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The Hurst Effect: The Scale of Fluctuation Approach

OSCAR J. MESA AND GERMAN POVEDA

Programas de Postgrado en Aprovechamiento de Recursos Hidraulicos, Universidad Nacional de Colombia, Medellin

After more than 40 years the so-called Hurst effect remains an open problem in stochastic hydrology. Historically, its existence has been explained either by preasymptotic behavior of the rescaled adjusted range R_n^*, certain classes of nonstationarity in time series, infinite memory, or erroneous estimation of the Hurst exponent. Various statistical tests to determine whether an observed time series exhibits the Hurst effect are presented. The tests are based on the fact that for the family of processes in the Brownian domain of attraction, $R_n^*/((\theta n))^{1/2}$ converges in distribution to a nondegenerate random variable with known distribution (functional central limit theorem). The scale of fluctuation θ, defined as the sum of the correlation function, plays a key role. Application of the tests to several geophysical time series seems to indicate that they do not exhibit the Hurst effect, although those series have been used as examples of its existence, and furthermore the traditional pox diagram method to estimate the Hurst exponent gives values larger than 0.5. It turned out that the coefficient in the relation of R_n^* versus n, which is directly proportional to the scale of fluctuation, was more important than the exponent. The Hurst effect motivated the popularization of $1/f$ noises and related ideas of fractals and scaling. This work illustrates how delicate the procedures to deal with infinity must be.

1. Introduction

The Hurst effect is one of the most important unsolved problems in stochastic hydrology. There is ample evidence to support this statement. Ever since *Hurst*'s [1951] original work, there has been a proliferation of papers about it. Some of the most important awards of the hydrologic community have gone to contribution toward its solution. Hydrologists have been divided into two schools in the attempt to interpret the alleged existence of this anomaly in geophysical records. Some of the most famous personalities in probability theory, like *Feller* [1951], have devoted time to this problem. *Mandelbrot* [1982] has declared that his original investigation into the Hurst effect was one of the sources of inspiration for his, now in vogue, fractal theory, whose importance in chaos theory is without doubt.

Despite the activity cited above, the problem in hydrology is stagnant. There are no clear winners between the short- and long-memory schools. Hurst's original motivation was the practical problem of reservoir design, but it was later discovered that only in some particular cases did the so-called Hurst effect have significant practical implications [*Kleměs et al.*, 1981]. Nevertheless, economic implications of the use of different hydrologic models are not insignificant, and the way the persistence is modeled is very important from an economic point of view [*Zapata*, 1987; *Mejía and Millán*, 1982; *Pereira et al.*, 1984].

We present various tests for the existence of the Hurst effect. Applications of these tests to Hurst's original data shows that either there is no Hurst effect or the series is not long enough to provide definite answers. Even in the latter case, the Hurst effect is not a natural interpretation. In addition to the importance of the study of Hurst's geophysical series, the proposed test may prove to be important in other applications where infinite memory models (nonsummable correlograms) have been proposed for physical problems such as turbulence (see, for instance, *Mandelbrot* [1974]).

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2. Antecedents

Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables representing, for instance, the inflows into an infinite reservoir. Denote the partial sum series (cumulative inflows) by

$$S_0 = 0 \qquad S_t = \sum_{m=1}^{t} X_m \qquad t = 1, 2, \cdots, n. \tag{1}$$

The sample mean represents the ideal release from the reservoir and therefore the adjusted partial sum sequence S_t^*, defined by

$$S_t^* = S_t - (t/n)S_n \qquad t = 0, 1, 2, \cdots, n, \tag{2}$$

represents the fluctuations in the content of this ideal reservoir. In his studies for the Aswan Dam on the Nile River, *Hurst* [1951] considered the "adjusted range," defined by

$$R_n = \max S_t^* - \min S_t^* \qquad \text{for } 0 \le t \le n. \tag{3}$$

He showed that the adjusted range is a measure of the reservoir capacity required under idealized conditions and therefore the study of its properties becomes very pertinent. He was particularly interested in the dependence of R_n on the sample size n. Obviously, R_n increases with n, but how fast? The fact that the length of the existing streamflow records is rarely of the order of 100 years motivated *Hurst* [1951] to look at various series of different geophysical phenomena. For that purpose, he defined the "rescaled adjusted range"

$$R_n^* = R_n/D_n \tag{4}$$

which is a dimensionless quantity; D_n is the sample standard deviation of the X_i.

Hurst [1951] used 690 different time series of 75 geophysical variables such as temperature, rainfall, solar spot numbers, mud varves, tree rings, etc. His empirical findings were that, in general, R_n^* grows like n to a power of the order of 0.72 with some variations for the different records, but in all the cases the exponent was larger than 0.5. This result was in contradiction with his own theoretical analysis which indicated that the exponent should be asymptotically 0.5. *Feller* [1951] was quick to provide firm theoretical computations for the case of independent identically distributed (IID) random variables with finite second moments. He computed the limit mean and variance of the rescaled adjusted range, and suggested that *Hurst*'s [1951] empirical findings might be explained by some kind of Markovian dependence. The discrepancy between the empirical observation showing the increase of the rescaled adjusted range like a power of 0.7 or so, and the theoretical expectation that for a wide class of processes the exponent is asymptotically 0.5 has become to be known as the Hurst effect. The exponent H in the empirical relation

$$R_n^* \sim n^H \tag{5}$$

is called the Hurst exponent.

