CP-strong problem and $U(1)'$ non-universal symmetry

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Niels Bohr: “What are you working on Mr. Dirac?”
Paul Dirac: “I’m trying to take the square root of something”

A mi familia

Resumen

El presente proyecto tiene como objetivo utilizar una extensión $U(1)_X$ no universal al modelo estándar que permita explicar el problema de jerarquía de masas. Posteriormente, aplicando una simetría global axial tipo Peccei-Quinn $U(1)_{PQ}$ se pretende obtener un modelo que permita además interpretar el problema CP-fuerte. El esquema propuesto permite distinguir entre familias fermiónicas sin introducir simetrías discretas adicionales, generando los ansatz de matrices de masa correctos para obtener el espectro de masas fermiónico observado experimentalmente en el modelo estándar. El rompimiento espontáneo de las simetrías del modelo es producido por dos dobletes escalares de Higgs y dos singletes, donde uno de estos últimos tiene la excitación asociada con el axion, el cual posee una rica fenomenología estudiada en la literatura. El sector exótico esta compuesto de un axion invisible a, un quark pesado T tipo up y dos quarks pesados $J^{1,2}$ tipo down, dos leptones pesados cargados $E, \mathcal{E}$ y un neutrino derecho $\nu_R^{e,\mu,\tau}$ adicional por familia. Además, la gran escala de energía asociada con el rompimiento espontáneo de la simetría de PQ permite generar masas para los neutrinos derechos. Así, a través de un mecanismo see-saw tipo I, los neutrinos activos adquieren masas del orden de los eV.

Palabras clave: Extensiones no universales, Simetría de Peccei-Quinn, Anomalías quirales, instantones, teorías efectivas, axión.

Abstract

We present a non-universal $U(1)_X$ extension and an additional global anomala Peccei-Quinn (PQ) symmetry to the standard model (SM). The scheme proposed allow us to distinguish among fermion families without introducing additional discrete symmetries and generating the correct ansatz of mass matrix to obtain the fermionic mass spectrum in SM. The symmetry breakdown is performed by two scalar Higgs doublets and two scalar singlets, where one of these has the excitation associated with the axion-particle which turns out to be a candidate for dark matter. The exotic sector is composed of an invisible axion a, one up-type T and two down-type $J^{1,2}$ heavy quarks, two heavy charged leptons $E, \mathcal{E}$ and one right-handed $\nu_R^{e,\mu,\tau}$ additional neutrino per family. In addition, the large energy scale associated with the spontaneously breaking (SSB) of the PQ-symmetry provides a solution to the strong CP-problem, also giving masses to the right neutrinos in such manner that the active neutrinos acquire eV-mass values due to the see-saw mechanism implementation.

Non-universal extensions, Peccei-Quinn symmetry, chiral anomalies, instantons, effective theories, axion.

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1 Introduction

One of the most successful current physical models in the agreement between experimental results and theoretical predictions is the standard model of particle physics (SM) proposed by Glashow, Weinberg and Salam [1, 2, 3] in the decade of the '60s. SM predicts with a high degree of precision the interaction between the different particles, explaining a wide variety of experimental results and predicting a large number of phenomena, such as the discovery of the W and Z gauge bosons and the prediction of the Higgs boson [4]. Despite all the successes in different fields, SM has some problems that can not be explained within the context of the theory:

- Within the framework of the SM, neutrinos appear as massless particles, contradicting the observed experimental results [5]. The introduction of right- -handed heavy neutrinos in order of$(10^{12} - 10^{15})GeV$ (See-Saw mechanism) [6] is a possible explanation. Adding Majorana neutrinos it is possible to introduce smaller masses [7, 8]

- The mass hierarchy of the fermions together with the concept of flavor and the existence of three different families of particles (table **1-1**) are known as the *flavor problem* [9]. The SM requires particles without mass because due to the presence of chiral interactions, the Lagrangian can not include them directly. The masses (table **1-2**) are generated via spontaneous symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_Q$ through the Higgs mechanism [10], ensuring a mass value for each of fermions on the same scale $(\nu = 246GeV)$. But experimentally it is observed there are three different hierarchies:
  - Mass hierarchy among fermionic families *i.e.* $m_\tau \gg m_\mu \gg m_e, m_b \gg m_s \gg m_d$ and $m_t \gg m_c \gg m_u$.
  - The second kind of hierarchy is within the families *i.e.* $m_d > m_u, m_c \gg m_s$ and $m_t \gg m_b$.
  - The third hierarchy is associated with the values of the quark-mixing angles *i.e.* $\sin\theta_{12} \gg \sin\theta_{23} \gg \sin\theta_{13}$,

  which requires a numerical adjustment of the coupling constants of Yukawa to obtain a correspondence between the predicted values and the experimental values. This problem is known as the *Fermionic mass hierarchy problem* and is closely related to the mechanism for generating masses. Within the theoretical structure of the SM it is not possible to find a proccess to produce the observed fermionic mass spectrum (**1-2**). Therefore, it becomes neccesary to use theories beyond the standard model. One of the first attepts to understand the three mass scales in the mass fermionic spectrum by means of mass matrices with suited textures was proposed by H. Fritzsch [11, 12]. C. Froggatt and H. Nielsen presented a model in which the heaviest fermions acquire mass through the vacuum expectation value (VEV) of the Higgs fields, while the lighter ones obtain mass through radiative corrections using degrees of freedom heavier than SM particles [13]. Another possibility to understand this problem is based on assuming that the light neutrinos are the particles which acquires masses through these radiative corrections [14, 15]. Similar methods involve analyzing the quarks mass spectrum and the mixing angles of the CKM matrix [16], extra dimensions [17] and anti-de Sitter space approaches in brane theories [18]. It is also possible to obtain the mass spectrum by introducing discrete symmetry and an anarchic mass texture [19]. Finally, the detection of the Higgs boson has allowed new schemes involving extended scalar sectors (2HDM) [20] and some extensions through additional scalar fields (N2HDM) [21], N3HDM [22]), which generate the additional VEVs required to produce the correct masss matrices textures in order to generate the desired hierarchies in the fermonic sector.

- Another problem that does not allow a direct solution through SM is the existing asymmetry between particles and antiparticles (baryonic asymmetry) in the universe. In 1967 Sakharov [23] discovered that in addition to requiring a violation in a discrete type C symmetry (symmetry of particle-antiparticle exchange), a violation of the CP symmetry is required (C transformation plus a spatial reflection). Electromagnetism and strong interactions are symmetric under C and P, while weak interactions are not symmetric under C, in addition to presenting a small violation of CP [24]. But this violation is not enough to explain the phenomenon of baryonic asymmetry, so it is necessary to find sources CP - violation in other sectors.

- There is an associated theoretical problem and it is the apparent absence of CP violation in the strong interaction. Quantum chromodynamics (QCD), which is the theory that explains the processes of strong interaction, is a theory that is symmetric under the group of transformations $SU(3)$, which is a non-abelian Lie group. QCD is in principle non-symmetric under CP, due to the presence of a term associated with non-conservation of chirality, that is, QCD is not symmetric under $U(1)_A$. The physical parameter that involves the breaking of the CP symmetry comes from two contributions that differ in their origin, so it is necessary to explain how the combination of these two anomalous terms that generate a particularly small (but not null) result can be compatible with the experimental fact that no such violation is observed. This is the so-called *Strong CP-problem*. Some of the most studied solutions in the literature for this problem are:
    - Non-conventional dynamics: Establish the fact that the θ-parameter is only a

result of exotic vacuum periodicity conditions [25] or state that the θ-contribution is due to the impossibility of selecting appropriate boundary conditions for the vacuum [26],

- Spontaneous CP-braking: If CP-symmetry is spontanously broken, $\theta = 0$ at tree-level [27, 28, 29]. However, in this kind of process θ is re-induced to 1-loop level. Then, to obtain the necessary limit $\theta \sim 10^{-9}$ in order to solve the CP-problem, it is required to ensure that θ is vanished at 1-loop. This situation requires complex VEV for the Higgs fiels, which leads to situations (Flavor changing neutral currents (FCNC) and domain wall problems [30]) that requires physical concepts more complicated than the problem to solve [31, 32, 33, 34],
- Introduction of an additional chiral symmetry: Adding an additional symmetry, it is possible to induce a spontaneous rupture process, generating an effective theory in which the static angle θ is replaced with a CP-conserving dynamical filed. This can be done at the limit where the mass of the lightest quark in the model is exactly zero [35] or expanding the SM symmetry group [36, 37]. The first option is ruled out mainly by an analisis of current algebra [38] and experimental limits [39]. Then, the best option is introduce an additional global chiral $U(1)$ symmetry known as the PQ-symmetry, resulting in a mechanism to solve the CP-problem known as the PQ-mechanism [37, 39, 40, 41]

- The standard cosmological model requires the existence of dark matter and dark energy ([42], [43])but the standard model of particles in its current form does not provide a good candidate. Structure formation depends on wheter dark matter is hot (particles whose momentum is much larger then their masses) or cold (slow-moving particles). A successful dark matter candidate should be electrically neutral with small self-interactions. It also should have a very long life-time. In order to quantify the amount of the components of the universe, the density parameter is defined as [44]:

$$\Omega_X = \frac{\rho_X}{\rho_c}, \tag{1-1}$$

where ρ_X is the density of the X-component (Dark matter, Dark energy, ordinary baryonic matter), and ρ_c is the critical density

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{1-2}$$

with $H \equiv \dot{a}/a$ is the Hubble parameter [45] and G is the gravitational constant. Using the reduced Hubble parameter h defined by $H_0 = 100h(\text{km/s})/\text{Mpc}$, (where H_0 is the current value of the Hubble parameter), it is possible to write a density parameter based in the anisotropies of the Cosmical Microwave Background [46] for the baryonic (b) and total amount of matter (m):

$$\Omega_b h^2 = 0{,}0226, \qquad \Omega_m h^2 = 0{,}133 \tag{1-3}$$

Therefore, the density parameter of non-baryonic dark matter (dm) is:

$$\Omega_{dm}h^2 = 0{,}112 \tag{1-4}$$

In SM, neutrinos are the only particle that meets these requirements, but, their density is not enough to satisfy the expected results (1-4) [47]:

$$\Omega_{\nu}h^2 < 0{,}066 \tag{1-5}$$

1^{st} Family	2^{nd} Family	3^{rd} Family
$q_L^1 = \begin{pmatrix} u^1 \\ d^1 \end{pmatrix}_L$	$q_L^1 = \begin{pmatrix} u^1 \\ d^1 \end{pmatrix}_L$	$q_L^3 = \begin{pmatrix} u^3 \\ d^3 \end{pmatrix}_L$
u_R^1	u_R^2	u_R^3
d_R^1	d_R^2	d_R^3
$\ell_L^e = \begin{pmatrix} \nu^e \\ e^e \end{pmatrix}_L$	$\ell_L^\mu = \begin{pmatrix} \nu^\mu \\ e^\mu \end{pmatrix}_L$	$\ell_L^\tau = \begin{pmatrix} \nu^\tau \\ e^\tau \end{pmatrix}_L$
e_R^e	e_R^μ	$e_R\tau$

Tab. **1-1**: Flavor fermionic families in SM

Family	Particle	Mass
1	u	$2{,}2^{+0{,}6}_{-0{,}4}$MeV
	d	$4{,}7^{+0{,}5}_{-0{,}4}$MeV
	e	0,511MeV
2	c	$1{,}27 \pm 0{,}03$ GeV
	s	96^{+8}_{-4}MeV
	μ	105,7MeV
3	t	$173{,}21 \pm 0{,}71$GeV
	b	$4{,}18^{+0{,}04}_{-0{,}03}$GeV
	τ	1,776GeV

Tab. **1-2**: Fermionic masses in the SM

The common feature of any scenario that allows a solution to the aforementioned problems is the need to propose models beyond SM. The flavor problem addresses different situations such as the number of families, the mass hierarchy for fermions among others. Although these observations should be obtained naturally from the theoretical background model, SM does not provide them, then the implementation of models beyond SM is necessary. The simplest extension and one of the most studied is done through the addition of $U(1)'$ gauge symmetries. There are many motivations to consider this type of models [48]. For example, supersymmetric extensions that provide mechanisms of effective mass generation through the addition of scalar singlets [49], non-supersymmetric extensions associated to models with dynamic symmetry breaking, extra dimensions, etc. [50, 51], which includes flavor physics [52], neutrino physics [53, 54], dark matter [55, 56] among others.

The present project proposes to use the non-universal extension $U(1)'$ to solve the problem of mass hierarchy, to subsequently apply a Peccei-Quinn $U(1)_{PQ}$ symmetry to explain the strong - CP problem. In addition, associated with the new energy scale introduced by the PQ mechanism, it will be possible to implement a see-saw mechanism to give mass to the active neutrinos. Chapter 2 reviews the concept of anomalies in Abelian theories, trying to address the perturbative approach from the Adler-Bell-Jackiw anomaly (ABJ Anomaly) and Fujikawa's non-perturbative approach. Then, in chapter 3 the presentation of the necessary concepts to address the anomaly in non-Abelian theories is made and consists of three parts. In the first part, an introduction to the basic concepts of quantum chromodynamics is made. After that, the problem known as the *"missing-meson"* or $U(1)_A$ -problem is addressed. A possible solution to this problem is shown by introducing the concept of instanton, which will allow for a new analysis of the vacuum of quantum chromodynamics, to continue with an analysis of the origin of anomalies in non-Abelian gauge theories. In the last part, the formulation of the CP-strong problem is presented and the solution associated with the Peccei-Quinn mechanism is shown. Chapter 4 presents the construction of an abelian extenion of the Standard model in order to study two problems: the hierarchy fermion spectrum and the strong CP-problem. The mass eigenvalues for the scalar and fermionic sector are obtained and the process to calculate the charges associated with the introduction to the Peccei-Quinn anomala symmetry. Finally, some conclusions are discussed related to the obtained results.

2 QFT anomalies

In common terms, an anomaly is a symmetry that becomes manifest through a process of conservation at a classical level, but, after performing the quantization process, it no longer exists. It is of crucial importance the study of anomalies in physics, since the consistency of a model depends exclusively on the appearance of some types of anomaly: global anomalies allow to explain physical phenomena as the decay of π^0, but the existence of local anomalies can damage the gauge invariance and therefore the renormalizability of the theory, making it totally inconsistent from the physical point of view. To build extensions beyond the standard model to solve problems that have no solution within the original framework, it is necessary to find conditions that require the total cancellation of anomalies of local type. We are interested in studying the origin of such anomalies in order to build extensions to the standard model that allows us to solve two specific problems: the strong CP problem and the mass hierarchy in the fermionic sector. For this, we need to analyze the emergence of the so-called chiral anomaly and understand from this the approach of conditions for anomalies of local type.

2.1. The chiral anomaly in 2 dimensions

Maybe a good strategy to build the anomaly into $3+1$ dimensions is to approximate into the $1+1$ anomaly. Our problem is related to study the behaviour of the ground state of a single-flavor theory with massless fermion in the presence of an electric field. This field is a fixed background field, not a fluctuating one, so the action of switch on this field has to be done adiabatically. The action related to the massless fermion if the background fixed field A_μ is not turned on is:

$$S = \int d^2x i\bar{\psi}\partial\!\!\!/\psi, \tag{2-1}$$

where the two-dimensional massless spinors are:

$$\begin{aligned}\psi(x,t)_i &= \frac{1}{\sqrt{2\pi}}\int dk \exp\left[-ikx\right] a_{i,k}(t),\\ \psi_i^*(x,t) &= \frac{1}{\sqrt{2\pi}}\int dk \exp\left[ikx\right] a_{i,k}^*(t)\end{aligned} \tag{2-2}$$

(with $i=1,2$) and the a's satisfies the anticommutation relations:

$$\left\{a_{i,k}^*(t), a_{j,l}(t')\right\}|_{t=t'} = \delta(k-l)\delta_{ij} \tag{2-3}$$

For study this problem we construct a Clifford algebra with the following commutation rule:

$$\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}; \quad \mu, \nu = 0, 1, \tag{2-4}$$

and we choose the 2-d Dirac matrices in the following way:

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^5 = \gamma^0\gamma^1 = \sigma_3, \tag{2-5}$$

where σ_i denote the familiar Pauli matrices. Thus, at classical level, we found that the lagrangian structure satisfies the expected conservation laws. Under the $U(1)$ symmetry $\psi \to e^{i\alpha}\psi$ the vector current and the associated electric charge have the form:

$$j_\mu = \bar{\psi}\gamma^\mu\psi, \quad \partial^\mu j_\mu = 0, \quad Q(t) = \int dx j_0(t,x) \to \dot{Q} = 0, \tag{2-6}$$

and under the $U(1)$-axial symmetry $\psi \to e^{i\alpha\gamma^5}\psi$ the conservation of the axial current and the associated axial charge can be written as:

$$j^5_\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial^\mu j^5_\mu = 0, \quad Q(t)^5 = \int dx j^5_0(t,x) \to \dot{Q}^5 = 0. \tag{2-7}$$

The selected basis (2-5) allows to write the massless Dirac Fermions into independent chiral components:

$$\psi_L = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}, \tag{2-8}$$

where the ψ_1 component is a left-moving fermion and the ψ_2 is a right-handed moving fermion. Written in terms of this chiral components, the action (2-1) becomes:

$$S = \int d^2x i\psi_1^\dagger\partial_-\psi_1 + i\psi_2^\dagger\partial_+\psi_2, \tag{2-9}$$

with $\partial_\pm = \partial_t \pm \partial_x$. Then, the chiral fermions ψ_1 satisfies the equation of motion $\partial_-\psi_1 = 0$ which has the solution $\psi_1 = \psi_1(t+x)$ (or, in other words, ψ_1 is a left-handed fermion) and ψ_2 satisfies $\partial_+\psi_2 = 0$, corresponding to a right-handed fermion $\psi_2 = \psi_2(t-x)$. The chiral components have a chirality ± 1:

$$\gamma^5\psi_{L,R} = \pm\psi_{L,R}. \tag{2-10}$$

The chiral charges have the form:

$$Q_{L,R} = \int dx \bar{\psi}_{L,R}\gamma_0\psi_{L,R} = \int dx \psi^\dagger_{L,R}\psi_{L,R}, \tag{2-11}$$

therefore, the vector and axial current can be written as:

$$Q = Q_L + Q_R \qquad Q^5 = Q_L - Q_R. \tag{2-12}$$

The axial charge for a left- or right-handed fermion corresponds to:

$$Q^5_{L,R} = \int dx \bar{\psi}_{L,R}\gamma^0\gamma^5\psi_{L,R} = \pm \int dx \psi^\dagger_{L,R}\psi_{L,R} = \pm Q_{L,R}. \tag{2-13}$$

Thus, the number of left-moving fermions and the right-moving fermions are separately conserved. This fact is known as a *chiral symmetry.*

Naively, we could expect these conservation laws to be maintained when the background field is turned on. Deforming our theory to include A_μ, the action has to be written as (considering A_μ as a not fluctuacting field):

$$S = \int d^2xi\bar{\psi}\not{\mathcal{D}}\psi, \tag{2-14}$$

with $\mathcal{D}_\mu = \partial_\mu - iA_\mu$. In this context is really useful to choose our vacuum states in the *Dirac sea* language: the vaccuum configuration consists of filling all negative energy states, and the corresponding states with $E < 0$ are unfilled. It is possible to propose a compactification over all the configuration space onto a $2-$dimensional cylinder related to an $x-$space turning around a S^1 circle in order to provide a better analysis. The boundary conditions are [57]:

$$\begin{aligned} A_\mu\left(t, x = -\frac{L}{2}\right) &= A_\mu\left(t, x = \frac{L}{2}\right) \\ \psi\left(t, x = -\frac{L}{2}\right) &= -\psi\left(t, x = \frac{L}{2}\right). \end{aligned} \tag{2-15}$$

The antiperiodic condition in the wave function is only a matter of convenience in order to reproduce the correct structure of the vacuum associated with the Dirac sea [58]. The gauge field is chosen such that $A_0 = 0$ and the electric field $A_1 = A_1(t)$ can be turned on adiabatically. This means that the field configuration is periodic inside the S^1 circle with length $2\pi/L$. The associated Dirac equation is:

$$\left(i\not{\partial} - \not{A}\right)\psi = 0. \tag{2-16}$$

Under the $A_0 = 0$ and boundary conditions, we can rewrite:

$$\left(i\sigma_2\partial_0 + i\sigma_1\left(i\partial_1 - A_1\right)\right)\psi = 0. \tag{2-17}$$

Multiplying by σ_2:

$$\left(i\partial_0 + \sigma_3\left(i\partial_1 - A_1\right)\right)\psi = 0. \tag{2-18}$$

Expressing the fermion wave function (2-2) into the Fourier series:

$$\psi(t,x) = \frac{1}{\sqrt{L}}\sum_k u(k)\exp\left(-iE_kt\right)\exp\left(ikx\right), \tag{2-19}$$

(where the $u(k)$'s have the expansion coefficients plus the contribution of the a's, and E, k are the corresponding energy and momentum of the fermions satisfying a dispersion relation) and applying the boundary conditions (2-15), we have:

$$\psi(t,x) = \frac{1}{\sqrt{L}} \sum_k u(k) \exp\left(-iE_k t\right) \exp\left[i\frac{2\pi}{L}\left(k+\frac{1}{2}\right)x\right]. \tag{2-20}$$

Using

$$\partial_0 \psi(t,x) = -iE_k\psi, \tag{2-21}$$

$$\partial_1 \psi(t,x) = \left[i\frac{2\pi}{L}\left(k+\frac{1}{2}\right)\right]\psi, \tag{2-22}$$

we obtain the energy solutions for the left- and right-handed fermions:

$$E_k^L = \frac{2\pi}{L}\left(k+\frac{1}{2}\right) + A_1, \tag{2-23}$$

$$E_k^R = -\frac{2\pi}{L}\left(k+\frac{1}{2}\right) - A_1, \tag{2-24}$$

with $k = 0, \pm 1, \pm 2, \cdots$. The energy spectrum are discrete because of the compactification (the conditions (2-15) generate this specific structure). Then, we can see that when the $A_1 = 0$, the ground states are degenerated (this state represent the ground state). If we turn on the field, the levels split: the left levels increase and the right levels decrease in the same amount of the background field. When $A_1 = 2\pi/L$, the structure arise to an equivalent state to the initial configuration (as we expected for a gauge equivalence), but we produce a left-handed particle and an right- handed hole. The electric charge of the left particle and the right hole are opposite, then there is no change in the total electric charge and the vector current is conserved:

$$\partial^\mu j_\mu = 0, \quad \dot{Q} = 0. \tag{2-25}$$

On the other hand, the axial charge ($Q^5 = Q_L - Q_R$) is identical for both the left particle ($Q_L = 1, Q_R = 0$) and the right hole ($Q_L = 0, Q_R = -1$). Therefore, the total axial charge changes:

$$\Delta Q^5 = 1 + 1 = 2. \tag{2-26}$$

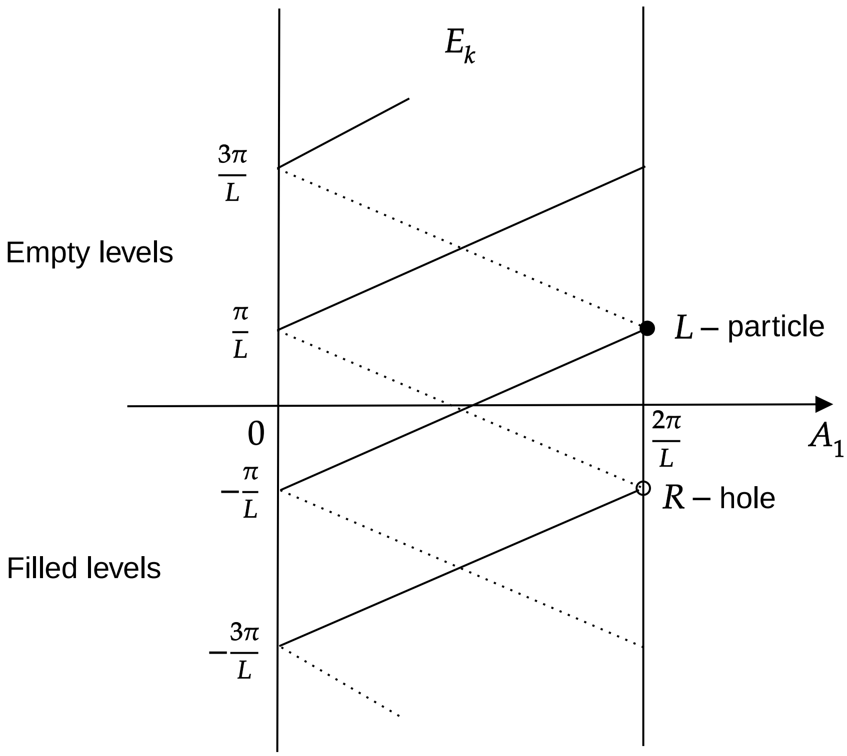
Taking into account that we increase the background field A_1 from 0 to $2\pi/L$ (gauge equivalence), it is possible to write:

$$\Delta A_1 = \frac{2\pi}{L} \quad \rightarrow \quad 2 = \frac{L\Delta A_1}{\pi}. \tag{2-27}$$

Thus:

$$\Delta Q^5 = 2 = \frac{L\Delta A_1}{\pi}, \tag{2-28}$$

Fig. **2-1**: Energy levels associated with the $A_1 \neq 0$ fixed background field



where we have used the change associated with the background field. The rate of change per time unit is:

$$\frac{\Delta Q^5}{\Delta t} = \frac{L \Delta A_1}{\pi \Delta t}. \tag{2-29}$$

Considering the local change in this rate, we have:

$$\frac{\partial}{\partial t} \int_0^t dx j_0^5(t, x) = \frac{1}{\pi} \frac{\partial}{\partial t} \int_0^L dx A_1(t). \tag{2-30}$$

Thus, for the axial current we have:

$$\partial_0 j_0^5 = \frac{1}{\pi} \partial_0 A_1, \tag{2-31}$$

or written in a Lorentz invariant way, we arrive to the *anomaly in 2-d* [59]:

$$\partial_\mu j_5^\mu = \frac{1}{\pi} \varepsilon_{\mu\nu} \partial^\mu A^\nu. \tag{2-32}$$

Therefore, the axial classical symmetry does not arise a conserved quantity under the new background field assumption. These extra fermions come from the infinite associated with the structure of the Dirac sea. The only solution would be to truncate the structure of the vacuum in some specific place, in order to compensate for the excess of effective fermionic states with the exhaustion of right - moving states. The Dirac sea is a infinite structure itself, so the origin of the anomaly expression is related with the infinite number of states.

