Matrix methods for projective modules over $\sigma - PBW$ extensions

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Métodos matriciales para módulos proyectivos sobre extensiones $\sigma - PBW$

Abstract: In this monograph, we study finitely generated projective modules defined on a certain type of noncommutative rings, called $\sigma - PBW$ extensions, also known as skew PBW extensions. This class of noncommutative rings of polynomial type include many important examples of algebras and rings of recent interest as Weyl algebras, enveloping algebras of Lie algebras of finite dimension, diffusion algebras, quantum algebras, quadratic algebras in three variables, among many others. The study of projective modules was developed from a constructive matrix approach that will allow us to make effective calculations using a powerful computational tool: noncommutative Gröbner bases. Specifically, we establish an equivalent constructive matrix interpretation for the notions of being a projective, stably free or free module. Because of the close relationship between these three kinds of modules, we investigate when a given finitely generated module belongs to one of these classes. In this regard, Stafford showed that any stably free module on the Weyl algebra $D = A_n(\mathbb{k})$ or $B_n(\mathbb{k})$, with rank ≥ 2, turns out to be free; in this direction, we present a constructive proof of such important theorem for arbitrary rings which satisfy the condition range.

On the other hand, we present several matrix descriptions of Hermite rings, various - characterizations of $\mathcal{PF}$ rings, and some subclasses of Hermite rings. However, since there is a variety of noncommutative rings that have nontrivial stably free modules, we use the Stafford's theorem, the stable range of a ring, and existing bounds for Krull dimension of a skew PBW extension, in order to set a value from which all stably free module are free.

In the second part of this thesis, we develop the theory of Gröbner bases for arbitrary bijective skew PBW extensions. Specifically, we extend Gröbner theory of quasi-commutative bijective skew extensions to arbitrary bijective skew PBW extensions. We construct Buchberger's algorithm for left (right) ideals and modules over these noncommutative rings, and we present elementary applications of this theory as the membership problem, calculation of the syzygy module, intersection of ideals and modules, the quotient ideal, presentation of a module, calculation of free resolutions and the kernel and image of a homomorphism. Finally, we use the constructive proofs established in the early chapters, in order to develop effective algorithms to compute the projective dimension of a given module, algorithms for testing stably-freeness, procedures for computing minimal presentations and bases for free modules.

Resumen: En esta monografía estudiamos los módulos proyectivos definidos sobre un cierto tipo de anillos no conmutativos, denominados extensiones $\sigma - PBW$, también conocidos como extensiones PBW torcidas. Esta clase de anillos no conmutativos de tipo polinomial incluye importantes ejemplos de álgebras y anillos de interés reciente tales como álgebras de Weyl, álgebras envolventes de álgebras de Lie de dimensión finita, álgebras cuánticas, álgebras cuadráticas en tres variables, entre muchos otros. El estudio de los módulos proyectivos lo desarrollamos desde una perspectiva constructiva-matricial, enfoque que nos permitirá hacer cálculos efectivos mediante el uso de una importante herramienta computacional: las bases de Gröbner no conmutativas. Específicamente, establecemos interpretaciones matriciales constructivas para la noción de módulo proyectivo, módulo establemente libre y módulo libre. Debido a la estrecha relación existente entre estas tres clases de módulos, investigamos cuándo un módulo finitamente generado dado pertenece a una de tales clases. En este sentido, Stafford demostró que cualquier módulo establemente libre sobre el álgebra de Weyl $D = A_n(\Bbbk)$ o $B_n(\Bbbk)$, de rango ≥ 2, resulta ser libre; a este respecto, presentamos una prueba constructiva de este importante teorema para anillos arbitrarios que satisfagan la condición de rango.

Por otra parte, presentamos descripciones matriciales de los anillos de Hermite, caracterizaciones de anillos $\mathcal{PF}$, y algunas subclases de anillos de Hermite. Ahora bien, puesto que existe una gran variedad de anillos no conmutativos que poseen módulos establemente libres no triviales, nosotros usamos el teorema de Stafford, el rango estable de un anillo, y las cotas existentes para la dimensión de Krull de una extensión PBW torcida, con el fin de establecer un valor a partir del cual todo módulo establemente libre resulta libre.

En la segunda parte de esta tesis desarrollamos la teoría de bases de Gröbner para extensiones PBW torcidas biyectivas arbitrarias. Concretamente, extendemos la teoría de Gröbner de las extensiones cuasi-conmutativas biyectivas al caso general biyectivo. - Construimos el algoritmo de Buchberger para ideales izquierdos (derechos) y para módulos sobre estos anillos, presentamos aplicaciones elementales de esta teoría como el problema de membresía, el calculo del módulo de sicigias, la intersección de ideales y módulos, el ideal cociente, la presentación de un módulo, el cálculo de resoluciones libres y el núcleo e imagen de un homomorfismo. Finalmente, usamos las demostraciones constructivas establecidas en los primeros capítulos, con la finalidad de elaborar algoritmos que permiten efectivamente calcular la dimensión proyectiva de un módulo dado, verificar si un módulo es establemente libre, calcular presentaciones minimales y bases para módulos libres.

Keywords: Skew PBW extensions. Projective, stably free and free modules. Hermite rings. $\mathcal{PF}$ rings. Stable range. Noncommutative Gröbner bases.

Palabras clave: Extensiones PBW torcidas. Módulos proyectivos, establemente libres y libres. Anillos de Hermite. Anillos $\mathcal{PF}$. Rango estable. Bases de Gröbner no conmutativas.

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Dedicated to

My grandmother, my parents, Dilan, my cute dogs and Paquita.

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Contents

Introduction

When a new type of rings arise, the study of finitely generated projective modules over them is a classical task in homological algebra. Investigating if these modules are free, or at least stably free, has occupied the attention of many mathematicians; one of the most famous cases is the Quillen-Suslin theorem about Serre's problem for the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field. In this particular example, Quillen and Suslin proved independently that the finitely generated projective modules are free (see [106] and [119]). However, for noncommutative rings of polynomial type it is easy to present examples where Quillen-Suslin theorem fails. In fact, if T is a division ring, then $S := T[x, y]$ has a module M such that $M \oplus S \cong S^2$, but M is not free ([62]). When this occurs, we can ask if the modules are stably free, and this situation, to investigate the minimum value of rank for which the modules start to be free. This is the content of Stafford's theorem on Weyl algebras (see [114]), or Artamonov's theorem for quantum polynomials ([4],[5] and [6]).

The origin of our interest in investigating projective modules over skew PBW extensions from a matrix constructive approach arises in a previous thesis (master's thesis) where we study the theory of Gröbner bases of left ideals for the particular class of skew PBW extensions that are quasi-commutative bijective (see Chapter 1). Skew PBW extensions are a wide class of noncommutative rings of polynomial type introduced in [40], and generalize the PBW extensions defined in [10]. Skew PBW extensions include many important classes of noncommutative rings and algebras as Weyl algebras, enveloping algebras of Lie algebras of finite dimension, important classes of Ore algebras, quantum algebras, Manin's algebra of quantum matrices, q-Heisenberg algebra, quantum Weyl algebras, quantum enveloping algebras, Witten's algebra, diffusion algebras, among many others. Some ring and homological properties of skew PBW extensions have been studied in the last years from a purely theoretic non-constructive approach, for example, global, Krull, Goldie and Gelfand-Kirillov dimensions of these rings have been computed as well as its K-theory gropus (see [83], [81] and [121]).

In this thesis, we investigate free, stably free, and in general, projective modules over skew PBW extensions from a matrix approach, complementing the results with the theory of Gröbner bases. In the first part of the thesis, we will present matrix criteria (theorems) for testing projectivity, stably freeness and freeness - in general - for finitely generated modules over arbitrary rings satisfying the rank condition (see Definition 2.1.3). In the second part, we will present algorithms for bijective skew PBW extensions making theorems constructive, and finally, we will apply the developed theory of Gröbner bases to illustrate all theorems and algorithms with concrete examples. We want to remark that the examples of skew PBW extensions selected are completely nontrivial and probably have not been considered before in the specialized literature in noncommutative Gröbner bases. The results presented in the monograph can be applied to any of types of noncommutative rings and algebras mentioned in the previous paragraph, in particular, our general theory of noncommutative Gröbner bases of skew PBW extensions can be used in different applications of such algebras as it is done in algebraic analysis (see [13], [22], [23], [24], [25], [35], [73], [99], [100], [101], [102], [103], [104], [105]). Actually, one of the main our motivations to study projective modules from a matrix constructive point of view resides in its future eventual application in algebraic analysis.

The thesis is divided into seven chapters. In the first chapter, we recall the definition and some basic properties of the skew PBW extensions. Some key and nontrivial examples of these rings are presented. These interesting examples will we used for illustrating the theorems and algorithms. Concrete matrix and Gröbner computations with this type of noncommutative rings probably have not been considered before in the literature.

Chapter 2 includes four sections. In Section 2.1, we recall some basic notions on linear algebra for left modules over arbitrary noncommutative rings. The $\mathcal{RC}$ condition (rank condition) and the $\mathcal{IBN}$ condition (Invariant Basis Number) are recalled. In Corollary 2.1.8 we prove that a skew PBW extension is $\mathcal{RC}$ if and only if its ring of coefficients is $\mathcal{RC}$. Many characterizations of stably free modules are given in Section 2.2. Section 2.3 is devoted to present a completely constructive proof of the general version of Stafford's theorem. This theorem was also considered in [105] but introducing an involution for the ring, our proof avoids this involution and is the main result of the chapter (Lemma 2.3.5 and Theorem 2.3.6). In Section 2.4, we present some theoretic results that give effective methods for computing the projective dimension of a module, and also for constructing minimal presentations.

In Chapter 3 are presented some matrix characterizations of Hermite rings (for which stably free modules are free), $\mathcal{PSF}$ rings (for which finitely generated projective modules are stably free) and $\mathcal{PF}$ rings (for which finitely generated projective modules are free). The main results are Theorem 3.1.2 and Corollary 3.2.4. Some subclasses of Hermite rings are characterized from a matrix point of view as well as its behavior under products, quotients and localizations (Theorem 3.4.1).

As it was observed above, it is easy to present examples of skew PBW extensions that are not Hermite rings. So, instead of this condition it is possible to study a weaker one, the d-Hermite condition, i.e., when any stably free module of rank $\geq d$ is free (see Definition 4.1.2). In Chapter 4, we investigate the d-Hermite condition for skew PBW extensions. We will give an upper bound for the stable range of a bijective skew PBW extension with finite left Krull dimension, and with this, in order to know a value d for which the extension is d-Hermite, i.e., for which every stably free module of rank $\geq d$ is free. Closely related to the stable range of a ring and its left Krull dimension is a Kronecker's theorem about the radical of finitely generated left ideals. In Section 4.3, we consider this theorem for bijective skew PBW extensions over left Noetherian domains, using the technique of Zariski lattice and boundary ideal that we found in [88], [89] and [123], but in the noncommutative framework. Thus, the main results of Chapter 4 are Proposition 4.2.2, Theorem 4.3.7 and Corollary 4.3.9.

Chapters 5, 6 and 7 conform the second part of the thesis. In Chapter 5, we complete the construction of the theory of Gröbner bases for general bijective skew PBW extensions. This construction was initiated in [40] for left ideals and in [58] for left modules, but under the assumption that the extension is quasi-commutative and bijective. In the present thesis we not only extend the theory to the general bijective case, eliminating the quasi-commutative restriction, but also we construct the theory for right ideals and modules. Thus, we can say that we construct a complete Gröbner theory for all quantum algebras mentioned at the beginning of this preface. The main results of Chapter 5 are Theorems 5.4.4, 5.5.13, 5.5.18 and 5.6.6.

In Chapter 6, we present some classical applications of Gröbner bases as the membership problem, the computations of syzygies, intersections, quotient modules, finite presentations of modules, kernel and images of homomorphisms and the construction of free resolutions. All of these constructions are illustrated for modules over nontrivial examples of skew PBW extensions. The main results are Theorem 6.2.12 and Corollary 6.2.7. This corollary establishes that if the rings of coefficients of a bijective skew PBW extension has a Gröbner theory, then the extension also satisfies this property.

The matrix-constructive theorems proved in the first chapters of the thesis will be interpreted by algorithms in the last chapter. Applying the Gröbner theory developed in Chapters 5 and 6, we obtain effective procedures for constructing left and right inverses of matrices over bijective skew PBW extensions, and with this, effective algorithms for testing stably freeness, freeness; effective procedures for computing the projective dimension of a module and for computing bases of free modules.

A Filter-graded transfer is presented in the appendix A, as a generalization of what was developed in this regard in [19] and [84].

CHAPTER 1

Skew PBW extensions

In this first chapter, we recall the definition of skew PWB extensions (also known as σ-PBW extensions), introduced by Lezama and Gallego in [40], as a generalization of the PBW (Poincaré-Birkhoff-Witt) extensions. Furthermore, we consider some of their structural properties and some important facts which are satisfied by them. We also establish some preliminary notation and necessary results for the subsequent sections. Finally, we present some examples of this class that includes well known classes of Ore algebras, operator algebras, and also many quantum rings and algebras.

1.1 Definitions and elementary examples

In this section, we present the definition of skew PBW extensions, some of their structural properties and some examples of these class of noncommutative rings. As we will see, the skew PBW extensions are a generalization of PBW extensions defined by Bell and Goodearl in 1988 in [10].

Definition 1.1.1. *Let R and A be rings, we say that A is a skew PBW extension of R (also called $\sigma - PBW$ extension), if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finite elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis*

$$Mon(A) := Mon\{x_1, \dots, x_n\} = \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case it is also said that A is a ring of a left polynomial type over R with respect to $\{x_1, \dots, x_n\}$ and $Mon(A)$ is the set of standard monomials of A. Moreover, $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r - c_{i,r} x_i \in R. \tag{1.1.1}$$

(iv) *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.1.2}$$

Under these conditions we will write $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Remark 1.1.2. (i) Since that $Mon(A)$ is a R-basis for A, the elements $c_{i,r}$ and $c_{i,j}$ in the above definition are unique.

(ii) If $r = 0$, then $c_{i,0} = 0$: in fact, $0 = x_i 0 = c_{i,0} x_i + s_i$, with $s_i \in R$, but since $Mon(A)$ is a R-basis, then $c_{i,0} = 0 = s_i$.

(iii) In (iv), $c_{i,i} = 1$: in fact, $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, hence $1 - c_{i,i} = 0 = s_i$.

(iv) Let $i < j$, by (1.1.2) there exist $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, but since $Mon(A)$ is a R-basis then $1 = c_{j,i} c_{i,j}$, i.e., for every $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.

(v) Each element $f \in A - \{0\}$ has a unique representation in the form $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R - \{0\}$ and $X_i \in Mon(A)$, $1 \leq i \leq t$.

The following proposition justifies the notation that we have introduced for the skew PBW extensions.

Proposition 1.1.3. *Let* A *be a skew* PBW *extension of* R. *Then, for every* $1 \leq i \leq n$, *there exist an injective ring endomorphism* $\sigma_i : R \to R$ *and a* σ_i*-derivation* $\delta_i : R \to R$ *such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [40], Proposition 3. □

A particular case of skew PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective and the constants c_{ij} are invertible. We have the following definition.

Definition 1.1.4. *Let* A *be a skew* PBW *extension.*

(a) *A is quasi-commutative if the conditions (iii) and (iv) in Definition 1.1.1 are replaced by*

(iii') *For every* $1 \leq i \leq n$ *and* $r \in R - \{0\}$ *there exists* $c_{i,r} \in R - \{0\}$ *such that*

$$x_i r = c_{i,r} x_i. \tag{1.1.3}$$

(iv') *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.1.4}$$

(b) *A is bijective if* σ_i *is bijective for every* $1 \leq i \leq n$ *and* $c_{i,j}$ *is invertible for any* $1 \leq i < j \leq n$.

Some familiar examples of skew PBW extensions are the following.

Example 1.1.5. (i) Any PBW extension is a bijective skew PBW extension since in this case $\sigma_i = i_R$ for each $1 \leq i \leq n$ and $c_{i,j} = 1$ for every $1 \leq i, j \leq n$.

(ii) Any *skew polynomial ring* $R[x; \sigma, \delta]$ *of injective type*, i.e., with σ injective, is a skew PBW extension; in this case we have $R[x; \sigma, \delta] = \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x; \sigma]$ is quasi-commutative.

(iii) Let $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be an *iterated skew polynomial ring of injective type*, i.e., if the following conditions hold:

For $1 \leq i \leq n$, σ_i is injective

For every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$

For $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$ and c has a left inverse.

For $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_i$.

Then, $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is a skew PBW extension. Under these conditions we have

$$R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] = \sigma(R)\langle x_1, \ldots, x_n\rangle.$$

In particular, any *Ore extension* $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ *of injective type*, i.e., for $1 \leq i \leq n$, σ_i is injective, is a skew PBW extension. In fact, in Ore extensions for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$, and for $i < j$, $\sigma_j(x_i) = x_i$ and $\delta_j(x_i) = 0$. An important subclass of Ore extension of injective type are the *Ore algebras of injective type*, i.e., when $R = \Bbbk[t_1, \ldots, t_m]$, $m \geq 0$, with $\Bbbk$ a field. Thus, we have

$$\Bbbk[t_1, \ldots, t_m][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] = \sigma(\Bbbk[t_1, \ldots, t_m])\langle x_1, \ldots, x_n\rangle.$$

Some concrete examples of Ore algebras of injective type are the following.

The algebra of shift operators: let $\Bbbk$ be a field and $h \in \Bbbk$, then the algebra of shift operators is defined by $S_h := \Bbbk[t][x_h; \sigma_h, \delta_h]$, where $\sigma_h(p(t)) := p(t - h)$, and $\delta_h := 0$ (observe that S_h can be considered also as a skew polynomial ring of injective type). Thus, S_h is a quasi-commutative bijective skew PBW extension.

The *mixed algebra* D_h: let again $\Bbbk$ be a field and $h \in \Bbbk$, then the mixed algebra D_h is defined by $D_h := \Bbbk[t][x; i_{\Bbbk[t]}, \frac{d}{dt}][x_h; \sigma_h, \delta_h]$, where $\sigma_h(x) := x$. Then, D_h is a quasi-commutative bijective skew PBW extension.

The *algebra for multidimensional discrete linear systems* is defined by $D := \Bbbk[t_1, \ldots, t_n][x_1; \sigma_1, 0] \cdots [x_n; \sigma_n, 0]$, where $\Bbbk$ is a field and

$$\sigma_i(p(t_1, \ldots, t_n)) := p(t_1, \ldots, t_{i-1}, t_i + 1, t_{i+1}, \ldots, t_n),\ \sigma_i(x_i) = x_i,\ 1 \leq i \leq n.$$

Thus, D is a quasi-commutative bijective skew PBW extension.

Observe that all of these examples are not PBW extensions.

(iv) *Additive analogue of the Weyl algebra*: let $\Bbbk$ be a field, the $\Bbbk$-algebra $A_n(q_1, \ldots, q_n)$ is generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and subject to the relations:

$$x_jx_i = x_ix_j, y_jy_i = y_iy_j,\ 1 \le i, j \le n,$$
$$y_ix_j = x_jy_i,\ i \ne j,$$
$$y_ix_i = q_ix_iy_i + 1,\ 1 \le i \le n,$$

where $q_i \in \Bbbk - \{0\}$. We observe that $A_n(q_1, \dots, q_n)$ is isomorphic to the iterated skew polynomial ring $\Bbbk[x_1, \dots, x_n][y_1; \sigma_1, \delta_1] \cdots [y_n; \sigma_n, \delta_n]$ over the commutative polynomial ring $\Bbbk[x_1, \dots, x_n]$:

$$\sigma_j(y_i) := y_i, \delta_j(y_i) := 0,\ 1 \le i < j \le n,$$
$$\sigma_i(x_j) := x_j, \delta_i(x_j) := 0,\ i \ne j,$$
$$\sigma_i(x_i) := q_ix_i, \delta_i(x_i) := 1,\ 1 \le i \le n.$$

Thus, $A_n(q_1, \dots, q_n)$ satisfies the conditions of (iii) and is bijective; we have

$$A_n(q_1, \dots, q_n) = \sigma(\Bbbk[x_1, \dots, x_n])\langle y_1, \dots, y_n\rangle.$$

(v) *Multiplicative analogue of the Weyl algebra*: let $\Bbbk$ be a field, the $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1, \dots, x_n$ and subject to the relations:

$$x_jx_i = \lambda_{ji}x_ix_j,\ 1 \le i < j \le n,$$

where $\lambda_{ji} \in \Bbbk - \{0\}$. We note that $\mathcal{O}_n(\lambda_{ji})$ is isomorphic to the iterated skew polynomial ring $\Bbbk[x_1][x_2; \sigma_2] \cdots [x_n; \sigma_n]$

$$\sigma_j(x_i) := \lambda_{ji}x_i,\ 1 \le i < j \le n.$$

Thus, $\mathcal{O}_n(\lambda_{ji})$ satisfies the conditions of (iii), and hence $\mathcal{O}_n(\lambda_{ji})$ is an iterated skew polynomial ring of injective type but is not Ore. Thus,

$$\mathcal{O}_n(\lambda_{ji}) = \sigma(\Bbbk[x_1])\langle x_2, \dots, x_n\rangle.$$

Moreover, note that $\mathcal{O}_n(\lambda_{ji})$ is quasi-commutative and bijective.

(vi) *q-Heisenberg algebra*: let $\Bbbk$ be a field, the $\Bbbk$-algebra $H_n(q)$ is generated by the elements $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n$ and subject to the relations:

$$x_jx_i = x_ix_j, z_jz_i = z_iz_j, y_jy_i = y_iy_j,\ 1 \le i, j \le n,$$
$$z_jy_i = y_iz_j, z_jx_i = x_iz_j, y_jx_i = x_iy_j,\ i \ne j,$$
$$z_iy_i = qy_iz_i, z_ix_i = q^{-1}x_iz_i + y_i, y_ix_i = qx_iy_i,\ 1 \le i \le n,$$

with $q \in \Bbbk - \{0\}$. Note that $H_n(q)$ is isomorphic to the iterated skew polynomial ring $\Bbbk[x_1, \dots, x_n][y_1; \sigma_1] \cdots [y_n; \sigma_n][z_1; \theta_1, \delta_1] \cdots [z_n; \theta_n, \delta_n]$ with coefficients in the commutative polynomial ring $\Bbbk[x_1, \dots, x_n]$:

$$\theta_j(z_i) := z_i,\ \delta_j(z_i) := 0, \sigma_j(y_i) := y_i,\ 1 \leq i < j \leq n,$$
$$\theta_j(y_i) := y_i,\ \delta_j(y_i) := 0, \theta_j(x_i) := x_i,\ \delta_j(x_i) := 0, \sigma_j(x_i) := x_i,\ i \neq j,$$
$$\theta_i(y_i) := qy_i,\ \delta_i(y_i) := 0, \theta_i(x_i) := q^{-1}x_i,\ \delta_i(x_i) := y_i, \sigma_i(x_i) := qx_i,\ 1 \leq i \leq n,$$

Since $\delta_i(x_i) = y_i \notin \Bbbk[x_1, \ldots, x_n]$, then $H_n(q)$ is not a skew PBW extension of $\Bbbk[x_1, \ldots, x_n]$, however, with respect to $\Bbbk$, $H_n(q)$ satisfies the conditions of (iii), and hence, $H_n(q)$ is a bijective skew PBW extension of $\Bbbk$:

$$H_n(q) = \sigma(\Bbbk)\langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n\rangle.$$

Remark 1.1.6. we want to emphasize that the skew PBW extensions are not a subclass of the collection of iterated skew polynomial rings: take for example $\mathcal{U}(\mathcal{G})$ or the diffusion algebra (see [83] and Section 1.3 below). On the other hand, the skew polynomial rings are not included in the class of skew PBW extensions: take $R[x; \sigma, \delta]$, with σ not injective.

1.2 Basic properties

In this section, some basic important properties of skew PBW extensions are presented. We start with some notation that we will use frequently in this thesis.

Definition 1.2.1. *Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 1.1.3.*

(i) *For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.*

(ii) *For $X = x^\alpha \in Mon(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.*

(iii) *Let $0 \neq f \in A$, $t(f)$ is the finite set of terms that shape f, i.e., if $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1X_1, \ldots, c_tX_t\}$.*

(iv) *Let f be as in* (iii), *then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.*

The skew PBW extensions can be characterized in a similar way as it was done in [18] for PBW rings (see Proposition 2.4 there in).

Theorem 1.2.2. *Let A be a left polynomial ring over R w.r.t. $\{x_1, \ldots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in Mon(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{1.2.1}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(b) *For every $x^\alpha, x^\beta \in Mon(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.2.2}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Proof. See [40], Theorem 7. □

Remark 1.2.3. (i) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.

(ii) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$, then we have the following identities:

$$\sigma^\theta(c_{\gamma,\beta}) c_{\theta,\gamma+\beta} = c_{\theta,\gamma} c_{\theta+\gamma,\beta},$$
$$\sigma^\theta(\sigma^\gamma(c)) c_{\theta,\gamma} = c_{\theta,\gamma} \sigma^{\theta+\gamma}(c).$$

In fact, since $x^\theta(x^\gamma x^\beta) = (x^\theta x^\gamma) x^\beta$, then

$$x^\theta(c_{\gamma,\beta} x^{\gamma+\beta} + p_{\gamma,\beta}) = (c_{\theta,\gamma} x^{\theta+\gamma} + p_{\theta,\gamma}) x^\beta,$$
$$\sigma^\theta(c_{\gamma,\beta}) c_{\theta,\gamma+\beta} x^{\theta+\gamma+\beta} + p = c_{\theta,\gamma} c_{\theta+\gamma,\beta} x^{\theta+\gamma+\beta} + q,$$

with $p = 0$ or $\deg(p) < |\theta + \gamma + \beta|$, and, $q = 0$ or $\deg(q) < |\theta + \gamma + \beta|$. From this we get the first identity. For the second, $x^\theta(x^\gamma c) = (x^\theta x^\gamma) c$, and hence

$$x^\theta(\sigma^\gamma(c) x^\gamma + p_{\gamma,c}) = (c_{\theta,\gamma} x^{\theta+\gamma} + p_{\theta,\gamma}) c,$$
$$\sigma^\theta(\sigma^\gamma(c)) c_{\theta,\gamma} x^{\theta+\gamma} + p = c_{\theta,\gamma} \sigma^{\theta+\gamma}(c) x^{\theta+\gamma} + q,$$

with $p = 0$ or $\deg(p) < |\theta + \gamma|$, and, $q = 0$ or $\deg(q) < |\theta + \gamma|$. This proves the second idenity.

(iii) If A is quasi-commutative, from the proof of Theorem 1.2.2, we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$. On the other hand, note that the evaluation function at 0, i.e., $A \to R$, $f \in A \mapsto f(0) \in R$, is a ring surjective homomorphism with kernel $\langle x_1, \ldots, x_n \rangle$ the two-sided ideal generated by $x_1, \ldots, x_n$. Thus, $A/\langle x_1, \ldots, x_n \rangle \cong R$.

(iv) If A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

(v) In $Mon(A)$ we define

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists\, i \text{ with } \quad \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases}$$

It is clear that this is a total order on $Mon(A)$, called *deglex* order. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Each element $f \in A - \{0\}$ can be represented in a unique way as

$f = c_1x^{\alpha_1} + \cdots + c_tx^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leader monomial* of f and we write $lm(f) := x^{\alpha_1}$; c_1 is the *leader coefficient* of f, $lc(f) := c_1$, and $c_1x^{\alpha_1}$ is the *leader term* of f denoted by $lt(f) := c_1x^{\alpha_1}$. If $f = 0$, we define $lm(0) := 0, lc(0) := 0, lt(0) := 0$, and we set $X \succ 0$ for any $X \in Mon(A)$. We observe that

$$x^{\alpha} \succ x^{\beta} \Rightarrow lm(x^{\gamma}x^{\alpha}x^{\lambda}) \succ lm(x^{\gamma}x^{\beta}x^{\lambda}), \text{ for every } x^{\gamma}, x^{\lambda} \in Mon(A).$$

The following properties are natural and useful results that will be used later.

Proposition 1.2.4. *Let A be a bijective skew PBW extension of a ring R. Then,*

(i) *If R is a domain, the A is a domain.*

(ii) *A_R is free with basis $Mon(A)$.*

Proof. See [83] Proposition 1.7 and Proposition 4.1. □

The next theorem shows how can be associated one quasi-commutative skew PBW extension to an arbitrary skew PBW extension.

Proposition 1.2.5. *Let A be a skew PBW extension of R. Then, there exists a quasi-commutative skew PBW extension A^{σ} of R in n variables $z_1, \ldots, z_n$ defined by*

$$z_ir = c_{i,r}z_i, \; z_jz_i = c_{i,j}z_iz_j, \; 1 \leq i, j \leq n,$$

where $c_{i,r}, c_{i,j}$ are the same constants that define A. If A is bijective then A^{σ} is also bijective.

Proof. See [83], Proposition 2.1. □

Before continuing, we need to recall the definition of a filtered ring. As we shall see, the skew PBW extensions are filtered rings; this last fact turns out to be essential in several important results that we present later.

Definition 1.2.6. *A ring S is called a filtered ring with filtration $F(S)$ if there is a sequence $F(S) = \{F_p(S)\}_{p\in\mathbb{Z}}$ of subgroups of the additive group of S such that:*

(i) $\bigcup_{p\in\mathbb{Z}} F_p(S) = S$.

(ii) $1 \in F_0(S)$.

(iii) *For $p < q$, $F_p(S) \subseteq F_q(S)$.*

(iv) *$F_p(S)F_q(S) \subseteq F_{p+q}(S)$ for all $p, q \in \mathbb{Z}$.*

We say that the filtration $F(S)$ is separated if $\bigcap_{p\in\mathbb{Z}} F_p(S) = 0$. Finally, if $F_{-1}(S) = 0$, then S is called a positively filtered ring, and $F(S)$ is called a positive filtration on S

Given a filtered ring S with filtration $F(S)$, the *associated graded ring* of S with respect to $F(S)$, is defined to be the graded ring $G(S) = \oplus_{p\in\mathbb{Z}} G(S)_p$ with $G(S)_p := F_p(S)/F_{p-1}(S)$ and the multiplication given by

$$F_p(S)/F_{p-1}(S) \times F_q(S)/F_{q-1}(S) \to F_{p+q}(S)/F_{p+q-1}(S)$$
$$(a + F_{p-1}(S), b + F_{q-1}(S)) \mapsto ab + F_{p+q-1}(S).$$

The following theorem shows that any skew PBW extension is a filtered ring, and presents a characterization of its associated graded ring.

Theorem 1.2.7. *Let A be an arbitrary skew PBW extension of the ring R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R, & \text{if } m = 0, \\ \{f \in A \mid \deg(f) \leq m\}, & \text{if } m \geq 1 \end{cases} \tag{1.2.3}$$

and the corresponding graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $Gr(A)$ is a quasi-commutative bijective skew PBW extension of R.

Proof. See [83], Theorem 2.2. □

The following theorem is an important result that characterizes the quasi-commutative skew PBW extensions.

Theorem 1.2.8. *Let A be a quasi-commutative skew PBW extension of a ring R. Then,*

(i) *A is isomorphic to an iterated skew polynomial ring of endomorphism type.*

(ii) *If A is bijective, then each endomorphism is bijective.*

Proof. See [83], Theorem 2.3. □

These last results allow to establish the Hilbert Basis Theorem for skew PBW extensions.

Theorem 1.2.9 (Hilbert Basis Theorem)**.** *Let A be a bijective skew PBW extension of R. If R is a left (right) Noetherian ring then A is also a left (right) Noetherian ring.*

Proof. See [83], Corollary 2.4. □

The task of studying properties of modules defined on skew PBW extensions should consider the computation of measures such as global dimension, Krull dimension or uniform dimension. More specifically, knowing such dimensions will allow us to make assertions about freeness of stably free modules, or more generally, of finitely generated projective modules. A initial approach in this sense provide us the following two theorems: the first theorem establishes sufficient conditions for a skew PBW extension to be a regular. The second theorem - that can be considered as Serre's theorem for these rings

- asserts that if the ring of coefficients is a PSF ring, then the extension also satisfies such property.

Recall that a noncommutative ring is said to be *left regular* if every left finitely generated module has a finite projective dimension or, equivalently, if every left cyclic module over this ring has a finite projective dimension (right regularity is defined analogously). Moreover, a ring is called *left PSF* if every left finitely generated projective module is stably free. This class of rings will be considered again in the Section 3.1, Chapter 3.

Theorem 1.2.10. *Let A be a bijective skew PBW extension of a ring R. If R is a left (right) regular and left (right) Noetherian ring, then A is left (right) regular.*

Proof. See [83], Corollary 2.6. □

Theorem 1.2.11 (Serre's theorem)**.** *Let A be a bijective skew PBW extension of a ring R such that R is left (right) Noetherian, left (right) regular and PSF. Then A is PSF.*

Proof. See [83], Corollary 2.8. □

1.3 More examples

Many other important and interesting examples of bijective skew PBW extensions, and some other classes of noncommutative rings of polynomial type closely related to such extensions, were presented and discussed in [108] and [83]. In this section, we recall some of these key examples that will be used later to illustrate the algorithms that will be presented in the thesis.

Example 1.3.1. The *Quantum Weyl Algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the variables $x_1, x_2, \partial_1, \partial_2$, with the relations (depending upon parameters $a, b \in \Bbbk$):

$$\begin{aligned}
x_1x_2 &= x_2x_1 + ax_1^2\\
\partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2\\
\partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2\\
\partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2\\
\partial_2x_1 &= x_1\partial_2\\
\partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2.
\end{aligned}$$

When $a = b = 0$, we have that $A_2(J_{0,0}) \cong A_2(\Bbbk)$ for any field $\Bbbk$ (see [38] for more properties). In [108] was shown that $A_2(J_{a,b}) \cong \sigma(\Bbbk[x_1, \partial_2])\langle x_2, \partial_1\rangle$.

Example 1.3.2. *The coordinate ring of the manifold of quantum* 2×2 *matrices* $M_q(2)$. This algebra is also known as *Manin algebra of* 2×2 *quantum matrices* (cf. [84], [93]). By definition, $M_q(2)$ is the $\Bbbk$-algebra generated by the variables x, y, u, v satisfying the relations

$$xu = qux, \qquad yu = q^{-1}uy, \qquad vu = uv, \tag{1.3.1}$$

and

$$xv = qvx, \qquad vy = qyv, \qquad yx - xy = -(q - q^{-1})uv, \tag{1.3.2}$$

where $q \in \Bbbk - \{0\}$. Thus, $M_q(2) \cong \sigma(\Bbbk[u])\langle x, y, v\rangle$. Due to the last relation in (1.3.2), we remark that it is not possible to consider $M_q(2)$ as a skew PBW extension of $\Bbbk$. See [19] for more details.

Example 1.3.3. According to [55], a *diffusion algebra* $\mathcal{D}$ over a field $\Bbbk$ is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ with relations

$$x_i x_j = x_j x_i, \;\; x_i D_j = D_j x_i, \;\; 1 \leq i, j \leq n.$$

$$c_{ij} D_i D_j - c_{ji} D_j D_i = x_j D_i - x_i D_j, \;\; i < j, c_{ij}, c_{ji} \in \Bbbk^*.$$

Thus, $\mathcal{D} \cong \sigma(\Bbbk[x_1, \dots, x_n])\langle D_1, \dots, D_n\rangle$ is a bijective non quasi-commutative skew PBW extension of $\Bbbk[x_1, \dots, x_n]$. Observe that $\mathcal{D}$ is not a PBW extension neither an iterated skew polynomial ring of bijective type (see Example 1.1.5).

Example 1.3.4. Viktor Levandovskyy defined in [73] the G-algebras and he constructed the theory of Gröbner bases for these rings (see Chapter 5 of the current monograph for the Gröbner theory of bijective skew PBW extensions). Let $\Bbbk$ be a field, a $\Bbbk$-algebra A is called a *G-algebra* if $\Bbbk \subset Z(A)$ (center of A) and A is generated by a finite set $\{x_1, \dots, x_n\}$ of elements that satisfy the following conditions: (a) the collection of standard monomials of A is a $\Bbbk$-basis of A. (b) $x_j x_i = c_{ij} x_i x_j + d_{ij}$, for $1 \leq i < j \leq n$, with $c_{ij} \in \Bbbk - \{0\}$ and $d_{ij} \in A$. (c) There exists a total order $<_A$ on $\mathrm{Mon}(A)$ such that for $i < j$, $lm(d_{ij}) <_A x_i x_j$. According to this definition, G-algebras appear like more general than skew PBW extensions since d_{ij} is not necessarily linear; however, in G-algebras the coefficients of polynomials are in a field and they commute with the variables $x_1, \dots, x_n$. Note that the class of G-algebras does not include the class of skew PBW extensions over fields. For example, consider the $\Bbbk$-algebra $\mathcal{A}$ generated by x, y, z subject to the relations

$$yx - q_2 xy = x, \qquad zx - q_1 xz = z, \qquad zy = yz, \quad q_1, q_2 \in \Bbbk.$$

Thus, $\mathcal{A}$ is not a G-algebra in the sense of [73]. Note that if $q_1, q_2 \neq 0$, then $\mathcal{A} \cong \sigma(\Bbbk)\langle x, y, z\rangle$ is a bijective non quasi-commutative skew PBW extension of $\Bbbk$.

Example 1.3.5. *Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2, \Bbbk)$.* E. Witten introduced and studied a 7-parameter deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \Bbbk))$ over the field $\Bbbk$, depending on a 7-tuple of parameters $\underline{\xi} = (\xi_1, \dots, \xi_7)$ of $\Bbbk$ and subject to relations

$$xz - \xi_1 zx = \xi_2 x, \quad zy - \xi_3 yz = \xi_4 y, \quad yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z.$$

The resulting algebra is denoted by $W(\underline{\xi})$ and it is assumed that $\xi_1\xi_3\xi_5 \neq 0$ (see [73]). Note that if $\xi_2\xi_4\xi_6 \neq 0$, then $W(\underline{\xi}) \cong \sigma(\boldsymbol{\sigma}(\Bbbk[\boldsymbol{x}])\langle \boldsymbol{z}\rangle)\langle y\rangle$ is a bijective non quasi-commutative skew PBW extension of $\sigma(\Bbbk[x])\langle z\rangle$, and consequently, $\sigma(\Bbbk[x])\langle z\rangle$ is a bijective non quasi-commutative skew PBW extension of $\Bbbk[x]$. In [73] is proved that the only way that $W(\underline{\xi})$ is a G-algebra is when $\xi_1 = \xi_3$ and $\xi_2 = \xi_4$. Thus, in general, $W(\underline{\xi})$ is a skew PBW extension but is not a G-algebra.

Example 1.3.6. In [18] (see also [19]) Bueso, Gómez-Torrecillas and Verschoren defined a type of rings and algebras called *left PBW rings*. Many of rings and algebras considered in [83] (see also [108]) can be interpreted also as left PBW rings. We present an example of skew PBW extension that is not a left PBW ring: let $\Bbbk$ be a field; for any $0 \neq q \in \Bbbk$, let $\mathcal{R}$ be an algebra generated by the variables a, b, c, d subject to the relations

$$\begin{aligned} ba &= qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd \\ bc &= \mu cb, \quad ad - da = (q^{-1} - q)bc. \end{aligned}$$

for some $\mu \in \Bbbk$. Then $\mathcal{R}$ is not a left PBW ring unless $\mu = 1$ (see [19]). Thus, for $\mu \neq 1$, $\mathcal{R} \cong \sigma(\Bbbk[b])\langle a, c, d\rangle$ is a bijective non quasi-commutative skew PBW extension of $\Bbbk[b]$ that is not a left PBW ring.

CHAPTER 2

Stably free modules

Serre's Theorem for bijective skew PBW extensions (see Theorem 1.2.11 and Corollary 2.8 in [83]) states that if M is a finitely generated projective module over a bijective skew PBW extension A of a left Noetherian, left regular PSF ring R, then M is stably free. In the same way, Remark 3.3. in [83] establishes that if M is a f.g. projective module over the ring $Q^{r,n}_{\mathbf{q},\sigma}(R)$ of skew quantum polynomials over R, where R satisfies the same above conditions, then M is stably free. The following natural question arises: when are stably free modules over A (or over $Q^{r,n}_{\mathbf{q},\sigma}(R)$) free? The first thing that we have to observe is that not any stably free module over a bijective skew PBW extension is free. The next trivial example shows this ([62], p. 36): If T is a division ring, then $S := T[x, y]$ has a module M such that $M \oplus S \cong S^2$, but M is not free. In a more general framework, and as preparatory material for posterior studies in next chapters, we are interested in studying when stably free modules over enough arbitrary noncommutative rings are free. A well known result in this direction is the Stafford's Theorem that we will prove in this chapter. Many characterizations of stably free modules will be presented also. There are different techniques to research stably free modules, we will combine homological and matrix constructive methods.

