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GRAVITATIONAL LENSING IN THE STRONG FIELD LIMIT FOR KAR'S METRIC

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Abstract

Gravitational lensing studies have been considered one of the most important applications of General Theory of Relativity. The full theory of gravitational lensing has been developed based on the scheme of the weak field approximation, which has been successfully used to explain all the physical observations. However, in the last years the scientific community has started to look this phenomenon from a different point of view: the strong field limit. The reason of studying this limit is that deviation of light rays in strong fields is one of the most promising areas where a theory of gravitation can be tested in its full form. In this work we have used the method proposed by V. Bozza to calculate the strong field limit deflection angle for a light ray passing near a scalar charged spherically symmetric object described by the metric proposed by Sayan Kar. This metric came from the low-energy limit of heterotic string theory equations of motion. Using Bozza's method, we solved the lens equation to calculate the parameters of the strong field limit expansion which are directly connected with observables such as the magnification of the images.

Keywords: light deviation, gravitational lensing, weak limit, Bozza's method,photon sphere, strong field limit.

Resumen

El estudio de las lentes gravitacionales ha sido considerada una de las aplicaciones más importantes de la Teoría General de la Relatividad. La teoría completa de las lentes gravitacionales ha sido desarrollada basada en la aproximación de campo débil, que ha sido utilizada con éxito para explicar todas las observaciones. Sin embargo, en los últimos años la comunidad científica ha comenzado a mirar este fenómeno desde un punto de vista diferente: el límite de campo fuerte. La razón de estudiar esta aproximación es que la desviación de los rayos de luz en campos fuertes es una de las áreas más prometedoras en que una teoría de la gravitación puede ser corroborada de manera completa. En este trabajo, se utilizó el método propuesto por V. Bozza para calcular el ángulo de deflexión en el límite de campo fuerte para un rayo de luz que pasa cerca de objeto esféricamente simétrico descrito por la métrica propuesta por Sayan Kar. Utilizando el método de Bozza, hemos resuelto la ecuación de la lente para el cálculo de los parámetros de la expansión los cuales están conectados directamente con los observables, como la magnificación de las imágenes.

Palabras clave: deflexión de la luz, lente gravitacional, límiti de campo débil, método de Bozza, esfera de fotones, límite de campo fuerte.

Contents

Chapter 1

GENERAL RELATIVITY

1.1 Introduction

Einstein's General Theory of Relativity requires a curved space for the description of the physical world. If one wishes to go beyond a superficial discussion of the physical relations one needs to set up a precise equations for handling curved space [1]. For this reason, the mathematical structure of space-time has been fundamentally built up using the concept of manifold [2]. In this sense, the set of "all events" is described by the the pair (M, g), where M is a $4-$dimensional smooth manifold and g is a lorentzian metric. But, how could we include physics on this mathematical structure? in others words what conditions must the mathematical structure of space-time satisfies to describe gravity? On the manifold we define several fields, such as the electromagnetic field, the neutrino field, etc, which describe the matter content of space-time. In this sense, Einstein's field equation will depend on the matter field we consider (the momentum-energy one chooses). There are two postulates on the nature of the equations obeyed by the field which are common to both the special and the general theory of relativity: local causality and local conservation of energy and momentum [3].

a) **Local causality:** The manifold is endowed with a causal structure [2]. In this sense, the equation governing the matter fields must be such that if $\mathcal{U}$ is a convex normal neighbourhood and p and q are points in $\mathcal{U}$ then a signal can be sent in $\mathcal{U}$ between p and q if and only if p and q can be joined by a C^1 curve lying entirely in $\mathcal{U}$, whose tangent vector is everywhere non-zero and is either time-like or null [3].

b) **Local conservation of energy and momentum:** The equations governing the matter fields are such that there exists a symmetric tensor $T_{\mu\nu}$, called the energy-momentum tensor, which depends on the fields, their covariant derivatives, and the metric, and which has the properties[3]:

   i T^{ab} vanish on an open set $\mathcal{U}$ if and only if all the matter fields vanish on $\mathcal{U}$.

   ii T^{ab} obeys the equation $\nabla_b T^{ab} = 0$.

Condition (i) expresses the principle that all fields have positive energy. Condition (ii) is very important because if the metric one is working with admits a Killing vector ξ, then the condition $\nabla_b T^{ab} = 0$ can be integrated to give a conservation law. To see this, define P^a to be the vector whose components are $P^a = T^{ab}\xi_b$. Then [3]

$$\nabla_a P^a = \nabla_a (T^{ab}\xi_b) = (\nabla_a T^{ab})\xi_b + T^{ab}\nabla_a \xi_b, \tag{1.1}$$

but $\nabla_a T^{ab} = 0$. Then

$$\nabla_a P^a = T^{ab}\nabla_a \xi_b. \tag{1.2}$$

If ξ is a killing vector it must satisfy

$$\begin{aligned} \nabla_a \xi_b + \nabla_b \xi_a = \mathcal{L}_\xi g_{ab} = 0 \\ 2\nabla_{(a}\xi_{b)} = 0. \end{aligned} \tag{1.3}$$

If $\mathcal{D}$ is a compact orientable region with boundary $\partial\mathcal{D}$, Gauss's theorem gives

$$\int_{\partial\mathcal{D}} P^a d\sigma_a = \int \nabla_a P^a dx^4 = 0. \tag{1.4}$$

This may be interpreted as saying that the local flux over a closed surface of the $\xi-$component of energy-momentum is zero [3].

In this chapter, we are going to obtain Einstein's field equations using the Lagrangian formulation of General Theory of Relativity. What is interesting about this formalism is that most part of physics, classical and quantum physics, can be expressed in terms of an action. Moreover, because of this formalism the physical meaning becomes clearer; for example, once physics has been written in terms of an action, it is easier to identify the conserved quantities [4]. On the other hand, one of the most important advantages of working with such a formalism is that it is possible to define the energy-momentum tensor. In practice one relies heavily on one's intuitive knowledge of what energy and momentum are. However, there is a definite and unique formula for this tensor in the case that the equations of the fields are obtained from a Lagrangian [3]. As a consequence, we will be able to obtain a system of differential equations in which the momentum-energy tensor is that of a massless scalar field [1].

1.2 Lagrangian formulation of Einstein's field equations:

In order to extend the idea of a variational principle to the field theory in curved spaces, one could consider a finite set of fields $\Phi^a(x^\mu)$[2] defined on a $4-$dimentional manifold parameterized in terms of the coordinates x^μ [4]. In this sense, when considering field theories defined on an arbitrary manifold it is necessary to include the components of the metric tensor in the set $\{\Phi^i\}$. For example, in electromagnetism, the complete set of fields would be $\{\Phi^i\} = \{A^\mu, g_{\mu\nu}\}$; where A^μ are the components of the potential vector. If the potentials Φ^i are not function of the components of the metric tensor, it is possible to express the lagrangian in terms of the fields, and its first or higher-order derivatives [4]. Therefore, the action takes the form

$$S = \int \widehat{\mathcal{L}}(\Phi^i, \nabla_\mu \Phi^i, \nabla_\nu \nabla_\mu \Phi^i, ..., g_{\mu\nu}, \partial_\sigma g_{\mu\nu})\sqrt{-g} d^n x. \tag{1.5}$$

For simplicity, let us assume that no second- or higher-order covariant derivatives appear in $\mathcal{L}$ and that Φ does not depend on the components of the metric tensor. This does not mean that the Lagrangian does not depend on those component, what this mean is that the variation with respect to Φ does not involve terms in which the variation of the components of the metric tensor are included. We are going to do the variation in such a way that

$$\begin{aligned} \Phi^i &\longrightarrow \Phi^i + \alpha\eta^i \\ \nabla_\mu \Phi^i &\longrightarrow \nabla_\mu \Phi^i + \alpha \nabla_\mu \eta^i, \end{aligned} \tag{1.6}$$

where α is a small parameter and η^i is a function of the coordinates. The variation of the action δS is calculated in the following way [5]

[1] What we mean by massless scalar field is going to be clear latter.

[2] The index is used to enumerate the fields considered. For example, If one considers a field theory in which exist k scalar fields then $\{\Phi^a\} = \{\phi^1, \phi^2, ..., \phi^k\}$

$$\delta S = d\alpha \left(\frac{\partial S}{\partial \alpha} \right)_{\alpha=0}$$

$$\begin{aligned} \frac{\partial S}{\partial \alpha} &= \int \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} \frac{\partial \Phi^i}{\partial \alpha} + \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \frac{\partial (\nabla_\mu \Phi^i)}{\partial \alpha} \right\} \sqrt{-g} d^n x \\ &= \int \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} \eta^i + \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \nabla_\mu (\eta^i) \right\} \sqrt{-g} d^n x \end{aligned} \tag{1.7}$$

$$\delta S = d\alpha \int_\Sigma \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} \eta^i + \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \nabla_\mu (\eta^i) \right\} \sqrt{-g} d^n x.$$

For $\delta S = 0$ we have that

$$\int_\Sigma \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} \eta^i + \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \nabla_\mu (\eta^i) \right\} \sqrt{-g} d^n x = 0. \tag{1.8}$$

Taking into account that

$$\nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \eta^i \right) = \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) \eta^i + \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \nabla_\mu (\eta^i), \tag{1.9}$$

and integrating by parts the second term in (1.8)

$$\int_\Sigma \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) \right\} \eta^i \sqrt{-g} d^n x + \int_\Sigma \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \eta^i \right) \sqrt{-g} d^n x = 0. \tag{1.10}$$

Because of Stokes's theorem, it is possible to transform the volume integral in the region Σ into a surface integral on the boundary $\partial\Sigma$

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|} d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} x \longrightarrow \text{Stokes's theorem.} \tag{1.11}$$

Therefore,

$$\begin{gathered} \int_\Sigma \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) \right\} \eta^i \sqrt{-g} d^n x + \int_\Sigma \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \eta^i \right) \sqrt{-g} d^n x = 0 \\ \int_\Sigma \left\{ \frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) \right\} \eta^i \sqrt{-g} d^n x + \int_{\partial\Sigma} n_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \eta^i \right) \sqrt{|\gamma|} d^{n-1} x = 0. \end{gathered} \tag{1.12}$$

The last integral in (1.12) vanish at $\partial\Sigma$. So that, the *Euler-lagrange* equation in curved spaces takes the final form

$$\frac{\partial \widehat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) = 0. \tag{1.13}$$

1.2.1 Field equations for a massless scalar field: Klein-Gordon Equation

As an example, we are going use the *Euler-Lagrange* equations to deduce the differential equation for a single scalar field $\Phi(x^\mu)$ defined on a space-time. Our deduction is going to be in such a way that no second- or higher-order derivatives of the field appear in the Lagrangian density. Therefore, the lagrangian density we are going to use is expressed as

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\nabla_\mu\Phi)(\nabla_\nu\Phi) - V(\Phi) \tag{1.14}$$

where $\frac{1}{2}g^{\mu\nu}(\nabla_\mu\Phi)(\nabla_\nu\Phi)$ can be considered as the "kinetic energy of the field" and $V(\Phi)$ as the "potential energy". In (1.14) we have used covariant derivatives rather than partial derivatives since $\mathcal{L}$ must itself be a scalar function of space-time position [4]. However, since the covariant derivative of a scalar quantity reduces to a partial derivative, in this case de later could be used. Nevertheless, it is usually wiser to retain the manifestly covariant notation [4][6]. Then, the action for the lagrangian (1.14) is:

$$S = \int_\Sigma \left[-\frac{1}{2}g^{\mu\nu}(\nabla_\mu\Phi)(\nabla_\nu\Phi) - V(\Phi)\right]\sqrt{-g}d^4x, \tag{1.15}$$

and using equation (1.13)

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)} &= \frac{\partial}{\partial(\nabla_\mu\Phi)}\left[-\frac{1}{2}g^{\rho\sigma}(\nabla_\rho\Phi)(\nabla_\sigma\Phi) - V(\Phi)\right] \\
&= -\frac{1}{2}g^{\rho\sigma}\left[\delta^\mu_\rho\nabla_\sigma\Phi + \delta^\mu_\sigma\nabla_\rho\Phi\right] \\
&= -\frac{1}{2}g^{\mu\sigma}\nabla_\sigma\Phi - \frac{1}{2}g^{\rho\mu}\nabla_\rho\Phi = -g^{\mu\nu}\nabla_\nu\Phi \\
\frac{\partial\mathcal{L}}{\partial\Phi} &= -\frac{dV}{d\Phi},
\end{aligned} \tag{1.16}$$

where in the second equation we have relabelled the dummy indices (ρ and σ) in order to make the differentiation more transparent. Evaluating this derivative explicitly

$$-\frac{dV}{d\Phi} - \nabla_\mu(-g^{\mu\nu}\nabla_\nu\Phi) = -\frac{dV}{d\Phi} + \nabla_\mu(g^{\mu\nu})\nabla_\nu\Phi + g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0; \tag{1.17}$$

however, recalling that $\nabla_\mu(g^{\mu\nu}) = 0$ and $\Box^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the last equation takes the form

$$\begin{aligned}
g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - \frac{dV}{d\Phi} &= 0 \\
\Box^2\Phi - \frac{dV}{d\Phi} &= 0.
\end{aligned} \tag{1.18}$$

Equation (1.18) is known as the *Klein-Gordon* equation. A common choice for the potential is $V = \frac{1}{2}m^2\Phi^2$, where m is a constant parameter that characterises the dynamics of the scalar field. Therefore, the field equation (1.18) reduces to

$$\Box^2\Phi - m^2\Phi = 0. \tag{1.19}$$

For $m = 0$ the last equation reduces to $\Box^2\Phi = 0$ which is the equation for a massless scalar field.

1.3 Einstein's field equations and stress-energy tensor:

To write down an action principle relativistically, we need an integral which is a scalar invariant [7]. This mean that its value has to have a value independent of the choice of the coordinate system [8]. One of the problems when considering the curvature tensor, is that one has all twenty distinct components to work with. Expressed in a local inertial frame, this twenty components are arbitrary to the extent of the six parameters of a local lorentz transformation. There are thus fourteen independent local features of the curvature that are coordinate-independent, any one of which one could imagine employing in the action principle. However, the Ricci scalar R is the only one of these fourteen quantities that is linear in the second derivatives of the metric coefficients.

Five days before Einstein presented his geometrodynamic law in its final and now standard form, Hilbert, animated by Einstein's earlier work, independently discovered how to formulate this law as the consequence of the simples action principle of the form

$$S_{HE} = \int \sqrt{-g} R d^4 x. \tag{1.20}$$

Any other choice of invariant other than Hilbert's complicates the geometrodynamic law, and destroys the simple correspondence with Newtonian theory of gravity[8].

1. The variation is made with respect to the metric.

2. It is convenient to make the variation[3] with respect to $g^{\mu\nu}$ because $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

$$\begin{aligned} g^{\mu\lambda} &\to g^{\mu\lambda} + \alpha\eta^{\mu\lambda} \\ \delta^\mu_\nu &= (g_{\mu\lambda} + \alpha\eta_{\mu\lambda})(g^{\nu\lambda} + \alpha\eta^{\lambda\nu}) \\ \delta^\mu_\nu &= \delta^\mu_\nu + \alpha(g^{\mu\lambda}\eta_{\lambda\nu} + g^{\nu\lambda}\eta_{\mu\lambda}) + \alpha^2\eta^{\mu\lambda}\eta_{\lambda\nu}. \end{aligned} \tag{1.21}$$

The terms multiplied by α^2 are not taken into account because we are working up to first order. Then,

$$\begin{aligned} \delta^\mu_\nu + \alpha(g^{\mu\lambda}\eta_{\lambda\nu} + g^{\nu\lambda}\eta_{\mu\lambda}) &= \delta^\mu_\nu \\ g^{\mu\lambda}\eta_{\lambda\nu} + g^{\nu\lambda}\eta_{\mu\lambda} &= 0. \end{aligned} \tag{1.22}$$

There is a sum on λ for both the first and the second terms of (1.22); so that,

$$\begin{aligned} (g^{\mu 1}\eta_{1\nu} + ... + g^{\mu n}\eta_{n\nu}) &= -(g_{1\nu}\eta^{\mu 1} + ... + g_{n\nu}\eta^{\mu n}) \\ g_{\mu\rho} g^{\mu\lambda}\eta_{\lambda\mu} &= -g_{\mu\rho} g_{\lambda\mu}\eta^{\mu\lambda} \\ \delta^\lambda_\rho \eta_{\lambda\mu} &= -g_{\mu\rho} g_{\lambda\mu}\eta^{\mu\lambda}. \end{aligned} \tag{1.23}$$

However, the remaining terms are those in which $\lambda = \rho$. therefore,

$$\eta_{\rho\nu} = -g_{\mu\rho} g_{\lambda\nu}\eta^{\mu\lambda}. \tag{1.24}$$

In equation (1.20) $R = g^{\mu\nu} R_{\mu\nu}$. Hence,

$$S_{HE} = \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x. \tag{1.25}$$

Finally, calculating δS_{HE} we obtain that

$$\delta S_{HE} = \int_\Sigma g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g} d^n x + \int_\Sigma R_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g} d^n x + \int_\Sigma g^{\mu\nu} R_{\mu\nu}\delta(\sqrt{-g}) d^n x. \tag{1.26}$$

[3] η is **not** the Minkowskian metric.

1.3.1 Variation of $\delta R_{\mu\nu}$

The *Riemann* tensor is expressed by

$$R^{\rho}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{1.27}$$

Contracting $R^{\rho}_{\mu\lambda\nu}$ we obtain $R_{\mu\nu}$. Therefore,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\mu\rho} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{1.28}$$

Calculating the variation of $\delta R^{\rho}_{\mu\lambda\nu}$ it is possible to obtain the variation of $R_{\mu\nu}$. Therefore, as (1.27) shows, it is necessary to calculate the variation of Γ^{r}_{ij}.

$$\begin{aligned} \Gamma^{r}_{ij} &= \frac{1}{2}g^{rk}[\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}] \\ g^{\mu\nu} &\longrightarrow g^{\mu\nu} + \alpha\eta^{\mu\nu} \\ g_{\mu\nu} &\longrightarrow g_{\mu\nu} + \alpha\eta_{\mu\nu}. \end{aligned} \tag{1.29}$$

$$\begin{aligned} \Gamma^{r}_{ij} &\longrightarrow \frac{1}{2}(g^{rk} + \alpha\eta^{rk})[\partial_j(g_{ik} + \alpha\eta_{ik}) + \partial_i(g_{jk} + \alpha\eta_{jk}) - \partial_k(g_{ij} + \alpha\eta_{ij})] \\ \Gamma^{r}_{ij} &\longrightarrow \Gamma^{r}_{ij} + \frac{\alpha}{2}g^{rk}[\partial_j\eta_{ik} + \partial_i\eta_{jk} - \partial_k\eta ij] + \frac{\alpha}{2}\eta^{rk}[\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}] + \mathcal{O}(\alpha^2). \end{aligned}$$

$$\delta\Gamma^{r}_{ij} = \alpha\left(\frac{\partial\Gamma^{r}_{ij}}{\partial\alpha}\right)_{\alpha=0} = \frac{\alpha}{2}g^{rk}[\partial_j\eta_{ik} + \partial_i\eta_{jk} - \partial_k\eta_{ij}] + \frac{\alpha}{2}\eta^{rk}[\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}]. \tag{1.30}$$

Now, making $\Gamma^{r}_{ij} \longrightarrow \Gamma^{r}_{ij} + \delta\Gamma^{r}_{ij}$ in (1.27):

$$\begin{aligned} R^{\rho}_{\mu\lambda\nu} &\longrightarrow \partial_{\lambda}(\Gamma^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}) + (\Gamma^{\rho}_{\lambda\sigma} + \delta\Gamma^{\rho}_{\lambda\sigma})(\Gamma^{\sigma}_{\nu\mu} + \delta\Gamma^{\sigma}_{\nu\mu}) - \partial_{\nu}(\Gamma^{\rho}_{\mu\lambda} + \delta\Gamma^{\rho}_{\mu\lambda}) \\ &\quad - (\Gamma^{\rho}_{\nu\sigma} + \delta\Gamma^{\rho}_{\nu\sigma})(\Gamma^{\sigma}_{\lambda\mu} + \delta\Gamma^{\sigma}_{\lambda\mu}) \\ &\quad \longrightarrow R^{\rho}_{\mu\lambda\nu} + \partial_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}(\delta\Gamma^{\rho}_{\mu\lambda}) - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} \\ &\quad - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \mathcal{O}(\alpha^2). \end{aligned} \tag{1.31}$$

Since $\delta\Gamma$ is a tensor[4], it is possible to calculate its covariant derivative; so that

$$\begin{aligned} \nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) &= \partial_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} \\ \nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} &= \partial_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} \\ \\ \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\nu}) &= \partial_{\nu}(\delta\Gamma^{\rho}_{\lambda\mu}) + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma}. \end{aligned} \tag{1.32}$$

Replacing into (1.31) we find that

$$\begin{aligned} R^{\rho}_{\mu\lambda\nu} &\longrightarrow R^{\rho}_{\mu\lambda\nu} + \partial_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - [\partial_{\nu}(\delta\Gamma^{\rho}_{\mu\lambda}) + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma}] \\ &\quad - \mathcal{O}(\alpha^2) \\ &\longrightarrow R^{\rho}_{\mu\lambda\nu} + \nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) - [\partial_{\nu}(\delta\Gamma^{\rho}_{\mu\lambda}) + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma}] - \mathcal{O}(\alpha^2) \\ &\longrightarrow R^{\rho}_{\mu\lambda\nu} + \nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\lambda}) - \mathcal{O}(\alpha^2). \end{aligned} \tag{1.33}$$

As a consequence, the variation of *Riemann* tensor is

$$\delta R^{\rho}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\lambda}). \tag{1.34}$$

[4]The difference of two connections is a tensor.

We had shown above that[5]

$$\begin{aligned}\delta\Gamma^r_{ij} &= \frac{\alpha}{2}g^{rk}[\partial_j\eta_{ik}+\partial_i\eta_{jk}-\partial_k\eta_{ij}]+\frac{\alpha}{2}\eta^{rk}[\partial_j g_{ik}+\partial_i g_{jk}-\partial_k g_{ij}]\\ \delta g_{\rho\nu} &= -g_{m\rho}g_{n\nu}\delta g^{mn}.\end{aligned} \tag{1.35}$$

Replacing we obtain

$$\begin{aligned}\delta\Gamma^\sigma_{\nu\mu} = \frac{1}{2}g^{\sigma k}\{&2\Gamma^\sigma_{\mu\nu}(\delta g^{\sigma k})+g_{m\mu}g_{nk}\partial_\nu(\delta g^{mn})+g_{nk}\partial_\nu(g_{m\mu})\delta g^{nm}\\ &+g_{m\mu}\partial_\nu(g_{nk})\delta g^{nm}+g_{m\nu}g_{nk}\partial_\mu(\delta g^{mn})+g_{nk}\partial_\mu(g_{m\nu})\delta g^{nm}\\ &+g_{m\nu}\partial_\mu(g_{nk})\delta g^{nm}-[g_{m\mu}g_{n\nu}\partial_k(\delta g^{mn})\\ &+g_{m\nu}\partial_k(g_{n\mu})\delta g^{nm}+g_{n\mu}\partial_k(g_{m\nu})\delta g^{nm}]\}.\end{aligned} \tag{1.36}$$

The easiest way to derive $\delta\Gamma^\sigma_{\nu\mu}$ is to note that since it is a tensor relation, it must be valid in any coordinate system. In particular, one could choose normal coordinates about a point p. For these coordinates the components $\delta\Gamma^\sigma_{\nu\mu}$ and the coordinate derivatives of the components $g_{\mu\nu}$ vanish at p [3][6]

$$\begin{aligned}\delta\Gamma^\sigma_{\nu\mu} &= -\frac{1}{2}\{g_{m\mu}\delta^\sigma_n\partial_\nu(\delta g^{nm})+g_{m\nu}\delta^\sigma_n\partial_\mu(\delta g^{nm})-g_{m\mu}g_{n\nu}g^{\sigma k}\partial_k(\delta g^{nm})\}\\ &= -\frac{1}{2}\{g_{m\mu}\partial_\nu(\delta g^{\sigma m})+g_{m\nu}\partial_\mu(\delta g^{\sigma m})-g_{m\mu}g_{n\nu}\partial^\sigma(\delta g^{nm})\}\\ &= -\frac{1}{2}\{g_{m\mu}\nabla_\nu(\delta g^{\sigma m})+g_{m\nu}\nabla_\mu(\delta g^{\sigma m})-g_{\alpha\mu}g_{\beta\nu}\nabla^\sigma(\delta g^{\alpha\beta})\}.\end{aligned} \tag{1.37}$$

The first integral in (1.26) can be expressed in terms of $\delta\Gamma$ as [6]

$$\begin{aligned}\int_\Sigma g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^nx &= \int_\Sigma g^{\mu\nu}\{\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu})-\nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda})\}\sqrt{-g}d^nx\\ &= \int_\Sigma \sqrt{-g}d^nx\nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu}-g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda}).\end{aligned}$$

But,

$$\begin{aligned}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} &= -\frac{1}{2}\left[\delta^\nu_m\nabla_\nu(\delta g^{\sigma m})+\delta^\mu_m\nabla_\mu(\delta g^{\sigma m})-g^{\mu\nu}g_{\alpha\mu}g_{\beta\nu}\nabla^\sigma(\delta g^{\alpha\beta})\right]\\ &= -\frac{1}{2}\left[\nabla_\nu(\delta g^{\sigma\nu})+\nabla_\mu(\delta g^{\sigma\mu})-\delta^\nu_\alpha g_{\beta\nu}\nabla^\sigma(\delta g^{\alpha\beta})\right]\\ &= -\nabla_\lambda(\delta g^{\sigma\lambda})+\frac{1}{2}g_{\beta\nu}\nabla^\sigma(\delta g^{\nu\beta}).\end{aligned}$$

Changing ν and μ by λ (dummy indices) we obtain:

$$\begin{aligned}g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda} &= -\frac{1}{2}\left[\delta^\sigma_m\nabla_\lambda(\delta g^{\lambda m})+g_{m\lambda}\nabla^\sigma(\delta g^{\lambda m})-g^{\mu\sigma}g_{\alpha\mu}g_{\beta\lambda}\nabla^\lambda(\delta g^{\alpha\beta})\right]\\ &= -\frac{1}{2}\left[\nabla_\lambda(\delta g^{\lambda\sigma})+g_{m\lambda}\nabla^\sigma(\delta g^{\lambda m})-\delta^\sigma_\alpha g_{\beta\lambda}\nabla^\lambda(\delta g^{\alpha\beta})\right]\\ &= -\frac{1}{2}\left[\nabla_\lambda(\delta g^{\lambda\sigma})+g_{m\lambda}\nabla^\sigma(\delta g^{\lambda m})-\nabla_\beta(\delta g^{\sigma\beta})\right].\end{aligned}$$

Putting λ instead of β [7], we have:

$$\begin{aligned}g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda} &= -\frac{1}{2}\left[\nabla_\lambda(\delta g^{\lambda\sigma})+g_{m\lambda}\nabla^\sigma(\delta g^{\lambda m})-\nabla_\lambda(\delta g^{\sigma\lambda})\right]\\ &= -\frac{1}{2}g_{m\lambda}\nabla^\sigma(\delta g^{\lambda m}).\end{aligned}$$

[5] We had into account that $\delta g^{\mu\nu}=\alpha\eta^{\mu\nu}$

[6] We have renamed $m\to\alpha$ y $n\to\beta$ because they are dummy indices.

[7] Because β is a dummy indice

So that

$$
\begin{aligned}
g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\lambda}_{\mu\lambda} &= \frac{1}{2}g_{m\lambda}\nabla^{\sigma}(\delta g^{\lambda m}) - \nabla_{\lambda}(\delta g^{\sigma\lambda}) + \frac{1}{2}g_{\beta\nu}\nabla^{\sigma}(\delta g^{\nu\beta}) \\
&= g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda}).
\end{aligned}
$$

Finally, The first integral in (1.26) can be expressed in terms of $\delta g^{\alpha\beta}$ as

$$
\int_{\Sigma} g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^n x = \int_{\Sigma}\sqrt{-g}d^n x \nabla_{\sigma}\left[g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda})\right]. \tag{1.38}
$$

Equation (1.38) is a integral with respect to the natural volume element of the covariant divergence of a vector; by Stoke's theorem, this is equal to a boundary contribution at infinity, which we can set to zero by making the variation vanish at infinity. therefore, this term contributes nothing to the total variation[6]. So that,

$$
\begin{aligned}
\int_{\Sigma} g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^n x &= \int_{\Sigma}\sqrt{-g}d^n x \nabla_{\sigma}\left[g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda})\right] \\
&= \int_{\partial\Sigma} n_{\sigma}[g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda}]\sqrt{|\gamma|}d^{n-1}x.
\end{aligned} \tag{1.39}
$$

However, the boundary term will include not only the metric variation, but also its first derivative, which is not traditionally set to zero [6]. In this sense, it is important to point out that the calculation of this integral is not a easy task. In [9] equation (2), Einstein shows us the possibility of express the Hilbert action in such a way that it considers the boundary term. This idea was considered in [10]. For our present purposes it does not matter, but in principle we might care about what happens at the boundary, and would have to an additional term in the action to take care of this subtlety.

1.3.2 Variation of $\delta\sqrt{-g}$

For any square matrix M it is true that

$$
\ln(\det M) = \mathrm{Tr}(\ln M). \tag{1.40}
$$

Calculating the variation of (1.40)

$$
\begin{aligned}
\delta(\ln(\det M)) &= \delta\mathrm{Tr}(\ln M) \\
\frac{1}{\det M}\delta(\det M) &= \mathrm{Tr}(M^{-1}\delta M);
\end{aligned} \tag{1.41}
$$

for $M = g^{\mu\nu}$ then $\det g^{\mu\nu} = g^{-1}$. Replacing we get

$$
\delta(g^{-1}) = \frac{1}{g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.42}
$$

Now, the variation of $\delta\sqrt{-g}$ is

$$
\delta\sqrt{-g} = -\frac{1}{2}(-g^{-1})^{-\frac{3}{2}}\delta(-g^{-1})
$$

$$
\delta(g^{-1}) = \frac{1}{g}g_{\mu\nu}\delta g^{\mu\nu} \tag{1.43}
$$

$$
\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\nu\mu}\delta g^{\nu\mu}.
$$

Therefore, the contribution of $\delta\sqrt{-g}$ to the total variation is

$$\int_{\Sigma} d^n x g^{\mu\nu} R_{\mu\nu} \delta\sqrt{-g} = -\int_{\Sigma} d^n x \sqrt{-g} \frac{R}{2} g_{\mu\nu} \delta g^{\mu\nu}. \tag{1.44}$$

Replacing in (1.26)

$$\begin{aligned} \delta S_H &= \int_{\Sigma} R_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^n x - \int_{\Sigma} d^n x \sqrt{-g} \frac{R}{2} g_{\mu\nu} \delta g^{\mu\nu} \\ \frac{\delta S_H}{\delta g^{\mu\nu}} &= \int_{\Sigma} d^n x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] = 0. \end{aligned} \tag{1.45}$$

from which the Einstein's field equations in vacuum are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{1.46}$$

As was mentioned in the introduction, the lagrangian formulation of General Relativity allow us to find a unique formula for the momentum-energy tensor. When we obtained equation (1.46), we do not considered the contribution of matter fields so that the components of the momentum-energy tensor were zero. In order to find the Einstein's field equations in presence of matter fields it is necessary to add an extra term in the Hilbert-Einstein action. Therefore,

$$S = \frac{1}{2\kappa} S_{HE} + S_M = \int_{\Sigma} \left(\frac{1}{2\kappa} \mathcal{L}_{HE} + \mathcal{L}_M \right) d^4x = \int_{\Sigma} \left(\frac{1}{2\kappa} \widehat{\mathcal{L}}_{HE} + \widehat{\mathcal{L}}_M \right) \sqrt{-g} d^4 x. \tag{1.47}$$

Where S_{HE} is the Hilbert-Einstein action, S_M is the "matter" action and $\kappa = 8\pi G/c^4$[8]. The factor $1/2\kappa$ in (1.47) has been chosen by convenience . Making the variation with respect to the metric we find that

$$\frac{1}{2\kappa} \frac{\delta \mathcal{L}_{HE}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} = 0; \tag{1.48}$$

from (1.45) we know that

$$\frac{\delta \mathcal{L}_{HE}}{\delta g^{\mu\nu}} = \sqrt{-g} G_{\mu\nu} = -2\kappa \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{1.49}$$

In this sense, and recalling that $G_{\mu\nu} \propto T_{\mu\nu}$, the momentum-energy tensor for any present field is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{1.50}$$

For the case we are interested: the solution to the field equations for a massless scalar field), we have found an action of the form (1.15). Using this action we can calculate the momentum-energy tensor and the field equations for this tensor,

$$\delta S_M = \int \left[\sqrt{-g} \left(-\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \right) + \delta\sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\nu \Phi \nabla_\mu \Phi - V(\Phi) \right) \right] d^4x, \tag{1.51}$$

recalling that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$

$$\begin{aligned} \delta S_M &= \int \sqrt{-g} \left[-\left(\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \right) - \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \left(-\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi - V(\Phi) \right) \right] d^4x \\ &= -\int \frac{1}{2} \sqrt{-g} \delta g^{\mu\nu} \left[\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi - g_{\mu\nu} V(\Phi) \right] d^4x. \end{aligned} \tag{1.52}$$

[8] λ^2 corresponds to this constant when we consider the Newtonian limit

From equation (1.50), the momentum-energy tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = \nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - g_{\mu\nu}V(\Phi); \tag{1.53}$$

setting $m = 0$ in $V(\Phi) = \frac{1}{2}m^2\Phi^2$, we found that

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi. \tag{1.54}$$

Finally, the Einstein's field equations for a massless scalar field are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\left[\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\right]. \tag{1.55}$$

Chapter 2

FROM JANIS-NEWMAN-WINICOUR TO S. KAR METRIC

2.1 The Janis, Newman, Winicour and Max Wyman Solution:

The line element for a static and spherically symmetric metric has the form

$$ds^2 = \mathrm{e}^{\alpha(r,t)}dt^2 - \mathrm{e}^{\beta(r,t)}dr^2 - r^2 d\Omega^2. \tag{2.1}$$

The Janis-Newman-Winicour metric **JNW**[11],[12] is the solution to the Field equations[1]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -\kappa\left[\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\right] \\ \Box\Phi &= 0. \end{aligned} \tag{2.2}$$

In this sense, we should calculate each of the terms involved in (2.2); it is to say:

1. The *Christoffel's* symbols $\Gamma^\rho_{\mu\nu}$.
2. The *Riemann* tensor $R^\rho_{\mu\lambda\nu}$.
3. The *Ricci* tensor $R^\rho_{\mu\rho\nu}$.

In order to do so, it is necessary to find the inverse of matrix $\mathbf{g}$ using the relation $\mathbf{g}\mathbf{g}^{-1} = \mathbf{g}^{-1}\mathbf{g} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. From the line element (2.1) we know that

$$[\mathbf{g}]_{\alpha\beta} = \begin{pmatrix} \mathrm{e}^{\alpha(r,t)} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{\beta(r,t)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{2.3}$$

Using $\mathbf{g}\mathbf{g}^{-1} = \mathbf{g}^{-1}\mathbf{g} = \mathbf{I}$ we have that

$$\begin{pmatrix} \mathrm{e}^{\alpha(r,t)} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{\beta(r,t)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \times \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

[1] In this chapter we are going to use de notation and the signature used by Wyman in [12] (+ - - -).

