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Gravitational radiation from the inspiral of compact binaries based on a Yukawa-type addition to the Newtonian potential

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Declaration of Authorship

I, Carlos Alfonso Conde Ocazionez, declare that this thesis titled, "Gravitational radiation from the inspiral of compact binaries based on a Yukawa-type addition to the Newtonian potential" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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UNIVERSIDAD NACIONAL DE COLOMBIA

Abstract

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Gravitational radiation from the inspiral of compact binaries based on a Yukawa-type addition to the Newtonian potential

by Carlos A. Conde O.

In this work, the gravitational radiation emitted from a compact binary system is analyzed in the context of general relativity and $f(R)$ gravity based on linearized theory. Besides the two standard polarizations of gravitational waves, an additional massive scalar mode is present in $f(R)$. At the Newtonian limit, it implies a Yukawa-like addition to the Newtonian potential. This kind of potential interaction has been studied in other scenarios. Here, the quadrupole radiation for the massless polarizations of a binary source in circular motion under such potential is determined. The back-reaction effect due to the emission of gravitational waves is discussed at linear and second order in $\Upsilon = 1/\lambda_g$ where λ_g is the Compton wavelength of the graviton. It is expected that in future measurements, slightly changes in the frequency waveform pattern of those systems may be put better constraints on the space parameters of alternative theories of gravity such as $f(R)$.

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To my family, for their love and patience...

Chapter 1

Introduction

Einstein special theory of relativity has revolutionized the understanding of space and time. As inseparable concepts, events takes place in a four dimensional entity: the *spacetime*. General relativity states that it becomes participant in the dynamics when including gravity as a geometrical structure with curvature. The Einstein field equations shows how matter and energy distribution characterize the spacetime properties that can be noticed by the motion of particles. In order to be compatible with the special theory, general relativity (GR) must be causal: the information about the changes in a gravitational source must propagate no faster than the speed of light, c [1]. This leads to the idea of *gravitational radiation*. The first theoretical intention to follow this idea was proposed by Einstein in 1916 and the strategy was based on the framework of linearized theory [2] . He considered that the gravitational field has the form of a slightly perturbed flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ with $\varepsilon \ll 1$. This assumption enables Einstein to derive a wave equation for the perturbation $h_{\mu\nu}$ by inserting $g_{\mu\nu}$ in the field equations and retaining only the linear terms in ε. Since in vacuum the solution are plane waves traveling at the speed of light, Einstein called this features of the spacetime *gravitational waves* (GWs). He also showed that these waves carry energy and are radiated by accelerated matter sources in a similar way as electromagnetic waves are produced due to the accelerate motion of charges. However, a fundamental problem posed initially by himself and later by others, was to proof their existence in the full theory of GR. After all, the theory is essentially nonlinear and must describe physical phenomena without any approximations. This issue bothered Einstein all his life and in fact he ended up believing against the real existence of GWs. Actually, the first attempt to define a plane GW in the full theory was due to Einstein and Rosen. They believed that they had found an exact solution of the vacuum field equations representing a plane GW. They argued that their solution had some unphysical singularities and so plane waves in linearized theory are merely a mathematical trick that does not represent the physical reality [3]. The Einstein-Rosen paper was refereed by Howard P. Robertson, who realized that the solution indeed was not coordinate independent. Only until the subsequent developments of Bondi, Pirani and Roberston a well defined GW in the full theory was in the right direction to be discovered. Their works gave an important advance mainly in the concept of a plane wave in the whole theory, its solution to the field equations and also their energy [4]. In spite of this progress, other questions remained unsolved, e.g. the extension to nonplanar front waves and the existence of radiative solutions from bounded sources. Andrzej Trautman was the person who deal with these subjects and established the definition of a GW in the full Einstein theory. Its general idea was to impose boundary conditions at infinity as a generalization of Sommerfled's radiation conditions [5]. Along with I. Robinson, they found a large class of exact solutions satisfying Trautman's conditions which can be interpreted as coming from bounded sources [6]. Additionally, further contributions began to emerge over time to give a robust theoretical approach to GWs[1].

One might think that if radiative metric solutions of the full Einstein field equations exist, linearized theory should be valid in the weak field regime at very large distances from the

[1]A very detailed and documented review about the history of gravitational waves can be found in [7].

sources [8]. Then, the approximate and the complete solution of the Einstein equations must coincide in this limit [4]. For instance, the quadrupole formula first derived by Einstein within the linearized gravity, trigger a very long discussion on its validity in this sense (among other issues). It states that the leading contribution of the energy loss due to the emission of GWs from a system, depends on the third time derivative of the quadrupole moment of the source. Furthermore, it tells that this waves are very difficult to produce and high masses moving at relativistic speeds are needed [1]. This means that the best candidates to generate them are compact binary systems, where the time variation of the quadrupole moment is a non-vanishing quantity. Unfortunately, linearized theory no longer stands for self-gravity systems such as binaries and that was first pointed out by Eddington[2]. However, with the advent of Landau & Lifshitz reformulation of the full Einstein equations in the 1980s, some close similarities with linearized theory began to arise. At the weak field limit and at spatial infinity, both theories agree in their equations. For self-gravity systems, this reformulation allows to extend the applicability of linearized theory. On the other hand, although the Landau & Lifshitz equations do not require any approximation method, so far there is no closed and exact analytical solution of the two-body problem in GR. Other approximation methods to face this problem like the post-Minkowskian and post-Newtonian formalisms are founded in the basis of the better well-posed arguments of the Landau & Lifshitz scheme [9]. Indeed, the quadrupole formula is deduced more formally by iterating two times the relaxed Einstein field equations in the post-Newtonian approach. From the practical point of view, only if there is such a well theoretical support on GWs is worthwhile to expend effort, time and money to detect them. If not, linearized theory by itself may be thought as a misleading approach to GWs even though it has its similarities with the full theory at the weak field limit. Notwithstanding, the first measurement of energy loss due to GWs emission in the 1980s, showed an indirect proof of the quadrupole formula by detecting the orbit decay of the binary pulsar of Hulse-Taylor PSR B1913+16 [10]. This was a great triumph of linearized theory because it was observed a rate of change in the orbital period which matches with the theoretical predictions done by Peters and Mathews in 1963, based on linear approximations and Keplerian orbits [11]. An interesting feature of the period formula is that it may be derived also with the post-Newtonian formalism [8]. Perhaps, this is the reason why linearized theory is still being used as a first description of GWs, because it behaves in agreement with many theoretical aspects at the weak field limit but also with experiments. Nowadays, the recent waves detected by LIGO (Laser Interferometer Gravitational-Wave Observatory) were so weak that they were treated as GWs in linearized theory [4][3].

The theory of GR is not the only theory of gravity. Just a few years after Einstein equations were published, new modifications to this theory started to be considered by including higher-order invariant quantities in the Einstein-Hilbert action [17]. Although at that epoch, such alternative theories have been studied for scientific curiosity or to understand the incoming vision of gravity as geometry, the motivation on these ideas rapidly began to emerge. In the 1960s, the first attempts to construct a quantized theory of gravity suggested that additional high-order terms corrections would allow the action to be renormalizable. More recently, the breakthrough of observational cosmology over the years has increased the incentive of modified theories of gravity. Various applications to recent problems such as inflation, dark matter, dark energy, cosmological perturbations, and also GWs, indicate that this models are worth alternatives that may be tested with experiments. Even if modifying GR is the way to go, this it is not an easy challenge. In particular, $f(R)$ gravity consists in not assume the lagrangian density of the action to be R as in GR but rather a function $f(R)$. For each particular function $f(R)$ a different model of gravity is obtained. Naturally, when $f(R) = R$,

[2]This will be explained in Chapter 6 with more detail.
[3]For more information about the recent detections of GWs see [12–16] .

then GR is recovered. For this family of theories and for others like scalar-tensor, scalar-tensor vector, Brans-Dicke, a natural Yukawa-like correction to the Newtonian potential arises at the Newtonian limit. Following this direction, some authors have been working on this *Yukawa gravitational potential* in other scenarios. Namely, the precession of bodies in the Solar System, the orbit of the $S2$ around Sgr A* at the Galactic Center, anomalistic period of celestial bodies, satellite dynamics, periastron shift, fifth force, GWs and so on [18–27].

This dissertation is based on the Yukawa-like potential applied to the gravitational radiation of a binary system in linearized $f(R)$ gravity [27]. The first part of the work is dedicated to the fundamentals of GWs in GR linearized theory. As was mentioned previously, this is a natural framework to study GWs in agreement with not only some aspects of the weak field limit of GR, but also with the experiments. The mathematical formality of linearized theory befall over the more deep understanding of perturbation theory in GR, which a brief introduction is given at the beginning of Chapter 2. Then, in order to study GWs at the weak field limit, perturbation theory is applied to flat Minkowski spacetime. The result is a linear wave equation for a perturbed quantity $h_{\mu\nu}$. The rest of this chapter deals with a discussion about gauge constraints and physical radiative degrees of freedom. In Chapter 3, the interaction of GWs with test masses is reviewed in order to see the effects of the two independent GW polarizations. This would be a preamble to define the energy-momentum tensor of GWs in Chapter 4. The generation of GWs in linearized theory is cover in Chapter 5, where some subtleties and assumptions about the sources and multipolar expansion techniques are explained. The extension of this developments to bound systems as is the case for a Newtonian compact binary is considered in Chapter 6. In Chapter 7, a discussion about GWs in linearized $f(R)$ gravity is presented in analogy to linearized GR. At the Newtonian limit, the Yukawa potential is obtained to be included in the interaction of a binary system at Chapter 8. Finally, some conclusions are exhibited in Chapter 9.

Chapter 2

Gravitational Waves in Linearized Theory

General relativity is the actual theory of gravity and is described essentially by the Einstein's Field Equations (EFE). These can be obtained from a variational principle with a suitable action $S[g]$ [28]. The stationary condition $\delta S = 0$, where the variation is taken with respect to the metric $g_{\mu\nu}$ gives rise to,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad . \tag{2.1}$$

To solve the EFE equations is a very hard task. Mathematically, is an impressive feat that people have found a lot of solutions [29]. Physically, just a few of them are useful such as the Schwarzschild, Kerr, FLRW, Vaidya, Weyl, etc. However, when natural phenomena is very hard to described based on these idealized solutions, a different approach is required. Perturbation theory is a useful technique to investigate realistic systems that are similar to the exact solutions. This section provides a very brief introduction to the general relativistic perturbation theory that will be of great significance for further developments.

2.1 Perturbation theory

Suppose one wish to know a slightly different solution of the Einstein's Field Equations (EFE) from a known one. This could be the case, for example, if one considers a system in which a small gravitational radiation is incident on a Schwarzschild black hole or a tiny deformation of it. In these cases, the effects could be viewed as perturbations of the original Schwarzschild spacetime. Then, it makes sense to find an approximate solution by postulating that the metric could be decomposed as

$$g_{\mu\nu}(\lambda, x) = \bar{g}_{\mu\nu}(0, x) + \lambda\tilde{h}^{(1)}_{\mu\nu}(x) + \frac{1}{2}\lambda^2\tilde{h}^{(2)}_{\mu\nu}(x) + \cdots \tag{2.2}$$

where

$$\tilde{h}^{(n)}_{\mu\nu} = \left.\frac{\partial^n g_{\mu\nu}(\lambda, x)}{\partial\lambda^n}\right|_{\lambda=0} \tag{2.3}$$

and $\bar{g}_{\mu\nu}(0, x)$ is the known solution called the background metric. The expansion parameter λ takes values from $[0, \epsilon]$ where $\epsilon \ll 1$. The equation (2.2) defines a one-parameter family of metrics [30]. Is usual to rewrite this equation in a more compact form as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots \qquad ; \qquad h^{(n)}_{\mu\nu} = \frac{1}{n!}\lambda^n\tilde{h}^{(n)}_{\mu\nu} \tag{2.4}$$

and so each of $h^{(n)}_{\mu\nu}$ gives the order of the perturbed metric. With this decomposition, the Einstein tensor can be computed until the desired order of accuracy. Furthermore, if one assumed for simplicity that the background metric satisfies the vacuum EFE, the problem can be solved by expanding the energy-momentum tensor in the right hand side of (2.1) in a similar way, setting $\lambda = \epsilon$ and solving order by order in ϵ. However, the equation (2.2) or (2.4) has a complication. The perturbed quantities must contain equal physical information under general coordinate transformations of the form

$$x^\mu \longrightarrow x'^\mu(\lambda) = x^\mu \; + \xi^\mu_1(x)\lambda + \frac{1}{2}\xi^\mu_2(x)\lambda^2 + \cdots \quad , \quad \xi^\mu_n \; = \; \frac{\partial^n x'^\mu}{\partial^n \lambda}\bigg|_{\lambda=0} \quad . \tag{2.5}$$

Or more formally, the solution needs to satisfy the diffeomorphism invariance of general relativity. To overcome this issue, a geometrical point of view of the perturbed solution is discussed.

Imagine there are two manifolds that represent equivalent spacetimes by means of a diffeomorphism ϕ. In each of the manifolds there are different tensorial fields, in one of them there is the background metric $\bar{g}_{\mu\nu}(0, x)$ and in the other the full metric $g_{\mu\nu}(\lambda, x)$ for a fixed value of λ, e.g. $\lambda = \epsilon$. They are often called the background spacetime $\mathcal{M}_\mathrm{B}$ and the physical spacetime $\mathcal{M}_\mathrm{P}$, accordingly [31]. Actually these are 4D submanifolds embedded in a 5D manifold $\mathcal{M}$ (with boundaries) that contains the entire one-parameter family of spacetimes for $\lambda \in [0, \epsilon]$. Therefore, the whole 5D manifold may be thought as a foliation of spacetimes of different values of λ for $\lambda \in [0, \epsilon]$ as shown in figure 2.1. To compare points between the physical and the background spacetimes, an identification map is required. This could be done through the integral curves defined by a vector field $\mathbf{u} = \partial_\lambda$. This vector field, called the generator of the diffeomorphism ϕ_λ belongs to the 5D manifold and it's always transverse to each spacetime hypersurface. Therefore, is possible to do comparisons in $\mathcal{M}_\mathrm{B}(0)$ between tensors that lie in $\mathcal{M}_\mathrm{P}(\lambda)$ via the integral curves. The perturbation of a tensor $Q(0, x)$ in $\mathcal{M}_\mathrm{B}$ is then defined as

$$\tilde{Q}(x) := \frac{\partial Q(\lambda, x)}{\partial \lambda}\bigg|_{\lambda=0} \; = \; \mathcal{L}_\mathbf{u} Q\bigg|_{\lambda=0} \quad . \tag{2.6}$$

This quantity could be for example the Riemann tensor or the metric itself. There is nothing to prevent the definition of the perturbation tensor in terms of another identification map Ψ_λ associated to a different integral curves that are generated by other vector field $\mathbf{v}$. Then, the difference between two equivalent perturbed tensors through different identification maps are related by

$$\begin{aligned} \tilde{Q}[\phi] \; - \; \tilde{Q}[\Psi] \; &= \; \mathcal{L}_\mathbf{u} Q \; - \; \mathcal{L}_\mathbf{v} Q \\ &= \; \mathcal{L}_{[\mathbf{u}-\mathbf{v}]} Q \\ &= \; \mathcal{L}_{\boldsymbol{\xi}} Q \quad , \end{aligned} \tag{2.7}$$

where $\boldsymbol{\xi} := [\mathbf{u} - \mathbf{v}]_{\lambda=0}$ is tangent to $\mathcal{M}_\mathrm{B}$. In particular if $\tilde{h}_{\mu\nu} = \tilde{h}_{\mu\nu}[\phi]$ and $\tilde{h'}_{\mu\nu} = \tilde{h}_{\mu\nu}[\Psi]$, then the difference between the metric perturbation tensors is

$$\begin{aligned} \tilde{h}_{\mu\nu} \; - \; \tilde{h'}_{\mu\nu} \; &= \; \mathcal{L}_{\boldsymbol{\xi}} g_{\mu\nu} \\ &= \; 2\nabla_{(\mu}\xi_{\nu)} \quad . \end{aligned} \tag{2.8}$$

Is important to mention the meanig of the lie derivative in the definition (2.6). It compares a tensor in a point $p(0)$ of $\mathcal{M}_B$. Thus, the perturbed metric $\tilde{h}_{\mu\nu}$ is just the difference between the background metric $\bar{g}_{\mu\nu}$ and the physical metric $g_{\mu\nu}$ after a pullback of the latter through the integral curves. Note that only if the gravitational fields in $\mathcal{M}_P$ are weak with respect to the background, the components of the perturbation tensor would be small. If one restrict only to those particular diffeomorphisms, then the physical metric $g_{\mu\nu}$ would be slighlty different of $\bar{g}_{\mu\nu}$ when the coordinates of the background are fixed[1]. The previous discussion is the geometrical view on what is called *gauge transformations* of perturbation theory.

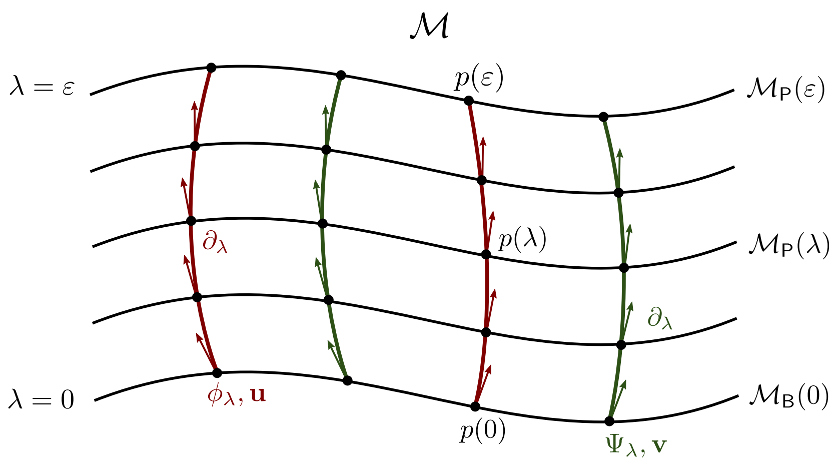

Figure 2.1: Schematic foliation of the spacetimes submanifolds in a 5D manifold $\mathcal{M}$. The vector fields $\mathbf{u}$ and $\mathbf{v}$ that defines de integral curves are within $\mathcal{M}$. A point $p(\lambda)$ on $\mathcal{M}_P(\lambda)$ is identified with a point $p(0)$ on $\mathcal{M}_B(0)$ if they lie in the same integral curve.

2.2 Expansion around flat spacetime

Consider the metric decomposition in (2.4) where the background is the flat metric. The first approach to gravitational waves is to consider the spacetime in a very large region, as a very slightly perturbation of the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad , \tag{2.9}$$

where $h_{\mu\nu}(x) = h^{(1)}_{\mu\nu}(x) = \varepsilon\tilde{h}^{(1)}_{\mu\nu}(x)$ with $\varepsilon \ll 1$ and $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. This kind of linear perturbation in ε around flat spacetime is the basis of **linearized theory**. By fixing the background coordinates one breaks the invariance of general relativity under coordinate transformations [32]. Moreover, the condition $\varepsilon \ll 1$ requires the gravitational field to be weak and the coordinate system to be approximately Lorentzian[1]. This assumption allows us to discard all higher order quantities that are not linear in ε. As a result, indices are raised and lowered with $\eta_{\mu\nu}$. At linear order, the inverse metric $g^{\mu\nu}$ is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad , \tag{2.10}$$

with $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. This can be proven from

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\ \nu} \tag{2.11}$$

[1]Consider for example the Minkowski background with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. The physical perturbed metric deviates very little from the original $\eta_{\mu\nu}$ in those Lorentzian coordinates.

by using the equations (2.9) and (2.10) in (2.11) i.e.,

$$\begin{aligned} g^{\mu\lambda}g_{\lambda\nu} &= \left(\eta^{\mu\lambda} - h^{\mu\lambda}\right)\left(\eta_{\lambda\nu} + h_{\lambda\nu}\right) \\ &= \eta^{\mu\lambda}\eta_{\lambda\nu} + \eta^{\mu\lambda}h_{\lambda\nu} - h^{\mu\lambda}\eta_{\lambda\nu} - h^{\mu\lambda}h_{\lambda\nu} \\ &= \delta^{\mu}{}_{\nu} + h^{\mu}{}_{\nu} - h^{\mu}{}_{\nu} - h^{\mu\lambda}h_{\lambda\nu} \\ &= \delta^{\mu}{}_{\nu} + \mathcal{O}(\varepsilon^2) \\ &\simeq \delta^{\mu}{}_{\nu} \quad . \end{aligned} \tag{2.12}$$

In the sense of special relativity, linearized theory is invariant under Lorentz transformations. This means that if the spacetime coordinates changes as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \tag{2.13}$$

the metric in (2.9) transforms into

$$\begin{aligned} g'_{\mu\nu}(x') &= \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}g_{\alpha\beta}(x) \\ &= \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}\left[\eta_{\alpha\beta} + h_{\alpha\beta}(x)\right] \\ &= \eta_{\mu\nu} + \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}h_{\alpha\beta} \quad , \end{aligned} \tag{2.14}$$

where the flat metric satisfy $\eta_{\mu\nu} = \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}\eta_{\alpha\beta}$. Using $g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x')$ then it is concluded that $h_{\mu\nu}$ is a tensor under Lorentz transformations[2],

$$h'_{\mu\nu}(x') = \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}h_{\alpha\beta}(x) \quad . \tag{2.15}$$

From the geometrical point of view, one may think linearized gravity as a perturbed symmetric tensor field propagating on a flat background spacetime[3]. Whereas the classical field approach consider $h_{\mu\nu}$ as any other field inside the flat spacetime without interpreting it as a metric perturbation. This work is based on the geometrical aspects of the linearized field equations that governs general relativity and $f(R)$ gravity. In order to obtain the linearized version of the field equations, all quantities related to the geometry of the spacetime are computed up to linear order in ε[4].

The connections

The connections are defined by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left[\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right] \quad . \tag{2.16}$$

Replacing the metric (2.9) in this equation gives

[2] It is straightforward to prove that $h_{\mu\nu}$ is also invariant under the Poincaré group.

[3] A further explanation will be given in the subsequent chapters.

[4] Unless the energy-mementum tensor of gravitational waves is considered, a second order terms needs to be included.

$$
\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2} g^{\alpha\sigma} \Big[\partial_\mu \Big(\eta_{\sigma\nu} + h_{\sigma\nu} \Big) + \partial_\nu \Big(\eta_{\sigma\mu} + h_{\sigma\mu} \Big) - \partial_\sigma \Big(\eta_{\mu\nu} + h_{\mu\nu} \Big) \Big] \\
&= \frac{1}{2} g^{\alpha\sigma} \Big[\partial_\mu \eta_{\sigma\nu} + \partial_\mu h_{\sigma\nu} + \partial_\nu \eta_{\sigma\mu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma \eta_{\mu\nu} - \partial_\sigma h_{\mu\nu} \Big] \\
&= \frac{1}{2} g^{\alpha\sigma} \Big[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \Big] \\
&= \frac{1}{2} \eta^{\alpha\sigma} \Big[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \Big] + \mathcal{O}(\varepsilon^2) \quad .
\end{aligned}
$$

Thus, one obtains the linearized connections,

$$
\boxed{\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \eta^{\alpha\sigma} \Big[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \Big]} \quad . \tag{2.17}
$$

The Riemann tensor

The Riemann curvature tensor is built from derivatives and products of connections. The only contribution will come from the derivatives of Γ's because the quadratic terms in Γ are of order $\mathcal{O}(\varepsilon^2)$,

$$
\begin{aligned}
R^{\sigma}{}_{\beta\mu\nu} &= \partial_\mu \Gamma^{\sigma}_{\nu\beta} - \partial_\nu \Gamma^{\sigma}_{\mu\beta} + \Gamma^{\sigma}_{\mu\lambda} \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\sigma}_{\nu\lambda} \Gamma^{\lambda}_{\mu\beta} \\
&\simeq \partial_\mu \Gamma^{\sigma}_{\nu\beta} - \partial_\nu \Gamma^{\sigma}_{\mu\beta} \quad .
\end{aligned} \tag{2.18}
$$

With all their covariant indices, the Riemann tensor could be also expressed as

$$
\begin{aligned}
R_{\alpha\beta\mu\nu} &= g_{\alpha\sigma} \Big\{ \partial_\mu \Gamma^{\sigma}_{\nu\beta} - \partial_\nu \Gamma^{\sigma}_{\mu\beta} \Big\} \\
&= \eta_{\alpha\sigma} \Big\{ \partial_\mu \Gamma^{\sigma}_{\nu\beta} - \partial_\nu \Gamma^{\sigma}_{\mu\beta} \Big\} \quad , \quad [g_{\alpha\sigma} \to \eta_{\alpha\sigma} \ \text{ at linear order in } \varepsilon] \\
&= \eta_{\alpha\sigma} \Big\{ \partial_\mu \Big[\frac{1}{2} \eta^{\sigma\lambda} \Big(\partial_\nu h_{\lambda\beta} + \partial_\beta h_{\lambda\nu} - \partial_\lambda h_{\nu\beta} \Big) \Big] - \partial_\nu \Big[\frac{1}{2} \eta^{\sigma\lambda} \Big(\partial_\mu h_{\lambda\beta} + \partial_\beta h_{\lambda\mu} - \partial_\lambda h_{\mu\beta} \Big) \Big] \Big\} \\
&= \frac{1}{2} \eta_{\alpha\sigma} \eta^{\sigma\lambda} \Big\{ \partial_\mu \Big(\partial_\nu h_{\lambda\beta} + \partial_\beta h_{\lambda\nu} - \partial_\lambda h_{\nu\beta} \Big) - \partial_\nu \Big(\partial_\mu h_{\lambda\beta} + \partial_\beta h_{\lambda\mu} - \partial_\lambda h_{\mu\beta} \Big) \Big\} \\
&= \frac{1}{2} \delta^{\lambda}_{\alpha} \Big\{ \underline{\partial_\mu \partial_\nu h_{\lambda\beta}} + \partial_\mu \partial_\beta h_{\lambda\nu} - \partial_\mu \partial_\lambda h_{\nu\beta} - \underline{\partial_\nu \partial_\mu h_{\lambda\beta}} - \partial_\nu \partial_\beta h_{\lambda\mu} + \partial_\nu \partial_\lambda h_{\mu\beta} \Big\} \\
&= \frac{1}{2} \delta^{\lambda}_{\alpha} \Big\{ \partial_\mu \partial_\beta h_{\lambda\nu} - \partial_\mu \partial_\lambda h_{\nu\beta} - \partial_\nu \partial_\beta h_{\lambda\mu} + \partial_\nu \partial_\lambda h_{\mu\beta} \Big\} \quad .
\end{aligned}
$$

Therefore, the Riemann curvature tensor is given at linear order by,

$$
\boxed{R_{\alpha\beta\mu\nu} = \frac{1}{2} \Big\{ \partial_\mu \partial_\beta h_{\alpha\nu} - \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\nu \partial_\beta h_{\alpha\mu} + \partial_\nu \partial_\alpha h_{\mu\beta} \Big\}} \quad . \tag{2.19}
$$

The Ricci tensor

The Ricci tensor comes from contracting the first contravariant index with the second covariant index of the Riemann tensor,

$$R_{\beta\nu} = g^{\mu\alpha} R_{\alpha\beta\mu\nu} \simeq \eta^{\mu\alpha} R_{\alpha\beta\mu\nu} \quad . \tag{2.20}$$

Using (2.19) in (2.20) gives,

$$\begin{aligned} R_{\beta\nu} &= \frac{1}{2}\eta^{\mu\alpha}\Big\{\partial_\mu\partial_\beta h_{\alpha\nu} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\alpha\mu} + \partial_\nu\partial_\alpha h_{\mu\beta}\Big\} \\ &= \frac{1}{2}\Big\{\partial^\alpha\partial_\beta h_{\alpha\nu} - \Big(\partial^\alpha\partial_\alpha\Big)h_{\nu\beta} - \partial_\nu\partial_\beta\Big(\eta^{\mu\alpha}h_{\alpha\mu}\Big) + \partial_\nu\partial^\mu h_{\mu\beta}\Big\} \qquad \mu : \textit{dummy index} \\ &= \frac{1}{2}\Big\{\partial^\alpha\partial_\beta h_{\alpha\nu} - \Box h_{\nu\beta} - \partial_\nu\partial_\beta h + \partial_\nu\partial^\alpha h_{\alpha\beta}\Big\} \qquad \mu \to \alpha \quad , \end{aligned}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation tensor and $\Box$ is the d'Alembertian operator in flat space, $\Box = \partial_\alpha\partial^\alpha = -(1/c^2)\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. The Ricci tensor takes the form

$$\boxed{R_{\beta\nu} = \frac{1}{2}\Big\{\partial_\beta\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\beta} - \partial_\beta\partial_\nu h - \Box h_{\beta\nu}\Big\}} \quad , \tag{2.21}$$

which is clearly symmetric in β and ν.

The Ricci scalar

The Ricci scalar is the trace of the Ricci tensor,

$$R^\nu{}_\nu = g^{\beta\nu} R_{\beta\nu} \simeq \eta^{\beta\nu} R_{\beta\nu} \quad . \tag{2.22}$$

Replacing (2.21) in (2.22) yields

$$\begin{aligned} R &= \frac{1}{2}\eta^{\beta\nu}\Big\{\partial_\beta\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\beta} - \partial_\beta\partial_\nu h - \Box h_{\beta\nu}\Big\} \\ &= \frac{1}{2}\Big\{\partial^\nu\partial^\alpha h_{\alpha\nu} + \partial^\beta\partial^\alpha h_{\alpha\beta} - \Box h - \Box h\Big\} \qquad \nu \to \beta \quad , \end{aligned}$$

$$\boxed{R = \partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h} \quad . \tag{2.23}$$

The Einstein tensor

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \simeq R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} \quad . \tag{2.24}$$

From equations (2.21) and (2.23) for the Ricci scalar and the scalar curvature, it is obtained

$$G_{\mu\nu} = \frac{1}{2}\Big\{\partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\Big\} - \frac{1}{2}\Big\{\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h\Big\}\eta_{\mu\nu}$$

$$\boxed{G_{\mu\nu} = \frac{1}{2}\Big\{\partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\}} \quad . \tag{2.25}$$

The linearized Einstein's field equations

The Einstein's field equations is given by

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad , \tag{2.26}$$

and using (2.25) in the left hand side of this equation gives the linearized version of the Einstein's field equations,

$$\boxed{\frac{1}{2}\Big\{\partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\} = \frac{8\pi G}{c^4}T_{\mu\nu}} \quad . \tag{2.27}$$

The form of the equations of motion in (2.27) can be simplified if it is introduced the *trace-reversed* perturbation tensor $\bar{h}_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad . \tag{2.28}$$

The name of this tensor follows from the trace property,

$$\begin{aligned} g^{\mu\nu}\bar{h}_{\mu\nu} &= \eta^{\mu\nu}\bar{h}_{\mu\nu} \\ \bar{h} &= \eta^{\mu\nu}\Big[h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\Big] \\ \bar{h} &= h - 2h = -h \quad . \end{aligned} \tag{2.29}$$

As a consequence, equation (2.28) can be inverted to get

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \quad . \tag{2.30}$$

Inserting $h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h$ in the Einstein tensor (2.25) all terms with the trace h cancel each other. That is,

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\Big\{\partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\} \\ &= \frac{1}{2}\Big\{\partial_\mu\partial^\rho\Big[\bar{h}_{\rho\nu} + \frac{1}{2}\eta_{\rho\nu}h\Big] + \partial_\nu\partial^\rho\Big[\bar{h}_{\rho\mu} + \frac{1}{2}\eta_{\rho\mu}h\Big] - \eta_{\mu\nu}\partial^\rho\partial^\sigma\Big[\bar{h}_{\rho\sigma} + \frac{1}{2}\eta_{\rho\sigma}h\Big] \\ &\quad - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\} \end{aligned}$$

$$G_{\mu\nu} = \frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \frac{1}{2}\underline{\partial_\mu\partial_\nu h} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} + \frac{1}{2}\underline{\partial_\mu\partial_\nu h} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\Box h$$
$$- \underline{\partial_\mu\partial_\nu h} + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\}$$
$$= \frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\Box h + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\}$$
$$= \frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} + \frac{1}{2}\eta_{\mu\nu}\Box h - \Box h_{\mu\nu}\Big\}$$
$$= \frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \Box\Big(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\Big)\Big\}$$
$$G_{\mu\nu} = \frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \Box\bar{h}_{\mu\nu}\Big\} \quad . \tag{2.31}$$

Finally, the linearized EFE can be rewritten in a more compact form as

$$\frac{1}{2}\Big\{\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \Box\bar{h}_{\mu\nu}\Big\} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.32}$$

i.e.,

$$\boxed{\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}} \quad . \tag{2.33}$$

Note that the trace-reversed $\bar{h}_{\mu\nu}$ has reduced the number of terms in (2.27) as can be seen in equation (2.33). Moreover, the form of this equation is almost a wave equation for $\bar{h}_{\mu\nu}$, except for the 4-divergence terms of $\bar{h}_{\mu\nu}$. The next section discuss how to obtain the wave equation by imposing a convenient gauge condition to fix the coordinates.

2.3 The Lorenz gauge and the wave equation

In section 2.1 a geometrical interpretation on gauge transformations has been analyzed. The freedom in the definition of an adequate mapping between the background and physical manifolds leads to equivalent perturbation tensors. That means that two different perturbation tensors are related by the gauge transformation given in equation (2.8). Since the background spacetime in linearized theory is the flat metric $\eta_{\mu\nu}$, the relation between $h_{\mu\nu}$ and $h'_{\mu\nu}$ in this context is given by

$$h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)} \quad . \tag{2.34}$$

This is the **gauge transformation of linearized theory** [31]. Now, however, we shall adopt a passive transformation perspective of linearized theory. This is to say that rather than perform active manifold transformations by means of diffeomorphisms, the difference between tensors is provided by an explicit coordinate transformation of the form given in (2.5). Hence, at linear order, the coordinates change by an infinitesimal shift,

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x) \qquad , \qquad \varepsilon^\mu = \varepsilon\xi^\mu(x) \quad . \tag{2.35}$$

Then, the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) \tag{2.36}$$

or equivalently by

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\, g'_{\alpha\beta}(x') \quad . \tag{2.37}$$

This is called the **gauge symmetry** of general relativity [8]. Although the choice of the Lorentizan coordinates of the flat metric $\eta_{\mu\nu}$ breaks the general invariance of general relativity, is expected to get a relation between the perturbation tensors $h_{\mu\nu}(x)$ and $h'_{\mu\nu}(x)$ which depends on the spacetime coordinates[5]. To do that, is possible to use either of the two equations in (2.36) and (2.37). If one uses the first one, there is a subtlety. To linear order in ε, the inverse to equation (2.35) is $x^\mu = x'^\mu - \xi^\mu(x')$ just because the difference between $\xi^\mu(x)$ and $\xi^\mu(x')$ is of order $\mathcal{O}(\varepsilon^2)$[9]. Then, compute $\partial x^\alpha/\partial x'^\mu$ to apply equation (2.36). Even though is more straightforward to follow equation (2.37). Due to the relation

$$\begin{aligned}\frac{\partial x'^\alpha}{\partial x^\mu} &= \frac{\partial[x^\alpha + \varepsilon^\alpha]}{\partial x^\mu} = \frac{\partial x^\alpha}{\partial x^\mu} + \frac{\partial \varepsilon^\alpha}{\partial x^\mu} \\ &= \delta^\alpha_\mu + \partial_\mu \varepsilon^\alpha \quad ,\end{aligned}$$

from (2.37) the metric changes as

$$\begin{aligned} g_{\mu\nu}(x) &= \left[\left(\delta^\alpha_\mu + \partial_\mu\varepsilon^\alpha\right)\left(\delta^\beta_\nu + \partial_\nu\varepsilon^\beta\right)\right] g'_{\alpha\beta}(x') \\ &= \left[\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\mu\partial_\nu\varepsilon^\beta + \delta^\beta_\nu\partial_\mu\varepsilon^\alpha + \partial_\mu\varepsilon^\alpha\partial_\nu\varepsilon^\beta\right] g'_{\alpha\beta}(x') \\ &= \left[\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\mu\partial_\nu\varepsilon^\beta + \delta^\beta_\nu\partial_\mu\varepsilon^\alpha + \partial_\mu\varepsilon^\alpha\partial_\nu\varepsilon^\beta\right]\left[\eta_{\alpha\beta} + h'_{\alpha\beta}(x')\right] \\ &= \eta_{\mu\nu} + h'_{\mu\nu}(x') + \partial_\nu\eta_{\mu\beta}\varepsilon^\beta + h'_{\mu\beta}(x')\partial_\nu\varepsilon^\beta + \partial_\mu\eta_{\alpha\nu}\varepsilon^\alpha \\ &\quad + h'_{\alpha\nu}(x')\partial_\mu\varepsilon^\alpha + \eta_{\alpha\beta}\partial_\mu\varepsilon^\alpha\partial_\nu\varepsilon^\beta + h'_{\alpha\beta}(x)\partial_\mu\varepsilon^\alpha\partial_\nu\varepsilon^\beta \\ g_{\mu\nu}(x) &= \eta_{\mu\nu} + h'_{\mu\nu}(x') + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^3) \\ \eta_{\mu\nu} + h_{\mu\nu}(x) &= \eta_{\mu\nu} + h'_{\mu\nu}(x') + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu \quad . \end{aligned} \tag{2.38}$$

Therefore, the gauge transformation of linearized theory shown in (2.34) is recover,

$$\boxed{h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \left[\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right]} \quad . \tag{2.39}$$

Bearing in mind the definition of $\bar{h}_{\mu\nu}$ in (2.28), the trace-reversed perturbation tensor change as follows. First, take the trace of (2.39) to obtain

[5] Recall that the coordinate system in linearized gravity is quasi-Lorentzian. There is a freedom on how the full metric deviates from the flat one and that's encoded in $h_{\mu\nu} = h_{\mu\nu}(x)$, but they must vary very slightly because $\eta_{\mu\nu}$ is fixed as $\mathrm{diag}(-1, 1, 1, 1)$.

$$\begin{aligned} h' &= \eta^{\mu\nu} h'_{\mu\nu} \\ &= \eta^{\mu\nu}\Big[h_{\mu\nu} - \Big(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\Big)\Big] \\ &= h - 2\partial_\mu \varepsilon^\mu \quad . \end{aligned} \tag{2.40}$$

Then, insert this result in (2.28) to get

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' \\ &= h_{\mu\nu} - \Big(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\Big) - \frac{1}{2}\eta_{\mu\nu} h' \\ &= h_{\mu\nu} - \Big(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\Big) - \frac{1}{2}\eta_{\mu\nu}\Big(h - 2\partial_\rho \varepsilon^\rho\Big) \\ &= \underline{h_{\mu\nu}} - \Big(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\Big) - \underline{\frac{1}{2}\eta_{\mu\nu} h} + \eta_{\mu\nu}\partial_\rho \varepsilon^\rho \end{aligned}$$

$$\boxed{\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \Big(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\Big) + \eta_{\mu\nu}\partial_\rho \varepsilon^\rho} \quad . \tag{2.41}$$

The importance of the gauge transformation in linearized theory given by equation (2.39) is that it leaves the curvature of the spacetime unchange. After the gauge transformation, the linearized Riemann tensor remains the same as shown below,

$$\begin{aligned} R'_{\mu\nu\rho\sigma} &= \frac{1}{2}\Big\{\partial_\nu \partial_\rho h'_{\mu\sigma} + \partial_\mu \partial_\sigma h'_{\nu\rho} - \partial_\mu \partial_\rho h'_{\nu\sigma} - \partial_\nu \partial_\sigma h'_{\mu\rho}\Big\} \\ &= \frac{1}{2}\Big\{\partial_\nu \partial_\rho \left[h_{\mu\sigma} - \partial_\mu \varepsilon_\sigma - \partial_\sigma \varepsilon_\mu\right] + \partial_\mu \partial_\sigma \left[h_{\nu\rho} - \partial_\nu \varepsilon_\rho - \partial_\rho \varepsilon_\nu\right] \\ &\quad - \partial_\mu \partial_\rho \left[h_{\nu\sigma} - \partial_\nu \varepsilon_\sigma - \partial_\sigma \varepsilon_\nu\right] - \partial_\nu \partial_\sigma \left[h_{\mu\rho} - \partial_\mu \varepsilon_\rho - \partial_\rho \varepsilon_\mu\right]\Big\} \\ &= \frac{1}{2}\Big\{\partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\nu \partial_\rho \partial_\mu \varepsilon_\sigma - \partial_\nu \partial_\rho \partial_\sigma \varepsilon_\mu + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\sigma \partial_\nu \varepsilon_\rho - \partial_\mu \partial_\sigma \partial_\rho \varepsilon_\nu \\ &\quad - \partial_\mu \partial_\rho h_{\nu\sigma} + \partial_\mu \partial_\rho \partial_\nu \varepsilon_\sigma + \partial_\mu \partial_\rho \partial_\sigma \varepsilon_\nu - \partial_\nu \partial_\sigma h_{\mu\rho} + \partial_\nu \partial_\sigma \partial_\mu \varepsilon_\rho + \partial_\nu \partial_\sigma \partial_\rho \varepsilon_\mu\Big\} \\ &= \frac{1}{2}\Big\{\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\Big\} = R_{\mu\nu\rho\sigma} \quad . \end{aligned} \tag{2.42}$$

Since the Ricci tensor and the scalar curvature comes from the Riemann tensor, they are invariant as well. Hence, the Einstein tensor is also invariant and the equations of motion represent the same physics under an infinitesimal transformation of the coordinates [33][6]. Taking advantage of the gauge freedom that is present in linearized theory through equation

[6] This can be understood with an analogy from electrodynamics. The electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by a gauge calibration of the form $A_\mu \to A_\mu - \partial_\mu \psi$.

(2.39), one could simplify (2.33) by imposing the following constraint,

$$\boxed{\partial^{\nu}\bar{h}_{\mu\nu} = 0} \quad . \tag{2.43}$$

This is called the **Lorenz gauge**[7]. To prove that this condition can always be imposed for some coordinate system, we first take the 4-divergence of equation (2.41) to obtain,

$$\begin{aligned} \partial^{\nu}\bar{h}'_{\mu\nu} &= \partial^{\nu}\bar{h}_{\mu\nu} - \partial^{\nu}\Big(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}\Big) + \eta_{\mu\nu}\partial^{\nu}\partial_{\rho}\varepsilon^{\rho} \\ &= \partial^{\nu}\bar{h}_{\mu\nu} - \underline{\partial_{\mu}\partial^{\nu}\varepsilon_{\nu}} - \partial^{\nu}\partial_{\nu}\varepsilon_{\mu} + \underline{\partial_{\mu}\partial_{\rho}\varepsilon^{\rho}} \end{aligned}$$

$$\boxed{\partial^{\nu}\bar{h}'_{\mu\nu} = \partial^{\nu}\bar{h}_{\mu\nu} - \Box\varepsilon_{\mu}} \quad . \tag{2.44}$$

Suppose that initially a coordinate system does not obey the Lorenz gauge (2.43). Therefore, any perturbation tensor can be put into this gauge by performing an infinitesimal transformation provided that also the new ”primed” system satisfy the relation

$$\boxed{\Box\varepsilon_{\mu}(x) = f_{\mu}(x)} \quad ; \quad f_{\mu}(x) = \partial^{\nu}\bar{h}_{\mu\nu} \quad . \tag{2.45}$$

One can always find solutions of equation (2.45) and thus always exist a coordinate system in which the Lorenz condition can be done [1, 8]. If $G(x, x')$ is the Green’s function of the d’Alembertian operator $\Box$, then it must satisfy

$$\Box_x G(x, x') = \delta^4(x - x') \quad . \tag{2.46}$$

Thus, the solution $\varepsilon_{\mu}(x)$ for the wave equation in (2.45) is

$$\varepsilon_{\mu}(x) = \int G(x, x')\, f_{\mu}(x')\, d^4x' \quad . \tag{2.47}$$

This solution can easily be verified by applying the d’Alembert operator

$$\begin{aligned} \Box\varepsilon_{\mu} &= \int \Box G(x, x')\, f_{\mu}(x')\, d^4x' \\ &= \int \delta^4(x - x')\, f_{\mu}(x')\, d^4x' \\ &= f_{\mu}(x) \quad , \end{aligned} \tag{2.48}$$

and this completes the proof[8]. Having shown that is always possible to find a coordinate system in which $\partial^{\nu}\bar{h}_{\mu\nu} = 0$, the linearized EFE in (2.33) for the trace-reversed perturbation

[7] There are other denominations for the same coordinate restriction such as harmonic gauge, Hilbert gauge, De Donder gauge, Lorentz gauge, Einstein’s gauge or De Sitter gauge.

[8] Compare the Lorenz gauge of the linearized theory with the gauge fixing of the electromagnetic vector potential giving by $\partial_{\mu}A^{\mu} = 0$. One can still make the gauge calibration $A_{\mu} \to A_{\mu} - \partial_{\mu}\psi$ and if ψ is harmonic the gauge is preserved.

tensor is reduced to,

$$\boxed{\Box \bar{h}_{\mu\nu} \;=\; -\frac{16\pi G}{c^4} T_{\mu\nu}} \quad . \tag{2.49}$$

This is the **gravitational wave equation** in linearized theory. Because the metric $g_{\mu\nu}$ is a symmetric tensor, $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ are also symmetric. In general, a second order symmetric tensor has 10 independent components. However, our gauge condition gives 4 additional constraints that can be used to reduce the number of independent components to 6. In the next section we will see that from these 6 degrees of freedom only 2 are physical radiative modes of propagation, the others depends actually on the choice of the coordinate system. Taking the covariant derivative on both sides of (2.49) one gets

$$\begin{aligned}
\Box \nabla^{\nu} \bar{h}_{\mu\nu} \;&=\; -\frac{16\pi G}{c^4} \nabla^{\nu} T_{\mu\nu} \\
\Box \eta^{\nu\sigma} \nabla_{\sigma} \bar{h}_{\mu\nu} \;&=\; -\frac{16\pi G}{c^4} \nabla^{\nu} \eta_{\alpha\mu} \eta_{\beta\nu} T^{\alpha\beta} \\
\Box \eta^{\nu\sigma} \left(\partial_{\sigma} \bar{h}_{\mu\nu} \,-\, \Gamma^{\lambda}_{\sigma\mu} \bar{h}_{\lambda\nu} \,-\, \Gamma^{\lambda}_{\sigma\nu} \bar{h}_{\mu\lambda} \right) \;&=\; -\frac{16\pi G}{c^4} \eta_{\alpha\mu} \nabla_{\beta} T^{\alpha\beta} \\
\Box \eta^{\nu\sigma} \partial_{\sigma} \bar{h}_{\mu\nu} \,+\, \mathcal{O}(\varepsilon^2) \;&=\; -\frac{16\pi G}{c^4} \eta_{\alpha\mu} \nabla_{\beta} T^{\alpha\beta} \\
\Box \partial^{\nu} \bar{h}_{\mu\nu} \;&=\; 0 \;=\; -\frac{16\pi G}{c^4} \eta_{\alpha\mu} \nabla_{\beta} T^{\alpha\beta} \qquad \text{[Lorenz gauge]}
\end{aligned} \tag{2.50}$$

Equation (2.50) ensures the conservation of the energy-momentum tensor,

$$\nabla_{\beta} T^{\alpha\beta} \;=\; 0 \quad . \tag{2.51}$$

Neverthless, in linearized theory, the components of $T^{\alpha\beta}$ must be very small. Otherwise, a strong curvature in the spacetime produced by the source would not allow to make a good approximation at linear order in the metric perturbation. Thus, the connection terms of the form ΓT can be neglected at linear order in $\mathcal{O}(\varepsilon)$. Therefore, an appropriate statement of energy-momentum conservation in linearized theory is actually

$$\boxed{\partial_{\beta} T^{\alpha\beta} \;=\; 0} \quad . \tag{2.52}$$

Physically, this equation implies that matter fields are allowed to exchange energy and momentum between themselves but not with the gravitational field [9]. The information of gravity is encoded in the connection terms. Hence, to include the effects of gravity, one must consider the full expression in (2.51). But again, in linearized theory the connection terms doesn't contribute. In conclusion, in this theory the dynamics of matter couldn't include exactly the effects of gravity such as self-gravitating systems like binary stars. Moreover, equation (2.52) as well as in special relativity, means that all bodies move on geodesics of Minkowski spacetime, i.e., straight lines [34]. The fact that the background metric is $\eta_{\mu\nu}$, suggest that for those gravitational bound systems one may introduce it's dynamics with Newtonian gravity rather than the full general relativity [8]. However, in the exact formulation of general relativity by Landau & Lifshitz is possible to introduce the effects of gravitational binding energy that in case of very weak fields can be neglected[9].

In particular, in the absence of matter, $T_{\mu\nu} = 0$ and it is obtained the vacuum wave equation

$$\boxed{\Box\bar{h}_{\mu\nu} = 0} \quad , \tag{2.53}$$

that can be expressed as

$$\nabla^2\bar{h}_{\mu\nu} = \frac{1}{c^2}\frac{\partial^2\bar{h}_{\mu\nu}}{\partial t^2} \quad . \tag{2.54}$$

Clearly, equation (2.54) shows that the perturbation components of $\bar{h}_{\mu\nu}$ propagates at c, the speed of light. The homogeneous wave equation admits the general solution

$$\boxed{\bar{h}_{\mu\nu}(\mathbf{x},t) = \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, d\mathbf{k}} \quad , \tag{2.55}$$

which are superpositions of plane waves moving in different directions. These solutions are simply gravitational waves (GWs) [1]. Equation (2.55) is often written in a more compact form,

$$\bar{h}_{\mu\nu}(\mathbf{x},t) = \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\, e^{ik_\alpha x^\alpha}\, d\mathbf{k} \quad , \tag{2.56}$$

where

$$k_\mu = \left(-\frac{\omega}{c}\,,\,\mathbf{k}\right) \qquad ; \qquad x^\mu = (ct\,,\,\mathbf{x}) \quad . \tag{2.57}$$

The quantity $A_{\mu\nu}$ is called the **polarization tensor**[10]. Another way to show the speed of GWs is by taking the d'Alembertian operator of the solution (2.56) and setting equal to zero,

$$\begin{aligned}
\Box\bar{h}_{\mu\nu}(\mathbf{x},t) &= \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\,\Box e^{ik_\alpha x^\alpha}\, d\mathbf{k} && [\Box := \partial^\mu\partial_\mu\,] \\
&= \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\,\partial^\mu\Big[\partial_\mu\Big(e^{ik_\alpha x^\alpha}\Big)\Big]\, d\mathbf{k} && [\,\partial^\mu = \eta^{\mu\sigma}\partial_\sigma] \\
&= \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\,\eta^{\mu\sigma}\partial_\sigma\Big[e^{ik_\alpha x^\alpha}\, ik_\alpha\partial_\mu x^\alpha\Big]\, d\mathbf{k} && [\partial_\mu x^\alpha = \delta^\alpha_\mu\,] \\
&= \mathrm{Re}\int ik_\mu A_{\mu\nu}(\mathbf{k})\,\eta^{\mu\sigma}\partial_\sigma\Big[e^{ik_\alpha x^\alpha}\,\Big]\, d\mathbf{k} \\
&= \mathrm{Re}\int ik_\mu A_{\mu\nu}(\mathbf{k})\,\eta^{\mu\sigma} e^{ik_\alpha x^\alpha}\, ik_\alpha\partial_\sigma x^\alpha\, d\mathbf{k} \\
&= \mathrm{Re}\int i^2\Big[\eta^{\mu\sigma}k_\mu k_\sigma\Big] A_{\mu\nu}(\mathbf{k})\, e^{ik_\alpha x^\alpha}\, d\mathbf{k} = 0 \quad .
\end{aligned} \tag{2.58}$$

It follows that,

$$\boxed{k_\mu k^\mu = 0} \quad . \tag{2.59}$$

[9] This problem is discussed in subsequent chapters.
[10] The reason for this name will become clear in the next section.

Thus, the wave four-vector k^μ is null and GWs propagates at c. Note that equation (2.59) also implies that there is no dispersion. This means that all GWs travels at the speed of light and from (2.57) the dispersion relation is given by

$$\omega = c|\mathbf{k}| \quad . \tag{2.60}$$

On the other hand, after applying the Lorenz gauge to the solution $\bar{h}_{\mu\nu}(\mathbf{x},t)$ one finds that these waves are also transverse,

$$\begin{aligned}
\partial^\mu \bar{h}_{\mu\nu}(\mathbf{x},t) &= \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\,\partial^\mu e^{ik_\alpha x^\alpha}\,d\mathbf{k} \\
&= \mathrm{Re}\int A_{\mu\nu}(\mathbf{k})\,\eta^{\mu\rho}\partial_\rho e^{ik_\alpha x^\alpha}\,d\mathbf{k} \\
&= \mathrm{Re}\int iA_{\mu\nu}(\mathbf{k})\,\eta^{\mu\rho}\,e^{ik_\alpha x^\alpha}\,k_\alpha\delta^\alpha_\rho\,d\mathbf{k} \\
&= \mathrm{Re}\int i\eta^{\mu\rho}A_{\mu\nu}(\mathbf{k})k_\rho\,e^{ik_\alpha x^\alpha}\,d\mathbf{k} = 0 \qquad \text{[Lorenz gauge]} \; .
\end{aligned} \tag{2.61}$$

This equation automatically gives the transverse condition,

$$\boxed{k^\mu A_{\mu\nu}(\mathbf{k}) = 0} \quad . \tag{2.62}$$

2.4 The transverse-traceless (TT) gauge

As stated earlier, the metric perturbation tensor $h_{\mu\nu}$ is symmetric as well as $\bar{h}_{\mu\nu}$. Then, the 10 independent components can be seen from the following matrix representation,

$$[\bar{h}_{\mu\nu}] = \begin{pmatrix} \bar{h}_{00} & \bar{h}_{01} & \bar{h}_{02} & \bar{h}_{03} \\ \bar{h}_{01} & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ \bar{h}_{02} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{23} \\ \bar{h}_{03} & \bar{h}_{13} & \bar{h}_{23} & \bar{h}_{33} \end{pmatrix} \qquad \text{[10 independent components]} \quad . \tag{2.63}$$

The Lorenz gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ allows us to reduce from these 10 to 6 independent components. Now, imagine that in some initial coordinate system the Lorenz gauge can be imposed. One might wonder if under an infinitesimal transformation the Lorenz gauge is still valid in the new system. This can be done as long as the generator functions ε_μ satisfy the homogeneous wave equation. From (2.44),

$$\begin{aligned}
\partial^\nu \bar{h}'_{\mu\nu} &= \partial^\nu \bar{h}_{\mu\nu} - \Box\varepsilon_\mu \\
&= -\Box\varepsilon_\mu \quad .
\end{aligned} \tag{2.64}$$

Thus,

$$\boxed{\Box\varepsilon_\mu = 0} \tag{2.65}$$

and the new system will preserve the Lorenz condition. It is expected that in this frame the linearized EFE also takes the same wave equation for $\bar{h}'_{\mu\nu}$. By applying the d'Alembertian operator at both sides of equation (2.41), one can check that this is indeed true,

$$\begin{aligned}\Box\bar{h}'_{\mu\nu} &= \Box\bar{h}_{\mu\nu} - \partial_\mu\Box\varepsilon_\nu - \partial_\nu\Box\varepsilon_\mu + \eta_{\mu\nu}\eta_{\sigma\rho}\partial^\sigma\eta^{\rho\sigma}\Box\varepsilon_\sigma \\ &= \Box\bar{h}_{\mu\nu} \quad . \end{aligned} \tag{2.66}$$

Therefore, the gravitational wave equation in linearized theory is invariant under an infinitesimal coordinate transformation if and only if $\Box\varepsilon_\mu = 0$. Both conditions, $\Box\varepsilon_\mu = 0$ and $\partial^\nu\bar{h}_{\mu\nu} = 0$, represent in total 8 constraints over the 10 independent components of $\bar{h}_{\mu\nu}$. As a result, there are only 2 components left, known as + and × polarizations of the gravitational wave[11]. To be convinced of this assertion, an explicit procedure is followed [35, 36]. First of all, pick up a coordinate system in which the Lorenz gauge is satisfied. Clearly, in this system the linearized EFE in vacuum is the homogeneous wave equation and the general solution is given by (2.55). As mentioned before, the polarization tensor is subject to the transverse condition $k^\mu A_{\mu\nu}(\mathbf{k}) = 0$, which follows from the Lorenz gauge. Now, for simplicity without loss of generality, consider a plane wave propagating in the $+z$-direction[12]. In this case, the components of the wave four-vector are

$$[k^\mu] = (\,k\,,\,0\,,\,0\,,\,k\,) \qquad\qquad [k_\mu] = (\,-k\,,\,0\,,\,0\,,\,k\,) \quad , \tag{2.67}$$

where $k = \omega/c$. The transverse condition, which in turn comes from the Lorenz gauge automatically gives,

$$\begin{aligned} k^\mu A_{\mu\nu} &= k^0 A_{0\nu} + k^3 A_{3\nu} \\ &= kA_{0\nu} + kA_{3\nu} \\ &= 0 \\ \Longrightarrow \quad & \boxed{A_{0\nu} = -A_{3\nu}} \quad . \end{aligned} \tag{2.68}$$

Due to the symmetry $\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$, the polarization tensor is also symmetric. Thus, with equation (2.68) one can express $A_{\mu\nu}$ in terms of 6 independent components, namely A_{00}, A_{01}, A_{02}, A_{11}, A_{12} and A_{13}:

$$[A_{\mu\nu}] = \begin{pmatrix} A_{00} & A_{01} & A_{02} & -A_{00} \\ A_{01} & A_{11} & A_{12} & -A_{01} \\ A_{02} & A_{12} & A_{22} & -A_{02} \\ -A_{00} & -A_{01} & -A_{02} & -A_{00} \end{pmatrix} \quad \text{[6 independent components]} \quad . \tag{2.69}$$

To simplify further the number of independent components in (2.69) one may perform an infinitesimal gauge transformation of the form in (2.41). To preserve the Lorenz gauge, the

[11] At this stage, the 'plus' and 'cross' polarizations are just names for the two degrees of freedom that are left. These names will gain meaning in the interaction of GWs with test masses in chapter 4.

[12] Because the metric perturbation is invariant under the Lorentz group, one can always align the z axis with the wavevector direction by making a rotation.

generators of the gauge transformation ε_μ must satisfy the homogeneous wave equation $\Box\varepsilon_\mu = 0$. The general solution to (2.65) is given by

$$\varepsilon_\mu = \mathrm{Re}\int b_\mu(\mathbf{k})\, e^{ik_\alpha x^\alpha}\, d\mathbf{k} \quad , \tag{2.70}$$

for some functions b_μ that depends only on $\mathbf{k}$. Inserting (2.70) and (2.55) on (2.41) gives,

$$\begin{aligned} \bar{h}'_{\mu\nu} &= \bar{h}_{\mu\nu} - \partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu + \eta_{\mu\nu}\partial^\rho\varepsilon_\rho \\ \mathrm{Re}\int A'_{\mu\nu}(\mathbf{k})\, e^{ik_\alpha x^\alpha}\, d\mathbf{k} &= \mathrm{Re}\int \Big[A_{\mu\nu}(\mathbf{k}) - ik_\mu b_\nu - ik_\nu b_\mu + i\eta_{\mu\nu}k^\rho b_\rho\Big]\, e^{ik_\alpha x^\alpha}\, d\mathbf{k} \\ \Longrightarrow &\boxed{A'_{\mu\nu} = A_{\mu\nu} - ik_\mu b_\nu - ik_\nu b_\mu + i\eta_{\mu\nu}k^\rho b_\rho} \quad . \end{aligned} \tag{2.71}$$

Since in this new coordinate system the Lorenz gauge is preserved, there are initially the same 6 independent components for $A'_{\mu\nu}$ as in (2.69). However, from the equation $\Box\varepsilon_\mu = 0$, we are able to choose 4 constraints over b_μ to further reduce from 6 to only 2 independent components. Using (2.67) in (2.71), the initially 6 independent components of $A'_{\mu\nu}$ are

$$\begin{aligned} A'_{00} &= A_{00} + ik(b_0 - b_3)\,, & A'_{11} &= A_{11} + ik(b_0 + b_3)\,, \\ A'_{01} &= A_{01} + ikb_1\,, & A'_{12} &= A_{12}\,, \\ A'_{02} &= A_{02} + ikb_2\,, & A'_{22} &= A_{22} + ik(b_0 + b_3)\,. \end{aligned} \tag{2.72}$$

Now, by choosing the functions b_μ as

$$\begin{aligned} b_0 &= i(2A_{00} + A_{11} + A_{22})/4k\,, & b_1 &= iA_{01}/k\,, \\ b_2 &= iA_{02}/k\,, & b_3 &= -i(2A_{00} - A_{11} - A_{22})/4k\,, \end{aligned} \tag{2.73}$$

one finds that

$$A'_{00} = A'_{01} = A'_{02} = 0 \qquad \text{and} \qquad A'_{11} = -A'_{22} \quad . \tag{2.74}$$

Denoting by $A'_{11} = A_+$ and $A'_{12} = A_\times$, the independent components of the polarization tensor becomes

$$[A'_{\mu\nu}] = \left[A^{\mathrm{TT}}_{\mu\nu}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{+z} \quad . \tag{2.75}$$

In conclusion, only two physical modes of the spacetime itself are propagating. They are called the "*plus*" and "*cross*" polarizations. The new system in which the polarization tensor takes this simple form (as shown in (2.75)), is called the *transverse-traceless* frame, or TT frame[13].

[13] In principle, one can use another coordinate system. However, the TT frame is extremely convenient because it fixes completely the coordinate freedom [1]. This allows to extract the two physical radiative modes of propagation.

Evidently, the trace of $A^{\mathrm{TT}}_{\mu\nu}$ vanish and it's transverse because of the condition $k^\mu A_{\mu\nu} = 0$, which actually follows from the Lorenz gauge. Thereby, the complete solution in the TT frame for a plane wave moving in the $+z$-direction is

$$\left[\bar{h}^{\mathrm{TT}}_{\mu\nu}(z,t)\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{+z} \cos(kz - \omega t) \quad . \tag{2.76}$$

Note from (2.30) that the traceless condition means that $h^{\mathrm{TT}}_{\mu\nu} = \bar{h}^{\mathrm{TT}}_{\mu\nu}$, and these quantites may be used interchangeably. What's more, in vacuum the tensors $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ contain the same physical information and one could use either, since $h_{\mu\nu}$ also satisfies the homogeneous wave equation. To see this, we first take the trace of the wave equation (2.49),

$$\begin{aligned} \eta^{\mu\nu}\Box\bar{h}_{\mu\nu} &= -\frac{16\pi G}{c^4}\eta^{\mu\nu}T_{\mu\nu} \\ \Box\Big(\eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h\Big) &= -\frac{16\pi G}{c^4}T \\ \Box\Big(h - 2h\Big) &= -\frac{16\pi G}{c^4}T \\ \Longrightarrow \qquad \Box h &= \frac{16\pi G}{c^4}T \quad . \end{aligned} \tag{2.77}$$

Then, the linearized EFE can be rewritten in the Lorenz gauge alternatively as,

$$\begin{aligned} \Box\bar{h}_{\mu\nu} &= -\frac{16\pi G}{c^4}T_{\mu\nu} \\ \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h &= -\frac{16\pi G}{c^4}T_{\mu\nu} \\ \Box h_{\mu\nu} &= -\frac{16\pi G}{c^4}T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box h \\ \Box h_{\mu\nu} &= -\frac{16\pi G}{c^4}\Big(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\Big) \\ \boxed{\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\bar{T}_{\mu\nu}} \quad , \end{aligned} \tag{2.78}$$

where $\bar{T}_{\mu\nu}$ is the trace-reversed of the energy-momentum tensor $T_{\mu\nu}$. Outside the source, $T_{\mu\nu} = 0$ and therefore $\bar{T}_{\mu\nu} = 0$. Hence, one gets the homogeneous wave equation for the perturbation tensor $h_{\mu\nu}$,

$$\boxed{\Box h_{\mu\nu} = 0} \quad . \tag{2.79}$$

The solution for $h_{\mu\nu}$ can be put also in the TT frame in a similar way as explained before. The only difference is that instead of using the rule transformation for $\bar{h}_{\mu\nu}$ in (2.41), one may use the rule transformation for $h_{\mu\nu}$ in (2.39). Again, assuming a plane wave in the $+z$-direction, one can use (2.39) to choose properly the functions b_μ to set $A'_{00} = A'_{13} = A'_{23} = A'_{33} = 0$. Likewise, by exploiting the Lorenz gauge in terms of $h_{\mu\nu}$, another 4 conditions can be obtained between the other components. The result is the same TT structure as shown in (2.75) [36]. Hence, the trace h vanishes and by the definition (2.28) one arrives at the same statement, $\bar{h}^{\rm TT}_{\mu\nu} = h^{\rm TT}_{\mu\nu}$. In any case, the solution of the homogeneous wave equation whether for $\bar{h}_{\mu\nu}$ or for $h_{\mu\nu}$, can be expressed in the TT frame by demanding what is called the **TT gauge** that is defined as follows,

$$\boxed{h_{0\mu} = 0 \ , \qquad \partial^j h_{ij} = 0 \ , \qquad h^i_i = h = 0 \ .} \tag{2.80}$$

There are 4 coordinate restrictions in the first equation of (2.80), 3 in the second one and 1 in the third one. These are the same 8 constraints that we had earlier with the Lorenz gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ and $\Box \varepsilon_\mu = 0$. Note that the first equation in (2.80) claims that all the time dependence of $h_{\mu\nu}$ vanishes. Thus, $A_{00} = A_{01} = A_{02} = A_{03} = 0$. The second is just the spatial part of the Lorenz gauge, that when considering a plane wave in $+z$-direction gives,

$$\begin{aligned} \partial^1 h_{11} + \partial^2 h_{12} + \partial^3 h_{13} &= 0 \\ k^1 A_{11} + k^2 A_{12} + k^3 A_{13} &= 0 \\ A_{13} &= 0 \ , \end{aligned} \tag{2.81}$$

$$\begin{aligned} \partial^1 h_{21} + \partial^2 h_{22} + \partial^3 h_{23} &= 0 \\ k^1 A_{21} + k^2 A_{22} + k^3 A_{23} &= 0 \\ A_{23} &= 0 \ , \end{aligned} \qquad \begin{aligned} \partial^1 h_{31} + \partial^2 h_{32} + \partial^3 h_{33} &= 0 \\ k^1 A_{31} + k^2 A_{32} + k^3 A_{33} &= 0 \\ A_{33} &= 0 \ . \end{aligned} \tag{2.82}$$

The traceless condition is,

$$\begin{aligned} h^i_i = h &= A_{11} + A_{22} + A_{33} = 0 \\ &= A_{11} + A_{22} = 0 \\ \Longrightarrow \qquad A_{11} &= -A_{22} \ , \end{aligned} \tag{2.83}$$

and in addition to the symmetry property $h_{\mu\nu} = h_{\nu\mu}$, the TT gauge ensures two independent degrees of freedom, i.e, the physical modes of GWs.