2.1 $\mathcal{RC}$ and $\mathcal{IBN}$ rings

In this section, we recall some notations and elementary properties well known of linear algebra for left modules. All rings are noncommutative and modules will be considered on the left; the letter S will represent an arbitrary noncommutative ring, thus S^r is the left S-module of columns of size $r \times 1$. If $S^s \xrightarrow{f} S^r$ is an S-homomorphism then there is a matrix associated to f in the canonical bases of S^r and S^s, denoted $F := m(f)$, and disposed by columns, i.e., $F \in M_{r\times s}(S)$. In fact, if f is given by

$$S^s \xrightarrow{f} S^r \ , \ \boldsymbol{e}_j \mapsto \boldsymbol{f}_j$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_s\}$ is the canonical basis of S^s, f can be represented by a matrix, i.e., if $\boldsymbol{f}_j := \begin{bmatrix} f_{1j} & \dots & f_{rj} \end{bmatrix}^T$, then the matrix of f in the canonical bases of S^s and S^r is

$$F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1s} \\ \vdots & & \vdots \\ f_{r1} & \cdots & f_{rs} \end{bmatrix} \in M_{r\times s}(S).$$

Note that $Im(f)$ is the column module of F, i.e., the left S-module generated by the columns of F, denoted by $\langle F\rangle$:

$$Im(f) = \langle f(\boldsymbol{e}_1), \ldots, f(\boldsymbol{e}_s)\rangle = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\rangle = \langle F\rangle.$$

Moreover, observe that if $\boldsymbol{a} := (a_1, \ldots, a_s)^T \in S^s$, then

$$f(\boldsymbol{a}) = (\boldsymbol{a}^T F^T)^T. \tag{2.1.1}$$

In fact,

$$\begin{aligned} f(\boldsymbol{a}) &= a_1 f(\boldsymbol{e}_1) + \cdots + a_s f(\boldsymbol{e}_s) = a_1 \boldsymbol{f}_1 + \cdots + a_s \boldsymbol{f}_s \\ &= a_1 \begin{bmatrix} f_{11} \\ \vdots \\ f_{r1} \end{bmatrix} + \cdots + a_s \begin{bmatrix} f_{1s} \\ \vdots \\ f_{rs} \end{bmatrix} \\ &= \begin{bmatrix} a_1 f_{11} + \cdots + a_s f_{1s} \\ \vdots \\ a_1 f_{r1} + \cdots + a_s f_{rs} \end{bmatrix} \\ &= (\begin{bmatrix} a_1 & \cdots & a_s \end{bmatrix} \begin{bmatrix} f_{11} & \cdots & f_{r1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{rs} \end{bmatrix})^T \\ &= (\boldsymbol{a}^T F^T)^T. \end{aligned}$$

Note that function $m : Hom_S(S^s, S^r) \to M_{r\times s}(S)$ is bijective; moreover, if $S^r \xrightarrow{g} S^p$ is a homomorphism, then the matrix of gf in the canonical bases is $m(gf) = (F^T G^T)^T$. Thus, $f : S^r \to S^r$ is an isomorphism if and only if $F^T \in GL_r(S)$. Finally, let $C \in M_r(S)$; the columns of C conform a basis of S^r if and only if $C^T \in GL_r(S)$.

We also recall that

$$Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) := \{\boldsymbol{a} := (a_1, \ldots, a_s)^T \in S^s | a_1 \boldsymbol{f}_1 + \cdots + a_s \boldsymbol{f}_s = \boldsymbol{0}\}.$$

Note that

$$Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) = \ker(f), \tag{2.1.2}$$

but $Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) \neq \ker(F)$ since we have

$$\boldsymbol{a} \in Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) \Leftrightarrow \boldsymbol{a}^T F^T = \boldsymbol{0}. \tag{2.1.3}$$

A matrix characterization of f.g. projective modules can be formulated in the following way.

Proposition 2.1.1. *Let S be an arbitrary ring and M a S-module. Then, M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F^T is idempotent and $M = \langle F\rangle$.*

Proof. $\Rightarrow$): If $M = 0$, then $F = 0$; let $M \neq 0$, there exists $s \geq 1$ and a M' such that $S^s = M \oplus M'$; let $f : S^s \to S^s$ be the projection on M and F the matrix of f in the canonical basis of S^s. Then, $f^2 = f$ and $(F^T F^T)^T = F$, so $F^T F^T = F^T$; note that $M = Im(f) = \langle F\rangle$.

$\Leftarrow$): Let $f : S^s \to S^s$ be the homomorphism defined by F (see (2.1.1)); from $F^T F^T = F^T$ we get that $f^2 = f$, moreover, since $M = \langle F\rangle$, then $Im(f) = M$ and hence M is direct summand of S^s, i.e., M is f.g. projective (observe that the complement M' of M is $\ker(f)$ and f is the projection on M). $\square$

Remark 2.1.2. (i) When S is commutative, or when we consider right modules instead of left modules, (2.1.1) asserts that $f(\boldsymbol{a}) = F\boldsymbol{a}$. Moreover, in such cases $Syz(\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}) = \ker(F)$ and the matrix of a compose homomorphism gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^r \to S^r$ is an isomorphism if and only if $F \in GL_r(S)$; besides, $C \in GL_r(S)$ if and only if its columns conform a basis of S^r. In addition, Proposition 2.1.1 states that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F\rangle$.

(ii) When the matrices of homomorphisms of left modules are disposed by rows instead of by columns, i.e., if $S^{1\times s}$ is the left free module of rows vectors of length s and the matrix of the homomorphism $S^{1\times s} \xrightarrow{f} S^{1\times r}$ is defined by

$$F' = \begin{bmatrix} f'_{11} & \cdots & f'_{1r} \\ \vdots & & \vdots \\ f'_{s1} & \cdots & f'_{sr} \end{bmatrix} := \begin{bmatrix} f_{11} & \cdots & f_{r1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{rs} \end{bmatrix} \in M_{s\times r}(S),$$

then

$$f(a_1, \dots, a_s) = (a_1, \dots, a_s)F', \tag{2.1.4}$$

i.e., $f(\boldsymbol{a}^T) = \boldsymbol{a}^T F^T$. Thus, the values given by (2.1.4) and (2.1.1) agree since $F' = F^T$. Moreover, the composed homomorphism gf means that g acts first and then acts f, and hence, the matrix of gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^{1\times r} \to S^{1\times r}$ is an isomorphism if and only if $m(f) \in GL_r(S)$; furthermore, $C \in GL_r(S)$ if and only if its rows conform a basis of $S^{1\times r}$. This left-row notation is used in [26]. Observe that with this notation, the proof of Proposition 2.1.1 claims that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F\rangle$, but in this case $\langle F\rangle$ represents the module generated by the rows of F. Note that Proposition 2.1.1 could have been formulated this way: In fact, the set of idempotents matrices of $M_s(S)$ coincides with the set $\{F^T | F \in M_s(S), F^T \text{ idempotent}\}$.

Definition 2.1.3 ([62]). *Let S be a ring.*

(i) *S satisfies the rank condition ($\mathcal{RC}$) if for any integers $r, s \geq 1$, given an epimorphism $S^r \xrightarrow{f} S^s$, then $r \geq s$.*

(ii) *S is an $\mathcal{IBN}$ ring (Invariant Basis Number) if for any integers $r, s \geq 1$, $S^r \cong S^s$ if and only if $r = s$.*

Proposition 2.1.4. *Let S be a ring.*

(i) *S is $\mathcal{RC}$ if and only if given any matrix $F \in M_{s\times r}(S)$ the following condition holds:*

if F has a right inverse then $r \geq s$.

(ii) *S is $\mathcal{RC}$ if and only if given any matrix $F \in M_{s\times r}(S)$ the following condition holds:*

if F has a left inverse then $s \geq r$.

Proof. (i) $\Rightarrow$): Let G be a right inverse of F, that is $FG = I_s$; let $f : S^r \to S^s$ and $g : S^s \to S^r$ such that $m(f) = F$ and $m(g) = G$. Thus $((F^T)^T(G^T)^T)^T = I_s$; let $f^T : S^s \to S^r$ and $g^T : S^r \to S^s$ such that $m(f^T) = F^T$ and $m(g^T) = G^T$, then $m(g^T f^T) = m(i_{S^s})$ and hence $g^T f^T = i_{S^s}$, i.e., g^T is surjective. Since S is $\mathcal{RC}$, then $r \geq s$.

$\Leftarrow$): Let $S^r \xrightarrow{f} S^s$ be an epimorphism, there exists $S^s \xrightarrow{g} S^r$ such that $fg = i_{S^s}$; let $F := m(f) \in M_{s\times r}(S)$ and $G := m(g) \in M_{r\times s}(S)$, then $m(fg) = (G^T F^T)^T = I_s$, so $G^T F^T = I_s$, i.e., G^T has right inverse, and by hypothesis $r \geq s$. This means that S is $\mathcal{RC}$.

(ii) $\Rightarrow$): Let $G \in M_{r\times s}(S)$ a left inverse of F, then G has right inverse, and by (i), $s \geq r$.

$\Leftarrow$): Let $S^r \xrightarrow{f} S^s$ be an epimorphism; as in (i), $G^T F^T = I_s$, so $F^T \in M_{r\times s}(S)$ has a left inverse and by the hypothesis $r \geq s$. Thus, S is $\mathcal{RC}$. □

The relation between the $\mathcal{RC}$ and $\mathcal{IBN}$ properties is established below.

Proposition 2.1.5. *$\mathcal{RC} \Rightarrow \mathcal{IBN}$.*

Proof. Let $S^r \xrightarrow{f} S^s$ be an isomorphism, then f is an epimorphism, and hence $r \geq s$; considering f^{-1} we get that $s \geq r$. □

Example 2.1.6. Most of rings considered in the literature are $\mathcal{RC}$, and hence, $\mathcal{IBN}$.

(i) Any field $\Bbbk$ is $\mathcal{RC}$: let $\Bbbk^r \xrightarrow{f} \Bbbk^s$ be an epimorphism, then $\dim(\Bbbk^r) = r = \dim(\ker(f)) + s$, so $r \geq s$.

(ii) Let S and T be rings and let $S \xrightarrow{f} T$ be a ring homomorphism, if T is a $\mathcal{RC}$ ring then S is also a $\mathcal{RC}$ ring. In fact, T is a right S-module, $t \cdot s := tf(s)$; suppose that $S^r \xrightarrow{f} S^s$ is an epimorphism, then $T \otimes_S S^r \xrightarrow{i_T \otimes f} T \otimes_S S^s$ is also an epimorphism of left T-modules, i.e., we have an epimorphism $T^r \to T^s$, so $r \geq s$ (a similar result and proof is valid for the $\mathcal{IBN}$ property).

(iii) We can apply the property proved in (ii) in many situations. For example, any commutative ring S is $\mathcal{RC}$: let J be a maximal ideal of S, then the canonical homomorphism $S \to S/J$ shows that S is $\mathcal{RC}$ since S/J is a field.

(iv) Any ring S with finite uniform dimension (Goldie dimension, see [95] and [51]) is $\mathcal{RC}$: in fact, suppose that $S^r \xrightarrow{f} S^s$ is an epimorphism, then $S^r \cong S^s \oplus M$ and hence $r\,\mathrm{udim}(S) = s\,\mathrm{udim}(S) + \mathrm{udim}(M)$, so $r \geq s$.

(v) Since any left Noetherian ring S has finite uniform dimension, then S is $\mathcal{RC}$. In particular, any left Artinian ring is $\mathcal{RC}$.

Since the objects studied in the present monograph are the skew PBW extensions, it is natural to investigate the $\mathcal{IBN}$ and $\mathcal{RC}$ properties for these rings.

Proposition 2.1.7. *Let B be a filtered ring. If $Gr(B)$ is $\mathcal{RC}$ ($\mathcal{IBN}$), then B is $\mathcal{RC}$ ($\mathcal{IBN}$).*

Proof. Let $\{B_p\}_{p\geq 0}$ be the filtration of B and $f : B^r \to B^s$ an epimorphism. For $M := B^r$ we consider the standard positive filtration given by

$$F_0(M) := B_0 \cdot e_1 + \cdots + B_0 \cdot e_r,\, F_p(M) := B_pF_0(M),\, p \geq 1,$$

where $\{e_i\}_{i=1}^r$ is the canonical basis of B^r. Let $e_i' := f(e_i)$, then B^s is generated by $\{e_i'\}_{i=1}^r$ and $N := B^s$ has an standard positive filtration given by

$$F_0(N) := B_0 \cdot e_1' + \cdots + B_0 \cdot e_r',\, F_p(N) := B_pF_0(N),\, p \geq 1.$$

Note that f is filtered and strict [1]: In fact, $f(F_p(M)) = B_pf(F_0(M)) = B_p(B_0 \cdot f(e_1) + \cdots + B_0 \cdot f(e_r)) = B_p(B_0 \cdot e_1' + \cdots + B_0 \cdot e_r') = B_pF_0(N) = F_p(N)$. This implies that $Gr(M) \xrightarrow{Gr(f)} Gr(N)$ is surjective (see [97], Theorem 4.4). If we prove that $Gr(M)$ and $Gr(N)$ are free over $Gr(B)$ with bases of r and s elements, respectively, then from the hypothesis we conclude that $r \geq s$ and hence B is $\mathcal{RC}$.

Since every $e_i \in F_0(M)$ and $F_p(M) = \sum_{i=1}^r \oplus B_p \cdot e_i$, M is filtered-free with filtered-basis $\{e_i\}_{i=1}^r$, so $Gr(M)$ is graded-free with graded-basis $\{\overline{e_i}\}_{i=1}^r$, $\overline{e_i} := e_i + F_{-1}(M) = e_i$ (recall that by definition of positive filtration, $F_{-1}(M) := 0$). For $Gr(N)$ note that N is also filtered-free with respect the filtration $\{F_p(N)\}_{p\geq 0}$ given above: Indeed, we will show next that the canonical basis $\{f_j\}_{j=1}^s$ of N is a filtered basis. If $f_j = x_{j1} \cdot e_1' + \cdots + x_{jr} \cdot e_r'$, with $x_{ji} \in B_{p_{ij}}$, let $p := \max\{p_{ij}\}$, $1 \leq i \leq r$, $1 \leq j \leq s$, then $f_j \in F_p(N)$, moreover, for every q, $B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s \subseteq B_{q-p}F_p(N) \subseteq F_q(N)$ (recall that for $k < 0$, $B_k = 0$); in turn, let $x \in F_q(N) \setminus F_{q-1}(N)$, then $x = b_1 \cdot f_1 + \cdots + b_s \cdot f_s$ and in $Gr(N)$ we have $\overline{x} \in Gr(N)_q$, $\overline{x} = \overline{b_1} \cdot \overline{f_1} + \cdots + \overline{b_s} \cdot \overline{f_s} \neq \overline{0}$, if $b_j \in B_{u_j}$, let $u := \max\{u_j\}$, so $\overline{b_j} \cdot \overline{f_j} \in Gr(N)_{u+p}$, so $q = u + p$, i.e., $u = q - p$ and hence $x \in B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s$, Thus, we have proved that $B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s = F_q(N)$, for every q, and consequently, $\{f_j\}_{j=1}^s$ is a filtered basis of N. From this we conclude that $Gr(N)$ is graded-free with graded-basis $\{\overline{f_j}\}_{j=1}^s$, $\overline{f_j} := f_j + F_{p-1}(N)$.

We can repeat the previous proof for the $\mathcal{IBN}$ property but assuming that f is an isomorphism. □

Corollary 2.1.8. *Let A be a skew PBW extension of a ring R. Then, A is $\mathcal{RC}$ ($\mathcal{IBN}$) if and only if R is $\mathcal{RC}$ ($\mathcal{IBN}$).*

[1] Remember that a homomorphism $f : M \to N$ between filtered modules is a *filtered homomorphism* if $f(F_p(M)) \subseteq F_p(N)$ for all p. Moreover, f is strict if $f(F_p(M)) = F_p(N) \cap Im(f)$.

Proof. We consider only the proof for $\mathcal{RC}$, the case $\mathcal{IBN}$ is completely analogous.

$\Rightarrow$): Since $R \hookrightarrow A$, Example 2.1.6 shows that if A is $\mathcal{RC}$, then R is $\mathcal{RC}$.

$\Leftarrow$): We consider first the skew polynomial ring $R[x;\sigma]$ of endomorphism type, then $R[x;\sigma] \to R$ given by $p(x) \to p(0)$ is a ring homomorphism, so $R[x;\sigma]$ is $\mathcal{RC}$ since R is $\mathcal{RC}$. By Theorem 1.2.8, $Gr(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$, so $Gr(A)$ is $\mathcal{RC}$. It only remains to apply Proposition 2.1.7. □

Remark 2.1.9. (i) The condition $\mathcal{IBN}$ for rings is independent of the side we are considering the modules. In fact, if we define left $\mathcal{IBN}$ rings and right $\mathcal{IBN}$ rings, depending on left or right free S-modules, then S is left $\mathcal{IBN}$ if and only if S is right $\mathcal{IBN}$ (see [79]). The same is true for the $\mathcal{RC}$ property.

(ii) Another property, closely related to $\mathcal{IBN}$ and $\mathcal{RC}$, is the *weakly finite* condition, denoted simply by $\mathcal{WF}$: a ring S is $\mathcal{WF}$ if any epimorphism $S^r \to S^r$ of free modules is an isomorphism (cf. [63], [26] or [20]). The $\mathcal{WF}$ rings satisfy similar properties that the $\mathcal{IBN}$ and $\mathcal{RC}$ rings. So, for example, if S is a filtered ring and $Gr(S)$ is $\mathcal{WF}$, then S is $\mathcal{WF}$ too. Thus, if A is a skew PBW extension of R, then R is $\mathcal{WF}$ if and only if A is $\mathcal{WF}$. Moreover, it is not difficult to show that every ring $\mathcal{WF}$ is $\mathcal{RC}$. Therefore, we have that

$$\mathcal{WF} \Longrightarrow \mathcal{RC} \Longrightarrow \mathcal{IBN},$$

and these implications are strict (see [28]).

(iii) **From now on we will assume that all rings considered in the present thesis are** $\mathcal{RC}$.

2.2 Characterizations of stably free modules

Definition 2.2.1. *Let M be a S-module and $t \geq 0$ an integer. M is stably free of rank $t \geq 0$ if there exist an integer $s \geq 0$ such that $S^{s+t} \cong S^s \oplus M$.*

The rank of M is denoted by rank(M). Note that any stably free module M is finitely generated and projective. Moreover, as we will show in the next proposition, rank(M) is well defined, i.e., rank(M) is unique for M.

Proposition 2.2.2. *Let $t, t', s, s' \geq 0$ integers such that $S^{s+t} \cong S^s \oplus M$ and $S^{s'+t'} \cong S^{s'} \oplus M$. Then, $t' = t$.*

Proof. We have $S^{s'} \oplus S^{s+t} \cong S^{s'} \oplus S^s \oplus M$ and $S^s \oplus S^{s'+t'} \cong S^s \oplus S^{s'} \oplus M$, then since S is an $\mathcal{IBN}$ ring, $s' + s + t = s + s' + t'$, and hence $t' = t$. □

Corollary 2.2.3. *M is stably free of rank $t \geq 0$ if and only if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$, with $r \geq s$ and $t = r - s$.*

Proof. If M is stably free of rank t, then $S^{s+t} \cong S^s \oplus M$ for some integers $s, t \geq 0$; taking $r := s + t$ we get the result. Conversely, if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$, with $r \geq s$, then $S^{s+r-s} \cong S^s \oplus M$, i.e., M is stably free of rank $r - s$. □

Proposition 2.2.4. *Let M be an S-module and let $r, s \geq 0$ integers such that $S^r \cong S^s \oplus M$. Then $r \geq s$.*

Proof. The canonical projection $S^r \to S^s$ is an epimorphism; since we are assuming that S is $\mathcal{RC}$, then $r \geq s$. $\square$

Corollary 2.2.5. *M is stably free if and only if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$.*

Proof. This is a direct consequence of Corollary 2.2.3 and Proposition 2.2.4. $\square$

Proposition 2.2.6. *Let M_1, M_2 be stably free modules of ranks p, q, respectively. Then, $M_1 \oplus M_2$ is stably free of rank $p + q$.*

Proof. We have $S^{s+p} \cong S^s \oplus M_1$, $S^{r+q} \cong S^r \oplus M_2$, then $S^{s+p} \oplus M_2 \cong S^s \oplus M_1 \oplus M_2$ and also $S^{s+p} \oplus S^r \oplus M_2 \cong S^s \oplus S^r \oplus M_1 \oplus M_2$. Hence, $S^{s+p} \oplus S^{r+q} \cong S^{s+r} \oplus M_1 \oplus M_2$, i.e., $S^{s+r+p+q} \cong S^{s+r} \oplus M_1 \oplus M_2$. $\square$

Remark 2.2.7. Let S be a ring with finite uniform dimension and let M be stably free, then

$$\operatorname{rank}(M) = \frac{\operatorname{udim}(M)}{\operatorname{udim}(S)}. \tag{2.2.1}$$

In fact, from $S^{s+t} \cong S^s \oplus M$ we have $(s + t)\operatorname{udim}(S) = s\operatorname{udim}(S) + \operatorname{udim}(M)$, and this proves the equality.

Next, we will prove many characterizations of stably free modules over noncommutative rings (compare with [69], Chapter 21, [86], and [95], Chapter 11).

Theorem 2.2.8. *Let M be an S-module. Then, the following conditions are equivalent*

(i) *M is stably free.*

(ii) *M is projective and has a finite free resolution:*

$$0 \to S^{t_k} \xrightarrow{f_k} S^{t_{k-1}} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} S^{t_1} \xrightarrow{f_1} S^{t_0} \xrightarrow{f_0} M \to 0.$$

In this case

$$rank(M) = \sum_{i=0}^{k}(-1)^i t_i. \tag{2.2.2}$$

(iii) *M is isomorphic to the kernel of an epimorphism of free modules: $M \cong \ker(\pi)$, $\pi : S^r \to S^s$.*

(iv) *M is projective and has a finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where $\ker(f_0)$ is stably free.*

(v) *M has a finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_1 has a left inverse.*

Proof. (i) $\Rightarrow$ (ii) If $S^{s+t} \cong S^s \oplus M$ for some integers $s, t \geq 0$, then M is projective and we have the finite free resolution

$$0 \to S^s \xrightarrow{\iota} S^{s+t} \xrightarrow{\pi} M \to 0,$$

where ι is the canonical inclusion and π is the canonical projection on M.

(ii) $\Rightarrow$ (i) Let

$$0 \to S^{t_k} \xrightarrow{f_k} S^{t_{k-1}} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} S^{t_1} \xrightarrow{f_1} S^{t_0} \xrightarrow{f_0} M \to 0$$

be a finite free resolution of M. By induction on k, we will prove that M is stably free and (2.2.2) holds.

If $k = 0$ then $M \cong S^{t_0}$ is free of finite dimension t_0, and hence, stably free of rank t_0. Let $k \geq 1$ and let $M_1 = \ker(f_0)$. We get the exact sequence

$$0 \to M_1 \xrightarrow{\iota} S^{t_0} \xrightarrow{f_0} M \to 0,$$

and hence $S^{t_0} \cong M \oplus M_1$ since M is a projective module. This implies that M_1 is also projective and we have the finite free resolution of M_1

$$0 \to S^{t_k} \xrightarrow{f_k} S^{t_{k-1}} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} S^{t_1} \xrightarrow{f_1} M_1 \to 0.$$

By induction, M_1 is stably free of $\mathrm{rank}(M_1) = \sum_{i=1}^{k}(-1)^{i-1}t_i := p$. There exists $q \geq 0$ such that $S^{q+p} \cong S^q \oplus M_1$, and hence, $S^{t_0} \oplus S^q \cong M \oplus M_1 \oplus S^q \cong M \oplus S^{q+p}$, i.e., $S^{t_0+q} \cong M \oplus S^{q+p}$. By Proposition 2.2.4, $t_0 + q \geq q + p$, i.e., $t_0 \geq p$, so $S^{q+p+(t_0-p)} \cong M \oplus S^{q+p}$, i.e., M is stably free of rank $t_0 - p = \sum_{i=0}^{k}(-1)^i t_i$.

(i) $\Rightarrow$ (iii) By Proposition 2.2.5 there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$, with $r \geq s$. Hence $M \cong \ker(\pi)$, where π is the canonical projection of S^r on S^s.

(iii) $\Rightarrow$ (i) Let $S^r \xrightarrow{\pi} S^s$ be an epimorphism such that $M \cong \ker(\pi)$. Then we have the exact sequence

$$0 \to M \xrightarrow{\iota} S^r \xrightarrow{\pi} S^s \to 0,$$

but S^s is projective and hence $S^r \cong S^s \oplus M$.

(i) $\Rightarrow$ (iv) Let $S^r \cong S^s \oplus M$ for some integers $r, s \geq 0$, then M is projective and we have the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, and also the finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_0 is the canonical projection and f_1 is the canonical injection of S^s in S^r. But $\ker(f_0) = Im(f_1) \cong S^s$, thus $\ker(f_0)$ is free, and hence, stably free.

(iv) $\Rightarrow$ (i) Let M be projective and $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ a finite presentation of M with $\ker(f_0)$ stably free. Then $S^r \cong M \oplus \ker(F_0)$. There exist some integers $p, q \geq 0$ with $p \geq q$ such that $S^p \cong S^q \oplus \ker(F_0)$ and hence $S^p \oplus M \cong S^{q+r}$; by Corollary 2.2.5, M is stably free.

(i) $\Rightarrow$ (v) Let $S^r \cong S^s \oplus M$ for some integers $r, s \geq 0$, then we have the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, and also the finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_0 is the canonical projection and f_1 is the canonical injection of S^s in S^r. Since M is projective there exists $h_0 : M \to S^r$ such that $f_0 h_0 = i_M$, and hence, $S^r = \ker(f_0) \oplus Im(h_0) = Im(f_1) \oplus Im(h_0)$. For $x \in S^r$ we have $x = f_1(y) + h_0(z)$ with $y \in S^s$ and $z \in M$, we note that y and z are unique for x since f_1 and h_0 are injective, so we define $g_1 : S^r \to S^s$ by $g_1(x) = y$. It is clear that g_1 is an S-homomorphism and $g_1 f_1 = i_{S^s}$.

(v) $\Rightarrow$ (i) Let $g_1 : S^r \to S^s$ such that $g_1 f_1 = i_{S^s}$, then f_1 is injective and M has the finite free resolution $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. M is projective since this sequence splits; by (ii) and (i) M is stably free. □

Definition 2.2.9. *A finite presentation*

$$S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0 \tag{2.2.3}$$

of a S-module M is minimal if f_1 has a left inverse.

Corollary 2.2.10. *Let M be an S-module. Then, M is stably free if and only if M has a minimal presentation.*

Proof. See the proof of Theorem 2.2.8, part (i)⇔(v). □

Unimodular matrices are closely related to the stably free modules.

Definition 2.2.11. *Let F be a matrix over S of size $r \times s$. Then*

(i) *Let $r \geq s$. F is unimodular if and only if F has a left inverse.*

(ii) *Let $s \geq r$. F is unimodular if and only if F has a right inverse.*

The set of unimodular column matrices of size $r \times 1$ is denoted by $Um_c(r, S)$. $Um_r(s, S)$ is the set of unimodular row matrices of size $1 \times s$.

Remark 2.2.12. Note that a column matrix is unimodular if and only if the left ideal generated by its entries coincides with S; in addition, a row matrix is unimodular if and only if the right ideal generated by its entries is S.

We can add some others characterizations of stably free modules (compare with [105], Lemma 16).

Corollary 2.2.13. *Let M be an S-module. Then the following conditions are equivalent:*

(i) *M is stably free.*

(ii) *M is projective and has a finite system of generators $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r$ such that $Syz\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r\}$ is the module generated by the columns of a matrix F_1 of size $r \times s$ such that F_1^T has a right inverse.*

(iii) *M is projective and has a finite system of generators $\boldsymbol{f}_1, \dots, \boldsymbol{f}_r$ such that $Syz\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_r\}$ is the module generated by the columns of a matrix F_1 of size $r \times s$ such that F_1^T is unimodular.*

Proof. (i) $\Rightarrow$ (ii) By (v) of Theorem 2.2.8, M is projective and has a finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_1 has a left inverse. Let $\boldsymbol{f}_i = f_0(\boldsymbol{e}_i)$, where $\{\boldsymbol{e}_i\}_{1 \leq i \leq r}$ is the canonical basis of S^r. Then $M = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_r \rangle$ and $Im(f_1) = \ker(f_0) = Syz\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_r\}$, but $Im(f_1)$ is the module generated by the columns of the matrix F_1 defined by f_1 in the canonical bases. Thus, let $g_1 : S^r \to S^s$ be a left inverse of f_1, then $g_1 f_1 = i_{S^s}$ and the matrix of $g_1 f_1$ in the canonical bases is $I_s = (F_1^T G_1^T)^T$, so $I_s = F_1^T G_1^T$.

(ii) $\Rightarrow$ (i) Let $\boldsymbol{f}_1, \dots, \boldsymbol{f}_r$ be a set of generators of M such that $Syz\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_r\}$ is the module generated by the columns of a matrix F_1 of size $r \times s$ such that F_1^T has a right inverse. We have the exact sequence $0 \to \ker(f_0) \xrightarrow{\iota} S^r \xrightarrow{f_0} M \to 0$, where ι is the canonical injection and f_0 is defined as above. We have $\ker(f_0) = Syz\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_r\} = \langle F_1 \rangle$, and thus we get the finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where $f_1(\boldsymbol{e}_j)$ is the j^{th} column of F_1, $1 \leq j \leq s$. By hypothesis F_1^T has a right inverse, $F_1^T G_1^T = I_s$, so $I_s = (F_1^T G_1^T)^T$. Let $g_1 : S^r \to S^s$ be the homomorphism defined by $G_1 \in M_{s \times r}(S)$ in the canonical bases, then $g_1 f_1 = i_{S^s}$ and f_1 is injective, this implies that the sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ is exact. By Theorem 2.2.8, M is stably free.

(ii) $\Leftrightarrow$ (iii) This is a direct consequence of Definition 2.2.11. $\square$

Corollary 2.2.14. *Let M be an S-module.*

(i) *If M is stably free, then for any free resolution of M,*

$$\cdots \xrightarrow{f_{k+1}} S^{s_k} \xrightarrow{f_k} S^{s_{k-1}} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} S^{s_1} \xrightarrow{f_1} S^{s_0} \xrightarrow{f_0} M \longrightarrow 0,$$

$Im(f_k)$ is stably free for each $k \geq 0$.

(ii) *If there exists a free resolution of M as in* (i) *such that $Im(f_k)$ is stably free for some $k \geq 0$ and $Im(f_{k-1}), \dots, Im(f_0)$ are projective, then M is stably free.*

Proof. (i) We will prove this by induction on k. For $k = 0$ we have $Im(f_0) = M$. For $k = 1$ we have the exact sequence $0 \to \ker(f_0) \to S^{s_0} \xrightarrow{f_0} M \to 0$, then $S^{s_0} \cong M \oplus \ker(f_0)$ since M is projective. But $S^q \oplus M = S^p$ since M is stably free, then $S^{s_0+q} \cong S^p \oplus \ker(f_0)$, thus $\ker(f_0) = Im(f_1)$ is stably free. We assume that $Im(f_{k-1})$ is stably free and we consider the exact sequence $0 \to \ker(f_{k-1}) \to S^{s_{k-1}} \xrightarrow{f_{k-1}} Im(f_{k-1}) \to 0$, then $S^{s_{k-1}} \cong Im(f_{k-1}) \oplus \ker(f_{k-1})$, and hence there exist $l, t \geq 0$ such that $S^l \oplus Im(f_{k-1}) \cong S^t$ and hence $S^{s_{k-1}+l} \cong S^t \oplus \ker(f_{k-1})$. Thus, $\ker(f_{k-1}) = Im(f_k)$ is stably free.

(ii) If $k = 0$ there is nothing to prove. Let $k \geq 1$, we consider the presentation $S^{s_k} \xrightarrow{f_k} S^{s_{k-1}} \xrightarrow{f_{k-1}} Im(f_{k-1}) \to 0$, by (iv) of Theorem 2.2.8, $Im(f_{k-1})$ is stably free. In the same way we prove that $Im(f_{k-2}), \dots, Im(f_1), Im(f_0) = M$ are stably free. $\square$

Another interesting result about stably free modules over arbitrary RC rings is presented next (see [23], Proposition 12). For this, we recall that if M is a finitely presented left S-module with presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ and F_1 is the matrix of f_1 in the canonical bases, then the right S-module M^T defined by $M^T := S^s/Im(f_1^T)$, where $f_1^T : S^r \to S^s$ is the homomorphism of right free S-modules induced by the matrix F_1^T, is called the *transposed module* of M. Thus, M^T is given by the presentation $S^r \xrightarrow{f_1^T} S^s \to M^T \to 0$.

Theorem 2.2.15. *Let M be an S-module with exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. Then, $M^T \cong Ext_S^1(M,S)$ and the following conditions are equivalent:*

(i) *M is stably free.*

(ii) *M is projective.*

(iii) *$M^T = 0$.*

(iv) *F_1^T has a right inverse.*

(v) *f_1 has a left inverse.*

Proof. We first prove that $M^T \cong Ext_S^1(M,S)$: from the left complex $0 \to S^s \xrightarrow{f_1} S^r \to 0$ we get the right complex

$$0 \to Hom_S(S^r,S) \xrightarrow{f_1^*} Hom_S(S^s,S) \xrightarrow{0} Hom_S(0,S) \to \cdots,$$

i.e.,

$$0 \to S^r \xrightarrow{f_1^*} S^s \xrightarrow{0} 0 \to \cdots,$$

so $Ext_S^1(M,S) = \ker(0)/Im(f_1^*) = S^s/Im(f_1^*)$. But $Im(f_1^*) \cong Im(f_1^T)$ under the isomorphisms $Hom_S(S^r,S) \cong S^r$ and $Hom_S(S^s,S) \cong S^s$. In fact, we have the following diagram

$$\begin{tikzcd} Hom_S(S^r,S) \arrow[r, "f_1^*"] \arrow[d, "\alpha"'] & Hom_S(S^s,S) \arrow[d, "\beta"] \\ S^r \arrow[r, "f_1^T"] & S^s \end{tikzcd} \tag{2.2.4}$$

where the vertical rows are isomorphisms of right S-modules defined by

$$\alpha(h) := (h(\boldsymbol{e}_1),\dots,h(\boldsymbol{e}_r))^T,$$
$$\beta(g) := (g(\boldsymbol{e}_1),\dots,g(\boldsymbol{e}_s))^T,$$

and moreover $f_1^*(h) := hf_1$ and $f_1^T((x_1,\dots,x_r)^T) := F_1^T(x_1,\dots,x_r)^T$. Note that the diagram is commutative:

$$\begin{aligned}\beta f_1^*(h) &= \beta(hf_1) = (hf_1(\boldsymbol{e}_1),\dots,hf_1(\boldsymbol{e}_s))^T = (h((\boldsymbol{e}_1^T F_1^T)^T),\dots,h((\boldsymbol{e}_s^T F_1^T)^T))^T\\ &= (h(\begin{bmatrix} f_{11}\\ \vdots \\ f_{r1}\end{bmatrix}),\dots,h(\begin{bmatrix} f_{1s}\\ \vdots \\ f_{rs}\end{bmatrix}))^T;\end{aligned}$$

$$\begin{aligned}f_1^T\alpha(h) &= f_1^T((h(\boldsymbol{e}_1),\dots,h(\boldsymbol{e}_r))^T) = F_1^T\begin{bmatrix} h(\boldsymbol{e}_1)\\ \vdots \\ h(\boldsymbol{e}_r)\end{bmatrix} = \begin{bmatrix} f_{11}h(\boldsymbol{e}_1)+\cdots+f_{r1}h(\boldsymbol{e}_r)\\ \vdots \\ f_{1s}h(\boldsymbol{e}_1)+\cdots+f_{rs}h(\boldsymbol{e}_r)\end{bmatrix}\\ &= \begin{bmatrix} h(f_{11}\boldsymbol{e}_1+\cdots+f_{r1}\boldsymbol{e}_r)\\ \vdots \\ h(f_{1s}\boldsymbol{e}_1+\cdots+f_{rs}\boldsymbol{e}_r)\end{bmatrix} = (h(\begin{bmatrix} f_{11}\\ \vdots \\ f_{r1}\end{bmatrix}),\dots,h(\begin{bmatrix} f_{1s}\\ \vdots \\ f_{rs}\end{bmatrix}))^T.\end{aligned}$$

From this, we conclude that $Ext_S^1(M,S) \cong S^s/Im(f_1^T) = M^T$.

(i)⇒(ii) This is obvious.

(ii)⇒(i) This is a direct consequence of Theorem 2.2.8.

(ii)⇒(iii) Since M is projective, then $Ext_S^1(M,S) = 0$ and hence $M^T = 0$.

(iii)⇒(i) If $M^T = 0$, then $Ext_S^1(M,S) = 0$. From the given exact sequence of left modules we get the exact sequence of right modules

$$0 \to Hom_S(M,S) \xrightarrow{f_0^*} Hom_S(S^r,S) \xrightarrow{f_1^*} Hom_S(S^s,S) \to Ext_S^1(M,S) \to \dots,$$

i.e., we have the exact sequence $0 \to M^* \to S^r \xrightarrow{f_1^T} S^s \to 0$; but since S^s is projective, this sequence splits, i.e., f_1^T has right inverse, says $S^s \xrightarrow{g_1^T} S^r$, i.e., $f_1^T g_1^T = i_{S^s}$. Let G_1 be a matrix of size $s \times r$ such that G_1^T is the matrix of the right homomorphism g_1^T, then $m(f_1^T g_1^T) = m(f_1^T)m(g_1^T) = m(i_{S^s})$, i.e., $F_1^T G_1^T = I_s$. Let $S^r \xrightarrow{g_1} S^s$ be the left homomorphism corresponding to G_1, then $m(g_1 f_1) = (F_1^T G_1^T)^T = I_s = m(i_{S^s})$, so $g_1 f_1 = i_{S^s}$, i.e., f_1 has left inverse. This means that the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ splits, so M is stably free.

(ii)⇔(iv): if M is projective, then the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ splits, so there exists g_1 such that $g_1 f_1 = i_{S^s}$, and hence, as before, F_1^T has a right inverse. Conversely, if $F_1^T G_1^T = I_s$, then $g_1 f_1 = i_{S^s}$, where $S^r \xrightarrow{g_1} S^s$ is the left homomorphism corresponding to G_1, so the previous sequence splits, and hence, M is projective.

(iv)⇔(v): from the above discussion, we get that f_1 has a left inverse if and only if F_1^T has a right inverse. □

Remark 2.2.16. (i) In Definition 2.2.1, if the finiteness restriction on s and t is not imposed, then every projective module is free: indeed, using the "trick of Eilenberg", we can prove that if P is a projective and Q is a a module such that $P \oplus Q = E$ is free, then $P \oplus F \cong F$, where $F := E \oplus E \oplus \cdots$. On the other hand, if $P \oplus S^s$ is free but P is not finitely generated, it is not difficult to prove that P is actually free (see [62], Proposition 4.2).

(ii) Theorem 2.2.15 gives procedures for testing stably freeness if we have algorithms for computing the module of syzygies of a finite set of vectors, the right inverse of a matrix and the Ext modules. These algorithms will be considered later.

2.3 Stafford's theorem: a constructive proof

A well known result due Stafford asserts that any left ideal of the Weyl algebras $D := A_n(\Bbbk)$ or $B_n(\Bbbk)$, with char$(\Bbbk) = 0$, is generated by two elements, (see [114] and [105]). From the Stafford's Theorem follows that any stably free left module M over D with rank$(M) \geq 2$ is free. In [105] is shown a constructive proof of this result that we want to study for arbitrary $\mathcal{RC}$ rings. Actually, we will consider the generalization given in [105] showing that any stably left S-module M with rank$(M) \geq$ sr(S) is free, where sr(S) denotes the stable rank of the ring S. Our proof have been adapted from [105], however we do not need the involution of ring S used in [105] because of our left notation for modules and column representation for homomorphism. This could justify our special left-column notation. In order to apply the main result of this section to bijective skew PBW extensions we will estimate the stable rank of such extensions. In Chapter 7, we will complement these results presenting algorithms for computing the corresponding free bases.

Definition 2.3.1. *Let S be a ring and $\boldsymbol{v} := \begin{bmatrix} v_1 & \ldots & v_r \end{bmatrix}^T \in Um_c(r, S)$ an unimodular column vector. $\boldsymbol{v}$ is called stable (reducible) if there exists $a_1, \ldots, a_{r-1} \in S$ such that $\boldsymbol{v}' := \begin{bmatrix} v_1 + a_1v_r & \ldots & v_{r-1} + a_{r-1}v_r \end{bmatrix}^T$ is unimodular. It says that the left stable rank of S is $d \geq 1$, denoted $\mathrm{sr}(S) = d$, if d is the least positive integer such that every unimodular column vector of length $d+1$ is stable. It says that $\mathrm{sr}(S) = \infty$ if for every $d \geq 1$ there exits a non stable unimodular column vector of length $d + 1$.*

Remark 2.3.2. In a similar way is defined the right stable rank of S, however, both ranks coincide; we list next some well known properties of the stable rank (see [5], [8], [20], [95], [105], [114], [115], [120], [66] , or also [48]).

(i) $\mathrm{sr}(S) = \mathrm{sr}(S^{op})$.

(ii) If T is a division ring, then $\mathrm{sr}(T) = 1$.

(iii) If I is a two sided ideal of S, then $\mathrm{sr}(S/I) \leq \mathrm{sr}(S)$. Moreover, if $1 + I \subseteq S^*$, then $\mathrm{sr}(S/I) = \mathrm{sr}(S)$. In particular, $\mathrm{sr}(S/Rad(S)) = \mathrm{sr}(S)$.

(iv) For any field $\Bbbk$, $\mathrm{sr}(\Bbbk[[x_1, \ldots, x_n]]) = 1$ (this follows from 2.3.2 (iii))

(v) If S is a local ring, then $\mathrm{sr}(S) = 1$.