Therefore,

$$\begin{pmatrix} a\times \mathrm{e}^{\alpha(r,t)} & b\times \mathrm{e}^{\alpha(r,t)} & c\times \mathrm{e}^{\alpha(r,t)} & d\times \mathrm{e}^{\alpha(r,t)} \\ -e\times \mathrm{e}^{\beta(r,t)} & -f\times \mathrm{e}^{\beta(r,t)} & -g\times \mathrm{e}^{\beta(r,t)} & -h\times \mathrm{e}^{\beta(r,t)} \\ -i\times r^2 & -j\times r^2 & -k\times r^2 & -l\times r^2 \\ -m\times r^2\sin^2\theta & -n\times r^2\sin^2\theta & -o\times r^2\sin^2\theta & -p\times r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

and finally $\mathbf{g}^{-1}$, is

$$\mathbf{g}^{-1} = \begin{pmatrix} \mathrm{e}^{-\alpha(r,t)} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{-\beta(r,t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{2.6}$$

2.1.1 Christoffel's symbols (Γ^m_{ij}):

The Christoffel's symbols are calculated by

$$\Gamma^m_{ij} = \frac{1}{2}g^{mk}[\partial_j(g_{ik}) + \partial_i(g_{jk}) - \partial_k(g_{ij})]. \tag{2.7}$$

However, for $m = k$ we have that $\Gamma^m_{ij} \neq 0$. Then,

$$\begin{aligned} \Gamma^t_{ij} &= \frac{1}{2}g^{tt}[2\partial_i g_{jt} - \partial_t g_{ij}] \quad & \Gamma^r_{ij} &= \frac{1}{2}g^{rr}[2\partial_i g_{jr} - \partial_r g_{ij}] \\ \Gamma^\theta_{ij} &= \frac{1}{2}g^{\theta\theta}[2\partial_i g_{j\theta} - \partial_\theta g_{ij}] \quad & \Gamma^\phi_{ij} &= \frac{1}{2}g^{\phi\phi}[2\partial_i g_{j\phi} - \partial_\phi g_{ij}]. \end{aligned} \tag{2.8}$$

The terms different from zero are

$$\begin{aligned} \Gamma^t_{tt} &= \frac{1}{2}\partial_t\alpha & \Gamma^t_{tr} &= \frac{1}{2}\partial_r\alpha & \Gamma^t_{rr} &= \frac{1}{2}\mathrm{e}^{(\beta-\alpha)}\partial_t\beta \\ \Gamma^r_{tt} &= \frac{1}{2}\mathrm{e}^{(\alpha-\beta)}\partial_r\alpha & \Gamma^r_{tr} &= \frac{1}{2}\partial_t\beta & \Gamma^r_{rr} &= \frac{1}{2}\partial_r\beta \\ \Gamma^r_{\theta\theta} &= -r\mathrm{e}^{-\beta} & \Gamma^\theta_{r\theta} &= \frac{1}{r} & \Gamma^r_{\phi\phi} &= -r\mathrm{e}^{-\beta}\sin^2\theta \\ \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta & \Gamma^\phi_{r\phi} &= \frac{1}{r} & \Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}. \end{aligned} \tag{2.9}$$

2.1.2 Components of $R^\rho_{\mu\lambda\nu}$

The Riemann tensor is calculated using

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - \partial_\nu(\Gamma^\rho_{\mu\lambda}) - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}. \tag{2.10}$$

The components different from zero are

$$R^t_{rtr} = \frac{1}{2}\mathrm{e}^{(\beta-\alpha)}\left[\partial_t^2\beta + \frac{1}{2}(\partial_t\beta)^2 - \frac{1}{2}\partial_t\alpha\partial_t\beta\right] - \frac{1}{2}\left[\partial_r^2\alpha + \frac{1}{2}(\partial_r\alpha)^2 - \frac{1}{2}\partial_r\alpha\partial_r\beta\right] \tag{2.11}$$

$$\begin{aligned} R^t_{\theta t\theta} &= -\frac{1}{2}r\mathrm{e}^{-\beta}\partial_r\alpha & R^t_{\phi t\phi} &= -\frac{1}{2}r\mathrm{e}^{-\beta}\sin^2\theta\partial_r\alpha & R^t_{\theta r\theta} &= -\frac{1}{2}r\mathrm{e}^{-\alpha}\partial_t\beta \\ R^t_{\phi r\phi} &= -\frac{1}{2}r\mathrm{e}^{-\alpha}\sin^2\theta\partial_t\beta & R^r_{\theta r\theta} &= \frac{1}{2}r\mathrm{e}^{-\beta}\partial_r\beta & R^r_{\phi r\phi} &= \frac{1}{2}r\mathrm{e}^{-\beta}\sin^2\theta\partial_r\beta \\ & & R^\theta_{\phi\theta\phi} &= (1-\mathrm{e}^{-\beta})\sin^2\theta & & . \end{aligned} \tag{2.12}$$

2.1.3 Components of $R_{\mu\nu}$

The components of the Ricci tensor $R_{\mu\nu}$ are calculated by

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu(\Gamma^{\lambda}_{\mu\lambda}) - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{2.13}$$

The components different from zero are[2]:

$$\begin{aligned}
R_{tt} &= \frac{1}{2}\left[\partial_t^2\beta + \frac{1}{2}(\partial_t\beta)^2 - \frac{1}{2}\partial_t\alpha\partial_t\beta\right] - \frac{1}{2}\mathrm{e}^{(\alpha-\beta)}\left[\partial_r^2\alpha + \frac{1}{2}(\partial_r\alpha)^2 - \frac{1}{2}\partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right]\\
R_{rr} &= \frac{1}{2}\left[\partial_r^2\alpha + \frac{1}{2}(\partial_r\alpha)^2 - \frac{1}{2}\partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] - \frac{1}{2}\mathrm{e}^{(\beta-\alpha)}\left[\partial_t^2\beta + \frac{1}{2}(\partial_t\beta)^2 - \frac{1}{2}\partial_t\alpha\partial_t\beta\right]\\
R_{tr} &= -\frac{1}{r}\partial_t\beta\\
R_{\theta\theta} &= \frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha - \partial_r\beta) + (\mathrm{e}^{-\beta} - 1)\\
R_{\phi\phi} &= \left[\frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha - \partial_r\beta) + (\mathrm{e}^{-\beta} - 1)\right]\sin^2\theta.
\end{aligned} \tag{2.14}$$

2.1.4 Integration of the Janis-Newman-Winicour-Wyman equations

The field equations we are going to solve have the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa\left[\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\right]. \tag{2.15}$$

This equation can be expressed in a more convenient way [13]. Multiplying by $g^{\mu\nu}$, recalling that $g^{\lambda\mu}g_{\nu\lambda} = \delta^{\mu}_{\nu}$, and having into account that $A = g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi = \nabla^\beta\Phi\nabla_\beta\Phi$, we find

$$\begin{aligned}
g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R &= -\kappa\nabla^\nu\Phi\nabla_\nu\Phi + \frac{\kappa}{2}g^{\mu\nu}g_{\mu\nu}A\\
R - 2R &= -\kappa\nabla^\nu\Phi\nabla_\nu\Phi + 2\kappa A.
\end{aligned} \tag{2.16}$$

In the last equation ν is a dummy index; so that, we can rename it as β. Therefore,

$$R = -\kappa\nabla^\beta\Phi\nabla_\beta\Phi = -\kappa A. \tag{2.17}$$

Replacing R in (2.15)

$$\begin{aligned}
R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\kappa A &= -\kappa\nabla_\mu\Phi\nabla_\nu\Phi + \frac{1}{2}g_{\mu\nu}\kappa A\\
R_{\mu\nu} &= -\kappa\nabla_\mu\Phi\nabla_\nu\Phi.
\end{aligned} \tag{2.18}$$

The condition $\Box^2\Phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0$ is already included in (2.18). In order to see this, we can use the Bianchi identities [13]

$$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0. \tag{2.19}$$

A useful form of this identities comes from contracting twice (2.19),

$$\begin{aligned}
0 &= g^{\nu\sigma}g^{\mu\lambda}(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu})\\
&= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu}\\
&= 2\nabla^\mu R_{\rho\mu} - \nabla_\rho R.
\end{aligned} \tag{2.20}$$

[2]This components are Known as the Takeno's formulas[14].

Therefore,

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R. \tag{2.21}$$

Using this identity, and recalling that $\nabla_\rho g^{\mu\nu} = 0$ it is possible to show that $\Box^2\Phi = g^{\mu\nu}\nabla_\nu\nabla_\mu\Phi = 0$

$$\begin{aligned}
\nabla^\mu R_{\rho\mu} &= -\kappa\nabla^\mu(\nabla_\mu\Phi\nabla_\rho\Phi) \\
\nabla^\mu R_{\rho\mu} &= -\kappa(\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi - \kappa(\nabla^\mu\nabla_\rho\Phi)\nabla_\mu\Phi \\
\frac{1}{2}\nabla_\rho R &= -\kappa(\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi - \kappa(\nabla^\mu\nabla_\rho\Phi)\nabla_\mu\Phi \\
\frac{1}{2}\nabla_\rho(\kappa g^{\mu\nu}T_{\mu\nu}) &= -\kappa(\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi - \kappa(\nabla^\mu\nabla_\rho\Phi)\nabla_\mu\Phi.
\end{aligned} \tag{2.22}$$

Replacing the mathematical expression for $T_{\mu\nu}$ (Eq.1.54)

$$\begin{aligned}
-(\nabla^\mu\nabla_\mu)\nabla_\rho\Phi - (\nabla^\mu\nabla_\rho)\nabla_\mu\Phi = &\frac{1}{2}(\nabla_\rho\nabla_\mu\Phi)\nabla^\mu\Phi + \frac{1}{2}(\nabla_\rho\nabla_\nu\Phi)\nabla^\nu\Phi \\
&- (\nabla_\rho\nabla_\alpha\Phi)\nabla^\alpha\Phi - (\nabla_\rho\nabla_\beta\Phi)\nabla^\beta\Phi;
\end{aligned} \tag{2.23}$$

In this expression α and β are dummy indeces; so that, we can rename them as μ and ν respectively. Thus,

$$-\frac{1}{2}(\nabla_\rho\nabla_\mu\Phi)\nabla^\mu\Phi - \frac{1}{2}(\nabla_\rho\nabla_\nu\Phi)\nabla^\nu\Phi = -(\nabla^\mu\nabla_\mu)\nabla_\rho\Phi - (\nabla^\mu\nabla_\rho)\nabla_\mu\Phi. \tag{2.24}$$

In a similar way, The index μ in the last equation is a dummy index; therefore, we rename it as ν and we have that

$$-(\nabla_\rho\nabla_\nu\Phi)\nabla^\nu\Phi = -(\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi - (\nabla_\nu\nabla_\rho\Phi)\nabla^\nu\Phi. \tag{2.25}$$

In general, the covariant derivatives do not commute as partial derivative do. However, since Φ is a scalar function, the covariant derivative can be expressed as a partial derivative. Therefore,

$$\begin{aligned}
&(\nabla_\rho\nabla_\nu\Phi)\nabla^\nu\Phi = (\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi + (\nabla_\rho\nabla_\nu)\nabla^\nu\Phi \\
&(\nabla^\mu\nabla_\mu\Phi)\nabla_\rho\Phi = 0;
\end{aligned} \tag{2.26}$$

having into account that $\partial_\rho\Phi \neq 0$, we finally obtain

$$\nabla^\mu\nabla_\mu\Phi = g^{\mu\nu}\nabla_\nu\nabla_\mu\Phi = \Box^2\Phi = 0. \tag{2.27}$$

Field equations:

From equation (2.18) and using the calculations of $R_{\mu\nu}$ in the last section we obtain a system of differential equations where Φ is the scalar potential. Therefore,

$$\begin{aligned}
-\kappa(\nabla_t\Phi)^2 &= \frac{1}{2}\left[\partial_t^2\beta+\frac{1}{2}(\partial_t\beta)^2-\frac{1}{2}\partial_t\alpha\partial_t\beta\right]-\frac{1}{2}\mathrm{e}^{(\alpha-\beta)}[\partial_r^2\alpha+\frac{1}{2}(\partial_r\alpha)^2-\frac{1}{2}\partial_r\alpha\partial_r\beta \\
&\quad +\frac{2}{r}\partial_r\alpha] \\
-\kappa(\nabla_r\Phi)^2 &= \frac{1}{2}\left[\partial_r^2\alpha+\frac{1}{2}(\partial_r\alpha)^2-\frac{1}{2}\partial_r\alpha\partial_r\beta-\frac{2}{r}\partial_r\beta\right]-\frac{1}{2}\mathrm{e}^{(\beta-\alpha)}[\partial_t^2\beta+\frac{1}{2}(\partial_t\beta)^2 \\
&\quad -\frac{1}{2}\partial_t\alpha\partial_t\beta] \\
-\kappa\nabla_r\Phi\nabla_t\Phi &= -\frac{1}{r}\partial_t\beta \\
-\kappa(\nabla_\theta\Phi\nabla)^2 &= \frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha-\partial_r\beta)+(\mathrm{e}^{-\beta}-1) \\
-\kappa(\nabla_\phi\Phi) &= \left[\frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha-\partial_r\beta)+(\mathrm{e}^{-\beta}-1)\right]\sin^2\theta \\
g^{\mu\sigma}\nabla_\sigma\nabla_\mu\Phi &= 0.
\end{aligned} \tag{2.28}$$

In [12] it is assumed that both α and β are functions of r only. Moreover, the assumption that the line element (2.1) is static an spherically symmetric, imply that $\nabla_\nu\Phi=\partial_\nu\Phi$ also has this properties.This does not, however, required that $\Phi=\Phi(r,t)$ be independent of t. For the moment, it will only be assumed that $\partial_\nu\Phi$ is spherically symmetric and has been placed into the form

$$\partial_\nu\Phi=(\partial_t\Phi,\partial_r\Phi,0,0) \tag{2.29}$$

In this sense, the differential equations system (2.28) can be expressed as

$$\begin{aligned}
-\kappa(\partial_t\Phi)^2 &= -\frac{1}{2}\mathrm{e}^{(\alpha-\beta)}\left[\partial_r^2\alpha+\frac{1}{2}(\partial_r\alpha)^2-\frac{1}{2}\partial_r\alpha\partial_r\beta+\frac{2}{r}\partial_r\alpha\right] \\
-\kappa(\partial_r\Phi)^2 &= \frac{1}{2}\left[\partial_r^2\alpha+\frac{1}{2}(\partial_r\alpha)^2-\frac{1}{2}\partial_r\alpha\partial_r\beta-\frac{2}{r}\partial_r\beta\right] \\
0 &= -\kappa\partial_r\Phi\partial_t\Phi \\
0 &= \frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha-\partial_r\beta)+(\mathrm{e}^{-\beta}-1) \\
0 &= \left[\frac{1}{2}r\mathrm{e}^{-\beta}(\partial_r\alpha-\partial_r\beta)+(\mathrm{e}^{-\beta}-1)\right]\sin^2\theta \\
0 &= g^{\mu\sigma}\nabla_\sigma\nabla_\mu\Phi.
\end{aligned} \tag{2.30}$$

The equations for $R_{\theta\theta}$ and $R_{\phi\phi}$ are the same. This leave us with a system of five differential equation. Moreover, as α and β depend only on r, the partial derivatives are total derivatives

$$\begin{aligned}
\kappa(\dot{\Phi})^2\mathrm{e}^{(\beta-\alpha)} &= \frac{1}{r}\alpha'+\frac{1}{2}\left[\alpha''+\frac{1}{2}(\alpha')^2-\frac{1}{2}\alpha'\beta'\right] \\
-\kappa(\partial_r\Phi)^2 &= -\frac{1}{r}\partial_r\beta+\frac{1}{2}\left[\alpha''+\frac{1}{2}(\alpha')^2-\frac{1}{2}\alpha'\beta'\right] \\
(\alpha'-\beta') &= \frac{2}{r}(\mathrm{e}^{\beta}-1) \\
\partial_r\Phi\dot{\Phi} &= 0 \\
\Box\Phi &= 0.
\end{aligned} \tag{2.31}$$

Where $'$ and $\dot{}$ are $\frac{d}{dr}$ and $\frac{\partial}{\partial t}$ respectively. The fourth equation in (2.31) has two possibilities:

1. The first possibility is to consider $\partial_r\Phi \neq 0$ and $\partial_t\Phi = 0$.
2. The second posibility is to consider $\partial_r\Phi = 0$ and $\partial_t\Phi \neq 0$.

We are interested in the first case. The second case can be studied from [12].

Case $\partial_r\Phi \neq 0$ y $\partial_t\Phi = 0$:

For $\partial_r\Phi \neq 0$ and $\partial_t\Phi = 0$, the first equation in (2.31) is:

$$\frac{1}{r}\alpha' + \frac{1}{2}\left[\alpha'' + \frac{1}{2}(\alpha')^2 - \frac{1}{2}\alpha'\beta'\right] = 0, \tag{2.32}$$

this equation can be expressed as[3]

$$\begin{aligned} \frac{2}{r}\alpha' \mathrm{e}^{(\alpha-\beta)} + \mathrm{e}^{(\alpha-\beta)}\left[\alpha'' + \frac{1}{2}(\alpha')^2 - \frac{1}{2}\alpha'\beta'\right] = 0 \\ \mathrm{e}^{(\alpha-\beta)/2}\alpha'' + \frac{1}{2}\mathrm{e}^{(\alpha-\beta)/2}(\alpha')^2 - \frac{1}{2}\beta'\alpha'\mathrm{e}^{(\alpha-\beta)/2} + \frac{2}{r}\mathrm{e}^{(\alpha-\beta)/2}\alpha' = 0. \end{aligned} \tag{2.33}$$

But,

$$\frac{d}{dr}\left(\mathrm{e}^{(\alpha-\beta)/2}\alpha'\right) = \mathrm{e}^{(\alpha-\beta)/2}\alpha'' + \frac{1}{2}\mathrm{e}^{(\alpha-\beta)/2}(\alpha')^2 - \frac{1}{2}\alpha'\beta'\mathrm{e}^{(\alpha-\beta)/2}. \tag{2.34}$$

then

$$\begin{aligned} \frac{d}{dr}\left(\mathrm{e}^{(\alpha-\beta)/2}\alpha'\right) + \frac{2}{r}\mathrm{e}^{(\alpha-\beta)/2}\alpha' = 0; \quad U = \mathrm{e}^{(\alpha-\beta)/2}\alpha' \\ \frac{dU}{dr} + \frac{2}{r}U = 0, \end{aligned} \tag{2.35}$$

whose solution is

$$r^2\alpha'\mathrm{e}^{(\alpha-\beta)/2} = h; \tag{2.36}$$

where h is a constant. For $\partial_r\Phi \neq 0$ and $\partial_t\Phi = 0$, the fifth equation in (2.31) is

$$\frac{\partial}{\partial r}(r^2\mathrm{e}^{(\alpha-\beta)/2}\partial_r\Phi) = 0, \tag{2.37}$$

whose solution is

$$r^2\mathrm{e}^{(\alpha-\beta)/2}\partial_r\Phi = k. \tag{2.38}$$

Dividing (2.36) and (2.38) between them an integrating [4], we find that[5]

$$\begin{aligned} \alpha' &= c\Phi' \\ \alpha &= c\Phi + c' \\ \alpha &= c\Phi. \end{aligned} \tag{2.39}$$

Where c' has been absorbed by the line element defining a new time $dt = \mathrm{e}^{c'}dt$. Therefore, the proportionality between α and Φ allow us to find e^{β} using (2.38)

$$\mathrm{e}^{\beta} = \frac{1}{k^2}r^4\mathrm{e}^{c\Phi}(\partial_r\Phi)^2. \tag{2.40}$$

[3] Where we have rewritten the term $\mathrm{e}^{(\beta-\alpha)/2}$.

[4] we made $c = \frac{h}{k}$ when dividing

[5] Here we have had into account that Φ only depends on r because $\dot{\Phi} = 0$. in this sense, $d\Phi = \partial_r\Phi dr + \partial_t\Phi dt$ is reduced to $\frac{d\Phi}{dr} = \partial_r\Phi$.

In this way, the line element is

$$\begin{aligned} ds^2 &= \mathrm{e}^{c\Phi}dt^2 - \frac{1}{k^2}r^4\mathrm{e}^{c\Phi}(\partial_r\Phi)^2dr^2 - r^2d\Omega^2 \\ ds^2 &= \mathrm{e}^{c\Phi}dt^2 - \frac{r^4\mathrm{e}^{c\Phi}}{k^2}(d\Phi)^2 - r^2d\Omega^2. \end{aligned} \tag{2.41}$$

This suggest the transformation of coordinates $\bar{r} = \Phi(r)$ to obtain $r = r(\bar{r})$ and $\partial_\nu\Phi = (0,1,0,0)$. So that the metric is

$$ds^2 = \mathrm{e}^{c\bar{r}}dt^2 - \frac{r^4\mathrm{e}^{c\bar{r}}}{k^2}d\bar{r}^2 - r^2d\Omega^2. \tag{2.42}$$

The constant k^2 can be absorbed by linear translation of $\bar{r}$. Therefore, $\bar{r} \to \bar{r} + b$, where b is a constant

$$\begin{aligned} ds^2 &= \mathrm{e}^{c(\bar{r}+b)}dt^2 - \frac{r^4\mathrm{e}^{c(\bar{r}+b)}}{k^2}d\bar{r}^2 - r^2d\Omega^2 \\ ds^2 &= \mathrm{e}^{c\bar{r}}\mathrm{e}^{cb}dt^2 - \frac{r^4\mathrm{e}^{c\bar{r}}\mathrm{e}^{cb}}{k^2}d\bar{r}^2 - r^2d\Omega^2 \\ ds^2 &= \mathrm{e}^{c\bar{r}}dt^2 - r^4(\bar{r})\mathrm{e}^{c\bar{r}}d\bar{r}^2 - r^2(\bar{r})d\Omega^2. \end{aligned} \tag{2.43}$$

The function $r = r(\bar{r})$ is unknown. Then, making the variable chance $r(\bar{r}) = 1/W(\bar{r})$ we have

$$ds^2 = \mathrm{e}^{cr}dt^2 - W^{-4}(r)\mathrm{e}^{cr}dr^2 - W^{-2}(r)d\Omega^2. \quad ^6 \tag{2.44}$$

The differential equations can be obtained from the components of the Riemann tensor. For the line element (2.44) the components different from zero are [7]

$$R_{rr}W = -2W'' \quad R_{\theta\theta}\mathrm{e}^{cr} = (W')^2 - WW'' - \mathrm{e}^{cr} \quad R_{\phi\phi} = [(W')^2 - WW'' - \mathrm{e}^{cr}]\sin^2\theta. \tag{2.45}$$

According to equation (2.18), the differential equations for (2.44) are [8]:

$$W'' - cW' + \frac{1}{2}\kappa W = 0 \quad WW'' - (W')^2 = -\mathrm{e}^{2cr}. \tag{2.46}$$

the first equation in (2.46) is a second order equation with constant coefficients. The solution has the form

$$W = A\mathrm{e}^{m_1 r} + B\mathrm{e}^{m_2 r}, \tag{2.47}$$

where A and B are arbitrary constants and m_1 y m_2 need to be found. Replacing (2.47) into first equation of (2.46)

$$\left(m_1^2 - cm_1 + \frac{\kappa}{2}\right)A\mathrm{e}^{m_1 r} + \left(m_2^2 - cm_2 + \frac{\kappa}{2}\right)B\mathrm{e}^{m_2 r} = 0. \tag{2.48}$$

This equation is satisfied when m_1 y m_2

$$\begin{aligned} m_+ &= \frac{\left[c + \sqrt{c^2 - 4\kappa}\right]}{2} \\ m_- &= \frac{\left[c - \sqrt{c^2 - 4\kappa}\right]}{2}. \end{aligned} \tag{2.49}$$

Therefore, (2.47) can be expressed as

[6] Al final se a cambiado $\bar{r}$ por r.

[7] We have used W' to express $\frac{dW}{dr}$

[8] Here we used that $\partial_i\Phi = (0,1,0,0)$. The differential equations for para $R_{\theta\theta}$ y $R_{\phi\phi}$ are the same

$$W = A\mathrm{e}^{rm_+} + B\mathrm{e}^{rm_-}. \tag{2.50}$$

The relation between the constants A y B can be found using the second equation of (2.46). Replacing we find

$$\begin{aligned} W &= A\mathrm{e}^{rm_+} + B\mathrm{e}^{rm_-} \\ W' &= Am_+\mathrm{e}^{rm_+} + Bm_-\mathrm{e}^{rm_-} \\ W'' &= Am_+^2\mathrm{e}^{rm_+} + Bm_-^2\mathrm{e}^{rm_-} \end{aligned} \tag{2.51}$$

$$WW'' - (W')^2 = (m_+ - m_-)^2 BA\mathrm{e}^{(m_+ + m_-)r} = -\mathrm{e}^{cr}.$$

Finally,

$$(c^2 + 2\kappa)BA = -1. \tag{2.52}$$

On the other hand, (2.50) is solution of the system (2.31) if and only if both A and B satisfy (2.52).

This solution will be asymptotically flat if (2.50) vanish at $r = 0$. Above, we had made $r = \Phi$ in (2.44). So that, the condition $r = 0$ is equivalent to $\Phi = 0$. Now, we want to find a coordinate system in which for $r \to \infty$ the space is asymptotically flat ($\Phi = 0$). In order to do so, we can make $r = 1/\bar{r}$ which is equivalent to $\Phi = 1/\bar{r}$. Therefore, the point at $r = 0$ has been transformed to the point at infinity. Then, when $r \to \infty$ equation (2.50) takes the form

$$\begin{aligned} W &= A\mathrm{e}^{m_+/\bar{r}} + B\mathrm{e}^{m_-/\bar{r}} = A + B = 0 \\ A &= -B. \end{aligned} \tag{2.53}$$

from equation (2.52) we find that

$$A = -B = \frac{1}{(c^2 + 2\kappa)^{1/2}}. \tag{2.54}$$

Therefore (2.50) is

$$\begin{aligned} W &= \frac{\mathrm{e}^{\frac{c}{2\bar{r}}}}{(c^2 + 2\kappa)^{1/2}} (\mathrm{e}^{\frac{(c^2+2\kappa)^{1/2}}{2\bar{r}}} - \mathrm{e}^{-\frac{(c^2+2\kappa)^{1/2}}{2\bar{r}}}) \\ &= 2\frac{\mathrm{e}^{\frac{c}{2\bar{r}}}}{(c^2 + 2\kappa)^{1/2}} \sinh\left(\frac{(c^2 + 2\kappa)^{1/2}}{2\bar{r}}\right) \\ \gamma &= \frac{(c^2 + 2\kappa)^{1/2}}{2} \\ W &= \frac{\mathrm{e}^{\frac{c}{2\bar{r}}}}{\gamma} \sinh(\gamma\bar{r}^{-1}). \end{aligned} \tag{2.55}$$

And the line element (2.44)

$$\begin{aligned} ds^2 &= \mathrm{e}^{c\bar{r}^{-1}} dt^2 - \left[\frac{\mathrm{e}^{\frac{c}{2\bar{r}}}}{\gamma} \sinh(\gamma\bar{r}^{-1})\right]^{-4} \frac{\mathrm{e}^{c\bar{r}^{-1}}}{\bar{r}^4} d\bar{r}^2 - \left[\frac{\mathrm{e}^{\frac{c}{2\bar{r}}}}{\gamma} \sinh(\gamma\bar{r}^{-1})\right]^{-2} \frac{1}{\bar{r}^2} d\Omega^2 \\ &= \mathrm{e}^{c\bar{r}^{-1}} dt^2 - \left[\frac{\gamma\bar{r}^{-1}}{\sinh(\gamma\bar{r}^{-1})}\right]^4 \mathrm{e}^{-c\bar{r}^{-1}} d\bar{r}^2 - \left[\frac{\gamma\bar{r}^{-1}}{\sinh(\gamma\bar{r}^{-1})}\right]^2 \mathrm{e}^{-c\bar{r}^{-1}} d\Omega^2. \end{aligned} \tag{2.56}$$

2.2 Sayan Kar metric:

The most general static and spherically symmetric solution to the Einstein massless scalar equations was independently obtained by Janis, Newman, and Winicour [11], as well as Wyman [12]. As both solutions were available in different coordinates, they were not know to be the same until Virbhabra [15] showed the equivalence between the two by a coordinate transformation. As Janis, Newman and Winicour obtained this solution about 13 years before Wyman, it is usually to call it the Janis-Newman-Winicour solution. However, in this work, we are going to call it as Janis-Newman-Winicour-Wyman solution. Thus, this solution,(characterized by constant and real parameters, the **ADM** mass m, and the scalar charge σ) is expressed by the line element

$$ds^2 = \left(1 - \frac{2\eta}{r}\right)^{\gamma} dt^2 - \left(1 - \frac{2\eta}{r}\right)^{-\gamma} dr^2 - \left(1 - \frac{2\eta}{r}\right)^{1-\gamma} dr^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.57)$$

and the massless scalar field

$$\Phi = \frac{\sigma}{2\eta}\ln\left(1 - \frac{2\eta}{r}\right), \quad (2.58)$$

with

$$\gamma = \frac{m}{\eta} \quad \text{and} \quad \eta = \sqrt{m^2 + \sigma^2} \quad (2.59)$$

The Kar's metric comes from the field equations generated by the action

$$S_{eff} = \int \left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - F_{\mu\nu}F^{\mu\nu}\right] d^4x\sqrt{-g}e^{-2\Phi}, \quad (2.60)$$

where Φ is the scalar potential, $F_{\mu\nu}$ is the electromagnetic tensor and $H_{\mu\nu\rho}$ is related to the antisymmetric tensor field, $B_{\mu\nu}$ by

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} - [\Omega_3(A)_{\mu\nu\rho}], \quad (2.61)$$

with the gauge Chern-Simons term

$$[\Omega_3(A)]_{\mu\nu\rho} = \frac{1}{4}(A_\mu F_\nu\rho + A_\nu F_{\rho\mu} + A_\rho F_\mu\nu). \quad (2.62)$$

The corresponding equations of motion obtained by performing variations with respect to the metric, $g_{\mu\nu}$; the antisymmetric tensor field, $B_{\mu\nu}$; the scalar potential, Φ; and the vector potential, A_μ, are given as follow[16]:

$$\begin{aligned}
R_{\mu\nu} &= -2\nabla_\mu\nabla_\nu\Phi + 2F_{\mu\lambda}F^{\lambda}_{\nu} + \frac{1}{4}H_{\mu\lambda\sigma}H^{\lambda\sigma}_{\nu} \\
&\nabla^\nu(\mathrm{e}^{-2\Phi}F_{\mu\nu}) + \frac{1}{12}\mathrm{e}^{-2\Phi}H_{\mu\nu\rho}F^{\mu\rho} = 0 \\
&\nabla^\nu(\mathrm{e}^{-2\Phi}H_{\mu\nu\rho}) = 0 \\
&4\nabla^2\Phi - 4(\nabla\Phi)^2 + R - F^2 - \frac{1}{12}H^2 = 0.
\end{aligned} \quad (2.63)$$

Assuming the antisymmetric tensor field to be zero they obtain the following spherically symmetric, static solution

$$\begin{aligned}
ds^2_{str} &= \left(1 - \frac{2\eta}{r}\right)^{(m+\sigma)/\eta} dt^2 - \left(1 - \frac{2\eta}{r}\right)^{(\sigma-m)/\eta} dr^2 - \left(1 - \frac{2\eta}{r}\right)^{1+(\sigma-m)/\eta} r^2 d\Omega^2 \\
\Phi &= \frac{\sigma}{2\eta}\ln\left(1 - \frac{2\eta}{r}\right),
\end{aligned} \quad (2.64)$$

where m is the mass, σ is the scalar charge and η is given by $\eta^2 = m^2 + \sigma^2$. For $\sigma = 0$, this solution reduces to the Schwarzschild solution.

In order to obtain equation (2.64) from equation (2.57), we are going to use the standard relations between the tow metrics described by

$$g_{\mu\nu}^{str} = e^{2\Phi} g_{\mu\nu}^{E}, \tag{2.65}$$

where, $g_{\mu\nu}^{E}$ is each component of equation (2.57). Therefore, from equation (2.58) we have

$$e^{2\Phi} = e^{\frac{\sigma}{\eta}\ln\left(1-\frac{2\eta}{r}\right)} = \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma}{\eta}}. \tag{2.66}$$

The equation (2.65) takes the form

$$g_{\mu\nu}^{str} = \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma}{\eta}} g_{\mu\nu}^{E}. \tag{2.67}$$

Finally,

$$ds_{str}^2 = g_{\mu\nu}^{str} dx^\mu dx^\nu = \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma}{\eta}} g_{\mu\nu}^{E} dx^\mu dx^\nu. \tag{2.68}$$

Which reduces to equation (2.64). The reality of the metric coefficients indicates that we confine ourselves to the domain $r \geq \eta$. Even for $m = p\eta$, $\sigma = q\eta$, with p,q as integers we end up with $p^2 + q^2 = 1$ which contradicts the assumption that p, q are integers. We also see later that the metric has a naked singularity at $r = 2\eta$ [16].

Chapter 3

BOZZA'S METHOD

3.1 Introduction:

General Theory of Relativity has been considered one of the most important theories human kind have ever created. As far as we know, this theory establishes a new conception of space and time describing how the curvature of space-time acts on matter to manifest itself as gravity, and how energy and momentum influence space-time to create curvature: *Space tells matter how to move; matter tells space how to curve*[8]. As a consequence, Einstein's theory of gravity has important astrophysical implications among which gravitational lensing is one of the most important. Although Newton had suggested the idea of light been deflected by gravity in 1704 and Johann Georg von Soldner calculated the amount of deflection of a light ray from a star under Newtonian gravity in 1804[1]; it was not until Einstein that this idea took place and a correct calculation of the amount of deflection was made. In [17] Einstein explains this idea by means of the equivalence principle. In his paper, Einstein considered two systems of co-ordinates: a stationary system of co-ordinates K, in a homogeneous gravitational field (acceleration of gravity ε), oriented so that the lines of force of the gravitational field run in the negative direction of the axis of z; and a second system of co-ordinates K' moving with uniform acceleration ε in the positive direction of its axis of z in a space free of gravitational fields. As far as we know, because of the Galileo's principle in the accelerated system K' and from the experience that all bodies are equally and uniformly accelerated in the system K, because it is at rest in a homogenous gravitational field, material points which are not subjected to the action of other material points move in keeping with equations $\ddot{x} = 0$, $\ddot{y} = 0$ and $\ddot{z} = -\varepsilon$ relative to both K and K'. This consideration brings important implications if we assume that the two co-ordinate systems are equivalent; that is, if we may regard K as been in a space free of gravitational fields and moving with uniform acceleration. Because of this equivalence, we can not differentiate one system to the other and it is impossible for us to speak of the absolute acceleration of the system of reference. Although, this conclusion is very important it will not have any deeper significance unless the systems K and K' are equivalent with respect to all physical processes; that is: *all physical laws with respect to K are in entire agreement with those in K'*. In his treatise, Einstein considers three important implications or consequences of this principle: the gravitation of energy, time and the velocity of light in the gravitational field and bending of light-rays.

The possibility of using the equivalence principle to study the behavior of light in presence of gravitational fields is very interesting. As the principle states, all physical laws with respect to K are in entire agreement with those with respect to K'. This means that we can study the behavior of light in one system of co-ordinates, for example: the one moving with uniform acceleration in a space free of gravitational field, and realize then

[1] Professor Castañeda's lecture on gravitational lensing

that the results of experiments performed in K' are the same to those performed in K. For example; it is known, because the special theory of relativity states so, that the inertia mass of a body increases with the energy it contains: the famous equation $E = mc^2$. Now ¿Is it possible for the gravitating mass to increase when the inertia mass does? As Einstein suggested:"*...the hypothesis of equivalence of the systems* $\boldsymbol{K}$ *and* $\boldsymbol{K}'$ *gives us gravitating of energy as a necessary condition*"[17] By means of this hypothesis, we are allowed to see the process of a definite quantity of energy E_2 being emitted from S_2 toward S_1 (rigidly separated along the z axis a distant h from each other) in the system of reference K', and judge the process of the transference of energy by radiation from a system K_0 which is to be free from acceleration. With this hypothesis Einstein concluded, by the ordinary theory of relativity, that the radiation arriving at S_1 does not possess the energy E_2, but a grater energy E_1[2], that is;

$$E_1 = E_2 \left(1 + \frac{\varepsilon h}{c^2}\right). \tag{3.1}$$

The same relation holds in the system K which is not moving with constant acceleration, but provided with a gravitational field; in this case, however, we have to change εh by the potential Φ of the gravitation vector on S_2[3]. In this way Einstein has shown us that the difference $E_1 - E_2$ is the potential energy of the mass E_2/c^2 in the gravitational field and we have to ascribe to the energy E_2, before emitted, the potential energy due to gravity for the fulfilment of the principle of energy. This increase in the potential energy correspond to an increase E_2/c^2 in the gravitating mass which is equal to the increase of the inertia mass, as given by the theory of relativity; for this reason "*energy must posses gravitational mass which is equal to its inertia mass*" and in consequence it gravitates[17].