To appreciate the different arguments that have been put forward as possible explanations of the Hurst effect, it is necessary to use a more precise definition. Such precision is important also for the proper understanding of the estimation problem of the next section. To that end, we will use the definition of *Bhattacharya et al.* [1983, p. 651]: "A sequence of random variables is said to exhibit the Hurst effect with exponent $H > 0.5$ if $(1/n^H)R_n^*$ converges in distribution, as n goes to infinity, to a non-zero random variable." This is in contrast with a very general result, known as the invariance principle or the functional central limit theorem [*Ibragimov*, 1962; *Billingsley*, 1968], which implies that under conditions of stationarity and weak dependence $R_n^*/n^{0.5}$ converges in distribution to a random variable $R_\infty^*/n^{0.5}$ with mean

$$E(n^{-0.5}R_\infty^*) = (\theta\pi/2)^{1/2} \tag{6}$$

and variance

$$\mathrm{Var}\,(n^{-0.5}R_\infty^*) = \theta(\pi^2/6 - \pi/2) \tag{7}$$

where θ is a positive constant, the so-called scale of fluctuation, or correlation length scale, a parameter first introduced by *Taylor* [1921]. It can be shown that θ is the sum of the correlation function

$$\theta = \sum_{m=-\infty}^{\infty} \rho(m) \tag{8}$$

where $\rho(m)$ denotes the correlation coefficient between X_n and X_{n+m}.

Equations (6) and (7) correspond to *Feller*'s [1951] previous result, for in that case θ is one. Besides the mean and the variance, the whole asymptotic distribution of the rescaled adjusted range are ready available from known results in the Brownian motion case [*Bhattacharya et al.*, 1983]. The condition of weak dependence implies a correlation function decreasing fast enough to ensure convergence of the series in (8). This explains the use of terms such as strong dependence, infinite memory (infinite θ), and short memory (finite θ). Normally, existence of finite second moments for the sequence of X_i is assumed. Then the ergodic theorem applied to X_i^2 implies that D_n converges in probability to the standard deviation of the X_i, say, σ. In that case, R_∞^* is scaled by σ because it is the limit of R_n^* which is scaled by D_n.

A consequence of the above definition and of the invariance principle is that for a sequence to exhibit the Hurst effect, it is necessary that at least one of the conditions of the functional central limit theorem be violated. Because of the generality of these hypotheses something quite dramatic must be happening from the physical point of view and a search for the cause of the violation is very much in order. The discovery of anomalies such as a nonstationarity or a strong dependence in that large a class of geophysical records will have profound physical implications.

As may be expected, from a retrospective viewpoint, the explanations proposed by different authors were related to the violation of the hypotheses of the invariance principle. There have been some theories essentially related to the dependence structure of the process, to its stationarity, to the existence of an infinite second moment, and to the preasymptotic behavior of the limiting process.

The strong dependence explanation was initiated by *Hurst* [1951] himself, and by *Feller*'s [1951] suggestion about the Markovian character of the series being responsible for the Hurst effect. However, soon *Barnard* [1956] showed the existence of a 0.5 convergence for that case. Nevertheless, *Matalas and Huzzen* [1967] presented reports of estimates of H between 0.58 and 0.87 for simulation of first-order autoregressive Markovian models with different correlation coefficients. At that time the Markovian explanation seemed adequate, and the autoregressive models were becoming fashionable. However, *Mandelbrot and Van Ness* [1968] put things back in proper perspective by recalling that for processes in the domain of attraction of the Brownian motion, the rescaled adjusted range grows like n to the 0.5 power (notice that Barnard's result goes back to 1956, and that the invariance principle dates back to *Ibragimov* [1962]). *Mandelbrot and Wallis* [1968, 1969] and *Mandelbrot and Van Ness* [1968] proposed an explanation of the Hurst effect by the strong dependence of the geophysical series and introduced the so-called fractional Brownian motion and the fractional Brownian noise ($1/f$ noise), which are processes with infinite memory, as models that may be used to simulate the Hurst effect. These important theoretical contributions were not accompanied by either a physical explanation nor an investigation of the structure of dependence of the different geophysical series. The lack of physical justification of this theory was pointed out by authors such as *Scheideger* [1970] and *Kleměs* [1974]. There has not been appreciation for this kind of question as the following quote shows: "I am prepared to argue that a lack of serious motivation in a model that fits and works well is much preferable to lack of fit in a model that seems well motivated" [*Mandelbrot*, 1982, p. 253].

On the estimation side, the contribution of *Mandelbrot and Wallis* [1968] was also important. They pointed out that the way *Hurst* [1951] estimated the exponent H was not adequate. *Hurst* [1951] used the equation

$$H_1 = \log\,(R_n^*)/\log\,(n/2) \tag{9}$$

which presupposes that the relation (R_n^*) versus n in logarithmic paper passes through the point $n = 2$, $R_n^* = 1$. This was motivated by weak empirical arguments, because it is easy to see that using the biased estimator for σ_n, one obtains $R_n^* = 1$ when $n = 2$. *Mandelbrot and Wallis* [1968] introduced the so-called "pox" diagram. This and other ways of estimating H will be discussed in the next section.