2.2. Classical conservation laws

In the previous section, we have seen that the origin of the anomaly term is related to the non-conservation of the axial charges in the presence of background electromagnetic fields. This affirmation seems to contradict the Noether's theorem, which affirms that the axial charge is a conserved quantity [60]. In this section we will remember the Noether's theorem, to later try to deduce the origin of the unconserved terms. For this, we will start with a general theory for scalar fields and then generalize to fermionic fields by finding the corresponding axial symmetry. Considering an infinitesimal transformation of the scalar field ϕ:

$$\delta\phi = \alpha X(\phi), \tag{2-33}$$

where α is a infinitesimal small parameter. We consider this trnsformation as a symmetry if:

$$\delta\mathcal{L} = 0. \tag{2-34}$$

$\mathcal{L}$ is the lagrangian density as usual. If the parameter α is an space-time dependent function $\alpha = \alpha(x)$, the lagrangian changes as:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\alpha X(\phi)) + \frac{\partial\mathcal{L}}{\partial\phi}\alpha X(\phi) \tag{2-35}$$

$$= (\partial_\mu\alpha)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) + \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu X(\phi) + \frac{\partial\mathcal{L}}{\partial\phi}x(\phi)\right]\alpha. \tag{2-36}$$

When α is constant, $\delta\mathcal{L} = 0$, therefore the square brackets must vanish. The remaining expression is:

$$\delta\mathcal{L} = (\partial_\mu\alpha)J^\mu \quad \text{with} \quad J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi). \tag{2-37}$$

Therefore, the action changes as:

$$\delta S = \int d^dx\delta\mathcal{L} = \int d^dx(\partial_\mu\alpha)J^\mu = -\int d^dx\alpha\partial_\mu J^\mu, \tag{2-38}$$

which holds for any field configuration. Then, if the parameter $\alpha(x)$ decays asymptotically it is possible to ignore surface contributions. In addition, if the field ϕ satisfies the classical equations of motion, $\delta S = 0$ for any variation $\delta\phi$ including (2-33) with $\alpha = \alpha(x)$, resulting in the conservation law:

$$\partial_\mu J^\mu = 0. \tag{2-39}$$

Considering the QED lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2-40}$$

Then, it is possible to build the currents

$$\text{Vectorial} \qquad j_\mu = \bar{\psi}\gamma_\mu\psi, \tag{2-41}$$

$$\text{Axial} \qquad j_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi, \tag{2-42}$$

$$\text{Pseudoescalar} \qquad P = \bar{\psi}\gamma_5\psi, \tag{2-43}$$

from the equations of motion associated with Lagrangian (2-40) (Dirac's equations)

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{2-44}$$

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0, \tag{2-45}$$

It is possible to write the following conservation laws:

$$\begin{aligned}\partial^\mu j_\mu &= \bar{\psi}\overleftarrow{\partial\!\!\!/}\psi + \bar{\psi}\partial\!\!\!/\psi \\ &= i\bar{\psi}m\psi + i\bar{\psi}(-m)\psi \\ &= 0,\end{aligned} \tag{2-46}$$

$$\begin{aligned}\partial^\mu j_\mu^5 &= i\bar{\psi}m\gamma_5\psi - i\bar{\psi}\gamma_5(-m)\psi \\ &= 2imP,\end{aligned} \tag{2-47}$$

where the definition of currents and the property of anticonmutativity of Dirac matrices have been used:

$$\{\gamma_5, \gamma_\mu\} = 0.$$

Then, it can be observed that the vector current is conserved. The axial current is preserved for massless fermions:

$$\partial^\mu j_\mu^5 = 0 \qquad \text{for} \quad m = 0.$$

2.3. Ward Identities

The laws of conservation in QFT are generated by explicit relationships between Green's functions. These relations, which also include the functional generator, are known as *Ward identities* and allow an internal consistency of the theory. It is possible to study the origin of the conservation laws in QFT through the Euclidean path integral, where the euclidean time is defined as:

$$\tau = it. \tag{2-48}$$

This transformation is known as a Wick rotation. Then, the path integral can be written as:

$$Z[K] = \int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^dx K\phi\right), \tag{2-49}$$

where $K(x)$ is a background source for the fields ϕ [60]. Under the transformation (2-33) written as:

$$\phi \to \phi' = \phi + \alpha(x) X(\phi), \tag{2-50}$$

the partition function (2-49) transforms as:

$$Z[K] = \int \mathcal{D}\phi' \exp\left(-S[\phi'] + \int d^dx K\phi'\right). \tag{2-51}$$

Then, as the field is a dummy variable, the transformation leaves the same partition function. Using (2-38) and (2-50), we can rewrite:

$$\begin{aligned} Z[K] &= \int \mathcal{D}\phi' \exp\left(-S[\phi] + \int d^dx K\phi\right) \exp\left(-\int d^dx \alpha(x)(\partial_\mu J^\mu - KX)\right) \\ &\approx \int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^dx K\phi\right)\left[1 - \int d^dx \alpha(x)(\partial_\mu J^\mu - KX)\right], \end{aligned} \tag{2-52}$$

where we assume that the symmetry is conserved in the measure *i.e.* $\mathcal{D}\phi = \mathcal{D}\phi$ (but the reality is completely different (2.6)). The first term in the squared brackets are the original partition function, therefore we obtain:

$$\int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^dx K\phi\right)\left[\int d^dx \alpha(x)(\partial_\mu J^\mu - KX)\right] = 0. \tag{2-53}$$

Since $\alpha(x)$ is an arbitrary parameter, we can lose the integral and obtain an expression for each spacetime point:

$$\int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^dx K\phi\right)(\partial_\mu J^\mu - KX) = 0. \tag{2-54}$$

Setting $K = 0$, we get:

$$\langle \partial_\mu J^\mu \rangle = 0 \tag{2-55}$$

We can derive correlation functions between $\partial_\mu J^\mu$ and ϕ differentiating with respect to $K(x')$ before setting $K = 0$:

$$\partial_\mu \langle J^\mu(x)\phi(x')\rangle = \delta(x - x')\langle X(\phi)\rangle \tag{2-56}$$

If we differentiate more times, we get:

$$\partial_\mu \left\langle J^\mu(x)\phi(x^1)\ldots\phi(x^n)\right\rangle = 0 \quad \text{for} \quad x \neq x^i. \tag{2-57}$$

Then, if x matches one of the insertion points x^i, the expression (2-56) pick up a term proportional to $\delta\phi$ on the righr-hand side. these expressions are known as *Ward identities*. These identities remark the fact that $\partial_\mu J^\mu = 0$ inside any correlation function when its position does not coincide with the insertion point of the other fields.

Then, in general, for any set of arbitrary operators O^i the correlation function has the form:

$$\left\langle 0|TJ^\mu(x)O^1(y_1)\cdots O^n(y_n)|0\right\rangle . \tag{2-58}$$

Differentiating this expression is obtained

$$\begin{aligned}\partial_\mu^x \left\langle 0|TJ^\mu(x)O^1(y_1)\cdots O^n(y_n)|0\right\rangle &= \left\langle 0|T\partial_\mu^x j^\mu(x)O^1(y_1)\cdots O^n(y_n)|0\right\rangle \\ +\sum_{i=1}^{n}\left\langle 0|T\left[j^0(x),O^i(y_i)\right]\delta(x_0-y_{i0})O^1\cdots O^{i-1}O^{i+1}\cdots O^n|0\right\rangle ,\end{aligned} \tag{2-59}$$

where the commutator and the Dirac delta come from the derivatives associated with the step function Θ in the temporal ordering (then x_0, y_{i0} corresponds to the temporal coordiantes) and the super-index x in the partial derivative corresponds to the insertion point. Inserting the conservation laws for the vector current (2-46), we obtain the Ward identity for general operators (2-59). These relations associated with the Green's functions must be satisfied in order to guarantee the renormalizability of the theory [57],[61].

To see more clearly the result in a particular process that will be of vital importance in the analysis of the anomaly, we will study a certain type of one-loop Feynman diagrams known as "triangle diagrams". These kind of diagrams are special because they involve both $U(1)_V$ current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and the axial $U(1)_A$ current $j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$. The anomaly arise in the fact that even in the free theory these diagrams need to be regulated, but any process of regularization implies the violation either the $U(1)_V$ symmetry or the $U(1)_A$ symmetry: there is no way to preserve at the same time both symmetries. In order to build the necessary background to study these diagrams, let us study the $3-$point correlation function:

$$\tau^\mu(x,y,z) = \left\langle 0|Tj^\mu(z)\psi(x)\bar{\psi}(y)|0\right\rangle , \tag{2-60}$$

where j^μ corresponds to (2-41). Therefore, to obtain the Ward identities we require the commutator to equal times:

$$\begin{aligned}\left[j^0(z),\psi(x)\right]\delta(z_0-x_0) &= \left[\psi^\dagger(z)\psi(z),\psi(x)\right]\delta(z_0-x_0) \\ &= -\left\{\psi(x),\psi^\dagger(z)\right\}\psi(z)\delta(z_0-x_0) \\ &= -\psi(z)\delta^4(z-x).\end{aligned} \tag{2-61}$$

Further $\left[j^0(z),\bar{\psi}(y)\right]\delta(z_0-y_0) = \bar{\psi}(z)\delta^4(z-y)$. Now, imposing the current conservation condition (2-46), we get using (2-59):

$$\begin{aligned}\partial_\mu^z\tau^\mu(x,y,z) &= \partial_\mu^z\left\langle 0|Tj^\mu(z)\psi(x)\bar{\psi}(y)|0\right\rangle \\ &= \left\langle 0|T\partial_\mu^z j^\mu(z)\psi(x)\bar{\psi}(y)|0\right\rangle + \left\langle 0|T\left[j^0(z),\psi(x)\right]\delta(z_0-x_0)\bar{\psi}(y)|0\right\rangle \\ &+ \left\langle 0|T\psi(x)\left[j^0(z),\bar{\psi}(y)\right]\delta(z_0-y_0)|0\right\rangle .\end{aligned} \tag{2-62}$$

Using the commutators (2-61), we get:

$$\partial^z_\mu \tau^\mu(x,y,z) = -\left\langle 0|T\psi(z)\bar{\psi}(y)|0\right\rangle \delta^4(z-x) + \left\langle 0|T\psi(x)\bar{\psi}(z)|0\right\rangle \delta^4(z-y)$$
$$= -iS_F(z-y)\delta^4(z-x) + iS_F(x-z)\delta^4(z-y), \tag{2-63}$$

where S_F is the usual fermion propagator [60]. The Ward identity for the conserved vector current takes a particularly simple form in the moment space:

$$(p_\mu - p'_\mu)\tau^\mu(p,p') = S_F(p) - S_F(p'). \tag{2-64}$$

Taking only the contributions of the amputated diagrams (that is, keeping only the vertices):

$$-\frac{\tau^\mu(p,p')}{S_F(p)S_F(p')} \equiv \Gamma^\mu(p,p'). \tag{2-65}$$

which allows us to finally express the principle of conservation as the *Takahashi's identity* [57]:

$$(p_\mu - p'_\mu)\Gamma^\mu(p,p') = S_F^{-1}(p) - S_F^{-1}(p'). \tag{2-66}$$

Then, taking the limit when $p \to p'$:

$$\Gamma^\mu(p,p') = \frac{S_F^{-1}(p) - S_F^{-1}(p')}{p_\mu - p'_\mu} \overset{p' \to p}{\equiv} \frac{\Delta S_F{-1}}{\Delta p_\mu}, \tag{2-67}$$

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p_\mu} S_F^{-1}(p), \tag{2-68}$$

we get *Ward's identity.*

2.4. Triangle diagrams

We are interested in the 3−point correlator function involving two vector currents (2-41) and one axial current[1] (2-42):

$$\Gamma^{\mu\nu\rho}(x_1,x_2,x_3) = \langle 0|T(j^\mu(x_1)j^\nu(x_2)j^\rho_A(x_3)|0\rangle \tag{2-69}$$

where T denotes again time-ordering. It is much easier to work in the space of the moment as we saw before. The Fourier transformation is:

$$\int d^3x_1 d^3x_2 d^3x_3 \Gamma^{\mu\nu\rho}(x_1,x_2,x_3) e^{ip_1.x_1 + ip_2.x_2 + iq.x_3} = \Gamma^{\mu\nu\rho}(p_1,p_2,q)\delta^3(p_1+p_2+q) \tag{2-70}$$

where the delta function on the right-hand side indicate the fact that our theory is translational invariant. According to (2-69), the momenta p_1 and p_2 are related to the vector current

[1]The relevance of this function is that in the end, the anomaly equation includes an axial current j_A and two gauge fields, which couple to the vector currents j^μ

while q refers to the axial current. We naively expect from the classical behavior that the currents are conserved. Consider:

$$k_{1\mu}\Gamma^{\mu\nu\rho}(k_1,k_2,q) = -i\int d^3x_1 d^3x_2 d^3x_3 \Gamma^{\mu\nu\rho}(x_1,x_2,x_3)\frac{\partial}{\partial x_1^\mu}e^{ik_1.x_1+ik_2.x_2+iq.x_3} \tag{2-71}$$

$$= +i\int d^3x_1 d^3x_2 d^3x_3 \frac{\partial\Gamma^{\mu\nu\rho}(x_1,x_2,x_3)}{\partial x_1^\mu}e^{ik_1.x_1+ik_2.x_2+iq.x_3}. \tag{2-72}$$

The Ward identity (2-56) tell us that $\partial_\mu j^\mu = 0$. Then, using the fact that j^μ and j_A^μ does not transform under the symmetry (2-50), we obtain a really simple form for the Ward identity in the momentum space:

$$k_{1\nu}\Gamma^{\mu\nu\rho}(p_1,p_2,q) = 0, \tag{2-73}$$

and

$$k_{2\mu}\Gamma^{\mu\nu\rho}(p_1,p_2,q) = 0. \tag{2-74}$$

Using the same strategy for the conservation of the axial current (2-47) we find:

$$q_\rho\Gamma^{\mu\nu\rho}(k_1,k_2,q) = 0 \leftrightarrow -(k_{1\rho}+k_{2\rho})\Gamma^{\mu\nu\rho}(k_1,k_2,q) = 2m\Gamma^{\mu\nu}, \tag{2-75}$$

where the equivalence of these expression arise form the 4-momentum conservation $k_1+k_2+q=0$ and the expresion $\Gamma^{\mu\nu}$ is:

$$\Gamma^{\mu\nu}(x_1,x_2,x_3) = \langle 0|T(j^\mu(x_1)j^\nu(x_2)P(x_3)|0\rangle\,, \tag{2-76}$$

with P corresponding to the pseudoscalar current (2-43). The leading order contribution comes from one-loop triangle diagrams showed in Fig. (2.4). Following the usual Feynman

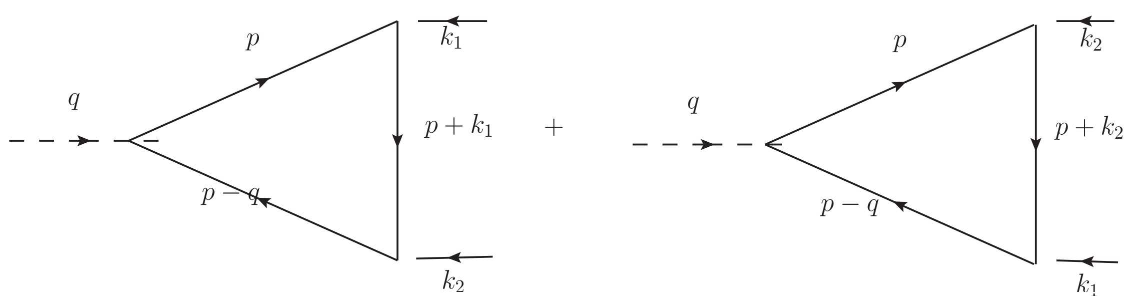


Fig. **2-2**: Triangle Diagrams

rules, we can write the amplitudes associated with the diagrams as in [57]:

$$-i\Gamma^{\mu\nu\rho} = -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,\frac{i}{\not{p}-m}\gamma^\rho\gamma^5\frac{i}{\not{p}-\not{q}-m}\gamma^\nu\frac{i}{\not{p}+\not{k}_1-m}\gamma^\mu + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu\end{pmatrix}, \tag{2-77}$$

$$-i\Gamma^{\mu\nu} = -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,\frac{i}{\not{p}-m}\gamma^5\frac{i}{\not{p}-\not{q}-m}\gamma^\nu\frac{i}{\not{p}+\not{k}_1-m}\gamma^\mu + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu\end{pmatrix}, \tag{2-78}$$

where has been taken again $k_1 + k_2 + q = 0$. In order to establish the Axial Ward Identity (2-75) , we use the relation:

$$\not{q}\gamma^5 = \gamma^5(\not{p} - \not{q} - m) + (\not{p} - m)\gamma^5 + 2m\gamma^5. \tag{2-79}$$

Replacing in (2-78) and (2-77) we obtain:

$$\begin{aligned} -iq_\rho\Gamma^{\mu\nu\rho}(k_1,k_2,q) &= i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not{p}-m}\not{q}\gamma^5\frac{1}{\not{p}-\not{q}-m}\gamma^\nu\frac{1}{\not{p}+\not{k}_1-m}\gamma^\mu\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{1}{\not{p}-m}\gamma^5 + \gamma^5\frac{1}{\not{p}-\not{q}-m}\right)\gamma^\nu\frac{1}{\not{p}+\not{k}_1-m}\gamma^\mu\right. \\ &\left. + \left(\frac{1}{\not{p}-m}\gamma^5 + \gamma^5\frac{1}{\not{p}-\not{q}-m}\right)\gamma^\mu\frac{1}{\not{p}+\not{k}_2-m}\gamma^\nu\right] + 2m\Gamma^{\mu\nu} \end{aligned} \tag{2-80}$$

We gather the terms before like:

$$q^\lambda T_{\mu\nu\lambda} = 2m\Gamma^{\mu\nu} + \Delta_1^{\mu\nu} + \Delta_2^{\mu\nu}, \tag{2-81}$$

where:

$$\begin{aligned} \Delta_1^{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma^5\gamma^\nu\frac{1}{\not{p}+\not{k}_1-m}\gamma^\mu + \gamma^5\frac{1}{\not{p}-\not{q}-m}\gamma^\mu\frac{1}{\not{p}+\not{k}_2-m}\gamma^\nu\right] \\ &= i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma^5\gamma^\nu\frac{1}{\not{p}+\not{k}_1-m}\gamma^\mu - \frac{1}{\not{p}+\not{k}_2-m}\gamma^5\gamma^\nu\frac{1}{\not{p}-\not{q}-m}\gamma^\mu\right] \end{aligned} \tag{2-82}$$

and

$$\begin{aligned} \Delta_2^{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4} tr\left[\gamma^5\frac{1}{\not{p}-\not{q}-m}\gamma^\nu\frac{1}{\not{p}+\not{k}_1-m}\gamma^\mu + \frac{1}{\not{p}-m}\gamma^5\gamma^\mu\frac{1}{\not{p}+\not{k}_2-m}\gamma^\nu\right] \\ &= i\int \frac{d^4p}{(2\pi)^4} tr\left[-\frac{1}{\not{p}+\not{k}_1-m}\gamma^5\gamma^\mu\frac{1}{\not{p}-\not{q}-m}\gamma^\nu + \frac{1}{\not{p}-m}\gamma^5\gamma^\mu\frac{1}{\not{p}+\not{k}_2-m}\gamma^\nu\right], \end{aligned} \tag{2-83}$$

where we have used the ciclicity of the trace and the commutator of the γ's matrices (2-61). Apparently, under a shift of the integration variables (*e.g.* $k \to p + k_2$ in the first term of $\Delta_1^{\mu\nu}$, $p \to p + k_1$ in $\Delta_2^{\mu\nu}$ and using the momentum conservation) we see that the two terms in each Δ-term cancel. But, looking more closely at the expressions, we see that they are linearly divergent, so it is not possible to make such a shift. That is to say, Ward's identities receive a net contribution from these rest-terms. It is possible to see that these differences between divergent integrals have the general form:

$$\tilde{\Delta} = i\int \frac{d^4p}{(2\pi)^4}\left[f(p) - f(p+a)\right], \tag{2-84}$$

where each integral is linearly divergent. Using a Taylor expansion for small a, we get:

$$\tilde{\Delta} = -i\int \frac{d^4p}{(2\pi)^4}\left[a^\mu\partial_{p^\mu}f + \frac{1}{2}a^\mu a^\nu\partial_{p^\mu}\partial_{p^\nu}f + \dots\right], \tag{2-85}$$

but, each term in the expansion is less and less divergent. Then, we need to keep only the first of these terms because our integral is linearly divergent:

$$\tilde{\Delta} = -\int_{\mathbf{S}^3_\infty} \frac{d\hat{p}_\mu}{(2\pi)^4} a^\mu |p|^3 f(p), \tag{2-86}$$

where we consider that, in fact, each term in the integral is a boundary term so the integral is taken over the boundary $\mathbb{S}^3$ at $|p| \to \infty$. To see what is the contribution of this new surface term associated with divergent integrals, we have to modify the triangular diagrams as shown in Figure (2.4). Then, the final answer with the surface contributions depends on

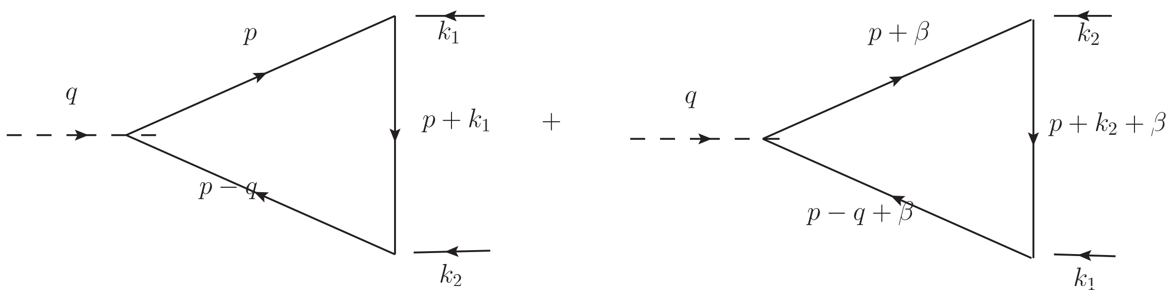


Fig. **2-3**: Modified Triangle Diagrams

this new arbitrary parameter β that will allow us to solve the apparent ambiguity. Then, the Axial Ward identity take the form:

$$-iq_\rho \Gamma^{\mu\nu\rho}(k_1, k_2, q) = 2m\Gamma^{\mu\nu} + \tilde{\Delta}_1^{\mu\nu} + \tilde{\Delta}_2^{\mu\nu} \tag{2-87}$$

where

$$\tilde{\Delta}_1^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[\frac{1}{\not{p} - m}\gamma^5\gamma^\nu \frac{1}{\not{p} + \not{k}_1 - m}\gamma^\mu - \frac{1}{\not{p} + \not{\beta} + \not{k}_2 - m}\gamma^5\gamma^\nu \frac{1}{\not{p} + \not{\beta} - \not{q} - m}\gamma^\mu\right] \tag{2-88}$$

and

$$\tilde{\Delta}_2^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4} tr\left[-\frac{1}{\not{p} + \not{k}_1 - m}\gamma^5\gamma^\mu \frac{1}{\not{p} - \not{q} - m}\gamma^\nu + \frac{1}{\not{p} + \not{\beta}}\gamma^5\gamma^\mu \frac{1}{\not{p} + \not{\beta} + \not{k}_2 - m}\gamma^\nu\right], \tag{2-89}$$

where each of these contributions has the form (2-84). For $\tilde{\Delta}_1^{\mu\nu}$, we have the difference of two divergent integrals with integrand:

$$\begin{aligned} f^{\mu\nu}(p) &= \operatorname{Tr}\left[\frac{1}{\not{p} - m}\gamma^5\gamma^\nu \frac{1}{\not{p} + \not{k}_1 - m}\gamma^\mu\right] \\ &= \frac{\operatorname{Tr}\left[(\not{p} - m)\gamma^5\gamma^\nu(\not{p} + \not{k}_1 - m)\gamma^\mu\right]}{(p - m)^2(p + k_1 - m)^2}. \end{aligned} \tag{2-90}$$

Using the gamma matrix identity [61]:

$$\mathrm{Tr}\left(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^5\right) = -4i\epsilon^{\nu\rho\mu\sigma}, \tag{2-91}$$

to write:

$$f^{\mu\nu}(p) = -4i\epsilon^{\nu\rho\mu\sigma}\frac{(p+k_1)^\rho p^\sigma}{p^2(p+k_1)^2} = -4i\epsilon^{\nu\rho\mu\sigma}\frac{k_1^\rho p^\sigma}{p^2(p+k_1)^2}, \tag{2-92}$$

where the anti-symmetry of the epsilon tensor was used in order to cancel the $p^\rho p^\sigma$ term. For $\tilde{\Delta}_1^{\mu\nu}$ the off-set is given by $a = \beta + k_2$, then:

$$\tilde{\Delta}_1^{\mu\nu} = -4\int_{\mathbf{S}^3_\infty}\frac{d\hat{p}^\lambda}{(2\pi)^4}\epsilon^{\nu\rho\mu\sigma}(\beta+k_2)_\lambda k_{1\rho}p_\sigma\frac{|p|^3}{p^2(p+k_1)^2}. \tag{2-93}$$

Using the integration formula:

$$\int_{S^3} d\hat{p}^\lambda p^\sigma = \frac{1}{4}\delta^{\lambda\sigma}\mathrm{Vol}(\mathbf{S}^3), \quad \mathrm{Vol}(\mathbf{S}^3) = 2\pi^2, \tag{2-94}$$

we find:

$$\tilde{\Delta}_1^{\mu\nu} = -\frac{1}{8\pi^2}\epsilon^{\nu\rho\mu\sigma}k_{1\rho}(\beta+k_2)_\sigma \tag{2-95}$$

We can go over the same steps to evaluate $\tilde{\Delta}_2^{\mu\nu}$ in (2-89) with the off-set $a = k_1 - \beta$. Then we obtain:

$$\tilde{\Delta}_2^{\mu\nu} = +\frac{1}{8\pi^2}\epsilon^{\nu\rho\mu\sigma}k_{2\rho}(k_1-\beta)_\sigma \tag{2-96}$$

Therefore, the Axial Ward identity has the form:

$$-iq_\rho\Gamma^{\mu\nu\rho}(k_1,k_2,q) = 2m\Gamma^{\mu\nu} - \frac{1}{8\pi^2}\epsilon^{\nu\rho\mu\sigma}\left[2k_{1\rho}k_{2\sigma} + (k_1+k_2)_\rho\beta_\sigma\right] \tag{2-97}$$

Under the inclusion of β and making the same procedure as with the Axial Ward identity, we realize that the Vector Ward identities (2-73, 2-74) have the form:

$$\begin{aligned} -ik_{1\mu}\Gamma^{\mu\nu\rho} &= \frac{1}{8\pi^2}\epsilon^{\nu\rho\mu\sigma}k_{1\mu}(\beta-k_2)_\sigma, \\ -ik_{2\nu}\Gamma^{\mu\nu\rho} &= \frac{1}{8\pi^2}\epsilon^{\rho\mu\nu\sigma}k_{2\nu}(\beta+k_1)_\sigma. \end{aligned} \tag{2-98}$$

Therefore, all the three Ward identities depend on the arbitrary 4-momentum β. Then, it is possible to fix the β-value insisting that the vector current survives quantization. Our choice of β must be such that the two vector Ward identities are non-anomalous. For this, we must have:

$$\beta - k_2 \sim -k_1 \quad \text{and} \quad \beta + k_1 \sim k_2 \quad \Rightarrow \quad \beta = k_2 - k_1. \tag{2-99}$$

With this choice

$$-ik_{1\mu}\Gamma^{\mu\nu\rho} = -ik_{2\mu}\Gamma^{\mu\nu\rho} = 0 \tag{2-100}$$

and the Axial Ward identity (2-97) becomes:

$$-iq_\rho\Gamma^{\mu\nu\rho} = 2m\Gamma^{\mu\nu} - \frac{1}{2\pi^2}\epsilon^{\nu\rho\mu\sigma}k_{1\rho}k_{2\sigma}, \tag{2-101}$$

turns out to be the anomaly.