Other important consequence, emerging from the equivalence principle, regards the velocity of light and time. Suppose we emit radiation from S_2 toward S_1 in the system of coordinates K'; the radiation emitted having the frequency ν_2 relative to a clock in S_2 ¿What happens with the frequency once the radiation arrives at S_1 if we judge the process from the unaccelerated system of coordinate K_0? From K_0 we see K' moving with constant acceleration ε; if K' has not velocity when emission occurs, then S_1, at the time of arrival will have, with respect to K_0, the velocity $\varepsilon h/c$. Einstein considered this experiment into first approximation; that is: the velocity of K', relative to K_0, each instant of time is smaller than that of light. If this is the case, then in $t = h/c$, S_1 moves toward S_2 with $v = \varepsilon h/c$ and we can calculate, via Doppler effect, what an observer, in a system of coordinates in which S_1 is at rest, see. As special Theory of relativity states, an observer who moves toward a source will see a frequency ν' relative to a system of coordinates in which the observer is at rest. The relation between the frequency in the source ν and the frequency ν', when calculated in the case of the observer approaching the source, is [18]

$$\nu' = \nu \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)^{\frac{1}{2}}. \tag{3.2}$$

In our case ν' is ν_1 and ν is ν_2. However, to a first approximation[4] and replacing v as $\varepsilon h/c$, equation (6.11) takes the form

$$\nu_1 = \left(1 + \varepsilon \frac{h}{c^2}\right) \nu_2. \tag{3.3}$$

As equation (3.3) shows, the radiation being emitted, in the uniformly accelerated system K', from S_2 toward S_1 has a grater frequency when it arrives to S_1. Using the equivalence principle, the same thing must happen in the stationary system of co-ordinates K, provided

[2]See equation (1) in [17]
[3]See equation (2) in [17]
[4]If we expand ν'/ν in a Taylor's series around $v/c = 0$ and then take terms up to first order

with a uniform gravitational field, if there the transference by radiation takes place as described. In this case, we have to replace εh by Φ in S_2, and equation (3.3) takes the form

$$\nu_1 = \left(1 + \frac{\Phi}{c^2}\right)\nu_2. \tag{3.4}$$

The last equation seems to have a serious problem regarding time. This "problem", as Einstein pointed out, was: *¿How can any other number of periods per second arrive in S_1 than is emitted in S_2?*[17] The key point to answer this question is that we can not consider the frequencies, ν_2 and ν_1, as the number of periods per second because we have not yet determined the time in the system K. All we know is that ν_2 is the number of periods per second with reference to the time-unit of the clock U in S_2 and ν_1 denotes the number of periods per second with reference to an identical clock in S_1. This means that we can not assume that clocks placed in different gravitation potentials must go at the same rate. Therefore, the two clocks in S_1 and S_2 do not both give the "time" correctly. In this sense, it is necessary to define what time is in such a way that the number of crests and troughs between S_2 and S_1 is independent of the absolute value of time[17]. For ν_1 is grater than ν_2, the clock U at S_2 takes less time to measured the signal coming from S_2 when compared to the clock at S_2. Then ¿How many times more slowly the clock at S_2 goes? As equation (3.4) shows,

$$t_2 = \left(1 + \frac{\Phi}{c^2}\right)t_1; \tag{3.5}$$

thus, a clock placed at S_2 goes $\left(1 + \frac{\Phi}{c^2}\right)$ times more slowly than a clock in S_1. In this sense, for measuring time at places which, relatively to the origin of the co-ordinates, has the gravitational potential Φ, we must employ a clock which, when remove to the origin of co-ordinates, goes $\left(1 + \frac{\Phi}{c^2}\right)$ times more slowly than the clock used for measuring time at the origin of coordinates. If we call the velocity of light at the origin c_0 then the velocity of light at a place with the gravitational potential Φ will be given by the relation

$$c = \left(1 + \frac{\Phi}{c^2}\right)c_0. \tag{3.6}$$

Therefore, the coordinate velocity of light in the gravitational field is a function of the place. As a consequence, Einstein estimated that a ray of light going past the Sun undergo deflexion to the amount of $4 \cdot 10^{-6} = 0.83$arcs. In this sense, the angular distant of the star from the center of the Sun appears to be increased by this amount. The consequence of the equivalent principle, regarding bending of light, opens the opportunity to compare his calculation with experience, as Einstein suggested:"*It would be a most desirable thing if astronomers would take up the question here raised. For apart from any theory there is the question whether it is possible with the equipment at present available to detect an influence of gravitational fields on the propagation of light.* " However, Einstein's prediction had to wait until 1919 to be confirmed during a solar eclipse. In the experiment, Arthur Eddington observed that light from stars passing close to the sun was slightly bent, so that stars appeared slightly out of position.

The lens-like action was proposed by Einstein in 1936. His idea was to consider the lens-like action of a star B by the deviation of light coming form a star A in the gravitational field of B. In [19], Einstein used the law of deviation to show that an observer situated exactly on the extension of the central line connecting the stars A and B will perceive a luminius circle of the angular radius β around the center of B. Einstein believed that such a phenomenon could not be seen directly because it is necessary to approach closely enough to the central line; furthermore, the light coming from the luminous circle can not be distinguished by an observer as geometrically different from that coming from the star

B because it will manifest itself as increased apparent brightness of B. However, it was not until 1979 that the first gravitational lens would be discovered. At first, it looked like two identical quasi-stellar objects so that it was known as the "Twin QSO". Now, It is officially named SBS 0957+561. In the last years, recent works have considered how gravitational lensing studies might possibly distinguish between Schwarzschild black holes and the naked singularities that can occur for example if there is a massless scalar field present in a spherically symmetric space-time[20]. Given that no proof is known for the cosmic censorship hypothesis, and the importance of this hypothesis for gravitational physics, it is a worthwhile research project to investigate the distinctive observational features of naked singularities and black holes[20]. All these studies have been performed from the weak field limit point of view. Nowadays, the scientific community has been interested in the lensing properties from the strong field limit point of view. These studies will be very important because in order to show deviations from General Relativity it is necessary to probe exact equations of motion in some way. Moreover, deviation of light rays in the strong field limit is one of the most promising grounds where a theory of gravitation can be tested in its full form.

In this chapter we are going to study the influence of gravitation on the propagation of light from the strong field limit point of view. Our main purpose here is to get an analytical expansion of the deflection angle. First, we deduce this deflection using the general theory of relativity. In order to do so, we begin with the equation of motion of the freely falling material particle or photon in a spherically symmetric metric. In this sense, we follow the procedure proposed in [21] to obtain a more general expression for de deflection angle. Then, we explain the method proposed by V. Bozza [22] in detail. Some examples studied in [22] are reproduced in the appendix A. Finally, we use this method to calculate the deflection angle for a metric proposed by Sayan Kar [16].

3.2 General equations of motion:

We now considered the motion of the freely falling material particle or photon in a static isotropic gravitational field. First let us consider the most general such metric[5]

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - D(r)r^2 d\theta^2 - D(r)r^2 \sin^2\theta d\phi^2. \tag{3.7}$$

The equations of free fall are [21]

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0, \tag{3.8}$$

where p is a parameter describing the trajectory. In general $d\tau$ is proportional to dp, so for a material particle we could normalize p so that $p = \tau$. However, for a photon the proportionality constant $d\tau/dp$ vanish, and since we wish to treat photons as well as massive particles, we shall find it convenient to reserve the right to fix the normalization of p independently from that of τ.

Using MAPLE 13 and the metric (3.7), the nonzero Christoffel's symbols in equation (3.8) are:

[5] We use the $(+---)$ signature.

$$\begin{aligned}
\Gamma^r_{\phi\phi} &= -\frac{1}{2}\frac{r^2\sin^2\theta D'(r) + 2r\sin^2\theta D(r)}{B(r)} \\
\Gamma^r_{rr} &= \frac{1}{2B(r)}\frac{dB(r)}{dr} \\
\Gamma^r_{\theta\theta} &= -\frac{1}{2}\frac{r^2 D'(r) + 2rD(r)}{B(r)} \\
\Gamma^r_{tt} &= \frac{1}{2B(r)}\frac{dA(r)}{dr}
\end{aligned}$$

$$\begin{aligned}
\Gamma^\theta_{r\theta} &= \frac{1}{2}\frac{rD'(r) + 2D(r)}{rD(r)} = \Gamma^\theta_{\theta r} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi r} &= \frac{1}{2}\frac{rD'(r) + 2D(r)}{rD(r)} = \Gamma^\phi_{\phi r} \\
\Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta} = \cot\theta = \Gamma^\phi_{\phi\theta} \\
\Gamma^t_{tr} &= \frac{1}{2A(r)}\frac{dA(r)}{dr} = \Gamma^t_{rt}.
\end{aligned} \tag{3.9}$$

From (3.8), the equations for r, θ, ϕ, t are [6]:

3.2.1 Equation for r:

For the r coordinate we have[7]

$$\begin{aligned}
0 &= \frac{d^2x^r}{dp^2} + \Gamma^r_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} \\
&= \frac{d^2r}{dp^2} + \Gamma^r_{rr}\left(\frac{dr}{dp}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{dp}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dp}\right)^2 + \Gamma^r_{tt}\left(\frac{dt}{dp}\right)^2 \\
&= \frac{d^2r}{dp^2} + \frac{1}{2}\frac{A'(r)}{B(r)}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2}\frac{B'(r)}{B(r)}\left(\frac{dr}{dp}\right)^2 - \frac{1}{2}\frac{r^2D'(r) + 2rD(r)}{B(r)}\left(\frac{d\theta}{dp}\right)^2 \\
&- \frac{1}{2}\frac{r^2\sin^2\theta D'(r) + 2r\sin^2\theta D(r)}{B(r)}\left(\frac{d\phi}{dp}\right)^2.
\end{aligned} \tag{3.10}$$

3.2.2 Equation for θ:

As all we know, the Christoffel's symbols are symmetric: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$; for this reason we have to consider twice in the sum of equation (3.8).

$$\begin{aligned}
0 &= \frac{d^2x^\theta}{dp^2} + \Gamma^\theta_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} \\
&= \frac{d^2\theta}{dp^2} + 2\Gamma^\theta_{r\theta}\frac{d\theta}{dp}\frac{dr}{dp} + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{dp}\right)^2 \\
&= \frac{d^2\theta}{dp^2} + \frac{2}{2}\frac{rD'(r) + 2D(r)}{rD(r)}\frac{dr}{dp}\frac{d\theta}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2.
\end{aligned} \tag{3.11}$$

[6] A prime denotes d/dr

[7] $x^r \equiv r$, $x^\phi \equiv \phi$, $x^t \equiv t$ and $x^\theta \equiv \theta$

3.2.3 Equation for ϕ:

$$\begin{aligned}
0 &= \frac{d^2x^\phi}{dp^2} + \Gamma^\phi_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} \\
&= \frac{d^2\phi}{dp^2} + 2\Gamma^\phi_{r\phi}\frac{d\phi}{dp}\frac{dr}{dp} + 2\Gamma^\phi_{\phi\theta}\frac{d\phi}{dp}\frac{d\theta}{dp} \\
&= \frac{d^2\phi}{dp^2} + \frac{2}{2}\frac{rD'(r)+2D(r)}{rD(r)}\frac{d\phi}{dp}\frac{dr}{dp} + 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp}.
\end{aligned} \tag{3.12}$$

3.2.4 Equation for t:

$$\begin{aligned}
0 &= \frac{d^2x^t}{dp^2} + \Gamma^t_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} \\
&= \frac{d^2t}{dp^2} + 2\Gamma^t_{tr}\frac{dt}{dp}\frac{dr}{dp} \\
&= \frac{d^2t}{dp^2} + \frac{A'(r)}{A(r)}\frac{dt}{dp}\frac{dr}{dp}.
\end{aligned} \tag{3.13}$$

3.2.5 Constants of motion:

Since the field is isotropic, we may consider the orbit of our particle to be confined to the equatorial plane,

$$\theta = \frac{\pi}{2}. \tag{3.14}$$

In this case, equation (3.11) is satisfied, and we can forget about θ as a dynamical variable. Finally, the set of equations, evaluated at $\theta = \pi/2$, take the form:

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{A'(r)}{B(r)}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2}\frac{B'(r)}{B(r)}\left(\frac{dr}{dp}\right)^2 - \frac{1}{2}\frac{r^2D'(r)+2rD(r)}{B(r)}\left(\frac{d\phi}{dp}\right)^2 = 0 \tag{3.15}$$

$$\frac{d^2\phi}{dp^2} + \frac{rD'(r)+2D(r)}{rD(r)}\frac{d\phi}{dp}\frac{dr}{dp} = 0 \tag{3.16}$$

$$\frac{d^2t}{dp^2} + \frac{A'(r)}{A(r)}\frac{dt}{dp}\frac{dr}{dp} = 0 \tag{3.17}$$

Making $v = \frac{d\phi}{dp}$ in (3.16) we find:

$$\begin{aligned}
0 &= \frac{dv}{dp} + \frac{rD'(r)+2D(r)}{rD(r)}\frac{dr}{dp}v \\
&= dv + \frac{D'(r)}{D(r)}v dr + \frac{2v}{r}dr.
\end{aligned} \tag{3.18}$$

Integrating

$$\begin{aligned}
\text{cte} &= \int\frac{1}{v}dv + \int\frac{D'(r)}{D(r)}dr + 2\int\frac{1}{r}dr \\
&= \ln(v) + \ln(D(r)) + \ln(r^2).
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
0 &= \frac{d}{dp}\left[\ln(v) + \ln(D(r)) + \ln(r^2)\right] \\
&= \frac{d}{dp}\left[\ln(vD(r)r^2)\right].
\end{aligned} \tag{3.20}$$

In the same way, making $u = \frac{dt}{dp}$ in (3.17), we find that

$$
\begin{aligned}
0 &= du + \frac{A'(r)}{A(r)} u dr \\
\text{cte} &= \int \frac{1}{u} du + \int \frac{A'(r)}{A(r)} dr \\
&= \ln u + \ln A(r)
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
0 &= \frac{d}{dp}\left[\ln u + \ln A(r)\right] \\
&= \frac{d}{dp}\left[\ln u A(r)\right]
\end{aligned}
\tag{3.22}
$$

Equations (3.20) and (3.22) yield two constant of the motion. One of them, the second one, will be absorbed immediately into the definition of p as we will see. Replacing $v = \frac{d\phi}{dp}$ in (3.20) and $u = \frac{dt}{dp}$ in (3.22), we get

$$
r^2 D(r) \frac{d\phi}{dp} = \text{cte} = J, \tag{3.23}
$$

$$
\frac{dt}{dp} = \frac{1}{A(r)}. \tag{3.24}
$$

The first constant of motion plays the role of an angular momentum per unit mass[21]. For the second constant, we choose to normalize p so that the solution is (3.24)[8]. Since $A(r)$ is close to unity, p is nearly equal to the coordinate time t. Inserting (3.23) and (3.24) in (3.15) we find

$$
\begin{aligned}
0 &= \frac{d^2 r}{dp^2} + \frac{1}{2}\frac{A'(r)}{B(r)A^2(r)} + \frac{1}{2}\frac{B'(r)}{B(r)}\left(\frac{dr}{dp}\right)^2 - \frac{1}{2}\frac{r^2 D'(r) + 2rD(r)}{B(r)}\frac{J^2}{r^4 D^2(r)} \\
&= 2B(r)\frac{d^2 r}{dp^2} + \frac{A'(r)}{A^2(r)} + B'(r)\left(\frac{dr}{dp}\right)^2 - (r^2 D'(r) + 2rD(r))\frac{J^2}{r^4 D^2(r)}.
\end{aligned}
\tag{3.25}
$$

But

$$
\frac{d}{dr}\left[B(r)\left(\frac{dr}{dp}\right)^2\right] = B'(r)\left(\frac{dr}{dp}\right)^2 + 2B(r)\frac{dr}{dp}\frac{d}{dp}\left(\frac{dr}{dp}\right)\frac{dp}{dr}. \tag{3.26}
$$

Inserting (3.26) in (3.25) we get

$$
\begin{aligned}
0 &= \frac{d}{dr}\left[B(r)\left(\frac{dr}{dp}\right)^2\right] + \frac{A'(r)}{A^2(r)} - \frac{(r^2 D'(r) + 2rD(r))J^2}{r^4 D^2(r)} \\
&= \frac{d}{dr}\left[B(r)\left(\frac{dr}{dp}\right)^2\right] - \frac{d}{dr}\left(\frac{1}{A(r)}\right) + J^2 \frac{d}{dr}\left(\frac{1}{r^2 D(r)}\right) \\
&= \frac{d}{dr}\left[B(r)\left(\frac{dr}{dp}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2 D(r)}\right].
\end{aligned}
\tag{3.27}
$$

Our last constant of motion is therefore

$$
\text{cte} = B(r)\left(\frac{dr}{dp}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2 D(r)} = -E. \tag{3.28}
$$

The proper time $d\tau^2$ may now be determined from (3.7) making $\theta = \pi/2$ and considering the equations (3.23), (3.24) and (3.28). Equation 3.7 states that

[8] First we changed p to $p* = \text{cte} \times p$ and then removed the prime.

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - D(r)r^2 d\theta^2 - D(r)r^2 \sin^2\theta d\phi^2; \tag{3.29}$$

for $\theta = \pi/2$ and dividing with dp^2 we get

$$\begin{aligned}\left(\frac{d\tau}{dp}\right)^2 &= A(r)\left(\frac{dt}{dp}\right)^2 - B(r)\left(\frac{dr}{dp}\right)^2 - D(r)r^2\left(\frac{d\phi}{dp}\right)^2 \\ &= A(r)\left(\frac{1}{A(r)}\right)^2 - \left[\frac{1}{A(r)} - \frac{J^2}{r^2D(r)} - E\right] - \frac{J^2}{r^2D(r)}\end{aligned} \tag{3.30}$$

finally we obtain

$$d\tau^2 = E dp^2 \tag{3.31}$$

in accordance with our earlier remark that (3.8) forces $d\tau/dp$ to be constant. We see that E must take the values

$$\begin{aligned} E &> 0 \qquad \text{for material particles} \\ E &= 0 \qquad \text{for photons}\end{aligned} \tag{3.32}$$

3.3 Deflexion angle:

In most applications of general relativity we are interested in the shape of orbits, that is, in r as a function of ϕ, than in their time history[21]. The orbit shape can be obtained directly by eliminating dp from (3.23) and (3.28). Previously, we have shown that

$$\begin{aligned}-E &= B(r)\left(\frac{dr}{dp}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)} \\ J &= r^2D(r)\frac{d\phi}{dp}.\end{aligned} \tag{3.33}$$

But $\frac{dt}{dp} = \frac{1}{A(r)}$, then

$$\begin{aligned}-E &= B(r)\left(\frac{dr}{dt}\frac{dt}{dp}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)} \\ &= \frac{B(r)}{A^2(r)}\left(\frac{dr}{dt}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)}\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}J &= r^2D(r)\frac{d\phi}{dt}\frac{dt}{dp} \\ &= r^2\frac{D(r)}{A(r)}\frac{d\phi}{dt}.\end{aligned} \tag{3.35}$$

We can express equation (3.34) in terms of ϕ;

$$-E = \frac{B(r)}{A^2(r)}\left(\frac{dr}{d\phi}\frac{d\phi}{dt}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)}. \tag{3.36}$$

From (3.35) $\frac{d\phi}{dt} = \frac{JA(r)}{r^2D(r)}$. Then equation (3.36) takes the form

$$\begin{aligned}-E &= \frac{B(r)}{A^2(r)}\frac{J^2A^2(r)}{r^4D^2(r)}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)} \\ &= \frac{J^2B(r)}{r^4D^2(r)}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{A(r)} + \frac{J^2}{r^2D(r)}.\end{aligned} \tag{3.37}$$

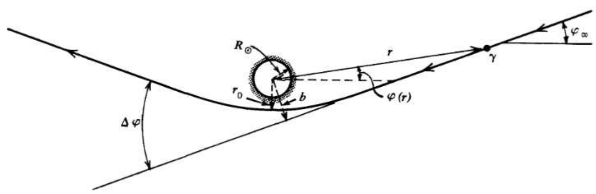
From (3.37) we get

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2} \frac{B^{1/2}(r)}{D(r)} \left[\frac{1}{J^2 A(r)} - \frac{1}{r^2 D(r)} - \frac{E}{J^2} \right]^{-1/2}; \tag{3.38}$$

and integrating

$$\phi = \pm \int \frac{1}{r^2} \frac{B^{1/2}(r)}{D(r)} \left[\frac{1}{J^2 A(r)} - \frac{1}{r^2 D(r)} - \frac{E}{J^2} \right]^{-1/2} dr + \text{cte.} \tag{3.39}$$

In order to obtain the deflection angle consider a particle or photon approaching from very great distances. At infinity the metric becomes Minkowskian, that is, $A(\infty) = B(\infty) = 1$, and we expect motion on a straight line at constant velocity V, from the figure [21]:



we obtain that,

$$\begin{aligned} b &\simeq r\sin(\phi - \phi_\infty) \simeq r(\phi - \phi_\infty) \\ -V &\simeq \frac{d}{dt}(r\cos(\phi - \phi_\infty)) \simeq \frac{dr}{dt}, \end{aligned} \tag{3.40}$$

Where b is the "impact parameter" and ϕ_∞ is the incident direction. Inserting these in (3.34) and (3.35), we see that they do satisfy the equation of motion at infinity, where $A = B = 1$, and that the constants of the motion are:

$$\begin{aligned} J &= bV \\ E &= 1 - V^2. \end{aligned} \tag{3.41}$$

As we can see from (3.41) a photon has $V = 1$, and as we have already seen, this gives $E = 0$. It is often more convenient to express J in terms of the distant r_0 of closest approach to the distribution of mass, rather than the impact parameter b. At r_0, $dr/d\phi$ vanishes so, from equations (3.37) and (3.41), we have

$$\begin{aligned} -\frac{E}{J^2} &= \frac{1}{r_0^2 D^2(r_0)} - \frac{1}{A(r_0) J^2} \\ J &= r_0 \left[\left(\frac{1}{A(r_0)} - 1 + V^2 \right) D(r_0) \right]^{1/2}. \end{aligned} \tag{3.42}$$

The orbit is then described by (3.39)

$$\phi(r) = \phi_\infty + \int_r^\infty \frac{1}{r^2} \frac{\sqrt{B(r)}}{D(r)} \left[\frac{1}{J^2 A(r)} - \frac{1}{r^2 D(r)} - \frac{E}{J^2} \right]^{-1/2} dr. \tag{3.43}$$

From (3.42)

$$\frac{1}{J^2} = \frac{1}{r_0^2} \left[\left(\frac{1}{A(r_0)} - 1 + V^2 \right) D(r_0) \right]^{-1}; \tag{3.44}$$

and we have

$$
\begin{aligned}
\frac{1}{J^2 A(r)} - \frac{(1-V^2)}{J^2} &= \frac{1}{r_0^2 A(r)} \left[\left(\frac{1}{A(r_0)} - 1 + V^2 \right) D(r_0) \right]^{-1} \\
&- \frac{1-V^2}{r_0^2} \left[\left(\frac{1}{A(r_0)} - 1 + V^2 \right) D(r_0) \right]^{-1} \\
&= \frac{D^{-1}(r_0)}{r_0^2} \left[\frac{1}{A(r)} - 1 + V^2 \right] \left[\frac{1}{A(r_0)} - 1 + V^2 \right]^{-1} .
\end{aligned} \tag{3.45}
$$

Inserting the last equation in (3.43)

$$
\begin{aligned}
\phi = \phi_\infty + \int_r^\infty \frac{1}{r^2} \frac{\sqrt{B(r)}}{D(r)} \left\{ \frac{1}{r_0^2 D(r_0)} \left[\frac{1}{A(r)} - 1 + V^2 \right] \left[\frac{1}{A(r_0)} - 1 + V^2 \right]^{-1} - \frac{1}{r^2 D(r)} \right\}^{-1/2} dr \\
= \phi_\infty + \int_r^\infty \frac{1}{r} \left[\frac{B(r)}{D(r)} \right]^{1/2} \left\{ \left(\frac{r}{r_0} \right)^2 \frac{D(r)}{D(r_0)} \left[\frac{1}{A(r)} - 1 + V^2 \right] \left[\frac{1}{A(r_0)} - 1 + V^2 \right]^{-1} - 1 \right\}^{-1/2} dr.
\end{aligned} \tag{3.46}
$$

The total change in ϕ as r decreases from infinity to its minimum value r_0 and then increases again to infinity is just twice its change from ∞ to r_0, that is, $2|\phi(r_0) - \phi_\infty|$. If the trajectory were a straight line, this would equal just π; hence the deflection of the orbit from straight line is[21]

$$
\Delta\phi = \widehat{\alpha}(r_0) = 2|\phi(r_0) - \phi_\infty| - \pi \tag{3.47}
$$

If this is positive, then the angle ϕ changes by more than $180°$, that is, the trajectory is bent toward the mass distribution; if $\Delta\phi$ es negative then the trajectory is bent away from the mass distribution. Inserting (3.46) in (3.47) we get

$$
\widehat{\alpha}(r_0) = 2 \int_{r_0}^\infty \frac{1}{r} \left[\frac{B(r)}{D(r)} \right]^{1/2} \left\{ \left(\frac{r}{r_0} \right)^2 \frac{D(r)}{D(r_0)} \frac{\left[\frac{1}{A(r)} - 1 + V^2 \right]}{\left[\frac{1}{A(r_0)} - 1 + V^2 \right]} - 1 \right\}^{-1/2} dr - \pi \tag{3.48}
$$

For a photon $E = 0$, that is $V^2 = 1$, equation (3.48) takes the form

$$
\widehat{\alpha}(r_0) = 2 \int_{r_0}^\infty \frac{1}{r} \left[\frac{B(r)}{D(r)} \right]^{1/2} \left\{ \left(\frac{r}{r_0} \right)^2 \frac{D(r)}{D(r_0)} \frac{A(r_0)}{A(r)} - 1 \right\}^{-1/2} dr - \pi. \tag{3.49}
$$

3.4 Photon sphere:

A photon sphere is a spherical region of space where gravity is strong enough that photons are forced to travel in orbits. This means that Einstein bending angle of a light ray with the closest distance of approach r_0 becomes unboundedly large as r_0 tends to r_{sp}[20]. In order to obtain the photon sphere equation we have to set $r = cte$ in (3.15); this condition makes (3.16) to take the form :

$$
\frac{A'(r)}{B(r)} \left(\frac{dt}{dp} \right)^2 - \frac{r^2 D'(r) + 2rD(r)}{B(r)} \left(\frac{d\phi}{dp} \right)^2 = 0, \tag{3.50}
$$

from which

$$
\left(\frac{d\phi}{dt} \right)^2 = \frac{A'(r)}{(r^2 D)'}. \tag{3.51}
$$

On the other hand, from equation (3.7) and making $\theta = \frac{\pi}{2}$, $r =$ cte. and $d\tau^2 = 0$ (because it is a photon) we get

$$0 = A(r)dt^2 - Dr^2 d\phi^2; \tag{3.52}$$

and finally

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{A(r)}{r^2 D(r)} \tag{3.53}$$

comparing (3.51) and (3.53) we find that

$$\frac{A(r)}{r^2 D(r)} = \frac{A'(r)}{(r^2 D(r))'} \tag{3.54}$$

and finally, using the relation $C(r) = r^2 D(r)$ we have

$$A(r)C'(r) = A'(r)C(r). \tag{3.55}$$

Equation (3.55) admits at least one positive solution. We shall call the largest root of it the radius of the photon sphere r_m. A, B, C, A' and C' must be positive for $r > r_m$. For metrics expressed by in standard coordinates ($C(r) = r^2$) a sufficient condition for the existence of r_m is the presence of a static limit (a radius r_s such that $A(r_s) = 0$). The strong field expansion takes the photon sphere as the starting point. In this study, we shall not consider naked singularities without a photon sphere[22].

3.5 Bozza's method:

The Bozza's method is used to calculate the deflection angle in the strong filed limit taking the photon sphere as the starting point. This method is universal and can be applied to any space-time in any theory of gravitation provided that the photons satisfy the standard geodesic relation, equation (3.8). The parameters of the strong field limit expansion ($\bar{a}$, $\bar{b}$ and u_m) are directly connected with the observables, providing an effective tool to discriminate among different metrics. In this section, we are going to show that the deflection angle diverges logarithmically as we approach the photon sphere.

3.5.1 Divergent term of the deflection angle

We defined two new variables[9]

$$\begin{aligned} y &= A(x) \\ z &= \frac{y - y_0}{1 - y_0}, \end{aligned} \tag{3.56}$$

where $y_0 = A_0$. The integral (3.49) becomes

[9]From now on $x \to r$ and $x_0 \to r_0$.

$$
\begin{aligned}
\hat{\alpha}(x_0) &= 2\int_{x_0}^{\infty}\left[\frac{B(x)}{C(x)}\right]^{1/2}\left[\frac{C(x)}{C(x_0)}\frac{y_0}{y}-1\right]^{-1/2}dx-\pi\\
&= 2\int_{x_0}^{\infty}\left[\frac{B(x)}{C(x)}\right]^{1/2}\left[\frac{C(x)}{C(x_0)}\frac{y_0}{z(1-y_0)+y_0}-1\right]^{-1/2}dx-\pi\\
&= 2\int_{x_0}^{\infty}\frac{\sqrt{B(x)}}{\sqrt{C(x)}}\left[y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}\right]^{-1/2}\left[\frac{C(x)}{C_0[(1-y_0)z+y_0]}\right]^{-1/2}dx-\pi\\
&= 2\int_{x_0}^{\infty}\frac{\sqrt{B(x)}}{\sqrt{C(x)}}\frac{\sqrt{C_0}}{\sqrt{C(x)}}\frac{\sqrt{y}dx}{\left[y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}\right]^{1/2}}-\pi\\
&= 2\int_{x_0}^{\infty}\frac{\sqrt{B(x)y}}{C(x)}\sqrt{C_0}\frac{dx}{\left[y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}\right]^{1/2}}-\pi.
\end{aligned}
\tag{3.57}
$$

From (3.56)

$$
\begin{aligned}
A' &= \frac{d}{dx}[(1-y_0)z+y_0]=(1-y_0)\frac{dz}{dx}\\
dx &= \frac{(1-y_0)}{A'}dz.
\end{aligned}
\tag{3.58}
$$

Then, equation (3.57) takes the form[10]

$$
\hat{\alpha}(x_0)=2\int_0^1\frac{\sqrt{B(x)y}}{C(x)A'(x)}(1-y_0)\sqrt{C_0}\frac{dz}{\left[y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}\right]^{1/2}}-\pi; \tag{3.59}
$$

where all functions without the subscript 0 are evaluated at $x=A^{-1}[(1-y_0)z+y_0]$. Calling

$$
\begin{aligned}
R(z,x_0) &= 2\frac{\sqrt{B(x)y}}{C(x)A'(x)}(1-y_0)\sqrt{C_0}\\
f(z,x_0) &= \frac{1}{\left[y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}\right]^{1/2}},
\end{aligned}
\tag{3.60}
$$

we find that (3.49) can be expressed as

$$
\hat{\alpha}(x_0)=\int_0^1 R(z,x_0)f(z,x_0)dz-\pi=I(x_0)-\pi. \tag{3.61}
$$

The function $R(z,x_0)$ is regular for values of z and x_0, while $f(z,x_0)$ diverges for $z\to 0$. To find out the order of divergence of integrand, we expand the argument of the square root in $f(z,x_0)$ to the second order in z. For $z_0=0$ the expansion of $l(z,x_0)=y_0-[(1-y_0)z+y_0]\frac{C_0}{C(x)}$ is:

$$
\begin{aligned}
l(z,x_0) &= \sum_{n=0}^{\infty}\frac{l^{(n)}(z_0)}{n!}(z-z_0)^n\\
&= l(0)+l'(0)z+\frac{l''(0)}{2}z^2+...\\
&= l(0)+\alpha z+\beta z^2+...
\end{aligned}
\tag{3.62}
$$

[10] For $x=x_0$, equation (3.56) gives $z(x_0)=0$. For $x\to\infty$, $A(x)\to 1$ because the metric is asymptotically flat then (3.56) gives $z=1$

Finding α

From equation (3.56) we notice that z is a function of x for this reason

$$
\begin{aligned}
l'(z) &= \frac{d}{dz} l(z) = \frac{dl}{dx}\left[\frac{dz}{dx}\right]^{-1} \\
&= \frac{d}{dx}\left\{y_0 - [(1-y_0)z + y_0]\frac{C_0}{C}\right\}\left[\frac{dz}{dx}\right]^{-1}
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
\frac{d}{dx}\left\{y_0 - [(1-y_0)z + y_0]\frac{C_0}{C}\right\} &= -\frac{C_0}{C}\frac{d}{dx}[(1-y_0)z + y_0] - C_0[(1-y_0)z + y_0]\frac{d}{dx}C^{-1} \\
&= -\frac{C_0}{C}(1-y_0)\frac{dz}{dx} - (-C^{-2})C'C_0[(1-y_0)z + y_0] \\
&= -\frac{(1-y_0)C_0}{C}\frac{dz}{dx} + \frac{C_0[(1-y_0)z + y_0]C'}{C^2}
\end{aligned}
\tag{3.64}
$$

$$
\begin{aligned}
l'(z) &= \left\{\frac{C_0[(1-y_0)z + y_0]C'}{C^2} - \frac{(1-y_0)C_0}{C}\frac{dz}{dx}\right\}\left[\frac{dz}{dx}\right]^{-1} \\
&= \frac{C_0[(1-y_0)z + y_0]C'}{C^2}\left[\frac{dz}{dx}\right]^{-1} - \frac{(1-y_0)C_0}{C}.
\end{aligned}
\tag{3.65}
$$

Using (3.58) we obtain

$$
l'(z) = \frac{C_0C'(1-y_0)[(1-y_0)z + y_0]}{C^2A'} - \frac{(1-y_0)C_0}{C}. \tag{3.66}
$$

For $z = 0$, $x = x_0$, the last equation gives α

$$
\begin{aligned}
\alpha = l'(z=0) &= \frac{C_0C_0'(1-y_0)y_0}{C_0^2A_0'} - \frac{(1-y_0)C_0}{C_0} \\
\alpha &= \frac{(1-y_0)}{C_0A_0'}[C_0'y_0 - C_0A_0'].
\end{aligned}
\tag{3.67}
$$

Finding β

As we have done for α in the last section, β can be expressed as[11]:

$$
l''(z, x_0) = \frac{d}{dz}\left(\frac{dl}{dz}\right) = \frac{d}{dx}\left(\frac{dl}{dz}\right)\frac{dx}{dz} \tag{3.68}
$$

then

[11] From equation (3.56) $\frac{dx}{dz} = \frac{1-y_0}{A'}$

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{dl}{dz}\right) &= \frac{d}{dx}\left\{\frac{(1-y_0)C_0}{C^2A'}\left[C'[(1-y_0)z+y_0]-CA'\right]\right\} \\
&= (1-y_0)C_0\{\frac{d}{dx}\left(\frac{1}{C^2A'}\right)[C'[(1-y_0)z+y_0]] \\
&+ \frac{1}{C^2A'}\frac{d}{dx}(C'[(1-y_0)z+y_0]) - \frac{d}{dx}\left(\frac{1}{C}\right)\} \\
&= (1-y_0)C_0\{-\frac{(2C'A'+A''C)(C'[(1-y_0)z+y_0])}{A'^2C^3} \\
&+ \frac{CA'}{C^3A'^2}((1-y_0)(C''z+C'z')+C''y_0) - \frac{(-1)C'CA'^2}{C^3A'^2}\} \\
&= \frac{(1-y_0)C_0}{C^3A'^2}\{-(2C'A'+A''C)C'[(1-y_0)z+y_0] \\
&+ CA'((1-y_0)(C''z+C'z')+y_0C'')+CC'A'^2\}.
\end{aligned} \tag{3.69}
$$

For $z=0$, that is $x=x_0$, and using (3.58) equation (3.68) gives

$$
\begin{aligned}
l''(0,x_0) &= \frac{(1-y_0)}{C_0^2A_0'^2}\{-(2C_0'A_0'+A_0''C_0)y_0C_0'+C_0A_0'(1-y_0)C_0'z_0' \\
&+ C_0A_0'y_0C_0''+C_0C_0'A_0'^2\}\frac{(1-y_0)}{A_0'} \\
&= \frac{(1-y_0)^2}{C_0^2A_0'^3}\{-(2C_0'A_0'+A_0''C_0)y_0C_0'+C_0A_0'(1-y_0)C_0'z_0' \\
&+ C_0A_0'y_0C_0''+C_0C_0'A_0'^2\} \\
&= \frac{(1-y_0)^2}{C_0^2A_0'^3}\{-2y_0C_0'^2A_0'-A_0''C_0y_0C_0'+C_0A_0'(1-y_0)\frac{C_0'A_0'}{(1-y_0)} \\
&+ C_0A_0'y_0C_0''+C_0C_0'A_0'^2\} \\
&= \frac{(1-y_0)^2}{C_0^2A_0'^3}\left\{2C_0C_0'A_0'^2+(C_0C_0''-2C_0'^2)y_0A_0'-C_0y_0C_0'A_0''\right\}.
\end{aligned} \tag{3.70}
$$

From which β is

$$
\beta = \frac{1}{2}l''(0,x_0) = \frac{(1-y_0)^2}{2C_0^2A_0'^3}\left\{2C_0C_0'A_0^2+(C_0C_0''-2C_0'^2)y_0A_0'-C_0C_0'y_0A_0''\right\}. \tag{3.71}
$$

Finally we can express $f(z,x_0)$ as

$$
f(z,x_0) \sim f_0(z,x_0) = \frac{1}{\sqrt{\alpha z+\beta z^2}}. \tag{3.72}
$$

When α is non zero, the leading order of the divergence in f_0 is $z^{-1/2}$, which can be integrated to give a finite result. When α vanishes, the divergence is z^{-1} which makes the integral diverge. Examining the form of α, we see that it vanishes at $x_0=x_m$ with x_m defined by Eq. (3.55). Each photon having $x_0<x_m$ is captured by the central object and cannot emerge back[22].