Attempts to explain the Hurst effect as a consequence of an infinite second moment were proposed by *Moran* [1964] and *Boes and Salas* [1973]. However, *Mandelbrot and Taqqu* [1979] demonstrated that an asymptotic relation with $H = 0.5$ holds for a sequence of IID random variables with stable distribution and characteristic exponent strictly less than 2 (see *Feller* [1971, p. 169] for a discussion of stable distributions).

Hurst [1951] recognized the nonstationarities in his original geophysical series. He even designed an experiment with "probability cards" which produced sudden changes in the mean of the process, and obtained empirical estimations of the exponent H near 0.71. *Klemeš* [1974] and *Potter* [1975] developed simulations with nonstationary models that produced empirical series exhibiting the Hurst effect. Something similar was obtained by *Boes and Salas* [1978] with the shifting levels model. It is worth noting that all of these nonstationary models belong to the Brownian domain of attraction.

Bhattacharya et al. [1983] provided clear mathematical demonstration of the existence of the Hurst effect for weakly dependent processes perturbed by small monotonic trends. As an example, let Y_n be a sequence of weakly dependent random variables, say IID normal variables with zero mean and unit variance, then

$$X_n = Y_n + c(m + n)^{\beta} \tag{10}$$

will exhibit the Hurst effect with exponent H dependent on the value of the parameter β as follows: for $-1/2 < \beta < 0$, H is equal to $1 + \beta$, for $\beta > 0$, the exponent H is 1, and for $\beta \leq -1/2$, and for $\beta = 0$, the Hurst exponent is 1/2 [see *Bhattacharya et al.*, 1983]; notice the discontinuity at $\beta = 0$. In all the cases, m and c are arbitrary parameters ($c > 0$ and $m \geq 0$). The significance of this demonstration is twofold. First, the class of processes with infinite memory is no longer the only theoretically proved class of processes exhibiting the Hurst effect. Second, the estimation problem becomes very important, for it is possible that very small trends may be responsible for the appearance of the Hurst effect.

In addition to the infinite memory and the nonstationarity explanations, there have been theories that present the Hurst effect only as a preasymptotic behavior. This means the convergence to the theoretical 0.5 exponent is slow, and therefore the empirical observations for finite sample sizes may give Hurst exponents larger than 0.5 [*Lloyd*, 1967]. *Salas and Boes* [1974] considered the equation for the expected value of R_n for finite n and the case of IID variables showing the preasymptotic behavior of the adjusted range, and proposed a different way of estimation of the exponent H. *Gomide* [1975, 1978] considered the case of Markovian processes and showed how the preasymptotic region is expanded with values of $\rho(1)$ near one, and proposed a new way of estimation of the exponent H. *Salas et al.* [1979*a*, *b*], using autoregressive moving average (ARMA)(1, 1) models, showed that the preasymptotic region is expanded because of either asymmetric marginal distribution, large but finite memory (large θ) and nonstationarity.

It is worth noting that parallel to the preasymptotic explanation of the Hurst effect, various short-memory models were used in hydrology to model Hurst's exponents larger than 0.5. The broken line model [*Rodríguez-Iturbe et al.*, 1972; *Mejia et al.*, 1972] and the ARMA(1, 1) model [*O'Connell*, 1974] are the standard practice. Indeed, *Hipel and McLeod* [1978] demonstrated that ARMA(p, q) models statistically preserve the rescaled adjusted range or equivalently the Hurst exponent estimated as (9). One can interpret the philosophy of these models as an attempt to increase the correlation length scale θ so as to obtain preasymptotic estimates of the Hurst coefficient similar to the observed in empirical records. The conclusions of the chapter on the Hurst effect in the textbook by *Bras and Rodríguez-Iturbe* [1984, p. 265] are very illustrative of the standard practice in engineering hydrology. This is most noteworthy if one considers that the physical problem is unsolved.

In fact, there is no physical explanation yet for the occurrence of infinite memory in geophysical series. There is no systematic (physical or empirical) study of the correlation structure of these processes. Even though the nonstationarity of some geophysical processes may be argued on some physical grounds [*Leopold et al.*, 1964, p. 61], it is only in very general terms and it remains to be explained why the trends produce the same Hurst exponent in various geophysical series. Moreover, the issues of estimation raised by the preasymptotic explanations are very relevant and have no definite answers. The importance of the estimation problem is reinforced if one considers the claims about the robustness of the range analysis in current literature on fractals [*Mandelbrot*, 1982, p. 382; *Feder*, 1988, p. 194].

3. Tests for the Existence of the Hurst Effect

Notice that the whole puzzle rests on the empirical evidence of the exponent H being larger than 0.5. Also, recall that the question is not related to minor things: $n^{0.7}$ is almost 4 times $n^{0.5}$ for $n = 1000$, and the factor keeps increasing with n. With all the history behind but the perspective of huge unsolved questions present, the least that can be done is to look further into the empirical evidence.