One is left wondering if the TT frame can always be found. For globally vacuum space-times, this is true [1]. Nevertheless, if $\bar{h}_{\mu\nu} \neq 0$, is not possible to choose conveniently the functions b_μ to reduce the polarization tensor to (2.75) [8]. To show the existence of the TT gauge in vacuum, it's enough to show that one can find a system in which $h_{0\mu} = 0$ and $h = 0$[14]. From (2.71), that is to say, finding an explicit solution to the following set of equations,

[14] The other condition $\partial^j h_{ij} = 0$ is actually the spatial part of the Lorenz gauge, but we have shown earlier that in fact this constraint can always be achieved under a infinitesimal gauge transformation.

$$0 \;=\; \eta^{\mu\nu} A'_{\mu\nu} \;=\; \eta^{\mu\nu} A_{\mu\nu} \;+\; 2ik^{\rho} b_{\rho} \tag{2.84}$$

$$0 \;=\; A'_{0\mu} = A_{0\mu} \;-\; ik_0 b_{\mu} \;-\; ik_{\mu} b_0 - i\delta^0_{\mu}\,(k^{\rho} b_{\rho}) \quad . \tag{2.85}$$

The solution is given by

$$b_{\mu} \;=\; \frac{A_{\alpha\beta} l^{\alpha} l^{\beta}}{8i(\omega/c)^4} k_{\mu} \;+\; \frac{\eta^{\alpha\beta} A_{\alpha\beta}}{4i(\omega/c)^2} l_{\mu} \;-\; \frac{1}{2i(\omega/c)^2} A_{\mu\nu} l^{\nu} \quad , \tag{2.86}$$

where $k^{\mu} = (\omega/c\,,\, \mathbf{k})$ and $l^{\mu} = (\omega/c\,,\, -\mathbf{k})$ [37]. In conclusion, one can *always* make a transformation to the TT gauge as long as $T_{\mu\nu} = 0$. For instance, if the GW is moving along the $+x$-direction or the $+y$-direction, a similar procedure will gives,

$$\left[A^{\mathrm{TT}}_{\mu\nu}\right] \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{+} & A_{\times} \\ 0 & 0 & A_{\times} & -A_{+} \end{pmatrix}_{+x} \quad , \quad \left[A^{\mathrm{TT}}_{\mu\nu}\right] \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{+} & 0 & A_{\times} \\ 0 & 0 & 0 & 0 \\ 0 & A_{\times} & 0 & -A_{+} \end{pmatrix}_{+y} \quad . \tag{2.87}$$

Finally, the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ can be written as

$$g_{\mu\nu} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;+\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{+} & A_{\times} & 0 \\ 0 & A_{\times} & -A_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{+z} \cos(kz - \omega t)$$

$$g^{(+z)}_{\mu\nu} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + A_{+}\cos(kz - \omega t) & A_{\times}\cos(kz - \omega t) & 0 \\ 0 & A_{\times}\cos(kz - \omega t) & 1 - A_{+}\cos(kz - \omega t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{2.88}$$

Thus, the line element becomes

$$\begin{aligned} ds^2 \;=\; & -c^2 dt^2 \;+\; [1 \;+\; A_{+}\cos(kz - \omega t)]\, dx^2 + [1 \;-\; A_{+}\cos(kz - \omega t)]\, dy^2 \\ & + \; 2A_{\times}\cos(kz - \omega t) dx dy \;+\; dz^2 \quad . \end{aligned} \tag{2.89}$$

2.5 Projection onto the TT gauge

As has been exhibited in the previous section, a solution $h_{\mu\nu}(x)$ can always be put onto the TT gauge. If a plane wave is moving along any of the coordinate axis, the polarization tensor takes the simple form as shown in equations (2.75) and (2.87). In general, if a plane wave solution outside the sources is propagating in the direction $\hat{\mathbf{n}}$, one is able to find the form of the solution in the TT gauge as follows. First, define the tensor

$$P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j \quad . \tag{2.90}$$

This tensor has the properties of being symmetric, transverse, its trace is $P_{ii} = 2$ and it's also a projector tensor. It's straightforward to verify such properties from the definition (2.90),

Symmetric:

$$P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j = \delta_{ji} - n_j n_i = P_{ji}(\hat{\mathbf{n}}) \quad , \tag{2.91}$$

Transverse:

$$\begin{aligned} n^i P_{ij}(\hat{\mathbf{n}}) &= n^i \delta_{ij} - n^i n_i n_j \\ &= n_j - (1) n_j = 0 \end{aligned} \tag{2.92}$$

Trace:

$$\begin{aligned} P_{ii}(\hat{\mathbf{n}}) &= \delta_{ii} - n_i n_i \\ &= 3 - 1 = 2 \quad , \end{aligned} \tag{2.93}$$

Projector:

$$\begin{aligned} P_{ik}(\hat{\mathbf{n}}) P_{kj}(\hat{\mathbf{n}}) &= \delta_{ik}\delta_{kj} - \delta_{ik} n_k n_j - \delta_{kj} n_i n_k + n_i n_k n_k n_j \\ &= \delta_{ij} - n_i n_j - n_i n_j + n_i n_j \\ &= \delta_{ij} - n_i n_j \\ &= P_{ij}(\hat{\mathbf{n}}) \quad . \end{aligned} \tag{2.94}$$

Now, the TT projector tensor $\Lambda_{ij|kl}$ is constructed from $P_{ij}(\hat{\mathbf{n}})$ as

$$\boxed{\Lambda_{ij|kl}(\hat{\mathbf{n}}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}} \quad . \tag{2.95}$$

This tensor is symmetric under the exchange between the first and second pair of indices, it is transverse on all indices, its trace with respect to the indices ij and jk vanishes and is still a projector tensor. These properties can be proven from the definition (2.95),

Symmetric:

$$\begin{aligned} \Lambda_{ij|kl} &= P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \\ &= P_{ki} P_{lj} - \frac{1}{2} P_{kl} P_{ij} \\ &= \Lambda_{kl|ij} \end{aligned} \tag{2.96}$$

Transverse:

$$
\begin{aligned}
n^i \Lambda_{ij|kl} &= \underline{n^i P_{ik}} P_{jl} - \frac{1}{2} \underline{n^i P_{ij}} P_{kl} = 0 \\
n^j \Lambda_{ij|kl} &= P_{ik} \underline{n^j P_{jl}} - \frac{1}{2} \underline{n^j P_{ij}} P_{kl} = 0 \\
n^k \Lambda_{ij|kl} &= \underline{n^k P_{ik}} P_{jl} - \frac{1}{2} P_{ij} \underline{n^k P_{kl}} = 0 \\
n^l \Lambda_{ij|kl} &= P_{ik} \underline{n^l P_{jl}} - \frac{1}{2} P_{ij} \underline{n^l P_{kl}} = 0
\end{aligned}
\tag{2.97}
$$

Trace:

$$
\begin{aligned}
\Lambda_{ii|kl} &= P_{ik} P_{il} - \frac{1}{2} P_{ii} P_{kl} = P_{kl} - P_{kl} = 0 \\
\Lambda_{ij|kk} & P_{ik} P_{jk} - \frac{1}{2} P_{ij} P_{kk} = P_{ij} - P_{ij} = 0
\end{aligned}
\tag{2.98}
$$

Projector:

$$
\begin{aligned}
\Lambda_{ij|kl} \Lambda_{kl|mn} &= \Big(P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \Big) \Big(P_{km} P_{ln} - \frac{1}{2} P_{kl} P_{mn} \Big) \\
&= P_{ik} P_{jl} P_{km} P_{ln} - \frac{1}{2} P_{ik} P_{jl} P_{kl} P_{mn} - \frac{1}{2} P_{ij} P_{kl} P_{km} P_{ln} + \frac{1}{4} P_{ij} P_{kl} P_{kl} P_{mn} \\
&= P_{im} P_{jn} - \frac{1}{2} P_{il} P_{lj} P_{mn} - \frac{1}{2} P_{ij} P_{mk} P_{kn} + \frac{1}{4} P_{ij} P_{kk} P_{mn} \\
&= P_{im} P_{jn} - \frac{1}{2} P_{ij} P_{mn} - \frac{1}{2} P_{ij} P_{mn} + \frac{1}{2} P_{ij} P_{mn} \\
&= P_{im} P_{jn} - \frac{1}{2} P_{ij} P_{mn} \\
&= \Lambda_{ij|mn}
\end{aligned}
\tag{2.99}
$$

In terms of the unit vector $\mathbf{\hat{n}}$, the TT *Lambda* tensor is

$$
\begin{aligned}
\Lambda_{ij|kl}(\mathbf{\hat{n}}) &= P_{ik}(\mathbf{\hat{n}}) P_{jl}(\mathbf{\hat{n}}) - \frac{1}{2} P_{ij}(\mathbf{\hat{n}}) P_{kl}(\mathbf{\hat{n}}) \\
&= \Big(\delta_{ik} - n_i n_k \Big) \Big(\delta_{jl} - n_j n_l \Big) - \frac{1}{2} \Big(\delta_{ij} - n_i n_j \Big) \Big(\delta_{kl} - n_k n_l \Big) \\
&= \delta_{ik} \delta_{jl} - \delta_{ik} n_j n_l - \delta_{jl} n_i n_k + n_i n_k n_j n_l \\
&\quad - \frac{1}{2} \Big(\delta_{ij} \delta_{kl} - \delta_{ij} n_k n_l - \delta_{kl} n_i n_j + n_i n_j n_k n_l \Big)
\end{aligned}
$$

$$\Lambda_{ij|kl}(\hat{\mathbf{n}}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l \ . \tag{2.100}$$

Therefore, given a plane wave solution $h_{\mu\nu}$ already in the Lorenz gauge but not in the TT gauge, the gravitational wave in the TT gauge is obtained by applying the *Lambda* operator,

$$\boxed{h_{ij}^{\mathrm{TT}} = \Lambda_{ij|kl} h_{kl}} \ . \tag{2.101}$$

The TT gauge conditions in the spatial part gives,

$$* \quad \partial^i h_{ij}^{\mathrm{TT}} = 0 \quad \Longrightarrow \quad n^i h_{ij}^{\mathrm{TT}} = 0 \tag{2.102}$$

$$* \quad h_{ii}^{\mathrm{TT}} = 0 \ . \tag{2.103}$$

Note that the expression in (2.101) satisfies the TT conditions in (2.102) and (2.103). By the transverse and the trace properties of the *Lambda* tensor, then

$$\begin{aligned} n^i h_{ij}^{\mathrm{TT}} &= \underline{n^i \Lambda_{ij|kl}} h_{kl} = 0 \ , \\ h_{ii}^{\mathrm{TT}} &= \underline{\Lambda_{ii|kl}} h_{kl} = 0 \ . \end{aligned} \tag{2.104}$$

In general, given a symmetric tensor S_{ij}, its transverse and traceless part is[15]

$$\boxed{S_{ij}^{\mathrm{TT}} = \Lambda_{ij|kl} S_{kl}} \ . \tag{2.105}$$

Observe that in this equation the quantity S_{ij}^{TT} remains symmetric.

[15] Recall that a tensor can be decomposed in a symmetric part plus an antisymmetric part. The action of the *Lambda* tensor over the antisymmetric part will vanish.

Chapter 3

Interaction of Gravitational Waves with Test Masses

Having studied the two physical modes of gravitational radiation, the interaction of GWs with a detector is given in the present chapter. A set of point particles will be assumed as the makeup of the detector and the effects of the GW passage will be devised from two coordinate systems, the TT frame and the detector frame. In order to accomplish this, the geodesic motion and geodesic deviation of particles are revisited in first place. Finally, the interaction of GWs with a ring of test masses is presented as an example to understand the meaning of the *plus* and *cross* polarizations.

3.1 Geodesic equation

Consider in some reference frame, a temporal curve $x^\mu(\tau)$ parametrized by the proper time τ. This means that, at each point of such a curve, the tangent vector is timelike. From all possible temporal curves that satisfies the stationary events $x^\mu(\tau_A) = x^\mu_A$ and $x^\mu(\tau_B) = x^\mu_B$, the shortest path between these points is called a *geodesic*. For a free particle, it follows from the action

$$S = -m \int_{\tau_A}^{\tau_B} d\tau \quad , \tag{3.1}$$

where m is the particles's rest mass. An extremal path is found when the variation of the action vanishes, $\delta S = 0$. If the spacetime is flat, one obtains the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \quad . \tag{3.2}$$

However, in general spacetime may have curvature. In this case, the geodesic motion for a free particle comes from considering the spacetime line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2 d\tau^2 \quad . \tag{3.3}$$

Since $x^\mu = x^\mu(\tau)$, then

$$dx^\mu = \frac{dx^\mu}{d\tau} d\tau \tag{3.4}$$

and equation (3.3) is written as

$$ds^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau^2 = -c^2 d\tau^2 \quad , \tag{3.5}$$

which is equal to

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2 \quad . \tag{3.6}$$

The action (3.1) is now,

$$S = -m\int_{\tau_A}^{\tau_B}(1)\,d\tau = -m\int_{\tau_A}^{\tau_B}\Big(-\frac{1}{c^2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)\,d\tau = \int_{\tau_A}^{\tau_B}\frac{m}{c^2}\Big(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)\,d\tau \quad . \tag{3.7}$$

Therefore, the geodesic motion is obtained if

$$\begin{aligned}\delta S &= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\delta\Big(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)\,d\tau = 0\\ &= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\delta g_{\mu\nu}\right) + 2\,g_{\mu\nu}\frac{d\left(\delta x^\mu\right)}{d\tau}\frac{dx^\nu}{d\tau}\right]d\tau = 0\\ &= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha + 2\,g_{\mu\nu}\frac{d\left(\delta x^\mu\right)}{d\tau}\frac{dx^\nu}{d\tau}\right]d\tau = 0 \quad .\end{aligned} \tag{3.8}$$

An integration by parts of the second term in (3.8) is performed by noting that

$$\begin{aligned}\frac{d}{d\tau}\left[\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] &= g_{\mu\nu}\frac{d\left(\delta x^\mu\right)}{d\tau}\frac{dx^\nu}{d\tau} + \delta x^\mu\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\\ \Longrightarrow\quad g_{\mu\nu}\frac{d\left(\delta x^\mu\right)}{d\tau}\frac{dx^\nu}{d\tau} &= \frac{d}{d\tau}\left[\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] - \delta x^\mu\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) \quad .\end{aligned} \tag{3.9}$$

Replacing (3.9) in (3.8) one gets

$$\begin{aligned}\delta S &= \frac{m}{c^2}\left\{\,2\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{d\tau}\Big|_{\tau_A}^{\tau_B} + \int_{\tau_A}^{\tau_B}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\delta x^\mu\right]d\tau\right\}\\ &= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\delta x^\mu\right]d\tau = 0 \quad .\end{aligned} \tag{3.10}$$

Expanding the term on the right hand side of the integrand results in

$$\begin{aligned}\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\delta x^\mu &= \frac{dg_{\mu\nu}}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\\ &= \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} \quad ,\end{aligned} \tag{3.11}$$

and inserting (3.11) into equation (3.10) gives,

$$
\begin{aligned}
\delta S &= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\mu - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu\right] d\tau \\
&= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\partial_\mu g_{\alpha\nu} - 2\partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\mu\, d\tau \\
&= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - \partial_\nu g_{\mu\alpha}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\mu\, d\tau \\
&= \frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[\left(\partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu} - \partial_\nu g_{\mu\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\mu\, d\tau \\
&= -\frac{m}{c^2}\int_{\tau_A}^{\tau_B}\left[2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\right]\delta x^\mu\, d\tau = 0 \ . \qquad (3.12)
\end{aligned}
$$

The expression in (3.12) implies that

$$
2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (3.13)
$$

and multiplying (3.13) by $\frac{1}{2}g^{\mu\sigma}$ one arrives at

$$
\begin{aligned}
\delta^\sigma_\nu\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}g^{\mu\sigma}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} &= 0 \\
\frac{d^2x^\sigma}{d\tau^2} + \frac{1}{2}g^{\mu\sigma}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} &= 0 \ . \qquad (3.14)
\end{aligned}
$$

Taking into account the Christoffel symbols definition, the **geodesic equation** for a free particle in a curved background is given by

$$
\boxed{\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0} \ . \qquad (3.15)
$$

3.2 Geodesic deviation equation

In Euclidean geometry, the defining property of the flat space is the concept of parallel lines. If initially close lines are parallel, the distance between them remains the same forever as they extend over the entire space. In a curved manifold this is not true. For instance, two initially lines over a sphere will approach each other and eventually cross together. In fact, a straight line is merely a geodesic in flat space. Thus, a natural way to study the concept of parallel lines in curved space is to measure how the distance is changing between nearby geodesics. From a general relativistic perspective, free particles follow geodesic motions due to the curvature of the spacetime produced by a source. Hence, the difference between geodesics in this context is interpreted as a manifestation of gravitational tidal forces[1].

To understand the concept of geodesic deviation, first consider two nearby geodesic trayectories. One is described by $x^\mu(\tau)$ and the other by $x^\mu(\tau) + \xi^\mu(\tau)$, where ξ^μ is a very slighlty displacement. From (3.15) they must satisfy

$$\frac{d^2x^\mu(\tau)}{d\tau^2} \;+\; \Gamma^\mu_{\alpha\beta}(x)\frac{dx^\alpha(\tau)}{d\tau}\frac{dx^\beta(\tau)}{d\tau} \;=\; 0 \quad , \tag{3.16}$$

$$\frac{d^2[x^\mu(\tau) \,+\, \xi^\mu(\tau)]}{d\tau^2} \;+\; \Gamma^\mu_{\alpha\beta}(x \,+\, \xi)\frac{d[x^\alpha(\tau) \,+\, \xi^\alpha(\tau)]}{d\tau}\frac{d[x^\beta(\tau) \,+\, \xi^\beta(\tau)]}{d\tau} \;=\; 0 \quad . \tag{3.17}$$

Assume also $|\xi^\mu|$ smaller than the typical scale variation of the gravitational field. A Taylor expansion of $\Gamma^\mu_{\alpha\beta}(x \,+\, \xi)$ around x at linear order gives

$$\Gamma^\mu_{\alpha\beta}(x+\xi) \;=\; \Gamma^\mu_{\alpha\beta}(x) \;+\; \xi^\sigma\partial_\sigma\Gamma^\mu_{\alpha\beta}(x) \;+\; \cdots \tag{3.18}$$

Making the substitution of (3.18) in (3.17) yields

$$\frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \left[\Gamma^\mu_{\alpha\beta} \;+\; \xi^\sigma\partial_\sigma\Gamma^\mu_{\alpha\beta}\right]\left(\frac{dx^\alpha}{d\tau} + \frac{d\xi^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau} + \frac{d\xi^\beta}{d\tau}\right) = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \left[\Gamma^\mu_{\alpha\beta} \;+\; \xi^\sigma\partial_\sigma\Gamma^\mu_{\alpha\beta}\right]\left(\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \frac{d\xi^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \cdots\right) = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \cdots = 0 \quad , \tag{3.19}$$

at first order in ξ. Substracting (3.19) from (3.16) the equation of the **geodesic deviation** reads

$$\boxed{\frac{d^2\xi^\mu}{d\tau^2} \;+\; 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} \;+\; \xi^\sigma\partial_\sigma\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \;=\; 0} \quad . \tag{3.20}$$

Because the geodesic deviation has everything to do with curvature, is often used to rewrite the expression (3.20) as a function of the Riemann tensor. To do this, consider a set of continuous timelike geodesic parametrized by the proper time τ, where each geodesic is labelled by a parameter α. This is called a *congruence* of timelike geodesics and is described by the parametric equations $x^\mu = x^\mu(\tau, \alpha)$ [9]. The displacement is along some geodesic when α is fixed and thus the tangent vector is $u^\mu = \partial x^\mu/\partial\tau$. Conversely, the displacement is across the geodesic when τ is fixed and $\xi^\mu = \partial x^\mu/\partial\alpha$ is a *deviation vector* that points from one geodesic to the other (see figure 3.1). One might wonder what is the evolution equation for the acceleration of ξ^μ when is transported along the vector field $\mathbf{u}$. Remind that the covariant derivative of a vector field V^μ along a curve $x^\mu(\tau)$ is,

[1]Contemplate the difference between geodesics of two particles in space that are in radial infall near the Earth. An observer in free fall with the particles would detect the inhomogeneities of the gravitational field by observing the geodesic deviation of such particles.

$$\nabla_{\mathbf{u}} V^{\mu} \;=\; u^{\rho}\nabla_{\rho}V^{\mu} \;=\; u^{\rho}\partial_{\rho}V^{\mu} \;+\; \Gamma^{\mu}_{\nu\rho}V^{\nu}u^{\rho} \qquad ; \qquad u^{\rho} \equiv \frac{\partial x^{\rho}(\tau,\alpha)}{\partial\tau} \quad . \tag{3.21}$$

Let $V^{\mu} = \nabla_{\mathbf{u}}\xi^{\mu}$, then equation (3.21) turns out to be

$$\begin{aligned}
\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\xi^{\mu} \;&=\; u^{\rho}\partial_{\rho}\left(u^{\beta}\partial_{\beta}\xi^{\mu} \,+\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}u^{\beta}\right) \,+\, \Gamma^{\mu}_{\nu\rho}\left(u^{\beta}\partial_{\beta}\xi^{\nu} \,+\, \Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}\right)u^{\rho} \\
&=\; u^{\rho}\partial_{\rho}u^{\beta}\partial_{\beta}\xi^{\mu} \,+\, \partial_{\rho}\Gamma^{\mu}_{\alpha\beta}u^{\rho}\xi^{\alpha}u^{\beta} \,+\, \Gamma^{\mu}_{\alpha\beta}u^{\beta}u^{\rho}\partial_{\rho}\xi^{\alpha} \\
&\quad +\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}u^{\rho}\partial_{\rho}u^{\beta} \,+\, \Gamma^{\mu}_{\nu\rho}u^{\rho}u^{\beta}\partial_{\beta}\xi^{\nu} \,+\, \Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}u^{\rho} \\
&=\; \frac{d^2\xi^{\mu}}{d\tau^2} \,+\, 2\Gamma^{\mu}_{\nu\rho}u^{\rho}u^{\beta}\partial_{\beta}\xi^{\nu} \,+\, \partial_{\rho}\Gamma^{\mu}_{\alpha\beta}u^{\rho}\xi^{\alpha}u^{\beta} \\
&\quad +\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}u^{\rho}\partial_{\rho}u^{\beta} \,+\, \Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}u^{\rho}
\end{aligned} \tag{3.22}$$

Using (3.20) we have

$$\frac{d^2\xi^{\mu}}{d\tau^2} \;=\; -2\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\sigma}\partial_{\sigma}\xi^{\beta} \,-\, \xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} \quad , \tag{3.23}$$

and inserting this result in (3.22) yields

$$\begin{aligned}
\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\xi^{\mu} \;&=\; -\,\xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} \,+\, \partial_{\rho}\Gamma^{\mu}_{\alpha\beta}u^{\rho}\xi^{\alpha}u^{\beta} \,+\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}u^{\rho}\partial_{\rho}u^{\beta} \,+\, \Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}u^{\rho} \\
&=\; -\,\xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} \,+\, \partial_{\rho}\Gamma^{\mu}_{\alpha\beta}u^{\rho}\xi^{\alpha}u^{\beta} \,+\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}\left(\frac{d^2x^{\beta}}{d\tau^2}\right) \,+\, \Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}u^{\rho} \\
&=\; -\,\xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\mu}_{\alpha\beta}\right)u^{\alpha}u^{\beta} \,+\, \left(\partial_{\rho}\Gamma^{\mu}_{\alpha\beta}\right)u^{\rho}\xi^{\alpha}u^{\beta} \,-\, \Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}\Gamma^{\beta}_{\nu\rho}u^{\nu}u^{\rho} \,+\, \Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}\xi^{\alpha}u^{\beta}u^{\rho} \\
&=\; \underbrace{\left(\partial_{\rho}\Gamma^{\mu}_{\alpha\beta}\right)u^{\rho}\xi^{\alpha}u^{\beta}}_{(\rho,\alpha,\beta)\longrightarrow(\nu,\rho,\sigma)} \,-\, \underbrace{\left(\partial_{\sigma}\Gamma^{\mu}_{\alpha\beta}\right)u^{\alpha}\xi^{\sigma}u^{\beta}}_{(\sigma,\alpha,\beta)\longrightarrow(\rho,\nu,\sigma)} \,+\, \underbrace{\Gamma^{\mu}_{\nu\rho}\Gamma^{\nu}_{\alpha\beta}u^{\beta}\xi^{\alpha}u^{\rho}}_{(\nu,\rho,\alpha,\beta)\longrightarrow(\lambda,\nu,\rho,\sigma)} \,-\, \underbrace{\Gamma^{\mu}_{\alpha\beta}\Gamma^{\beta}_{\nu\rho}u^{\nu}\xi^{\alpha}u^{\rho}}_{(\beta,\alpha,\rho)\longrightarrow(\lambda,\rho,\sigma)} \\
&=\; \partial_{\nu}\Gamma^{\mu}_{\rho\sigma}u^{\nu}\xi^{\rho}u^{\sigma} \,-\, \partial_{\rho}\Gamma^{\mu}_{\nu\sigma}u^{\nu}\xi^{\rho}u^{\sigma} \,+\, \Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma}u^{\nu}\xi^{\rho}u^{\sigma} \,-\, \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma}u^{\nu}\xi^{\rho}u^{\sigma} \\
&=\; -\left(\partial_{\rho}\Gamma^{\mu}_{\nu\sigma} \,-\, \partial_{\nu}\Gamma^{\mu}_{\rho\sigma} \,+\, \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} \,-\, \Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma}\right)u^{\nu}\xi^{\rho}u^{\sigma} \\
&=\; -\,R^{\mu}_{\sigma\rho\nu}u^{\sigma}\xi^{\rho}u^{\nu} \qquad (\nu \longleftrightarrow \sigma) \\
&=\; -\,R^{\mu}_{\nu\rho\sigma}u^{\nu}\xi^{\rho}u^{\sigma} \quad .
\end{aligned}$$

Finally, the equation of geodesic deviation is given by,

$$\boxed{\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\xi^{\mu} \;=\; -\,R^{\mu}_{\nu\rho\sigma}u^{\nu}\xi^{\rho}u^{\sigma}} \quad . \tag{3.24}$$

As expected, this equation says that the relative acceleration between two neighboring time-like geodesics is proportional to the Riemann tensor, which has all the information of the spacetime curvature.

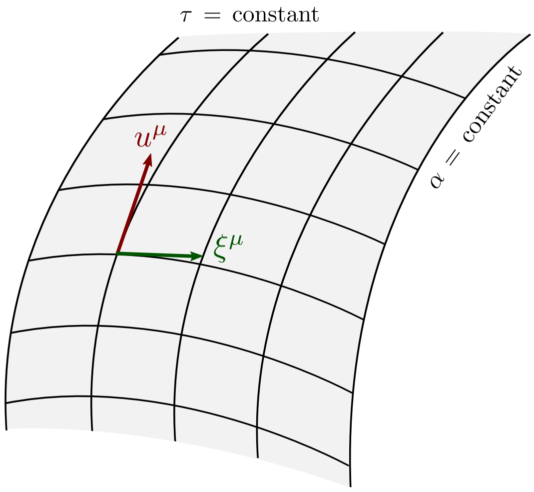

Figure 3.1: Congruence of timelike geodesics.

3.3 Local flatness

Before discussing the TT frame and the detector frame, two local coordinate systems are of great importance to develop further aspects in the interaction of GWs with test masses. These are the *Riemann normal coordinates* and the *Fermi normal coordinates*. The main results is that in the former at a selected event $\mathcal{P}$ in the spacetime, the metric is the flat metric $\eta_{\mu\nu}$ and the connections $\Gamma^{\mu}_{\alpha\beta}$ vanish. In the latter, given a timelike geodesic γ, at each point along the geodesic the metric is given by the flat metric $\eta_{\mu\nu}$ and the connections vanish.

Riemann normal coordinates

Consider a curved spacetime with a metric $g_{\mu\nu}$ in some arbitrary coordinates. Select an event $\mathcal{P}$ in the spacetime and adapt a tetrad $e^{\alpha}_{(\mu)} = \{e^{\alpha}_{(0)}, e^{\alpha}_{(1)}, e^{\alpha}_{(2)}, e^{\alpha}_{(3)}\}$ to this point. The bracketed index (μ) label each basis vector of the tetrad and the index α is refered to the usual components of a selected basis vector. The orthonormality condition of the tetrad is given by

$$\mathbf{e}_{(\mu)} \cdot \mathbf{e}_{(\nu)} = g_{\alpha\beta} e^{\alpha}_{(\mu)} e^{\beta}_{(\nu)} = \eta_{\mu\nu} \quad , \tag{3.25}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Now, take another event $\mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{Q}$ lie in a small enough region where the spacetime is almost flat, there exists a unique geodesic that connects both events. The geodesic may be parametrized by the proper distance s if it's a spacelike geodesic and by the proper time τ in case of a timelike geodesic. A unit and tangent vector $\hat{\mathbf{n}}$ to the geodesic at $\mathcal{P}$ can be decomposed in terms of the basis vectors. Hence, the components of $\hat{\mathbf{n}}$ are

$$n^{\alpha}(\mathcal{P}) = n^{\mu} e^{\alpha}_{(\mu)} \quad . \tag{3.26}$$

The *Riemann normal coordinates* of $\mathcal{Q}$ at $\mathcal{P}$ are defined as

$$\zeta^{\mu}(\mathcal{P}) = \kappa n^{\mu} \quad \begin{cases} \kappa = s_{\mathcal{Q}} \quad , & \text{for a spacelike geodesic} \\ \kappa = \tau_{\mathcal{Q}} \quad , & \text{for a timelike geodesic} \quad . \end{cases} \tag{3.27}$$

The following figure summarizes the construction of the Riemann normal coordinates at the spacetime event $\mathcal{P}$.

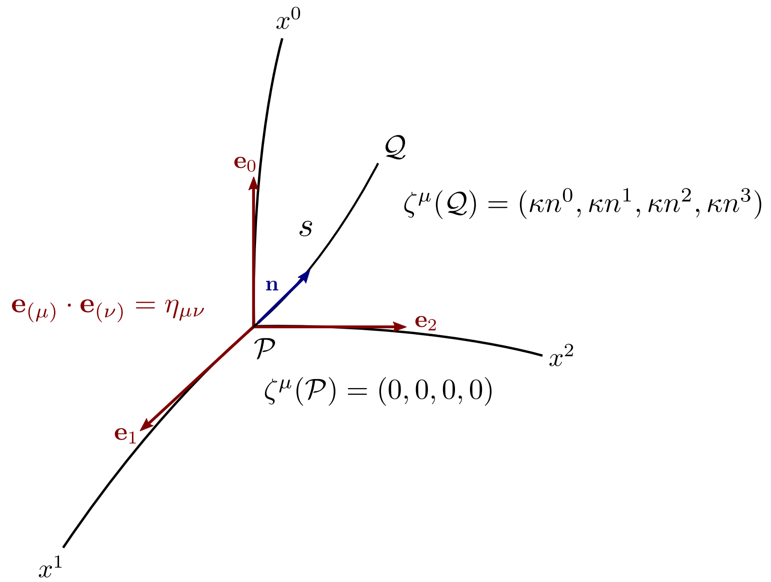

Figure 3.2: Definition of the Riemann normal coordinates about an event $\mathcal{P}$.

The strong consequence of this coordinate system is that at the event $\mathcal{P}$ the metric $g_{\mu\nu}$ reduces to the flat metric $\eta_{\mu\nu}$ and the connections vanish,

$$\boxed{ g_{\mu\nu}(\zeta)\Big|_{\mathcal{P}} = \eta_{\mu\nu} \qquad , \qquad \Gamma^{\mu}_{\ \alpha\beta}(\zeta)\Big|_{\mathcal{P}} = 0 } \ . \tag{3.28}$$

To show that $g_{\mu\nu}(\zeta) = \eta_{\mu\nu}$ at $\mathcal{P}$, bear in mind the fact that n^μ is a unit vector. In Riemann normal coordinates this implies that $g_{\mu\nu}(\zeta) n^\mu n^\nu = 1$, which is valid along the entire geodesic that goes from $\mathcal{P}$ to $\mathcal{Q}$. However, in the initially arbitrary coordinates the invariant magnitude of the tangent vector $\hat{\mathbf{n}}$ is $g_{\alpha\beta}(x) n^\alpha n^\beta = 1$. But equation (3.25) can be inverted to get,

$$g_{\alpha\beta}(x) = \eta_{(\mu)(\nu)} e^{(\mu)}_{\alpha} e^{(\nu)}_{\beta} \ . \tag{3.29}$$

Thus, using (3.29) and (3.26) into $g_{\alpha\beta}(x) n^\alpha n^\beta$ then,

$$\begin{aligned} g_{\mu\nu}(\zeta) n^\mu n^\nu &= g_{\alpha\beta}(x) n^\alpha n^\beta \\ &= \left[\eta_{(\mu)(\nu)} e^{(\mu)}_{\alpha} e^{(\nu)}_{\beta}\right] \left[n^\sigma e^{\alpha}_{(\sigma)}\right] \left[n^\rho e^{\beta}_{(\rho)}\right] \\ &= \eta_{(\mu)(\nu)} \left[e^{(\mu)}_{\alpha} e^{\alpha}_{(\sigma)}\right] \left[e^{(\nu)}_{\beta} e^{\beta}_{(\rho)}\right] n^\sigma n^\rho \\ &= \eta_{(\mu)(\nu)} \delta^\mu_\sigma \delta^\nu_\rho n^\sigma n^\rho \\ g_{\mu\nu}(\zeta) n^\mu n^\nu &= \eta_{(\mu)(\nu)} n^\mu n^\nu \ . \end{aligned} \tag{3.30}$$

Because n^μ is arbitrary, then $g_{\mu\nu}(\zeta) = \eta_{\mu\nu}$ at $\mathcal{P}$. To show that in Riemann normal coordinates the connections vanish, consider the geodesic equation at the event $\mathcal{P}$,

$$\frac{d^2\zeta^\mu}{d\kappa^2}\bigg|_{\mathcal{P}} + \left[\Gamma^{\mu}_{\alpha\beta}(\zeta) \frac{d\zeta^\alpha}{d\kappa} \frac{d\zeta^\beta}{d\kappa}\right]_{\mathcal{P}} = 0 \ . \tag{3.31}$$

Inserting equation (3.27) in (3.31) gives

$$\Gamma^{\mu}_{\alpha\beta}\Big|_{\mathcal{P}} n^{\alpha} n^{\beta} = 0 \quad . \tag{3.32}$$

Again, because n^{μ} si arbitrary one concludes that $\Gamma^{\mu}_{\alpha\beta}(\zeta) = 0$ at $\mathcal{P}$. Observe that the partial derivative of the metric can be obtained from [9]

$$\partial_{\rho} g_{\alpha\beta} = g_{\alpha\lambda}\Gamma^{\lambda}{}_{\beta\rho} + g_{\beta\lambda}\Gamma^{\lambda}{}_{\alpha\rho} \quad , \tag{3.33}$$

and therefore is also true that $\partial_{\rho} g_{\alpha\beta} = 0$ at $\mathcal{P}$. The metric can be expanded about the event $\mathcal{P}$ in Riemann normal coordinates [9]. The result is,

$$g_{\mu\nu}(\zeta) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\lambda\nu\rho}(\zeta)\Big|_{\mathcal{P}} \zeta^{\lambda}\zeta^{\rho} + O(\zeta^{3}) \quad . \tag{3.34}$$

From figure (3.2) clearly $\zeta^{\mu} = 0$ at $\mathcal{P}$. Then, the metric (3.34) becomes the flat metric $\eta_{\mu\nu}$ at $\mathcal{P}$. On the other hand, because there are not linear terms in ζ^{μ} in the metric expansion, each term of the first derivative depend on ζ^{μ}. Then, the first derivative of the metric is zero at $\mathcal{P}$ and the connections vanish at that point.

Fermi normal coordinates

To define the Fermi normal coordinates, select an entire timelike geodesic γ parametrized by the proper time τ . Adapt a tetrad $e^{\alpha}_{(\mu)} = \{e^{\alpha}_{(0)}, e^{\alpha}_{(1)}, e^{\alpha}_{(2)}, e^{\alpha}_{(3)}\}$, which satisfies the orthonormality condition (3.25) everywhere on γ. Assume that $e^{\alpha}_{(0)}$ is aligned with γ's tangent vector and that all the basis vectors of the tetrad are being parallel transported along the geodesic γ. Now, consider an event $\mathcal{Q}$ away from γ, and construct a spacelike geodesic segment β from $\mathcal{O}$ to $\mathcal{Q}$ as shown in figure 3.3[2]. This segment is orthogonal to γ at $\mathcal{O}$ and is parameterized by the proper distance s. Thus, $s = 0$ at $\mathcal{O}$ and $s = s_{\mathcal{Q}}$ at $\mathcal{Q}$. If n^{α} is the tangent vector in β, it can be decomposed in terms of the tetrad as,

$$n^{\alpha}(\mathcal{O}) = n^{j} e^{\alpha}_{(j)} \quad . \tag{3.35}$$

Observe that $e^{\alpha}_{(j)}$ are the spatial members of the tetrad in $\mathcal{O}$. The basis vector $e^{\alpha}_{(0)}$ is not involved in the decomposition of n^{α} because β is precisely orthogonal to γ at the event $\mathcal{O}$. The *Fermi normal coordinates* of $\mathcal{Q}$ at $\mathcal{O}$ is given by

$$\xi^{\mu}(\mathcal{Q}) = (\, c\tau_{\mathcal{O}} \,,\, s_{\mathcal{Q}} n^{1} \,,\, s_{\mathcal{Q}} n^{2} \,,\, s_{\mathcal{Q}} n^{3} \,) \quad . \tag{3.36}$$

Of course, the coordinates at the event $\mathcal{O}$ are $\xi^{\mu}(\mathcal{O}) = (c\tau_{\mathcal{O}}, 0, 0, 0)$. It can be shown that the spacetime metric near γ can be written in these coordinates as [9, 38]

[2] The orthonormality condition ensures that β is unique.

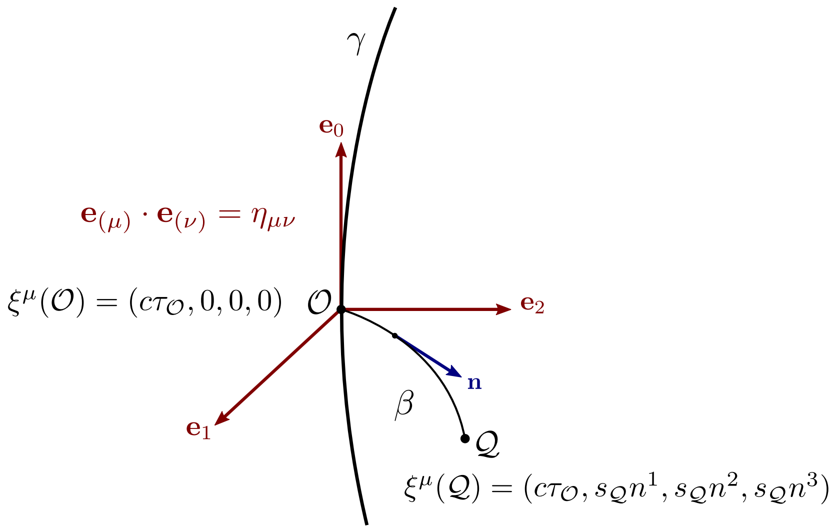

Figure 3.3: Definition of the Fermi normal coordinates along a timelike geodesic. The segment β is a spacelike geodesic and is orthogonal to γ.

$$\begin{aligned} g_{00}(\xi) &= -1 - R_{0p0q}(\xi^0)\Big|_\gamma \xi^p\xi^q + O(\xi^3) , \\ g_{0j}(\xi) &= \frac{2}{3} R_{jpq0}(\xi^0)\Big|_\gamma \xi^p\xi^q + O(\xi^3) , \\ g_{jk}(\xi) &= \delta_{jk} - \frac{1}{3} R_{jpkq}(\xi^0)\Big|_\gamma \xi^p\xi^q + O(\xi^3) \ , \end{aligned} \tag{3.37}$$

where the Riemann tensor depends on $\tau = \xi^0/c$, it's also evaluted on γ and the indices p and q are spatial. It should be noted that just like the coordinates at $\mathcal{O}$, the spatial part of the Fermi normal coordinates vanishes everywhere on γ. This implies that $g_{00}(\xi)\big|_\gamma = -1$, $g_{0j}(\xi)\big|_\gamma = 0$, $g_{jk}(\xi)\big|_\gamma = \delta_{jk}$. Thus, $g_{\mu\nu}(\xi)\big|_\gamma = \eta_{\mu\nu}$ and the metric reduces to the flat metric in the entire timelike geodesic γ. Likewise, because the metric is constant in γ and the components of the metric in (3.37) does not depend on linear terms in ξ^p, each term of the first derivative of the metric components depends on ξ^p. Therefore, the derivative of the metric in γ is zero and the connections $\Gamma^\mu{}_{\alpha\beta}(\xi)$ vanish in γ. These results are summarized as follows,

$$\boxed{ g_{\mu\nu}(\xi)\Big|_\gamma = \eta_{\mu\nu} \qquad , \qquad \Gamma^\mu{}_{\alpha\beta}(\xi)\Big|_\gamma = 0 } \ . \tag{3.38}$$

3.4 The TT frame

In chapter 2, it has been emphasized that in some coordinate system the *plus* and *cross* polarizations of the GW represent the two independent radiative degrees of freedom. This system was the TT frame. The present section discuss the physical implications of being in the TT frame when a gravitational wave interact with a test mass. To understand the GW effects on the motion of a point particle one must invoke the geodesic equation (3.15). Let assume a test particle A initially at rest at time $\tau = 0$. Then, the spatial part of the geodesic equation in $\tau = 0$ is

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} + \left[\Gamma^i_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} = 0$$

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} + \left[\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0} = 0 \quad , \tag{3.39}$$

where $(dx^i/d\tau)\big|_{\tau=0} = 0$ because the particle is initially at rest. Using the TT gauge, the connections Γ^i_{00} are obtained from (2.17) and gives,

$$\begin{aligned}\Gamma^i_{00} &= \frac{1}{2}\eta^{i\sigma}\left(\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}\right)\\ &= \frac{1}{2}\delta_{ij}\left(\partial_0 h_{0j} + \partial_0 h_{0j} - \partial_j h_{00}\right)\\ &= \frac{1}{2}\left(2\partial_0 h_{0i} - \partial_i h_{00}\right)\end{aligned} \tag{3.40}$$

Keep in mind that in the TT gauge $h_{0\mu} = 0$, so from (3.40) the connections Γ^i_{00} are zero at $\tau = 0$[3]. The conclusion is that if at time $\tau = 0$, $dx^i/d\tau = 0$, also $d^2x^i/d\tau^2 = 0$ and the particle that is initially at rest before the passage of the GW, remains at rest even after the arrival of the wave. Physically, this does not mean that there are no effects of the GW on the particle. The interpretation is that the coordinates stretch themselves in such a way that the position of the particle initially at rest doesn't change with time.
Now, if two test masses are initially at rest, they will remain at rest for all times. Therefore, their separation also does not change with time. This can be seen by employing the geodesic deviation between two neighboring test particles. The spatial part of (3.20) reads

$$\frac{d^2\xi^i}{d\tau^2} + 2\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \quad . \tag{3.41}$$

Evaluating this expression at time $\tau = 0$ one finds,

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} + \left[2\Gamma^i_{0\beta}\frac{dx^0}{d\tau}\frac{d\xi^\beta}{d\tau}\right]_{\tau=0} + \left[\xi^\sigma\partial_\sigma\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\right]_{\tau=0} = 0$$

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} + \left[2\Gamma^i_{0\beta}\frac{dx^0}{d\tau}\frac{d\xi^\beta}{d\tau}\right]_{\tau=0} = 0$$

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} + \left[2c\Gamma^i_{0j}\frac{d\xi^j}{d\tau}\right]_{\tau=0} = 0 \quad . \tag{3.42}$$

Again, from the first to the second line in equations (3.42), the quantity $\Gamma^i_{00} = 0$ in the TT gauge. Furthermore, if the particle is initially at rest, its 4-velocity can be written as $(dx^\mu/d\tau) = (c, 0)$ at $\tau = 0$ and thus $(dx^0/d\tau) = c$. The connections Γ^i_{0j} are obtained directly

[3]This is valid only at order $\mathcal{O}(\varepsilon)$ in the connections. However, it is expected a value of $h \sim 10^{-21}$ in the detection of GWs on Earth, so the linear order approximation is very worthwhile [8].

from their linearized definition,

$$\begin{aligned}
\Gamma^i_{0j} &= \frac{1}{2}\eta^{i\sigma}\left(\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j}\right) \\
&= \frac{1}{2}\delta_{ik}\partial_0 h_{jk} \\
&= \frac{1}{2c}\dot{h}_{ij} \qquad \left[\dot{h}_{ij} \equiv \frac{\partial h_{ij}}{\partial t}\right] \quad .
\end{aligned} \tag{3.43}$$

The equation (3.42) results in

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} + \left[\dot{h}_{ij}\frac{d\xi^j}{d\tau}\right]_{\tau=0} = 0 \quad . \tag{3.44}$$

As a consequence, if at time $\tau = 0$, $d\xi^i/d\tau = 0$, then $d^2\xi^i/d\tau^2 = 0$ and the coordinate separation ξ^i remains constant at all times. Figure (3.4) shows a spacetime diagram for a single test particle and for two test particles. In both cases, the worldline of each particle is a straight line because they remain in the same spatial position at all times. Therefore, the evolution of the geodesics is given along the temporal direction.
Is important to recall that although the position of the particles does not change with time, this is just a particular feature of the TT coordinates. What is not changing is the *coordinate* separation, but indeed the particles physically move when the GW passes by. General relativity is an invariant theory and all coordinate systems must represent the same physics. Therefore, all frames of reference agree in what is actually changing is proper distances and proper times.

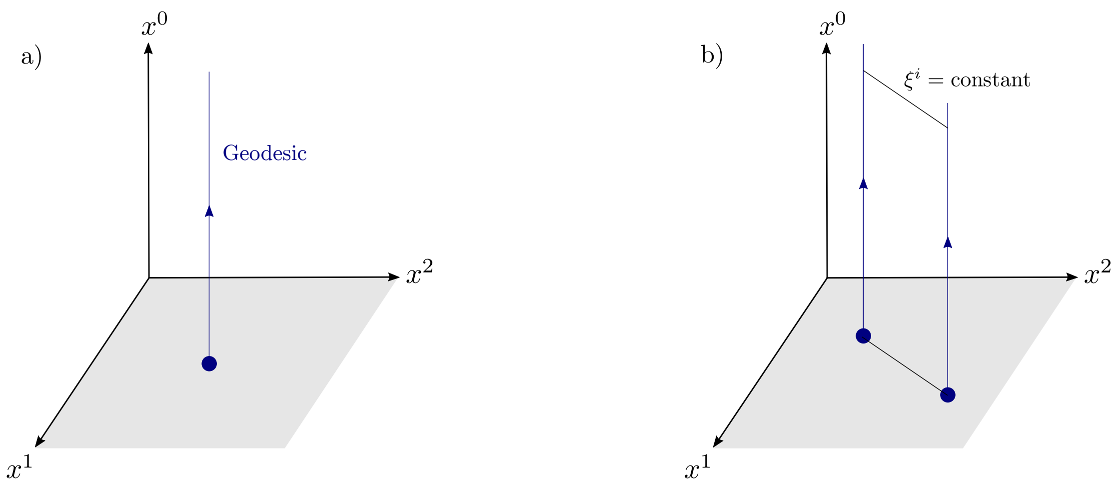

Figure 3.4: Interaction of GW with test masses in the TT frame. a) A test particle initially at rest remains in the same position at all times even after the arrival of the GW, so its worldline is a straight line. b) Two test particles initially at rest remains in the same position at all times, so the coordinate separation ξ^μ between them does not change with time.

Proper Time

The proper time of a timelike geodesic $x^\mu(\tau)$ can be determined by using the line element in the TT frame,

$$
\begin{aligned}
g_{\mu\nu} dx^\mu(\tau) dx^\nu(\tau) &= -c^2 d\tau^2 \\
\left(\eta_{\mu\nu} + h^{\mathrm{TT}}_{\mu\nu}\right) dx^\mu(\tau) dx^\nu(\tau) &= -c^2 d\tau^2 \\
-c^2 dt^2(\tau) + \left(\delta_{ij} + h^{\mathrm{TT}}_{ij}\right) dx^i(\tau) dx^j(\tau) &= -c^2 d\tau \\
-c^2 dt^2(\tau) + \left(\delta_{ij} + h^{\mathrm{TT}}_{ij}\right) \frac{dx^i(\tau)}{d\tau}\frac{dx^i(\tau)}{d\tau} d\tau^2 &= -c^2 d\tau \quad .
\end{aligned} \tag{3.45}
$$

However, if a particle is initially at rest it will be in the same position forever in the TT frame and $(dx^i/d\tau) = 0$ for all τ. Then, from (3.45) it is concluded that,

$$
\begin{aligned}
-c^2 dt^2(\tau) &= -c^2 d\tau^2 \\
t(\tau) &= \tau \quad .
\end{aligned} \tag{3.46}
$$

Hence, if an observer in the TT frame is sitting initially with a clock on the rest mass, the measure of the coordinate time t is the proper time τ.

Proper Distance

Consider two particles initially along the x-direction at $(ct, x_1, 0, 0)$ and $(ct, x_2, 0, 0)$. If the spatial separation between these events is denoted by $L = x_2 - x_1$, the proper distance between the particles is given by

$$
s = \int_0^L \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad . \tag{3.47}
$$

If a GW is propagating in the $+z$-direction, from equation (2.89) the line element in the TT frame reads,

$$
g_{\mu\nu} dx^\mu dx^\nu = \left[1 + A_+ \cos(\omega t)\right] dx^2 \quad . \tag{3.48}
$$

Note that in the last expression, because initially the separation in the y and z directions is zero, it will remain the same for all times in the TT gauge [1]. The only contribution comes from the x-direction. Inserting this equation into (3.47) one obtains,

$$
\begin{aligned}
s &= \int_0^L \left[1 + A_+ \cos(\omega t)\right]^{1/2} dx \\
&= L \left[1 + A_+ \cos(\omega t)\right]^{1/2} \\
&\boxed{s \simeq L \left[1 + \frac{1}{2} A_+ \cos(\omega t)\right]} \quad ,
\end{aligned} \tag{3.49}
$$

where in the last line was used the approximation $(1+x)^n \backsimeq 1+nx$ because $A_+ \ll 1$. Recall that the quantity L is the initial and unperturbed separation between the particles along the x-direction before the GW arrival. This value is constant in the TT gauge and the axis move with the GW in such frame.
More generally, if the particles are set down at coordinates (ct, x_1, y_1, z_1) and (ct, x_2, y_2, z_2), the initial separation between them is given by the vector $\mathbf{L}$, which remains constant in time for an observer in the TT gauge. The proper distance (3.47) can be obtained from the line element given by,

$$\begin{aligned} s^2 &= -c^2\cancel{\Delta t^2} + (\delta_{ij} + h_{ij}^{\mathrm{TT}})\Delta x^i \Delta x^j \\ s^2 &= \delta_{ij}\Delta x^i \Delta x^j + h_{ij}^{\mathrm{TT}}\Delta x^i \Delta x^j \\ s^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2 + h_{ij}^{\mathrm{TT}}\Delta x^i \Delta x^j \\ s^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + h_{ij}^{\mathrm{TT}}\Delta x^i \Delta x^j \\ s^2 &= L_x^2 + L_y^2 + L_z^2 + h_{ij}^{\mathrm{TT}}\Delta x^i \Delta x^j \\ s^2 &= L^2 + h_{ij}^{\mathrm{TT}} L_i L_j \quad , \end{aligned} \tag{3.50}$$

where $L_x = x_2 - x_1$, $L_y = y_2 - y_1$, $L_z = z_2 - z_1$ and $L^2 = L_x^2 + L_y^2 + L_z^2$. To linear order in h, i.e $\mathcal{O}(\varepsilon)$, the equation (3.50) becomes,

$$\begin{aligned} s^2 &= L^2\left(1 + \frac{h_{ij}(t)L_iL_j}{L^2}\right) \\ s &= L\left(1 + \frac{h_{ij}(t)L_iL_j}{L^2}\right)^{1/2} \backsimeq L\left(1 + \frac{h_{ij}(t)L_iL_j}{2L^2}\right) \quad . \end{aligned} \tag{3.51}$$

Thus, the proper distance is

$$\boxed{s \backsimeq L + h_{ij}(t)\left(\frac{L_iL_j}{2L}\right)} \quad . \tag{3.52}$$

Differentiating the last expression with respect to t one gets,

$$\ddot{s} \backsimeq \frac{1}{2}\ddot{h}_{ij}(t)\frac{L_i}{L}L_j \quad . \tag{3.53}$$

Writing $(L_i/L) = n_i$ and defining s_i from $s = n_i s_i$, equation (3.53) results in,

$$\begin{aligned} n_i\ddot{s}_i &\backsimeq \frac{1}{2}\ddot{h}_{ij}(t)n_iL_j \\ \ddot{s}_i &\backsimeq \frac{1}{2}\ddot{h}_{ij}(t)L_j \quad . \end{aligned} \tag{3.54}$$

Observe that $L_j = s_j$ to lowest order in h [8]. Therefore, a geodesic equation in terms of the proper distance is obtained,

$$\boxed{\ddot{s}_i \backsimeq \frac{1}{2}\ddot{h}_{ij}(t) s_j} \quad . \tag{3.55}$$

If the two particles are in fact test mirrors in which light travels back and forth, the proper distance determines a complete cycle (see figure 3.5). Moreover, since the proper distance is changing, is possible to compare the time of a cycle before and after the GW passes by.

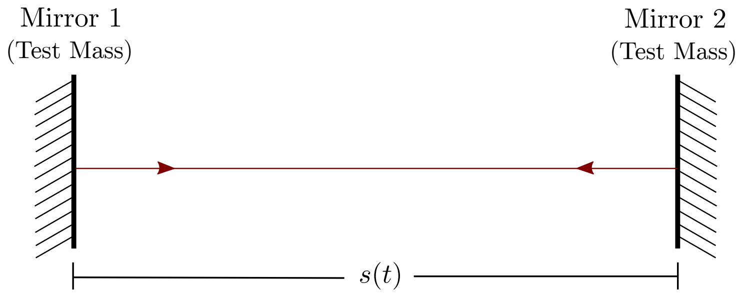

Figure 3.5: Proper distance between mirrors in a GW interferometer.

3.5 The proper detector frame

As a worthwhile approximation, the proper detector frame uses a freely falling observer whose coordinates are the Fermi normal coordinates. Extrictly speaking, for experiments on Earth this is not completely true. The interferometer is subjected to an acceleration $\mathbf{a} = -\mathbf{g}$ and also rotates due to the Earth's gravity and motion[4]. For this reason, the laboratory actually is not in free fall [8, 32]. The exact metric that describes a system in which these effects are included is quite complicated. Such a system is called the *proper detector frame*. However, thanks to the arrangement of the experiment and the frequency window in which the GWs are detected, is possible to extract the signal by neglecting such effects. Therefore, the approximate metric that describes the local proper detector frame is given by a freely-falling observer in Fermi normal coordinates. From (3.37) this metric is given by,

$$ds^2 \simeq -c^2dt^2\left[1 + R_{0i0j}x^ix^j\right] - 2cdtdx^i\left[\frac{2}{3}R_{0jik}x^jx^k\right] + dx^idx^j\left[\delta_{ij} - \frac{1}{3}R_{ikjl}x^kx^l\right] . \tag{3.56}$$

To analize the interaction of GWs with test masses in this frame, is better to work with the geodesic deviation equation. By choosing the Fermi normal coordinates, the connections vanish everywhere in an entire timelike geodesic γ. Due to this, and because the temporal derivative is defined along the timelike geodesic, one obtains inmediately [38],

$$\partial_0 \Gamma^{\mu}_{\alpha\beta}\Big|_{\gamma} = 0 \quad . \tag{3.57}$$

The previous statement also can be seen directly from the metric. As said previously in the last section, the metric depends explicitly only on the spatial Fermi coordinates, but implictly on the temporal Fermi coordinate through the Riemann tensor. Nevertheless, the Riemann tensor always appears multiplied by a cuadratic term in the spatial coordinates as shown

[4] There are other effects that may be considered in the laboratory frame. For instance, see [8].

in (3.56). Hence, taking the temporal derivative of the connections gives again terms with spatial components, which in turns vanish when evaluating at γ.
Bearing in mind that $\Gamma^{\mu}_{\alpha\beta}$ and $\partial_0\Gamma^{\mu}_{\alpha\beta}$ vanish everywhere in γ, the spatial part of the deviation geodesic equation reads

$$\frac{d^2\xi^i}{d\tau^2} + 2\underbrace{\Gamma^i_{\alpha\beta}}_{=0}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d^2\xi^i}{d\tau^2} + \xi^0\underbrace{\partial_0\Gamma^i_{\alpha\beta}}_{=0}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \xi^j\partial_j\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d^2\xi^i}{d\tau^2} + \xi^j\partial_j\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \quad . \tag{3.58}$$

If additionally one assumes that the detector performs a non-relativistic motion, then

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \quad . \tag{3.59}$$

As a consequence, equation (3.58) turns out into

$$\frac{d^2\xi^i}{d\tau^2} + \xi^j\partial_j\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0 \quad . \tag{3.60}$$

This equation can be expressed in terms of the Riemann tensor. For this, observe that the components R^i_{0j0} evaluated at the timelike geodesic γ results in[5]

$$R^i_{0j0} = \partial_j\Gamma^i_{00} - \underbrace{\partial_0\Gamma^i_{0j}}_{=0} = \partial_j\Gamma^i_{00} \quad , \tag{3.61}$$

and after inserting (3.61) in (3.60) yields

$$\frac{d^2\xi^i}{d\tau^2} + \xi^j R^i_{0j0}\left(\frac{dx^0}{d\tau}\right)^2 = 0 \quad . \tag{3.62}$$

On the other hand, when considering the metric nearly flat at the position of the detector, the time coordinate and proper time are related by

$$\eta_{\mu\nu}dx^\mu dx^\nu = -c^2 d\tau^2 \quad . \tag{3.63}$$

If a test particle is initially at rest, then acquires a velocity of order $(dx^i/d\tau) = c\mathcal{O}(\varepsilon^2)$ when the GW passes by [8]. Then, expanding (3.63) gives,

$$-c^2dt^2 + \delta_{ij}dx^i dx^j = -c^2 d\tau^2$$

$$c^2dt^2 = c^2d\tau^2 + \delta_{ij}dx^i dx^j$$

$$dt^2 = d\tau^2 + \frac{1}{c^2}\delta_{ij}dx^i dx^j$$

[5] The $\Gamma\Gamma$ terms in the definition of the Riemann tensor can be set to zero at γ in these coordinates.

$$
\begin{aligned}
dt^2 &= d\tau^2 \left[1 + \frac{1}{c^2}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right] \\
dt^2 &= d\tau^2 \left[1 + \frac{1}{c^2}\frac{dx^i}{d\tau}\frac{dx^i}{d\tau}\right] \\
dt^2 &= d\tau^2 \left[1 + \mathcal{O}(h^2)\right] \\
\implies t &= \tau \quad .
\end{aligned}
\tag{3.64}
$$

From (3.64) one has $dx^0/d\tau = c$ and the equation (3.62) can be rewritten as

$$
\ddot{\xi}^i + c^2 R^i_{0j0}\xi^j = 0 \qquad ; \qquad \left[\ddot{\xi}^i \equiv \frac{d^2\xi^i}{dt^2}\right] \quad . \tag{3.65}
$$

Measurements of GWs at the detector are of order $h \sim \mathcal{O}(10^{-21})$. Thus, linearized theory is appropriate in this context and one is able to compute the Riemann tensor at linear order in the equation (3.65). As mentioned before, apart from being covariant, the Riemann tensor is also invariant in linearized theory. So, is more suitable to compute it in a simple system, the TT frame. The result is,

$$
\begin{aligned}
R_{i0j0} &= \frac{1}{2}\left\{\partial_0\partial_j \underbrace{h^{\rm TT}_{i0}}_{=0} + \partial_i\partial_0 \underbrace{h^{\rm TT}_{0j}}_{=0} - \partial_i\partial_j \underbrace{h^{\rm TT}_{00}}_{=0} - \partial_0\partial_0 h^{\rm TT}_{ij}\right\} \\
R_{i0j0} &= -\frac{1}{2c^2}\ddot{h}^{\rm TT}_{ij} \quad .
\end{aligned}
\tag{3.66}
$$

Finally, using (3.66) in (3.65) one obtains

$$
\boxed{\ddot{\xi}_i = \frac{1}{2}\ddot{h}^{\rm TT}_{ij}\xi^j} \quad . \tag{3.67}
$$

In conclusion, in the freeling-falling frame the effect of a GW on a test mass can be described as a Newtonian force given by,

$$
F_i = \frac{m}{2}\ddot{h}^{\rm TT}_{ij}\xi^j \quad . \tag{3.68}
$$

3.6 Ring of test masses

In this section we shall study the interaction of GWs with a ring of test masses. This will allow to understand the meaning of the *plus* and *cross* polarizations. Consider a GW propagating along the $+z$-direction and a ring constituted by test particles as is shown in figure 3.6. The ring is located at the xy plane, so is expected that the particles do not move in the direction of the wave[6]. Conversely, the particles will stretch in the xy plane and the task is to describe the motion of the *deviation vector* $\xi_i = (\xi_x, \xi_y)$ in the proper detector frame.

[6]This is because GWs are transverse waves.

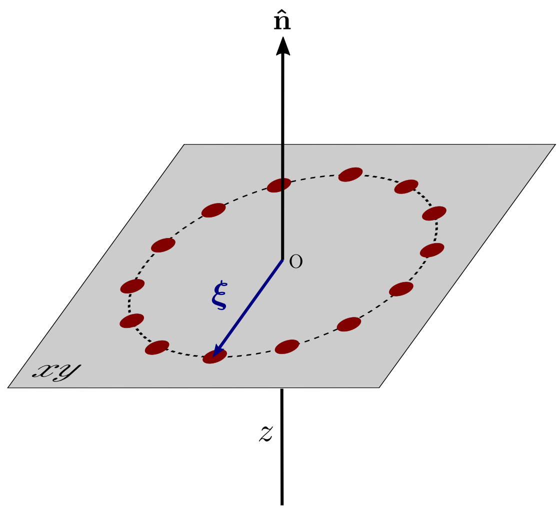

Figure 3.6: Ring of test masses located at the xy plane. The free-falling frame is attached to the center of the ring and the GW is propagating along the $+z$ direction.

In the TT gauge, a general GW in the $+z$-direction could be expressed as

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(kz - \omega t + \phi) \quad , \tag{3.69}$$

where ϕ is a phase that depends on the initial conditions. Inserting this solution in the equation (3.67) we have

$$\begin{aligned} \ddot{\xi}_z &= \frac{1}{2}\ddot{h}_{zj}^{\mathrm{TT}}\xi^j = 0 \qquad [h_{zj}^{\mathrm{TT}} = 0] \\ \dot{\xi}_z &= \text{constant} \quad . \end{aligned} \tag{3.70}$$

If initially the particles are at rest, then $\dot{\xi}_z(0) = 0$ and therefore $\dot{\xi}_z(t) = 0$. This is just the statement of the transversality property. If initially the particles set down at $z = 0$, they will remain in the plane even after the GW arrival. Moreover, assume that at the instant $t = 0$ the GW has not yet reached the particles and so $h_{ij}^{\mathrm{TT}}(0) = 0$. By imposing this initial condition over (3.69) one gets

$$h_{ij}^{\mathrm{TT}}(0,0) = \begin{pmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(\phi) = 0 \qquad \Longrightarrow \qquad \cos(\phi) = 0 \qquad \left[\phi = -\frac{\pi}{2}\right]$$

$$h_{ij}^{\mathrm{TT}}(t,0) = \begin{pmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left(\omega t - \frac{\pi}{2}\right) \quad ; \qquad \left[\cos\left(\omega t - \frac{\pi}{2}\right) = \sin(\omega t)\right] \quad .$$

Hence, the solution (3.69) when the particles are initially at rest in the plane xy becomes

$$h_{ij}^{\mathrm{TT}}(t,0) = \begin{pmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin(\omega t) \quad . \tag{3.71}$$

It's appropriate to split this solution as a sum of the *plus* and *cross* contributions,

$$h_{ij}^{\mathrm{TT}}(t,0) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_+ \sin(\omega t)}_{h_{ij}^{(+)}(t)} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_\times \sin(\omega t)}_{h_{ij}^{(\times)}(t)} \tag{3.72}$$

The ring will be deformed in the xy plane after the GW arrival. The changes in the relative positions of the test particles are described by the deviation vector $\boldsymbol{\xi}$ that can be written as

$$\xi_i(t) = \Big(x_0 + \delta x(t) \, , \, y_0 + \delta y(t) \, , \, 0 \Big) \quad , \tag{3.73}$$

where x_0 and y_0 are the unperturbed initial position of the particles at $t = 0$. The deviation vector at this instant is simply $\xi_i(0) = (x_0, y_0, 0)$ before the wave reach the ring. The problem consists to find the functions $\delta x(t)$ and $\delta y(t)$ to see the effects of the GW. In order to accomplish this, the polarizations are considered first separately.

Plus Polarization (+)

Suppose an incoming GW with only the *plus* polarization. Equation (3.67) reads as follows,

$$\begin{aligned}
\ddot{\xi}_x &= \frac{1}{2}\ddot{h}_{xj}^{(+)}\xi^j \\
\ddot{\xi}_x &= \frac{1}{2}\left[\ddot{h}_{xx}^{(+)}\xi^x + \ddot{h}_{xy}^{(+)}\xi^y \right] \\
\ddot{\xi}_x &= \frac{1}{2}\ddot{h}_{xx}^{(+)}\xi^x \qquad ; \qquad h_{xy}^{(+)} = 0 \ \ , \ \ \xi^z = 0 \\
\frac{d^2\delta x}{dt^2} &= -\frac{1}{2}\omega^2 A_+ \sin(\omega t) \left[x_0 + \delta x \right] \qquad ; \qquad \delta x \sim \mathcal{O}(\varepsilon) \\
\frac{d^2\delta x}{dt^2} &\backsimeq -\frac{1}{2}\omega^2 x_0 A_+ \sin(\omega t) \quad .
\end{aligned} \tag{3.74}$$

Solving for δx in (3.74) yields,

$$\delta x = \frac{1}{2} x_0 A_+ \sin(\omega t) \quad . \tag{3.75}$$

Similarly, for the y component one gets,

$$
\begin{aligned}
\ddot{\xi}_y &= \frac{1}{2}\ddot{h}^{(+)}_{yj}\xi^j \\
\ddot{\xi}_y &= \frac{1}{2}\left[\ddot{h}^{(+)}_{yx}\xi^x + \ddot{h}^{(+)}_{yy}\xi^y\right] \\
\ddot{\xi}_y &= \frac{1}{2}\ddot{h}^{(+)}_{yy}\xi^y \quad ; \quad h^{(+)}_{yx} = 0 \ , \ h^{(+)}_{yz} = 0 \\
\frac{d^2\delta y}{dt^2} &= \frac{1}{2}\omega^2 A_+ \sin(\omega t)\left[y_0 + \delta y\right] \quad ; \quad \delta y \sim \mathcal{O}(\varepsilon) \\
\frac{d^2\delta y}{dt^2} &\backsimeq \frac{1}{2}\omega^2 y_0 A_+ \sin(\omega t) \ .
\end{aligned} \tag{3.76}
$$

After integration of (3.76) one finds,

$$
\delta y = -\frac{1}{2} y_0 A_+ \sin(\omega t) \ . \tag{3.77}
$$

Using (3.75) and (3.77) in (3.73) it is obtained the displacement $\boldsymbol{\xi}$ of the particles that make up the ring with respect to the origin,

$$
\boxed{\xi_i = \left(x_0 + \frac{1}{2}x_0 A_+ \sin(\omega t) \ , \ y_0 - \frac{1}{2}y_0 A_+ \sin(\omega t) \ , \ 0\right) \ .} \tag{3.78}
$$

The behaviour of a GW only with the *plus* polarization is shown in figure 3.7. At the instant $t = 0$, the ring is a perfect unperturbed circle. At later times for $\omega t = (0, \pi/2]$, the ring starts to deform along the x-direction until it reaches the time given by $\omega t = \pi/2$, in which the ring is at its maximum elongation. After that, the ring shrinks along the same x-direction between the phase values $\omega t = (\pi/2, \pi]$, recovering its original shape at $\omega t = \pi$. Then, at times given by $\omega t = (\pi, 3\pi/2]$, the wave stretches the ring along the y-direction and the maximum elongation occurs at $\omega t = 3\pi/2$. Finally, from $\omega t = (3\pi/2, 2\pi]$ the ring shrinks in the same y-direction until the time given by $\omega t = 2\pi$, in which the ring is again a perfect circle and starts the cycle over again. Observe that the overall shape that is taking the ring with time, is an oscillatory (+) pattern and the name *plus* polarization is now justified.

Cross Polarization (×)

If the GW only has the *cross* polarization, equation (3.67) gives for the x component,

$$
\begin{aligned}
\ddot{\xi}_x &= \frac{1}{2}\ddot{h}^{(\times)}_{xj}\xi^j = \frac{1}{2}\left[\ddot{h}^{(\times)}_{xx}\xi^x + \ddot{h}^{(\times)}_{xy}\xi^y\right] \\
\ddot{\xi}_x &= \frac{1}{2}\ddot{h}^{(\times)}_{xy}\xi^y \quad ; \quad h^{(\times)}_{xx} = 0 \ , \ \xi^z = 0 \\
\frac{d^2\delta x}{dt^2} &= -\frac{1}{2}\omega^2 A_\times \sin(\omega t)\left[y_0 + \delta y\right] \quad ; \quad \delta y \sim \mathcal{O}(A_\times) \\
\frac{d^2\delta x}{dt^2} &\backsimeq -\frac{1}{2}\omega^2 y_0 A_\times \sin(\omega t) \ .
\end{aligned} \tag{3.79}
$$

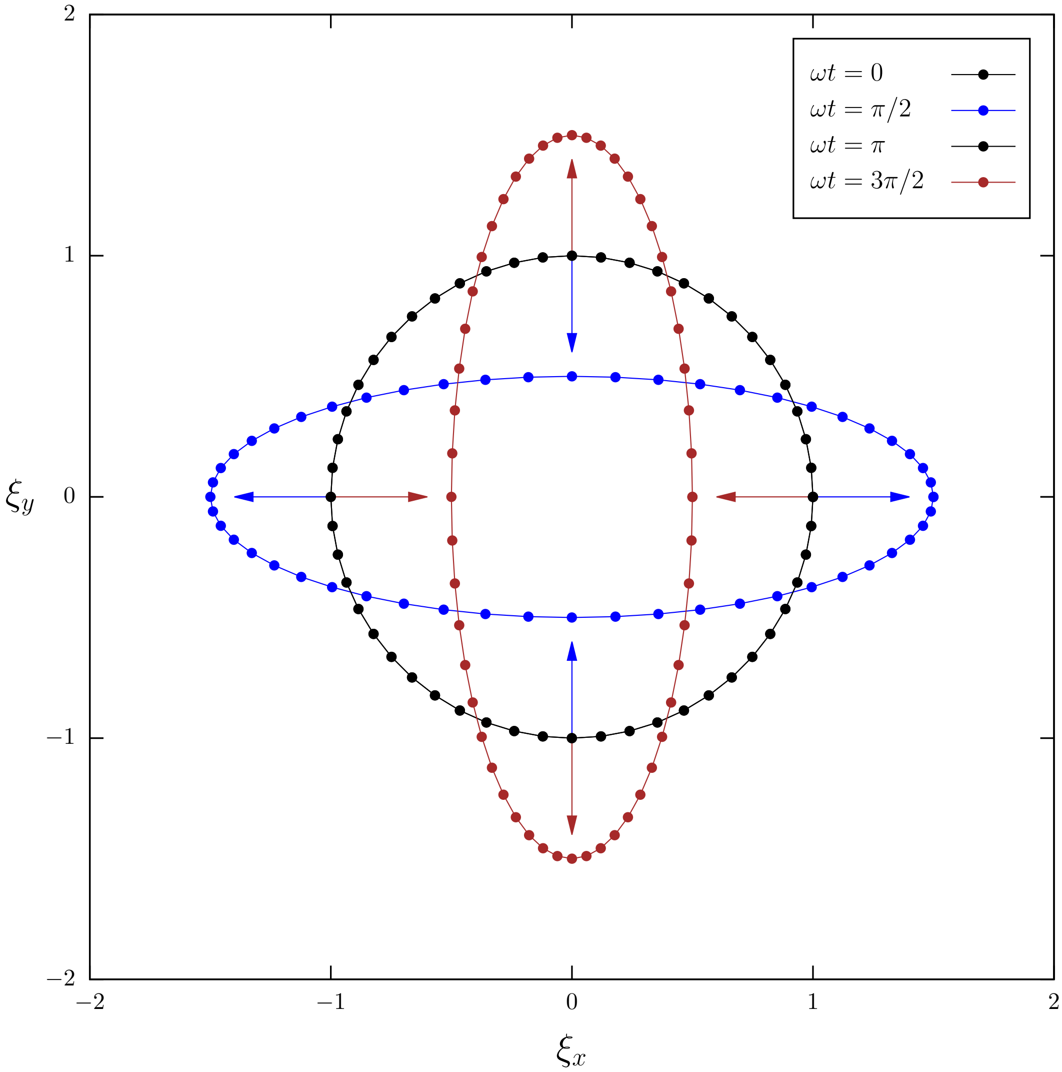

Figure 3.7: Deformation of a ring of test masses under the effect of the *plus* polarization.