2.5. Pauli-Villars Regularization

It is possible to understand the emergence of the anomaly through the regularization of the triangular diagrams analyzed in the previous section. In this case we will use Pauli-Villars (PV) regularization process. We closely follow the development of [57] The amplitude of regularized PV is the difference between the amplitude generated by the diagrams and the amplitude evaluated at a regulator mass M:

$$\Gamma^{\mu\nu\lambda}_{\text{reg}} = \Gamma^{\mu\nu\lambda}(m) - \Gamma^{\mu\nu\lambda}(M). \tag{2-102}$$

The physical amplitude follows from take $M \to \infty$ in the regularized amplitude:

$$\Gamma^{\mu\nu\lambda}_{\text{phys}} = \lim_{M\to\infty} \Gamma^{\mu\nu\lambda}_{\text{reg}}. \tag{2-103}$$

In the case of $\Gamma_{\mu\nu}$ amplitude (2-78), we have:

$$\Gamma^{\mu\nu}_{\text{phys}} = \lim_{M\to\infty} \Gamma^{\mu\nu\lambda}_{\text{reg}} = \lim_{M\to\infty} \left[\Gamma^{\mu\nu}(m) - \Gamma^{\mu\nu}(M)\right] = \Gamma^{\mu\nu}(m), \tag{2-104}$$

since $\Gamma^{\mu\nu} \sim \frac{1}{M}$ in the replacement $m \to M$ in (2-78). Then, $\Gamma^{\mu\nu}$ is convergent and does not require regularization. The vectorial Ward identities (2-98) are automatically fulfilled because there are not an explicit dependence of the mass term, thus:

$$\Gamma^{\mu\nu\lambda}(m) = \Gamma^{\mu\nu\lambda}(M) \quad \to \quad \begin{aligned} T^{\text{reg}}_{\mu\nu\lambda} &= 0, \\ T^{\text{phys}}_{\mu\nu\lambda} &= 0, \end{aligned}$$

then:

$$-ik_{1\mu}\Gamma^{\mu\nu\rho} = -ik_{2\mu}\Gamma^{\mu\nu\rho} = 0. \tag{2-105}$$

For the Axial Ward identity we have:

$$q_\rho \Gamma^{\mu\nu\rho}_{\text{phys}} = \lim_{M\to\infty} q_\rho \Gamma^{\mu\nu\rho}_{\text{reg}} = 2m\Gamma^{\mu\nu}(m) - \lim_{M\to\infty} 2M\Gamma^{\mu\nu}(M). \tag{2-106}$$

It is possible to prove that the anomaly is generated in the limit:

$$\lim_{M\to\infty} 2M\Gamma^{\mu\nu}(M) = -\mathcal{A}_{\mu\nu}. \tag{2-107}$$

For develop this, we rewrite here the amplitude $\Gamma^{\mu\nu}$ (2-78):

$$-i\Gamma^{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4} \operatorname{Tr} \frac{i}{\not{p} - m}\gamma^5 \frac{i}{\not{p} - \not{q} - m}\gamma^\nu \frac{i}{\not{p} + \not{k}_1 - m}\gamma^\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}. \tag{2-108}$$

For the denominator we introduce the Feynman integral [57]:

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\left[a_1 x_2 + a_2(1 - x_1 - x_2) + a_3 x_1\right]^3}, \tag{2-109}$$

thus:

$$\Gamma^{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times$$
$$\times \frac{\mathrm{Tr}(\not{p}+m)\gamma^5(\not{p}-\not{q}+m)\gamma^\nu(\not{p}+\not{k}_1+m)\gamma_\mu}{[(p^2-m^2)x^2+[(p-q)^2-m^2](1-x_1-x_2)+[(p+k_1)^2-m^2]x_1]^3} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}. \tag{2-110}$$

Recalling that axial traces (involving γ^5) with 1, 2, 3, or 5 matrices γ^μ are canceled and replacing the identity:

$$\epsilon^{\mu\nu\alpha\beta} p^\alpha p^\beta = 0, \tag{2-111}$$

only remains the term:

$$m \,\mathrm{Tr}\, \gamma^5 \not{q} \gamma^\nu \not{k}_1 \gamma^\mu = 4im\epsilon^{\beta\nu\alpha\mu} k_{2\beta} k_{1\alpha} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}. \tag{2-112}$$

Therefore:

$$\Gamma^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{2m4i\epsilon^{\beta\nu\alpha\mu} k_{2\beta} k_{1\alpha}}{[p^2-2pk-\bar{m}^2]^3}, \tag{2-113}$$

with:

$$k = q(1-x_1-x_2) + k_1 x_1, \tag{2-114}$$
$$\bar{m}^2 = m^2 - q^2(1-x_1-x_2). \tag{2-115}$$

In this expression are already included the terms associated with the exchange $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Taking the 't Hooft-Veltmann integration formula:

$$\int \frac{d^np}{(p^2-2pk-\bar{m}^2)^\alpha} = i^{1-2\alpha}\pi^{n/2} \frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} \frac{1}{(k^2-\bar{m}^2)^{\alpha-n/2}} \equiv J_0. \tag{2-116}$$

In our case $\alpha = 3$ y $n = 4$, so we obtain:

$$J_0 = \frac{\pi^2}{2i} \frac{1}{m^2 + f(x_1,x_2)}, \tag{2-117}$$

where $f(x_1, x_2)$ does not depend of m. In the large masses limit:

$$\lim_{M\to\infty} J_0(M) = \frac{\pi^2}{2i} \lim_{M\to\infty} \frac{1}{M^2}. \tag{2-118}$$

Replacing we get:

$$\begin{aligned}\lim_{M\to\infty} 2M\Gamma^{\mu\nu}(M) &= \lim_{M\to\infty} \frac{1}{(2\pi)^4} \frac{\pi^2}{2i} \frac{1}{M^2} 2M2M4i\varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\ &= \frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} = -\mathcal{A}_{\mu\nu},\end{aligned} \tag{2-119}$$

which coincides with (2-101) in te limit $m \to 0$. In the coordinate space the anomalous divergence of the axial current is expressed as:

$$\begin{aligned}\left\langle 0|\partial\mu j^{\mu}_{5reg}(x)|0\right\rangle &= 2im\left\langle 0|P(x)|0\right\rangle + \mathcal{A}(x)\\ \to \partial_{\mu} j^{\mu}_{5reg}(x) &= 2imP(x) + \mathcal{A}(x),\end{aligned} \tag{2-120}$$

with

$$\mathcal{A}(x) = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x). \tag{2-121}$$

2.6. Non-perturbative approximation to the anomaly

Another way to understand the chiral anomaly comes from analyzing the law of conservation of the axial current from the functional integral for the fermionic field. Starting from the functional generator:

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left[i\int d^4x\bar{\psi}(i\not{D})\psi\right]. \tag{2-122}$$

Performing the chiral transformation:

$$\psi(x) \to \psi'(x) = (1+\alpha(x)\gamma^5)\psi(x), \tag{2-123}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)(1+\alpha(x)\gamma^5). \tag{2-124}$$

Then:

$$\int d^4x\bar{\psi}'(i\not{D})\psi' = \int d^4x\left[\bar{\psi}(i\not{D}\psi - \partial_{\mu}\alpha(x)\bar{\psi}\gamma^{\mu}\gamma^5\psi\right] = \int d^4x\left[\bar{\psi}(i\not{D})\psi + \alpha(x)\partial_{\mu}(\bar{\psi}\gamma^{\mu}\gamma^5\psi)\right], \tag{2-125}$$

(where part integration has been applied to obtain the last equality). Then, varying the Lagrangian with respect to $\alpha(x)$ we derive the classical conservation law for the axial current:

$$dS = \int \alpha(x)\partial_{\mu}(\bar{\psi}\gamma^{\mu}\gamma^5\psi) = 0 \tag{2-126}$$
$$\to \partial_{\mu}(\bar{\psi}\gamma^{\mu}\gamma^5\psi) = 0, \tag{2-127}$$

since α is an arbitrary parameter of the transformation. It is clear that to obtain this result, we assume that the measure is conservative under the change $\psi \to \psi'$, but it is possible to verify that the measure is not invariant under axial transformations. To test this, let's expand the fermion fields in an eigenstate base of $\not{D}$:

$$\psi(x) = \sum_n a_n\varphi_n(x) = \sum_n a_n\left\langle x|n\right\rangle, \tag{2-128}$$
$$\bar{\psi}(x) = \sum_m \varphi^{\dagger}_m\bar{b}_m = \sum_m \left\langle m|x\right\rangle\bar{b}_m. \tag{2-129}$$

The coefficients $a_n, \bar{b}_m$ are independent Grassmann variables. In addition, the Dirac operator $\not{D}$ accomplish the eigenvalue equation:

$$\not{D}\varphi_n(x) = \lambda_n \varphi_n(x) \qquad \lambda_n \in \mathbb{R}. \tag{2-130}$$

Since the eigenfunctions are orthonormal, it is possible to show that:

$$\varphi_m'(x)\sum_n a_n'\varphi_n(x) = \varphi_m(x)\sum_n a_m\varphi_m(x) + \varphi_m(x)\left(i\alpha(x)\gamma_5\right)\sum_m a_m\varphi_m(x). \tag{2-131}$$

Integrating we get:

$$a_m' = a_m + \sum_n i\int d^4x\varphi_m^\dagger(x)\alpha(x)\gamma_5\varphi_m(x)a_n. \tag{2-132}$$

Then, it is possible to express the independent Grassmann variables as:

$$a_n' = \sum_n C_{nm}a_m, \tag{2-133}$$

where:

$$C_{nm} = \delta_{nm} + i\int dx\beta(x)\varphi_n^\dagger(x)\gamma_5\varphi_m(x), \tag{2-134}$$

and analogously for the rotated spinor $\bar{\psi}$.

$$\bar{b}_m' = \sum_n C_{nm}\bar{b}_n. \tag{2-135}$$

Now, taking into account the transformation of the Grassmann variables, we obtain for the axial transformation:

$$\prod_n da_n' = (\det C)^{-1}\prod_n da_n, \tag{2-136}$$

$$\prod_m d\bar{b}_m' = (detC)^{-1}\prod_m d\bar{b}_m, \tag{2-137}$$

it is possible to express the transformation of the functional measure in the path integral as:

$$d\psi' d\bar{\psi}' = (\det C)^{-2}\, d\psi d\bar{\psi} = J[\alpha]d\psi d\bar{\psi}, \tag{2-138}$$

where $J[\alpha]$ represents the Jacobian of the transformation. It is possible to rewrite this Jacobian as:

$$J[\alpha] = (\det C)^{-2} = \exp\left[-2\,\mathrm{Tr}\ln C\right], \tag{2-139}$$

where $\det C = \exp \mathrm{Tr}\ln C$ has been used. Replacing the value of C_{mn} and using the first-order approximation for the logarithm $\ln x \approx x$ we have:

$$\begin{aligned} J[\alpha] &= \exp\left[-2\,\mathrm{Tr}\ln\left(\delta_{mn} + i\int dx\alpha(x)\varphi_n^\dagger(x)\gamma_5\varphi_m(x)\right)\right] \\ &= \exp\left[-2\,\mathrm{Tr}\, i\int dx\alpha(x)\varphi_n(x)^\dagger\gamma_5\varphi_m(x)\right] \\ &= \exp\left[-2i\int dx\alpha(x)\sum_n\varphi_n^\dagger(x)\gamma_5\varphi_n(x)\right]. \end{aligned} \tag{2-140}$$

Although at this point it would seem that the problem of the transformation had been solved, the sum within the exponential is not well defined, since it would give something similar to:

$$\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) \approx \mathrm{Tr}\,\gamma_5 \cdot \delta(0), \tag{2-141}$$

thus, it is necessary to regularize it. Fujikawa [62] proposed a regularization based on a Gaussian cut-off that allows reducing the large eigenvalues contributions. This cut-off has the form:

$$\begin{aligned}\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) &\to \lim_{M\to\infty}\sum_n \varphi_n^\dagger(x)\exp\left[-\frac{\not{D}^2}{m^2}\right]\gamma_5\varphi_n(x)\\ &= \lim_{M\to\infty}\sum_n \varphi_n^\dagger(x)\exp\left[-\frac{\lambda^2}{m^2}\right]\gamma_5\varphi_n(x),\end{aligned} \tag{2-142}$$

$$\tag{2-143}$$

where the limit over M allow us to regularize the sum. Introducing the Fourier components:

$$\varphi_n(x) = \int \frac{d^4x}{(2\pi)^2} e^{ikx}\tilde{\varphi}_n(k). \tag{2-144}$$

Using again the completeness of the eigenfunctions:

$$\sum_n \tilde{\varphi}_n^\dagger(l)\Gamma\tilde{\varphi}_n(k) = \mathrm{Tr}\,\Gamma\delta(l-k), \tag{2-145}$$

and integrating over the momentum l, we have

$$\begin{aligned}\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) &= \lim_{M\to\infty}\int \frac{d^4l\,d^4k}{(2\pi)^4}\sum_n \tilde{\varphi}_n^\dagger(l)e^{-ilx}\gamma_5\exp\left[-\frac{\not{D}^2}{M^2}\right]e^{ikx}\tilde{\varphi}_n(k)\\ &= \lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,e^{-ikx}\gamma_5\exp\left[-\frac{\not{D}^2}{M^2}\right]e^{ikx}.\end{aligned} \tag{2-146}$$

The trace is taken over the Dirac matrices and the group generators T^a. Now, we decompose the Dirac operator:

$$\begin{aligned}\not{D}^2 &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu D_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu]D_\mu D_\nu\\ &= D_\mu D^\mu - \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu},\end{aligned} \tag{2-147}$$

moving e^{-ikx} through the differential operator:

$$e^{-ikx}f(\partial_\mu)e^{ikx} = f(\partial_\mu + ik_\mu), \tag{2-148}$$

and rescaling the momentum like $K_\mu \to Mk_\mu$, we have:

$$\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = \lim_{M\to\infty}\int \frac{d^4x}{(2\pi)^4}\operatorname{Tr}\gamma_5 \exp\left[-\frac{(D_\mu + ik_\mu)(D^\mu + ik^\mu)}{M^2} - \frac{ig\gamma^\mu\gamma^\nu F_{\mu\nu}}{M^2}\right]$$
$$= \lim_{M\to\infty} M^4 \int \frac{d^4x}{(2\pi)^4} e^{k^\mu k_\mu} \operatorname{Tr}\gamma_5 \exp\left[-\frac{2ik-\mu D^\mu}{M} - \frac{D_\mu D^\mu}{M^2} - \frac{ig\gamma^\mu\gamma^\nu F_{\mu\nu}}{M^2}\right]. \tag{2-149}$$

Under the trace properties of the Dirac matrices:

$$\operatorname{Tr}\gamma_5 = \operatorname{Tr}\gamma_5\gamma^\mu\gamma^\nu = 0, \tag{2-150}$$
$$\operatorname{Tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = -4\varepsilon^{\mu\nu\alpha\beta}. \tag{2-151}$$

Then, expanding the exponential function, the only contribution different from zero is the $F_{\mu\nu}$ term on the second order. So:

$$\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = \lim_{M\to\infty}\frac{(ig)^2}{2!}\frac{M^4}{4M^4}\int \frac{d^4x}{(2\pi)^4} e^{-k_\mu k_\mu}\operatorname{Tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta F_{\mu\nu}F_{\alpha\beta}, \tag{2-152}$$

where $k_\mu k^\mu = -k_\mu k_\mu$ under the usual euclidean conventions. Thus, replacing the Gaussian integral, we have:

$$\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = \frac{1}{8}\frac{g^2}{16\pi^2}(4\varepsilon^{\mu\nu\alpha\beta})\operatorname{Tr}[F_{\mu\nu}F_{\alpha\beta}]$$
$$= \frac{g^2}{32\pi^2}\operatorname{Tr}[\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]. \tag{2-153}$$

Finally, the jacobian for the chiral transformation is:

$$J = \exp\left[-2i\int d^4x\alpha(x)\frac{g^2}{32\pi^2}\operatorname{Tr}[\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right]. \tag{2-154}$$

The axial ward identity could be written as (without sources):

$$\partial_\mu\left\langle\bar{\psi}\gamma^\mu\gamma_5\psi\right\rangle = \left\langle 2im\bar{\psi}\gamma_5\psi + 2i\frac{g^2}{32\pi^2}\operatorname{Tr}[\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right\rangle. \tag{2-155}$$

The concepts addressed in this section should be sufficient to be able to study some of the problems present in the QFT, associated with the impossibility of canceling anomalous terms within the theory, both in its abelian and non-abelian form. This will allow us to recognize the strong CP-problem and its possible solutions.Then, the final answer with the surface contributions depends on this new arbitrary parameter β that will allow us to solve the apparent ambiguity

3 Strong CP-problem + $U(1)_A$ «symmetry» and missing meson problem

3.1. QCD introduction

Quantum chromodynamics (QCD) is a gauge theory that is invariant under the $SU(3)$ group of symmetry. In this theory every quark field comes in three different colors (red, green and blue). The lagrangian density for such a theory with different fermions can be written as:

$$\mathcal{L}_{\text{Global}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi, \qquad \psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}. \tag{3-1}$$

In this particular case, the unitary transformation has to be a 3×3 matrix. This lagrangian is invariant under global transformation, but it is not the situation under a local one. To build the correct form of the lagrangian we must take into account the structure of the transformation: $SU(3)$ is non - Abelian group defined by 3×3 matrices with determinant 1. The vector of the 3 quark colors transforms under the fundamental representation of the group conformed by this matrices. This representation can be parameterized by 8 real numbers, $\zeta_a, a = 1, \cdots, 8$. The local transformation can be specified by 8 real fields:

$$\psi \to U(x)\psi, \qquad U(x) = \exp\left(i\zeta_a \frac{\lambda^a}{2}\right), \tag{3-2}$$

where $T^a = \lambda^a/2$ are the generators of the group. This generators satisfied the algebra:

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f^{abc} \frac{\lambda^c}{2}, \tag{3-3}$$

where f_{abc} are the structure constants of $SU(3)$. But the global lagrangian (3-1) is not symmetric under local $SU(3)$ transformations:

$$\mathcal{L}_{\text{Global}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \to \quad \mathcal{L}' = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + i\bar{\psi}U^\dagger(x)\left(\gamma^\mu \partial_\mu U(x)\right)\psi. \tag{3-4}$$

The problem is the derivative does not transform in the same way as the field vector. We can construct a covariant derivative:

$$D_\mu = \partial_\mu - igT^a A^a_\mu \equiv \partial_\mu - igA_\mu, \tag{3-5}$$

where we define the 3×3 matrix as:

$$(A_\mu)_{\alpha\beta} = \left(\frac{\lambda^a}{2}\right)_{\alpha\beta} A^a_\mu. \tag{3-6}$$

The construction of a covariant derivative implies that under the $U(x)$ transformation:

$$D_\mu\psi \to (D_\mu\psi)' = U(x)D_\mu\psi. \tag{3-7}$$

Then, using the properties of $U(x)$, we have:

$$D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' = U(x)D_\mu\psi \tag{3-8}$$
$$D'_\mu U\psi = UD_\mu\psi \tag{3-9}$$
$$\to D'_\mu U = UD_\mu. \tag{3-10}$$

Therefore, the transformation rule for a covariant derivative is:

$$D_\mu \to (D_\mu)' = UD_\mu U^\dagger \tag{3-11}$$

Replacing the expression for the covariant derivative we can see easily the transformation law for the gauge field A_μ:

$$A^a_\mu T^a = A_\mu \to A'_\mu = U(x)\left(A^a_\mu T^a + \frac{i}{g}\partial_\mu\right)U^\dagger(x). \tag{3-12}$$

Thus, the covariant derivative transforms as:

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' &= (\partial_\mu - igA'_\mu)\psi' \\ &= \left[\partial_\mu - ig\left(U(x)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger(x)\right)\right]U(x)\psi \\ &= \partial_\mu\left(U(x)\psi\right) - igU(x)A_\mu\psi + U(x)\left(\partial_\mu U^\dagger(x)\right)U(x)\psi \\ &= U(x)\left(\partial_\mu - igA_\mu\right)\psi + \left[\partial_\mu U(x) + U(x)(\partial_\mu U^\dagger(x))U(x)\right]\psi \\ &= U(x)\left(\partial_\mu - igA_\mu\right)\psi, \end{aligned} \tag{3-13}$$

where in the last line we use the unitarity of $U(x)$:

$$\begin{aligned} \partial_\mu(U(x)U^\dagger(x)) &= \left(\partial_\mu U(x)\right)U^\dagger(x) + U(x)\left(\partial_\mu U^\dagger(x)\right) = 0 \\ &\to \partial_\mu U(x) = -U(x)\left(\partial_\mu U^\dagger(x)\right)U(x), \end{aligned}$$

that coincides with the way that the field vector transforms. Replacing the covariant derivative in the global lagrangian we obtain an invariant lagrangian under $SU(3)$ transformation. But, we can add more terms to the new lagrangian. If we maintain the renormalization criteria (not involving terms of order higher than four) and the Lorentz invariance, we can add combinations of the covariant derivative and fermion fields that also be invariant under

$SU(3)$. Our first attempt to mix terms might be taking the commutator of the covariant derivative. We define the 3×3 matrix $F^{\mu\nu}$ as:

$$F_{\mu\nu} = -\frac{i}{g}\left[D_\mu, D_\nu\right] \equiv T^a F^a_{\mu\nu}. \tag{3-14}$$

Then, the structure of the $F_{\mu\nu}$ matrix is:

$$\begin{aligned}
F_{\mu\nu}\psi &= -\frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right]\psi \\
&= -\frac{i}{g}\left[\left(\partial_\mu - igA_\mu\right)\left(\partial_\nu - igA_\nu\right)\psi - \left(\partial_\nu - igA_\nu\right)\left(\partial_\mu - igA_\mu\right)\psi\right] \\
&= -\frac{i}{g}\left\{\partial_\mu\partial_\nu\psi - g^2 A_\mu A_\nu\psi - ig\left[\partial_\mu\left(A_\nu\psi\right) + A_\mu\partial_\nu\psi\right] - \partial_\nu\partial_\mu\psi + g^2 A_\nu A_\mu\psi + \right. \\
&\qquad \left. + ig\left[\partial_\nu(A^\mu\psi) + A_\nu\partial_\mu\psi\right]\right\} \\
&= \left\{\partial_\mu A_\nu - \partial_\nu A_\mu - ig\left(A_\mu A_\nu - A_\nu A_\mu\right)\right\}\psi.
\end{aligned} \tag{3-15}$$

Thus, we can write $F_{\mu\nu}$ as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{3-16}$$

From (3-14), we easily see the way in which way $F_{\mu\nu}$ transforms. Taking the transformation of the covariant derivative (3-11):

$$\begin{aligned}
F_{\mu\nu} \to & -\frac{i}{g}[D'_\mu, D'_\nu] \\
&= -\frac{i}{g}[UD_\mu U^\dagger, UD_\nu U_\dagger] \\
&= UF_{\mu\nu}U^\dagger.
\end{aligned} \tag{3-17}$$

Therefore we have two options to build our Lorentz invariant quantity contracting the indices. One way is to take the trace of the product of two strength matrices:

$$\begin{aligned}
\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \to \mathrm{Tr}(F'^{\mu\nu}F'_{\mu\nu}) &= \mathrm{Tr}\left(UF^{\mu\nu}F_{\mu\nu}U^\dagger UF_{\mu\nu}U^\dagger\right) \\
&= \mathrm{Tr}\left(UF^{\mu\nu}F_{\mu\nu}U^\dagger\right) \\
&= \mathrm{Tr}\left(U^\dagger UF^{\mu\nu}F_{\mu\nu}\right) \\
&= \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right),
\end{aligned} \tag{3-18}$$

which turns out to be a scalar. The other option is built taking the trace over contraction with a Levi-Civita tensor resulting in a pseudoscalar term $\mathrm{Tr}\left(\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)$. This pseudoscalar can be written as a full derivative:

$$\partial_\mu K^\mu_{CS} = \mathrm{Tr}\left(\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right), \tag{3-19}$$

where $K^{\mu}_{CS} = 2\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma]$ is known as the Chern-Simons term. Therefore, we can write down in the euclidean space, using the Gauss theorem in 4−d:

$$\int d^4x\,\mathrm{Tr}[\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}] = \int d^4x\partial_\mu K^{\mu}_{CS} = \int_{S^3} dA_\mu K^{\mu}_{CS}, \tag{3-20}$$

where S^3 is the 3−sphere in the euclidean space. Therefore, if $K^\mu(x) \to 0$ fast enough as $x \to \infty$ the integral vanish and the contribution associated to this term could be left out of the lagrangian. Under this structure, seems very reasonable to assume that the A_μ potentials vanish at infinity. In that way, every infinite integral related to gauge field configurations does not contribute to the path integral e.g. the term $F_{\mu\nu}F^{\mu\nu}$ in the two-point-two function contributes like:

$$\lim_{T\to\infty(1-i\epsilon)} \exp\left[-i\int_{-T}^{T} d^4x\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\right]. \tag{3-21}$$

After Wick rotation, the domain of integration belong to the euclidean space. It is possible to write this term in the form:

$$\exp\left[-\int d^4x\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\right]. \tag{3-22}$$

Then, the contribution of divergent terms to the path integral is zero, so, we are only interested in field configurations that result in finite contributions to the path integral. Thus, under the naive assumption that field configurations on the form (3-19) do not generate any finite contribution to the path integral (but as we will see later, this condition is not true), the most general lagrangian that we could write under the analyzed conditions has the form:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}\mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}]. \tag{3-23}$$

3.1.1. QCD symmetries

The QCD lagrangian density for N quark flavors can be written in a very compact form like:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{\psi}\left(i\not{D} - \mathcal{M}\right)\psi, \tag{3-24}$$

where, in the most general case, the mass matrix M is flavour-diagonal with complex entries:

$$\mathcal{M} = \begin{pmatrix} m_u e^{i\theta_\lambda} & 0 \\ 0 & m_d e^{i\theta_\lambda} \end{pmatrix}. \tag{3-25}$$

ψ_j are the quark fields and $F^a_{\mu\nu}$ is the gluon field-strength tensor. The term $\not{D} = \gamma^\mu D_\mu$ represents the covariant derivative and transform in the same way as the field vector itself:

$$D_\mu = \partial_\mu - igA^a_\mu T^a.$$

We can define new conserved currents in function of left- and right - handed quark fields:

$$\psi_R = P_R\psi, \qquad \psi_L = P_L\psi, \tag{3-26}$$

with

$$P_L = \frac{1}{2}(1-\gamma_5), \qquad P_R = \frac{1}{2}(1+\gamma_5). \tag{3-27}$$

The lagrangian density can be rewritten in the massless approximation as:

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} = \bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_L i \not{D} \psi_L - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}. \tag{3-28}$$

Restricting our analysis only to two flavors $N = 2$, we see that the QCD lagrangian is invariant under the global unitary transformation $U(2)_L \otimes U(2)_R$, where $U(N)$ is the unitary group associated to the $N \times N$ unitary squared matrices. Each Dirac spinor with different chirality transform in a different way:

$$\psi_{L\alpha i} \rightarrow U^k_{Li} \psi_{L\alpha k} \tag{3-29}$$

$$\bar{\psi}^{R\alpha j} \rightarrow U^{*j}_{Rm} \bar{\psi}^{R\alpha m} \tag{3-30}$$

where α's are spinor indices and i, j are flavor indices. Another set of symmetries in the massless approximation are:

$$\psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad SU(2)_R : \quad \psi_R \rightarrow \psi'_R = e^{-i\Theta^a_R T_a} \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{3-31}$$

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad SU(2)_L : \quad \psi_L \rightarrow \psi'_L = e^{-i\Theta^a_L T_a} \begin{pmatrix} u_L \\ d_L \end{pmatrix}. \tag{3-32}$$

This $SU(2)_L \times SU(2)_R$ symmetry group is generated because the right- and left-handed component on the massless lagrangian does not mix[1].