3.1. *Estimators of H*

As was pointed out before, *Hurst* [1951] originally estimated H by means of (9), but this practice was shortly abandoned and substituted by the least squares slope in the linear relation of log (n) versus log R_n^*, along with some other variations. How good are those estimators? Very little theoretical work has been done along these lines. As an alternative for the complex theoretical issues involved, extensive computer experiments were performed by *Poveda* [1987] using *Bhattacharya et al.*'s [1983] nonstationary model (equation (10)), given its capacity to produce values of H, at will, by fixing β. This analysis considered most of the estimators reported in the literature [see also *Poveda and Mesa*, 1993]; Table 1 presents *Poveda*'s [1987] results for one case and some of the estimators. The performance of all estimators in all cases was poor. Slightly better results were obtained with a new estimator proposed by *Poveda* [1987], which consists of the slope s_n of the regression of sample values of R_m^* versus m taking only values of m larger than n, but some degree of arbitrariness remains regarding the choice of n.

A conclusion of these computer experiments is that all the estimators of the *Hurst*'s [1951] exponent H performed

TABLE 1. Estimated Values of the Hurst Exponent H for a Simulated Sequence of the *Bhatthacharya et al.*'s [1983] Nonstationary Model, Equation (10), With $\beta = -0.3$, $c = 1$, and $m = 1{,}000$ and 20,000 Record Length

n	*Hurst* [1951]	*Wallis and Matalas* [1970]	*Gomide* [1975]	*Poveda* [1987]
5	0.7172		0.1343	0.5408
10	0.6901	0.6544	0.3773	0.5370
25	0.6571	0.6216	0.4137	0.5357
100	0.6195	0.5889	0.5487	0.5399
250	0.6049	0.5760	0.4788	0.5508
500	0.5939	0.5661	0.4660	0.5760
1,000	0.5753	0.5517	0.5020	0.6257
2,500	0.5563	0.5346	0.4966	0.7086
5,000	0.5474	0.5223	0.4710	0.7998
10,000	0.5637	0.5248	0.5037	0.7921
15,000	0.5855	0.5372	0.5198	0.4528
20,000	0.5812	0.5421	0.5177	

poorly. This contradicts the alleged robustness of the range analysis [*Mandelbrot*, 1982, p. 386].

3.2. *Visual Tests*

The behavior of R_n^*/n^H ($H = 0.5$ and $H > 0.5$) is the most natural thing to examine to test for the Hurst effect in a geophysical time series instead of the logarithmic regression of R_n^* on n. A useful set of diagrams was designed for that purpose, the so-called "GEOS" (geophysical record) diagrams, with n on the abscissa and $R_n^*/n^{0.5}$ on the ordinates. Recall that if there exists the Hurst effect $R_n^*/n^{0.5}$ will eventually diverge to infinity, whereas if there is no Hurst effect $R_n^*/n^{0.5}$ will converge to a finite limit, with small random variation around it. Therefore sample points for a time series which exhibits the Hurst effect will increase indefinitely in the GEOS diagram. Instead, a time series will not possess the Hurst effect when its GEOS diagram converges to a finite limit.

In a similar way, it is possible to check the convergence of R_n^*/n^H, $H > 0.5$, by scaling the vertical axis by n^H. In this case a geophysical time series which exhibits the Hurst effect, with exponent H, will converge to a nonzero limit, whereas a time series without the Hurst effect will converge to zero.

GEOS diagrams are visual tests for the existence of the Hurst effect in any time series with a long enough record. These diagrams become more powerful tools than the so-called pox diagrams. This superiority is due to the fact that the diagram is scaled down properly, not only with respect to the mean of R_n^*, but also with respect to the variance and other moments as well. Therefore deviations from the expected behavior have the proper significance through the whole range of n values. Besides, no slope estimation is involved and, as was pointed out, factors of the order of 4 or more are involved providing a magnifying view that should help discriminate the existence of the Hurst effect. In applications, the main limitation of these visual tests is that if the length of the record is not long enough it may not be easy to draw definite conclusions. For instance a convergence from below may be wrongly interpreted as a continuous increase.

3.3. *Statistical Tests*

The visual test in the GEOS diagram may be improved substantially if an independent estimate of the limit of the sequence $\{R_n^*/n^{0.5}\}$ is known. Under the hypotheses of the functional central limit theorem this limit is a random variable with known distribution; therefore sample values should be around the mean, with deviations of the order of the standard deviation. In fact, for short memory stationary processes the mean and the variance are given by (6) and (7). As a consequence, standard statistical techniques may be employed to test the hypothesis of absence of the Hurst effect. The only extra parameter needed is the scale of fluctuation or correlation length scale θ. Clearly, because of the emphasis on the exponent previous studies have over looked the proportionality constant in the asymptotic expression (5).

3.3.1. *Test 1.* Given a sequence $X_1, X_2, \cdots$ of random variables, with known scale of fluctuation θ, a sample sequence $x_1, x_2, \cdots$, and a level of confidence α, the sequence does not exhibit the Hurst effect if the sample values of $R_n^*/n^{0.5}$ remains in the interval (q_α^-, q_α^+) for large enough n (where q_α^- and q_α^+ are the $1 - \alpha/2$ and $\alpha/2$ quantiles of the asymptotic distribution of $R_n^*/n^{0.5}$, respectively).