(vi) If $\{S_i\}_{i \in \mathcal{C}}$ is a non empty family of rings, then $\mathrm{sr}(\prod_{i \in \mathcal{C}} S_i) = \sup\{\mathrm{sr}(S_i)\}_{i \in \mathcal{C}}$.

(vii) If $\mathrm{sr}(S) = 1$, then $\mathrm{sr}(M_n(S)) = 1$, for any $n \geq 1$.

(viii) If S is simple Artinian, semisimple or semilocal, then $\mathrm{sr}(S) = 1$.

(ix) If S is a Dedekind domain, then $\mathrm{sr}(S) = 2$. In particular, if $\Bbbk$ is a field, then $\mathrm{sr}(\Bbbk[x]) = 2$; thus, $\mathrm{sr}(\mathbb{Q}[x]) = \mathrm{sr}(\mathbb{R}[x]) = \mathrm{sr}(\mathbb{C}[x]) = 2$.

(x) If $\Bbbk$ is a field with $\mathrm{char}(\Bbbk) = 0$ then $\mathrm{sr}(A_n(\Bbbk)) = 2 = \mathrm{sr}(B_n(\Bbbk))$.

(xi) If $S = T[x; \sigma, \delta]$, with T a division ring and σ is an automorphism, then $\mathrm{sr}(S) = 2$.

(xii) If S is a left Noetherian ring, then $\mathrm{sr}(S) \leq \mathrm{Kdim}(S) + 1$. In particular, if S is a left Artinian ring, then $\mathrm{sr}(S) = 1$.

(xiii) Let $n \geq 3$. If $n > \mathrm{sr}(S)$, then $E_n(S) \trianglelefteq GL_n(S)$.

Proposition 2.3.3. *Let S be a ring and $\boldsymbol{v} := \begin{bmatrix} v_1 & \dots & v_r \end{bmatrix}^T$ an unimodular stable column vector over S, then there exists $U \in E_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$.*

Proof. There exist elements $a_1, \dots, a_{r-1} \in S$ such that

$$\boldsymbol{v}' := (v'_1, \dots, v'_{r-1})^T \in Um_c(r-1, S), \text{ with } v'_i := v_i + a_i v_r, 1 \leq i \leq r-1. \tag{2.3.1}$$

Consider the matrix

$$E_1 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{r-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in E_r(S); \tag{2.3.2}$$

then $E_1 v = (v'_1, \dots, v'_{r-1}, v_r)^T$. Since that $\boldsymbol{v}' := (v'_1, \dots, v'_{r-1}) \in Um_c(r-1, S)$, there exists $b_1, \dots, b_{r-1} \in S$ such that $\sum_{i=1}^{r-1} b_i v'_i = 1$, and hence, $\sum_{i=1}^{r-1}(v'_1 - 1 - v_r) b_i v'_i = v'_1 - 1 - v_r$. Let $v''_i := (v'_1 - 1 - v_r) b_i$, $1 \leq i \leq r-1$ and

$$E_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ v''_1 & v''_2 & v''_3 & \cdots & v''_{r-1} & 1 \end{bmatrix} \in E_r(S); \tag{2.3.3}$$

then $E_2 E_1 v = (v'_1, \dots, v'_{r-1}, v'_1 - 1)^T$. Moreover, let

$$E_3 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in E_r(S), \tag{2.3.4}$$

then $E_3 E_2 E_1 v = (1, v'_2, \dots, v'_{r-1}, v'_1 - 1)^T$. Finally, let

$$E_4 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -v'_2 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -v'_{r-1} & 0 & 0 & \cdots & 1 & 0 \\ -v'_1 + 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in E_r(S), \tag{2.3.5}$$

then $E_4E_3E_2E_1v = \boldsymbol{e}_1$ and $U := E_1E_2E_3E_4 \in E_r(S)$. □

As was presented in [105], the proof of above lemma allows us to calculate effectively the matrix $U \in E_r(S)$. An algorithm to compute this elementary matrix will be considered in Section 7.5.

Next we present two lemmas that give some elementary matrix characterizations of free modules, the second one is needed for the proof of the main theorem of the present section.

Lemma 2.3.4. *Let S be a ring and let $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\rangle$ be a finitely generated S-module. Then,*

(i) *M is free with basis $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ if and only if $Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) = 0$.*

(ii) *M is free if and only if there exist matrices P of size $r \times s$ and Q of size $s \times r$ such that $M \cong \langle P\rangle$ and $Q^TP^T = I_r$, with $s \geq r$, i.e., M is isomorphic to the column module of a matrix such that its transpose is unimodular. Thus, M is isomorphic to the image of a S-module epimorphism of free modules of finite dimension.*

Proof. (i) Evident.

(ii)$\Rightarrow$) There exists an isomorphism $M \xrightarrow{g} S^r$; from this we get the epimorphism $S^s \xrightarrow{gh} S^r$, where $S^s \xrightarrow{h} M$ is defined by $h(\boldsymbol{e}_i) := \boldsymbol{f}_i$, $1 \leq i \leq s$, and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\}$ is the canonical basis of S^s. Thus, we get the epimorphism $p := gh : S^s \to S^r$; let P be the matrix of p in the canonical bases of S^s and S^r, then P is of size $r \times s$ and $\langle P\rangle \cong M$. In fact, let $\{x_1, \ldots, x_r\}$ a basis of M, we choose $z_j \in S^s$ such that $h(z_j) = x_j$, $1 \leq j \leq r$. We define the homomorphism $t : M \to Im(p) = \langle P\rangle$ by $t(x_j) := p(z_j)$. t is injective since if $t(a_1 \cdot x_1 + \cdots + a_r \cdot x_r) = 0$ with $a_j \in A$, then $a_1 \cdot p(z_1) + \cdots + a_r \cdot p(z_r) = 0$ and hence $a_1 \cdot gh(z_1) + \cdots + a_r \cdot gh(z_r) = 0$, so $g(a_1 \cdot h(z_1) + \cdots + a_r \cdot h(z_r)) = 0$, but g is injective, then $a_1 \cdot h(z_1) + \cdots + a_r \cdot h(z_r) = 0$, i.e., $a_1 \cdot x_1 + \cdots + a_r \cdot x_r = 0$ and from this $a_1 = \cdots = a_r = 0$. Now, if $p(z) \in Im(p)$, with $z \in S^s$, then $p(z) = gh(z) = g(b_1 \cdot x_1 + \cdots + b_r \cdot x_r)$ for some $b_j \in A$, so $p(z) = g(b_1 \cdot h(z_1) + \cdots b_r \cdot h(z_r)) = b_1 \cdot gh(z_1) + \cdots + b_r \cdot gh(z_r) = b_1 \cdot p(z_1) + \cdots + b_r \cdot p(z_r) = t(b_1 \cdot x_1 + \cdots + b_r \cdot x_r)$, and this proves that t is surjective.

Since S^r is projective there exists an homomorphism $S^r \xrightarrow{q} S^s$ such that $pq = i_{S^r}$ and hence $Q^TP^T = I_r$, with $s \geq r$.

$\Leftarrow$) Now we assume that $\langle P\rangle \cong M$ and $Q^TP^T = I_r$, where P of size $r \times s$ and Q of size $s \times r$, with $s \geq r$. If p, q are the homomorphisms defined by P and Q, we have $pq = i_{S^r}$ and $S^r = Im(i_{S^r}) \subseteq Im(p) \subseteq S^r$, i.e., $M \cong Im(p) = S^r$. □

Lemma 2.3.5. *Let S be a ring and M a stably free S-module given by a minimal presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. Let $g_1 : S^r \to S^s$ such that $g_1f_1 = i_{S^s}$. Then the following conditions are equivalent:*

(i) *M is free of dimension $r - s$.*

(ii) *There exists a matrix $U \in GL_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, where G_1 is the matrix of g_1 in the canonical bases. In such case, the last $r-s$ columns of U^T conform a basis for M. Moreover, the first s columns of U^T conform the matrix F_1 of f_1 in the canonical bases.*

(iii) *There exists a matrix $V \in GL_r(S)$ such that G_1^T coincides with the first s columns of V, i.e., G_1^T can be completed to an invertible matrix V of $GL_r(S)$.*

Proof. By the hypothesis, the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ splits, so F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$, i.e., $F_1^T G_1^T = I_s$. Moreover, there exists $g_0 : M \to S^r$ such that $f_0 g_0 = i_M$. From this we get also the split sequence $0 \to M \xrightarrow{g_0} S^r \xrightarrow{g_1} S^s \to 0$. Note that $M \cong \ker(g_1)$.

(i) $\Rightarrow$ (ii): We have $S^r = \ker(g_1) \oplus Im(f_1)$; by the hypothesis $\ker(g_1)$ is free. If $s = r$ then $\ker(g_1) = 0$ and hence f_1 is an isomorphism, so $f_1 g_1 = i_{S^s}$, i.e., $G_1^T F_1^T = I_s$. Thus, we can take $U := F_1^T$.

Let $r > s$; if $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\}$ is the canonical basis of S^s, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s\}$ is a basis of $Im(f_1)$ with $\boldsymbol{u}_i := f_1(\boldsymbol{e}_i)$, $1 \leq i \leq s$; let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be a basis of $\ker(g_1)$ with $p = r - s$. Then, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_s\}$ is a basis of S^r. We define $S^r \xrightarrow{h} S^r$ by $h(\boldsymbol{e}_i) := \boldsymbol{u}_i$ for $1 \leq i \leq s$, and $h(\boldsymbol{e}_{s+j}) = \boldsymbol{v}_j$ for $1 \leq j \leq p$. Clearly h is bijective; moreover, $g_1 h(\boldsymbol{e}_i) = g_1(\boldsymbol{u}_i) = g_1 f_1(\boldsymbol{e}_i) = \boldsymbol{e}_i$ and $g_1 h(\boldsymbol{e}_{s+j}) = g_1(\boldsymbol{v}_j) = \boldsymbol{0}$, i.e., $H^T G_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$. Let $U := H^T$, so we observe that the last p columns of U^T conform a basis of $\ker(g_1) \cong M$ and the first s columns of U^T conform F_1.

(ii) $\Rightarrow$ (i): Let $U_{(k)}$ the k-th row of U, then $UG_1^T = [U_{(1)} \cdots U_{(s)} \cdots U_{(r)}]^T G_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, so $U_{(i)} G_1^T = \boldsymbol{e}_i^T$, $1 \leq i \leq s$, $U_{(s+j)} G_1^T = \boldsymbol{0}$, $1 \leq j \leq p$ with $p := r - s$. This means that $(U_{(s+j)})^T \in \ker(g_1)$ and hence $\langle (U_{(s+j)})^T | 1 \leq j \leq p \rangle \subseteq \ker(g_1)$. On the other hand, let $\boldsymbol{c} \in \ker(g_1) \subseteq S^r$, then $\boldsymbol{c}^T G_1^T = \boldsymbol{0}$ and $\boldsymbol{c}^T U^{-1} U G_1^T = \boldsymbol{0}$, thus $\boldsymbol{c}^T U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix} = \boldsymbol{0}$ and hence $(\boldsymbol{c}^T U^{-1})^T \in \ker(l)$, where $l : S^r \to S^s$ is the homomorphism with matrix $\begin{bmatrix} I_s & 0 \end{bmatrix}$. Let $\boldsymbol{d} = [d_1, \ldots, d_r]^T \in \ker(l)$, then $[d_1, \ldots, d_r] \begin{bmatrix} I_s \\ 0 \end{bmatrix} = \boldsymbol{0}$ and from this we conclude that $d_1 = \cdots = d_s = 0$, i.e., $\ker(l) = \langle \boldsymbol{e}_{s+1}, \boldsymbol{e}_{s+2}, \ldots, \boldsymbol{e}_{s+p} \rangle$. From $(\boldsymbol{c}^T U^{-1})^T \in \ker(l)$ we get that $(\boldsymbol{c}^T U^{-1})^T = a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p}$, so $\boldsymbol{c}^T U^{-1} = (a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p})^T$, i.e., $\boldsymbol{c}^T = (a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p})^T U$ and from this we get that $\boldsymbol{c} \in \langle (U_{(s+j)})^T | 1 \leq j \leq p \rangle$. This proves that $\ker(g_1) = \langle (U_{(s+j)})^T | 1 \leq j \leq p \rangle$; but since U is invertible, then $\ker(g_1)$ is free of dimension p. We have proved also that the last p columns of U^T conform a basis for $\ker(g_1) \cong M$.

(ii) $\Leftrightarrow$ (iii): $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ if and only if $G_1^T = U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, but the first s columns of $U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ coincides with the first s columns of U^{-1}; taking $V := U^{-1}$ we get the result. $\square$

Theorem 2.3.6. *Let S be a ring. Then any stably free S-module M with* $\operatorname{rank}(M) \geq \operatorname{sr}(S)$ *is free with dimension equals to* $\operatorname{rank}(M)$.

Proof. Since M is stably free it has a minimal presentation, and hence, it is given by an exact sequence

$$0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0;$$

moreover, note that $\operatorname{rank}(M) = r - s$. Since this sequence splits, F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$. The idea of the proof is to find a matrix $U \in GL_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ and then apply Lemma 2.3.5.

We have $F_1^T G_1^T = I_s$ and from this we get that the first column $\boldsymbol{g}_1$ of G_1^T is unimodular, but since $r > r - s \geq \operatorname{sr}(S)$, then $\boldsymbol{g}_1$ is stable, and by Proposition 2.3.3, there exists $U_1 \in E_r(S)$ such that $U_1 \boldsymbol{g}_1 = \boldsymbol{e}_1$. If $s = 1$, we finish since $G_1^T = \boldsymbol{g}_1$.

Let $s \geq 2$; we have

$$U_1 G_1^T = \begin{bmatrix} 1 & * \\ 0 & F_2 \end{bmatrix}, F_2 \in M_{(r-1)\times(s-1)}(S).$$

Note that $U_1 G_1^T$ has a left inverse (for instance $F_1^T U_1^{-1}$), and the form of this left inverse is

$$L = \begin{bmatrix} 1 & * \\ 0 & L_2 \end{bmatrix}, L_2 \in M_{(s-1)\times(r-1)}(S),$$

and hence $L_2 F_2 = I_{s-1}$. The first column of F_2 is unimodular and since $r - 1 > r - s \geq \operatorname{sr}(S)$ we apply again Proposition 2.3.3 and we obtain a matrix $U_2' \in E_{r-1}(S)$ such that

$$U_2' F_2 = \begin{bmatrix} 1 & * \\ 0 & F_3 \end{bmatrix}, F_3 \in M_{(r-2)\times(s-2)}(S).$$

Let

$$U_2 := \begin{bmatrix} 1 & 0 \\ 0 & U_2' \end{bmatrix} \in E_r(S),$$

then we have

$$U_2 U_1 G_1^T = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & F_3 \end{bmatrix}.$$

By induction on s and multiplying on the left by elementary matrices we get a matrix $U \in E_r(S)$ such that

$$UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}.$$

□

Corollary 2.3.7 (Stafford). *Let $D := A_n(\Bbbk)$ or $B_n(\Bbbk)$, with* $\mathrm{char}(\Bbbk) = 0$. *Then, any stably free left D-module M satisfying* $\mathrm{rank}(M) \geq 2$ *is free.*

Proof. The results follows from Theorem 2.3.6 since $\mathrm{sr}(D) = 2$. □

2.4 Projective dimension of a module

Closely related to the study of stably free modules is the computation of the projective dimension of a given module M. Later, we will expose some theoretical results that will be used in Chapter 7 for computing the projective dimension of a finitely presented left module over certain classes of skew PBW extensions. The first one only requires the computation of arbitrary free resolutions of M; the second one allows additionally to compute a minimal presentation of a finitely presented module M when a finite free resolution of M is given, and also, it allows to check whether M is stably free or not(see [105]). Remember that S denotes an arbitrary noncommutative $\mathcal{RC}$ ring.

We start with the following theorem which can be used for testing if a finitely presented module is projective (compare with [77], Theorem 4).

Theorem 2.4.1. *Let M be an S-module given by a presentation*

$$0 \to K \to S^n \xrightarrow{f_0} M \to 0,$$

where K is f.g. Then, the following conditions are equivalent:

(i) *M is projective.*

(ii) $Ext_S^1(M, K) = 0$.

Proof. (i) $\Rightarrow$ (ii) This implication is well known, see [111].

(ii) $\Rightarrow$ (i) From the given sequence we get the exact sequence

$$0 \to Hom_S(M, K) \to Hom_S(M, S^n) \xrightarrow{(f_0)_*} Hom_S(M, M) \to Ext_S^1(M, K) = 0,$$

see [111], Theorem 7.3. Then, $(f_0)_*$ is surjective and there exists $f \in Hom_S(M, S^n)$ such that $(f_0)_*(f) = i_M$, i.e., $f_0 f = i_M$. This means that $S^n \cong K \oplus M$, i.e., M is projective. □

Let

$$\cdots \xrightarrow{f_{r+1}} P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M; recall that $\ker(f_i)$ is called the i-th syzygy of M. When $P_i := S^{s_i}$ is free of finite dimension, we get a free resolution of M.

Theorem 2.4.2. *Let M be an S-module and*

$$\cdots \xrightarrow{f_{r+1}} P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{2.4.1}$$

a projective resolution of M. Let r be the smallest integer such $Im(f_r)$ is projective. Then r does not depend on the resolution and $pd(M) = r$.

Proof. It is well known that $pd(M) \le r$ if and only if there exists a projective resolution of M where the $(r-1)$-th syzygy is projective if and only if for every projective resolution of M the $(r-1)$-th syzygy is projective (see [111]), Theorem 9.5). Let r be the smallest integer such $Im(f_r)$ is projective, since $Im(f_r) = \ker(f_{r-1}) = (r-1)$-th syzygy, then $pd(M) \le r$. Suppose that $pd(M) = t < r$, then the $(t-1)$-th syzygy of (2.4.1) is projective, but this means that r is not minimum. Thus, $pd(M) = r$.

Let

$$\cdots \xrightarrow{f'_{s+1}} P'_s \xrightarrow{f'_s} P'_{s-1} \xrightarrow{f'_{s-1}} \cdots \xrightarrow{f'_2} P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} M \longrightarrow 0$$

another projective resolution of M, where s is the smallest integer such $Im(f'_s)$ is projective. Then $pd(M) \le s$ and hence $r \le s$. Suppose that $r < s$, the $(r-1)$-th syzygy of M in the previous resolution is projective since $pd(M) = r$, but this is impossible since s is minimum, hence $r = s$. $\square$

Next we present the second result of this section that allows also to compute the projective dimension of a module given by a finite free resolution. For this we follow [105].

Theorem 2.4.3. *Let M be an S-module and*

$$0 \to P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} P_{m-2} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{2.4.2}$$

a projective resolution of M. If $m \ge 2$ and there exists a homomorphism $g_m : P_{m-1} \to P_m$ such that $g_m f_m = i_{P_m}$, then we have the following projective resolution of M:

$$0 \to P_{m-1} \xrightarrow{h_{m-1}} P_{m-2} \oplus P_m \xrightarrow{h_{m-2}} P_{m-3} \xrightarrow{f_{m-3}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{2.4.3}$$

with

$$h_{m-1} := \begin{bmatrix} f_{m-1} \\ g_m \end{bmatrix}, \quad h_{m-2} := \begin{bmatrix} f_{m-2} & 0 \end{bmatrix}.$$

Proof. $Im(h_{m-1}) \subseteq \ker(h_{m-2})$: we have

$$h_{m-2}h_{m-1} = \begin{bmatrix} f_{m-2} & 0 \end{bmatrix} \begin{bmatrix} f_{m-1} \\ g_m \end{bmatrix} = 0.$$

$\ker(h_{m-2}) \subseteq Im(h_{m-1})$: let $(a,b)^T \in \ker(h_{m-2})$, then $a \in P_{m-2}$, $b \in P_m$ and $h_{m-2}[(a,b)^T] = 0 = f_{m-2}(a)$. Then there exists $c \in P_{m-1}$ such that $a = f_{m-1}(c)$; we define

$$d := \begin{bmatrix} i_{P_{m-1}} - f_m g_m & f_m \end{bmatrix} (c, b)^T = c - (f_m g_m)(c) + f_m(b) \in P_{m-1}.$$

Then, the image of d under h_{m-1} is

$$\begin{bmatrix} f_{m-1}(c) - f_{m-1}(f_m(g_m(c))) + f_{m-1}(f_m(b)) \\ g_m(c) - ((g_m f_m)g_m)(c) + g_m f_m(b) \end{bmatrix} = \begin{bmatrix} f_{m-1}(c) \\ g_m(c) - g_m(c) + b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

h_{m-1} is injective: if $d \in \ker(h_{m-1})$, then $h_{m-1}(d) = 0$, so $f_{m-1}(d) = 0$ and $g_m(d) = 0$; we consider the exact sequence

$$0 \to P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} Im(f_{m-1}) \to 0,$$

since $g_m f_m = i_{P_m}$ this sequence splits, i.e., there exists a homomorphism $k_{m-1} : Im(f_{m-1}) \to P_{m-1}$ such that $i_{P_{m-1}} = f_m g_m + k_{m-1} f_{m-1}$. Hence, $d = f_m g_m(d) + k_{m-1} f_{m-1}(d) = 0$.

Finally, $Im(h_{m-2}) = h_{m-2}(P_{m-2} \oplus P_m) = f_{m-2}(P_{m-2}) = Im(f_{m-2}) = \ker(f_{m-3})$. ☐

Corollary 2.4.4. *Let M be an S-module and*

$$0 \to S^{s_m} \xrightarrow{f_m} S^{s_{m-1}} \xrightarrow{f_{m-1}} S^{s_{m-2}} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_2} S^{s_1} \xrightarrow{f_1} S^{s_0} \xrightarrow{f_0} M \longrightarrow 0 \tag{2.4.4}$$

a finite free resolution of M. Let F_i be the matrix of f_i in the canonical bases, $1 \leq i \leq m$. Then,

(i) *If $m \geq 3$ and there exists a homomorphism $g_m : S^{s_{m-1}} \to S^{s_m}$ such that $g_m f_m = i_{S^{s_m}}$, then we have the following finite free resolution of M:*

$$0 \to S^{s_{m-1}} \xrightarrow{h_{m-1}} S^{s_{m-2}+s_m} \xrightarrow{h_{m-2}} S^{s_{m-3}} \xrightarrow{f_{m-3}} \cdots \xrightarrow{f_1} S^{s_0} \xrightarrow{f_0} M \longrightarrow 0 \tag{2.4.5}$$

with

$$h_{m-1} := \begin{bmatrix} f_{m-1} \\ g_m \end{bmatrix}, \quad h_{m-2} := \begin{bmatrix} f_{m-2} & 0 \end{bmatrix}.$$

In a matrix notation, if G_m is the matrix of g_m and H_j is the matrix of h_j in the canonical bases, $j = m-1, m-2$, then

$$H_{m-1}^T := \begin{bmatrix} F_{m-1}^T & G_m^T \end{bmatrix}, \quad H_{m-2}^T := \begin{bmatrix} F_{m-2}^T \\ 0 \end{bmatrix}.$$

(ii) *If $m = 2$ and there exists a homomorphism $g_2 : S^{s_1} \to S^{s_2}$ such that $g_2 f_2 = i_{S^{s_2}}$, then we have the following finite presentation of M:*

$$0 \to S^{s_1} \xrightarrow{h_1} S^{s_0+s_2} \xrightarrow{h_0} M \to 0, \tag{2.4.6}$$

with

$$h_1 := \begin{bmatrix} f_1 \\ g_2 \end{bmatrix}, \quad h_0 := \begin{bmatrix} f_0 & 0 \end{bmatrix}.$$

In a matrix notation,

$$H_1^T := \begin{bmatrix} F_1^T & G_2^T \end{bmatrix}, \ \ H_0^T := \begin{bmatrix} f_0 \\ 0 \end{bmatrix}.$$

Proof. This is an obvious consequence of the previous theorem. □

Theorem 2.4.5. *Let M be an S-module and $n \geq 1$. $\mathrm{pd}(M) = n$ if and only if there exists a finite projective resolution of M as (2.4.2) where f_n is non-split, i.e., there exists no homomorphism $g_n : P_{n-1} \to P_n$ such that $g_n f_n = i_{P_n}$.*

Proof. $\Rightarrow$): there exists a finite projective resolution of M as in (2.4.2) with $m = n$; we have the exact sequence $0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} Im(f_{n-1}) \to 0$. If f_n splits, then $Im(f_{n-1})$ is projective, and by Theorem 2.4.2, $\mathrm{pd}(M) \leq n-1$, false. Thus, f_n is non-split.

$\Leftarrow$): if M has a finite projective resolution as in in (2.4.2), with $m = n$, which is non-split, then $\mathrm{pd}(M) \leq n$ and $Im(f_{n-1})$ in not projective. Suppose that there exists $k \leq n-2$ such that $Im(f_k)$ is projective; we have the exact sequence $0 \to Im(f_{k+1}) \xrightarrow{\iota} P_k \xrightarrow{f_k} Im(f_k) \to 0$, where ι is the canonical inclusion, and hence, $Im(f_{k+1})$ is also projective. We can repeat this reasoning and we get that $Im(f_{n-1})$ is projective, false. Thus, the smallest r such that $Im(f_r)$ is projective is $r = n$, and by Theorem 2.4.2, $\mathrm{pd}(M) = n$. □

Remark 2.4.6. The results above will be used in Chapter 7 for constructing algorithms for computing the projective dimension of modules over bijective skew PBW extensions, and also for computing minimal presentations and testing stably-freeness.

CHAPTER 3

Hermite rings

Rings for which all stably free modules are free have occupied special attention in homological algebra. In this chapter, we will consider matrix-constructive interpretation of such rings and some other classes closely related. We will study also some classical algebraic constructions as quotients, products and rings of fractions of these rings. The material presented here can be considered as preparatory for the next chapter where we will study the Hermite condition for skew PBW extensions. Recall that all rings considered are $\mathcal{RC}$ (see Remark 2.1.9).

3.1 Matrix descriptions of Hermite rings

Definition 3.1.1. *Let S be a ring.*

(i) *S is a PF ring if every f.g. projective S-module is free.*

(ii) *S is a PSF ring if every f.g. projective S-module is stably free.*

(iii) *S is a Hermite ring, property denoted by H, if any stably free S-module is free.*

The right versions of the above rings (i.e., for right modules) are defined in a similar way and denoted by PF_r, PSF_r and H_r, respectively. We say that S is a $\mathcal{PF}$ ring if S is PF and PF_r simultaneously; similarly, we define the properties $\mathcal{PSF}$ and $\mathcal{H}$. However, we will prove below later that these properties are left-right symmetric, i.e., they can be denoted simply by $\mathcal{PF}$, $\mathcal{PSF}$ and $\mathcal{H}$. For domains we will write $\mathcal{PFD}$, $\mathcal{PSFD}$ and $\mathcal{HD}$.

From Definition 3.1.1 we get that

$$H \cap PSF = PF. \tag{3.1.1}$$

The following theorem gives a matrix description of H rings (see [26] and compare with [78] for the particular case of commutative rings. In [20] is presented a different and independent proof of this theorem for right modules).

Theorem 3.1.2. *Let S be a ring. Then, the following conditions are equivalent.*

(i) *S is H.*

(ii) *For every $r \geq 1$, any unimodular row matrix $\boldsymbol{u}$ over S of size $1 \times r$ can be completed to an invertible matrix of $GL_r(S)$ adding $r-1$ new rows.*

(iii) *For every $r \geq 1$, if $\boldsymbol{u}$ is an unimodular row matrix of size $1 \times r$, then there exists a matrix $U \in GL_r(S)$ such that $\boldsymbol{u}U = (1, 0, \ldots, 0)$.*

(iv) *For every $r \geq 1$, given an unimodular matrix F of size $s \times r$, $r \geq s$, there exists $U \in GL_r(S)$ such that*

$$FU = \begin{bmatrix} I_s & \mid & 0 \end{bmatrix}.$$

Proof. (i) $\Rightarrow$ (ii): Let $\boldsymbol{u} := [u_1 \cdots u_r]$ and $\boldsymbol{v} := [v_1 \cdots v_r]^T$ such that $\boldsymbol{uv} = 1$, i.e., $u_1v_1 + \cdots + u_rv_r = 1$; we define

$$S^r \xrightarrow{\alpha} S$$
$$\boldsymbol{e}_i \mapsto v_i$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ is the canonical basis of the left free module S^r of columns vectors. Observe that $\alpha(\boldsymbol{u}^T) = 1$; we define the homomorphism $\beta : S \to S^r$ by $\beta(1) := \boldsymbol{u}^T$, then $\alpha\beta = i_S$. From this we get that $S^r = Im(\beta) \oplus \ker(\alpha)$, β is injective, $\langle \boldsymbol{u}^T \rangle = Im(\beta) \cong S$ and $Im(\beta)$ is free with basis $\{\boldsymbol{u}^T\}$. This implies that $S^r \cong S \oplus \ker(\alpha)$, i.e., $\ker(\alpha)$ is stably free of rank $r-1$, so by hypothesis, $\ker(\alpha)$ is free of dimension $r-1$; let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r-1}\}$ be a basis of $\ker(\alpha)$, then $\{\boldsymbol{u}^T, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r-1}\}$ is a basis of S^r. This means that $\begin{bmatrix} \boldsymbol{u}^T & \boldsymbol{x}_1 \cdots \boldsymbol{x}_{r-1} \end{bmatrix}^T \in GL_r(S)$, i.e., $\boldsymbol{u}$ can be completed to an invertible matrix of $GL_r(S)$ adding $r-1$ rows.

(ii) $\Rightarrow$) (i): Let M be a stably free S-module, then there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$. It is enough to prove that M is free for the case when $s = 1$. In fact, $S^r \cong S^s \oplus M = S \oplus (S^{s-1} \oplus M)$ is free and hence $S^{s-1} \oplus M$ is free; repeating this reasoning we conclude that $S \oplus M$ is free, so M is free.

Let $r \geq 1$ such that $S^r \cong S \oplus M$, let $\pi : S^r \longrightarrow S$ be the canonical projection with kernel isomorphic to M and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ be the canonical basis of S^r; there exists $\mu : S \longrightarrow S^r$ such that $\pi\mu = i_S$ and $S^r = \ker(\pi) \oplus Im(\mu)$. Let $\mu(1) := \boldsymbol{u}^T := [u_1 \cdots u_r]^T \in S^r$, then $\pi(\boldsymbol{u}^T) = 1 = u_1\pi(\boldsymbol{e}_1) + \cdots + u_r\pi(\boldsymbol{e}_r)$, i.e., $\boldsymbol{v} := [\pi(\boldsymbol{e}_1) \cdots \pi(\boldsymbol{e}_r)]^T$ is such that $\boldsymbol{uv} = 1$, moreover, $S^r = \ker(\pi) \oplus \langle \boldsymbol{u}^T \rangle$. By hypothesis, there exists $U \in GL_r(S)$ such that $\boldsymbol{e}_1^T U = \boldsymbol{u}$.

Let $f^T : S^r \longrightarrow S^r$ be the homomorphism defined by U^T, then $f^T(\boldsymbol{e}_1) = \boldsymbol{u}^T$ and $f^T(\boldsymbol{e}_i) = \boldsymbol{u}_i$ for $i \geq 2$, where $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_r$ are the others columns of U^T (i.e., the transpose of the other rows of U). Since $U = (U^T)^T$ then f^T is an isomorphism. If we prove that $f^T(\boldsymbol{e}_i) \in \ker(\pi)$ for each $i \geq 2$, then $\ker(\pi)$ is free, and consequently, M is free. In fact, let f' be the restriction of f^T to $\langle \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r \rangle$, i.e., $f' : \langle \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r \rangle \longrightarrow \ker(\pi)$. Then f' is bijective: of course f' is injective; let $\boldsymbol{w}$ be any vector of S^r, then there exists $\boldsymbol{x} \in S^r$ such that $f^T(\boldsymbol{x}) = \boldsymbol{w}$, we write $\boldsymbol{x} := [x_1 \cdots x_r]^T = x_1\boldsymbol{e}_1 + \boldsymbol{z}$, with $\boldsymbol{z} = x_2\boldsymbol{e}_2 + \cdots + x_r\boldsymbol{e}_r$. We have $f^T(\boldsymbol{x}) = f^T(x_1\boldsymbol{e}_1 + \boldsymbol{z}) = x_1 f^T(\boldsymbol{e}_1) + f^T(\boldsymbol{z}) = x_1\boldsymbol{u}^T + f^T(\boldsymbol{z}) = \boldsymbol{w}$. In particular, if $\boldsymbol{w} \in \ker(\pi)$, then $\boldsymbol{w} - f^T(\boldsymbol{z}) \in \ker(\pi) \cap \langle \boldsymbol{u}^T \rangle = 0$, so $\boldsymbol{w} = f^T(\boldsymbol{z})$ and hence $\boldsymbol{w} = f'(\boldsymbol{z})$, i.e., f' is surjective.

In order to conclude the proof, we will show that $f^T(\boldsymbol{e}_i) \in \ker(\pi)$ for each $i \geq 2$. Since f^T was defined by U^T, the idea is to change U^T in a such way that its first column was

$\boldsymbol{u}^T$ and for the others columns were $\boldsymbol{u}_i \in \ker(\pi)$, $2 \leq i \leq r$. Let $\pi(\boldsymbol{u}_i) := r_i \in S$, $i \geq 2$, and $\boldsymbol{u}_i' := \boldsymbol{u}_i - r_i\boldsymbol{u}^T$; then adding to column i of U^T the first column multiplied by $-r_i$ we get a new matrix U^T such that its first column is again $\boldsymbol{u}^T$ and for the others we have $\pi(\boldsymbol{u}_i') = \pi(\boldsymbol{u}_i) - r_i\pi(\boldsymbol{u}^T) = r_i - r_i = 0$, i.e., $\boldsymbol{u}_i' \in \ker(\pi)$.

(ii) $\Leftrightarrow$ (iii): $\boldsymbol{u}$ can be completed to an invertible matrix of $GL_r(S)$ if and only if there exists $V \in GL_r(S)$ such that $(1, 0, \ldots, 0)V = \boldsymbol{u}$ if and only if $(1, 0, \ldots, 0) = \boldsymbol{u}V^{-1}$; thus $U := V^{-1}$.

(iii) $\Rightarrow$) (iv): The proof will be done by induction on s. For $s = 1$ the result is trivial. We assume that (iv) is true for unimodular matrices with $l \leq s - 1$ rows. Let F be an unimodular matrix of size $s \times r, r \geq s$, then there exists a matrix B such that $FB = I_s$. This implies that the first row $\boldsymbol{u}$ of F is unimodular; by (iii) there exists $U' \in GL_r(S)$ such that $\boldsymbol{u}U' = (1, 0, \ldots, 0) = \boldsymbol{e}_1^T$, and hence $FU' = F''$,

$$F'' = \begin{bmatrix} \boldsymbol{e}_1^T \\ F' \end{bmatrix},$$

with F' a matrix of size $(s-1) \times r$. Since $FB = I_s$, then $I_s = F''(U'^{-1}B)$, i.e., F'' is an unimodular matrix; let F''' be the matrix eliminating the first column of F', then F''' is unimodular of size $(s-1) \times (r-1)$, with $r - 1 \geq s - 1$, since the right inverse of F'' has the form $\begin{bmatrix} * & 0 \\ * & G''' \end{bmatrix}$. By induction, there exists a matrix $C \in GL_{r-1}(S)$ such that $F'''C = \begin{bmatrix} I_{s-1} & | & 0 \end{bmatrix}$. From this we get,

$$FU' = F'' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a'_{11} & a'_{12} & \cdots & a'_{1r} \\ \vdots & \vdots & & \vdots \\ a'_{s-11} & a'_{s-12} & \cdots & a'_{s-1r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix},$$

and hence

$$FU' \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & I_{s-1} & 0 \end{bmatrix}.$$

Multiplying the last matrix on the right by elementary matrices we get (iv).

(iv) $\Rightarrow$) (iii): Taking $s = 1$ and $F = \boldsymbol{u}$ in (iv) we get (iii). $\square$

From the proof of the previous theorem we get the following result.

Corollary 3.1.3. *Let S be a ring. Then, S is H if and only if any stably free S-module M of type $S^r \cong S \oplus M$ is free.*

Remark 3.1.4. (a) If we consider right modules and the right S-module structure on the module S^r of columns vectors, the conditions of the previous theorem can be formulated in the following way:

(i)$^{\mathrm{r}}$ S is H_r.

(ii)$^{\mathrm{r}}$ For every $r \geq 1$, any unimodular column matrix $\boldsymbol{v}$ over S of size $r \times 1$ can be completed to an invertible matrix of $GL_r(S)$ adding $r-1$ new columns.

(iii)$^{\mathrm{r}}$ For every $r \geq 1$, given an unimodular column matrix $\boldsymbol{v}$ over S of size $r \times 1$ there exists a matrix $U \in GL_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$.

(iv)$^{\mathrm{r}}$ For every $r \geq 1$, given an unimodular matrix F of size $r \times s$, $r \geq s$, there exists $U \in GL_r(S)$ such that

$$UF = \begin{bmatrix} I_s \\ 0 \end{bmatrix}.$$

The proof is as in the commutative case, see [78]. Corollary 3.1.3 can be formulated in this case as follows: S is H_r if and only if any stably free right S-module M of type $S^r \cong S \oplus M$ is free.

(b) Considering again left modules and disposing the matrices of homomorphisms by rows and composing homomorphisms from the left to the right (see Remark 2.1.2), we can repeat the proof of Theorem 3.1.2 and obtain the equivalence of conditions (i)-(iv). With this notation we do not need to take transposes in the proof of Theorem 3.1.2.

(c) If S is a commutative ring, of course, left and right conditions are equivalent, see [78]. This follows from the fact that $(FG)^T = G^T F^T$ for any matrices $F \in M_{r\times s}(S), G \in M_{s\times r}(S)$. However, as we remarked before, the Hermite condition is left-right symmetric for general rings (Proposition 3.2.7). Another independent proof of this fact can be found in [20], Theorem 11.4.4.

3.2 Matrix characterization of PF rings

In [26] are given some matrix characterizations of projective-free rings. In this section, we present another matrix interpretation of this important class of rings. The main result presented here (Corollary 3.2.4) extends Theorem 6.2.2 in [78]. This result has been proved independently also in [20], Proposition 11.4.9. A matrix proof of a Kaplansky theorem about finitely generated projective modules over local rings is also included.

Theorem 3.2.1. *Let S be a Hermite ring and M a f.g. projective module given by the column module of a matrix $F \in M_s(S)$, with F^T idempotent. Then, M is free with $dim(M) = r$ if and only if there exists a matrix $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and*

$$(U^T)^{-1} F^T U^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T. \tag{3.2.1}$$

In such case, a basis of M is given by the last r rows of $(U^T)^{-1}$.

Proof. $\Rightarrow$): As in the proof of Proposition 2.1.1, let $f : S^s \to S^s$ be the homomorphism defined by F and $S^s = M \oplus M'$ with $Im(f) = M$ and $M' = \ker(f)$; by the hypothesis M es free with dimension r, so $r \leq s$ (recall that S is $\mathcal{RC}$). Let $h : M \to S^r$ an isomorphism and $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r\} \subset M$ such that $h(\boldsymbol{z}_i) = \boldsymbol{e}_i$, $1 \leq i \leq r$, then $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r\}$ is a basis of M.

Since S is an Hermite ring, M' is free, let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{s-r}\}$ be a basis of M' (recall that S is $\mathcal{IBN}$). Then $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{s-r}; \boldsymbol{z}_1, \ldots, \boldsymbol{z}_r\}$ is a basis for S^s. With this we define u in the following way:

$$u(\boldsymbol{w}_j) := \boldsymbol{e}_j, \text{ for } 1 \leq j \leq s-r,$$
$$u(\boldsymbol{z}_i) := \boldsymbol{e}_{s-r+i}, \text{ for } 1 \leq i \leq r.$$

Note that u is an isomorphism and we get the following commutative diagram

$$\begin{tikzcd} S^s \arrow[r, "f"] \arrow[d, "u"'] & S^s \arrow[d, "u"] \\ S^s \arrow[r, "t_0"'] & S^s \end{tikzcd}$$

where t is given by $t_0(\boldsymbol{e}_j) := \boldsymbol{0}$ if $1 \leq j \leq s-r$, and $t_0(\boldsymbol{e}_{s-r+i}) = \boldsymbol{e}_{s-r+i}$ if $1 \leq i \leq r$; thus, the matrix of t_0 in the canonical basis is

$$T_0 = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}.$$

Thus, $uf = t_0u$ and hence $F^TU^T = U^TT_0^T$. Note that $(U^T)^{-1}$ exists since u is an isomorphism, hence $(U^T)^{-1}F^TU^T = T_0^T$. From $u(\boldsymbol{z}_i) := \boldsymbol{e}_{s-r+i}$ we get that $(\boldsymbol{z}_i^TU^T)^T = \boldsymbol{e}_{s-r+i}$, so $\boldsymbol{z}_i^TU^T = \boldsymbol{e}_{s-r+i}^T$ and hence $\boldsymbol{z}_i = \boldsymbol{e}_{s-r+i}^T(U^T)^{-1}$, i.e., the basis of M coincides with the last r rows of $(U^T)^{-1}$.

$\Leftarrow$): Let f, u be the homomorphisms defined by F and U, then $m(uf) = m(t_0u)$, where t_0 is the homomorphism defined by T_0, this means that $uf = t_0u$, but by the hypothesis U^T is invertible, so u is an isomorphism; from this we conclude that $Im(f) \cong Im(t_0)$, i.e., $M = Im(f) \cong Im(t_0) = \langle T_0 \rangle \cong S^r$. Note that this part of the proof does not use that S is an Hermite ring. □

From the previous theorem we get the following matrix description of PF rings.