In order to solve $I(x_0)$, we split it into two pieces

$$
I(x_0) = I_D(x_0) + I_R(x_0), \tag{3.73}
$$

where

$$I_D(x_0) = \int_0^1 R(0, x_m) f_0(z, x_0) dz, \tag{3.74}$$

contains the divergence and

$$I_R(x_0) = \int_0^1 g(z, x_0) dz$$
$$g(z, x_0) = R(z, x_0) f(z, x_0) - R(0, x_m) f_0(z, x_0), \tag{3.75}$$

is the original integral with the divergence subtracted. We shall solve both integrals separately and then sum up their results to rebuilt the deflection angle. Here we deal with I_D and its divergence, while in the next section we shall verify that I_R is indeed regular[22].

We can solve the integral $I_D(x_0)$ exactly. Using the Taylor expansion for $f(z, x_0)$ we have

$$I_D(x_0) = \int_0^1 R(0, x_m) f_0(x_0, z) dz = \int_0^1 \frac{R(0, x_m)}{\sqrt{\alpha z + \beta z^2}} dz. \tag{3.76}$$

In this integral $R(0, x_m)$ is a constant so

$$I_D(x_0) = R(0, x_m) \int_0^1 \frac{1}{\sqrt{\alpha z + \beta z^2}} dz. \tag{3.77}$$

Using the list of integrals from [23] we find, for $\beta > 0$, that

$$\int \frac{1}{\sqrt{\alpha z + \beta z^2}} dz = \frac{1}{\sqrt{\beta}} \ln |2\beta z + \alpha + 2\sqrt{\beta}\sqrt{\beta z^2 + \alpha z}|. \tag{3.78}$$

Inserting this result in $I_D(x_0)$ we get

$$\begin{aligned} I_D(x_0) &= \frac{R(0, x_m)}{\sqrt{\beta}} \ln |2\beta + \alpha + 2\sqrt{\beta}\sqrt{\beta + \alpha}| - \ln |\alpha| \\ &= \frac{R(0, x_m)}{\sqrt{\beta}} \ln \left| \frac{2\beta + \alpha + 2\sqrt{\beta}\sqrt{\beta + \alpha}}{\alpha} \right|. \end{aligned} \tag{3.79}$$

But

$$(\sqrt{\beta} + \sqrt{\beta + \alpha})^2 = 2\beta + \alpha + 2\sqrt{\beta}\sqrt{\beta + \alpha}. \tag{3.80}$$

Then the integral $I_D(x_0)$ takes the final form [22]

$$\begin{aligned} I_D(x_0) &= \frac{R(0, x_m)}{\sqrt{\beta}} \ln \left| \left(\frac{\sqrt{\beta} + \sqrt{\beta + \alpha}}{\sqrt{\alpha}} \right)^2 \right| \\ &= R(0, x_m) \frac{2}{\sqrt{\beta}} \ln \left(\frac{\sqrt{\beta} + \sqrt{\beta + \alpha}}{\sqrt{\alpha}} \right). \end{aligned} \tag{3.81}$$

Since we are interested in the terms up to $\mathcal{O}(x_0 - x_m)$ we expand α as

$$\begin{aligned} \alpha(x_0) &= \sum_{n=0}^{\infty} \frac{\alpha^{(n)}(x_m)}{n!} (x_0 - x_m)^n \\ &= \alpha(x_m) + \frac{d\alpha}{dx_0}|_{x_0 = x_m} (x_0 - x_m) + \mathcal{O}(x_0 - x_m)^2 + ... \end{aligned} \tag{3.82}$$

which means that we are working near to the photon sphere. Recalling equation (3.67), we see that $\alpha(x_m)$ is zero because x_m is a solution of (3.55). The next term in (3.82) is not zero.

$$\frac{d\alpha}{dx_0} = \frac{d}{dx_0}\left[\frac{(1-y_0)}{C_0A_0'}\right](C_0y_0' - C_0A_0') + \frac{(1-y_0)}{C_0A_0'}\frac{d}{dx_0}(C_0y_0' - C_0A_0') \tag{3.83}$$

When the first term in (3.83) is evaluated at $x_0 = x_m$ its value is zero; the second term, however, can be calculated easily as [12]

$$\begin{aligned}\frac{(1-y_0)}{C_0A_0'}\frac{d}{dx_0}(C_0y_0' - C_0A_0') &= \frac{(1-y_0)}{C_0A_0'}(C_0''y_0 + C_0'y_0' - C_0'A_0' - C_0A_0'') \\ &= \frac{(1-y_0)}{C_0A_0'}\left[C_0''y_0 - C_0A_0''\right],\end{aligned} \tag{3.84}$$

and evaluating at $x_0 = x_m$ we get[13]

$$\begin{aligned}\frac{d\alpha}{dx_0} &= \frac{(1-y_m)}{C_mA_m'}\left[C_m''y_m - C_mA_m''\right] \\ &= \frac{2(1-y_m)^2C_mA_m'}{2(1-y_m)C_m'^2A_m^2}\left[y_mC_m'' - C_mA''(x_m)\right],\end{aligned} \tag{3.85}$$

and (3.82) takes the form

$$\alpha = \frac{2\beta_mA_m'}{(1-y_m)}(x_0 - x_m) + \mathcal{O}(x_0 - x_m)^2 \tag{3.86}$$

where[14]

$$\beta_m = \beta|_{x_0=x_m} = \frac{C_m(1-y_m)^2(y_mC_m'' - C_mA''(x_m))}{2y_m^2C_m'^2}. \tag{3.87}$$

As mentioned before, we are interested in working with terms up to first order in $(x_0 - x_m)$; for this reason we are going to expand equation (3.81) up to first order in $(x_0 - x_m)$. In order to do so, the lowest order comes out by taking α at the first order in $(x_0 - x_m)$ and taking $\beta = \beta_m$; this is the zero order for β. If we expand α (equation 3.82) without expanding β, then you still get a residual dependence on x_0 trough beta, and the coefficient a will not be a number but a function of x_0, which is incorrect as the power expansion is not fulfilled[15].

$$I_D(x_0) = \sum_{n_0}^{\infty}\frac{I_D^{(n)}(x_m)}{n!}(x_0 - x_m)^n, \tag{3.88}$$

then

$$\begin{aligned}I_D(x_0) &= I_D(x_m) + \mathcal{O}(x_0 - x_m) \\ &= \frac{R(0,x_m)}{\sqrt{\beta_m}}2\ln(\sqrt{\beta_m} + \sqrt{\alpha+\beta} - \ln\sqrt{\alpha}) + \mathcal{O}(x_0 - x_m)\end{aligned} \tag{3.89}$$

Inserting (3.86) we get

$$\begin{aligned}I_D &= \frac{R(0,x_m)}{\sqrt{\beta_m}}2\ln\left(\frac{\sqrt{\beta_m} + \sqrt{\beta_m + 2\frac{\beta_mA_m'}{(1-y_m)}(x_0 - x_m)}}{\sqrt{\frac{2\beta_mA_m'}{1-y_m}(x_0 - x_m)}}\right) \\ &= \frac{R(0,x_m)}{\sqrt{\beta_m}}\ln\left(\frac{\left(1 + \sqrt{1 + 2\frac{A_m'}{(1-y_m)}(x_0 - x_m)}\right)^2}{\frac{2A_m'}{1-y_m}(x_0 - x_m)}\right).\end{aligned} \tag{3.90}$$

[12] It is important to take into account that $y_0 = A_0$[22].
[13] We had into account that $C_m'A_m - C_mA_m' = 0$
[14] Remember that $y_0 = A_0$ and $y_m = A_m$.
[15] A discussion with professor V. Bozza via e-mail.

Expanding the term $\left(1+\sqrt{1+2\frac{A'_m}{(1-y_m)}(x_0-x_m)}\right)^2$ we have that

$$I_D = -\frac{R(0,x_m)}{\sqrt{\beta_m}} \ln\left(\frac{x_0}{x_m}-1\right) + \frac{R(0,x_m)}{\sqrt{\beta_m}} \ln\frac{2(1-y_m)}{x_m A'_m} + \mathcal{O}(x_0-x_m). \tag{3.91}$$

Rewriting

$$I_D(x_0) = -a\ln\left(\frac{x_0}{x_m}-1\right) + b_D + \mathcal{O}(x_0-x_m), \tag{3.92}$$

where

$$\begin{aligned} a &= \frac{R(0,x_m)}{\sqrt{\beta_m}} \\ b_D &= \frac{R(0,x_m)}{\sqrt{\beta_m}} \ln\left(\frac{2(1-y_m)}{A'_m x_m}\right). \end{aligned} \tag{3.93}$$

Equation (3.92) yields the leading order in the divergence of the deflection angle, which is logarithmic, as anticipated before. The coefficient a is given by equation (3.93).

3.5.2 Regular term of the deflection angle

In order to find the correct coefficient b in (3.92), we have to add to the term b_D of equation (3.93), an analogous term coming from the regular part of the original integral, defined by (3.75). To do so, we can expand $I_R(x_0)$ in powers of (x_0-x_m)

$$I_R(x_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(x_0-x_m)^n \int_0^1 \frac{\partial^n g}{\partial x_0}|_{x_0=x_m} dz, \tag{3.94}$$

and evaluate the single coefficients.

If we had not subtracted the singular part from $R(z,x_0)f(z,x_0)$, we would have an infinite coefficient for $n=0$, while all other coefficients would be finite. However, the function $g(z,x_0)$ is regular in $z=0$, $x_0=x_m$ as can be easily checked by a power expansion, recalling that $\alpha_m=0$.
Since we are interested to terms up to $\mathcal{O}(x_0-x_m)$, we will just retain the $n=0$ term

$$I_R(x_0) = \int_0^1 g(z,x_m)dz + \mathcal{O}(x_0-x_m) \tag{3.95}$$

and then

$$b_R = I_R(x_m), \tag{3.96}$$

is the term we need to add to b_D in order to get the regular coefficient. Recalling also the term $-\pi$ in the deflection angle, we have

$$b = b_D + b_R - \pi. \tag{3.97}$$

The coefficient b_R can be easily evaluated numerically for all metrics, since the integrand has no divergence. However, in many cases it is also possible to built a completely analytical formula for b_R as well. In fact, in Schwarzschild metric, the integral (3.95) is solved exactly. Then, in most metrics, we can expand (3.95) in powers of their parameters, starting from the Schwarzschild limit, and evaluate each coefficient separately.

3.6 From $\hat{\alpha}(x_0)$ to $\hat{\alpha}(\theta)$

From conservation of angular momentum, there is a relation between J and the impact parameter u [21]

$$J = uV. \tag{3.98}$$

As equation (3.42) states

$$J = x_0\sqrt{\left[\frac{1}{A_0} - 1 + V^2\right] D_0}; \tag{3.99}$$

then

$$uV = x_0\sqrt{\left[\frac{1}{A_0} - 1 + V^2\right] D_0}. \tag{3.100}$$

For a photon $E = 0$ ($V = 1$) [21]; so that, the last equation takes the form

$$u = x_0\sqrt{\frac{D_0}{A_0}} = \sqrt{\frac{x_0^2 D_0}{A_0}}, \tag{3.101}$$

but $C = x_0^2 D$ (Using the notation in [22]) and finally we have

$$u = \sqrt{\frac{C_0}{A_0}}. \tag{3.102}$$

Making an expansion in Taylor series around x_m

$$u = \sum_{n=0}^{\infty} \frac{u^{(n)}(x_m)}{n!}(x_0 - x_m)^n \tag{3.103}$$

up to second order we get

$$u(x_0) = u(x_0 = x_m) + \frac{du(x_0 = x_m)}{dx_0}(x_0 - x_m) + \frac{d^2u(x_0 = x_m)}{dx_0^2}(x_0 - x_m)^2$$

$$u(x_0) - u_m = \frac{du(x_0 = x_m)}{dx_0}(x_0 - x_m) + \frac{d^2u(x_0 = x_m)}{dx_0}(x_0 - x_m)^2 \tag{3.104}$$

then calculating each coefficient in the expansion

$$\begin{aligned}\frac{du}{dx_0} &= \frac{d}{dx_0}\left(\sqrt{\frac{C_0}{A_0}}\right) = \frac{1}{2}\sqrt{\frac{A_0}{C_0}}\frac{d}{dx_0}\left(\frac{C_0}{A_0}\right) \\ &= \frac{1}{2}\sqrt{\frac{A_0}{C_0}}\frac{(C_0'A_0 - C_0A_0')}{A_0^2}\end{aligned} \tag{3.105}$$

$$\frac{du(x_0 = x_m)}{dx_0} = \frac{1}{2}\sqrt{\frac{A_m}{C_m}}\frac{(C_m'A_m - C_mA_m')}{A_m^2} = 0$$

and

$$
\begin{aligned}
\frac{d^2u}{dx_0} &= \frac{1}{2}\frac{d}{dx_0}\left[\sqrt{\frac{A_0}{C_0}}\frac{C_0'A_0 - C_0A_0'}{A_0^2}\right] \\
&= \frac{1}{2}\left[\underbrace{\frac{d}{dx_0}\left(\sqrt{\frac{A_0}{C_0}}\right)\frac{C_0'A_0 - C_0A_0'}{A_0^2}}_{\text{at } x_0 = x_m \text{ this term vanishes}} + \sqrt{\frac{A_0}{C_0}}\frac{d}{dx_0}\left[\frac{C_0'A_0 - C_0A_0'}{A_0^2}\right]\right] \qquad (3.106) \\
&= \frac{1}{2}\left[\bigcirc + \sqrt{\frac{A_0}{C_0}}\frac{d}{dx_0}\left[\frac{C_0'A_0 - C_0A_0'}{A_0^2}\right]\right],
\end{aligned}
$$

where $\bigcirc \equiv \frac{d}{dx_0}\left(\sqrt{\frac{A_0}{C_0}}\right)\frac{C_0'A_0 - C_0A_0'}{A_0^2}$. Then,

$$
\begin{aligned}
\frac{d^2u}{dx_0} &= \frac{1}{2}\left[\bigcirc + \sqrt{\frac{A_0}{C_0}}\frac{(C_0''A_0 + C_0'A_0' - C_0'A_0' - C_0A_0'')A_0^2 - (C_0'A_0 - C_0A_0')(2A_0A_0')}{A_0^4}\right] \\
&= \frac{1}{2}\left[\bigcirc + \sqrt{\frac{A_0}{C_0}}\frac{(C_0''A_0 - C_0A_0'')A_0^2 - (C_0'A_0 - C_0A_0')(2A_0A_0')}{A_0^4}\right] \\
&= \frac{1}{2}\left[\bigcirc + \sqrt{\frac{1}{C_0A_0^3}}\left((C_0''A_0 - C_0A_0'') - \frac{(C_0'A_0 - C_0A_0')(2A_0A_0')}{A_0}\right)\right]
\end{aligned}
\qquad (3.107)
$$

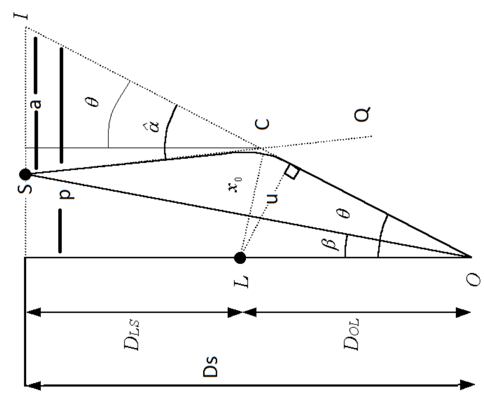
evaluating at $x_0 = x_m$ we get

$$
\begin{aligned}
c = \frac{1}{2}\frac{d^2u(x_0 = x_m)}{dx_0} &= \frac{1}{4}\left[\sqrt{\frac{1}{C_mA_m^3}}\left((C_m''A_m - C_mA_m'') - \frac{\overbrace{(C_m'A_m - C_mA_m')}^{\text{at } x_0 = x_m \text{ this term vanishes}}\ (2A_mA_m')}{A_m}\right)\right]. \\
&= \beta_m\sqrt{\frac{y_m}{C_m^3}}\frac{C_m'^2}{(1-y_m)^2}.
\end{aligned}
\qquad (3.108)
$$

Finally we get that the impact parameter is

$$
\begin{aligned}
u - u_m &= \frac{1}{4}\frac{(C_m''A_m - C_mA_m'')}{\sqrt{C_mA_m^3}}(x_0 - x_m)^2 \\
&= c(x_0 - x_m)^2.
\end{aligned}
\qquad (3.109)
$$

Using this relation, we can write the deflection angle as a function of θ. To do so, we have to remember that $\theta = u/D_{oL}$ as can be seen in the next figure[22]



so equation (3.109) gets the form

$$\theta D_{oL} - u_m = c\left(\frac{x_0}{x_m} - 1\right)x_m^2. \tag{3.110}$$

Replacing in

$$\widehat{\alpha}(x_0) = -a\ln\left(\frac{x_0}{x_m} - 1\right) + b_D + b_R - \pi. \tag{3.111}$$

we get

$$\begin{aligned}\widehat{\alpha}(\theta) &= -\frac{a}{2}\ln\left(\frac{\theta D_{oL} - u_m}{cx_m^2} - 1\right) + b_D + b_R - \pi \\ &= -\frac{a}{2}\ln\left[\left(\frac{\theta D_{oL}}{u_m} - 1\right)\right] + \frac{a}{2}\ln\left(\frac{cx_m^2}{u_m}\right) + b_D + b_R - \pi.\end{aligned} \tag{3.112}$$

This conclude our general discussion of the form of the deflection angle in the strong field limit. Even if the proof is somewhat tricky, the application to concrete cases is very straightforward, as we shall see in the applications. In fact, once we write the metric, it is sufficient to:

1. Solve $\alpha(x_m) = 0$.
2. Write β_m and $R(0, x_m)$
3. Compute b_R numerically or by a proper expansion in the parameters of the metric.
4. Compute the coefficients u_m, $\bar{a}$ and $\bar{b}$.

The crucial step is the calculation of b_R, since it is the only integral involved in the whole procedure.

Chapter 4

GRAVITATIONAL LENSING FOR S. KAR METRIC

The metric proposed by S. Kar in [16] has the form,

$$
\begin{aligned}
ds^2 &= +\left(1-\frac{2\eta}{r}\right)^{(m+\sigma)/\eta} dt^2 - \left(1-\frac{2\eta}{r}\right)^{(\sigma-m)/\eta} dr^2 - \left(1-\frac{2\eta}{r}\right)^{1+(\sigma-m)/\eta} r^2 d\Omega^2 \\
\phi &= \frac{\sigma}{2\eta}\ln\left(1-\frac{2\eta}{r}\right) \\
\eta^2 &= m^2+\sigma^2.
\end{aligned}
\tag{4.1}
$$

Where m is the mass and σ is the scalar charge. For $r = 2\eta$ this metric has a naked singularity [16]. In order to study the strong field limit approach, using the method described in the last chapter, we have expressed the metric coefficient as

$$
A(r) = \left(1-\frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \quad C(r) = \left(1-\frac{2\eta}{r}\right)^{1+\frac{\sigma-m}{\eta}} r^2 \quad B(r) = \left(1-\frac{2\eta}{r}\right)^{\frac{\sigma-m}{\eta}} \tag{4.2}
$$

For $\sigma = 0$ this solution reduces to the Schwarzschild solution. The reality of the metric coefficients indicates that we confine ourselves to the domain $r \geq \eta$. In order to use the Bozza's method, it is convenient to define $x \equiv \frac{r}{2\eta}$ and rename $\frac{m+\sigma}{\eta}$ and $\frac{\sigma-m}{\eta}$ as k and p respectively, so that equation (4.2) takes the form

$$
\begin{aligned}
A(x) &= \left(1-\frac{1}{x}\right)^{k} \\
B(x) &= \left(1-\frac{1}{x}\right)^{p} \\
C(x) &= \left(1-\frac{1}{x}\right)^{1+p} x^2
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\phi &= \frac{\sigma}{2\eta}\ln\left(1-\frac{1}{x}\right) \\
\eta^2 &= m^2+\sigma^2.
\end{aligned}
$$

The first and second derivatives of $A(x)$ and $C(x)$ with respect to x are[1]

[1] It is important to take into account that $' = \frac{d}{dx}$

$$
\begin{aligned}
A'(x) &= \frac{k}{x^2}\left(1-\frac{1}{x}\right)^{k-1} \\
A''(x) &= \left(1-\frac{1}{x}\right)^{k-1}\left[\frac{k(k-1)-2k(x-1)}{x^3(x-1)}\right] \\
C'(x) &= \left(1-\frac{1}{x}\right)^{p}[2x+p-1] \\
C''(x) &= \left(1-\frac{1}{x}\right)^{p}\left[\frac{2x^2+2(p-1)x+p(p-1)}{x(x-1)}\right]
\end{aligned}
\tag{4.4}
$$

4.1 Solving the photon sphere equation

From equation (3.67) and (4.4)

$$
\begin{aligned}
\alpha &= \frac{1-\left(1-\frac{1}{x_0}\right)^k}{k\left(1-\frac{1}{x_0}\right)^{k+p}}\left\{(1+p)\left(1-\frac{1}{x_0}\right)^{k+p}+2x_0\left(1-\frac{1}{x_0}\right)^{k+p+1}-k\left(1-\frac{1}{x_0}\right)^{p+k}\right\} \\
&= \frac{1-\left(1-\frac{1}{x_0}\right)^k}{k}\left\{(1+p)+2x_0\left(1-\frac{1}{x_0}\right)-k\right\} \\
&= \frac{1-\left(1-\frac{1}{x_0}\right)^k}{k}\left\{2x_0+p-k-1\right\}.
\end{aligned}
\tag{4.5}
$$

For $\sigma = 0$ we see that $k = 1$ and $p = -1$; therefore, α reduces to equation (6.27). The solution of equation $\alpha = 0$ from (4.5) give us the radio of photon sphere x_m.

$$
\begin{aligned}
0 &= 2x_m + p - k - 1 \\
x_m &= \frac{k-p+1}{2}.
\end{aligned}
\tag{4.6}
$$

For $\sigma = 0$, x_m reduces to equation (6.30).

4.2 Finding β_m and $R(0, x_m)$

In order to calculate β and β_m we are going to calculate each term of equation (3.71).

$$
\begin{aligned}
2C_0C_0'A_0'^2 &= 2\left(1-\frac{1}{x_0}\right)^{1+p}x_0^2\left(1-\frac{1}{x_0}\right)^{p}(2x_0+p-1)\frac{k^2}{x_0^4}\left(1-\frac{1}{x_0}\right)^{2k-2} \\
&= \frac{2k^2}{x_0^2}\left(1-\frac{1}{x_0}\right)^{2(p+k)-1}(2x_0+p-1),
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
-C_0y_0C_0'A_0'' &= -\left(1-\frac{1}{x_0}\right)^{1+p}x_0^2\left(1-\frac{1}{x_0}\right)^{k}\left(1-\frac{1}{x_0}\right)^{k-1}\left[\frac{k(k-1)-2k(x_0-1)}{x_0^3(x_0-1)}\right] \\
&\quad\times\left(1-\frac{1}{x_0}\right)^{p}(2x_0+p-1) \\
&= -\frac{k(k-1)-2k(x_0-1)}{x_0(x_0-1)}\left(1-\frac{1}{x_0}\right)^{2(p+k)}(2x_0+p-1)
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
(C_0C_0'' - 2C_0'^2)y_0A_0' &= \{\left(1-\frac{1}{x_0}\right)^{1+p} x_0^2 \left(1-\frac{1}{x_0}\right)^{p} \left[\frac{2x_0^2+2(p-1)x_0+p(p-1)}{x_0(x_0-1)}\right] \\
&- 2\left(1-\frac{1}{x_0}\right)^{2p} [2x_0+(p-1)]^2\} \times \left(1-\frac{1}{x_0}\right)^{2k-1} \frac{k}{x_0^2} \\
&= \{\left(1-\frac{1}{x_0}\right)^{2p+1} x_0 \left[\frac{2x_0^2+2(p-1)x_0+p(p-1)}{(x_0-1)}\right] \\
&- 2\left(1-\frac{1}{x_0}\right)^{2p} [4x_0+4(p-1)x_0+(p-1)^2]\} \times \left(1-\frac{1}{x_0}\right)^{2k-1} \frac{k}{x_0^2} \\
&= \{\left(\frac{x_0-1}{x_0}\right) x_0 \left[\frac{2x_0^2+2(p-1)x_0+p(p-1)}{(x_0-1)}\right] \\
&- 2[4x_0+4(p-1)x_0+(p-1)^2]\} \times \left(1-\frac{1}{x_0}\right)^{2(k+p)-1} \frac{k}{x_0^2} \\
&= \frac{-6x_0^2-6(p-1)x_0+(p-1)(2-p)}{x_0(x_0-1)} \left(1-\frac{1}{x_0}\right)^{2(p+k)}
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\frac{(1-y_0)^2}{2C_0^2A_0'^3} &= \frac{\left(1-\left(1-\frac{1}{x_0}\right)^k\right)^2}{2\left(1-\frac{1}{x_0}\right)^{2+2p} x_0^4\frac{k^3}{x_0^6}\left(1-\frac{1}{x_0}\right)^{3k-3}} \\
&= \frac{\left(1-\left(1-\frac{1}{x_0}\right)^k\right)^2}{2\frac{k^3}{x_0^2}\left(1-\frac{1}{x_0}\right)^{2(k+p)}\left(1-\frac{1}{x_0}\right)^{k-1}}.
\end{aligned}
\tag{4.10}
$$

Finally β is

$$
\begin{aligned}
\beta &= \frac{\left(1-\left(1-\frac{1}{x_0}\right)^k\right)^2}{2\frac{k^3}{x_0^2}\left(1-\frac{1}{x_0}\right)^{k-1}} \times \\
&\{\frac{2k^2(2x_0+p-1)-k(6x_0^2+6(p-1)x_0-(p-1)(2-p))-(2x_0+p-1)(k(k-1)-2k(x_0-1))}{x_0(x_0-1)}\} \\
&= \frac{\left(1-\left(1-\frac{1}{x_0}\right)^k\right)^2}{2k^3\left(1-\frac{1}{x_0}\right)^{k}}[2k^2(2x_0+p-1)-k(6x_0^2+6(p-1)x_0-(p-1)(2-p)) \\
&-(2x_0+p-1)(k(k-1)-2k(x_0-1))] \\
&= \frac{\left(1-\left(1-\frac{1}{x_0}\right)^k\right)^2}{2k^3\left(1-\frac{1}{x_0}\right)^{k}}[-2kx_0^2+(2k(k+1)-4kp)x_0+k(p-1)(1-p+k)]
\end{aligned}
\tag{4.11}
$$

For $\sigma = 0$, $k = 1$ and $p = -1$; thus β reduces to equation (6.28). In order to calculate β_m we replace $x_0 = x_m$ in equation (4.11) [2].

[2] It is important to recall that $k - p = 2x_m - 1$

$$\begin{aligned}
\beta_m &= \frac{\left(1-\left(1-\frac{1}{x_m}\right)^k\right)^2}{2k^3\left(1-\frac{1}{x_m}\right)^k}[-2kx_m^2+(2k(k+1)-4kp)x_0+k(p-1)(1-p+k)] \\
&= \frac{[-2x_m^2+(2(k+1)-4p)x_m+2(p-1)x_m]}{(x_m-1)^k}\frac{(x_m^k-(x_m-1)^k)^2}{2k^2x_m^k} \\
&= \frac{[-2x_m^2+2(k-p)x_m]}{(x_m-1)^k}\frac{(x_m^k-(x_m-1)^k)^2}{2k^2x_m^k} \\
&= \frac{[-2x_m^2+2(2x_m-1)x_m]}{(x_m-1)^k}\frac{(x_m^k-(x_m-1)^k)^2}{2k^2x_m^k} \\
&= \frac{[2x_m^2-2x_m]}{(x_m-1)^k}\frac{(x_m^k-(x_m-1)^k)^2}{2k^2x_m^k} \\
&= \frac{(x_m^k-(x_m-1)^k)^2}{k^2(x_m-1)^{k-1}x_m^{k-1}}.
\end{aligned} \tag{4.12}$$

Replacing x_m in equation (4.11) we obtain

$$\begin{aligned}
\beta_m &= \frac{[(k-p+1)^k-(k-p-1)^k]^2}{4k^2(k-p-1)^{k-1}(k-p+1)^{k-1}} \\
&= \frac{[(k-p+1)^k-(k-p-1)^k]^2}{4k^2((k-p)^2-1)^{k-1}}
\end{aligned} \tag{4.13}$$

For $\sigma=0$, β_m reduces to equation (6.28).

From equation (3.60)we obtain

$$\begin{aligned}
R(z,x_0) &= 2\frac{\left(1-\frac{1}{x}\right)^{\frac{p+k}{2}}\left(1-\left(1-\frac{1}{x_0}\right)^k\right)\left(1-\frac{1}{x_0}\right)^{\frac{1+p}{2}}x_0}{k\left(1-\frac{1}{x}\right)^{p+k}} \\
&= \frac{2x_0}{k}\frac{\left(\left(1-\frac{1}{x_0}\right)^{\frac{1+p}{2}}-\left(1-\frac{1}{x_0}\right)^{\frac{2k+p+1}{2}}\right)}{\left(1-\frac{1}{x_0}\right)^{\frac{p+k}{2}}}.
\end{aligned} \tag{4.14}$$

For $\sigma=0$, $R(z,x_0)$ reduces to equation (6.24). In order to calculate $R(0,x_0)$, notice that $z=0$ for $x=x_0$ as equation (3.56) shows, and thus

$$\begin{aligned}
R(0,x_0) &= \frac{2x_0}{k}\frac{\left(\left(1-\frac{1}{x_0}\right)^{\frac{1+p}{2}}-\left(1-\frac{1}{x_0}\right)^{\frac{2k+p+1}{2}}\right)}{\left(1-\frac{1}{x_0}\right)^{\frac{p+k}{2}}} \\
&= \frac{2x_0}{k}\left(\left(1-\frac{1}{x_0}\right)^{\frac{1-k}{2}}-\left(1-\frac{1}{x_0}\right)^{\frac{k+1}{2}}\right).
\end{aligned} \tag{4.15}$$

For $\sigma=0$, $R(0,x_0)$ reduces to equation (6.32).

4.3 Computing b_R

In order to calculate b_R we have defined $\zeta = \frac{\sigma}{\eta}$ in such a way that

$$\lambda^2 + \zeta^2 = 1; \tag{4.16}$$

where $\lambda = \frac{m}{\eta}$ and $\zeta = \frac{\sigma}{\eta}$. Therefore, $R(z, x_m)$, $R(0, x_m)$, $f(z, x_m)$ and $f_0(z, x_m)$ for S. Kar's metric takes the form

$$R(z,x_m) = \frac{2\sqrt{1-\zeta^2}+1}{\sqrt{1-\zeta^2}+\zeta}\left\{\frac{\left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{2}} - \left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{3\zeta-\sqrt{1-\zeta^2}+1}{2}}}{[(1-y_m)z+y_m]^{\frac{\zeta}{\sqrt{1-\zeta^2}+\zeta}}}\right\}$$

$$R(0,x_m) = \frac{2\sqrt{1-\zeta^2}+1}{\sqrt{1-\zeta^2}+\zeta}\left\{\frac{\left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{2}} - \left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{3\zeta-\sqrt{1-\zeta^2}+1}{2}}}{y_m^{\frac{\zeta}{\sqrt{1-\zeta^2}+\zeta}}}\right\}$$

$$f(z,x_m) = \frac{1}{\sqrt{y_m - [(1-y_m)z+y_m]\frac{C_m}{C}}}$$

$$f_0(z,x_m) = \frac{1}{\sqrt{\beta_m(\zeta)}|z|} = \frac{2(\sqrt{1-\zeta^2}+\zeta)(3-4\zeta^2)^{\frac{\sqrt{1-\zeta^2}+\zeta-1}{2}}}{(2\sqrt{1-\zeta^2}+1)^{\sqrt{1-\zeta}+\zeta} - (2\sqrt{1-\zeta^2}-1)^{\sqrt{1-\zeta}+\zeta}}\frac{1}{|z|}, \tag{4.17}$$

where

$$\begin{aligned}
y_m &= \left(\frac{2\sqrt{1-\zeta}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\sqrt{1-\zeta^2}+\zeta} \\
C_m &= \left[\frac{2\sqrt{1-\zeta^2}+1}{2}\right]^2\left[\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right]^{1+\zeta-\sqrt{1-\zeta^2}} \\
C &= \frac{[(1-y_m)z+y_m]^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{\zeta+\sqrt{1-\zeta^2}}}}{[1-[(1-y_m)z+y_m]^{\frac{1}{\zeta+\sqrt{1-\zeta^2}}}]^2} \\
\beta_m &= \frac{1}{4}\frac{[(2\sqrt{1-\zeta^2}+1)^{\sqrt{1-\zeta^2}+\zeta} - (2\sqrt{1-\zeta^2}-1)^{\sqrt{1-\zeta^2}+\zeta}]^2}{(\sqrt{1-\zeta^2}+\zeta)^2(3-4\zeta^2)^{\sqrt{1-\zeta^2}+\zeta-1}}.
\end{aligned} \tag{4.18}$$

We have expressed C as a function of z using (3.56). For $\zeta = 0$ these expressions reduce to those of Schwarzschild (Cfr. Appendix A).

