e c u_e}{\rho f' r_e}(\sigma\boldsymbol{E}\times\boldsymbol{B}\cdot\boldsymbol{d})\right]\hat{\jmath}_e\psi,$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{4}{3}\frac{m_e c u_e}{r_e}(\boldsymbol{E}\cdot\boldsymbol{\mu}) - \frac{2}{3}\frac{\hbar u_e}{r_e c}(\sigma|\boldsymbol{E}|^2)\right]\hat{\jmath}_e\psi.$$

These equations were obtained using equations (2.8) - (2.11), Ohm's law for electromagnetic conductors $\boldsymbol{j} = \sigma\boldsymbol{E}$ (see [14]), and assuming that fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular.

For this case, the energy of an electron in terms of the electrical conductivity σ in the atomic model is calculated using the coupling of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with the magnetic moment $\boldsymbol{\mu}$ and the angular orbital moment $\boldsymbol{L}$. With this result, the energy for the ground state $n = 1$ and for $n = 2$ (which corresponds to the helium atom) is obtained, predicting a possible value of the electrical conductivity despite not having recorded data.

From the point of view of classical electrodynamics, when a charged particle accelerates, it emits an energy flux of the radiation described with the Poynting vector, denoted by $\boldsymbol{S}_0$ (see [19]). With this principle the following equations are given

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{1}{6}\frac{m_e c^2 u_e}{\pi\rho f' r_e}(\sigma\boldsymbol{S}_0\cdot\boldsymbol{d})\right]\hat{\jmath}_e\psi, \tag{5.1}$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\boldsymbol{p}\cdot\boldsymbol{\psi}_0 + \left[m_e c^2 - \frac{1}{3}\frac{m_e c^2 u_e}{\pi\rho r_e}(\sigma\boldsymbol{S}_0\cdot\boldsymbol{\mu})\right]\hat{\jmath}_e\psi. \tag{5.2}$$

The equations (5.1) and (5.2) describe the behavior of an electron in a conductor medium coupling the magnetic and electrical moments with the Poynting vector, despite the absence of experimental results on the behavior of the electrical dipole momentum in the atom. Theoretically, models of conductors can be built with potentials that depend on Poynting vector, the electric dipole moment $\boldsymbol{d}$ and the magnetic moment $\boldsymbol{\mu}$, together with the delayed and advanced fields $\boldsymbol{E}$ and $\boldsymbol{B}$. A quasi-classical electrodynamics for radiant electron can be formulated from Maxwell equations (in the quantum case) for the moments $\boldsymbol{\mu}$ and $\boldsymbol{d}$ as a function of fields $\boldsymbol{E}$ and $\boldsymbol{B}$. With respect to the description of the experiment of the electron that emits radio waves, the advanced case can be considered replacing the potentials of Lienard-Whiechert with the magnetic field and the current density using (2.8) (see [9]).

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7 Appendix A: Solution for Free Electron II

In this section we show the solution of the equation (3.1). For that it will be found each component of the matricial form (3.1)

$$\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & c|\boldsymbol{p}| + m_e c^2 \\ 0 & 0 & -c|\boldsymbol{p}| - m_e c^2 & 0 \\ 0 & -c|\boldsymbol{p}| - m_e c^2 & 0 & 0 \\ c|\boldsymbol{p}| + m_e c^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_+^{(+)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_+^{(-)} \end{bmatrix}.$$

Then, the following system of equations for the energy appears

$$\begin{aligned} E\psi_+^{(+)} &= (c|\boldsymbol{p}| + m_e c^2)\psi_+^{(-)}, \\ E\psi_-^{(+)} &= -(c|\boldsymbol{p}| + m_e c^2)\psi_-^{(-)}, \\ E\psi_-^{(-)} &= -(c|\boldsymbol{p}| + m_e c^2)\psi_-^{(+)}, \\ E\psi_+^{(-)} &= (c|\boldsymbol{p}| + m_e c^2)\psi_+^{(+)}. \end{aligned}$$

Proposing for each component its respective solution in a similar way as shown in Schiff's work [11], it is obtained

$$\begin{aligned} \psi_+^{(+)} &= \frac{E}{(c|\boldsymbol{p}| + m_e c^2)}\psi_+^{(-)}, \\ \psi_-^{(+)} &= \frac{E}{-(c|\boldsymbol{p}| + m_e c^2)}\psi_-^{(-)}, \\ \psi_-^{(-)} &= \frac{E}{-(c|\boldsymbol{p}| + m_e c^2)}\psi_-^{(+)}, \\ \psi_+^{(-)} &= \frac{E}{(c|\boldsymbol{p}| + m_e c^2)}\psi_+^{(+)}. \end{aligned}$$