Corollary 3.2.2. *Let S be a ring. S is PF if and only if for each $s \geq 1$, given a matrix $F \in M_s(S)$, with F^T idempotent, there exists a matrix $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and*

$$(U^T)^{-1}F^TU^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T, \tag{3.2.2}$$

where $r = dim(\langle F \rangle)$, $0 \leq r \leq s$.

Proof. $\Rightarrow$): Let $F \in M_s(S)$, with F^T idempotent, and let M be the S-module generated by the columns of F. By Proposition 2.1.1, M is a f.g. projective module, and by the hypothesis, M is free. Since S is H, we can apply Theorem 3.2.1. If $r = \dim(M)$, then $r = dim(\langle F \rangle)$.

$\Leftarrow$): Let M be a finitely generated projective S-module, so there exists $s \geq 1$ such that $S^s = M \oplus M'$; let $S^s \xrightarrow{f} S^s$ be the canonical projection on M, so F^T is idempotent and, by the hypothesis, there exists $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and (3.2.2) holds. From the second part of the proof of Theorem 3.2.1 we get that M is free. □

Remark 3.2.3. (i) If we consider right modules instead of left modules, then the previous corollary can be reformulated in the following way: S is PF_r if and only if for each $s \geq 1$, given an idempotent matrix $F \in M_s(S)$, there exists a matrix $U \in GL_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{3.2.3}$$

where $r = dim(\langle F \rangle)$, $0 \leq r \leq s$, and $\langle F \rangle$ represents the right S-module generated by the columns of F. The proof is as in the commutative case, see [78].

(ii) Considering again left modules and disposing the matrices of homomorphisms by rows and composing homomorphisms from the left to the right (see Remark 2.1.2), we can repeat the proofs of Theorem 3.2.1 and Corollary 3.2.2 and get the characterization (3.2.3) for the PF property; with this row notation we do not need to take transposes in the proofs. However, observe that in this case $\langle F \rangle$ represents the left S-module generated by the rows of F. Note that Corollary 3.2.2 could have been formulated this way: In fact,

$$\begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$

and we can rewrite (3.2.2) as (3.2.3) changing F^T by F (see Remark 2.1.2) and $(U^T)^{-1}$ by U.

(iii) If S is a commutative ring, of course $PF = PF_r = \mathcal{PF}$. However, we will prove in Corollary 3.2.5 that the projective-free property is left-right symmetric for general rings.

Corollary 3.2.4. *S is PF if and only if for each $s \geq 1$, given an idempotent matrix $F \in M_s(S)$, there exists a matrix $U \in GL_s(S)$ such that*

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{3.2.4}$$

where $r = dim(\langle F \rangle)$, $0 \leq r \leq s$, and $\langle F \rangle$ represents the left S-module generated by the rows of F.

Proof. This is the content of the part (ii) in the previous remark. □

Corollary 3.2.5. *Let S be a ring. S is PF if and only if S is PF_r, i.e., $PF = PF_r = \mathcal{PF}$.*

Proof. Let $F \in M_s(S)$ be an idempotent matrix. If S is PF, then there exists $P \in GL_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

where r is the dimension of the left S-module generated by the rows of F. Observe that UFU^{-1} is also idempotent, moreover, the matrices $X := UF$ and $Y := U^{-1}$ satisfy $UFU^{-1} = XY$ and $F = YX$, then from Proposition 0.3.1 in [26] we conclude that the left S-module generated by the rows of UFU^{-1} coincides with the left S-module generated by the rows of F, and also, the right S-module generated by the columns of UFU^{-1} coincides with the right S-module generated by the columns of F. This implies that the

S-module generated by the rows of F coincides with the right S-module generated by the columns of F. This means that S is PF_r. The symmetry of the problem completes the proof. □

Another interesting matrix characterization of $\mathcal{PF}$ rings is given in [26], Proposition 0.4.7: a ring S is $\mathcal{PF}$ if and only if given an idempotent matrix $F \in M_s(S)$ there exist matrices $X \in M_{s\times r}(S), Y \in M_{r\times s}(S)$ such that $F = XY$ and $YX = I_r$. A similar matrix interpretation can be given for PSF rings using Proposition 0.3.1 in [26] and Corollary 2.2.5.

Proposition 3.2.6. *Let S be a ring. Then,*

(i) *S is PSF if and only if given an idempotent matrix $F \in M_r(S)$ there exist $s \geq 0$ and matrices $X \in M_{(r+s)\times r}(S), Y \in M_{r\times(r+s)}(S)$ such that*

$$\begin{bmatrix} F & 0 \\ 0 & I_s \end{bmatrix} = XY \text{ and } YX = I_r.$$

(ii) $PSF = PSF_r = \mathcal{PSF}$.

Proof. Direct consequence of Proposition 0.3.1 in [26] and Corollary 2.2.5. □

For the H property we have a similar characterization that proves the symmetry of this condition.

Proposition 3.2.7. *Let S be a ring. Then,*

(i) *S is H if and only if given an idempotent matrix $F \in M_r(S)$ with factorization*

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} = XY \text{ and } YX = I_r, \text{for some matrices } X \in M_{(r+1)\times r}(S), Y \in M_{r\times(r+1)}(S),$$

there exist matrices $X' \in M_{r\times(r-1)}(S), Y' \in M_{(r-1)\times r}(S)$ such that $F = X'Y'$ and $Y'X' = I_{r-1}$.

(ii) $H = H_r = \mathcal{H}$.

Proof. Direct consequence of Propositions 0.3.1 and 0.4.7 in [26], and Corollary 3.1.3. □

Remark 3.2.8. By Theorem 3.1.2, S is H if and only if given $\boldsymbol{u} \in Um_r(n, S)$ there exist $U \in GL_n(S)$ such $\boldsymbol{u}U = (1, 0, \ldots, 0)$. This last implies that $GL_n(S)$ acts transitively on $Um_r(n, S)$, which is equivalent to say that $GL_n(S)$ acts transitively on $Um_c(n, S)$ (see Lemma 11.1.13 in [95]). Therefore, given $\boldsymbol{v} \in Um_r(n, S)$ there exist $V \in GL_n(S)$ such $Vv = \boldsymbol{e}_1$; i.e., S is H_r. Hence, we have obtained an alternative proof of Proposition 3.2.7.

We conclude this section given a matrix constructive proof of a well known Kaplansky's theorem.

Proposition 3.2.9. *Any local ring S is $\mathcal{PF}$.*

Proof. Let M a projective left S-module. By Remark 2.1.2, part (ii), there exists an idempotent matrix $F = [f_{ij}] \in M_s(S)$ such that the module generated by the rows of F coincides with M. According to Corollary 3.2.4, we need to show that there exists $U \in GL_s(S)$ such that the relation (3.2.4) holds. The proof is by induction on s.

$s = 1$: In this case $F = [f_{ij}] = [f]$; since S is local, its idempotents are trivial, then $f = 1$ or $f = 0$ and hence M is free.
$s = 2$: In view of fact that S is local, two possibilities may arise:

f_{11} is invertible. Then, one can find $G \in GL_2(S)$ such that $GFG^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}$, for some $f \in S$. For this it is enough to take $G = \begin{bmatrix} 1 & f_{11}^{-1}f_{12} \\ -f_{21}f_{11}^{-1} & 1 \end{bmatrix}$; to show that this matrix is invertible with inverse $G^{-1} = \begin{bmatrix} f_{11} & -f_{12} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12} + 1 \end{bmatrix}$ we can use the relations that exist between the entries of F. See for example that $GG^{-1} = I_2$:

$$f_{11} + f_{11}^{-1}f_{12}f_{21} = 1 \text{ because } f_{11}^2 + f_{12}f_{21} = f_{11} \text{ and } f_{11} \text{ is invertible;}$$

$$-f_{12} - f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{12} + f_{11}^{-1}f_{12} = -f_{12} + (1 - f_{11}^{-1}f_{12}f_{21})f_{11}^{-1}f_{12}$$

$$= -f_{12} + f_{11}f_{11}^{-1}f_{12} = 0;$$

$$-f_{21}f_{11}^{-1}f_{11} + f_{21} = 0;$$

$$f_{21}f_{11}^{-1}f_{12} - f_{21}f_{11}^{-1}f_{12} + 1 = 1.$$

Similar calculations show that $G^{-1}G = I_2$. Since F is idempotent, f so is; applying the case $s = 1$ we get the result.

$1 - f_{11}$ is invertible. In the same way, we can find $H \in GL_2(S)$ such that $HFH^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}$; for this it is enough to take $H = \begin{bmatrix} 1 & -(1 - f_{11})^{-1}f_{12} \\ f_{21} & -f_{21}(1 - f_{11})^{-1}f_{12} + 1 \end{bmatrix}$; note that $H^{-1} = \begin{bmatrix} 1 - f_{11} & (1 - f_{11})^{-1}f_{12} \\ -f_{21} & 1 \end{bmatrix}$. Indeed $HH^{-1} = I_2$:

$$1 - f_{11} + (1 - f_{11})^{-1}f_{12}f_{21} = 1 - f_{11} + f_{11} = 1 \text{ because } f_{12}f_{21} = (1 - f_{11})f_{11};$$

$$(1 - f_{11})^{-1}f_{12} - (1 - f_{11})^{-1}f_{12} = 0;$$

$$f_{21}(1 - f_{11}) + f_{21}(1 - f_{11})^{-1}f_{12}f_{21} - f_{21} = f_{21}(1 - f_{11}) + f_{21}f_{11} - f_{21} = 0;$$

$$f_{21}(1 - f_{11})^{-1}f_{12} - f_{21}(1 - f_{11})^{-1}f_{12} + 1 = 1.$$

An analogous calculation shows that $H^{-1}H = I_2$. Note that g is an idempotent of S, then $g = 0$ or $g = 1$ and the statement follows.

Now suppose that the result holds for $s - 1$; considering both possibilities for f_{11} we have:

If f_{11} is invertible, taking

$$G=\begin{bmatrix} 1 & f_{11}^{-1}f_{12} & f_{11}^{-1}f_{13} & \cdots & f_{11}^{-1}f_{1s} \\ -f_{21}f_{11}^{-1} & 1 & 0 & \cdots & 0 \\ -f_{31}f_{11}^{-1} & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \\ -f_{s1}f_{11}^{-1} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

we have that $G \in GL_s(S)$ and its inverse is:

$$G^{-1}=\begin{bmatrix} f_{11} & -f_{12} & -f_{13} & \cdots & -f_{1s} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12}+1 & -f_{21}f_{11}^{-1}f_{13} & \cdots & -f_{21}f_{11}^{-1}f_{1s} \\ f_{31} & -f_{31}f_{11}^{-1}f_{12} & -f_{31}f_{11}^{-1}f_{13}+1 & \cdots & -f_{31}f_{11}^{-1}f_{1s} \\ \vdots & & & \cdots & \\ f_{s1} & -f_{s1}f_{11}^{-1}f_{12} & -f_{s1}f_{11}^{-1}f_{13} & \cdots & -f_{s1}f_{11}^{-1}f_{1s}+1 \end{bmatrix}.$$

In fact, see that $GG^{-1}=I_s$:

$f_{11}+f_{11}^{-1}f_{12}f_{21}+\cdots+f_{11}^{-1}f_{1s}f_{s1}=1$ because $f_{11}^2+f_{12}f_{21}+\cdots+f_{1s}f_{s1}=f_{11}$;

$-f_{12}-f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{12}+f_{11}^{-1}f_{12}-f_{11}^{-1}f_{13}f_{31}f_{11}^{-1}f_{12}-\cdots-f_{11}^{-1}f_{1s}f_{s1}f_{11}^{-1}f_{12}=-f_{12}+(1-f_{11}^{-1}\sum_{i=2}^{s}f_{1i}f_{i1})f_{11}^{-1}f_{12}=-f_{12}+f_{11}f_{11}^{-1}f_{12}=0$;

$\vdots$

$-f_{1s}-f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{1s}-f_{11}^{-1}f_{13}f_{31}f_{11}^{-1}f_{1s}-\cdots-f_{11}^{-1}f_{1s}f_{s1}f_{11}^{-1}f_{1s}+f_{11}^{-1}f_{1s}=-f_{1s}+(1-f_{11}^{-1}\sum_{i=2}^{s}f_{1i}f_{i1})f_{11}^{-1}f_{1s}=-f_{1s}+f_{11}f_{11}^{-1}f_{1s}=0$;

$-f_{21}f_{11}^{-1}f_{11}+f_{21}=0$; $f_{21}f_{11}^{-1}f_{12}-f_{21}f_{11}^{-1}f_{12}+1=1$; $f_{21}f_{11}^{-1}f_{1i}-f_{21}f_{11}^{-1}f_{1i}=0$ for every $3\leq i\leq s$;

$\vdots$

$-f_{s1}f_{11}^{-1}f_{11}+f_{s1}=0$; $f_{s1}f_{11}^{-1}f_{1i}-f_{s1}f_{11}^{-1}f_{1i}=0$ for every $2\leq i\leq s-1$ and, finally, $f_{s1}f_{11}^{-1}f_{1s}-f_{s1}f_{11}^{-1}f_{1s}+1=1$.

Similarly, $G^{-1}G=I_s$. Moreover, $GFG^{-1}=\begin{bmatrix} 1 & 0_{1,s-1} \\ 0_{s-1,1} & F_1 \end{bmatrix}$ where $F_1\in M_{s-1}(S)$ is an idempotent matrix. Only remains to apply the induction hypothesis.

If $1-f_{11}$ is invertible, taking

$$H=\begin{bmatrix} 1 & -(1-f_{11})^{-1}f_{12} & -(1-f_{11})^{-1}f_{13} & \cdots & -(1-f_{11})^{-1}f_{1s} \\ f_{21} & -f_{21}(1-f_{11})^{-1}f_{12}+1 & -f_{21}(1-f_{11})^{-1}f_{13} & \cdots & -f_{21}(1-f_{11})^{-1}f_{1s} \\ f_{31} & -f_{31}(1-f_{11})^{-1}f_{12} & -f_{31}(1-f_{11})^{-1}f_{13}+1 & \cdots & -f_{31}(1-f_{11})^{-1}f_{1s} \\ \vdots & & & \cdots & \\ f_{s1} & -f_{s1}(1-f_{11})^{-1}f_{12} & -f_{s1}(1-f_{11})^{-1}f_{13} & \cdots & -f_{s1}(1-f_{11})^{-1}f_{1s}+1 \end{bmatrix}$$

we have that $H \in GL_s(S)$ with inverse given by:

$$H^{-1} = \begin{bmatrix} 1-f_{11} & (1-f_{11})^{-1}f_{12} & (1-f_{11})^{-1}f_{13} & \cdots & (1-f_{11})^{-1}f_{1s} \\ -f_{21} & 1 & 0 & \cdots & 0 \\ -f_{31} & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \\ -f_{s1} & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In fact, note that $HH^{-1} = I_s$:

$1-f_{11}+(1-f_{11})^{-1}\sum_{i=2}^{s} f_{1i}f_{i1} = 1-f_{11}+f_{11} = 1$ because $\sum_{i=2}^{s} f_{1i}f_{i1} = (1-f_{11})f_{11}$ and $(1-f_{11})$ is invertible; also $(1-f_{11})^{-1}f_{1i} - (1-f_{11})^{-1}f_{1i}$ for $2 \le i \le s$;

$f_{21}(1-f_{11}) + f_{21}\sum_{i=1}^{s}(1-f_{11})^{-1}f_{1i}f_{i1} - f_{21} = -f_{21}f_{11} + f_{21}f_{11} = 0$; $f_{21}(1-f_{11})^{-1}f_{12} - f_{21}(1-f_{11})^{-1}f_{12} + 1 = 1$; and $f_{21}(1-f_{11})^{-1}f_{1i} - f_{21}(1-f_{11})^{-1}f_{1i} = 0$ for $3 \le i \le s$.

$\vdots$

$f_{s1}(1-f_{11})+f_{s1}\sum_{i=1}^{s}(1-f_{11})^{-1}f_{1i}f_{i1}-f_{s1} = -f_{s1}f_{11}+f_{21}f_{11} = 0$; $f_{s1}(1-f_{11})^{-1}f_{1i} - f_{s1}(1-f_{11})^{-1}f_{1i} = 0$ for $3 \le i \le s-1$ and, finally, $f_{s1}(1-f_{11})^{-1}f_{1s} - f_{s1}(1-f_{11})^{-1}f_{1s} + 1 = 1$.

Similarly, we can to show that $H^{-1}H = I_s$. Furthermore, we have also $HFH^{-1} = \begin{bmatrix} 0 & 0_{1,s-1} \\ 0_{s-1,1} & F_2 \end{bmatrix}$ with $F_2 \in M_{s-1}(S)$ an idempotent matrix. One more time we apply the induction hypothesis. □

3.3 Some important subclasses of Hermite rings

There are some other classes of rings closely related to Hermite rings that we will recall next (see [26], [60], [62] and [125]).

Definition 3.3.1. *Let S be a ring.*

(i) *S is an elementary divisor ring ($\mathcal{ED}$) if for any $r, s \ge 1$, given a rectangular matrix $F \in M_{r\times s}(S)$ there exist invertible matrices $P \in GL_r(S)$ and $Q \in GL_s(S)$ such that PFQ is a Smith normal diagonal matrix, i.e., there exist $d_1, d_2, \ldots, d_l \in S$, with $l = \min\{r, s\}$, such that*

$$PFQ = \operatorname{diag}(d_1, d_2, \ldots, d_l), \text{ with } Sd_{i+1}S \subseteq Sd_i \cap d_iS \text{ for } 1 \le i \le l,$$

where SdS denotes the two-sided ideal generated by d.

(ii) *S is an $\mathcal{ID}$ ring if for any $s \ge 1$, given an idempotent matrix $F \in M_s(S)$ there exists an invertible matrix $P \in GL_s(S)$ such that PFP^{-1} is a Smith normal diagonal matrix.*

(iii) *S is a left K-Hermite ring (KH) if given $a, b \in S$ there exist $U \in GL_2(S)$ and $d \in S$ such that $U \begin{bmatrix} a & b \end{bmatrix}^T = \begin{bmatrix} d & 0 \end{bmatrix}^T$. S is a right K-Hermite ring (KH_r) if $\begin{bmatrix} a & b \end{bmatrix} U = \begin{bmatrix} d & 0 \end{bmatrix}$. The ring S is $\mathcal{KH}$ if S is KH and KH_r.*

(iv) *S is a left Bézout ring (B) if every f.g. left ideal of S is principal. S is a right Bézout ring (B_r) if every f.g. right ideal of S is principal. S is a $\mathcal{B}$ ring if S is B and B_r.*

(v) *S is a left cancellable ring (C) if for any f.g. projective left S-modules P, P' holds: $P \oplus S \cong P' \oplus S \Leftrightarrow P \cong P'$. S is right cancellable (C_r) if for any f.g. projective right S-modules P, P' holds: $P \oplus S \cong P' \oplus S \Leftrightarrow P \cong P'$. S is cancellable ($\mathcal{C}$) if S is (C) and (C_r).*

From Proposition 0.3.1 of [26] it is easy to give a matrix interpretation of C rings, and also, we can deduce that $C = C_r = \mathcal{C}$.

Proposition 3.3.2. *Let S be a ring. Then,*

(i) *S is C if and only if given idempotent matrices $F \in M_s(S)$, $G \in M_r(S)$ the following statement is true: The matrices*

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized as

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} = X'Y', \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} = Y'X', \text{for some matrices } X' \in M_{(s+1)\times(r+1)}(S), \\ Y' \in M_{(r+1)\times(s+1)}(S)$$

if and only if $F = XY$, $G = YX$, for some matrices $X \in M_s(S)$, $Y \in M_r(S)$.

(ii) *$C = C_r = \mathcal{C}$.*

Proof. Direct consequence of Proposition 0.3.1 in [26]. □

For domains, the above classes of rings are denoted by $\mathcal{EDD}$, $\mathcal{IDD}$, KHD, KHD_r, $\mathcal{KHD}$, BD, BD_r, $\mathcal{BD}$ and $\mathcal{CD}$, respectively.

Theorem 3.3.3. (i) *$\mathcal{ED} \subseteq KH \subseteq B$.*

(ii) *$KHD = BD \subseteq \mathcal{PFD}$.*

(iii) *$\mathcal{PF} \subseteq \mathcal{ID}$*

(iv) *$\mathcal{ID} = \mathcal{PF}$ for rings without nontrivial idempotents. Thus, $\mathcal{IDD} = \mathcal{PFD}$.*

(v) *$\mathcal{PF} \subseteq \mathcal{C} \subseteq \mathcal{H}$.*

Similar relations are valid for $KH_r, \mathcal{KH}, B_r$ and $\mathcal{B}$.

Proof. (i) It is clear that $\mathcal{ED} \subseteq KH$. Let $a, b \in S$, we want to proof that any left ideal $Sa + Sb$ is principal. There exist $U \in GL_2(S)$ and $d \in S$ such that $U \begin{bmatrix} a & b \end{bmatrix}^T = \begin{bmatrix} d & 0 \end{bmatrix}^T$, this implies that $Sd \subseteq Sa + Sb$, but since $\begin{bmatrix} a & b \end{bmatrix}^T = U^{-1} \begin{bmatrix} d & 0 \end{bmatrix}^T$, then $Sa + S \subseteq Sd$. This proved that $KH \subseteq B$.

(ii) $KHD = BD$ was proved by Amitsur in [3]. We include the proof by completeness.

In order to prove the inclusion $BD \subseteq \mathcal{PFD}$ we show first that if S is BD then each finitely generated left ideal of S is free: Let I be a left ideal of S, if $I = 0$, so I is free; let $I \neq 0$, then $I = Sa$, for some $a \neq 0$, but since S has no zero divisors, then I is free with basis $\{a\}$.

Next we will prove that each finitely generated submodule of a free S-module is free: Let M be a free S-module with basis X and let $N = Sz_1 + \cdots + Sz_t$ be a finitely generated submodule of M (if $M = 0$ or $N = 0$ there is nothing to prove). Each z_i defines a finite subset X_i of X, $1 \leq i \leq t$, so $N \subseteq \langle \cup_{i=1}^{t} X_i \}$, and hence, there exists a finite sequence $x_1, \ldots, x_n$ of elements of X such that $N \subseteq Sx_1 \oplus \cdots \oplus Sx_n$, i.e., N is a submodule of a free module with a basis of n elements, so we can complete the proof of freeness of N by induction: For $n = 1$ we have $N \subseteq Sx_1 \cong S$, so N is isomorphic to a finitely generated left ideal of S, hence N is free. Consider again that $N \subseteq Sx_1 \oplus \cdots \oplus Sx_n$ and we define the function $f : N \to S$ by $x = s_1x_1 + \cdots + s_nx_n \mapsto s_n$. Note that f is a homomorphism and $f(N)$ is a finitely generated left ideal of S, i.e., $f(N)$ is free. We have the exact sequence $0 \to N \cap (Sx_1 \oplus \cdots \oplus Sx_{n-1}) \to N \to f(N) \to 0$, but since $f(N)$ is projective, then this sequence splits, so $N \cong f(N) \oplus (N \cap (Sx_1 \oplus \cdots \oplus Sx_{n-1}))$. Note that $N \cap (Sx_1 \oplus \cdots \oplus Sx_{n-1})$ is a finitely generated submodule of a free module with a basis of $n-1$ elements, by induction $N \cap (Sx_1 \oplus \cdots \oplus Sx_{n-1})$ is free, and hence N is free. Now we are able to prove that S is $\mathcal{PF}$: Let M be a finitely generated projective S-module, then M is a finitely generated submodule (as a free summand) of a free module, hence M is free.

(iii) Using permutation matrices it is clear that $\mathcal{PF} \subseteq \mathcal{ID}$ (see Corollary 3.2.4).

(iv) Let S be an $\mathcal{ID}$ ring and let $F = [f_{ij}] \in M_s(S)$ be an idempotent matrix over S; by the hypothesis, there exists $P \in GL_s(S)$ such that PFP^{-1} is diagonal, let $D := PFP^{-1} = \operatorname{diag}(d_1, d_2, \ldots, d_s)$; since PFP^{-1} is idempotent, then each d_i is idempotent, so $d_i = 0$ or $d_i = 1$ for each $1 \leq i \leq s$. By permutation matrices we can assume that

$$PFP^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

in addition, note that r is the dimension of the left S-module generated by the rows of F. Then, S is $\mathcal{PF}$.

(v) Let P, P' be f.g. S-modules such that $P \oplus S \cong P' \oplus S$; since S is $\mathcal{PF}$ there exists n, n' such that $P \cong S^n$, $P' \cong S^{n'}$ and hence $S^n \oplus S \cong S^{n'} \oplus S$, so $n + 1 = n' + 1$, i.e., $P \cong P'$.

Let now M be a stably free module, $M \oplus S^s \cong S^r$, since $r \geq s$ and S is left cancellable, then $M \cong S^{r-s}$. $\square$

From Theorem 3.3.3 we conclude that for domains the following inclusions hold:

$$\mathcal{EDD} \subseteq KHD = BD \subseteq \mathcal{PFD} = \mathcal{IDD} \subseteq \mathcal{CD} \subseteq \mathcal{HD}. \tag{3.3.1}$$

Similar relations are valid for the right side.

The next proposition gives an alternative characterization of KH rings and will be used to prove that $\mathcal{KH} \subseteq \mathcal{H}$ for commutative rings.

Proposition 3.3.4. *Let S be a ring. S is KH if and only if for every $r \geq 2$, given elements $b_1, \ldots, b_r \in S$, there exists $U \in Gl_r(S)$ and $d \in S$ such that $U \begin{bmatrix} b_1 & \cdots & b_r \end{bmatrix}^T = \begin{bmatrix} d & \cdots & 0 \end{bmatrix}^T$. Similar characterization holds for KH_r rings.*

Proof. $\Rightarrow$): By induction over r. The case $r = 2$ is direct consequence from the definition. Suppose that the result holds for any row of size $< r$ and let $U_0 \in GL_2(S)$ such that $U_0 \begin{bmatrix} b_{r-1} & b_r \end{bmatrix}^T = \begin{bmatrix} d' & 0 \end{bmatrix}^T$, for some $d' \in S$. We have $U_1 \begin{bmatrix} b_1 & \cdots & b_{r-2} & b_{r-1} & b_r \end{bmatrix}^T = \begin{bmatrix} b_1 & \cdots & b_{r-2} & d' & 0 \end{bmatrix}^T$, with $U_1 := \begin{bmatrix} I_{r-2} & 0 \\ 0 & U_0 \end{bmatrix} \in GL_r(S)$. Applying the induction hypothesis to $b_1, \ldots, b_{r-2}, d'$ we find $U_2 \in GL_{r-1}(S)$ such that $U_2 \begin{bmatrix} b_1 & \cdots & b_{r-2} & d' \end{bmatrix}^T = \begin{bmatrix} d & \cdots & 0 \end{bmatrix}^T$ for some $d \in S$. Let $U' := \begin{bmatrix} U_2 & 0 \\ 0 & 1 \end{bmatrix} \in GL_r(S)$, then $U := U'U_1 \in GL_r(S)$ satisfies $U \begin{bmatrix} b_1 & \cdots & b_r \end{bmatrix}^T = \begin{bmatrix} d & \cdots & 0 \end{bmatrix}^T$.

$\Leftarrow$): Trivial. □

Corollary 3.3.5. *For commutative rings, $\mathcal{KH} \subseteq \mathcal{H}$.*

Proof. Let S be a commutative $\mathcal{KH}$ ring and let $\boldsymbol{u} = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}^T$ be an unimodular column vector, by Proposition 3.3.4 there exists $U \in GL_r(S)$ such that $U\boldsymbol{u} = \begin{bmatrix} d & \cdots & 0 \end{bmatrix}^T$, for some $d \in S$. This implies that $Sd = Su_1 + \cdots + Su_r = S$, i.e., d is left invertible, and hence, invertible. From this we get that $d^{-1}U\boldsymbol{u} = \boldsymbol{e}_1$. □

The following characterization of $\mathcal{ID}$ rings for which all idempotents are central will be used below (see [90] and [78] for the particular case of commutative rings).

Proposition 3.3.6. *Let S be a ring such that all idempotents are central. Then the following conditions are equivalent*

(i) *S is $\mathcal{ID}$.*

(ii) *Any idempotent matrix over S is similar to a diagonal matrix.*

(iii) *Given an idempotent matrix $F \in M_r(S)$ there exists an unimodular vector $\boldsymbol{v} = [v_1, \ldots, v_r]^T$ over S and an invertible matrix $U \in GL_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$ and $F\boldsymbol{v} = a\boldsymbol{v}$, for some $a \in S$.*

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Let $F \in M_r(S)$ be idempotent, there exists $P \in GL_r(S)$ such that $PFP^{-1} = \operatorname{diag}(d_1, \ldots, d_r)$, note that each d_i is idempotent (see the proof of the part (iv) in Theorem 3.3.3); the canonical vector $\boldsymbol{e}_1$ is unimodular, moreover $PFP^{-1}\boldsymbol{e}_1 = d_1\boldsymbol{e}_1$. Let $\boldsymbol{v} := P^{-1}\boldsymbol{e}_1$, then $\boldsymbol{v}$ is unimodular, $F\boldsymbol{v} = d_1\boldsymbol{v}$ and $P\boldsymbol{v} = \boldsymbol{e}_1$. Thus, the result is valid with $U = P$ and $a = d_1$.

(iii) $\Rightarrow$ (ii): Let $F \in M_r(S)$ be idempotent, we will prove that there exists $Q \in GL_r(S)$ such that QFQ^{-1} is diagonal. The proof is by induction on r. For $r = 1$, if $f \in S$ with $f^2 = f$, then there exist $v, u \in S^*$ such that $uv = 1$ and $fv = av$, for some $a \in S$, hence $f = a$, i.e., $1f1^{-1} = a$.

Suppose that any idempotent matrix of size $< r$ is similar to a diagonal matrix. Let $F \in M_r(S)$ idempotent; if $F = 0$ there is nothing to prove. Let $F \neq 0$. By the hypothesis, there exist an unimodular vector $\boldsymbol{v} = [v_1, \ldots, v_r]^T$ over S and an invertible matrix $U \in GL_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$ and $F\boldsymbol{v} = d_1\boldsymbol{v}$, for some $d_1 \in S$. Then, F is similar to the matrix $\widetilde{F} := UFU^{-1}$, and $\widetilde{F}$ has the form

$$\widetilde{F} = \begin{bmatrix} d_1 & a_{12} & \cdots & a_{1r} \\ 0 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{r2} & \ldots & a_{rr} \end{bmatrix}.$$

In fact, $\widetilde{F}\boldsymbol{e}_1 = UFU^{-1}\boldsymbol{e}_1 = UF\boldsymbol{v} = Ud_1\boldsymbol{v} = d_1U\boldsymbol{v} = d_1\boldsymbol{e}_1$. But $\widetilde{F}$ is idempotent since F is idempotent, so $d_1^2 = d_1$ and the submatrix $H := [a_{ij}]$, with $2 \leq i, j \leq r$, is idempotent of size $(r-1) \times (r-1)$. By induction, there exists $Q' \in GL_{r-1}(S)$ and $d_2, d_3, \ldots, d_r \in S$ such that $Q'HQ'^{-1} = \operatorname{diag}(d_2, d_3, \ldots, d_r)$. From this we get that F is similar to the matrix $\widehat{F}$, where

$$\widehat{F} := \begin{bmatrix} 1 & 0 \\ 0 & Q' \end{bmatrix} \widetilde{F} \begin{bmatrix} 1 & 0 \\ 0 & Q'^{-1} \end{bmatrix} = \begin{bmatrix} d_1 & b_2 & b_3 & \cdots & b_r \\ 0 & d_2 & 0 & \ldots & 0 \\ 0 & 0 & d_3 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & d_r \end{bmatrix},$$

for some $b_2, \ldots, b_r \in S$. Since F is idempotent, then $\widehat{F}$ is idempotent, and hence, $d_i^2 = d_i$, for each $1 \leq i \leq r$, moreover, for each $2 \leq j \leq r$,

$$b_j(d_1 + d_j - 1) = 0. \tag{3.3.2}$$

Now we consider for a moment S^r as the right S-module of column vectors (see Remark 2.1.2 (i)); the idea is to make a change of basis of S^r and to prove that F is similar to the matrix $\operatorname{diag}(d_1 \ldots, d_r)$. For this we have to construct a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ of S^r such that $\widehat{F}\boldsymbol{u}_i = d_i\boldsymbol{u}_i$, $1 \leq i \leq r$. We consider the vectors $\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{u}_2 = (a_2, 1, 0, \ldots, 0)^T$, $\boldsymbol{u}_3 = (a_3, 0, 1, \ldots, 0)^T, \ldots, \boldsymbol{u}_r = (a_r, 0, 0, \ldots, 1)^T$, where $a_2, \ldots, a_r \in S$ must be defined. For $2 \leq j \leq r$, from condition $\widehat{F}\boldsymbol{u}_j = d_j\boldsymbol{u}_j$, the a_j's must satisfy

$$b_j = (d_j - d_1)a_j. \tag{3.3.3}$$

(3.3.2) implies that $b_j(d_1 - d_j + 2d_j - 1) = 0$, and hence $b_j(d_1 - d_j) = b_j(1 - 2d_j)$, but $(1 - 2d_j)^2 = 1$, so $b_j(d_1 - d_j)(1 - 2d_j) = b_j$, thus $a_j := b_j(2d_j - 1)$ satisfies (3.3.3). With this change of basis we get $H\widehat{F}H^{-1} = \operatorname{diag}(d_1 \ldots, d_r)$, where

$$H := \begin{bmatrix} 1 & -a_2 & -a_3 & \cdots & -a_r \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ with } a_j := b_j(2d_j - 1),\ 2 \leq j \leq r.$$

Thus, we have proved that F is similar to the matrix $\operatorname{diag}(d_1, d_2, \dots, d_r)$, i.e., there exists $P \in GL_r(S)$ such that $PFP^{-1} = \operatorname{diag}(d_1, d_2, \dots, d_r)$.

(ii) $\Rightarrow$ (i): Let $F \in GL_r(S)$ be an idempotent matrix. Then there exists $Q \in GL_r(S)$ such that $QFQ^{-1} = D := \operatorname{diag}(d_1, d_2, \dots, d_r)$; as we saw before, each d_i is idempotent. We will prove that there exists $P \in GL_r(S)$ such that PDP^{-1} is a diagonal Smith normal matrix. We divide this proof in some steps.

Step 1. We observe first that there exist idempotents $f_1, \dots, f_r \in S$ and $a \in S$ such that $\boldsymbol{f} = \begin{bmatrix} f_1 & \cdots & f_r \end{bmatrix}^T$ is unimodular and $af_i = d_i$, for $1 \leq i \leq r$. In fact, we define

$$\begin{aligned} a :=& d_1 + \cdots + d_r + \sum_{j=2}^{r} (-1)^{j+1} \Big(\prod_{i_1 < i_2 < \cdots < i_j} d_{i_1} \cdots d_{i_j} \Big), \\ f_i :=& 1 - a + d_i,\ 1 \leq i \leq r \end{aligned}$$

(for example, for $r = 3$, $a = d_1 + d_2 + d_3 - d_1d_2 - d_1d_3 - d_2d_3 + d_1d_2d_3$, $f_1 = 1 - d_2 - d_3 + d_1d_2 + d_1d_3 + d_2d_3 - d_1d_2d_3$, $f_2 = 1 - d_1 - d_3 + d_1d_2 + d_1d_3 + d_2d_3 - d_1d_2d_3$ and $f_3 = 1 - d_1 - d_2 + d_1d_2 + d_1d_3 + d_2d_3 - d_1d_2d_3$). By a direct computation can be proved that a is idempotent and $d_i = ad_i$, for $1 \leq i \leq r$. From this, $af_i = a(1 - a + d_i) = a - a^2 + ad_i = ad_i = d_i$; moreover, $f_i^2 = (1 - a + d_i)(1 - a + d_i) = 1 - a + d_i - a + a^2 - ad_i + d_i - ad_i + d_i^2 = 1 - a + d_i = f_i$. The proof of unimodularity of $\boldsymbol{f}$ can be done by direct computation, $1 = g_1f_1 + g_2f_2 + \cdots + g_rf_r$, with

$$\begin{aligned} g_i :=& d_i - \sum_{l=i+1}^{r} d_l + \sum_{j=2}^{r-2} (-1)^j \Big(\prod_{i < i_1 < i_2 < \cdots < i_j} d_{i_1} \cdots d_{i_j} \Big), \text{ for } 1 \leq i \leq r-1 \\ g_r :=& 1 + (-1)^{r-1} d_1 \cdots d_{r-1}. \end{aligned}$$

Step 2. Now we want to prove that there exists $U \in GL_r(S)$ such that $U\boldsymbol{f} = \boldsymbol{e}_1$. We consider the matrix $H := [h_{ij}] \in M_r(S)$, with $h_{ij} := f_ig_j$ central, $1 \leq i, j \leq r$ (remember that all idempotents are central). Note that $H^2 = H$; by the hypothesis there exists $V \in GL_r(S)$ such that VHV^{-1} is diagonal, let $D' := VHV^{-1} = \operatorname{diag}(b_1, b_2, \dots, b_r)$; since VHV^{-1} is idempotent, then each b_i is idempotent; moreover, since each h_{ij} is central, then $\operatorname{tr}(D') = \operatorname{tr}(H) = 1$ and hence $b_1 + \cdots + b_r = 1$. Let $\boldsymbol{w} := [b_1, \dots, b_r]^T$, then $\boldsymbol{w}$ is unimodular and $D'\boldsymbol{w} = \boldsymbol{w}$, additionally, $W\boldsymbol{e}_1 = \boldsymbol{w}$, where

$$W := \begin{bmatrix} b_1 & -1 & -1 & \dots & -1 \\ b_2 & 1 & 0 & \dots & 0 \\ b_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_r & 0 & 0 & \dots & 1 \end{bmatrix} \in GL_r(S).$$

Let $\boldsymbol{z} := [z_1 \cdots z_r]^T := V^{-1}\boldsymbol{w}$, then $\boldsymbol{z}$ is unimodular and $VH\boldsymbol{z} = VHV^{-1}\boldsymbol{w} = D'\boldsymbol{w} = \boldsymbol{w}$, so $H\boldsymbol{z} = \boldsymbol{z}$. Hence, $\sum_{j=1}^{r} f_i g_j z_j = z_i$, for each i, i.e., $(\sum_{j=1}^{r} g_j z_j)f_i = z_i$, thus $(\sum_{j=1}^{r} g_j z_j)[f_1 \cdots f_r]^T = [z_1 \cdots z_r]^T$. But since $f_1, \ldots, f_r$ are central and $\boldsymbol{z}$ is unimodular, then $c := \sum_{j=1}^{r} g_j z_j$ is left invertible and $c'c = 1$ for some $c' \in S$; observe that cc' is idempotent, so central, and by the hypothesis there exists $x \in S^*$ such that $xcc'x^{-1} = d$, with $d \in S$ idempotent, from this we get that $cc' = d$ and $c' = c'd$, i.e., $c'(1-d) = 0$, so $(1-d)c' = 0$ and consequently $1 - d = 0$, i.e, $cc' = 1$. This means that c is invertible. Note that $V^{-1}W\boldsymbol{e}_1 = \boldsymbol{z}$, so $c^{-1}V^{-1}W\boldsymbol{e}_1 = \boldsymbol{f}$. Taking $U := W^{-1}Vc$ we get the claimed.

Step 3. $D\boldsymbol{f} = \begin{bmatrix} d_1 f_1 & \cdots & d_r f_r \end{bmatrix}^T = \begin{bmatrix} a f_1^2 & \cdots & a f_r^2 \end{bmatrix}^T = a \begin{bmatrix} f_1 & \cdots & f_r \end{bmatrix}^T = a\boldsymbol{f}$. Thus, we have an idempotent matrix D, an unimodular vector $\boldsymbol{f}$, an invertible matrix U and an element $a \in S$ such that $D\boldsymbol{f} = a\boldsymbol{f}$ and $U\boldsymbol{f} = \boldsymbol{e}_1$. Then, as in the proof (iii) $\Rightarrow$ (ii), there exists $L \in GL_r(S)$ such that $LDL^{-1} = \operatorname{diag}(a, a'_2, \ldots, a'_r)$, and hence $T'FT'^{-1} = \operatorname{diag}(a, a'_2, \ldots, a'_r)$, with $T' := LQ \in GL_r(S)$. Since $\operatorname{diag}(a'_2, \ldots, a'_r)$ is idempotent, then by induction there exists $T \in GL_r(S)$ such that $T\operatorname{diag}(a, a'_2, \ldots, a'_r)T^{-1} = \operatorname{diag}(a, a_2, \ldots, a_r)$ is a diagonal Smith normal matrix. If $a = 0$ or $a_2 = 0$, we have finished. Let $a, a_2 \neq 0$, since $a, a_2, \ldots, a_r$ are central, we must prove that $Sa_2 \subseteq Sa$, i.e., a divides a_2. Since a divides each d_i, then a divides each entry of D, and hence, a divides each entry of LDL^{-1}, thus a divides each a'_j, $2 \leq j \leq r$. From this we get that a divides each entry of $T\operatorname{diag}(a, a'_2, \ldots, a'_r)T^{-1}$, so in particular, a divides a_2.