The regular term b_R can not be calculated analytically. However, we can expand the integrand in equation (3.95) in powers of ζ and evaluate the single coefficients. So that, b_R up to first order in ζ is

$$\begin{aligned}
b_R &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^{(n)}}{d\zeta^n}I_R(x_m)(\zeta-0)^n \\
&= I_R(x_m)_{\zeta=0} + \left(\frac{d}{d\zeta}I_R(x_m)\right)_{\zeta=0}\zeta
\end{aligned} \tag{4.19}$$

For $\zeta = 0$ the value of $I_R(x_m)$ reduces to that of Schwarzschild (Crf. Appendix A). Therefore, for $0 \leq z \leq 1$ ($|z| = z$) we obtain

$$I_R(x_m)_{\zeta=0} = 2\int_0^1 \left[\frac{1}{|z|\sqrt{1-\frac{3}{2}z}} - \frac{1}{|z|}\right] dz = 2\ln 6(2-\sqrt{3}) = 0.9496. \tag{4.20}$$

On the other hand,

$$\begin{aligned}
\frac{d}{d\zeta}I_R(x_m) &= \int_0^1 \left[\frac{d}{d\zeta}(R(z,x_m)f(z,x_m)) - \frac{d}{d\zeta}(R(0,x_m)f_0(z,x_m))\right]_{\zeta=0} dz \\
&= \int_0^1 [f(z,x_m)\frac{d}{d\zeta}R(z,x_m) + R(z,x_m)\frac{d}{d\zeta}f(z,x_m) \\
&\quad - f_0(z,x_m)\frac{d}{d\zeta}R(0,x_m) - R(0,x_m)\frac{d}{d\zeta}f_0(z,x_m)]_{\zeta=0}dz \\
&= \int_0^1 \{f_S(z,x_m)\left[\frac{d}{d\zeta}R(z,x_m)\right]_{\zeta=0} + 2\left[\frac{d}{d\zeta}f(z,x_m)\right]_{\zeta=0} \\
&\quad - f_{0S}(z,x_m)\frac{d}{d\zeta}R(0,x_m) - 2\left[\frac{d}{d\zeta}f_0(z,x_m)\right]_{\zeta=0}\}dz
\end{aligned} \tag{4.21}$$

where $f_S(z,x_m)$, $f_{0S}(z,x_m)$ are those of Schwarzschild. This gives

$$\frac{d}{d\zeta}I_R(x_m) = \int_0^1 \left[\frac{\frac{d}{d\zeta}R(z,x_m)}{z\sqrt{1-\frac{2}{3}z}} - \frac{\frac{d}{d\zeta}R(0,x_m)}{z} + 2\frac{d}{d\zeta}f(z,x_m) + \frac{\frac{d\beta_m}{d\zeta}}{z}\right] dz, \tag{4.22}$$

where all derivatives, evaluated at $\zeta = 0$, are:

$$\begin{aligned}
\left[\frac{d}{d\zeta}R(z,x_m)\right]_{\zeta=0} &= -2 - 2\ln\left(\frac{2}{3}z + \frac{1}{3}\right) \\
\left[\frac{d}{d\zeta}R(0,x_m)\right]_{\zeta=0} &= -2 + 2\ln(3) \\
\left[\frac{d}{d\zeta}f_0(z,x_m)\right]_{\zeta=0} &= \frac{\ln(3)-1}{|z|} \\
\left[\frac{d}{d\zeta}f(z,x_m)\right]_{\zeta=0} &= -\frac{1}{2}\frac{\ln 3\left[\frac{7}{3}z^3 - 2z^2\right] + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}}.
\end{aligned} \tag{4.23}$$

Finally we have that

$$\begin{aligned}
\frac{d}{d\zeta}I_R(x_m) &= \int_0^1 \left[\frac{2-2\ln(3)}{z} - \frac{2+2\ln\left(\frac{2}{3}z+\frac{1}{3}\right)}{z\sqrt{1-\frac{2}{3}z}}\right] dz \\
&\quad + \int_0^1 \left[\frac{\ln 3\left[\frac{7}{3}z^3 - 2z^2\right] + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}} + \frac{2\ln(3)-2}{z}\right] dz.
\end{aligned} \tag{4.24}$$

Using a numerical method, we calculate the integrals

$$\begin{aligned}
\int_0^1 \left[\frac{2-2\ln(3)}{z} - \frac{2+2\ln\left(\frac{2}{3}z+\frac{1}{3}\right)}{z\sqrt{1-\frac{2}{3}z}}\right] dz &= -3.457723875 \\
\frac{7\ln(3)}{3}\int_0^1 \frac{dz}{\left(1-\frac{2}{3}z\right)^{\frac{3}{2}}} &= \frac{14\sqrt{3}\ln(3)}{(3+\sqrt{3})}
\end{aligned} \tag{4.25}$$

In order to calculate de remaining integral

$$i = \int_0^1 \left[\frac{-2\ln(3)z^2 + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}} + \frac{2\ln(3)-2}{z} \right] dz, \tag{4.26}$$

we expressed $\ln(2z+1)$ in a Taylor series around $z = 0.5$. The detail of this expansion is shown in appendix B. Therefore, b_R is

$$b_R = 2\ln(6(2-\sqrt{3})) + 3.005480454\zeta. \tag{4.27}$$

4.4 Finding u_m, $\bar{a}$ and $\bar{b}$

To calculate a we use equation (3.93):

$$\begin{aligned} a &= \frac{2x_m}{k} \frac{\left(\left(1-\frac{1}{x_m}\right)^{\frac{1-k}{2}} - \left(1-\frac{1}{x_m}\right)^{\frac{k+1}{2}}\right)}{[(k-p+1)^k - (k-p-1)^k]} [2k((k-p)^2-1)^{\frac{k-1}{2}}] \\ &= 2(k-p+1)\left(\left(1-\frac{2}{k-p+1}\right)^{\frac{1-k}{2}} - \left(1-\frac{2}{k-p+1}\right)^{\frac{k+1}{2}}\right) \\ &\times \frac{((k-p)^2-1)^{\frac{k-1}{2}}}{[(k-p+1)^k-(k-p-1)^k]} \\ &= 2(k-p+1)\left(\frac{(k-p-1)^{\frac{1-k}{2}}}{(k-p+1)^{\frac{1-k}{2}}} - \frac{(k-p-1)^{\frac{k+1}{2}}}{(k-p+1)^{\frac{k+1}{2}}}\right) \frac{((k-p)^2-1)^{\frac{k-1}{2}}}{[(k-p+1)^k-(k-p-1)^k]} \\ &= 2(k-p+1)\frac{[(k-p-1)^{\frac{1-k}{2}}(k-p+1)^{\frac{k+1}{2}} - (k-p-1)^{\frac{k+1}{2}}(k-p+1)^{\frac{1-k}{2}}]}{(k-p+1)} \\ &\times \frac{((k-p)^2-1)^{\frac{k-1}{2}}}{[(k-p+1)^k-(k-p-1)^k]} \\ &= 2\frac{(k-p+1)^k-(k-p-1)^k}{(k-p+1)^k-(k-p-1)^k} = 2. \end{aligned} \tag{4.28}$$

Then

$$\bar{a} = \frac{R(0,x_m)}{2\sqrt{\beta_m}} = \frac{2}{2} = 1. \tag{4.29}$$

In order to compute $\bar{b}$ we must use the relation

$$\bar{b} = -\pi + b_R + \bar{a}\ln\left(\frac{2\beta_m}{y_m}\right), \tag{4.30}$$

where

$$\begin{aligned} \frac{2\beta}{y_m} &= 2\frac{[(k-p+1)^k-(k-p-1)^k]^2}{4k^2((k-p)^2-1)^{k-1}} \frac{(k-p+1)^k}{(k-p-1)^k} \\ &= \frac{[(k-p+1)^k-(k-p-1)^k]^2}{2k^2(k-p+1)^{k-1}(k-p-1)^{k-1}} \frac{(k-p+1)^k}{(k-p-1)^k} \\ &= \frac{[(k-p+1)^k-(k-p-1)^k]^2}{2k^2(k-p-1)^{2k-1}}(k-p+1). \end{aligned} \tag{4.31}$$

Finally, from equation (3.102),

$$
\begin{aligned}
u_m &= \left[\frac{\left(\frac{k-p-1}{k-p+1}\right)^{1+p}}{\left(\frac{k-p-1}{k-p+1}\right)^{k}}\right]^{\frac{1}{2}} \frac{k-p+1}{2} \\
&= \frac{(k-p+1)}{2}\left(\frac{k-p-1}{k-p+1}\right)^{\frac{p-k+1}{2}} \\
&= \frac{1}{2}\frac{(k-p-1)^{\frac{p-k+1}{2}}}{(k-p+1)^{\frac{p-k-1}{2}}}.
\end{aligned} \tag{4.32}
$$

Chapter 5

GRAVITATIONAL LENSING IN THE STRONG FIELD LIMIT

5.1 Introduction:

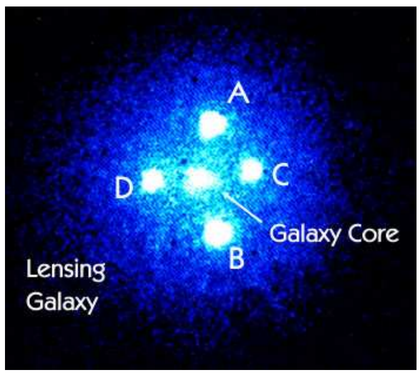
As explained in the last chapter, One of the consequences of Einstein's General Theory of Relativity is that light rays are deflected by gravity. Although this discovery was made only in the las century, the possibility that there could be such a deflection had been suspected much earlier, by Newton and Laplace among others. Furthermore, Soldner (1804) calculated the magnitude of the deflection due to the Sun, assuming that light consist of material particles and using Newtonian gravity. Later, Einstein [17] employed the equivalence principle to calculate the deflection angle and re-derived Soldner's formula. Later yet, Einstein (1915) applied the full field equations of general relativity and discovered that the deflection angle is actually twice his previous result, the factor of two arising because of the curvature of the metric. Einstein's final result was confirmed in 1919 when the apparent angular shift of stars close to the limb of the Sun was measured during a total eclipse. The quantitative agreement between the measured shift and Einstein's prediction was immediately perceived as compelling evidence in support of the theory of General Relativity. The deflection of light by massive bodies, and the phenomena resulting therefrom, are now referred to as *Gravitational lensing*[24].

Eddington (1920) noted that under certain conditions there may be multiple light paths connecting a source and an observer. This implies that gravitational lensing can give rise to multiple images of a single source[1]. Chwolson (1924) considered the creation of fictitious double stars by gravitational lensing of stars by stars, but did not comment on whether the phenomenon could actually be observer[24]. Einstein [19] discussed the same problem and concluded that there is little chance of observing lensing phenomena caused by the stellar-mass lenses. His reason was that the angular image splitting caused by a stellar-mass lens is to small to be resolved by an optical telescope.

Zwicky (1937) elevated gravitational lensing from a curiosity to a field with grate potential when he pointed out that galaxies can split images of background sources by a large enough angle to be observed. He argued that the deflection of light by galaxies which would otherwise remain undetected, and would allow accurate determination of galaxy masses. In this sense, gravitational lensing studies have been considered an important tool. For example, the most accurate mass determination of the central regions of galaxies are due to to gravitational lensing, and the cosmic telescope effect of gravitational lenses has enable us to study faint and distant galaxies which happened to be strongly magnified

[1] As we will see later, this is a consequence of the lens equation.

by galaxy clusters[24]. Other important example was proposed by Refsdal (1964). In his paper he described how the Hubble constant H_0 could in principle be measured through gravitational lensing of a variable source. Since the light travel times for the various images are unequal, intrinsic variations of the source would be observed at different times in the images. The time delay between images is proportional to the difference in the absolute lengths of the light path, which in turn is proportional to H_0^{-1}. Thus, if the time delay is measured and if an accurate model of the lenses source is developed, the Hubble constant could be measured. However, all this ideas on gravitational lensing remained mere speculation until Walsh, Carswell and Weymann (1979) discovered the first example of gravitational lensing, the quasar $QSO0957 + 561A, B$. Quasars are ideal for studying the effects of gravitational lensing because they are distant, and the probability they are lensed by intervening galaxies is sufficiently large. Yet, they are bright enough to be detected even at cosmological distances[24]. Moreover, their optical emission region is very compact, much smaller than the typical scales of galaxy lenses. The resulting magnification can therefore be very large, and multiple image components are well separated and easily detected [24].



(a)

(b)

Figure 5.1: (a) A quasar (taken from The European Space Agency's Faint Object Camera on board NASA's Hubble Space Telescope) (b) Einstein Ring (taken from Hubble space telescope)

But, how they did realize that the quasar $QSO0957 + 561A, B$ were indeed a twin lensed images of a single QSO? the answer is provided by (i)the similarity of the spectra of the two images, (ii) the fact that the flux ratio between the images is similar in the optical and ratio wave-bands, (iii) the presence of a foreground galaxy between the images, and (iv) VLBI observations which show detailed correspondence between various knots of emission in the two radio images.

The full theory of gravitational lensing has been developed following the scheme of the weak field approximation. For almost all cases of relevance to gravitational lensing, we can assume that the overall geometry of the universe is well described by the Friedmann-Lemaître-Robertson-Walker metric and that the mater inhomogeneities which cause the lensing are no more than local perturbation[24]. In this sense, we can broke up the light paths propagating from the source past the lens to the observer into three distinct zones. In the first zone, light travels from the source to a point close to the lens through unperturbed space-time. In the second zone, near the lens, light is deflected. Finally, in the third zone, light again travels through unperturbed space-time. In order to study the deflection close the lens, we can assume a locally flat, Minkowskian space-time which is weakly perturbed by the Newtonian gravitational potential of the mass distribution constituting the lens. This approach is legitimate if the Newtonian potential Φ is small, $| \Phi | \ll c^2$, and the peculiar velocity ν of the lens is small, $\nu \ll c$ [24]. This approach has been successfully employed to explain all the physical observations[22]. Consider for instance, a galaxy cluster at redshift ~ 0.3 which deflects light from a source at redshift ~ 1. The distances from the source and the lens and from the lens to the observer are ~ 1Gpc, or about three orders of magnitude larger than the diameter of the cluster. Thus the second zone is limited to a small local segment of the total light path. The relative peculiar velocities in a galaxy cluster are $\sim 10^3$Km$s^{-1} \ll c$, and the Newtonian potential are is $\mid \Phi \mid < 10^{-4}c^2 \ll c^2$. In agrement with the conditions stated about[24].

Several studies about light rays close to the Schwarzschild horizon has been lead. For example, Viergutz (1993) made a semi-analytical investigation about geodesics in Kerr geometry. Bardeen (1973) and Falcke (1999) studied the appearance of a black hole in front a uniform background. Virbhabra and Ellis (1999) faced the simplest strong field problem, represented by deflection in Schwarzschild space-time, by numerical techniques. The existence of an infinity set of relativistic images has been enlightened and the results have been applied to the black hole at the center of the galaxy. Recently (2002), Virbhabra and Ellis in [20] distinguished the main features of gravitational lensing by normal black holes and by naked singularities, analyzing the Janis-Newman-Winicour metric (also obtained by Max Wyman). They remarked the importance of these studies in providing a test for the cosmic censorship hypothesis. In the last years, however, the scientific community is starting to look at this phenomenon from the strong field limit point of view. The reason for such an interest in gravitational lensing in strong fields is that by the properties of the relativistic images it may be possible to investigate the regions immediately outside of the event horizon. Moreover, since alternative theories of gravitation must agree with General Relativity in the weak field limit, in order to show deviations from General Relativity it is necessary to probe strong fields in some way. In this sense, the possibility of testing the full general relativity in a regime where the differences with non-standard theories would be manifest, would help to discriminate among the various theories of gravitation [29]. Indeed, deviation of light rays in strong fields is one of the most promising grounds where a theory of gravitation can be tested in its full form [22].

In this chapter we are going to study the gravitational lensing using the method proposed by V. Bozza in [22]. As mentioned by the author: "*The method proposed is universal and can be applied to any space-time in any theory of gravitation, provided that photons satisfy the standard geodesics equation*". As was shown in chapter II V. Bozza has proved that the strong field limit approximation can be used to obtain a simple and reliable formula for the deflection angle, which contains a logarithmic and a constant term. This relation is expressed by equation,

$$\alpha(\theta) = -\bar{a}\ln\left(\frac{\theta D_{OL}}{u_m} - 1\right) + \bar{b} + \mathcal{O}(u - u_m). \tag{5.1}$$

Now, we are going to replace this formula into the lens equation and establish direct relations between the position and the magnification of the relativistic images and the deflection angel, calculated according to the strong field limit approximation. In this sense, We are going to use this method on the line element for a spherically symmetric metric proposed by S. Kar in [16].

5.2 The weak limit approach:

The propagation of light in arbitrary curved space-times is in general a complicated theoretical problem. However, for almost all cases of relevant to gravitational lensing, we can assume that the overall geometry of the universe is well described by the Friedman-Lemaître-Robertson-Walker metric and that the matter inhomogeneities which causes the lensing are not more than the matter inhomogeneities which causes the lensing are no more than local perturbations [24]. In this sense, light paths propagating from the source to the observer can be broken up into three distinct zones. In the first zone, light travels from the source to a point close to the lens trough unperturbed space-time. In the second zone, near the lens, light is deflected. Finally in the third zone light again travels trough unperturbed space-time.

The weak limit approach emerges from studying light deflection close to the lens (third zone). As mentioned above, the matter inhomogeneities which cause the lensing are local perturbations. Hence, we can assume a locally flat, Minkowskian space-time which is weakly perturbed by the Newtonian potential of the mass distribution constituting the lens [24]. In order to study this approximation, we assume that the metric $g_{\alpha\beta}$ differs little from the flat Minkowskian metric [6][25]

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.2}$$

in orthonormal coordinates $x^0 = ct$, $\mathbf{x} = x^i$. Then we can write,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}; \tag{5.3}$$

here $h := \eta^{\alpha\beta} h_{\alpha\beta}$ and $|h_{\alpha\beta}| \ll 1$ [6][25]. The assumption that $h_{\alpha\beta}$ is small allows us to ignore anything that is higher than first order in this quantity, from which we immediately obtain,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}, \tag{5.4}$$

where $h^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma}h_{\rho\sigma}$.

In linear approximation with respect to the metric deviation components $h_{\alpha\beta}$, one can, without loss of generality choose the coordinate such that

$$h^{\alpha\beta}{}_{,\beta} = \partial_\beta h^{\alpha\beta} = 0. \tag{5.5}$$

5.2.1 The linearized field equations

Replacing equation (5.3) into Einstein's field equations,

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta},$$

we obtain that,

$$R_{\alpha\beta} - \frac{1}{2} R(\eta_{\alpha\beta} + h_{\alpha\beta}) = \frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{5.6}$$

In order to calculate the form of (5.6) under the approximation $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, it is necessary to find the Christoffel symbols $\Gamma^\rho_{\mu\nu}$; which are represented by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\mu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{5.7}$$

Replacing equation (5.3) we obtain[2]

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda}) \left(\partial_\mu(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu(\eta_{\mu\lambda} + h_{\mu\lambda}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu})\right),$$

then[3]

[2] We had into account that $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$

[3] Because $\eta_{\mu\nu}$ is constant, $\partial_\lambda \eta_{\mu\nu} = 0$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) - \frac{1}{2}h^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right)$$

Finally[4],

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right). \tag{5.8}$$

Since the connection coefficients are first order quantities, the only contribution to the Riemann tensor will come from the derivatives of the $\Gamma's$, not the Γ^2 terms [6]. Lowering and index

$$R_{\mu\nu\rho\sigma} = \eta_{\nu\lambda}\partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}_{\nu\rho} \tag{5.9}$$

where,

$$\begin{aligned}
\eta_{\mu\lambda}\partial_\rho\Gamma^{\lambda}_{\nu\sigma} &= \frac{1}{2}\left[\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\rho\partial_\mu h_{\nu\sigma}\right] \\
\eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}_{\nu\rho} &= \frac{1}{2}\left[\partial_\sigma\partial_\nu h_{\rho\mu} + \partial_\sigma\partial_\rho h_{\mu\nu} - \partial_\sigma\partial_\mu h_{\nu\rho}\right].
\end{aligned} \tag{5.10}$$

Then, replacing (5.10) into equation (5.9) we obtain[5]

$$\begin{aligned}
R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left[\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\rho\partial_\sigma h_{\nu\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu} - \partial_\sigma\partial_\rho h_{\mu\nu} + \partial_\sigma\partial_\mu h_{\nu\rho}\right] \\
&= \frac{1}{2}\left[\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu}\right].
\end{aligned} \tag{5.11}$$

The Ricci tensor comes from contracting over μ and ρ. We know that

$$R^{\alpha}_{\nu\rho\sigma} = \eta^{\mu\alpha}R_{\mu\nu\rho\sigma}, \tag{5.12}$$

then, for $\alpha = \rho$ we have

$$R_{\nu\sigma} = R^{\alpha}_{\nu\alpha\sigma}. \tag{5.13}$$

So, from equation (5.12)

$$\begin{aligned}
\eta^{\mu\alpha}R_{\mu\nu\rho\sigma} &= \frac{1}{2}\eta^{\mu\alpha}\left[\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\rho\mu} - \partial_\rho\partial_\mu h_{\nu\sigma}\right] \\
&= \frac{1}{2}\left[\partial_\rho\partial_\nu h^{\alpha}_{\sigma} + \partial_\sigma\partial^{\alpha} h_{\nu\rho} - \partial_\sigma\partial_\nu h^{\alpha}_{\rho} - \partial_\rho\partial^{\alpha} h_{\nu\sigma}\right].
\end{aligned} \tag{5.14}$$

Finally, from equation (5.13) (making $\alpha = \rho$) we obtain[6]

$$\begin{aligned}
R_{\nu\sigma} &= \frac{1}{2}\left[\partial_\alpha\partial_\nu h^{\alpha}_{\sigma} + \partial_\alpha\partial_\sigma h^{\alpha}_{\nu} - \partial_\sigma\partial_\nu h - \partial_\alpha\partial^{\alpha} h_{\nu\sigma}\right] \\
&= \frac{1}{2}\left[\partial_\alpha\partial_\sigma h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\sigma} - \partial_\nu\partial_\sigma h - \partial_\alpha\partial^{\alpha} h_{\nu\sigma}\right],
\end{aligned} \tag{5.15}$$

which is manifestly symmetric in ν and σ. In this expression we have defined the trace of the perturbation as $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}_{\mu}$. Moreover, we can rewrite 5.14 using $\Box = \partial_\alpha\partial^\alpha$, then

$$R_{\nu\sigma} = \frac{1}{2}\left[\partial_\alpha\partial_\sigma h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\sigma} - \partial_\nu\partial_\sigma h - \Box h_{\nu\sigma}\right]. \tag{5.16}$$

[4]Because $h_{\alpha\beta} \ll 1$

[5]We have used $h_{\alpha\beta} = h_{\beta\alpha}$ and $\partial_\rho\partial_\sigma = \partial_\sigma\partial_\rho$

[6]We have used that $h = \eta^{\alpha\beta}h_{\alpha\beta} = h^{\alpha}_{\alpha}$

Contracting (5.16) again to obtain the Ricci scalar,

$$\begin{aligned} R =& \eta^{\nu\sigma} R_{\nu\sigma} = R^{\sigma}_{\sigma} \\ =& \frac{1}{2}\eta^{\nu\sigma}\left[\partial_\alpha\partial_\sigma h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\sigma} - \partial_\nu\partial_\sigma h - \Box h_{\nu\sigma}\right] \\ =& \frac{1}{2}\left(\partial_\alpha\partial_\sigma h^{\sigma\alpha} + \partial_\alpha\partial_\nu h^{\nu\alpha} - \partial^{\sigma}\partial_\sigma h - \Box h\right) \\ =& \frac{1}{2}\left[2\partial_\alpha\partial_\sigma h^{\sigma\alpha} - 2\Box h\right]. \end{aligned} \tag{5.17}$$

Finally,

$$R = \partial_\nu\partial_\sigma h^{\sigma\nu} - \Box h. \tag{5.18}$$

Putting it all together we obtain the Einstein tensor,

$$\begin{aligned} G_{\nu\sigma} &= R_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}R \\ &= \frac{1}{2}\left[\partial_\alpha\partial_\sigma h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\sigma} - \partial_\nu\partial_\sigma h - \Box h_{\nu\sigma} - \eta_{\nu\sigma}\partial_\nu\partial_\sigma h^{\sigma\nu} + \eta_{\nu\sigma}\Box h\right]. \end{aligned} \tag{5.19}$$

Using condition (5.5) we obtain,

$$\Box h_{\nu\sigma} = \partial^{\alpha}\partial_\alpha h_{\nu\sigma} = \eta^{\beta\alpha}\partial_\beta\partial_\alpha h_{\nu\sigma} = -\frac{16\pi G}{c^4}T_{\nu\sigma}$$

and finally[7],

$$\nabla^2 h_{\nu\sigma} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}h_{\nu\sigma} = -\frac{16\pi G}{c^4}T_{\nu\sigma} \tag{5.20}$$

It is also possible to consider an a approximation of Einstein field equations with a cosmological constant in a flat Background (cfr.[26]). However, we are going to consider only the case in which the cosmological constant is zero.

The linearized Einstein field equation were also obtained by [25] using a different perturbation for the metric[8]

$$g_{\alpha\beta} = \left(1 - \frac{1}{2}h\right)\eta_{\alpha\beta} + h_{\alpha\beta}. \tag{5.21}$$

In the same way as before, we obtain that Einstein field equation in this approximation for $h_{\alpha\beta}$ reads

$$\nabla^2 h^{\alpha\beta} - \frac{1}{c^2}\frac{\partial^2 h^{\alpha\beta}}{\partial t^2} = \frac{16\pi G}{c^4}T^{\alpha\beta}. \tag{5.22}$$

The solution of equation (5.22) for an isolated source without incoming gravitational radiation is the retarded one[25],

$$h^{\alpha\beta}(t, \mathbf{x}) = -\frac{4G}{c^4}\int \frac{T^{\alpha\beta}\left(t - \frac{|\mathbf{y}|}{c}, \mathbf{x} + \mathbf{y}\right)}{|\mathbf{y}|}d^3\mathbf{y}. \tag{5.23}$$

[7] He had into account that $x^0 = ct$, therefore $\partial_0^2 = \frac{\partial^2}{\partial(x^0)^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

[8] In this reference $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$

5.2.2 Specialization to slowly moving, perfect fluid source:

The Einstein's field equation attains a physical meaning only if the matter tensor $T^{\alpha\beta}$ is specified; for example in vacuo, $T^{\alpha\beta} = 0$. The interpretation of $T^{\alpha\beta}$ is generally as follow: in any local inertial frame, T^{00} represents the energy density, the spatial vector cT^{0i} represents the energy flux density which equals $c^2 \times$ momentum density and $-T^{ij}$ represents the stress tensor which equals the momentum flux density (these quantities refer to matter, whose energy-momentum is considered as localizable in General Relativity, not to the gravitational field)[25].

For most astrophysical purposes, one idealizes bulk matter as a perfect fluid, for which

$$T^{\alpha\beta} = (\rho c^2 + p)U^\alpha U^\beta - pg^{\alpha\beta}, \tag{5.24}$$

where ρ denotes the mass density (which includes the mass-equivalents of short range interaction and thermal energies) and p the pressure, both measured by a co-moving observer, and U^α is the $4-$velocity normalized to one,

$$g_{\alpha\beta}U^\alpha U^\beta = 1. \tag{5.25}$$

In order to study the weak field limit, we have assumed a matter tensor of the form (5.24) and that[25]

* Matter moves slowly with respect to the coordinate system x^α; it is to say: $v^i := \frac{dx^i}{dt}$ obeys $|\mathbf{v}| \ll c$.

** $\mid p \mid \ll \rho c^2$.

Hence, equation (5.24) implies

$$T^{00} \approx \rho c^2, \quad T^{0i} \approx c\rho v^i, \quad T^{ij} \approx \rho v^i v^j + p\delta^{ij} \tag{5.26}$$

where $\approx$ indicates that terms of relative order $\frac{v^2}{c^2}, \frac{p^2}{\rho c^2}$ have been neglected. From solution (5.23) and using the approximations in equation (5.26), the solutions for h^{00} and h^{0i} are

$$\begin{aligned} h^{00} &= -\frac{4G}{c^4}\int \frac{c^2\rho(t-\frac{|\mathbf{y}|}{c}, \mathbf{x}+\mathbf{y})}{\mid \mathbf{y} \mid} d^3y = \frac{4}{c^2}\Phi(\mathbf{x},t) \\ h^{0i} &= -\frac{4G}{c^4}\int \frac{c(\rho v^i)(t-\frac{|\mathbf{y}|}{c}, \mathbf{x}+\mathbf{y})}{\mid \mathbf{y} \mid} d^3y = \frac{4}{c^3}V^i(\mathbf{x},t). \end{aligned} \tag{5.27}$$

The line element ds^2 is expressed by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta. \tag{5.28}$$

Then, using equation (5.21) we have

$$ds^2 = \left(1 - \frac{h}{2}\right)\eta_{\alpha\beta}dx^\alpha dx^\beta + h_{\alpha\beta}dx^\alpha dx^\beta.$$

In this approximation, the stresses T^{ij} do not affect the metric. In this sense, we only consider terms h_{00} and h_{0i}.

$$\begin{aligned} ds^2 = &\left(1-\frac{1}{2}h\right)\eta_{00}(dx^0)^2 + \left(1-\frac{1}{2}h\right)\eta_{11}(dx^1)^2 + \left(1-\frac{1}{2}h\right)\eta_{22}(dx^2)^2 + \left(1-\frac{1}{2}h\right)\eta_{33}(dx^3)^2 \\ &+ h_{00}(dx^0)^2 + 2h_{01}dx^1dx^0 + 2h_{02}dx^2dx^0 + 2h_{03}dx^3dx^0. \end{aligned} \tag{5.29}$$

Using equations (5.27) we obtain that[9]

$$\begin{aligned} ds^2 &= \left(1-\frac{1}{2}h+h\right)(dx^0)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^1)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^2)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^3)^2 \\ &- 2dx^0\left(\frac{4V^1}{c^3}dx^1+\frac{4V^2}{c^3}dx^2+\frac{4V^3}{c^3}dx^3\right) \\ &= \left(1+\frac{2\Phi}{c^2}\right)(dx^0)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^1)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^2)^2-\left(1-\frac{2\Phi}{c^2}\right)(dx^3)^2 \\ &- \frac{8}{c^3}dx^0(V^1dx^1+V^2dx^2+V^3dx^3). \end{aligned} \tag{5.30}$$

Finally, using $x^0 = c^2$ we obtain that,

$$ds^2 = \left(1+\frac{2\Phi}{c^2}\right)c^2(dt)^2-\left(1-\frac{2\Phi}{c^2}\right)[(dx^1)^2+(dx^2)^2(dx^3)^2]-\frac{8c}{c^3}dt(V^1dx^1+V^2dx^2+V^3dx^3) \tag{5.31}$$

or, as expressed in [25]

$$ds^2 = \left(1+\frac{2\Phi}{c^2}\right)c^2(dt)^2-\left(1-\frac{2\Phi}{c^2}\right)d\mathbf{x}^2-\frac{8c}{c^3}dt\mathbf{V}.d\mathbf{x}. \tag{5.32}$$

In the near zone of as system of slowly moving bodies, the retardation in (5.27) can be neglected; there,

$$\begin{aligned} h^{00} &\approx -\frac{4G}{c^4}\int\frac{c^2\rho(t,\mathbf{x}+\mathbf{y})}{\mid\mathbf{y}\mid}d^3y = \frac{4}{c^2}\Phi(\mathbf{x},t) \\ h^{0i} &\approx -\frac{4}{c^4}\int\frac{c(\rho v^i)(t-,\mathbf{x}+\mathbf{y})}{\mid\mathbf{y}\mid}d^3y = \frac{4}{c^3}V^i(\mathbf{x},t). \end{aligned} \tag{5.33}$$

The "post-Minkowskian" metric (5.32) satisfies the weak-field condition $h_{\alpha\beta} \ll 1$ if, and only if, in addition to assumptions (*) and (**) above, the Newtonian potential Φ of the mass distribution ρ obeys

$$\mid\Phi\mid \ll c^2; \tag{5.34}$$

then

$$\mid\mathbf{V}/c^3\mid \lesssim \mid\mathbf{v}/c\mid .U/c^2 \mid\mid \ll 1. \tag{5.35}$$

For spherical bodies, the first relation implies[10] $\frac{2GM}{c^2} = R_S \ll R$; hence, compact objects as black holes and neutron stars have to be excluded. Since the gravitational vector potential $\mathbf{V}$ is smaller than Φ/c^2 by one order in v/c, equation (5.32) shows that, in near zone, the metric at one instant t is completely determined in lowest order by the density at the same instant, just like the potential in Newton's Theory [25].

5.3 Fermat's principle and light deflection:

As discussed in last section, the weak limit approach is valid when $|\Phi| << c^2$ and $\mid\mathbf{v}\mid << c$. Therefore, light propagation close to gravitational lenses can be described in a locally Minkowskian space-time by the gravitational potential of the lens to first post-Newtonian order [24]. Hence, the metric (5.32), can be expressed as[27]

[9] Here $\tilde{h}^{\alpha\beta} = -h^{\alpha\beta}$

[10] R_S is the schwarzschild radius

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 (dt)^2 - \left(1 - \frac{2\Phi}{c^2}\right) d\mathbf{x}^2 \tag{5.36}$$

In this sense, a weak lens perturbs the Minkowskian metric such that,

$$\eta_{\alpha\beta} \longrightarrow g_{\alpha\beta} = \begin{pmatrix} \left(1+\frac{2\Phi}{c^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2\Phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -\left(1-\frac{2\Phi}{c^2}\right) & 0 \\ 0 & 0 & 0 & -\left(1-\frac{2\Phi}{c^2}\right) \end{pmatrix}. \tag{5.37}$$

It is possible to studied the behavior of light in this space-time by making $ds^2 = 0$. This condition corresponds to a null vector in the manifold described by equation (5.36). Hence, we have

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 = \left(1 - \frac{2\Phi}{c^2}\right) (d\mathbf{x})^2. \tag{5.38}$$

Therefore, the speed of light in the gravitational field es thus[11],

$$c' = \frac{|d\mathbf{x}|}{dt} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} = c\sqrt{\frac{(1+2\frac{\Phi}{c^2})^2}{1-4\frac{\Phi^2}{c^4}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right) \tag{5.39}$$

This means light changes its velocity as a consequence of the perturbation in the Minkowskian metric. A. Einstein predicted in [17]. In this sense, the change in velocity of light, as a consequence of the effect of space-time curvature on the light paths, can then be interpreted as an effective index of refraction n. In order to calculate n, we are going to use the Fermat's principle as our starting point. In its simplest form the Fermat's principle says that light waves of a given frequency traverse the path between two points which takes the least time. The speed of light in a medium with refractive index n is $\frac{c}{n}$, where c is the speed of light in a vacuum. Thus, the time required to for light to go some distance in such a medium is n times the time light takes to go the same distance in a vacuum[27]. Therefore, the index of refraction is[12]

$$n = \frac{c}{c'} = \left(1 + \frac{2\Phi}{c^2}\right)^{-1} \approx 1 - \frac{2\Phi}{c^2}. \tag{5.40}$$

As in normal geometrical optics, a refractive index $n > 1$ ($\Phi \leq 0$) implies that light travels slower than in free vacuum.

Mathematically, Fermat's principle is expressed as a theorem[25]: Let S be a event ("source") and l a time-like world line ("Observer") in a space-time (M,$g_{\alpha\beta}$). Then, a smooth null curve γ from S to l is a light ray (null geodesic) if, and only if, its arrival time τ on l is stationary under first-order variations of γ with in the set of smooth null curves from S to l, $\delta\tau = 0$. It is important to point out two things: (i) this version of Fermat's principle does not refer to the "time" a light ray needs to travel from the source to the observer (which in general has no intrinsic meaning in General Relativity) but states a stationary property of the time of arrival at the observer who, in contrast to the assumptions made in classical optics, may be moving relative to the source, in a time-dependent optical field; (ii) no preferred parameters on l or γ enter the theorem; on l one may use proper time τ or any monotonic function of it; and (iii) the assertion is conformally invariant. In this sense, light will follow a path a long which the travel time,

$$\int \frac{n}{c} dl, \tag{5.41}$$

[11] We had into account that $\frac{\Phi}{c^2} \ll 1$

[12] In order to calculate this approximation we considered that $(1+x)^b \approx (1+bx)$ for $x \ll 1$. We can use this approximation because of the weak field approach.

will be extremal[27]. The index of refraction n will typically depends on the spacial coordinates $\mathbf{x}$ and perhaps also on time t. Let $\mathbf{x}(l)$ be a light path. Then the light travel time is proportional to $\int_A^B n[\mathbf{x}(l)]dl$. Hence, we are looking for a path $\mathbf{x}(l)$ for which the variation

$$\delta \int_A^B n(\mathbf{x}(l))dl = 0, \tag{5.42}$$

where the starting point A and the end point B are kept fixed[27]. If we choose λ as the curve parameter (which is yet arbitrary), we have that

$$dl = n[\mathbf{x}(\lambda)] \mid \frac{d\mathbf{x}}{d\lambda} \mid d\lambda \tag{5.43}$$

This is a standard variational problem. Hence, using Euler-Lagrange equation,

$$\frac{d}{d\lambda}\left(\frac{\partial}{\partial \dot{x}_i} L\right) - \frac{\partial L}{\partial x_i} = 0,$$

where,

$$\begin{aligned} L(\dot{\mathbf{x}}, \mathbf{x}, \lambda) &\equiv n[\mathbf{x}(\lambda)] \mid \frac{d\mathbf{x}}{d\lambda} \mid \\ &= n[\mathbf{x}(\lambda)] \mid \dot{\mathbf{x}} \mid \\ &= n[\mathbf{x}(\lambda)] \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}. \end{aligned} \tag{5.44}$$

For each x_i we have that,

$$\frac{d}{d\lambda}\left(n \frac{\dot{x}_i}{\mid \dot{\mathbf{x}} \mid}\right) + \mid \dot{\mathbf{x}} \mid \frac{\partial n}{\partial x_i} = 0. \tag{5.45}$$

Putting each component together we obtain [27],

$$\frac{d}{d\lambda}\left(n \frac{\dot{\mathbf{x}}}{\mid \dot{\mathbf{x}} \mid}\right) + (\nabla n \mid \dot{\mathbf{x}} \mid) = 0. \tag{5.46}$$

$\dot{\mathbf{x}}$ is a tangent vector to the light path, which we can assume to be normalized by a suitable choice for the curve parameter λ. We thus assume $\mid \dot{\mathbf{x}} \mid = 1 \equiv \mathbf{e}$. Then, for the tangent vector to the light path, equation (5.46) we have,

$$\frac{d}{d\lambda}(n\mathbf{e}) - \nabla n = 0. \tag{5.47}$$

Then[13],

$$n\mathbf{e} + \mathbf{e}[\nabla n \cdot \dot{\mathbf{x}}] - \nabla n = 0 \tag{5.48}$$

or

$$\dot{\mathbf{e}} = \frac{\nabla n}{n} - \dot{\mathbf{e}}(\nabla n \cdot \dot{\mathbf{e}}). \tag{5.49}$$

The second term on the right had side is the derivative along the light path, thus the whole right hand side is the gradient of n perpendicular to the light path. Thus,

$$\dot{\mathbf{e}} = \frac{1}{n}\nabla_{\perp} n = \nabla_{\perp} \ln n. \tag{5.50}$$

As $n = 1 - \frac{2\Phi}{c^2}$ and $\frac{\Phi}{c^2} \ll 1$, $\ln(n) \approx -\frac{2\Phi}{c^2}$. Then,

[13] We used $\frac{dn}{d\lambda} = \sum_{i=1}^{3} \frac{\partial n}{\partial x_i}\frac{dx_i}{d\lambda} = \dot{\mathbf{x}}\nabla n$

$$\dot{\mathbf{e}} \approx -\frac{2}{c^2}\nabla_\perp\Phi. \tag{5.51}$$

The total deflection angle of the light path is now the integral over $-\dot{\mathbf{e}}$ along the light path[27],

$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B}\nabla_\perp\Phi d\lambda. \tag{5.52}$$

This deflection is thus the integral over the "pull" of the gravitational potential perpendicular to the light path. Note that $\nabla_\perp\Phi$ points away from the lens center, so de deflection angle points towards it. As it stands, equation (5.52) is not useful, as we would have to integrate over the actual light path. However, since $\frac{\Phi}{c^2} \ll 1$, we expect the deflection angle to be small. Then, we can adopt the Born approximation familiar from scattering theory and integrate over the unperturbed light path[27]. Using this approximation the deflection angle reduces to (Cfr. [24][27]),

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{\infty}\nabla_\perp\Phi dz, \tag{5.53}$$

where b is the impact parameter. Using the Newtonian potential

$$\Phi = -\frac{GM}{r}, \tag{5.54}$$

where $r = \sqrt{x^2+y^2+z^2} = \sqrt{b^2+z^2}$, $b = \sqrt{x^2+y^2}$ we find that the deflection angle reduces to,

$$\mid \hat{\vec{\alpha}} \mid = 4\frac{GM}{c^2 b} = 2\frac{R_S}{b}, \tag{5.55}$$

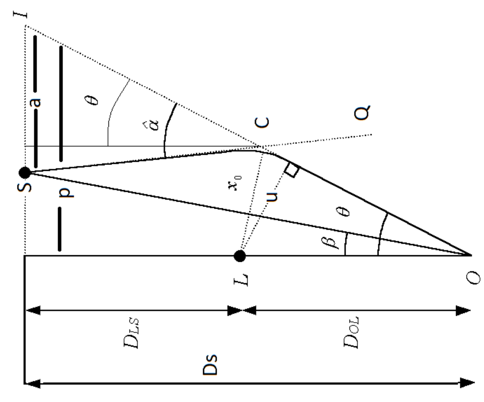
where R_S is the Schwarzschild radius of a (point) Mass. It is important to point out that the $\mid \hat{\vec{\alpha}} \mid$ is linear in M, thus the deflection angles of an array of lenses can linearly be superposed. The deflection angle found here in the framework of General relativity exceeds by a factor of two that calculated by using Newtonian gravity.