Now replacing in (3.2), four linearly independent spinors were found

$$\psi(x) := \psi(\boldsymbol{x}, t) = \left\{ u_+^{(+)}(p)e^{ip\cdot x}, u_-^{(-)}(p)e^{-ip\cdot x}, u_+^{(-)}(p)e^{-ip\cdot x}, u_-^{(+)}(p)e^{ip\cdot x} \right\}$$

$$= \left\{ \begin{bmatrix} \psi_+^{(+)} \\ 0 \\ 0 \\ \frac{E}{c|\boldsymbol{p}|+m_e c^2}\psi_+^{(+)} \end{bmatrix} e^{ip\cdot x}, \begin{bmatrix} 0 \\ \frac{-E}{c|\boldsymbol{p}|+m_e c^2}\psi_-^{(-)} \\ \psi_-^{(-)} \\ 0 \end{bmatrix} e^{-ip\cdot x}, \begin{bmatrix} \frac{E}{c|\boldsymbol{p}|+m_e c^2}\psi_+^{(-)} \\ 0 \\ 0 \\ \psi_+^{(-)} \end{bmatrix} e^{-ip\cdot x}, \begin{bmatrix} 0 \\ \psi_-^{(+)} \\ \frac{-E}{c|\boldsymbol{p}|+m_e c^2}\psi_-^{(+)} \\ 0 \end{bmatrix} e^{ip\cdot x} \right\}.$$

$p \cdot x = \boldsymbol{p} \cdot \boldsymbol{x} - \omega t$. Applying the operator $\hat{j}_e$ on each spinor, it is arrived at

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{E}{c|\boldsymbol{p}|+m_e c^2} \end{bmatrix} \psi_+^{(+)} e^{ip\cdot x} = \begin{bmatrix} \frac{E}{c|\boldsymbol{p}|+m_e c^2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \psi_+^{(-)\dagger} e^{ip\cdot x},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \frac{E}{c|\boldsymbol{p}|+m_e c^2} \\ 0 \end{bmatrix} \psi_-^{(+)} e^{ip\cdot x} = \begin{bmatrix} 0 \\ \frac{-E}{c|\boldsymbol{p}|+m_e c^2} \\ 1 \\ 0 \end{bmatrix} \psi_-^{(-)\dagger} e^{ip\cdot x},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{-E}{c|\boldsymbol{p}|+m_e c^2} \\ 1 \\ 0 \end{bmatrix} \psi_-^{(-)} e^{-ip\cdot x} = \begin{bmatrix} 0 \\ -1 \\ \frac{E}{c|\boldsymbol{p}|+m_e c^2} \\ 0 \end{bmatrix} \psi_-^{(+)\dagger} e^{-ip\cdot x},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-E}{c|\boldsymbol{p}|+m_e c^2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \psi_+^{(-)} e^{-ip\cdot x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{E}{c|\boldsymbol{p}|+m_e c^2} \end{bmatrix} \psi_+^{(+)\dagger} e^{-ip\cdot x}.$$

Thus, the equations (3.3) - (3.6) are obtained

$$\hat{j}_e u_+^{(+)}(p) = -u_+^{(-)\dagger}(p), \tag{7.1}$$
$$\hat{j}_e u_-^{(+)}(p) = -u_-^{(-)\dagger}(p), \tag{7.2}$$
$$\hat{j}_e u_-^{(-)}(p) = -u_-^{(+)\dagger}(p), \tag{7.3}$$
$$\hat{j}_e u_+^{(-)}(p) = u_+^{(+)\dagger}(p). \tag{7.4}$$

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8 Appendix B: K - eigen-values

In this section, the eigen-values of the operator K are found by developing $\hbar K = \hat{\boldsymbol{j}}_e \cdot \boldsymbol{L} + \hbar$ in terms of the eigen-values of the operator $\boldsymbol{J}$. Then,

$$\begin{aligned} \hbar^2 K^2 &= (\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L})^2 + 2\hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \hbar^2 \\ &= L^2 + 2\hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \hbar^2 \\ &= L^2 + 2\hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \hbar^2 + \frac{\hbar^2}{4} - \frac{\hbar^2}{4} \\ &= L^2 + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \frac{\hbar^2}{4} + \frac{3\hbar^2}{4} \\ &= \left(\boldsymbol{L} + \frac{\hbar}{2}\right)^2 + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \frac{3}{4}\hbar^2 \\ &= \boldsymbol{J}^2 + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \frac{3}{4}\hbar^2, \end{aligned}$$