Hence, we can conclude that there exists a matrix $P \in GL_r(S)$ such that $PFP^{-1} = \operatorname{diag}(a, a_2, \ldots, a_r)$ is a Smith normal diagonal matrix. □

In (3.3.1) we saw that $\mathcal{IDD} \subseteq \mathcal{HD}$, moreover $\mathcal{ID} \subseteq \mathcal{H}$ for commutative rings (see [118], [90], and also [78]). These results can be extended using some ideas in the proof of the previous proposition, and also the following elementary fact.

Remark 3.3.7. If $\boldsymbol{u}$ is an unimodular row of size $1 \times r$ and $P \in GL_r(S)$, then $\boldsymbol{u}$ is completable to an invertible matrix if and only if $\boldsymbol{u}P$ is completable.

Proposition 3.3.8. *Let S be a ring such that all idempotents are central. Then, $\mathcal{ID} \subseteq \mathcal{H}$.*

Proof. Let $\boldsymbol{u} = [u_1 \cdots u_r]$ be an unimodular row matrix of size $1 \times r$, there exists $\boldsymbol{v} = [v_1 \cdots v_r]^T$ such that $u_1v_1 + \cdots + u_rv_r = 1$; we consider the matrix $F = [f_{ij}] \in M_r(S)$, with $f_{ij} := v_iu_j$, $1 \leq i, j \leq r$. Note that $F^2 = F$; by the hypothesis there exists $P \in GL_r(S)$ such that PFP^{-1} is diagonal, let $D := PFP^{-1} = \operatorname{diag}(d_1, d_2, \ldots, d_r)$; since PFP^{-1} is idempotent, then each d_i is idempotent. Let $\boldsymbol{w} := \boldsymbol{u}P^{-1}$ and $\boldsymbol{x} := P\boldsymbol{v}$, then $\boldsymbol{wx} = \boldsymbol{u}P^{-1}P\boldsymbol{v} = 1$ and $\boldsymbol{xw} = P\boldsymbol{vu}P^{-1} = PFP^{-1} = D$. By the above remark, $\boldsymbol{u}$ is completable if and only if $\boldsymbol{w}$ is. Thus, we will show that $\boldsymbol{w}$ is completable. From $\boldsymbol{xw} = D$ we obtain that $x_iw_i = d_i$ is idempotent for all $1 \leq i \leq r$ and $x_iw_j = 0$ for $i \neq j$. But $\sum_{k=1}^{r} w_ix_i = 1$, then $w_i = w_ix_iw_i$ and $x_i = x_iw_ix_i$. Let $f_i := w_ix_i$ for $1 \leq i \leq r$, hence each f_i is idempotent. By the hypothesis d_i, f_i are central, then $d_i = d_i^2 = x_if_iw_i = f_id_i$ and $f_i = f_i^2 = d_if_i$, so that $d_i = f_i$ and $x_iw_i = w_ix_i$ for $1 \leq i \leq r$. Therefore, $(\sum_{i=1}^{r} x_i)(\sum_{i=1}^{r} w_i) = 1$, hence $\sum_{i=1}^{r} w_i$ is left invertible, and as we saw in the step 2 in the proof of the previous proposition, $\sum_{i=1}^{r} w_i$ is invertible; thereby, the matrix

$$V := \begin{bmatrix} w_1 & w_2 & w_3 & \cdots & w_r \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

is invertible, i.e., $\boldsymbol{w}$ is completable. □

3.4 Products and quotients

Next we will study the properties introduced in Definition 3.3.1 with respect to some algebraic standard constructions.

Theorem 3.4.1. *Let S be a ring and $I \subseteq Rad(S)$ an ideal of S. Let $\{S_i\}_{i\in C}$ be a family of rings. Then,*

(i) *S is $\mathcal{H}$ if and only if S/I is $\mathcal{H}$.*

(ii) *$\prod_{i\in C} S_i$ is $\mathcal{H}$ if and only if each S_i is $\mathcal{H}$.*

(iii) *If $\prod_{i\in C} S_i$ is $\mathcal{PF}$, then each S_i is $\mathcal{PF}$.*

(iv) *If S is $\mathcal{ED}$, then S/I is $\mathcal{ED}$ for any proper ideal I of S.*

(v) *$\prod_{i\in C} S_i$ is $\mathcal{ED}$ if and only if each S_i is $\mathcal{ED}$.*

(vi) *If S is B, then S/I is B for any proper ideal I of S which is f.g. as left ideal.*

(vii) *$\prod_{i\in C} S_i$ is B if and only if each S_i is B.*

(viii) *Suppose that in S all idempotents are central and I is a nilideal. If S/I is $\mathcal{ID}$, then S is $\mathcal{ID}$.*

(ix) *$\prod_{i\in C} S_i$ is $\mathcal{ID}$ if and only if each S_i is $\mathcal{ID}$.*

(x) *If S is KH, then S/I is KH for any proper ideal I of S.*

(xi) *$\prod_{i\in C} S_i$ is KH if and only if each S_i is KH.*

(xii) *$\prod_{i\in C} S_i$ is $\mathcal{C}$ if and only if each S_i is $\mathcal{C}$.*

(xiii) *If $\prod_{i\in C} S_i$ is $\mathcal{PSF}$, then each S_i is $\mathcal{PSF}$.*

Similar relations are valid for the right side.

Proof. In this proof we will use the following facts: (a) if $\overline{S} := S/I$, then $U := [u_{ij}] \in GL_r(S)$ if and only if $\overline{U} = [\overline{u_{ij}}] \in GL_r(\overline{S})$. Moreover, $(\overline{U})^{-1} = \overline{U^{-1}}$. In fact, the necessary condition is trivial. Now let $\overline{U} \in GL_r(\overline{S})$, then there exists $\overline{V} \in GL_r(\overline{S})$ such that $\overline{U}\,\overline{V} = \overline{I_r} = \overline{V}\,\overline{U}$, where $\overline{I_r}$ is the identical matrix over $\overline{S}$; from this we get that $UV - I_r, VU - I_r \in$

$M_r(Rad(S)) = Rad(M_r(S))$, and hence, there exist $C, D \in M_r(S)$ such that $UVC = I_r$ and $DVU = I_r$, so $U \in GL_r(S)$.

(b) On the other hand, let $B := \prod_{i\in\mathcal{C}} S_i$, then $M_s(B) \cong \prod_{i\in\mathcal{C}} M_s(S_i)$, where the isomorphism is defined by $F \mapsto (F^{(i)})$, with $F = [f_{uv}]$, $f_{uv} = (f_{uv}^{(i)})$, $F^{(i)} = [f_{uv}^{(i)}]$. From this we obtain that $M_s(B)^* = GL_s(B) \cong \prod_{i\in\mathcal{C}} GL_s(S_i) = \prod_{i\in\mathcal{C}} M_s(S_i)^*$.

(i) We will use the characterization given in Theorem 3.1.2 (iii).

$\Rightarrow$): Let $\overline{\boldsymbol{u}} = [\overline{v_1}, \dots, \overline{v_r}]$ be an unimodular row matrix of size $1 \times r$ over $\overline{S}$. There exist $v_1, \dots, v_r \in S$ such that $\overline{u_1}\,\overline{v_1} + \cdots + \overline{u_r}\,\overline{v_r} = \overline{1}$, i.e., $u_1v_1 + \cdots + u_rv_r - 1 \in Rad(S)$. This means that $u_1v_1 + \cdots + u_rv_r \in S^*$, and hence, $\boldsymbol{u} = [u_1, \dots, u_r]^T \in S^r$ is unimodular. By the hypothesis, there exists $U = [u_{ij}] \in GL_r(S)$ such that $\boldsymbol{u}U = \boldsymbol{e}_1^T$. From this we get that $\overline{\boldsymbol{u}}\overline{U} = \overline{\boldsymbol{e}_1}^T$, with $\overline{U} = [\overline{u_{ij}}] \in GL_r(\overline{S})$. This proves that $\overline{S}$ is $\mathcal{H}$.

$\Leftarrow$): Let $\boldsymbol{u} = [u_1, \dots, u_r]$ be unimodular over S, then $\overline{\boldsymbol{u}}$ is unimodular over $\overline{S}$. By the hypothesis, there exists $\overline{U} \in GL_r(\overline{S})$ such that $\overline{\boldsymbol{u}}\overline{U} = \overline{\boldsymbol{e}_1}^T$. We get that

$$\begin{aligned} u_1u_{11} + \cdots + u_ru_{r1} - 1 &= p_1, \\ u_1u_{12} + \cdots + u_ru_{r2} &= p_2, \\ &\vdots \\ u_1u_{1r} + \cdots + u_ru_{rr} &= p_r, \end{aligned}$$

with $p_1, \dots, p_r \in Rad(S)$. Let $z = (1 + p_1)^{-1}$, then $z \in S^*$ and hence

$$\boldsymbol{u}UD = [1, p_2, \dots, p_r],$$

where D is the diagonal matrix $D = \operatorname{diag}(z, 1 \dots, 1)$. Finally, $\boldsymbol{u}UDH = [1, 0, \dots, 0]$ with $H := E_{12}(-p_2)E_{13}(-p_3)\cdots E_{1r}(-p_r)$. Note that $UDH \in GL_r(S)$.

(ii) $\Leftarrow$): Let $B := \prod_{i\in\mathcal{C}} S_i$ and $\boldsymbol{u} = [u_1, \dots, u_r]$ an unimodular row over B, then there exists $v_1, \dots, v_r \in B$ such that $u_1v_1 + \cdots + u_rv_r = 1$, let $u_j := (u_j^{(i)})$, $u_j^{(i)} \in S_i$, $i \in \mathcal{C}$, $1 \le j \le r$. Then, $\boldsymbol{u}^{(i)} := [u_1^{(i)}, \dots, u_r^{(i)}]$ is unimodular over S_i for each i, and there exists $U^{(i)} := [u_{pq}^{(i)}] \in GL_r(S_i)$ such that $\boldsymbol{u}^{(i)}U^{(i)} = [1^i, 0, \dots, 0]$ (the first canonical vector over S_i). Let $U = [u_{pq}]$ with $u_{pq} = (u_{pq}^{(i)}) \in B$, then $U \in GL_r(B)$ and $\boldsymbol{u}U = \boldsymbol{e}_1^T$ (the first canonical vector over B).

$\Rightarrow$): Let $k \in \mathcal{C}$, we will prove that S_k is $\mathcal{H}$. Let $\boldsymbol{u}^{(k)} := [u_1^{(k)}, \dots, u_r^{(k)}]$ be unimodular over S_k, then there exists $\boldsymbol{v}^{(k)} = [v_1^{(k)}, \dots, v_r^{(k)}]^T$ such that $u_1^{(k)}v_1^{(k)} + \cdots + u_r^{(k)}v_r^{(k)} = 1$. Note that $\boldsymbol{u} := [u_1, \dots, u_r]$ is unimodular, with

$$u_1 := (\dots, 1, u_1^{(k)}, 1, \dots), u_2 := (\dots, 0, u_2^{(k)}, 0, \dots), \dots, u_r := (\dots, 0, u_r^{(k)}, 0, \dots).$$

In fact, let

$$v_1 := (\dots, 1, v_1^{(k)}, 1, \dots), v_2 := (\dots, 0, v_2^{(k)}, 0, \dots), \dots, v_r := (\dots, 0, v_r^{(k)}, 0, \dots),$$

then $u_1v_1+\cdots+u_rv_r = (\dots, 1, 1, 1, \dots,)$, and hence, there exists $U = [u_{pq}] \in GL_r(B)$, with $u_{pq} = (u_{pq}^{(i)})$, such that $\boldsymbol{u}U = \boldsymbol{e}_1^T$. Thus, for $U^{(k)} = [u_{pq}^{(k)}] \in GL_r(S_k)$ we have $\boldsymbol{u}^{(k)}U^{(k)} = [1^k, 0, \dots, 0]$.

(iii) Let $k \in \mathcal{C}$, we will prove that S_k is $\mathcal{PF}$. Let $F^{(k)} = [f_{uv}^{(k)}] \in M_s(S_k)$ idempotent, then $F \in M_s(B)$ is idempotent, where $F = [f_{uv}]$, with $f_{uv} = (f_{uv}^{(i)})$ and $f_{uv}^{(i)} = 0$ for $i \neq k$. There exists $P \in GL_s(B)$ such that

$$PFP^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

hence for $P^{(k)} \in GL_s(S_k)$ we have

$$P^{(k)}F^{(k)}(P^{(k)})^{-1} = \begin{bmatrix} 0^{(k)} & 0^{(k)} \\ 0^{(k)} & I_r^{(k)} \end{bmatrix},$$

where $I_r^{(k)}$ is the identical matrix over S_k of size $r \times r$ and the $0^{(k)}$ are null matrices over S_k, thus S_k is $\mathcal{PF}$.

(iv) Let $\overline{F}$ be a rectangular matrix over $\overline{S}$, then F is a rectangular matrix over S and there exist invertible matrices $P \in GL_r(S), Q \in GL_s(S)$ and $d_1, d_2, \ldots, d_l$ in S, with $0 \leq l \leq \min\{r, s\}$, such that $PFQ = \operatorname{diag}(d_1, d_2, \ldots, d_l, 0)$ and $Sd_{i+1}S \subseteq Sd_i \cap d_iS$, for $1 \leq i \leq l$. From this we obtain that $\overline{P} \in GL_r(\overline{S}), \overline{Q} \in GL_s(S)$ and $\overline{P}\ \overline{F}\ \overline{Q} = \operatorname{diag}(\overline{d_1}, \overline{d_2}, \ldots, \overline{d_l})$ and $\overline{S}\ \overline{d_{i+1}}\ \overline{S} \subseteq \overline{S}\ \overline{d_i} \cap \overline{d_i}\ \overline{S}$, for $1 \leq i \leq l$.

(v) $\Rightarrow$): Let $k \in \mathcal{C}$, we will prove that S_k is $\mathcal{ED}$. Let $F^{(k)} = [f_{uv}^{(k)}] \in M_{r\times s}(S_k)$ a rectangular matrix, then $F \in M_{r\times s}(B)$ is a rectangular matrix over B, where $F = [f_{uv}]$, with $f_{uv} = (f_{uv}^{(i)})$ and $f_{uv}^{(i)} = 0$ for $i \neq k$. There exist $P \in GL_r(B), Q \in GL_s(G)$, and $(d_1^{(i)}), (d_2^{(i)}), \ldots, (d_l^{(i)})$ in B, $l = \min\{r, s\}$, such that

$$PFQ = \operatorname{diag}((d_1^{(i)}), (d_2^{(i)}), \ldots, (d_l^{(i)})), B(d_{j+1}^{(i)})B \subseteq B(d_j^{(i)}) \cap (d_j^{(i)})B, 1 \leq j \leq l.$$

Then, $P^{(k)} \in GL_r(S_k), Q^{(k)} \in GL_s(S_k)$ and

$$P^{(k)}F^{(k)}Q^{(k)} = \operatorname{diag}(d_1^{(k)}, d_2^{(k)}, \ldots, d_l^{(k)}), S_kd_{j+1}^{(k)}S_k \subseteq S_kd_j^{(k)} \cap d_j^{(k)}S_k, 1 \leq j \leq l,$$

$\Leftarrow$) Let $F = [f_{uv}] \in M_{r\times s}(B)$ be a rectangular matrix, with $f_{uv} = (f_{uv}^{(i)})$, $f_{uv}^{(i)} \in S_i$; then $F^{(i)} = [f_{uv}^{(i)}] \in M_{r\times s}(S_i)$ and there exist matrices $P^{(i)} \in GL_r(B), Q^{(i)} \in GL_s(B)$ and $d_1^{(i)}, d_2^{(i)}, \ldots, d_{l_i}^{(i)}$ in S_i, $l_i = \min\{r, s\}$, such that

$$P^{(i)}F^{(i)}Q^{(i)} = \operatorname{diag}(d_1^{(i)}, d_2^{(i)}, \ldots, d_{l_i}^{(i)}), S_id_{j+1}^{(i)}S_i \subseteq S_id_j^{(i)} \cap d_j^{(i)}S_i, 1 \leq j \leq l_i.$$

Since for each i, $l_i = \min\{r, s\}$, let $l := \min\{r, s\}$ and then

$$PFQ = \operatorname{diag}((d_1^{(i)}), (d_2^{(i)}), \ldots, (d_l^{(i)})), B(d_{j+1}^{(i)})B \subseteq B(d_j^{(i)}) \cap (d_j^{(i)})B, 1 \leq j \leq l.$$

(vi) and (vii) are direct consequence of the form of left ideals in S/I and $\prod_{i\in\mathcal{C}} S_i$.

(viii) We preserve the previous notation. Let $F \in M_s(S)$ be an idempotent matrix, then $\overline{F} \in M_s(\overline{S})$ is idempotent and there exists $\overline{P} \in GL_s(\overline{S})$ such that

$$\overline{D} = \overline{P}\,\overline{F}\,(\overline{P})^{-1} = \operatorname{diag}(\overline{d_1}, \ldots, \overline{d_r}), \text{ with } \overline{S}\,\overline{d_{i+1}}\,\overline{S} \subseteq \overline{S}\,\overline{d_i} \cap \overline{d_i}\,\overline{S}.$$

Note that $\overline{D}$ is idempotent, so each $\overline{d_i}$ is idempotent, $1 \leq i \leq r$; let $\overline{d} := \overline{d_1} \cdots \overline{d_r}$, then $\overline{d}^2 = \overline{d}$. Since I is nilideal we can assume that d is idempotent (see [68]), and hence, central; moreover since each $\overline{d_i}$ is central, $\overline{d_i}|\overline{d_{i+1}}$, and then $\overline{d} = \overline{d_r}$ (this can be easy prove by induction on r). Note that $\overline{D}\overline{\boldsymbol{e}_r} = \overline{d}\overline{\boldsymbol{e}_r}$, so $\overline{F}\overline{\boldsymbol{v}} = \overline{d}\overline{\boldsymbol{v}}$, with $\overline{\boldsymbol{v}} := (\overline{P})^{-1}\overline{\boldsymbol{e}_r}$ unimodular over $\overline{S}$, and hence, $\boldsymbol{v}$ is unimodular over S. Moreover, there exists $V \in GL_r(S)$ such that $V\boldsymbol{v} = \boldsymbol{e}_1$. In fact, we have $\boldsymbol{v} - P^{-1}\boldsymbol{e}_r = \boldsymbol{u} = [u_1, \ldots, u_r]^T$, with $u_i \in Rad(S)$, $1 \leq i \leq r$. Then, $\boldsymbol{v} = P^{-1}\boldsymbol{e}_r + \boldsymbol{u}$, and hence, $P\boldsymbol{v} = \boldsymbol{e}_r + P\boldsymbol{u}$ is a column matrix with the last component invertible, so multiplying by elementary and permutation matrices we get $V \in GL_r(S)$ such that $V\boldsymbol{v} = \boldsymbol{e}_1$.

We have $F\boldsymbol{v} = d\boldsymbol{v} + \boldsymbol{z}$, with $\boldsymbol{z} = [z_1, \ldots, z_r]^T, z_i \in Rad(S)$, $1 \leq i \leq r$. From this we get that $F^2\boldsymbol{v} = F\boldsymbol{v} = dF\boldsymbol{v} + F\boldsymbol{z}$, so $F\boldsymbol{z} = (1-d)F\boldsymbol{v} = (1-d)(d\boldsymbol{v} + \boldsymbol{z}) = (1-d)\boldsymbol{z}$ since $(1-d)d = 0$. Then, $F(\boldsymbol{v} + (2d-1)\boldsymbol{z}) = F\boldsymbol{v} + (2d-1)F\boldsymbol{z} = d\boldsymbol{v} + \boldsymbol{z} + (2d-1)(1-d)\boldsymbol{z} = d\boldsymbol{v} + d\boldsymbol{z} = d(\boldsymbol{v} + (2d-1)\boldsymbol{z})$. Thus, given the idempotent matrix F we have found a vector $\boldsymbol{w} := \boldsymbol{v} + (2d-1)\boldsymbol{z}$ and an element $d \in S$ such that $F\boldsymbol{w} = d\boldsymbol{w}$, moreover $\boldsymbol{w}$ is unimodular since $\boldsymbol{v}$ is unimodular and $z_i \in Rad(S)$, $1 \leq i \leq r$. In addition, the first component of the vector $V\boldsymbol{w} = \boldsymbol{e}_1 + V(2d-1)\boldsymbol{z}$ is invertible, so by elementary operations we found a matrix $W \in GL_r(S)$ such that $W\boldsymbol{w} = \boldsymbol{e}_1$. From Proposition 3.3.6 we get that S is an $\mathcal{ID}$ ring.

(ix) The proof is completely similar to the proof of (v).

(x) Evident.

(xi) The proof is as in (v).

(xii) $\Rightarrow$): We will apply Proposition 3.3.2. Let $k \in \mathcal{C}$ and $F^{(k)} = [f_{uv}^{(k)}] \in M_s(S_k)$, $G^{(k)} = [g_{uv}^{(k)}] \in M_r(S_k)$ idempotent matrices, then $F \in M_s(B)$, $G \in M_r(B)$ are idempotent, where $F = [f_{uv}]$, $G = [g_{uv}]$, with $f_{uv} = (f_{uv}^{(i)})$, $g_{uv} = (g_{uv}^{(i)})$ and $f_{uv}^{(i)} = 0 = g_{uv}^{(i)}$ for $i \neq k$. Since B is a $\mathcal{C}$ ring, the enlarged matrices

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized as in Proposition 3.3.2 if and only if the matrices F, G can be factorized. This implies that the matrices

$$\begin{bmatrix} F^{(k)} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G^{(k)} & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized if and only if the matrices $F^{(k)}, G^{(k)}$ can be factorized. This proves that S_k is a $\mathcal{C}$ ring.

$\Leftarrow$): Let $F = [f_{uv}] \in M_s(B)$, $G = [g_{uv}] \in M_r(B)$ be idempotent matrices, with $f_{uv} = (f_{uv}^{(k)})$, $g_{uv} = (g_{uv}^{(k)})$, $f_{uv}^{(k)}, g_{uv}^{(k)} \in S_k$; since each ring S_k is $\mathcal{C}$, we can repeat the previous reasoning, but in the inverse order, and conclude that B is a $\mathcal{C}$ ring.

(xiii) The proof is analogous to the first part of (xii). □

Proposition 3.4.2. *Given a ring S, if S is $\mathcal{H}$ ($\mathcal{C}$) then $M_n(S)$ is $\mathcal{H}$ ($\mathcal{C}$) for every $n \geq 1$.*

Proof. Let P be a stably free $M_n(S)$-module, then there exist integers $r, s \geq 0$ such that $P \oplus (M_n(S))^s \cong (M_n(S))^r$. From this we have

$$S^{1\times n} \otimes_{M_n(S)} P \oplus S^{s(1\times n)} \cong S^{r(1\times n)}$$

and, hence, $S^{1\times n} \otimes_{M_n(S)} P$ is a stably free S-module. Since $S \in \mathcal{H}$, this module turns out free with rank $n(r-s)$, i.e., $S^{1\times n} \otimes_{M_n(S)} P \cong S^{(1\times n)(r-s)}$. Thus,

$$S^n \otimes_S S^{1\times n} \otimes_{M_n(S)} P \cong S^n \otimes_S S^{(1\times n)(r-s)},$$

which implies that

$$P \cong M_n(S) \otimes_{M_n(S)} P \cong M_n(S)^{r-s},$$

this is, P is a free $M_n(S)$-module of rank $r-s$.
Now, given P, Q left finitely generated projective $M_n(S)$-modules such that $P \oplus M_n(S) \cong Q \oplus M_n(S)$, we have that

$$S^{1\times n} \otimes_{M_n(S)} P \oplus S^{1\times n} \cong S^{1\times n} \otimes_{M_n(S)} Q \oplus S^{1\times n}.$$

It is not difficult to show that $S^{1\times n} \otimes_{M_n(S)} P$ and $S^{1\times n} \otimes_{M_n(S)} Q$ are finitely generated S-modules and, therefore,

$$S^{1\times n} \otimes_{M_n(S)} P \cong S^{1\times n} \otimes_{M_n(S)} Q.$$

Whereby, applying $S^n \otimes_S -$ to this last isomorphism, we get $P \cong Q$, i.e., $M_n(S) \in \mathcal{C}$. $\square$

Remark 3.4.3. The problem of computing the matrices U in Theorem 3.1.2 and Corollary 3.2.4 has been considered in various contexts. For example, in the commutative setting, Yengui in [2] presents an algorithm for unimodular completion over Laurent polynomial ring, whereas in [96] a method for unimodular completion over Noetherian rings is developed. Of course, the constructive proofs of Quillen-Suslin Theorem include algorithms for the calculation of such matrices in the case $S = \Bbbk[x_1, \ldots, x_n]$ (see [86]). In [71] Laubenbacher regarded the unimodular completion problem for quotient polynomial rings by monomial ideals. Interesting examples about completion unimodular in particular cases are shown by Lam in [62], Examples 5.10 - 5.14.

3.5 Localizations

Now we will consider the localizations of rings introduced in Definition 3.3.1.

Proposition 3.5.1. *Let S be a ring and T a multiplicative system of S such that $T^{-1}S$ exits. If S is $\mathcal{ED}$ (KH, B), then $T^{-1}S$ is $\mathcal{ED}$ (KH, B). Similar properties are valid for the right side.*

Proof. Let S a $\mathcal{ED}$ ring and $F \in M_{r\times s}(T^{-1}S)$, then $F = [f_{ij}]$ with $f_{ij} = t_{ij}^{-1}s_{ij}$, where $t_{ij} \in T$ and $s_{ij} \in S$, for $1 \leq i \leq r$, $1 \leq j \leq s$. By Proposition 2.1.16 in [95], there exist $t \in T$ and $l_{ij} \in S$ such that $f_{ij} = t^{-1}l_{ij}$, then $tF = [l_{ij}] \in M_{r\times s}(S)$, hence tF admits a diagonal reduction, i.e., there exist $P \in GL_r(S)$ and $Q \in GL_s(S)$ such that $P(tF)Q = \operatorname{diag}(d_1, \ldots, d_l)$, with $d_1, \ldots, d_l \in S$, $l = \min\{r, s\}$ and $Sd_{i+1}S \subseteq Sd_i \cap d_iS$. Note that

$Pt, Q \in GL_r(T^{-1}S)$. Thus, $(Pt)FQ = P(tF)Q = D$, moreover, $T^{-1}Sd_{i+1}T^{-1}S \subseteq T^{-1}Sd_i \cap d_iT^{-1}S$. This proves that $T^{-1}S$ is $\mathcal{ED}$.

The proof for KH is completely analogous.

Suppose now that S is a B ring and let J be a f.g. left ideal of $T^{-1}S$, then $J = \langle q_1, \ldots, q_r\}$ where $q_i = t_i^{-1}s_i$ with $t_i \in T$ and $s_i \in S$ for $1 \leq i \leq r$. Let $t \in T$ and $a_i \in S$ such that $q_i = t^{-1}q_i$, then $tq_i = a_i$. Therefore, $J' := T^{-1}S\frac{a_1}{1} + \cdots + T^{-1}S\frac{a_r}{1} \subseteq J$; but $J \subseteq J'$: in fact, let $x = \frac{b_1}{t_1}q_1 + \cdots + \frac{b_r}{t_r}q_r \in J$, then $x = t_1^{-1}b_1t^{-1}\frac{a_1}{1} + \cdots + t_r^{-1}b_rt^{-1}\frac{a_r}{1}$; since $b_it^{-1} \in T^{-1}S$ exist, $b_i' \in S$ and $l_i \in T$ such that $b_it^{-1} = l_i^{-1}b_i'$, $1 \leq i \leq r$, hence $x = t_1^{-1}l_1^{-1}b_1'\frac{a_1}{1} + \cdots + t_r^{-1}l_r^{-1}b_r'\frac{a_r}{1} = (l_1t_1)^{-1}b_1'\frac{a_1}{1} + \cdots + (l_rt_r)^{-1}b_r'\frac{a_r}{1} \in J'$. Thus, $J = J'$.

Now note that $J' = T^{-1}I$, where $I := Sa_1 + \cdots + Sa_r$: clearly $T^{-1}I \subseteq J'$; let $y \in J'$, then $y = \frac{b_1}{s_1}\frac{a_1}{1} + \cdots + \frac{b_r}{s_r}\frac{a_r}{1} = \frac{b_1a_1}{s_1} + \cdots + \frac{b_ra_r}{s_r} = \frac{c_1b_1a_1+\cdots+c_rb_ra_r}{u}$ for some $c_i \in S$ and $u \in T$. Hence $y = u^{-1}(c_1b_1a_1 + \cdots + c_rb_ra_r) \in T^{-1}I$. But I is a f.g. left ideal of S, then $I = \langle a\}$ for some $a \in S$, and therefore $J = T^{-1}S\frac{a}{1}$, i.e., J is principal. $\square$

Remark 3.5.2. (i) We observe that if S is $\mathcal{B}$ and T a multiplicative system of S such that $T^{-1}S$ and ST^{-1} exist, then $T^{-1}S$ is $\mathcal{B}$ since $ST^{-1} \cong T^{-1}S$.

(ii) In general, if S is $\mathcal{H}$ ($\mathcal{PF}$, $\mathcal{PSF}$) not always $T^{-1}S$ has the correspondent property (see [26]).

For the localization by primes ideals we need to recall a definition. Let S be a left Noetherian ring and P a prime ideal of S. It says that P is *left localizable* if the set

$$S(P) := \{a \in S | \overline{a} \in S/P \text{ is not a zero divisor}\}$$

is a multiplicative system of S and $S(P)^{-1}S$ exists; we will write $S_P := S(P)^{-1}S$. Right localizable prime ideals are defined similarly (see [11]).

Proposition 3.5.3. *Let S be a left Noetherian ring.*

(i) *If P is a left (right) localizable prime ideal, then S_P is $\mathcal{H}$.*

(ii) *If P is a left (right) localizable completely prime ideal, then S_P is $\mathcal{PF}$, and hence, $\mathcal{C}$ and $\mathcal{PSF}$.*

Proof. (i) It is well known (see for example [11], and also [80]) that S_p has a unique maximal ideal $PS_P := \{\frac{a}{s} \mid a \in P, s \in S(P)\}$; moreover, $Rad(S_P) = PS_P$ and S_p/PS_p is simple Artinian, therefore, S_P is a semilocal ring and hence S_P is $\mathcal{H}$ (Proposition 3.4.1).

(ii) If P is completely prime, S/P is a domain, so that $Q_l(S/P)$ is a division ring, and therefore, S_P is a local ring. From [26], Corollary 0.3.8, we get that S_P is $\mathcal{PF} \subseteq \mathcal{C} \cap \mathcal{PSF}$. $\square$

3.6 Examples, remarks and open problems

Example 3.6.1. (a) Probably the most classical example of $\mathcal{PF}$ (and hence of $\mathcal{PSF}$ and $\mathcal{H}$) ring is $S[x_1, \ldots, x_n]$, where S is a commutative principal ideal domain (this is the content of the Quillen-Suslin Theorem).

(b) Any principal ideal commutative ring (PIR) is $\mathcal{KH}$, and hence, $\mathcal{H}$ ([62], Theorem I.4.31).

(c) Any commutative von Neumann regular ring is $\mathcal{KH}$, and hence, $\mathcal{H}$ ([62], Theorem I.4.34).

(d) Any Dedekind domain is $\mathcal{H}$ (see [78], Remark 6.7.14).

(e) Any local ring (in the sense that $S/Rad(S)$ is a division ring) is $\mathcal{PF}$ (see [26], Corollary 0.3.8), and hence, it is also $\mathcal{C}$ and $\mathcal{H}$.

(f) Any semilocal ring is $\mathcal{H}$. This follows from Theorem 3.4.1.

(g) Note that $\mathcal{PF}, \mathcal{PSF} \neq \mathcal{H}$: $\mathbb{Z}_6$ (see [78], Example 6.1.2).

Example 3.6.2. Let T be a division ring. Then, any (f.g.) projective left (right) module over $T[x]$ is free. Thus, $T[x]$ is $\mathcal{PF}$, and hence, $\mathcal{H}$ ([62], p. 2 and p. 73). However, $S := T[x_1, x_2]$ has a module M such that $M \oplus S \cong S^2$, but M is not free, i.e., S is not $\mathcal{H}$, and hence, is not $\mathcal{PF}$ ([62], p. 3 and p. 74; [5], Corollary 6.3).

Example 3.6.3. (a) We exhibit a commutative ring that is not $\mathcal{H}$ (see [107]). Let $S = \mathbb{R}[x, y, z]$ and $\overline{S} = \mathbb{R}[x, y, z]/I$, with $I = \langle x^2+y^2+z^2-1\rangle$, then $\boldsymbol{u} = \begin{bmatrix}\overline{x} & \overline{y} & \overline{z}\end{bmatrix}$ is unimodular with right inverse $\boldsymbol{u}^T$, however $\boldsymbol{u}$ cannot be completed to an unimodular matrix: In fact, suppose that exists $U \in GL_3(\overline{S})$ such that $\boldsymbol{u}U = \begin{bmatrix}\overline{1} & \overline{0} & \cdots & \overline{0}\end{bmatrix}$. Note that makes sense to evaluate elements of $\overline{S}$ at points $(v_1, v_2, v_3) \in \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, since if $\overline{f} = \overline{g}$ then $f - g \in I$ and hence $f(v_1, v_2, v_3) - g(v_1, v_2, v_3) = 0$, i.e., $f(v_1, v_2, v_3) = g(v_1, v_2, v_3)$. Moreover, an unit in $\overline{S}$ takes nonzero values everywhere on the sphere: in fact, if $\overline{f}\overline{g} = \overline{1}$, by above, $f(v_1, v_2, v_3)g(v_1, v_2, v_3) = 1$ for every $(v_1, v_2, v_3) \in \mathbb{S}^2$, In particular, since $\det U^{-1}$ is an unit, then $\det U^{-1} \neq 0$ in every point on $\mathbb{S}^2$. So, if $U^{-1} = [\overline{f}_{ij}] \in GL_3(\overline{S})$, then $\varphi(v) := (f_{12}(v), f_{22}(v), f_{32}(v)) \in \mathbb{R}^3 \setminus \{0\}$ for all $v \in \mathbb{S}^2$. But $\boldsymbol{u}\begin{bmatrix}\overline{f}_{12} & \overline{f}_{22} & \overline{f}_{32}\end{bmatrix}^T = \overline{0}$, so that $v \cdot \varphi(v) = 0$ and hence, $\varphi(v)$ is a tangent vector to $\mathbb{S}^2$ that results also continuous (and differentiable) since each f_{ij} is a polynomial. Thus, the map $\varphi : \mathbb{S}^2 \to \mathbb{R}^3$ is a nowhere zero vector field on $\mathbb{S}^2$. But this is a contradiction, because the hairy ball theorem in topology says every continuous vector field on the sphere vanishes at least once, (see [62], Chapter III).

(b) This example also shows that if S is $\mathcal{H}$ not always S/I is $\mathcal{H}$, with I an arbitrary proper ideal of S. In the same way, this example also shows that if S is $\mathcal{ID}$ not always S/I is $\mathcal{ID}$.

Example 3.6.4. The product of $\mathcal{PF}$ rings is not necessarily $\mathcal{PF}$. In fact, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are $\mathcal{PF}$, but $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is not $\mathcal{PF}$ (see Example 3.6.1, literal (d)). This example also shows that quotients of $\mathcal{PF}$ rings are not necessarily $\mathcal{PF}$: $\mathbb{Z}$ is $\mathcal{PF}$. In addition, from Theorem 3.4.1 we obtain that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is $\mathcal{C}$, so $\mathcal{PF} \neq \mathcal{C}$.

Example 3.6.5. $\mathcal{H}$ and $\mathcal{PF}$ are not (in general) preserved by localizations by arbitrary multiplicative systems ([62], Remark I.4.19. See also see [26], Exercise 0.7.15).

Example 3.6.6. It is well known that $B \neq B_r$, a classical example is given by the skew polynomial ring $T[x; \sigma]$, where T is a division ring a σ is an endomorphism of T that is not automorphism. Every left ideal of this ring is principal, hence, it is a left Bézout ring; but if $a \notin \sigma(T)$, then the right ideal generated by x and ax is not principal. In fact, suppose that there exists $f \in T[x;\sigma]$ such that $xT[x;\sigma] + axT[x;\sigma] = fT[x;\sigma]$, we have $x = fh$ and $ax = fg$, for some polynomials $f, g \in T[x;\sigma]$; f is not a constant polynomial since $f \in xT[x;\sigma] + axT[x;\sigma]$, so $x = (f_1x + f_0)h_0$, from this we get that $f_0 = 0, h_0 \neq 0$ and $f_1 = \sigma(h_0^{-1})$. From $ax = fg$ we conclude that $ax = f_1xg_0$, i.e., $a = \sigma(h_0^{-1}g_0)$, a contradiction.

This example shows also that $KH \neq KH_r$. In fact, as we saw $T[x;\sigma]$ is $BD = KHD$, but $T[x;\sigma]$ is not $KHD_r = BD_r$.

Example 3.6.7. Note that if $\Bbbk$ is a field, then $\Bbbk[x,y]$ is $\mathcal{PFD}$ but is not BD. Thus, $B \neq \mathcal{PF}$, and consequently, $B \neq \mathcal{C}$, $B \neq \mathcal{H}$, $KH \neq \mathcal{PF}$, $KH \neq \mathcal{C}$, $KH \neq \mathcal{H}$, $\mathcal{ED} \neq \mathcal{PF}$, $\mathcal{ED} \neq \mathcal{C}$, $\mathcal{ED} \neq \mathcal{H}$.

Example 3.6.8. In (3.3.1) we observed that $BD \subseteq \mathcal{PFD}$, note that in general $B \nsubseteq \mathcal{PF}$. In fact, consider $\mathbb{Z}_6$. This example also shows that $\mathcal{PF} \neq \mathcal{ID}$ since $\mathbb{Z}_6$ is semilocal and commutative semilocal rings are $\mathcal{ID}$ (see [118]).

Example 3.6.9. $\mathbb{Z}[\sqrt{-5}]$ shows that $\mathcal{ID} \neq \mathcal{H}$, see [78], Example 6.6.1 and Remark 6.7.14.

Example 3.6.10. Note that if $\Bbbk$ is a field, then $S := M_2(\Bbbk) \in \mathcal{C}$ by Proposition 3.4.2; nevertheless $S \notin \mathcal{PSF}$: indeed, we have that

$$M_2(\Bbbk) = \begin{bmatrix} \Bbbk & 0 \\ \Bbbk & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \Bbbk \\ 0 & \Bbbk \end{bmatrix},$$

thus $P := \begin{bmatrix} \Bbbk & 0 \\ \Bbbk & 0 \end{bmatrix}$ is a finitely generated projective S-module. If P was stably free, then there exist integers $r, s \geq 0$ such that $P \oplus S^s \cong S^r$ (S-isomorphism). But every S-isomorphism is a $\Bbbk$-isomorphism, hence $P \oplus S^s \cong S^r$ as vectorial spaces. From this, it follows $\dim_{\Bbbk}(P \oplus S^s) = 2 + 4s = \dim_{\Bbbk}(S^r) = 4r$, and whence, $2 = 4(r-s) \geq 4$, a contradiction. Therefore, $\mathcal{C} \nsubseteq \mathcal{PSF}$. On the other hand, $A_1(\Bbbk) \in \mathcal{PSF}$ but this ring is not $\mathcal{C}$ (see Example 11.1.4 in [95]). So, $\mathcal{PSF} \nsubseteq \mathcal{C}$.

Remark 3.6.11. (a) In [50] it is proved that $\mathcal{ED} \neq \mathcal{KH} \neq \mathcal{B}$.

(b) In [60], Theorem 3.2, Kaplansky proved that a commutative Bézout ring is $\mathcal{KH}$ when all zero divisors of the ring are in the Jacobson radical, establishing in particular that if S is local then $\mathcal{KH} = \mathcal{B}$ (see also [3] and [62], Theorem I.4.27).

(c) In [125], Theorem 2, it is proved that every commutative Bézout ring with compact minimal prime spectrum is $\mathcal{KH}$.

(d) In [126], Theorem 1, Zabavsky showed that a commutative Bézout ring S is $\mathcal{KH}$ if and only if $\mathrm{sr}(S) \leq 2$.

(e) [126], Theorem 2, shows that a B_r ring with stable range 1 is KH_r. Moreover, Corollary 1 in [110] shows that a B_r ring with stable range 1 is $\mathcal{H}$ (see also Corollary 4.1.5 in the next chapter). In a similar direction, in [52] is proved that if S is B_r and $Rad(S)$ contains a completely prime ideal, then S is KH_r.

(f) For noncommutative rings, Zabavsky in [126], Corollary 2, proved that any semilocal right Bézout ring is KH_r.

(g) In [125], Proposition 2, it is proved that a $n - KH_r$ ring has stable range $\leq n$ (let $n \geq 2$, a ring S is $n - KH_r$ if given a row matrix $\boldsymbol{u}$ of size $1 \times n$ there exist $U \in GL_n(S)$

and $d \in S$ such that $\boldsymbol{u}U = \begin{bmatrix} d & 0 & \cdots & 0 \end{bmatrix}$; in a similar way the rings $n - KH$ and $n - \mathcal{KH}$ are defined; note that $2 - KH_r = KH_r$. In Lemma 3.3.4 we have proved that a ring S is KH if and only if S is $n - KH$, for all $n \geq 2$).

(h) If S is B_r with $\mathrm{sr}(S) = n$ then S is $m - KH_r$, for all $m \geq n + 1$ ([125], Corollary 1).

(i) If S is $n - KH_r$, then S is B_r ([125], Proposition 4).