5.4 Lens equation in the weak limit:

In this section we derived a lens equation that allows us to study the bending of light in the Strong field limit approximation. Figure 2 shows the geometrical configuration of gravitational lensing. Light rays emitted by the source S are deflected by the lens L and reach the observer O with an angle θ, instead of β. The total deflection angle is $\hat{\alpha}$. x_0 is the closest approach distance and u is the impact parameter. D_{OL} is the distance between the lens and the observer. D_{LS} is the distance between the lens and the projection of the source on the optical axis OL. $D_{OS} = D_{OL} + D_{LS}$ [14]. The space-time under consideration, with the lens (deflector) causing strong curvature, is asymptotically flat; the observer as well as the source are situated in the flat space-time region [28].

SQ and OI are tangents to the null geodesic at the source and image positions, respectively; C is where their point of intersection would be if there were no lensing object present. The angular positions of the source and the image are measured from the optic axis OL. The null geodesic and the broken geodesic OCS will be almost identical, except near the lens where most of the bending will take place. Given the vast distances from observer to lens and from lens to source, this will be a good approximation, even if the light goes round and round the lens before reaching the observer. We assume that line

[14] In General $D_{OS} \neq D_{OL} + DOS$ [24]. However, we have consider $D_{OS} = D_{OL} + DOS$ as considered in [22]



(a)

Figure 5.2: Geometrical configuration of gravitational lensing

joining the point C and the location of the lens L is perpendicular to the optic axis. This is a good approximation for small values of β (for observationally significant lensing β is small)[28]. From the figure,

$$\begin{aligned}\tan(\beta) &= \frac{P-a}{D_{OS}} = \frac{P}{D_{OS}} + \frac{a}{D_{OS}} \\ \tan(\theta) &= \frac{P}{D_{OS}}.\end{aligned} \tag{5.56}$$

Then,

$$\tan(\beta) = \tan\theta - \frac{a}{D_{OS}}. \tag{5.57}$$

From figure (2),

$$a = D_{LS}[\tan(\theta) + \tan(\hat{\alpha} - \theta)] \tag{5.58}$$

Finally the relation among the source position, the image position and the deflection angle $\hat{\alpha}$ is

$$\tan(\beta) = \tan(\theta) - \frac{D_{LS}}{D_{OS}}[\tan(\theta) + \tan(\hat{\alpha} - \theta)]. \tag{5.59}$$

This is what is called the lens equation. Given a source position β, the values of θ, solving this equation, give the position of the images observed by O.

In the weak field limit, several standard approximations are performed. The tangents are expanded to the first order in the angles since they are, at most, of the order of *arcsec*. The weak field assumption reduces to de deflection angle to $\frac{4GM}{c^2 x_0}$. Then the lens equation can be solved exactly and two images are fund: one on the same side of the source and one on the opposite. We will put our attention on situations where the source is almost perfectly aligned with the lens. In fact, this is the case where the relativistic images are most prominent[29]. In this case we are allowed to expand $\tan(\beta)$ y $\tan(\theta)$ to the first order[29]. Therefore, expanding the tangents to first order we obtain that

$$\tan(\beta) = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{d^{(n)}\tan(\beta)}{d\beta^n}|_{\beta=0}\beta^n \simeq \beta + \mathcal{O}(\beta^2)$$
$$\tan(\theta) = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{d^{(n)}\tan\theta}{d\theta^n}|_{\beta=0}\beta^n \simeq \theta + \mathcal{O}(\theta^2). \quad (5.60)$$

Then the lens equation, takes the form

$$\beta = \theta - \frac{D_{LS}}{D_{OS}}[\theta - \tan(\hat{\alpha} - \theta)]. \quad (5.61)$$

Some more words are needed for the term $\tan(\hat{\alpha} - \theta)$. Even if θ is small $\hat{\alpha}$ is not small in the situations of our interest. However, if a ray of light emitted by the source S is going to reach the observer after turning around the black hole, $\hat{\alpha}$ must be very close to a multiple of 2π. In this sense, writing[15] $\hat{\alpha} = 2n\pi + \triangle\alpha_n$, with n integer, we can perform the expansion $\tan(\hat{\alpha} - \theta) \sim \triangle\alpha_n - \theta$ [29]. Finally the lens equation becomes

$$\beta = \theta - \frac{D_{LS}}{D_{OS}}\triangle\alpha_n. \quad (5.62)$$

Taking a positive β, this equation describes only images on the same side of the source ($\theta > 0$). To obtain the images on the opposite side, we can solve the same equation with the source placed in $-\beta$. Taking the opposite of these solutions, we obtain the full set of secondary images.

5.4.1 Lens equation in the strong field limit:

As was shown in the last chapter, the deflection angle in the strong field limit takes the form

$$\hat{\alpha}(\theta) = -\bar{a}\ln\left(\frac{\theta D_{OL}}{u_m} - 1\right) + \bar{b} + \mathcal{O}(u - u_m). \quad (5.63)$$

Our intention now is to study the lens equation in the strong field limit using equation

$$\beta = \theta - \frac{D_{LS}}{D_{OS}}\triangle\alpha_n, \quad (5.64)$$

where D_{LS} is the distance between the lens and the source, $D_{OS} = D_{OL} + D_{LS}$[16], β is the angular separation between the source and the lens, θ is the angular separation between the lens and the image (Cfr. figure 2), $\triangle\alpha_n = \hat{\alpha}(\theta) - 2n\pi$ is the offset of the deflection angle, once we subtracted all the loops done by the photon [22]. To obtain the offset $\triangle\alpha_n$ we have to expand $\hat{\alpha}(\theta)$ around θ_n^0. Hence,

$$\hat{\alpha}(\theta) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^{(k)}\alpha(\theta)}{d\theta}|_{\theta=\theta_n^0}(\theta - \theta_n^0)^k$$
$$\hat{\alpha}(\theta) \approx \hat{\alpha}(\theta_n^0) + \frac{d\hat{\alpha}(\theta)}{d\theta}|_{\theta=\theta_n^0}(\theta - \theta_n^0). \quad (5.65)$$

Calculating $\frac{d\hat{\alpha}(\theta)}{d\theta}$,

[15] In the next subsection, we will show where the relation $\alpha = 2n\pi + \triangle\alpha_n$ comes from.

[16] In General $D_{OS} \neq D_{OL} + DOS$[24]. However, we have consider, as considered in [22], $D_{OS} = D_{OL} + DOS$

$$\begin{aligned}\frac{d\hat{\alpha}}{d\theta} &= \frac{d}{d\theta}\left[-\bar{a}\ln\left(\frac{\theta D_{OL}}{u_m}-1\right)+\bar{b}\right] \\ &= -\frac{\bar{a}D_{OL}}{\theta D_{OL}-u_m};\end{aligned} \tag{5.66}$$

evaluating at $\theta = \theta_n^0$ we have

$$\frac{d\hat{\alpha}}{d\theta}\,|_{\theta=\theta_0} = -\frac{\bar{a}D_{OL}}{\theta_n^0 D_{OL}-u_m}. \tag{5.67}$$

Then,

$$\hat{\alpha}(\theta) = \hat{\alpha}(\theta_n^0) - \frac{\bar{a}D_{OL}}{\theta_n^0 D_{OL}-u_m}(\theta-\theta_n^0). \tag{5.68}$$

Finally[17],

$$\hat{\alpha}(\theta) - \hat{\alpha}(\theta_n^0) = -\frac{\bar{a}D_{OL}}{\theta_n^0 D_{OL}-u_m}\triangle\theta_n. \tag{5.69}$$

To pass from the deflection angle $\hat{\alpha}(\theta)$ to the offset $\triangle\alpha_n$, we need to find the values θ_n^0 such that $\hat{\alpha}(\theta_n^0) = 2n\pi$. In this sense, the offset $\triangle\alpha_n$ is

$$\triangle\alpha_n = \hat{\alpha}(\theta) - 2n\pi = -\frac{\bar{a}D_{OL}}{\theta_n^0 D_{OL}-u_m}\triangle\theta_n. \tag{5.70}$$

Solving the strong field limit expansion of the deflection angle we find that,

$$2n\pi = -\bar{a}\ln\left(\frac{\theta_n^0}{u_m}-1\right)+\bar{b}, \tag{5.71}$$

then,

$$\frac{2n\pi-\bar{b}}{\bar{a}} = -\ln\left(\frac{\theta_n^0 D_{OL}}{u_m}-1\right). \tag{5.72}$$

Therefore,

$$\begin{aligned} e^{\frac{\bar{b}-2n\pi}{\bar{a}}} &= \frac{\theta_n^0 D_{OL}}{u_m}-1 \\ \theta_n^0 &= \frac{u_m}{D_{OL}}(e^{\frac{\bar{b}-2n\pi}{u_m}}+1),\end{aligned} \tag{5.73}$$

calling

$$e_n = e^{\frac{\bar{b}-2n\pi}{a}}, \tag{5.74}$$

we obtain that.

$$\theta_n^0 = \frac{u_m}{D_{OL}}(e_n+1) \tag{5.75}$$

Using the expression for θ_n^0 in the expression for $\triangle\alpha_n$ we have,

$$\triangle\alpha_n = -\frac{\bar{b}D_{OL}}{u_m e_n}\triangle\theta_n. \tag{5.76}$$

Replacing into the lens equation we obtain that

[17] We have defined $\triangle\theta_n = \theta - \theta_n^0$.

$$\begin{aligned}\beta &= \theta + \frac{D_{LS}}{D_{OS}} \frac{\bar{a} D_{OL}}{u_m e_n} \triangle\theta_n \\ &= (\theta_n^0 + \triangle\theta_n) + \left(\frac{\bar{a} D_{OL}}{u_m e_n} \frac{D_{LS}}{D_{OS}} \right) \triangle\theta_n .\end{aligned} \tag{5.77}$$

5.5 Images position and Einstein rings:

From equation (5.77) we can observe that D_{OL}, D_{LS}, D_{OS} (since $u_m \ll D_{OL}$) are all much grater than unity (remember that all distances are measured in Schwarzschild radii). This means that the last term in the lens equation prevails on the $\triangle\theta_n$ at the second place in the right hand side. Neglecting this term, we get

$$\beta = \theta_n^0 + \left(\frac{\bar{a} D_{OL}}{u_m e_n} \frac{D_{LS}}{D_{OS}} \right) \triangle\theta_n , \tag{5.78}$$

and the position of the n^{th} image is

$$\theta_n = \theta_n^0 + \frac{u_m e_n (\beta - \theta_n^0) D_{OS}}{\bar{a} D_{LS} D_{OL}} . \tag{5.79}$$

Where the correction to θ_n^0 is much smaller than θ_n^0. Equation (5.79) is valid both for the images on the same side of the source and the images on the opposite side. In fact, to find the later, it is sufficient to take a negative β [22]. When β equals θ_n^0, there is no correction to the position of the n^{th} image, that remains in θ_n^0 simply. In this particular case, the image position coincides with the source position. It is worthwhile to note that the second term term in the las equation is much smaller than the first one. For practical purposes, θ_n^0 are already a good approximation for the position of relative images[29].

Making $\beta = 0$ in equation 5.79 it is possible to obtain the radius of the Einstein ring for the n^{th} image. Einstein ring correspond to a source perfectly aligned with the lens. Therefore, the radius is expressed by

$$\theta_{n,E} = \left(1 - \frac{u_m e_n D_{OS}}{\bar{a} D_{LS} D_{OL}} \right) \theta_n^0 . \tag{5.80}$$

Einstein rings, a particularly interesting manifestation of gravitational lensing, were discovered first in the radio waveband by known and these sources permit the most detailed modeling yet of the mass distribution of lensing galaxies [24].

5.6 Magnification:

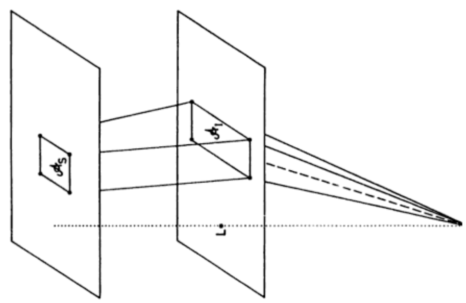
Light deflection in a gravitational field not only changes the direction of a light ray (as we have shown in last chapter and section 3 using General Relativity and the weak field limit approach), but also the cross-section of a bundle of rays [25]. This means that deflection also affects the properties of the image of a source. In particular, the flux of images is influenced since the cross sectional area of a light bundle is distorted by the deflection. Since the photon number is conserved, the flux of an image is determined by this area distortion. Consider an infinitesimal source with surface brightness I_ν (ν is the observer frequency), which, in the absence of gravitational light deflection, subtends a solid angle $(\triangle\omega)_0$ on the sky. Hence, the monochromatic flux from this source is the product of its surface brightness and the solid angle $(\triangle\omega)_0$ it subtends on the sky [25]. Therefore,

$$S_\nu^0 = I_\nu (\triangle\omega)_0 . \tag{5.81}$$

If the light bundle undergoes a deflection, the solid angle $\triangle\omega$ of the image will differ from $(\triangle\omega)_0$. Since gravitational light deflection is not connected with emission or absorption because of Liouville's theorem[24], the specific intensity I_ν is constant along a ray, if measured by observers with no frequency shift relative to each other. Moreover, gravitational light deflection by a localized, near static deflector does not introduce an observer; therefore, the surface brightness I for an image is identical to that of source in the absence of the lens (both per unit frequency, and as an integral over frequency). Since I is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in absence on the lens, is given by

$$\mu = \frac{\triangle\omega}{(\triangle\omega)_0}, \tag{5.82}$$

where 0−subscripts denote undeflected quantities.



(a)

Figure 5.3: The distortion of the solid angle subtended by a source. taken from [25]

In figure 3 we shows a typical gravitational lensing arrangement. This figure represents schematically the distortion of a light bundle. The source spans an area A_s and thus subtends a solid angle $(\triangle\omega)_0 = \frac{A_s}{D_s^2}$ at the observer O in the absence of lensing. If the lensing takes place, the solid angle of the image is $\triangle\omega = \frac{A_I}{D_S^2}$, in general different from $(\triangle\omega)_0$. Since the surface brightness of the source is unchanged by the light deflection, the apparent brightness of the source is magnified in proportion to the solid angle $\triangle\omega$,

$$\mu = \frac{I_\nu \frac{A_I}{D_d^2}}{I_\nu \frac{A_S}{D_s^2}} = \frac{A_I}{A_2}\left(\frac{D_s}{D_d}\right)^2 \tag{5.83}$$

Consider now an infinitesimal source at β that subtends a solid angle $(\triangle\omega)_0$ on the source sphere and also on the sphere of vision of the observer. Let θ be the angular position of an image with solid angle $\triangle$. The relation of the two solid angles is determined by the area-distortion of the lens and given by [25]

$$\frac{(\triangle\omega)_0}{\triangle\omega} = \frac{1}{|\det\frac{\partial\beta}{\partial\theta}|} = \frac{A_s}{A_I}\left(\frac{D_d}{D_s}\right)^2; \tag{5.84}$$

This means that the distortion caused by the deflection is given by the determinant of the Jacobian matrix of the lens mapping $\theta \to \beta$. From (5.81) the magnification factor is[22][25]

$$\mu = \frac{1}{|\det\frac{\partial\beta}{\partial\theta}|}. \tag{5.85}$$

For circularly symmetric lens, the magnification factor μ is given by [22][24]

$$\mu = \frac{\theta}{\beta \frac{\partial \beta}{\partial \theta}}. \tag{5.86}$$

5.6.1 Magnification in the strong field limit:

The Magnification of images is given by equation (6.7) evaluated at the position of the image. For simplicity, we approximate the position of the images by θ_n^0. Since, $u_m \ll D_{OL}$ we have that [22]

$$\mu_n = \frac{1}{\frac{\sin(\beta)}{\sin(\theta)} \frac{\partial \beta}{\partial \theta}} \,|_{\theta_n^0} = \frac{\theta}{\beta \frac{\partial \beta}{\partial \theta}} \,|_{\theta_n^0} \,. \tag{5.87}$$

From equation (5.77),

$$\frac{\partial \beta}{\partial \theta}|_{\theta_n^0} = 1 + \frac{\bar{a} D_{OL}}{u_m e_n} \frac{D_{LS}}{D_{OS}}. \tag{5.88}$$

Where the first term is small compared to the second and can be neglected. Therefore, we use $\frac{\partial \beta}{\partial \theta}|_{\theta_n^0} \approx \frac{\bar{a} D_{OL}}{u_m e_n} \frac{D_{LS}}{D_{OS}}$ in equation (5.87)

$$\mu_n = e_n \frac{\theta_n^0 u_m e_n D_{OS}}{\bar{a} \beta D_{OL} D_{LS}}.$$

Recalling that $\theta_n^0 = \frac{u_m}{D_{OL}}(1 + e_n)$, we finally obtain that

$$\mu_n = e_n \frac{u_m^2 (1 + e_n) D_{OS}}{\bar{a} \beta D_{OL}^2 D_{LS}}, \tag{5.89}$$

which decreases very quickly in n.

Chapter 6

ANALYSIS AND DISCUSSION

The general theory of relativity has passed experimental tests scenario in a weak gravitational field limit; however, the theory has not been tested in a strong gravitational field. Testing the gravitational field in the vicinity of a compact massive object, such as a black hole or a neutron star, could be a possible avenue for such investigations. Dynamical observations of several galaxies show that their centers contain massive dark objects. This observations suggest that these are super-massive black holes at least in the galaxy as well as in NGC4258 [28]. In this sense, these objects could be possible observational targets to test the Einstein Theory of relativity in a strong gravitational field scenario through gravitational lensing.

In 2000 Virbhabra and George Ellis studied black hole lensing using Schwarzschild's solution. As we know, the Schwarzschild solution is a static spherically symmetric asymptotically flat vacuum solution to the Einstein equations which has an event horizon when maximally extended. In this sense, it represents the gravitational field of a spherically symmetric black hole. In order to study black hole lensing, the authors obtained a lens equation that allows for the large bending of light near a black hole, model the Galactic super-massive "Black hole" as a Schwarzschild lens and studied point source lensing in the strong gravitational field region, when the bending angle can be very large. This lens equation, as shown in the last chapter, was used by V. Bozza in [22] [29] to calculate the position of relativistic images and magnification. Apart from a primary image and a secondary image (which are observed due to small bending of light in a weak gravitational field) the authors obtained a theoretically infinite sequence of images on both sides close to the optic axis. This images are named by the authors as *relativistic images*. These images are formed due to the large bending of light in a strong gravitational field in the vicinity of $3M$[1] (the photon sphere $x_m = \frac{3}{2}$ [22]), and are usually greatly demagnified. Thought the observation of relativistic images is a very difficult task, if it ever were accomplished it would support the general theory of relativity in a strong gravitational field.

It is known that the Schwarzschild gravitational lensing in the weak approximation gives rise to an Einstein ring when the source, lens and observer are aligned on the optical axis; i.e. the value of the angular position of the source is zero $(\beta = 0)$[2]. The Einstein ring is given by [24][25]

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_{OL}D_S}}. \tag{6.1}$$

On the other hand, a pair of images (primary and secondary) of opposite parities are formed when the lens components are misaligned. This images are solution to the lens

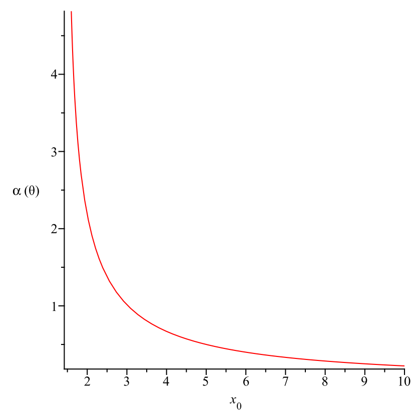
[1] In [28] the authors use geometrized units: $G = 1$, $c = 1$ so that $M \equiv \frac{MG}{c^2}$

[2] Cfr. Figure2 last chapter

equation. The angular position of these images are given by [24][25]

$$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + \frac{8R_s D_{LS}}{D_{OL} D_S}}\right). \tag{6.2}$$

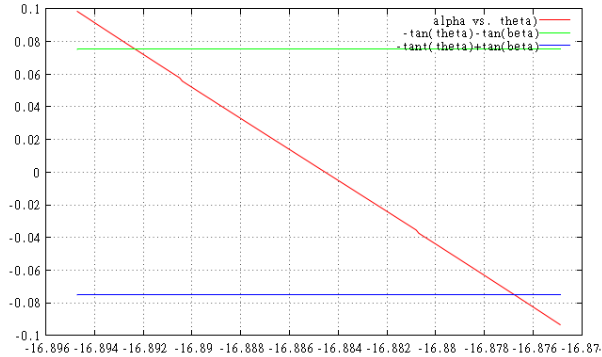
However, when the lens is a massive object, a light ray can pass close to the photon sphere and go around the lens many times before reaching the observer depending on the impact parameter u. In figure 1, for example, we have plotted the behavior of the deflection angle $\hat{\alpha}$ vs. x_0. For $x_0 > x_m = \frac{3}{2}$ $(u > u_m)$ a light ray reaches the observer; when $x_0 \to \infty$, $\hat{\alpha} \to 0$ and for $x_0 \to x_m = \frac{3}{2}$ $(u \to u_m)$, $\hat{\alpha}$ diverges[22]. This means that the photon is captured on the photon sphere. When $x_0 \to x_m = \frac{3}{2}$ $(u \to u_m)$ a light ray can pass close the photon sphere and go around the lens once, twice, thrice, or many times. Therefore, a massive compact lens gives rise, in addition to the primary and secondary images, to a a large number of images on both sides of the optical axis. These images are called by the authors as *relativistic images* and the rings which are formed by bending of light rays more that 2π, *relativistic Einstein rings.*



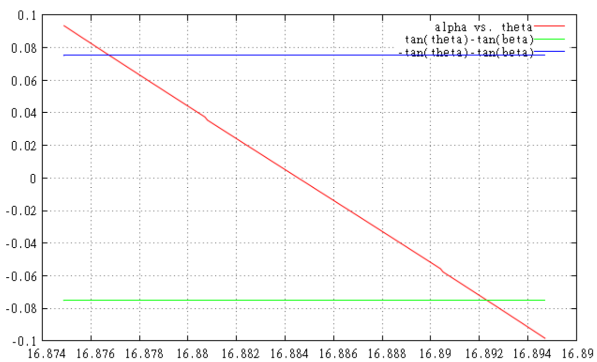
(a)

Figure 6.1: (a) $\hat{\alpha}$ (radians) vs. x_0 For Schwarzschild.

In the study, the authors modeled the Galactic super-massive "black hole" as a Schwarzschild lens. This has mass $M = 2.8 \times 10^6 M_{\odot}$ and the distance $D_{OL} = 8.5$Kpc; therefore, the ratio of the mass to the distance $\frac{M}{D_{OL}} \approx 1.57 \times 10^{-11}$. For calculations they considered a point source, with the lens situated half way between the source and the observer $(\frac{D_{LS}}{D_S} = \frac{1}{2})$. They allowed the angular position of the source to change keeping D_{LS} fixed.



(a)



(b)

Figure 6.2: (a) A plot of $-\alpha = -\frac{D_{LS}}{D_S}[\tan(\theta) + \tan(\hat{\alpha} - \theta)]$ (red line) and $-\tan(\theta) - \tan(\beta)$ (b) A plot of $\alpha = \frac{D_{LS}}{D_S}[\tan(\theta) + \tan(\hat{\alpha} - \theta)]$ (red line) and $\tan(\theta) - \tan(\beta)$

Figure 2 shows the relativistic image positions for $\beta = \mp 0.075$ radian ($\approx 4.29718°$). The points of intersection of the red line (α) (the two outermost ones on each side being shown) with the two other lines (blue and green) give the angular position of *the relativistic images*. In order to find the angular positions of images on the same side of the source we plot α and $\tan(\theta) - \tan(\beta)$ against θ (expressed in microarcoseconds) for the given value of the source position β; the points of intersection give the image position. Similarly, to find the image position on the opposite side of the source, we plot $-\alpha$ and $-\tan(\theta) - \tan(\beta)$ against θ. In fact there is a sequence of theoretically an infinite number of continuous curves which intersect with a given horizontal curve (blue and green lines) giving rise to a sequence of an infinite number of images on both sides of the optic axis [28]. For $\beta = 0$ the lens equation reduce to (Cfr. the graphic in section "**Lens equation**" in the last chapter)

$$\tan\theta = \alpha = \tan(\hat{\alpha} - \theta). \tag{6.3}$$

In this sense, the points of intersection of α with $\tan\theta$ give a sequence of infinite number of *Relativistic Einstein rings*. As β increase any image on the same side of the source moves away from the optic axis, whereas any image on the opposite side of the source moves away from the optic axis. As can be seen from figure 2, the displacement of relativistic images with respect to a change in the source position is very small. In the figure the two sets of outermost relativistic images are formed at about 17 microarcoseconds from the optic axis.

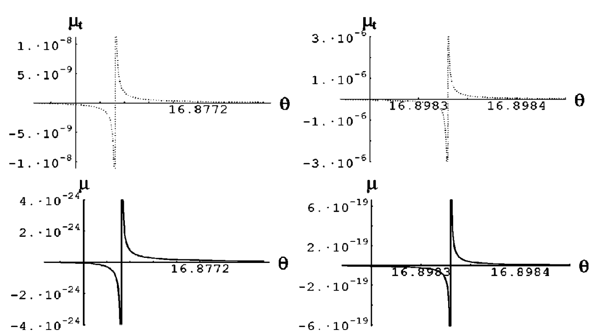
The magnification is defined as the ration between the flux of the image to the flux on the unlensed source. As explained in the last chapter, because of the Liouville's theorem, the magnification reduces to the ratio of the solid angles of the image and of the unlensed source (at the observer). For a circularly symmetric gravitational lensing, the magnification is given by [22][24][25]

$$\mu = \left(\frac{\sin\beta}{\sin\theta}\frac{d\beta}{d\theta}\right)^{-1}. \tag{6.4}$$

In order to study the magnification it is usually defined the tangential and radial magnifications as,

$$\mu_t \equiv \left(\frac{\sin\beta}{\sin\theta}\right)^{-1} \quad \mu_r \quad \equiv \left(\frac{d\beta}{d\theta}\right)^{-1}. \tag{6.5}$$

The singularities in these give the *tangential critical curves* and *radial critical curves*; respectively; the corresponding values in the source plane are know as *tangential caustic* and *radial caustics*, respectively. For example, $\beta = 0$ gives tangential critical curves which correspond to relativistic Einstein rings.



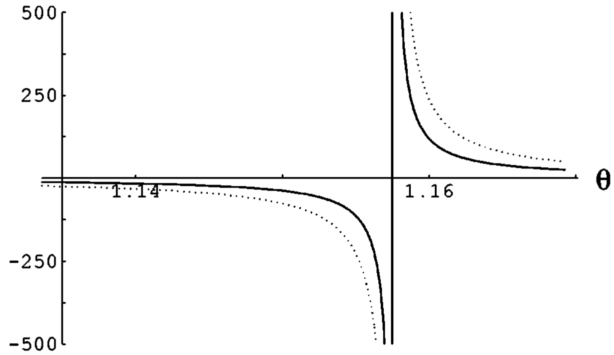
(a)

Figure 6.3: Tangential magnification μ_t and total magnification as a function of θ (expressed in microarcoseconds)

In figure 3 (taken from [28]) the authors plotted the tangential magnification μ_t and the total magnification as a function of θ (expressed in microarcoseconds). As can be seen from the figure, the singularities in μ_t give the angular radii of the two relativistic Einstein rings. The first relativistic Einstein ring has an angular radii of 16.898μas and the second one a radii of 16.877μas. The angular radius of the Einstein ring in Schwarzschild is expressed by equation 6.1; therefore, the value of θ_E is 1.157544 arcseconds. When we compare the magnification of the primary and secondary images (Cfr. figure 4 taken from [28]), we concluded that the magnification for the relativistic images falls very fast as the source position increase from perfect alignment.

It is important to point out that the values of the Einstein and relativistic rings correspond to the singularities in μ_t (cfr. figure 3 and 4). On the other hand, the sing of μ_t as well as the sing of μ are positive for all images on the same side of the source. Furthermore, the sing of μ_r is positive for all the images in the Schwarzschild lensing.

As observation suggest, the super-massive "black hole " at the center of $NGC3115$ and $NGC4486$ have $\frac{M}{D_{OL}} \approx 1.14 \times 10^{-11}$ and $1.03 \times 10{-}3$, respectively. These values are very



(a)

Figure 6.4: the tangential magnification μ_t and the total magnification as a function of θ (expressed in microarcoseconds)

close to that used by the authors in [28]. Therefore, the results obtained from modeling the galactic super-massive "black hole" as a Schwarzschild lens will be very similar to those using the galactic center of $NGC3115$ or $NGC4486$. In this sense, there is the possibility to study gravitational lensing in the strong field limit for real astrophysical objects.

As explained before, for a given source position the magnification for relativistic images decrease very fast as the angular position θ decrease; therefore the outermost set of images, one on each side of the optic axis, is observationally the most significant. The angular separation among relativistic images are to small to be resolved, so that all these images would be at the same position; however, these relativistic images will be resolved from the primary and secondary images and thus resolution is not a problem for observation of relativistic images [28].

In a similar paper [20] published in 2002. Virbhabra and Ellis studied gravitational lensing using the Janis-Newman-Winicour-Wyman metric. In this paper they obtained a photon sphere equation for a general static spherically symmetric metric which was deduced in chapter 3. Using this equation, they classify a naked singularity in two categories:

a) **Weakly naked singularities (WNS):** Naked singularities contained within at least one photon sphere.

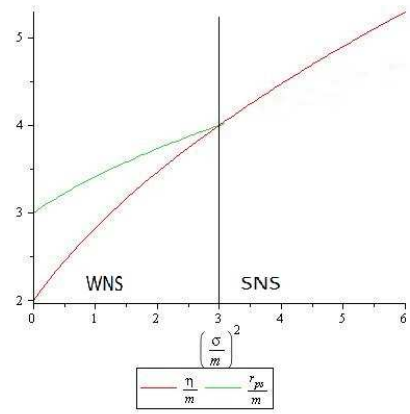
b) **Strong naked singularities (SNS):** Naked singularities which are not covered within any photon spheres.

These photon spheres are solution to the photon sphere equation. Therefore, a naked singularity may or may not be covered within a photon sphere. The existence or non existence of a photon sphere with a space-time acting as a gravitational lens has important implications for gravitational lensing.

The solution to photon sphere equation for Janis-Newman-Winicour-Wyman metric (Cfr. chapter 2) is expressed by [20]

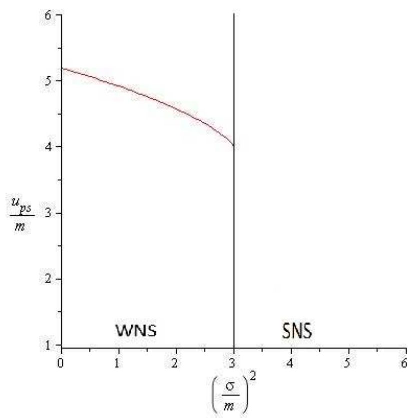
$$
\begin{aligned}
x_m &= \frac{(1+2\gamma)}{2} \\
x_m &= \frac{r_{ps}}{2\eta}.
\end{aligned}
\tag{6.6}
$$

The Janis-Newman-Winicour-Wyman metric has a photon sphere only for $\frac{1}{2} \leq \gamma \leq 1$ $(0 \leq \left(\frac{\sigma}{m}\right)^2 < 3)$. In figure 5, it is plotted $\frac{r_{sp}}{m}$ and $\frac{2\eta}{m}$ vs. the square of the scalar charge. As figure 5 shows, $\frac{r_{p}s}{m}$ and $\frac{2\eta}{m}$ increase with an increase in $\left(\frac{\sigma}{m}\right)^2$ and meet at $\left(\frac{\sigma}{m}\right)^2 = 3$. For $0 < \left(\frac{\sigma}{m}\right)^2 < 3$ there is always one photon sphere which covers the naked singularities and thus these singularities are weakly naked; nevertheless, for $\left(\frac{\sigma}{m}\right)^2 \geq 3$ the singularities ar not covered by any photon sphere and there fore these are strongly naked.



(a)

Figure 6.5: $\frac{r_{sp}}{m}$ and $\frac{2\eta}{m}$ vs. $\left(\frac{\sigma}{m}\right)^2$



(a)

Figure 6.6: Impact parameter $\frac{u_{ps}}{m}$ vs. $\left(\frac{\sigma}{m}\right)^2$

In figure 6 it is plotted the impact parameter $\frac{u_{ps}}{m}$[3] of the photon sphere against the square of the scalar charge $\left(\frac{q}{m}\right)^2$. $\frac{u_{ps}}{m}$ is real only for $0 \leqslant \left(\frac{q}{m}\right)^2 < 3$ and it decrease monotonically in the range with an increase in the value of $\left(\frac{q}{m}\right)^2$.

In order to study the implications of the existence or non existence of a photon sphere with a space-time acting as a gravitational lens for gravitational lensing, it is important to point out that one has $lim_{r_0 \to \infty} \hat{\alpha}(r_0) = 0$ for all values of γ, $lim_{r_0 \to r_m} \hat{\alpha}(r_0) = \infty$ for $\frac{1}{2} < \gamma \leqslant 1$ (Schwarzschild black hole and weakly naked singularity (**WNS**)) and $lim_{r_0 \to 2\eta} \hat{\alpha} = -\pi$

[3] u_{ps} is the same as u_m in [22]

for $0 \leq \gamma \leq \frac{1}{2}$ (strongly naked singularities (**SNS**)). In this sense, the authors in [20] discuss the behavior of the bending angle $\hat{\alpha}$ for the Schwarzschild black hole ($\gamma = 1$), WNS, and SNS. When they plotted the behavior of $\hat{\alpha}$ in the **WNS** region, which correspond a values of γ in the interval $\frac{1}{2} < \gamma \leqslant 1$ (Cfr. figure 5), they found that $\hat{\alpha}$ strictly increases with a decrease in the impact parameter and becomes unboundedly large as the impact parameter approaches the impact parameter for their respective photon sphere. Thus lensing by **WNS** would not give rise to a radial critical curve, instead they would give relativistic Einstein rings as in the case of the Schwarzschild black hole lensing. On the other hand, the authors plotted the behavior of $\hat{\alpha}$ in the **SNS** region (for $0 \leq \gamma \leq \frac{1}{2}$). In this region $\hat{\alpha}$ first increase with a decrease in the impact parameter and further decrease to the minimum value $-\pi$ as the impact parameter $u \to 0$. Because of this behavior of the deflection angle gravitational lensing with **SNS** would give either two or nil Einstein rings and one radial critical curve. As there are no photon spheres of **SNS** the deflection angles for these cases are never unboundedly large. Gravitational lensing by **SNS** would not give rise to relativistic images.