In practical applications q_α^- and q_α^+ can be approximated by the mean asymptotic value (equation (6)) ± 2.3 times the standard deviation (square root of (7)). The value 2.3 surely exceeds the values corresponding to the confidence level of 0.95 in the asymptotic distribution. More precise values of q_α^- and q_α^+ may be computed if desired.

Test 1 is an immediate consequence of the definition of the Hurst effect. Obviously, a test with a not known value of θ is needed. Various alternative ways for estimating θ from stationary random time series have been presented in the literature [*Vanmarcke*, 1988, p. 327]. A short summary is presented next.

The first procedure is by means of the sample correlation function using (8). However, this estimator is inconsistent, since its variance does not vanish when the record length becomes very large; indeed, it exhibits a high coefficient of variation [*Vanmarcke*, 1988, p. 325]. On the other hand, consistent estimators of θ can be obtained by using the variance function $\Gamma(\ )$ and the known fact that under a condition of weak dependence (finite first moment of $\rho(m)$ [*Vanmarcke*, 1988, p. 188]) the scale of fluctuation is also given by

$$\theta = \lim_{T \to \infty} T\Gamma(T). \tag{11}$$

Recall that the variance function $\Gamma(T)$ is simply the variance of the T average of the original process.

Nevertheless, the ordinary variance function estimator is biased and a correction is required for the estimation of θ. Following *Vanmarcke* [1988, p. 336], the expected value of the estimator of the variance function for an n long zero mean unit variance sample is

$$E[\Gamma^*(T)] \approx \frac{\Gamma(T) - \Gamma(n)}{1 - \Gamma(n)} \qquad T \leq n. \tag{12}$$

This motivates a corrected estimate Γ_c^* as follows:

$$\Gamma_c^*(T) = \Gamma(n) + \Gamma^*(T)[1 - \Gamma(n)]. \tag{13}$$

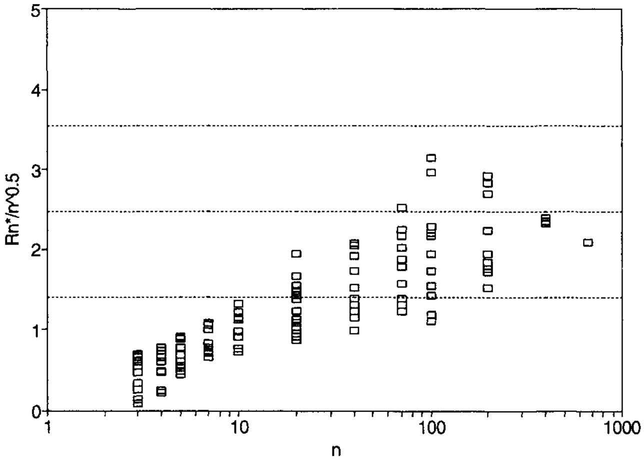

Fig. 1*a*. GEOS diagram for a time series of tree rings of a Douglas fir (Snake River). Horizontal dashed lines correspond to the asymptotic mean and ±2 standard deviations.

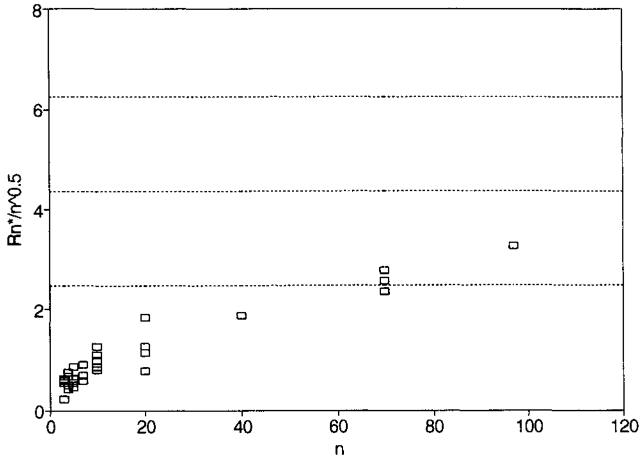

Fig. 1*c*. GEOS diagram for St. Lawrence river discharges.

Since $\Gamma(n)$ is not known, finding an estimate θ^* of the scale of fluctuation will require some iteration. Using an approximate model for the variance function ($\Gamma(T) \approx \theta/T$, for large T) provides the following expression [*Vanmarcke*, 1988, p. 337]:

$$\theta^* = \frac{\Gamma^*(T)Tn}{[1 - \Gamma^*(T)]n - T} \qquad T < n. \tag{14}$$

Another possible way of estimating θ is by means of the one-side unit area spectral density function at zero. Also, if a short memory theoretical model is adjusted to the data, θ could be estimated from the model, according to the theoretical expressions for θ in terms of the parameters of the model.