(j) If S is $n - KH$ and B_r, then S is right $n - KH_r$ ([125], Proposition 3).

(k) Let S be an integral domain, i.e., a commutative domain. If S is $\mathcal{BD}$ with enumerable many maximal ideals or with Krull dimension 1, then S is $\mathcal{EDD}$. If S is $\mathcal{BD}$ such that given a proper invertible integral ideal I of S there exists a non-empty finite set of finitely generated maximal ideals that contain I, then S is $\mathcal{EDD}$ ([78], Remark 6.7.7).

Remark 3.6.12. A very close notion to the task of studying when stably free modules are free is that of power-free modules. We say that a stably free S-module P with rank t is *power-free* if exists a positive integer n such that $P^n \cong S^{tn}$. In [64], Theorem 5.10 and Theorem 5.11, Lam proved that if S is a right (left) noetherian ring or a commutative ring, then every stably free module is power-free. From this, we can conclude that if A is a bijective skew PBW extension of a right (left) noetherian ring R, then every stably free A-module is power-free.

Problem 3.6.13. (a) In general, $\mathcal{ID} \subseteq \mathcal{C}$? (b) In general, $\mathcal{ID} \subseteq \mathcal{H}$? (d) $\mathcal{C} \neq \mathcal{H}$? (see [26], Exercise 0.4.7).

Conjecture 3.6.14 (Kaplansky)**.** *For commutative domains, $\mathcal{BD} = \mathcal{EDD}$.*

CHAPTER 4

d-Hermite rings and skew PBW extensions

As we saw at the beginning of Chapter 2, under suitable conditions on the ring R of coefficients, most of skew PBW extensions are $\mathcal{PSF}$. It was also remarked that if R is a left Noetherian, left regular $\mathcal{PSF}$ ring, then the ring of skew quantum polynomials $Q_{\mathbf{q},\sigma}^{r,n}(R)$ is also $\mathcal{PSF}$. In particular, if $\Bbbk$ is a field, the $\Bbbk$-algebra of skew quantum polynomials $Q_{\mathbf{q},\sigma}^{r,n}(\Bbbk)$ is a $\mathcal{PSF}$ ring. Related to the $\mathcal{H}$ property that we study in the previous chapter, there exists an important example of skew polynomial ring that satisfies this condition: let T be a division ring and $T[x;\sigma,\delta]$ the ring of skew polynomials ring over T, where σ is an automorphism, then it is well known that $T[x;\sigma,\delta]$ is a principal ideal domain ($\mathcal{PID}$), i.e., it has non zero divisors and all left and right ideals are principal (see [26], Theorem 1.3.2, see also [80]), but any $\mathcal{PID}$ is $\mathcal{EDD}$ ([26], Theorem 1.4.7), so by (3.3.1), $T[x;\sigma,\delta]$ is $\mathcal{HD}$. For example, $B_1(\Bbbk)$ is $\mathcal{HD}$. However, it is easy to show examples of skew PBW extensions $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ that are not $\mathcal{H}$ rings (and hence, are not $\mathcal{PF}$): if T is a division ring, then $S:=T[x,y]$ has a module M such that $M\oplus S\cong S^2$, but M is not free, i.e., S is not $\mathcal{H}$ (see [62], p. 36 and [98], Proposition 1). Let $R=\mathbb{H}[[x,y]]$ be the power series ring in x,y over the division ring $\mathbb{H}$ of the real quaternions, and let $A:=R[t]$. Then, R is a noncommutative local ring but $R[t]$ is not $\mathcal{H}$ (see [62], p. 325). Another example occurs in Weyl algebras: let $\Bbbk$ be a field, with $\operatorname{char}(\Bbbk)=0$, the Weyl algebra $A_1(\Bbbk)=\Bbbk[t][x;\frac{d}{dt}]$ is not $\mathcal{H}$ since there exist stably free modules of rank 1 over $A_n(\Bbbk)$ that are not free ([26], Corollary 1.5.3; see also [95], Example 11.1.4). Note that $\Bbbk[t]$ is $\mathcal{H}$. In general, if R is a left Noetherian domain, then $A_n(R)$ is not $\mathcal{H}$ ([95], Corollary 11.2.11). In this chapter, we will study a weaker condition than the $\mathcal{H}$ property for skew PBW extensions: the d-Hermite condition. Recall that we always assume that all rings are $\mathcal{RC}$.

4.1 d-Hermite rings

There is a famous conjecture in commutative algebra which asserts that if R is a commutative $\mathcal{H}$-ring, then the polynomial ring $R[x]$ is $\mathcal{H}$ (see [62]). As we observed at the beginning of the chapter, this conjecture for skew PBW extensions is not true. Thus, instead of considering the $\mathcal{H}$ condition and the conjecture for skew PBW extensions, we will study a weakly property, the d-Hermite property. The following proposition induces the definition of d-Hermite rings.

Proposition 4.1.1. *Let S be a ring. For any integer $d \geq 0$, the following statements are equivalent:*

(i) *Any stably free module of rank $\geq d$ is free.*

(ii) *Any unimodular row matrix over S of length $\geq d+1$ can be completed to an invertible matrix over S.*

(iii) *For every $r \geq d+1$, if $\boldsymbol{u}$ is an unimodular row matrix of size $1 \times r$, then there exists a matrix $U \in GL_r(S)$ such that $\boldsymbol{u}U = (1, 0, \ldots, 0)$, i.e., $GL_r(S)$ acts transitively on $Um_r(r, S)$.*

(iv) *For every $r \geq d+1$, given an unimodular matrix F of size $s \times r$, $r \geq s$, there exists $U \in GL_r(S)$ such that*

$$FU = \begin{bmatrix} I_s & | & 0 \end{bmatrix}.$$

Proof. We can repeat the proof of Theorem 3.1.2 taking $r \geq d+1$. □

Definition 4.1.2. *Let S be a ring and $d \geq 0$ an integer. S is d-Hermite, property denoted by d-$\mathcal{H}$, if S satisfies any of conditions in Proposition 4.1.1.*

The next result extends Proposition 3.2.7.

Proposition 4.1.3. *The d-Hermite condition is left-right symmetric.*

Proof. We can repeat the proof of Proposition 3.2.7 taking $r \geq d+1$. See also [95], Lemma 11.1.13. □

Corollary 4.1.4. *Let S be a ring. Then, S is $\mathrm{sr}(S)$-$\mathcal{H}$.*

Proof. This follows from Definition 4.1.2 and Theorem 2.3.6. □

Corollary 4.1.5. *Let S be a ring. If $\mathrm{sr}(S) = 1$, then S is $\mathcal{H}$.*

Proof. According to Corollary 4.1.4 S is 1-$\mathcal{H}$, however, it is well known that rings with stable rank 1 are cancellable (see [34]), so by Theorem 3.3.3, S is $\mathcal{H}$. □

Remark 4.1.6. (i) Observe that 0-Hermite rings coincide with $\mathcal{H}$ rings, and for commutative rings, 1-Hermite coincides also with $\mathcal{H}$ (see [62], Theorem I.4.11). If K is a field with $\mathrm{char}(\Bbbk) = 0$, by Corollary 2.3.7, $A_1(\Bbbk)$ is 2-$\mathcal{H}$ but, as we observed at the beginning of the chapter, $A_1(\Bbbk)$ is not 1-$\mathcal{H}$. In general, $\mathcal{H} \subsetneq 1$-$\mathcal{H} \subsetneq 2$-$\mathcal{H} \subsetneq \cdots$ (see [26]).

(ii) Note that $\mathcal{H} = 1$-$\mathcal{H} \cap \mathcal{WF}$ (a ring S is $\mathcal{WF}$, *weakly finite*, if for all $n \geq 0$, $P \oplus S^n \cong S^n$ if and only if $P = 0$. See Remark 2.1.9).

(iii) Any left Artinian ring S is $\mathcal{H}$ since $\mathrm{sr}(S) = 1$, see Remark 2.3.2. In particular, semisimple and semilocal rings are $\mathcal{H}$.

(iv) Rings with big stable rank can be Hermite, for example $\mathrm{sr}(\mathbb{R}[x_1, \ldots, x_n]) = n+1$ ([95], Theorem 11.5.9), but by Quillen-Suslin Theorem, $\mathbb{R}[x_1, \ldots, x_n]$ is $\mathcal{H}$.

4.2 Stable rank

Corollaries 2.3.7 and 4.1.4 motivate the task of computing the stable rank of bijective skew PBW extensions. For this purpose, we need to recall the famous stable range theorem. This theorem relates the stable rank and the Krull dimension of a ring. The original version of this classical result is due to Bass (1968, [8]) and states that if S is a commutative Noetherian ring and $\mathrm{Kdim}(S) = d$ then $\mathrm{sr}(S) \leq d+1$. Heitmann extends the theorem for arbitrary commutative rings (1984, [53]). Lombardi et. al. in 2004 ([30], Theorem 2.4; see also [88]) proved again the theorem for arbitrary commutative rings using the Zariski lattice of a ring and the boundary ideal of an element. This proof is elementary and constructive. Stafford in 1981 ([115]) proved a noncommutative version of the theorem for left Noetherian rings.

Proposition 4.2.1 (Stable range theorem)**.** *Let S be a left Noetherian ring and $\mathrm{lKdim}(S) = d$, then $\mathrm{sr}(S) \leq d+1$.*

Proof. See [115]. □

From this we get the following modest result.

Proposition 4.2.2. *Let R be a left Noetherian ring with finite left Krull dimension and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ a bijective skew PBW extension of R, then*

$$1 \leq \mathrm{sr}(A) \leq \mathrm{lKdim}(R) + n + 1,$$

and A is d-$\mathcal{H}$, with $d := (\mathrm{lKdim}(R) + n + 1)$.

Proof. The inequalities follow from Proposition 4.2.1 and Theorem 4.2 in [83]. The second statement follows from Corollary 4.1.4. □

Example 4.2.3. The results in [83] for the Krull dimension of bijective skew PBW extensions can be combined with Proposition 4.2.2 in order to get an upper bound for the stable rank. With this, we can estimate also the d-Hermite condition. The following table gives such estimations:

Ring	**U. B.**
Habitual polynomial ring $R[x_1, \dots, x_n]$	$\dim(R) + n + 1$
Ore extension of bijective type $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$	$\dim(R) + n + 1$
Weyl algebra $A_n(K)$	$2n + 1$
Extended Weyl algebra $B_n(K)$	$n + 1$
Universal enveloping algebra of a Lie algebra $\mathfrak{g}, \mathcal{U}(\mathfrak{g})$, K commutative ring	$\dim(K) + n + 1$
Tensor product $R \otimes_K \mathcal{U}(\mathcal{G})$	$\dim(R) + n + 1$
Crossed product $R * \mathcal{U}(\mathcal{G})$	$\dim(R) + n + 1$
Algebra of q-differential operators $D_{q,h}[x, y]$	3
Algebra of shift operators S_h	3
Mixed algebra D_h	4
Discrete linear systems $\Bbbk[t_1, \dots, t_n][x_1, \sigma_1] \cdots [x_n; \sigma_n]$	$2n + 1$
Linear partial shift operators $\Bbbk[t_1, \dots, t_n][E_1, \dots, E_n]$	$2n + 1$
Linear partial shift operators $\Bbbk(t_1, \dots, t_n)[E_1, \dots, E_n]$	$n + 1$
L. P. Differential operators $\Bbbk[t_1, \dots, t_n][\partial_1, \dots, \partial_n]$	$2n + 1$
L. P. Differential operators $\Bbbk(t_1, \dots, t_n)[\partial_1, \dots, \partial_n]$	$n + 1$
L. P. Difference operators $\Bbbk[t_1, \dots, t_n][\Delta_1, \dots, \Delta_n]$	$2n + 1$
L. P. Difference operators $\Bbbk(t_1, \dots, t_n)[\Delta_1, \dots, \Delta_n]$	$n + 1$
L. P. q-dilation operators $\Bbbk[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$	$n + m + 1$
L. P. q-dilation operators $\Bbbk(t_1, \dots, t_n)[H_1^{(q)}, \dots, H_m^{(q)}]$	$m + 1$
L. P. q-differential operators $\Bbbk[t_1, \dots, t_n][D_1^{(q)}, \dots, D_m^{(q)}]$	$n + m + 1$
L. P. q-differential operators $\Bbbk(t_1, \dots, t_n)[D_1^{(q)}, \dots, D_m^{(q)}]$	$m + 1$
Diffusion algebras	$2n + 1$
Additive analogue of the Weyl algebra $A_n(q_1, \dots, q_n)$	$2n + 1$
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	$n + 1$
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$	4
3-dimensional skew polynomial algebras	4
Dispin algebra $\mathcal{U}(osp(1, 2))$	4
Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, \Bbbk))$	4
Complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$	11
Algebra **U**	$3n + 1$
Manin algebra $\mathcal{O}_q(M_2(\Bbbk))$	5
Coordinate algebra of the quantum group $SL_q(2)$	5
q-Heisenberg algebra $\mathbf{H}_n(q)$	$3n + 1$
Quantum enveloping algebra of $\mathfrak{sl}(2, \Bbbk), \mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$	4
Hayashi algebra $W_q(J)$	$3n + 1$
Differential operators on a quantum space $S_{\mathbf{q}}$, $D_{\mathbf{q}}(S_{\mathbf{q}})$	$2n + 1$
Witten's Deformation of $\mathcal{U}(\mathfrak{sl}(2, \Bbbk)$	4
Quantum Weyl algebra of Maltsiniotis $A_n^{\mathbf{q}, \lambda}$, K commutative ring	$\dim(K) + 2n + 1$
Quantum Weyl algebra $A_n(q, p_{i,j})$	$2n + 1$
Quantum Weyl algebra $A_2(J_{a,b})$, $a \neq b$	4
Multiparameter Weyl algebra $A_n^{Q,\Gamma}(\Bbbk)$	$2n + 1$
Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(\Bbbk^{2n}))$	$2n + 1$
Quadratic algebras in 3 variables	4

Table 4.1: Stable rank for some examples of bijective skew PBW extensions.

Remark 4.2.4. (i) The values presented in Table 4.1 can be improved for some particular classes of skew PBW extensions. For example, it is well known that $\mathrm{sr}(A_n(\Bbbk)) = 2$ if $\mathrm{char}(\Bbbk) = 0$ (see Remark 2.3.2). A challenging problem is to give exact values for the stable rank of all examples of bijective PBW extensions presented in [83].

(ii) For the algebra of quantum polynomials $\mathcal{O}_{\mathbf{q}}$, Artamonov proved that under ceratin conditions on the system of parameters $\mathbf{q} := [q_{ij}]$, if P is a f.g. projective module over $\mathcal{O}_{\mathbf{q}}$ of rank at least 2, then P is free (the *rank* of P is the dimension of $Q(\mathcal{O}_{\mathbf{q}}) \otimes_{\mathcal{O}_{\mathbf{q}}} P$; see [5], Theorem 5.3 and Corollary 5.39; [4], Theorems 4.1 and 4.2; [6], Theorems 1.3 and 1.12). Thus, $\mathcal{O}_{\mathbf{q}}$ is 2-$\mathcal{H}$.

4.3 Kronecker's theorem

Closely related to the stable range theorem is the Kronecker's theorem stating that if S is a commutative ring with $\mathrm{Kdim}(S) < d$, then every finitely generated ideal I of S has the same radical as an ideal generated by d elements. In this section, we want to investigate this theorem for noncommutative rings using the Zariski lattice and the boundary ideal, but generalizing these tools and their properties to noncommutative rings. The main result will be applied to skew PBW extensions.

Definition 4.3.1. *Let S be a ring and $Spec(S)$ the set of all prime ideals of S. The Zariski lattice of S is defined by*

$$Zar(S) := \{D(X)|X \subseteq S\}, \text{ with } D(X) := \bigcap_{X \subseteq P \in Spec(S)} P.$$

$Zar(S)$ is ordered with respect to the inclusion. The description of the Zariski lattice is presented in the next proposition, $\langle X\}, \langle X\rangle, \{X\rangle$ will represent the left, two-sided, and right ideal of S generated by X, respectively. $\vee$ denotes the $\sup$ and $\wedge$ the $\inf$.

Proposition 4.3.2. *Let S be a ring, I, I_1, I_2, I_3 two-sided ideals of S, $X \subseteq S$, and $x_1, \dots, x_n, x, y \in S$. Then,*

(i) $D(X) = D(\langle X\}) = D(\langle X\rangle) = D(\{X\rangle)$.

(ii) $D(I) = rad(S)$ *if and only if* $I \subseteq rad(S)$. *In particular,* $D(0) = rad(S)$.

(iii) $D(I) = S$ *if and only if* $I = S$.

(iv) $I \subseteq D(I)$ *and* $D(D(I)) = D(I)$. *Moreover, if* $I_1 \subseteq I_2$, *then* $D(I_1) \subseteq D(I_2)$.

(v) *Let* $\{I_j\}_{j\in\mathcal{J}}$ *a family of two-sided ideals of S. Then,* $D(\sum_{j\in\mathcal{J}} I_j) = \vee_{j\in\mathcal{J}} D(I_j)$. *In particular,* $D(x_1, \dots, x_n) = D(x_1) \vee \cdots \vee D(x_n)$.

(vi) $D(I_1 I_2) = D(I_1) \wedge D(I_2)$. *In particular,* $D(\langle x\rangle\langle y\rangle) = D(x) \wedge D(y)$.

(vii) $D(x + y) \subseteq D(x, y)$.

(viii) *If* $\langle x\rangle\langle y\rangle \subseteq D(0)$, *then* $D(x, y) = D(x + y)$.

(ix) *If* $x \in D(I)$, *then* $D(I) = D(I, x)$.

(x) *If* $\overline{S} := S/I$, *then* $D(\overline{J}) = \overline{D(J)}$, *for any two-sided ideal J of S containing I.*

(xi) $u \in D(I)$ *if and only if* $\overline{u} \in rad(S/I)$. *In such case, if* $u \in D(I)$, *there exists* $k \geq 1$ *such that* $u^k \in I$.

(xii) $Zar(S)$ *is distributive:*

$$D(I_1) \wedge [D(I_2) \vee D(I_3)] = [D(I_1) \wedge D(I_2)] \vee [D(I_1) \wedge D(I_3)],$$
$$D(I_1) \vee [D(I_2) \wedge D(I_3)] = [D(I_1) \vee D(I_2)] \wedge [D(_1) \vee D(I_3)].$$

Proof. (i), (ii), (iv), (ix) and (x) are evident from the definitions.

(iii) If $I = S$ there is no prime ideal containing I, so the intersection of prime ideals containing I is taken equals S (see [51], p. 51). Conversely, if $I \neq S$ the intersection of proper ideals containing I is proper (this collection is not empty since I is contained in at least one prime ideal), thus $D(I) \neq S$.

(v) We prove, first, that $\vee_{j\in\mathcal{J}} D(I_j) = D(\sum_{j\in\mathcal{J}} D(I_j))$: for every $j \in \mathcal{J}$, $D(I_j) \subseteq \sum_{j\in\mathcal{J}} D(I_j) \subseteq D(\sum_{j\in\mathcal{J}} D(I_j))$; let $D(I) \supseteq D(I_j)$ for every $j \in \mathcal{J}$, then $D(I) \supseteq \sum_{j\in\mathcal{J}} D(I_j)$ and hence $D(I) = D(D(I)) \supseteq D(\sum_{j\in\mathcal{J}} D(I_j))$.

It only remains to show that $D(\sum_{j\in\mathcal{J}} D(I_j)) = D(\sum_{j\in\mathcal{J}} I_j)$: since $I_j \subseteq \sum_{j\in\mathcal{J}} I_j$, then $D(I_j) \subseteq D(\sum_{j\in\mathcal{J}} I_j)$, so $D(\sum_{j\in\mathcal{J}} I_j) \supseteq \vee_{j\in\mathcal{J}} D(I_j) = D(\sum_{j\in\mathcal{J}} D(I_j))$; on the other hand, $D(\sum_{j\in\mathcal{J}} D(I_j)) \supseteq \sum_{j\in\mathcal{J}} D(I_j) \supseteq \sum_{j\in\mathcal{J}} I_j$, so $D(D(\sum_{j\in\mathcal{J}} D(I_j))) \supseteq D(\sum_{j\in\mathcal{J}} I_j)$, thus $D(\sum_{j\in\mathcal{J}} D(I_j)) \supseteq D(\sum_{j\in\mathcal{J}} I_j)$.

(vi) It is clear that $D(I_1I_2) \subseteq D(I_1), D(I_2)$. Let I be a two-side ideal of S such that $D(I) \subseteq D(I_1), D(I_2)$, then $D(I) \subseteq D(I_1) \cap D(I_2) \subseteq D(I_1I_2)$. The last inclusion follows from the fact that if P is a prime ideal containing I_1I_2, then $I_1 \subseteq P$ or $I_2 \subseteq P$, thus if $x \in D(I_1)\cap D(I_2)$, then $x \in P$, i.e., $x \in D(I_1I_2)$. This implies that $D(I_1)\wedge D(I_2) = D(I_1I_2)$.

(vii) Since $\langle x + y\rangle \subseteq \langle x, y\rangle$, then the result follows from (iv).

(viii) According to (vii), $D(x + y) \subseteq D(x, y)$; for the other inclusion, note first that $D(x, y) = D(x+y, \langle x\rangle\langle y\rangle)$: the inclusion $D(x+y, \langle x\rangle\langle y\rangle) \subseteq D(x, y)$ is clear since any prime ideal containing x, y contains $x + y, \langle x\rangle\langle y\rangle$. Let P be a prime that contains $x + y, \langle x\rangle\langle y\rangle$, so $x \in P$ or $y \in P$, in the first case $x \in P$ and $y \in P$ and the same it is true for the second case. This implies that $D(x, y) \subseteq D(x + y, \langle x\rangle\langle y\rangle)$.

By the hypothesis and numeral (ii), $\langle x\rangle\langle y\rangle \subseteq rad(S)$, i.e., $\langle x\rangle\langle y\rangle$ is contained in all primes, so $D(x + y, \langle x\rangle\langle y\rangle) = D(x + y)$ and hence $D(x, y) = D(x + y)$.

(xi) The first assertion is clear from the definition of $D(I)$ and $rad(S/I)$. If $u \in D(I)$, then $\overline{u} \in rad(S/I)$ and hence $\overline{u}$ is strongly nilpotent, but this implies that $\overline{u}$ is nilpotent (see [95]), i.e., there exists $k \geq 1$ such that $\overline{u}^k = \overline{0}$, i.e., $u^k \in I$.

(xii) For the first identity we have:

$$D(I_1) \wedge [D(I_2) \vee D(I_3)] = D(I_1) \wedge D(I_2 + I_3) = D[I_1(I_2 + I_3)] = D(I_1I_2 + I_1I_3) = \\ D(I_1I_2) \vee D(I_1I_3) = [D(I_1) \wedge D(I_2)] \vee [D(I_1) \wedge D(I_3].$$

For the second relation we have

$$D(I_1) \vee [D(_2) \wedge D(I_3)] = D(I_1) \vee D(I_2I_3) = D(I_1 + I_2I_3) \supseteq D[(I_1 + I_2)(I_1 + I_3)] = \\ [D(I_1) \vee D(I_2)] \wedge [D(I_1) \vee D(I_3)];$$

the other inclusion follows from the fact that $D(I_1 + I_2I_3) \subseteq D[(I_1 + I_2)(I_1 + I_3)]$ since if P is a prime ideal that contains $(I_1 + I_2)(I_1 + I_3)$, then $P \supseteq (I_1 + I_2)$ or $P \supseteq (I_1 + I_3)$, thus $P \supseteq I_1$ and $P \supseteq I_2 \supseteq I_2I_3$, or, $P \supseteq I_1$ and $P \supseteq I_3 \supseteq I_2I_3$, i.e., $P \supseteq I_1 + I_2I_3$. □

Definition 4.3.3. *Let S be a ring and $v \in S$, the boundary ideal of v is defined by $I_v := \langle v\rangle + (D(0) : \langle v\rangle)$, where $(D(0) : \langle v\rangle) := \{x \in S|\langle v\rangle x \subseteq D(0)\}$.*

Note that $I_v \neq 0$ for every $v \in S$. On the other hand, if v is invertible or if $v = 0$, then $I_v = S$. If S is a domain and $v \neq 0$, then $I_v = \langle v\rangle$.

Definition 4.3.4. *Let S be a ring such that* $\mathrm{lKdim}(S)$ *exists. We say the S satisfies the boundary condition if for any $d \geq 0$ and every $v \in S$,*

$$\mathrm{lKdim}(S) \leq d \Rightarrow \mathrm{lKdim}(S/I_v) \leq d - 1.$$

Example 4.3.5. (i) Any commutative Noetherian ring satisfies the boundary condition: indeed, for commutative Noetherian rings, the classical Krull dimension and the Krull dimension coincide, so we can apply Theorem 13.2 in [88].

(ii) Any prime ring S with left Krull dimension satisfies the boundary condition: in fact, for prime rings, any non-zero two sided ideal is essential, so $\mathrm{lKdim}(S/I_v) < \mathrm{lKdim}(S)$ (see [95], Proposition 6.3.10).

(iii) Any domain with left Krull dimension satisfies the boundary condition: indeed, any domain is a prime ring.

Remark 4.3.6. In [29], a constructive notion of classical Krull dimension for commutative rings is presented. Such concept is used to give a constructive proof of Stable Range Theorem for commutative case. Since in right FBN rings [1] the classical Krull dimension and module theoretic left (right) Krull dimension coincides (see e.g., [51], Theorem 15.13), we could think that this constructive notion holds over these rings. Nevertheless, for this, the boundary condition must be satisfied which, in general, is not true for FBN rings: let $S = M_2(\Bbbk)$, with $\Bbbk$ a field. Thus S is semisimple and, hence, an artinian ring. Since S has not essential ideals, S is a FBN ring. Now, note that $Rad(S) = rad(S) = 0$; so, if $v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $I_v \neq S$ and $\mathrm{lKdim}(S/I_v) = 0$, i.e., S does not satisfy the boundary condition: indeed, if $u = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in (rad(S) : \langle v \rangle)$, in particular we must have that $vu = 0$, i.e., $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The latter implies $x = 0$ and $y = 0$ and $u = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}$, with $z, w \in \Bbbk$ arbitraries. But, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \langle v \rangle$, and thus $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, i.e., $z = 0$ and $w = 0$, therefore, if $u \in (rad(S) : \langle v \rangle)$, then $u = 0$. As $v \notin M_2(\Bbbk)^*$, then $I_v \neq S$ and $\mathrm{lKdim}(S/I_v) = 0$, since S/I_v is artinian.

Theorem 4.3.7 (Kronecker). *Let S be a domain such that $\mathrm{lKdim}(S)$ exists. If $\mathrm{lKdim}(S) < d$ and $u_1, \ldots, u_d, u \in S$, then there exist $x_1, \ldots, x_d \in S$ such that*

$$D(u_1, \ldots, u_d, u) = D(u_1 + x_1 u, \ldots, u_d + x_d u).$$

Proof. The proof is by induction on d. Let $d = 1$ and $u_1, u \in S$, if $\mathrm{lKdim}(S) = -1$, then by definition $S = 0$ and $u_1, u = 0$, so we take $x_1 := 0$. Let $\mathrm{lKdim}(S) = 0$; by the boundary condition, $\mathrm{lKdim}(S/I_{u_1}) = -1$, i.e., $S = I_{u_1} = \langle u_1 \rangle + (D(0) : \langle u_1 \rangle)$. There exist $c_1, c_1', \ldots, c_l, c_l' \in S$ and $x_1 \in (D(0) : \langle u_1 \rangle)$ such that $1 = c_1 u_1 c_1' + \cdots + c_l u_1 c_l' + x_1$, then $\langle u_1 \rangle \langle x_1 \rangle \subseteq D(0)$ and $u = c_1 u_1 c_1' u + \cdots + c_l u_1 c_l' u + x_1 u$, thus $u \in \langle u_1, x_1 u \rangle$ and hence $u \in D(u_1, x_1 u)$ (Proposition 4.3.2, part (iv)). Moreover, $\langle u_1 \rangle \langle x_1 u \rangle \subseteq D(0)$, then by Proposition 4.3.2, part (viii), $D(u_1, x_1 u) = D(u_1 + x_1 u)$. Thus, $u \in D(u_1 + x_1 u)$, so $D(u_1 + x_1 u) = D(u_1 + x_1 u, u)$ (Proposition 4.3.2, part (ix)), but $D(u_1 + x_1 u, u) = D(u_1, u)$ since $\langle u_1 + x_1 u, u \rangle = \langle u_1, u \rangle$, so $D(u_1, u) = D(u_1 + x_1 u)$.

Now, let us assume that the proposition is true for rings with left Krull dimension $< d - 1$, $d \geq 2$, and let S be a ring with $\mathrm{lKdim}(S) < d$. Let $u_1, \ldots, u_d, u \in S$. We consider two cases.

[1] A prime ring is **right bounded** if every essential right ideal contains a nonzero ideal; a ring S is **right fully bounded** if S/P is right bounded for each prime ideal P of S. Thus, bounded or fully bounded, means the ring also has the left-handed property. A ring S is **right FBN** (respectively **FBN**) is a right Noetherian ring fully bounded (respectively, a Noetherian fully bounded ring).

Case 1. If $u_d = 0$, then the theorem is trivial with $x_1 = \cdots = x_{d-1} = 0$, $x_d = 1$.

Case 2. Let $u_d \neq 0$. Let I be the boundary ideal of u_d, then $D(I) = \langle u_d \rangle$. We consider the elements $\overline{u_1}, \ldots, \overline{u_{d-1}}, \overline{u} \in \overline{S}$, with $\overline{S} := S/I$. By the hypothesis, $\mathrm{lKdim}(\overline{S}) < d-1$ and hence there exist elements $\overline{x_1}, \ldots, \overline{x_{d-1}} \in \overline{S}$ such that $D(\overline{u_1}, \ldots, \overline{u_{d-1}}, \overline{u}) = D(\overline{u_1} + \overline{x_1}\,\overline{u}, \ldots, \overline{u_{d-1}} + \overline{x_{d-1}}\,\overline{u})$. From this, we get that

$$D(\overline{\langle u_1, \ldots, u_{d-1}, u\rangle + I}) = D(\overline{\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u\rangle + I}),$$

but by Proposition 4.3.2, part (x),

$$\overline{D(\langle u_1, \ldots, u_{d-1}, u\rangle + I)} = \overline{D(\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u\rangle + I)}, \text{ i.e.,}$$

$$D(\langle u_1, \ldots, u_{d-1}, u\rangle + I) = D(\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u\rangle + I).$$

Since $u \in \langle u_1, \ldots, u_{d-1}, u\rangle + I \subseteq D(\langle u_1, \ldots, u_{d-1}, u\rangle + I)$, then $u \in D(\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u\rangle + I) = D(\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u\rangle \vee D(I) = D(\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d)$. Taking $x_d := 0$ we get that $u \in D(u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d + x_d u)$. From this, and using Proposition 4.3.2, part (ix), we conclude that

$$D(u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d + x_d u) = D(u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d + x_d u, u)$$

however note that

$$\langle u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d + x_d u, u\rangle = \langle u_1, \ldots, u_{d-1}, u_d, u\rangle,$$

so $D(u_1 + x_1 u, \ldots, u_{d-1} + x_{d-1} u, u_d + x_d u) = D(u_1, \ldots, u_{d-1}, u_d, u)$. □

Corollary 4.3.8. *Let S be a domain such that* $\mathrm{lKdim}(S)$ *exists. If* $\mathrm{lKdim}(S) < d$ *and* $u_1, \ldots, u_{d+1} \in S$ *are such that* $\langle u_1, \ldots, u_{d+1}\rangle = S$*, then there exist elements* $x_1, \ldots, x_d \in S$ *such that* $\langle u_1 + x_1 u_{d+1}, \ldots, u_d + x_d u_{d+1}\rangle = S$.

Proof. The statement follows directly from Proposition 4.3.2, part (iii), and Theorem 4.3.7. □

Corollary 4.3.9. *Let* $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ *be a bijective skew PBW extension of a left Noetherian domain R. If* $\mathrm{lKdim}(R) < d$ *and* $u_1, \ldots, u_{d+n}, u \in A$*, then there exist* $y_1, \ldots, y_{d+n} \in A$ *such that*

$$D(u_1, \ldots, u_{d+n}, u) = D(u_1 + y_1 u, \ldots, u_{d+n} + y_{d+n} u).$$

Proof. This follows directly from Proposition 1.2.4, Theorem 1.2.9, Theorem 4.2 in [83], and Theorem 4.3.7. □

CHAPTER 5

Gröbner bases for skew PBW extensions

In order to make constructive the theory of projective modules, stably free modules and Hermite rings studied in the previous chapters, we will study the theory of Gröbner bases of left (right) ideals and modules for bijective skew PBW extensions in the current chapter. This theory was initially investigated in [40], [57] and [58] for the particular case of quasi-commutative bijective skew PBW extensions. We will extend the theory to arbitrary bijective skew PBW extensions, in particular, Buchberger's algorithm will be established for general bijective extensions. We start recalling the basic facts of Gröbner theory for arbitrary skew PBW extensions; we will use the notation given in Definition 1.2.1.

5.1 Monomial orders in skew PBW extensions

Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be an arbitrary skew PBW extension of R and let $\succeq$ be a total order defined on $Mon(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$ we will write $x^\alpha \succ x^\beta$. Further, $x^\beta \preceq x^\alpha$ means that $x^\alpha \succeq x^\beta$. Let $f \neq 0$ be a polynomial of A, if

$$f = c_1X_1 + \cdots + c_tX_t,$$

with $c_i \in R - \{0\}$ and $X_1 \succ \cdots \succ X_t$ are the monomials of f, then $lm(f) := X_1$ is the *leading monomial* of f, $lc(f) := c_1$ is the *leading coefficient* of f and $lt(f) := c_1X_1$ is the *leading term* of f. If $f = 0$, we define $lm(0) := 0, lc(0) := 0, lt(0) := 0$, and we set $X \succ 0$ for any $X \in Mon(A)$. Thus, we extend $\succeq$ to $Mon(A) \cup \{0\}$.

Definition 5.1.1. *Let $\succeq$ be a total order on $Mon(A)$, it says that $\succeq$ is a monomial order on $Mon(A)$ if the following conditions hold:*

(i) *For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in Mon(A)$*

$$x^\beta \succeq x^\alpha \Rightarrow lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda).$$

(ii) *$x^\alpha \succeq 1$, for every $x^\alpha \in Mon(A)$.*

(iii) $\succeq$ *is degree compatible, i.e.,* $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

Monomial orders are also called *admissible orders*. The condition (iii) of the previous definition is needed for the proof of the following proposition, and this one will be used in the division algorithm (Theorem 5.2.6).

Proposition 5.1.2. *Every monomial order on $Mon(A)$ is a well-order. Thus, there are not infinite decreasing chains in $Mon(A)$.*

Proof. See [40], Proposition 12. □

From now on, we will assume that $Mon(A)$ is endowed with some monomial order.

Definition 5.1.3. *Let $x^\alpha, x^\beta \in Mon(A)$, we say that x^α divides x^β, denoted by $x^\alpha | x^\beta$, if there exists $x^\gamma, x^\lambda \in Mon(A)$ such that $x^\beta = lm(x^\gamma x^\alpha x^\lambda)$. We will also say that any monomial $x^\alpha \in Mon(A)$ divides the polynomial zero.*

Proposition 5.1.4. *Let $x^\alpha, x^\beta \in Mon(A)$ and $f, g \in A - \{0\}$. Then,*

(a) $lm(x^\alpha g) = lm(x^\alpha lm(g)) = x^{\alpha+\exp(lm(g))}$, *i.e.,* $\exp(lm(x^\alpha g)) = \alpha + \exp(lm(g)$. *In particular,*

$$lm(lm(f)lm(g)) = x^{\exp(lm(f))+\exp(lm(g))}, \text{ i.e.,}$$
$$\exp(lm(lm(f)lm(g))) = \exp(lm(f)) + \exp(lm(g))$$

and

$$lm(x^\alpha x^\beta) = x^{\alpha+\beta}, \ i.e., \ \exp(lm(x^\alpha x^\beta)) = \alpha + \beta. \tag{5.1.1}$$

(b) *The following conditions are equivalent:*

(i) $x^\alpha | x^\beta$.

(ii) *There exists a unique $x^\theta \in Mon(A)$ such that $x^\beta = lm(x^\theta x^\alpha) = x^{\theta+\alpha}$ and hence $\beta = \theta + \alpha$.*

(iii) *There exists a unique $x^\theta \in Mon(A)$ such that $x^\beta = lm(x^\alpha x^\theta) = x^{\alpha+\theta}$ and hence $\beta = \alpha + \theta$.*

(iv) $\beta_i \geq \alpha_i$ *for $1 \leq i \leq n$, with $\beta := (\beta_1, \ldots, \beta_n)$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$.*

Proof. See [40], Proposition 14. □

Remark 5.1.5. We note that a least common multiple of monomials of $Mon(A)$ there exists: in fact, let $x^\alpha, x^\beta \in Mon(A)$, then $lcm(x^\alpha, x^\beta) = x^\gamma \in Mon(A)$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i := \max\{\alpha_i, \beta_i\}$ for each $1 \leq i \leq n$.

5.2 Reduction in skew PBW extensions

Some natural computational conditions on R will be assumed in the remaining sections of this thesis (see [75]).

Definition 5.2.1. *A ring R is left Gröbner soluble (LGS) if the following conditions hold:*

(i) *R is left Noetherian.*

(ii) *Given $a, r_1, \dots, r_m \in R$ there exists an algorithm which decides whether a is in the left ideal $Rr_1 + \cdots + Rr_m$, and if so, find $b_1, \dots, b_m \in R$ such that $a = b_1r_1 + \cdots + b_mr_m$.*

(iii) *Given $r_1, \dots, r_m \in R$ there exists an algorithm which finds a finite set of generators of the left R-module*

$$Syz_R[r_1 \ \cdots \ r_m] := \{(b_1, \dots, b_m) \in R^m | b_1r_1 + \cdots + b_mr_m = 0\}.$$

Remark 5.2.2. The three above conditions imposed to R are needed in order to guarantee a Gröbner theory in the rings of coefficients, in particular, to have an effective solution of the membership problem in R (see (ii) in Definition 5.2.3 below). **From now on we will assume that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is a LGS ring and $Mon(A)$ is endowed with some monomial order.**

Definition 5.2.3. *Let F be a finite set of non-zero elements of A, and let $f, h \in A$, we say that f reduces to h by F in one step, denoted $f \xrightarrow{F} h$, if there exist elements $f_1, \dots, f_t \in F$ and $r_1, \dots, r_t \in R$ such that*

(i) *$lm(f_i)|lm(f)$, $1 \leq i \leq t$, i.e., there exists $x^{\alpha_i} \in Mon(A)$ such that $lm(f) = lm(x^{\alpha_i}lm(f_i))$, i.e., $\alpha_i + \exp(lm(f_i)) = \exp(lm(f))$.*

(ii) *$lc(f) = r_1\sigma^{\alpha_1}(lc(f_1))c_{\alpha_1,f_1} + \cdots + r_t\sigma^{\alpha_t}(lc(f_t))c_{\alpha_t,f_t}$, where c_{α_i,f_i} are defined as in Theorem 1.2.2, i.e., $c_{\alpha_i,f_i} := c_{\alpha_i,\exp(lm(f_i))}$.*

(iii) *$h = f - \sum_{i=1}^{t} r_ix^{\alpha_i}f_i$.*

We say that f reduces to h by F, denoted $f \xrightarrow{F}_+ h$, if there exist $h_1, \dots, h_{t-1} \in A$ such that

$$f \xrightarrow{F} h_1 \xrightarrow{F} h_2 \xrightarrow{F} \cdots \xrightarrow{F} h_{t-1} \xrightarrow{F} h$$

f is reduced (also called minimal) w.r.t.. F if $f = 0$ or there is no one step reduction of f by F, i.e., one of the first two conditions of Definition 5.2.3 fails. Otherwise, we will say that f is reducible w.r.t. F. If $f \xrightarrow{F}_+ h$ and h is reduced w.r.t. F, then we say that h is a remainder for f w.r.t. F.

Remark 5.2.4. (i) By Theorem 1.2.2, the coefficients c_{α_i,f_i} in the previous definition are unique and satisfy

$$x^{\alpha_i}lm(f_i) = c_{\alpha_i,f_i}x^{\alpha_i+\exp(lm(f_i))} + p_{\alpha_i,f_i},$$

where $p_{\alpha_i,f_i} = 0$ or $\deg(p_{\alpha_i,f_i}) < |\alpha_i + \exp(lm(f_i))|$, $1 \leq i \leq t$.

(ii) $lm(f) \succ lm(h)$ and $f - h \in \langle F\}$, where $\langle F\}$ denotes the left ideal of A generated by F.

(iii) The remainder of f is not unique.

(iv) By definition we will assume that $0 \xrightarrow{F} 0$.

From the reduction relation we get the following interesting properties.