Taking into account the studies by Virbhadra and Ellis, gravitational lensing must not be conceived as a weak field phenomenon, since high bending and looping of light rays in strong field is one of the most well-known and amazing prediction of general relativity. The importance of gravitational lensing in strong fields is highlighted by the possibility of testing the full general relativity in a regime where the differences with non-standard theories would be manifest, helping the discriminations among the various theories of gravitation. For this reason, the scientific community has been interested in the lensing properties near the photon sphere (the strong field limit). The first attempt to study light deflection near the photon sphere, as suggested in [28], was made in [29] by V. Bozza, S. Capozziello, G. Iovane and G. Scarpetta (2001). In order to study gravitational lensing in the strong field limit, the authors started from the lens equation proposed by Virbhadra and Ellis. In [29], they performed a set of expansions exploiting the source-lens-observer geometry (Cfr. [28]) and the properties of highly deflected light rays. In this analysis, they considered situations where the source is almost aligned with the lens (when the relativistic images are more prominent). Thus, they could expand $\tan\beta$ and $\tan\theta$ to the first order. However, because $\hat{\alpha}$ is not small in this situation[4], the authors expressed $\tan(\hat{\alpha} - \theta)$ as $\triangle\alpha_n - \theta$ and finally they obtained the lens equation (Cfr. chapter 5)

$$\beta = \theta - \frac{D_L}{D_{OS}} \triangle\alpha_n, \tag{6.7}$$

where $\triangle\alpha_n$ (the offset) is $\triangle\alpha_n = \hat{\alpha} - 2n\pi$. This offset was named as "effective deflection angle" by Virbhadra and Ellis. As studied in [28], the effective deflection angle decreases with the decrease in the angular position radius for Schwarzschild metric.

For Schwarzschild, the deflection angle can be evaluated exactly with an integral (Cfr. [21]), but its expression does not allow the resolution of (6.7). However, it is possible to make some approximations which reduce the deflection angle to an expression easier to handle. As mentioned above, the authors were interested just into small closest approach (x_0), since they correspond to the high deflection angles producing relativistic images. In this sense, all approximation are essentially based on the proximity of the closest approach distance x_0 to its minimum value which is $\frac{3}{2}$ (The photon sphere for Schwarzschild). Therefore, for $x_0 = \frac{3}{2} + \epsilon$, the authors found that the leading order of the deflection angle is logarithmic in ϵ, that is,

$$\hat{\alpha} \sim -2 \ln\left(\frac{(2+\sqrt{3})\epsilon}{18}\right) - \pi. \tag{6.8}$$

[4] Near to photon sphere the deflection angle could be very large

This approximation in terms of θ is (Cfr. [29])

$$\hat{\alpha} \approx -\ln\left(\theta D_{OL} - \frac{3\sqrt{3}}{2}\right) + A, \tag{6.9}$$

where $A = -\ln\left(\frac{5+3\sqrt{3}}{1944}\right) - \pi$.

Equation (6.9) was replaced into the lens equation (6.7) to study gravitational lensing in the strong field limit for Schwarzschild. With this approximation of the deflection angle the authors could calculate the position of the relativistic images and the magnification (Cfr. chapter 5).

The formulae derived in [29] provide a complete characterization of the two infinite set of relativistic images surrounding a black hole or, in general, any compact object acting as a lens whose size is comparable with its Schwarzschild radius (Cfr. figure 2). Nevertheless, in [22] (2008) professor V. Bozza provide an analytical method to discriminate among different types of black holes on the ground of their strong field gravitational lensing properties. In this paper, he expand the deflection angle of the photon in the neighbourhood of complete capture, defining a strong field limit, in opposition to the standard weak limit. This expansion was worked out for a complete generic spherically symmetric space-time, without any reference to the field equations and just assuming that the light ray follows the geodesic equation. In this analysis, V. Bozza proved that the deflection angle always diverges logarithmically when the minimum impact parameter is reached ($u = u_m$ or $x_0 = x_m$). All this ideas were studied in detail in chapter 3. Using this method we calculated the deflection angles for Schwarzschild, Reissner-Nordstrom and Janis-Newman-Winicour (Cfr. appendix A). However, the values of β_m and $\bar{b}$ obtained for Janis-Newman-Winicour are different from equations (71) and (74) of [22]. The expressions obtained in appendix A are

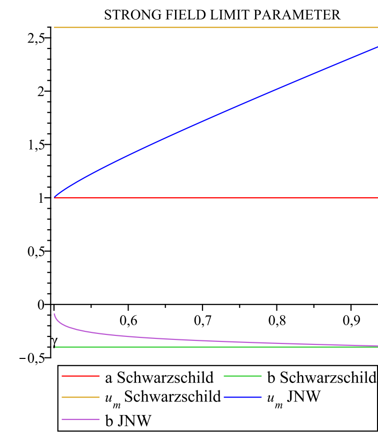
$$\begin{aligned}\bar{b} &= -\pi + b_R + \ln\left(\frac{(2\gamma+1)[(2\gamma+1)^{\gamma} - (2\gamma-1)^{\gamma}]^2}{2\gamma^2(2\gamma-1)^{2\gamma-1}}\right)\\ \beta_m &= \frac{[(2\gamma+1)^{\gamma} - (2\gamma-1)^{\gamma}]^2}{4\gamma^2(4\gamma^2-1)^{\gamma-1}},\end{aligned} \tag{6.10}$$

and those obtained by V. Bozza are

$$\begin{aligned}\bar{b} &= -\pi + b_R - 2\ln\left(\frac{(2\gamma+1)[(2\gamma+1)^{\gamma} - (2\gamma-1)^{\gamma}]^2}{2\gamma^2(2\gamma-1)^{2\gamma-1}}\right)\\ \beta_m &= \frac{[(2\gamma+1)^{\gamma} - (2\gamma-1)^{\gamma}]^2}{4\gamma^2(4\gamma^2-1)^{2\gamma-1}}.\end{aligned} \tag{6.11}$$

For $\gamma = 1$ equation (6.10) reduces to $\beta_m = 1$ and $\bar{b} = -\pi + 0.9496 + \ln(6)$. These are the values reported for Schwarzschild in [22]. Nevertheless, for $\gamma = 1$, equation (2) (equations (71) and (75) of [22]) does not reduce to those of Schwarzschild. In consequence, the expression describing the deflection angle in the strong field limit does not reduce to that of Schwarzschild when $\gamma = 1$. For this reason, we included the Janis-Newman-Winicour in the discussion.

In chapter 3 we explained that u_m is the value of the impact parameter at $x_0 = x_m$, where x_0 is the closest approach distance. Moreover, we showed that for $u = u_m$ ($x_0 = x_m$) the deflection angle diverges. The strong field limit expansion studies the behavior of the deflection angle near the photon sphere ($x_0 = x_m$); in this sense, using the impact parameter as $u_m + 0.003$ we ensure that our analysis will be constrained to this limit.



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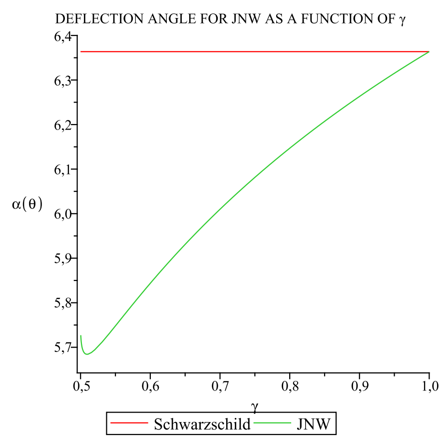


(b)

Figure 6.7: (a) Strong field limit parameters as a function of γ for Janis-Newman-Winicour (b) Deflection angle in the strong field limit $u = u_m + 0.003$.

In figure 7, we have plotted $\bar{a}$, $\bar{b}$, u_m and the deflection angle in the strong field limit for Janis-Newman-Winicour considering $u = u_m + 0.003$. Figure 7.a shows us the behavior of the strong field limit parameters. In this plot, $\bar{a}$ has the same constant value as Schwarzschild, e.i. 1 and the impact parameter u_m increase for $0.5 < \gamma \leq 1$. At $\gamma = 1$ the value of u_m reaches that of Schwarzschild, e.i. $\frac{3\sqrt{3}}{2}$ (cfr. Appendix A). This value correspond to the brown line in the plot. In the same figure, the $\bar{b}$ parameter decrease for the same interval and for $\gamma = 1$ its value is the same as that of Schwarzschild, e.i. -0.4002 (cfr. Appendix A). This limit corresponds to the green line in the plot. On the other hand, Figure 1.b shows the behavior of the deflection angle as a function of γ. In the plot, the angle only has values in the interval $0.5 \leq \gamma \leq 1$. The reason for such a behavior is that the photon sphere equation for Janis-Newman-Winicour

$$x_m = \frac{2\gamma + 1}{2}, \tag{6.12}$$

does not have solution unless $\gamma : 0.5 \leq \gamma \leq 1$ if we consider $m \geq 0$ [33]. For this interval, a photon coming from infinity is deflected through an unboundedly large angle, i.e. the photon passes increasingly many times around the singularity as the closest distance of approach tends to x_m. In order to understand this idea, we plotted the deflection angle for

Janis-Newman-Winicour as a function of u, Figure 8. This figure, shows how the deflection angle in the strong field limit changes for $\gamma = 0.6; 0.8; 1$(Schwarzschild) as a function of the impact parameter u. Although the behavior of the deflection angle for greater values of u in this approximation does not fit well, when we get closer to u_m (near the photon sphere) the plot reproduce very well the behavior in this limit: the strong limit. As can be inferred from the plot, when we decrease the impact parameter, the deflection angle increase. At some point, the deflection angle will exceed 2π, resulting in a complete loop around the black hole. Decreasing u further, the photon will wind several times before emerging. Finally, for $u_m = u$ ($x_0 = x_m$) the deflection angle diverges and the photon is captured.

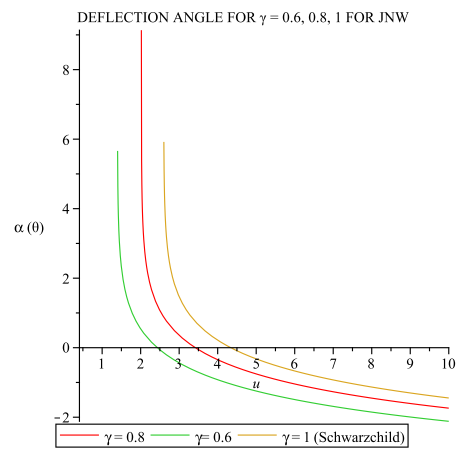


Figure 6.8: Deflection angle for Janis-Newman-Winicour as a function of u for $\gamma = 0.6; 0.8; 1$

In order to discuss S. Kar lensing (no charge case), it is necessary to define the interval in which we are going to work. As considered in [33] $r : 2\eta < r < \infty$. Making $x = r/2\eta$ this interval reduces to $1 < x < \infty$. In chapter 4 we rewrite this metric defining $\zeta = \frac{\sigma}{\eta}$ in such a way that[5]

$$\lambda^2 + \zeta^2 = 1; \tag{6.13}$$

where $\lambda = \frac{m}{\eta}$ and $\zeta = \frac{\sigma}{\eta}$. Hence, the S. Kar metrics in terms of ζ takes the form,

$$ds^2 = \left(1 - \frac{1}{x}\right)^{\zeta + \sqrt{1-\zeta^2}} dt^2 - \left(1 - \frac{1}{x}\right)^{\zeta - \sqrt{1-\zeta^2}} dx^2 - \left(1 - \frac{1}{x}\right)^{1+\zeta - \sqrt{1-\zeta^2}} x^2 d\Omega^2 \tag{6.14}$$

As considered in [33], we assume $\eta > 0$. In this sense, in order to obtain $m \geq 0$ one must assume $0 \leq \zeta \leq 1$. However, the photon sphere equation for S. Kar metric, calculated in Chapter 3, change the interval of ζ because we have to consider $x > 1$ (naked singularity). In terms of ζ equation (4.5) takes the form,

$$x_m = \sqrt{1 - \zeta^2} + \frac{1}{2}. \tag{6.15}$$

Using the analysis from [33], we found that equation (6.15) has solution for $\zeta : 0 \leq \zeta < \frac{\sqrt{3}}{2}$. To obtain this interval, we calculated the value of ζ when $x_m = 1$. Therefore, $\sqrt{1 - \zeta^2} + \frac{1}{2} = 1$ or $\zeta = \frac{\sqrt{3}}{2}$. In consequences, if we consider $x > 1$ it is necessary that

[5] This expression comes from $\eta^2 = m^2 + \sigma^2$

$0 \leq \zeta < \frac{\sqrt{3}}{2}$.

The deflection angle in the strong field limit is expressed by

$$\alpha(\theta) = -\bar{a} \ln \left(\frac{\theta D_{OL}}{u_m} - 1 \right) + \bar{b}. \tag{6.16}$$

For $u = \theta D_{OL} = u_m + 0.003$ and using equations (4.29), (4.30) and (4.32) the strong field limit expression for the deflection is

$$\alpha(\theta) = -\ln \left(\frac{0.003}{u_m} \right) + 0.9496 + 3.005480454\zeta - \pi + \ln \left(\frac{[(k-p+1)^k - (k-p-1)^k]^2 (k-p+1)}{2k^2 (k-p-1)^{2k-1}} \right). \tag{6.17}$$

In order to plot the behavior of $\alpha(\theta)$ as a function of the parameter ζ, we express $k = \frac{m+\sigma}{\eta}$, $p = \frac{\sigma - m}{\eta}$ and u_m in terms of ζ. These values are,

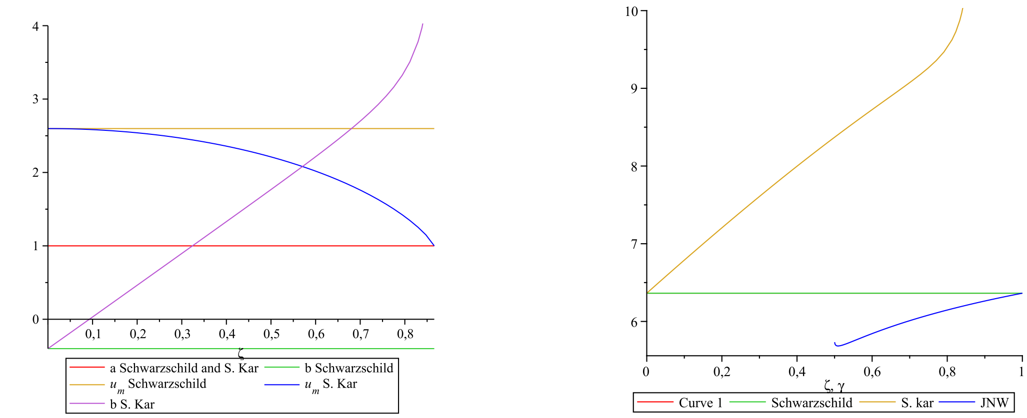
$$k = \sqrt{1-\zeta^2} + \zeta, \quad p = \zeta - \sqrt{1-\zeta^2}, \quad u_m = \frac{1}{2} \frac{\left(2\sqrt{1-\zeta^2} - 1\right)^{\frac{1}{2} - \sqrt{1-\zeta^2}}}{\left(2\sqrt{1-\zeta^2} + 1\right)^{-\frac{1}{2} - \sqrt{1-\zeta^2}}}, \tag{6.18}$$

then $\alpha(\theta)$ is

$$\alpha(\theta) = -\ln \left[0.006 \frac{\left(2\sqrt{1-\zeta^2} + 1\right)^{-\frac{1}{2} - \sqrt{1-\zeta^2}}}{\left(2\sqrt{1-\zeta^2} - 1\right)^{\frac{1}{2} - \sqrt{1-\zeta^2}}} \right] + 0.9496 + 3.005480454\zeta - \pi + \ln [\Box], \tag{6.19}$$

where $\Box \equiv \frac{1}{2} \frac{\left((2\sqrt{1-\zeta^2}+1)^{2\zeta+2\sqrt{1-\zeta^2}} - 2\left(3-4\zeta^2\right)^{\sqrt{1-\zeta^2}+\zeta} + \left(2\sqrt{1-\zeta^2}-1\right)^{2\sqrt{1-\zeta^2}+2\zeta} \right) (2\sqrt{1-\zeta^2}+1)}{(\sqrt{1-\zeta^2}+\zeta)^2 (2\sqrt{1-\zeta^2}-1)^{2\zeta+2\sqrt{1-\zeta^2}-1}}$. For $\zeta = 0$ de deflection angle for S. Kar reduces to that of Schwarzschild [22].

The behavior of $\bar{b}$, u_m and $\alpha(\theta)$ as a function of ζ is shown in figure 9.a, 9.b and figure 10.a respectively. From figure 9.a, we see that $\bar{b}$ and u_m reduce to those of Schwarzschild when $\zeta = 0$. In this sense, as ζ tends to 0 we see that $\bar{b} \to -0.4002$ and $u_m \to \frac{3\sqrt{3}}{2}$. This values are represented as green and yellow lines in the plot. As occurred in Janis-Newman-Winicour, the value of $\bar{a}$ is the same to that of Schwarzschild [22]. On the other hand, for $\zeta \to \frac{\sqrt{3}}{2}$ we see that $\bar{b}$ diverges. In the same plot, we see that u_m only takes values in the interval $0 \leq \zeta < \frac{\sqrt{3}}{2}$. In figure 9.b we plotted the deflection angle $\alpha(\theta)$ for Janis-Newman-Winicour, Schwarzschild and S. Kar as a function of γ (for JNW) and ζ (for S. Kar). When $\gamma = 1$, as was discussed previously, the Janis-Newman-Winicour-Wyman deflection angle reduces to that of Schwarzschild. For $\gamma = 0.5$ it diverges. This means that we are at the naked singularity [20][22][33]. On the other hand, For $\zeta = 0$ the S. Kar deflection angle reduces to $-\ln\left(\frac{2}{3\sqrt{3}}u - 1\right) + 0.9496 - \pi + \ln(6)$ which is the value for Schwarzschild (cfr. Appendix A). However, for $\zeta \to \frac{\sqrt{3}}{2}$ the deflection angle diverges because we are at $r = 2\eta$: the naked singularity. In figure 10.a it is possible to see this behavior more closely. The deflection angle $\alpha(\theta)$ is positive for $0 < \zeta < \frac{\sqrt{3}}{2}$. This means that S. Kar metric always acts like a converging lens.



(a) (b)

Figure 6.9: (a) Strong field limit parameters $\bar{a}$, $\bar{b}$ y u_m for S. Kar as a function of ζ (b) Deflection angle as a function of γ, ζ for Schwarzschild, JNW and S. Kar ($u = u_m + 0.003$)

Finally, in figure 10.b, 10.c, 10.d we plotted the behavior of the deflection angle for S. Kar as a function of the impact parameter u for $\zeta = 0; 0,6; 0,8$. As happened in the case of Janis-Newman-Winicour, the plot of the deflection angle seems to be wrong for bigger values of u. It is know that the bigger the value of u the smaller the value of deflection and for $u \to \infty$ the deflection angle tens to zero. As figure 10.b shows, the deflection angle becomes negative when u increase which contradicts the idea of not deflection for bigger values of u. However, as was explained before, it is necessary to recall that we are working near the photon sphere: the strong field limit (Cfr. [29]). In this sense, the behavior of the deflection angle fits well only for values of u near to u_m; e.i. when x_0 tends to x_m (photon sphere). As occurred before for Janis-Newman-Winicour lensing, de deflection angle for S. Kar diverges when $u \to u_m$. For example, for $\zeta = \frac{\sqrt{3}}{2}$ the deflection angle diverges (the green line in the plot) when u tends to $u_m = 1.398079777$.

As explained in chapter 5, the magnification is expressed by

$$\mu_n = e_n \frac{u_m^2(1+e_n)D_{OS}}{\bar{a}\beta D_{OL} D_{LS}}. \tag{6.20}$$

Then, the tangential magnification μ_n^t is defined as

$$\mu_n^t = \left(\frac{\beta}{\theta_n^0}\right)^{-1} \tag{6.21}$$

In order to study the behavior of the magnification as a function of β (the source position) we have chosen for this analysis, as the lens, the galactic "Black hole" with mass $M = 2.8 \times 10^6 M_\odot$ and $D_{OL} = 8.5$kpc so that $\frac{M}{D_{OL}} \approx 1.57 \times 10^{-11}$ (geometrized units Cfr. [28]). From equation (6.21), and making $\zeta = 0$, we obtain the magnification for the relativistic images as a function of β (expressed in microarcoseconds). The tangential magnification for Schwarzschild is expressed by

$$\begin{aligned}\mu_n^t &= \frac{\frac{3\sqrt{3}}{2}}{6.37 \times 10^{10}\beta}(1 + e^{\bar{b}-2n\pi}) \\ &= \frac{\frac{3\sqrt{3}}{2}}{6.37 \times 10^{10}\beta}\left(1 + e^{0.9496-\pi+\ln(6)-2n\pi}\right).\end{aligned} \tag{6.22}$$

In figure 11 we plotted the behavior of u_m^t as a function of the source position β for $n = 1$.

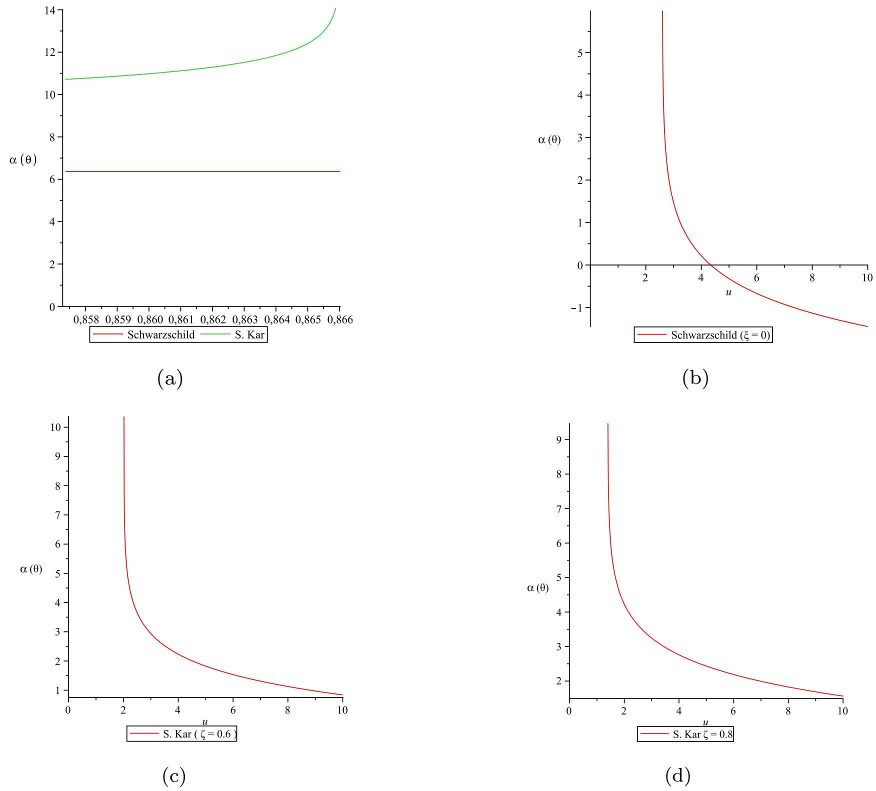


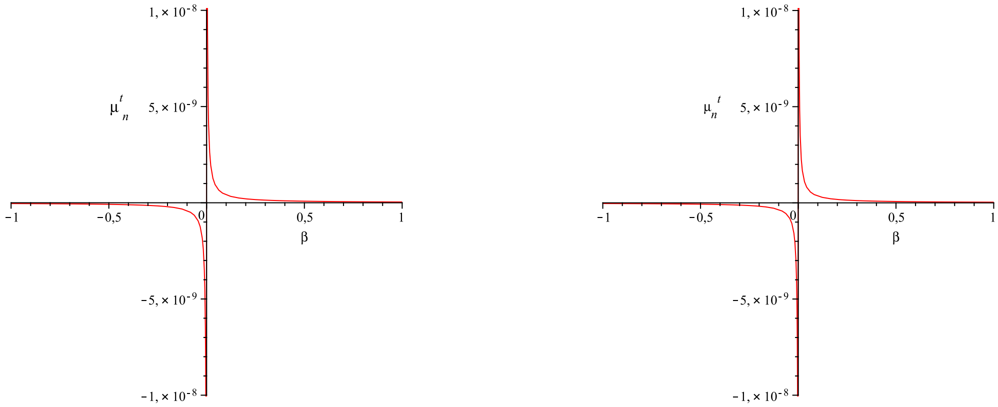
Figure 6.10: (a) Deflection angle near $\frac{\sqrt{3}}{2}$ (b) Deflection angle as a function of u for S. Kar for $\zeta = 0$ (Schwarzschild) (c) Deflection angle as a function of u for S. Kar for $\zeta = 0.6$ (d) Deflection angle as a function of u for S. Kar for $\zeta = 0.8$

As can be seen from the figure the relativistic images decreases very fast as the position of the source increases; which agrees with the results obtained in [28]. Furthermore, the order of magnitude of μ_m^t in figure 11.a agrees with those in figure 3. In this sense, relativistic images are very much demagnified unless the source, lens and observer are highly aligned (when $\beta = 0$). When the position β decreases the magnification in creases rapidly and therefore one may possible get observable relativistic images.

6.1 Conclusions

1. In this work we studied a general method to compute the coefficient of the leading order divergent term and the first regular term ($\bar{a}$ and $\bar{b}$ respectively). When the latter can not be calculated analytically, we have seen that it can be well approximated by a simple series expansion starting from Schwarzschild space-time (Cfr. appendix B). As was shown in this thesis, the method proposed by V. Bozza is very general and can be applied to any spherically symmetric metric representing a "black hole".

2. In this work we used the strong field limit approach to study the light deflection for S. Kar metric and we compare it with Schwarzschild and Janis-Newman-Winicour-Wyman lensing. This study enable us to see, by general arguments, that the deflection angle diverges logarithmically as we approach the photon sphere (Cfr. chapter 3). In this sense, this method opens the possibility to compare the gravitational lensing behaviour of "Black holes" in different theories of gravitation near the zone



(a) (b)

Figure 6.11: (a) Tangential magnification for Schwarzschild ($\zeta = 0$ and $n = 1$) (b) Tangential magnification ($\zeta = 0.5$ and $n = 1$)

in which $x_0 \rightarrow x_m$.

3. Gravitational lensing is undoubtedly a potential powerful tool for the investigation of strong fields. However, there are some difficulties hindering the observation of the primary and secondary images (weak limit) pair near a galactic center; the observation of relativistic images is even much more difficult. For one hand, these images are very much demagnified unless the source, lens and observer are highly aligned. When the position β decreases the magnification in creases rapidly and therefore one may possible get observable relativistic images, but only when the source, lens and observer are highly aligned ($\beta \ll 1$ microarcsec) and the source has a large surface brightness. Quasar and supernovas would be ideal sources for observations of relativistic images. Nevertheless, the number of quasars is low and therefore the probability that a quasar will be highly aligned along the direction of any galactic center is a very small probability that a supernova will be strongly aligned with any galactic center. If relativistic images were observed it would be for a short period of time because the magnification decreases very fast with increase in the source position. On the other hand, The extinction of electromagnetic radiation near the line of sight to galactic nuclei would be appreciable; the smaller the wavelength, the larger the extinction. The radiation at several frequencies from the material accreting on the "Black hole" would make these observations more difficult. Due to theses obstacles no lensing event near a galactic center has been observed till now, but it seems this is a very worth while project.

Appendix A

6.2 Schwarzschild lensing

This is the simplest spherically symmetric metric describing the outer solution for a black hole. It only depends on the mass of the central object (by Birkhoff's theorem). It is convenient to define the Schwarzschild radius $x_s = 2M$ as the measure of distances; them in standard coordinates, the functions in the metric (3.7) become

$$\begin{aligned} A(x) &= 1 - \frac{1}{x} \\ B(x) &= \left(1 - \frac{1}{x}\right)^{-1} \\ C(x) &= x^2, \end{aligned} \tag{6.23}$$

Which obviously satisfy all hypotheses required by the method, with static limit $x_s = 1$. Before following the Bozza's method, we must find $R(z, x_0)$ and $f(z, x_0)$. From equation (3.60) we have

$$R(z, x_0) = 2\frac{\sqrt{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x}\right)^{-1}}}{x^2\frac{1}{x^2}}\left[1-\left(1-\frac{1}{x_0}\right)\right]\sqrt{x_0^2} = 2 \tag{6.24}$$

and[6]

$$\begin{aligned} f(z, x_0) &= \frac{1}{\sqrt{\left(1-\frac{1}{x_0}\right) - \left[\left(1-\left(1-\frac{1}{x_0}\right)\right)z + \left(1-\frac{1}{x_0}\right)\right]\frac{x_0^2}{x^2}}} \\ &= \frac{1}{\sqrt{\left(1-\frac{1}{x_0}\right) - \left[\frac{z}{x_0} + \left(1-\frac{1}{x_0}\right)\right](1-z)^2}} \\ &= \frac{1}{\sqrt{\left(1-\frac{1}{x_0}\right) - \left[\frac{z}{x_0} + \left(1-\frac{1}{x_0}\right)\right](1-2z+z^2)}} \\ &= \frac{1}{\sqrt{\left(2-\frac{3}{x_0}\right)z + \left(\frac{3}{x_0}-1\right)z^2 - \frac{z^3}{x_0}}} \end{aligned} \tag{6.25}$$

6.2.1 Solving the photon sphere equation

Before solving the photon sphere equation we should find $f_0(z.x_0)$. Ones we have found this expansion up to second order for $f(z, x_0)$ we can compute x_m using the condition $\alpha(x_m) = 0$ which is equivalent to solve equation (3.55). From equation (3.72) we have that

$$f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}}; \tag{6.26}$$

[6]Here we have used the relation between z and x: $(1-z) = \frac{x_0}{x}$

where

$$
\begin{aligned}
\alpha &= \frac{(1-y_0)}{C_0 A_0'}(C_0' y_0 - C_0 A_0') \\
&= \frac{\left(1-\left(1-\frac{1}{x_0}\right)\right)}{x_0^2 \frac{1}{x_0^2}}\left(2x_0\left(1-\frac{1}{x_0}\right) - x_0^2\frac{1}{x_0^2}\right) \\
&= \frac{1}{x_0}(2x_0-3) = 2 - \frac{3}{x_0}
\end{aligned} \tag{6.27}
$$

and

$$
\begin{aligned}
\beta &= \frac{(1-y_0)^2}{C_0^2 A_0'^3}\left\{2C_0 C_0' A_0'^2 + (C_0 C_0'' - 2C_0'^2) y_0 A_0' - C_0 y_0 C_0' A_0''\right\} \\
&= \frac{1}{2}\frac{1/x_0^2}{1/x_0^2}\left[4x_0^2 x_0 \frac{1}{x_0^4} + (2x_0^2 - 8x_0^2)\frac{1}{x_0^2}\left(1-\frac{1}{x_0}\right) - 2x_0^2 x_0\left(1-\frac{1}{x_0}\right)\left(-\frac{2}{x_0^3}\right)\right] \\
&= \frac{1}{2}\left[\frac{4}{x_0} - 2\left(1-\frac{1}{x_0}\right)\right] \\
&= \frac{3}{x_0} - 1.
\end{aligned} \tag{6.28}
$$

Finally $f_0(z, x_0)$ can be expressed as

$$
f_0(z, x_0) = \frac{1}{\sqrt{\left(2-\frac{3}{x_0}\right)z + \left(\frac{3}{x_0}-1\right)z^2}} \tag{6.29}
$$

Once we have calculated the α coefficient from the expansion, we can now solve the the condition $\alpha(x_m) = 0$ to find the photon sphere radius x_m

$$
\begin{aligned}
0 &= \alpha(x_m) \\
&= 2 - \frac{3}{x_m} \\
\\
x_m &= \frac{3}{2}
\end{aligned} \tag{6.30}
$$

6.2.2 Finding β_m and $R(0, x_m)$

To find β_m and $R(0, x_m)$ we replace the value $x_m = 3/2$, equation (6.30), into equations (6.28) and (6.24) respectively

$$
\beta_m = \frac{3}{x_m} - 1 = 1 \tag{6.31}
$$

and

$$
R(0, x_m) = 2 \tag{6.32}
$$

6.2.3 Computing b_R

For this simple case, it is possible to solve the integral (3.96) exactly. To do so we evaluate $x_0 = x_m$ in equations (3.75) and (6.29), taking into account that $R(z, x_0)$ is a constant as can be seen from equation (3.60). The regular term in the deflection angle is

$$\begin{aligned} b_r &= \int_0^1 [R(z,x_m)f(z,x_m) - R(0,x_m)f_0(z,x_m)] \\ &= \int_0^1 \left\{ \frac{2}{\sqrt{z^2 - \frac{2}{3}z^3}} - \frac{2}{\sqrt{z^2}} \right\} dz \\ &= \int_0^1 \left\{ \frac{2}{|z|\sqrt{1 - \frac{2}{3}z}} - \frac{2}{|z|} \right\} dz. \end{aligned} \tag{6.33}$$

However, z is positive in the interval of integration then $|z| = z$. We compute the integral's value using a list of integrals from [23]

$$\begin{aligned} b_R &= \int_0^1 \left\{ \frac{2}{z\sqrt{1 - \frac{2}{3}z}} - \frac{2}{z} \right\} dz; \\ &= 2\ln\left(\frac{\sqrt{1-\frac{2}{3}z} - 1}{\sqrt{1-\frac{2}{3}z} + 1} \right)_0^1 - 2\ln(z)|_0^1 \\ &= 2\ln\left(\frac{\sqrt{9-6z} - 3}{\sqrt{9-6z} + 3} \right)_0^1 - 2\ln(z)|_0^1 \\ &= 2\ln(2) + 2\ln(3) - 2\ln(\sqrt{3}+3) + 2\ln(3-\sqrt{3}) \\ &= 2\ln 6(2+\sqrt{3}). \end{aligned} \tag{6.34}$$

6.2.4 Finding $\bar{a}$, $\bar{b}$ and u_m

From equations (3.93) the coefficients are

$$a = \frac{R(0,x_m)}{\sqrt{\beta}} = 2, \tag{6.35}$$

then

$$\bar{a} = \frac{a}{2} = 1. \tag{6.36}$$

From equation (3.93)

$$\begin{aligned} b_D &= \frac{R(0,x_m)}{\sqrt{\beta}} \ln\left(\frac{2(1-y_m)}{A'_m x_m} \right) \\ &= 2\ln\frac{2(1-1/3)}{(4/9)(3/2)} \\ &= 2\ln 2, \end{aligned} \tag{6.37}$$

in consequence

$$\begin{aligned} b &= -\pi + b_D + b_R \\ &= -\pi + 2\ln 2 + 2\ln 6(2+\sqrt{3}), \end{aligned} \tag{6.38}$$

and

$$\begin{aligned} \bar{b} &= b + \bar{a}\ln\frac{cx_m^2}{u_m} = -\pi + b_R + \bar{a}\ln\frac{2\beta_m}{y_m} \\ &= -\pi + 2\ln 6(2+\sqrt{3}) + \ln\frac{2}{(1-\frac{2}{3})} \\ &= -\pi + 2\ln 6(2+\sqrt{3}) + \ln 6 \end{aligned} \tag{6.39}$$

Finally from equation (3.112), where $u_m = \sqrt{\frac{C_m}{y_m}} = \frac{3\sqrt{3}}{2}$, The Schwarzschild deflection angle, in the strong field limit, is

$$\begin{aligned}\alpha(\theta) &= -\bar{a}\ln\left(\frac{2\theta D_{OL}}{3\sqrt{3}} - 1\right) + \bar{b} \\ &= -\ln\left(\frac{2\theta D_{OL}}{3\sqrt{3}} - 1\right) + 2\ln 6(2+\sqrt{3}) + \ln 6 - \pi \\ &= -\ln\left(\frac{2\theta D_{OL}}{3\sqrt{3}} - 1\right) + \ln 216(7+4\sqrt{3}) - \pi.\end{aligned} \tag{6.40}$$

6.3 Reissner-Nordstrom lensing

The Reissner-Nordstrom metric describes the gravitational field of a spherically symmetric massive object endowed with an electric charge q. The metric functions in standard coordinates are

$$\begin{aligned}A(x) &= 1 - \frac{1}{x} + \frac{q^2}{x^2} \\ B(x) &= \left(1 - \frac{1}{x} + \frac{q^2}{x^2}\right)^{-1} \\ C(x) &= x^2.\end{aligned} \tag{6.41}$$

They satisfy the hypotheses required in the introduction, only when $q \leq \frac{3}{4\sqrt{2}}$. However, beyond the critical value $q = 0.5$, there is no event horizon and causality violations appear. We shall restrict to $q = 0.5$.