Using the Noether's theorem (2.2), we can define six currents that are conserved:

$$j^\mu_H = \bar{\psi}_H \gamma^\mu T_a \psi_H, \tag{3-33}$$

where $H = L, R$. The linear combinations define the vector currents:

$$j^\mu_a = j^\mu_{La} + j^\mu_{Ra} = \bar{\psi}\gamma^\mu T_a \psi, \tag{3-34}$$

and the axial currents:

$$j^\mu_{5a} = j^\mu_{La} - j^\mu_{Ra} = \bar{\psi}\gamma^\mu \gamma^5 T_a \psi. \tag{3-35}$$

This vector and axial currents transform under parity as their name indicate. The vector transformations are given by:

$$SU(2)_V : \psi \rightarrow \psi' = e^{-i\alpha^a T_a}\psi. \tag{3-36}$$

[1]This symmetry is also known as *chiral symmetry*.

We also can define $U(1)$ flavor transformations for each specific flavor (e.g u-quark):

$$U(1)_u : \quad u \to u' = e^{i\delta} u, \tag{3-37}$$

and an equal $U(1)$ transformation on every flavour is defined as a $U(1)_V$:

$$U(1)_V : \quad \psi \to \psi' = e^{i\delta} \otimes \mathcal{I}_2 \psi. \tag{3-38}$$

where $\mathcal{I}_2$ corresponds to the 2×2 identity matrix. Also, it is possible to define a $U(1)_A$ transformation (1):

$$U(1)_A : \quad \psi \to \psi' = e^{i\delta\gamma_5} \otimes \mathcal{I}_2 \psi. \tag{3-39}$$

Therefore, our QCD lagrangian is invariant under certain mix of symmetries depending on how the quark masses appears (**3-1**). In order to obtain quantum amplitudes related with

Condition	**Symmetry**
m_u, m_d (Arbitrary)	$U(1)_u \otimes U(1)_d$
$m_u = m_d$ (Degenerate)	$SU(2)_V \otimes U(1)_V$
$m_u \sim m_d \sim 0$ (Massless approximation)	$SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A$

Tab. **3-1**: Symmetries QCD Lagrangian

this lagrangian, we use path integral formalism, but the process is only approximate (It is necessary to take perturbative approximations). An approximate method is taking perturbations on the minimal energy point and expand over the coupling constant g_s, under the assumption that higher order terms in the expansion contributes less and less.

We are interested in hadronic physics, so we can manage a energetic limit under $\Lambda < 1GeV$. Since the confinement limit of QCD is $\Lambda_{QCD} < 0,2GeV$ [63], it is possible to begin our study with the approximation $\Lambda_{QCD} \gg m_u \sim m_d \sim 0$. In this case, the QCD lagrangian is symmetric under the general unitary group[2]:

$$U(2) \otimes U(2) \sim SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A, \tag{3-40}$$

where the 2 in the unitary group comes from the number of flavors studied in the theory. If the quarks have masses, the mass term $\bar{\psi}\mathcal{M}\psi$ breaks chiral symmetry explicitly. Writing the mass matrix as:

$$\begin{aligned}\mathcal{M} &= \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \\ &= \frac{1}{2}(m_u + m_d)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_u - m_d)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2}(m_u + m_d)\mathcal{I}_2 + \frac{1}{2}(m_u - m_d)\tau_3. \end{aligned} \tag{3-41}$$

[2]This separation are related with the structure of the generators more than with the formal irreducible decomposition [60]

The vectorial subgroup represents a manifestly symmetry of the QCD Lagrangian when $m_u \sim m_d$. This is equivalent to make $\theta_R = \theta_L$ in (3-32):

$$SU(2)_L \otimes SU(2)_R \xrightarrow{\theta_R=\theta_L} SU(2)_V. \tag{3-42}$$

In fact, we can write:

$$SU(2)_V \otimes U(1)_V = SU(N)_I \otimes U(1)_B, \tag{3-43}$$

where I is for isospin symmetry and is related with the hadronical spectrum: For $N = 2$ the lightest spin-$\frac{1}{2}$ hadrons form a doublet (proton-neutron) while te lightest spin-0 hadrons form a triplet $(\pi^{\pm}, \pi^0)$. The $U(1)_B$ is associated with the conservation of the baryon number.

On the other hand, the axial subgroup:

$$SU(2)_L \otimes SU(2)_R \xrightarrow{\theta_R^{\dagger}=\theta_L} SU(2)_A, \tag{3-44}$$

does not seem to produce a multiplet classification different from the $SU(2)_V$ sector. Thus, the only possibility is that the axial generators are spontaneously broken for an unknown operator. We are looking for some operator that acquire VEV under this transformation, but we have to avoid the breaking of Lorentz symmetry and $SU(3)$-gauge color symmetry. Our best chance is to build a scalar color-singlet operator. Since we have not singlet scalar states with VEV, the best option is generate a composite field. The simplest combination is [61]:

$$\left\langle 0|\psi_{L\alpha i}\bar{\psi}^{R\alpha j}|0\right\rangle \neq 0. \tag{3-45}$$

This “quark-condensate”formation can be explained in a similar way than cooper pairs formation. In superconductivity theory, the electron-fonon interaction allows that two electrons are attracted one to each other in a interaction mediated by a positive ion. In QCD, quarks and anti-quarks have strong attractive interactions. In the massless quark limit, it is possible to create quark-antiquark pairs with a relatively small energy cost, with zero total linear and angular momentum. It is possible to see that this chiral condensate vaccum configuration changes under $U(2)_L \otimes U(2)_R$ group as:

$$\left\langle 0\right| \left(\psi_{L\alpha i}\bar{\psi}^{R\alpha j}\right)' \left|0\right\rangle = -U_{Li}^{k} U_{Rm}^{*j} \left\langle 0|\psi_{L\alpha k}\bar{\psi}^{R\alpha m}|0\right\rangle \tag{3-46}$$

$$= -v^3 U_{Li}^{k} U_{Rm}^{*j} \delta_k^m, \tag{3-47}$$

where i, j, k, m are flavor indices. v^3 is a constant with mass dimensions related with the renormalization scheme. It is easy to see that the chiral condensate breaks the axial sector $(U_{Li}^k = U_{Rm}^{*j})$ but is invariant under the vector transformation $(U_{Li}^k = U_{Rm}^{j})$. Then, it is needed to consider the axial sector as a spontaneously breaking sector in order to reproduce the last analysis.

According to the Goldstone theorem, every spontaneously broken symmetry must have associated massless Goldstone bosons. However, the mass term explicitly breaks axial symmetry and the Goldstone bosons acquire masses (they are pseudo-goldstone bosons). The number of broken axial generators for $N = 2$ is 4 (3 for $SU(2)_A$ and 1 for $U(1)_A$) so the mass spectrum should have 4 particles. But, even when the $(\pi^{\pm}, \pi^0)$ pions are really light, there are no signs of more light particles in the same spectrum, because $m_{\eta'} \gg m_\pi$. The fact that there is no goldstone boson associated with the $U(1)_A$ breaking is known as the problem of the missing meson or $U(1)_A$ problem [64].

3.2. Effective Chiral symmetry

As mentioned in the previous section, the quark condensates breaks chiral $SU(2)_L \otimes SU(2)_R$ symmetry as just as $U(1)_A$ symmetry. Is is possible to construct a low-energy effective theory. When the quark condensate break the axial generators at low energies, the quark- gluon interaction behave in a non-perturbative way, so the relevant degrees of freedom are the pions (pseudo-Goldstone bosons) and not the quarks or gluons. The trick [65] is make the VEV associated to the quark condensate a function of spacetime:

$$\langle 0|\psi_{L\alpha k}\bar{\psi}^{R\alpha m}|0\rangle = -v^3 U(x), \tag{3-48}$$

where $U(x)$ is a spacetime dependent unitary matrix:

$$U(x) = \exp\left\{\frac{i\pi^0}{f_0} + \frac{i\pi^a\sigma^a}{f_\pi}\right\}. \tag{3-49}$$

f_0, f_π are parameters with dimension of mass. In principle there are no differences between this two constants, so we take $f_0 = f_\pi = f$. The σ^a with $a = 1, 2, 3$ are the generators of $SU(2)_A$ symmetry and the π^a are related with the pseudo-Goldstone bosons to be identified with the pions. The π^0 field is proportional to the identity and would be correspond to the pseudo-Goldstone boson associated with the $U(1)_A$ group. We require $\det U(x) = 1$ and $U^\dagger U = 1$ in order to ensure $U(x)$ is a unitary transformation. Due to the above conditions, the lagrangian related with this new $U(x)$-field only have derivatives. We can write down all the terms in the form:

$$\mathcal{L} = -\frac{1}{4} f^2 \operatorname{Tr} \partial^\mu U^\dagger \partial_\mu U. \tag{3-50}$$

(All the other terms result equivalents after integration by parts). Upon expanding U in terms of the pion field, the kinetic and quartic - interaction terms for pions are:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi_a + \frac{1}{6}f_\pi^{-2}\left(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b - \pi^a\pi^b\partial\mu\pi^b\partial_\mu\pi^a\right) + \cdots \tag{3-51}$$

We can realize that the interaction term generate vertices that contain factors of the form p/f, which can be thought as an expansion parameter. This f-parameter is known as the *pion decay constant* and has an approximate experimental value $f \approx 92{,}4MeV$ [66]. If we compare this value with the informed value for the mass of the pions seem low. However, the real cutoff for each loop momentum diagram is imposed in the value $4\pi f \approx 1GeV$ due to the fact that tree and loop diagrams contribute to any process in this particular scale.

What happens with the mass term in this new regime? This term transforms under $U(x)$, providing a potential for the pseudo-Goldstone bosons. We can write:

$$\mathcal{L}_{mass} = \bar{\psi}\mathcal{M}\psi \rightarrow \begin{pmatrix}\bar{u} & \bar{d}\end{pmatrix} U^{\dagger}\mathcal{M}U \begin{pmatrix}u \\ d\end{pmatrix}. \tag{3-52}$$

The mass matrix $\mathcal{M}$ is a complex matrix. Under a $SU(2)_L \otimes SU(2)_R$ chiral transformation, is possible to bring $\mathcal{M}$ to the form:

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2}, \tag{3-53}$$

but, we cannot remove the phase θ, because this would require a type $U(1)_A$ transformation that is prohibited. For this particular basis we will set $\theta = 0$ (This is valid under this first approach since there is no explicit dependence on θ in the Lagrangian). On the other hand, the expansion of the $\mathcal{L}_{mass}$-term do not require all our expertise: due to the fact that the quark condensate is flavor-diagonal (is only dependent to color indices and we are working in the flavor basis, therefore the only non-zero components are in the diagonal) we can see that:

$$\langle \bar{u}d \rangle = \langle \bar{d}u \rangle = 0. \tag{3-54}$$

Then, taking

$$\Sigma \equiv U^{\dagger}\mathcal{M}U = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}, \tag{3-55}$$

the only terms that contribute are in the diagonal of the $\mathcal{M}$ matrix. It is possible to write:

$$\begin{aligned} \mathcal{L}_{mass} &= \langle \bar{d}d \rangle \mathcal{M}_{11} + \langle \bar{u}u \rangle \mathcal{M}_{22} \\ &= \langle \bar{u}u \rangle \operatorname{Tr}\Sigma \\ &= -v^3 \operatorname{Tr}\left(U^{\dagger}\mathcal{M}U\right) \\ &= -v^3 \operatorname{Tr}\left(\mathcal{M}U^2\right), \end{aligned} \tag{3-56}$$

where we use the fact that $\langle \bar{d}d \rangle \sim \langle \bar{u}u \rangle$. Then, any shift under the pseudo-Goldstone bosons costs energy. In fact, this $\mathcal{L}_{mass}$ corresponds to a potential term, because does not appear derivatives. In this potential it is necessary to take into account the contribution of the right quarks ($\langle \bar{q}q \rangle = \langle \bar{q}_L q_R \rangle + \langle \bar{q}_R q_L \rangle$) so we must write the $h.c$ part. The charged sector is obtained making $\pi_0 = \pi_3 = 0$ and

$$\pi_1, \pi_2 \rightarrow \pi_{\pm} = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2). \tag{3-57}$$

In this approximation, we see that all the odd-exponent terms vanish when we sum up the terms of the $h.c$ part. The product in the $U(x)$-matrix it is written as:

$$U(x) = \exp\left\{\frac{i\pi^0}{f_0} + \frac{i\pi^a\sigma^a}{f_\pi}\right\} \to \exp\left\{\frac{i}{f}\left(\pi_1\sigma_1 + \pi_2\sigma_2\right)\right\}$$
$$= \exp\left\{i\begin{pmatrix} 0 & \theta_- \\ \theta_+ & 0\end{pmatrix}\right\} \equiv \exp i\Theta, \tag{3-58}$$

with $\theta_\pm := \frac{\pi_\pm}{f}$. It is easy to see that the $\mathcal{O}(2)$ in the exponential expansion generate a diagonal matrix:

$$\mathcal{O}(\Theta^2) \sim \begin{pmatrix}\theta_-\theta_+ & 0 \\ 0 & \theta_-\theta_+\end{pmatrix}. \tag{3-59}$$

The remaining terms in the exponential addition (3-58) can be written as:

$$\begin{aligned} e^{i\Theta} + e^{-i\Theta} &= \sum_{\text{n-even}} \frac{(i\Theta)^n}{n!} \\ &= \sum_n \frac{(-\Theta^2)^n}{(2n)!} \\ &= \sum_n \frac{\left(-\sqrt{\Theta^2}\right)^{2n}}{(2n)!} \\ &= -\begin{pmatrix}\cos\sqrt{\theta_+\theta_-} & 0 \\ 0 & \cos\sqrt{\theta_+\theta_-}\end{pmatrix}, \end{aligned} \tag{3-60}$$

and the mass potential is:

$$V = -(m_u + m_d)v^3\cos\sqrt{\theta_-\theta_+} \approx (m_u + m_d)v^3 + \frac{(m_u + m_d)v^3}{2f^2}\pi_-\pi_+ + \cdots \tag{3-61}$$

Taking the quadratic term, we can set the charged pion mass:

$$m^2_{\pi_\pm} = \frac{(m_u + m_d)v^3}{f^2}. \tag{3-62}$$

The neutral sector mass spectrum is calculated doing $\pi_\pm = 0$. We write $\theta_0 = \frac{\Pi_0}{f}, \theta_3 = \frac{\Pi_3}{f}$. Due to σ_3 is diagonal, the cosine argument is a sum with different sign in each component:

$$V_{03} = -m_u v^3\cos(\theta_0 + \theta_3) - m_d v^3\cos(\theta_0 - \theta_3). \tag{3-63}$$

In order to generate the mass spectrum we expand the potential over the minimum:

$$M^2 = \frac{\partial^2 V_{03}}{\partial\pi_0\partial\pi_3} = \frac{v^3}{f^2}\begin{pmatrix} m_u + m_d & m_u - m_d \\ m_u - m_d & m_u + m_d\end{pmatrix}. \tag{3-64}$$

Diagonalizing, we obtain:

$$\mathrm{Tr}\, M^2 = m_{\eta'}^2 + m_{\pi_0}^2 = \frac{2(m_u + m_d)v^3}{f^2}. \tag{3-65}$$

Then, comparing (3-62) with (3-65), we obtain:

$$2m_{\pi_\pm}^2 = m_{\eta'}^2 + m_{\pi_0}^2. \tag{3-66}$$

But, taking the experimental values[3] [66]

$$m_{\eta'} = 957{,}78 \pm 0,06 MeV, \tag{3-67}$$
$$m_{\pi_0} = 134{,}9770 \pm 0,0005 MeV, \tag{3-68}$$
$$m_{\pi_\pm} - m_{\pi_0} = 4,5936 \pm 0,0005 MeV, \tag{3-69}$$

it is impossible for this equality to be satisfied. Even if we take $f_0 \neq f_\pi, \beta = \frac{f_\pi}{f_0}$, the relation obtained:

$$m_{\eta'}^2 + m_{\pi_0}^2 = (1+\beta^2)m_{\pi_\pm}^2, \tag{3-70}$$

could adjust the mass difference, but seems very unnatural think that $\beta \gg 1$ because f_0 and f_π have the same origin. So the only natural explanation is to think that exists another source of $U(1)_A$-violation that contributes to the $m_{\eta'}^2$-term. the theoretical way to understand the origin of the η'-meson is introducing to our model the contribution of the strange quark:

$$U = \exp\left\{\frac{i}{f_\pi}\left(\sum_{a=1}^{8}\pi^a\lambda^a + \mathcal{I}_{3\times 3}\frac{\eta_0}{\sqrt{2}}\right)\right\} \quad \text{with} \quad \sum_{a=1}^{8}\pi^a\lambda^a = \begin{pmatrix} \pi^0 + \frac{\eta^8}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta^8}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\frac{\eta^8}{\sqrt{3}} \end{pmatrix}, \tag{3-71}$$

where λ^a-matrices are the *Gell-Mann* matrices related with the $SU(3)$ group. There are no relevant contributions associated to the π^0 state (because the new mixing terms are proportional to the factor $(m_u - m_d)$). The physical states η and η' has the form (associated to the rotation of the previous ones):

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{pmatrix}\begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix}, \tag{3-72}$$

with $\theta_m \approx 17^o$ [67]. Even with a effective contribution of kaons for the η' mass, the real value is even larger. So there is no possible to identify even in this situation the η' meson with the Goldstone boson of the $U(1)_A$ symmetry.

[3]The electromagnetic interaction explain raise up the masses of the charged pions

3.3. QCD vacuum

The missing messon problem does not have a true solution between the context of the classic theory. t' Hooft realized that this problem has a clear solution in the axial anomaly. In the last section we calculate the pion mass associated with the term that break explicitly axial symmetry but in the classical regime. We obtain that the η'-meson does not have the correct value of mass, so we are looking for a new source for the correct value. As we mention in (3.1) the contribution of the $\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[F_{\mu\nu}F_{\rho\sigma}]$ term can not be ignored. We are going to study why it is necessary to take this term into account

3.3.1. Homotopy classes

There are one mathematical notion that could be allow us to analyze the behavior of the gauge fields corresponding to the different configurations in the ground state. Under the gauge transformations, we saw that the strength tensor to obtain a finite-euclidean action is:

$$F_{\mu\nu} = 0 \qquad \text{as} \quad r \to \infty. \tag{3-73}$$

Thus, this imply that our most general conjecture over the $A_\mu = A_\mu^a T^a$ is that this corresponds to a gauge transformation of zero. Fixing our temporal gauge to the condition $A_0 = 0$, we are restricting our attention to the gauge transformation independent of time $U = U(x)$. Therefore, the condition

$$A_\mu = 0, \tag{3-74}$$

is too general. It is possible to impose the same boundary condition under the strength field only making the $U(x)$ matrix approaches to a constant matrix as $|x| \to \infty$. This condition is equivalent to adding a spatial *"point at infinity"*. Therefore, the space has the same topology than S^3-sphere (because U has a definite value regardless the direction).
But, can we establish an equivalence between the different configurations associated by the selection of different values for U? To study this possible equivalences, we parameterize the different kind of functions through classes. Thus, a class of functions is defined by the set of functions that can be deformed in other by a smoothly continuous transformation, e.g. it is possible to identify all the points under the unity circle S^1 via a map that identify each point with a complex number of module 1. Each possible function can be expressed as:

$$f(\theta) = \exp\left[i\left(\nu\theta + \alpha\right)\right]. \tag{3-75}$$

Every function with the same value of α belong to the same homotopic class:

$$H(\theta, t) = \exp\left[i\left(\nu\theta + (1-t)\alpha_1 + t\alpha_2\right)\right], \tag{3-76}$$

such that, changing the value of t we obtain equivalent homotopic classes e.g. for $t = 0, t = 1$:

$$f_0(\theta) = \exp\left[i(\nu\theta + \alpha_1)\right], \tag{3-77}$$

$$f_1(\theta) = \exp\left[i(\nu\theta + \alpha_2)\right]. \tag{3-78}$$

Thus, the transformation is equal to map a circle into another [68]. But the situation is different for different values of ν: we can think on a map associating n points of the first circle with 1 point of the second circle. Then, the ν-number characterize the number of *winding* of one function around another and each homotopic class has its own winding number. From (3-76) it is possible to obtain the value of ν as:

$$\nu = \int_0^{2\pi} \frac{-i}{2\pi} \left[d\theta (f(\theta))^{-1} \frac{df(\theta)}{d\theta} \right]. \tag{3-79}$$

For our analysis we require transformations from S^3 to representations in $SU(3)$. In general, maps V that provide this kind of transformations are labeled by one integer ν, that could be expressed as [69]:

$$\nu = -\frac{1}{24\pi^2} \int d^3\theta \varepsilon^{ijk} \, \mathrm{Tr}[(V(\theta)\partial_i V(\theta)^\dagger)(V(\theta)\partial_j V(\theta)^\dagger)(V(\theta)\partial_k V(\theta)^\dagger)], \tag{3-80}$$

which is invariant under smooth deformations and change of coordinates [63]. It is possible to compound individual maps under product to obtain a new map

$$V_1(\theta_1, \theta_2, \theta_3) V_2(\theta_1, \theta_2, \theta_3) \equiv V(\theta_1, \theta_2, \theta_3), \tag{3-81}$$

which has a winding number resulting of sum the individual winding numbers of the initial maps $\nu_1 + \nu_2 = \nu$. So, if we going back to the euclidean action

$$S_E = -\frac{1}{2} \int d^4x_E \, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \int_0^\infty drr^3 \int d\Omega \, \mathrm{Tr}[F_{\mu\nu}(r,\Omega)F^{\mu\nu}(r,\Omega)], \tag{3-82}$$

where we divide the radial (r) and angular (Ω). Therefore, in order to maintain only finite contributions, we require that $G_{\mu\nu}$ go to zero faster than $1/r^2$, so $G_{\mu\nu} \sim \mathcal{O}(1/r^3)$ for $r \to \infty$. Then, the $A_\mu \equiv a^a_\mu T^a$ must remain fixed under gauge transformation conserving the wanted boundary conditions. t' Hooft realized that A_μ must be a pure gauge field [70] (a pure gauge transformation of zero) is the most general boundary to produce zero strength fields:

$$A_\mu = \frac{i}{g} V(\Omega)\partial_\mu V(\Omega)^{-1} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{3-83}$$

where $V(\Omega)$ is a continuous and differentiable map which is only function on the angular variables:

$$V(\Omega) : S^3 \to SU(3). \tag{3-84}$$

Applying a gauge transformation under the boundary condition for A_μ (3-12), we have that the euclidean potential transforms as:

$$V(\Omega) \to U(\Omega)V(\Omega) + \mathcal{O}(1/r^2). \tag{3-85}$$

Then, the only way to reestablish the boundary condition is assuming that $U(\Omega) = V(\Omega)^{-1}$ $A_\mu \sim \mathcal{O}(1/r^2)$ at $r \to \infty$. But the problem is that the nature of each transformation are really different: $U(\Omega)$ is a truly gauge transformation, so has to be continuous and well defined in every point. This mean that has to be independent of the angular variables in the origin to be continuous and should be constant in the point. So it is possible to establish a continuous deformation from $r=0$ to $r\to\infty$. This is equivalent to say that every map belongs to the trivial homotopic class $\nu=0$. Then, the only way to put the equivalence between U and V is that they has the same winding number, or, in other words, that they belong to the same trivial homotopic class. But, $V(\Omega)$ can be of any homotopic class because is a explicit function of the angular variables. Therefore, it is not possible to accomplish the needed condition, so we can not vanish the contributions related to the $\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$-term. The presence of this non-trivial potentials that cannot be obtained from gauge transformation to trivial field configurations i.e. $\mathcal{O}(1/r^2)$ does not generate any problem in abelian theory. because there is only one trivial map $S^3\to U(1)$, but affect extremely the structure of the non-abelian theories. These non-trivial potentials are characterized by non-zero winding numbers. It is possible to write explicitly the winding number as a integral of $V(\Omega)$. To see how, we write the Chern-Simmons current in a different way:

$$\begin{aligned} K_\mu^{CS} &= 2\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma\right] = 2\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[\frac{A_\nu}{2}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) - \frac{ig}{3}A_\nu\left[A_\rho, A_\sigma\right]\right] \\ &= \varepsilon_{\mu\nu\rho\sigma}\left[A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right] \xrightarrow{r\to\infty} \frac{2}{3g^2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[(V\partial_\nu V^{-1})(V\partial_\rho V^{-1})(V\partial_\sigma V^{-1})\right], \end{aligned} \tag{3-86}$$

where we have used the asymptotic limit for $F_{\rho\sigma}, A_\mu$. Replacing the last result into the winding number definition (3-80), we can rewrite the winding number in the euclidean space as:

$$\nu = -\frac{1}{24\pi^2}\int dS_{E_\mu}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[(V\partial_\nu V^{-1})(V\partial_\rho V^{-1})(V\partial_\sigma V^{-1})\right]. \tag{3-87}$$

Making the definition of the dual-strength field $\tilde{F}_{\mu\nu}$ through:

$$\mathrm{Tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[F_{\mu\nu}\tilde{F}_{\rho\sigma}\right], \tag{3-88}$$

it is possible to write:

$$\int d^4x_E\,\mathrm{Tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] = -\frac{16\pi^2\nu}{g^2}. \tag{3-89}$$

Therefore, our boundary conditions are only valid for $\nu=0$. But, what happen with the A_μ potentials if $\nu\neq 0$?