Solving for δx from (3.79) gives,

$$\delta x^{(\times)} = \frac{1}{2} y_0 A_\times \sin(\omega t) \quad . \tag{3.80}$$

For the y-component we have

$$\begin{aligned}
\ddot{\xi}_y &= \frac{1}{2}\ddot{h}^{(\times)}_{yj} x i^j = \frac{1}{2}\left[\ddot{h}^{(\times)}_{yx}\xi^x + \ddot{h}^{(\times)}_{yy}\xi^y\right] \\
\ddot{\xi}_y &= \frac{1}{2}\ddot{h}^{(\times)}_{yx}\xi^x \qquad ; \qquad h^{(\times)}_{yy} = 0 \;\; , \;\; \xi^z = 0 \\
\frac{d^2\delta y}{dt^2} &= -\frac{1}{2}\omega^2 A_\times \sin(\omega t)\,[\,x_0 + \delta x\,] \qquad ; \qquad \delta x \sim \mathcal{O}(A_\times) \\
\frac{d^2\delta y}{dt^2} &\backsimeq -\frac{1}{2}\omega^2 x_0 A_\times \sin(\omega t) \quad .
\end{aligned} \tag{3.81}$$

From (3.81) it is obtained,

$$\delta y \;=\; \frac{1}{2} x_0 A_\times \sin(\omega t) \quad . \tag{3.82}$$

Using (3.80) and (3.82) in (3.73) the solution of the deviation vector $\boldsymbol{\xi}$ is now,

$$\boxed{\xi_i \;=\; \Big(x_0 \;+\; \frac{1}{2} y_0 A_\times \sin(\omega t)\;,\; y_0 \;+\; \frac{1}{2} x_0 A_\times \sin(\omega t)\;,\; 0\Big)} \quad . \tag{3.83}$$

The situation here is similar to the plus polarization case. The only thing to bear in mind is the stretching directions in which the GW is acting on the ring. In the plus polarization, this axis were the x and y axis. In this case, the axis are perpendicular each other as well but are rotated 45° with respect to the x and y directions (see figure 3.8).

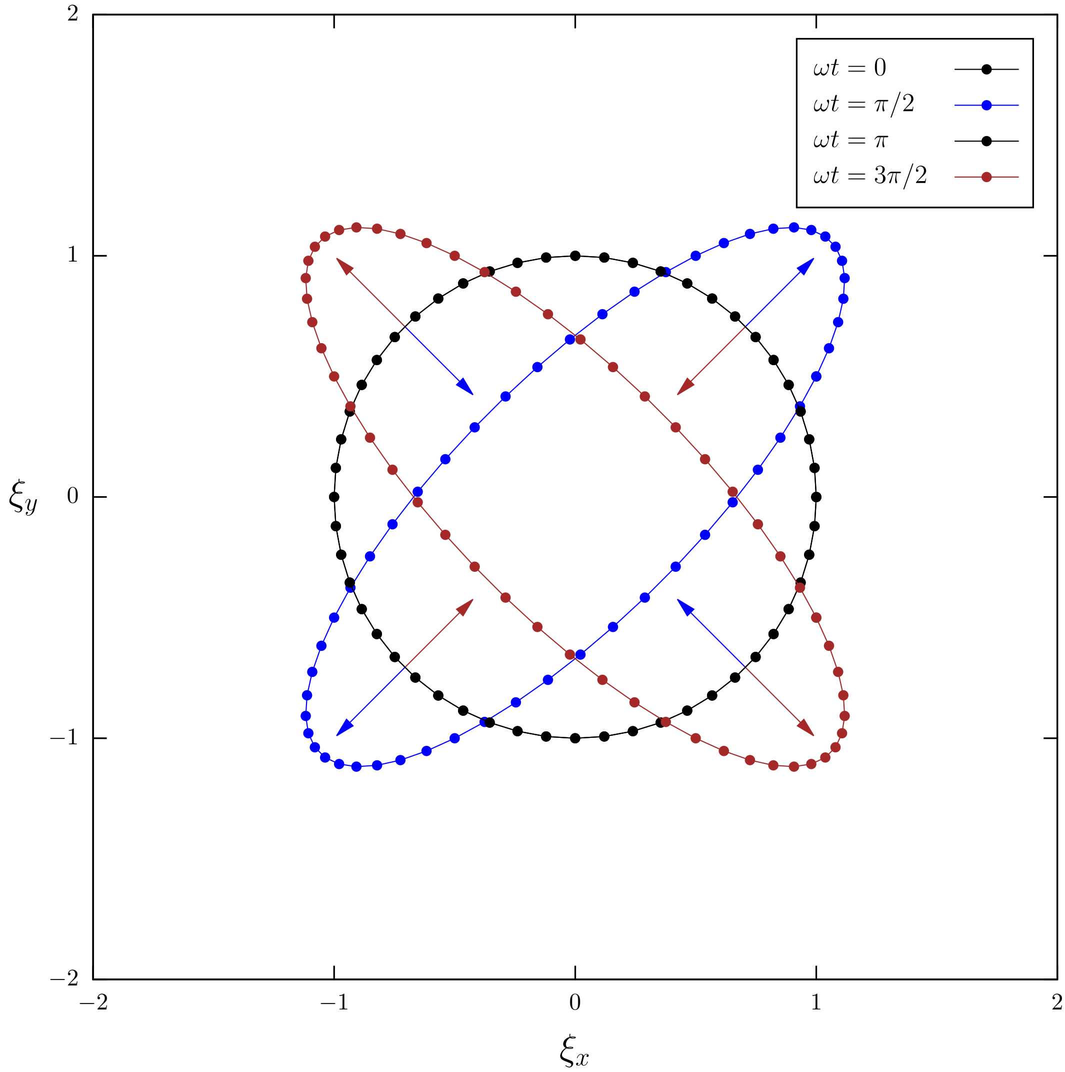

Figure 3.8: Deformation of a ring of test masses under the effect of the *cross* polarization.

Of course, if the GW has both polarizations, the equations to solve becomes in

$$\frac{d^2\delta x}{dt^2} \simeq -\frac{1}{2}\omega^2 \sin(\omega t)\,[A_+ x_0 \,+\, A_\times y_0] \tag{3.84}$$

$$\frac{d^2\delta y}{dt^2} \simeq \frac{1}{2}\omega^2 \sin(\omega t)\,[A_+ y_0 \,-\, A_\times x_0] \quad , \tag{3.85}$$

The deviation vector $\boldsymbol{\xi}$ in this case is

$$\boxed{\xi_i = \left(x_0 + \frac{1}{2}\sin(\omega t)\big[A_+ x_0 + A_\times y_0\big] \;,\; y_0 - \frac{1}{2}\sin(\omega t)\big[A_+ y_0 - A_\times x_0\big] \;,\; 0\right)} \;, \tag{3.86}$$

and figure 3.9 illustrates the effect of both polarizations.

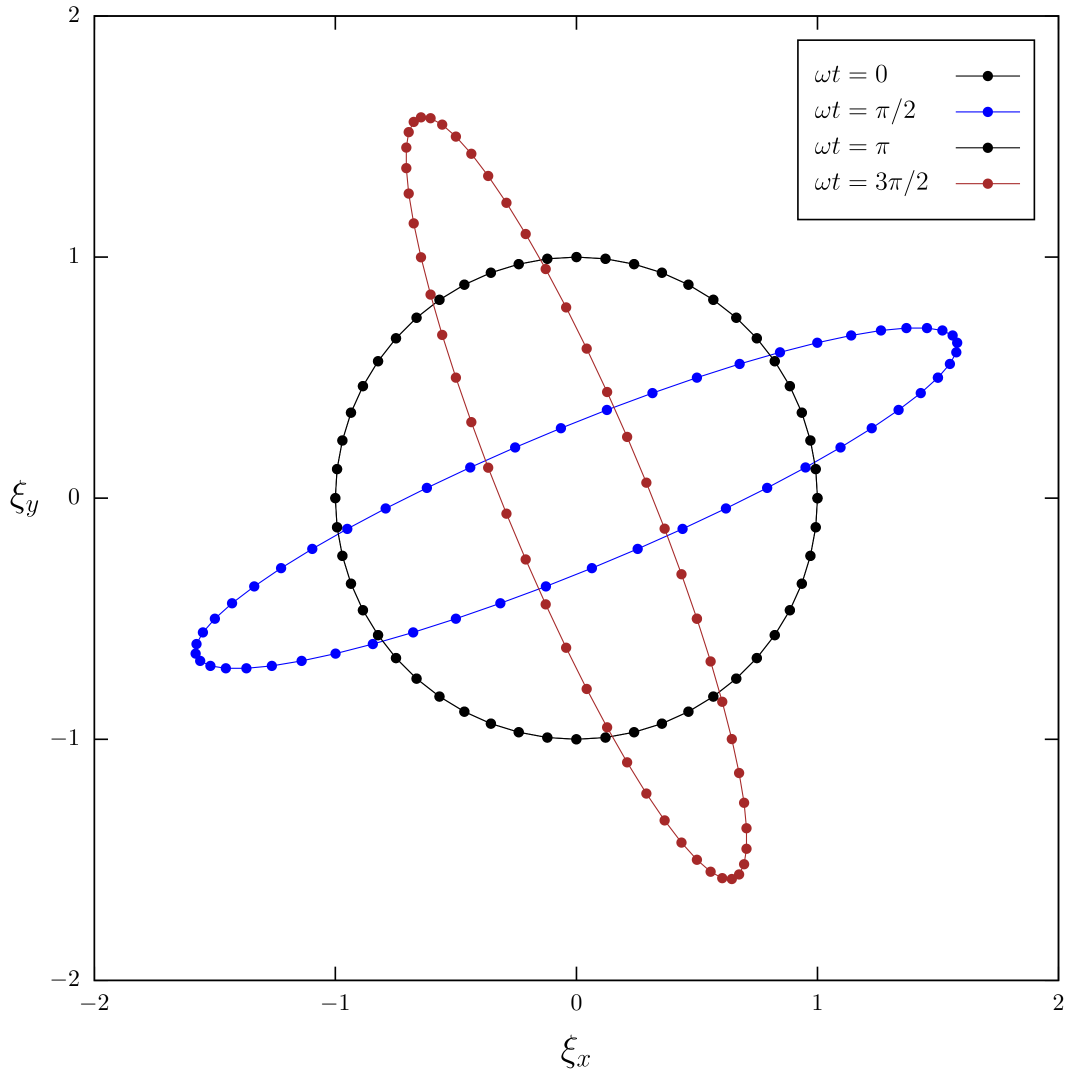

Figure 3.9: *Plus* and *cross* polarizations contributions on a ring of test masses.

Chapter 4

The Energy-Momentum Tensor of Gravitational Waves

The previous chapter demonstrates that GWs carries energy and momentum. The proper distance between particles are indeed changing in the laboratory. If these particles are joined together by a spring with friction, the kinetic energy may dissipate into heat. This means that GWs can do work and therefore the kinetic energy of the test particles must come from the energy of GWs. General relativity predicts the curvature of the spacetime by the presence of energy and matter. If the energy of a GW is propagating through space, the curvature of the spacetime itself is propagating. Neverthless, there is no local measure of the gravitational field in general relativity. As has been mentioned before, is always possible to turn off gravity at one point in spacetime through an appropiate coordinate system[1]. This implies that there is no local gravitational energy. To extract the energy of GWs, the way to proceed is to consider a region of the spacetime that is large enough to capture many wavelengths of the wave, but small enough so the associate energy in this region comes only from the small ripples that generates the wave and not from the background curvature. This is only possible when there is a clearly distinction between the curvature scales of the perturbations and the background. A good example is an orange. As a whole could represent a S^2 sphere, but locally it has tiny lumps. The overall sphere plays the role of the smooth background and the tiny lumps the ripples of the GWs. In the case of linearized theory, this was a straightforward task beacause any disturbance of the spacetime was interpretated as GWs. However, if there is an incoming GW near a spherical symmetric object is not easy to distinguish between the Schwarzschild curvature and the ripples.

4.1 The shortwave approximation in perturbation theory

To ensure the separation of scales and identify the ripples from the smooth background one must impose the condition

$$\lambda\!\!\!^{-} \ll L_B \quad , \tag{4.1}$$

where L_B is typical scale of variation of the background and $\lambda\!\!\!^{-}$ is the reduced wavelength of the GW [8]. Alternatively a formulation in terms of frequencies is also valid,

$$f \gg f_B \quad . \tag{4.2}$$

The length scales and the frequency scales between the waves and the background are *apriori* unrelated and one may use equivalently equation (4.1) or (4.2) to extract the energy of the GW. The formalism that allows to separate the background from the ripples is called the **shortwave approximation**. At the heart of this formalism, is to capture enough physical

[1]The Riemann normal coordinates and the Fermi normal coordinates at one point in the spacetime is the flat metric and thus there is no gravitational field.

curvature in a small region by implementing an average scheme [31]. The inability to define a local measure of the gravitational field energy, suggest to consider a neighborhood about a point in spacetime such that the typical size $\bar{l}$ should be of many wavelengths but not too large compare to the background scale of variation,

$$\lambda\!\!\!^{-} \ll \bar{l} \ll L_B \quad . \tag{4.3}$$

If an average procedure is perform over a volume of size $\bar{l}$, the radiative contributions of GWs may be distinguish as follows,

1) Slowly-varying modes in spacetime of order L_B remains constant when averaging,

2) Rapidly-varying modes in spacetime of order $\lambda\!\!\!^{-}$ average to zero.

The formal details of the spacetime average process are not as important as their properties [33, 39]. For a spacetime with a general background described by a metric $\bar{g}_{\mu\nu}$, the shortwave averaging linear operator $\langle\cdot\rangle$ satisfied the following useful properties. For general tensors A and B, then

$$\text{a) Covariant derivatives conmute, } \big\langle \bar{\nabla}_\mu \bar{\nabla}_\nu (A) \big\rangle = \big\langle \bar{\nabla}_\nu \bar{\nabla}_\mu (A) \big\rangle \quad . \tag{4.4}$$

$$\text{b) Gradients average to zero, } \big\langle \bar{\nabla}_\mu (A) \big\rangle = 0 \quad . \tag{4.5}$$

$$\text{c) As a corollary of the above property, } \big\langle A(\bar{\nabla}_\mu B) \big\rangle = -\big\langle B(\bar{\nabla}_\mu A) \big\rangle \quad . \tag{4.6}$$

A formal justification of such properties can be found in [40, 41]. However, when computing explicitly the energy-momentum tensor in a flat background, these properties will gain meaning[2]. For the moment, we shall focus on how would be the energy-momentum tensor of GWs when a clearly separation of scales are given from the beggining in the shortwave approximation. An intuitive strategy to imagine the schematic form that should take this tensor, may be inspired by the energy-momentum tensor from other fields [31]. In electromagnetism or scalar field theories, the energy-momentum tensors are constructed from quadratic terms in the relevant fields[3]. If one is interested in extracting the energy-momentum tensor of GWs, the relevant field in this case must be $h_{\mu\nu}$. Thus, our framework of perturbation theory must be extended beyond the linear order[4]. With this motivation, a natural way to express the metric decomposition should be as,

$$g_{\mu\nu}(x) \;=\; \bar{g}_{\mu\nu}(x) \;+\; h^{(1)}_{\mu\nu}(x) \;+\; h^{(2)}_{\mu\nu}(x) \;+\; \mathcal{O}(\varepsilon^3) \quad . \tag{4.7}$$

One is actually interested in studying the energy of GWs when they are propagating in vacuum curved background. Therefore, by using the perturbed metric in (4.7), all geometrical quantities can be obtained at order $\mathcal{O}(\varepsilon^2)$. This allow us to expand the vacuum EFE as,

$$G_{\mu\nu}[g] = \bar{G}_{\mu\nu}[\bar{g}] \;+\; \underbrace{\overset{(1)}{G}_{\mu\nu}[h^{(1)}]}_{\mathcal{O}(\varepsilon)} + \underbrace{\overset{(2)}{G}_{\mu\nu}[h^{(1)}]}_{\mathcal{O}(\varepsilon^2)} + \underbrace{\overset{(1)}{G}_{\mu\nu}[h^{(2)}]}_{\mathcal{O}(\varepsilon^2)} + \mathcal{O}(\varepsilon^3) \;=\; 0 \quad . \tag{4.8}$$

The notation of each term in (4.8) is explained as follows,

[2] This work is based on a flat background, so this properties will be justified later in this context.
[3] For instance, the electromagnetic energy-momentum tensor is quadratic in $F_{\mu\nu}$.
[4] Is expected a nonlinear behaviour of the gravitational field with itslef due to the energy of GWs.

$\bar{G}_{\mu\nu}$: Is the usual Einstein operator acting on the background metric $g_{\mu\nu}$.

$\overset{(1)}{G}_{\mu\nu}$: Is a linear operator in their covariant derivatives acting on a perturbation tensor $h^{(n)}_{\mu\nu}$ of some order $\mathcal{O}(\varepsilon^n)$.

$\overset{(2)}{G}_{\mu\nu}$: Is the quadratic part in their covariant derivatives, acting on a perturbation tensor $h^{(n)}_{\mu\nu}$ of some order $\mathcal{O}(\varepsilon^n)$.

Thus, the term of order $\mathcal{O}(\varepsilon)$ in (4.8) has terms of the form $\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{(1)}_{\mu\nu}$. The first term of $\mathcal{O}(\varepsilon^2)$ has terms as $\bar{\nabla}_\alpha h^{(1)}_{\mu\nu}\bar{\nabla}_\beta h^{(1)}_{\rho\sigma}$ and $h^{(1)}_{\sigma\rho}\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{(1)}_{\mu\nu}$. The second term of $\mathcal{O}(\varepsilon)$ has terms of the form $\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{(2)}_{\mu\nu}$[5]. Then, the next step is to solve the EFE order by order in ε. By the initial assumption, the background Einstein's tensor must satisfy by itself the equation $\bar{G}_{\mu\nu}[\bar{g}] = 0$[6]. The first and second order perturbative equations are [31, 42],

$$\overset{(1)}{G}_{\mu\nu}[h^{(1)}] = 0 \quad , \tag{4.9}$$

$$\overset{(1)}{G}_{\mu\nu}[h^{(2)}] = -\overset{(2)}{G}_{\mu\nu}[h^{(1)}] \quad . \tag{4.10}$$

Solving this equations, gives the first order ($h^{(1)}_{\mu\nu}$) and second order ($h^{(2)}_{\mu\nu}$) corrections to the background metric. In particular, if such background is flat, the first order equation in (4.9) is just the linearized EFE with the Einstein's tensor given by (2.25). However, if one retained the metric expansion up to second order, the quantity at the right hand side of (4.10) acts as a source term for the second order metric perturbation $h^{(2)}_{\mu\nu}$. Therefore, one may write (4.10) as

$$\overset{(1)}{G}_{\mu\nu}[h^{(2)}] = \frac{8\pi G}{c^4} t_{\mu\nu} \quad , \tag{4.11}$$

where,

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\overset{(2)}{G}_{\mu\nu}[h^{(1)}] \quad . \tag{4.12}$$

Then, $t_{\mu\nu}$ *is interpretated as the energy-momentum tensor created by the first order metric perturbation* $h^{(1)}_{\mu\nu}$ or GWs. According to our initial motivation, the expression for $t_{\mu\nu}$ in (4.12) is a symmetric tensor as all energy-momentum tensors and is also quadratic in the relevant field $h^{(1)}_{\mu\nu}$ that describes the GW. Moreover, is a conserved quantity by the Bianchi identities. Nevertheless, this is not a gauge invariant quantity[7]. What is gauge invariant is the shortwave averaging of $t_{\mu\nu}$. Thus, the *effective* energy-momentum tensor of GWs is defined as,

$$\boxed{t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle \overset{(2)}{G}_{\mu\nu}[h^{(1)}]\right\rangle} \quad . \tag{4.13}$$

[5]Note the functional form of the equations given in (A.26) and (A.30).

[6]Actually, the background Einstein's tensor is of order $\mathcal{O}(\varepsilon^2)$. We will retake this statement in section 4.2.

[7]If the background is flat, this is refered to infinitesimal gauge transformation in linearized theory. In general one may consider the propagation of a GW in a curved background. This can be done by solving the EFE in vacuum, $R^{(1)}_{\mu\nu} = 0$ by choosing a suitable gauge.

4.2 The coarse-grained form of the Einstein's Field Equations

In the last section, it was mentioned the shortwave approximation to clarify the distintion between the typical scales of the background and GWs. The construction of the energy-momentum tensor $t_{\mu\nu}$ was given through the perturbative EFE based on a formal expansion of the metric. From the beggining, this was made without any assumption about the scale variation of each term in (4.8). Only at the condition of gauge invariance, the shortwave averaging was required. This section explains how to split the EFE at first glance, when the radiative modes of propagation are classified into high and low frequencies. At the end, a macroscopic version of the EFE are obtained in which $t_{\mu\nu}$ works as a source of the background curvature. This is called the *coarse-grained* form of the EFE.
Rather than using the vacuum EFE given in (4.8), is better to rewritte these equations in terms of the Ricci tensor. Note that the trace of the EFE is given by

$$\begin{aligned} g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} &= \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} \\ R - \frac{1}{2}R(4) &= \frac{8\pi G}{c^4}T \\ R &= -\frac{8\pi G}{c^4}T \quad , \quad . \end{aligned} \tag{4.14}$$

where T is the trace of $T_{\mu\nu}$. Using (4.14) in (2.1), an alternative form of the EFE is

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big) \quad . \tag{4.15}$$

Considering the perturbative metric up to second order, the left hand side of (4.15) can be expanded as

$$R_{\mu\nu}[g] = \bar{R}_{\mu\nu}[\bar{g}] + \overset{(1)}{R}_{\mu\nu}[h^{(1)}] + \overset{(2)}{R}_{\mu\nu}[h^{(1)}] + \text{error} \quad . \tag{4.16}$$

The quantity $\bar{R}_{\mu\nu}$ is the usual Ricci tensor definition refered to the background metric. The other terms are the Ricci tensor at first and second order in ε. However, for the purpose of this procedure, one may include just the terms of the form $R^{(2)}_{\mu\nu}[h^{(1)}]$ and not $R^{(1)}_{\mu\nu}[h^{(2)}]$ [33][8]. They have the following characteristics according to their frequency,

$\bar{R}_{\mu\nu}$: *It is constructed from the metric $\bar{g}_{\mu\nu}$ which has a low frequency scale variation. Then, this term contains low-frequency modes.*

$\overset{(1)}{R}_{\mu\nu}$: *By definition is linear order in $h_{\mu\nu}$ which has a high frequency scale variation. Therefore, this term contains high-frequency modes.*

$\overset{(2)}{R}_{\mu\nu}$: *Contain terms of the form $h_{\mu\nu}h_{\rho\sigma}$ and may have low and high frequency scales variation. For instance, the superposition of two waves of high-frequency in opposite directions can results in a low-frequency wave.*

In view of (4.16), the EFE in (4.15) reads,

[8] This can be obtained by considering the metric $g_{\mu\nu}$ at order $\mathcal{O}(\varepsilon)$ but $g^{\mu\nu}$ up to order $\mathcal{O}(\varepsilon^2)$. This is the lowest expansion next to leading order in ε. For more details, see appendix B.

$$\underbrace{\bar{R}_{\mu\nu}[\bar{g}]}_{\text{Low } f} \quad + \quad \underbrace{\overset{(1)}{R}_{\mu\nu}[h^{(1)}]}_{\text{High } f} \quad + \quad \underbrace{\overset{(2)}{R}_{\mu\nu}[h^{(1)}]}_{\text{Low \& High } f} \quad = \quad \frac{8\pi G}{c^4} \underbrace{\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}_{\text{Low \& High } f} \quad . \tag{4.17}$$

These equations can be split into low and high modes as follows,

$$\bar{R}_{\mu\nu} \;=\; -\Big[\overset{(2)}{R}_{\mu\nu}\Big]^{\text{Low} f} \;+\; \frac{8\pi G}{c^4}\Big[T_{\mu\nu} \;-\; \frac{1}{2} g_{\mu\nu} T\Big]^{\text{Low} f} \quad , \tag{4.18}$$

$$\overset{(1)}{R}_{\mu\nu} \;=\; -\Big[\overset{(2)}{R}_{\mu\nu}\Big]^{\text{High} f} \;+\; \frac{8\pi G}{c^4}\Big[T_{\mu\nu} \;-\; \frac{1}{2} g_{\mu\nu} T\Big]^{\text{High} f} \quad . \tag{4.19}$$

The equation (4.18) is the projection of the EFE onto the low frequency modes. This can be formalized by introducing the shortwave approximation. By recalling the properties given in the last section and the frequency distintion given in (4.17), the average of the EFE over a volume $\bar{l}$ turns out to be,

$$\underbrace{\left\langle \bar{R}_{\mu\nu} \right\rangle}_{\sim L_B} + \underbrace{\left\langle \overset{(1)}{R}_{\mu\nu} \right\rangle}_{\sim \lambda} + \underbrace{\left\langle \overset{(2)}{R}_{\mu\nu} \right\rangle}_{\sim L_B \,\&\, \lambda} = \frac{8\pi G}{c^4} \underbrace{\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle}_{\sim L_B \,\&\, \lambda}$$

$$\bar{R}_{\mu\nu} + \Big[\overset{(2)}{R}_{\mu\nu}\Big]^{\text{Low} f} \;=\; \frac{8\pi G}{c^4}\Big[T_{\mu\nu} \;-\; \frac{1}{2} g_{\mu\nu} T \;\;\Big]^{\text{Low} f} \quad , \tag{4.20}$$

and equation (4.18) is recovered. In vacuum, the EFE projected onte the low frequency modes implies that,

$$\bar{R}_{\mu\nu} \;=\; -\Big[\overset{(2)}{R}_{\mu\nu}\Big]^{\text{Low} f} \quad . \tag{4.21}$$

This expression suggest that $\bar{R}_{\mu\nu}$ is of order $\mathcal{O}(\varepsilon^2)$. Likewise, comparing with (4.15), equation (4.21) also tells that the term at the right hand side is a source term for the background curvature. Denoting the low frequency modes of the right hand side in (4.20) by

$$\left\langle T_{\mu\nu} \;-\; \frac{1}{2} g_{\mu\nu} T \right\rangle \;=\; \bar{T}_{\mu\nu} \;-\; \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \quad , \tag{4.22}$$

where $\bar{T} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu}$. By definition, $\bar{T}_{\mu\nu}$ is a low frequency quantity and thus can be interpretated as macroscopic version of $T_{\mu\nu}$ when the scales of variation are fixed[9]. The expression for $t_{\mu\nu}$ in (4.13) yields,

$$t_{\mu\nu} \;=\; -\frac{c^4}{8\pi G} \left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \;-\; \frac{1}{2} \bar{g}_{\mu\nu} \overset{(2)}{R}[h^{(1)}] \right\rangle \quad , \tag{4.23}$$

where $\overset{(2)}{R} = \bar{g}^{\mu\nu} \overset{(2)}{R}_{\mu\nu}$. The trace of (4.23) is

[9] Remember that the bracket operation is catching the information over $\bar{l}$. Then, the resulting low frequency modes is just an average or macroscopic version of $T_{\mu\nu}$.

$$\begin{aligned}
t &= \bar{g}^{\mu\nu} t_{\mu\nu} \\
&= \bar{g}^{\mu\nu} \left[-\frac{c^4}{8\pi G} \left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] - \frac{1}{2} \bar{g}_{\mu\nu} \overset{(2)}{R}[h^{(1)}] \right\rangle \right] \\
&= -\frac{c^4}{8\pi G} \left\langle \bar{g}^{\mu\nu} \overset{(2)}{R}_{\mu\nu}[h^{(1)}] - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} \overset{(2)}{R}[h^{(1)}] \right\rangle \\
t &= +\frac{c^4}{8\pi G} \left\langle \overset{(2)}{R}[h^{(1)}] \right\rangle
\end{aligned} \tag{4.24}$$

Inserting (4.24) into (4.23) we get

$$\begin{aligned}
t_{\mu\nu} &= -\frac{c^4}{8\pi G} \left[\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \right\rangle - \frac{1}{2} \bar{g}_{\mu\nu} \left\langle \overset{(2)}{R}[h^{(1)}] \right\rangle \right] \\
&= -\frac{c^4}{8\pi G} \left[\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \right\rangle - \frac{1}{2} \bar{g}_{\mu\nu} \frac{8\pi G}{c^4} t \right]
\end{aligned} \tag{4.25}$$

Now, solving for $\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \right\rangle$ it is obtained,

$$\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \right\rangle = -\frac{8\pi G}{c^4} \left[t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right] \tag{4.26}$$

Using (4.26), the projected EFE onto the low frequency modes are given by

$$\begin{aligned}
\bar{R}_{\mu\nu} &= -\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] \right\rangle + \frac{8\pi G}{c^4} \left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle \\
&= \frac{8\pi G}{c^4} \left[t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right] + \frac{8\pi G}{c^4} \left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle \\
&= \frac{8\pi G}{c^4} \left[t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right] + \frac{8\pi G}{c^4} \left[\bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \right] \\
&= \frac{8\pi G}{c^4} \left[\left(\bar{T}_{\mu\nu} + t_{\mu\nu} \right) - \frac{1}{2} \bar{g}_{\mu\nu} \left(\bar{T} + t \right) \right] \quad .
\end{aligned} \tag{4.27}$$

Taking the trace of (4.27) one obtains,

$$\begin{aligned}
\bar{R} &= \frac{8\pi G}{c^4} \left[\left(\bar{T} + t \right) - 2 \left(\bar{T} + t \right) \right] \\
\bar{R} &= -\frac{8\pi G}{c^4} \left(\bar{T} + t \right) \quad .
\end{aligned} \tag{4.28}$$

Finally, substituting (4.28) into (4.27) and reorganizing terms, the **coarse-grained** version of the EFE is obtained,

$$\boxed{\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)} \quad . \tag{4.29}$$

These equations determined the background dynamics and represent a macroscopic version of the EFE. Observe that $t_{\mu\nu}$ acts a source for the background spacetime. By virtue of the Bianchi identities, from (4.29) one ensures the conservation of the total energy-momentum tensor,

$$\boxed{\bar{\nabla}^{\nu}\left(T_{\mu\nu} + t_{\mu\nu}\right) = 0} \tag{4.30}$$

This implies that there is in general an exchange of energy and momentum between the matter sources and GWs.
As a final remark, consider for simplicity a nearly flat background[10]. Note from (4.21) that $\bar{R}_{\mu\nu} \sim \mathcal{O}(\varepsilon^2)$ because the right hand side of this equation has terms of the form $\partial_\mu h^{(1)}_{\alpha\beta}\partial_\nu h^{(1)}_{\rho\sigma}$, $h^{(1)}_{\alpha\beta}\partial_\mu\partial_\nu h^{(1)}_{\rho\sigma}$ and both terms are of order $\mathcal{O}(\varepsilon^2)$. Apart from that, $\bar{g}_{\mu\nu}$ has a scale of variation L_B and $h_{\mu\nu}$ vary as $\lambda\!\!\!^{-}$. Now, keeping in mind that $R_{\mu\nu}$ comes from the second derivatives of the background metric it is true that,

$$\bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2} \qquad \text{and} \qquad \bar{R}_{\mu\nu} \sim (\partial h_{\mu\nu})^2 \sim \left(\frac{\varepsilon}{\lambda\!\!\!^{-}}\right)^2 \tag{4.31}$$

Therefore, it is concluded that

$$\boxed{\varepsilon \sim \frac{\lambda\!\!\!^{-}}{L_B}} \qquad \text{(Curvature determined by GWs)} \quad . \tag{4.32}$$

If $T_{\mu\nu} \neq 0$ and the contribution of GWs to the background is very weak compared to matter sources, then

$$\frac{1}{L_B^2} \sim \left(\frac{\varepsilon}{\lambda\!\!\!^{-}}\right)^2 + \text{Typical value of } T_{\mu\nu} \gg \left(\frac{\varepsilon}{\lambda\!\!\!^{-}}\right)^2 \quad . \tag{4.33}$$

In this case we have,

$$\boxed{\varepsilon \ll \frac{\lambda\!\!\!^{-}}{L_B}} \qquad \text{(Curvature determined by matter)} \quad . \tag{4.34}$$

From the previous expressions, is possible to understand why linearized theory is not possible to be extended beyond the linear order in a systematic way. If $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ then $(1/L_B) \to 0$ and equation (4.32) implies that $\varepsilon \to 0$. Even if ε is arbitrarily small, equation (4.34) is not valid. This means that general relativity can not be promoted in a systematic expansion of ε if the background is the flat metric [8]. The notion that $\varepsilon \ll 1$ is also understood from the previous discussion. If one assumes that $\varepsilon = 1$, from $\lambda\!\!\!^{-}/L_B$ there is no distintion between GWs and the background. Thus, GWs are slightly ripples in the fabric of spacetime and there is nothing like a GW with an arbitrary amplitude.

[10]For example, consider the background metric as $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}(\varepsilon^2)$.

4.3 The energy-momentum tensor of gravitational waves

An explicit expression for $t_{\mu\nu}$ at the detector is obtained in this section. Very far away from the source, i.e, at the laboratory, xthe background spacetime may be consider as flat. This implies that $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\nabla \equiv \partial$. Recalling the definition of the energy-momentum tensor from the previous sections, the energy momentum-tensor is given at the detector by,

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle \overset{(2)}{R}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\overset{(2)}{R}[h^{(1)}] \right\rangle \quad . \tag{4.35}$$

Because the second order perturbation tensor $h^{(2)}_{\mu\nu}$ is not involved in this computations, from now on $h^{(1)}_{\mu\nu} \equiv h_{\mu\nu}$. Based on the calculations shown in appendix B the first term that is required from (4.35) gives,

$$\begin{aligned} \overset{(2)}{R}_{\mu\nu} = \; & \frac{1}{2}\eta^{\rho\sigma}\eta^{\alpha\beta}\Big[\frac{1}{2}\partial_\mu h_{\rho\alpha}\partial_\nu h_{\sigma\beta} + \Big(\partial_\rho h_{\nu\alpha}\Big)\Big(\partial_\sigma h_{\mu\beta} - \partial_\beta h_{\mu\sigma}\Big)\Big] \\ & + h_{\rho\alpha}\Big(\partial_\nu\partial_\mu h_{\sigma\beta} + \partial_\beta\partial_\sigma h_{\mu\nu} - \partial_\beta\partial_\nu h_{\mu\sigma} - \partial_\beta\partial_\mu h_{\nu\sigma}\Big) \\ & + \Big(\frac{1}{2}\partial_\alpha h_{\rho\sigma} - \partial_\rho h_{\alpha\sigma}\Big)\Big(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu}\Big) \end{aligned} \tag{4.36}$$

$$\begin{aligned} \overset{(2)}{R}_{\mu\nu} = \; & \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta} + \partial^\sigma h_\nu^{\ \beta}\partial_\sigma h_{\mu\beta} - \partial^\sigma h_\nu^{\ \beta}\partial_\beta h_{\mu\sigma} \\ & + h^{\sigma\beta}\partial_\mu\partial_\nu h_{\sigma\beta} + h^{\sigma\beta}\partial_\sigma\partial_\beta h_{\mu\nu} - h^{\sigma\beta}\partial_\nu\partial_\beta h_{\mu\sigma} - h^{\sigma\beta}\partial_\mu\partial_\beta h_{\nu\sigma} \\ & + \frac{1}{2}\partial^\beta h\partial_\nu h_{\mu\beta} + \frac{1}{2}\partial^\beta h\partial_\mu h_{\nu\beta} - \frac{1}{2}\partial^\beta h\partial_\beta h_{\mu\nu} \\ & - \partial_\sigma h^{\beta\sigma}\partial_\nu h_{\mu\beta} - \partial_\sigma h^{\beta\sigma}\partial_\mu h_{\nu\beta} + \partial_\sigma h^{\beta\sigma}\partial_\beta h_{\mu\nu}\Big] \quad . \end{aligned} \tag{4.37}$$

In general, $t_{\mu\nu}$ is a symmetric tensor, so it has 10 independent components. Only two of them are physical modes that comes from the *plus* and *cross* polarizations. The other components depends on the choice of coordinates. To throw out spurious degrees of freedom, is worthwhile to impose the Lorenz gauge. By doing this, 6 independent components remains, 2 physical modes and 4 gauge modes that depends on ε_μ through $\Box\varepsilon_\mu = 0$ (see equation (2.65)). If initially the functions ε_μ are choosen such that $h = 0$, then

$$\bar{h}_{\mu\nu} = h_{\mu\nu} \qquad \text{and} \qquad \partial^\nu \bar{h}_{\mu\nu} = \partial^\nu h_{\mu\nu} = 0 \quad . \tag{4.38}$$

Therefore, the last six terms in (4.37) vanish and $t_{\mu\nu}$ becomes,

$$\begin{aligned} \overset{(2)}{R}_{\mu\nu} = \; & \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta} + \partial^\sigma h_\nu^{\ \beta}\partial_\sigma h_{\mu\beta} - \partial^\sigma h_\nu^{\ \beta}\partial_\beta h_{\mu\sigma} \\ & + h^{\sigma\beta}\partial_\mu\partial_\nu h_{\sigma\beta} + h^{\sigma\beta}\partial_\sigma\partial_\beta h_{\mu\nu} - h^{\sigma\beta}\partial_\nu\partial_\beta h_{\mu\sigma} - h^{\sigma\beta}\partial_\mu\partial_\beta h_{\nu\sigma}\Big] \quad . \end{aligned} \tag{4.39}$$

The bracket operator $\langle\cdot\rangle$ that appears in (4.35) implies that the shortwave average of some terms in (4.39) is given by,

$$\divideontimes \ \frac{1}{\bar{l}^3}\int_V \partial^\sigma h_\nu{}^\beta \partial_\sigma h_{\mu\beta}\, d^3x \quad ; \quad \partial_\sigma h_{\mu\beta}\partial^\sigma h_\nu{}^\beta = \partial_\sigma\Big(h_{\mu\beta}\partial^\sigma h_\nu^\beta\Big) - h_{\mu\beta}\Box h_\nu{}^\beta$$

$$\frac{1}{\bar{l}^3}\int_V \Big[\partial_\sigma\Big(h_{\mu\beta}\partial^\sigma h_\nu{}^\beta\Big) - h_{\mu\beta}\underbrace{\Box h_\nu{}^\beta}_{=0}\Big]\, d^3x = \frac{1}{\bar{l}^3}\oint_{\partial V} h_{\mu\beta}\partial^\sigma h_\nu{}^\beta d^2x \simeq 0 \tag{4.40}$$

$$\divideontimes \ \frac{1}{\bar{l}^3}\int_V \partial^\sigma h_\nu{}^\beta \partial_\beta h_{\mu\sigma}\, d^3x \quad ; \quad \partial^\sigma h_\nu{}^\beta\partial_\beta h_{\mu\sigma} = \partial_\beta\Big(h_{\mu\sigma}\partial^\sigma h_\nu{}^\beta\Big) - h_{\mu\sigma}\partial_\beta\partial^\sigma h_\nu{}^\beta$$

$$\frac{1}{\bar{l}^3}\int_V \Big[\partial_\beta\Big(h_{\mu\sigma}\partial^\sigma h_\nu{}^\beta\Big) - h_{\mu\sigma}\partial^\sigma\underbrace{\partial_\beta h_\nu{}^\beta}_{=0}\Big]\, d^3x = \frac{1}{\bar{l}^3}\oint_{\partial V} h_{\mu\sigma}\partial^\sigma h_\nu{}^\beta d^2x \simeq 0 \tag{4.41}$$

$$\divideontimes \ \frac{1}{\bar{l}^3}\int_V h^{\sigma\beta}\partial_\sigma\partial_\beta h_{\mu\nu}\, d^3x \quad ; \quad h^{\sigma\beta}\partial_\sigma\partial_\beta h_{\mu\nu} = \partial_\beta\Big(h^{\sigma\beta}\partial_\sigma h_{\mu\nu}\Big) - \partial_\beta h^{\sigma\beta}\partial_\sigma h_{\mu\nu}$$

$$\frac{1}{\bar{l}^3}\int_V \Big[\partial_\beta\Big(h^{\sigma\beta}\partial_\sigma h_{\mu\nu}\Big) - \underbrace{\partial_\beta h^{\sigma\beta}}_{=0}\partial_\sigma h_{\mu\nu}\Big]\, d^3x = \frac{1}{\bar{l}^3}\oint_{\partial V} h^{\sigma\beta}\partial_\sigma h_{\mu\nu} d^2x \simeq 0 \tag{4.42}$$

Similarly,

$$\divideontimes \ \frac{1}{\bar{l}^3}\int_V h^{\sigma\beta}\partial_\nu\partial_\beta h_{\mu\sigma}\, d^3x = 0 \qquad \divideontimes \ \frac{1}{\bar{l}^3}\int_V h^{\sigma\beta}\partial_\mu\partial_\beta h_{\nu\sigma}\, d^3x = 0 \ . \tag{4.43}$$

Is important to remark that although to average is spatial, it could be done also over the temporal component due to the Lorenz gauge. In this reference frame one has the wave equation $\Box h_{\mu\nu} = 0$, whose solution depends on the agument $(x^0 - z)$. The action of the temporal derivative over the solution $h_{\mu\nu}(x^0 - z)$ implies that $\partial_0 h_{\mu\nu}(x^0 - z) = -\partial_z h_{\mu\nu}(x^0 - z)$. As a consequence, the integration by parts is also possible when a temporal derivative is involved by changing $\partial_0 \longrightarrow -\partial_z$. The resulting surface integral drops to zero by assuming that the size of the box in which the integration takes place is infinitely larger than $\lambda\!\!\!^{-}$ [8]. By (4.39) one is left with,

$$\begin{aligned}\Big\langle \overset{(2)}{R}_{\mu\nu} \Big\rangle &= \frac{1}{2}\Big\langle \frac{1}{2}\partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta} + h^{\sigma\beta}\partial_\mu\partial_\nu h_{\sigma\beta}\Big\rangle \\ &= \frac{1}{2}\Big\langle \frac{1}{2}\partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta} + \underbrace{\partial_\mu\Big(h^{\sigma\beta}\partial_\nu h_{\sigma\beta}\Big)}_{\text{boundary term}} - \partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta}\Big\rangle \\ \Big\langle \overset{(2)}{R}_{\mu\nu} \Big\rangle &= -\frac{1}{4}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta}\Big\rangle \ .\end{aligned} \tag{4.44}$$

Equation (4.44) gives the first term in the definition of $t_{\mu\nu}$ as shown in (4.35). To obtain the other term, one must contract the result in (4.44) with the background metric, $\eta_{\mu\nu}$ as follows,

$$\begin{aligned}
\eta^{\alpha\beta}\left\langle \overset{(2)}{R}_{\alpha\beta}\right\rangle &= -\frac{1}{4}\eta^{\alpha\beta}\Big\langle \partial_\alpha h^{\sigma\rho}\partial_\beta h_{\sigma\rho}\Big\rangle \\
\left\langle \eta^{\alpha\beta}\overset{(2)}{R}_{\alpha\beta}\right\rangle &= -\frac{1}{4}\Big\langle \partial_\alpha h^{\sigma\rho}\partial^\alpha h_{\sigma\rho}\Big\rangle \\
\left\langle \overset{(2)}{R}\right\rangle &= -\frac{1}{4}\Big\langle \underbrace{\partial_\alpha\Big(h^{\sigma\rho}\partial^\alpha h_{\sigma\rho}\Big)}_{\text{boundary}} - h^{\sigma\rho}\underbrace{\Box h_{\sigma\rho}}_{=0}\Big\rangle = 0 \quad . \qquad (4.45)
\end{aligned}$$

Finally, inserting (4.44) and (4.45) into (4.35) one gets,

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu h_{\sigma\beta}\Big\rangle} \quad . \qquad (4.46)$$

The conditions given by the Lorenz gauge and $h = 0$ allowed to obtain equation (4.46). However this 5 constraints reduce from 10 to 5 independent components. These are 2 physical and 3 gauge dependent. Is possibe to check that the 3 residual gauge modes that comes from ε_μ do not contribute to the expression (4.46). This can be done by veryfing that (4.46) is gauge invariant in the linearized theory or computing the variation $\delta t_{\mu\nu}$. As mentioned in chapter 2, the invariance gauge in linearized theory is,

$$\delta h_{\mu\nu} \equiv h_{\mu\nu} - h'_{\mu\nu} = \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu \quad . \qquad (4.47)$$

Applying the Leibniz rule, the variation of $t_{\mu\nu}$ gives,

$$\delta t_{\mu\nu} = \frac{c^4}{32\pi G}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu \delta h_{\mu\nu} + \partial_\mu \delta h_{\mu\nu}\partial_\nu h_{\sigma\beta}\Big\rangle \quad . \qquad (4.48)$$

Therefore,

$$\begin{aligned}
\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu\Big(\partial_\sigma\varepsilon_\beta + \partial_\beta\varepsilon_\sigma\Big) + \partial_\mu\Big(\partial^\sigma\varepsilon^\beta + \partial^\beta\varepsilon^\sigma\Big)\partial_\nu h_{\sigma\beta}\Big\rangle \\
&= \frac{c^4}{32\pi G}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu\partial_\sigma\varepsilon_\beta + \partial_\mu h^{\sigma\beta}\partial_\nu\partial_\beta\varepsilon_\sigma + \partial_\mu\partial^\sigma\varepsilon^\beta\partial_\nu h_{\sigma\beta} + \partial_\mu\partial^\beta\varepsilon^\sigma\partial_\nu h_{\sigma\beta}\Big\rangle \\
&= \frac{c^4}{32\pi G}\Big\langle 2\partial_\mu h^{\sigma\beta}\partial_\nu\partial_\sigma\varepsilon_\beta + 2\partial_\mu\partial^\sigma\varepsilon^\beta\partial_\nu h_{\sigma\beta}\Big\rangle \\
&= \frac{c^4}{16\pi G}\Big\langle \partial_\mu h^{\sigma\beta}\partial_\nu\partial_\sigma\varepsilon_\beta + \partial_\mu\partial^\sigma\varepsilon^\beta\partial_\nu h_{\sigma\beta}\Big\rangle \\
&= \frac{c^4}{16\pi G}\Big\langle \Big[\underbrace{\partial_\sigma\Big(\partial_\mu h^{\sigma\beta}\partial_\nu\varepsilon_\beta\Big)}_{\text{boundary}} - \partial_\mu\underbrace{\partial_\sigma h^{\sigma\beta}}_{=0}\partial_\nu\varepsilon_\beta\Big] + \Big[\underbrace{\partial^\sigma\Big(\partial_\mu\varepsilon^\beta\partial_\nu h_{\sigma\beta}\Big)}_{\text{boundary}} - \partial_\mu\varepsilon^\beta\partial_\nu\underbrace{\partial^\sigma h_{\sigma\beta}}_{=0}\Big]\Big\rangle \\
&= 0 \quad . \qquad (4.49)
\end{aligned}$$

The previous result shows that $t_{\mu\nu}$ depends only on the physical modes. Thus, equation (4.46) can be put as,

$$\boxed{t_{\mu\nu} \;=\; \frac{c^4}{32\pi G}\left\langle \partial_\mu h^{ij}_{\mathrm{TT}} \partial_\nu h^{\mathrm{TT}}_{ij} \right\rangle} \quad . \tag{4.50}$$

In particular, the t_{00} component is the energy density and reads,

$$t_{00} \;=\; \frac{c^2}{32\pi G}\left\langle \dot{h}^{ij}_{\mathrm{TT}}\, \dot{h}^{\mathrm{TT}}_{ij} \right\rangle \quad . \tag{4.51}$$

This expression can be rewritten in terms of the two physical degrees of freedom. For simplicity, consider a generic TT-wave propagating along some of the spatial axis, e.g., $+z$-direction[11]. Hence, t_{00} becomes

$$t_{00} \;=\; \frac{c^2}{32\pi G}\left\langle \dot{h}^{\mathrm{TT}}_{11}\dot{h}^{11}_{\mathrm{TT}} \,+\, \dot{h}^{\mathrm{TT}}_{12}\dot{h}^{12}_{\mathrm{TT}} \,+\, \dot{h}^{\mathrm{TT}}_{21}\dot{h}^{21}_{\mathrm{TT}} \,+\, \dot{h}^{\mathrm{TT}}_{22}\dot{h}^{22}_{\mathrm{TT}} \right\rangle$$

$$\boxed{t_{00} \;=\; \frac{c^2}{16\pi G}\left\langle \dot{h}^2_+ \,+\, \dot{h}^2_\times \right\rangle} \quad , \tag{4.52}$$

where $h^{\mathrm{TT}}_{11} = h_+$ and $h^{\mathrm{TT}}_{22} = h_\times$. For instance, let's compute the energy-momntum tensor $t_{\mu\nu}$ for a plane wave travelling along the $+z$-direction. This wave is described by

$$h^{\mathrm{TT}}_{ij}(t,z) \;=\; \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \begin{cases} h_+ \;=\; A_+ \cos(kz - \omega t) \\ h_\times \;=\; A_\times \cos(kz - \omega t) \end{cases} \quad . \tag{4.53}$$

Observe in (4.53) that h^{TT}_{ij} depends on the argument $(kz - \omega t)$. As a consequence, one obtains immediately,

$$\partial_1 h^{\mathrm{TT}}_{ij} \;=\; 0 \quad , \quad \partial_2 h^{\mathrm{TT}}_{ij} \;=\; 0 \ . \tag{4.54}$$

Thus,

$$t_{01} \;=\; \frac{c^4}{32\pi G}\left\langle \partial_0 h^{ij}_{\mathrm{TT}} \partial_1 h^{\mathrm{TT}}_{ij} \right\rangle \;=\; 0 \quad , \quad t_{02} \;=\; \frac{c^4}{32\pi G}\left\langle \partial_0 h^{ij}_{\mathrm{TT}} \partial_2 h^{\mathrm{TT}}_{ij} \right\rangle \;=\; 0 \ . \tag{4.55}$$

For the t_{03} component, first note that

$$\partial_3 h^{\mathrm{TT}}_{ij}\left[\omega\left(\frac{z}{c} - t\right)\right] \;=\; -\frac{1}{c}\partial_t h^{\mathrm{TT}}_{ij}\left[\omega\left(\frac{z}{c} - t\right)\right] \quad . \tag{4.56}$$

Therefore,

[11] This may be a plane wave in vacuum or a wave generated by a source like a binary system.

$$
\begin{aligned}
t_{03} &= \frac{c^4}{32\pi G}\left\langle \partial_0 h_{\mathrm{TT}}^{ij} \partial_3 h_{ij}^{\mathrm{TT}} \right\rangle \\
t_{03} &= \frac{c^4}{32\pi G}\left\langle \partial_0 h_{\mathrm{TT}}^{ij} \Big(- \partial_0 h_{ij}^{\mathrm{TT}} \Big) \right\rangle = -t_{00} \quad . \qquad (4.57)
\end{aligned}
$$

Using (4.56) the t_{33} component gives,

$$
\begin{aligned}
t_{33} &= \frac{c^4}{32\pi G}\left\langle \partial_3 h_{\mathrm{TT}}^{ij} \partial_3 h_{ij}^{\mathrm{TT}} \right\rangle \\
t_{33} &= \frac{c^4}{32\pi G}\left\langle \Big(- \partial_0 h_{\mathrm{TT}}^{ij} \Big) \Big(- \partial_0 h_{ij}^{\mathrm{TT}} \Big) \right\rangle = t_{00} \quad . \qquad (4.58)
\end{aligned}
$$

Lastly, note that $\left\langle \sin^2 x \right\rangle = 1/2$, so $\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle = \omega^2 (A_+^2 + A_\times^2)$. Thus,

$$
t_{\mu\nu}^{(+z)} = \frac{c^2\omega^2}{16\pi G}\left(A_+^2 + A_\times^2\right) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} , \qquad (4.59)
$$

$$
t^{\mu\nu}_{(+z)} = \frac{c^2\omega^2}{16\pi G}\left(A_+^2 + A_\times^2\right) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} . \qquad (4.60)
$$

Likewise, similar expressions can be obtained for a plane wave propagating along the x and y-directions.
The energy-momentum tensor of gravitational waves is in general an invariant gauge quantity. The preceding development was calculated specifically in the TT frame and at the location of the laboratory. The usage of the TT conditions was done not only to simplify the expressions, but also to extract the physical degrees of freedom. However, this does not mean that one cannot use other coordinate systems. Certainly, these systems must describe perturbations in the sense of gauge transformations (see chapter 2). For this reason, a well defined momentum-tensor is requiered under these types of gauge transformations that represent perturbations. If one chooses a coordinate system without fixing a gauge, the tensor $t_{\mu\nu}$ at the position of the detector is given by [31, 33][12],

$$
t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu \bar{h} \partial_\nu \bar{h} - \partial_\rho \bar{h}^{\rho\sigma} \partial_\nu \bar{h}_{\mu\sigma} - \partial_\rho \bar{h}^{\rho\sigma} \partial_\mu \bar{h}_{\nu\sigma} \right\rangle . \qquad (4.61)
$$

It can be shown that equation (4.61) with the bracket operation, is indeed invariant under infinitesimal gauge transformations. More generally, if the background is not flat, the energy-momentum tensor yields [33],

[12] To perform this explicit computation, one may use the shortwave average properties as well as the propagation equation, i.e, the linearized EFE in vacuum.

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \bar{\nabla}_\mu \bar{h}_{\alpha\beta}\bar{\nabla}_\nu \bar{h}^{\alpha\beta} - \frac{1}{2}\bar{\nabla}_\mu \bar{h}\bar{\nabla}_\nu \bar{h} - 2\bar{\nabla}_\beta \bar{h}^{\alpha\beta}\bar{\nabla}_{(\nu}\bar{h}_{\mu)\alpha}\right\rangle \ , \tag{4.62}$$

where $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}h$. The expression (4.62) is also invariant under perturbed gauge transformations (see equation (2.8)). This means that $t_{\mu\nu}$ is also a well defined quantity in a curved background spacetime. To proof equation (4.62), one should use the general definition of $t_{\mu\nu}$ in (4.13), the equation (A.30) and the propagation equation for waves on curved background. This latter equation is given by setting to zero the expression (A.26) that can be rewritten also in terms of $\bar{h}_{\mu\nu}$. The result is,

$$\bar{\Box}\bar{h}_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{h}^{\alpha\beta} - 2\bar{\nabla}_{(\nu}\bar{\nabla}^\alpha\bar{h}_{\mu)\alpha} + 2\bar{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} - 2\bar{R}_{\alpha(\mu}\bar{h}^\alpha_{\nu)} = 0 \ . \tag{4.63}$$

On the other hand, besides it's gauge invariance, the energy-momenutm tensor $t_{\mu\nu}$ is also a conserved quantity in vacuum. If one is outside the matter sources, then $T_{\mu\nu} = 0$. Thus, from (4.30) one obtains,

$$\boxed{\bar{\nabla}^\nu t_{\mu\nu} = 0} \ . \tag{4.64}$$

At the detector location $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{\nabla} \equiv \partial$. Therefore, equation (4.64) becomes,

$$\boxed{\partial^\nu t_{\mu\nu} = 0} \ . \tag{4.65}$$

As a final comment, it is important to mentioned that the energy-momentum tensor can be also studied from other approaches. For instance, if one considers linearized gravity as a classical field theory, is possible to obtained $t_{\mu\nu}$ from Noether's theorem by taking the shortwave average at the end. Another way to get the same expression for $t_{\mu\nu}$ is from the Landau-Lifshitz pseudotensor. After making the operation $\langle t^{\mu\nu}_{\rm LL}\rangle$ results the same expression as in (4.46)[13].

4.4 The energy flux of gravitational waves

In terms of its components, the energy-momentum tensor $t^{\alpha\beta}$ represents the flux of momentum α across a constant surface $x^\beta = (ct, \mathbf{x})$ [43]. The component $\alpha = 0$ of momentum is the energy and the constant surface x^0 is some instant t. This means that the component t^{00} is just the energy density of GWs. Thus, the energy E that is contained within a volume V reads,

$$E_V = \int_V t^{00}\ d^3x \ . \tag{4.66}$$

At large distances from the source, the conservation equation for $t^{\mu\nu}$ is given by the equation (4.65). Then, the energy E may be expressed as,

$$\frac{1}{c}\frac{dE_V}{dt} = \frac{1}{c}\frac{d}{dt}\int_V t^{00}\ d^3x = \int_V \frac{1}{c}\frac{\partial t^{00}}{\partial t}\ d^3x = \int_V \partial_0 t^{00}\ d^3x$$

[13] For more details on different approaches, see [39].

$$
\begin{aligned}
\frac{1}{c}\frac{dE_V}{dt} &= -\int_V \partial_i t^{0i}\; d^3x && \Big[\partial_0 t^{00} + \partial_i t^{0i} = 0\Big] \\
\frac{1}{c}\frac{dE_V}{dt} &= -\int_{\partial V} t^{0i} n_i\; dA && [\text{Divergence theorem}] \quad , \qquad (4.67)
\end{aligned}
$$

where n_i is the outward pointing unit normal vector field of the boundary ∂V. The integration volume V is assumed to be a spherical shell centered on the source that generates the GW, where the boundary is located at spatial infinity [8, 31][14]. If $n_i = \hat{r}_i$ is the radial unit vector at each point of the shell which has a surface element $dA = r^2 d\Omega$, the equation (4.67) turns out to be,

$$
\frac{dE_V}{dt} = -c\, r^2 \int_{\partial V} t^{0r}\; d\Omega \qquad (4.68)
$$

where

$$
t^{0r} = \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \frac{\partial h^{\mathrm{TT}}_{ij}}{\partial r}\right\rangle \quad . \qquad (4.69)
$$

Let's anticipate that for a GW propagating radially outwards at the far-field zone, the perturbation solution has a functional dependence given by[15],

$$
h^{\mathrm{TT}}_{ij}(t,r) = \frac{1}{r} f_{ij}(t - r/c) \quad . \qquad (4.70)
$$

First, note that

$$
\frac{\partial f_{ij}(t-r/c)}{\partial r} = -\frac{1}{c}\frac{\partial f_{ij}(t-r/c)}{\partial t} = -\partial_0 f_{ij}(t-r/c) \quad . \qquad (4.71)
$$

Then, at very large distances one has

$$
\begin{aligned}
\frac{\partial h^{\mathrm{TT}}_{ij}(t,r)}{\partial r} &= \underbrace{-\frac{1}{r^2} f_{ij}(t-r/c)}_{\mathcal{O}(1/r^2)\sim 0} + \frac{1}{r}\frac{\partial f_{ij}(t-r/c)}{\partial r} \\
&= -\partial_0 h^{\mathrm{TT}}_{ij}(t,r) \qquad (4.72)
\end{aligned}
$$

Therefore, using this result in (4.72) it is obtained,

$$
\begin{aligned}
t^{0r} &= \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \frac{\partial h^{\mathrm{TT}}_{ij}}{\partial r}\right\rangle = \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \left(-\frac{1}{c}\frac{\partial h^{\mathrm{TT}}_{ij}}{\partial t}\right)\right\rangle \\
&= \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \left(-\partial_0 h^{\mathrm{TT}}_{ij}\right)\right\rangle = \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \partial^0 h^{\mathrm{TT}}_{ij}\right\rangle = t^{00} = t_{00} \quad . \qquad (4.73)
\end{aligned}
$$

[14] At spatial infinity the gravitational field is given only by GWs. This allows to neglect the contributions of the matter sources and capture only the energy of GWs.

[15] A detailed explanation of this functional form can be found in Chapter 5.

Now, by taking into account the expression for t_{00} previously mentioned in (4.52), the energy flux yields,

$$\begin{aligned} \frac{dE_V}{dt} &= -c\,r^2 \int_{\partial V} t^{00}\, d\Omega \\ &= -\frac{c^3 r^2}{16\pi G} \int_{\partial V} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle d\Omega \quad . \end{aligned} \tag{4.74}$$

The minus sign in the previous equation means that the energy E inside the volume V is decreasing with time. From the energy conservation principle, the energy loss per unit time inside the volume implies that GWs carry an energy per unit time given by[16]

$$P^{\mathsf{GW}} = -\frac{dE_V}{dt} \quad . \tag{4.75}$$

Thus, the power emitted by GWs is

$$\boxed{\frac{dE^{\mathsf{GW}}}{dt} = \frac{c^3 r^2}{16\pi G} \int_{\partial V} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle d\Omega} \quad . \tag{4.76}$$

Likewise, the energy flux carried away by the waves reads,

$$\mathcal{F}^{\mathsf{GW}} = \frac{dE^{\mathsf{GW}}}{dAdt} = \frac{c^3}{16\pi G} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \quad . \tag{4.77}$$

The total energy across the surface dA from the instant $t = -\infty$ to $t = \infty$ is, [17]

$$\begin{aligned} \frac{dE_{\mathsf{GW}}}{dA} &= \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle_{\text{time}} dt \\ \frac{dE_{\mathsf{GW}}}{dA} &= \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} \left(\dot{h}_+^2 + \dot{h}_\times^2 \right) dt \quad . \end{aligned} \tag{4.78}$$

With a similar procedure, is possible to calculate the flux of linear momentum across the boundary ∂V that delimits the volume V. The total momentum within the volume V reads,

$$P^k = \frac{1}{c} \int_V t^{0k}\, d^3x \quad . \tag{4.79}$$

Thus,

[16]In Chapter 6 the mechanical energy loss of a compact binary system due to the emission of GWs is discussed. A balance energy condition will gives a description on how is increasing the binary frequency when the compact objects approach each other.

[17]The shortwave average can be ommited for the following reason. Recall that the operation $\langle\cdot\rangle$ may be done over many wavelenghts but also over a few periods if one assumes $f \gg f_{\mathrm{B}}$. Thus, by performing first the integral over dt from $-\infty$ to ∞ one eliminates the temporal dependence (i.e. the result is a constant) and the average is the same argument. Hence, the $\langle\cdot\rangle$ symbol can be ignored.

$$\begin{aligned}
\frac{dP^k}{dt} &= \frac{1}{c}\frac{d}{dt}\int_V t^{0k}\ d^3x = \int_V \partial_0 t^{0k}\ d^3x \\
&= -\int_V \partial_i t^{ik}\ d^3x \qquad ; \qquad \Big[\partial_0 t^{0k} + \partial_i t^{ik} = 0\Big] \\
&= -\int_{\partial V} t^{ik} n_i\ dA \qquad \text{[Divergence theorem]} \\
&= -\int_{\partial V} t^{rk}\ r^2 d\Omega \quad . \qquad (4.80)
\end{aligned}$$

The quantity t^{rk} follows from (4.50) and (4.72), i.e.,

$$\begin{aligned}
t^{rk} &= \frac{c^4}{32\pi G}\left\langle \partial^r h^{ij}_{\mathrm{TT}} \partial^k h^{\mathrm{TT}}_{ij}\right\rangle = \frac{c^4}{32\pi G}\left\langle \partial_r h^{ij}_{\mathrm{TT}} \partial^k h^{\mathrm{TT}}_{ij}\right\rangle \\
&= \frac{c^4}{32\pi G}\left\langle \Big(-\partial_0 h^{ij}_{\mathrm{TT}}\Big) \partial^k h^{\mathrm{TT}}_{ij}\right\rangle \\
&= \frac{c^4}{32\pi G}\left\langle \partial^0 h^{ij}_{\mathrm{TT}} \partial^k h^{\mathrm{TT}}_{ij}\right\rangle = t^{0k} \quad . \qquad (4.81)
\end{aligned}$$

Replacing (4.81) into (4.80) gives the momentum's rate of change carried away by the GW in the outward direction. The result is,

$$\boxed{\frac{dP^k_{\mathsf{GW}}}{dt} = -\frac{c^3 r^2}{32\pi G}\int \left\langle \dot{h}^{\mathrm{TT}}_{ij} \partial^k h^{\mathrm{TT}}_{ij}\right\rangle\ d\Omega} \quad . \qquad (4.82)$$

Chapter 5

Generation of Gravitational Waves in Linearized Theory

In this chapter, the generation of GWs in linearized theory is revisited. So far, the discussions made in previous chapters shows that at linearized level, the perturbation of the metric could be interpreted as GWs that carries energy and momentum. It was mentioned that the solution of the homogeneous wave equation is in general a superposition of plane waves that travels at the speed of light. However, it was assumed that this waves indeed exists in vacuum and one can detect them at spatial infinity. The following sections intend to review the main aspects of the production of GWs to linear order at very large distances from the sources, i.e., at the far-field zone.

5.1 Solution of the wave equation in linearized theory

If one is interested in study how the waves are produced, is not possible to assume vacuum. The matter sources generates the curvature in spacetime which is propagating itself due to the dynamics of the source. At linear order, this is to solve the linearized EFE in the presence of matter or the inhomogeneous wave equation given by

$$\Box \bar{h}_{\mu\nu} \;=\; -\frac{16\pi G}{c^4} T_{\mu\nu} \quad . \tag{5.1}$$

To solve equation (5.1), the Green's function method is implemented as follows. A Green's function in 4-dimensions is some function $G = G(x - x')$ that satisfies the following equation,

$$\Box_x G(x - x') \;=\; \delta^4(x - x') \quad . \tag{5.2}$$

Thus, the solution of (5.1) is given by

$$\bar{h}_{\mu\nu}(x) \;=\; -\frac{16\pi G}{c^4} \int G(x - x') T_{\mu\nu}(x')\, d^4x' \quad . \tag{5.3}$$

This can be checked as follows,

$$\begin{aligned}
\Box \bar{h}_{\mu\nu} \;&=\; -\frac{16\pi G}{c^4} \int \Box G(x - x') T_{\mu\nu}(x')\, d^4x' \\
&=\; -\frac{16\pi G}{c^4} \int \delta^4(x - x') T_{\mu\nu}(x')\, d^4x' \;=\; -\frac{16\pi G}{c^4} T_{\mu\nu}(x) \quad .
\end{aligned}$$

The aim of this method is to find the Green's function by solving equation (5.2) and then use it in (5.3) to find the solution $\bar{h}_{\mu\nu}(x)$. To do this, is easier to solve the partial differential equation (5.2) in the Fourier space. This means that the Green's function and the Dirac delta function may be expressed as,

$$G(x-x') \;=\; \frac{1}{(2\pi)^4}\int \tilde{G}(k-k')e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \quad , \tag{5.4}$$

$$\delta^4(x-x') \;=\; \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \quad . \tag{5.5}$$

Replacing these definitions into (5.2) gives,

$$\begin{aligned}
\frac{1}{(2\pi)^4}\int \tilde{G}(k-k')\eta^{\mu\nu}\partial_\mu\partial_\nu e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \;&=\; \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \\
\int \tilde{G}(k-k')\eta^{\mu\nu}\partial_\mu\Big[ik_\nu e^{ik_\alpha(x^\alpha - x'^\alpha)}\Big]\, d^4k \;&=\; \int e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \\
\int \tilde{G}(k-k')\Big[-\eta^{\mu\nu}k_\mu k_\nu\Big]e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \;&=\; \int e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \quad .
\end{aligned} \tag{5.6}$$

Recall that the 4-wavevector reads as $k^\mu = (\omega/c\,,\,\mathbf{k})$. Then,

$$\eta^{\mu\nu}k_\mu k_\nu \;=\; k^\nu k_\nu \;=\; -\frac{\omega^2}{c^2} + \mathbf{k}^2 \tag{5.7}$$

and using (5.6) one obtains,

$$\boxed{\tilde{G}(k-k') \;=\; -\frac{1}{\left(\mathbf{k}^2 - \dfrac{\omega^2}{c^2}\right)}} \quad . \tag{5.8}$$

The Green's function is found by substituting (5.8) into (5.4) ,

$$\begin{aligned}
G(x-x') \;&=\; -\frac{1}{(2\pi)^4}\int \frac{e^{ik_\alpha(x^\alpha - x'^\alpha)}}{\mathbf{k}^2 - \omega^2/c^2}\, d^4k \\
&=\; -\frac{1}{(2\pi)^4}\int \frac{e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t')]}}{\frac{1}{c^2}\left(k^2c^2 - \omega^2\right)}\, d(\omega/c)\, d^3k \quad .
\end{aligned} \tag{5.9}$$

Defining $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $\tau = t - t'$, the integral in (5.9) may be solved in spherical coordinates, i.e.

$$G(x-x') \;=\; -\frac{c}{(2\pi)^4}\int \frac{e^{i(kR\cos\theta\, -\, \omega\tau)}}{k^2c^2 - \omega^2}\, k^2\sin\theta d\theta dk d\phi d\omega \quad , \tag{5.10}$$

where $R \equiv |\mathbf{x} - \mathbf{x}'|$ and θ is the angle between $\mathbf{R}$ and $\mathbf{k}$ after aligning the z-axis with the vector $\mathbf{R}$. Now, making the substitution

$$u \;=\; ikR\cos\theta \qquad \Longrightarrow \qquad du \;=\; -ikR\sin\theta d\theta \quad , \tag{5.11}$$

the integral in (5.10) becomes,

$$G(x-x') = \frac{c}{(2\pi)^3 iR}\int \frac{e^{-i\omega\tau}}{k^2c^2 - \omega^2}\left[\int_{ikR}^{-ikR} e^u\, du\right] kdk\, d\omega \quad . \tag{5.12}$$

Thus, the Green's function in (5.12) simplifies to

$$\begin{aligned}
G(x-x') &= \frac{c}{(2\pi)^3 iR}\int \frac{e^{-i\omega\tau}}{k^2c^2 - \omega^2}\left(e^{-ikR} - e^{ikR}\right) kdk\, d\omega \\
&= -\frac{c}{(2\pi)^3 iR}\int_{-\infty}^{\infty}\int_0^{\infty}\left[\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{-ikR} - \frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{ikR}\right] kdk\, d\omega \\
&= -\frac{c}{(2\pi)^3 iR}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{-ikR}\, kdk - \int_0^{\infty}\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{ikR}\, kdk\right] d\omega \\
&= -\frac{c}{(2\pi)^3 iR}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{0}\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{-ikR}\, kdk + \int_0^{\infty}\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}e^{-ikR}\, kdk\right] d\omega \\
&= -\frac{c}{(2\pi)^3 iR}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}}{\omega^2 - k^2c^2}\, d\omega\right] e^{-ikR}\, kdk
\end{aligned} \tag{5.13}$$

The integral inside the square brackets may be solved in the complex plane and using the residue theorem[1]. The result is,

$$\begin{aligned}
G(x-x') &= -\frac{c}{(2\pi)^3 iR}\int_{-\infty}^{\infty}\left[\frac{i\pi e^{ikc\tau}}{kc}\right] e^{-ikR}\, kdk \quad , \quad \tau > 0 \\
&= -\frac{1}{4\pi R}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(R-c\tau)}\, dk\right] \\
&= -\frac{1}{4\pi R}\delta(R - c\tau) = -\frac{1}{4\pi R}\delta(c\tau - R) \quad , \quad [\delta(x) = \delta(-x)] \\
&= -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left[c\left(t - |\mathbf{x}-\mathbf{x}'|/c\right) - ct'\right] \quad .
\end{aligned} \tag{5.14}$$

Finally, the Green's function yields

$$\boxed{G(x-x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left(x^0_{\text{ret}} - x'^0\right)} \quad , \tag{5.15}$$

where x^0_{ret} is defined as

$$x^0_{\text{ret}} = ct_{\text{ret}} = c\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \quad . \tag{5.16}$$

[1]A more general integral of this kind is solved with more detail in chapter 7.