For Reissner-Nordstrom, the functions $R(z, x_0)$ and $f(z, x_0)$ are calculated using equation (3.60). Initially the functions $R(z, x)$ and $f(z, x_0)$ do not depend on z but on x because of the metric as equation (6.41) shows. However, solving equation (3.56) we obtain x as a function of z. It is important to point out that the solution of (3.56) is the positive one.

$$\begin{aligned}R(x, x_0) &= 2\frac{\sqrt{\left(1-\frac{1}{x}+\frac{q^2}{x^2}\right)\left(1-\frac{1}{x}+\frac{q^2}{x^2}\right)^{-1}}}{x^2\left(\frac{1}{x^2} - 2\frac{q^2}{x^3}\right)}\left(1 - \left(1 - \frac{1}{x_0} + \frac{q^2}{x_0^2}\right)\right)x_0 \\ &= 2\frac{\left(1 - \frac{q^2}{x_0}\right)}{\left(1 - 2\frac{q^2}{x}\right)}.\end{aligned} \tag{6.42}$$

From equation (3.56)

$$\begin{aligned}z &= \frac{\left(1-\frac{1}{x}+\frac{q^2}{x^2}\right) - \left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)}{1 - \left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)} \\ &= \frac{\left(\frac{1}{x_0} - \frac{q^2}{x_0^2}\right) - \frac{1}{x} + \frac{q^2}{x^2}}{\frac{1}{x_0} - \frac{q^2}{x_0}},\end{aligned} \tag{6.43}$$

then

$$\left(\frac{1}{x_0} - \frac{q^2}{x_0^2}\right)(z-1)x^2 + x - q^2 = 0, \quad x > 0, \tag{6.44}$$

and solving we have

$$\begin{aligned}x_+ &= \frac{1}{2}\frac{(x_0 + \sqrt{x_0^2 + 4q^2x_0z - 4q^2x_0 - 4q^4z + 4q^4})x_0}{-zx_0 + q^2z + x_0 - q^2} \\ x_- &= -\frac{1}{2}\frac{(-x_0 + \sqrt{x_0^2 + 4q^2x_0z - 4q^2x_0 - 4q^4z + 4q^4})x_0}{-zx_0 + q^2z + x_0 - q^2}.\end{aligned} \tag{6.45}$$

For $x = x_+(z)$, $R(z, x_0)$ is

$$R(z,x_0) = 2\frac{(x_0 - q^2)(x_0 + \sqrt{x_0^2 + 4q^2x_0z - 4q^2x_0 - 4q^2z + 4q^4})}{x_0(x_0 + \sqrt{x_0^2 + 4q^2x_0z - 4q^2x_0 - 4q^2z + 4q^4}) - 4q^2(-x_0z + q^2z + x_0 - q^2)}. \tag{6.46}$$

From equation (3.60)

$$f(z,x_0) = \frac{1}{\sqrt{y_0\left(1 - \frac{x_0^2}{x_+^2}\right) + \frac{(y_0-1)x_0^2}{x_+^2}z}}. \tag{6.47}$$

If we set $q = 0$ in $f(z, x_0)$ and $R(z, x_0)$ we obtain those for Schwarzschild; it is to say equations (6.25) and (6.32).

6.3.1 Solving the photon sphere equation

In order to solve the photon sphere for Reissner-Nordstrom we have to solve the equation $\alpha = 0$ as we did in the last section for Schwarzschild's metric. From equation (3.67) we have

$$\begin{aligned}\alpha &= \frac{1-\left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)}{x_0^2\left(\frac{1}{x_0}-2\frac{q^2}{x_0^3}\right)}\left(2x_0\left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)-x_0^2\left(\frac{1}{x_0^2}-2\frac{q^2}{x_0^3}\right)\right)\\ &= \left(2-\frac{3}{x_0}+4\frac{q^2}{x_0^2}\right)\frac{(x_0-q^2)}{(x_0-2q^2)},\end{aligned} \tag{6.48}$$

then the equation $\alpha = 0$ for $x_0 = x_m$

$$\begin{aligned} 0 &= \left(2-\frac{3}{x_0}+4\frac{q^2}{x_0^2}\right)\frac{(x_0-q^2)}{(x_0-2q^2)}\\ 0 &= 2x_m^2 - 3x_m + 4q^2 \qquad x_m > 0\end{aligned} \tag{6.49}$$

these equation has two solutions

$$x_m = \frac{3+\sqrt{9-32q^2}}{4} \qquad x_{m1} = \frac{3-\sqrt{9-32q^2}}{4}. \tag{6.50}$$

We chose x_m as the photo sphere radius because it is the largest root.

6.3.2 Finding β_m and $R(0, x_m)$

To calculate β_m we evaluate β at $x_0 = x_m$. From equation (3.71) the coefficient β is[7]

$$\begin{aligned}\beta &= \frac{\left(1-\left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)\right)^2}{2x_0^4\left(\frac{1}{x_0^2}-2\frac{q^2}{x_0^3}\right)^3}\{4x_0^3\left(\frac{1}{x_0^2}-2\frac{q^2}{x_0^3}\right)^2-6x_0^2\left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)\left(\frac{1}{x_0^2}-2\frac{q^2}{x_0^3}\right)\\ &-2x_0^3\left(1-\frac{1}{x_0}+\frac{q^2}{x_0^2}\right)\left(6\frac{q^2}{x_0^4}-\frac{2}{x_0^3}\right)\}\\ &= \frac{1}{2}\frac{x_0(x_0-q^2)^2}{(x_0-2q^2)^3}\left\{-2-\frac{18q^2}{x_0^2}+\frac{6}{x_0}+\frac{16q^2}{x_0^3}\right\}\\ &= \left\{\frac{3}{x_0}-1-9\frac{q^2}{x_0^2}+8\frac{q^4}{x_0^3}\right\}\frac{x_0(x_0-q^2)^2}{(x_0-2q^2)^3}.\end{aligned} \tag{6.51}$$

[7] There was a mistake in [22] it is $(x_0 - 2q^2)^3$ instead of $(x_0^3 - 2q^2)^3$.

For $x_0 = x_m$

$$\beta_m = \left\{ \frac{3}{x_m} - 1 - 9\frac{q^2}{x_m^2} + 8\frac{q^4}{x_m^3} \right\} \frac{x_m(x_m - q^2)^2}{(x_m - 2q^2)^3}$$
$$= [-9 + 32q^2 - 144q^4 + 512q^6 + \sqrt{9 - 32q^2}(3 + 16q^2 - 80q^4)][8(q - 4q^3)]^{-2} \tag{6.52}$$

In order to calculate $R(0, x_m)$ it is necessary to set $z = 0$ and $x_0 = x_m$ in (6.46)

$$\begin{aligned} R(0, x_m) &= 2\frac{(x_m - q^2)(x_m + \sqrt{x_m^2 - 4q^2x_m + 4q^4})}{x_m(x_m + \sqrt{x_m^2 - 4q^2x_m + 4q^4}) - 4q^2(x_m - q^2)} \\ &= 2\frac{(x_m - q^2)(x_m + (x_m - 2q^2))}{x_m(x_m + (x_m - 2q^2)) - 4q^2(x_m - q^2)} \\ &= 2\frac{(x_m - q^2)}{(x_m - 2q^2)}. \end{aligned} \tag{6.53}$$

For $q = 0$ we get (6.32).

6.3.3 Computing b_R

The regular term b_R can not be calculated analytically. However, we can expand the integrand in equation (3.95) in powers of q and evaluate the single coefficients.

$$\begin{aligned} g(z, x_m) &= R(z, x_m)f(z, x_m) - R(0, x_m)f_0(z, x_m) \\ &\approx [R(z, x_m)f(z, x_m) - R(0, x_m)f_0(z, x_m)]|_{q=0} + q[\dot{R}(z, x_m)f(z, x_m) \\ &+ R(z, x_m)\dot{f}(z, x_m) - \dot{R}(0, x_m)f_0(z, x_m) - R(z, x_m)\dot{f}_0(z, x_m)]|_{q=0} \\ &+ \frac{q^2}{2}[\ddot{R}(z, x_m)f(z, x_m) + 2\dot{R}(z, x_m)\dot{f}(z, x_m) \\ &+ R(z, x_m)\ddot{f}(z, x_m) - \ddot{R}(0, x_m)f_0(z, x_m) + 2\dot{R}(0, x_m)\dot{f}_0(z, x_m) \\ &- R(0, x_m)\ddot{f}_0(z, x_m)]|_{q=0} + \mathcal{O}(q^4). \end{aligned} \tag{6.54}$$

Calculating $g(z, x_m)$ we obtain that the expression for b_R is

$$\begin{aligned} b_R &= \frac{1}{2}\int_0^1 \left[\frac{8}{9z} - \frac{\left(\frac{16}{9}z^4 - \frac{80}{27}z^3 + \frac{24}{27}z^2\right)}{(z\sqrt{1 - \frac{2}{3}z})^3} \right] dz \\ &= \frac{8}{9}[\ln 6(2 - \sqrt{3}) + \sqrt{3} - 4] \end{aligned} \tag{6.55}$$

6.3.4 Finding $\bar{a}$, $\bar{b}$ and u_m

From equation (3.93)

$$\begin{aligned} a &= \frac{R(0, x_m)}{\sqrt{\beta_m}} \\ &= 2\frac{(x_m - q^2)}{(x_m - 2q^2)} \frac{(x_m - 2q^2)\sqrt{x_m - 2q^2}}{\sqrt{(3 - x_m)x_m^2 - 9q^2x_m + 8q^4}\left(1 - \frac{q^2}{x_m}\right)} \\ &= 2\frac{x_m\sqrt{x_m - 2q^2}}{\sqrt{(3 - x_m)x_m^2 - 9q^2x_m + 8q^4}} \end{aligned} \tag{6.56}$$

then

$$\bar{a} = \frac{a}{2} = \frac{x_m\sqrt{x_m - 2q^2}}{\sqrt{(3 - x_m)x_m^2 - 9q^2x_m + 8q^4}}. \tag{6.57}$$

To calculate $\bar{b}$ we use

$$\bar{b} = -\pi + b_R + \bar{a} \ln \frac{2\beta_m}{y_m} \tag{6.58}$$

but

$$\begin{aligned} \frac{\beta_m}{y_m} &= \frac{[(3-x_m)x_m^2 - 9q^2 x_m + 8q^4]\left(1 - \frac{q^2}{x_m}\right)}{\left(1 - \frac{1}{x_m} + \frac{q^2}{x_m^2}\right)(x_m - q^2)^3} \\ &= \frac{[(3-x_m)x_m^2 - 9q^2 x_m + 8q^4](x_m - q^2)^2}{(x_m^2 - x_m + q^2)(x_m - 2q^2)^3} \end{aligned} \tag{6.59}$$

then $\bar{b}$ is

$$\bar{b} = -\pi + b_R + \bar{a} \ln \frac{2(x_m - q^2)^2[(3-x_m)x_m^2 - 9q^2 x_m + 8q^4]}{(x_m^2 - x_m + q^2)(x_m - 2q^2)^3}. \tag{6.60}$$

Finally from equation u_m

$$\begin{aligned} u_m &= \sqrt{\frac{C_m}{A_m}} \\ &= \sqrt{\frac{x_m^2}{\left(1 - \frac{1}{x_m} + \frac{q^2}{x_m^2}\right)}} \\ &= \frac{x_m^2}{\sqrt{(x_m^2 - x_m + q^2)}}, \end{aligned} \tag{6.61}$$

replacing $x_m = \frac{3+\sqrt{9-32q^2}}{4}$ in the last equation we get

$$\begin{aligned} u_m &= \frac{(3+\sqrt{9-32})^2}{16\sqrt{\frac{9+6\sqrt{9-32q^2}+9-32q^2}{16} - \frac{12+4\sqrt{9-32q^2}}{16} + \frac{16q^2}{16}}} \\ &= \frac{(3+\sqrt{9-32q^2})^2}{4\sqrt{2}\sqrt{3-8q^2+\sqrt{9-32q^2}}}. \end{aligned} \tag{6.62}$$

6.4 Janis-Newman-Winicour lensing

The spherically symmetric solution to the Einstein massless scalar equations[8] can be written in Janis-Newman-Winicour (JNW) coordinates

$$\begin{aligned} A(x) &= \left(1 - \frac{1}{x}\right)^{\gamma} \\ B(x) &= \left(1 - \frac{1}{x}\right)^{-\gamma} \\ C(x) &= \left(1 - \frac{1}{x}\right)^{1-\gamma} x^2 \\ \\ \Phi(x) &= \frac{q}{2\sqrt{M^2+q^2}} \ln\left(1 - \frac{1}{x}\right) \\ \gamma &= \frac{M}{\sqrt{M^2+q^2}}. \end{aligned} \tag{6.63}$$

where all distances are measured in terms of $x_s = 2\sqrt{M^2+q^2}$ and q is the scalar charge of the central object. This metric admits a photon sphere external to the static limit when

[8] See chapter I equation (71)

$\gamma > \frac{1}{2}$, i.e. when $q < M$. We shall thus restrict our investigations to the objects with scalar charge lower than their mass.

Using equation (3.60) and (6.63) we obtain

$$\begin{aligned} R(z,x_0) &= 2\frac{\left(1-\left(1-\frac{1}{x_0}\right)^{\gamma}\right)\left(1-\frac{1}{x_0}\right)^{\frac{1-\gamma}{2}}x_0}{\left(1-\frac{1}{x}\right)^{1-\gamma}x^2\frac{\gamma}{x^2}\left(1-\frac{1}{x}\right)^{\gamma-1}} \\ &= 2\frac{\left(1-\left(1-\frac{1}{x_0}\right)^{\gamma}\right)\left(1-\frac{1}{x_0}\right)^{\frac{1-\gamma}{2}}x_0}{\gamma} \\ &= \frac{2x_0}{\gamma}\left(\left(1-\frac{1}{x_0}\right)^{\frac{1-\gamma}{2}}-\left(1-\frac{1}{x_0}\right)^{\frac{1+\gamma}{2}}\right). \end{aligned} \tag{6.64}$$

6.4.1 Solving the photon sphere equation

As in the previous cases, we compute the α coefficient using equation (3.67). However, as it shows we have to calculate C_0' and A_0' first from (6.63)

$$\begin{aligned} A_0' &= \frac{\gamma}{x_0^2}\left(1-\frac{1}{x_0}\right)^{\gamma-1} \\ C_0' &= (1-\gamma)\left(1-\frac{1}{x_0}\right)^{-\gamma}+2x_0\left(1-\frac{1}{x_0}\right)^{1-\gamma}; \end{aligned} \tag{6.65}$$

then

$$\begin{aligned} (1-y_0) &= 1-\left(1-\frac{1}{x_0}\right)^{\gamma} \\ C_0 y_0' &= \left(1-\frac{1}{x_0}\right)^{1-\gamma}x_0\gamma\left(1-\frac{1}{x_0}\right)^{\gamma-1}\frac{1}{x_0^2}=\gamma \\ C_0' y_0 &= \left[(1-\gamma)\left(1-\frac{1}{x_0}\right)^{-\gamma}+2\left(1-\frac{1}{x_0}\right)^{1-\gamma}x_0\right]\left(1-\frac{1}{x_0}\right)^{\gamma} \\ &= (1-\gamma)+2\left(1-\frac{1}{x_0}\right)x_0. \end{aligned} \tag{6.66}$$

Replacing in (3.67) we obtain

$$\begin{aligned} \alpha &= \frac{1-(1-\frac{1}{x_0})^{\gamma}}{\gamma}\left((1-\gamma)+2x_0-2-\gamma\right) \\ &= \frac{1-\left(1-\frac{1}{x_0}\right)^{\gamma}}{\gamma}(2x_0-1-2\gamma) \\ &= \frac{1-x_0^{-\gamma}\left(x_0-1\right)\gamma}{\gamma}\left(2-\frac{1+2\gamma}{x_0}\right)x_0 \\ &= \left(2-\frac{1+2\gamma}{x_0}\right)\frac{1}{\gamma x_0^{\gamma-1}}(x_0^{\gamma}-(x_0-1)^{\gamma}). \end{aligned} \tag{6.67}$$

When $\gamma = 1$ (6.67) reduces to (6.27). From $\alpha = 0$, we derive the radius of the photon sphere.

$$0 = \left(2-\frac{1+2\gamma}{x_m}\right)\frac{1}{\gamma x_m^{\gamma-1}}(x_m^{\gamma}-(x_m-1)^{\gamma}) \tag{6.68}$$

Therefore

$$\begin{aligned} 0 &= \left(2 - \frac{1+2\gamma}{x_m}\right) \\ x_m &= \frac{2\gamma+1}{2} \end{aligned} \tag{6.69}$$

6.4.2 Finding β_m and $R(0, x_m)$

$$\beta = -\frac{[(2\gamma+1)(\gamma+1) - 2x_0(3\gamma+1) + 2x_0^2]}{(x_0-1)^\gamma}\frac{[x_0^\gamma - (x_0-1)^\gamma]^2}{2\gamma^2 x_0^\gamma}. \tag{6.70}$$

For $x_0 = x_m$ we obtain[9]

$$\begin{aligned} \beta_m &= -\frac{[(2\gamma+1)(\gamma+1) - 2x_m(3\gamma+1) + 2x_m^2]}{(x_m-1)^\gamma}\frac{[x_m^\gamma - (x_m-1)^\gamma]^2}{2\gamma^2 x_m^\gamma} \\ &= -\frac{[2(2\gamma^2+3\gamma+1) - 2(6\gamma^2+5\gamma+1)] + (4\gamma^2+4\gamma+1)[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(4\gamma^2-1)^\gamma} \\ &= -\frac{[-4\gamma+1][(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(4\gamma^2-1)^\gamma} \\ &= \frac{[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(4\gamma^2-1)^{\gamma-1}}. \end{aligned} \tag{6.71}$$

setting $z = 0$ in equation (6.64) we obtain

$$R(0, x_0) = \frac{2x_0}{\gamma}\left(\left(1 - \frac{1}{x_0}\right)^{\frac{1-\gamma}{2}} - \left(1 - \frac{1}{x_0}\right)^{\frac{1+\gamma}{2}}\right). \tag{6.72}$$

For $\gamma = 1$ we obtain (6.32).

6.4.3 Finding u_m, $\bar{a}$ and $\bar{b}$

The coefficients of the strong field limit

$$\begin{aligned} a &= \frac{R(0, x_m)}{\sqrt{\beta_m}} \\ &= \left[\frac{(2\gamma-1)^{\frac{1-\gamma}{2}}(2\gamma+1)^{\frac{1+\gamma}{2}} - (2\gamma-1)^{\frac{1+\gamma}{2}}(2\gamma+1)^{\frac{1-\gamma}{2}}}{\gamma}\right]\frac{2\gamma(2\gamma-1)^{\frac{\gamma-1}{2}}(2\gamma+1)^{\frac{\gamma-1}{2}}}{(2\gamma+1)^\gamma - (2\gamma-1)^\gamma} \\ &= 2\frac{(2\gamma+1)^\gamma - (2\gamma-1)^\gamma}{(2\gamma+1)^\gamma - (2\gamma-1)^\gamma} = 2, \end{aligned} \tag{6.73}$$

then

$$\bar{a} = \frac{a}{2} = 1. \tag{6.74}$$

To calculate $\bar{b}$ we use

$$\bar{b} = -\pi + b_R + \bar{a}\ln\left(\frac{2\beta_m}{y_m}\right) \tag{6.75}$$

[9] There was a mistake in equation (71) of [22]. It is $(4\gamma^2-1)^{\gamma-1}$ instead of $(4\gamma^2-1)^{2\gamma-1}$

but

$$
\begin{aligned}
\frac{2\beta_m}{y_m} &= 2\frac{[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(4\gamma^2-1)^{\gamma-1}}\left(\frac{2\gamma-1}{2\gamma+1}\right)^{-\gamma} \\
&= 2\frac{[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(2\gamma-1)^{\gamma-1}(2\gamma+1)^{\gamma-1}}\left(\frac{2\gamma+1}{2\gamma-1}\right)^{\gamma} \\
&= 2\frac{[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{4\gamma^2(2\gamma-1)^{2\gamma-1}}(2\gamma+1)
\end{aligned} \tag{6.76}
$$

in consequence we obtain for $\bar{b}$

$$
\bar{b} = -\pi + b_R + \ln\left(\frac{(2\gamma+1)[(2\gamma+1)^\gamma - (2\gamma-1)^\gamma]^2}{2\gamma^2(2\gamma-1)^{2\gamma-1}}\right) \tag{6.77}
$$

Finally, u_m is

$$
\begin{aligned}
u_m &= \sqrt{\frac{C_m}{y_m}} \\
&= x_m\left[\frac{\left(1-\frac{1}{x_m}\right)^{1-\gamma}}{\left(1-\frac{1}{x_m}\right)^{\gamma}}\right]^{\frac{1}{2}} \\
&= x_m\left(1-\frac{1}{x_m}\right)^{\frac{1-2\gamma}{2}} \\
&= \frac{(2\gamma+1)}{2}\left(\frac{(2\gamma-1)}{(2\gamma+1)}\right)^{\frac{1-2\gamma}{2}} \\
&= \frac{(2\gamma+1)^{\frac{1}{2}+\gamma}}{2(2\gamma-1)^{\gamma-\frac{1}{2}}}.
\end{aligned} \tag{6.78}
$$

Appendix B: Finding b_R for S. Kar metric

In order to calculate b_R we have defined $\zeta = \frac{\sigma}{\eta}$ in such a way that

$$\lambda^2 + \zeta^2 = 1; \tag{6.79}$$

where $\lambda = \frac{m}{\eta}$ and $\zeta = \frac{\sigma}{\eta}$. Therefore, $R(z, x_m)$, $R(0, x_m)$, $f(z, x_m)$ and $f_0(z, x_m)$ for S. Kar metric take the form

$$R(z, x_m) = \frac{2\sqrt{1-\zeta^2}+1}{\sqrt{1-\zeta^2}+\zeta} \left\{ \frac{\left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{2}} - \left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{3\zeta-\sqrt{1-\zeta^2}+1}{2}}}{[(1-y_m)z+y_m]^{\frac{\zeta}{\sqrt{1-\zeta^2}+\zeta}}} \right\}$$

$$R(0, x_m) = \frac{2\sqrt{1-\zeta^2}+1}{\sqrt{1-\zeta^2}+\zeta} \left\{ \frac{\left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{2}} - \left(\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\frac{3\zeta-\sqrt{1-\zeta^2}+1}{2}}}{y_m^{\frac{\zeta}{\sqrt{1-\zeta^2}+\zeta}}} \right\}$$

$$f(z, x_m) = \frac{1}{\sqrt{y_m - [(1-y_m)z+y_m]\frac{C_m}{C}}}$$

$$f_0(z, x_m) = \frac{1}{\sqrt{\beta_m(\zeta)}|z|} = \frac{2(\sqrt{1-\zeta^2}+\zeta)(3-4\zeta^2)^{\frac{\sqrt{1-\zeta^2}+\zeta-1}{2}}}{(2\sqrt{1-\zeta^2}+1)^{\sqrt{1-\zeta}+\zeta} - (2\sqrt{1-\zeta^2}-1)^{\sqrt{1-\zeta}+\zeta}} \frac{1}{|z|}, \tag{6.80}$$

where

$$\begin{aligned}
y_m &= \left(\frac{2\sqrt{1-\zeta}-1}{2\sqrt{1-\zeta^2}+1}\right)^{\sqrt{1-\zeta^2}+\zeta} \\
C_m &= \left[\frac{2\sqrt{1-\zeta^2}+1}{2}\right]^2 \left[\frac{2\sqrt{1-\zeta^2}-1}{2\sqrt{1-\zeta^2}+1}\right]^{1+\zeta-\sqrt{1-\zeta^2}} \\
C &= \frac{[(1-y_m)z+y_m]^{\frac{1+\zeta-\sqrt{1-\zeta^2}}{\zeta+\sqrt{1-\zeta^2}}}}{[1-[(1-y_m)z+y_m]^{\frac{1}{\zeta+\sqrt{1-\zeta^2}}}]^2} \\
\beta_m &= \frac{1}{4}\frac{[(2\sqrt{1-\zeta^2}+1)^{\sqrt{1-\zeta^2}+\zeta} - (2\sqrt{1-\zeta^2}-1)^{\sqrt{1-\zeta^2}+\zeta}]^2}{(\sqrt{1-\zeta^2}+\zeta)^2(3-4\zeta^2)^{\sqrt{1-\zeta^2}+\zeta-1}}.
\end{aligned} \tag{6.81}$$

We have expressed C as a function of z using 3.56. For $\zeta = 0$ these expressions reduce to those of Schwarzschild (Cfr. Appendix A).

The regular term b_R can not be calculated analytically. However, we can expand the integrand in equation (3.95) in powers of ζ and evaluate the single coefficients. So that, b_R up to first order in ζ is

$$\begin{aligned} b_R &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^{(n)}}{d\zeta^n} I_R(x_m)(\zeta - 0)^n \\ &= I_R(x_m)_{\zeta=0} + \left(\frac{d}{d\zeta} I_R(x_m) \right)_{\zeta=0} \zeta. \end{aligned} \tag{6.82}$$

For $\zeta = 0$ the value of $I_R(x_m)$ reduces to that of Schwarzschild (Crf. Appendix A). Therefore, for $0 \leq z \leq 1$ ($|z| = z$) we obtain

$$I_R(x_m)_{\zeta=0} = 2 \int_0^1 \left[\frac{1}{|z|\sqrt{1-\frac{3}{2}z}} - \frac{1}{|z|} \right] dz = 2 \ln 6(2-\sqrt{3}) = 0.9496. \tag{6.83}$$

On the other hand,

$$\begin{aligned} \frac{d}{d\zeta} I_R(x_m) &= \int_0^1 \left[\frac{d}{d\zeta}(R(z,x_m)f(z,x_m)) - \frac{d}{d\zeta}(R(0,x_m)f_0(z,x_m)) \right]_{\zeta=0} dz \\ &= \int_0^1 [f(z,x_m)\frac{d}{d\zeta}R(z,x_m) + R(z,x_m)\frac{d}{d\zeta}f(z,x_m) - f_0(z,x_m)\frac{d}{d\zeta}R(0,x_m) \\ &\quad - R(0,x_m)\frac{d}{d\zeta}f_0(z,x_m)]_{\zeta=0} dz \\ &= \int_0^1 \{ f_S(z,x_m) \left[\frac{d}{d\zeta}R(z,x_m) \right]_{\zeta=0} + 2 \left[\frac{d}{d\zeta}f(z,x_m) \right]_{\zeta=0} \\ &\quad - f_{0S}(z,x_m)\frac{d}{d\zeta}R(0,x_m) - 2 \left[\frac{d}{d\zeta}f_0(z,x_m) \right] \} dz, \end{aligned} \tag{6.84}$$

where $f_S(z,x_m)$, $f_{0S}(z,x_m)$ are those of Schwarzschild. Therefore,

$$\frac{d}{d\zeta} I_R(x_m) = \int_0^1 \left[\frac{\frac{d}{d\zeta}R(z,x_m)}{z\sqrt{1-\frac{2}{3}z}} - \frac{\frac{d}{d\zeta}R(0,x_m)}{z} + 2\frac{d}{d\zeta}f(z,x_m) + \frac{\frac{d\beta_m}{d\zeta}}{z} \right] dz, \tag{6.85}$$

where all derivatives, evaluated at $\zeta = 0$, are:

$$\begin{aligned} \left[\frac{d}{d\zeta}R(z,x_m) \right]_{\zeta=0} &= -2 - 2\ln\left(\frac{2}{3}z + \frac{1}{3} \right) \\ \left[\frac{d}{d\zeta}R(0,x_m) \right]_{\zeta=0} &= -2 + 2\ln(3) \\ \left[\frac{d}{d\zeta}f_0(z,x_m) \right]_{\zeta=0} &= \frac{\ln(3)-1}{|z|} \\ \left[\frac{d}{d\zeta}f(z,x_m) \right]_{\zeta=0} &= -\frac{1}{2} \frac{\ln 3\left[\frac{7}{3}z^3 - 2z^2\right] + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}}. \end{aligned} \tag{6.86}$$

Thus,

$$\frac{d}{d\zeta}I_R(x_m) = \int_0^1 \left[\frac{2-2\ln(3)}{z} - \frac{2+2\ln\left(\frac{2}{3}z+\frac{1}{3}\right)}{z\sqrt{1-\frac{2}{3}z}}\right] dz$$
$$+ \int_0^1 \left[\frac{\ln 3\left[\frac{7}{3}z^3 - 2z^2\right] + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}} + \frac{2\ln(3)-2}{z}\right] dz. \tag{6.87}$$

Using a numerical method, we calculate the integrals

$$\int_0^1 \left[\frac{2-2\ln(3)}{z} - \frac{2+2\ln\left(\frac{2}{3}z+\frac{1}{3}\right)}{z\sqrt{1-\frac{2}{3}z}}\right] dz = -3.457723875$$
$$\frac{7\ln(3)}{3}\int_0^1 \frac{dz}{\left(1-\frac{2}{3}z\right)^{\frac{3}{2}}} = \frac{14\sqrt{3}\ln(3)}{(3+\sqrt{3})} \tag{6.88}$$

In order to calculate de remaining integral

$$i = \int_0^1 \left[\frac{-2\ln(3)z^2 + z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}} + \frac{2\ln(3)-2}{z}\right] dz, \tag{6.89}$$

we can express $\ln(2z+1)$ in a Taylor series around $z = 0.5$. This expansion is

$$\ln(2z+1) = \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(z-\frac{1}{2}\right)^n = \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n n}\left(2z-1\right)^n. \tag{6.90}$$

A plot of $\ln(2z+1)$ and (6.90) is shown in the next figure

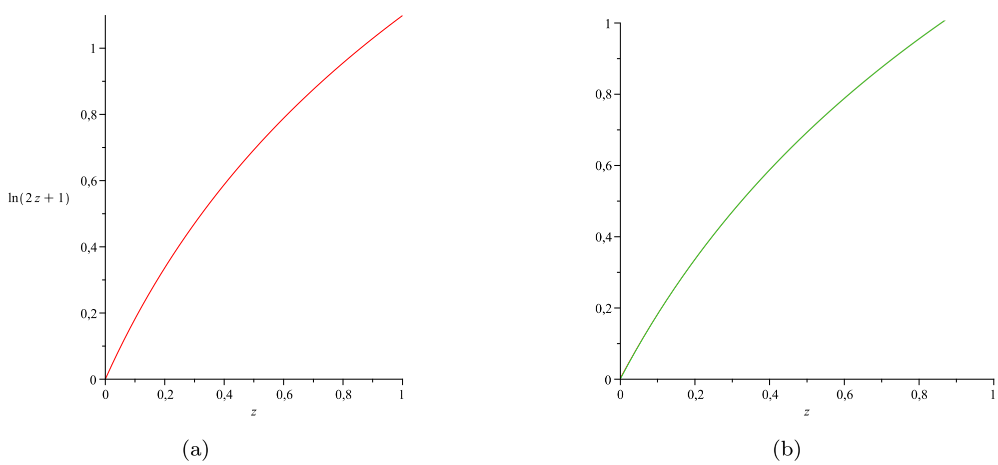


Figure 6.12: (a) $\ln(2z+1)$ (b) Approximation $\ln(2z+1) = \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n n}(2z-1)^n$

In order to express $\ln(2z+1)$ as a Taylor series in the interval $0 \le z \le 1$, it is necessary to demonstrate that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(z-\frac{1}{2}\right)^n, \tag{6.91}$$

converges in that interval. As we see this series is alternating; therefore, if the series $\sum_{n=1}^{\infty} a_n$ is alternating, then it converges absolutely if the series $\sum_{n=1}^{\infty}|a_n|$ converges. In this sense, it is necessary to show that

$$\sum_{n=1}^{\infty} \frac{\mid \left(z - \frac{1}{2}\right)^n \mid}{n}, \tag{6.92}$$

converges for $0 \le z \le 1$. Using the criterion of D´Alembert for a series of positive terms and the H´Lopital rule we obtain that

$$lim_{n \to \infty} \frac{|z - \frac{1}{2}|^{n+1}}{n+1} \frac{n}{|z - \frac{1}{2}|^n} = |z - \frac{1}{2}| < 1. \tag{6.93}$$

Hence, the series converges for $-\frac{1}{2} < z < \frac{3}{2}$. This means that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(z - \frac{1}{2}\right)^n$ converges absolutely for $0 \le z \le 1$.

Using the Newton binomial theorem,

$$(x + y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k \tag{6.94}$$

we can express equation (6.90) as

$$\begin{aligned}
\ln(2z+1) &= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{2^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} 2^{n-k} z^{n-k} (-1)^k \\
&= \ln(2) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!} \frac{1}{2^k} z^{n-k} \\
&= \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!} \frac{1}{2^k} z^{n-k} + z \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{2^2 n} + \ln(2).
\end{aligned} \tag{6.95}$$

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series of the form

$$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}, \tag{6.96}$$

for $a = 1$ and $r = \frac{1}{2}$ we have that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2. \tag{6.97}$$

On the other hand, we know that

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{2^n n}. \tag{6.98}$$

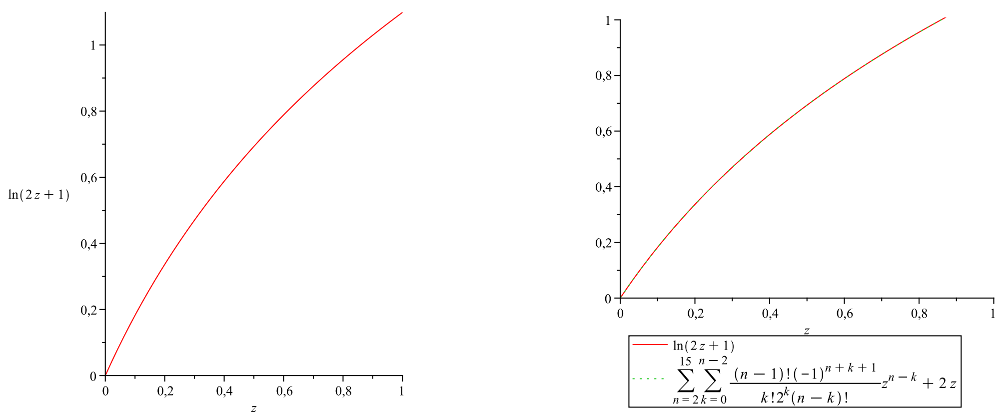
Finally, equation (6.95) is

$$\ln(2z+1) = 2z + \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!} \frac{1}{2^k} z^{n-k}. \tag{6.99}$$

A plot of this expansion is shown in figure 2.

In this sense, the integrant of (6.89) can be expressed as

$$\frac{z(2z+1)(1-z)\ln(2z+1)}{z^3(1-\frac{2}{3}z)^{\frac{3}{2}}} = \frac{(-2z^2+z+1)}{z^2\left(\sqrt{1-\frac{2}{3}z}\right)^3}\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!}\frac{1}{2^k}z^{n-k}$$
$$-\frac{4z}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}+\frac{2}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}+\frac{2}{z\left(\sqrt{1-\frac{2}{3}z}\right)^3}$$
$$=\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!}\frac{1}{2^k}\frac{(-2z^{n-k}+z^{n-k-1}+z^{n-k-2})}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}$$
$$-\frac{4z}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}+\frac{2}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}+\frac{2}{z\left(\sqrt{1-\frac{2}{3}z}\right)^3}. \tag{6.100}$$



(a) (b)

Figure 6.13: (a) $\ln(2z+1)$ (b) Approximation $\ln(2z+1) = 2z + \sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!}\frac{1}{2^k}z^{n-k}$

Finally, the integral (6.89) is

$$i=\int_0^1\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!}\frac{1}{2^k}\frac{(-2z^{n-k}+z^{n-k-1}+z^{n-k-2})}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}dz$$
$$+2\int_0^1\frac{dz}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}-4\int_0^1\frac{zdz}{\left(\sqrt{1-\frac{2}{3}z}\right)^3}+(2\ln(3)-2)\int_0^1\left[\frac{1}{z}-\frac{1}{z\left(\sqrt{1-\frac{2}{3}z}\right)^3}\right]dz \tag{6.101}$$

Approximating up to a 20^{th} polynomial we have that

$$i = \int_0^1 \sum_{n=2}^{20} \sum_{k=0}^{n-2} \frac{(-1)^{n+k+1}(n-1)!}{k!(n-k)!} \frac{1}{2^k} \frac{(-2z^{n-k} + z^{n-k-1} + z^{n-k-2})}{\left(\sqrt{1-\frac{2}{3}z}\right)^3} dz$$
$$+ 2\int_0^1 \frac{dz}{\left(\sqrt{1-\frac{2}{3}z}\right)^3} - 4\int_0^1 \frac{zdz}{\left(\sqrt{1-\frac{2}{3}z}\right)^3} + (2\ln(3)-2)\int_0^1 \left[\frac{1}{z} - \frac{1}{z\left(\sqrt{1-\frac{2}{3}z}\right)^3}\right] dz$$

$$i = -1.064885740 + \frac{12\sqrt{3}}{3+\sqrt{3}} - \frac{72\sqrt{3}}{(3+\sqrt{3})^2} - 0.3823993616 \tag{6.102}$$

Finally, from (6.88) and (6.102) we obtain that

$$\frac{d}{d\zeta} I_R(x_m)|_{\zeta=0} = 3.005480454 \tag{6.103}$$

Then

$$b_R = 2\ln(6(2-\sqrt{3})) + 3.005480454\zeta \tag{6.104}$$

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