3.3.2. *Test 2 (Outline).* Suppose the scale of fluctuation θ is not given, estimate it by means of θ^* using any of the methods discussed above, estimate also the size of the variance of $R_n^*/(\theta^* n)^{0.5}$ and perform a test in classical terms. Even though some technical details need to be worked out, the test is in the same spirit of test 1. For this reason, formal substitution of θ^* for the scale of fluctuation in test 1 is proposed. The idea is that this will not affect the power of the test significantly. Only small modifications in the factors multiplying q_α^- and q_α^+ are affected by the new situation of θ^* being an estimate. In a process exhibiting the Hurst effect, the increase of $R_n^*/n^{0.5}$ with n will eventually dominate. The hypothesis of the test is also weak dependence and stationarity.

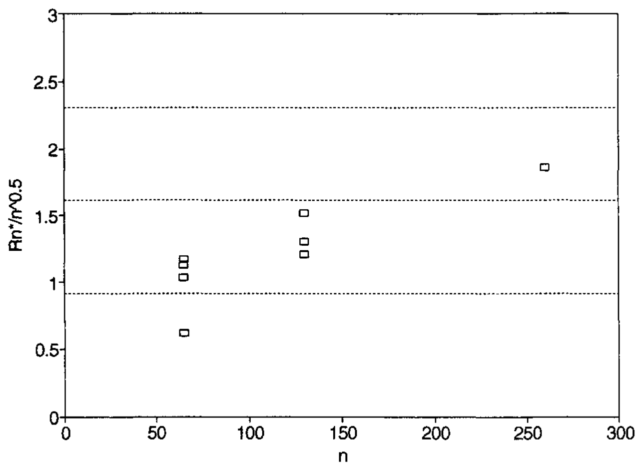

Fig. 1*b*. GEOS diagram for temperature in central England series.

Notice that the estimation of θ is by itself a test of the existence of the Hurst effect for stationary processes. In fact, because of the functional central limit theorem, if θ is finite the exponent h is 0.5 and there is no Hurst effect. Otherwise, if θ is infinite then there is Hurst effect and it is not necessary to perform the tests. In fact, for long memory time series the estimation of the scale of fluctuation in (14) does not converge to a finite limit and therefore test 2 is not suitable. For the nonstationary process of *Bhattacharya et al.* [1983] the estimation of θ by any of the means presented above also show divergence to infinity, in concordance with the theoretical result about the existence of the Hurst effect. For those processes test 2 is not applicable either.

In view of the above, the recommendation to deal with an observed time series is to proceed first to the estimation of θ using for instance the variance function approach of (14) (see Figure 3*b*). Stabilization of the estimator with n indicates finite memory and test 2 may be performed. If there is no stabilization there may be three possible causes: the series comes from an infinite memory process, or it comes from a nonstationary process or the length of the record is insufficient to estimate θ. As an easy cheek, if the value of n (the length of the record) divided by the estimated θ is less than, say, 15, the record is short. In these cases, if possible, the length of the record should be increased, and the estimation of θ repeated. On the other side, there are various ways of testing and removing nonstationarities. If the problem remains, no conclusion can be inferred from the data alone. In fact, all extrapolations of the behavior of either the range or the estimator of θ are equally arbitrary from a statistical point of view, and any decision should be based on physical reasoning.

In a related problem, *Burg* [1967] observed that the problem with conventional Fourier spectral analysis of finite time series is that only a finite number of correlation lags are estimable and that the truncation in lag space results in a smoothing of the true spectral function in frequency space. *Burg* [1967] argued that the criterion for extrapolation should be to obtain the spectral density estimate that corresponds to

TABLE 2. Estimation of H According to *Siddiqui* [1976]

Series Code	n	θ	*Poveda* [1987]	*Hipel and McLeod* [1978]
Mstouis	96	1.6	0.451	0.591
Neumunas	132	1.4	0.499	0.591
Danubio	120	1.0	0.495	0.495
Rhin	150	1.0	0.4984	0.484
Odgen	97	12.1	0.436	0.929
Gota	150	1.6	0.504	0.636
Española	350	44.8	0.455	0.927
Temp	255	1.6	0.521	0.640
Precip	100	1.0	0.473	0.473
Minimum	848	24.6	0.462	0.746
Snake	669	3.9	0.475	0.663
Exshaw	506	3.9	0.420	0.580
Naramata	515	2.1	0.435	0.543
Dell	655	3.8	0.475	0.667
Lakeview	544	5.9	0.499	0.706
Ninemile	771	7.4	0.466	0.642
Eaglecol	858	9.3	0.485	0.701
Navajo	700	2.9	0.468	0.584
Brice	625	4.0	0.513	0.727
Tioga	661	3.5	0.498	0.691
Bigcone	509	3.6	0.404	0.691
Whitment	1164	2.5	0.53	0.627

the most random or unpredictable stochastic process whose correlation function is consistent with the given information. Using the maximum entropy method he derived a widely used procedure for the estimation of the spectral density. The technique is based on obtaining a data model that is least informative with respect to data that are not available. This maximum entropy method for estimation of the spectral density is equivalent to the linear prediction method that assume that the underlying data model is an autoregressive (finite memory) model [*Roy et al.*, 1991].