Having the Green's function, the solution is obtained from (5.3). Hence,

$$\begin{aligned}
\bar{h}_{\mu\nu}(x) &= -\frac{16\pi G}{c^4}\int G(x-x')T_{\mu\nu}(x')\,d^4x' \\
&= \frac{16\pi G}{c^4}\int \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\Big(x^0_{\text{ret}}-x'^0\Big)T_{\mu\nu}(x')\,dx'^0 d^3x' \\
&= \frac{4G}{c^4}\int\left[\int_{-\infty}^{\infty}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\Big(x^0_{\text{ret}}-x'^0\Big)T_{\mu\nu}(x'^0,\mathbf{x}')\,dx'^0\right]d^3x' \\
&= \frac{4G}{c^4}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{\mu\nu}(ct_{\text{ret}},\mathbf{x}')\,d^3x' \quad .
\end{aligned}$$

Finally, the retarded solution reads,

$$\bar{h}_{\mu\nu}(x) = \frac{4G}{c^4}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{\mu\nu}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)d^3x' \quad . \tag{5.17}$$

Of course, as well as in electrodynamics, another solution is also acceptable for $\tau < 0$[2]. However, this solution violates causality. It's important to remark the interpretation of the retarded solution in (5.17). The disturbance in the gravitation field at $(ct,\mathbf{x})$ is the sum of all influences from the sources present in $T_{\mu\nu}$ at the point $(ct_{\text{ret}},\mathbf{x}')$ on the *past light cone* [35]. This is depicted in figure 5.1.

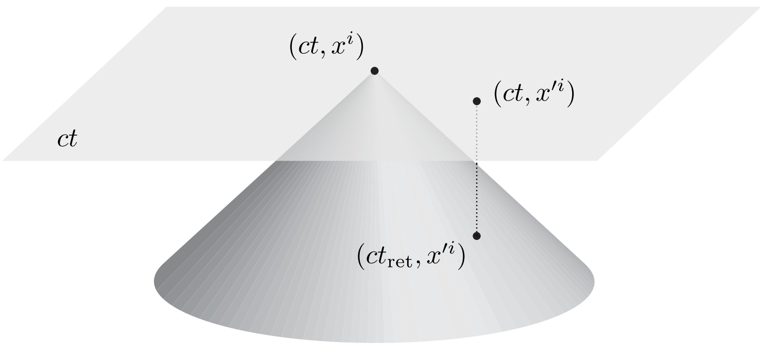

Figure 5.1: Retarded solution of the perturbation in terms of the past light cone. Adapted from [9, 31].

As mentioned before, outside the sources the solution may be projected onto the TT gauge using the tensor $\Lambda_{ij|kl}$. Observe that,

$$h^{\text{TT}}_{ij} = \Lambda_{ij|kl}h_{kl} = \Lambda_{ij|kl}\bar{h}_{kl} \quad . \tag{5.18}$$

The previous expression follows from the explicit definition of h_{kl} in terms of the trace-reversed form and the trace-free property of the Lambda tensor,

[2]This is called the *advanced* solution.

$$\begin{aligned}\Lambda_{ij|kl}h_{kl} &= \Lambda_{ij|kl}\left(\bar{h}_{kl} - \frac{1}{2}\delta_{kl}\bar{h}\right) \\ &= \Lambda_{ij|kl}\bar{h}_{kl} - \frac{1}{2}\Lambda_{ij|ll}\bar{h} \\ &= \Lambda_{ij|kl}\bar{h}_{kl} \quad . \end{aligned} \tag{5.19}$$

Therefore,

$$\boxed{h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)d^3x'} \quad . \tag{5.20}$$

Furthermore, a Taylor expansion of $|\mathbf{x}-\mathbf{x}'|^{-1}$ is

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} + (-x'^i)\partial_i\left(\frac{1}{r}\right) + \frac{1}{2!}\left(-x'^i\right)\left(-x'^j\right)\partial_i\partial_j\left(\frac{1}{r}\right) + \ldots \\ &= \frac{1}{r} + x'^i\frac{x_i}{r^3} + x'^i x'^j\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) + \ldots \end{aligned} \tag{5.21}$$

Then, the solution in (5.20) becomes,

$$\begin{aligned}h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\Bigg[&\frac{1}{r}\int T^{kl}(t_{\mathrm{ret}},\mathbf{x}')\,d^3x' + \frac{x^i}{r^3}\int T^{kl}(t_{\mathrm{ret}},\mathbf{x}')x'^i\,d^3x' \\ &+ \frac{3x_ix_j - r^2\delta_{ij}}{r^5}\int T^{kl}(t_{\mathrm{ret}},\mathbf{x}')x'^ix'^j\,d^3x' + \ldots\Bigg] \quad . \end{aligned} \tag{5.22}$$

The *multiple moments* of the energy-momentum tensor of the source are defined as

$$S^{kl,j_1j_2\ldots j_l}(t,\mathbf{x}) = \int T^{kl}(t,\mathbf{x}')x'^{j_1}x'^{j_2}\ldots x'^{j_l}\,d^3x' \quad . \tag{5.23}$$

Then, the solution in (5.22) yields

$$\boxed{h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}S^{kl,L}(t_{\mathrm{ret}})\partial_L\left(\frac{1}{r}\right)} \quad , \tag{5.24}$$

where $L \equiv j_1j_2\ldots j_l$. The fall-off with distance depends on $1/r^{l+1}$ for each lth multipole moment. Thus, the gravitational field at large distances is well approximated by a few terms of the expansion [35].

5.2 Weak-field sources and the far-field zone

Let's consider a weak gravitational field generated by a source, so the linearized approximation in the metric perturbation is justified at spatial infinity [8]. One may wonder what is the solution that connects the gravitational radiation very far away from the source with the internal dynamics of the objects that generate it. This is called the far-field zone solution [42].

At these distances, is reasonable to approximate the expression given in (5.22) by retaining just the first term contribution of the expansion. The next to leading term falls-off as $1/r^3$ so can be neglected as well as the subsequents. Therefore,

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}}) \int T^{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right) d^3x' \quad . \tag{5.25}$$

If is not possible to neglect the relative retardation effects of one region of the source from another, a slow motion of the constituents fails [42]. Thus, the retarded time cannot be approximated properly as $t_{\mathrm{ret}} = t - r/c$. Instead, consider figure 5.2 in which the far-zone region from the source is illustrated. Assume an isolated source in the spacetime with a typical size d. At the far away region, the distance between any point located at the source and the point in which the field is evaluated is much larger than the typical size d.

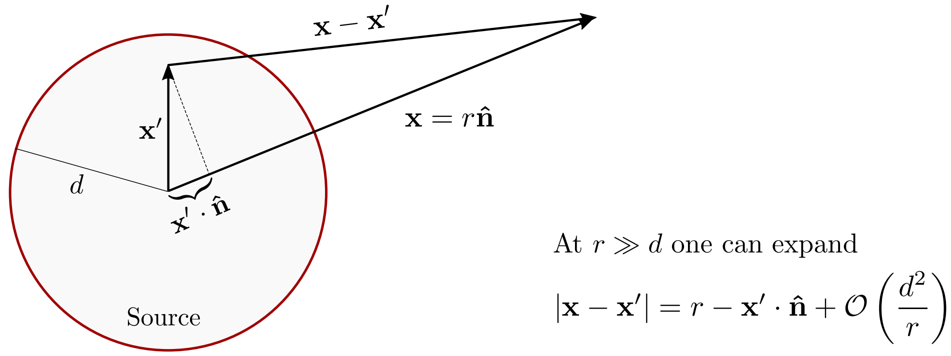

Figure 5.2: Graphical illustration of the far-field zone approximation. The particles that makeup the whole source can move with arbitrary velocity.

Then, a good approximation to the solution for a source with arbitrary velocity is

$$\boxed{h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) \simeq \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}}) \int_{|\mathbf{x}'|<d} T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}, \mathbf{x}'\right) d^3x'} \quad . \tag{5.26}$$

Is better to rewrite (5.26) in terms of the frequency of the source ω by performing a Fourier transform of the energy-momentum tensor,

$$T_{kl}(t,\mathbf{x}) = \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, d^4x \quad . \tag{5.27}$$

Then,

$$\begin{aligned} T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\, , \mathbf{x}'\right) &= \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k}) \exp\left\{ i\left[\mathbf{k}\cdot\mathbf{x} - \omega\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\right)\right]\right\} \left(\frac{d\omega}{c}\right) d^3k \\ &= \frac{1}{(2\pi)^4 c}\int \tilde{T}_{kl}(\omega,\mathbf{k}) \exp\left\{ i\left[\left(\mathbf{k} - \frac{\omega\hat{\mathbf{n}}}{c}\right)\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c}\right)\right]\right\} d\omega d^3k \end{aligned}$$

and

$$\int T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\,,\,\mathbf{x}'\right)d^3x'$$
$$=\frac{1}{(2\pi)^4c}\int\int\tilde{T}_{kl}(\omega,\mathbf{k})\exp\left\{i\left[\left(\mathbf{k}-\frac{\omega\hat{\mathbf{n}}}{c}\right)\cdot\mathbf{x}'-\omega\left(t-\frac{r}{c}\right)\right]\right\}d\omega d^3k d^3x' \quad . \tag{5.28}$$

With the Dirac delta definition

$$\int\exp\left\{i\left(\mathbf{k}-\frac{\omega\hat{\mathbf{n}}}{c}\right)\cdot\mathbf{x}'\right\}d^3x' = (2\pi)^3\delta^{(3)}\left(\mathbf{k}-\frac{\omega\hat{\mathbf{n}}}{c}\right) \quad , \tag{5.29}$$

the integral in (5.28) yields

$$\int T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\,,\,\mathbf{x}'\right)d^3x'$$
$$=\frac{1}{(2\pi)^4c}\int\tilde{T}_{kl}(\omega,\mathbf{k})(2\pi)^3\delta^{(3)}\left(\mathbf{k}-\frac{\omega\hat{\mathbf{n}}}{c}\right)\exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}d\omega d^3k$$
$$=\frac{1}{2\pi c}\int\int\tilde{T}_{kl}(\omega,\mathbf{k})\delta^{(3)}\left(\mathbf{k}-\frac{\omega\hat{\mathbf{n}}}{c}\right)\exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}d\omega d^3k$$
$$=\frac{1}{2\pi c}\int\tilde{T}_{kl}\left(\omega,\frac{\omega\hat{\mathbf{n}}}{c}\right)\exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}d\omega \quad . \tag{5.30}$$

Inserting (5.30) in the solution (5.26) gives,

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij|kl}(\hat{\mathbf{n}})\int_{-\infty}^{\infty}\tilde{T}_{kl}\left(\omega,\frac{\omega\hat{\mathbf{n}}}{c}\right)\exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}\frac{d\omega}{2\pi} \quad . \tag{5.31}$$

The expression in (5.31) is valid for sources with arbitrary velocity. No assumptions has been made about the internal motion of the source and thus it may be relativistic or non-relativistic as long as linearized theory applies. Recall the energy per unit area in equation (4.78),

$$\frac{dE^{\mathsf{GW}}}{dA} = \frac{c^3}{32\pi G}\int_{-\infty}^{\infty}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}\,dt = \frac{c^3}{32\pi G}\int_{-\infty}^{\infty}\left(\dot{h}_+^2+\dot{h}_\times^2\right)dt \quad . \tag{5.32}$$

The area element is just $dA = r^2d\Omega$ and so the energy per unit solid angle is

$$\frac{dE}{d\Omega} = \frac{r^2c^3}{32\pi G}\int_{-\infty}^{\infty}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}\,dt \quad . \tag{5.33}$$

We wish to get an explicit expression for (5.33). From equation (5.31) is possible to determine $\dot{h}_{ij}^{\mathrm{TT}}$,

$$\dot{h}_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij|kl}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}(-i\omega)\,\frac{d\omega}{2\pi} \quad . \tag{5.34}$$

The integrand in (5.33) results as follows,

$$\begin{aligned}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}} =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}(-i\omega)\,d\omega \\ & \times \Lambda_{ij|mn}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{mn}\left(\varpi, \frac{\varpi\hat{\mathbf{n}}}{c}\right) \exp\left\{-i\varpi\left(t-\frac{r}{c}\right)\right\}(-i\varpi)\,d\varpi \quad .\end{aligned} \tag{5.35}$$

Making the substitution $\varpi = -\omega'$ in the second integral of (5.35) gives

$$\begin{aligned}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}} =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \exp\left\{-i\omega\left(t-\frac{r}{c}\right)\right\}(-i\omega)\,d\omega \\ & \times \Lambda_{ij|mn}(\hat{\mathbf{n}}) \int_{\infty}^{-\infty} \tilde{T}_{mn}\left(-\omega', -\frac{\omega'\hat{\mathbf{n}}}{c}\right) \exp\left\{i\omega'\left(t-\frac{r}{c}\right)\right\}(i\omega')\,(-d\omega') \quad . \\ =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}})\Lambda_{ij|mn}(\hat{\mathbf{n}}) \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega', \frac{\omega'\hat{\mathbf{n}}}{c}\right) e^{-i(\omega'-\omega)r/c} e^{i(\omega'-\omega)t} \omega\omega'\,d\omega d\omega' \quad ,\end{aligned} \tag{5.36}$$

where $\tilde{T}^*(\omega, \omega\hat{\mathbf{n}}/c) = \tilde{T}(-\omega, -\omega\hat{\mathbf{n}}/c)$. After using (5.36) in (5.33) it is obtained,

$$\begin{aligned}\int_{-\infty}^{\infty} \dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}\,dt =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}})\Lambda_{ij|mn}(\hat{\mathbf{n}}) \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega', \frac{\omega'\hat{\mathbf{n}}}{c}\right) e^{-i(\omega'-\omega)r/c} e^{i(\omega'-\omega)t} \omega\omega'\,d\omega d\omega'\,dt\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}\,dt =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}})\Lambda_{ij|mn}(\hat{\mathbf{n}}) \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega', \frac{\omega'\hat{\mathbf{n}}}{c}\right) e^{-i(\omega'-\omega)r/c} 2\pi\delta(\omega-\omega')\omega\omega'\,d\omega d\omega'\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}\,dt =& \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{ij|kl}(\hat{\mathbf{n}})\Lambda_{ij|mn}(\hat{\mathbf{n}}) \\ & \times \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \int_{-\infty}^{\infty} \tilde{T}^*_{mn}\left(\omega', \frac{\omega'\hat{\mathbf{n}}}{c}\right) e^{-i(\omega'-\omega)r/c} 2\pi\delta(\omega'-\omega)\omega\omega'\,d\omega' d\omega \quad .\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{\mathrm{TT}}^{ij}\, dt \;=\; \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{kl|mn}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \quad . \tag{5.37}$$

The integral at the right hand side of equation (5.37) is rewritten as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \\
&= \int_{-\infty}^{0} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \;+\; \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \\
&= \int_{\infty}^{0} \tilde{T}_{kl}\left(-\omega, -\frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(-\omega, -\frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi(-\omega)^2 d(-\omega) \;+\; \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \\
&= \int_{0}^{\infty} \tilde{T}^*_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \;+\; \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \quad .
\end{aligned}$$

Using this last result into (5.37) one gets,

$$\begin{aligned}
&\left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{kl|mn}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \\
&= \left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{kl|mn}(\hat{\mathbf{n}}) \\
&\quad \times \left\{ \int_{0}^{\infty} \tilde{T}^*_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \;+\; \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \right\} \quad .
\end{aligned}$$

Due to the contraction of the indices kl and mn between the *Lambda* tensor and the energy-momentum tensor of the sources in the frequency domain, they represent dummy indices. This means that by relabelling the indices of the first term as $kl \leftrightarrow mn$ yields,

$$\begin{aligned}
\int_{-\infty}^{\infty} \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{\mathrm{TT}}^{ij}\, dt \;&=\; 2\left(\frac{4G}{2\pi c^5 r}\right)^2 \Lambda_{kl|mn}(\hat{\mathbf{n}}) \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) 2\pi\omega^2 d\omega \\
&= \frac{1}{\pi}\left(\frac{4G}{c^5 r}\right)^2 \Lambda_{kl|mn}(\hat{\mathbf{n}}) \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \omega^2 d\omega \quad .
\end{aligned} \tag{5.38}$$

Finally, replacing (5.38) into (5.33) one obtains the energy of GWs per unit solid angle,

$$\boxed{\frac{dE^{\mathsf{GW}}}{d\Omega} \;=\; \frac{G}{2\pi^2 c^7} \Lambda_{kl|mn}(\hat{\mathbf{n}}) \int_{0}^{\infty} \tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right) \omega^2 d\omega} \quad . \tag{5.39}$$

Differentiating equation (5.39) with respect to ω and then integrating over the solid angle, it is obtained the **espectral energy** of GWs,

$$\boxed{\frac{dE^{\mathsf{GW}}}{d\omega} = \frac{G\omega^2}{2\pi^2 c^7}\int \Lambda_{kl|mn}(\hat{\mathbf{n}})\tilde{T}_{kl}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right)\tilde{T}^*_{mn}\left(\omega, \frac{\omega\hat{\mathbf{n}}}{c}\right)d\Omega} \quad . \tag{5.40}$$

5.3 Low-velocity expansion

So far, nothing has been assumed about the internal motions of the source. In this section a slow motion approximation is considered. To do this, some scaling quantities that describe the system are defined.

d : Size of the source.

ω_s : Typical frequency of the internal motions of the source.

v : Typical internal velocity of the source $v \sim \omega_s d$.

ω : Gravitational radiation frequency emmited by the source.

We shall see later that the dominant contribution to the gravitational radiation comes from the lowest multipoles. For those, the frequency of the radiation are the same order of the frequency of the source, i.e., $\omega \sim \omega_{\mathrm{s}}$[3]. Thus,

$$\lambda\!\!\!^{-} = \frac{\lambda}{2\pi} = \frac{c}{2\pi f} = \frac{c}{\omega} \sim \frac{c}{\omega_{\mathrm{s}}} = \frac{c}{v}d \quad . \tag{5.41}$$

For a non-relativistic source $v \ll c$ and equation (5.41) implies,

$$\boxed{\lambda\!\!\!^{-} \gg d} \qquad \text{(non-relativistic sources)} \quad . \tag{5.42}$$

The scenario for a non-relativistic source is illustrated in figure 5.3. When the condition $\lambda\!\!\!^{-} \gg d$ is fulfilled is not neccesary to know the exact details of the internal motions of the particles, but only the coarse features of matter [8]. Thus, the leading contributions to the radiation are the lowest multiple moments of the energy-momentum tensor[4]. To see this, first consider the Fourier version of the solution given in (5.26),

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}, \mathbf{x}'\right) = \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega, \mathbf{k})e^{i\left[\mathbf{k}\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\right)\right]}d^4k \quad . \tag{5.43}$$

Recall that the energy-momentum tensor is non-vanishing only in the integration domain for $|\mathbf{x}'| < d$. Therefore, low velocities means that $\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}} \ll 1$. This can be seen because,

$$\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}} \leq \frac{\omega d}{c} \sim \frac{\omega_s d}{c} = \frac{v}{c} \ll 1 \quad . \tag{5.44}$$

[3]For instance, the radiation frequency for a binary system is $\omega = 2\omega_{\mathrm{s}}$. For more details see Chapter 6

[4]This can be compared with the Newtonian case. In the two-body problem if the separation between the objects is very large, the mutual effects due to the non-sphericity and the internal dynamics that are encoded in the higher mutipole moments are negligible.

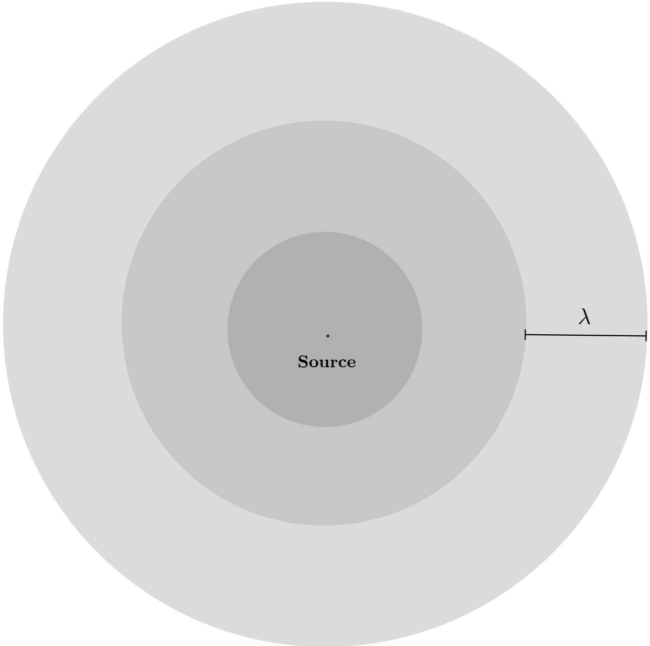

Figure 5.3: Schematic representation of a non-relativistic source. The wavelength of the radiation is much larger than the typical size of the source ($\lambda \gg d$).

Then, is suitable to expand the exponential term with $\frac{\omega}{c}\mathbf{x}' \cdot \hat{\mathbf{n}}$ in (5.43). The Taylor series for the exponential is

$$e^{-ix} = 1 - ix - \frac{x^2}{2} + \frac{ix^3}{6} + \cdots \quad . \tag{5.45}$$

Thus,

$$\begin{aligned} e^{-i\omega\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\right)} &= e^{-i\omega\left(t - \frac{r}{c}\right)} e^{-i\omega\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}} \\ &= e^{-i\omega\left(t - \frac{r}{c}\right)} \left[1 - i\left(\frac{\omega}{c}\right) x'^i n_i - \frac{1}{2}\left(\frac{\omega}{c}\right)^2 x'^i x'^j n_i n_j + \cdots\right] \quad . \end{aligned} \tag{5.46}$$

Replacing the expansion (5.46) into (5.43) gives,

$$\begin{aligned} T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}, \mathbf{x}'\right) &= \frac{1}{(2\pi)^4} \int \tilde{T}_{kl}(\omega, \mathbf{k}) e^{i\left[\mathbf{k}\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c}\right)\right]} d^4k \\ &+ \frac{1}{(2\pi)^4} \int \tilde{T}_{kl}(\omega, \mathbf{k}) e^{i\left[\mathbf{k}\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c}\right)\right]} \left[-i\left(\frac{\omega}{c}\right) x'^i n_i\right] d^4k \\ &+ \frac{1}{(2\pi)^4} \int \tilde{T}_{kl}(\omega, \mathbf{k}) e^{i\left[\mathbf{k}\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c}\right)\right]} \left[-\frac{1}{2}\left(\frac{\omega}{c}\right)^2 x'^i x'^j n_i n_j\right] d^4k + \cdots \end{aligned} \tag{5.47}$$

Observe that from the definition of Fourier transform,

$$T_{kl}\left(t - \frac{r}{c}, \mathbf{x}'\right) = \frac{1}{(2\pi)^4} \int \tilde{T}_{kl}(\omega, \mathbf{k}) e^{i\left[\mathbf{k}\cdot\mathbf{x}' - \omega\left(t - \frac{r}{c}\right)\right]} d^4k \quad . \tag{5.48}$$

Then, the expression in (5.47) is equivalent to

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}\,,\,\mathbf{x}'\right) = \Big[\,T_{kl}(t,\mathbf{x}) + \frac{x'^i n_i}{c}\partial_t T_{kl}(t,\mathbf{x}) + \frac{1}{2c^2}x'^i x'^j n_i n_j \partial_t^2 T_{kl}(t,\mathbf{x}) + \cdots\,\Big]_{(t-r/c\,,\,\mathbf{x}')} \,. \tag{5.49}$$

Is possible to verify that indeed, the equations (5.47) and (5.49) are the same,

$$\begin{aligned} T_{kl}(t,\mathbf{x})\Big|_{(t-r/c\,,\,\mathbf{x}')} &= \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\,d^4k\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &= \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}'\,-\,i\omega\left(t-\frac{r}{c}\right)}\,d^4k \quad , \end{aligned}$$

$$\begin{aligned} \frac{x'^i n_i}{c}\partial_t\, T_{kl}(t,\mathbf{x})\Big|_{(t-r/c\,,\,\mathbf{x}')} &= \frac{1}{(2\pi)^4}\frac{x'^i n_i}{c}\int \tilde{T}_{kl}(\omega,\mathbf{k})\partial_t e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\,d^4k\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &= \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\Big[-i\left(\frac{\omega}{c}\right)x'^i n_i\Big]\,d^4k\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &= \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}'\,-\,i\omega\left(t-\frac{r}{c}\right)}\Big[-i\left(\frac{\omega}{c}\right)x'^i n_i\Big]\,d^4k \quad , \end{aligned}$$

$$\begin{aligned} &\frac{1}{2c^2}x'^i n_i x'^j n_j \partial_t^2\, T_{kl}(t,\mathbf{x})\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &\quad = \frac{1}{(2\pi)^4}\frac{1}{2c^2}x'^i n_i x'^j n_j\int \tilde{T}_{kl}(\omega,\mathbf{k})\partial_t^2 e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\,d^4k\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &\quad = \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\left[-\frac{1}{2}\left(\frac{\omega}{c}\right)^2 x'^i x'^j n_i n_j\right]\,d^4k\Big|_{(t-r/c\,,\,\mathbf{x}')} \\ &\quad = \frac{1}{(2\pi)^4}\int \tilde{T}_{kl}(\omega,\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}'\,-\,\omega(t-r/c)]}\left[-\frac{1}{2}\left(\frac{\omega}{c}\right)^2 x'^i x'^j n_i n_j\right]\,d^4k \quad . \end{aligned}$$

Note that the result in (5.49) could be obtained as well by making a formal Taylor expansion in the parameter $\mathbf{x}'\cdot\hat{\mathbf{n}}/c$. However, this procedure emphasize the explicit condition of low velocities given by $\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}} \ll 1$. If one uses such a result into the solution h_{ij}^{TT} in (5.26) yields,

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\int\Big[T_{kl}(t,\mathbf{x}') + \frac{x'^i n_i}{c}\partial_t T_{kl}(t,\mathbf{x}') + \frac{1}{2c^2}x'^i n_i x'^j n_j \partial_t^2 T_{kl}(t,\mathbf{x}') + \cdots\,\Big]_{t-r/c} d^3x'$$

$$h_{ij}^{\rm TT}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\Bigg[\int T_{kl}(t,\mathbf{x}')\,d^3x' + \frac{1}{c}n_i\frac{d}{dt}\int T_{kl}(t,\mathbf{x}')x'^i\,d^3x' \tag{5.50}$$

$$+ \frac{1}{2c^2}n_in_j\frac{d^2}{dt^2}\int T_{kl}(t,\mathbf{x}')x'^ix'^j\,d^3x' + \cdots\Bigg]_{t-r/c} \quad . \tag{5.51}$$

Remind the multiple moments of the energy-momentum tensor in (5.23)[5]. The first moments are given by

$$S^{ij}(t) = \int T^{ij}(t,\mathbf{x})\,d^3x \quad , \tag{5.52}$$

$$S^{ij,k}(t) = \int T^{ij}(t,\mathbf{x})x^k d^3x \quad , \tag{5.53}$$

$$S^{ij,kl}(t) = \int T^{ij}(t,\mathbf{x})x^kx^l\,d^3x \quad . \tag{5.54}$$

Using these expressions in (5.51), the solution $h_{ij}^{\rm TT}(t,\mathbf{x})$ is written as an expansion of the multiple moments of the energy-momentum tensor,

$$\boxed{h_{ij}^{\rm TT}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_i\dot{S}^{kl,i} + \frac{1}{2c^2}n_in_j\ddot{S}^{kl,ij} + \cdots\right]_{t-r/c}} \quad . \tag{5.55}$$

Observe that in the expression (5.51) the moment $S^{kl,i}$ has an additional factor of $x'^i \sim \mathcal{O}(d)$. Furthermore, each time derivative gives a factor $\mathcal{O}(\omega) \sim \mathcal{O}(\omega_s)$ as can be seen for example in (5.47). One concludes that,

$$\dot{S}^{kl,i} \sim \mathcal{O}(\omega_s d) = \mathcal{O}(v) \quad \Longrightarrow \quad \frac{1}{c}n_i\dot{S}^{kl,i} \sim \mathcal{O}\left(\frac{v}{c}\right) \tag{5.56}$$

$$\ddot{S}^{kl,ij} \sim \mathcal{O}\left[(\omega_s d)^2\right] = \mathcal{O}(v^2) \quad \Longrightarrow \quad \frac{1}{2c^2}n_in_j\ddot{S}^{kl,ij} \sim \mathcal{O}\left(\frac{v^2}{c^2}\right) \quad . \tag{5.57}$$

Therefore, the solution given in (5.55) is actually an expansion in v/c where the first term is the leading term. An important remark of this procedure is that the correction in v/c in the expanded solution is only possible if the internal velocities of the source and the flat background can be treated independently [8]. For instance, consider a compact binary in orbital motion due to the mutual gravitational interaction. Extrictly speaking, this system would *not be possible* in linearized gravity. The reason is that the objects will follow geodesics in the flat spacetime due to $\partial^\nu T_{\mu\nu} = 0$[6]. As we shall see later, in the exact formulation of general relativity by Landau & Lifshtiz, weak gravitational fields implies the same equations of linearized theory, but instead of having only the energy-momentum of the matter sources, it is included the pseudotensor that accounts for the motion of bounded orbits [1, 32, 34]. If the binding energy of the self-gravity interaction can be neglected in this formulation, then at leading order one can argue to use the first term in (5.55). If this is the case, note that even if the binary system is Newtonian, from the virial theorem one has

[5] $S^{ij} = S^{ji}$ due to $T^{ij} = T^{ji}$. Also $S^{ij,k} = S^{ji,k}$, $S^{ij,kl} = S^{ji,kl} = S^{ji,lk}$. However, $S^{ij,kl} \neq S^{kl,ij}$.

[6] This equation will be discussed for a closed system of particles at the end of this chapter.

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{G\mu m}{r} \qquad \Longrightarrow \qquad \frac{v^2}{c^2} = \frac{R_\mathrm{s}}{2r} \quad , \tag{5.58}$$

where μ is the reduced mass of the system, m is the total mass and R_s is the Schwarzscild radius associated to the mass m. Hence, a weak gravitational field means $R_\mathrm{s}/r \ll 1$, so $v \ll c$. In this case, the velocity of the source is related directly with the strength nature of the gravitational field. If the velocity of the system is high compared with the speed of light, the background cannot be assumed as flat and linearized theory is no longer valid. Even if more terms are considered in the (v/c) expansion, is not reasonable to matained the background as flat because this contributions means strong fields. In conclusion, for the binary system exmaple, if one uses the equation (5.55) at leading order, the gravitational field must be *weak* to approximate the background spacetime as flat, but also the objects requiere *slow motions* to preserve a systematic expansion in (v/c) while the background remains flat. An example to use equation (5.55) for a source with arbitrary velocity should be a beam of charged particles accelerated by an electric field, where the particles do not contribute to the background curvature [8]. In this case, the particles can move at relativistic speeds. Thus, more terms in the expansion (v/c) must be considered.
As a final comment, note that at leading order the approximation of *weak* gravitational field at the far-field zone and *slow motion* of the source can also arise from the solution given in (5.22). Weak field at spatial infinity means that the leading term is the first one that depends on $1/r$ and slow motions means that $t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \simeq t - \frac{r}{c}$ over the entire source, i.e., disregarding the relative retardation effects of one region of the source relative to another [42]. From (5.23) it follows that,

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}}) \int T^{kl}(t - r/c, \mathbf{x}')d^3x' \quad . \tag{5.59}$$

5.4 Conservation equations

The physical interpretation of the expansion (5.55) becomes more clear if the solution is expressed in terms of the multipole moments of the energy density and the multiple moments of the momentum density. These are defined as,

$$M = \frac{1}{c^2}\int T^{00}(t,\mathbf{x})\, d^3x \qquad P^i = \frac{1}{c}\int T^{0i}(t,\mathbf{x})\, d^3x \tag{5.60}$$

$$M^i = \frac{1}{c^2}\int T^{00}(t,\mathbf{x})x^i\, d^3x \qquad P^{i,j} = \frac{1}{c}\int T^{0i}(t,\mathbf{x})x^j\, d^3x \tag{5.61}$$

$$M^{ij} = \frac{1}{c^2}\int T^{00}(t,\mathbf{x})x^i x^j\, d^3x \qquad P^{i,jk} = \frac{1}{c}\int T^{0i}(t,\mathbf{x})x^j x^k\, d^3x \tag{5.62}$$

$$M^{ijk} = \frac{1}{c^2}\int T^{00}(t,\mathbf{x})x^i x^j x^k\, d^3x \qquad P^{i,jkl} = \frac{1}{c}\int T^{0i}(t,\mathbf{x})x^j x^k x^l\, d^3x \quad . \tag{5.63}$$

Consider a volume V delimited by a boundary ∂V such that $T_{\mu\nu}$ vanishes at ∂V. Then, by the conservation of the energy-momentum tensor in linearized theory it follows that,

$$\partial_0 T^{00} = -\partial_i T^{0i} \quad . \tag{5.64}$$

Integrating over the volume V and using the definition of M in (5.60) then,

$$\begin{aligned}
\int_V \partial_0 T^{00}(t,\mathbf{x}) d^3x &= -\int_V \partial_i T^{0i}(t,\mathbf{x}) d^3x \\
\frac{1}{c}\frac{d}{dt}\Big(c^2 M\Big) &= -\oint_{\partial V} T^{0i}(t,\mathbf{x})\, dS_i \qquad \text{[Divergence theorem]} \\
c\dot{M} &= 0 \qquad [T^{0i} = 0 \text{ in } \partial V] \ . \qquad (5.65)
\end{aligned}$$

Thus, one obtains the conservation of the total mass M of the system[7],

$$\boxed{\dot{M} = 0} \ . \qquad (5.66)$$

Similarly, multiplying equation (5.64) by x^i and integrating over the volume V yields

$$\begin{aligned}
\int_V x^i \partial_0 T^{00}(t,\mathbf{x})\, d^3x &= -\int_V x^i \partial_j T^{0j}(t,\mathbf{x})\, d^3x \\
\frac{1}{c}\frac{d}{dt}\int_V x^i T^{00}(t,\mathbf{x})\, d^3x &= -\int_V x^i \partial_j T^{0j}(t,\mathbf{x})\, d^3x \\
c\frac{d}{dt}\left[\frac{1}{c^2}\int_V T^{00}(t,\mathbf{x}) x^i\, d^3x\right] &= -\int_V \left[\partial_j\left(x^i T^{0j}(t,\mathbf{x})\right) - \left(\partial_j x^i\right) T^{0j}(t,\mathbf{x})\right] d^3x \\
c\frac{dM^i}{dt} &= -\oint_{\partial V} x^i T^{0j}(t,\mathbf{x})\, dS_j + \int_V T^{0i}(t,\mathbf{x})\, d^3x \ , \qquad (5.67)
\end{aligned}$$

and one get the identity

$$\boxed{\dot{M}^i = P^i} \ . \qquad (5.68)$$

Now, multiplying equation (5.64) by $x^i x^j$ and integrating over the volume V,

$$\begin{aligned}
\int_V x^i x^j \partial_0 T^{00}\, d^3x &= -\int_V x^i x^j \partial_m T^{0m}\, d^3x \\
\frac{1}{c}\frac{d}{dt}\int_V x^i x^j T^{00}\, d^3x &= -\int_V \left[\partial_m\left(x^i x^j T^{0m}\right) - \partial_m\left(x^i x^j\right) T^{0m}\right] d^3x \\
c\frac{d}{dt}\left[\frac{1}{c^2}\int_V x^i x^j T^{00}\, d^3x\right] &= -\int_V \left[\partial_m\left(x^i x^j T^{0m}\right) - \left(\partial_m x^i\right) x^j T^{0m} - x^i\left(\partial_m x^j\right) T^{0m}\right] d^3x \\
c\frac{dM^{ij}}{dt} &= -\int_V \left[\partial_m\left(x^i x^j T^{0m}\right) - x^j T^{0i} - x^i T^{0j}\right] d^3x \\
c\dot{M}^{ij} &= -\oint_{\partial V} x^i x^j T^{0m}\, dS_m + cP^{i,j} + cP^{j,i} \ . \qquad (5.69)
\end{aligned}$$

[7] This is valid only at linear order. In general, a system that radiates gravitational waves lose mass.

Hence,

$$\boxed{\dot{M}^{ij} = P^{i,j} + P^{j,i}} \quad . \tag{5.70}$$

By multiplying (5.64) by $x^i x^j x^k$ and integrating over the volume V gives

$$\begin{aligned}
\int_V x^i x^j x^k \partial_0 T^{00}\, d^3x &= -\int_V x^i x^j x^k \partial_m T^{0m}\, d^3x \\
\frac{1}{c}\frac{d}{dt}\int_V x^i x^j x^k T^{00}\, d^3x &= -\int_V \left[\partial_m\left(x^i x^j x^k T^{0m}\right) - \partial_m\left(x^i x^j x^k\right) T^{0m}\right] d^3x \\
c\frac{d}{dt}\left[\frac{1}{c^2}\int_V x^i x^j x^k T^{00}\, d^3x\right] &= -\oint_{\partial V} x^i x^j x^k T^{0m}\, dS_m + \int_V \partial_m\left(x^i x^j x^k\right) T^{0m}\, d^3x \\
c\frac{dM^{ijk}}{dt} &= \int_V \left[\left(\partial_m x^i\right) x^j x^k T^{0m} + x^i \left(\partial_m x^j\right) x^k T^{0m} + x^i x^j \left(\partial_m x^k\right) T^{0m}\right] d^3x \\
c\dot{M}^{ijk} &= \int_V \left[x^j x^k T^{0i} + x^i x^k T^{0j} + x^i x^j T^{0k}\right] \quad ,
\end{aligned} \tag{5.71}$$

and another identity is obtained,

$$\boxed{\dot{M}^{ijk} = P^{i,jk} + P^{j,ik} + P^{k,ij}} \quad . \tag{5.72}$$

Other identities for the multipole moments of the mementum density may be derived with a similar procedure. The temporal part of $\partial_\nu T^{\mu\nu} = 0$ is

$$\partial_0 T^{i0} = -\partial_j T^{ij} \tag{5.73}$$

An integration of (5.73) over the volume V leads to,

$$\begin{aligned}
\int_V \partial_0 T^{0i}\, d^3x &= -\int_V \partial_j T^{ij}\, d^3x \\
\dot{P}^i &= -\oint_{\partial V} T^{ij}\, dS_j \qquad \text{[Divergence theorem]} \quad .
\end{aligned} \tag{5.74}$$

As a consequence,

$$\boxed{\dot{P}^i = 0} \quad . \tag{5.75}$$

If equation (5.73) is multiplied by x^j and integrated over the volume V one obtains,

$$\begin{aligned}
\int_V x^j \partial_0 T^{i0}\, d^3x &= -\int_V x^j \partial_m T^{im}\, d^3x \\
\frac{d}{dt}\left[\frac{1}{c}\int_V x^j T^{i0}\, d^3x\right] &= -\int_V \left[\partial_m\left(x^j T^{im}\right) - \left(\partial_m x^j\right) T^{im}\right] d^3x \\
\frac{dP^{i,j}}{dt} &= -\oint_{\partial V} x^j T^{im}\, dS_m + \int_V T^{ij}\, d^3x \quad .
\end{aligned} \tag{5.76}$$

Therefore ,

$$\boxed{\dot{P}^{i,k} \; = \; S^{ij}} \quad . \tag{5.77}$$

Another identity is deduced by making,

$$\begin{aligned}
\int_V x^j x^k \partial_0 T^{i0} \, d^3x \; &= \; - \int_V x^j x^k \partial_m T^{im} \, d^3x \\
\frac{d}{dt}\left[\frac{1}{c}\int x^j x^k T^{i0\,d^3x}\right] \; &= \; - \int_V \left[\partial_m \left(x^j x^k T^{im}\right) \; - \; \partial_m \left(x^j x^k\right) T^{im}\right] d^3x \\
\frac{dP^{i,jk}}{dt} \; &= \; \int_V \left[\left(\partial_m x^j\right) x^k T^{im} \; + \; x^j \left(\partial_m x^k\right) T^{im}\right]
\end{aligned} \tag{5.78}$$

i.e.,

$$\boxed{\dot{P}^{i,jk} \; = \; S^{ij,k} \; + \; S^{ik,j}} \quad . \tag{5.79}$$

The angular momentum is also conserved. This can be seen from the symmetry property of S^{ij} and using (5.78). The result is,

$$\begin{aligned}
\dot{P}^{i,j} \; - \; \dot{P}^{j,i} \; &= \frac{d}{dt}\left[\frac{1}{c}\int \left(x^j T^{0i} \; - \; x^i T^{0j}\right) d^3x\right] \\
&= \; S^{ij} \; - \; S^{ji} \\
&= \; 0 \quad .
\end{aligned} \tag{5.80}$$

Is also possible to write the multipole moments of the energy-momentum tensor in terms of the multiple moments of the energy density and the momentum density. For instance, taking (5.77) and using the symmetry property of S^{ij}, after differentiating equation (5.70) it is obtained

$$\ddot{M}^{ij} \; = \; \dot{P}^{i,j} \; + \; \dot{P}^{j,i} \; = \; S^{ij} \; + \; S^{ji} \; = \; 2S^{ij}$$

or,

$$\boxed{S^{ij} \; = \; \frac{1}{2}\ddot{M}^{ij}} \quad . \tag{5.81}$$

From equations (5.72) and (5.79) we have

$$\dddot{M}^{ijk} \; = \; \ddot{P}^{i,jk} \; + \; \ddot{P}^{j,ki} \; + \; \ddot{P}^{k,ij} \tag{5.82}$$

$$\ddot{P}^{i,jk} \; = \; \dot{S}^{ij,k} \; + \; \dot{S}^{ik,j} \quad , \tag{5.83}$$

and substituting (5.83) into (5.82) gives,

$$\begin{aligned}\dddot{M}^{ijk} &= \dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i} + \dot{S}^{ji,k} + \dot{S}^{ki,j} + \dot{S}^{kj,i} \\ &= 2\dot{S}^{ij,k} + 2\dot{S}^{ik,j} + 2\dot{S}^{jk,i} \quad .\end{aligned} \tag{5.84}$$

Thus,

$$\boxed{\dddot{M}^{ijk} = 2\left(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i}\right)} \quad . \tag{5.85}$$

A last expression may be deduced by first solving $\dot{S}^{ij,k}$ from (5.85) and rewriting it as,

$$\begin{aligned}\dot{S}^{ij,k} &= \frac{1}{2}\dddot{M}^{ijk} - \dot{S}^{ik,j} - \dot{S}^{jk,i} \\ &= \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\dddot{M}^{ijk} - \dot{S}^{ik,j} - \dot{S}^{jk,i} \\ &= \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}2\left(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i}\right) - \dot{S}^{ik,j} - \dot{S}^{jk,i} \\ &= \frac{1}{6}\dddot{M}^{ijk} - \frac{1}{3}\dot{S}^{ik,j} - \frac{1}{3}\dot{S}^{jk,i} + \frac{2}{3}\dot{S}^{ij,k} \\ &= \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\dot{S}^{ik,j} - 2\dot{S}^{ki,j}\right) + \frac{1}{3}\left(\dot{S}^{jk,i} - 2\dot{S}^{kj,i}\right) + \frac{1}{3}\left(\dot{S}^{ij,k} + \dot{S}^{ji,k}\right) \\ &= \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{ji,k} + \dot{S}^{jk,i} - 2\dot{S}^{kj,i} - 2\dot{S}^{ki,j}\right) \quad .\end{aligned} \tag{5.86}$$

Using the identity in (5.83), the previous result becomes

$$\boxed{\dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right)} \quad . \tag{5.87}$$

Similar relations can be obtained for higher order multipole moments. In particular, the identities (5.81) and (5.87) allows us to rewrite the solution h_{ij}^{TT} in terms of the energy and mementum densities multipole moments.

5.5 Mass quadrupole radiation

We shall focus only on the leading term contribution of the radiation. Recall that for *weak* gravitational fields and *slow motions* the solution of the inhomegeneous wave equation in the far-field zone is given by

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_i\dot{S}^{kl,i} + \frac{1}{2c^2}n_i n_j\ddot{S}^{kl,ij} + \cdots\right]_{t-r/c} \quad . \tag{5.88}$$

The leading term in the expanded solution shown in (5.88) comes from S^{kl}. By using the identity (5.81) the leading contribution takes the form

$$\boxed{h_{ij}^{\rm TT}(t,\mathbf{x}) \;=\; \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\ddot{M}^{kl}\left(t-\frac{r}{c}\right)} \quad . \tag{5.89}$$

The tensor M^{kl} may be decomposed as a traceless part plus the trace part[8],

$$\begin{aligned} M^{kl} \;&=\; \left(M^{kl} \;-\; \frac{1}{3}\delta^{kl}M_{ii}\right) \;+\; \frac{1}{3}\delta^{kl}M_{ii} \\ &=\; Q^{kl} \;+\; \frac{1}{3}\delta^{kl}M_{ii} \quad , \end{aligned} \tag{5.90}$$

where the quantity Q^{kl} is called the *quadrupole moment* and is defined as

$$Q^{kl} \;:=\; M^{kl} \;-\; \frac{1}{3}\delta^{kl}M_{ii} \quad . \tag{5.91}$$

Clearly, the trace of Q^{kl} in (5.91) vanish, i.e.,

$$\begin{aligned} Tr(Q^{kl}) \;&=\; Tr\left(M^{kl} \;-\; \frac{1}{3}\delta^{kl}M_{ii}\right) \;=\; Tr\left(M^{kl}\right) \;-\; \frac{1}{3}Tr\left(\delta^{kl}\right)M_{ii} \\ &=\; M_{ii} \;-\; \frac{1}{3}(3)M_{ii} \;=\; 0 \quad . \end{aligned} \tag{5.92}$$

On the other hand, due to the traceless property in their first and second pair of indices of the *Lambda* tensor, one has

$$\begin{aligned} \Lambda_{ij|kl}M^{kl} \;&=\; \Lambda_{ij|kl}\left(Q^{kl} \;+\; \frac{1}{3}\delta^{kl}M_{ii}\right) \\ &=\; \Lambda_{ij|kl}Q^{kl} \;+\; \frac{1}{3}\Lambda_{ij|kk}M_{ii} \\ &=\; \Lambda_{ij|kl}Q^{kl} \quad . \end{aligned} \tag{5.93}$$

This results implies that $\Lambda_{ij|kl}(\hat{\mathbf{n}})\ddot{M}^{kl} \;=\; \Lambda_{ij|kl}(\hat{\mathbf{n}})\ddot{Q}^{kl}$ and the leading contribution of the solution in (5.89) is equivalent to

$$\boxed{\left[h_{ij}^{\rm TT}(t,\mathbf{x})\right]_{\rm quad} \;=\; \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\ddot{Q}_{kl}\left(t-\frac{r}{c}\right)} \quad . \tag{5.94}$$

The equation (5.94) is called the **Einstein's quadrupole formula**[9]. The name of the tensor Q^{kl} is more clear when is written in its explicit form,

$$\begin{aligned} Q^{kl} \;&=\; \frac{1}{c^2}\int T^{00}x^kx^l\,d^3x \;-\; \frac{1}{3}\delta^{kl}\frac{1}{c^2}\int T^{00}x^ix^i\,d^3x \\ &=\; \int \frac{1}{c^2}T^{00}\left(x^kx^l \;-\; \frac{1}{3}\delta^{kl}x^ix^i\right)\,d^3x \quad . \end{aligned} \tag{5.95}$$

---

[8] Formally, this is the irreducible representation of a symmetric tensor of two indices under $SO(3)$.

[9] After a long history of debate, this formula was confirmed with the observation of the Hulse-Taylor binary in 1974. A formal derivation in the post-Newtonian formalism is a better scenario to justify this formula. For more details see [9].

Recall that T^{00}/c^2 is the energy density and $x^i x^i = |x|^2 = r^2$. Thus, the quadrupolar moment yields

$$\boxed{Q^{kl} = \int \rho(t, \mathbf{x}) \left[x^k x^l - \frac{1}{3}\delta^{kl} r^2 \right] d^3x} \quad . \tag{5.96}$$

Observe that the first and leading contribution of the gravitation radiation is related to the variation of the quadrupole moment. This is not the case for electromagnetic radiation in which the leading contribution comes from the variation of the dipole moment. This result can be understood by appealing to Einstein's equivalence principle and the conservation laws discussed earlier [37]. The radiation $h_{ij}^{\rm TT}$ depends on derivatives of the multipole moments. For a system of point masses, the monopole term is just the total mass which is conserved. Thus the time derivative of the total mass M vanish as is shown in equation (5.66) and thus there cannot be monopolar radiation. The dipole moment of a system of point masses is

$$\mathbf{d} = \sum_i m_i^{(g)} \mathbf{x}_i = \sum_i m_i \mathbf{x}_i \quad , \tag{5.97}$$

where $m_i^{(g)}$ is the gravitational mass and m_i is the inertial mass. The time derivative of the dipole moment gives,

$$\dot{\mathbf{d}} = \sum_i m_i \dot{\mathbf{x}}_i = \mathbf{P} \quad . \tag{5.98}$$

The second time derivative of the dipole moment vanish due to the conservation of linear momentum of isolated particles[10]. If a magnetic-dipole analog of gravitation radiation is considered, the second time derivative vanish because of the conservation of the total angular momentum,

$$\boldsymbol{\mu} = \sum_i \mathbf{x}_i \times (m_i \mathbf{v}_i) = \mathbf{L} \quad . \tag{5.99}$$

The absense of the monopolar radiation can be understood also from the *Birkhoff's Theorem* [44]. The monopolar term means a spherically symmetric source. But any spherically symmetric solution of the EFE must be static so there cannot be changes in the field outside the source and monopolar gravitational radiation is not emmited[11]. In conclusion, there cannot be monopolar nor dipolar gravitational radiation of any sort in GR[12] [8].

5.6 Angular distribution of quadrupole radiation

One is able to obtain the functional waveform of the quadrupole radiation from (5.94) by using the explicit definition of the *Lambda* tensor and making the contraction with $\ddot{Q}_{kl}$. However, in order to get the gravitational radiation emitted in an arbitrary direction $\hat{\mathbf{n}}$ is worthwhile to proceed as follows. Consider first a GW propagating along the $+z$-direction. This means that $\hat{\mathbf{n}} = (0, 0, 1)$. Recall that the projector tensor P_{ij} is defined as

$$P_{ij} = \delta_{ij} - n_i n_j \quad . \tag{5.100}$$

[10] Equivalently, the dipole moment can be also rewritten as $\mathbf{d} = \sum_i m_i \mathbf{x}_i = M\mathbf{R}_{\rm CM}$, i.e., is proportional to the center of mass position. For an insolated body, due to the conservation of the total linear momentum, the center of mass moves with uniform velocity and thus the second time derivative of the dipole term vanish.

[11] This is also true even if the source is pulsating spherically symmetric. Thus, a perfect spherically symmetric collapse of a star does not emit GWs.

[12] Of course, these results are also valid for extended bodies.

Thus, the matrix representation of P_{ij} is

$$[P_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{5.101}$$

The effect of applying the projector tensor in (5.101) on any vector is to substract the z-component. This means that if $v_j = (a, b, c)$ is a generic vector in space, then $v'_i = P_{ij}v_j = (a, b, 0)$. Thus, P_{ij} projects the vector onto the xy plane which is perpendicular to $\hat{\mathbf{n}}$. Then, the TT projection of a generic tensor A_{ij} can be computed by using the definition of the *Lambda* tensor,

$$\begin{aligned} \Lambda_{ij|kl}A_{kl} &= \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)A_{kl} \\ &= P_{ik}A_{kl}P_{jl} - \frac{1}{2}P_{ij}P_{kk}A_{kl} \\ &= P_{ik}A_{kl}P_{lj} - \frac{1}{2}P_{ij}P_{lk}A_{kl} \\ A^{\mathrm{TT}}_{ij} &= (PAP)_{ij} - \frac{1}{2}P_{ij}Tr\,(PA) \quad . \end{aligned} \tag{5.102}$$

In matrix form, the first term of (5.102) gives,

$$\begin{aligned} [PAP] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.103}$$

and the second term

$$\begin{aligned} -\frac{1}{2}Tr\,[PA]\,[P] &= -\frac{1}{2}Tr \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \end{aligned} \tag{5.104}$$

By adding equations (5.103) and (5.104) one obtains A^{TT}_{ij},

$$\left[A^{\mathrm{TT}}\right]_{ij} = \begin{pmatrix} (A_{11} - A_{22})/2 & A_{21} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \quad . \tag{5.105}$$

Since $\Lambda_{ij|kl}\ddot{Q}^{kl} = \Lambda_{ij|kl}\ddot{M}^{kl}$ and making $A = \ddot{Q}$ in equation (5.105) results in

$$\left[\Lambda\ddot{Q}\right]_{ij} = \begin{pmatrix} \frac{1}{2}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & -\frac{1}{2}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{5.106}$$

and the quadrupolar radiation contribution is given by

$$\left[h^{\mathrm{TT}}(t,\mathbf{x})\right]^{\mathrm{quad}}_{ij} = \frac{1}{r}\frac{2G}{c^4} \begin{pmatrix} \frac{1}{2}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & -\frac{1}{2}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{t-r/c} . \tag{5.107}$$

Therefore, the radiation of the two polarizations in the quadrupole approximation for a GW propagating in the $+z$ direction gives,

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left[\ddot{M}_{11}\left(t - \frac{r}{c}\right) - \ddot{M}_{22}\left(t - \frac{r}{c}\right)\right] , \tag{5.108}$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}\left(t - \frac{r}{c}\right) . \tag{5.109}$$

The next step is to find out the quadrupole radiation of a GW propagating in a generic direction $\hat{\mathbf{n}}$ with respect to a coordinate system (x, y, z). From figure 5.4, the components of the unit vector $\hat{\mathbf{n}}$ in the system (x, y, z) are

$$\hat{\mathbf{n}} = \Big(\sin\theta\sin\phi \ , \ \sin\theta\cos\phi \ , \ \cos\theta\Big) \tag{5.110}$$

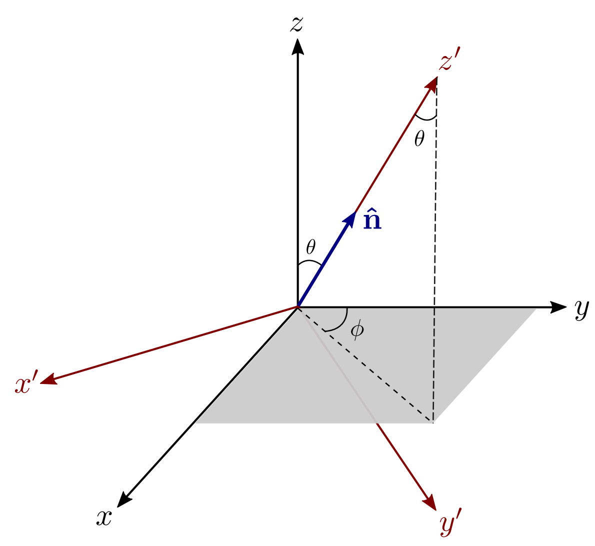

Figure 5.4: Two reference frames describing the direction $\hat{\mathbf{n}}$ of a GW.

Note that these components can also be obtained after making two rotations in order to align the axis of the systems (x', y', z') and (x, y, z). Such rotations are described as follows,

$R_{x'}(\theta)$: Align the axis z' with z.

$R_z(\phi)$: Align the axis x' with x and y' with y.

Then, the unit vector $\hat{\mathbf{n}}$ is given by $n_i = \mathcal{R}_{ij} n'_j$ with $\mathcal{R}_{ij} = [R_{x'} R_z]_{ij}$. i.e.,

$$\begin{aligned}
\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} n_{x'} \\ n_{y'} \\ n_{z'} \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ -\sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}
\end{aligned} \tag{5.111}$$

as expected. Furthermore, rotation matrices are orthogonal and thus their inverse are just their transpose. Therefore, $n'_i = \mathcal{R}^{\mathsf{T}}_{ij} n_j$ with $\mathcal{R}^{\mathsf{T}}_{ij} = [R_{x'} R_z]^{\mathsf{T}}_{ij} = [R^{\mathsf{T}}_z R^{\mathsf{T}}_{x'}]_{ij}$. Then,

$$\begin{aligned}
\begin{pmatrix} n_{x'} \\ n_{y'} \\ n_{z'} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad .
\end{aligned} \tag{5.112}$$

In a similar way, is possible to extend these results to tensors. The coordinates of a tensor $\ddot{M}^{kl}$ changes between the two references frames under a rotation $\mathcal{R}$ as

$$\begin{aligned}
\ddot{M}^{kl} &= \mathcal{R}^k{}_i \mathcal{R}^l{}_j \ddot{M}'^{ij} && \text{[In the } (x, y, z) \text{ frame]} \quad , \\
\ddot{M}'^{ij} &= \mathcal{R}_k{}^i \mathcal{R}_l{}^j \ddot{M}^{kl} && \text{[In the } (x', y', z') \text{ frame]} \quad .
\end{aligned} \tag{5.113}$$

with

$$\mathcal{R}^i{}_j = \begin{pmatrix} \cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ -\sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad , \quad \mathcal{R}_i{}^j = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Since the $+z'$-direction is align with $\hat{\mathbf{n}}$, the aim is to find the solution $[h^{\rm TT}]^{\rm quad}_{ij}$ in the (x', y', z') frame using the expression (5.113). For this, observe that from equations (5.108) and (5.109) only the quantities $\ddot{M}'^{11}$, $\ddot{M}'^{22}$ and $\ddot{M}'^{12}$ are required. Expanding (5.113) yields

$$\begin{aligned}
\ddot{M}'^{11} &= R_1{}^1 R_l{}^1 \ddot{M}^{1l} + R_2{}^1 R_l{}^1 \ddot{M}^{2l} + R_3{}^1 R_l{}^1 \ddot{M}^{3l} \\
&= \Big(R_1{}^1 R_1{}^1 \ddot{M}^{11} + \underline{R_1{}^1 R_2{}^1 \ddot{M}^{12}} + \underline{R_1{}^1 R_3{}^1 \ddot{M}^{13}}\Big) + \\
&\quad \Big(\underline{R_2{}^1 R_1{}^1 \ddot{M}^{21}} + R_2{}^1 R_2{}^1 \ddot{M}^{22} + \underline{R_2{}^1 R_3{}^1 \ddot{M}^{23}}\Big) + \\
&\quad \Big(\underline{R_3{}^1 R_1{}^1 \ddot{M}^{31}} + \underline{R_3{}^1 R_2{}^1 \ddot{M}^{32}} + R_3{}^1 R_3{}^1 \ddot{M}^{33}\Big) \\
&= R_1{}^1 R_1{}^1 \ddot{M}^{11} + 2R_1{}^1 R_2{}^1 \ddot{M}^{12} + 2R_1{}^1 \cancel{R_3{}^1} \ddot{M}^{13} + \\
&\quad R_2{}^1 R_2{}^1 \ddot{M}^{22} + 2R_2{}^1 \cancel{R_3{}^1} \ddot{M}^{23} + \cancel{R_3{}^1} R_3{}^1 \ddot{M}^{33} \\
&= R_1{}^1 R_1{}^1 \ddot{M}^{11} + 2R_1{}^1 R_2{}^1 \ddot{M}^{12} + R_2{}^1 R_2{}^1 \ddot{M}^{22} \\
&= \cos^2\phi \ddot{M}^{11} - 2\cos\phi\sin\phi \ddot{M}^{12} + \sin^2\phi \ddot{M}^{22} \quad ,
\end{aligned} \tag{5.114}$$

$$\begin{aligned}
\ddot{M}'^{22} &= R_1{}^2 R_1{}^2 \ddot{M}^{11} + 2R_1{}^2 R_2{}^2 \ddot{M}^{12} + R_1{}^2 R_3{}^2 \ddot{M}^{13} + \\
&\quad R_2{}^2 R_2{}^2 \ddot{M}^{22} + 2R_2{}^2 R_3{}^2 \ddot{M}^{23} + R_3{}^2 R_3{}^2 \ddot{M}^{33} \\
&= \cos^2\theta\sin^2\phi \ddot{M}^{11} + 2\cos^2\theta\cos\phi\sin\phi \ddot{M}^{12} - 2\cos\theta\sin\theta\sin\phi \ddot{M}^{13} \\
&\quad + \cos^2\theta\cos^2\phi \ddot{M}^{22} - 2\cos\theta\sin\theta\cos\phi \ddot{M}^{23} + \sin^2\theta \ddot{M}^{33} \quad ,
\end{aligned} \tag{5.115}$$

$$\begin{aligned}
\ddot{M}'^{12} &= \Big(R_1{}^1 R_1{}^2 \ddot{M}^{11} + R_1{}^1 R_2{}^2 \ddot{M}^{12} + R_1{}^1 R_3{}^2 \ddot{M}^{13}\Big) + \\
&\quad \Big(R_2{}^1 R_1{}^2 \ddot{M}^{21} + R_2{}^1 R_2{}^2 \ddot{M}^{22} + R_2{}^1 R_3{}^2 \ddot{M}^{23}\Big) + \\
&\quad \Big(\cancel{R_3{}^1} R_1{}^2 \ddot{M}^{31} + \cancel{R_3{}^1} R_2{}^2 \ddot{M}^{32} + \cancel{R_3{}^1} R_3{}^2 \ddot{M}^{33}\Big) \\
&= R_1{}^1 R_1{}^2 \ddot{M}^{11} + R_1{}^1 R_2{}^2 \ddot{M}^{12} + R_1{}^1 R_3{}^2 \ddot{M}^{13} + \\
&\quad R_2{}^1 R_1{}^2 \ddot{M}^{21} + R_2{}^1 R_2{}^2 \ddot{M}^{22} + R_2{}^1 R_3{}^2 \ddot{M}^{23} \\
&= \cos\theta\cos\phi\sin\phi \ddot{M}^{11} + \cos\theta\cos^2\phi \ddot{M}^{12} - \sin\theta\cos\phi \ddot{M}^{13} \\
&\quad - \cos\theta\sin^2\phi \ddot{M}^{21} - \cos\theta\cos\phi\sin\phi \ddot{M}^{22} + \sin\theta\sin\phi \ddot{M}^{23} \quad .
\end{aligned} \tag{5.116}$$

Now,

$$\begin{aligned}
\ddot{M}'^{11} - \ddot{M}'^{22} &= \cos^2\phi \underline{\ddot{M}^{11}} - 2\cos\phi\sin\phi\underline{\ddot{M}^{12}} + \sin^2\phi\underline{\ddot{M}^{22}} \\
&\quad - \cos^2\theta\sin^2\phi\underline{\ddot{M}^{11}} - 2\cos^2\theta\cos\phi\sin\phi\underline{\ddot{M}^{12}} + 2\cos\theta\sin\theta\sin\phi\ddot{M}^{13} \\
&\quad - \cos^2\theta\cos^2\phi\underline{\ddot{M}^{22}} + 2\cos\theta\sin\theta\cos\phi\ddot{M}^{23} - \sin^2\theta\ddot{M}^{33} \\
&= \ddot{M}^{11}\Big(\cos^2\phi - \sin^2\phi\cos^2\theta\Big) + \ddot{M}^{22}\Big(\sin^2\phi - \cos^2\phi\cos^2\theta\Big) \\
&\quad - \ddot{M}^{33}\sin^2\theta - \ddot{M}^{12}\Big(2\cos\phi\sin\phi + 2\cos^2\theta\cos\phi\sin\phi\Big) \\
&\quad + \ddot{M}^{13}\Big(2\cos\theta\sin\theta\sin\phi\Big) + \ddot{M}^{23}\Big(2\cos\theta\sin\theta\cos\phi\Big)
\end{aligned} \tag{5.117}$$

Using the relation $\sin(2\phi) = \sin\phi\cos\phi + \sin\phi\cos\phi = 2\sin\phi\cos\phi$ then,

$$\begin{aligned}
2\cos\phi\sin\phi + 2\cos^2\theta\cos\phi\sin\phi &= \sin(2\phi) + \sin(2\phi)\cos^2\theta \\
&= \sin(2\phi)(1+\cos^2\theta) \quad .
\end{aligned} \tag{5.118}$$

Thus,

$$\begin{aligned}
\ddot{M}'^{11} - \ddot{M}'^{22} = \; & \ddot{M}^{11}\Big(\cos^2\phi - \sin^2\phi\cos^2\theta\Big) + \ddot{M}^{22}\Big(\sin^2\phi - \cos^2\phi\cos^2\theta\Big) \\
& - \ddot{M}^{33}\sin^2\theta - \ddot{M}^{12}\sin(2\phi)(1+\cos^2\theta) + \ddot{M}^{13}\sin\phi\sin(2\theta) \\
& + \ddot{M}^{23}\cos\phi\sin(2\theta) \quad .
\end{aligned} \tag{5.119}$$

The component $\ddot{M}^{12}$ in equation (5.116) can be simplified to

$$\begin{aligned}
\ddot{M}^{12} &= \Big(\ddot{M}^{11} - \ddot{M}^{22}\Big)\cos\theta\cos\phi\sin\phi + \ddot{M}^{12}\Big(\cos\theta\cos^2\phi - \cos\theta\sin^2\phi\Big) \\
&\quad - \ddot{M}^{13}\sin\theta\cos\phi + \ddot{M}^{23}\sin\theta\sin\phi \\
\ddot{M}^{12} &= \frac{1}{2}\Bigg[\Big(\ddot{M}^{11} - \ddot{M}^{22}\Big)\sin(2\phi)\cos\theta + 2\ddot{M}^{12}\cos(2\phi)\cos\theta \\
&\quad - 2\ddot{M}^{13}\cos\phi\sin\theta + 2\ddot{M}^{23}\sin\phi\sin\theta\Bigg] \quad .
\end{aligned} \tag{5.120}$$

Finally using (5.119) and (5.120) into (5.108) and (5.109) yields,

$$\begin{aligned}
h_+(t;\theta,\phi) = \; & \frac{1}{r}\frac{G}{c^4}\Bigg[\ddot{M}^{11}\Big(\cos^2\phi - \sin^2\phi\cos^2\theta\Big) \\
& + \ddot{M}^{22}\Big(\sin^2\phi - \cos^2\phi\cos^2\theta\Big) \\
& - \ddot{M}^{33}\sin^2\theta - \ddot{M}^{12}\sin(2\phi)(1+\cos^2\theta) \\
& + \ddot{M}^{13}\sin\phi\sin(2\theta) + \ddot{M}^{23}\cos\phi\sin(2\theta)\Bigg]_{t-r/c} \quad ,
\end{aligned} \tag{5.121}$$

$$h_\times(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\Big[\Big(\ddot{M}^{11} - \ddot{M}^{22}\Big)\sin(2\phi)\cos\theta + 2\ddot{M}^{12}\cos(2\phi)\cos\theta - 2\ddot{M}^{13}\cos\phi\sin\theta + 2\ddot{M}^{23}\sin\phi\sin\theta\Big]_{t-r/c} \quad . \tag{5.122}$$

5.7 Radiated energy

Using (4.73) into (4.68), the power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{c^3 r^2}{32\pi G}\left\langle \dot{h}^{ij}_{\mathrm{TT}}\dot{h}^{\mathrm{TT}}_{ij}\right\rangle \quad . \tag{5.123}$$

The quadrupolar radiation was obtained previously as shown in (5.94). Then,

$$\left[\dot{h}^{\mathrm{TT}}_{ij}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\dddot{Q}_{kl}\left(t-\frac{r}{c}\right) \quad . \tag{5.124}$$

Replacing (5.124) into (5.123) and using the *Lambda* projection property, the quadrupolar power per unit solid angle gives[13],

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{quad}} = \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}})\left\langle \dddot{Q}^{kl}\dddot{Q}^{mn}\right\rangle\Big|_{t-r/c} \quad . \tag{5.125}$$

Power

To get the total radiated power one must perform an integration of (5.125) over the entire solid angle,

$$P_{\mathrm{quad}} = \frac{G}{8\pi c^5}\left\langle \dddot{Q}^{kl}\dddot{Q}^{mn}\right\rangle\Big|_{t-r/c}\int \Lambda_{kl|mn}(\hat{\mathbf{n}})\, d\Omega \quad . \tag{5.126}$$

The integral that appears in (5.126) may be computed by using two useful properties. Writing the unit vector $\hat{\mathbf{n}}$ as in equation (5.110), is possible to show that

$$\int n_i n_j \, d\Omega = \frac{4\pi}{3}\delta_{ij} \quad , \tag{5.127}$$

$$\int n_i n_j n_k n_l \, d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \quad . \tag{5.128}$$

From the explicit definition of $\Lambda_{kl|mn}(\hat{\mathbf{n}})$ in terms of the propagation unit vector $\hat{\mathbf{n}}$, the integral to be found is

$$\int \Big[\delta_{km}\delta_{ln} - \frac{1}{2}\delta_{kl}\delta_{mn} - n_l n_n\delta_{km} - n_k n_m\delta_{ln} + \frac{1}{2}n_m n_n\delta_{kl} + \frac{1}{2}n_k n_l\delta_{mn} + \frac{1}{2}n_k n_l n_m n_n\Big] d\Omega \quad .$$

[13] The average $\langle\cdot\rangle$ is over several periods of the GW.