3.3.2. Non-trivial vacuum configurations

We are concerned about the possibility of different gauge transformations of the potential fields could be transformed smoothly or not into another non-trivial configurations. As we have seen, in general there is no possible to define such trivial transformations. For example, considering two different gauge transformations of zero [63]:

$$A_\mu = \frac{i}{g} U \partial_\mu U^\dagger, \qquad \tilde{A}_\mu = \frac{i}{g} \tilde{U} \partial_\mu \tilde{U}^\dagger. \tag{3-90}$$

The consequence related with the impossibility of develop trivial transformations between different homotopic classes are intrinsically related with the behavior of the vacuum states. If we try to deform A_μ into $\tilde{A}_\mu$, the transformation involves pass through vector potentials that belong to different homotopic classes (so they are not gauge transformations of zero). Therefore, the associated strength tensors do not vanish and there will be different energy eigenstates associated with each configuration. so, each vector potential represent two different vacuum states in the quantum field theory separated by energy barriers (each A_μ is associated with a different minima of the hamiltonian). It is possible to see a similar behavior in the semi-classical theory for a scalar potential of the form:

$$V(\varphi) = \lambda v^4 \left[1 - \cos\left(\frac{2\pi\varphi}{v}\right)\right],$$

where the transition probability amplitude has the form [68]:

$$\langle n'|V|n\rangle \sim e^{-S_E}. \tag{3-91}$$

S is the euclidean action related to the classical solution of the field equations. This action mediates the configurations from n at $t \to \infty$ to n' at $t \to +\infty$. In this classical situation, the action increases at the infinite limit volume, so the probability amplitude vanishes at the infinity. Therefore the minima of V remain degenerate. But the situation is really different in QCD: the presence of classical solutions that can mediate between two vacuum states of different winding numbers generate a transition probability that is not zero. This non-trivial configuration allow tunnelling between states even in the limit of large volume space. To see how to arise this kind of configurations, we can start analyzing the object:

$$\frac{1}{2}\,\mathrm{Tr}\left[d^4x_E \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2\right] = \int d^4x_E \left(\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu} \pm \mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}]\right) \geq 0, \tag{3-92}$$

(where we have used $\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} = 2\left(\delta_{\rho\alpha}\delta_{\sigma\beta} - \delta_{\rho\beta}\delta_{\sigma\alpha}\right)$, therefore $\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}$. The left-hand side of (3-92) is non negative, so:

$$-\int d^4x_E\,\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] \geq \left|\int d^4x_E\,\mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}]\right|. \tag{3-93}$$

The left-hand term corresponds to the euclidean action times 2. The right hand part can be rewritten in function of the winding number using (3-89), so we can write:

$$S_E \geq \frac{8\pi^2|\nu|}{g^2}, \tag{3-94}$$

where $\nu = n' - n$ implies the action stays fixed and finite in the infinite volume space. For $\nu = 1$, the solution is the *instanton*. For $\nu = -1$ the solution is known as the *anti-instanton*. For $|\nu| > 1$ the solution is a dilute gas of $|\nu|$ instantons or anti-instantons if $\nu < 0$. For $\nu = 1$, the vector potential has the form [71]:

$$A_\mu = \left(\frac{x^2}{x^2 + a^2}\right) U(x)\partial_\mu U^{-1}(x), \tag{3-95}$$

where a is the size of instanton. Far away from the center of the instanton, the potential behave as:

$$A_\mu(x) \to U(x)\partial_\mu U^{-1}(x). \tag{3-96}$$

Thus, the contribution to the action is equal to $\frac{8\pi^2}{g^2}$ (in the case when $|\nu| > 1$, it is possible to build solutions putting instantons together, whose centers are separated by their sizes). Calculating the instanton solution in the temporal gauge $A_0(x) = 0$, we have [68]:

$$A_\mu(x) \to V(x)A_\mu(x)V^{-1}(x) + V(x)\partial_\mu V^{-1}(x). \tag{3-97}$$

Then, the condition $A_0(x) = 0$ implies:

$$\frac{\partial}{\partial x_0} V^{-1}(x) = -A_0(x)V^{-1}(x) = \frac{-ix\cdot\sigma}{x_0^2 + x^2 + a^2} V^{-1}(x), \tag{3-98}$$

where we have used the identity map in the boundary like $U(x) = \frac{x_0 + ix\cdot\sigma}{|x|}$. It is possible to determine the explicit form of V satisfying the boundary conditions as:

$$V(x_0 \to -\infty) = \exp\left[i\pi \frac{x\cdot\sigma}{\sqrt{x^2+a^2}} n\right], \qquad V(x_0 \to +\infty) = \exp\left[i\pi \frac{x\cdot\sigma}{\sqrt{x^2+a^2}} (n+1)\right]. \tag{3-99}$$

Then, the instanton solution $\nu = 1$ connects vacua which differ by one unit of winding number.

3.3.3. θ-vacua

If we replace the saturation condition of (3-94) in (3-91), we see that transition amplitude depend exclusively of the field instantons. Moreover, there is a explicit dependence on the g coupling constant. We can see easily that the tunnelling process is relevant for the non-perturbative regime, so ($g \gg 1$). In order to construct an invariable vacuum structure under gauge transformation , we are interested in take some general state labeled by time-independent quantity, with states labeled in such a way that they do not overlap. For a non-trivial G_m gauge transformation (non-trivial in the toplogical sense, its mean that connects states which differ by a winding number m) on the vacuum state $|n>$,we have a change in the vacua winding number:

$$G_m |n\rangle = |n + m\rangle\,. \tag{3-100}$$

It is possible to build a good vacuum state as a sum over winding-number vacua:

$$|\theta\rangle = \sum_n e^{in\theta} |n\rangle . \tag{3-101}$$

The states are labeled by a real parameter θ as we desired. So, it is easy to see that this kind of state is invariant under gauge transformation

$$G_m |\theta\rangle = \sum_n e^{in\theta} |n+m\rangle = e^{-im\theta} \sum_{n'} e^{in'\theta} = e^{-im\theta} |\theta\rangle , \tag{3-102}$$

up to a change of phase. Also, the different vacua states do not overlap:

$$\begin{aligned} \langle \theta'_{out}|\theta_{in}\rangle &= \sum_{n,m} e^{-im\theta'} e^{in\theta} \langle m_{out}|n_{in}\rangle \\ &= \sum_{\nu,k} e^{\frac{i}{2}(\theta'+\theta)}\nu e^{\frac{i}{2}(\theta'-\theta)}k \langle \nu_{out}|0_{in}\rangle = \delta(\theta'-\theta) \sum_{\nu} e^{i\theta\nu} \langle \nu_{out}|0_{in}\rangle . \end{aligned} \tag{3-103}$$

Using the definition for winding number, we can write the integration over all euclidean field configurations in the vacuum to vacuum transition as:

$$\begin{aligned} \langle \theta'_{out}|\theta_{in}\rangle &\propto \sum_{\nu} e^{i\theta\nu} \langle \nu_{out}|0_{in}\rangle = \sum_{\nu} \int [dA_\mu]_\nu e^{i\nu\theta} e^{\int d^4x \mathcal{L}_E} \\ &= \sum_{\nu} \int [dA_\mu]_\nu \exp\left[\int d^4x \left(\mathcal{L}_E - \frac{i\theta g^2}{32\pi^2} \mathrm{Tr}[\varepsilon_{\mu\alpha\rho\sigma} F_{\mu\alpha} F_{\rho\sigma}]\right)\right] . \end{aligned} \tag{3-104}$$

We have been working in the euclidean space, so we need back away to the Minkowski space. To do this we need to do an *anti-Wick* rotation taking into account that we will get an extra i factor from the four space measure, an extra $-i$ factor associated with the time-like derivative term in $\mathrm{Tr}[F_{\mu\alpha}\tilde{F}^{\mu\alpha}]$ and a minus sign related with the Levi-Civita term, because $\varepsilon^{4123} = -1$ but $\varepsilon^{0123} = 1$. So, under this new ideas, the presence of instantons make mandatory to take account the contributions of the $\mathrm{Tr}\left(\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right)$ term in the lagrangian:

$$\mathcal{L}_{eff} = \mathcal{L} + \frac{\theta g^2}{32\pi^2} \mathrm{Tr}[F_{\mu\alpha}\tilde{F}^{\mu\alpha}]. \tag{3-105}$$

Therefore, the only way to avoid the instanton contributions is related to use a simple vacuum ($\nu = 0$) in order to avoid non-trivial topological transitions. But if we sum over all possible vacuum configurations, θ is a new parameter related to a topological property of the vacuum.

3.4. Chirality issues

As we have seen in the Chapter 2, the fact that there is a manifest symmetry in the Lagrangian does not necessarily imply that it remains at quantum level. The conservation laws implies some modifications in the Ward-Takahashi identities (**??**, **??**) and the linear divergent integrals related to the triangular diagrams do not allow axial and vector currents to be conserved at the same time. To study that bad behavior in the non-perturbative level, it is impossible to study the problem from perturbative methods. To make this generalization, we have to study the generating functional $Z[J]$ that results to be the fundamental object instead of lagrangian. For a theory with one fermion, the lagrangian density has the form (3-23)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]. \tag{3-106}$$

The chiral transformation (3-39)

$$\psi \to \quad e^{i\alpha\gamma^5}, \tag{3-107}$$

$$\bar{\psi} \to \quad \left(e^{i\alpha\gamma^5}\psi\right)\dagger\gamma^0 = \bar{\psi}e^{i\alpha\gamma^5}, \tag{3-108}$$

leaves invariant the classical lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu D_\mu - m)\psi - \frac{1}{4}Tr[F_{\mu\nu}F^{\mu\nu}], \tag{3-109}$$

when $m = 0$. If the chiral transformation is local i.e. $\alpha = \alpha(x)$, the lagrangian density with $m = 0$ transforms in the following way:

$$\int d^4x\mathcal{L} \to \int d^4x[\mathcal{L} + \alpha(x)\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)]. \tag{3-110}$$

Since the Lagrangian must be stationary under $\alpha(x)$ variations, we obtain

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = 0. \tag{3-111}$$

The last expression represents the conservation of the chiral current:

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{3-112}$$

This chiral current can be written as (3-35). The zero-component has the form:

$$j^{05} = \psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L, \tag{3-113}$$

which represents the difference between the number density of handed particles. The associated conserved charge is:

$$Q^5 \equiv \int d^3xj^{05}, \tag{3-114}$$

is the difference of right handed particles minus left handed particles. According to the previously studied topics, it is easy to guess that the behavior for the QFT will be different The conservation laws can be derived of the generating functional:

$$Z = \int [d\psi][d\bar{\psi}][dA_\mu]e^{i\int d^4x\mathcal{L}}. \tag{3-115}$$

As we saw in (2-154), the measure change under the local chiral transformation. Then, the aditional contribution spoil the classical symmetry transforming as:

$$[d\psi][d\bar{\psi}] \to \quad \exp\left[i\int d^4x\alpha(x)\mathcal{A}(x)\right][d\psi][d\bar{\psi}], \tag{3-116}$$

where $\mathcal{A}$ is the anomaly function:

$$\mathcal{A}(x) = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr[F_{\mu\nu}F_{\alpha\beta}]. \tag{3-117}$$

Therefore, the chiral transformation over Z is determined adding the contribution of the transformation law for the action plus the effect on the functional integral:

$$Z \to \quad \int [d\psi][d\bar{\psi}][dA_\mu]\exp^{i\int d^4x\mathcal{L}+\alpha(x)\partial_\mu j^{\mu^5}+\alpha(x)\mathcal{A}(x)}. \tag{3-118}$$

Again, the functional Z has to be stationary under arbitrary variations of α. This condition generates a expectation value for the divergence of the chiral current:

$$\left.\frac{\delta Z}{\delta\alpha(x)}\right|_{\alpha=0} = \int [d\psi][d\bar{\psi}][dA_\mu]i(\mathcal{A}(x)+\partial_\mu j^{\mu 5})e^{i\int d^4x\mathcal{L}} = 0, \tag{3-119}$$

$$\left\langle \partial_\mu j^{\mu 5}\right\rangle = -\left\langle \mathcal{A}\right\rangle = \frac{g^2}{16\pi^2}\left\langle \epsilon^{\mu\nu\alpha\beta}Tr[F_{\mu\nu}F_{\alpha\beta}]\right\rangle. \tag{3-120}$$

The expression (3-120) allow us to see that, even in absence of mass terms, the divergence of the chiral current does not correspond to a exact symmetry of the theory and the expected value turns out to be equal to the anomaly ABJ studied previously. Perhaps we could think that the result is canceled at higher orders in the perturbative contributions associated with the triangular diagrams, but in this case, we use a different treatment that does not involve lower order corrections. Considering now the charge associated with the axial current:

$$\int d^4x j^{05} = Q_f^5 - Q_i^5 = \int d^4x\frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[F_{\mu\nu}F_{\alpha\beta}]. \tag{3-121}$$

Comparing with (3-89), we have:

$$Q_f^5 - Q_i^5 = -2\nu, \tag{3-122}$$

where $= Q_f^5 - Q_i^5$ represents the change of chirality in function of the final and initial difference in right and left handed particles respectively. So, the presence of instantons make that particles change their chirality converting right handed particles into left handed. The expectation value is the divergence of the chiral current vanish only when we take into account trivial gauge field configurations. It is worth noting that the contribution associated with the chiral transformation has the same form as the term added to the Lagrangian because of the non-trivial structure of the QCD vacuum. This fact will be important for later analyzes.

3.5. Non-Abelian generalization

Considering the action of a non-abelian group symmetry over non-abelian gauge fields, it is possible to generalize the previous results. For example, making a non-abelian transformation like (3-32) over Weyl spinors:

$$\psi_L \to e^{i\Theta_L^a \tau^a}\psi_L, \quad \psi_R \to e^{i\Theta_R^a \tau^a}\psi_R, \tag{3-123}$$

(τ^a are the generators of some arbitrary group), the transformation associated to non-abelian gauge fields generates an extra term on the anomaly contribution:

$$\frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F^c_{\mu\nu}G^d_{\alpha\beta}\,\mathrm{Tr}[\tau^a T^c T^d]. \tag{3-124}$$

The object that allows us to really know if a transformation represented by the generators τ^a coupling with spinor fields presents or not quantum corrections is then the factor:

$$d^{abc} = \mathrm{Tr}[\tau^a\left\{T^a, T^b\right\}], \tag{3-125}$$

where we use:

$$\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[\tau^a T^b T^c] = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[\tau^a\left\{T^b, T^c\right\}]. \tag{3-126}$$

We use the anticommutator expression because the presence of fermions induce different signs. In general, if $d^{abc} = 0$, there are no anomalies and the classical symmetry associated with τ^a can be extended to the quantum level. But if $d^{abc} \neq 0$, the current is not conserved at quantum level. It is possible to see that the SM is an anomaly free theory, and this criterion will be used to construct our $U(1)_X$ extended model.

3.6. Solution to the $U(1)_A$ problem: more problems

Then, it is possible to analyze under the behavior of d^{abc} if the axial group $SU(2)_A \otimes U(1)_A$ takes or not quantum correction when interact with gluon fields. For $SU(2)_A$, the corresponding generators $\tau^a = \sigma^a/2$ and for the gluonic fields the generators are the Gell-Mann matrices $T^c = \lambda^c$. therefore, the behavior of this isospin axial current is:

$$\partial_\mu j^{\mu 5a} = \frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F^c_{\mu\nu}F^d_{\alpha\beta}\,\mathrm{Tr}[\tau^a\lambda^c\lambda^d], \tag{3-127}$$

where $F^c_{\mu\nu}$ is a gluon strength field, τ^a is an isospin matrix and λ^c is a color matrix. The trace is taken over colors and flavors:

$$\mathrm{Tr}[\tau^a\lambda^c\lambda^d] = \mathrm{Tr}[\tau^a]\,\mathrm{Tr}[\lambda^c\lambda^d] = 0. \tag{3-128}$$

Then, as we have seen, the axial current associated to $SU(2)_A$ is only broken explicitly for quark mass terms and does not have quantum correction. For the $U(1)_A$ subgroup, the τ^a generator corresponds to the identity on flavors (3-39), then the axial singlet current :

$$\partial_\mu j^{\mu 5}(x) = \frac{g^2 n_f}{32\pi^2}\varepsilon^{\mu\nu\alpha\beta}F^c_{\mu\nu}F^c_{\alpha\beta}. \tag{3-129}$$

Therefore, the singlet axial current has quantum corrections and is not conserved. The quantum correction is directly associated with the tricky structure of QCD vacuum (after all $\mathrm{Tr}[F_{\mu\nu}F_{\alpha\beta}]$ is nothing but the instanton topological term). So, the solution for the $U(1)_A$ problem has to be related with the anomaly term itself. Indeed, the anomalous triangle diagram which couples to $j^{\mu 5}$ will directly provide mass to the golsdtone-boson η'. To see how, we have to analyze one more thing. If we raise again the mass terms into the QCD lagrangian, the divergence of the singlet axial current read as:

$$\partial_\mu j^{\mu 5} = -2m_u\bar{u}i\gamma_5 u - 2m_d\bar{d}i\gamma_5 d + 4\frac{\alpha_s}{8\pi}F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a, \tag{3-130}$$

where we replace the definition of the dual strength field and define $\alpha_s = \frac{g^2}{4\pi}$ for our $n_f = 2$ theory. The first two terms remain us that the mass terms violate the symmetry too. Thus, the most general QCD lagrangian can be written as:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + i\bar{q}\not{D}q - (\bar{q}m_q e^{i\theta_Y}q_R + h.c.) - \frac{\alpha_s}{8\pi}F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\theta_{QCD}, \tag{3-131}$$

where $\theta_Y = \theta$ in (3-53). We expect that the $F\tilde{F}$ term that violates the $U(1)_A$ symmetry generate mass for the π_0-field as m_u generates mass for the $\theta_0 + \theta_3$ combination. Under a $U(1)_A$ transformation,

$$q_L \to e^{-i\beta}, \quad q_R \to e^{i\beta}q_R, \tag{3-132}$$

leading to an anomaly contribution of

$$\mathcal{L}_{Anom} = N_g\beta\frac{\alpha_s}{4\pi}\,\mathrm{Tr}[F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a]. \tag{3-133}$$

If $\beta = \theta_Y$ we can shift the phase θ_Y to θ_{QCD} through an axial transformation. So, the combination

$$\theta_{SM} = \theta_{QCD} + 2\theta_Y, \tag{3-134}$$

appears now multiplying the $G\tilde{G}$ term for $N_g = 2$. As we have mention, this term violates P and T parities, as:

$$\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \xrightarrow{P,T} -\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{3-135}$$

This implies that QCD theory is invariant under $P-, T-$ and $CP-$ iff $\theta = -\theta$, i.e. θ mod $\pi = 0$. But we do not have any restriction on θ, therefore QCD does not naturally conserve CP . Thus, every CP violation observable has to depend on the physical θ_{SM}. So, this term impose a CP-violation in QCD (we talk more about this fact later).

As a last ingredient, we need to remember that when we defined the transformations (3-32), we are effectively making a local $U(1)_A$ shift. Thus, we have another contribution to the lagrangian that can be read as:

$$\frac{\alpha_s}{8\pi} F\tilde{F} \times 2\theta_0. \tag{3-136}$$

Under this new ideas, we can write new contributions for the meson potential (3-63) as

$$V \sim -m_u v^3 \cos(\theta_0 + \theta_3) - m_d v^3 \cos(\theta_0 - \theta_3) - \Lambda^4_{QCD} \cos(2\theta_0 - \theta_{SM}), \tag{3-137}$$

where the new contribution relies the fact that has a minimum in the additional factor that accompanies the anomalous structure, it has to give a large mass to η' so must have a non-zero second derivative at the minimum and has to be periodic. Thus, the meson mass matrix has the form [72]:

$$M^2 = \frac{v^3}{f^2} \begin{pmatrix} m_u + m_d & m_u - m_d \\ m_u - m_d & m_u + m_d \end{pmatrix} + 4\Lambda^4_{QCD} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3-138}$$

Thus, η' takes its mass from the new term and pions from the chiral symmetry breaking by quark masses and the quark condensate:

$$m^2_{\pi^0} = \frac{(m_u + m_d)v^3}{f^2}; \quad m^2_{\eta'} \sim 4\Lambda^4_{QCD} + \mathcal{O}(m_q v^3). \tag{3-139}$$

Then, we were able to solve the problem associated with the meson η' mass, but we introduced another problem: Our QCD theory is now CP violating because we add a new CP-violating term.

Another method to isolate anomalous contributions in a single term is rotating the fermions with a phase $-\theta_{QCD}/N_g$, to be able to cancel the θ- term leaving a complex mass quark matrix. Then, there are two phases with independent origins, which, when mixed, generate an explicit violation of the CP symmetry in strong interactions. This is the origin of the so-called *strong CP-problem*: From the experimental point of view QCD preserves CP, that is, all bound states must be eigenestates of the parity operator. This leads to the fact that if parity is conserved, Neutron Electric Dipole Momentum (NEDM) must be equal to zero. Therefore, the presence of the θ - term contributes to NEDM. From chirality techniques [73], we have:

$$d_n = 2{,}4 \times 10^{-3} \theta e \text{fm}, \tag{3-140}$$

where e is the charge of the electron. Compared with the experimental boundary $|d_n| < 3{,}0 \times 10^{-13} e$fm [39], we have an upper bound of:

$$|\theta| < 1{,}3 \times 10^{-10}. \tag{3-141}$$

This is a tiny value. θ can carry inside the $[0, 2\pi]$ interval. Therefore, the *CP-problem* could be enunciated as: Why is it possible to think that this small value, which comes from two totally different phases, is compatible with zero?

3.7. Vafa-Witten Theorem

The proposition associated to make the $\bar{\theta}$-term related to the spontaneously breaking of the $U(1)_A$ group a dynamical field is related with the CP-conservation in vector-like theories. Following the Vafa - Witten theorem [74], in parity-conserving vector like theories such as QCD, parity conservation is not spontaneously broken. We are allowed to use the same way to proof the necessity of use a dynamical field instead of a static parameter.
The effective euclidean action in the QCD θ-vacuum has the form:

$$e^{-\mathcal{V}_4 E(\theta)} = \sum_{\nu} [dA_\mu]_\nu \exp\left[-\int d^4x \left(\frac{1}{4}\operatorname{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{i\theta g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]\right)\right]. \quad (3\text{-}142)$$

It is possible to see that there are a specific order in the values for the energies in this integral:

- For $\theta = 0$ the (3-142) is real
- For $\theta \neq 0$ the θ factor is only a phase that reduce the value of the integral

so we can write in general:

$$E(\theta) \geq E(0) \qquad \forall\theta. \quad (3\text{-}143)$$

But, θ-term is a explicit CP violation term, so we can not use the Vafa-Witten theorem in order to generate the later affirmation, because θ is only a fixed parameter. But, if we change the nature of theta to be now a dynamical field, this could relax itself until reach the minimum state of energy, corresponding to $\theta = 0$. That is the basis of a good mechanism to avoid the CP violation in QCD

3.8. Peccei-Quinn Mechanism

An elegant way to avoid the CP problem was proposed by Robert Peccei and Helen Quinn in 1977 [40]. The propose is based in the assumption of a new $U(1)$ global anomalous symmetry. The symmetry has to be anomalous in order to cancel the θ-term using the color anomaly produced by the rotation of the fields. The spontaneously broken of this new symmetry at certain energy scale produce an extra degree of freedom associated with a new goldstone boson. But, because this $U(1)$ is anomalous at quantum level our boson obtain mass through topological effects associated with the interaction with instanton fields and couple with gluons. This additional particle is called *axion*, and its detection depends on the energy scale of the particular model. So, in the same way as we do for the pion fields, we can write an effective lagrangian below the energy breaking scale for the axion field $a(x)$:

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a + \left(\theta_{QCD} + \frac{a}{f_a}\right)\frac{\alpha_s}{8\pi}F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a. \quad (3\text{-}144)$$

Then, we can suppose that the $U(1)_{PQ}$ acts in the same way as the usual axial symmetry making $\theta_a = \frac{a}{f_a}$ as in (3-49). So, the new symmetry adds a new contribution to the meson scalar potential

$$V \sim -m_u v^3 \cos(\theta_0 + \theta_3) - m_d v^3 \cos(\theta_0 - \theta_3) - \Lambda_{QCD}^4 \cos(2\theta_0 - \theta_{SM} + \theta_a). \tag{3-145}$$

Thus, in the minimum, we can avoid any possible value of θ_{SM} making $\theta_0 = \theta_3 = 0$ and $\theta_a = \theta_{SM}$. But, the spontaneously breaking of the $U(1)_{PQ}$ symmetry can destroy our model, because it can induce again CP violating terms. However, considering that $a(x)$ is a dynamical field, the Vafa-Witten theorem ensures that the axion field evolves towards $\langle a \rangle = 0$, so there is no new CP violating phases. Even in the presence of external sources of CP violation (as complex phases in CKM matrix), observables associated to θ- term imposes constrictions over the maximum possible value $\sum |\theta_k| < 1,3 \times 10^{-10}$.