4. EMPIRICAL EVIDENCE

At this point an obvious question arises, Do the classical time series which have been used to illustrate the existence of the Hurst effect really possess it? To elucidate the question GEOS diagrams were plotted for several geophysical time series. Basically, the formal version of test 2 was performed using GEOS diagrams with θ estimators based on the parameters of short memory models fitted to the observations. Indeed, for ARMA(p, q) models in the sense of *Box and Jenkins* [1970], the scale of fluctuation is given by [*Siddiqui*, 1976]

$$\theta^* = \frac{1}{\gamma_0} \frac{\left(1 - \sum_{j=1}^{q} \alpha_j\right)^2}{\left(1 - \sum_{i=1}^{p} \phi_i\right)^2} \tag{15}$$

where γ_0 is the ratio of the variance of the process to the variance of the noise, and ϕ and α are the ARMA(p, q) parameters. This procedure to estimate θ was used because the original series were not available. Values of R_n^* versus n for various geophysical series were taken from the extensive work by *Hurst et al.* [1965]. Additionally, results by *Hipel and McLeod* [1978] on ARMA(p, q) models fitted to some of those series allowed estimation of θ using (15). In Figures 1*a*, 1*b*, and 1*c* three of those GEOS diagrams are shown. For the majority of the cases, sample values of $R_n^*/n^{0.5}$ seem to indicate convergence to the asymptotic theoretical distribution of the fitted ARMA(p, q) model. Moreover, estimation of the Hurst exponent H in the way suggested by *Siddiqui* [1976] also shows the lack of existence of the Hurst effect in the set of geophysical time series analyzed by *Hipel and McLeod* [1978] (see Table 2). According to the estimated scale of fluctuation all the series but Odgen and Española have ratios of n over θ larger than 30, indicating adequate length of the records.

Nevertheless, time series corresponding to mud varves exhibit GEOS diagrams that always increase with the value of n (see for instance Figure 2), although the asymptotic value given by (6) for the fitted short memory model is unknown in these cases. Two facts could explain this situation: the time series is not long enough to reach the asymptotic mean of $R_n^*/n^{0.5}$ or those time series actually exhibit the Hurst effect either because of nonstationarity or long dependence.

For further illustration of the ideas presented, a 18,000 long series of vertical wind velocity sampled every 0.1 s collected with a very precise instrument was analyzed. Figure 3*a* shows the pox diagram with the exponent H estimated using traditional estimation indicating the presence of the Hurst effect (least squares slope of 0.773). Figure 3*b* shows the estimation of the scale of fluctuation; θ can be estimated to be of the order of 55 notwithstanding sampling fluctuations and after observing stabilization. Figure 3*c* shows the GEOS diagram and the proposed test indicates that there is no Hurst effect that can be inferred from this record (asymptotic mean around 9.3). A straight line of slope 0.5 and intercept equal to $((\theta\pi/2))^{1/2}$ was also drawn in Figure 3*a* for further clarification of the test. The least squares line in the pox diagram has both the slope and intercept free and fits the preasymptotic domain. The 0.5 slope line has no fitting parameter and is supposed to predict the asymptotic behavior of the rescaled adjusted range. Nevertheless, some may extrapolate the 0.773 power law increase and others may consider that the length of the series does not permit definite conclusions yet. In any case, the test may be repeated if a longer series is available.

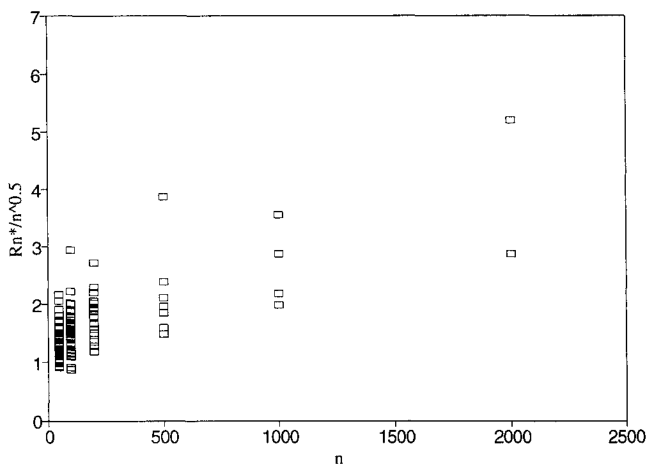

Fig. 2. GEOS diagram for Lake Saki mud varves.

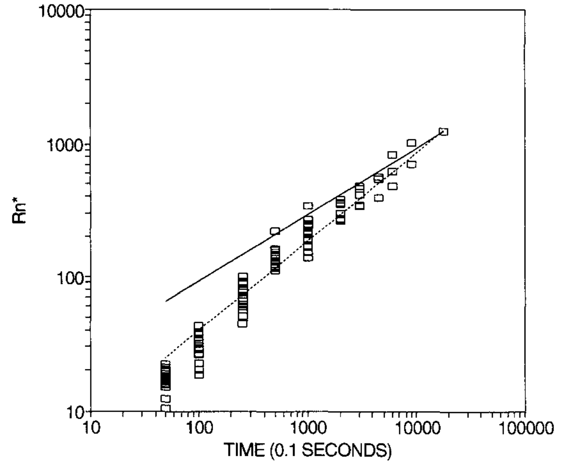

Fig. 3*a*. Pox diagram for a vertical wind velocity time series. The dotted line is a least squares fit with slope of 0.773, and the solid line is a 0.5 slope with $((\pi\theta/2))^{1/2}$ intercept.