Employing the properties (5.127) and (5.128) the integration reads as follows,

$$\begin{aligned}\int \Lambda_{kl|mn}(\hat{\mathbf{n}})\, d\Omega \;&=\; 4\pi\underline{\delta_{km}\delta_{ln}} - 2\pi\underline{\delta_{kl}\delta_{mn}} - \frac{4\pi}{3}\underline{\delta_{ln}\delta_{km}} - \frac{4\pi}{3}\underline{\delta_{km}\delta_{ln}} \\ &\quad + \frac{2\pi}{3}\underline{\delta_{mn}\delta_{kl}} + \frac{2\pi}{3}\underline{\delta_{kl}\delta_{mn}} + \frac{4\pi}{30}\left(\underline{\delta_{kl}\delta_{mn}} + \underline{\delta_{km}\delta_{ln}} + \delta_{kn}\delta_{lm}\right) \\ &= \frac{22\pi}{15}\delta_{km}\delta_{ln} - \frac{8\pi}{15}\delta_{kl}\delta_{mn} + \frac{2\pi}{15}\delta_{kn}\delta_{lm} \\ &= \frac{2\pi}{15}\Big(11\delta_{km}\delta_{ln} - 4\delta_{kl}\delta_{mn} + \delta_{kn}\delta_{lm}\Big) \quad . \end{aligned} \tag{5.129}$$

Retaking the equation (5.126) and using the result in (5.129) one obtains the total quadrupolar power,

$$\begin{aligned} P_{\text{quad}} \;&=\; \frac{G}{8\pi c^5}\left\langle \dddot{Q}^{kl}\dddot{Q}^{mn}\right\rangle\Big|_{t-r/c}\left[\frac{2\pi}{15}\Big(11\delta_{km}\delta_{ln} \;-\; 4\delta_{kl}\delta_{mn} \;+\; \delta_{kn}\delta_{lm}\Big)\right] \\ &= \frac{G}{60c^5}\left[11\left\langle \dddot{Q}^{kl}\dddot{Q}_{kl}\right\rangle - 4\left\langle \dddot{Q}^{k}_{\ k}\dddot{Q}^{m}_{\ m}\right\rangle \;+\; \left\langle \dddot{Q}^{kl}\dddot{Q}_{lk}\right\rangle\right]_{t-r/c} \\ &= \frac{G}{60c^5}\left[11\left\langle \dddot{Q}^{kl}\dddot{Q}_{kl}\right\rangle \;+\; \left\langle \dddot{Q}^{kl}\dddot{Q}_{kl}\right\rangle\right]_{t-r/c} \end{aligned}$$

i.e.,

$$\boxed{P_{\text{quad}} \;=\; \frac{G}{5c^2}\left\langle \dddot{Q}^{kl}\dddot{Q}_{kl}\right\rangle\Big|_{t-r/c}} \quad . \tag{5.130}$$

In some cases is more convenient to express (5.130) in terms of M_{kl},

$$\dddot{Q}_{kl} \;=\; \dddot{M}_{kl} \;-\; \frac{1}{3}\delta_{kl}\dddot{M}_{ii} \quad . \tag{5.131}$$

Inserting (5.131) into (5.130) it is obtained,

$$\begin{aligned} P_{\text{quad}} \;&=\; \frac{G}{5c^2}\left\langle\Big(\dddot{M}^{kl} \;-\; \frac{1}{3}\delta^{kl}\dddot{M}^{ii}\Big)\Big(\dddot{M}_{kl} \;-\; \frac{1}{3}\delta_{kl}\dddot{M}_{ii}\Big)\right\rangle\Big|_{t-r/c} \\ &= \frac{G}{5c^2}\left\langle \dddot{M}^{kl}\dddot{M}_{kl} \;-\; \frac{2}{3}\dddot{M}^2_{ii} \;+\; \frac{1}{9}\delta^{kl}\delta_{kl}\dddot{M}^2_{ii}\right\rangle\Big|_{t-r/c} \\ &= \frac{G}{5c^2}\left\langle \dddot{M}^{kl}\dddot{M}_{kl} \;-\; \frac{2}{3}\dddot{M}^2_{ii} \;+\; \frac{1}{3}\dddot{M}^2_{ii}\right\rangle\Big|_{t-r/c} \end{aligned}$$

and therefore,

$$\boxed{P_{\text{quad}} \;=\; \frac{G}{5c^2}\left\langle \dddot{M}^{kl}\dddot{M}_{kl} \;-\; \frac{1}{3}\dddot{M}^2_{ii}\right\rangle\Big|_{t-r/c}} \quad . \tag{5.132}$$

To obtain the emitted energy per unit solid angle, first consider the Fourier transform of the quadrupolar moment,

$$Q_{kl}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{Q}_{kl}(\omega) e^{-i\omega t}\, d\omega \quad . \tag{5.133}$$

On the other hand, it is also true that

$$\dddot{Q}_{kl}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{Q}_{kl}(\omega)\,(-i\omega)^3\; e^{-i\omega t}\, d\omega \quad , \quad \begin{cases} \omega' = -\omega \\ d\omega' = -d\omega \end{cases} \tag{5.134}$$

$$= -\frac{1}{2\pi}\int_{\infty}^{-\infty} \tilde{Q}_{kl}(-\omega')\,\left(i\omega'\right)^3\; e^{i\omega' t}\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{Q}_{kl}(-\omega')\,\left(i\omega'\right)\; e^{i\omega' t}\, d\omega' \quad . \tag{5.135}$$

From (5.125), the energy per unit solid angle is given by

$$\int \left(\frac{dP}{d\Omega}\right)_{\text{quad}} dt = \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}}) \int \left\langle \dddot{Q}_{kl}(t)\dddot{Q}_{mn}(t)\right\rangle\, dt$$

$$\left(\frac{dE}{d\Omega}\right)_{\text{quad}} = \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}}) \left\langle \int \dddot{Q}_{kl}(t)\dddot{Q}_{mn}(t)\, dt\right\rangle$$

$$= \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}}) \int \dddot{Q}_{kl}(t)\dddot{Q}_{mn}(t)\, dt$$

$$= \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}}) \int\int\int \tilde{Q}_{kl}(-\omega')\,\left(i\omega'\right)^3 \tilde{Q}_{mn}(\omega)\,(-i\omega)^3\, e^{i(\omega-\omega')t}\, dt\, \frac{d\omega'}{2\pi}\frac{d\omega}{2\pi}$$

$$= \frac{G}{8\pi c^5}\Lambda_{kl|mn}(\hat{\mathbf{n}}) \int\int \tilde{Q}_{kl}(-\omega')\,\left(i\omega'\right)^3 \tilde{Q}_{mn}(\omega)\,(-i\omega)^3\, 2\pi\delta(\omega'-\omega)\, \frac{d\omega'}{2\pi}\frac{d\omega}{2\pi}$$

$$= \frac{G}{8\pi c^5}\Lambda_{mn|kl}(\hat{\mathbf{n}}) \int \tilde{Q}_{mn}(\omega)\,(-i\omega)^3 (i\omega)^3\, \tilde{Q}_{kl}(-\omega)\, \frac{d\omega}{2\pi}$$

$$= \frac{G}{8\pi c^5}\Lambda_{mn|kl}(\hat{\mathbf{n}}) \int \tilde{Q}_{mn}(\omega)\omega^6 \tilde{Q}^*_{kl}(\omega)\, \frac{d\omega}{2\pi} \quad , \tag{5.136}$$

where all the previous integral symbols have limits from $-\infty$ to ∞ and $\tilde{Q}^*_{kl}(\omega) = \tilde{Q}_{kl}(-\omega)$. The integral in (5.136) is twice the same integral from 0 to ∞. Thus,

$$\boxed{\left(\frac{dE}{d\Omega}\right)_{\text{quad}} = \frac{G}{8\pi^2 c^5}\Lambda_{mn|kl}(\hat{\mathbf{n}}) \int_0^{\infty} \tilde{Q}_{mn}(\omega)\omega^6 \tilde{Q}^*_{kl}(\omega)\, d\omega} \tag{5.137}$$

Finally, integrating equation (5.137) over the whole solid angle gives the total radiated energy

$$E_{\text{quad}} = \frac{G}{8\pi^2 c^5}\int \Lambda_{mn|kl}(\hat{\mathbf{n}})\, d\Omega \int_0^{\infty} \tilde{Q}_{mn}(\omega)\omega^6 \tilde{Q}^*_{kl}(\omega)\, d\omega$$

$$= \frac{G}{8\pi c^5}\int_0^{\infty} \frac{2}{15}\left[11\tilde{Q}_{mn}(\omega)\omega^6\tilde{Q}^*_{mn}(\omega) + \tilde{Q}_{mn}(\omega)\omega^6\tilde{Q}^*_{mn}(\omega)\right] d\omega \tag{5.138}$$

Hence,

$$\boxed{E_{\text{quad}} \; = \; \frac{G}{5\pi c^5} \int_0^\infty \tilde{Q}_{mn}(\omega)\omega^6 \tilde{Q}^*_{mn}(\omega)\, d\omega} \quad . \tag{5.139}$$

Therefore, the spectral energy is

$$\boxed{\left(\frac{dE}{d\omega}\right)_{\text{quad}} \; = \; \tilde{Q}_{mn}(\omega)\omega^6 \tilde{Q}^*_{mn}(\omega)} \quad . \tag{5.140}$$

5.8 Conservation of $T^{\mu\nu}$ for a closed system of particles

This section shows the conservation of the energy-momentum tensor $T^{\mu\nu}$ for a closed system of particles. We first construct the tensor $T^{\mu\nu}$ assuming an isolated set of non-interacting particles in flat spacetime. The four-momentum density of the system is defined as [36]

$$\frac{T^{0\nu}(\mathbf{x},t)}{c} \; \equiv \; \sum_{a=1}^{n} m_a\, u^\nu_a(\tau)\, \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{5.141}$$

$$= \; \sum_{a=1}^{n} \left[\gamma_a m_a \frac{dx^\nu_a(t)}{dt}\right] \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))$$

$$= \; \sum_{a=1}^{n} p^\nu_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \quad , \tag{5.142}$$

where the four-momentum of a particle is given by

$$p^\nu \; = \; m u^\nu = m\frac{dx^\nu}{d\tau} = m\frac{dx^\nu}{dt}\frac{dt}{d\tau} = \gamma m \frac{dx^\nu}{dt} = (E/c\, ,\, \mathbf{p}) \quad . \tag{5.143}$$

On the other hand, the current density of the four-momentum reads

$$T^{\mu i}(\mathbf{x},t) \; \equiv \; \sum_{a=1}^{n} p^\mu_a(t) \frac{dx^i_a(t)}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{5.144}$$

$$= \; \sum_{a=1}^{n} \left[\gamma_a m_a \frac{dx^\mu_a(t)}{dt}\right] \frac{dx^i_a(t)}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \quad . \tag{5.145}$$

The equations (5.142) and (5.144) can be rewritten in a compact form as

$$T^{\mu\nu}(\mathbf{x},t) \; = \; \sum_{a=1}^{n} \frac{p^\mu_a p^\nu_a}{\gamma_a m_a} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{5.146}$$

$$= \; \sum_{a=1}^{n} \gamma_a m_a \frac{dx^\mu_a}{dt}\frac{dx^\nu_a}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \quad . \tag{5.147}$$

In particular, the components T^{00} and T^{0i} are

$$T^{00}(\mathbf{x},t) \;=\; \sum_{a=1}^{n}\gamma_a m_a c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \tag{5.148}$$

$$T^{0i}(\mathbf{x},t) \;=\; \sum_{a=1}^{n}\gamma_a m_a c\dot{x}^i \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \tag{5.149}$$

Is important to recall that the energy-momentum tensor is conserved only if each of the free particles of the isolated system moves along geodesics in the flat spacetime. This can be seen by taking the divergence of equation (5.144).

$$\begin{aligned}
\partial_i T^{\mu i}(\mathbf{x},t) \;&=\; \sum_{a=1}^{n} p^{\mu}_a(t)\frac{dx^i_{\ a}(t)}{dt}\frac{\partial}{\partial x^i}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \\
&=\; -\sum_{a=1}^{n} p^{\mu}_a(t)\left(\frac{dx^i_{\ a}(t)}{dt}\frac{\partial}{\partial x^i_{\ a}}\right)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \;\;,\;\; \left[\frac{\partial}{\partial x^i}\to-\frac{\partial}{\partial x^i_{\ a}}\right] \\
&=\; -\sum_{a=1}^{n} p^{\mu}_a(t)\frac{\partial}{\partial t}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \qquad , \qquad \frac{\partial}{\partial t}\equiv\left(\frac{dx^i_{\ a}(t)}{dt}\frac{\partial}{\partial x^i_{\ a}}\right) \;\; .
\end{aligned} \tag{5.150}$$

Furthermore, notice that

$$\begin{aligned}
\frac{\partial}{\partial t}\left[p^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\right] \;&=\; \dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \;+\; p^{\mu}_a(t)\frac{\partial}{\partial t}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \\
p^{\mu}_a(t)\frac{\partial}{\partial t}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \;&=\; \frac{\partial}{\partial t}\left[p^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\right] \;-\; \dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))
\end{aligned} \tag{5.151}$$

and substituting this result in the expression (5.150) one obtains,

$$\begin{aligned}
\partial_i T^{\mu i}(\mathbf{x},t) \;&=\; -\sum_{a=1}^{n}\left\{\frac{\partial}{\partial t}\left[p^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\right] \;-\; \dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\right\} \\
\partial_i T^{\mu i}(\mathbf{x},t) \;&=\; -\frac{\partial}{\partial t}\sum_{a=1}^{n} p^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \;+\; \sum_{a=1}^{n}\dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \\
\partial_i T^{\mu i}(\mathbf{x},t) \;&=\; -\frac{1}{c}\frac{\partial}{\partial t}T^{\mu 0}(\mathbf{x},t) \;+\; \sum_{a=1}^{n}\dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \;\; .
\end{aligned}$$

Thus, the 4-divergence of the energy-momentum tensor may be written as

$$\partial_0 T^{\mu 0}(\mathbf{x},t) \;+\; \partial_i T^{\mu i}(\mathbf{x},t) \;=\; \underbrace{\sum_{a=1}^{n}\dot{p}^{\mu}_a(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))}_{\text{Force Density}} \tag{5.152}$$

From the previous equation is clear that only if $\dot{p}^{\mu}_a(t) = 0$ for all a then

$$\boxed{\partial_\nu T^{\mu\nu}(\mathbf{x},t) \;=\; 0} \;\; . \tag{5.153}$$

The statement $\dot{p}^{\mu}_{a}(t) = 0$ means that the particles follow geodesics in the flat spacetime, i.e, straight lines. Therefore, in principle is not possible to estimate the energy-momentum tensor for a set of interacting particles only just with the expression (5.146). To ensure the full conservation of the energy-momentum tensor one should include terms that describe all types of interactions and also external forces that change the geodesic trajectories of the particles. A priori, is not valid to use a general arbitrary trajectory $\mathbf{x}(t)$ and inserting it into (5.146) because the energy-momentum wolud not be conserved. For instance, if a single particle follows a trajectory that is not a geodesic of the flat spacetime then there must be and external force which needs to be included in the full definition of the energy-momentum tensor of the system to guarantee the conservation equation (5.153). If the force is the electromagnetic force, then the term at the right hand side of equation (5.152) does not vanish and thus the conservation ot the total energy-momentum of the system is

$$\partial_\nu \left(T^{\mu\nu} + T^{\mu\nu}_{\mathsf{EM}} \right) = 0 \quad . \tag{5.154}$$

In the next chapter we will discuss the construction of the energy-momentum tensor for a binary system which is gravitational bound at linear order in v/c.

Chapter 6

Newtonian Binary System

The goal of this chapter is to discuss the quadrupole radiation of GWs due to the accelerate motion of compact objects in a binary system. As a first approach, the objects will be treated as point particles in mutual Newtonian gravitational interaction. The formalism developed in Chapter 5 would be extended for a binary source in which each particle does not follows a geodesic in the flat spacetime. First off, the emitted gravitational radiation for a fix circular orbit motion is considered. Then, the *back-reaction* of GWs will be included by means of the energy balance equation which represents the energy loss that causes the circular radius to skrink and the orbital frecuency to increase. At the end, the waveforms for the h_+ and $h_\times$ polarizations are obtained during the inspiral phase of the binary at Newtonian order until the plunge of the particles is reached.

6.1 The *effective* one-body problem

The problem of finding the motion of two interacting point masses under the influence of a mutual central force is equivalent to solve the problem of two fictitious particles with masses $m = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$. To see this, let be $\mathbf{x}_1$ and $\mathbf{x}_2$ the vector positions of the particles with masses m_1 and m_2 as depicted in figure (6.1). The relative coordinate that points from m_1 to m_2 is given by

$$\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1 \tag{6.1}$$

and the position of the center of mass of the system is

$$\mathbf{x}_{\rm CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \quad . \tag{6.2}$$

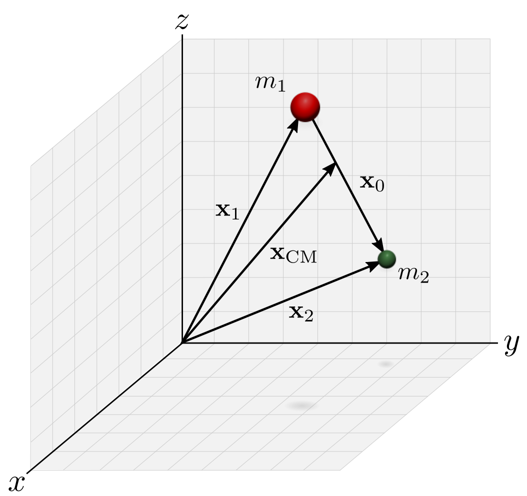

Figure 6.1: The two-body problem diagram

The position vectors of the particles $\mathbf{x}_1$ and $\mathbf{x}_2$ may be expressed in terms of the relative coordinate $\mathbf{x}_0$ and the position of the center of mass $\mathbf{x}_{\rm CM}$ by using equations (6.1) and (6.2),

$$\mathbf{x}_1 = \mathbf{x}_{\rm CM} - \frac{m_2}{m_1+m_2}\mathbf{x}_0 \qquad , \qquad \mathbf{x}_2 = \mathbf{x}_{\rm CM} + \frac{m_1}{m_1+m_2}\mathbf{x}_0 \ . \tag{6.3}$$

In terms of the six generalized coordinates from $\mathbf{x}_0$ and $\mathbf{x}_{\rm CM}$ the lagrangian is expressed as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m_1\dot{\mathbf{x}}_1^2 \ + \ \frac{1}{2}m_2\dot{\mathbf{x}}_2^2 \ - U(|\mathbf{x}_2-\mathbf{x}_1|) \\ &= \frac{1}{2}(m_1+m_2)\dot{\mathbf{x}}_{\rm CM}^2 \ + \ \frac{1}{2}\Big(\frac{m_1 m_2}{m_1+m_2}\Big)\dot{\mathbf{x}}_0^2 \ - \ U(|\mathbf{x}_0|) \\ &= \frac{1}{2}m\dot{\mathbf{x}}_{\rm CM}^2 \ + \ \frac{1}{2}\mu\dot{\mathbf{x}}_0^2 \ - \ U(|\mathbf{x}_0|)\end{aligned} \tag{6.4}$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\Big) \ = \ \frac{\partial\mathcal{L}}{\partial x^a} \ . \tag{6.5}$$

For $\mathbf{x}_{\rm CM}$ we get

$$\frac{d}{dt}\left[m\dot{\mathbf{x}}_{\rm CM}\right] \ = \ 0 \qquad \Longrightarrow \qquad \dot{\mathbf{x}}_{\rm CM} = \text{constant} \tag{6.6}$$

and the center of mass of the system is at rest or moves with constant velocity. Using the Newtonian potential energy the lagrangian becomes

$$\mathcal{L} \ = \ \text{constant} \ + \ \frac{1}{2}\mu\dot{\mathbf{x}}_0^2 + \ G\frac{\mu m}{|\mathbf{x}_0|} \ . \tag{6.7}$$

An additional constant in the lagrangian does not affect the equations of motion. Therefore, the problem is to find and solve the equations of motion for the fictitious particle of reduced mass μ under a central gravitational potential generated by another fictitious particle of mass $m = m_1+m_2$. In particlar, in the center of mass reference frame $\mathbf{x}_{\rm CM} = 0$ and the lagrangian takes the form

$$\mathcal{L} \ = \ \frac{1}{2}\mu\dot{\mathbf{x}}_0^2 + \ G\frac{\mu m}{|\mathbf{x}_0|} \ . \tag{6.8}$$

Solving the Euler-Lagrange equations for the relative coordinate, the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ can be obtained by using the equations in (6.3).

6.2 Radiation from sources with non-negligible self gravity

In previous chapters, it was emphasized that the formalism that describes the generation of GWs in linearized theory cannot be applied to systems that are gravitationally bound. The reason is that the conservation of the energy-momentum tensor for the matter sources is given by the expression $\partial_\nu T^{\mu\nu} = 0$. As it was shown in Chapter 5, this conservation statement implies that for an isolated and free system of particles, each one of these are forced to move along geodesics of the flat spacetime. For a binary system, the particles are bound together by the mutual gravitational interaction and thus the particles do not follow geodesic motions[1]. In order to take into account sources with non-negligible self gravity as in the case of binary

[1]This issue was first pointed out by Eddington.

systems, one must rewrite the EFE in the Landau-Lifshitz formulation. With a suitable global gauge fixing, the EFE reduces to a nonlinear wave-like equation called the relaxed EFE.

The Landau-Lifshitz formulation of general relativity

Hereunder a very concise summary of the EFE in the Landau-Lifshitz formulation is presented. The framework of this development is based on a new quantity called the *gothic inverse metric* and is defined as[2]

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \quad , \tag{6.9}$$

with g the metric determinant and $g^{\mu\nu}$ the inverse metric[3]. From the definition in (6.9) a new object is built as follows,

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \quad . \tag{6.10}$$

An essential identity in this formulation is given by [9]

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{\mathrm{LL}} \quad , \tag{6.11}$$

where $G^{\alpha\beta}$ is the Einstein tensor and $(-g)t^{\alpha\beta}_{\mathrm{LL}}$ is the Landau-Lifshitz pseudotensor [45],

$$\begin{aligned}(-g)t^{\alpha\beta}_{\mathrm{LL}} := \frac{c^4}{16\pi G}\Big\{ & \partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu g^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\ & - g^{\alpha\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\ & + \frac{1}{8}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)\left(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\right)\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma}\Big\} \quad . \end{aligned} \tag{6.12}$$

The identity (6.11) implies that the EFE can be rewritten as

$$\boxed{\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right) \quad .} \tag{6.13}$$

The right-hand side of the previous equation suggest that $t^{\alpha\beta}_{\mathrm{LL}}$ represent the gravitational field pseudotensor which is added to the matter distribution energy-momentum tensor[4]. Furthermore, from the definition of $H^{\alpha\mu\beta\nu}$ in (6.10) clearly $H^{\alpha\mu\beta\nu} = -H^{\alpha\mu\nu\beta}$. Using the fact that partial derivatives commute one obtains,

$$\partial_\beta \partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 0 \quad . \tag{6.14}$$

Applying the partial derivative at both sides of equation (6.13) then,

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right)\right] = 0 \tag{6.15}$$

and this equation is the conservation of the total energy-momentum pseudotensor which is equivalent to $\nabla_\beta T^{\alpha\beta} = 0$[5].

[2] Tensor quantities that have a prefactor of $\sqrt{-g}$ are known as tensor densities and transform diferent.

[3] Note that from equation (6.9) if the gothic inverse metric is known, then the metric itself may be determined. This is due to the relation $\det(\mathfrak{g}^{\mu\nu}) = g$.

[4] Remind that the quantity $t^{\alpha\beta}_{\mathrm{LL}}$ does not transform as a tensor. Likewise, the quantity at the left-hand side of equation (6.13) is also a pseudotensor.

[5] Actually, this equation is more fundamental than (6.15) in the sense that it is independent of the validity of GR [9].

The relaxed Einstein's Field Equations

The previous development is an exact reformulation of general relativity. Now, if a new potential is defined as [1]

$$\mathsf{h}^{\alpha\beta} := \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta} \quad , \tag{6.16}$$

where $\eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ and $\mathfrak{g}^{\alpha\beta}$ follows from equation (6.9). The EFE in the Landau-Lifshitz formulation takes a nonlinear wave-like form with a suitable global coordinate condition. To see this, such a condition is introduced,

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0 \quad . \tag{6.17}$$

This is a coordinate constraint and is called the *harmonic* or *deDonder* gauge. Observe that in terms of the potential in (6.16), the expression in (6.17) reads

$$\partial_\beta \mathsf{h}^{\alpha\beta} = 0 \quad . \tag{6.18}$$

Using the harmonic gauge and the definition of the potential $\mathsf{h}^{\alpha\beta}$ it is obtained

$$\begin{aligned}
\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} &= \partial_\mu\partial_\nu \left(\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}\right) \\
&= \partial_\mu \left(\mathfrak{g}^{\mu\nu}\partial_\nu\mathfrak{g}_{\alpha\beta} + \mathfrak{g}^{\alpha\beta}\partial_\nu\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\beta\mu}\partial_\nu\mathfrak{g}^{\alpha\nu} - \mathfrak{g}^{\alpha\nu}\partial_\nu\mathfrak{g}^{\beta\mu}\right) \\
&= \partial_\mu\mathfrak{g}^{\mu\nu}\partial_\nu\mathfrak{g}^{\alpha\beta} + \mathfrak{g}^{\mu\nu}\partial_\mu\partial_\nu\mathfrak{g}^{\alpha\beta} - \partial_\mu\mathfrak{g}^{\alpha\nu}\partial_\nu\mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\nu}\partial_\mu\partial_\nu\mathfrak{g}^{\beta\mu} \\
&= \mathfrak{g}^{\mu\nu}\partial_\mu\partial_\nu\mathfrak{g}^{\alpha\beta} - \partial_\mu\mathfrak{g}^{\alpha\nu}\partial_\nu\mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\nu}\partial_\mu\partial_\nu\mathfrak{g}^{\beta\mu} \\
&= -\left(\eta^{\mu\nu} - \mathsf{h}^{\mu\nu}\right)\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} - \partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} + \left(\eta^{\alpha\nu} - \mathsf{h}^{\alpha\nu}\right)\partial_\mu\partial_\nu\mathsf{h}^{\beta\mu} \\
&= -\Box\mathsf{h}^{\alpha\beta} + \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} - \partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} + \eta^{\alpha\nu}\partial_\nu\partial_\mu\mathsf{h}^{\beta\mu} \\
&= -\Box\mathsf{h}^{\alpha\beta} + \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} - \partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} \quad .
\end{aligned} \tag{6.19}$$

Thus, the EFE in the Landau-Lifshitz formulation becomes [9, 46, 47],

$$-\Box\mathsf{h}^{\alpha\beta} + \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} - \partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} = \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right) \tag{6.20}$$

or,

$$\Box\mathsf{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right) + \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} - \partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} \quad . \tag{6.21}$$

Defining $(-g)t^{\alpha\beta}_{\mathrm{H}}$ as

$$(-g)t^{\alpha\beta}_{\mathrm{H}} := \frac{c^4}{16\pi G}\left(\partial_\mu\mathsf{h}^{\alpha\nu}\partial_\nu\mathsf{h}^{\beta\mu} - \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta}\right) \tag{6.22}$$

the relaxed EFE are

$$\boxed{\Box\mathsf{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}} \tag{6.23}$$

where $\tau^{\alpha\beta}$ is the effective energy-momentum pseudotensor,

$$\tau^{\alpha\beta} := (-g)\left(T^{\alpha\beta}[\mathsf{m}, g] + t^{\alpha\beta}_{\mathrm{LL}}[\mathsf{h}] + t^{\alpha\beta}_{\mathrm{H}}[\mathsf{h}]\right) \quad . \tag{6.24}$$

The wave-like equation in (6.23) is still equivalent to the exact Einstein equations but rewritten in terms of the potential $\mathsf{h}^{\alpha\beta}$ and using the harmonic gauge. One might wonder if this gauge can always be imposed over some reference frame to get the expression (6.23). To prove that indeed this gauge exist first consider a system in which $\partial_\beta \mathfrak{g}^{\alpha\beta} \neq 0$. By making a transformation to a new coordinates $x'^\mu = f^\mu(x^\alpha)$, then in the new system we have [9]

$$\partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \sqrt{-g'} \Box_g f^\mu(x^\alpha) \qquad , \qquad \Box_g := g^{\mu\nu}\nabla_\mu \nabla_\nu \ . \tag{6.25}$$

Choosing each of the four scalar functions f^μ to be harmonic, then $\Box_g f^\mu = 0$ and the harmonic gauge will holds in the new coordinate system. Thus, in harmonic coordinates the EFE in the Landau-Lifshitz formalism are given by equation (6.23).
An interesting coincidence between the Landau-Lifshitz formalism and linearized theory in the weak field limit will allow us to extend the derivation of the quadrupole formula to sources with non-negligible self-gravity [1, 32, 34]. The first thing to point out is that the definition of the potential $\mathsf{h}^{\alpha\beta}$ in equation (6.16) is the same definition of the trace-reversed of the perturbation tensor in linearized theory for weak fields [8]. So far, no approximations have been made. But if one considers weak fields, the metric is defined as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| << 1$. The weak field limit expansion of the term $\sqrt{-g}$ that is involved in the definition (6.16) can be computed as follows. First, write the metric as

$$g^\alpha_\beta = \delta^\alpha_\beta + h^\alpha_\beta \equiv (I + H)^\alpha_\beta \ . \tag{6.26}$$

Note that $g_{\alpha\beta} = \eta_{\alpha\rho} g^\rho_\beta$ and so

$$\begin{aligned} -\det(g_{\alpha\beta}) &= -\det\Big(\eta_{\alpha\rho} g^\rho_\beta\Big) \\ -g &= -\det(\eta_{\alpha\rho}) \det\Big(g^\rho_\beta\Big) \\ -g &= \det(I+H)^\alpha_\beta \end{aligned} \tag{6.27}$$

For a non-degenerate matrix A the identity $\log\det(A) = Tr(\log A)$ holds. Furthermore, if A is sufficiently close to the identity matrix, then

$$\log A = (A - I) - \frac{(A-I)^2}{2} + \frac{(A-I)^3}{3} - \cdots \tag{6.28}$$

Hence, the right-hand side of (6.27) may be expanded as,

$$\begin{aligned} -g &= \exp\{\log\det(I+H)\} \\ &= \exp\{\mathrm{Tr}\log(I+H)\} && \text{Using the identity} \\ &= \exp\{\mathrm{Tr}\left[H + \mathcal{O}(H^2)\right]\} && \text{Expanding } \log A \\ &= 1 + h + \mathcal{O}(h^2) && \text{Expanding } e^x \end{aligned} \tag{6.29}$$

and thus $\sqrt{-g} = 1 + \frac{1}{2}h + \mathcal{O}(h^2)$. Then, the potential $\mathsf{h}^{\alpha\beta}$ that is defined in 6.16 coincides with the ususal definition of the trace-reversed perturbation tensor in linearized theory as is shown,

$$
\begin{aligned}
\mathsf{h}^{\alpha\beta} &= \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta} \\
&= \eta^{\alpha\beta} - \left(1 + \frac{1}{2}h\right)\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) + \mathcal{O}(h^2) \\
&= \eta^{\alpha\beta} - \left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) - \frac{1}{2}h\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right) + \mathcal{O}(h^2) \\
&= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h + \mathcal{O}(h^2) \\
&= \bar{h}^{\alpha\beta} + \mathcal{O}(h^2)
\end{aligned} \tag{6.30}
$$

Observe that after inserting the potential $\mathsf{h}^{\alpha\beta}$ in (6.30) into (6.18), the Lorenz gauge of linearized theory is recovered at linear order,

$$\partial_\beta \bar{h}^{\alpha\beta} = 0 \ . \tag{6.31}$$

Finally, the relaxed Einstein equations at leading order in the weak gravitational field limit yields,

$$\boxed{\Box \bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}\left(T^{\alpha\beta} + t_{\rm B}^{\alpha\beta}\right)} \tag{6.32}$$

where $t_{\rm B}^{\alpha\beta} = t_{\rm LL}^{\alpha\beta} + t_{\rm H}^{\alpha\beta}$. By virtue of the harmonic gauge condition, the effective energy-momentum pseudotensor $\tau^{\alpha\beta}$ is conserved. This can be seen after taking the 4-divergence of the wave-like equation in (6.23),

$$\partial_\beta \tau^{\alpha\beta} = 0 \ . \tag{6.33}$$

The same argument for the weak field limit in equation (6.32) gives,

$$\partial_\beta \left(T^{\alpha\beta} + t_{\rm B}^{\alpha\beta}\right) = 0 \ . \tag{6.34}$$

This conservation equation implies that the particles in a gravitationally bound system no longer moves along geodesics of the flat spacetime[6]. Indeed, particles follow non-geodesic trajectories due to the gravitational interaction that is encoded in $t_{\rm B}^{\alpha\beta}$. Recall that to derive the quadrupole formula in linearized theory one uses the Lorenz gauge, the wave equation and energy momentum tensor conservation equation. In the weak field limit of the relaxed Einstein equations, the same expressions were recovered, except for the important fact that the energy-momentum tensor $T^{\alpha\beta}$ is replaced by $T^{\alpha\beta} + t_{\rm B}^{\alpha\beta}$. Then, the mass density moment M^{ij} that is involved in the quadrupole formula changes to

$$M^{ij}(t) = \frac{1}{c^2}\int \left[T^{00}(t,\mathbf{x}) + t_{\rm B}^{00}(t,\mathbf{x})\right] x^i x^j d^3x \ . \tag{6.35}$$

Roughly speaking the term $t_{\rm B}^{00}$ represents the gravitational binding energy between the particles [1]. Nevertheles, for weak fields this term is negligible in comparison with T^{00} which describes the rest mass distribution of the system. Thus, one is left with the same definition

[6] Compare this conservation equation with the one in linearized theory $\partial_\beta T^{\alpha\beta} = 0$.

for M^{ij} mentioned before in linearized theory,

$$M^{ij}(t) \backsimeq \frac{1}{c^2}\int T^{00}(t,\mathbf{x})x^i x^j\, d^3x \quad . \tag{6.36}$$

In conclusion, the quadrupole formula is still valid in the more general situation for gravitationally binding objects[7]. But is important to insist that this is possible due to the strong analogy between the weak field limit at linear order of the relaxed EFE and linearized theory.

6.3 The mass density moment M^{ij} of a binary system

The mass density moment for a binary system can be computed assuming that the binding gravitational energy between the point particles is negligible in comparison with T^{00}. From equation (6.36) one has

$$\begin{aligned} M^{ij} &= \frac{1}{c^2}\int T^{00}(t,\mathbf{x})x^i x^j\, d^3x \\ &= \frac{1}{c^2}\int \Big[m_1 c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_1(t)) \,+\, m_2 c^2\delta^{(3)}(\mathbf{x}-\mathbf{x}_2(t))\Big]x^i x^j\, d^3x \\ &= m_1 x_1^i(t)x_1^j(t) \,+\, m_1 x_2^i(t)x_2^j(t) \quad . \end{aligned} \tag{6.37}$$

This relation can be expressed in terms of the relative coordinate of the binary $\mathbf{x}_0$ and the coordinate position of the center of mass of the system $\mathbf{x}_{\mathrm{CM}}$ as follows,

$$\begin{aligned} M^{ij}(t) &= m_1 x_1^i(t)x_1^j(t) \,+\, m_2 x_2^i(t)x_2^j(t) \\ &= m_1\Big(x^i_{\mathrm{CM}} - \frac{m_2}{m}x_0^i\Big)\Big(x^j_{\mathrm{CM}} - \frac{m_2}{m}x_0^j\Big) + m_2\Big(x^i_{\mathrm{CM}} + \frac{m_1}{m}x_0^i\Big)\Big(x^j_{\mathrm{cm}} + \frac{m_1}{m}x_0^j\Big) \\ &= m_1 x^i_{\mathrm{CM}}x^j_{\mathrm{CM}} \,-\, \mu x^i_{\mathrm{CM}}x_0^j \,-\, \mu x_0^i x^j_{\mathrm{CM}} \,+\, \mu\frac{m_2}{m}x_0^i x_0^j + \\ &\quad m_2 x^i_{\mathrm{CM}}x^j_{\mathrm{CM}} \,+\, \mu x^i_{\mathrm{CM}}x_0^j \,+\, \mu x_0^i x^j_{\mathrm{CM}} \,+\, \mu\frac{m_1}{m}x_0^i x_0^j \\ &= (m_1+m_2)x^i_{\mathrm{CM}}x^j_{\mathrm{CM}} \,+\, \mu\frac{(m-m_1)}{m}x_0^i x_0^j \,+\, \mu\frac{m_1}{m}x_0^i x_0^j \\ &= m x^i_{\mathrm{CM}}x^j_{\mathrm{CM}} \,+\, \mu x_0^i x_0^j \quad . \end{aligned} \tag{6.38}$$

Remind that the quadrupole radiation approximation depends on the second derivative of the mass density moment. From equation (6.38) is clear that if the the system is isolated, then $\mathbf{x}_{\mathrm{CM}}$ is moving at constant velocity and the first term does not contributes to the production of GWs. In the center of mass reference frame the previous expression turns out to be

$$\boxed{M^{ij} = \mu x_0^i(t)x_0^j(t)} \tag{6.39}$$

and the mass density that give rise to (6.39) reads,

$$\rho(\mathbf{x},t) = \mu\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t)) \quad . \tag{6.40}$$

[7] Formally, to obtain the quadrupole formula from the relaxed EFE see references [9, 48] for discussion.

Circular orbits

Consider a binary system with masses m_1 and m_2 in circular orbits. This means that each particle follows a circular motion around the barycenter of the system. As was discussed before, the two-body problem of the real particles is equivalent to the effective one-body problem of two fictitious particles of masses μ and m as shown in figure 6.2. At the center of mass reference frame and by using equations (6.3), the particles of mass m_1 and m_2 moves in circles of radius

$$r_1 = \frac{m_2}{m_1 + m_2} R \qquad , \qquad r_2 = \frac{m_1}{m_1 + m_2} R \tag{6.41}$$

where $r_1 = |\mathbf{x}_1|$, $r_2 = |\mathbf{x}_2|$ and $R = |\mathbf{x}_0|$. Of course the radius of the particle of mass m_2 will be always greater than the radius of the circle followed by the particle with mass m_1 provided that $m_2 < m_1$[8]. The problem translates to the trivial motion of a fictitious particle of mass $m = m_1 + m_2$ which is at rest or moving with constant velocity, and another fictitious particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_1}$ moving around the particle of mass m in a circle of radius $R = |\mathbf{x}_0|$. Due to the conservation of angular momentum, the motion of the real particles occurs in a plane. Let this plane to be the (x, y) plane, then the fictitious particle of mass μ experiences a circular motion with coordinates

$$\begin{cases} x_0(t) &= R\cos\left(\omega_{\mathrm{s}} t + \frac{\pi}{2}\right) \\ y_0(t) &= R\sin\left(\omega_{\mathrm{s}} t + \frac{\pi}{2}\right) \\ z_0(t) &= 0 \quad , \end{cases} \tag{6.42}$$

where ω_{s} is the angular frequency of the binary, R is the separation between the particles and an initial phase of $\frac{\pi}{2}$ has been taken for convenience.

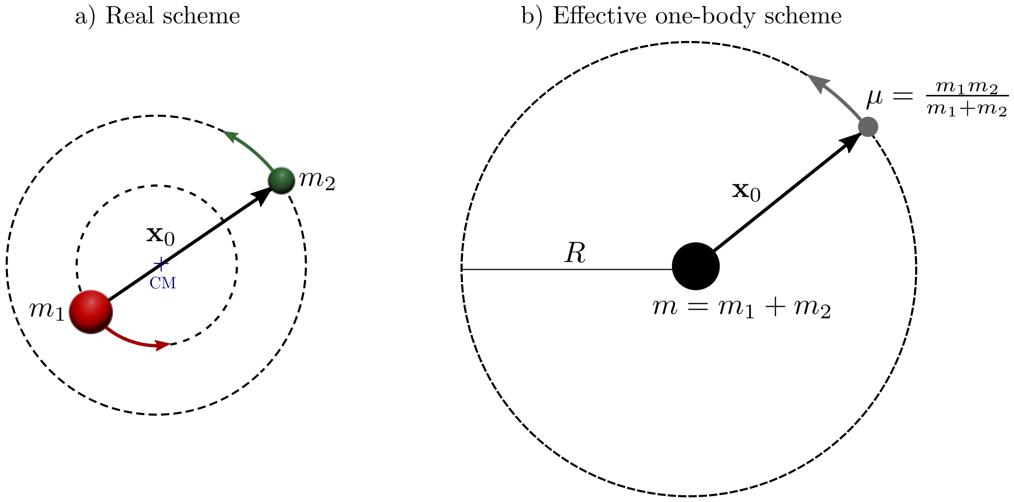

Figure 6.2: a) A real binary system moving in circular orbits. The relative coordinate always passes through the center of mass position and perform a circular path of radius $R = |\mathbf{x}_0|$. The particles of masses m_1 and m_2 follow circular motions of radius $r_1 = \frac{m_2}{m_1+m_2}R$ and $r_2 = \frac{m_1}{m_1+m_2}R$, respectively. b) An effective one-body representation of the real two body problem for circular orbits. The fictitious particle with reduced mass μ follows a circular motion due to the potential generated by another fictitious particle of mass m which is fixed at the center of the circle of radius R or at least is moving with constant velocity.

[8]From equations (6.41) we have $r_2 = \frac{m_1}{m_2} r_1$.

In the center of mass reference frame, the mass density moment can be computed by using equations (6.39) and (6.42). Each component of the tensor M^{ij} gives,

$$\begin{aligned} M^{11}(t) &= \mu R^2 \cos^2(\omega_\mathrm{s} t + \tfrac{\pi}{2}) & M^{22}(t) &= \mu R^2 \sin^2(\omega_\mathrm{s} t + \tfrac{\pi}{2}) \\ &= \mu R^2 \left[\frac{1+\cos(2\omega_\mathrm{s} t + \pi)}{2}\right] & &= \mu R^2 \left[\frac{1-\cos(2\omega_\mathrm{s} t + \pi)}{2}\right] \\ &= \mu R^2 \left[\frac{1-\cos(2\omega_\mathrm{s} t)}{2}\right] , & &= \mu R^2 \left[\frac{1+\cos(2\omega_\mathrm{s} t)}{2}\right] , \\ M^{12}(t) &= \mu R^2 \sin\big(\omega_\mathrm{s} t + \tfrac{\pi}{2}\big)\cos\big(\omega_\mathrm{s} t + \tfrac{\pi}{2}\big) & M^{13}(t) &= M^{31}(t) = 0 , \\ &= \frac{1}{2}\mu R^2 \sin(2\omega_\mathrm{s} t + \pi) & M^{23}(t) &= M^{32}(t) = 0 , \\ &= -\frac{1}{2}\mu R^2 \sin(2\omega_\mathrm{s} t) = M^{21}(t) , & M^{33}(t) &= 0 . \end{aligned} \tag{6.43}$$

The quadrupole radiation depends on the second derivative with respect to time of the mass density moment. Thus, the non-vanishing contributions to the emission of GWs come from

$$\ddot{M}^{11}(t) = 2\mu R^2 \omega_\mathrm{s}^2 \cos(2\omega_\mathrm{s} t) , \tag{6.44}$$

$$\ddot{M}^{22}(t) = -2\mu R^2 \omega_\mathrm{s}^2 \cos(2\omega_\mathrm{s} t) , \tag{6.45}$$

$$\ddot{M}^{12}(t) = 2\mu R^2 \omega_\mathrm{s}^2 \sin(2\omega_\mathrm{s} t) . \tag{6.46}$$

6.4 Quadrupole radiation from a binary in circular motion

The quadrupole radiation for a binary system moving in circular orbits is obtained by evaluating the expressions (6.44), (6.45) and (6.46) at the instant $t_\mathrm{r} = t - \frac{r}{c}$ and inserting them into the equations for h_+ and $h_\times$ in (5.121) and (5.122). Since the motion of the particles is restricted to the (x, y) plane there is no mass distribution along the z-direction and all components of M^{ij} with $i, j = 3$ are zero. This simplifies the equations for h_+ and $h_\times$. The radiation for the *plus* polarization yields,

$$\begin{aligned} h_+(t;\theta,\phi) &= \frac{1}{r}\frac{G}{c^4}\Big[\ddot{M}^{11}\Big(\cos^2\phi - \sin^2\phi\cos^2\theta\Big) + \ddot{M}^{22}\Big(\sin^2\phi - \cos^2\phi\cos^2\theta\Big) \\ &\qquad - \ddot{M}^{12}\sin(2\phi)(1+\cos^2\theta)\Big]_{t_\mathrm{r}} \\ &= \frac{1}{r}\frac{G}{c^2}\Big[\ddot{M}^{11}\Big(\cos^2\phi - \cos^2\theta\sin^2\phi - \sin^2\phi + \cos^2\theta\cos^2\phi\Big) \\ &\qquad - \ddot{M}^{12}\sin(2\phi)\big(1+\cos^2\theta\big)\Big]_{t_\mathrm{r}} \\ &= \frac{1}{r}\frac{G}{c^4}\Big[\ddot{M}^{11}\Big(\cos^2\phi\big(1+\cos^2\theta\big) - \sin^2\phi\big(1+\cos^2\theta\big) - \ddot{M}^{12}\sin(2\phi)\big(1+\cos^2\theta\big)\Big]_{t_\mathrm{r}} \end{aligned}$$

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\big(1+\cos^2\theta\big)\Big[\ddot{M}^{11}\big(\cos^2\phi - \sin^2\phi\big) - \sin(2\phi)\ddot{M}^{12}\Big]_{t_\mathrm{r}}$$

Using the explicit functions for $\ddot{M}^{11}$ and $\ddot{M}^{12}$ from equations (6.44) and (6.46) we get

$$\begin{aligned} h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\left(1+\cos^2\theta\right)\Big[& 2\mu R^2\omega_s^2\cos(2\omega_s t_r)\left(\cos^2\phi-\sin^2\phi\right) \\ & - \sin(2\phi)2\mu R^2\omega_s^2\sin(2\omega_s t_r)\Big] \\ = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\Big[& \cos(2\omega_s t_r)\left(\cos^2\phi-\sin^2\phi\right) \\ & - \sin(2\phi)\sin(2\omega_s t_r)\Big] \\ = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\Big[& \cos(2\omega_s t_r)\cos(2\phi) - \sin(2\phi)\sin(2\omega_s t_r)\Big] \end{aligned}$$

i.e.,

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\Big[\cos(2\omega_s t_r + 2\phi)\Big] \quad . \tag{6.47}$$

Similarly, the *cross* polarization is

$$\begin{aligned} h_\times(t;\theta,\phi) &= \frac{1}{r}\frac{G}{c^4}\Big[4\mu\omega_s^2R^2\cos(2\omega_s t_r)\sin(2\phi)\cos\theta \\ &\quad + 4\mu\omega_s^2R^2\sin(2\omega_s t_r)\cos(2\phi)\cos\theta\Big] \\ &= \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\Big[\cos(2\omega_s t_r)\sin(2\phi)\cos\theta + \sin(2\omega_s t_r)\cos(2\phi)\cos\theta\Big] \end{aligned}$$

then,

$$h_\times(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\Big(\cos\theta\Big)\Big[\sin(2\omega_s t_r + 2\phi)\Big] \quad . \tag{6.48}$$

The results obtained in (6.47) and (6.48) shows that *the frequency of the quadrupole radiation is twice the frequency of the source* [8]. It is important to highlight that the angular distribution dependence of the polarizations are $h_+ \sim 1+\cos^2\theta$ and $h_\times \sim \cos\theta$. This is a general outcome provided that $M^{13} = M^{23} = M^{33} = 0$ and $\ddot{M}^{22} = -\ddot{M}^{11}$[9].

The functions h_+ and $h_\times$ can also be rewritten in terms of the frequency of the circular orbital motion. Solving for the separation distance R from the Kepler's third law one obtains,

$$R = \left(\frac{Gm}{\omega_s^2}\right)^{1/3} \quad . \tag{6.49}$$

Inserting equation (6.49) into (6.47) and (6.48) gives,

[9] There are other problems in which the angular dependence of the polarizations are the same. For instance, consider a rotating rigid bar in the (x, y) plane or a rigid body rotating around one of its principal axis [8, 42, 49] .

$$h_+(t) = \frac{4}{r}\frac{G\mu}{c^4}\left(\frac{Gm}{\omega_s^2}\right)^{2/3}\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\pi f_{gw}t_r + 2\phi) \quad , \tag{6.50}$$

$$h_\times(t) = \frac{4}{r}\frac{G\mu}{c^4}\left(\frac{Gm}{\omega_s^2}\right)^{2/3}\omega_s^2\Big(\cos\theta\Big)\sin(2\pi f_{gw}t_r + 2\phi) \quad . \tag{6.51}$$

Moreover, if a new quantity called the "*chirp mass*" is introduced as

$$M_c \;=\; \mu^{3/5}m^{2/5} \;=\; \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}} \quad , \tag{6.52}$$

and using the fact that $\omega_{gw} = 2\pi f_{gw} = 2\omega_s$ then,

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\pi f_{gw}t_r + 2\phi) \tag{6.53}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\Big(\cos\theta\Big)\sin(2\pi f_{gw}t_r + 2\phi) \tag{6.54}$$

Note that in this Newtonian approximation, the amplitudes of h_+ and $h_\times$ depend on the masses m_1 and m_2 through the combination M_c. From the observational point of view, the gravitational radiation is measured along the line of sight as is depicted in figure 6.3.

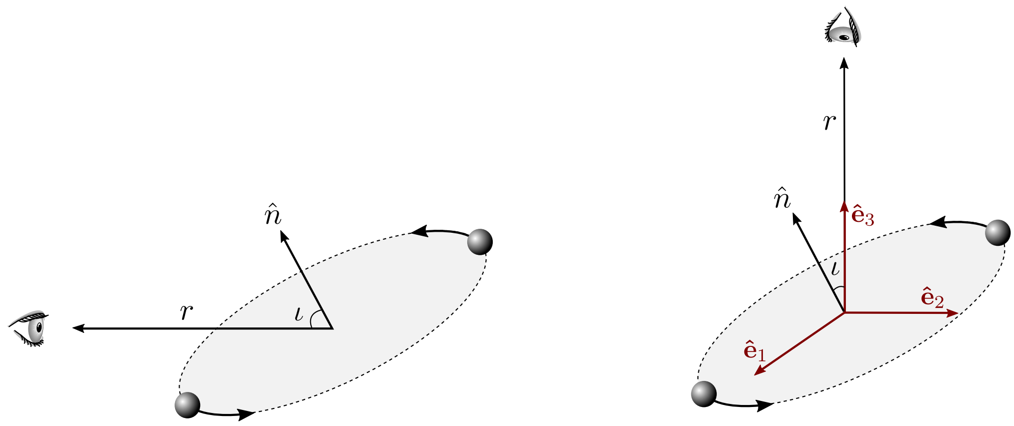

Figure 6.3: Observation of a binary system at a distance r from the source. The quantity ι is the inclination angle for a fixed value of constant θ. The line of sight is align with the direction $\hat{\mathbf{e}}_3$ and the unit vector that describes the orientation of the orbital plane is denoted by $\hat{n}$.

Besides the constant value of $\theta = \iota$, the azimuthal angle ϕ could be fixed if the proper motion of the source is negligible. In such a case, the argument of the temporal dependence in equations (6.53) and (6.54) turns out to be

$$\begin{aligned} 2\omega_s t_r \,+\, 2\phi \;&=\; 2\omega_s\left(t - \frac{r}{c}\right) \,+\, 2\phi \\ &=\; 2\omega_s t \,+\, 2\alpha \quad , \end{aligned} \tag{6.55}$$

where

$$\alpha \;=\; -\,\frac{\omega_{gw} r}{c} \,+\, \phi \quad . \tag{6.56}$$

The origin of time can be shifted by assuming a constant value of ϕ. In particular, if $\phi = -\omega_{\rm gw} r/c$, then $\alpha = 0$ and $2\omega_{\rm s} t_{\rm r} + 2\phi \longrightarrow 2\omega_{\rm s} t$. The observer would detect the polarizations of the radiation as

$$h_+(t) = \mathcal{A}\left(\frac{1+\cos^2\iota}{2}\right)\cos(2\omega_{\rm s} t) \tag{6.57}$$

$$h_\times(t) = \mathcal{A}\Big(\cos\iota\Big)\sin(2\omega_{\rm s} t) \tag{6.58}$$

with

$$\mathcal{A} = \frac{4}{r}\left(\frac{GM_{\rm c}}{c^2}\right)^{5/3}\left(\frac{\omega_{\rm s}}{c}\right)^{2/3} \quad . \tag{6.59}$$

Depending on the orientation and the line of sight to the source, there are two extremal cases to be considered as explained in figure 6.4. If the source is edge-on, i.e. $\iota = \frac{\pi}{2}$ only the *plus* polarization is detected and the radiation arrives with *linear polarization*. But if the source is face-on, i.e. $\iota = 0$ and the radiation comes with *circular polarization*. For other values of ι the GW has an elliptical polarization.

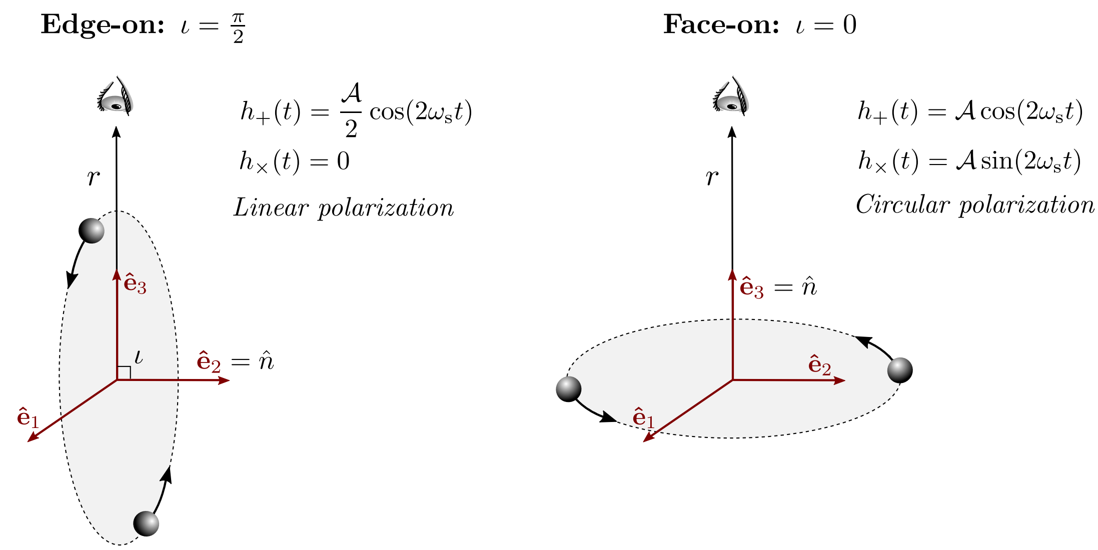

Figure 6.4: Extremal cases for the orientation of the binary system when is observed.

Radiated power

From equation (5.123), the radiated power per unit solid angle reads

$$\frac{dP^{\rm GW}_{\rm quad}}{d\Omega} = \frac{c^3 r^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle_{\rm quad} \tag{6.60}$$

For the binary system we have

$$\dot{h}_+^2(t,\theta,\phi) = -\mathcal{A}\left(\frac{1+\cos^2\theta}{2}\right)(2\pi f_{\rm gw})\sin\left(2\pi f_{\rm gw} t_{\rm r} + 2\phi\right) \quad , \tag{6.61}$$

$$\dot{h}_\times^2(t,\theta,\phi) = \mathcal{A}\left(\cos\theta\right)(2\pi f_{\rm gw})\cos\left(2\pi f_{\rm gw} t_{\rm r} + 2\phi\right) \quad . \tag{6.62}$$

Hence,

$$\frac{dP^{\mathsf{GW}}_{\text{quad}}}{d\Omega} = \frac{c^3 r^2}{16\pi G}\left(2\pi f_{\text{gw}}\right)^2 \mathcal{A}^2 \Bigg[\left(\frac{1+\cos^2\theta}{2}\right)^2 \left\langle \sin^2\left(2\pi f_{\text{gw}} t_{\text{r}} + 2\pi\right)\right\rangle$$
$$+ \left(\cos^2\theta\right)\left\langle \cos^2\left(2\pi f_{\text{gw}} t_{\text{r}} + 2\pi\right)\right\rangle \Bigg] \quad . \tag{6.63}$$

If the average $\langle\cdot\rangle$ is taken over several periods of the GW then,

$$\left\langle \sin^2\left(2\pi f_{\text{gw}} t_{\text{ret}} + 2\pi\right)\right\rangle = \left\langle \cos^2\left(2\pi f_{\text{gw}} t_{\text{ret}} + 2\pi\right)\right\rangle = \frac{1}{2} \tag{6.64}$$

and the radiated power is given by

$$\frac{dP^{\mathsf{GW}}_{\text{quad}}}{d\Omega} = \frac{c^3 r^2}{32\pi G}\left(2\pi f_{\text{gw}}\right)^2 \mathcal{A}^2 \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right] \quad . \tag{6.65}$$

With a little of algebra, one gets

$$\frac{c^3 r^2}{32\pi G}\left(2\pi f_{\text{gw}}\right)^2 \mathcal{A}^2 = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{G M_{\text{c}} \omega_{\text{gw}}}{2c^3}\right)^{10/3} \quad , \tag{6.66}$$

and therefore the power per unit solid angle produced by a binary system is

$$\frac{dP^{\mathsf{GW}}_{\text{quad}}}{d\Omega} = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{G M_{\text{c}} \omega_{\text{gw}}}{2c^3}\right)^{10/3} \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right] \quad . \tag{6.67}$$

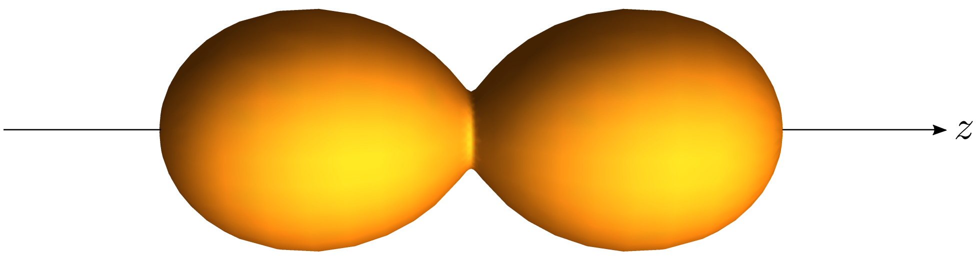

Figure 6.5: Angular distribution of the power emitted from a binary source.

Finally, integrating this expression over the entire solid angle it is obtained the total radiated power,

$$\boxed{P^{\mathsf{GW}}_{\text{quad}} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G M_{\text{c}} \omega_{\text{gw}}}{2c^3}\right)^{10/3}} \quad . \tag{6.68}$$

This result is often rewritten in terms of the *symmetric mass ratio* $\nu = \mu/m$ and the dimensionless variable $x = (Gm\omega_{\rm s}/c^3)^{2/3}$ as

$$P^{\sf GW}_{\rm quad} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5 \quad . \tag{6.69}$$

6.5 Evolution of the circular orbit under back-reaction

As we have seen, a binary of point masses is a source of GWs. Due to the mutual interaction of gravity, the particles are being accelerating and the variation of the mass quadrupole moment of the system is a non-vanishing quantity. The energy loss of the binary when gravitational radiation are emitted means that there must exist a *back-reaction* force which acts on the point masses producing the necessary work to account for such energy loss[10]. At linear approximation there is no back-reaction of GWs. However, this effect will be included by assuming that there exist a balance between the total orbital energy of the system and the energy carried away by GWs[11]. Mathematically this is described by the **balance equation**,

$$\boxed{\frac{dE_{\rm orbit}}{dt} = -P^{\sf GW}} \quad . \tag{6.70}$$

The minus sign in (6.70) clearly indicates that the rate in the orbital energy decreases as GWs (which carry energy) are emitted away from the source. The left-hand side of the balance equation at the quadrupole approximation is given by (6.68) whereas the right-hand side can be computed from the mechanical energy of the circular orbit,

$$\begin{aligned} E_{\rm orbit} &= E_0 + E_{\rm kin} + E_{\rm pot} \\ &= mc^2 + \frac{1}{2}\mu v^2 - \frac{Gm\mu}{R} \quad ; \quad v^2 = \frac{Gm}{R} \\ &= mc^2 - \frac{1}{2}\frac{Gm\mu}{R} \quad . \end{aligned} \tag{6.71}$$

Inserting equation (6.49) for the separation distance R into the orbital energy $E_{\rm orbit}$, by using the chirp mass $M_{\rm c}$ definition in (6.52) and considering $\omega_{\rm gw} = 2\omega_{\rm s}$ one obtains

$$E_{\rm orbit} = mc^2 - \left(\frac{G^2 M_{\rm c}^5 \omega_{\rm gw}^2}{32}\right)^{1/3} \tag{6.72}$$

and the mechanical power of the binary is expressed as,

$$\frac{dE_{\rm orbit}}{dt} = -\frac{2}{3}\left(\frac{G^2 M_{\rm c}^2}{32}\right)^{1/3} \frac{\dot{\omega}_{\rm gw}(t)}{\omega_{\rm gw}^{1/3}(t)} \quad . \tag{6.73}$$

Time evolution of the gravitational wave frequency $\omega_{\rm gw}$

Using (6.73) and (6.68) in the balance equation (6.70) gives

[10] This fact follows from the conservation of energy.

[11] For a formal derivation of the energy balance equation in the post-newtonian formalism see [Blanchet & Faye 2018].

$$\frac{2}{3}\left(\frac{G^2 M_c^2}{32}\right)^{1/3}\frac{\dot{\omega}_{gw}(t)}{\omega_{gw}^{1/3}(t)} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{gw}(t)}{2c^3}\right)^{10/3} \qquad (6.74)$$

and solving for $\dot{\omega}_{gw}(t)$ a differential equation for the GW frequency is obtained,

$$\dot{\omega}_{gw}(t) = \frac{12}{5}2^{1/3}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega_{gw}^{11/3}(t) \qquad (6.75)$$

which can be rewritten as

$$\dot{f}_{gw}(t) = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{gw}^{11/3}(t) \quad . \qquad (6.76)$$

By integrating the previous expression with respect to time we get the time dependence of the GW frequency,

$$\boxed{f_{gw}(t) = \frac{1}{\pi}\left[\frac{5}{256}\frac{1}{(t_{coal}-t)}\right]^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}} \qquad (6.77)$$

where t_{coal} is an integration constant. Observe that for $t = t_{coal}$ then $f_{gw} \longrightarrow \infty$ and therefore t_{coal} corresponds to the coalescence time. Therefore, the *time to coalescence* is defined as $\tau := t_{coal} - t$. Inserting the numerical constants in (6.77) and assuming a typical value for a binary system conformed by neutron stars with masses $m_1 = m_2 = 1.4\,M_\odot$ then,

$$f_{gw}(\tau) \simeq 130\,\text{Hz}\left(\frac{1,2M_\odot}{M_c}\right)^{5/8}\left(\frac{1\text{s}}{\tau}\right)^{3/8} \qquad (6.78)$$

or

$$\tau = \left(\frac{130\,\text{Hz}}{f_{gw}}\right)^{8/3}\left(\frac{1,2M_\odot}{M_c}\right)^{5/3}\text{s} \quad . \qquad (6.79)$$

For the frequency values of 10Hz, 100Hz and 1000Hz, equation (6.79) predicts times to coalescence of $\sim$ 17 min, 2 s and 1 ms [32, 34]. The separation distance R for such frequencies gives $\sim$ 711, 153 and 33 km and can be estimated from Kepler's third law in (6.49) with $\omega_{gw} = 2\omega_s$. Furthermore, if the period of the GW is a function that varies very slowly with time, the number of cycles in an intervale dt is found from $d\mathcal{N}_{cyc} = \frac{dt}{T(t)} = f_{gw}(t)dt$. Hence,

$$\mathcal{N}_{cyc} = \int_{t_{min}}^{t_{max}} f_{gw}(t)\,dt = \int_{f_{min}}^{f_{max}}\frac{f_{gw}}{\dot{f}_{gw}}\,df_{gw} \quad . \qquad (6.80)$$

Using (6.76) to express $\dot{f}_{gw}$ as a function of f_{gw} and substituting the result in (6.80) yields,

$$\begin{aligned}\mathcal{N}_{cyc} &= \frac{1}{32\pi^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3}\left(f_{min}^{-5/3} - f_{max}^{-5/3}\right) \\ &\simeq 1.6\times 10^4\left(\frac{10\,\text{Hz}}{f_{min}}\right)^{5/3}\left(\frac{1.2M_\odot}{M_c}\right)^{5/3} \quad .\end{aligned} \qquad (6.81)$$

where $f_{min}^{-5/3} - f_{max}^{-5/3} \backsimeq f_{min}^{-5/3}$. For the LISA (*Laser Interferometer Space Antenna*) detector, $f_{min} \sim 10^{-4}$ Hz and signals in this frequency band correspond to the inspiral of supermassive black holes with masses of order $\mathcal{O}(10^6)M_\odot$. For black holes with masses $m_1 = m_2 = 10^6 M_\odot$ the number of cycles is $\mathcal{N}_{\rm cyc} \sim 600$, but for $m_1 = 10^6 M_\odot$ and $m_2 = 10 M_\odot$ the number of cycles is $\mathcal{N}_{\rm cyc} \sim 10^7$. In such cases, very precise predictions beyond the Newtonian order must be considered in the modelling of the GW signal.

Time evolution of the separation distance R

In order to compensate the energy loss of the system, the separation distance R between the mass particles must decrease in time as long as GWs are radiated away from the source. From the Keplerian frequency relation in (6.49) and the radiated power in (6.68) the coalescence process is understood as follows. Since $\omega_{\rm s} \propto R^{-3/2}$, if R decreases then $\omega_{\rm s}$ increases. As $P_{\rm quad} \propto (2\omega_{\rm s})^{10/3}$, the increment of $\omega_{\rm s}$ leads to a rise in $P_{\rm quad}$. But if the power $P_{\rm quad}$ becomes greater, the separation distance R decreases much further generating a coalescence process. The change in R could be obtained by differentiating the Keplerian frequency,

$$\begin{aligned} \frac{d}{dt}\left(\omega_{\rm s}^2\right) &= Gm\frac{d}{dt}\left(\frac{1}{R^3}\right) \\ 2\omega_{\rm s}\dot{\omega}_{\rm s} &= -3\omega_{\rm s}^2\frac{\dot{R}}{R} \end{aligned}$$

i.e.,

$$\dot{R}(t) = -\frac{2}{3}\left(\omega_{\rm s}R\right)\frac{\dot{\omega}_{\rm s}}{\omega_{\rm s}^2} \quad . \tag{6.82}$$

If $|\dot{R}| \ll \omega_{\rm s}R$ the change in the separation distance is much less than the tangencial speed and the result is a quasi-circular orbital motion. From (6.82) this is possible as long as

$$\dot{\omega}_{\rm s} \ll \omega_{\rm s}^2 \quad . \tag{6.83}$$

Recall that the time to coalescence is $\tau = t_{\rm coal} - t$ so $d(\cdot)/dt = -d(\cdot)/d\tau$. Moreover, using the fact that $\omega_{\rm gw} = 2\omega_{\rm s}$ along with the equation (6.77), the time evolution of the separation distance R is described by the following separable differential equation,

$$\begin{aligned} \frac{1}{R}\frac{dR}{d\tau} &= -\frac{2}{3}\frac{1}{f_{\rm gw}}\frac{df_{\rm gw}(\tau)}{d\tau} \\ &= -\frac{2}{3}\tau^{3/8}\frac{d}{d\tau}\left(\tau^{-3/8}\right) \\ &= \frac{1}{4\tau} \quad . \end{aligned} \tag{6.84}$$

The solution of (6.84) gives $R(\tau) = R_0(\tau/\tau_0)^{1/4}$ with $\tau_0 = t_{\rm coal} - t_0$ and the temporal dependence of the separation distance is given by

$$\boxed{R(t) = R_0\left(\frac{t_{\rm coal} - t}{t_{\rm coal} - t_0}\right)^{1/4}} \quad , \tag{6.85}$$

where R_0 is the value of R at the initial time $t = t_0$. Evaluating the equation (6.77) at the instant τ_0 and using the Keplerian frequency then,

$$\frac{1}{\pi^2}\left(\frac{5}{256}\frac{1}{\tau_0}\right)^{3/4}\left(\frac{GM_c}{c^3}\right)^{-5/4} = \frac{Gm}{\pi^2 R_0^3}$$

or

$$\tau_0 = \frac{5}{256}\frac{c^5 R_0^4}{G^3 \mu m^2} \quad . \tag{6.86}$$

Finally, the initial radius R_0 may be expressed in terms of the initial period $T_0 = 2\pi/\omega_s(\tau_0)$ by means of Kepler's third law, i.e., $R_0^2 = Gm(T_0/2\pi)^2$. From (6.86) it follows that

$$\tau_0 \simeq 9.829 \times 10^6\,\mathrm{yr}\left(\frac{T_0}{1\,\mathrm{hr}}\right)\left(\frac{M_\odot}{m}\right)^{2/3}\left(\frac{M_\odot}{\mu}\right) \quad . \tag{6.87}$$

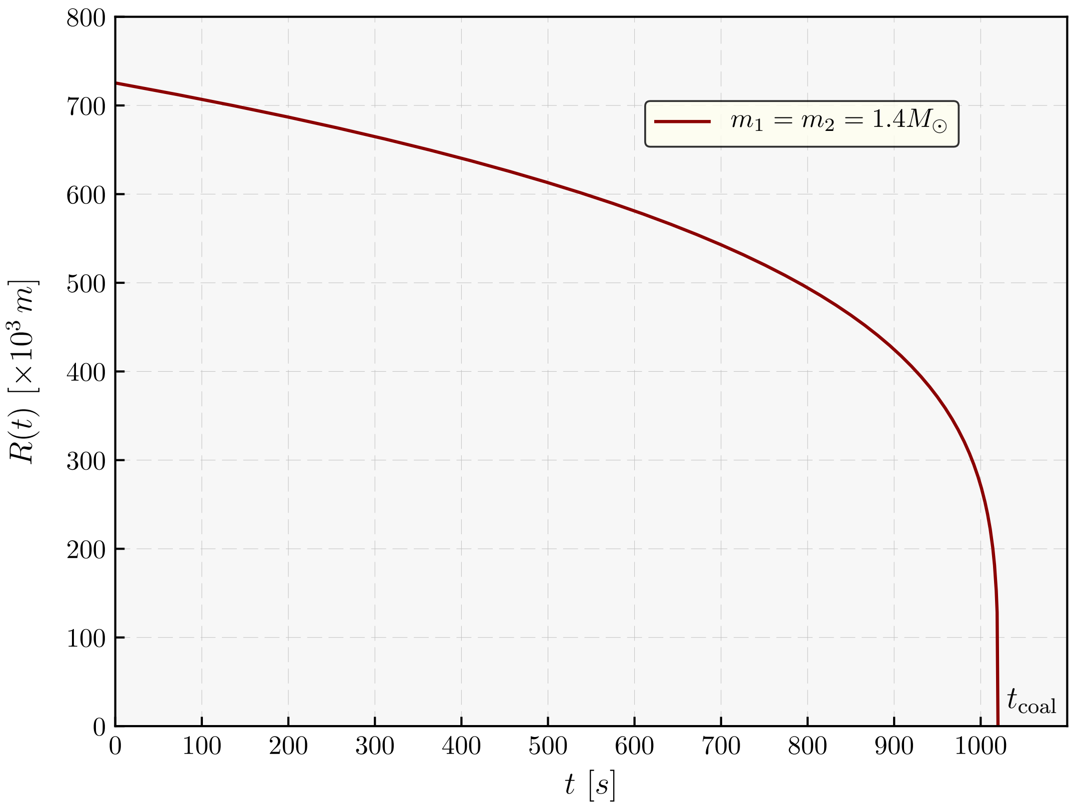

Figure 6.6: Time evolution of the separation distance R for a binary in circular motion.