Thus, in terms of eigen-values, it follows that

$$\begin{aligned}\hbar^2 k^2 &= j(j+1)\hbar^2 + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) + \frac{3}{4}\hbar^2 \\ &= j(j+1)\hbar^2 + \frac{\hbar^2}{4} + \frac{2}{4}\hbar^2 + \hbar(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) \\ &= j(j+1)\hbar^2 + \frac{\hbar^2}{4} + \frac{2\hbar^2}{4} \pm \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}}\hbar^2,\end{aligned}$$

where it is used the equality

$$(\hat{\boldsymbol{j}}_e \cdot \boldsymbol{L}) = \pm\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}}.$$

Therefore, the eigen-values of the operator K are given by

$$k = \sqrt{\left(j+\frac{1}{2}\right)^2 + \frac{1}{2} \pm \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}}}.$$

9 Appendix C: Formulation of Energy Levels in the Atomic Model

In this section, the energy associated with the atomic levels of an accelerated electron will be formulated. Replacing (4.10) and (4.11) in (4.6) and (4.9), respectively, it holds

$$\begin{aligned}Ei\sum_{m=0}^{\infty} a_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} =& -\hbar c \sum_{m=0}^{\infty} b_m \left[\frac{m}{r_0}\left(\frac{r}{r_0}\right)^{m-1} - \frac{1}{r_0}\left(\frac{r}{r_0}\right)^m\right] e^{-r/r_0} \\ & - i\left(m_e c^2 + \frac{i\hbar c k}{r} + \frac{\beta}{r^2}\right)\sum_{m=0}^{\infty} b_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0},\end{aligned} \tag{9.1}$$

$$\begin{aligned}-Ei\sum_{m=0}^{\infty} b_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} =& \hbar c \sum_{m=0}^{\infty} a_m \left[\frac{m}{r_0}\left(\frac{r}{r_0}\right)^{m-1} - \frac{1}{r_0}\left(\frac{r}{r_0}\right)^m\right] e^{-r/r_0} \\ & + i\left(m_e c^2 + \frac{i\hbar c k}{r} + \frac{\beta}{r^2}\right)\sum_{m=0}^{\infty} a_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0}.\end{aligned} \tag{9.2}$$

Taking the imaginary parts of (9.1) and (9.2), it is obtained

$$Ea_m = -\left(m_e c^2 + \frac{\beta}{r^2}\right) b_m, \tag{9.3}$$

$$-Eb_m = \left(m_e c^2 + \frac{\beta}{r^2}\right) a_m. \tag{9.4}$$

From (9.3) and (9.4) is given that

$$E = \pm\left(m_e c^2 + \frac{\beta}{r^2}\right). \tag{9.5}$$

Now, taking the real parts of (9.1) and (9.2), it is true that

$$r = r_0(m \pm k).$$

Substituting the above equality in (9.5) and rewriting again β, the energy levels of an accelerated relativistic electron in a hydrogen atom associated with degenerated states are obtained

$$E \equiv E_{nlm;j} = \pm\left[m_e c^2 + \frac{(m_e c)^2\alpha^2}{\hbar^2 n^4 (m \pm k)^2}\left(\frac{4}{3}\frac{e^2 m_e u_e Z k_C}{r_e}\mu_B\sqrt{l(l+1)} + \frac{2}{3}\frac{Z k_C \hbar u_e e^2}{r_e}\right)\right].$$

Performing the same procedure when replacing (4.12) and (4.13) in (4.7) and (4.8), respectively, the spinor can be rewritten as follows

$$\psi_{nlm;j}(r) := \begin{bmatrix} i\sum_{m=0}^{\infty} a_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} \\ i\sum_{m=0}^{\infty} \frac{E}{\left[m_e c^2 + \frac{\beta}{r_0^2 (m+k)^2}\right]} a_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} \\ i\sum_{m=0}^{\infty} c_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} \\ i\sum_{m=0}^{\infty} \frac{E}{\left[m_e c^2 + \frac{\beta}{r_0^2 (m-k)^2}\right]} c_m \left(\frac{r}{r_0}\right)^m e^{-r/r_0} \end{bmatrix}.$$

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