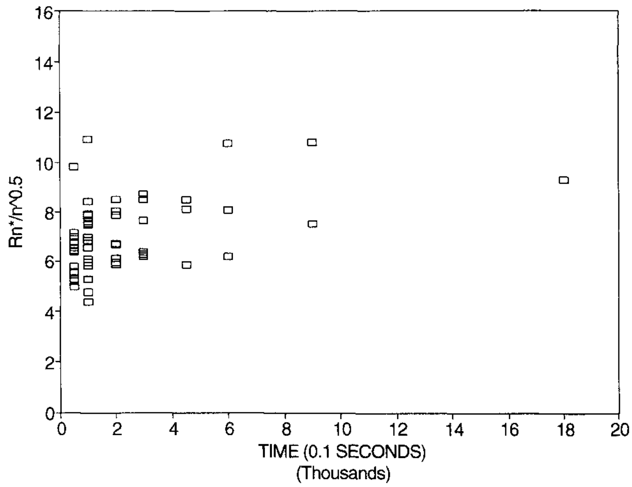

Fig. 3*c*. GEOS diagram for a vertical wind velocity time series.

5. CONCLUSIONS

To determine if a finite time series exhibits the Hurst effect is a delicate matter. Pox diagrams alone are not sufficient and further tests are proposed in that respect. Various statistical tests were developed to determine the existence of the Hurst effect. They are formulated in precise mathematical terms. Their basis is the so-called GEOS diagrams of $R_n^*/n^{0.5}$ versus n that for a short memory stationary process will converge to a known distribution, whereas for a series exhibiting the Hurst effect will diverge to infinity.

The estimation of the scale of fluctuation is itself a way of testing for short memory stationarity and for computing the asymptotic value for the GEOS diagram. Additionally, the scale of fluctuation provides a way of determining the length of a time series.

The tests introduced in this work do not have adjustable parameters. The scale of fluctuation needs to be estimated by independent means, but even without it GEOS diagrams may serve to indicate the presence of the Hurst effect. If the estimation of the scale of fluctuation does not show convergence to a finite limit, the tests presented here are not conclusive. In these cases any extrapolation of the asymptotic behavior of the rescaled adjusted range is not supported by observations and should be based on physical evidence.

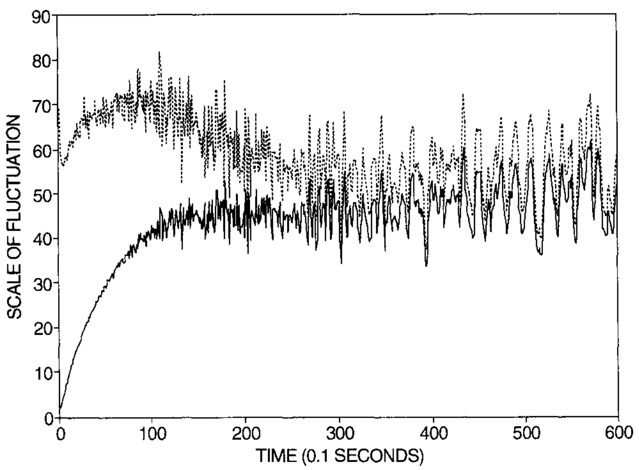

Fig. 3*b*. Scale of fluctuation estimation for a vertical wind velocity time series. The solid curve is $T\Gamma(T)$, and the dotted curve is the estimator given by (14). Units of the scale of fluctuation are in 0.1 s.

Application of the tests to data used by Hurst do not show existence of the Hurst effect. Only in the case of mud varves is there space for some speculation. However, even in that case, long memory does not appear to be the natural explanation of the evidence which might be related to nonstationarity of the records.

The scale of fluctuation is, with the mean and the variance, one of the most important parameters in hydrological stochastic modeling, since they completely describe statistical properties of a time series such as central tendency, fluctuations around the mean, the whole correlation structure, the "Hurst characteristics" of the time series, the low- and high-frequency components of the signal power spectrum, the so-called Noah and Joseph effects [*Mandelbrot and Wallis*, 1968], etc. Additional advantages of a better knowledge of the scale of fluctuation of a stochastic process, besides its role in the definition of the Hurst effect, lies on its importance to study some characteristics of a time series such as runs, run lengths, level crossings, passage times and, in general, probability distribution of extreme values.

The so-called Hurst effect and other related anomalities in geophysical time series are probably the result of a mixture of scales more than infinite memory. Knowledge of those scales is a more fundamental issue from a physical viewpoint. Modeling that mixture of scales is more simple and down to earth than infinite memory modeling.

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O. J. Mesa and G. Poveda, Programas de Postgrado en Aprovechamiento de Recursos Hidraulicos, Universidad Nacional de Colombia, Apartado Aéreo 1027, Medellin, Colombia.

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