6.6 The waveform of a binary source in circular motion

It was discussed that the *back-reaction* effect in a binary system of point masses increases the orbital frequency of the circular motion over time and thereby the GW frequency evolves as shown in (6.77). In addition, the emitted gravitational radiation at the quadrupole approximation is described by the functions $h_+(t)$ and $h_\times(t)$ in equations (6.53) and (6.54). One might note that the amplitudes and phases of such functions depend directly on the frequency ω_{gw} which indeed is increasing with time. As the particles approach to each other, the waveform signal of the emitted radiation begins to rise not only in frequency but also in amplitude. Hereafter, we shall describe these changes in the *plus* and the *cross* modes of propagation.

Consider an effective one-particle with reduced mass μ moving in a quasi-circular orbit of radius $R = R(t)$ with frequency $\omega_{\rm s} = \omega_{\rm s}(t)$ due to the gravitational potential generated by a particle with mass m. If the motion of the fictitious particle of mass μ occurs in the (x, y) plane its cartesian coordinates are given by

$$\begin{cases} x_0(t) & = R(t)\cos\left(\frac{\Phi(t)}{2} + \frac{\pi}{2}\right) \ , \\ y_0(t) & = R(t)\sin\left(\frac{\Phi(t)}{2} + \frac{\pi}{2}\right) \ , \\ z_0(t) & = 0 \ , \end{cases} \tag{6.88}$$

where the phase $\frac{\pi}{2}$ is included for convenience and the quantity $\Phi(t)$ is defined as

$$\Phi(t) \ = \ 2\int_{t_0}^{t}\omega_{\rm s}(t')\,dt' \ = \ \int_{t_0}^{t}\omega_{\rm gw}(t')\,dt' \quad . \tag{6.89}$$

The mass density moment of the system is $M^{ij} = \mu x_0^i(t)x_0^j(t)$ which gives

$$\begin{cases} M^{11}(t) & = \mu R^2(t)\dfrac{1 \ - \ \cos\left[\Phi(t)\right]}{2} \ , \\ M^{22}(t) & = \mu R^2(t)\dfrac{1 \ + \ \cos\left[\Phi(t)\right]}{2} \ , \\ M^{12}(t) & = -\dfrac{1}{2}\mu R^2(t)\sin\left[\Phi(t)\right] \quad . \end{cases} \tag{6.90}$$

The quadrupole radiation depends on the second deerivative with respecto to time of the mass density moment. Thus,

$$\dot{M}^{11} \ = \ 2\mu R\dot{R}\frac{1 \ - \ \cos\left(\Phi\right)}{2} \ + \ \mu R^2\frac{\dot{\Phi}}{2}\sin\left(\Phi\right) \tag{6.91}$$

$$\begin{aligned} \ddot{M}^{11} \ &= \ \mu\left[\dot{R}^2 \ + \ R\ddot{R}\right]\left[1 \ - \ \cos\left(\Phi\right)\right] \ + \ \mu R\dot{R}\dot{\Phi}\sin\left(\Phi\right) \\ &+ \ \mu\left[R\dot{R}\dot{\Phi} \ + \ \frac{1}{2}R^2\ddot{\Phi}\right]\sin\left(\Phi\right) \ + \ \frac{1}{2}\mu R^2\dot{\Phi}^2\cos\left(\Phi\right) \end{aligned} \tag{6.92}$$

If a **quasi-circular** motion assumption is imposed, then $\dot{R}(t)$ is negligible as long as $\dot{\omega}_{\rm s} \ll \omega_{\rm s}^2$. From (6.76) this condition translates into [8]

$$\frac{GM_{\rm c}\omega_{\rm s}}{c^3} \ll 0.5 \ \ , \tag{6.93}$$

or in terms of the binary frequency,

$$f_{\rm s} \ \ll \ 13\,{\rm kHz}\left(\frac{1.2M_\odot}{M_{\rm c}}\right) \quad . \tag{6.94}$$

The so-called *plunge* phase of the binary where the bodies are very close together before the merger occurs earlier than the given condition in (6.94). The underline reason is that when the objects approach to each other, the gravitational field near those objects are very strong and the dynamics of the system is more complicated because the background cannot be considered as flat. Actually, circular orbits are forbidden beyond the *Innermost Stable Circular Orbit* (ISCO) and one should stop the quasi-circular approximation at this point. In a Scharzschild background, the ISCO reads in Schwarzschild coordinates as

$$r_{\mathrm{ISCO}} = \frac{6Gm}{c^2} \quad . \tag{6.95}$$

An upper limit estimation of the frequency at the inspiral phase where the emission of GWs in the quasi-circular approximation can be found by using Kepler's third law. Thereby,

$$(f_{\mathrm{s}})_{\mathrm{ISCO}} = \frac{1}{6\sqrt{6}(2\pi)}\frac{c^3}{Gm} \quad . \tag{6.96}$$

For a binary neutron star with masses $m_1 = m_2 = 1.4M_{\odot}$, then $(f_{\mathrm{s}})_{\mathrm{ISCO}} \sim 800$ and the upper limit for the inspiral is less than the condition (6.94). In conclusion, the cuasi-circular approximation is suitable provided that the binary frequency is far from reach $(f_{\mathrm{s}})_{\mathrm{ISCO}}$ during the inspiral phase[12]. In this case the terms $\dot{R}$ and $\dot{\omega}_{\mathrm{gw}}$ involved in the expression (6.92) can be neglected. Since $\dot{\Phi} = \omega_{\mathrm{gw}}$, then the terms that have $\ddot{\Phi}$ do not contribute i.e.,

$$\ddot{M}^{11}(t) = \frac{1}{2}\mu R^2(t)\dot{\Phi}^2(t)\cos[\Phi(t)] = 2\mu R^2(t)\omega_{\mathrm{s}}^2(t)\cos[\Phi(t)] \quad . \tag{6.97}$$

Likewise, for the other components one obtains[13]

$$\begin{cases} \ddot{M}^{11}(t) &= 2\mu R^2(t)\omega_{\mathrm{s}}^2(t)\cos[\Phi(t)] \ , \\ \ddot{M}^{22}(t) &= -2\mu R^2(t)\omega_{\mathrm{s}}^2(t)\cos[\Phi(t)] \ , \\ \ddot{M}^{12}(t) &= 2\mu R^2(t)\omega_{\mathrm{s}}^2(t)\sin[\Phi(t)] \ , \end{cases} \tag{6.98}$$

and therefore the functions $h_+(t)$ y $h_\times(t)$ yields,

$$h_+(t) = \frac{4}{r}\left(\frac{GM_{\mathrm{c}}}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t_{\mathrm{r}})}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos[\Phi(t)] \ , \tag{6.99}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_{\mathrm{c}}}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t_{\mathrm{r}})}{c}\right)^{2/3}\Big(\cos\theta\Big)\sin[\Phi(t)] \quad . \tag{6.100}$$

In terms of the time to coalescence τ measured by the observer, the amplitudes in equations (6.99) and (6.100) are rewritten by using (6.77) where

$$\frac{\pi f_{\mathrm{gw}}}{c} = \frac{1}{c}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_{\mathrm{c}}}{c^3}\right)^{-5/8} \quad . \tag{6.101}$$

Similarly, using (6.77) in the definition for $\Phi(t)$ in (6.89) one finds,

$$\Phi(t) = -2\left(\frac{5GM_{\mathrm{c}}}{3}\right)^{-5/8}[t_{\mathrm{coal}} - t]^{5/8} + \Phi_0 \tag{6.102}$$

where,

$$\Phi_0 = 2\left(\frac{5GM_{\mathrm{c}}}{c^3}\right)^{-5/8}[t_{\mathrm{coal}} - t_0]^{5/8} \quad . \tag{6.103}$$

[12] Of course, for better predictions of the inspiral phase when the bodies are approaching one must use the post-Newtonian formalism rather than only a Newtonian approach [9, 48].

[13] Compare these results with equations (6.44), (6.45) and (6.46).

Finally, inserting (6.101) into the expressions (6.99) and (6.100) it is obtained the time evolution for the emitted radiation at the quadrupole approximation,

$$h_+(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\,(t_{\rm coal}-t)}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\Phi(t)\right]\ , \tag{6.104}$$

$$h_\times(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\,(t_{\rm coal}-t)}\right)^{1/4}(\cos\iota)\sin\left[\Phi(t)\right]\quad . \tag{6.105}$$

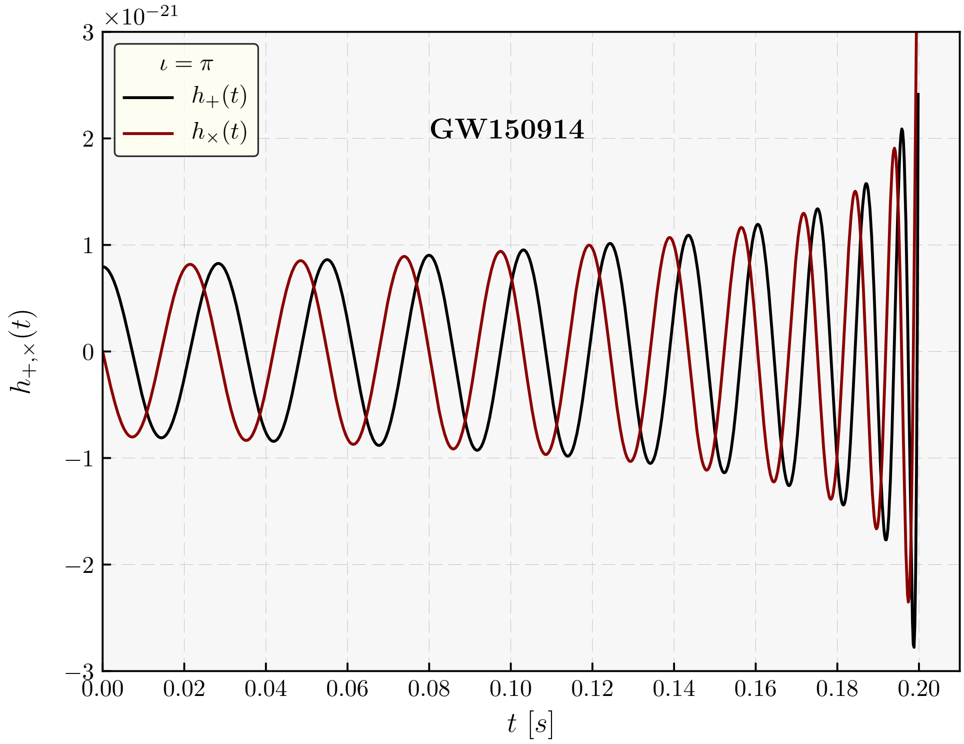

Figure 6.7: Time evolution of GWs as the orbit of a binary system decays. Since the frequency and the amplitude sweeps upwards with time, this plot is often known as the *chirp waveform*. The data to reproduce this figure was obtained from the source GW150914 [13].

Chapter 7

GWs in Linearized $f(R)$ Gravity

This chapter aims to investigate the linearized equations of $f(R)$ gravity as a natural initial framework to study GWs, following essentially the work done in [27]. A massless wave equation for a modified perturbation potential $\bar{h}_{\mu\nu}$ would be discussed with some similarities and differences with the procedure developed in Chapter 1. Unlike the two polarizations that arise in GR, an extra scalar and massive radiative degree of freedom appears in this theory for the Ricci scalar at linear order. The solution of the gravitational radiation emitted by a stationary point source at the Newtonian limit is presented. As in GR this kind of source does not generates radiation for the massless propagation modes. However, the solution for the massive mode exhibit a Yukawa-like functional dependence. As a consequence, it is shown that the metric for a stationary point mass in $f(R)$ gravity to first order, implies that such a source generates a Yukawa gravitational potential. The form of this potential consists in an addition of an exponential term to the usual Newtonian potential. At the end, the energy-momentum tensor of GWs will be revisited in this context in order to give a consistent description of a binary system in Chapter 8 with some assumptions.

7.1 Brief introduction to $f(R)$ gravity

At the beginning of Chapter 1 it was mentioned that the EFE could be derived from a variational principle. The full action that leads to the EFE is [38, 50],

$$S[\Psi, g] = \frac{1}{2\kappa}\left(I_{\rm H}[g] + I_{\rm B}[g]\right) + I_{\rm M}[\Psi, g] \ , \tag{7.1}$$

with $\kappa = 8\pi G c^{-3}$. The quantity $I_{\rm H}[g]$ along with $I_{\rm B}[g]$ represents the action of the gravitational field $g_{\mu\nu}$ and consists in the Einstein-Hilbert term plus the Gibbons-York-Hawking boundary term. The contribution $I_{\rm M}[\Psi, g]$ is the action of the matter fields, which are collectively denoted by Ψ. Explicitly, the GR action reads as

$$S[\Psi, g] = \int_{\mathcal{V}} \left(\frac{R}{2\kappa} + \mathcal{L}\right)\sqrt{-g}\, d^4x + \frac{1}{\kappa}\oint_{\partial\mathcal{V}} \epsilon K |h|^{1/2} d^3y \ , \tag{7.2}$$

where

$$I_H = \int_{\mathcal{V}} R\sqrt{-g}\, d^4x \ , \qquad I_{\rm B} = 2\oint_{\partial\mathcal{V}} \epsilon K|h|^{1/2} d^3y \ , \qquad I_{\rm M} = \int_{\mathcal{V}} \mathcal{L}\sqrt{-g}\, d^4x \ , \tag{7.3}$$

Here, R is the Ricci scalar on $\mathcal{V}$, $g = \det(g_{\mu\nu})$ is the determinant of the metric, K is the trace of the extrinsic curvature, ϵ takes the value of $+1$ if $\partial\mathcal{V}$ is timelike and -1 if $\partial\mathcal{V}$ is spacelike, h is the determinant of the induced metric on the hypersurface $\partial\mathcal{V}$ and $\mathcal{L}$ is some matter lagrangian density. Coordinates are denoted as x^α and y^α on $\mathcal{V}$ and $\partial\mathcal{V}$, respectively. The EFE are obtained by varying the action (7.2) with respect to $g^{\mu\nu}$ when the condition $\delta g_{\mu\nu}\big|_{\partial\mathcal{V}} = 0$ is imposed. The role of the surface integral term in the gravitational action is to cancels out exactly a surface term that appears in the variation of the Einstein-Hilbert functional. The result is the EFE equations given in (2.1).

It has been accepted that GR is the actual theory of gravity. The EFE describe a widely astronomical phenomena tested with a high level of accuracy in different regimes. However, a great variety of alternative theories have been proposed even since the birth of GR in order to explain the behavior of gravity in other scenarios [17, 51]. The simplest modification is not to consider the Ricci scalar R as the lagrangian density of the Einstein-Hilbert action, but a general function $f(R)$. The result is a family of gravity theories, each one with some functional form $f(R)$[1]. A well-defined variational action in this context is given by [50]

$$S_{\rm FR}[\Psi, g] = \int_{\mathcal{V}} \left(\frac{f(R)}{2\kappa} + \mathcal{L} \right) \sqrt{-g} d^4x + \frac{1}{\kappa} \oint_{\partial \mathcal{V}} \epsilon K f'(R) |h|^{1/2} d^3 y \ , \tag{7.4}$$

where $f' \equiv \frac{f(R)}{dR}$. Since the action includes higher-order derivatives of the metric one has the freedom to impose the additional condition $\delta R\big|_{\partial \mathcal{V}} = 0$ in order to subtract a boundary term similar to GR [27, 50]. Varying the action (7.4) with respect to $g^{\mu\nu}$ gives the $f(R)$ field equations in the metric formalism[2],

$$\boxed{f' R_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' - \frac{1}{2} f g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \ . \tag{7.5}$$

In analogy to GR, defining a modified Einstein's tensor by

$$\mathcal{G}_{\mu\nu} \equiv f' R_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' - \frac{1}{2} f g_{\mu\nu} \ , \tag{7.6}$$

the field equations in (7.5) are written as

$$\mathcal{G}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{7.7}$$

so that in vacuum $\mathcal{G}_{\mu\nu} = 0$. In particular, when $f(R) = R$ the EFE are recover from (7.5). In vacuum, the trace condition of the field equations in (7.5) gives

$$\begin{aligned} f' g^{\mu\nu} R_{\mu\nu} - g^{\mu\nu} \nabla_\mu \nabla_\nu f' + g^{\mu\nu} g_{\mu\nu} \Box f' - \frac{1}{2} f g^{\mu\nu} &= 0 \\ f' R + 3 \Box f' - 2f &= 0 \ . \end{aligned} \tag{7.8}$$

If one assumes that $f(R)$ is analytic about $R = 0$, it can be expressed as a power series[3],

$$f(R) = a_0 + a_1 R + \frac{a_2}{2!} R^2 + \frac{a_3}{3!} R^3 + \cdots \tag{7.9}$$

and for a uniform flat spacetime $(R = 0)$, equation (7.8) implies that

$$a_0 = 0 \ . \tag{7.10}$$

[1]This means that GR is a particlar case when $f(R) = R$.

[2]There are other approaches to obtain the field equations in $f(R)$ which may yield different results [17, 51]. For instance, in the Palatini formalism the metric $g^{\mu\nu}$ and the connections $\Gamma^\sigma_{\mu\nu}$ are treated independently because the connection is not the Levi-Civita connection [27, 52]. Moreover, the metric-affine formalism not only consider the independence between the metric and the connections, but also assumes that the matter lagrangian density depends on the connections.

[3]The dimensions of $f(R)$ must be the same as of R and thus $[a_n] = [R]^{(1-n)}$.

Conservation of the Energy-Momentum tensor $T_{\mu\nu}$

From the geometrical point of view, the energy-momentum tensor $T_{\mu\nu}$ is conserved in GR due to the Bianchi identities. In this sense, the conservation of such tensor in $f(R)$ gravity is also ensured by means of an additional identity. The contraction of the field equations in (7.5) with the covariant derivative gives,

$$\begin{aligned}
\nabla^{\mu}\left(f'R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu} + g_{\mu\nu}\Box f' - \frac{1}{2}fg_{\mu\nu}\right) &= 8\pi G\nabla^{\mu}T_{\mu\nu} \\
\left(\nabla^{\mu}f'\right)R_{\mu\nu} + f'\nabla^{\mu}R_{\mu\nu} - \Box\nabla_{\nu}f' + g_{\mu\nu}\nabla^{\mu}\Box f' - \frac{1}{2}g_{\mu\nu}\nabla^{\mu}f &= 8\pi G\nabla^{\mu}T_{\mu\nu} \\
R_{\mu\nu}\nabla^{\mu}f' + f'\nabla^{\mu}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla^{\mu}f - \Box\nabla_{\nu}f' + \nabla_{\nu}\Box f' &= 8\pi G\nabla^{\mu}T_{\mu\nu} \\
R_{\mu\nu}\nabla^{\mu}f' + f'\nabla^{\mu}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f'\nabla^{\mu}R - \left(\Box\nabla_{\nu} - \nabla_{\nu}\Box\right)f' &= 8\pi G\nabla^{\mu}T_{\mu\nu} \\
R_{\mu\nu}\nabla^{\mu}f' + f'\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \left(\Box\nabla_{\nu} - \nabla_{\nu}\Box\right)f' &= 8\pi G\nabla^{\mu}T_{\mu\nu} \\
R_{\mu\nu}\nabla^{\mu}f' - \left(\Box\nabla_{\nu} - \nabla_{\nu}\Box\right)f' &= 8\pi G\nabla^{\mu}T_{\mu\nu} \ , \qquad (7.11)
\end{aligned}$$

where in the last step we applied the Bianchi identities. The left-hand side in (7.11) vanish because of the identity

$$\left(\Box\nabla_{\nu} - \nabla_{\nu}\Box\right)f' = R_{\mu\nu}\nabla^{\mu}f' \ . \qquad (7.12)$$

This relation can be proved as follows. Recall that the covariant derivative of a scalar field ϕ is just the partial derivative, $\nabla_{\mu}\phi = \partial_{\mu}\phi$. Hence,

$$\begin{aligned}
\nabla_{\nu}\nabla_{\mu}\phi &= \nabla_{\nu}\left(\partial_{\mu}\phi\right) = \partial_{\nu}\partial_{\mu}\phi - \Gamma^{\alpha}_{\nu\mu}\partial_{\alpha}\phi \\
&= \partial_{\mu}\partial_{\nu}\phi - \Gamma^{\alpha}_{\mu\nu}\partial_{\alpha}\phi \qquad \text{[no torsion]} \\
&= \nabla_{\mu}\nabla_{\nu}\phi \ . \qquad (7.13)
\end{aligned}$$

Additionally, the Riemann curvature tensor is related by $[\nabla_{\mu},\nabla_{\nu}]V^{\alpha} = R^{\alpha}_{\beta\mu\nu}V^{\beta}$. Therefore,

$$\begin{aligned}
R_{\mu\nu}\nabla^{\mu}f' &= \left(\Box\nabla_{\nu} - \nabla_{\nu}\Box\right)f' \\
&= \left(\nabla_{\mu}\nabla^{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\nabla^{\mu}\right)f' \\
&= \left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right)\nabla^{\mu}f' \\
&= \left[\nabla_{\mu}\, ,\, \nabla_{\nu}\right]\nabla^{\mu}f' \\
&= R^{\mu}_{\alpha\mu\nu}\nabla^{\alpha}f' \\
&= R_{\alpha\nu}\nabla^{\alpha}f' \ . \qquad (7.14)
\end{aligned}$$

Then, the left-hand side in (7.11) is zero and the energy-momentum tensor is conserved in $f(R)$ gravity,

$$\boxed{\nabla^{\mu}T_{\mu\nu} = 0} \ . \qquad (7.15)$$

7.2 Linearized $f(R)$ gravity

The field equations in (7.5) are even more complicated than the EFE. If we consider a very weak gravitational field, the metric decomposition within the framework of the perturbation theory, allows us to linearized the field equations in $f(R)$ gravity[4]. If the background is the Minkwoski spacetime, a linear perturbation to the metric reads,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad . \tag{7.16}$$

The perturbation tensor is defined as $h^{\mu\nu}(x) = h^{(1)}_{\mu\nu}(x) = \varepsilon\tilde{h}_{\mu\nu}$ with $\varepsilon \ll 1$ and $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. To order $\mathcal{O}(\varepsilon)$ the inverse metric is as usual given by,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{7.17}$$

and indices are raised and lowered with $\eta_{\mu\nu}$. We recall here the linearized connection, Riemannn tensor, Ricci tensor and Ricci scalar,

$$\overset{(1)}{\Gamma}{}^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right) \ , \tag{7.18}$$

$$\overset{(1)}{R}{}^{\lambda}{}_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^{\lambda}{}_{\rho} + \partial^\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\rho h^{\lambda}{}_{\nu} - \partial^\lambda\partial_\nu h_{\mu\rho}\right) \ , \tag{7.19}$$

$$\overset{(1)}{R}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) \ , \tag{7.20}$$

$$\overset{(1)}{R} = \partial_\mu\partial_\rho h^{\mu\rho} - \Box h \quad . \tag{7.21}$$

where the d'Alembertian operator is $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. In order to get the linearized version of the field equations in (7.5), the function $f(R)$ to order $\mathcal{O}(\varepsilon)$ and its first derivative is required. From (7.9) this is,

$$f(R) = a_0 + a_1\overset{(1)}{R} + \cdots \ , \tag{7.22}$$

$$f'(R) = a_1 + a_2 R + \cdots \quad . \tag{7.23}$$

A linear perturbation to the Minkowski metric implies that the spacetime is slightly different from the flat where the Ricci scalar vanishes. This means that the Ricci scalar is almost zero and one can assume that $a_0 \simeq 0$ as mentioned in (7.10). Additionally, to link the results in this theory with GR we will set $a_1 = 1$[5]. Thus, equations (7.22) and (7.23) becomes

$$f(R) = \overset{(1)}{R} + \cdots \ , \tag{7.24}$$

$$f'(R) = 1 + a_2\overset{(1)}{R} + \cdots \quad . \tag{7.25}$$

Inserting (7.25) and (7.24) into $\mathcal{G}_{\mu\nu}$ given by (7.6) then,

$$\mathcal{G}_{\mu\nu} = \left(1 + a_2\overset{(1)}{R}\right)\overset{(1)}{R}_{\mu\nu} - \nabla_\mu\nabla_\nu\left(1 + a_2\overset{(1)}{R}\right) + g_{\mu\nu}\Box\left(1 + a_2\overset{(1)}{R}\right) - \frac{1}{2}\overset{(1)}{R}g_{\mu\nu} + \cdots$$

Inserting the metric decomposition (7.16) in this expression and using the fact that at linear order the covariant derivative of a quantity of order $\mathcal{O}(\varepsilon)$ is the same as its partial derivative, the modified Einstein tensor to first order in ε is,

[4] See Chapter 2 for a review on perturbation theory.
[5] Any rescaling can be absorbed into the definition of the gravitational constant G.

$$\overset{(1)}{\mathcal{G}}_{\mu\nu} = \overset{(1)}{R}_{\mu\nu} - a_2\partial_\mu\partial_\nu\overset{(1)}{R} + a_2\eta_{\mu\nu}\Box\overset{(1)}{R} - \frac{1}{2}\overset{(1)}{R}\eta_{\mu\nu} \quad . \tag{7.26}$$

and the field equations in linearized $f(R)$ gravity are[6]

$$\boxed{\overset{(1)}{R}_{\mu\nu} - a_2\partial_\mu\partial_\nu\overset{(1)}{R} + a_2\eta_{\mu\nu}\Box\overset{(1)}{R} - \frac{1}{2}\overset{(1)}{R}\eta_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}} \quad , \tag{7.27}$$

where the Ricci tensor and the Ricci scalar are given in equations (7.20) and (7.21). Observe that if $a_2 \longrightarrow 0$, linearized theory in GR is recover as can be seen from equations (7.24), (7.25) and (7.27). Taking the trace of equation (7.27) we get

$$\begin{aligned} \overset{(1)}{R} - a_2\Box\overset{(1)}{R} + 4a_2\Box\overset{(1)}{R} - 2\overset{(1)}{R} &= \frac{8\pi G}{c^4}T \\ 3a_2\Box\overset{(1)}{R} - \overset{(1)}{R} &= \frac{8\pi G}{c^4}T \end{aligned} \tag{7.28}$$

Defining the parameter $\Upsilon^2 = \frac{1}{3a_2}$, the previous equation is rewritten as

$$\boxed{\left(\Box - \Upsilon^2\right)\overset{(1)}{R} = \frac{8\pi G}{c^4}\Upsilon^2 T} \quad . \tag{7.29}$$

This is just the inhomogeneous Klein-Gordon equation for the Ricci scalar at linear order. It represents a massive wave equation for a new scalar degree of freedom in the theory [53–57]. In vacuum, $T = 0$ and then

$$\left(\Box - \Upsilon^2\right)\overset{(1)}{R} = 0 \quad . \tag{7.30}$$

In GR the trace-reversed form of $h_{\mu\nu}$ satisfies the wave equation in the harmonic gauge. One might wonder if there is a massless wave equation for the perturbation tensor $h_{\mu\nu}$ in linearized $f(R)$ gravity as in GR. A clever way to do this is to propose the following ansatz,

$$\bar{h}_{\mu\nu} = \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) + B_{\mu\nu} \quad , \tag{7.31}$$

with $B_{\mu\nu} = ba_2\overset{(1)}{R}\eta_{\mu\nu}$[7]. Thus,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \left(ba_2\overset{(1)}{R} - \frac{1}{2}h\right)\eta_{\mu\nu} \tag{7.32}$$

which can be inverted to obtain

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right)\eta_{\mu\nu} \quad . \tag{7.33}$$

The constant a_2 has been included in the ansatz to ensure dimensional consistency and b is a dimensionless number. Contracting equations (7.32) and (7.33) with $\eta^{\mu\nu}$ yields,

[6] The energy-momentum tensor should be of order $\mathcal{O}(\varepsilon)$. Otherwise, is not possible to consider a perturbed Minkowski metric.

[7] Again, when $a_2 = 0$ the usual trace-reversed form in linearized GR is recovered.

$$\bar{h} = 4ba_2\overset{(1)}{R} - h \ , \tag{7.34}$$

$$h = 4ba_2\overset{(1)}{R} - \bar{h} \ . \tag{7.35}$$

The Lorenz gauge can also be imposed in this context over the the new perturbation potential tensor $\bar{h}_{\mu\nu}$. To see how, remind that the invariant gauge in linearized theory is

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \tag{7.36}$$

and the trace of this equation gives

$$h' = h - 2\partial_\mu \xi^\mu \ . \tag{7.37}$$

It was mentioned in Chapter 2 that the Riemann tensor is an invariant quantity under the gauge of linearized theory and the Ricci scalar also inherits this property. Using equations (7.36) and (7.37) into the linearized Ricci scalar in (7.21) it follows that

$$\begin{aligned} \overset{(1)}{R'} &= \partial_\mu \partial_\nu h'^{\mu\nu} - \Box h' \\ &= \partial_\mu \partial_\nu \left[h^{\mu\nu} - (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu)\right] - \Box\left(h - 2\partial_\rho \xi^\rho\right) \\ &= \partial_\mu \partial_\nu h^{\mu\nu} - \Box \partial_\nu \xi^\nu - \Box \partial_\mu \xi^\mu - \Box h + 2\Box \partial_\rho \xi^\rho \\ &= \partial_\mu \partial_\nu h^{\mu\nu} - \Box h = \overset{(1)}{R} \end{aligned} \tag{7.38}$$

and the Ricci scalar is indeed invariant under the gauge of linearized theory. Now, the perturbation potential $\bar{h}_{\mu\nu}$ defined in (7.32) changes as

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu} + ba_2\overset{(1)}{R'}\eta_{\mu\nu} \\ &= \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + \eta_{\mu\nu}\partial_\rho \xi^\rho + a_2 b\overset{(1)}{R}\eta_{\mu\nu} \end{aligned} \tag{7.39}$$

Then,

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} - \Box \xi_\nu + ba_2 \partial_\nu \overset{(1)}{R} \ . \tag{7.40}$$

Assume that in an initial system the Lorenz gauge is not satisfied and at the right-hand side of (7.40) we have $\partial^\mu \bar{h}_{\mu\nu} \neq 0$. The gauge transformed system must satisfies the Lorenz condition if

$$\Box \xi_\nu = g_\nu \tag{7.41}$$

with $g_\nu = \partial^\mu \bar{h}_{\mu\nu} + ba_2\partial_\nu \overset{(1)}{R}$. This equation can always be solved with a well defined boundary conditions. Therefore, the Lorenz gauge can be achieved in some coordinate system and

$$\partial_\nu \bar{h}_{\mu\nu} = 0 \ , \tag{7.42}$$

where $\bar{h}_{\mu\nu}$ is given by (7.32). Analog to GR, the way forward is to simplify the field equations in (7.27) using the Lorenz gauge condition over the new potential $\bar{h}_{\mu\nu}$. The first term at the left-hand side in (7.27) is the linearized Ricci tensor. Then,

$$
\begin{aligned}
\overset{(1)}{R}_{\mu\nu} &= \frac{1}{2}\Big\{\partial_\mu\partial_\rho h^\rho{}_\nu + \partial_\nu\partial_\rho h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\Big\} \\
&= \frac{1}{2}\Big\{\partial_\mu\partial_\rho\left[\bar{h}^\rho{}_\nu + \left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right)\delta^\rho{}_\nu\right] + \partial_\nu\partial_\rho\left[\bar{h}^\rho{}_\mu + \left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right)\delta^\rho{}_\mu\right] \\
&\quad - \partial_\mu\partial_\nu\left(4ba_2\overset{(1)}{R} - \bar{h}\right) - \Box\left[\bar{h}_{\mu\nu} + \left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right)\right]\Big\} \\
&= \frac{1}{2}\Big\{\underbrace{\partial_\mu\partial_\rho\bar{h}^\rho{}_\nu}_{=0} + \partial_\mu\partial_\nu\left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right) + \underbrace{\partial_\nu\partial_\rho\bar{h}^\rho{}_\mu}_{=0} + \partial_\mu\partial_\nu\left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right) \\
&\quad - \partial_\mu\partial_\nu\left(4ba_2\overset{(1)}{R} - \bar{h}\right) - \Box\left[\bar{h}_{\mu\nu} + \left(ba_2\overset{(1)}{R} - \frac{1}{2}\bar{h}\right)\right]\Big\} \\
&= -\frac{1}{2}\Big\{2a_2b\partial_\mu\partial_\nu\overset{(1)}{R} + \Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) + a_2b\Box\overset{(1)}{R}\eta_{\mu\nu}\Big\} \quad . \qquad (7.43)
\end{aligned}
$$

Observe that the left-hand side of (7.28) is just the trace of the modified Einstein tensor $\mathcal{G}$ at linear order. Thus, from this expression we have

$$
a_2\Box\overset{(1)}{R} = \frac{1}{3}\left(\overset{(1)}{\mathcal{G}} + \overset{(1)}{R}\right) \qquad (7.44)
$$

and using this result in (7.43), the linearized Ricci tensor becomes

$$
\overset{(1)}{R}_{\mu\nu} = -\frac{1}{2}\left[2a_2b\partial_\mu\partial_\nu\overset{(1)}{R} + \Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) + \frac{b}{3}\left(\overset{(1)}{\mathcal{G}} + \overset{(1)}{R}\right)\eta_{\mu\nu}\right] \quad . \qquad (7.45)
$$

Using (7.45) and once again (7.44) into (7.26), one obtains the modified Einstein tensor $\mathcal{G}_{\mu\nu}$ at linear order,

$$
\begin{aligned}
\overset{(1)}{\mathcal{G}}_{\mu\nu} &= \overset{(1)}{R}_{\mu\nu} - a_2\partial_\mu\partial_\nu\overset{(1)}{R} + a_2\eta_{\mu\nu}\Box\overset{(1)}{R} - \frac{1}{2}\overset{(1)}{R}\eta_{\mu\nu} \\
&= -\frac{1}{2}\left[2a_2b\partial_\mu\partial_\nu\overset{(1)}{R} + \Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) + \frac{b}{3}\left(\overset{(1)}{\mathcal{G}} + \overset{(1)}{R}\right)\eta_{\mu\nu}\right] \\
&\quad - a_2\partial_\mu\partial_\nu\overset{(1)}{R} + a_2\eta_{\mu\nu}\Box\overset{(1)}{R} - \frac{1}{2}\overset{(1)}{R}\eta_{\mu\nu} \\
&= -a_2b\partial_\mu\partial_\nu\overset{(1)}{R} - \frac{1}{2}\Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) - \frac{b}{6}\left(\overset{(1)}{\mathcal{G}} + \overset{(1)}{R}\right)\eta_{\mu\nu} \\
&\quad - a_2\partial_\mu\partial_\nu\overset{(1)}{R} + \frac{1}{3}\eta_{\mu\nu}\left(\overset{(1)}{\mathcal{G}} + \overset{(1)}{R}\right) - \frac{1}{2}\overset{(1)}{R}\eta_{\mu\nu} \\
\overset{(1)}{\mathcal{G}}_{\mu\nu} &= \frac{2-b}{6}\overset{(1)}{\mathcal{G}}\eta_{\mu\nu} - \frac{1}{2}\Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) - (b+1)\left(a_2\partial_\mu\partial_\nu\overset{(1)}{R} - \frac{1}{6}\overset{(1)}{R}\eta_{\mu\nu}\right) \qquad (7.46)
\end{aligned}
$$

By setting $b = -1$ in (7.46) the final term vanishes and then

$$
\overset{(1)}{\mathcal{G}}_{\mu\nu} = \frac{1}{2}\overset{(1)}{\mathcal{G}}\eta_{\mu\nu} - \frac{1}{2}\Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) \quad . \qquad (7.47)
$$

The expression in (7.47) is almost the left-hand side of a wave equation. To show that actually the first and the last term cancel each other, consider the linearized Ricci tensor. From (7.21),

$$\begin{aligned}\overset{(1)}{R} &= \partial_\mu \partial_\rho h^{\mu\rho} - \Box h \\ &= \partial_\mu \partial_\rho \left[\bar{h}^{\mu\rho} - \left(a_2 \overset{(1)}{R} + \frac{1}{2}\bar{h} \right) \eta^{\mu\rho} \right] + \Box \left(4a_2 \overset{(1)}{R} + \bar{h} \right) \\ &= \underbrace{\partial_\mu \partial_\rho \bar{h}^{\mu\rho}}_{=0} - \Box \left(a_2 \overset{(1)}{R} + \frac{1}{2}\bar{h} \right) + \Box \left(4a_2 \overset{(1)}{R} + \bar{h} \right) \\ &= 3\Box \left(a_2 \overset{(1)}{R} \right) + \frac{1}{2}\Box \bar{h} \quad . \end{aligned} \tag{7.48}$$

Comparing (7.48) with (7.28) it is concluded that

$$\overset{(1)}{\mathcal{G}} = -\frac{1}{2}\Box \bar{h} \tag{7.49}$$

and inserting (7.49) into (7.47) the modified Einstein tensor $G_{\mu\nu}$ at linear order gives,

$$\overset{(1)}{\mathcal{G}}_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu} \quad . \tag{7.50}$$

Finally, using the result in (7.50) into the field equations in (7.7) we obtain a wave equation for the potential $\bar{h}_{\mu\nu}$,

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}} \tag{7.51}$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \left(a_2 \overset{(1)}{R} + \frac{1}{2} h \right) \eta_{\mu\nu} \quad . \tag{7.52}$$

7.3 Gravitational waves in vacuum

In the previous section two wave equations have been found as shown in (7.29) and (7.29). This section seeks to identify the physical radiative degrees of freedom in linearized $f(R)$ gravity [27, 53, 57]. If we specialize in vacuum, it follows that $T_{\mu\nu} = 0$ and therefore we obtain

$$\Box \bar{h}_{\mu\nu} = 0 \ , \tag{7.53}$$

$$\left(\Box - \Upsilon^2 \right) \overset{(1)}{R} = 0 \quad . \tag{7.54}$$

The general solution of the equation in (7.53) is a superposition of plane waves of the form

$$\bar{h}_{\mu\nu} = \hat{h}_{\mu\nu}(k_\mu) e^{i k_\mu x^\mu} \ . \tag{7.55}$$

Certainly, by inserting the solution $\bar{h}_{\mu\nu}$ in the wave equation one gets

$$k_\mu k^\mu = 0 \tag{7.56}$$

and the wave four-vector is null. This means that GWs propagates at the speed of light as in GR. Explicitly, using $k^\mu = (\omega/c, \mathbf{k})$ this is to say that there is no dispersion,

$$\omega = ck \quad . \tag{7.57}$$

Observe that in this case the phase velocity v_p and the group velocity v_g are the same,

$$v_\mathrm{p} = \frac{\omega}{k} = c \qquad , \qquad v_\mathrm{g} = \frac{\partial \omega}{\partial k} = c \quad . \tag{7.58}$$

For the scalar mode, the solution of (7.54) gives,

$$\overset{(1)}{R} = \hat{R}(q_\mu) e^{i q_\mu x^\mu} \tag{7.59}$$

and inserting (7.59) into the homogeneous Klein-Gordon equation yields

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta \left[\hat{R}(q_\mu)e^{iq_\mu x^\mu}\right] - \Upsilon^2 \left[\hat{R}(q_\mu)e^{iq_\mu x^\mu}\right] = 0$$
$$i^2\eta^{\alpha\beta}q_\alpha q_\beta \left[\hat{R}(q_\mu)e^{iq_\mu x^\mu}\right] - \Upsilon^2 \left[\hat{R}(q_\mu)e^{iq_\mu x^\mu}\right] = 0$$

i.e.,

$$q^\mu q_\mu = -\Upsilon^2 \quad . \tag{7.60}$$

Thus, there is dispersion and q^μ is time-like. If $q^\mu \equiv (\Omega/c, \mathbf{q})$ where Ω is the angular frequency and $\mathbf{q}$ is a wave vector of this scalar mode, the expression in (7.60) gives

$$q = \sqrt{\frac{\Omega^2}{c^2} - \Upsilon^2} \; . \tag{7.61}$$

Using the dispersion relation in (7.61), the phase velocity and group velocity are given by[8]

$$v_\mathrm{p} = \frac{\Omega}{q} = \frac{c\Omega}{\sqrt{\Omega^2 - c^2\Upsilon^2}} \qquad , \qquad v_\mathrm{g} = \frac{\partial \Omega}{\partial q} = \frac{\sqrt{c^2\Omega^2 - c^4\Upsilon^2}}{\Omega} \tag{7.62}$$

and the plane wave packet of the massive particle propagates at $v_\mathrm{g} < c$. Instead, if one assumes that $\frac{\Omega^2}{c^2} < \Upsilon^2$ an evanescent wave is found. However, an important remark about the propagation of the scalar mode arises at the limit of GR when $a_2 \longrightarrow 0$, so that $\Upsilon^2 \longrightarrow \infty$. From equation (7.61), if indeed $\Upsilon^2 \longrightarrow \infty$ then it is required an infinity frequency Ω to excite this Ricci scalar mode to propagate and evanescent waves would decay infinitely fast [27]. Thus, the Ricci mode should not be detectable and one is left only with the massless modes. In fact, as well as in GR, the perturbation tensor $h_{\mu\nu}$ has only two polarization massless radiative modes that actually comes from $\bar{h}_{\mu\nu}$[9]. To show this, consider the perturbation tensor $h_{\mu\nu}$ by inverting equation (7.52). Then,

$$h_{\mu\nu} = \left(\bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\right) - a_2\overset{(1)}{R}\eta_{\mu\nu} \quad . \tag{7.63}$$

The term in brackets represents the massless radiative modes and the remain term the massive scalar mode. We will show that the tensor $\bar{h}_{\mu\nu}$ can be put into the TT form to extract the *plus* and *cross* polarizations. As a consequence, $\bar{h} = 0$ and the solution for the perturbation

[8] Observe that if $\Upsilon = 0$ the wave propagates at the speed of light.

[9] The massive scalar mode of propagation comes from the Ricci tensor.

tensor $h_{\mu\nu}$ will remain only with two massless radiative modes plus a massive scalar mode. To extract the polarizations, is worthwhile to introduce a new quantity,

$$H_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{7.64}$$

where $H_{\mu\nu}$ represent the trace-reversed form of $\bar{h}_{\mu\nu}$[10]. Using this definition, the independent components of $\bar{h}_{\mu\nu}$ are the same of $H_{\mu\nu}$. This can be shown by writing the vacuum wave equation in terms of $H_{\mu\nu}$ and using the expression (7.49),

$$\begin{aligned} \Box\bar{h}_{\mu\nu} &= 0 \\ \Box\left(H_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) &= 0 \qquad (7.65) \\ \Box H_{\mu\nu} - \overset{(1)}{\mathcal{G}}\eta_{\mu\nu} &= 0 \\ \Box H_{\mu\nu} &= 0 \ . \qquad (7.66) \end{aligned}$$

Then, the problem is equivalent to find out the independent components of $H_{\mu\nu}$. To achieve this, consider the following infinitesimal diffeomorphism

$$x^{\mu} \longrightarrow x^{\mu} - \xi^{\mu} \ . \tag{7.67}$$

With this transformation the quantities $\bar{h}_{\mu\nu}$, $\bar{h}$ and $H_{\mu\nu}$ change as

$$\bar{h}^{(\text{new})}_{\mu\nu} = \bar{h}_{\mu\nu} + (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}) \tag{7.68}$$

$$\bar{h}^{(\text{new})} = \bar{h} + 2\partial_{\mu}\xi^{\mu} \tag{7.69}$$

$$H^{(\text{new})}_{\mu\nu} = H_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial_{\rho}\xi^{\rho} \tag{7.70}$$

With a similar argument explained in Chapter 2, the Lorenz gauge $\partial^{\mu}H_{\mu\nu} = 0$ exists in the transformed coordinate system and therefore always can be imposed by choosing this new frame. Taking the 4-divergence of equation (7.70) one gets

$$\partial^{\mu}\bar{H}^{(\text{new})}_{\mu\nu} = \partial^{\mu}\bar{H}_{\mu\nu} + \Box\xi_{\nu} \tag{7.71}$$

and the Lorenz gauge in the new system always is satisfied provided that $\Box\xi_{\nu} = s_{\nu}$ with $s_{\nu} = -\partial^{\mu}H_{\mu\nu}$[11]. Now, suppose that this new coordinate system is chosen so that the Lorenz gauge is valid. If one wish preserved the Lorenz condition under the transformation in (7.67) in other coordinate system, only if $\Box\xi_{\nu} = 0$ then $\partial^{\mu}H^{(\text{other})}_{\mu\nu} = 0$[12]. Of course in all systems the physics is the same and the vacuum wave equation is preserved under coordinate transformations[13]. Selecting a coordinate system that preserves the Lorenz gauge, the rule transformation (7.70) reads explicitly as

$$\hat{H'}_{\mu\nu} = \hat{H}_{\mu\nu} + i\left(k_{\mu}\hat{\xi}_{\nu} + k_{\nu}\hat{\xi}_{\mu} - \eta_{\mu\nu}k^{\rho}\hat{\xi}_{\rho}\right) \ . \tag{7.72}$$

where

$$H_{\mu\nu} = \hat{H}_{\mu\nu}(k_{\rho})e^{ik_{\rho}x^{\rho}} \quad , \quad \xi_{\mu} = \hat{\xi}_{\mu}(k_{\rho})e^{ik_{\rho}x^{\rho}} \ . \tag{7.73}$$

[10] This means that $H = -\bar{h}$ with $H = \eta^{\mu\nu}H_{\mu\nu}$.
[11] This wave equation for the functions ξ_{ν} always can be solved and the right-hand side of (7.71) vanishes.
[12] Note that $\partial^{\mu}H^{(\text{other})} = \partial^{\mu}H^{(\text{new})} + \Box\xi_{\nu} = \Box\xi_{\nu}$.
[13] This can be proved by taking the d'Alembertian of the rule transformation between systems in (7.70).

As developed in Chapter 2, the Lorenz gauge reduces the 10 initial independent components to 6 and the functions ξ_ν simplify further to only two independent components. By assuming a GW propagating in the $+z$-direction, the Lorenz condition gives $\hat{H}'_{0\nu} = -\hat{H}_{3\nu}$. Additionally, from equation (7.72) one can choose the functions ξ_ν in order to get finally[14]

$$\left[\hat{H}'_{\mu\nu}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left[\hat{h}_{\mu\nu}\right] \tag{7.74}$$

where $\hat{h}_{\mu\nu}$ is the polarization tensor of $\bar{h}_{\mu\nu}$. Observe from (7.74) that the trace vanishes, i.e, $\bar{h} = 0$ and the tensors $H_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ are equivalent as expected[15]. Therefore, the perturbation tensor $h_{\mu\nu}$ becomes [27],

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - a_2 \overset{(1)}{R} \eta_{\mu\nu} \tag{7.75}$$

and can be written as [53, 54]

$$\boxed{h_{\mu\nu}(t,z) = h_+(t - z/c)\mathbf{e}^{(+)}_{\mu\nu} + h_\times(t - z/c)\mathbf{e}^{(\times)}_{\mu\nu} - a_2 \overset{(1)}{R}(t - v_g z/c^2)\eta_{\mu\nu}} \tag{7.76}$$

with

$$\mathbf{e}^{(+)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{e}^{(\times)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.77}$$

A remarkable consequence of the Ricci scalar mode in linearized $f(R)$ gravity is that is impossible to set the trace h equal to zero. Actually, only $\bar{h} = 0$ can be fixed in some reference frame. Indeed, is not necessary to choose a TT frame to set $\bar{h} = 0$. For instance, setting $b = -1$ in (7.39) and taking the trace it is obtained,

$$\bar{h}' = \bar{h} + 2\partial_\mu \xi^\mu - 4a_2 \overset{(1)}{R} \tag{7.78}$$

and the condition $\bar{h}' = 0$ could be fixed with adequately choosing the functions ξ^μ by imposing the relation

$$\partial_\mu \xi^\mu = 2a_2 \overset{(1)}{R} - \frac{1}{2}\bar{h} \ . \tag{7.79}$$

However, even choosing $\bar{h} = 0$ in any coordinate system, $h \neq 0$ as long as the Ricci mode is nonzero. This can be proved after taking the trace in equation (7.52) and assuming that in some reference frame $\bar{h} = 0$. Then,

$$h = -4a_2 \overset{(1)}{R} \ . \tag{7.80}$$

This is a great difference with linearized theory in GR where always is possible to set $h = 0$ in some reference frame[16]. In particular, from equations (7.76) and (7.80) when $a_2 \longrightarrow 0$ the GR limit is recovered.

In summary, the main purpose of this section was to highlight the differences that arise in linearized $f(R)$ gravity in contrast to gravitational radiation in GR. The conclusion is

[14] The procedure is very similar as was done in Chapter 2 with some changes in the signs of the expressions.
[15] This follows from equation (7.64)
[16] Not necessary only in the TT frame.

that aside from the *plus* and *cross* polarizations, there is another extra Ricci scalar mode of propagation of a *massive* wave. Moreover, the trace of the perturbation tensor $h_{\mu\nu}$ is always nonzero but one is able to fix $\bar{h} = 0$ in some reference frame. At the end of this chapter, a coordinate system where $\bar{h} = 0$ will be assumed in order to simplify the computations of the energy-momentum tensor $t_{\mu\nu}$ in this context.

7.4 Newtonian limit of $f(R)$

In vacuum, all propagating modes of GWs in linearized $f(R)$ gravity are described by plane waves. Nevertheless, to understand how this waves are produced one has to include the energy momentum tensor $T_{\mu\nu}$ of the matter sources. The inhomogeneous wave equation that describes the massless GWs is shown in (7.51). This equation is exactly the same as in the case of GR and thus all methods and developments that were introduced in Chapter 5 can be applied in this context [27]. Unfortunately, to find a solution of the inhomogeneous Klein-Gordon equation for the massive propagating mode is not an easy task. The goal of this section is to find a general solution of the Klein-Gordon equation and investigate the gravitational field generated by a stationary point source at the Newtonian limit. In general, an explicit solution for other sources like a binary system is very demanding [56].

Green's function for the Klein-Gordon equation

To express a general solution of the Ricci scalar mode in an integral form, the Green's function method is implemented. If $G_\Upsilon(x, x')$ is a Green's function of the inhomogeneous Klein-Gordon equation, it must satisfies the following relation,

$$\left(\Box_x - \Upsilon^2\right) G_\Upsilon(x, x') = \delta^{(4)}(x - x') \quad . \tag{7.81}$$

Then, the solution of (7.29) is given by

$$\overset{(1)}{R}(x) = \frac{8\pi G}{c^4}\Upsilon^2 \int G_\Upsilon(x, x')T(x')\, d^4x' \quad . \tag{7.82}$$

Is straightforward to check that indeed this a solution to the problem. By acting the Klein-Gordon operator $(\Box - \Upsilon^2)$ over the previous expression yields

$$\begin{aligned}
(\Box_x + \Upsilon^2)\overset{(1)}{R}(x) &= \frac{8\pi G}{c^4}\Upsilon^2 \int (\Box_x - \Upsilon^2) G_\Upsilon(x, x')T(x')\, d^4x' \\
&= \frac{8\pi G}{c^4}\Upsilon^2 \int \delta^{(4)}(x - x')T(x')\, d^4x' \\
&= \frac{8\pi G}{c^4}\Upsilon^2 T(x) \quad ,
\end{aligned} \tag{7.83}$$

as expected. In the Fourier space, the Green's function and the Dirac delta function are

$$G_\Upsilon(x, x') = \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\tilde{G}_\Upsilon(k, k')\, d^4k \quad , \tag{7.84}$$

$$\delta^{(4)}(x - x') = \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\, d^4k \tag{7.85}$$

and inserting them into the Green's equation (7.81) gives

$$(\Box_x - \Upsilon^2)\left[\frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\tilde{G}_\Upsilon(k,k')\,d^4k\right] = \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\,d^4k$$

$$\left[\frac{1}{(2\pi)^4}\int\left(-\eta^{\mu\nu}k_\mu k_\nu - \Upsilon^2\right)e^{ik_\alpha(x^\alpha - x'^\alpha)}\tilde{G}_\Upsilon(k,k')\,d^4k\right] = \frac{1}{(2\pi)^4}\int e^{ik_\alpha(x^\alpha - x'^\alpha)}\,d^4k \quad .$$

Thus,

$$\boxed{\tilde{G}_\Upsilon(k,k') = -\frac{1}{k^2 - \left(\frac{\omega^2}{c^2} - \Upsilon^2\right)}} \tag{7.86}$$

Replacing equation (7.86) into the definition (7.84) one gets the Klein-Gordon propagator [58],

$$G_\Upsilon(x,x') = -\frac{1}{(2\pi)^4}\int\frac{e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)}}{k^2 - \tilde{\omega}^2}d^4k \quad , \tag{7.87}$$

where $\tilde{\omega}^2 = (\omega^2/c^2) - \Upsilon^2$, $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $\tau = t - t'$. If we align the z axis with the direction of $\mathbf{R}$, the integrand takes an azimuthal symmetry and it is possible to perform the integration in spherical coordinates for the variable k,

$$G_\Upsilon(x,x') = -\frac{1}{(2\pi)^4}\int\frac{e^{-i\omega\tau}e^{ikR\cos\theta}}{k^2 - \tilde{\omega}^2}k^2\sin\theta d\theta d\phi dk d\left(\frac{\omega}{c}\right) \tag{7.88}$$

with $|\mathbf{R}| = R$. Making the substitution $u = ikR\cos\theta$ and $du = -ikR\sin\theta d\theta$, the integral becomes

$$\begin{aligned}
\tilde{G}_\Upsilon(x,x') &= \frac{1}{(2\pi)^3 iRc}\int\frac{e^{-i\omega\tau}}{k^2 - \tilde{\omega}^2}\left[\int_{ikR}^{-ikR}e^u\,du\right]kdkd\omega \\
&= \frac{1}{(2\pi)^3 iRc}\int_{-\infty}^{\infty}\left\{\int_0^\infty\frac{e^{-i\omega\tau}ke^{-ikR}}{k^2 - \tilde{\omega}^2}\,dk - \underbrace{\int_0^\infty\frac{e^{-i\omega\tau}ke^{ikR}}{k^2 - \tilde{\omega}^2}\,dk}_{k\longrightarrow -k}\right\} \\
&= \frac{1}{(2\pi)^3 iRc}\int_{-\infty}^{\infty}e^{-i\omega\tau}\left[\int_{-\infty}^{\infty}\frac{ke^{-ikR}}{k^2 - \tilde{\omega}^2}\,dk\right]\,d\omega \quad .
\end{aligned} \tag{7.89}$$

The integral in brackets can be solved in the complex plane if we consider the following integral

$$\oint\frac{ze^{-izR}\,dz}{z^2 - (\tilde{\omega} + i\epsilon)^2} = \lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{ze^{-izR}\,dk}{z^2 - (\tilde{\omega} + i\epsilon)^2} + \lim_{\rho\to\infty}\int_{C_\rho^-}\frac{ze^{-izR}\,dk}{z^2 - (\tilde{\omega} + i\epsilon)^2} \tag{7.90}$$

around the closed path shown in figure (7.1). The second integral in equation (7.90) vanish by virtue of the Jordan's lemma [59]. It says that if $m < 0$ and P/Q is the quotient of two polynomials such that *degree* $Q \geq 1 +$ *degree* P then,

$$\lim_{\rho\to\infty}\int_{C_\rho^-}e^{imz}\frac{P(z)}{Q(z)}\,dz = 0 \tag{7.91}$$

where C_ρ^- is the lower half-circle of radius ρ.

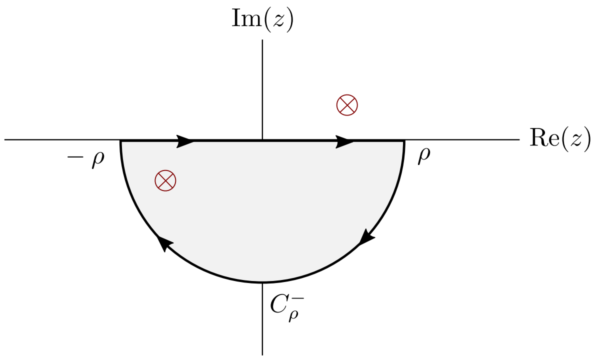

Figure 7.1: Closed loop for complex integration.

The poles of the integral are located at $z = \pm(\tilde{\omega} + i\epsilon)$. Using the Jordan's lemma along with the residue theorem, the integral is simply

$$\begin{aligned}
\oint \frac{ze^{-izR}\, dz}{z^2 - (\tilde{\omega} + i\epsilon)^2} &= \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{ze^{-izR}\, dk}{[z - (\tilde{\omega} + i\epsilon)][z + (\tilde{\omega} + i\epsilon)]} = -2\pi i \mathrm{Res}[-(\tilde{\omega} + i\epsilon)] \\
&= -2\pi i \lim_{z\to[-(\tilde{\omega}+i\epsilon)]} [z + (\tilde{\omega} + i\epsilon)] \frac{ze^{-izR}}{[z - (\tilde{\omega} + i\epsilon)][z + (\tilde{\omega} + i\epsilon)]} \\
&= -2\pi i \lim_{z\to[-(\tilde{\omega}+i\epsilon)]} \frac{-(\tilde{\omega} + i\epsilon)e^{i(\tilde{\omega}+i\epsilon)R}}{-2(\tilde{\omega} + i\epsilon)} \\
&= -\pi i e^{i(\tilde{\omega}+i\epsilon)R} \quad . \qquad (7.92)
\end{aligned}$$

Taking the limit where $\epsilon \to 0$, the integral in (7.89) gives

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{ke^{-ikR}}{k^2 - \tilde{\omega}^2}\, dk &= -\pi i e^{i\tilde{\omega}R} \\
&= -\pi i e^{i[(\omega^2/c^2) - \Upsilon^2]^{1/2} R} \qquad (7.93)
\end{aligned}$$

Finally, the Green's function for the Klein-Gordon equation is [27]

$$\boxed{G_\Upsilon(x, x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|c} \int e^{-i\omega(t-t')} e^{i[(\omega^2/c^2)-\Upsilon^2]^{1/2}|\mathbf{x}-\mathbf{x}'|} \frac{d\omega}{2\pi}} \qquad (7.94)$$

Especially, when $\Upsilon^2 = 0$ the Green's function simplifies to

$$\begin{aligned}
G_0(x, x') &= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|c} \int e^{-i\omega(t-t')} e^{i\omega|\mathbf{x}-\mathbf{x}'|/c} \frac{d\omega}{2\pi} \\
&= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|c} \int e^{i\omega\left[|\mathbf{x}-\mathbf{x}'|c^{-1} - (t-t')\right]} \frac{d\omega}{2\pi} \\
&= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|c} \delta\left[\frac{|\mathbf{x}-\mathbf{x}'|}{c} - (t - t')\right] \\
&= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta\left[c\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) - ct'\right] \quad , \qquad (7.95)
\end{aligned}$$

where in the last step we have used the properties $\delta(x) = \delta(-x)$ and $\delta(cx) = |c|^{-1}\delta(x)$. This is the same Green's function that was found in equation (5.15) for the wave equation. Now, substituting (7.94) into (7.82) one obtains the integral solution for the massive scalar mode,

$$\boxed{\overset{(1)}{R}(x) = \frac{8\pi G}{c^4}\Upsilon^2 \int \left[\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|c} \int e^{-i\omega(t-t')} e^{i[(\omega^2/c^2)-\Upsilon^2]^{1/2}|\mathbf{x}-\mathbf{x}'|} \frac{d\omega}{2\pi}\right] T(x')d^4x'} \quad (7.96)$$

A stationary Newtonian point source

The energy-momentum tensor of a stationary source has the property that is constant in time, i.e. $\partial_0 T^{\mu\nu} = 0$ [35]. This does not necessarily mean that the particles that make up the source are not moving[17]. A particular case of a stationary source is a *static* source. In this case, the velocity field of the matter distribution is zero and the particles actually are not in motion. As a consequence, the energy-momentum tensor must be also invariant under a time reversal transformation $t \to -t$, i.e. $T^{\mu\nu}(t) = T^{\mu\nu}(-t)$. In such a case, the only contribution of the energy-momentum tensor comes from the rest energy of the matter distribution. In fact, the *Newtonian limit* of a source is when the only important contribution of the energy-momentum tensor is T_{00} and thus,

$$T_{00} = \rho(\mathbf{x})c^2 \quad , \quad |T_{00}| \gg |T_{0i}| \quad , \quad |T_{00}| \gg |T_{ij}| \ . \quad (7.97)$$

The advantage when finding solutions for this kind of sources is that the time dependence of the energy-momentum vanishes and $T_{\mu\nu} = T_{\mu\nu}(\mathbf{x}')$. From equation (5.17), it follows that[18]

$$\bar{h}_{\mu\nu}(\mathbf{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \ . \quad (7.98)$$

For the Newtonian source limit the components in (7.98) read

$$\bar{h}_{00} = -\frac{4\Phi}{c^2} \ , \quad \bar{h}_{i0} = 0 \ , \quad \bar{h}_{ij} = 0 \quad (7.99)$$

where

$$\Phi \equiv -G\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \ . \quad (7.100)$$

As mentioned in Chapter 2, the quantity $\bar{h}_{\mu\nu}$ is known as the trace-reversed of the perturbation tensor $h_{\mu\nu}$ in linearized GR. Thus the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ may be rewritten as

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \ . \quad (7.101)$$

Since our signature is $(-+++)$, then for the Newtonian limit point source we have $\bar{h} = -\bar{h}_{00}$ and the line element from the metric in (7.101) gives,

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \ . \quad (7.102)$$

This equation is often referred to as the *line element* in the **Newtonian limit**. In particular, the density distribution of a point mass is given by $\rho(\mathbf{x}') = M\delta(\mathbf{x}')$ and therefore equation

[17] For example, a uniform rotating sphere is a stationary source and still is moving. What is constant is the *matter distribution* at each point in space.

[18] This expression is also valid for stationary sources.

(7.98) yields,

$$\bar{h}_{00} = \frac{4G}{c^4}\int \frac{Mc^2\delta(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \qquad \bar{h}_{0i} = 0\,,$$
$$= \frac{4GM}{c^2 r}\,, \qquad \bar{h}_{ij} = 0\,, \tag{7.103}$$

with $r = |\mathbf{x}|$. Then, the Newtonian line element becomes,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)\left(dx^2 + dy^2 + dz^2\right) \;. \tag{7.104}$$

which is identical as the expansion of the Schwarzschild line element to first order in M. Observe that the definition for the quantity Φ makes sense because it corresponds to the Newtonian potential of a point particle. Nevertheless, if one extends the previous results to linearized $f(R)$ gravity, the metric solution should be different because of the extra scalar mode present in the theory. Indeed, the perturbation tensor does not follow a trace-reversed form with $\bar{h}_{\mu\nu}$. As we saw in the last section, the form of $h_{\mu\nu}$ is given by equation (7.63). Consequently, the metric solution in linearized $f(R)$ gravity is given by

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} - a_2\overset{(1)}{R}\eta_{\mu\nu}} \;. \tag{7.105}$$

The massless sector in this theory is the same as in GR. So, the solution $\bar{h}_{\mu\nu}$ given in equation (7.103) for a static point source is the same. On the other hand, using $T(x') = -Mc^2\delta(\mathbf{x}')$ in equation (7.96) one obtains the solution of the Ricci scalar propagating mode[19]

$$\begin{aligned}
\overset{(1)}{R}(x) &= -\frac{8\pi G}{c^4}\Upsilon^2\int\left[\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|c}\int e^{-i\omega(t-t')}e^{i[(\omega^2/c^2)-\Upsilon^2]^{1/2}|\mathbf{x}-\mathbf{x}'|}\frac{d\omega}{2\pi}\right]Mc^2\delta^{(3)}(\mathbf{x}')\,d^4x' \\
&= -\frac{2GM\Upsilon^2}{c^2}\int\left[\int e^{-i\omega t}\left(\int e^{i\omega t'}\,dt'\right)e^{i[(\omega^2/c^2)-\Upsilon^2]^{1/2}|\mathbf{x}-\mathbf{x}'|}\frac{d\omega}{2\pi}\right]\frac{\delta^{(3)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \\
&= -\frac{2GM\Upsilon^2}{c^2}\int\left[\int e^{-i\omega t}e^{i[(\omega^2/c^2)-\Upsilon^2]^{1/2}|\mathbf{x}-\mathbf{x}'|}\delta(\omega)d\omega\right]\frac{\delta^{(3)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \\
&= -\frac{2GM\Upsilon^2}{c^2}\int\frac{e^{-\Upsilon|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\delta^{(3)}(\mathbf{x}')\,d^3x'
\end{aligned} \tag{7.106}$$

i.e.,

$$\boxed{\overset{(1)}{R} = -\frac{2GM\Upsilon^2}{c^2}\frac{e^{-\Upsilon r}}{r}} \;. \tag{7.107}$$

Finally, using $\Upsilon^2 = \frac{1}{3a_2}$ and inserting equations (7.107) and (7.103) into the metric $g_{\mu\nu}$ in (7.105) one obtains[20],

$$\boxed{ds^2 = -\left[1 - \frac{2GM}{c^2 r}\left(1 + \frac{1}{3}e^{-\Upsilon r}\right)\right]c^2dt^2 + \left[1 - \frac{2GM}{c^2 r}\left(1 - \frac{1}{3}e^{-\Upsilon r}\right)\right]dl^2} \;. \tag{7.108}$$

[19]The minus sign in $T(x') = -Mc^2\delta(\mathbf{x}')$ is due to the trace in our signature convention.
[20]Here we have defined $dl^2 = dx^2 + dy^2 + dz^2$.

7.5 The Energy-Momentum tensor of GWs

In Chapter 3, the interaction of GWs with test masses has been examined. The physical implications of the massless modes shows that in fact GWs carry energy. For linearized $f(R)$ gravity, the extra scalar mode implies not only a transverse contribution but also a *longitudinal* force [60]. This means that massive GWs can do work on point particles. Thus, it is expected that the total energy of GWs should be the energy of the massless modes plus a correction that comes from the Ricci scalar. Since this information is contained in the energy-momentum tensor, we shall derive it by following the same approach given in Chapter 4. As in GR, one is able to use perturbation theory to solve the field equations order by order. At second order, the energy of GWs act as a source for the gravitational field itself, i.e.

$$\overset{(1)}{\mathcal{G}}_{\mu\nu}[h^{(2)}] = \frac{8\pi G}{c^4}\tilde{t}_{\mu\nu} \tag{7.109}$$

where

$$\tilde{t}_{\mu\nu} = -\frac{c^4}{8\pi G}\overset{(2)}{\mathcal{G}}_{\mu\nu}[h^{(1)}] \quad . \tag{7.110}$$

Indeed, the quantity $\tilde{t}_{\mu\nu}$ is conserved by means of the Bianchi identities and equation (7.12)[21]. However, this is not an invariant tensor under gauge transformations unless a shortwave average is performed and

$$\tilde{t}_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle \overset{(2)}{\mathcal{G}}_{\mu\nu}[h^{(1)}] \right\rangle \quad . \tag{7.111}$$

Henceforth, an expression for $\tilde{t}_{\mu\nu}$ will be deduced. Lets begin with the full modified Einstein tensor from equation (7.6). It is equivalent to

$$\mathcal{G}_{\mu\nu} = f' R_{\mu\nu} - \nabla_\mu\left(\partial_\nu f'\right) + g_{\mu\nu}g^{\rho\sigma}\nabla_\rho(\partial_\sigma f') - \frac{1}{2}f g_{\mu\nu} \tag{7.112}$$

One might expand up to second order the connections, the Ricci tensor, the Ricci scalar, the function $f(R)$ and its derivative in the following form[22]

$$\Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} + \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} \, , \tag{7.113}$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu} \, , \tag{7.114}$$

$$R = \bar{R} + \overset{(1)}{R} + \overset{(2)}{R} \, , \tag{7.115}$$

$$f(R) = \bar{f}(R) + \overset{(1)}{f}(R) + \overset{(2)}{f}(R) \, , \tag{7.116}$$

$$f'(R) = \overset{(0)}{f'} + \bar{f}'(R) + \overset{(1)}{f'}(R) + \overset{(2)}{f'}(R) \quad . \tag{7.117}$$

where the terms with an upper bar at the righ-hand side of these equations are referred to the background spacetime. These expressions are obtained explicitly in Appendix B. To get $\mathcal{G}_{\mu\nu}$ to second order, we will focus on each of the terms of equation (7.112) separately. For simplicity, the following computations assumes a reference frame at spatial infinity from the sources such that the metric is written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{7.118}$$

[21] This is proved after contracting equation (7.109) with the covariant derivative.