3.9. Extensions to SM

It is easy to see that is impossible to implement this model into the SM. Writing the anomala PQ symmetry as

$$\phi \to e^{ix_\phi \alpha}, \quad u_R \to e^{ix_u \alpha} u_R, \quad d_R \to e^{ix_d \alpha} d_R. \tag{3-146}$$

We assume that the left-handed quarks does not transform under PQ (this assumption maintain the condition of the anomala nature for the PQ symmetry).Writing down the SM yukawa lagrangian:

$$\mathcal{L}_Y = -y_u \bar{q}_L \tilde{\phi} u_R - y_d \bar{q}_L \phi u_R, \tag{3-147}$$

where we use the usual definition for the fields ($\bar{q}_L$ corresponding to the left quarks and ψ to the higgs field are $SU(2)$ doublets and q_R are right-handed fields that are singlets to the isospin symmetry. In this way, under the definition $\tilde{\psi} = i\sigma_2 \psi^*$, we see that the PQ symmetry is accomplished by all the terms under the condition:

$$-x_\phi + x_u = 0, \quad x_\phi + x_d = 0. \tag{3-148}$$

Therefore, the anomalous term transforms as:

$$\frac{\alpha_s \theta_{QCD}}{8\pi} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a \to \frac{\alpha_s \left[\theta_{QCD} - \alpha(x_u + x_d/2\right]}{8\pi} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a. \tag{3-149}$$

Thus, there is no possibility to absorb the θ- term. through the PQ mechanism, Also, there is no enough degrees of freedom to put the axion field. Therefore, is necessary to extend the usual SM with extra fields.
The first attempt to implement the PQ mechanism in a extended SM was proposed by

Peccei, Quinn, Weinberg and Wilzeck (PQWW). They added an extra Higgs doublet to the original content of SM fields. Thus, the yukawa term has the form:

$$\mathcal{L}_Y = -y_u \bar{q}_L \phi_2 d_R - y_d \bar{q}_L \bar{\phi}_1 u_R. \tag{3-150}$$

The VEV developed by the two higgs fields must fulfill the relationship $\sqrt{v_1^2 + v_2^2} = v = 246GeV$ in order to reproduce the electroweak scale. Two higgs doublets correspond to 8 degrees of freedom, then it is possible to introduce a non-trivial PQ charges assignment as:

$$\phi_i \to e^{ix_i\alpha}, \quad u_R \to e^{ix_u\alpha} u_R, \quad d_R \to e^{ix_d\alpha} d_R. \tag{3-151}$$

Then $x_1 = x_u$ and $x_2 = -x_d$. Three degrees of freedom are eaten by the gauge bosons and the axion corresponds to the pseudoscalar state among the added higgs, then the axion decay constant f_a in the PQWW model is proportional to the electroweak scale v . Axion interactions are proportional to the factor $1/f_a$, which implies that a lighter axion interacts more weakly than a heavier one. Experimentally, the value $f_a \approx v$ turns out to be to small, generating an axion that interacts too strongly with matter, which is completely ruled out by experimental results such as the branching ratio [75]:

$$\mathcal{B}(K^+ \to \pi^+ + nothing) < 7{,}3 \times 10^{-11} \tag{3-152}$$

but, the so-called *visible axion* predicts a significant larger value $\sim 10^{-8}$, then, the PQWW axion is excluded.

Additional constraints can be obtained from astrophysical considerations. The energy loss in process associated to axion emission by hot dense plasma is inversely proportional to f_a^2. Axions have to interact weakly enough in order not to affect the stellar evolution. Therefore, the lower bound on the axion decay constant has the value [76]:

$$f_a < 10^7 GeV, \tag{3-153}$$

which, for proper PQ axions can be translated to an upper bound on its mass $m_a < 0{,}1$eV. On the other hand, cosmology places an upper bound of f_a. Even if the axions are not the main component of dark matter, their density can not exceed the observed dark matter density (1-4. The axion density parameter takes the form [77]:

$$\Omega_a h^2 = \kappa_a \left(\frac{f_a}{10^{12}\text{GeV}} \right)^{7/6}. \tag{3-154}$$

Imposing that $\Omega_a \leq \Omega_{dm}$, we get:

$$f_a < 10^{11}\text{GeV}, \tag{3-155}$$

or, in terms of the mass $m_a > 10^{-5}$eV. Then, a window for dark matter axions could be:

$$10^7\text{GeV} < f_a < 10^{11}\text{GeV}, \qquad 10^{-5}\text{eV} < m_a < 0{,}1\text{eV} \tag{3-156}$$

But these limits should not be taken too strict, since they are model dependent [76]. This scale is larger than v preventing a strong coupling of the axions with ordinary matter. This kind of axions are so-called *invisible axions*.
One of those invisible axion models was proposed by Kim, Schifman, Vainshtein and Zakharov (KSVZ) [78], [79]. In this model the exotic added fields are a complex scalar singlet coupling to a new heavy additional quark. the only fields charged through the PQ transformation are the exotic ones and the scalar acquires VEV through a “Mexican-hat”potential. Another type of extension proposed by Dine, Fischler, Srednicki and Zhitnitsky (DFSZ) [80], [81]. This model increase the PQWW model with a scalar singlet, requiring that all fields in the theory have a PQ charge (except the gauge bosons). This extension is the one used in this work to solve the CP-problem.

4 $G_{SM} \otimes U(1)_X \times U(1)_{PQ}$ model

In order to solve some of the problems of the SM (mass hierarchy, massive neutrinos, strong CP - problem, etc), study scenarios beyond the standard model. As mentioned in chapter (1), one of the most preferred extensions to the SM the enlargement of the scalar sector by adding new Higgs doublets (and also Higgs singlets) in order to understand some facts such as the top/bottom mass ratio or to provide the spontaneously breaking of new symmetries. A new model is built assuming the existence of a new abelian interaction $U(1)_X$. In addition, as was explained in (3.8), the solution of the strong CP - problem requires the introduction of an anomalous $U(1)_{PQ}$ symmetry that allows canceling the term associated with the CP - violation by means of the color anomaly. Then, the scalar sector is extended in order to break the new symmetries and, under the premise of the existence of invisible axions, a DFSZ-type extension will be used which involves an additional Higgs doublet (3.9).

As mentioned in (3.9), in order to the pseudo-Goldstone boson associated with the axion would be invisible, the PQ-symmetry has to be broken at an scale energy much greater than the electroweak scale and in fact much larger than the $U(1)_X$ scale. The introduction of the PQ symmetry guarantees, after spontaneously symmetry breaking (SSB), the existence of a remanent Z_2 symmetry [82] that will be used in order to distinguish between doublets with the same X charge, where the mass matrices fermionic textures are produced in a suitable way, producing the necessary zeros in order to obtain the observed mass fermionic hierarchy.

4.1. Scalar sector

The model consists on a DFSZ type axion on which the additional symmetry $U(1)_{PQ}$ is structured. Some properties of the scalar sector are:

- The scalar singlet that allows the SSB of the $U(1)_X$ is χ with VEV in order $\sim TeV$ and the the singlet that generates the SSB of $U(1)_{PQ}$-symmetry is $S \sim f_a$ as was mentioned in (3.9).

- The need for two additional Higgs doublets ϕ_1, ϕ_2 is shared by the DFSZ model and the search of suitable fermionic mass matrices texture. The VEV are v_1, v_2 respectively in such a way that are related to the VEV electroweak scale by $v = \sqrt{v_1^2 + v_2^2}$.

Scalar bosons	X	$U(1)_{PQ}$
Higgs Doublets		
$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \frac{h_1 + v_1 + i\eta_1}{\sqrt{2}} \end{pmatrix}$	$2/3$	x_1
$\phi_2 = \begin{pmatrix} \phi_2^+ \\ \frac{h_2 + v_2 + i\eta_2}{\sqrt{2}} \end{pmatrix}$	$1/3$	x_2
Higgs Singlets		
$\chi = \frac{\xi_\chi + v_\chi + i\zeta_\chi}{\sqrt{2}}$	$-1/3$	x_χ
σ	$-1/3$	x_σ
$S = \frac{\xi_S + v_S + i\zeta_S}{\sqrt{2}}$	$-2/3$	x_S

Tab. **4-1**: Non-universal X quantum number and $U(1)_{PQ}$ for Higgs fields.

- An additional scalar singlet σ is introduced. It will useful later to produce masses through radiative corrections.

- In order to give suitable texture in the fermionic mass matrices, we will use the argument of [13] related to the imposition of restrictions in the mass structures through the assignation of X and PQ charges. Following the model proposed in ref [20], it is possible to create hierarchical mass structures identifying zero-type mass matrices that allow production of hierarchycal mass eigenvalues related to the vacuum expectation values of the scalar field involved in spontaneously symmetry breaking. Our model is based in the same yukawa lagrangians, but we impose restrictions over the yukawa couplings through PQ charges instead of a Z_2 discrete symmetry with the same values of the X-charges in order to differentiate among different families.

The table (**4-1**) shows the scalar content of the model, including the two doublets ϕ_1 y ϕ_2 and the three singlets χ, σ, S in addition to the $U(1)_{PQ}$ labels.

4.1.1. Gauge boson masses (W^3_μ, B_μ, Z'_μ)

We can write the associated kinetic lagrangian as:

$$\mathcal{L}_{kin} = \sum_i (D_\mu \Psi_i)^\dagger (D^\mu \Psi_i), \tag{4-1}$$

where $\Psi_i = \phi_{1,2}, \chi, S$ The covariant derivatives are:

$$D_\mu \phi_i = \partial_\mu \phi_i - igW^a_\mu T_a \phi_i - ig'\frac{Y}{2}B_\mu \phi_i + ig_X X Z'_\mu \phi_i,$$
$$D_\mu \chi = \partial_\mu \chi - \frac{ig_\chi}{3} Z'_\mu \chi_i, \quad D_\mu S = \partial_\mu S, \quad D_\mu \sigma = \partial_\mu \sigma. \tag{4-2}$$

After the symmetry breaking, the $W^\pm_\mu = (W^1_\mu \mp W^2_\mu)/\sqrt{2}$ acquires masses $M_W = \frac{gv}{2}$. The neutral gauge bosons (W^3_μ, B_μ, Z'_μ) masses are obtained for the following matrix:

$$M_0^2 = \frac{1}{4}\begin{pmatrix} g^2v^2 & -gg'v^2 & -\frac{2}{3}gg_X v^2(1+c^2_\beta) \\ * & g'^2v^2 & \frac{2}{3}g'g_X v^2(1+c^2_\beta) \\ * & * & \frac{4}{9}g^2_X v^2_\chi \left[1+(1+3c^2_\beta)\frac{v^2}{V^2_\chi}\right], \end{pmatrix},$$

The M_0^2 has null determinant as is expected because the existence of the photon A_μ, which is a massless gauge boson So, the neutral masses are given by:

$$M_Z \approx \frac{gv}{2\cos\theta_W}, \qquad M_{Z'} \approx \frac{g_X v_\chi}{3}, \tag{4-3}$$

The matrix that diagonalized M_0 is given in [20] and has the form:

$$R_0 = \begin{pmatrix} s_W & c_W & 0 \\ c_W c_Z & -s_W c_Z & s_Z \\ -c_W s_Z & s_W s_Z & c_Z \end{pmatrix}, \tag{4-4}$$

where $\tan\theta_W = \frac{g'}{g}$ is the Weinberg angle and s_Z is the mixing angle between Z and Z' gauge bosons:

$$s_Z \approx \left(1+s^2_\beta\right)\frac{2g_X c_W}{3g}\left(\frac{m_Z}{m_{Z'}}\right)^2. \tag{4-5}$$

In order to define the mass eigenstates associated with the Goldstone bosons of the Z and Z' gauge fields (4-3), it is necessary to use the bilinear terms $Z_\mu \partial^\mu G_Z$ that coming from the kinetic term of the scalar fields. These contributions are expected to be canceled out with the bilinear terms originated in the gauge fixing. The gauge fixing condition has the form:

$$\mathcal{L}_{GF} = -\frac{1}{2}\left(\partial_\mu Z^\mu + M_Z G_Z\right)^2 - \frac{1}{2}\left(\partial_\mu Z'^\mu + M_{Z'} G_{Z'}\right)^2. \tag{4-6}$$

But, we are only interested in the mixed terms:

$$-M_Z G_Z \partial_\mu Z^\mu - M_{Z'} G_{Z'} \partial_\mu Z'^\mu. \tag{4-7}$$

Integrating by part we have the relevant components as:

$$M_Z Z^\mu \partial_\mu G_Z + M_{Z'} Z'^\mu \partial_\mu G_{Z'}. \tag{4-8}$$

These terms are expected to be canceled with a contribution from covariant derivative. In order to get the Goldstone boson mass eigenstates it is necessary to rotate the expression (4-2) in function of the mass eigenstates as:

$$\begin{aligned}
D_\mu = \partial_\mu &- ig\left(s_W A_\mu + c_W c_Z Z_\mu - c_W s_Z Z'_\mu\right) T_{3L}\\
&- ig\frac{s_W}{c_W}\left(c_W A_\mu - s_W c_Z Z_\mu + s_W s_Z Z'_\mu\right)\frac{Y}{2}\\
&- ig_X\left(s_Z Z_\mu + c_Z Z'_\mu\right),\\
= \partial_\mu &- igs_W\left(T_{3L}+\frac{Y}{2}\right)A_\mu\\
&-\left(\frac{ig}{c_W}c_Z\left(c_W^2 T_{3L} - s_W^2\frac{Y}{2}\right) + ig_X s_Z X\right)Z_\mu\\
&+\left(\frac{ig}{c_W}s_Z\left(c_W^2 T_{3L} - s_W^2\frac{Y}{2}\right) - ig_X c_Z X\right)Z'_\mu.
\end{aligned}$$

where we take into account the relation $g'/g = \tan W = t_W$. Ignoring the associated photon-term and taking $c_W^2 T_{3L} - s_W^2 = T_{3L} - s_W^2 Q$, we have:

$$\begin{aligned}
D_\mu = \partial_\mu &- \left(\frac{ig}{c_W}c_Z\left(T_{3L} - s_W^2 Q\right) + \frac{g_X}{g}c_W s_Z X\right)Z_\mu\\
&- ig_X\left(-\frac{g}{g_X}\frac{s_Z}{c_W}\left(T_{3L} - s_W^2 Q\right) + c_Z X\right)Z'_\mu.
\end{aligned}\tag{4-9}$$

Taking the scalar ϕ_1 field, the contribution of the covariant derivative applied to the neutral components of the scalar fields (since the charged components contribute nothing to the neutral Goldstone) is:

$$\begin{aligned}
D_\mu\phi_1 = \frac{i\partial_\mu\eta_1}{\sqrt{2}} &+ \left(-\frac{igc_Z}{c_W}\left(T_{3L} - s_W^2 Q\right) - ig_X X s_Z\right)Z_\mu\phi_1\\
&+\left(\frac{igs_Z}{c_W}\left(T_{3L} - s_W^2 Q\right) - ig_X X c_Z\right)Z'_\mu\phi_1.
\end{aligned}$$

In order to obtain the mass contributions, we can avoid the radial components, leaving only the contributions related to the VEV of the fields. Thus, taking into account that the action of the Q operator over the neutral components is equal to zero and replacing the corresponding values of the T_{3L} and hypercharge quantum numbers, it is possible to write:

$$\begin{aligned}
D_\mu\phi_1 = \frac{i\partial_\mu\eta_1}{\sqrt{2}} &+ \left(-\frac{ig}{c_W}c_Z\left(-\frac{1}{2}\right) - ig_X\left(\frac{2}{3}\right)s_Z\right)Z_\mu\frac{v_1}{\sqrt{2}}\\
&+\left(\frac{ig}{c_W}s_Z\left(-\frac{1}{2}\right) - ig_X\left(\frac{2}{3}\right)c_Z\right)Z'_\mu\frac{v_1}{\sqrt{2}}.
\end{aligned}$$

In the same way we have:

$$(D_\mu\phi_1)^\dagger = -\frac{i\partial_\mu\eta_1}{\sqrt{2}} + \left(\frac{ig}{c_W}c_Z\left(-\frac{1}{2}\right) + ig_X\left(\frac{2}{3}\right)s_Z\right)Z_\mu\frac{v_1}{\sqrt{2}} + \left(-\frac{ig}{c_W}s_Z\left(-\frac{1}{2}\right) + ig_X\left(\frac{2}{3}\right)c_Z\right)Z'_\mu\frac{v_1}{\sqrt{2}}.$$

In the kinetic term $(D_\mu\phi_1)^2$, we are only interested in the mixing products:

$$(D_\mu\phi_1)(D^\mu\phi_1)^\dagger \approx -v_1 Z_\mu\partial_\mu\eta_1\left(-\frac{g}{2c_W}c_Z + \frac{2g_X}{3}s_Z\right) - v_1 Z'_\mu\partial_\mu\eta_1\left(\frac{g}{2c_W}s_Z + \frac{2g_X}{3}c_Z\right). \tag{4-10}$$

Similarly:

$$(D_\mu\phi_2)(D^\mu\phi_2)^\dagger \approx -v_2 Z_\mu\partial_\mu\eta_2\left(-\frac{g}{2c_W}c_Z + \frac{g_X}{3}s_Z\right) - v_2 Z'_\mu\partial_\mu\eta_2\left(\frac{g}{2c_W}s_Z + \frac{g_X}{3}c_Z\right), \tag{4-11}$$

and, for the singlet χ

$$D_\mu\chi \approx \frac{i\partial_\mu\zeta_\chi}{\sqrt{2}} - ig_X\left(-\frac{1}{3}\right)s_Z\frac{v_\chi}{\sqrt{2}}Z_\mu - ig_X\left(-\frac{1}{3}\right)c_Z\frac{v_\chi}{\sqrt{2}}Z'_\mu \tag{4-12}$$

$$(D_\mu\chi)^\dagger \approx -\frac{i\partial_\mu\zeta_\chi}{\sqrt{2}} + ig_X\left(-\frac{1}{3}\right)s_Z\frac{v_\chi}{\sqrt{2}}Z_\mu + ig_X\left(-\frac{1}{3}\right)c_Z\frac{v_\chi}{\sqrt{2}}Z'_\mu \tag{4-13}$$

$$(D_\mu\chi)(D^\mu\chi)^\dagger \approx -v_\chi Z_\mu\partial_\mu\zeta_X g_X\left(-\frac{1}{3}\right)s_Z - v_\chi Z'_\mu\partial_\mu\zeta_X g_X\left(-\frac{1}{3}\right)c_Z. \tag{4-14}$$

Thus, matching the contributions of the covariant derivatives with the bilinear terms from the gauge fixing and replacing $M_Z = \frac{gv}{2c_W}$, $M_{Z'_\mu} = \frac{g_{v_X}}{3}$, we obtain for Z_μ and Z'_μ:

$$\partial_\mu G_{Z_\mu} = \frac{2c_W c_Z}{gv}\left[\frac{g}{2c_W}\left(v_1\partial_\mu\eta_1 + v_2\partial_\mu\eta_2\right)\right] + \frac{2c_W s_Z}{gv}\left[-\frac{g_X}{3}\left(2v_1\partial_\mu\eta_1 + v_2\partial_\mu\eta_2\right)\right] + \frac{2c_W}{gv}v_\chi\partial_\mu\zeta_\chi\frac{g_X}{3}s_Z, \tag{4-15}$$

$$\partial_\mu G_{Z'_\mu} = \frac{3}{g_\chi v_\chi}\left[\frac{g}{2c_W}s_Z\left(-v_1\partial_\mu\eta_1 - v_2\partial_\mu\eta_2\right) + \frac{g_\chi c_Z}{3}\left(-2\frac{v_1}{v_\chi}\partial_\mu\eta_1 - \frac{v_2}{v_\chi}\partial_\mu\eta_2\right)\right] + \frac{3}{g_\chi v_\chi}\left(v_\chi\partial_\mu\zeta\chi\frac{g_X}{3}c_Z\right). \tag{4-16}$$

Under the approximation $s_Z \sim 0, c_Z \sim 1$, it is possible to write:

$$G_{Z_\mu} = s_\beta\eta_1 + c_\beta\eta_2 + \frac{M_{Z'}}{M_Z}s_Z\zeta_\chi, \tag{4-17}$$

$$G_{Z'_\mu} = \zeta_\chi - 2\frac{v_1}{v_\chi}\eta_1 - \frac{v_2}{v_\chi}\eta_2. \tag{4-18}$$

The definition of Goldstone bosons allows us to impose new conditions for PQ-charges in order to decouple the axion.

4.1.2. PQ coupling to gauge bosons

The only PQ charged objects in the present model are the fermions and the higgs fields. There are no direct PQ coupling with the gauge bosons, so this implies that the phases eaten by these bosons are PQ neutral.Under the $U(1)_{PQ}$ symmetry, the scalar fields transforms as:

$$\phi_1 \to e^{ix_1\alpha}\phi_1, \quad \phi_2 \to e^{ix_2\alpha}\phi_2, \quad \chi \to e^{ix_\chi\alpha}\chi, \quad S \to e^{ix_S\alpha}S, \quad \sigma \to e^{ix_\sigma\alpha}\sigma, \tag{4-19}$$

and the current associated with the PQ-transformation is given by:

$$\begin{aligned} J_\mu^{PQ} &= \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi_1} + \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi_2} + \frac{\delta\mathcal{L}}{\delta\partial^\mu\chi}\frac{\delta\mathcal{L}}{\delta\partial^\mu S} + h.c \\ &= x_S v_S i\partial_\mu\zeta_S + x_\chi v_\chi i\partial_\mu\zeta_\chi + x_2 v_2\partial_\mu\eta_2 + x_1 v_1\partial_\mu\eta_1, \end{aligned} \tag{4-20}$$

which must be orthogonal with neutral currents at low energy

$$\left\langle J_\mu^{PQ}|G_Z\right\rangle = x_1 v_1 s_\beta \left\langle\partial_\mu|\eta_1\right\rangle + x_2 v_2 c_\beta \left\langle\partial_\mu\eta_2|\eta_2\right\rangle + \frac{m_{Z'_\mu}}{m_{Z_\mu}} s_Z x_Z v_\chi \left\langle\partial_\mu\zeta_\chi|\zeta_\chi\right\rangle = 0, \tag{4-21}$$

$$\left\langle J_\mu^{PQ}|G_{Z'}\right\rangle = -\frac{2v_1}{v_\chi}x_1 v_1 \left\langle\partial_\mu\eta_1|\eta_1\right\rangle - \frac{v_2}{v_\chi}x_2 v_2 \left\langle\partial_\mu\eta_2|\eta_2\right\rangle + x_\chi v_\chi \left\langle\partial_\mu\zeta_\chi|\zeta_\chi\right\rangle = 0. \tag{4-22}$$

We can simplify the last expression using s_Z using (4-5):

$$\frac{m_{Z'}}{m_Z}s_Z \approx \left(\frac{2v_1^2+v_2^2}{v^2}\right)\frac{v}{v_\chi}. \tag{4-23}$$

Therefore, we can write the following restrictions:

$$\begin{aligned} 0 &= v_1^2 x_1 + v_2^2 x_2 + (2v_1^2+v_2^2)x_\chi, \\ 0 &= 2v_1^2 x_1 + v_2^2 x_2 - v_\chi^2 x_\chi. \end{aligned} \tag{4-24}$$

In addition, the λ_{14} term in the scalar potential (4-28) generates the following equation:

$$x_\chi - x_S - x_1 + x_2 = 0. \tag{4-25}$$

So using eqs. (4-24), (4-25) and choosing the normalization condition

$$x_S - x_\chi = 1, \tag{4-26}$$

it is possible to write:

$$\begin{aligned} x_1 &= -\frac{v_2^2(2v_1^2+v_2^2+v_\chi^2)}{(2v_1^2+v_2^2)^2+v^2v_\chi^2}, \qquad & x_\chi &= -\frac{v_1^2v_2^2}{(2v_1^2+v_2^2)^2+v^2v_\chi^2}, \\ x_2 &= 1+x_1, & x_S &= 1+x_\chi. \end{aligned} \tag{4-27}$$

4.1.3. Higgs potential

The most general potential that could be written with the particle content showed in (**4-1**) and the scalar PQ-restrictions (4-27) is:

$$\begin{aligned}V &= \mu_1^2\phi_1^\dagger\phi_1 + \mu_2^2\phi_2^\dagger\phi_2 + \mu_\chi^2\chi^*\chi + \mu_\sigma^2\sigma^*\sigma + \mu_S S^* S + \lambda_1\left(\phi_1^\dagger\phi_1\right)^2 \\ &+ \lambda_2\left(\phi_2^\dagger\phi_2\right)^2 + \lambda_3\left(\chi^*\chi\right)^2 + \lambda_4\left(\sigma^*\sigma\right)^2 + \lambda_5\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2\right) \\ &+ \lambda_5'\left(\phi_1^\dagger\phi_2\right)\left(\phi_2^\dagger\phi_1\right) + \left(\phi_1^\dagger\phi_1\right)\left[\lambda_6\left(\chi^*\chi\right) + \lambda_6'\left(\sigma^*\sigma\right)\right] \\ &+ \left(\phi_2^\dagger\phi_2\right)\left[\lambda_7\left(\chi^*\chi\right) + \lambda_7'\left(\sigma^*\sigma\right)\right] + \lambda_8\left(\chi^*\chi\right)\left(\sigma^*\sigma\right) + \lambda_9(S^*S)^2 \\ &+ (S^*S)\left[\lambda_{10}\left(\phi_1^\dagger\phi_1\right) + \lambda_{11}\left(\phi_2^\dagger\phi_2\right) + \lambda_{12}\left(\chi^*\chi\right) + \lambda_{13}\left(\sigma^*\sigma\right)\right] \\ &+ \lambda_{14}\left(\chi S^*\phi_1^\dagger\phi_2 + h.c.\right). \end{aligned} \tag{4-28}$$

where the term proportional to λ_{14} is necessary to avoid trivial PQ charges for the scalar sector. At this point it is worth mentioning that the VEV of S singlet is the one that generates the mass of the right-handed neutrinos.

4.1.4. Potential minimization

Replacing the definition of the scalar fields in (4-28), we obtain the following minimization conditions:

$$\begin{aligned}-\mu_1^2 &= \lambda_1 v_1^2 + \frac{1}{2}\left(\bar{\lambda}_5 v_2^2 + \lambda_6 v_\chi^2 + \lambda_{10} v_S^2 + \lambda_{14}\frac{v_2 v_\chi v_S}{v_1}\right), \\ -\mu_2^2 &= \lambda_2 v_2^2 + \frac{1}{2}\left(\bar{\lambda}_5 v_1^2 + \lambda_7 v_\chi^2 + \lambda_{11} v_S^2 + \lambda_{14}\frac{v_1 v_\chi v_S}{v_2}\right), \\ -\mu_\chi^2 &= \lambda_3 v_\chi^2 + \frac{1}{2}\left(\lambda_6 v_1^2 + \lambda_7 v_2^2 + \lambda_{12} v_S^2 + \lambda_{14}\frac{v_1 v_2 v_S}{v_\chi}\right), \\ -\mu_S^2 &= \lambda_9 v_S^2 + \frac{1}{2}\left(\lambda_{10} v_1^2 + \lambda_{11} v_2^2 + \lambda_{12} v_\chi^2 + \lambda_{14}\frac{v_1 v_2 v_\chi}{v_S}\right). \end{aligned} \tag{4-29}$$

with $\bar{\lambda}_5 = \lambda_5 + \lambda_5^{'}$. The scalar mass spectrum is obtained expanding the scalar potential to second order terms around the VEV given by the above conditions.