Proposition 5.2.5. *Let A be a skew PBW extension such that $c_{\alpha,\beta}$ is invertible for each $\alpha, \beta \in \mathbb{N}^n$. Let $f, h \in A$, $\theta \in \mathbb{N}^n$ and $F = \{f_1, \ldots, f_t\}$ be a finite set of non-zero polynomials of A. Then,*

(i) *If $f \xrightarrow{F} h$, then there exists $p \in A$ with $p = 0$ or $lm(x^\theta f) \succ lm(p)$ such that $x^\theta f + p \xrightarrow{F} x^\theta h$. In particular, if A is quasi-commutative, then $p = 0$.*

(ii) *If $f \xrightarrow{F}_+ h$ and $p \in A$ is such that $p = 0$ or $lm(h) \succ lm(p)$, then $f + p \xrightarrow{F}_+ h + p$.*

(iii) *If $f \xrightarrow{F}_+ h$, then there exists $p \in A$ with $p = 0$ or $lm(x^\theta f) \succ lm(p)$ such that $x^\theta f + p \xrightarrow{F}_+ x^\theta h$. If A is quasi-commutative, then $p = 0$.*

(iv) *If $f \xrightarrow{F}_+ 0$, then there exists $p \in A$ with $p = 0$ or $lm(x^\theta f) \succ lm(p)$ such that $x^\theta f + p \xrightarrow{F}_+ 0$. If A is quasi-commutative, then $p = 0$.*

Proof. See [40], Proposition 20. □

The next theorem is the theoretical support of the division algorithm for skew PBW extensions.

Theorem 5.2.6. *Let $F = \{f_1, \ldots, f_t\}$ be a finite set of non-zero polynomials of A and $f \in A$, then the division algorithm below produces polynomials $q_1, \ldots, q_t, h \in A$, with h reduced w.r.t. F, such that $f \xrightarrow{F}_+ h$ and*

$$f = q_1 f_1 + \cdots + q_t f_t + h,$$

with

$$lm(f) = \max\{lm(lm(q_1)lm(f_1)), \ldots, lm(lm(q_t)lm(f_t)), lm(h)\}.$$

Division algorithm in A

INPUT: $f, f_1, \ldots, f_t \in A$ *with* $f_j \neq 0\,(1 \leq j \leq t)$

OUTPUT: $q_1, \ldots, q_t, h \in A$ *with* $f = q_1 f_1 + \cdots + q_t f_t + h$, h *reduced w.r.t.* $\{f_1, \ldots, f_t\}$ *and*
$lm(f) = \max\{lm(lm(q_1)lm(f_1)), \ldots, lm(lm(q_t)lm(f_t)), lm(h)\}$

INITIALIZATION: $q_1 := 0, q_2 := 0, \ldots, q_t := 0, h := f$

WHILE $h \neq 0$ *and there exists* j *such that* $lm(f_j)$ *divides* $lm(h)$ **DO**

Calculate $J := \{j \mid lm(f_j)$ *divides* $lm(h)\}$

FOR $j \in J$ **DO**

Calculate $\alpha_j \in \mathbb{N}^n$ *such that* $\alpha_j + \exp(lm(f_j)) = \exp(lm(h))$

IF *the equation* $lc(h) = \sum_{j \in J} r_j \sigma^{\alpha_j}(lc(f_j)) c_{\alpha_j, f_j}$ *is soluble, where* c_{α_j, f_j} *are defined as in the Theorem 1.2.2* **THEN**

Calculate one solution $(r_j)_{j \in J}$

$h := h - \sum_{j \in J} r_j x^{\alpha_j} f_j$

FOR $j \in J$ **DO**

$q_j := q_j + r_j x^{\alpha_j}$

ELSE

Stop

Proof. See [40], Theorem 21. □

The following example illustrates the above procedure.

Example 5.2.7. For this example, we consider the Manin algebra (see Example 1.3.2) with $\Bbbk := \mathbb{Q}$, the order deglex on $Mon(\mathcal{O}_q(M_2(\mathbb{Q})))$ with $x \succ y \succ v$, and $q = \frac{-1}{2}$. Let $f = (3u^3+2u)x^2y^2v+(u-2)xyv+2uyv \in \mathcal{O}_q(M_2(\mathbb{Q}))$ and $G := \{f_1 := (u^2+1)xyv+2uv^2, f_2 := uxy + 3v, f_3 := (u-1)yv\}$. We will divide f by G using the above algorithm.
Step 1. We start with $h := f$, $q_1 := 0$, $q_2 := 0$, $q_3 := 0$. Since $lm(f_j) \mid lm(f)$ for$j = 1, 2, 3$, we compute $\alpha = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3}) \in \mathbb{N}^3$ such that $\alpha_j + \exp(lm(f_j)) = \exp(lm(h))$ and the

corresponding value of $\sigma^{\alpha_j}(lc(f_j))c_{\alpha_j,\beta_j}$, where $\beta_j = \exp(lm(f_j))$:

$$\begin{aligned}
&(\alpha_{11},\alpha_{12},\alpha_{13}) + (1,1,1) = (2,2,1) \Rightarrow \alpha_{11} = 1, \alpha_{12} = 1, \alpha_{13} = 0,\\
&\sigma^{\alpha_1}(lc(f_1))c_{\alpha_1,\beta_1} = \sigma_1\sigma_2\sigma_3^0(u^2+1) = u^2+1,\\
&(\alpha_{21},\alpha_{22},\alpha_{23}) + (1,1,0) = (2,2,1) \Rightarrow \alpha_{21} = 1, \alpha_{22} = 1, \alpha_{23} = 1,\\
&\sigma^{\alpha_2}(lc(f_2))c_{\alpha_2,\beta_2} = \sigma_1\sigma_2\sigma_3(u) = u,\\
&(\alpha_{31},\alpha_{32},\alpha_{33}) + (0,1,1) = (2,2,1) \Rightarrow \alpha_{31} = 2, \alpha_{32} = 1, \alpha_{33} = 0,\\
&\sigma^{\alpha_3}(lc(f_3))c_{\alpha_3,\beta_3} = \sigma_1^2\sigma_2\sigma_3^0(u-1) = -\frac{1}{2}u - 1.
\end{aligned}$$

Now, we solve the equation

$$lc(h) = 3u^3 + 2u = r_1(u^2+1) + r^2(u) + r_3(-\tfrac{1}{2}u - 1) \Rightarrow r_1 = 3u,\ r_2 = -1xc,\ r_3 = 0,$$

and with the relations defining $\mathcal{O}_q(M_2(\mathbb{Q}))$, we compute

$$\begin{aligned}
h =& h - (r_1x^{\alpha_1}f_1 + r_2x^{\alpha_2}f_2 + r_3x^{\alpha_3}f_3)\\
=& h - (r_1[(u^2+1)x^2y^2v + \left(-\frac{3}{8}u^3 - \frac{3}{8}u + 2u\right)xyv^2] + r_2[ux^2y^2v + \left(-\frac{3}{8}u^2 + 3\right)xyv^2] + 0)\\
=& \left(\frac{9}{8}u^4 - \frac{21}{4}u^2 + 3\right)xyv^2 + (u-2)xyv + 2uyv.
\end{aligned}$$

We compute also

$$q_1 := 3uxy,\ q_2 := -xyv,\ q_3 := 0.$$

Step 2. $lm(h) = xyv^2$, $lc(h) = \frac{9}{8}u^4 - \frac{21}{4}u^2 + 3$. Again, $lm(f_j) \mid lm(f)$ for $j = 1,2,3$, we compute $\alpha = (\alpha_{j1},\alpha_{j2},\alpha_{j3}) \in \mathbb{N}^3$ such that $\alpha_j + \exp(lm(f_j)) = \exp(lm(h))$ and $\sigma^{\alpha_j}(lc(f_j))c_{\alpha_j,\beta_j}$:

$$\begin{aligned}
&(\alpha_{11},\alpha_{12},\alpha_{13}) + (1,1,1) = (1,1,2) \Rightarrow \alpha_{11} = 0, \alpha_{12} = 0, \alpha_{13} = 1,\\
&\sigma^{\alpha_1}(lc(f_1))c_{\alpha_1,\beta_1} = \sigma_1^0\sigma_2^0\sigma_3(u^2+1) = u^2+1,\\
&(\alpha_{21},\alpha_{22},\alpha_{23}) + (1,1,0) = (1,1,2) \Rightarrow \alpha_{21} = 0, \alpha_{22} = 0, \alpha_{23} = 2,\\
&\sigma^{\alpha_2}(lc(f_2))c_{\alpha_2,\beta_2} = \sigma_1^0\sigma_2^0\sigma_3^2(u) = u,\\
&(\alpha_{31},\alpha_{32},\alpha_{33}) + (0,1,1) = (1,1,2) \Rightarrow \alpha_{31} = 1, \alpha_{32} = 0, \alpha_{33} = 1,\\
&\sigma^{\alpha_3}(lc(f_3))c_{\alpha_3,\beta_3} = \sigma_1\sigma_2^0\sigma_3^0(u-1)c_{\alpha_3,\beta_3} = \frac{1}{4}u + \frac{1}{2}.
\end{aligned}$$

We resolve the equation

$$lc(h) = \tfrac{9}{8}u^4 - \tfrac{21}{4}u^2 + 3 = r_1(u^2+1) + r^2(u) + r_3(\tfrac{1}{4}u + \tfrac{1}{2}) \Rightarrow r_1 = \tfrac{9}{8}u^2 - \tfrac{51}{8},\ r_2 = -\tfrac{75}{16},\ r_3 = \tfrac{75}{4};$$

we have:

$$\begin{aligned}
h =& h - (r_1x^{\alpha_1}f_1 + r_2x^{\alpha_2}f_2 + r_3x^{\alpha_3}f_3)\\
=& h - (r_1[(u^2+1)xyv^2 + 2uv^3] + r_2[uxyv^2 + 3v^3] + r_3[\left(\frac{1}{4} + \frac{1}{2}\right)xyv^2])\\
=& (u-2)xyv - \left(\frac{9}{4}u^3 - \frac{51}{4}u - \frac{225}{16}\right)v^3 + 2uyv.
\end{aligned}$$

and

$$q_1 := 3uxy + \left(\tfrac{9}{8}u^2 - \tfrac{51}{8}\right)v,\ q_2 := -xyv - \tfrac{75}{16}v^2,\ q_3 := \tfrac{75}{4}xv.$$

Step 3. Note that $lm(h) = xyv$ and $lm(f_j) \mid lm(h)$ for $j = 1, 2, 3$. For this case we have:

$$\begin{aligned}
&(\alpha_{11}, \alpha_{12}, \alpha_{13}) + (1, 1, 1) = (1, 1, 1) \Rightarrow \alpha_{11} = 0, \alpha_{12} = 0, \alpha_{13} = 0,\\
&\sigma^{\alpha_1}(lc(f_1))c_{\alpha_1,\beta_1} = \sigma_1^0\sigma_2^0\sigma_3^0(u^2 + 1) = u^2 + 1,\\
&(\alpha_{21}, \alpha_{22}, \alpha_{23}) + (1, 1, 0) = (1, 1, 1) \Rightarrow \alpha_{21} = 0, \alpha_{22} = 0, \alpha_{23} = 1,\\
&\sigma^{\alpha_2}(lc(f_2))c_{\alpha_2,\beta_2} = \sigma_1^0\sigma_2^0\sigma_3(u) = u,\\
&(\alpha_{31}, \alpha_{32}, \alpha_{33}) + (0, 1, 1) = (1, 1, 1) \Rightarrow \alpha_{31} = 1, \alpha_{32} = 0, \alpha_{33} = 0,\\
&\sigma^{\alpha_3}(lc(f_3))c_{\alpha_3,\beta_3} = \sigma_1\sigma_2^0\sigma_3^0(u - 1) = -\frac{1}{2}u - 1.
\end{aligned}$$

We solve,

$$u - 2 = r_1(u^2 + 1) + r_2(u) + r_3(-\tfrac{1}{2}u - 1) \Rightarrow r_1 = 0,\ r_2 = 2,\ r_3 = 2;$$

thus,

$$\begin{aligned}
h =&h - (r_1x^{\alpha_1}f_1 + r_2x^{\alpha_2}f_2 + r_3x^{\alpha_3}f_3)\\
=&h - (r_2[uxyv + 3v^2] + r_3[\left(-\frac{1}{2}u - 1\right)xyv])\\
=&-\left(\frac{9}{4}u^3 - \frac{51}{4}u - \frac{225}{16}\right)v^3 + 2uyv - 6v^2.
\end{aligned}$$

and also

$$q_1 := 3uxy + \left(\tfrac{9}{8}u^2 - \tfrac{51}{8}\right)v,\ q_2 := -xyv - \tfrac{75}{16}v^2 + 2v,\ q_3 := \tfrac{75}{4}xv + 2x.$$

Step 4. Since $lm(h) = v^3$ is not divisible by $lm(f_j)$ for $j = 1, 2, 3$, then h is reduced with respect to G, and we can check that $f = q_1f_1 + q_2f_2 + q_3f_3 + h$; i.e.,

$$\begin{aligned}
f =&\left(3uxy + \left(\frac{9}{8}u^2 - \frac{51}{8}\right)v\right)f_1 + \left(-xyv - \frac{75}{16}v^2 + 2v\right)f_2 + \left(\frac{75}{4}xv + 2x\right)f_3\\
&-\left(\frac{9}{4}u^3 - \frac{51}{4}u - \frac{225}{16}\right)v^3 + 2uyv - 6v^2;
\end{aligned}$$

we also see that,

$$\begin{gathered}
\max\{lm(lm(q_1)lm(f_1)), lm(lm(q_2)lm(f_2)), lm(lm(q_3)lm(f_3))\}\\
= \max\{x^2y^2v, x^2y^2v, xyv^2, v^3\} = x^2y^2v = lm(f).
\end{gathered}$$

5.3 Gröbner bases of left ideals

Our next purpose is to recall the definition of a Gröbner bases for the left ideals of the skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$. Remember that if $\varnothing \neq F \subseteq A$, with $\langle F\}$ we are denoting the left ideal of A generated by F.

Definition 5.3.1. *Let $I \neq 0$ be a left ideal of A and let G be a non empty finite subset of non-zero polynomials of I, we say that G is a Gröbner basis for I if each element $0 \neq f \in I$ is reducible w.r.t. G.*

We will say that $\{0\}$ is a Gröbner basis for $I = 0$.

Theorem 5.3.2. *Let $I \neq 0$ be a left ideal of A and let G be a finite subset of non-zero polynomials of I. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for I.*

(ii) *For any polynomial $f \in A$,*

$$f \in I \text{ if and only if } f \xrightarrow{G}_+ 0.$$

(iii) *For any $0 \neq f \in I$ there exist $g_1, \dots, g_t \in G$ such that $lm(g_j)|lm(f)$, $1 \leq j \leq t$, (i.e., there exist $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(lm(g_j)) = \exp(lm(f))$) and*

$$lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \dots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}.$$

(iv) *For $\alpha \in \mathbb{N}^n$, let $\langle \alpha, I\}$ be the left ideal of R defined by*

$$\langle \alpha, I\} := \langle lc(f)|f \in I, \exp(lm(f)) = \alpha\}.$$

Then, $\langle \alpha, I\} = J$, with

$$J := \langle \sigma^{\beta}(lc(g))c_{\beta,g}|g \in G, \text{ with } \beta + \exp(lm(g)) = \alpha\}.$$

Proof. See [40], Theorem 24. □

From this theorem we get the following consequences.

Corollary 5.3.3. *Let $I \neq 0$ be a left ideal of A. Then,*

(i) *If G is a Gröbner basis for I, then $I = \langle G\}$.*

(ii) *Let G be a Gröbner basis for I, if $f \in I$ and $f \xrightarrow{G}_+ h$, with h reduced, then $h = 0$.*

(iii) *Let $G = \{g_1, \dots, g_t\}$ be a set of non-zero polynomials of I with $lc(g_i) \in R^*$ for each $1 \leq i \leq t$. Then, G is a Gröbner basis of I if and only if given $0 \neq r \in I$ there exists i such that $lm(g_i)$ divides $lm(r)$.*

Proof. (i) This is a direct consequence of Theorem 5.3.2.

(ii) Let $f \in I$ and $f \xrightarrow{G}_+ h$, with h reduced; since $f - h \in \langle G\} = I$, then $h \in I$; if $h \neq 0$ then h can be reduced by G, but this is not possible since h is reduced.

(iii) If G is a Gröbner basis of I, then given $0 \neq r \in I$, r is reducible w.r.t. G, hence there exists i such that $lm(g_i)$ divides $lm(r)$. Conversely, if this condition holds for some i, then r is reducible w.r.t. G since the equation $lc(r) = r_1\sigma^{\alpha_i}(lc(g_i)c_{\alpha_i,g_i}$, with $\alpha_i + \exp(lm(g_i)) = \exp(lm(r))$, is soluble with solution $r_1 = lc(r)c'_{\alpha_i,g_i}(\sigma^{\alpha_i}(lc(g_i)))^{-1}$, where c'_{α_i,g_i} is a left inverse of c_{α_i,g_i}. □

Corollary 5.3.4. *Let G be a Gröbner basis for a left ideal I. Given $g \in G$, if g is reducible w.r.t. $G' = G - \{g\}$, then G' is a Gröbner basis for I.*

Proof. According to Theorem 5.3.2, it is enough to show that all $f \in I$ is reducible w.r.t G'. Let f be a nonzero polynomial in I; since G is a Gröbner basis for I, f is reducible w.r.t G and there exist elements $g_1, \dots, g_t \in G$ satisfying the conditions (i), (ii) and (iii) in the Definition 5.2.3. If $g \neq g_i$ for each $1 \leq i \leq t$, then we finished. Suppose that $g = g_j$ for some $j \in \{1, \dots, t\}$ and let $\beta_i = \exp(g_i)$ for $i \neq j$, $\beta = \exp(g)$, and $\alpha_i, \alpha \in \mathbb{N}^n$ such that $\alpha_i + \beta_i = \exp(f) = \alpha + \beta$. Thus,

$$lc(f) = r_1\sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,\beta_1} + \cdots + r_j\sigma^{\alpha}(lc(g))c_{\alpha,\beta} + \cdots + r_t\sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,\beta_t}.$$

On the other hand, since g is reducible w.r.t. G', there exist $g'_1, \dots, g'_s \in G'$ such that $lm(g'_l) \mid lm(g)$ and $lc(g) = \sum_{l=1}^{s} r'_l\sigma^{\alpha'_l}(lc(g'_l))c_{\alpha'_l,\beta'_l}$, where $\beta'_l = \exp(g'_l)$, $\alpha'_l \in \mathbb{N}^n$ and $\alpha'_l + \beta'_l = \exp(g) = \beta$. So, $lm(g'_l) \mid lm(f)$ for $1 \leq i \leq s$; moreover, using the identities of Remark 1.2.3, we have that

$$\begin{aligned}
\sigma^{\alpha}(lc(g))c_{\alpha,\beta} =& \sigma^{\alpha}(\sum_{l=1}^{s} r'_l\sigma^{\alpha'_l}(lc(g'_l))c_{\alpha'_l,\beta'_l})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)\sigma^{\alpha}\sigma^{\alpha'_1}(lc(g'_1))\sigma^{\alpha}(c_{\alpha'_1,\beta'_1})c_{\alpha,\beta} + \cdots + \sigma^{\alpha}(r'_s)\sigma^{\alpha}\sigma^{\alpha'_s}(lc(g'_s))\sigma^{\alpha}(c_{\alpha'_s,\beta'_s})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)c_{\alpha,\alpha'_1}\sigma^{\alpha+\alpha'_1}(lc(g'_1))c^{-1}_{\alpha,\alpha'_1}\sigma^{\alpha}(c_{\alpha'_1,\beta'_1})c_{\alpha,\beta} + \cdots + \\
& \sigma^{\alpha}(r'_s)c_{\alpha,\alpha'_s}\sigma^{\alpha+\alpha'_s}(lc(g'_s))c^{-1}_{\alpha,\alpha'_s}\sigma^{\alpha}(c_{\alpha'_s,\beta'_s})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)c_{\alpha,\alpha'_1}\sigma^{\alpha+\alpha'_1}(lc(g'_1))c_{\alpha+\alpha'_1,\beta'_1} + \cdots + \sigma^{\alpha}(r'_s)c_{\alpha,\alpha'_s}\sigma^{\alpha+\alpha'_s}(lc(g'_s))c_{\alpha+\alpha'_s,\beta'_s}.
\end{aligned}$$

Since $\alpha + \beta = \exp(f)$, then $\alpha + \alpha'_l + \beta'_l = \exp(f)$. Further, if $g_k \in \{g_1, \dots, g_t\}$ exists such that $g_k = g'_l$ for some $l \in \{1, \dots, s\}$, then $\beta'_l = \beta_k$ and $\alpha + \alpha'_l = \alpha_k$; therefore, in the representation of $lc(f)$ would appear the term $(r_k + r_j\sigma^{\alpha}(r'_l)c_{\alpha,\alpha'_l})\sigma^{\alpha_k}(lc(g_k))c_{\alpha_k,\beta_k}$. From above it follows that f is reducible w.r.t. G' and, hence, G' is a Gröbner basis for I. □

5.4 Buchberger's algorithm for left ideals

In [40] was constructed the Buchberger's algorithm for computing Gröbner bases of left ideals for the particular case of quasi-commutative bijective skew PBW extensions. In this section, we extend the Buchberger's procedure to the general case of bijective skew PBW extensions without assuming that they are quasi-commutative. Complementing Remark 5.2.2, **from now on we will assume that** $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ **is bijective**.

We start fixing some notation and proving a preliminary key result for bijective skew PBW extensions.

Definition 5.4.1. *Let $F := \{g_1, \dots, g_s\} \subseteq A$, X_F the least common multiple of $\{lm(g_1), \dots, lm(g_s)\}$, $\theta \in \mathbb{N}^n$, $\beta_i := \exp(lm(g_i))$ and $\gamma_i \in \mathbb{N}^n$ such that $\gamma_i + \beta_i = \exp(X_F)$, $1 \leq i \leq s$. $B_{F,\theta}$ will denote a finite set of generators of*

$$S_{F,\theta} := Syz_R[\sigma^{\gamma_1+\theta}(lc(g_1))c_{\gamma_1+\theta,\beta_1} \ \cdots \ \sigma^{\gamma_s+\theta}(lc(g_s))c_{\gamma_s+\theta,\beta_s})].$$

For $\theta = \mathbf{0} := (0, \dots, 0)$, $S_{F,\theta}$ will be denoted by S_F and $B_{F,\theta}$ by B_F.

Remark 5.4.2. Let $(b_1, \ldots, b_s) \in S_{F,\theta}$. If A is a bijective skew PBW extension, then there exists an unique $(b'_1, \ldots, b'_s) \in S_F$ such that $b_i = \sigma^\theta(b'_i)c_{\theta,\gamma_i}$ for $1 \leq i \leq s$: in fact, the existence and uniqueness of $(b'_1, \ldots, b'_s)$ it follows of the bijectivity of A. Now, since $(b_1, \ldots, b_s) \in S_{F,\theta}$, then $\sum_{i=1}^s b_i\sigma^{\theta+\gamma_i}(lc(g_i))c_{\theta+\gamma_i,\beta_i} = 0$. Replacing b_i by $\sigma^\theta(b'_i)c_{\theta,\gamma_i}$ in the last equation, we obtain $\sum_{i=1}^s \sigma^\theta(b'_i)c_{\theta,\gamma_i}\sigma^{\theta+\gamma_i}(lc(g_i))c_{\theta,\gamma_i}^{-1}c_{\theta,\gamma_i}c_{\theta+\gamma_i,\beta_i} = 0$; multiplying by $c_{\theta,\gamma_i+\beta_i}^{-1}$ we get $\sum_{i=1}^s \sigma^\theta(b'_i)c_{\theta,\gamma_i}\sigma^{\theta+\gamma_i}(lc(g_i))c_{\theta,\gamma_i}^{-1}c_{\theta,\gamma_i}c_{\theta+\gamma_i,\beta_i}c_{\theta,\gamma_i+\beta_i}^{-1} = 0$; now we can use the identities of Remark 1.2.3, so $\sum_{i=1}^s \sigma^\theta(b'_i)\sigma^\theta(\sigma^{\gamma_i}(lc(g_i)))\sigma^\theta(c_{\gamma_i,\beta_i}) = 0$, and since σ^θ is injective then $\sum_{i=1}^s b'_i\sigma^{\gamma_i}(lc(g_i))c_{\gamma_i,\beta_i} = 0$, i.e., $(b'_1, \ldots, b'_s) \in S_F$.

Lemma 5.4.3. *Let $g_1, \ldots, g_s \in A$, $c_1, \ldots, c_s \in R - \{0\}$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}^n$ be such that $\alpha_i + \exp(g_i) = \delta$. If $lm(\sum_{i=1}^s c_ix^{\alpha_i}g_i) \prec x^\delta$, then there exist $r_1, \ldots, r_k \in R$ and $l_1, \ldots, l_s \in A$ such that*

$$\sum_{i=1}^s c_ix^{\alpha_i}g_i = \sum_{j=1}^k r_jx^{\delta-\exp(X_F)}\left(\sum_{i=1}^s b_{ji}x^{\gamma_i}g_i\right) + \sum_{i=1}^s l_ig_i,$$

where X_F is the least common multiple of $lm(g_1), \ldots, lm(g_s)$, $\gamma_i \in \mathbb{N}^n$ is such that $\gamma_i+\exp(g_i) = \exp(X_F)$, $1 \leq i \leq s$, and $(b_{j1}, \ldots, b_{js}) \in B_F$. Moreover, we have that $lm(x^{\delta-\exp(X_F)}\sum_{i=1}^s b_{ji}x^{\gamma_i}g_i) \prec x^\delta$ and $lm(lm(l_i)lm(g_i)) \prec x^\delta$.

Proof. Let $x^{\beta_i} := lm(g_i)$ for $1 \leq i \leq s$; since $x^\delta = lm(x_i^\alpha lm(g_i))$, then $lm(g_i) \mid x^\delta$ and hence $X_F \mid x^\delta$, so there exists $\theta \in \mathbb{N}^n$ such that $\exp(X_F) + \theta = \delta$. On the other hand, $\gamma_i + \beta_i = \exp(X_F)$ and $\alpha_i + \beta_i = \delta$, so $\alpha_i = \gamma_i + \theta$ for every $1 \leq i \leq s$. Now, $lm(\sum_{i=1}^s c_ix^{\alpha_i}g_i) \prec x^\delta$ implies that $\sum_{i=1}^s c_i\sigma^{\alpha_i}(lc(g_i))c_{\alpha_i,\beta_i} = 0$. So we have $\sum_{i=1}^s c_i\sigma^{\theta+\gamma_i}(lc(g_i))c_{\theta+\gamma_i,\beta_i} = 0$. Hence, we have that $(c_1, \ldots, c_s) \in S_{F,\theta}$; from Remark 5.4.2 we know that exists an unique $(c'_1, \ldots, c'_s) \in S_F$ such that $c_i = \sigma^\theta(c'_i)c_{\theta,\gamma_i}$. Then,

$$\sum_{i=1}^s c_ix^{\alpha_i}g_i = \sum_{i=1}^s \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\alpha_i}g_i.$$

Now,

$$\begin{aligned} x^\theta c'_ix^{\gamma_i} &= (\sigma^\theta(c'_i)x^\theta + p_{c'_i,\theta})x^{\gamma_i} \\ &= \sigma^\theta(c'_i)x^\theta x^{\gamma_i} + p_{c'_i,\theta}x^{\gamma_i} \\ &= \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} + \sigma^\theta(c'_i)p_{\theta,\gamma_i} + p_{c'_i,\theta}x^{\gamma_i} \\ &= \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} + p'_i \end{aligned}$$

where $p'_i := \sigma^\theta(c'_i)p_{\theta,\gamma_i} + p_{c'_i,\theta}x^{\gamma_i}$; note that $p'_i = 0$ or $lm(p'_i) \prec x^{\theta+\gamma_i}$ for each $1 \leq i \leq s$. Thus, $\sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} = x^\theta c'_ix^{\gamma_i} + p_i$, with $p_i = 0$ or $lm(p_i) \prec x^{\theta+\gamma_i}$. Hence,

$$\begin{aligned} \sum_{i=1}^s c_ix^{\alpha_i}g_i &= \sum_{i=1}^s \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\alpha_i}g_i \\ &= \sum_{i=1}^s (x^\theta c'_ix^{\gamma_i} + p_i)g_i \\ &= \sum_{i=1}^s x^\theta c'_ix^{\gamma_i}g_i + \sum_{i=1}^s p_ig_i, \end{aligned}$$

con $p_i g_i = 0$ or $lm(lm(p_i)lm(g_i)) \prec x^{\theta+\gamma_i+\beta_i} = x^\delta$. On the other hand, let $B_F := \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k\} := \{(b_{11}, \ldots, b_{1s}), \ldots,$
$(b_{k1}, \ldots, b_{ks})\}$ be a set of generators of S_F; as $(c'_1, \ldots, c'_s) \in S_F$, then there exist $r'_1, \ldots, r'_k \in R$ such that $(c'_1, \ldots, c'_s) = r'_1\boldsymbol{b}_1 + \cdots + r'_k\boldsymbol{b}_k = r'_1(b_{11}, \ldots, b_{1s}) + \cdots + r'_k(b_{k1}, \ldots, b_{ks})$, thus $c'_i = \sum_{j=1}^k r'_j b_{ji}$. Using this, we have

$$\begin{aligned}
\sum_{i=1}^s x^\theta c'_i x^{\gamma_i} g_i &= \sum_{i=1}^s x^\theta \big(\sum_{j=1}^k r'_j b_{ji}\big) x^{\gamma_i} g_i \\
&= \sum_{i=1}^s \big(\sum_{j=1}^k x^\theta r'_j b_{ji}\big) x^{\gamma_i} g_i \\
&= \sum_{i=1}^s \big(\sum_{j=1}^k (\sigma^\theta(r'_j) x^\theta + p_{r'_j,\theta}) b_{ji}\big) x^{\gamma_i} g_i \\
&= \sum_{i=1}^s \big(\sum_{j=1}^k \sigma^\theta(r'_j) x^\theta b_{ji} x^{\gamma_i} g_i + \sum_{j=1}^k p_{r'_j,\theta} b_{ji} x^{\gamma_i} g_i\big) \\
&= \sum_{j=1}^k \sum_{i=1}^s \sigma^\theta(r'_j) x^\theta b_{ji} x^{\gamma_i} g_i + \sum_{i=1}^s \sum_{j=1}^k p_{r'_j,\theta} b_{ji} x^{\gamma_i} g_i \\
&= \sum_{j=1}^k \sigma^\theta(r'_j) x^\theta \sum_{i=1}^s b_{ji} x^{\gamma_i} g_i + \sum_{i=1}^s q_i g_i,
\end{aligned}$$

where $q_i := \sum_{j=1}^k p_{r'_j,\theta} b_{ji} x^{\gamma_i} = 0$ or $lm(q_i) \prec x^{\theta+\gamma_i}$. Therefore,

$$\sum_{i=1}^s c_i x^{\alpha_i} g_i = \sum_{j=1}^k r_j x^\theta \sum_{i=1}^s b_{ji} x^{\gamma_i} g_i + \sum_{i=1}^s l_i g_i,$$

with $l_i := p_i + q_i$ for $1 \leq i \leq s$ and $r_j := \sigma^\theta(r'_j)$ for $1 \leq j \leq k$. Finally, is easy to see $lm(x^\theta(\sum_{i=1}^s b_{ji} x^{\gamma_i} g_i)) \prec x^\delta$ since that $lm(\sum_{i=1}^s b_{ji} x^{\gamma_i} g_i) \prec x^{\gamma_i+\beta_i}$, and $lm(lm(l_i)lm(g_i)) \leq \max\{lm(lm(p_i)lm(g_i)), lm(lm(q_i)lm(g_i))\} \prec x^\delta$. $\square$

With the notation of Definition 5.4.1 and Lemma 5.4.3, we can prove the main result of the present section.

Theorem 5.4.4. *Let $I \neq 0$ be a left ideal of A and let G be a finite subset of non-zero generators of I. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of I.*

(ii) *For all $F := \{g_1, \ldots, g_s\} \subseteq G$, and for any $(b_1, \ldots, b_s) \in B_F$,*

$$\textstyle\sum_{i=1}^s b_i x^{\gamma_i} g_i \xrightarrow{G}_+ 0.$$

Proof. (i) ⇒ (ii): We observe that $f := \sum_{i=1}^{s} b_i x^{\gamma_i} g_i \in I$, so by Theorem 5.3.2 $f \xrightarrow{G}_+ 0$.

(ii) ⇒ (i): Let $G := \{g_1, \dots, g_t\}$, then there exist $h_1, \dots, h_t \in A$ such that $f = h_1 g_1 + \cdots + h_t g_t$ and we can choose $\{h_i\}_{i=1}^{t}$ such that $x^\delta := \max\{lm(lm(h_i)lm(g_i))\}_{i=1}^{t}$ is minimal. Let $x^{\alpha_i} := lm(h_i)$, $c_i := lc(h_i)$, $x^{\beta_i} := lm(g_i)$ for $1 \leq i \leq t$ and $F := \{g_i \in G \mid lm(lm(h_i)lm(g_i)) = x^\delta\}$; renumbering the elements of G we can assume that $F = \{g_1, \dots, g_s\}$. We will consider two possible cases.

Case 1: $lm(f) = x^\delta$. Then $lm(g_i) \mid lm(f)$ for $1 \leq i \leq s$ and

$$lc(f) = c_1 \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,\beta_1} + \cdots + c_s \sigma^{\alpha_s}(lc(g_s))c_{\alpha_s,\beta_s},$$

i.e., the condition (iii) of Theorem 5.3.2 holds.

Case 2: $lm(f) \prec x^\delta$. We will prove that this produces a contradiction. To begin, note that f can be written as

$$f = \sum_{i=1}^{s} c_i x^{\alpha_i} g_i + \sum_{i=1}^{s}(h_i - c_i x^{\alpha_i})g_i + \sum_{i=s+1}^{t} h_i g_i; \tag{5.4.1}$$

we have that $lm(lm(h_i - c_i x^{\alpha_i})lm(g_i)) \prec x^\delta$ for each $1 \leq i \leq s$, and $lm(lm(h_i)lm(g_i)) \prec x^\delta$ for every $s+1 \leq i \leq t$, so

$$lm(\textstyle\sum_{i=1}^{s} c_i x^{\alpha_i} g_i) \prec x^\delta \text{ and } lm(\sum_{i=s+1}^{t} h_i g_i) \prec x^\delta,$$

and hence $lm(\sum_{i=1}^{s} c_i x^{\alpha_i} g_i) \prec x^\delta$. By lemma 5.4.3 (and its notation), we have

$$\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{j=1}^{k} r_j x^{\delta - \exp(X_F)}\big(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i\big) + \sum_{i=1}^{s} l_i g_i, \tag{5.4.2}$$

where $lm(x^{\delta-\exp(X_F)} \sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i) \prec x^\delta$ for every $1 \leq j \leq k$ and $lm(lm(l_i)lm(g_i)) \prec x^\delta$ for $1 \leq i \leq s$. By hypothesis, $\sum_{i=1}^{s} b_{ji} x^{\gamma_i + \theta} g_i \xrightarrow{G}_+ 0$, and according to Theorem 5.2.6, there exist $q_1, \dots, q_t \in A$ such that $\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i = \sum_{i=1}^{t} q_i g_i$, with $lm(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i) = \max\{lm(lm(q_i)lm(g_i))\}_{i=1}^{t}$, but $(b_{j1}, \dots, b_{js}) \in B_F$, so $lm(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i) \prec X_F$ and hence $lm(lm(q_i)lm(g_i)) \prec X_F$ for every $1 \leq i \leq t$. Thus,

$$\begin{aligned}\sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)}\big(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i\big) &= \sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)}\big(\sum_{i=1}^{t} q_i g_i\big)\\ &= \sum_{i=1}^{t}\sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)} q_i g_i\\ &= \sum_{i=1}^{t} \widetilde{q}_i g_i,\end{aligned}$$

with $\widetilde{q}_i := \sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)} q_i$ and $lm(lm(\widetilde{q}_i)lm(g_i)) \prec x^\delta$. Substituting $\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{i=1}^{t} \widetilde{q}_i g_i + \sum_{i=1}^{s} l_i g_i$ into equation 5.4.1, we obtain

$$f = \sum_{i=1}^{t} \widetilde{q}_i g_i + \sum_{i=1}^{s}(h_i - c_i x^{\alpha_i})g_i + \sum_{i=1}^{s} l_i g_i + \sum_{i=s+1}^{t} h_i g_i,$$

and so we have expressed f as a combination of polynomials $g_1, \dots, g_t$, where every term has leading monomial $\prec x^\delta$. This contradicts the minimality of x^δ and we finish the proof. $\square$

Corollary 5.4.5. *Let $F = \{f_1, \dots, f_s\}$ be a set of non-zero polynomials of A. The algorithm below produces a Gröbner basis for the left ideal $\langle F\}$ of A ($P(X)$ denotes the set of subsets of the set X):*

Buchberger's algorithm for
bijective skew PBW extensions

INPUT: $F := \{f_1, \dots, f_s\} \subseteq A$, $f_i \neq 0$, $1 \leq i \leq s$

OUTPUT: $G = \{g_1, \dots, g_t\}$ *a Gröbner basis for* $\langle F\}$

INITIALIZATION: $G := \emptyset, G' := F$

WHILE $G' \neq G$ **DO**

$D := P(G') - P(G)$

$G := G'$

FOR *each* $S := \{g_{i_1}, \dots, g_{i_k}\} \in D$ **DO**

Compute B_S

FOR *each* $\boldsymbol{b} = (b_1, \dots, b_k) \in B_S$ **DO**

Reduce $\sum_{j=1}^k b_j x^{\gamma_j} g_{i_j} \xrightarrow{G'}_+ r$, *with* r *reduced with respect to* G' *and* γ_j *defined as in Definition 5.4.1*

IF $r \neq 0$ **THEN**

$G' := G' \cup \{r\}$

From Theorem 1.2.9 and the previous corollary we get the following direct conclusion.

Corollary 5.4.6. *Each left ideal of A has a Gröbner basis.*

Example 5.4.7. For this example, we consider a *diffusion algebra* described in Example 1.3.3. Let $n = 2$, $\Bbbk = \mathbb{Q}$, $d_{12} = -2$ and $d_{21} = -1$. In this ring, $D_2D_1 = 2D_1D_2 + x_2D_1 - x_1D_2$ and the automorphisms σ_1 and σ_2 are the identity. We consider the order deglex with $D_1 \succ D_2$ and the polynomials $f_1 = x_1^2x_2D_1^2D_2$, $f_2 = x_2^2D_1D_2^2$. We will calculate a Gröbner basis for the left ideal generated by f_1 and f_2.
We start taking $G := \varnothing$ and $G' := \{f_1, f_2\}$.
Step 1. Since $G' \neq G$, we have $D = \{S_1, S_2, S_{1,2}\}$.
We make $G = G'$.
Since R has not zero divisors, S_1 and S_2 do not add any polynomial to G'. For $S_{1,2}$, we compute $B_{S_{1,2}}$, a generator set of $Syz_R[\sigma^{\gamma_1}(lc(f_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_2}(lc(f_2))c_{\gamma_2,\beta_2}]$: $X_{1,2} = lcm\{D_1^2D_2, D_1D_2^2\} = D_1^2D_2^2$, so $\gamma_1 = (0,1)$ and $D_2(D_1^2D_2) = 4D_1^2D_2^2 + 3x_2D_1^2D_2 -$

$3x_1D_1D_2^2 - x_1x_2D_1D_2 + x_1D_2^2$, thus $c_{\gamma_1,\beta_1} = 4$; in a similar way, $\gamma_2 = (1,0)$ and $c_{\gamma_2,\beta_2} = 1$. Whence, $B_{S_{1,2}} = \{(\frac{1}{4}x_2, -x_1^2)\}$ and we have

$$\tfrac{1}{4}x_2D_2f_1 - x_1^2D_1f_2 = \tfrac{3}{4}x_1^2x_2^3D_1^2D_2 - x_1^3x_2^2D_1D_2^2 - \tfrac{1}{4}x_1^3x_2^3D_1D_2 + \tfrac{1}{4}x_1^4x_2^2D_2^2;$$

Since that

$$\tfrac{3}{4}x_1^2x_2^3D_1^2D_2 - x_1^3x_2^2D_1D_2^2 - \tfrac{1}{4}x_1^3x_2^3D_1D_2 + \tfrac{1}{4}x_1^4x_2^2D_2^2 \xrightarrow{G}_+ -\tfrac{1}{4}x_1^3x_2^3D_1D_2 + \tfrac{1}{4}x_1^4x_2^2D_2^2 =: f_3$$

and f_3 is reduced with respect to G, we add the polynomial f_3 and we make $G' := \{f_1, f_2, f_3\}$.

Step 2. Since $G' \neq G$, we compute $D = P(G') - P(G)$ and we make $G = G'$. In D we only need to consider three subsets:

$$S_{1,3} = \{f_1, f_3\},\ S_{2,3} = \{f_2, f_3\},\ S_{1,2,3} = \{f_1, f_2, f_3\}.$$

For $S_{1,3}$, $X_{S_{1,3}} = D_1^2D_2$ so $\gamma_1 = (0,0)$, $c_{\gamma_1,\beta_1} = 1$; in the same way, $\gamma_3 = (1,0)$ and $c_{\gamma_3,\beta_3} = 1$. Thus, we must calculate a generator set for $Syz_R[x_1^2x_2, -\frac{1}{4}x_1^3x_2^3]$. We have $B_{S_{1,3}} = \{(x_1x_2^2, 4)\}$ and, therefore,

$$x_1x_2^2f_1 + 4D_1f_3 = x_1^4x_2^2D_1D_2^2$$

that can be reduced to 0 by f_2.