[22] The zero order in the first term of equation (7.117) is introduced for convenience but is actually equal to the unit. See Appendix B.

with

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}(\varepsilon^2) \quad . \tag{7.119}$$

From the discussion in Chapter 4, since $\bar{\mathcal{G}}_{\mu\nu}$ is of order $\mathcal{O}(\varepsilon^2)$, the term $j_{\mu\nu}$ in (7.119) is of order $\mathcal{O}(\varepsilon^2)$ to ensure that $\bar{R}_{\mu\nu}$ is of order $\mathcal{O}(\varepsilon^2)$ from which $\bar{\mathcal{G}}_{\mu\nu}$ is constructed. The first term in (7.112) gives

$$\begin{aligned} f'R_{\mu\nu} &= \left(\overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)\left(\bar{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu}\right) + \cdots \\ &= \underbrace{\bar{R}_{\mu\nu}}_{(B)} + \underbrace{\overset{(1)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon)} + \underbrace{\overset{(2)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^2)} + \underbrace{\bar{f}'\bar{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^4)} + \underbrace{\bar{f}'\overset{(1)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\bar{f}'\overset{(2)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^4)} \\ &\quad + \underbrace{\overset{(1)}{f'}\bar{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\overset{(1)}{f'}\overset{(1)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^2)} + \underbrace{\overset{(1)}{f'}\overset{(2)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\overset{(2)}{f'}\bar{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^4)} + \underbrace{\overset{(2)}{f'}\overset{(1)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\overset{(2)}{f'}\overset{(2)}{R}_{\mu\nu}}_{\mathcal{O}(\varepsilon^4)} + \cdots \end{aligned}$$

Retaining only the second order terms yields

$$\left[f'R_{\mu\nu}\right]^{(2)} = \overset{(2)}{R}_{\mu\nu} + \overset{(1)}{f'}\overset{(1)}{R}_{\mu\nu} \quad . \tag{7.120}$$

The second term in (7.112) is

$$\begin{aligned} -\nabla_\mu\left(\partial_\nu f'\right) &= -\partial_\mu\partial_\nu f' + \Gamma^\rho_{\nu\mu}\partial_\rho f' \\ &= -\partial_\mu\partial_\nu\left(\overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right) + \\ &\quad \left(\bar{\Gamma}^\rho_{\nu\mu} + \overset{(1)}{\Gamma}{}^\rho_{\nu\mu} + \overset{(2)}{\Gamma}{}^\rho_{\nu\mu}\right)\partial_\rho\left(\overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right) + \cdots \end{aligned} \tag{7.121}$$

After expanding (7.121) and leaving only the second order terms then,

$$[-\nabla_\mu(\partial_\nu f')]^{(2)} = -\partial_\mu\partial_\nu\overset{(2)}{f'} + \overset{(1)}{\Gamma}{}^\rho_{\nu\mu}\partial_\rho\overset{(1)}{f'} \quad . \tag{7.122}$$

At second order, is equivalent to write the background metric as $\eta_{\mu\nu}$ if it appears multiplying a term of order $\mathcal{O}(\varepsilon)$ or greater[23]. Thus, from now on and for clarity, when an expression includes a product with the background metric, the order of $\bar{g}_{\mu\nu}$ is not taking into account in the following notation. Making the order of the background metric to be $\mathcal{O}(0)$, then an expression of order $\mathcal{O}(n)$ is equal to the order of such expression without counting the background metric order. So for example, a term of order $\mathcal{O}(2)$ consist actually in two terms, one of order $\mathcal{O}(\varepsilon^2)$ and other of order $\mathcal{O}(\varepsilon^4)$ [24]. The third term in (7.112) is given by

$$\begin{aligned} g_{\mu\nu}\Box f' &= g_{\mu\nu}\nabla^\sigma\left(\partial_\sigma f'\right) \\ &= g_{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma f' - g_{\mu\nu}g^{\rho\sigma}\Gamma^\lambda_{\sigma\rho}\partial_\lambda f' \quad . \end{aligned} \tag{7.123}$$

Then,

[23] For instance, $\bar{g}_{\mu\nu}h^{\rho\delta}h^\sigma_\delta = (\eta_{\mu\nu} + j_{\mu\nu})h^{\rho\delta}h^{gma}_\delta = \eta_{\mu\nu}h^{\rho\delta}h^\sigma_\delta + \mathcal{O}(\varepsilon^4)$.

[24] We follow the work in [27] using the background metric instead of the flat one by retaining all terms of order $\mathcal{O}(2)$ that are not fully from the background as in GR.

$$g_{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma f' = (\bar{g}_{\mu\nu} + h_{\mu\nu})(\bar{g}^{\rho\sigma} - h^{\rho\sigma} + h^{\rho\delta}h^{\sigma}_{\delta})\partial_\rho\partial_\sigma\left(\overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$= \left(\underbrace{\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}}_{\mathcal{O}(0)} - \underbrace{\bar{g}_{\mu\nu}h^{\rho\sigma}}_{\mathcal{O}(1)} + \underbrace{\bar{g}_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}}_{\mathcal{O}(2)} + \underbrace{h_{\mu\nu}\bar{g}^{\rho\sigma}}_{\mathcal{O}(1)} - \underbrace{h_{\mu\nu}h^{\rho\sigma}}_{\mathcal{O}(\varepsilon^2)} + \underbrace{h_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}}_{\mathcal{O}(\varepsilon^3)}\right)$$

$$\times\,\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$= \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right) - \bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$+ \bar{g}_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right) + h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$- h_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right) + h_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\left(\bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$g_{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma f' = \underbrace{\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\text{Background}} + \underbrace{\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(1)} + \underbrace{\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(2)} - \underbrace{\bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\mathcal{O}(3)}$$

$$- \underbrace{\bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(2)} - \underbrace{\bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(3)} + \underbrace{\bar{g}_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\mathcal{O}(4)} + \underbrace{\bar{g}_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(3)}$$

$$+ \underbrace{\bar{g}_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(4)} + \underbrace{h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\mathcal{O}(3)} + \underbrace{h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(2)} + \underbrace{h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(3)}$$

$$- \underbrace{h_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\mathcal{O}(4)} - \underbrace{h_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(3)} - \underbrace{h_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(4)} + \underbrace{h_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\bar{f}'}_{\mathcal{O}(5)}$$

$$+ \underbrace{h_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'}}_{\mathcal{O}(4)} + \underbrace{h_{\mu\nu}h^{\rho\delta}h_{\delta}{}^{\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'}}_{\mathcal{O}(5)}$$

$$\left[g_{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma f'\right]^{(2)} = \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'} - \bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'} + h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'} \tag{7.124}$$

The second term in (7.123) is a bit longer, but there is only one contribution at second order. The reader can check it in a similar way like the previous one. The result is,

$$-g_{\mu\nu}g^{\rho\sigma}\Gamma^{\lambda}{}_{\sigma\rho}\partial_\lambda f' = (\bar{g}_{\mu\nu} + h_{\mu\nu})\left(\bar{g}^{\rho\sigma} - h^{\rho\sigma} + h^{\rho\delta}h_{\delta}{}^{\sigma}\right)\left(\bar{\Gamma}^{\lambda}{}_{\sigma\rho} + \overset{(1)}{\Gamma}{}^{\lambda}_{\sigma\rho} + \overset{(2)}{\Gamma}{}^{\lambda}_{\sigma\rho}\right)$$

$$\times\,\partial_\lambda\left(\overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\right)$$

$$\left[-g_{\mu\nu}g^{\rho\sigma}\Gamma^{\lambda}{}_{\sigma\rho}\partial_\lambda f'\right]^{(2)} = -\,\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\overset{(1)}{\Gamma}{}^{\lambda}_{\sigma\rho}\partial_\lambda\overset{(1)}{f'}\;. \tag{7.125}$$

Therefore,

$$[g_{\mu\nu}\Box f']^{(2)} = \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(2)}{f'} - \bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'} + h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{f'} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\overset{(1)}{\Gamma}{}^{\lambda}_{\sigma\rho}\partial_\lambda\overset{(1)}{f'}\;. \tag{7.126}$$

The last term in (7.112) yields,

$$
\begin{aligned}
-\frac{1}{2} f g_{\mu\nu} &= -\frac{1}{2}\left(\overset{(0)}{f} + \bar{f} + \overset{(1)}{f} + \overset{(2)}{f}\right)\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) \\
&= -\frac{1}{2}\left(\underbrace{\bar{f}\bar{g}_{\mu\nu}}_{(B)} + \underbrace{\bar{f}h_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\overset{(1)}{f}\bar{g}_{\mu\nu}}_{\mathcal{O}(1)} + \underbrace{\overset{(1)}{f}h_{\mu\nu}}_{\mathcal{O}(\varepsilon^2)} + \underbrace{\overset{(2)}{f}\bar{g}_{\mu\nu}}_{\mathcal{O}(2)} + \underbrace{\overset{(2)}{f}h_{\mu\nu}}_{\mathcal{O}(\varepsilon^3)}\right)
\end{aligned}
$$

$$
\left[-\frac{1}{2}\overset{(1)}{f} g_{\mu\nu}\right]^{(2)} = -\frac{1}{2}\overset{(1)}{f} h_{\mu\nu} - \frac{1}{2}\overset{(2)}{f}\bar{g}_{\mu\nu} \tag{7.127}
$$

Now, inserting equations (7.127), (7.126), (7.122) and (7.120) into (7.112), it is obtained the modified Einstein tensor at second order,

$$
\begin{aligned}
\overset{(2)}{\mathcal{G}}_{\mu\nu} = {} & \overset{(2)}{R}_{\mu\nu} + \overset{(1)}{f'}\overset{(1)}{R}_{\mu\nu} - \partial_\mu\partial_\nu \overset{(2)}{f'} + \overset{(1)}{\Gamma}{}^{\rho}_{\nu\mu}\partial_\rho \overset{(1)}{f'} + \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma \overset{(2)}{f'} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\overset{(1)}{\Gamma}{}^{\lambda}_{\sigma\rho}\partial_\lambda \overset{(1)}{f'} \\
& - \bar{g}_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma \overset{(1)}{f'} + h_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\rho\partial_\sigma \overset{(1)}{f'} - \frac{1}{2}\overset{(1)}{f} h_{\mu\nu} - \frac{1}{2}\overset{(2)}{f}\bar{g}_{\mu\nu} \ .
\end{aligned} \tag{7.128}
$$

From Appendix B, using the equations for f, f', $\bar{\Gamma}^{\alpha}_{\mu\nu}$, $\bar{g}_{\mu\nu}$, $h_{\mu\nu}$ which are shown in (B.35), (B.37), (B.3), (7.119) and (7.63) along with the Klein-Gordon equation in (B.31), it follows that

$$
\begin{aligned}
\overset{(2)}{\mathcal{G}}_{\mu\nu} = {} & \overset{(2)}{R}_{\mu\nu} + a_2\overset{(1)}{R}\overset{(1)}{R}_{\mu\nu} - a_2\partial_\mu\partial_\nu\overset{(2)}{R} - \frac{1}{2}a_3\partial_\mu\partial_\nu\overset{(1)}{R}{}^2 + \frac{1}{2}\Bigg[\left(\partial_\mu\bar{h}_{\rho\nu} + \partial_\nu\bar{h}_{\rho\mu} - \partial_\rho\bar{h}_{\mu\nu}\right) \\
& - a_2\left(\bar{g}_{\rho\mu}\partial_\nu\overset{(1)}{R} + \bar{g}_{\rho\nu}\partial_\mu\overset{(1)}{R} - \bar{g}_{\mu\nu}\partial_\rho\overset{(1)}{R}\right)\Bigg] a_2\partial^\rho\overset{(1)}{R} + a_2\bar{g}_{\mu\nu}\bar{\Box}\overset{(2)}{R} + \frac{1}{2}a_3\bar{g}_{\mu\nu}\bar{\Box}\overset{(1)}{R}{}^2 \\
& - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\Bigg[\left(\partial_\sigma\bar{h}_{\delta\rho} + \partial_\rho\bar{h}_{\delta\sigma} - \partial_\delta\bar{h}_{\sigma\rho}\right) - a_2\overset{(1)}{R}\left(\partial_\sigma j_{\delta\rho} + \partial_\rho j_{\delta\sigma} - \partial_\delta j_{\sigma\rho}\right)\Bigg] a_2\partial^\delta\overset{(1)}{R} \\
& - a_2\bar{g}_{\mu\nu}\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{R} + a_2^2\bar{g}_{\mu\nu}\overset{(1)}{R}\bar{\Box}\overset{(1)}{R} + a_2\bar{h}_{\mu\nu}\underbrace{\bar{\Box}\overset{(1)}{R}}_{\text{(KG)}} - a_2^2\bar{g}_{\mu\nu}\overset{(1)}{R}\bar{\Box}\overset{(1)}{R} - \frac{1}{2}\overset{(2)}{R}\bar{g}_{\mu\nu} - \frac{1}{4}a_2\bar{g}_{\mu\nu}\overset{(1)}{R}{}^2 \\
& - \frac{1}{2}\overset{(1)}{R}\bar{h}_{\mu\nu} + \frac{1}{2}a_2\overset{(1)}{R}{}^2 \ .
\end{aligned} \tag{7.129}
$$

To further simplify the result in (7.129), we choose a reference frame where the Lorenz gauge $\partial^\nu\bar{h}_{\mu\nu} = 0$ and the traceless condition $\bar{h} = 0$ are valid. Thus,

$$
\begin{aligned}
\overset{(2)}{\mathcal{G}}_{\mu\nu} = {} & \overset{(2)}{R}_{\mu\nu} + a_2\left(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\right)\overset{(2)}{R} - \frac{1}{2}\overset{(2)}{R}\bar{g}_{\mu\nu} + \frac{a_3}{2}\left(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\right)\overset{(1)}{R}{}^2 - \frac{1}{6}\bar{h}_{\mu\nu}\overset{(1)}{R} \\
& - a_2\bar{g}_{\mu\nu}\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{R} + \frac{a_2}{2}\partial^\rho\overset{(1)}{R}\left(\partial_\mu\bar{h}_{\rho\nu} + \partial_\nu\bar{h}_{\rho\mu} - \partial_\rho\bar{h}_{\mu\nu}\right) + a_2\left(\overset{(1)}{R}\overset{(1)}{R}_{\mu\nu} + \frac{1}{4}\overset{(1)}{R}{}^2\bar{g}_{\mu\nu}\right) \\
& - a_2^2\left(\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{1}{2}\bar{g}_{\mu\nu}\partial^\rho\overset{(1)}{R}\partial_\rho\overset{(1)}{R}\right) + \frac{1}{2}\bar{g}_{\mu\nu}\left(\underbrace{\partial^\rho\bar{h}_{\delta\rho}}_{=0} + \underbrace{\partial^\sigma\bar{h}_{\delta\sigma}}_{=0} - \underbrace{\partial_\delta\bar{h}}_{=0}\right)a_2\partial^\delta\overset{(1)}{R} + \mathcal{O}(\varepsilon^4)
\end{aligned}
$$

The modified Einstein tensor $\mathcal{G}_{\mu\nu}$ with only second order terms are given by

$$
\begin{aligned}
\overset{(2)}{\mathcal{G}}_{\mu\nu} \;=\; & \overset{(2)}{R}_{\mu\nu} + a_2\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(2)}{R} - \frac{1}{2}\overset{(2)}{R}\bar{g}_{\mu\nu} + \frac{a_3}{2}\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(1)}{R}{}^2 - \frac{1}{6}\bar{h}_{\mu\nu}\overset{(1)}{R} \\
& - a_2\bar{g}_{\mu\nu}\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{R} + \frac{a_2}{2}\partial^\rho\overset{(1)}{R}\Big(\partial_\mu\bar{h}_{\rho\nu} + \partial_\nu\bar{h}_{\rho\mu} - \partial_\rho\bar{h}_{\mu\nu}\Big) + a_2\Big(\overset{(1)}{R}\overset{(1)}{R}_{\mu\nu} + \frac{1}{4}\overset{(1)}{R}{}^2\bar{g}_{\mu\nu}\Big) \\
& - a_2^2\Big(\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{1}{2}\bar{g}_{\mu\nu}\partial^\rho\overset{(1)}{R}\partial_\rho\overset{(1)}{R}\Big) \;. \qquad (7.130)
\end{aligned}
$$

The energy-momentum $\tilde{t}_{\mu\nu}$ can be obtained using the equation (7.130) into (7.111). Thereby,

$$
\begin{aligned}
t_{\mu\nu} \;=\; & -\frac{c^4}{8\pi G}\left\langle \overset{(2)}{\mathcal{G}}_{\mu\nu} \right\rangle \\
& - \frac{c^4}{8\pi G}\Big\langle \overset{(2)}{R}_{\mu\nu} + a_2\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(2)}{R} - \frac{1}{2}\overset{(2)}{R}\bar{g}_{\mu\nu} + \frac{a_3}{2}\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(1)}{R}{}^2 - \frac{1}{6}\bar{h}_{\mu\nu}\overset{(1)}{R} \\
& - a_2\bar{g}_{\mu\nu}\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma\overset{(1)}{R} + \frac{a_2}{2}\partial^\rho\overset{(1)}{R}\Big(\partial_\mu\bar{h}_{\rho\nu} + \partial_\nu\bar{h}_{\rho\mu} - \partial_\rho\bar{h}_{\mu\nu}\Big) + a_2\Big(\overset{(1)}{R}\overset{(1)}{R}_{\mu\nu} + \frac{1}{4}\overset{(1)}{R}{}^2\bar{g}_{\mu\nu}\Big) \\
& - a_2^2\Big(\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{1}{2}\bar{g}_{\mu\nu}\partial^\rho\overset{(1)}{R}\partial_\rho\overset{(1)}{R}\Big)\Big\rangle \qquad (7.131)
\end{aligned}
$$

where $\langle\cdot\rangle$ is the linear integral operator that average over several wavelengths. We compute each term in equation (7.131) using the properties

$$
\langle\partial_\mu V\rangle = 0 \qquad , \qquad \langle U\partial_\mu V\rangle = -\langle V\partial_\mu U\rangle \qquad (7.132)
$$

along with the gauge condition and wave equations. This allows us to simplify many terms in (7.131). Using the Ricci tensor and the Ricci scalar to second order given in Appendix B, each term in (7.131) are calculated as follows,

$$
\begin{aligned}
\left\langle \overset{(2)}{R}_{\mu\nu} \right\rangle \;=\; & \Big\langle \frac{1}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} - \frac{1}{2}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} - \underbrace{\partial_\sigma\bar{h}^{\sigma\rho}\partial_\rho\bar{h}_{\mu\nu}}_{=0} + \frac{a_2}{2}\underbrace{\bar{g}_{\mu\nu}\partial_\sigma\bar{h}^{\sigma\rho}\partial_\rho\overset{(1)}{R}}_{=0} \\
& + \frac{1}{2}\underbrace{\partial_\rho\bar{h}^{\sigma\rho}\partial_\nu\bar{h}_{\sigma\mu}}_{=0} - \frac{a_2}{2}\underbrace{\bar{g}_{\sigma\mu}\partial_\rho\bar{h}^{\sigma\rho}\partial_\nu\overset{(1)}{R}}_{=0} + \frac{1}{2}\underbrace{\partial_\rho\bar{h}^{\sigma\rho}\partial_\mu\overset{(1)}{R}}_{=0} - \frac{a_2}{2}\underbrace{\bar{g}_{\sigma\nu}\partial_\rho\bar{h}^{\sigma\rho}\partial_\mu\overset{(1)}{R}}_{=0} \\
& - \frac{1}{2}\underbrace{\bar{h}^\sigma{}_\nu\bar{\Box}\bar{h}_{\sigma\mu}}_{=0} + \frac{1}{2}\underbrace{\bar{h}^\sigma{}_\nu\partial_\sigma\partial^\rho\bar{h}_{\rho\mu}}_{=0} + \frac{a_2}{2}\underbrace{\overset{(1)}{R}\bar{\Box}\bar{h}_{\mu\nu}}_{=0} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} \\
& + \frac{3a_2}{2}\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{a_2^2}{2}\overset{(1)}{R}\underbrace{\bar{\Box}\overset{(1)}{R}}_{(\mathrm{KG})}\bar{g}_{\mu\nu}\Big\rangle \\
=\; & \Big\langle -\frac{1}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{3}{2}a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{a_2}{6}\bar{g}_{\mu\nu}\overset{(1)}{R}{}^2\Big\rangle \\
\left\langle \overset{(2)}{R}_{\mu\nu} \right\rangle \;=\; & \Big\langle -\frac{1}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} + \frac{a_2^2}{2}\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \frac{a_2}{6}\bar{g}_{\mu\nu}\overset{(1)}{R}{}^2\Big\rangle \;, \qquad (7.133)
\end{aligned}
$$

$$\left\langle a_2\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(2)}{R}\right\rangle = \left\langle a_2\bar{g}_{\mu\nu}\bar{\Box}\overset{(2)}{R} - a_2\partial_\mu\partial_\nu\overset{(2)}{R}\right\rangle$$

$$= \left\langle a_2\bar{g}_{\mu\nu}\partial_\rho\left(\partial^\rho\overset{(2)}{R}\right)\right\rangle = -\left\langle a_2\partial^\rho\overset{(2)}{R}\partial_\rho\bar{g}_{\mu\nu}\right\rangle = 0 \ , \tag{7.134}$$

$$\left\langle -\frac{1}{2}\overset{(2)}{R}\bar{g}_{\mu\nu}\right\rangle = \left\langle -\frac{3}{4}\underbrace{\bar{h}^{\sigma\rho}\bar{\Box}\bar{h}_{\sigma\rho}}_{=0} + \frac{1}{2}\underbrace{\bar{h}_{\rho\mu}\partial^\rho\partial_\sigma\bar{h}^{\sigma\mu}}_{=0} + \underbrace{2a_2\partial_\nu\overset{(1)}{R}\partial_\mu\bar{h}^{\mu\nu}}_{=0}\right.$$

$$\left. + 2a_2\overset{(1)}{R}{}^2 - \frac{3}{2}a_2^2\overset{(1)}{R}\underbrace{\bar{\Box}\overset{(1)}{R}}_{\text{(KG)}}\right\rangle$$

$$= \left\langle -\frac{3}{4}a_2\overset{(1)}{R}{}^2\bar{g}_{\mu\nu}\right\rangle \ , \tag{7.135}$$

$$\left\langle \frac{a_3}{2}\Big(\bar{g}_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu\Big)\overset{(1)}{R}{}^2\right\rangle = \left\langle \frac{a_3}{2}\bar{g}_{\mu\nu}\bar{\Box}\overset{(1)}{R}{}^2 - \frac{a_3}{2}\partial_\mu\partial_\nu\overset{(1)}{R}{}^2\right\rangle$$

$$= \left\langle \frac{a_3}{2}\bar{g}_{\mu\nu}\partial_\rho\left(\partial^\rho\overset{(1)}{R}{}^2\right)\right\rangle = -\left\langle a_2\partial^\rho\overset{(1)}{R}{}^2\partial_\rho\bar{g}_{\mu\nu}\right\rangle = 0 \ , \tag{7.136}$$

$$\left\langle -\frac{1}{6}\bar{h}_{\mu\nu}\overset{(1)}{R}\right\rangle = \left\langle -\frac{1}{6}\bar{h}_{\mu\nu}\left(3a_2\bar{\Box}\overset{(1)}{R}\right)\right\rangle = \left\langle -\frac{a_2}{2}\bar{h}_{\mu\nu}\partial_\rho\left(\partial^\rho\overset{(1)}{R}\right)\right\rangle$$

$$= \left\langle \frac{a_2}{2}\left(\partial^\rho\overset{(1)}{R}\right)\partial_\rho\bar{h}_{\mu\nu}\right\rangle = \left\langle \frac{a_2}{2}\partial^\rho\bar{h}_{\mu\nu}\partial_\rho\overset{(1)}{R}\right\rangle$$

$$-\left\langle \frac{a_2}{2}\overset{(1)}{R}\bar{\Box}\bar{h}_{\mu\nu}\right\rangle = 0 \ , \tag{7.137}$$

$$\left\langle -a_2\bar{g}_{\mu\nu}\bar{h}^{\sigma\rho}\partial_\sigma\left(\partial_\rho\overset{(1)}{R}\right)\right\rangle = \left\langle a_2\partial_\rho\overset{(1)}{R}\underbrace{\partial_\sigma\bar{h}^{\sigma\rho}}_{=0}\right\rangle = 0 \ ,$$

$$\left\langle \frac{a_2}{2}\left(\partial^\rho\overset{(1)}{R}\partial_\mu\bar{h}_{\rho\nu} + \partial^\rho\overset{(1)}{R}\partial_\nu\bar{h}_{\rho\mu} - \partial^\rho\overset{(1)}{R}\partial_\rho\bar{h}_{\mu\nu}\right)\right\rangle$$

$$= \left\langle \frac{a_2}{2}\left(\overset{(1)}{R}\partial_\mu\underbrace{\partial^\rho\bar{h}_{\rho\nu}}_{=0} + \overset{(1)}{R}\partial_\nu\underbrace{\partial^\rho\bar{h}_{\rho\mu}}_{=0} - \overset{(1)}{R}\underbrace{\bar{\Box}\bar{h}_{\mu\nu}}_{=0}\right)\right\rangle = 0 \ , \tag{7.138}$$

$$
\begin{aligned}
\left\langle a_2 \overset{(1)}{R}\overset{(1)}{R}_{\mu\nu} \right\rangle &= \left\langle a_2 \left(a_2 \overset{(1)}{R} \partial_\mu \partial_\nu \overset{(1)}{R} + \frac{1}{6} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 \right) \right\rangle \\
&\quad \left\langle - a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2}{6} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 \right\rangle \ , \qquad (7.139)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle - a_2^2 \left(\partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{1}{2} \bar{g}_{\mu\nu} \partial^\rho \overset{(1)}{R} \partial_\rho \overset{(1)}{R} \right) \right\rangle &= \left\langle - a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2^2}{2} \bar{g}_{\mu\nu} \overset{(1)}{R} \bar{\Box} \overset{(1)}{R} \right\rangle \\
&= \left\langle - a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2^2}{6} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 \right\rangle \ . \qquad (7.140)
\end{aligned}
$$

Finally, adding all previous nonvanishing terms in the definition of the energy momentum tensor gives,

$$
\begin{aligned}
t_{\mu\nu} &= -\frac{c^4}{8\pi G} \left\langle - \frac{1}{4} \partial_\mu \bar{h}_{\sigma\rho} \partial_\nu \bar{h}^{\sigma\rho} + \frac{a_2^2}{2} \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2}{6} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 - \frac{3a_2}{4} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 \right. \\
&\quad \left. - a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2}{6} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 + \frac{a_2}{4} \bar{g}_{\mu\nu} \overset{(1)}{R}{}^2 - a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} + \frac{a_2^2}{6} \bar{g}_{\mu\nu} \overset{(1)}{R} \right\rangle \\
&= -\frac{c^4}{8\pi G} \left\langle - \frac{1}{4} \partial_\mu \bar{h}_{\sigma\rho} \partial_\nu \bar{h}^{\rho\sigma} - \frac{3}{2} a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} \right\rangle \ .
\end{aligned}
$$

Thus,

$$
\boxed{ t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu \bar{h}_{\sigma\rho} \partial_\nu \bar{h}^{\rho\sigma} + 6 a_2^2 \partial_\mu \overset{(1)}{R} \partial_\nu \overset{(1)}{R} \right\rangle } \ . \qquad (7.141)
$$

In the limit $a_2 \to 0$ the GR expression for $t_{\mu\nu}$ is recovered as expected. This result is also obtained when $R^{(1)} = 0$ as would be the case if the Ricci mode of propagation was not excited [27]. The last expression can be also rewritten in terms of the original perturbation tensor $h_{\mu\nu}$,

$$
t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu h_{\sigma\rho} \partial_\nu h^{\sigma\rho} + \frac{1}{8} \partial_\mu h \partial_\nu h \right\rangle \ . \qquad (7.142)
$$

The main conclusion of this chapter is that the energy-momentum tensor $\tilde{t}_{\mu\nu}$ at spatial infinity has an extra contribution that comes from the Ricci scalar mode. Therefore, the energy as well as the power radiated from a source is not exactly as in GR. However, is worthwhile to assume that the frequency of the Ricci mode is below the cutoff frequency Υ as a good approximation[25]. In this case, one are left with the same expressions as in GR but with a modified potential at the Newtonian limit.

[25] This is reasonable because the GR limit is recovered with $a_2 \to 0$ so that $\Upsilon \to \infty$

Chapter 8

Yukawa-like Binary System

The main goal of this chapter is to introduce the Yukawa-like gravitational potential in a binary system. The solution of the two-body problem is presented with an approximation method, showing that it admits circular trajectories as expected. The two polarizations for the massless modes of linearized $f(R)$ gravity for a circular orbit in a binary system, is plotted initially without back-reaction. Likewise, the chirp waveform by the emission of GWs from this source is obtained only at linear order in Υ. In a similar way, GWs emitted by elliptical orbits may be extended in the basis of the Yukawa-like two body solution for future works.

8.1 Yukawa-like potential

The Newtonian limit of $f(R)$ gravity for a static point particle was devised in section 7.4 of Chapter 7. The line element in (7.108) has the same functional form of (7.102). Thus, it can be rewritten as

$$ds^2 = -\left(1 + \frac{2\Psi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Psi}{c^2}\right)(dx^2 + dy^2 + dz^2) \tag{8.1}$$

where

$$\Psi \equiv -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-\Upsilon r}\right) \quad . \tag{8.2}$$

Furthermore, realize that in the Newtonian limit the spatial part of the geodesic equation is given by

$$\frac{dx^i}{dt} = \frac{1}{2}c^2 \partial_i h_{00} \quad . \tag{8.3}$$

From the line element in (8.1) it follows that $h_{00} = -2\Psi/c^2$ and thus the equation (8.3) reads in vectorial form as

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Psi \quad . \tag{8.4}$$

This is just the equation of motion of a particle in a potential Ψ. Therefore, the expression in (8.2) represents the potential of a static point particle at the Newtonian limit of $f(R)$ gravity. Observe that this potential is very similar to the Newtonian one but with a Yukawa-like addition. In fact, this type of potential has been studied in other contexts[]. We shall use this modified potential in the mutual interaction of a two point particles in circular motion.

8.2 The two-body problem

In Chapter 6 it was mentioned that the problem of two interacting bodies under a mutual gravitational interaction is equivalent to an effective one body problem. If the bodies are point particles with masses m_1 and m_2, the Lagrangian in the center of mass of the binary system is

$$\mathcal{L} = \frac{1}{2}\mu\mathbf{x}_0^2 - \mu\Psi \quad , \tag{8.5}$$

where $\mathbf{x}_0$ is the relative coordinate between the particles, μ is the reduced mass and Ψ is the Yukawa-like potential generated by a particle with mass $m = m_1 + m_2$. In polar coordinates, equation (8.5) reads

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \mu\Psi(r) \quad . \tag{8.6}$$

The Euler-Lagrange equations for the radial and polar coordinates are

$$\begin{cases} \mu\ddot{r} = \mu r\dot{\phi}^2 - \mu\dfrac{d\Psi(r)}{dr} \; , \\ \dfrac{d}{dt}\left(\mu r^2\dot{\phi}\right) = 0 \; . \end{cases} \tag{8.7}$$

The first equation in (8.7) describes the dynamics of the particle in the radial direction and the second is the conservation of angular momentum $L = \mu r^2\dot{\phi}$ in the z-direction. Eliminating $\dot{\phi}$ as a function of L, the radial equation becomes

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - \mu\frac{d\Psi(r)}{dr} \quad . \tag{8.8}$$

Multiplying (8.8) by $\dot{r}$ and integrating with respect to time gives

$$\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + \mu\Psi(r) = \text{constant} \quad . \tag{8.9}$$

This is just the conservation of the total energy of the system. This can be seen by recovering the same equation in terms of $L = \mu r^2\dot{\phi}^2$, i.e.

$$\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \mu\Psi(r) = \text{constant} \equiv E \quad . \tag{8.10}$$

The expression in (8.9) only involves the radial coordinate and so is almost the total energy of the particle of reduced mass μ in one dimension, if it was not for the term with L. However, such term can be recast into an *effective* potential and thus the energy E yields

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \tag{8.11}$$

with

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + \mu\Psi(r) \tag{8.12}$$

For convenience, rather than work with E is usual to define the energy and angular momentum per unit reduced mass as $\mathcal{E} = E/\mu$ and $h = L/\mu$. Then,

$$\mathcal{E} = \frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) \tag{8.13}$$

where

$$V_{\text{eff}}(r) = \frac{h^2}{2r^2} + \Psi(r) \quad . \tag{8.14}$$

The quantity $\mathcal{E}$ may be interpreted as the total energy of a fictitious particle moving in one dimension with a reduced mass equal to the unit under an effective potential Ψ[1].

Circular orbits

In particular, if the trajectory is a circular orbit, the distance from the origin to the particle does not change with time and so $\dot{r} = 0$. From (8.13) this implies that $\mathcal{E} = V_{\text{eff}}(r)$ and thus[2],

$$\frac{dV_{\text{eff}}(r)}{dr} = 0 \qquad \text{[Circular motion condition]} \quad . \tag{8.15}$$

Replacing (8.14) into (8.15) yields

$$\frac{d\Psi(r)}{dr} = r\dot{\phi}^2 \tag{8.16}$$

and solving for $\omega_\phi = \dot{\phi}$ one obtains de angular frequency of the binary system,

$$\boxed{\omega_\phi = \left(\frac{1}{r}\frac{d\Psi(r)}{dr}\right)^{1/2}} \quad . \tag{8.17}$$

This is a general result for an arbitrary central potential Ψ(r). In particular, the circular frequency for the Yukawa-like gravitational potential in (8.2) is[3]

$$\boxed{\omega_{\text{y}} = \left\{\frac{Gm}{r^3}\left[1 + \frac{1}{3}e^{-\Upsilon r}\left(1 + \Upsilon r\right)\right]\right\}^{1/2}} \quad . \tag{8.18}$$

Nevertheless, a circular motion may be unstable. To guarantee its stability the effective potential must satisfy the additional condition,

$$\frac{d^2V_{\text{eff}}(r)}{dr^2} > 0 \qquad \text{[Circular motion stability]} \quad . \tag{8.19}$$

The condition (8.19) means that a circular orbit is stable as long as exists a minimum in the effective potential V_{eff}. To explore the orbit stability, is suitable to consider a linear perturbation approach. Suppose initially a stable circular orbit of radius R. A slightly perturbation on this orbit could give variations on the radius affecting the motion of the particle. If the perturbation produce tiny oscillations around the radius R then the orbit may be stable. By contrast, if the perturbation makes the initial radius to expand arbitrarily with time, the orbit would be unstable. Mathematically, this is to say that the solution $r(t)$ that describes the circular motion can be written as

$$r(t) = R + \epsilon(t) \quad , \tag{8.20}$$

where $\epsilon(t)$ is a small linear perturbation on the initial orbit of radius R with $\epsilon^2(t) \ll R^2$. Since the unperturbed circular orbit is stable then,

$$\left.\frac{dV_{\text{eff}}(r)}{dr}\right|_R = 0 \quad . \tag{8.21}$$

[1]The term $h^2/2r^2$ prevents the particle to approach very close to the origin of the central force. This is known in the literature as a centrifugal potential energy [61, 62].

[2]Observe that the motion is allowed provided that $\mathcal{E} \geq V_{\text{eff}}(r)$ which follows from $\dot{r}^2 \geq 0$ from equation (8.13).

[3]If the potential is the Newtonian one, we get the Keplerian frequency $\omega = (Gm/r^3)^{1/2}$.

To analyze the behavior of the function $\epsilon(t)$, we shall use the energy equation (8.13) in terms of (8.20). First, expand the effective potential as a Taylor series around $r = R$,

$$\begin{aligned} V_{\text{eff}}(R+\epsilon) &= V_{\text{eff}}(R) + \left.\frac{dV_{\text{eff}}(r)}{dr}\right|_R \epsilon + \frac{1}{2}\left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_R \epsilon^2 + \cdots \\ &= V_{\text{eff}}(R) + \frac{1}{2}\left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_R \epsilon^2 + \cdots \end{aligned} \tag{8.22}$$

and using this result in (8.13) along with $\epsilon^2 \ll R^2$ we have

$$\mathcal{E} - V_{\text{eff}}(R) = \frac{1}{2}\dot{\epsilon}^2(t) + V_{\text{eff}}(R) + \frac{1}{2}\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_R \epsilon^2 + \cdots \tag{8.23}$$

Note that the left-hand side of (8.23) is a constant. Hence, differentiating this equation and dividing by $\dot{\epsilon}$ yields

$$\ddot{\epsilon}(t) + \left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_R \epsilon(t) = 0 \quad . \tag{8.24}$$

Identifying the radial frequency of the perturbation as

$$\omega_r^2 = \left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_R \quad , \tag{8.25}$$

the expression in (8.24) is just the equation of a simple oscillator with frequency ω_r, i.e.

$$\ddot{\epsilon}(t) + \omega_r^2\epsilon(t) = 0 \quad . \tag{8.26}$$

Depending on the sign of ω_r^2 is possible to have oscillatory solutions or exponential ones. To ensure stability, the perturbation must vary between the central value of radius R. This is the case when $\omega_r^2 > 0$ which is the same equation in (8.19). Replacing (8.14) in (8.19), the stability condition reads

$$\omega_r^2 = \left[\frac{3h^2}{r^4} + \frac{d^2\Psi(r)}{dr^2}\right]_R = \left[3\omega_\phi^2 + \frac{d^2\Psi(r)}{dr^2}\right]_R \tag{8.27}$$

and using ω_ϕ^2 from (8.17) into (8.27) one obtains,

$$\omega_r^2 = \left[\frac{d^2\Psi(r)}{dr^2} + \frac{3}{r}\frac{d\Psi(r)}{dr}\right]_R > 0 \qquad \text{[Circular stable orbit]} \tag{8.28}$$

For the Yukawa potential, their first and second derivatives are

$$\begin{cases} \dfrac{d\Psi}{dr} &= \dfrac{Gm}{r^2} + \dfrac{1}{3}\dfrac{Gm}{r^2}e^{-\Upsilon r} + \dfrac{1}{3}\dfrac{Gm}{r}\Upsilon e^{-\Upsilon r} \\ \dfrac{d^2\Psi}{dr^2} &= -2\dfrac{Gm}{r^3} - \dfrac{2}{3}\dfrac{Gm}{r^3}e^{-\Upsilon r} - \dfrac{1}{3}\dfrac{Gm}{r^2}\Upsilon e^{-\Upsilon r} - \dfrac{1}{3}\dfrac{Gm}{r^2}\Upsilon e^{-\Upsilon r} - \dfrac{1}{3}\dfrac{Gm}{r}\Upsilon^2 e^{-\Upsilon r} \end{cases} \tag{8.29}$$

Thus, using $\omega_r^2 > 0$ it is obtained the following constraint

$$\omega_r^2 = \frac{Gm}{r^3}\left[1 + \frac{1}{3}e^{-\Upsilon r}\left(1 + \Upsilon r - \Upsilon^2 r^2\right)\right] > 0 \tag{8.30}$$

which implies that

$$1 + \Upsilon r - \Upsilon^2 r^2 > -1 \tag{8.31}$$

and thus,

$$\boxed{\Upsilon r < \frac{\sqrt{5}+3}{2}} \ . \tag{8.32}$$

This is a remarkable result. Some recent works in $f(R)$ boundings about the mass of the graviton suggest an associated Compton wavelength of $\lambda_g > 1.6 \times 10^{10}$ km from the orbital decay of the Hulse-Taylor binary system[25, 63, 64]. Although it was not mentioned before, the Klein-Gordon equation in Chapter 7 means that $\Upsilon = m_g c/\hbar = 1/\lambda\!\!\!^{-}_g$. The Hulse-Taylor binary system has a semi-major axis equal to $r \sim a \sin \iota = 7 \times 10^5$ km [65–67]. Hence, $\Upsilon r = 2.7 \times 10^{-4}$ and the condition in equation (8.32) is satisfied.

Equation of motion

The common configuration of an astrophysical system such as binary pulsars or planetary motion around the Sun, occurs when the typical size of the system is much smaller than the Compton wavelength of the graviton [25]. This justifies an expansion of the Yukawa potential in a Taylor series because Υr is very small. To make a link with the Newtonian case, we introduce a new varible δ in the Yukawa potential as,

$$\Psi(r) = -\frac{Gm}{r}\left(1 + \frac{1}{3}\delta e^{-\Upsilon r}\right) \tag{8.33}$$

where

$$\delta = \begin{cases} 1 & \text{Yukawa potential ,} \\ 0 & \text{Newton potential .} \end{cases} \tag{8.34}$$

This does not affect the equations of motion of a particle under a Yukawa interaction when setting δ to 1, but one can compare the results with the Newtonian case making $\delta = 0$. Remind that the Taylor expansion of the exponential function is of the form

$$e^{-x} = 1 - x + \frac{x^2}{2} + \mathcal{O}(x^3) \tag{8.35}$$

and therefore the effective potential $V_{\text{eff}}(r)$ in (8.14) is expanded as

$$\begin{aligned} V_{\text{eff}}(r) &= \frac{h^2}{2r^2} - \frac{Gm}{r} - \frac{1}{3}\frac{Gm}{r}\delta\left[1 - \Upsilon r + \frac{1}{2}\Upsilon^2 r^2 + \mathcal{O}(\Upsilon^3)\right] \\ &= \frac{h^2}{2r^2} - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{r} + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}Gm\Upsilon^2 r + \mathcal{O}(\Upsilon^3) \end{aligned} \tag{8.36}$$

With the potential in (8.36), one can construct the orbit equation that describes the motion of a particle with reduced mass μ under the influence of a Yukawa interaction[4]. Inserting (8.36) in the energy equation given in (8.13) yields

$$\mathcal{E} = \frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{r} + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}Gm\Upsilon^2 r + \mathcal{O}(\Upsilon^3) \ . \tag{8.37}$$

[4] This is an approximation method to the original problem provided that $r \ll \lambda_g$. For more details see [25].

Since $r = r(\phi)$ we have

$$\dot{r} = \frac{dr}{dt} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}r' \tag{8.38}$$

and the expression for the energy $\mathcal{E}$ in (8.37) becomes

$$\mathcal{E} = \frac{1}{2}\frac{h^2}{r^4}(r')^2 + \frac{h^2}{2r^2} - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{r} + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}Gm\Upsilon^2 r + \mathcal{O}(\Upsilon^3) \tag{8.39}$$

Defining,

$$u = \frac{1}{r} \implies u' = -\frac{1}{r^2}r' \tag{8.40}$$

the energy equation is rewritten as

$$\mathcal{E} = \frac{1}{2}h^2(u')^2 + \frac{1}{2}h^2u^2 - \left(1 + \frac{\delta}{3}\right)Gmu + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}\frac{Gm\Upsilon^2}{u} + \mathcal{O}(\Upsilon^3) \tag{8.41}$$

Differentiating the previous equation and dividing by h^2u' one gets,

$$\boxed{u'' + u = \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2} - \frac{\delta}{6}\frac{Gm\Upsilon^2}{h^2u^2}} \; . \tag{8.42}$$

Observe that when $\delta = 0$ the orbit equation for the Newtonian case is obtained.

8.3 Solution of the orbit equation

To solve the orbit equation (8.42) we follow an approximate method based on a similar procedure given in a recent work [25]. The strategy is to make an *ansatz* solution with the same functional form as the solution in the Newtonian case,

$$u = \frac{1}{p}(1 + e\cos\phi) \quad . \tag{8.43}$$

Then, the semilatus rectum and the eccentricity are found by using the orbit and energy equations up to some order. Although the solution has the same structure as in the Newtonian problem, the orbital parameters p and e contain the information of the Yukawa strength mediated by the constant Υ. Here we solve (8.42) at first and second order in Υ.

First order solution

The orbit equation (8.42) to first order in Υ is obtained by expanding the effective potential to linear order in Υ and using the energy equation to build the differential equation for the variable u. Note from (8.41) that the linear term in Υ is a constant and after taking the derivative of this expression the resulting orbit equation does not depends on Υ,

$$u'' + u = \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2} \; . \tag{8.44}$$

This equation differs from the Newtonian one only by a factor of $(4/3)$ when setting $\delta = 1$ for the Yukawa interaction. Inserting the ansatz solution in (8.44) gives

$$-\frac{e}{p}\cos\phi + \frac{1}{p}(1 + e\cos\phi) = \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2} \tag{8.45}$$

and solving for the semilatus rectum we have,

$$p = \frac{h^2}{Gm\left(1 + \frac{\delta}{3}\right)} \quad . \tag{8.46}$$

This is the same semilatus rectum for the Newtonian limit when $\delta = 0$ [61]. However, for the Yukawa case we have

$$\boxed{p = \frac{3h^2}{4Gm}} \quad . \tag{8.47}$$

To obtain the eccentricity, the energy equation in (8.41) is required. A further substitution of (8.43) into (8.41) and using (8.46) one get at first order,

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}h^2(u')^2 + \frac{1}{2}h^2u^2 - \left(1 + \frac{\delta}{3}\right)Gmu + \frac{\delta}{3}Gm\Upsilon \\
&= \frac{1}{2}h^2\left(\frac{e}{p}\sin\phi\right)^2 + \frac{1}{2}h^2\left[\frac{1}{p}(1 + e\cos\phi)\right]^2 - \frac{h^2}{p^2}(1 + e\cos\phi) + \frac{\delta}{3}Gm\Upsilon \\
&= \frac{h^2e^2}{2p^2} + \frac{h^2}{2p^2} + \frac{h^2}{p^2}e\cos\phi - \frac{h^2}{p^2}(1 + e\cos\phi) + \frac{\delta}{3}Gm\Upsilon \\
&= \frac{h^2e^2}{2p^2} - \frac{h^2}{2p^2} + \frac{\delta}{3}Gm\Upsilon \quad .
\end{aligned} \tag{8.48}$$

Solving for e and substituting p from (8.46) we obtain,

$$e^2 = 1 + \frac{2\mathcal{E}h^2}{G^2m^2\left(1 + \frac{\delta}{3}\right)^2} - \frac{2\Upsilon h^2\delta}{3Gm\left(1 + \frac{\delta}{3}\right)^2} \tag{8.49}$$

Unlike the semilatus rectum p, the eccentricity e has a linear dependence on Υ at first order. Again, equation (8.49) recovers the Newtonian eccentricity when $\delta = 0$. While for $\delta = 1$, the Yukawa eccentricity to first order in Υ yields,

$$e^2 = 1 + \frac{9\mathcal{E}h^2}{8G^2m^2} - \frac{3\Upsilon h^2}{8Gm} \quad . \tag{8.50}$$

Second order solution

The full differential equation at second order in Υ is shown in (8.42). Inserting the ansatz solution in this equation produces

$$\frac{1}{p} = \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2} - \frac{\delta}{6}\frac{Gm}{h^2}\frac{\Upsilon^2p^2}{(1 + e\cos\phi)^2} \quad . \tag{8.51}$$

This is not an easy equation to solve for p but one can proceed using a trick. Is convenient to rewrite this equation as

$$\begin{aligned}
&(1 + e\cos\phi)^2 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}(1 + e\cos\phi)^2 p + \frac{\delta}{6}\frac{Gm}{h^2}\Upsilon^2p^3 = 0 \\
&\left(1 + 2e\cos\phi + e^2\cos^2\phi\right)\left[1 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}p\right] + \frac{\delta}{6}\frac{Gm}{h^2}\Upsilon^2p^3 = 0
\end{aligned} \tag{8.52}$$

or equivalently,

$$\left[1 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}p\right] e^2\cos^2\phi + 2\left[1 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}p\right] e\cos\phi \tag{8.53}$$

$$+\left[1 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}p + \frac{\delta}{6}\frac{Gm}{h^2}\Upsilon^2 p^3\right] = 0 \quad . \tag{8.54}$$

Since the ansatz solution must satisfied the orbit equation in every point of the trajectory, lets evaluate (8.54) in the apsides $\phi = 0$ and $\phi = \pi$ to get two equations,

$$\left[1 - \left(1 + \frac{\delta}{3}\frac{Gm}{h^2}p\right)\right] e^2 + 2\left[1 - \left(1 + \frac{\delta}{3}\frac{Gm}{h^2}p\right)\right] e + \left[1 - \left(1 - \frac{\delta}{3}\right)\frac{Gm}{h^2}p + \frac{\delta}{6}\frac{Gm}{h^2}\Upsilon^2 p^3\right] = 0 \ ,$$

$$\left[1 - \left(1 + \frac{\delta}{3}\frac{Gm}{h^2}p\right)\right] e^2 - 2\left[1 - \left(1 + \frac{\delta}{3}\frac{Gm}{h^2}p\right)\right] e + \left[1 - \left(1 - \frac{\delta}{3}\right)\frac{Gm}{h^2}p + \frac{\delta}{6}\frac{Gm}{h^2}\Upsilon^2 p^3\right] = 0 \ .$$

Substracting these equations implies that

$$4\left[1 - \left(1 + \frac{\delta}{3}\right)\frac{Gm}{h^2}p\right] e = 0 \tag{8.55}$$

and for $e \neq 0$ we obtain the semilatus rectum

$$p = \frac{h^2}{Gm\left(1 + \frac{\delta}{3}\right)} \quad . \tag{8.56}$$

This is the same answer as in (8.46) for the first order solution. Similarly, to obtain the eccentricity we use the same trick by inserting the ansatz solution into the energy equation. The result is,

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}h^2(u')^2 + \frac{1}{2}h^2u^2 - \left(1 + \frac{\delta}{3}\right)Gmu + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}\frac{Gm\Upsilon^2}{u} \\ &= \frac{h^2e^2}{2p^2} - \frac{h^2}{2p^2} + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}\frac{Gmp\Upsilon^2}{(1 + e\cos\phi)} \quad .\end{aligned} \tag{8.57}$$

Lets express this equation as

$$e^2 - 1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2 - \frac{\delta}{3}\frac{Gm}{h^2}p^3\frac{\Upsilon^2}{(1 + e\cos\phi)} = 0$$

$$(1 + e\cos\phi)\left[e^2 - 1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2\right] - \frac{\delta}{3}\frac{Gm}{h^2}p^3\Upsilon^2 = 0$$

i.e.,

$$e^3\cos\phi + e^2 + \left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\tilde{E}}{h^2}p^2\right] e\cos\phi$$

$$+\left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\tilde{E}}{h^2}p^2 - \frac{\delta}{3}\frac{Gm}{h^2}p^3\Upsilon^2\right] = 0 \tag{8.58}$$

Evaluating (8.58) at the apsides, $\phi = 0$ and $\phi = \pi$ we obtain

$$e^3 + e^2 + \left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2\right]e + \left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2 - \frac{\delta}{3}\frac{Gm}{h^2}p^3\Upsilon^2\right] = 0\,,$$

$$-e^3 + e^2 - \left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2\right]e + \left[-1 + \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon - \frac{2\mathcal{E}}{h^2}p^2 - \frac{\delta}{3}\frac{Gm}{h^2}p^3\Upsilon^2\right] = 0\,.$$

Adding these equations and solving for e^2 gives the eccentricity up to second order in Υ,

$$e^2 = 1 + \frac{2\mathcal{E}}{h^2}p^2 - \frac{2}{3}\delta\frac{Gm}{h^2}p^2\Upsilon + \frac{1}{3}\delta\frac{Gm}{h^2}p^3\Upsilon^2 \tag{8.59}$$

Replacing the solution for p shown in (8.56) into this expression yields

$$e^2 = 1 + \frac{2\mathcal{E}h^2}{G^2m^2\left(1+\frac{\delta}{3}\right)^2} - \frac{2\Upsilon h^2\delta}{3Gm\left(1+\frac{\delta}{3}\right)^2} + \frac{\Upsilon^2h^4\delta}{3G^2m^2\left(1+\frac{\delta}{3}\right)^3} \tag{8.60}$$

For the Yukawa case $\delta = 1$ and the eccentricity becomes,

$$\boxed{e^2 = 1 + \frac{9\mathcal{E}h^2}{8G^2m^2} - \frac{3\Upsilon h^2}{8Gm} + \frac{9\Upsilon^2h^4}{64G^2m^2}}\,. \tag{8.61}$$

With the semilatus rectum and the eccentricity in hands, is also possible to find the major semi-axis a using the relation [61, 62],

$$a = \frac{p}{1-e^2} \tag{8.62}$$

Plugging the eccentricity in (8.59) into (8.62) yields

$$a = \frac{h^2}{-2\mathcal{E}p + \frac{2}{3}\delta Gmp\Upsilon - \frac{1}{3}\delta Gmp^2\Upsilon^2} \tag{8.63}$$

and using p from (8.47) we have the major semi-axis for the Yukawa case up to second order,

$$\boxed{a = \left(-\frac{3\mathcal{E}}{2Gm} + \frac{1}{2}\Upsilon - \frac{3h^2}{16Gm}\Upsilon^2\right)^{-1}}\,. \tag{8.64}$$

8.4 Quadrupole waveform

Using the results in Chapter 6, the quadrupole radiation from a compact binary in circular motion is obtained for linearized $f(R)$ gravity. We have seen that for sources with non-negligible self gravity, an appropriate extension to these systems require the inclusion of a bounding interaction term in the total energy-momentum tensor of the system. In linearized GR, this was possible due to the great analogy with the weak field limit of the EFE in the Landau & Lifshitz formalism. However, is worth to consider another argument to justify the validity of linearized techniques in the generation of GWs for bounded sources in this context. Recall from Chapter 5 that a weak gravitational field implies slow velocities and thus terms of order $\mathcal{O}(v^2/c^2)$ may be neglected. The energy-momentum tensor of a binary system should be of the form

$$T^{\mu\nu} = \sum_{a=1}^{2} \gamma_a m_a \frac{dx^\mu_a}{dt}\frac{dx^\nu_a}{dt}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \quad + \quad \underbrace{\text{Interaction}}_{\mathcal{O}(v^2/c^2)} \ , \tag{8.65}$$

where the order of the interaction term can be seen from the virial theorem as is shown in equation (5.58). Thus, is reasonable to disregard the interaction term. Furthermore, for the components of the energy-momentum tensor we have,

$$T^{00}(t,\mathbf{x}) = \sum_{a=1}^{n} \gamma_a m_a c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \sim \mathcal{O}(v^0) \tag{8.66}$$

$$T^{0i}(t,\mathbf{x}) = \sum_{a=1}^{n} \gamma_a m_a c \dot{x}^i \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \sim \mathcal{O}(v/c) \tag{8.67}$$

$$T^{ij}(t,\mathbf{x}) = \sum_{a=1}^{n} \gamma_a m_a \dot{x}^i_a \dot{x}^j_a \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \sim \mathcal{O}(v^2/c^2) \tag{8.68}$$

The quantities T^{00} and T^{0i} can be obtained at lowest order. However, T^{ij} is a quadratic order term, but this is not essential for the following reason. Consider the solution $\bar{h}^{\mathrm{TT}}_{ij}$ for the inhomogeneous wave equation in terms of the multiple moments of the energy-momentum tensor,

$$\bar{h}^{\mathrm{TT}}_{ij} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij|kl}(\hat{\mathbf{n}})\Big[S^{kl} \ + \ \frac{1}{c}n_m \dot{S}^{kl,m} \ + \ \dots\Big]_{t_r} \tag{8.69}$$

with

$$S^{kl}(t) = \int T^{kl}(\mathbf{x},t)\, d^3x \ , \tag{8.70}$$

$$S^{kl,m}(t) = \int T^{kl}(\mathbf{x},t) x^m \, d^3x \ . \tag{8.71}$$

Since T^{kl} is of order $\mathcal{O}(v^2/c^2)$ and also is required to compute S^{kl} and $S^{kl,m}$, in principle the full expression of the energy-momentum tensor is needed. Nevertheless, from the conservation equation $\partial_\nu T^{\mu\nu} = 0$ it was proven in Chapter 5 the identities,

$$S^{kl} = \frac{1}{2}\ddot{M}^{kl} \ , \tag{8.72}$$

$$\dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\Big(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\Big) \ . \tag{8.73}$$

where

$$M^{kl} = \frac{1}{c^2}\int T^{00}x^k x^l \, d^3x \quad , \quad M^{ijk} = \frac{1}{c^2}\int T^{00}x^i x^j x^k \, d^3x \tag{8.74}$$

and

$$P^{i,jk} = \frac{1}{c}\int T^{0i}x^j x^k \, d^3x \ . \tag{8.75}$$

The conclusion of this analysis is that to first order is not necessary to know explicitly the form of the interaction term. At the end, the quantities S^{kl} and $\dot{S}^{kl,m}$ do not depend on T^{ij} and may be computed to lowest order. In addition, note also that the expression $\partial_0 T^{00} + \partial_i T^{0i} = 0$

is satisfied independently of the trajectory, i.e.

$$\partial_0 T^{00} = \frac{1}{c}\sum_a m_a c^2 \frac{\partial}{\partial t}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) = \sum_a m_a c \frac{dx_a^i}{dt}\frac{\partial}{\partial x_a^i}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))$$
$$= -\sum_a m_a c\, \dot{x}_a^i \partial_i \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) = -\partial_i T^{0i} \quad . \tag{8.76}$$

Thus, at linear order the quadrupole radiation of the massless modes h_+ and $h_\times$ in $f(R)$ gravity is obtained following the same procedure of Chapter 6. The only difference is that rather than use the Keplerian frequency for the circular orbit motion, one must use the Yukawa-like frequency ω_{y} given in (8.18). Then, we have

$$\bar{h}_+(t) = \frac{1}{r}\frac{4G\mu\omega_{\mathrm{y}}^2 R^2}{c^4}\left(\frac{1+\cos^2\iota}{2}\right)\cos(2\omega_{\mathrm{y}} t) \ , \tag{8.77}$$

$$\bar{h}_\times(t) = \frac{1}{r}\frac{4G\mu\omega_{\mathrm{y}}^2 R^2}{c^4}(\cos\iota)\sin(2\omega_{\mathrm{y}} t) \ . \tag{8.78}$$

The functions (8.77) and (8.78) for the Hulse-Taylor binary are plotted in figure 8.1 using the data in [67]. An upper limit for the value of Υ in this theory at this scales is taken from [56]. Note that at first order, the back reaction of GWs does not appear naturally and therefore both, the amplitude and the phase of the radiation are constant in time.

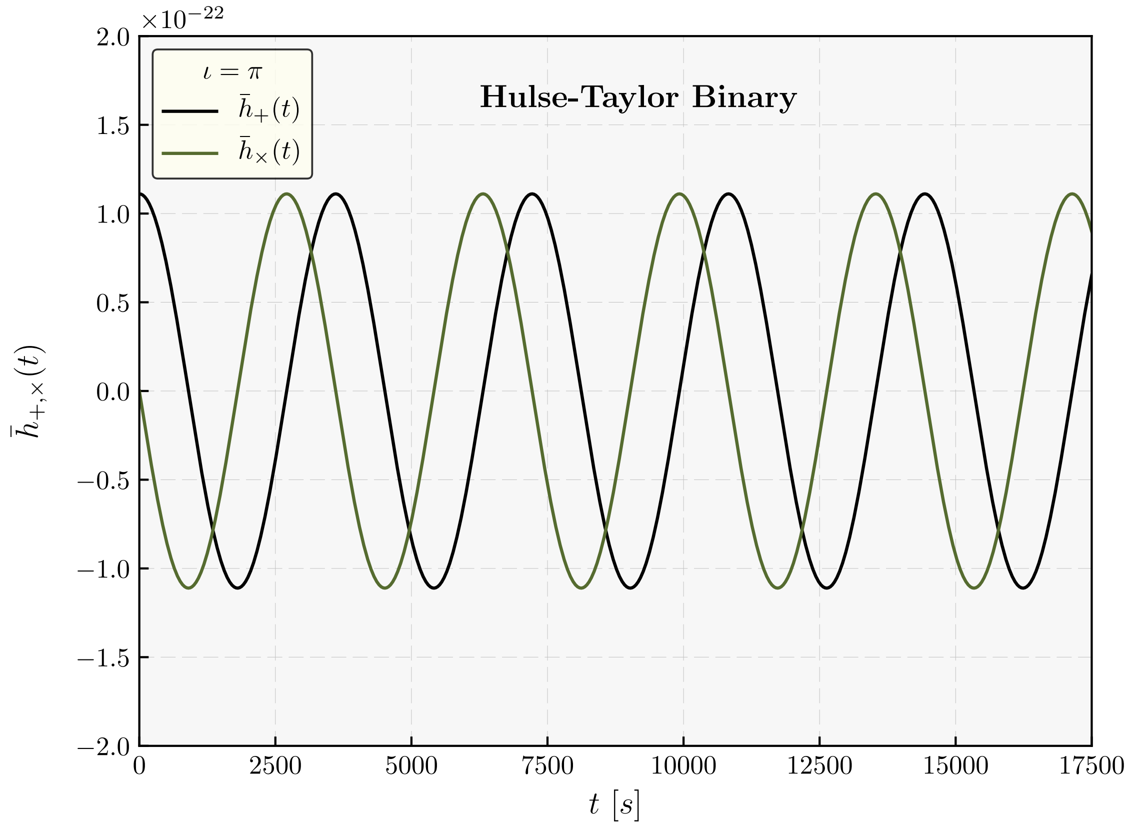

Figure 8.1: Waveform for the *plus* and *cross* polarizations in $f(R)$ linearized gravity for the Hulse-Taylor binary system. The value of Υ was fixed using $\lambda_{\mathrm{g}} = 1.63\times 10^{13}\,[m]$ based on [63, 64]. The back-reaction of GWs does not arise in linearized theory.

When GWs are emitted away from a binary system moving in a circular orbit, the frequency starts to increase because the particles takes less time to perform a complete revolution as the radius begins to shrink. To account for the change in frequency, the back-reaction effect is introduced by means of the energy balance equation that we recall here,

$$\frac{dE_{\rm orbit}}{dt} = -P^{\sf GW} \quad . \tag{8.79}$$

The left-hand side of this equation can be obtained from the total mechanical energy of the orbit $E_{\rm orbit}$ which is

$$E_{\rm orbit} = mc^2 + \frac{1}{2}\mu v^2 + \mu\Psi(r) \quad . \tag{8.80}$$

To second order in Υ, the Yukawa potential Ψ and the frequency $\omega_{\sf y}$ are

$$\Psi(r) \simeq -\left(1 + \frac{\delta}{3}\right)\frac{Gm}{r} + \frac{\delta}{3}Gm\Upsilon - \frac{\delta}{6}Gm\Upsilon^2 r \ , \tag{8.81}$$

$$\omega_{\sf y}^2 \simeq \left(1 + \frac{\delta}{3}\right)\frac{Gm}{r^3} - \frac{\delta}{6}\frac{Gm}{r}\Upsilon^2 \ . \tag{8.82}$$

For circular orbits we have $v^2 = r^2\omega_{\sf y}^2$. Inserting (8.82) and (8.81) into (8.80) and evaluating at the radius of the orbit $r = R$ yields,

$$E_{\rm orbit} = mc^2 - \frac{1}{2}\left(1 + \frac{\delta}{3}\right)\frac{Gm\mu}{R} + \frac{\delta}{3}Gm\mu\Upsilon - \frac{\delta}{4}Gm\mu\Upsilon^2 R \quad . \tag{8.83}$$

Is possible to write this expression only in terms of $\omega_{\sf y}$. For this, realize that the equation (8.82) may be put into Cardano's form [68]. Defining $x = 1/R$ then,

$$x^3 - 3{\sf p}x - 2{\sf q} = 0 \tag{8.84}$$

where

$${\sf p} = \frac{\delta\Upsilon^2}{18\left(1 + \frac{\delta}{3}\right)} \qquad , \qquad {\sf q} = \frac{\omega_{\sf y}^2}{2\left(1 + \frac{\delta}{3}\right)Gm} \ . \tag{8.85}$$

The solution to (8.84) is given by $x = {\sf a} + {\sf b}$ with

$${\sf a}^3 = {\sf q} + \sqrt{{\sf q}^2 - {\sf p}^3} \qquad \text{and} \qquad {\sf b}^3 = {\sf q} - \sqrt{{\sf q}^2 - {\sf p}^3} \quad . \tag{8.86}$$

This means that $R = ({\sf a} + {\sf b})^{-1}$ and substituting this result in (8.83) one can express the energy $E_{\rm orbit}$ in terms of the Yukawa frequency. In addition by assuming that $\omega_{\sf y} = \omega_{\sf y}(t)$, the left-hand side of equation in (8.79) is obtained when performing the time derivative of $E_{\rm orbit}$. For the right-hand side, we assume that the massive scalar mode is not excited so only the massless radiative modes h_+ and $h_\times$ are important. This implies to consider the same emitted energy from those modes as in the case of GR, i.e.

$$P^{\sf GW}_{\rm quad} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_{\rm c}\omega_{\sf y}}{c^3}\right)^{10/3} \quad . \tag{8.87}$$

Equating (8.87) with the time derivative of $E_{\rm orbit} = E_{\rm orbit}(\omega_{\sf y})$ in (8.83) through the balance equation (8.79), gives a differential equation for $\omega_{\sf y}$ at second order in Υ. However, this equation is not easy to solve, even numerically. For simplicity, we stop the expansion up to linear order in Υ. At the end, we are interested in typical astrophysical systems where $r \gg \lambda_g$. In this case, the total energy of the orbit is

$$E_{\rm orbit} \simeq mc^2 - \frac{1}{2}\left(1+\frac{\delta}{3}\right)\frac{Gm\mu}{R} + \frac{\delta}{3}Gm\mu\Upsilon \tag{8.88}$$

and the Yukawa frequency

$$\omega_{\rm y}^2 \simeq \left(1+\frac{\delta}{3}\right)\frac{Gm}{R^3} \ . \tag{8.89}$$

Solving for R in (8.89) and replacing the result in (8.88) we obtain

$$E_{\rm orbit}(t) = mc^2 - \frac{\mu}{2}\left[\left(1+\frac{\delta}{3}\right)Gm\omega_{\rm y}(t)\right]^{2/3} + \frac{\delta}{3}Gm\mu\Upsilon \ . \tag{8.90}$$

In terms of the chirp mass $M_{\rm c} = \mu^{3/5}m^{2/5}$ and the GW frequency $\omega_{\rm gw} = 2\omega_{\rm s}$ this equation is rewritten as,

$$E_{\rm orbit}(t) = mc^2 - \left[\left(1+\frac{\delta}{3}\right)^2\frac{G^2M_{\rm c}^5\omega_{\rm gw}^2(t)}{32}\right]^{1/3} + \frac{\delta}{3}Gm\mu\Upsilon \ . \tag{8.91}$$

Thus, taking the time derivative we have

$$\frac{dE_{\rm orbit}(t)}{dt} = -\frac{2}{3}\left[\left(1+\frac{\delta}{3}\right)^2\frac{G^2M_{\rm c}^5}{32}\right]^{1/3}\frac{\dot\omega_{\rm gw}(t)}{\omega_{\rm gw}^{1/3}} \ . \tag{8.92}$$

On the other hand, the GW power is given in (8.87) that in terms of $\omega_{\rm gw}$ is

$$P_{\rm quad}^{\sf GW}(t) = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_{\rm c}\omega_{\rm gw}}{2c^3}\right)^{10/3} \ . \tag{8.93}$$

Using (8.79) the differential equation for $\omega_{\rm gw}$ reads

$$\frac{2}{3}\left[\left(1+\frac{\delta}{3}\right)^2\frac{G^2M_{\rm c}^5}{32}\right]^{1/3}\frac{\dot\omega_{\rm gw}(t)}{\omega_{\rm gw}^{1/3}(t)} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_{\rm c}\omega_{\rm gw}(t)}{2c^3}\right)^{10/3}$$

or

$$\dot f_{\rm gw}(t) = \frac{96}{5}\pi^{8/3}\left(\frac{GM_{\rm c}}{c^3}\right)^{5/3}\left(1+\frac{\delta}{3}\right)^{-2/3} f_{\rm gw}^{11/3}(t) \ . \tag{8.94}$$

Integrating by parts, the solution of (8.94) gives

$$f_{\rm gw}(t) = \frac{1}{\pi}\left[\frac{5}{256}\frac{1}{(t_{\rm coal}-t)}\right]^{3/8}\left(\frac{GM_{\rm c}}{c^3}\right)^{-5/8}\left(1+\frac{\delta}{3}\right)^{1/4} \tag{8.95}$$

Following a similar procedure of Chapter 6 we find that

$$\Phi_{\rm y}(t) = -2\left(\frac{5GM_{\rm c}}{c^3}\right)^{-5/8}\left(1+\frac{\delta}{3}\right)^{1/4}(t_{\rm coal}-t)^{5/8} + \Phi_{\rm y0} \tag{8.96}$$

and

$$\Phi_{\rm y0} = 2\left(\frac{5GM_{\rm c}}{c^3}\right)^{-5/8}\left(1+\frac{\delta}{3}\right)^{1/4}(t_{\rm coal}-t_0)^{5/8} \ . \tag{8.97}$$

Finally, the quadrupole radiation for the *plus* and *cross* polarizations at linear order in Υ is given by

$$\bar{h}_{+}(t) = \frac{1}{r}\left(\frac{GM_{\rm c}}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{\rm coal}-t)}\right)^{1/4}\left(1+\frac{\delta}{3}\right)^{1/6}\left(\frac{1+\cos^2\iota}{2}\right)\cos[\Phi_{\rm y}(t)]\,, \tag{8.98}$$

$$\bar{h}_{\times}(t) = \frac{1}{r}\left(\frac{GM_{\rm c}}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{\rm coal}-t)}\right)^{1/4}\left(1+\frac{\delta}{3}\right)^{1/6}(\cos\iota)\sin[\Phi_{\rm y}(t)]\;. \tag{8.99}$$

In particular, for $\delta = 0$ the expressions for the Newtonian case is recovered as is shown in equations (6.104) and (6.105). We see that at this order the Yukawa-like potential affect the quadrupole radiation only as a numerical factor in the amplitude and in the phase. The following figure shows the chirp waveform evolution for source GW150914.

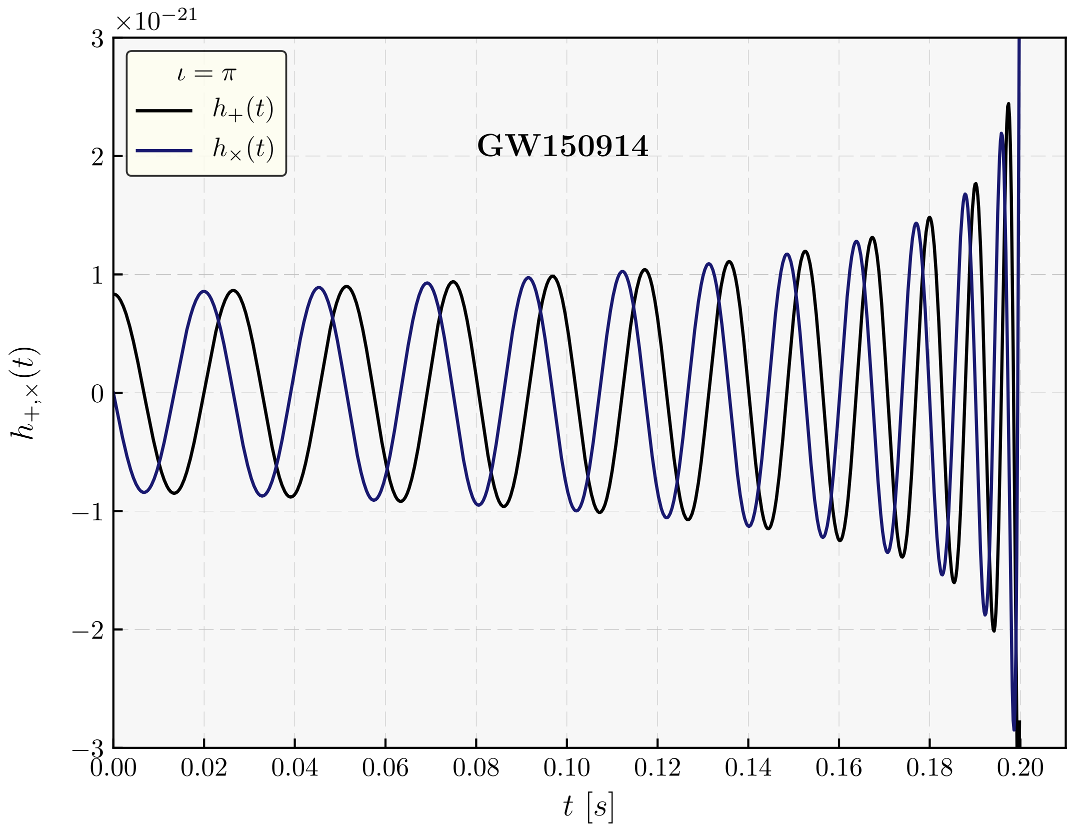

Figure 8.2: Chirp waveform for linearized $f(R)$ at linear order in Υ when $\delta = 1$. The data to reproduce this figure was obtained from the source GW150914 [13].