4.1.5. Charged scalar sector

The mass matrix for the charged scalar bosons expressed in the $\left(\phi_1^\pm, \phi_2^\pm\right)$ basis is

$$M_C^2 = \frac{1}{4}\begin{pmatrix} \lambda_5^{'} v_2^2 - \lambda_{14}\frac{v_2 v_\chi v_S}{v_1} & \lambda_5^{'} v_1 v_2 + \lambda_{14} v_\chi v_S \\ * & \lambda_5^{'} v_1^2 - \lambda_{14}\frac{v_1 v_\chi v_S}{v_2} \end{pmatrix}, \tag{4-30}$$

which is a Rank 1 matrix, so there are two Goldstone bosons which provides mass to the $W_\mu^\pm$ gauge bosons and two physical massive states corresponding to charged higgs bosons:

$$m^2_{G^\pm_W} = 0, \tag{4-31}$$

$$m^2_{H^\pm} = \frac{1}{2}\left(\lambda_5' v^2 - \lambda_{14}\frac{2v_\chi v_S}{s_{2\beta}}\right) \approx \frac{v^2}{2}\lambda_5' + m^2_{A_0}, \tag{4-32}$$

that corresponds to two physical charged Higgs.

4.1.6. CP-even scalars $(h_1, h_2, \xi_\chi, \xi_S)$

The CP-even sector has the mass matrix:

$$M_R^2 = \begin{pmatrix} \lambda_1 v_1^2 - \frac{\lambda_{14}}{4}\frac{v_2 v_\chi v_S}{v_1} & \frac{\bar{\lambda}_5}{2}v_1 v_2 + \frac{\lambda_{14}}{4}v_\chi v_S & \frac{\lambda_6}{2}v_1 v_\chi + \frac{\lambda_{14}}{4}v_2 v_S & \frac{\lambda_{14}}{4}v_2 v_\chi + \frac{\lambda_{10}}{2}v_1 v_S \\ * & \lambda_2 v_2^2 - \frac{\lambda_{14}}{4}\frac{v_1 v_\chi v_S}{v_2} & \frac{\lambda_7}{2}v_2 v_\chi + \frac{\lambda_{14}}{4}v_1 v_S & \frac{\lambda_{14}}{4}v_1 v_\chi + \frac{\lambda_{11}}{2}v_2 v_S \\ * & * & \lambda_3 v_\chi^2 - \frac{\lambda_{14}}{4}\frac{v_1 v_2 v_S}{v_\chi} & \frac{\lambda_{14}}{4}v_1 v_2 + \frac{\lambda_{12}}{2}v_\chi v_S \\ * & * & * & \lambda_9 v_S^2 - \frac{\lambda_{14}}{4}\frac{v_1 v_2 v_\chi}{v_S} \end{pmatrix}. \tag{4-33}$$

This matrix has Rank $M_R^2 = 4$ and we define $\bar{\lambda}_5 = \lambda_5 + \lambda_5'$. In order to obtain the eigenvalues, we use the VEV hierarchy $v_S \gg v_\chi \gg v$ to calculate them perturbatively. Through the scaling of couplings in the scalar potential in eq. (4-28), it is possible to made our model technically natural generating an explicit decoupling between SM and the neutral singlet on the PQ-scale. So, requiring the relations:

$$\lambda_6 \equiv a_6\frac{v_1^2}{v_\chi^2}, \quad \lambda_7 \equiv a_7\frac{v_2^2}{v_\chi^2}, \quad \lambda_{10} \equiv a_{10}\frac{v_1^2}{v_S^2}, \quad \lambda_{11} \equiv a_{11}\frac{v_2^2}{v_S^2},$$
$$\lambda_{12} \equiv a_{12}\frac{v_\chi^2}{v_S^2}, \quad \lambda_{14} \equiv a_{14}\frac{v^2}{v_\chi v_S},$$

is possible to build a natural hierarchy between the PQ and electroweak scale, without unpleasant fine tuning [83]. Thus, the leading contribution to the M_R^2-matrix can be approximated by the contributions related to the terms $\gtrsim \mathcal{O}^2$:

$$M_{\mathrm{R}}^2 = \begin{pmatrix} \lambda_1 v_1^2 - \frac{a_{14}}{4}\frac{v^2 v_2}{v_1} & \frac{\bar{\lambda}_5}{2}v_1 v_2 + \frac{a_{14}}{4}v^2 & 0 & 0 \\ \frac{\bar{\lambda}_5}{2}v_1 v_2 + \frac{a_{14}}{4}v^2 & -\frac{a_{14}}{4}\frac{v^2 v_1}{v_2} + \lambda_2 v_2^2 & 0 & 0 \\ 0 & 0 & \lambda_3 v_\chi^2 & 0 \\ 0 & 0 & 0 & \lambda_9 v_S^2 \end{pmatrix}, \tag{4-34}$$

Then, at LO, the heavier eigenvalues can be written as $\lambda_3 v_\chi^2$ and $\lambda_9 v_S^2$. In order to obtain the largest eigenvalue of $\mathcal{O} \sim v^2$ it is possible to neglect non-dominant terms from the condition $v_\chi v_S \gg v_1^2, v_2^2, v_1 v_2$ in the 2×2 superior matrix:

$$M_{\mathcal{O}\sim v^2}|_{v_\chi v_S \gg v_1^2, v_2^2, v_1 v_2} = -a_{14} v^2 \begin{pmatrix} \cot\beta & -1 \\ 1 & \tan\beta \end{pmatrix}. \tag{4-35}$$

The largest eigenvalue can be written as:

$$m_H^2 \approx Tr\left[M_{\mathcal{O}\sim v^2}|_{v_\chi v_S \gg v_1^2, v_2^2, v_1 v_2}\right] \approx -a_{14}\frac{v^2}{s_\beta c_\beta}, \tag{4-36}$$

and the lightest eigenvalue correspond to:

$$\frac{\text{Det}\left[M_{\mathcal{O}\sim v^2}\right]}{Tr\left[M_{\mathcal{O}\sim v^2}|_{v_\chi v_S \gg v_1^2, v_2^2, v_1 v_2}\right]} = \lambda_1 v^2, \tag{4-37}$$

where (4-37) can be identified as the SM Higgs, $t_\beta = v_1/v_2, \lambda_1, \lambda_3, \lambda_9 > 0$ and $a_{14} < 0$.

4.1.7. Cp-odd scalars $(\eta_1, \eta_2, \zeta_\chi, \zeta_S)$

The neutral pseudo-scalar in the basis $(\eta_1, \eta_2, \zeta_\chi, \zeta_S)$ has the mass matrix:

$$M_I^2 = -\frac{\lambda_{14}}{4}\begin{pmatrix} \frac{v_2 v_\chi v_S}{v_1} & -v_\chi v_S & -v_2 v_S & v_2 v_\chi \\ -v_\chi v_S & \frac{v_1 v_\chi v_S}{v_2} & v_1 v_S & v_1 v_\chi \\ -v_2 v_S & v_1 v_S & \frac{v_1 v_2 v_S}{v_\chi} & -v_1 v_2 \\ v_2 v_\chi & -v_1 v_\chi & -v_1 v_2 & \frac{v_1 v_2 v_\chi}{v_S} \end{pmatrix}, \tag{4-38}$$

where Rank $M_I^2 = 1$, so there are three zero modes, which implies the existence of three would-be Goldstone bosons associated with the bosons "eaten"by the vector bosons Z_μ and Z'_μ and the other corresponding to the axion related with the breaking of the $U(1)_{PQ}$ which obtains mass by non-perturbative QCD effects. The massive state is related with the pseudoscalar boson A^0 with mass:

$$m_{A^0}^2 = \frac{\lambda_{14}}{2}\left(\frac{2v_\chi v_S}{s_{2\beta}} + \frac{v^2(v_\chi^2 + v_S^2)s_{2\beta}}{2v_\chi v_S}\right) \approx -\frac{a_{14}v^2}{s_{2\beta}}. \tag{4-39}$$

4.2. Fermionic sector

The fermionic sector is build taking into account the cancellation of the local anomalies associated with the interaction term arising from the non-universal $U(1)_X$ and suppressed by the remanent symmetry after the breaking of the $U(1)_{PQ}$. The best method to implement the breaking $U(1)_{PQ} \to Z_2$ is to break the PQ symmetry by the VEV of the S - singlet. The singlet associated with the spontaneous breaking contain the axion field:

$$S = \frac{1}{\sqrt{2}}(f_a + \rho)e^{ia(x)/f_a}, \tag{4-40}$$

where, under the same effective mechanism described in (3-50) and adding the mass term in (3-144) it is possible to obtain the axion mass value:

$$m_a^2 = \frac{m_u + m_d}{(m_u + m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} = 5{,}70\mu eV\frac{10^{12}GeV}{f_a}. \tag{4-41}$$

This is the reason for take $f_a \sim v_S$. The SSB of the $U(1)_{PQ}$ through the VEV of the S-singlets generates a condition of periodicity of the effective potential in the variable a/f_a under a proportionality relation $\sim \sin\left(\mathcal{N}_{DW}\frac{a_s}{2v_S}\right)$. The effective potential is minimized for a collection of vacua:

$$\left\langle \frac{a_S}{2v_S} \right\rangle = 0, \frac{\pi}{\mathcal{N}_{DW}}, \cdots \frac{\pi(\mathcal{N}_{DW} - 1)}{\mathcal{N}_{DW}} \in [0, \pi], \tag{4-42}$$

with $\mathcal{N}_{DW}$ associated with the remain Z_2 symmetry. The hierarchical groups present in the fermionic sector induce us to think that the methods for mass acquisition for every group has to be the same, so our extension promotes the formation of mass textures that restrict the lagrangian terms to achieve this structure using the quantum numbers X, X_{PQ}, and Z_2. The constraints over the X - quantum number for the fermion sector have the origin in the behaviour of the scalar sector plus the cancellation of the chiral anomalies. Therefore, the non - universal X charges must vanish under the interaction with the other group generators presents in the theory. The restrictions coming from the triangular diagrams analyzed in chapter (2) taking into account the $U(1)_X$ and the exotic particles contribution reads as:

$$[\mathrm{SU}(3)_C]^2\,\mathrm{U}(1)_X \rightarrow \quad A_C = \sum_Q X_{Q_L} - \sum_Q X_{Q_R}, \tag{4-43}$$

$$[\mathrm{SU}(2)_L]^2\,\mathrm{U}(1)_X \rightarrow \quad A_L = \sum_\ell X_{\ell_L} + 3\sum_Q X_{Q_L}, \tag{4-44}$$

$$[\mathrm{U}(1)_Y]^2\,\mathrm{U}(1)_X \rightarrow \quad A_{Y^2} = \sum_{\ell,Q}\left[Y_{\ell_L}^2 X_{\ell_L} + 3Y_{Q_L}^2 X_{Q_L}\right] - \sum_{\ell,Q}\left[Y_{\ell_R}^2 X_{L_R} + 3Y_{Q_R}^2 X_{Q_R}\right], \tag{4-45}$$

$$\mathrm{U}(1)_Y\,[\mathrm{U}(1)_X]^2 \rightarrow \quad A_Y = \sum_{\ell,Q}\left[Y_{\ell_L} X_{\ell_L}^2 + 3Y_{Q_L} X_{Q_L}^2\right] - \sum_{\ell,Q}\left[Y_{\ell_R} X_{\ell_R}^2 + 3Y_{Q_R} X_{Q_R}^2\right], \tag{4-46}$$

$$[\mathrm{U}(1)_X]^3 \rightarrow \quad A_X = \sum_{\ell,Q}\left[X_{\ell_L}^3 + 3X_{Q_L}^3\right] - \sum_{\ell,Q}\left[X_{\ell_R}^3 + 3X_{Q_R}^3\right], \tag{4-47}$$

$$[\mathrm{Grav}]^2\,\mathrm{U}(1)_X \rightarrow \quad A_\mathrm{G} = \sum_{\ell,Q}\left[X_{\ell_L} + 3X_{Q_L}\right] - \sum_{\ell,Q}\left[X_{\ell_R} + 3X_{Q_R}\right]. \tag{4-48}$$

The last set of equations is accomplished by the fermionic spectrum showed in table (**4-2**), where the X_{PQ} charge are showed. The values of this quantum number are restricted by the X_{PQ} charges obtained by the scalar sector and the normalization condition for the S - singlet. The basis of our model is to restrict the values of the PQ-charges necessary to produce a Lagrangian that allows obtaining mass matrices with the appropriate texture according to what is studied in [20] to generate the SM fermionic mass hierarchy.

4.2.1. Quark sector lagrangian

According to the $SU(2)_L \otimes U(1)_Y \otimes U(1)_X \times U(1)_{PQ}$ symmetry, the most general lagrangian for the quark sector that produces suitable fermionic mass textures in order to produce the SM fermionic hierarchy is [20]:

$$\begin{aligned}
-\mathcal{L}_Q = {} & \overline{q_L^1}\left(\widetilde{\phi}_2 h_2^U\right)_{12} U_R^2 + \overline{q_L^1}\left(\widetilde{\phi}_2 h_2^T\right)_1 T_R + \overline{q_L^2}(\widetilde{\phi}_1 h_1^U)_{22} U_R^2 + \overline{q_L^2}(\widetilde{\phi}_1 h_1^T)_2 T_R \\
& + \overline{q_L^3}(\widetilde{\phi}_1 h_1^U)_{31} U_R^1 + \overline{q_L^3}(\widetilde{\phi}_1 h_1^U)_{33} U_R^3 + \overline{T_L}\left(\chi h_\chi^U\right)_2 U_R^2 + \bar{T}_L\left(\sigma h_\sigma^U\right)_{1,3} U_R^{1,3} \\
& + \overline{T_L}\left(\chi h_\chi^T\right) T_R + \overline{q_L^1}(\phi_1 h_1^J)_{1n} J_R^n + \overline{q_L^2}\left(\phi_2 h_2^J\right)_{2n} J_R^n + \overline{q_L^3}\left(\phi_2 h_2^D\right)_{33} D_R^3 \\
& + \overline{J_L^n}\left(\sigma^* h_\sigma^D\right)_{n(1,2)} D_R^{1,2} + \overline{J_L^n}\left(\chi^* h_\chi^J\right)_{nn} J_R^n + h.c.,
\end{aligned} \tag{4-49}$$

with $n = 1, 2$ and $\tilde{\phi}_{1,2} = i\sigma_2\phi_{1,2}^*$ are conjugate fields. The table **4-2** shows the fermionic content of the model and the notation used for the $U(1)_{PQ}$ charges.

In the last lagrangian we do not have take account terms like $\overline{q_L^3}\left(\phi_2 h_2^D\right)_{33} D_R^{(1,2)}$ because the charge for the $D_R^{1,2}$ quarks is generated by another mechanism will be explained later (related to the term $\overline{J_L^n}\sigma^* D_R^{1,2}$). The quark lagrangian (4-49) plus the scalar potential (4-28) restrict the values of the PQ charges. Then, in order to calculate the most general set of PQ-charges

Quarks	X	PQ-label	Leptons	X	PQ-label
SM Fermionic Isospin Doublets					
$q_L^1 = \begin{pmatrix} U^1 \\ D^1 \end{pmatrix}_L$	$+1/3$	$x_{q_L^1}$	$\ell_L^e = \begin{pmatrix} \nu^e \\ e^e \end{pmatrix}_L$	0	$x_{\ell_L^e}$
$q_L^2 = \begin{pmatrix} U^2 \\ D^2 \end{pmatrix}_L$	0	$x_{q_L^2}$	$\ell_L^\mu = \begin{pmatrix} \nu^\mu \\ e^\mu \end{pmatrix}_L$	0	$x_{\ell_L^\mu}$
$q_L^3 = \begin{pmatrix} U^3 \\ D^3 \end{pmatrix}_L$	0	$x_{q_L^3}$	$\ell_L^\tau = \begin{pmatrix} \nu^\tau \\ e^\tau \end{pmatrix}_L$	-1	$x_{\ell_L^\tau}$
SM Fermionic Isospin Singlets					
$U_R^{1,2,3}$	$+2/3$	$x_{U_R^{1,2,3}}$	$e_R^{e,\tau}$	$-4/3$	$x_{e_R^{e,\tau}}$
$D_R^{1,2,3}$	$-1/3$	$x_{D_R^{1,2,3}}$	e_R^μ	$-1/3$	$x_{e_R^\mu}$
Non-SM Quarks			Non-SM Leptons		
			$\nu_R^{e,\mu,\tau}$	$1/3$	$x_{\nu_R^{e,\mu,\tau}}$
T_L	$+1/3$	x_{T_L}	E_L	-1	x_{E_L}
T_R	$+2/3$	x_{T_R}	E_R	$-2/3$	x_{E_R}
$J_L^{1,2}$	0	$x_{J_L^{1,2}}$	$\mathcal{E}_L$	$-2/3$	$x_{\mathcal{E}_L}$
$J_R^{1,2}$	$-1/3$	$x_{J_R^{1,2}}$	$\mathcal{E}_R$	-1	$x_{\mathcal{E}_R}$

Tab. **4-2**: Non-universal X quantum number and PQ-labels for SM and non-SM fermions.

that allow the lagrangian (4-49), it is necessary to calculate the PQ-charges that restrict the model. Taking into account the restrictions imposed by (4-49), (4-28) and the values of the scalar PQ-charges (4-27), it is possible to write for the up sector:

$$-x_{q_L^1} - x_2 + x_{u_R^2} = 0, \qquad -x_{q_L^1} - x_2 + x_{T_R} = 0, \tag{4-50a-b}$$
$$-x_{q_L^2} - x_1 + x_{u_R^2} = 0, \qquad -x_{q_L^2} - x_1 + x_{T_R} = 0, \tag{4-50c-d}$$
$$-x_{q_L^3} - x_1 + x_{u_R^1} = 0, \qquad -x_{q_L^3} - x_1 + x_{u_R^3} = 0, \tag{4-50e-f}$$
$$-x_{T_L} + x_\chi + x_{u_R^2} = 0, \qquad -x_{T_L} + x_\chi + x_{T_R} = 0, \tag{4-50g-h}$$
$$-x_{T_L} + x_\sigma + x_{u_R^1} = 0. \tag{4-50i}$$

From eqs (4-50a), (4-50b), (4-50c) and (4-50d) it is possible to infer that $x_{u_R^2} = x_{T_R}$ and from (4-50e),(4-50f) we have $x_{u_R^1} = x_{u_R^3}$. Leaving the terms $x_{q_L^1}$ and $x_{q_L^3}$ free and without loss of generality we can use $x_{q_L^1} = x_{q_L^3}$. Replacing this conditions in the system we have:

$$x_{q_L^1} = -x_2 + x_{u_R^2} = -x_1 + x_{u_R^1}, \tag{4-51}$$
$$x_{q_L^2} = -x_1 + x_{u_R^2}, \tag{4-52}$$
$$x_{T_L} = x_\chi + x_{u_R^2} = x_\sigma + x_{u_R^1}. \tag{4-53}$$

Solving in function of the free parameters adding the last restrictions and the equation (4-50f), we can write:

$$x_{q_L^2} = -x_1 + x_2 + x_{q_L^1}, \qquad x_{u_R^1} = x_{q_L^1} + x_1, \tag{4-54}$$
$$x_{u_R^2} = x_{q_L^1} + x_2, \qquad x_{u_R^3} = x_{q_L^1} + x_1, \tag{4-55}$$
$$x_{T_L} = x_\chi + x_{q_L^1} + x_2, \qquad x_{T_R} = x_{q_L^1} + x_2, \tag{4-56}$$
$$x_\sigma = x_\chi + x_2 - x_1 = x_S. \tag{4-57}$$

Therefore, we get the set of PQ-charges related to the up-sector. The values of the PQ-charges allow the $T_L \sigma U_R^1$-vertex which is used to induce radiative corrections to 1-loop and generate the up quark mass.

In the same way (4-49) and (4-28) enforce the following restrictions for the down-sector:

$$-x_{q_L^1} + x_1 + x_{J_R^a} = 0, \qquad -x_{q_L^2} + x_2 + x_{J_R^a} = 0, \tag{4-58a-b}$$
$$-x_{q_L^3} + x_2 + x_{D_R^3} = 0, \qquad -x_{J_L^a} - x_\chi + x_{J_R^b} = 0, \tag{4-58c-d}$$
$$-x_{J_L^a} - x_\sigma + x_{D_R^1} = 0, \qquad -x_{J_L^a} - x_\sigma + x_{D_R^2} = 0. \tag{4-58e-f}$$

where $a, b = 1, 2$. The $J_L \sigma D_R$ couplings are necessary to find the masses of the down and strange quarks at 1-loop level. From eqs. (4-58a),(4-58c), (4-58d) and (4-58e-f) we obtain, respectively:

$$x_{J_R^1} = x_{J_R^2} = x_{q_L^1} - x_1, \tag{4-59}$$

$$x_{D_R^3} = x_2 - x_{q_L^3}, \tag{4-60}$$

$$x_{J_L^1} = x_{J_L^2} = x_{J_R^a} - x_\chi = x_{q_L^1} - x_1 - x_\chi, \tag{4-61}$$

$$x_{D_R^1} = x_{D_R^2} = -2x_1 + x_2 + 2x_{q_L^1} - x_{q_L^3}, \tag{4-62}$$

where we use the value of the x_σ charge given in (4-57).

Using the fact that the PQ-current is axial, we can put $x_{q_L^1} = x_{q_L^3} = 0$ without loss of generality. Therefore, we can obtain the whole set of the values of the PQ-charges in the fermionic sector as is shown in table (**4-3**).

PQ-label	PQ-charge	PQ-label	PQ-charge
SM Fermionic Isospin Doublets			
$x_{q_L^1}$	0	$x_{\ell_L^e}$	$-\frac{x_S}{2} - x_2$
$x_{q_L^2}$	1	$x_{\ell_L^\mu}$	$-\frac{x_S}{2} - x_2$
$x_{q_L^3}$	0	$x_{\ell_L^\tau}$	$-x_1 + \frac{x_S}{2} + x_\chi$
SM Fermionic Isospin Singlets			
$x_{U_R^{1,3}}$	x_1		
$x_{U_R^2}$	x_2	$x_{e_R^{e,\tau}}$	$-x_1 - x_2 + x_\chi + \frac{x_S}{2}$
$x_{D_R^{1,2}}$	$1 - x_1$	$x_{e_R^\mu}$	$-\frac{x_S}{2} - 2x_2$
$x_{D_R^3}$	$-x_2$		
Non-SM Quarks		Non-SM Leptons	
		$x_{\nu_R^{e,\mu,\tau}}$	$-\frac{x_S}{2}$
x_{T_L}	$x_\chi + x_2$	x_{E_L}	$-x_1 - x_2 + x_\chi - \frac{x_S}{2}$
x_{T_R}	x_2	x_{E_R}	$-x_1 - x_2 - \frac{x_S}{2}$
$x_{J_L^{1,2}}$	$-x_1 - x_\chi$	$x_{\mathcal{E}_L}$	$-2x_2 + \frac{x_S}{2}$
$x_{J_R^{1,2}}$	$-x_1$	$x_{\mathcal{E}_R}$	$-2x_2 + x_\chi + \frac{x_S}{2}$

Tab. **4-3**: Fermionic PQ-charge assignement according to the proposed lagrangian densities

4.2.2. Leptonic sector lagrangian

The most general lagrangian for the charged plus neutral leptonic sector that produces suitable zero textures for the mass matrices following the structure of ref. [20] is:

$$\begin{aligned}-\mathcal{L}_{Y,E} = & g^{2e}_{e\mu}\overline{\ell^e_L}\phi_2 e^\mu_R + g^{2e}_{\mu\mu}\overline{\ell^\mu_L}\phi_2 e^\mu_R + g^{2e}_{\tau e}\overline{\ell^\tau_L}\phi_2 e^e_R + g^{2e}_{\tau\tau}\overline{\ell^\tau_L}\phi_2 e^\tau_R + g^1_{Ee}\overline{\ell^e_L}\phi_1 E_R \\ & + g^1_{E\mu}\overline{\ell^\mu_L}\phi_1 E_R + h^{\sigma e}_E\overline{E_L}\sigma^* e^e_R + h^{\sigma\mu}_{\mathcal{E}}\overline{\mathcal{E}_L}\sigma e^\mu_R + h^{\sigma\tau}_E\overline{E_L}\sigma^* e^\tau_R + h^{\chi E}\overline{E_L}\chi E_R \\ & + h^{\chi\mathcal{E}}\overline{\mathcal{E}_L}\chi^*\mathcal{E}_R + h^{\nu i}_{2e}\overline{\ell^e_L}\tilde{\phi}_2\nu^i_R + h^{\nu i}_{2\mu}\overline{\ell^\mu_L}\tilde{\phi}_2\nu^i_R + h^{\nu j}_{Si}\overline{\nu^{i\,C}_R}S\nu^j_R, \end{aligned} \tag{4-63}$$

with $i, j = e, \mu, \tau$. The main difference with the cited reference is that we do not introduce the Majorana fields $N_R^{e,\mu,\tau}$ and $\nu_R^{e,\mu,\tau}$ get masses through the VEV of the S scalar field at PQ scale.

For the charged leptonic sector, the restrictions followed from the Yukawa lagrangians associated with the PQ-charges are:

$$-x_{\ell^e_L} + x_2 + x_{e^\mu_R} = 0, \qquad -x_{\ell^\tau_L} + x_2 + x_{e^e_R} = 0, \qquad -x_{\ell^e_L} + x_1 + x_{E_R} = 0, \tag{4-64a-c}$$

$$-x_{\ell^\mu_L} + x_2 + x_{e^\mu_R} = 0, \qquad -x_{\ell^\tau_L} + x_2 + x_{e^\tau_R} = 0, \qquad -x_{\ell^\mu_L} + x_1 + x_{E_R} = 0, \tag{4-64d-f}$$

$$-x_{E_L} - x_\sigma + x_{e^e_R} = 0, \qquad -x_{\mathcal{E}_L} + x_\sigma + x_{e^\mu_R} = 0, \qquad -x_{E_L} + x_\chi + x_{E_R} = 0, \tag{4-64g-i}$$

$$-x_{E_L} - x_\sigma + x_{e^\tau_R} = 0, \qquad -x_{\mathcal{E}_L} - x_\chi + x_{\mathcal{E}_R} = 0. \tag{4-64j-k}$$

In this case we take $x_{\ell^\mu_L}, x_{\ell^e_L}$ as free parameters. Thus, the other additional charges will be expressed in function of these ones. From eqs. (4-64a) and (4-64d) it is possible to see that $x_{\ell^\mu_L} = x_{\ell^e_L}$ and from eqs (4-64b) and (4-64e) we obtain $x_{e^e_R} = x_{e^\tau_R}$. In table (**4-3**), we summarize the PQ charges for charged leptons.