Chapter 9

Conclusions

The gravitational radiation from the inspiral of a compact binary source was studied in GR and $f(R)$ gravity. The theoretical tool to achieve this problem was based on the framework of linearized theory which represents a first approach to GWs. It was shown that two massless physical modes of propagation arise in both theories of gravity, namely the *plus* and *cross* polarizations. However, an extra massive mode emerge naturally in $f(R)$ directly from the field equations. A strong consequence of this difference befall on the Newtonian limit of a static point particle. Unlike GR, the result is a Yukawa-like addition to the usual Newtonian potential. Such modified potential has been studied in many different scenarios. Here, we have considered its application to the emission of GWs from a binary source in circular motion. If the Compton wavelength of the graviton is much greater than the scale of the source ($\lambda_g \gg r$), or at least of the same order, circular orbits are guaranteed. For many astrophysical phenomena this condition is satisfied. For instance, the waveform without back-reaction for the Hulse-Taylor binary system was obtained. To compute the radiation with the back-reaction effect, the exponential function of the Yukawa potential was expanded in a Taylor series. It was found a differential equation for the Yukawa frequency at second order in $\Upsilon = 1/\lambda_g$, however the solution was not easy to handle even numerically. In spite of this and for simplicity, a first order solution in Υ was presented for the h_+ and $h_\times$ polarizations. In order to extend these results to elliptical orbits, an approximate method was implemented to solve the two-body problem under the Yukawa-like interaction. It is expected that slightly changes in the frequency waveform pattern may be measured by future experiments from those kind of sources. This would allow to put better constraints in the space parameters of alternative theories of gravity such as $f(R)$ and in particular the validity of the Yukawa-like interaction. Finally, is important to mention that in all calculations it was assumed that the scalar mode was below the cutoff frequency Υ and thus it was not excited. Nevertheless, if this mode of propagation is not negligible, it will carry additional energy-momentum away from the source. In this case, the total power of GWs will not be the same as in the case of GR in it should be taken into account in the construction of the frequency time evolution through the balance equation.

Appendix A

Second order perturbative expansion

The main goal of this appendix is to compute the quantities $R^{(1)}_{\mu\nu}[h^{(1)}]$ and $R^{(2)}_{\mu\nu}[h^{(1)}]$. Setting the former to zero gives the first order perturbation EFE in vacuum (see equation (4.9))[1]. The latter is required to compute the energy-momentum tensor of GW (see equation (4.13))[2]. Because these quantities only depends on $h^{(1)}_{\mu\nu}$, the identification $h^{(1)}_{\mu\nu} \equiv h_{\mu\nu}$ is assumed to simplify the notation. Likewise, for this procedure only the inverse metric is required to second order in $h^{(1)}_{\mu\nu}$. A general computation of the mentioned tensors for a curved background is obtained.

Inverse metric at $\mathcal{O}(\varepsilon^2)$

The inverse metric at second order reads,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_{\lambda}^{\ \nu} \ , \tag{A.1}$$

where $\bar{g}^{\mu\nu}$ is the inverse background metric. The metric expansion in (A.1) is justified as follows,

$$\begin{aligned} g^{\mu\sigma}g_{\sigma\nu} &= \Big(\bar{g}^{\mu\sigma} - h^{\mu\sigma} + h^{\mu\lambda}h_{\lambda}^{\ \sigma}\Big)\Big(\bar{g}_{\sigma\nu} + h_{\sigma\nu}\Big) \\ &= \delta^{\mu}_{\nu} + h^{\mu}_{\ \nu} - h^{\mu}_{\ \nu} - h^{\mu\sigma}h_{\sigma\nu} + h^{\mu\lambda}h_{\lambda}^{\ \sigma}\bar{g}_{\sigma\nu} + \mathcal{O}(\varepsilon^3) \\ &= \delta^{\mu}_{\nu} - h^{\mu\sigma}h_{\sigma\nu} + h^{\mu\lambda}h_{\lambda\nu} + \mathcal{O}(\varepsilon^3) \\ &= \delta^{\mu}_{\nu} + \mathcal{O}(\varepsilon^3) \ , \end{aligned} \tag{A.2}$$

where indices are raised and lowered with $\bar{g}_{\mu\nu}$.

Connections at order $\mathcal{O}(\varepsilon^2)$

The connections are given by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\Big[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\Big] \tag{A.3}$$

Inserting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and the inverse metric (A.1) into (A.3) gives,

[1] The EFE in vacuum is given by $R_{\mu\nu} = 0$. Therefore, the first order perturbation EFE in vacuum is given by $R^{(1)}_{\mu\nu}[h^{(1)}] = 0$.

[2] Recall that $G^{(2)}_{\mu\nu}[h^{(1)}] = R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}_{\mu\nu}[h^{(1)}]$ is requiered in the computation of $t_{\mu\nu}$.

$$
\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}g^{\alpha\sigma}\Big[\partial_\mu\bar{g}_{\sigma\nu} + \partial_\mu h_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\mu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma\bar{g}_{\mu\nu} - \partial_\sigma h_{\mu\nu}\Big] \\
&= \frac{1}{2}\Big[\bar{g}^{\alpha\sigma} - h^{\alpha\sigma} + h^{\alpha\lambda}h_{\lambda}{}^{\sigma}\Big]\Big[\Big(\partial_\mu\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\mu} - \partial_\sigma\bar{g}_{\mu\nu}\Big) + \Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big)\Big] \\
&= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\sigma}\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) - \frac{1}{2}h^{\alpha\sigma}\Big(\partial_\mu\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\mu} - \partial_\sigma\bar{g}_{\mu\nu}\Big) \\
&\quad - \frac{1}{2}h^{\alpha\sigma}\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) + \frac{1}{2}h^{\alpha\lambda}h_{\lambda}{}^{\sigma}\Big(\partial_\mu\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\mu} - \partial_\sigma\bar{g}_{\mu\nu}\Big) + \mathcal{O}(\varepsilon^3) \\
&= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\Big(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\Big) - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}^{\sigma\delta}h_{\beta\delta}\Big(\partial_\mu\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\mu} - \partial_\sigma\bar{g}_{\mu\nu}\Big) \\
&\quad - \frac{1}{2}\bar{h}^{\alpha\sigma}\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) + \frac{1}{2}\bar{h}^{\alpha\sigma}h_{\sigma}{}^{\lambda}\Big(\partial_\mu\bar{g}_{\lambda\nu} + \partial_\nu\bar{g}_{\lambda\mu} - \partial_\lambda\bar{g}_{\mu\nu}\Big) + \mathcal{O}(\varepsilon^3) \\
&= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\Big[\Big(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\Big) - 2\bar{\Gamma}^{\delta}_{\mu\nu}h_{\beta\delta}\Big] \\
&\quad - \frac{1}{2}\bar{h}^{\alpha\sigma}\Big[\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) - \bar{g}^{\gamma\lambda}\Big(\partial_\mu\bar{g}_{\lambda\nu} + \partial_\nu\bar{g}_{\lambda\mu} - \partial_\lambda\bar{g}_{\mu\nu}\Big)h_{\sigma\gamma}\Big] + \mathcal{O}(\varepsilon^3) \\
&= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\Big[\Big(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\Big) - 2\bar{\Gamma}^{\delta}_{\mu\nu}h_{\beta\delta}\Big] \\
&\qquad - \frac{1}{2}\bar{h}^{\alpha\sigma}\Big[\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) - 2\bar{\Gamma}^{\gamma}_{\mu\nu}h_{\sigma\gamma}\Big] + \mathcal{O}(\varepsilon^3) \quad . \qquad \text{(A.4)}
\end{aligned}
$$

The terms in square brackets of the last result can be rewritten in a covariant form. For the first term in (A.4), observe that

$$\bar{\nabla}_\mu h_{\beta\nu} = \partial_\mu h_{\beta\nu} - \bar{\Gamma}^{\delta}_{\mu\beta}h_{\delta\nu} - \bar{\Gamma}^{\delta}_{\mu\nu}h_{\beta\delta} \quad , \qquad \text{(A.5)}$$

$$\bar{\nabla}_\nu h_{\beta\mu} = \partial_\nu h_{\beta\mu} - \bar{\Gamma}^{\delta}_{\nu\beta}h_{\delta\mu} - \bar{\Gamma}^{\delta}_{\mu\nu}h_{\beta\delta} \quad , \qquad \text{(A.6)}$$

$$\bar{\nabla}_\beta h_{\mu\nu} = \partial_\beta h_{\mu\nu} - \bar{\Gamma}^{\delta}_{\beta\mu}h_{\delta\nu} - \bar{\Gamma}^{\delta}_{\beta\nu}h_{\mu\delta} \quad . \qquad \text{(A.7)}$$

Therefore, using (A.5), (A.6) and (A.7) one obtains,

$$\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu} = \partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu} - 2\bar{\Gamma}^{\delta}_{\mu\nu}h_{\beta\delta} \quad . \qquad \text{(A.8)}$$

Smilarly, for the second term in (A.4),

$$\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu} = \partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} - 2\bar{\Gamma}^{\gamma}_{\mu\nu}h_{\sigma\gamma} \quad . \qquad \text{(A.9)}$$

Using (A.8) and (A.9) into (A.4) we have,

$$\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\Big(\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\Big) - \frac{1}{2}\bar{g}^{\alpha\rho}\bar{g}^{\sigma\delta}h_{\rho\delta}\Big(\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\Big) \quad .$$

Relabelling $\sigma \to \beta$, the connections up to order $\mathcal{O}(\varepsilon^2)$ are given by,

$$\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\Big(\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\Big) - \frac{1}{2}\bar{g}^{\alpha\rho}\bar{g}^{\beta\delta}h_{\rho\delta}\Big(\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\Big) \ .$$

Now, if the following tensors are defined as[3]

$$\overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\beta}\Big(\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\Big) \tag{A.10}$$

$$\overset{(2)}{\Gamma}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\rho}\bar{g}^{\beta\delta}h_{\rho\delta}\Big(\bar{\nabla}_\mu h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\Big) \ , \tag{A.11}$$

equation (A.10) is expressed in a compact form as

$$\boxed{\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}{}_{\mu\nu}} \ , \tag{A.12}$$

where $\overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ is of order $\mathcal{O}(\varepsilon)$ and $\overset{(2)}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ of order $\mathcal{O}(\varepsilon^2)$.

The Riemann tensor at order $\mathcal{O}(\varepsilon^2)$

The Riemann tensor comes from terms like $\partial\Gamma$ and $\Gamma\Gamma$. Therefore, is suitable to make the following splitting,

$$R^{\sigma}{}_{\beta\mu\nu} = \underbrace{\bar{R}^{\sigma}{}_{\beta\mu\nu}}_{\text{Background}} + \underbrace{\overset{(1)}{R}{}^{\sigma}{}_{\beta\mu\nu}}_{\mathcal{O}(\varepsilon)} + \underbrace{\overset{(2)}{R}{}^{\sigma}{}_{\beta\mu\nu}}_{\mathcal{O}(\varepsilon^2)} \ . \tag{A.13}$$

Using the connections (A.12) into the definition (2.18) gives,

$$\begin{aligned}
R^{\sigma}{}_{\beta\mu\nu} &= \partial_\mu\Gamma^{\sigma}_{\nu\beta} - \partial_\nu\Gamma^{\sigma}_{\mu\beta} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \\
&= \partial_\mu\Big(\bar{\Gamma}^{\sigma}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta}\Big) - \partial_\nu\Big(\bar{\Gamma}^{\sigma}_{\mu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta}\Big) \\
&\quad + \Big(\bar{\Gamma}^{\sigma}_{\mu\lambda} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\Big)\Big(\bar{\Gamma}^{\lambda}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta}\Big) \\
&\quad - \Big(\bar{\Gamma}^{\sigma}_{\nu\lambda} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\Big)\Big(\bar{\Gamma}^{\lambda}_{\mu\beta} + \overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} + \overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta}\Big) \\
&= \underline{\partial_\mu\bar{\Gamma}^{\sigma}_{\nu\beta}} + \underline{\partial_\mu\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta}} + \underline{\partial_\mu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta}} - \underline{\partial_\nu\bar{\Gamma}^{\sigma}_{\mu\beta}} - \underline{\partial_\nu\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta}} - \underline{\partial_\nu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta}} \\
&\quad + \underline{\bar{\Gamma}^{\sigma}_{\mu\lambda}\bar{\Gamma}^{\lambda}_{\nu\beta}} + \underline{\bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta}} + \underline{\bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta}} + \underline{\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta}} + \underline{\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta}} + \underline{\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta}} \\
&\quad - \underline{\bar{\Gamma}^{\sigma}_{\nu\lambda}\bar{\Gamma}^{\lambda}_{\mu\beta}} - \underline{\bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta}} - \underline{\bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta}} - \underline{\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}} - \underline{\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta}} - \underline{\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}} \ ,
\end{aligned}$$

[3] These objects are tensors because are formed by covariant derivatives.

$$R^{\sigma}{}_{\beta\mu\nu} = \bar{R}^{\sigma}{}_{\beta\mu\nu} + \Big(\partial_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \partial_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}\Big)$$
$$+ \Big(\partial_{\mu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \partial_{\nu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta}$$
$$- \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}\Big) \ . \tag{A.14}$$

The quantities defined in (A.13) are given from (A.14) as,

$$\bar{R}^{\sigma}{}_{\beta\mu\nu} = \partial_{\mu}\bar{\Gamma}^{\sigma}_{\nu\beta} - \partial_{\nu}\bar{\Gamma}^{\sigma}_{\mu\beta} + \bar{\Gamma}^{\sigma}_{\mu\lambda}\bar{\Gamma}^{\lambda}_{\nu\beta} - \bar{\Gamma}^{\sigma}_{\nu\lambda}\bar{\Gamma}^{\lambda}_{\mu\beta} \ , \tag{A.15}$$

$$\overset{(1)}{R}{}^{\sigma}{}_{\beta\mu\nu} = \partial_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \partial_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta} \ , \tag{A.16}$$

$$\overset{(2)}{R}{}^{\sigma}{}_{\beta\mu\nu} = \partial_{\mu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \partial_{\nu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta}$$
$$- \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta} \ . \tag{A.17}$$

Is desirable to express (A.16) and (A.17) in terms of the background geometry. To do this, remind that the quantities given in (A.10) and (A.11) are actually tensors. As tensors, their covariant derivatives are well defined. In particular,

$$\bar{\nabla}_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} = \partial_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} + \bar{\Gamma}^{\sigma}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\lambda}_{\mu\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} - \bar{\Gamma}^{\lambda}_{\mu\beta}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} \ , \tag{A.18}$$

and solving for the first term at the right hand side of this equation, one gets the first term of (A.16). With a similar argument is obtained the second term. Then, these terms yields,

$$\partial_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} = \bar{\nabla}_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \bar{\Gamma}^{\lambda}_{\mu\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\mu\beta}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} \tag{A.19}$$

$$\partial_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} = \bar{\nabla}_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} - \bar{\Gamma}^{\sigma}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} + \bar{\Gamma}^{\lambda}_{\nu\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\nu\beta}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda} \ . \tag{A.20}$$

Substituting the equations (A.19) and (A.20) into (A.16), the Riemann tensor at order $\mathcal{O}(\varepsilon)$ is obtained when using the definition in (A.10). The result is,

$$\overset{(1)}{R}{}^{\sigma}{}_{\beta\mu\nu} = \bar{\nabla}_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \bar{\Gamma}^{\lambda}_{\mu\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\mu\beta}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} - \bar{\nabla}_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \bar{\Gamma}^{\sigma}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta}$$
$$- \bar{\Gamma}^{\lambda}_{\nu\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} - \bar{\Gamma}^{\lambda}_{\nu\beta}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}$$
$$= \bar{\nabla}_{\mu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\nabla}_{\nu}\overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\beta}$$
$$= \bar{\nabla}_{\mu}\left[\frac{1}{2}\bar{g}^{\sigma\lambda}\Big(\bar{\nabla}_{\nu}h_{\lambda\beta} + \bar{\nabla}_{\beta}h_{\lambda\nu} - \bar{\nabla}_{\lambda}h_{\nu\beta}\Big)\right]$$
$$- \bar{\nabla}_{\nu}\left[\frac{1}{2}\bar{g}^{\sigma\lambda}\Big(\bar{\nabla}_{\mu}h_{\lambda\beta} + \bar{\nabla}_{\beta}h_{\lambda\mu} - \bar{\nabla}_{\lambda}h_{\mu\beta}\Big)\right] \ . \tag{A.21}$$

Thus,

$$\overset{(1)}{R}{}^{\sigma}{}_{\beta\mu\nu} = \frac{1}{2}\bar{g}^{\sigma\lambda}\Big(\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\lambda\beta} + \bar{\nabla}_\mu\bar{\nabla}_\beta h_{\lambda\nu} - \bar{\nabla}_\mu\bar{\nabla}_\lambda h_{\nu\beta} - \bar{\nabla}_\nu\bar{\nabla}_\mu h_{\lambda\beta} - \bar{\nabla}_\nu\bar{\nabla}_\beta h_{\lambda\mu} + \bar{\nabla}_\nu\bar{\nabla}_\lambda h_{\mu\beta}\Big)\,. \tag{A.22}$$

The Riemann tensor at order $\mathcal{O}(\varepsilon^2)$ follows from a similar procedure. The first terms of (A.17) are expressed as

$$\partial_\mu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} = \bar{\nabla}_\mu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \bar{\Gamma}^{\lambda}_{\mu\nu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\mu\beta}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} \tag{A.23}$$

$$\partial_\nu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} = \bar{\nabla}_\nu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} - \bar{\Gamma}^{\sigma}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta} + \bar{\Gamma}^{\lambda}_{\nu\mu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\nu\beta}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\,. \tag{A.24}$$

Inserting equations (A.23) and (A.24) into (A.17) it is obtained,

$$\begin{aligned}\overset{(2)}{R}{}^{\sigma}{}_{\beta\mu\nu} =\; & \bar{\nabla}_\mu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \bar{\Gamma}^{\lambda}_{\mu\nu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} + \bar{\Gamma}^{\lambda}_{\mu\beta}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda} - \bar{\nabla}_\nu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} \\ & + \bar{\Gamma}^{\sigma}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \bar{\Gamma}^{\lambda}_{\nu\mu}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\lambda\beta} - \bar{\Gamma}^{\lambda}_{\nu\beta}\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda} + \bar{\Gamma}^{\sigma}{}_{\mu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\nu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} \\ & + \overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\beta} - \bar{\Gamma}^{\sigma}{}_{\nu\lambda}\overset{(2)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta} - \overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta} \\ \overset{(2)}{R}{}^{\sigma}{}_{\beta\mu\nu} =\; & \bar{\nabla}_\mu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\nu\beta} - \bar{\nabla}_\nu\overset{(2)}{\Gamma}{}^{\sigma}{}_{\mu\beta} + \overset{(1)}{\Gamma}{}^{\sigma}{}_{\mu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\nu\beta} - \overset{(1)}{\Gamma}{}^{\sigma}{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^{\lambda}{}_{\mu\beta}\,. \end{aligned} \tag{A.25}$$

From the definitions (A.10) and (A.11) one is able to rewrite (A.25) in terms of the background geometry.

The Ricci tensor at order $\mathcal{O}(\varepsilon^2)$

Contracting σ and μ in (A.22) we get the Ricii tensor at order $\mathcal{O}(\varepsilon^2)$, i.e.,

$$\overset{(1)}{R}_{\beta\nu} = \frac{1}{2}\Big(\bar{\nabla}^\lambda\bar{\nabla}_\nu h_{\lambda\beta} + \bar{\nabla}^\lambda\bar{\nabla}_\beta h_{\lambda\nu} - \bar{\nabla}^\lambda\bar{\nabla}_\lambda h_{\nu\beta} - \bar{\nabla}_\nu\bar{\nabla}^\lambda h_{\lambda\beta} - \bar{\nabla}_\nu\bar{\nabla}_\beta h^\lambda{}_\lambda + \bar{\nabla}_\nu\bar{\nabla}_\lambda h^\lambda{}_\beta\Big)\,.$$

Now, making $\bar{\Box} := \bar{\nabla}^\lambda\bar{\nabla}_\lambda$ and $h := h^\lambda_\lambda = \bar{g}^{\lambda\mu}h_{\lambda\mu}$, the previous equation becomes,

$$\boxed{\overset{(1)}{R}_{\beta\nu}[h] = \frac{1}{2}\Big(\bar{\nabla}^\lambda\bar{\nabla}_\nu h_{\lambda\beta} + \bar{\nabla}^\lambda\bar{\nabla}_\beta h_{\lambda\nu} - \bar{\nabla}_\nu\bar{\nabla}_\beta h - \bar{\Box}h_{\nu\beta}\Big)}\,. \tag{A.26}$$

Observe that when comparing (A.26) with (2.21) the equation given in (2.21) is recover when the background is flat. On the other hand, the Ricci tensor at order $\mathcal{O}(\varepsilon^2)$ can be found from the contraction of the Riemann tensor given in (A.25). Using (A.10) and (A.11) one has,

$$\overset{(2)}{R}_{\beta\nu} = \bar\nabla_\sigma \overset{(2)}{\Gamma}{}^\sigma{}_{\nu\beta} - \bar\nabla_\nu \overset{(2)}{\Gamma}{}^\sigma{}_{\sigma\beta} + \overset{(1)}{\Gamma}{}^\sigma{}_{\sigma\lambda}\overset{(1)}{\Gamma}{}^\lambda{}_{\nu\beta} - \overset{(1)}{\Gamma}{}^\sigma{}_{\nu\lambda}\overset{(1)}{\Gamma}{}^\lambda{}_{\sigma\beta}$$

$$\begin{aligned}
\overset{(2)}{R}_{\beta\nu} = {} & \frac{1}{2}\Big\{\bar\nabla_\nu\Big[h^{\sigma\gamma}\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big)\Big] - \bar\nabla_\sigma\Big[h^{\sigma\gamma}\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big)\Big]\Big\} \\
& + \frac{1}{4}\bar g^{\sigma\alpha}\bar g^{\lambda\gamma}\Big(\bar\nabla_\sigma h_{\alpha\lambda} + \bar\nabla_\lambda h_{\alpha\sigma} - \bar\nabla_\alpha h_{\sigma\lambda}\Big)\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big) \\
& - \frac{1}{4}\bar g^{\sigma\alpha}\bar g^{\lambda\gamma}\Big(\bar\nabla_\nu h_{\alpha\lambda} + \bar\nabla_\lambda h_{\alpha\nu} - \bar\nabla_\alpha h_{\nu\lambda}\Big)\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big) \\
= {} & \frac{1}{2}\Big\{\Big[\bar\nabla_\nu h^{\sigma\gamma}\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big) + h^{\sigma\gamma}\Big(\underline{\bar\nabla_\nu\bar\nabla_\sigma h_{\gamma\beta}} + \bar\nabla_\nu\bar\nabla_\beta h_{\gamma\sigma} - \underline{\bar\nabla_\nu\bar\nabla_\gamma h_{\sigma\beta}}\Big)\Big] \\
& - \Big[\bar\nabla_\sigma h^{\sigma\gamma}\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big) + h^{\sigma\gamma}\Big(\bar\nabla_\sigma\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\sigma\bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\sigma\bar\nabla_\gamma h_{\nu\beta}\Big)\Big]\Big\} \\
& + \frac{1}{4}\Big(\cancel{\bar\nabla_\sigma h^{\sigma\gamma}} + \bar\nabla^\gamma h - \cancel{\bar\nabla_\alpha h^{\alpha\gamma}}\Big)\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big) \\
& - \frac{1}{4}\Big(\bar\nabla_\nu h^{\sigma\gamma} + \bar\nabla^\gamma h^\sigma{}_\nu - \bar\nabla^\sigma h^\gamma{}_\nu\Big)\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big) \ .
\end{aligned} \tag{A.27}$$

The underlined terms in the previous expression cancel each other by relabelling some indices and using the symmetry property of $h_{\mu\nu}$,

$$h^{\sigma\gamma}\Big(\bar\nabla_\nu\bar\nabla_\gamma h_{\sigma\beta}\Big) - \underbrace{h^{\sigma\gamma}\Big(\bar\nabla_\nu\bar\nabla_\sigma h_{\gamma\beta}\Big)}_{(\gamma\longleftrightarrow\sigma)} = h^{\sigma\gamma}\Big(\bar\nabla_\nu\bar\nabla_\gamma h_{\sigma\beta}\Big) - \underbrace{h^{\gamma\sigma}}_{\text{Symmetric}}\Big(\bar\nabla_\nu\bar\nabla_\gamma h_{\sigma\beta}\Big) = 0 \ .$$

Then, from (A.27) the Riemann tensor at order $\mathcal{O}(\varepsilon^2)$ yields,

$$\begin{aligned}
\overset{(2)}{R}_{\beta\nu} = {} & \frac{1}{2}\Big\{\underline{\bar\nabla_\nu h^{\sigma\gamma}\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big)} - \bar\nabla_\sigma h^{\sigma\gamma}\underline{\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big)} \\
& + h^{\sigma\gamma}\Big(\bar\nabla_\nu\bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\sigma\bar\nabla_\nu h_{\gamma\beta} - \bar\nabla_\sigma\bar\nabla_\beta h_{\gamma\nu} + \bar\nabla_\sigma\bar\nabla_\gamma h_{\nu\beta}\Big) \\
& + \frac{1}{2}\bar\nabla^\gamma h\underline{\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big)} \\
& - \frac{1}{2}\Big(\underline{\bar\nabla_\nu h^{\sigma\gamma}} + \bar\nabla^\gamma h^\sigma{}_\nu - \bar\nabla^\sigma h^\gamma{}_\nu\Big)\underline{\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big)}\Big\} \\
= {} & \frac{1}{2}\Big\{\frac{1}{2}\bar\nabla_\nu h^{\sigma\gamma}\Big(\underline{\bar\nabla_\sigma h_{\gamma\beta}} + \bar\nabla_\beta h_{\gamma\sigma} - \underline{\bar\nabla_\gamma h_{\sigma\beta}}\Big) + \Big(\frac{1}{2}\bar\nabla^\gamma h - \bar\nabla_\sigma h^{\sigma\gamma}\Big)\Big(\bar\nabla_\nu h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\nu} - \bar\nabla_\gamma h_{\nu\beta}\Big) \\
& + h^{\sigma\gamma}\Big(\bar\nabla_\nu\bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\sigma\bar\nabla_\nu h_{\gamma\beta} - \bar\nabla_\sigma\bar\nabla_\beta h_{\gamma\nu} + \bar\nabla_\sigma\bar\nabla_\gamma h_{\nu\beta}\Big) \\
& - \frac{1}{2}\Big(\bar\nabla^\gamma h^\sigma{}_\nu - \bar\nabla^\sigma h^\gamma{}_\nu\Big)\Big(\bar\nabla_\sigma h_{\gamma\beta} + \bar\nabla_\beta h_{\gamma\sigma} - \bar\nabla_\gamma h_{\sigma\beta}\Big)\Big\}
\end{aligned} \tag{A.28}$$

$$
\begin{aligned}
\overset{(2)}{R}_{\beta\nu} \;=\; & \frac{1}{2}\bigg\{\frac{1}{2}\bar{\nabla}_\beta h_{\sigma\gamma}\bar{\nabla}_\nu h^{\sigma\gamma} + \Big(\frac{1}{2}\bar{\nabla}^\gamma h - \bar{\nabla}_\sigma h^{\sigma\gamma}\Big)\Big(\bar{\nabla}_\nu h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\nu} - \bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + h^{\sigma\gamma}\Big(\bar{\nabla}_\nu\bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h_{\gamma\beta} - \bar{\nabla}_\sigma\bar{\nabla}_\beta h_{\gamma\nu} + \bar{\nabla}_\sigma\bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + \frac{1}{2}\bar{\nabla}^\sigma h^\gamma{}_\nu\Big(\bar{\nabla}_\sigma h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\gamma h_{\sigma\beta}\Big) - \frac{1}{2}\underbrace{\bar{\nabla}^\gamma h^\sigma{}_\nu\Big(\bar{\nabla}_\sigma h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\gamma h_{\sigma\beta}\Big)}_{(\gamma\longleftrightarrow\sigma)}\bigg\} \\
=\; & \frac{1}{2}\bigg\{\frac{1}{2}\bar{\nabla}_\beta h_{\sigma\gamma}\bar{\nabla}_\nu h^{\sigma\gamma} + \Big(\frac{1}{2}\bar{\nabla}^\gamma h - \bar{\nabla}_\sigma h^{\sigma\gamma}\Big)\Big(\bar{\nabla}_\nu h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\nu} - \bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + h^{\sigma\gamma}\Big(\bar{\nabla}_\nu\bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h_{\gamma\beta} - \bar{\nabla}_\sigma\bar{\nabla}_\beta h_{\gamma\nu} + \bar{\nabla}_\sigma\bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + \frac{1}{2}\bar{\nabla}^\sigma h^\gamma{}_\nu\Big(\underline{\bar{\nabla}_\sigma h_{\gamma\beta}} + \cancel{\bar{\nabla}_\beta h_{\gamma\sigma}} - \underline{\bar{\nabla}_\gamma h_{\sigma\beta}} - \underline{\bar{\nabla}_\gamma h_{\sigma\beta}} - \cancel{\bar{\nabla}_\beta h_{\gamma\sigma}} + \underline{\bar{\nabla}_\sigma h_{\gamma\beta}}\Big)\bigg\} \\
=\; & \frac{1}{2}\bigg\{\frac{1}{2}\bar{\nabla}_\beta h_{\sigma\gamma}\bar{\nabla}_\nu h^{\sigma\gamma} + \bar{\nabla}^\sigma h^\gamma{}_\nu\Big(\bar{\nabla}_\sigma h_{\gamma\beta} - \bar{\nabla}_\gamma h_{\sigma\beta}\Big) \\
& + h^{\sigma\gamma}\Big(\bar{\nabla}_\nu\bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h_{\gamma\beta} - \bar{\nabla}_\sigma\bar{\nabla}_\beta h_{\gamma\nu} + \bar{\nabla}_\sigma\bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + \Big(\frac{1}{2}\bar{\nabla}^\gamma h - \bar{\nabla}_\sigma h^{\sigma\gamma}\Big)\Big(\bar{\nabla}_\nu h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\nu} - \bar{\nabla}_\gamma h_{\nu\beta}\Big)\bigg\} \;.
\end{aligned} \tag{A.29}
$$

Finally, the Ricci tensor at second order with all its covariant indices may be written as,

$$
\boxed{
\begin{aligned}
\overset{(2)}{R}_{\beta\nu}[h] \;=\; & \frac{1}{2}\bar{g}^{\sigma\mu}\bar{g}^{\gamma\alpha}\bigg\{\frac{1}{2}\bar{\nabla}_\beta h_{\sigma\gamma}\bar{\nabla}_\nu h_{\mu\alpha} + \bar{\nabla}_\mu h_{\alpha\nu}\Big(\bar{\nabla}_\sigma h_{\gamma\beta} - \bar{\nabla}_\gamma h_{\sigma\beta}\Big) \\
& + h_{\mu\alpha}\Big(\bar{\nabla}_\nu\bar{\nabla}_\beta h_{\gamma\sigma} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h_{\gamma\beta} - \bar{\nabla}_\sigma\bar{\nabla}_\beta h_{\gamma\nu} + \bar{\nabla}_\sigma\bar{\nabla}_\gamma h_{\nu\beta}\Big) \\
& + \Big(\frac{1}{2}\bar{\nabla}_\alpha h_{\mu\sigma} - \bar{\nabla}_\sigma h_{\mu\alpha}\Big)\Big(\bar{\nabla}_\nu h_{\gamma\beta} + \bar{\nabla}_\beta h_{\gamma\nu} - \bar{\nabla}_\gamma h_{\nu\beta}\Big)\bigg\} \;.
\end{aligned}
} \tag{A.30}
$$

Appendix B

Second order perturbative expansion in $f(R)$ gravity

Here we develop a detailed computation of the required quantities to find the energy-momentum tensor $\tilde{t}_{\mu\nu}$ in $f(R)$ gravity. To the order of accuracy, the partial and the covariant derivatives are indistinguishable because $\bar{\nabla}_\mu$ may commute and so behaves like ∂_μ[27]. For simplicity we also assumed that in some reference frame the Lorenz gauge and the traceless condition are valid. This means that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - a_2 \overset{(1)}{R} \bar{g}_{\mu\nu} \quad , \quad \partial^\nu \bar{h}_{\mu\nu} = 0 \quad , \quad \bar{h} = 0 \ . \tag{B.1}$$

The connections at order $\mathcal{O}(\varepsilon^2)$

The connections to second order in ε are given by

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \overset{(1)}{\Gamma}{}^\rho_{\mu\nu} + \overset{(2)}{\Gamma}{}^\rho_{\mu\nu} \tag{B.2}$$

where,

$$\bar{\Gamma}^\rho_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\lambda\mu} - \partial_\lambda \bar{g}_{\mu\nu}\right) \ , \tag{B.3}$$

$$\begin{aligned}\overset{(1)}{\Gamma}{}^\rho_{\mu\nu} &= \frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right) \\ &= \frac{1}{2}\bar{g}^{\rho\lambda}\left[\partial_\mu\left(\bar{h}_{\lambda\nu} - a_2 \overset{(1)}{R}\bar{g}_{\lambda\nu}\right) + \partial_\nu\left(\bar{h}_{\lambda\mu} - a_2 \overset{(1)}{R}\bar{g}_{\lambda\mu}\right) - \partial_\lambda\left(\bar{h}_{\mu\nu} - a_2 \overset{(1)}{R}\bar{g}_{\mu\nu}\right)\right] \ ,\end{aligned} \tag{B.4}$$

$$\begin{aligned}\overset{(2)}{\Gamma}{}^\rho_{\mu\nu} &= -\frac{1}{2}h^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right) \\ &= -\frac{1}{2}\left(\bar{h}^{\rho\lambda} - a_2 \overset{(1)}{R}\bar{g}^{\rho\lambda}\right)\left[\partial_\mu\left(\bar{h}_{\lambda\nu} - a_2 \overset{(1)}{R}\bar{g}_{\lambda\nu}\right) + \partial_\nu\left(\bar{h}_{\lambda\mu} - a_2 \overset{(1)}{R}\bar{g}_{\lambda\mu}\right)\right. \\ &\quad \left. - \partial_\lambda\left(\bar{h}_{\mu\nu} - a_2 \overset{(1)}{R}\bar{g}_{\mu\nu}\right)\right] \ .\end{aligned} \tag{B.5}$$

The Ricci tensor at order $\mathcal{O}(\varepsilon^2)$

The Ricci tensor to order $\mathcal{O}(\epsilon^2)$ can be decomposed as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu} \tag{B.6}$$

with

$$\bar{R}_{\mu\nu} = \partial_\sigma \bar{\Gamma}^\sigma_{\mu\nu} - \partial_\nu \bar{\Gamma}^\sigma_{\mu\sigma} + \bar{\Gamma}^\sigma_{\gamma\sigma}\bar{\Gamma}^\gamma_{\nu\mu} - \bar{\Gamma}^\sigma_{\gamma\nu}\bar{\Gamma}^\gamma_{\sigma\mu} \tag{B.7}$$

$$\overset{(1)}{R}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^\rho{}_\nu + \partial_\nu\partial_\rho h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) \tag{B.8}$$

$$\begin{aligned}\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\Big[&\frac{1}{2}\partial_\mu h_{\sigma\rho}\partial_\nu h^{\sigma\rho} + h^{\sigma\rho}\partial_\mu\partial_\nu h_{\sigma\rho} - h^{\sigma\rho}\partial_\nu\partial_\rho h_{\sigma\mu} - h^{\sigma\rho}\partial_\mu\partial_\rho h_{\sigma\nu} + h^{\sigma\rho}\partial_\sigma\partial_\rho h_{\mu\nu} \\ &\partial^\rho h^\sigma{}_\nu\partial_\rho h_{\sigma\mu} - \partial^\rho h^\sigma{}_\nu\partial_\sigma h_{\rho\mu} - \partial_\rho h^{\sigma\rho}\partial_\nu h_{\sigma\mu} + \partial_\rho h^{\sigma\rho}\partial_\sigma h_{\mu\nu} - \partial_\rho h^{\sigma\rho}\partial_\mu h_{\sigma\nu} \\ &-\frac{1}{2}\partial^\sigma h\partial_\sigma h_{\mu\nu} + \frac{1}{2}\partial^\sigma h\partial_\nu h_{\sigma\mu} + \frac{1}{2}\partial^\sigma h\partial_\mu h_{\sigma\nu}\Big]\end{aligned} \tag{B.9}$$

Explicitly, the expression in (B.8) are given in (7.45). Furthermore, to simplify we use equations (7.50) and (7.49). Then,

$$\begin{aligned}\overset{(1)}{R}_{\mu\nu} &= \partial_\mu\partial_\nu\left(a_2\overset{(1)}{R}\right) - \frac{1}{2}\Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\bar{g}_{\mu\nu}\right) + \frac{1}{6}\left(\overset{(1)}{R} + \overset{(1)}{\mathcal{G}}\right)\bar{g}_{\mu\nu} \\ &= a_2\partial_\mu\partial_\nu\overset{(1)}{R} + \frac{1}{6}\overset{(1)}{R}\bar{g}_{\mu\nu} - \frac{1}{2}\Box\bar{h}_{\mu\nu} + \frac{1}{4}\Box\bar{h}\bar{g}_{\mu\nu} + \frac{1}{6}\overset{(1)}{\mathcal{G}}\bar{g}_{\mu\nu} \\ &= a_2\partial_\mu\partial_\nu\overset{(1)}{R} + \frac{1}{6}\overset{(1)}{R}\bar{g}_{\mu\nu} + \overset{(1)}{\mathcal{G}}_{\mu\nu} - \frac{2}{3}\overset{(1)}{\mathcal{G}}\bar{g}_{\mu\nu} \ .\end{aligned}$$

Assuming vacuum, so that $\mathcal{G}_{\mu\nu} = 0$ then,

$$\boxed{\overset{(1)}{R}_{\mu\nu} = a_2\partial_\mu\partial_\nu\overset{(1)}{R} + \frac{1}{6}\overset{(1)}{R}\bar{g}_{\mu\nu}} \ . \tag{B.10}$$

Now, inserting the definition of $h_{\mu\nu}$ into the equation (B.9) one obtains,

$$\begin{aligned}\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\Big[&\frac{1}{2}\partial_\mu\left(\bar{h}_{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}_{\sigma\rho}\right)\partial_\nu\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right) + \left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\partial_\nu\left(\bar{h}_{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}_{\sigma\rho}\right) \\ &-\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\nu\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) - \left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\partial_\rho\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right) \\ &+\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\sigma\partial_\rho\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) + \partial^\rho\left(\bar{h}^\sigma{}_\nu - a_2\overset{(1)}{R}\delta^\sigma{}_\nu\right)\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) \\ &-\partial^\rho\left(\bar{h}^\sigma{}_\nu - a_2\overset{(1)}{R}\delta^\sigma{}_\nu\right)\partial_\sigma\left(\bar{h}_{\rho\mu} - a_2\overset{(1)}{R}\bar{g}_{\rho\mu}\right) - \partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\nu\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) \\ &+\partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\sigma\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) - \partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right) \\ &-\frac{1}{2}\partial^\sigma\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\partial_\sigma\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) + \frac{1}{2}\partial^\sigma\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\partial_\nu\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) \\ &+\frac{1}{2}\partial^\sigma\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\partial_\mu\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right)\Big]\end{aligned} \tag{B.11}$$

Expanding each term of equation (B.11) gives

$$
\begin{aligned}
&\frac{1}{2}\partial_\mu\left(\bar{h}_{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}_{\sigma\rho}\right)\partial_\nu\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right) = \frac{1}{2}\left(\partial_\mu\bar{h}_{\sigma\rho} - a_2\bar{g}_{\sigma\rho}\partial_\mu\overset{(1)}{R}\right)\left(\partial_\nu\bar{h}^{\sigma\rho} - a_2\bar{g}^{\sigma\rho}\partial_\nu\overset{(1)}{R}\right)\\
&= \frac{1}{2}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} - \frac{1}{2}\partial_\mu\bar{h}_{\sigma\rho}\left(a_2\bar{g}^{\sigma\rho}\partial_\nu\overset{(1)}{R}\right) - \frac{1}{2}\partial_\nu\bar{h}^{\sigma\rho}\left(a_2\bar{g}_{\sigma\rho}\partial_\mu\overset{(1)}{R}\right) + 2a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&= \frac{1}{2}\partial_\mu\bar{h}_{\sigma\rho}\partial_\nu\bar{h}^{\sigma\rho} - \frac{1}{2}a_2\partial_\mu\bar{h}\partial_\nu\overset{(1)}{R} - \frac{1}{2}a_2\partial_\nu\bar{h}\partial_\mu\overset{(1)}{R} + 2a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\,,
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
&\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\partial_\nu\left(\bar{h}_{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}_{\sigma\rho}\right) = \left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\left(\partial_\mu\partial_\nu\bar{h}_{\sigma\rho} - a_2\bar{g}_{\sigma\rho}\partial_\mu\partial_\nu\overset{(1)}{R}\right)\\
&= \bar{h}^{\sigma\rho}\partial_\mu\partial_\nu\bar{h}_{\sigma\rho} - \bar{h}^{\sigma\rho}\left(a_2\bar{g}^{\sigma\rho}\partial_\mu\partial_\nu\overset{(1)}{R}\right) - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\partial_\mu\partial_\nu\bar{h}_{\sigma\rho} + a_2^2\overset{(1)}{R}\bar{g}^{\sigma\rho}\bar{g}_{\sigma\rho}\partial_\mu\partial_\nu\overset{(1)}{R}\\
&= \bar{h}^{\sigma\rho}\partial_\mu\partial_\nu\bar{h}_{\sigma\rho} - a_2\bar{h}\partial_\mu\partial_\nu\overset{(1)}{R} - a_2\overset{(1)}{R}\partial_\mu\partial_\nu\bar{h} + 4a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\,,
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
&-\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\nu\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) = -\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\left(\partial_\nu\partial_\rho\bar{h}_{\sigma\mu} - a_2\bar{g}_{\sigma\mu}\partial_\nu\partial_\rho\overset{(1)}{R}\right)\\
&= -\bar{h}^{\sigma\rho}\partial_\nu\partial_\rho\bar{h}_{\sigma\mu} + \bar{h}^{\sigma\rho}\partial_\nu\partial_\rho\left(a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) + a_2\overset{(1)}{R}\partial_\nu\partial_\rho\bar{h}^{\rho}{}_{\mu} - a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\\
&= -\bar{h}^{\sigma\rho}\partial_\nu\partial_\rho\bar{h}_{\sigma\mu} + a_2\bar{h}^{\rho}{}_{\mu}\partial_\nu\partial_\rho\overset{(1)}{R} + a_2\overset{(1)}{R}\partial_\nu\partial_\rho\bar{h}^{\rho}{}_{\mu} - a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\,,
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
&-\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\partial_\rho\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right) = -\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\left(\partial_\mu\partial_\rho\bar{h}_{\sigma\nu} - a_2\bar{g}_{\sigma\nu}\partial_\mu\partial_\rho\overset{(1)}{R}\right)\\
&= -\bar{h}^{\sigma\rho}\partial_\mu\partial_\rho\bar{h}_{\sigma\nu} + a_2\bar{h}^{\sigma\rho}\bar{g}_{\sigma\nu}\partial_\mu\partial_\rho\overset{(1)}{R} + a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\partial_\mu\partial_\rho\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\bar{g}_{\sigma\nu}\partial_\mu\partial_\rho\overset{(1)}{R}\\
&= -\bar{h}^{\sigma\rho}\partial_\mu\partial_\rho\bar{h}_{\sigma\nu} + a_2\bar{h}^{\rho}{}_{\nu}\partial_\mu\partial_\rho\overset{(1)}{R} + a_2\overset{(1)}{R}\partial_\mu\partial_\rho\bar{h}^{\rho}{}_{\nu} - a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\,,
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
&\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\sigma\partial_\rho\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) = \left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\left(\partial_\sigma\partial_\rho\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\partial_\sigma\partial_\rho\overset{(1)}{R}\right)\\
&= \bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\bar{h}_{\mu\nu} - a_2\bar{h}^{\sigma\rho}\bar{g}_{\mu\nu}\partial_\sigma\partial_\rho\overset{(1)}{R} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\partial_\sigma\partial_\rho\bar{h}_{\mu\nu} + a_2^2\overset{(1)}{R}\bar{g}^{\sigma\rho}\bar{g}_{\mu\nu}\partial_\sigma\partial_\rho\overset{(1)}{R}\\
&= \bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} - a_2\overset{(1)}{R}\bar{\Box}\bar{h}_{\mu\nu} + a_2^2\overset{(1)}{R}\bar{g}_{\mu\nu}\bar{\Box}\overset{(1)}{R}\,,
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
&\partial^\rho\left(\bar{h}^{\sigma}{}_{\nu} - a_2\overset{(1)}{R}\delta^{\sigma}{}_{\nu}\right)\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) = \left(\partial^\rho\bar{h}^{\sigma}{}_{\nu} - a_2\delta^{\sigma}{}_{\nu}\partial^\rho\overset{(1)}{R}\right)\left(\partial_\rho\bar{h}_{\sigma\mu} - a_2\bar{g}_{\sigma\mu}\partial_\rho\overset{(1)}{R}\right)\\
&= \partial^\rho\bar{h}^{\sigma}{}_{\nu}\partial_\rho\bar{h}_{\sigma\mu} - a_2\partial^\rho\bar{h}_{\mu\nu}\partial_\rho\overset{(1)}{R} - a_2\partial^\rho\overset{(1)}{R}\partial_\rho\bar{h}_{\mu\nu} + a_2^2\bar{g}_{\mu\nu}\partial^\rho\overset{(1)}{R}\partial_\rho\overset{(1)}{R}\,,
\end{aligned}
\tag{B.17}
$$

$$-\,\partial^\rho\left(\bar{h}^\sigma{}_\nu - a_2\overset{(1)}{R}\delta^\sigma{}_\nu\right)\partial_\sigma\left(\bar{h}_{\rho\mu} - a_2\overset{(1)}{R}\bar{g}_{\rho\mu}\right) = -\left(\partial^\rho\bar{h}^\sigma{}_\nu - a_2\delta^\sigma_\nu\partial^\rho\overset{(1)}{R}\right)\left(\partial_\sigma\bar{h}_{\rho\mu} - a_2\bar{g}_{\rho\mu}\partial_\sigma\overset{(1)}{R}\right)$$
$$= -\,\partial^\rho\bar{h}^\sigma{}_\nu\partial_\sigma\bar{h}_{\rho\mu} + a_2\partial_\mu\bar{h}^\sigma{}_\nu\partial_\sigma\overset{(1)}{R} + a_2\partial^\rho\overset{(1)}{R}\partial_\nu\bar{h}_{\rho\mu} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\,, \tag{B.18}$$

$$-\,\partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\nu\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) = -\left(\partial_\rho\bar{h}^{\sigma\rho} - a_2\bar{g}^{\sigma\rho}\partial_\rho\overset{(1)}{R}\right)\left(\partial_\nu\bar{h}_{\sigma\mu} - a_2\bar{g}_{\sigma\mu}\partial_\nu\overset{(1)}{R}\right)$$
$$= -\,\partial_\rho\bar{h}^{\sigma\rho}\partial_\nu\bar{h}_{\sigma\mu} + a_2\partial_\rho\bar{h}^\rho{}_\mu\partial_\nu\overset{(1)}{R} + a_2\partial_\rho\overset{(1)}{R}\partial_\nu\bar{h}^\rho{}_\mu - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\,, \tag{B.19}$$

$$\partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\sigma\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) = \left(\partial_\rho\bar{h}^{\sigma\rho} - a_2\bar{g}^{\sigma\rho}\partial_\rho\overset{(1)}{R}\right)\left(\partial_\sigma\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\partial_\sigma\overset{(1)}{R}\right)$$
$$= \partial_\rho\bar{h}^{\sigma\rho}\partial_\sigma\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\partial_\rho\bar{h}^{\sigma\rho}\partial_\sigma\overset{(1)}{R} - a_2\bar{g}^{\sigma\rho}\partial_\rho\overset{(1)}{R}\partial_\sigma\bar{h}_{\mu\nu} + a_2^2\bar{g}^{\sigma\rho}\bar{g}_{\mu\nu}\partial_\rho\overset{(1)}{R}\partial_\sigma\overset{(1)}{R}$$
$$= \partial_\rho\bar{h}^{\sigma\rho}\partial_\sigma\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\partial_\rho\bar{h}^{\sigma\rho}\partial_\sigma\overset{(1)}{R} - a_2\partial^\sigma\overset{(1)}{R}\partial_\sigma\bar{h}_{\mu\nu} + a_2^2\bar{g}_{\mu\nu}\partial^\sigma\overset{(1)}{R}\partial_\sigma\overset{(1)}{R}\,, \tag{B.20}$$

$$-\,\partial_\rho\left(\bar{h}^{\sigma\rho} - a_2\overset{(1)}{R}\bar{g}^{\sigma\rho}\right)\partial_\mu\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right) = -\left(\partial_\rho\bar{h}^{\sigma\rho} - a_2\bar{g}^{\sigma\rho}\partial_\rho\overset{(1)}{R}\right)\left(\partial_\mu\bar{h}_{\sigma\nu} - a_2\bar{g}_{\sigma\nu}\partial_\mu\overset{(1)}{R}\right)$$
$$= -\,\partial_\rho\bar{h}^{\sigma\rho}\partial_\mu\bar{h}_{\sigma\nu} + a_2\bar{g}_{\sigma\nu}\partial_\rho\bar{h}^{\sigma\rho}\partial_\mu\overset{(1)}{R} + a_2\bar{g}^{\sigma\rho}\partial_\rho\overset{(1)}{R}\partial_\mu\bar{h}_{\sigma\nu} - a_2^2\delta^\rho{}_\nu\partial_\rho\overset{(1)}{R}\partial_\mu\overset{(1)}{R}$$
$$= -\,\partial_\rho\bar{h}^{\sigma\rho}\partial_\mu\bar{h}_{\sigma\nu} + a_2\partial_\rho\bar{h}^\rho{}_\nu\partial_\mu\overset{(1)}{R} + a_2\partial^\sigma\overset{(1)}{R}\partial_\mu\bar{h}_{\sigma\nu} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\,, \tag{B.21}$$

$$-\frac{1}{2}\partial^\sigma\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\partial_\sigma\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) = \left(-\frac{1}{2}\partial^\sigma\bar{h} + 2a_2\partial^\sigma\overset{(1)}{R}\right)\left(\partial_\sigma\bar{h}_{\mu\nu} - a_2\bar{g}_{\mu\nu}\partial_\sigma\overset{(1)}{R}\right)$$
$$= -\frac{1}{2}\partial^\sigma\bar{h}\partial_\sigma\bar{h}_{\mu\nu} + \frac{1}{2}a_2\bar{g}_{\mu\nu}\partial^\sigma\bar{h}\partial_\sigma\overset{(1)}{R} + 2a_2\partial^\sigma\overset{(1)}{R}\partial_\sigma\bar{h}_{\mu\nu} - 2a_2^2\bar{g}_{\mu\nu}\partial^\sigma\overset{(1)}{R}\partial_\sigma\overset{(1)}{R}\,, \tag{B.22}$$

$$\frac{1}{2}\partial^\sigma\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\partial_\nu\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) = \left(\frac{1}{2}\partial^\sigma\bar{h} - 2a_2\partial^\sigma\overset{(1)}{R}\right)\left(\partial_\nu\bar{h}_{\sigma\mu} - a_2\bar{g}_{\sigma\mu}\partial_\nu\overset{(1)}{R}\right)$$
$$= \frac{1}{2}\partial^\sigma\bar{h}\partial_\nu\bar{h}_{\sigma\mu} - \frac{1}{2}a_2\bar{g}_{\sigma\mu}\partial^\sigma\bar{h}\partial_\nu\overset{(1)}{R} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\nu\bar{h}_{\sigma\mu} + 2a_2^2\bar{g}_{\sigma\mu}\partial^\sigma\overset{(1)}{R}\partial_\nu\overset{(1)}{R}$$
$$= \frac{1}{2}\partial^\sigma\bar{h}\partial_\nu\bar{h}_{\sigma\mu} - \frac{1}{2}a_2\partial_\mu\bar{h}\partial_\nu\overset{(1)}{R} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\nu\bar{h}_{\sigma\mu} + 2a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\,, \tag{B.23}$$

$$\begin{aligned}
\frac{1}{2}\partial^\sigma\left(\bar h - 4a_2\overset{(1)}{R}\right)\partial_\mu\left(\bar h_{\sigma\nu} - a_2\overset{(1)}{R}\bar g_{\sigma\nu}\right) &= \left(\frac{1}{2}\partial^\sigma\bar h - 2a_2\partial^\sigma\overset{(1)}{R}\right)\left(\partial_\mu\bar h_{\sigma\nu} - a_2\bar g_{\sigma\nu}\partial_\mu\overset{(1)}{R}\right)\\
&= \frac{1}{2}\partial^\sigma\bar h\partial_\mu\bar h_{\sigma\nu} - \frac{1}{2}a_2\bar g_{\sigma\nu}\partial^\sigma\bar h\partial_\mu\overset{(1)}{R} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\mu\bar h_{\sigma\nu} + 2a_2^2\bar g_{\sigma\nu}\partial^\sigma\overset{(1)}{R}\partial_\mu\overset{(1)}{R}\\
&= \frac{1}{2}\partial^\sigma\bar h\partial_\mu\bar h_{\sigma\nu} - \frac{1}{2}a_2\partial_\nu\bar h\partial_\mu\overset{(1)}{R} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\mu\bar h_{\sigma\nu} + 2a_2^2\partial_\nu\overset{(1)}{R}\partial_\mu\overset{(1)}{R}\,.
\end{aligned} \tag{B.24}$$

Adding all the terms from equation (B.12) to (B.24) and consider the Lorenz gauge, $\bar h = 0$ and vacuum spacetime. The result is,

$$\begin{aligned}
\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\Bigg[&\frac{1}{2}\partial_\mu\bar h_{\sigma\rho}\partial_\nu\bar h^{\sigma\rho} - \frac{1}{2}\underbrace{a_2\partial_\mu\bar h\partial_\nu\overset{(1)}{R}}_{=0} - \frac{1}{2}\underbrace{a_2\partial_\nu\bar h\partial_\mu\overset{(1)}{R}}_{=0} + 2a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&+ \bar h^{\sigma\rho}\partial_\mu\partial_\nu\bar h_{\sigma\rho} - \underbrace{a_2\bar h\partial_\mu\partial_\nu\overset{(1)}{R}}_{=0} - \underbrace{a_2\overset{(1)}{R}\partial_\mu\partial_\nu\bar h}_{=0} + 4a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\\
&- \bar h^{\sigma\rho}\partial_\nu\partial_\rho\bar h_{\sigma\mu} + a_2\bar h^\rho{}_\mu\partial_\nu\partial_\rho\overset{(1)}{R} + \underbrace{a_2\overset{(1)}{R}\partial_\nu\partial_\rho\bar h^\rho{}_\mu}_{=0} - a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\\
&- \bar h^{\sigma\rho}\partial_\mu\partial_\rho\bar h_{\sigma\nu} + a_2\bar h^\rho{}_\nu\partial_\mu\partial_\rho\overset{(1)}{R} + \underbrace{a_2\overset{(1)}{R}\partial_\mu\partial_\rho\bar h^\rho{}_\nu}_{=0} - a_2^2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R}\\
&+ \bar h^{\sigma\rho}\partial_\sigma\partial_\rho\bar h_{\mu\nu} - a_2\bar g_{\mu\nu}\bar h^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} - \underbrace{a_2\overset{(1)}{R}\bar\Box\bar h_{\mu\nu}}_{=0} + a_2^2\overset{(1)}{R}\bar g_{\mu\nu}\bar\Box\overset{(1)}{R}\\
&+ \partial^\rho\bar h^\sigma{}_\nu\partial_\rho\bar h_{\sigma\mu} - a_2\partial^\rho\bar h_{\mu\nu}\partial_\rho\overset{(1)}{R} - a_2\partial^\rho\overset{(1)}{R}\partial_\rho\bar h_{\mu\nu} + a_2^2\bar g_{\mu\nu}\partial^\rho\overset{(1)}{R}\partial_\rho\overset{(1)}{R}\\
&- \partial^\rho\bar h^\sigma{}_\nu\partial_\sigma\bar h_{\rho\mu} + a_2\partial_\mu\bar h^\sigma{}_\nu\partial_\sigma\overset{(1)}{R} + a_2\partial^\rho\overset{(1)}{R}\partial_\nu\bar h_{\rho\mu} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&- \underbrace{\partial_\rho\bar h^{\sigma\rho}\partial_\nu\bar h_{\sigma\mu}}_{=0} + \underbrace{a_2\partial_\rho\bar h^\rho{}_\mu\partial_\nu\overset{(1)}{R}}_{=0} + a_2\partial_\rho\overset{(1)}{R}\partial_\nu\bar h^\rho{}_\mu - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&+ \underbrace{\partial_\rho\bar h^{\sigma\rho}\partial_\sigma\bar h_{\mu\nu}}_{=0} - \underbrace{a_2\bar g_{\mu\nu}\partial_\rho\bar h^{\sigma\rho}\partial_\sigma\overset{(1)}{R}}_{=0} - a_2\partial^\sigma\overset{(1)}{R}\partial_\sigma\bar h_{\mu\nu} + a_2^2\bar g_{\mu\nu}\partial^\sigma\overset{(1)}{R}\partial_\sigma\overset{(1)}{R}\\
&- \underbrace{\partial_\rho\bar h^{\sigma\rho}\partial_\mu\bar h_{\sigma\nu}}_{=0} + \underbrace{a_2\partial_\rho\bar h^\rho{}_\nu\partial_\mu\overset{(1)}{R}}_{=0} + a_2\partial^\sigma\overset{(1)}{R}\partial_\mu\bar h_{\sigma\nu} - a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&- \frac{1}{2}\underbrace{\partial^\sigma\bar h\partial_\sigma\bar h_{\mu\nu}}_{=0} + \frac{1}{2}\underbrace{a_2\bar g_{\mu\nu}\partial^\sigma\bar h\partial_\sigma\overset{(1)}{R}}_{=0} + 2a_2\partial^\sigma\overset{(1)}{R}\partial_\sigma\bar h_{\mu\nu} - 2a_2^2\bar g_{\mu\nu}\partial^\sigma\overset{(1)}{R}\partial_\sigma\overset{(1)}{R}\\
&+ \frac{1}{2}\underbrace{\partial^\sigma\bar h\partial_\nu\bar h_{\sigma\mu}}_{=0} - \frac{1}{2}\underbrace{a_2\partial_\mu\bar h\partial_\nu\overset{(1)}{R}}_{=0} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\nu\bar h_{\sigma\mu} + 2a_2^2\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R}\\
&+ \frac{1}{2}\underbrace{\partial^\sigma\bar h\partial_\mu\bar h_{\sigma\nu}}_{=0} - \frac{1}{2}\underbrace{a_2\partial_\nu\bar h\partial_\mu\overset{(1)}{R}}_{=0} - 2a_2\partial^\sigma\overset{(1)}{R}\partial_\mu\bar h_{\sigma\nu} + 2a_2^2\partial_\nu\overset{(1)}{R}\partial_\mu\overset{(1)}{R}\Bigg]\,.
\end{aligned} \tag{B.25}$$

Finally, the Ricci tensor to order $\mathcal{O}(\varepsilon^2)$ is given by

$$\begin{aligned}
\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\Bigg\{ & \frac{1}{2}\partial_\mu \bar{h}_{\sigma\rho}\partial_\nu \bar{h}^{\sigma\rho} + \bar{h}^{\sigma\rho}\Bigg[\partial_\mu\partial_\nu \bar{h}_{\sigma\rho} + \partial_\sigma\partial_\rho\left(\bar{h}_{\mu\nu} - a_2\overset{(1)}{R}\bar{g}_{\mu\nu}\right) \\
& - \partial_\nu\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) - \partial_\mu\partial_\rho\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right)\Bigg] \\
& + \partial^\rho \bar{h}^\sigma{}_\nu\left(\partial_\rho \bar{h}_{\sigma\mu} - \partial_\sigma \bar{h}_{\rho\mu}\right) - a_2\partial^\sigma \overset{(1)}{R}\partial_\sigma \bar{h}_{\mu\nu} \\
& + a_2^2\left(2\overset{(1)}{R}\partial_\mu\partial_\nu\overset{(1)}{R} + 3\partial_\mu\overset{(1)}{R}\partial_\nu\overset{(1)}{R} + \overset{(1)}{R}\bar{\Box}\overset{(1)}{R}\bar{g}_{\mu\nu}\right)\Bigg\} \ .
\end{aligned} \tag{B.26}$$

The Ricci scalar is obtained when contracting the Ricci tensor with the full metric,

$$\begin{aligned}
R &= \bar{R} + \overset{(1)}{R} + \overset{(2)}{R} \\
&= g^{\mu\nu}\left(\bar{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu}\right) \\
&= \left(\bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu\right)\left(\bar{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu}\right) \\
&= \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\overset{(1)}{R}_{\mu\nu} + \bar{g}^{\mu\nu}\overset{(2)}{R}_{\mu\nu} - h^{\mu\nu}\bar{R}_{\mu\nu} - h^{\mu\nu}\overset{(1)}{R}_{\mu\nu} - h^{\mu\nu}\overset{(2)}{R}_{\mu\nu} \\
&\quad h^{\mu\lambda}h_\lambda{}^\nu\bar{R}_{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu\overset{(1)}{R}_{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu\overset{(2)}{R}_{\mu\nu} \\
&= \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\overset{(1)}{R}_{\mu\nu} + \left(\bar{g}^{\mu\nu}\overset{(2)}{R}_{\mu\nu} - h^{\mu\nu}\overset{(1)}{R}_{\mu\nu}\right) + \cdots
\end{aligned} \tag{B.27}$$

where,

$$\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} \ , \tag{B.28}$$

$$\overset{(1)}{R} = \bar{g}^{\mu\nu}\overset{(1)}{R}_{\mu\nu} \ , \tag{B.29}$$

$$\overset{(2)}{R} = \bar{g}^{\mu\nu}\overset{(2)}{R}_{\mu\nu} - h^{\mu\nu}\overset{(1)}{R}_{\mu\nu} \ . \tag{B.30}$$

The equation (B.28) is just the Ricci scalar of the background. Equation (B.29) reproduce the Klein-Gordon equation with respect to the background

$$\begin{aligned}
\overset{(1)}{R} &= \bar{g}^{\mu\nu}\overset{(1)}{R}_{\mu\nu} \\
&= \bar{g}^{\mu\nu}\left(a_2\partial_\mu\partial_\nu\overset{(1)}{R} + \frac{1}{6}\overset{(1)}{R}\bar{g}_{\mu\nu}\right) \\
&= a_2\bar{\Box}\overset{(1)}{R} + \frac{2}{3}\overset{(1)}{R} \ ,
\end{aligned}$$

i.e.,

$$\boxed{\left(\bar{\Box} - \Upsilon^2\right)\overset{(1)}{R} = 0} \ . \tag{B.31}$$

The Ricci scalar to second order is obtained as follows,

$$
\begin{aligned}
\overset{(1)}{R} &= \bar{g}^{\mu\nu}\overset{(2)}{R}_{\mu\nu} - h^{\mu\nu}\overset{(1)}{R}_{\mu\nu} \\
&= \frac{1}{2}\Bigg\{\frac{1}{2}\partial_\mu \bar{h}_{\sigma\rho}\partial^\mu \bar{h}^{\sigma\rho} + \bar{h}^{\sigma\rho}\left[\bar{\Box}\bar{h}_{\sigma\rho} + \partial_\sigma\partial_\rho\left(\bar{h} - 4a_2\overset{(1)}{R}\right)\right] - \partial^\mu\partial_\rho\left(\bar{h}_{\sigma\mu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\mu}\right) \\
&\quad - \partial^\nu\partial_\rho\left(\bar{h}_{\sigma\nu} - a_2\overset{(1)}{R}\bar{g}_{\sigma\nu}\right)\Bigg] + \partial^\rho\bar{h}^{\sigma\mu}\left(\partial_\rho\bar{h}_{\sigma\mu} - \partial_\sigma\bar{h}_{\rho\mu}\right) - a_2\partial^\sigma\overset{(1)}{R}\partial\bar{h} \\
&\quad + a_2^2\left(2\overset{(1)}{R}\bar{\Box}\overset{(1)}{R} + 3\partial_\mu\overset{(1)}{R}\partial^\mu\overset{(1)}{R} + 4\overset{(1)}{R}\bar{\Box}\overset{(1)}{R}\right)\Bigg\} - \Bigg\{a_2\bar{h}^{\mu\nu}\partial_\mu\partial_\nu\overset{(1)}{R} + a_2^2\overset{(1)}{R}\bar{\Box}\overset{(1)}{R} \\
&\quad - \frac{1}{6}\overset{(1)}{R}\bar{h} + \frac{2}{3}a_2\overset{(1)}{R}{}^2\Bigg\}\,, \qquad \text{(B.32)}
\end{aligned}
$$

after inserting the full definition of $h_{\mu\nu}$ from equation (7.52) with $\bar{g}_{\mu\nu}$. Using the the Lorenz gauge, the trace free of $\bar{h}_{\mu\nu}$, the wave equation and the Klein-Gordon equation one obtains,

$$
\begin{aligned}
\overset{(2)}{R} &= \frac{3}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial^\mu\bar{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho\bar{h}^{\sigma\mu}\partial_\sigma\bar{h}_{\rho\mu} - 2a_2\bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} + \frac{1}{2}a_2\bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} \\
&\quad + \frac{1}{2}a_2\bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} - a_2\bar{h}^{\mu\nu}\partial_\mu\partial_\nu\overset{(1)}{R} + 4a_2^2\overset{(1)}{R}\bar{\Box}\overset{(1)}{R} + \frac{3}{2}a_2^2\partial_\mu\overset{(1)}{R}\partial^\mu\overset{(1)}{R} + \frac{2}{3}a_2\overset{(1)}{R}{}^2 \\
&= \frac{3}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial^\mu\bar{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho\bar{h}^{\sigma\mu}\partial_\sigma\bar{h}_{\rho\mu} - 2a_2\bar{h}^{\sigma\rho}\partial_\sigma\partial_\rho\overset{(1)}{R} \\
&\quad + 4a_2^2\overset{(1)}{R}\left(\frac{1}{3a_2}\overset{(1)}{R}\right) + \frac{3}{2}a_2^2\partial_\mu\overset{(1)}{R}\partial^\mu\overset{(1)}{R} + \frac{2}{3}a_2\overset{(1)}{R}{}^2\ .
\end{aligned}
$$

Hence,

$$
\boxed{
\begin{aligned}
\overset{(2)}{R} &= \frac{3}{4}\partial_\mu\bar{h}_{\sigma\rho}\partial^\mu\bar{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho\bar{h}^{\sigma\mu}\partial_\sigma\bar{h}_{\rho\mu} - 2a_2\bar{h}^{\mu\nu}\partial_\mu\partial_\nu\overset{(1)}{R} \\
&\quad + 2a_2\overset{(1)}{R}{}^2 + \frac{3}{2}a_2^2\partial_\mu\overset{(1)}{R}\partial^\mu\overset{(1)}{R}\ .
\end{aligned}
} \qquad \text{(B.33)}
$$

For the function $f(R)$ we have,

$$
\begin{aligned}
f(R) &= R + \frac{1}{2!}a_2R^2 + \cdots \\
&= \left(\bar{R} + \overset{(1)}{R} + \overset{(2)}{R}\right) + \frac{1}{2}a_2\left(\bar{R} + \overset{(1)}{R} + \overset{(2)}{R}\right)^2 + \cdots \\
&= \left(\bar{R} + \overset{(1)}{R} + \overset{(2)}{R}\right) + \frac{1}{2}a_2\left(\underbrace{\bar{R}^2}_{\mathcal{O}(\varepsilon^4)} + \underbrace{2\bar{R}\overset{(1)}{R}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{2\bar{R}\overset{(2)}{R}}_{\mathcal{O}(\varepsilon^4)} + \overset{(1)}{R}{}^2 + \underbrace{2\overset{(1)}{R}\overset{(2)}{R}}_{\mathcal{O}(\varepsilon^3)} + \underbrace{\overset{(2)}{R}{}^2}_{\mathcal{O}(\varepsilon^4)}\right) + \cdots \\
&= \bar{R} + \overset{(1)}{R} + \left(\overset{(2)}{R} + \frac{1}{2}a_2\overset{(1)}{R}{}^2\right) + \cdots \\
&= \bar{f} + \overset{(1)}{f} + \overset{(2)}{f} + \cdots \qquad \text{(B.34)}
\end{aligned}
$$

where

$$\bar{f} = \bar{R}\,, \qquad \overset{(1)}{f} = \overset{(1)}{R}\,, \qquad \overset{(2)}{f} = \overset{(2)}{R} + \frac{1}{2}a_2\overset{(1)}{R}{}^2\,. \tag{B.35}$$

To find the corresponding derivatives at each order, we need to expand the function $f(R)$ up to the third power in R. After differentiate with respect to R one have

$$\begin{aligned} f'(R) &= 1 + a_2 R + \frac{1}{2}a_3 R^2 + \cdots \\ &= 1 + a_2\left(\bar{R} + \overset{(1)}{R} + \overset{(2)}{R}\right) + \frac{1}{2}a_3\left(\bar{R} + \overset{(1)}{R} + \overset{(2)}{R}\right)^2 + \ldots \\ &= 1 + a_2\bar{R} + a_2\overset{(1)}{R} + \left(\frac{1}{2}a_3\overset{(1)}{R}{}^2 + a_2\overset{(2)}{R}\right) + \cdots \\ &= \overset{(0)}{f'} + \bar{f}' + \overset{(1)}{f'} + \overset{(2)}{f'}\,, \end{aligned} \tag{B.36}$$

with

$$\overset{(0)}{f'} = 1\,, \qquad \bar{f}' = a_2\bar{R}\,, \qquad \overset{(1)}{f'} = a_2\overset{(1)}{R}\,, \qquad \overset{(2)}{f'} = a_2\overset{(2)}{R} + \frac{1}{2}a_3\overset{(1)}{R}{}^2\,. \tag{B.37}$$

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