In order to give masses to neutrinos, the restrictions over the PQ-charges are:

$$-x_{\ell^e_L} - x_2 + x_{\nu^e_R} = 0, \qquad -x_{\ell^\mu_L} - x_2 + x_{\nu^e_R} = 0, \tag{4-65a-b}$$

$$-x_{\ell^e_L} - x_2 + x_{\nu^\mu_R} = 0, \qquad -x_{\ell^\mu_L} - x_2 + x_{\nu^\mu_R} = 0, \tag{4-65c-d}$$

$$-x_{\ell^e_L} - x_2 + x_{\nu^\tau_R} = 0, \qquad -x_{\ell^\mu_L} - x_2 + x_{\nu^\tau_R} = 0, \tag{4-65e-f}$$

$$x_{\nu^i_R} + x_S + x_{\nu^j_R} = 0. \tag{4-65g}$$

From eqs. (4-65a), (4-65c) and (4-65d) we conclude easily that $x_{\nu^e_R} = x_{\nu^\mu_R} = x_{\nu^\tau_R}$. The eqs. (4-65b), (4-65d) and (4-65f) are equivalents because $x_{\ell^e_L} = x_{\ell^\mu_L}$. Therefore:

$$x_{\nu^i_R} = x_2 + x_{\ell^\mu_L}, \quad i = e, \mu, \tau, \tag{4-66}$$

$$x_{\nu^i_R} = -\frac{x_S}{2}, \tag{4-67}$$

$$x_{\ell^\mu_L} = x_{\ell^e_L} = x_{\nu^i_R} - x_2 = -\frac{x_S}{2} - x_2. \tag{4-68}$$

Finally, a set of PQ-charges that reproduce the same Lagrangian densities given in [20] due to the Z_2-symmetry are obtained as shown in Table **4-3**.

4.3. Mass matrices

4.3.1. Quark mass matrices

The restrictions imposed by the scalar potential (4-28) the X-charges and the $U(1)_{PQ}$-charges, allow us to build after the symmetry breaking the following mass matrices with zero structures:

$$M_U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & (h_2^U)_{12}v_2 & 0 \\ 0 & (h_1^U)_{22}v_1 & 0 \\ (h_1^U)_{31}v_1 & 0 & (h_1^U)_{33}v_1 \end{pmatrix}, \qquad M_{UT} = \frac{1}{\sqrt{2}}\begin{pmatrix} (h_2^T)_1 v_2 \\ (h_1^T)_2 v_1 \\ 0 \end{pmatrix},$$
$$M_{TU} = \frac{v_\chi}{\sqrt{2}}\begin{pmatrix} 0 & (h_\chi^U)_2 & 0 \end{pmatrix}, \qquad M_T = \frac{v_\chi}{\sqrt{2}}(h_\chi^T)_1. \tag{4-69}$$

and the down sector is structured like:

$$M_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (h_2^D)_{33}v_2 \end{pmatrix}, \qquad M_{DJ} = \frac{1}{\sqrt{2}}\begin{pmatrix} (h_1^J)^1_{11}v_1 & (h_1^J)^1_{12}v_1 \\ (h_2^J)^2_{21}v_2 & (h_2^J)^2_{22}v_2 \\ 0 & 0 \end{pmatrix}, \tag{4-70}$$

$$M_J = \frac{v_\chi}{\sqrt{2}}\begin{pmatrix} (h_\chi^J)^\chi_{11} & (h_\chi^J)^\chi_{12} \\ (h_\chi^J)^\chi_{21} & (h_\chi^J)^\chi_{22} \end{pmatrix}, \qquad M_{DJ} = 0. \tag{4-71}$$

Then, we obtain the following extended matrices

$$\mathcal{M}_U = \begin{pmatrix} M_U & M_{UT} \\ M_{TU} & M_T \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & (h_2^U)_{12}v_2 & 0 & | & (h_2^T)_1 v_2 \\ 0 & (h_1^U)_{22}v_1 & 0 & | & (h_1^T)_2 v_1 \\ (h_1^U)_{31}v_1 & 0 & (h_1^U)_{33}v_1 & | & 0 \\ - & - & - & - & - \\ 0 & (h_\chi^U)_2 v_\chi & 0 & | & h_\chi^T v_\chi \end{pmatrix} \tag{4-72}$$

$$\mathcal{M}_D = \begin{pmatrix} M_D & M_{DJ} \\ M_{JD} & M_J \end{pmatrix}, = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & | & (h_1^J)_{11}v_1 & (h_1^J)_{12}v_1 \\ 0 & 0 & 0 & | & (h_2^J)_{21}v_2 & (h_2^J)_{22}v_2 \\ 0 & 0 & (h_2^D)_{33}v_2 & | & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & | & (h_\chi^J)_{11}v_\chi & 0 \\ 0 & 0 & 0 & | & 0 & (h_\chi^J)_{22}v_\chi \end{pmatrix}. \tag{4-73}$$

4.3.2. Up sector

Considering the extended matrix structure $\mathcal{M}_U$, we can calculate the symmetrical quadratic form as $M_U^2 = \mathcal{M}_U(\mathcal{M}_U)^T$:

$$M_U^2 = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{4-74}$$

where

$$A = \frac{1}{2}\begin{pmatrix} av_2^2 & bv_1v_2 & 0 \\ * & cv_1^2 & 0 \\ * & * & dv_1^2 \end{pmatrix}, \tag{4-75}$$

$$B = \frac{1}{2}\begin{pmatrix} ev_2v_\chi \\ mv_1v_\chi \\ 0 \end{pmatrix}, \qquad C = n\frac{1}{2}v_\chi^2. \tag{4-76}$$

Thus, as each block have the order $A \sim v_{1,2}^2, B \sim v_{1,2}v_\chi$ and $C \sim v_\chi^2$ it is possible to use a see-saw mechanism to obtain the block diagonalized matrix:

$$m_U^2 = V_U^T M_U^2 V_U = \begin{pmatrix} m_u^2 & 0 \\ 0 & m_T^2 \end{pmatrix}, \tag{4-77}$$

with $m_U^2 \approx A - BC^{-1}B^T$ and $m_T^2 \approx D$. Because C is only a number, we obtain the first mass eigenvalue from the exotic heavy T quark:

$$m_T^2 \approx \frac{1}{2}nv_\chi^2. \tag{4-78}$$

In order to obtain the other eigenvalues, we see that the m_U^2 is a matrix with the form:

$$m_U^2 = \frac{1}{2}\begin{pmatrix} h_1^2v_2^2 & h_1h_2v_1v_2 & 0 \\ * & h_2^2v_1^2 & 0 \\ * & * & dv_1^2 \end{pmatrix}. \tag{4-79}$$

The $(m_U^2)_{33}$ component is decoupled, so we can assume that corresponds to the mass of the top quark:

$$m_t^2 = \frac{1}{2}dv_1^2, \tag{4-80}$$

leaving a 2×2 matrix

$$m_{UC}^2 = \frac{1}{2}\begin{pmatrix} h_1^2v_2^2 & h_1h_2v_1v_2 \\ * & h_2^2v_1^2 \end{pmatrix}, \tag{4-81}$$

which has null determinant, associated to the fact that in this model $m_u = 0$. The only eigenvalue correspond to the mass of the charm quark is proportional to the trace of the m_{UC}^2 matrix:

$$m_c^2 \approx \frac{1}{2}h_2^2v_1^2, \tag{4-82}$$

where we choose $v_1 \gg v_2$ to generate the correct hierarchy.

In order to generate mass for the lightest quark is necessary to consider radiative 1-loop correction as indicate the first diagram in the fig. (**4-1**).

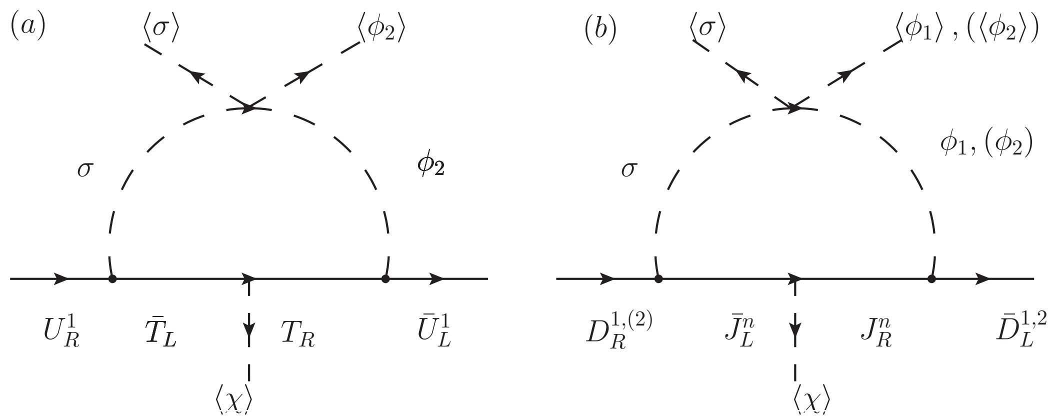


Fig. **4-1**: Radiative corrections for generate mass to the (a) up and (b) down sector

The 1- loop contribution obey the analytical expression given by:

$$\Sigma_{11}^{(u)} = \frac{-1}{16\pi^2}\frac{\lambda_7' \langle\sigma\rangle v_2 \left(h_\sigma^U\right)_1 \left(h_2^T\right)_1}{\sqrt{2}M_T} C_0\left(\frac{M_2}{M_T}, \frac{M_\sigma}{M_T}\right), \tag{4-83}$$

where:

$$C_0\left(x_1, x_2\right) = \frac{1}{\left(1-x_1^2\right)\left(1-x_2^2\right)\left(x_1^2-x_2^2\right)}\left[x_1^2 x_2^2 \ln\left(\frac{x_1^2}{x_2^2}\right) - x_1^2 \ln x_1^2 + x_2^2 \ln x_2^2\right], \tag{4-84}$$

adding new entries to the $(m_U^2)_{11,13,31}$ components. The $1-3$ and its symmetrical partner component only add corrections to the top-mass quark, but the correction is really small compared with the obtained value, so we neglect the correction. This leave us only with corrections in the 11 component, leading to a net contribution from (4-83) for the mass of the up quark. Going back to the original Lagrangian variables and summarizing:

$$\begin{aligned} m_u^2 &\sim \Sigma_{11}^{(u)}, & m_c^2 &\approx \frac{1}{2}v_1^2 \frac{(h_1^U)_2 h_\chi^T - (h_2^T)_1 (h_\chi^U)_2}{((h_\chi^U)_2)^2 + (h_\chi^T)^2}, \\ m_t^2 &\approx \frac{1}{2}v_1^2\left[(h_1^U)_{31})^2 + ((h_1^U)_{33})^2\right], & m_T^2 &\approx \frac{1}{2}v_\chi^2\left[((h_\chi^U)_2)^2 + (h_\chi^T)^2\right]. \end{aligned} \tag{4-85}$$

4.3.3. Down sector

Under the same approach for the up- sector, we obtain a quadratic extended matrix $M_D M_D^\dagger$ with similar structure to the matrix obtained in the up-sector:

$$M_D^2 = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{4-86}$$

Then, after block diagonalization making

$$m_D^2 \approx A - BC^{-1}B^T. \tag{4-87}$$

We can write M_D^2 as:

$$M_D^2 = \begin{pmatrix} m_D^2 & 0 \\ 0 & m_J^2 \end{pmatrix}. \tag{4-88}$$

Thus, there are two uncoupled components that corresponds to the mass of the two heavy exotic quarks $J^{1,2}$:

$$m_{J^1}^2 = \frac{1}{2} v_\chi^2 \left[(h_\chi^J)_{11}\right]^2, \qquad m_{J^2}^2 == \frac{1}{2} v_\chi^2 \left[(h_\chi^J)_{22}\right]^2. \tag{4-89}$$

The matrix m_D^2 only has one eigenvalue corresponding to the bottom quark:

$$m_b^2 = \frac{1}{2} \left[(h_2^D)_{33}\right]^2 v_2^2. \tag{4-90}$$

Then, there are two massless particles corresponding to the lightest d and s quarks. In order to assign finite values to this masses, we again use a radiative 1-loop correction as the shown in the diagram (b) of the figure (**4-1**). This diagram generates non-null entries for the m_D^2 matrix through the interaction with the σ-singlet. The self-energies generated at one-loop for down and strange quarks have the form:

$$\Sigma_{1a}^{(d)} = \frac{-1}{16\pi^2} \sum_{n=1,2} \frac{\lambda_6' \langle\sigma\rangle v_1 \left(h_1^J\right)_{1n} \left(h_\sigma^D\right)_{na}}{\sqrt{2} M_{J^n}} C_0 \left(\frac{M_1}{M_{J^n}}, \frac{M_\sigma}{M_{J^n}}\right), \tag{4-91}$$

$$\Sigma_{2a}^{(d)} = \frac{-1}{16\pi^2} \sum_{n=1,2} \frac{\lambda_7' \langle\sigma\rangle v_2 \left(h_2^J\right)_{2n} \left(h_\sigma^D\right)_{na}}{\sqrt{2} M_{J^n}} C_0 \left(\frac{M_2}{M_{J^n}}, \frac{M_\sigma}{M_{J^n}}\right), \tag{4-92}$$

with $a = 1, 2$. Thus, we can summarize the down quark masses as:

$$m_d \sim \Sigma_{1a}^{(d)}, \qquad m_s \sim \Sigma_{2a}^{(d)}, \tag{4-93}$$

$$m_b = \frac{1}{\sqrt{2}} (h_2^D)_{33} v_2, \qquad m_J^i = \frac{1}{\sqrt{2}} (h_\chi^J)_{ii} v_\chi. \tag{4-94}$$

4.3.4. Charged leptonic sector

After the spontaneous symmetry breaking, the lagrangian (4-63) allow to build the following extended mass matrix:

$$\mathcal{M}_{LC} = \begin{pmatrix} 0 & g_{e\mu}^{2e} & 0 & | & g_{Ee}^1 & 0 \\ 0 & g_{\mu\mu}^{2e} & 0 & | & g_{E\mu}^1 & 0 \\ g_{\tau e}^{2e} & 0 & g_{\tau\tau}^{2e} & | & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & | & h^{\chi E} & 0 \\ 0 & 0 & 0 & | & 0 & h^{\chi\mathcal{E}} \end{pmatrix}. \tag{4-95}$$

The squared mass matrix calculated as $\mathcal{M}_{LC}\mathcal{M}_{LC}^{\dagger}$ generates a $\mathcal{E}$ term decoupled. Then we can separate the terms in the flavor basis $\mathbf{E} = (e^e, e^\mu, e^\tau, E)$ as:

$$-\mathcal{L}_{Y,E} = \overline{\mathbf{E}_L}\mathbb{M}_E\mathbf{E}_R + \frac{H_2 v_\chi}{\sqrt{2}}\overline{\mathcal{E}_L}\mathcal{E}_R + \text{h.c.} \tag{4-96}$$

The see saw mechanism produces one massless lepton. Thus, it is necessary again use radiative 1-loop correction taking in account the interactions with the σ-singlet as showed in the figure (**4-2**) which add a contribution:

$$\Delta\mathbb{M}_E = \frac{v_2}{\sqrt{2}}\begin{pmatrix} \Sigma_{11} & 0 & \Sigma_{13} & | & 0 \\ \Sigma_{12} & 0 & \Sigma_{23} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & | & 0 \end{pmatrix}. \tag{4-97}$$

After the corresponding rotations, it is possible to diagonalize the squared mass matrix $\mathbb{M}_E\mathbb{M}_E^{\dagger}$ to obtain the following eigenvalues:

$$\begin{aligned} m_e &\approx \Sigma_{11}, & m_\mu &= \frac{v_2}{\sqrt{2}}\left[\left(g^{2e}_{e\mu}\right)^2 + \left(g^{2e}_{\mu\mu}\right)^2\right]^{1/2}, \\ m_\tau &= \frac{v_2}{\sqrt{2}}\left[\left(g^{2e}_{\tau e}\right)^2 + \left(g^{2e}_{\tau\tau}\right)^2\right]^{1/2}, & m_E &= \left(h^{\chi E}\right)\frac{v_\chi}{\sqrt{2}}, \\ m_{\mathcal{E}} &= \left(h^{\chi\mathcal{E}}\right)\frac{v_\chi}{\sqrt{2}}. \end{aligned} \tag{4-98}$$

where the self-energy is given by:

$$\Sigma_{11}^{(e)} = -\frac{1}{16\pi^2}\frac{\lambda_6' \langle\sigma\rangle v_1(g^1_{Ee})(h^{\sigma e}_E)}{M_E}C_0\left(\frac{M_1}{M_E}, \frac{M_\sigma}{M_E}\right). \tag{4-99}$$

4.3.5. Neutrino sector

Due to the energy scale of the energy break of the $U(1)_{PQ}$, the mass of the right neutrinos is in the order of v_S, so it is possible to find the mass of the active neutrinos through a typical see-saw mechanism. The mass lagrangian for the neutral leptonic sector in the basis $N = \left(\nu_L^{e,\mu,\tau}, (\nu_R^{e,\mu,\tau})^C\right)^T$ has the form:

$$-\mathcal{L} = h^{\nu i}_{2e}\bar{\ell}^e_L\tilde{\phi}_2\nu^i_R + h^{\nu i}_{2\mu}\bar{\ell}^\mu_L\tilde{\phi}_2\nu^i_R + h^{ij}_S\bar{\nu}^{Ci}_R S\nu^j_R, \tag{4-100}$$

where $i, j = e, \mu, \tau$. After the SSB, the mass lagrangian has the structure:

$$-\mathcal{L}_{\text{mass}} = h^{\nu i}_{2e}\frac{v_2}{\sqrt{2}}\bar{\nu}^e_L\nu^i_R + h^{\nu i}_{2\mu}\frac{v_2}{\sqrt{2}}\bar{\nu}^\mu_L\nu^i_R + h^{ij}_S\frac{v_S}{\sqrt{2}}\bar{\nu}^{Ci}_L\nu^j_R. \tag{4-101}$$

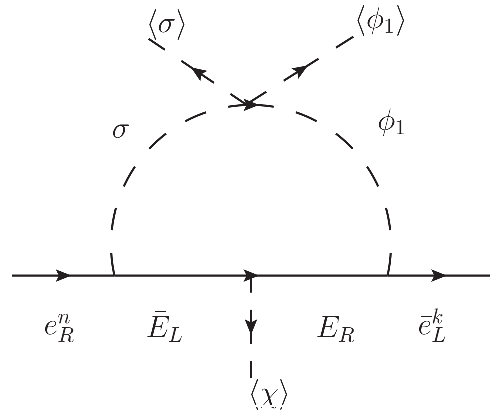


Fig. **4-2**: Radiative 1-loop correction for charged leptons

Without loss of generality, we consider h_S^{ij} a diagonal matrix, where:

$$\frac{h_S^{ij} v_S}{\sqrt{2}} = M^i \delta_i^j, \tag{4-102}$$

is the right-handed neutrino mass. Thus, in the $\left(\nu_L, \nu_R^C\right)$ basis the mass matrix can be written down as:

$$M_\nu = \begin{pmatrix} 0 & m_D^T \\ m_D & M_M \end{pmatrix}, \tag{4-103}$$

where

$$m_D = \frac{v_2}{\sqrt{2}} \begin{pmatrix} h_{2e}^{\nu e} & h_{2e}^{\nu\mu} & h_{2e}^{\nu\tau} \\ h_{2\mu}^{\nu e} & h_{2\mu}^{\nu\mu} & h_{2\mu}^{\nu\tau} \\ 0 & 0 & 0 \end{pmatrix}, \tag{4-104}$$

$$M_M = \frac{v_S}{\sqrt{2}} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}. \tag{4-105}$$

Then, under the see-saw mechanism, the light masses are of the form $m_\nu = M_D^T M_S M_D \sim \mathcal{O}\left(\frac{v_2^2}{v_S}\right)$. The diagonal of M_ν determine the mass eigenvalues where the light states has eigenvalue 0 and the squared mass differences $\Delta m_{12}^2, \Delta m_{23}^2$ depend on the Yukawa couplings. By diagonalizing M_ν, the mixing θ-angles that allows diagonalize the PMNS matrix are obtained. Using experimental data is possible to find a parameter region consistent with the neutrino oscillations. Performing the see-saw mechanism, the active neutrino mass matrix is given by:

$$
\begin{aligned}
m_{light} &\approx -m_D^T M_M^{-1} m_D \\
&\approx \frac{v_2^2}{\sqrt{2} h_1 v_S} \begin{pmatrix} (h_{2e}^{\nu e})^2 + (h_{2\mu}^{\nu e})^2 \rho & * & * \\ h_{2e}^{\nu e} h_{2e}^{\nu \mu} + h_{2\mu}^{\nu e} h_{2\mu}^{\nu \mu} \rho & (h_{2e}^{\nu \mu})^2 + (h_{2\mu}^{\nu \mu})^2 \rho & * \\ h_{2e}^{\nu e} h_{2e}^{\nu \tau} + h_{2\mu}^{\nu e} h_{2\mu}^{\nu \tau} \rho & h_{2e}^{\nu \mu} + h_{2e}^{\nu \tau} \rho & (h_{2e}^{\nu \tau})^2 + (h_{2\mu}^{\nu \tau})^2 + (h_{2\mu}^{\nu \tau})^2 \rho \end{pmatrix}.
\end{aligned}
\tag{4-106}
$$

In the reference [20], light neutrinos obtained mass through inverse see-saw mechanism and in this case through see-saw type I mechanism. It is possible to identify the parameters of the two models (inverse with type I) in the following way:

$$
\frac{\mu_N v_2^2}{h_{N\chi 1} v_\chi^2} \quad \rightarrow \quad \frac{v_2^2}{\sqrt{2} h_1 v_S} \approx \frac{m_\tau^2}{M_1}. \tag{4-107}
$$

Assuming that $v_2 \approx m_\tau$ and $v_S \approx 10^{10} GeV$, then light neutrinos will have a mass of the order of eV and the squared mass differences will be fixed by the $h_{2e(\mu)}^{\nu i}$ Yukawa couplings. We define the ρ-parameter as:

$$
\rho = \frac{M_1}{M_2} = \frac{h_1}{h_2}, \tag{4-108}
$$

where we assume the hierarchycal behaviour $M_1 < M_2 < M_3$, so the light-neutrino masses does not depend on the M_3 component. Taking into account this labelling, the neutrino oscillation treatment is reduced to that already studied in [?]. Thus, in order to make the model consistent with the neutrino oscillation data, the values of the Yukawa parameters are restricted to the same region. In particular for NO, we have:

$$
\rho = 0{,}5, \tag{4-109}
$$

and, for IO:

$$
\rho = 0{,}625. \tag{4-110}
$$

The tables (4) and (5) in [20] show the NO and IO respectively, where is obtained assuming that $|h_2| \approx 0{,}1$ and the right-handed neutrinos obey the mass hierarchy:

$$
M_1 \approx 10^{10} GeV = 0{,}5 M_2. \tag{4-111}
$$

5 Conclusions

The standard particle model presents some problems (fermion hierarchy, strong CP-problem, massless neutrino, etc.) that require the use of extended scenarios. During the develop of this work we have reviewed the concepts associated with the origin of the chiral anomaly, its emergence and subsequent development in the frames of abelian and non-abelian theories in order to explain the origin of the strong CP-problem and the origin of the axion. The need to use BSM theories was evident due to the impossibility of implementing an additional PQ type symmetry in the SM. We present a model BSM with an additional $U(1)_X \otimes U(1)_{PQ}$ symmetry in order to solve the three mentioned problems.

The $U(1)_X$ was built in such a way that the chiral anomalies was cancelled, guaranteeing the renormalizability of the theory. This model contains an extended scalar sector including 2 Higgs doublets $\phi_{1,2}$ with VEV on electroweak scale and 3 singlets χ, S, σ with VEV in order $TeV, 10^9 GeV, 0$ respectively. The singlet χ allows the SSB of the additional $U(1)_X$ symmetry giving mass to the new Z'_μ-boson. The S-singlet breaks the $U(1)_{PQ}$ symmetry generating a pseudo-Goldstone boson that is identified with the axion that obtain mass under non-perturbative methods . In order to forbid the interaction of the axion with the gauge bosons, a ortogonalization condition is imposed to guarantee that the phases eaten by this sector are neutral PQ charged . The scalar potential that is restricted by the judicious assignment of PQ loads allows us to find the mass spectrum analytically.

The fermionic sector was constructed as the usual SM adding exotic species that generate the complete cancellation of the chiral anomalies. The exotic set get heavy because the interactions with the χ-singlet generating the correct mass matrix textures for obtain three different energy scales: First, after SSB of the $U(1)_X$ symmetry, we obtain heavy masses to the exotic quarks J^n and T, with $M_{J^n} \approx M_T \approx v_\chi$. At tree level,the masses of the c, t and b quarks was obtained, with $M_{c,t,u} \approx v_{1,2}$. And using radiative corrections, we obtain masses for the u, d and s quarks, with $M_{u,d,s} \approx v_{1,2}^2/v_\chi$. For the leptonic sector, we also obtain the same hierarchical structure, where the extra leptons $m_E, m_{\mathcal{E}} \approx v_\chi$, $m_\mu, m_\tau \approx v$ and the m_e was obtained through loop corrections suppressed as $v_{1,2}^2/v_\chi$. The neutrino interact with the heavy S, so was possible to use a see-saw mechanism, generating masses on the order v_2/v_S. Thus, adding this two new symmetries to the SM lagrangian and a set of exotic particles, it was possible to solve three of the most important problems of fine tuning of the SM. In the case of active neutrinos, we changed the original inverse see-saw mechanism of the model by a see-saw type I mechanism, eliminating the Majorana fields $N_R^{e,\mu,\tau}$. The neutrino $\nu_R^{e,\mu,\tau}$ acquires mass through VEV of the scalar field S at the PQ scale *i.e.* $\langle S\rangle \sim 10^{10}$GeV. The mass structure of active neutrinos is the same as the presented in [20] if we identify:

$$\frac{\mu_N}{h_{N\chi 1} v_\chi^2} \to \frac{1}{\sqrt{2} h_1 v_S}. \tag{5-1}$$

Therefore, the regions allowed for the Yukawa couplings $h_{2e}^{\nu j}, h_{2\mu}^{\nu j}$ in the NO and IO orderings is the same as that obtained in Tables (4) and (5) of ref. [20], respectively.

As possible extended scenarios to continue developing this type of models, this work did not analyze the problem of the formation of domain walls. As we mention, the Z_2 symmetry is related to the formation of a collection of vacua, that could generate an interpolation between two different regions associated with different $\mathcal{N}_{\mathcal{DW}}$. This kind of field configuration is called a domain wall. Thus, it is necessary to implement several restrictions in order to solve this kind of non-trivial domains. There are possible solutions to this problem [77], that could contribute to the necessary restrictions to be able to generate masses at tree level in the complete model, besides being in complete agreement with cosmological restrictions, allowing a greater predictive power of the model.

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