For $S_{2,3}$, $X_{S_{2,3}} = D_1D_2^2$, so $\gamma_2 = (0,0)$ and $c_{\gamma_2,\beta_2} = 1$; in the same way, $\gamma_3 = (0,1)$ and, since $D_2D_1D_2 = 2D_1D_2^2 + x_2D_1D_2 - x_1D_2^2$, then $c_{\gamma_3,\beta_3} = 2$. Thus, a set of generators for $Syz_R[x_2^2, -\frac{1}{2}x_1^3x_2^3]$ is $B_{S_{2,3}} = \{(x_1^3x_2, 2)\}$, and

$$x_1^3x_2f_2 + 2D_2f_3 = \tfrac{1}{2}x_1^4x_2^2D_2^3 - \tfrac{1}{2}x_1^3x_2^4D_1D_2 + \tfrac{1}{2}x_1^4x_2^3D_2^2 =: f_4.$$

Since that f_4 is reduced with respect to G, then we add f_4 and we make $G' := \{f_1, f_2, f_3, f_4\}$.

For $S_{1,2,3}$, $X_{S_{1,2,3}} = D_1^2D_2^2$ and hence $\gamma_1 = (0,1)$, $\gamma_2 = (1,0)$ and $\gamma_3 = (1,1)$. So, $c_{\gamma_1,\beta_1} = 4$, $c_{\gamma_2,\beta_2} = 1$ and, since $D_1D_2D_1D_2 = 2D_1^2D_2^2 + x_2D_1^2D_2 - x_1D_1D_2^2$, then $c_{\gamma_3,\beta_3} = 2$. Therefore, a system of generators for $Syz_R[4x_1^2x_2, x_2^2, -\frac{1}{2}x_1^3x_2^3]$ is $B_{S_{1,2,3}} = \{(\frac{1}{4}x_2, -x_1^2, 0), (\frac{1}{4}x_1x_2^2, 0, 2)\}$; for the first generator we obtain a polynomial that can be reduced to 0 by f_1, f_2 and f_3 (in this case, we have the same calculations than step one). For the second generator, we obtain the following polynomial:

$$\tfrac{1}{4}x_1x_2^2D_2f_1 + 2D_1D_2f_3 = \tfrac{1}{4}x_1^3x_2^4D_1^2D_2 - \tfrac{1}{2}x_1^4x_2^3D_1D_2^2 - \tfrac{1}{4}x_1^4x_2^4D_1D_2 + \tfrac{1}{4}x_1^5x_2^3D_2^2$$

which can be reduced to 0 by f_1, f_2 and f_3. In consequence, we do not add any polynomial.

Step 3. Again, $G \neq G'$. Thus, we compute $D = P(G') - P(G)$ and we make $G = G'$. In this case, we only need to consider the following subsets:

$$S_{1,4},\ S_{2,4},\ S_{3,4},\ S_{1,2,4},\ S_{1,3,4},\ S_{2,3,4},\ S_{1,2,3,4}.$$

For $S_{1,4}$, $X_{S_{1,4}} = D_1^2D_2^3$, and $\gamma_1 = (0,2)$, $\gamma_4 = (2,0)$. Now, since

$$D_2^2D_1^2D_2 =$$
$$16D_1^2D_2^3+24x_2D_1^2D_2^2-24x_1D_1D_2^3+9x_2^2D_1^2D_2-26x_1x_2D_1D_2^2+9x_1^2D_2^3-4x_1x_2^2D_1D_2+4x_1^2x_2D_2^2,$$

then $c_{\gamma_1,\beta_1} = 16$. As $c_{\gamma_4\beta_4} = 1$, a generator set for $Syz_R[16x_1^2x_2, \frac{1}{2}x_1^4x_2^2]$ is $B_{S_{1,4}} = \{(\frac{1}{16}x_1^2x_2, -2)\}$. With this single generator, we obtain

$$\tfrac{1}{16}x_1^2x_2D_2^2f_1 - 2D_1^2f_4 = x_1^3x_2^4D_1^3D_2 - \tfrac{1}{2}x_1^4x_2^3D_1^2D_2^2 - \tfrac{3}{2}x_1^5x_2^2D_1D_2^3 + \tfrac{9}{16}x_1^4x_2^4D_1^2D_2 - \tfrac{13}{8}x_1^5x_2^3D_1D_2^2 + \tfrac{9}{16}x_1^6x_2^2D_2^3 - \tfrac{1}{4}x_1^5x_2^4D_1D_2 + \tfrac{1}{4}x_1^6x_2^3D_2,$$

a polynomial reducible to 0 by f_1, f_2, f_3 and f_4.
For $S_{2,4}$, $X_{S_{2,4}} = D_1D_2^3$, so $\gamma_2 = (0,1)$ and $\gamma_4 = (1,0)$. As $D_2D_1D_2^2 = 2D_1D_2^3 + x_2D_1D_2^2 - x_1D_2^3$, then $c_{\gamma_2,\beta_2} = 2$. Thus, $B_{S_{2,4}} = \{(\frac{1}{2}x_1^4, -2)\}$ is a system of generators of $Syz_R[2x_2^2, \frac{1}{2}x_1^4x_2^2]$, and we have

$$\tfrac{1}{2}x_1^4D_2f_2 - 2D_1f_4 = x_1^3x_2^4D_1^2D_2 + \tfrac{1}{2}x_1^4x_2^3D_1D_2^2 - \tfrac{1}{2}x_1^5x_2^2D_2^3,$$

which is also reducible to 0 w.r.t. f_1, f_2, f_3 and f_4.
For $S_{3,4}$, $X_{S_{3,4}} = D_1D_2^3$, whence $\gamma_3 = (0,2)$ and $\gamma_4 = (1,0)$. Seeing that $D_2^2D_1D_2 = 4D_1D_2^3 + 4x_2D_1D_2^2 - 3x_1D_2^3 + x_2^2D_1D_2 - x_1x_2D_2^2$, then $c_{\gamma_3,\beta_3} = 4$. Thus, a generator set for $Syz_R[-x_1^3x_2^3, \frac{1}{2}x_1^4x_2^2]$ is $B_{S_{3,4}} = \{(-x_1, -2x_2)\}$; therefore,

$$-x_1D_2^2f_3 - 2x_2D_1f_4 = -\tfrac{1}{4}x_1^5x_2^2D_2^4 + x_1^3x_2^5D_1^2D_2 - \tfrac{3}{4}x_1^5x_2^3D_2^3 + \tfrac{1}{4}x_1^4x_2^5D_1D_2 - \tfrac{1}{4}x_1^5x_2^4D_2^2.$$

Since this last polynomial is reducible to 0 through f_2, f_3 and f_4, then no polynomial is added.
For $S_{1,2,4}$ we have $X_{S_{1,2,4}} = D_1^2D_2^3$, hence $\gamma_1 = (0,2)$, $\gamma_2 = (1,2)$ and $\gamma_4 = (2,0)$. Thus, $c_{\gamma_1,\beta_1} = 16$, $c_{\gamma_2,\beta_2} = 2$, $c_{\gamma_4,\beta_4} = 1$ and, hence, $B_{S_{1,2,4}} = \{(\frac{1}{16}x_2, -\frac{1}{2}x_1^2, 0), (\frac{1}{16}x_1^2x_2, 0, -2)\}$. For these generators, we obtain polynomial that are reducible to 0 by f_1, f_2, f_3, and f_4.
For $S_{1,3,4}$, $X_{S_{1,3,4}} = D_1^2D_2^3$; thus $\gamma_1 = (0,2)$, $\gamma_3 = (1,2)$ and $\gamma_4 = (2,0)$. In consequence, $c_{\gamma_1,\beta_1} = 16$, $c_{\gamma_3,\beta_3} = 4$, $c_{\gamma_4,\beta_4} = 1$ and a set of generators for $Syz_R[16x_1^2x_2, -x_1^3x_2^3, \frac{1}{2}x_1^4x_2^2]$ is $B_{S_{1,3,4}} = \{(\frac{1}{16}x_1x_2^2, 1, 0),$
$(\frac{1}{16}x_1^2x_2, 0, -2)\}$. It is not difficult to show that these generators produce polynomials which can be reducible to 0 w.r.t. f_1, f_2, f_3, and f_4.
For $S_{2,3,4}$, we obtain a similar situation,
Finally, for $S_{1,2,3,4}$ we have that $X_{S_{1,2,3,4}} = D_1^2D_2^3$, $\gamma_1 = (0,2)$, $\gamma_2 = (1,1)$, $\gamma_3 = (1,2)$ and $\gamma_4 = (2,0)$. Thus $c_{\gamma_1,\beta_1} = 16$, $c_{\gamma_2,\beta_2} = 2$, $c_{\gamma_3,\beta_3} = 4$, $c_{\gamma_4,\beta_4} = 1$, and $B_{S_{1,2,3,4}} = \{(\frac{1}{16}x_2, -\frac{1}{2}x_1^2, 0, 0), (\frac{1}{16}x_1x_2^2, 0, 1, 0),$
$(\frac{1}{16}x_1^2x_2, 0, 0, -2)\}$. Once again, the polynomials obtained through these generators are reducible to 0 by f_1, f_2, f_3 and f_4. Therefore, $G = \{f_1, f_2, f_3, f_4\}$ is a Gröbner basis for $I := \langle f_1, f_2\rangle$.

Example 5.4.8. For this example, we consider the ring $\mathcal{R}$ described in the Example 1.3.6. For computational reasons, we rewrite the generators and relations for this algebra in the following way:

$$x := b, \quad y := a, \quad z := c, \quad w := d,$$

and the relations in this ring as:

$$yx = q^{-1}xy, \quad wx = qxw, \quad zy = qyz, \quad wz = qzw$$
$$zx = \mu^{-1}xz, \quad wy = yw + (q - q^{-1})xz.$$

Thus, $\mathcal{R} \cong \sigma(\Bbbk[x])\langle y, z, w\rangle$. On $Mon(\mathcal{R})$, we consider the order deglex with $y \succ z \succ w$; further, we will take $\Bbbk = \mathbb{Q}$, $\mu = \frac{1}{2}$ and $q = 3$. From above relations, we obtain that $\sigma_1(x) = \frac{1}{3}x$, $\sigma_2(x) = 2x$ and $\sigma_3(x) = 3x$. Given the polynomials $f_1 = x^2y^2zw^2$ and $f_2 = y^2z^2w$, we will calculate a Gröbner basis for the left ideal $I := \langle f_1, f_2\}$.
We start taking $G := \varnothing$ and $G' := \{f_1, f_2\}$.
Step 1. Since $G' \neq G$, we have $D = \{S_1, S_2, S_{1,2}\}$.
We make $G = G'$.
Since $\mathcal{R}$ does not have zero divisors, S_1 and S_2 do not add any polynomial to G'. For $S_{1,2}$, we have $X_{S_{1,2}} = y^2z^2w^2$ and, therefore, $\gamma_1 = (0, 1, 0)$ and $\gamma_2 = (0, 0, 1)$. Since that $zy^2zw^2 = 9y^2z^2w^2$ and $wy^2z^2w = 9y^2z^2w^2 + \frac{80}{9}xyz^3w$, we obtain that $c_{\gamma_1,\beta_1} = 9 = c_{\gamma_2,\beta_2}$. Moreover, $\sigma^{\gamma_1}(lc(f_1)) = 4x^2$ and $\sigma^{\gamma_2}(lc(f_2)) = 1$ and, whence, we must calculate a generator set of $Syz_R[\sigma^{\gamma_1}(lc(f_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_2}(lc(f_2))c_{\gamma_2,\beta_2}] = Syz_R[36x^2, 9]$. It is not hard to see that we can take $B_{S_{1,2}} = \{(\frac{1}{36}x_2, -\frac{1}{9}x^2)\}$. So,

$$\tfrac{1}{36}zf_1 - \tfrac{1}{9}x^2wf_2 = -\tfrac{80}{81}x^3yz^3 =: f_3$$

and, since f_3 is reduced with respect to G, we add the polynomial f_3 and we make $G' := \{f_1, f_2, f_3\}$.
Step 2. Since $G' \neq G$, we compute $D = P(G') - P(G)$ and we make $G = G'$. In D we only need to consider three subsets:

$$S_{1,3} = \{f_1, f_3\},\ S_{2,3} = \{f_2, f_3\},\ S_{1,2,3} = \{f_1, f_2, f_3\}.$$

For $S_{1,3}$, $X_{S_{1,3}} = y^2z^3w2$ so $\gamma_1 = (0, 2, 0)$ and $\gamma_3 = (1, 0, 1)$. Since $z^2y^2zw^2 = 81y^2z^3w^2$ and $ywyz^3w = 27y^2z^3w^2 + \frac{8}{9}xyz^4w$, we have that $c_{\gamma_1,\beta_1} = 81$ and $c_{\gamma_3,\beta_3} = 27$. On the other hand, $\sigma^{\gamma_1}(lc(f_1)) = 16x^2$ and $\sigma^{\gamma_3}(lc(f_3)) = -\frac{80}{81}x^3$; thus, we must calculate a generator set for $Syz_R[1296x^2, -\frac{80}{3}x^3]$. We have $B_{S_{1,3}} = \{\big(\frac{1}{1296}x, \frac{3}{80}\big)\}$ and, therefore,

$$\tfrac{1}{1296}xf_1 + \tfrac{3}{80}ywf_2 = -\tfrac{8}{243}x^4yz^4w$$

that can be reduced to 0 by f_3.
For $S_{2,3}$, $X_{S_{2,3}} = y^2z^3w$, so $\gamma_2 = (0, 1, 0)$ and $\gamma_3 = (1, 0, 0)$. Since $zyz^2w = 9y^2z^3w$ then $c_{\gamma_2,\beta_2} = 9$; in the same way, $c_{\gamma_3,\beta_3} = 1$ and $\sigma^{\gamma_2}(lc(f_2)) = 1$, $\sigma^{\gamma_3}(lc(f_3)) = -\frac{80}{2187}x^3$. Hence, a set of generators for $Syz_R[9, -\frac{80}{2187}x^3]$ is $B_{S_{2,3}} = \{\big(\frac{1}{9}x^3, -\frac{2187}{80}\big)\}$, and

$$\tfrac{1}{9}x^3zf_2 - \tfrac{2187}{80}yf_3 = \tfrac{1}{9}x^3z(y^2z^2w) + \tfrac{2187}{80}y(-\tfrac{80}{81}x^3yz^3w) = 0.$$

For $S_{1,2,3}$, $X_{S_{1,2,3}} = y^2z^3w^2$ and hence $\gamma_1 = (0, 2, 0)$, $\gamma_2 = (0, 1, 1)$ and $\gamma_3 = (1, 0, 1)$. Since $z^2y^2zw^2 = 81y^2z^3w^2$, $zwy^2z^2w = 81y^2z^3w^2 + \frac{160}{3}xyz^4$ and $ywyz^3w = 27y^2z^3w^2 + \frac{8}{9}xyz^4w$, then $c_{\gamma_1,\beta_1} = 81$, $c_{\gamma_2,\beta_2} = 81$ and $c_{\gamma_3,\beta_3} = 27$. Further, $\sigma^{\gamma_1}(lc(f_1)) = 16x^2$, $\sigma^{\gamma_2}(lc(f_2)) = 1$ and $\sigma^{\gamma_3}(lc(f_3)) = -\frac{80}{81}x^3$. Therefore, a system of generators for $Syz_R[1296x^2, 81, -\frac{80}{3}x^3]$ is $B_{S_{1,2,3}} = \{(\frac{1}{1296}, -\frac{1}{81}x^2, 0), (0, \frac{1}{81}x^3, \frac{3}{80})\}$; for both generators we obtain a polynomial that can be reduced to 0 by f_3. In consequence, we do not add any polynomial, and therefore, $G = \{f_1, f_2, f_3\}$ is a Gröbner basis for $I := \langle f_1, f_2\}$.

Remark 5.4.9. If I is a left ideal a bijective skew PBW extension A and $G = \{g_1, \ldots, g_t\}$ is a subset of nonzero polynomials in I, then Corollary 5.3.3 gives us a tool to verify if

G is a Gröbner basis for I when $lc(g_i) \in R^*$ for each $1 \leq i \leq t$. For example, let $\mathcal{A}$ be the ring described in Example 1.3.4, with $\Bbbk = \mathbb{Q}$, $q_1 = \frac{5}{4}$, $q_2 = \frac{2}{3}$, and $I = {}_A\langle f_1, f_2\}$, where $f_1 = y^2z + 3xz$ and $f_2 = x^2z - yz$. Employing the Buchberger's algorithm and the Corollary 5.3.4, we have that $G = \{xz, yz\}$ is a Gröbner basis for I. To verify this, note that given $f \in I$, $lm(f) = x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$ with $\alpha_3 \geq 1$, $\alpha_1 \geq 1$ or $\alpha_2 \geq 1$; in either case, $lm(f)$ will be divisible by xz or yz.

5.5 Gröbner bases of modules

In this section, we recall the general theory of Gröbner bases for submodules of A^m, $m \geq 1$, where A^m is the left free A-module of column vectors of length m, $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension of R, with R a LGS ring (see Definition 5.2.1) and $Mon(A)$ is endowed with some monomial order (see Definition 5.1.1). Since A is a left Noetherian ring (Theorem 1.2.9), we have that A is an IBN ring (Invariant Basis Number, see [79]), and hence, all bases of the free module A^m have m elements. Note also that A^m is a left Noetherian, and hence, any submodule of A^m is finitely generated. This theory was studied in [57] and [58], but now we will extend Buchberger's algorithm to the general bijective case without assuming that A is quasi-commutative. The goal is to establish and calculate Gröbner bases for submodules of A^m; for this, we will define the monomials in A^m, orders on the monomials, the concept of reduction, we will construct a division algorithm, give equivalent conditions in order to define Gröbner bases, and finally, we will compute Gröbner bases using a similar procedure to Buchberger's algorithm for the general case of bijective skew PBW extensions (not necessarily quasi-commutative as was assumed in [57] and [58]). The results presented in this section are an easy generalization of those of the previous sections, i.e., taking $m = 1$ we get the theory of Gröbner bases for the left ideals of A developed before. We will include only some proofs since most of them can be consulted in [57] and [58] or they are an easy adaptation of those of the previous sections. The theory presented in this section has been also studied by Gómez-Torrecillas et al. (see [18] , [19]) for left PBW algebras over division rings and assuming some special commutative conditions.

5.5.1 Monomial orders on $Mon(A^m)$

In the remainder of this section, we will write the elements of A^m as row vectors, if this not represent confusion. We recall that the canonical basis of A^m is

$$\boldsymbol{e}_1 = (1, 0, \dots, 0), \boldsymbol{e}_2 = (0, 1, 0, \dots, 0), \dots, \boldsymbol{e}_m = (0, 0, \dots, 1).$$

Definition 5.5.1. *A monomial in A^m is a vector $\boldsymbol{X} = X\boldsymbol{e}_i$, where $X = x^\alpha \in Mon(A)$ and $1 \leq i \leq m$, i.e.,*

$$\boldsymbol{X} = X\boldsymbol{e}_i = (0, \dots, X, \dots, 0),$$

where X is in the i-th position, named the index of $\boldsymbol{X}$, $ind(\boldsymbol{X}) := i$. A term is a vector $c\boldsymbol{X}$, where $c \in R$. The set of monomials of A^m will be denoted by Mon(A^m). Let $\boldsymbol{Y} = Y\boldsymbol{e}_j \in Mon(A^m)$, we say that $\boldsymbol{X}$ divides $\boldsymbol{Y}$ if $i = j$ and X divides Y. We will say that any monomial $\boldsymbol{X} \in Mon(A^m)$

divides the null vector $\mathbf{0}$*. The least common multiple of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, denoted by* $lcm(\boldsymbol{X},\boldsymbol{Y})$*, is* $\mathbf{0}$ *if* $i \neq j$*, and* $U\boldsymbol{e}_i$*, where* $U = lcm(X,Y)$*, if* $i = j$*. Finally, we define* $\exp(\boldsymbol{X}) := \exp(X) = \alpha$ *and* $\deg(\boldsymbol{X}) := \deg(X) = |\alpha|$.

Next, we define monomial orders on $Mon(A^m)$.

Definition 5.5.2. *A monomial order on* $Mon(A^m)$ *is a total order* $\succeq$ *satisfying the following three conditions:*

(i) $lm(x^\beta x^\alpha)\boldsymbol{e}_i \succeq x^\alpha \boldsymbol{e}_i$*, for every monomial* $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i \in Mon(A^m)$ *and any monomial* x^β *in* $Mon(A)$.

(ii) *If* $\boldsymbol{Y} = x^\beta \boldsymbol{e}_j \succeq \boldsymbol{X} = x^\alpha \boldsymbol{e}_i$*, then* $lm(x^\gamma x^\beta)\boldsymbol{e}_j \succeq lm(x^\gamma x^\alpha)\boldsymbol{e}_i$ *for every monomial* $x^\gamma \in Mon(A)$.

(iii) $\succeq$ *is degree compatible, i.e.,* $\deg(\boldsymbol{X}) \geq \deg(\boldsymbol{Y}) \Rightarrow \boldsymbol{X} \succeq \boldsymbol{Y}$.

If $\boldsymbol{X} \succeq \boldsymbol{Y}$ *but* $\boldsymbol{X} \neq \boldsymbol{Y}$ *we will write* $\boldsymbol{X} \succ \boldsymbol{Y}$*.* $\boldsymbol{Y} \preceq \boldsymbol{X}$ *means that* $\boldsymbol{X} \succeq \boldsymbol{Y}$*.*

Proposition 5.5.3. *Every monomial order on* $Mon(A^m)$ *is a well-order.*

Proof. We can repeat the proof of Proposition 5.1.2: Suppose that we have a monomial order $\succeq$ on $Mon(A^m)$ that is not a well order. This means that we have an infinite sequence of monomials

$$\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \boldsymbol{X}_3 \succ \cdots$$

and since $\succeq$ is degree compatible, then we have the an infinite subsequence

$$\deg(\boldsymbol{X}_{i_1}) > \deg(\boldsymbol{X}_{i_2}) > \deg(\boldsymbol{X}_{i_3}) > \cdots,$$

but this is impossible since $\deg(\boldsymbol{X}_{i_1})$ is finite. □

Given a monomial order $\succeq$ on $Mon(A)$, we can define two natural orders on $Mon(A^m)$.

Definition 5.5.4. *Let* $\boldsymbol{X} = X\boldsymbol{e}_i$ *and* $\boldsymbol{Y} = Y\boldsymbol{e}_j \in Mon(A^m)$.

(i) *The TOP (term over position) order is defined by*

$$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ or \\ X = Y and \quad i > j. \end{cases}$$

(ii) *The TOPREV order is defined by*

$$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ or \\ X = Y and \quad i < j. \end{cases}$$

Remark 5.5.5. (i) Note that with TOP we have

$$\boldsymbol{e}_m \succ \boldsymbol{e}_{m-1} \succ \cdots \succ \boldsymbol{e}_1$$

and

$$\boldsymbol{e}_1 \succ \boldsymbol{e}_2 \succ \cdots \succ \boldsymbol{e}_m$$

for TOPREV.

(ii) The POT (position over term) and POTREV orders defined in [1] and [75] for modules over classical polynomial commutative rings are not degree compatible.

(iii) Other examples of monomial orders in $Mon(A^m)$ are considered in [19], e.g, orders with weight.

We fix a monomial order on $Mon(A)$, let $\boldsymbol{f} \neq \boldsymbol{0}$ be a vector of A^m, then we may write $\boldsymbol{f}$ as a sum of terms in the following way

$$\boldsymbol{f} = c_1\boldsymbol{X}_1 + \cdots + c_t\boldsymbol{X}_t,$$

where $c_1, \ldots, c_t \in R - 0$ and $\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \cdots \succ \boldsymbol{X}_t$ are monomials of $Mon(A^m)$.

Definition 5.5.6. *With the above notation, we say that*

(i) $lt(\boldsymbol{f}) := c_1\boldsymbol{X}_1$ *is the leading term of* $\boldsymbol{f}$.

(ii) $lc(\boldsymbol{f}) := c_1$ *is the leading coefficient of* $\boldsymbol{f}$.

(iii) $lm(\boldsymbol{f}) := \boldsymbol{X}_1$ *is the leading monomial of* $\boldsymbol{f}$.

(iv) $ind(\boldsymbol{f}) := ind(lm(\boldsymbol{f}))$ *is the index of* $\boldsymbol{f}$.

For $\boldsymbol{f} = \boldsymbol{0}$ we define $lm(\boldsymbol{0}) = \boldsymbol{0}, lc(\boldsymbol{0}) = 0, lt(\boldsymbol{0}) = \boldsymbol{0}$, and if $\succeq$ is a monomial order on $Mon(A^m)$, then we define $\mathbf{X} \succ \mathbf{0}$ for any $\mathbf{X} \in Mon(A^m)$. So, we extend $\succeq$ to $Mon(A^m)\bigcup\{\mathbf{0}\}$.

5.5.2 Division algorithm in A^m

The reduction process in A^m is defined as follows.

Definition 5.5.7. *Let F be a finite set of non-zero vectors of A^m, and let $\boldsymbol{f}, \boldsymbol{h} \in A^m$, we say that $\boldsymbol{f}$ reduces to $\boldsymbol{h}$ by F in one step, denoted $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$, if there exist elements $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in F$ and $r_1, \ldots, r_t \in R$ such that*

(i) $lm(\boldsymbol{f}_i)|lm(\boldsymbol{f})$, $1 \leq i \leq t$, *i.e.,* $ind(lm(\boldsymbol{f}_i)) = ind(lm(\boldsymbol{f}))$ *and there exists* $x^{\alpha_i} \in Mon(A)$ *such that* $\alpha_i + \exp(lm(\boldsymbol{f}_i)) = \exp(lm(\boldsymbol{f}))$.

(ii) $lc(\boldsymbol{f}) = r_1\sigma^{\alpha_1}(lc(\boldsymbol{f}_1))c_{\alpha_1,\boldsymbol{f}_1} + \cdots + r_t\sigma^{\alpha_t}(lc(\boldsymbol{f}_t))c_{\alpha_t,\boldsymbol{f}_t}$, *where* $c_{\alpha_i,\boldsymbol{f}_i} := c_{\alpha_i,\exp(lm(\boldsymbol{f}_i))}$.

(iii) $\boldsymbol{h} = \boldsymbol{f} - \sum_{i=1}^{t} r_i x^{\alpha_i} \boldsymbol{f}_i$.

We say that $\boldsymbol{f}$ *reduces to* $\boldsymbol{h}$ *by* F*, denoted* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$*, if and only if there exist vectors* $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{t-1} \in A^m$ *such that*

$$\boldsymbol{f} \xrightarrow{\quad F \quad} \boldsymbol{h}_1 \xrightarrow{\quad F \quad} \boldsymbol{h}_2 \xrightarrow{\quad F \quad} \cdots \xrightarrow{\quad F \quad} \boldsymbol{h}_{t-1} \xrightarrow{\quad F \quad} \boldsymbol{h}$$

$\boldsymbol{f}$ *is reduced* (*also called minimal*) *w.r.t.* F *if* $\boldsymbol{f} = \boldsymbol{0}$ *or there is no one step reduction of* $\boldsymbol{f}$ *by* F*, i.e., one of the first two conditions of Definition 5.5.7 fails. Otherwise, we will say that* $\boldsymbol{f}$ *is reducible w.r.t.* F*. If* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$ *and* $\boldsymbol{h}$ *is reduced w.r.t.* F*, then we say that* $\boldsymbol{h}$ *is a remainder for* $\boldsymbol{f}$ *w.r.t.* F*.*

Remark 5.5.8. Related to the previous definition we have the following remarks:

(i) By Theorem 1.2.2, the coefficients $c_{\alpha_i, \boldsymbol{f}_i}$ in the previous definition are unique and satisfy

$$x^{\alpha_i} x^{\exp(lm(\boldsymbol{f}_i))} = c_{\alpha_i, \boldsymbol{f}_i} x^{\alpha_i + \exp(lm(\boldsymbol{f}_i))} + p_{\alpha_i, \boldsymbol{f}_i},$$

where $p_{\alpha_i, \boldsymbol{f}_i} = 0$ or $\deg(lm(p_{\alpha_i, \boldsymbol{f}_i})) < |\alpha_i + \exp(lm(\boldsymbol{f}_i))|$, $1 \leq i \leq t$.

(ii) $lm(\boldsymbol{f}) \succ lm(\boldsymbol{h})$ and $\boldsymbol{f} - \boldsymbol{h} \in \langle F \rangle$, where $\langle F \rangle$ is the submodule of A^m generated by F.

(iii) The remainder of $\boldsymbol{f}$ is not unique.

(iv) By definition we will assume that $\boldsymbol{0} \xrightarrow{F} \boldsymbol{0}$.

(v)

$$lt(\boldsymbol{f}) = \sum_{i=1}^{t} r_i lt(x^{\alpha_i} lt(\boldsymbol{f}_i)),$$

Proposition 5.5.9. *Let* $\boldsymbol{f}, \boldsymbol{h} \in A^m$*,* $\theta \in \mathbb{N}^n$ *and* $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ *be a finite set of non-zero vectors of* A^m*. Then,*

(i) *If* $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$*, then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F} x^\theta \boldsymbol{h}$*. In particular, if* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$*.*

(ii) *If* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$ *and* $\boldsymbol{p} \in A^m$ *is such that* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(\boldsymbol{h}) \succ lm(\boldsymbol{p})$*, then* $\boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{h} + \boldsymbol{p}$*.*

(iii) *If* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$*, then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_{+} x^\theta \boldsymbol{h}$*. If* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$*.*

(iv) *If* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{0}$*, then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{0}$*. If* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$*.*

Proof. This proof is an easy adaptation of the ideal case. See [58], Proposition 22. □

Theorem 5.5.10. *Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ be a set of non-zero vectors of A^m and $\boldsymbol{f} \in A^m$, then the the division algorithm below produces polynomials $q_1, \ldots, q_t \in A$ and a reduced vector $\boldsymbol{h} \in A^m$ w.r.t.. F such that $\boldsymbol{f} \xrightarrow{F}_+ \boldsymbol{h}$ and*

$$\boldsymbol{f} = q_1\boldsymbol{f}_1 + \cdots + q_t\boldsymbol{f}_t + \boldsymbol{h}$$

with

$$lm(\boldsymbol{f}) = \max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), \ldots, lm(lm(q_t)lm(\boldsymbol{f}_t)), lm(\boldsymbol{h})\}.$$

Division algorithm in A^m

INPUT: $\boldsymbol{f}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in A^m$ *with* $\boldsymbol{f}_j \neq 0\,(1 \leq j \leq t)$

OUTPUT: $q_1, \ldots, q_t \in A$, $\boldsymbol{h} \in A^m$ *with* $\boldsymbol{f} = q_1\boldsymbol{f}_1 + \cdots + q_t\boldsymbol{f}_t + \boldsymbol{h}$, $\boldsymbol{h}$ *reduced w.r.t..* $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ *and* $lm(\boldsymbol{f}) = \max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), \ldots, lm(lm(q_t)lm(\boldsymbol{f}_t)), lm(\boldsymbol{h})\}$

INITIALIZATION: $q_1 := 0, q_2 := 0, \ldots, q_t := 0, \boldsymbol{h} := \boldsymbol{f}$

WHILE $\boldsymbol{h} \neq \boldsymbol{0}$ *and there exists j such that $lm(\boldsymbol{f}_j)$ divides $lm(\boldsymbol{h})$* **DO**

  Calculate $J := \{j \mid lm(\boldsymbol{f}_j)$ *divides* $lm(\boldsymbol{h})\}$

  FOR $j \in J$ **DO**

    Calculate $\alpha_j \in \mathbb{N}^n$ *such that* $\alpha_j + \exp(lm(\boldsymbol{f}_j)) = \exp(lm(\boldsymbol{h}))$

  IF *the equation* $lc(\boldsymbol{h}) = \sum_{j \in J} r_j \sigma^{\alpha_j}(lc(\boldsymbol{f}_j))c_{\alpha_j,\boldsymbol{f}_j}$ *is soluble, where* $c_{\alpha_j,\boldsymbol{f}_j}$ *are defined as in Definition 5.5.7*

  THEN

    Calculate one solution $(r_j)_{j \in J}$

    $\boldsymbol{h} := \boldsymbol{h} - \sum_{j \in J} r_j x^{\alpha_j}\boldsymbol{f}_j$

    FOR $j \in J$ **DO**

      $q_j := q_j + r_j x^{\alpha_j}$

  ELSE

    Stop

Proof. The proof is an easy adaptation of the proof of Theorem 21 in [40]. See [58]. □

Example 5.5.11. We illustrate the above algorithm for $\mathcal{A}$, the diffusion algebra used in Example 1.3.3. In this case, we will take $\Bbbk = \mathbb{Q}$, $m = 2$, $d_{12} = -2$, $d_{21} = -1$, deglex order on $Mon(\mathcal{A})$ with $D_1 \succ D_2$, and TOPREV on $Mon(\mathcal{A}^2)$, with $\boldsymbol{e}_1 > \boldsymbol{e}_2$. Note that in this ring the endomorphism σ_i are the identity. Let $\boldsymbol{f}_1 = (D_1D_2^2, D_1^2 + x_1D_1D_2)$, $\boldsymbol{f}_2 = (x_1D_1D_2 + x_1D_1, D_2^2)$, $\boldsymbol{f}_3 = (x_1D_1, D_2^2 + x_2)$, $\boldsymbol{f}_4 = (D_2, D_1^2)$ and $\boldsymbol{f} = ((x_1x_2 + 1)D_1^2D_2^2 +$

$x_1D_1^2, D_1D_2 + x_2D_2^2)$. Then, we divide $\boldsymbol{f}$ by $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$ and $\boldsymbol{f}_4$.
Step 1. We start with $h := f$, $q_1 := 0$, $q_2 := 0$, $q_3 := 0$, $q_4 := 0$. Since $lm(\boldsymbol{f}_j) \mid lm(\boldsymbol{f})$ for $j = 1, 2$, we compute $\alpha = (\alpha_{j1}, \alpha_{j2}) \in \mathbb{N}^2$ such that $\alpha_j + \exp(lm(\boldsymbol{f}_j)) = \exp(lm(\boldsymbol{h}))$ and the corresponding value of $\sigma^{\alpha_j}(lc(\boldsymbol{f}_j))c_{\alpha_j,\beta_j}$, where $\beta_j := \exp(lm(\boldsymbol{f}_j))$:

$$\begin{aligned}
&(\alpha_{11}, \alpha_{12}) + (1, 2) = (2, 2) \Rightarrow \alpha_{11} = 1, \alpha_{12} = 0,\\
&D_1D_1D_2^2 = D_1^2D_2^2 \Rightarrow c_{\alpha_1,\beta_1} = 1,\\
&(\alpha_{21}, \alpha_{22}) + (1, 1) = (2, 2) \Rightarrow \alpha_{21} = 1, \alpha_{22} = 1,\\
&D_1D_2D_1D_2 = 2D_1^2D_2^2 + x_2D_1^2D_2 - x_1D_1D_2^2 \Rightarrow c_{\alpha_2,\beta_2} = 2.
\end{aligned}$$

Now, we solve the equation

$$lc(\boldsymbol{h}) = x_1x_2 + 1 = r_1 + 2r_2x_1 \Rightarrow r_1 = 1,\ r_2 = \tfrac{1}{2}x_2,$$

and with the relations defining $\mathcal{A}$, we compute

$$\begin{aligned}
\boldsymbol{h} =&\boldsymbol{h} - (r_1x^{\alpha_1}\boldsymbol{f}_1 + r_2x^{\alpha_2}\boldsymbol{f}_2)\\
=&\boldsymbol{h} - D_1(D_1D_2^2\boldsymbol{e}_1 + x_1D_1D_2\boldsymbol{e}_2 + D_2^2\boldsymbol{e}_2) - \frac{1}{2}x_2D_1D_2(x_1D_1D_2\boldsymbol{e}_1 + D_2^2\boldsymbol{e}_2 + x_1D_1\boldsymbol{e}_1)\\
=&-\frac{1}{2}x_2D_1D_2^3\boldsymbol{e}_2 - (\frac{1}{2}x_1x_2^2 + x_1x_2)D_1^2D_2\boldsymbol{e}_1 - x_1D_1^2D_2\boldsymbol{e}_2 + \frac{1}{2}x_1^2x_2D_1D_2^2\boldsymbol{e}_1 - D_2^3\boldsymbol{e}_2\\
&-\frac{1}{2}x_1x_2^2D_1^2\boldsymbol{e}_1 + \frac{1}{2}x_1^2x_2D_1D_2\boldsymbol{e}_1.
\end{aligned}$$

We also compute

$$q_1 := D_1,\ q_2 := \tfrac{1}{2}x_2D_1D_2,\ q_3 := 0,\ q_4 := 0.$$

Step 2. $lm(\boldsymbol{h}) = D_1D_2^3\boldsymbol{e}_2$, $lc(h) = -\frac{1}{2}x_2$. In this case, $lm(f_j) \mid lm(f)$ just for $j = 3$, and we must compute $\alpha = (\alpha_{31}, \alpha_{32}, \alpha_{33}) \in \mathbb{N}^3$ such that $\alpha_3 + \exp(lm(f_3)) = \exp(lm(h))$:

$$\begin{aligned}
&(\alpha_{31}, \alpha_{32}) + (0, 2) = (1, 3) \Rightarrow \alpha_{31} = 1, \alpha_{32} = 1,\\
&D_1D_2D_2^2 = D_1D_2^3 \Rightarrow c_{\alpha_3,\beta_3} = 1,
\end{aligned}$$

and we have $lc(\boldsymbol{h}) = -\frac{1}{2}x_2 = r_3$. Thus,

$$\begin{aligned}
\boldsymbol{h} =&\boldsymbol{h} + \frac{1}{2}x_2D_1D_2\boldsymbol{f}_3\\
=&-\frac{1}{2}x_1x_2^2D_1^2D_2\boldsymbol{e}_1 - x_1D_1^2D_2\boldsymbol{e}_2 + \frac{1}{2}x_1^2x_2D_1D_2^2\boldsymbol{e}_1 - D_2^3\boldsymbol{e}_2 + \frac{1}{2}x_2^2D_1D_2\boldsymbol{e}_2,
\end{aligned}$$

and

$$q_1 := D_1,\ q_2 := \tfrac{1}{2}x_2D_1D_2,\ q_3 := -\tfrac{1}{2}x_2D_1D_2,\ q_4 := 0.$$

Step 3. Note that $lm(\boldsymbol{h}) = D_1^2D_2\boldsymbol{e}_1$ and $lm(\boldsymbol{f}_j) \mid lm(\boldsymbol{h})$ for $j = 2$. In this case, we have:

$$\begin{aligned}
&(\alpha_{21}, \alpha_{22}) + (1, 1) = (2, 1) \Rightarrow \alpha_{21} = 1, \alpha_{22} = 0,\\
&D_1D_1D_2 = D_1^2D_2 \Rightarrow c_{\alpha_2,\beta_2} = 1.
\end{aligned}$$

and $r_2 = -\frac{1}{2}x_2^2$. Therefore,

$$
\begin{aligned}
\boldsymbol{h} =& \boldsymbol{h} + \frac{1}{2}x_2^2 D_1 \boldsymbol{f}_2 \\
=& -x_1 D_1^2 D_2 \boldsymbol{e}_2 + \frac{1}{2}x_1^2 x_2 D_1 D_2^2 \boldsymbol{e}_1 + \frac{1}{2}x_2^2 D_1 D_2^2 \boldsymbol{e}_2 - D_2^3 \boldsymbol{e}_2 + \frac{1}{2}x_2^2 D_1 D_2 \boldsymbol{e}_2 + \frac{1}{2}x_2 D_1^2 \boldsymbol{e}_1,
\end{aligned}
$$

and,

$$q_1 := D_1,\, q_2 := \tfrac{1}{2}x_2 D_1 D_2 - \tfrac{1}{2}x_2^2 D_1,\, q_3 := -\tfrac{1}{2}x_2 D_1 D_2,\, q_4 := 0.$$

Step 4. $lm(\boldsymbol{h}) = D_1^2 D_2 \boldsymbol{e}_2$ and $lm(\boldsymbol{f}_j) \mid lm(\boldsymbol{h})$ just for $j = 4$. So,

$$
\begin{aligned}
&(\alpha_{41}, \alpha_{42}) + (2, 0) = (2, 1) \Rightarrow \alpha_{21} = 0, \alpha_{22} = 1, \\
&D_2 D_1^2 = 4D_1^2 D_2 + 3x_2 D_1^2 - 4x_1 D_1 D_2 - x_1 x_2 D_1 + x_1^2 D_2 \Rightarrow c_{\alpha_4,\beta_4} = 4.
\end{aligned}
$$

and $r_4 = \frac{1}{4}x_1$. Therefore,

$$
\begin{aligned}
\boldsymbol{h} =& \boldsymbol{h} + \frac{1}{4}x_1 D_2 \boldsymbol{f}_4 \\
=& \frac{1}{2}x_1^2 x_2 D_1 D_2^2 \boldsymbol{e}_1 + \frac{1}{2}x_2^2 D_1 D_2^2 \boldsymbol{e}_2 - D_2^3 \boldsymbol{e}_2 + \frac{3}{4}x_1 x_2 D_1^2 \boldsymbol{e}_2 + \big(\frac{1}{2}x_2^2 - x_1^2\big) D_1 D_2 \boldsymbol{e}_2 + \\
& \frac{1}{2}x_2 D_1^2 \boldsymbol{e}_1 - \frac{1}{4}x_1^2 x_2 D_1 \boldsymbol{e}_2 + \frac{1}{4}x_1^3 D_2 \boldsymbol{e}_2,
\end{aligned}
$$

and

$$q_1 := D_1,\, q_2 := \tfrac{1}{2}x_2 D_1 D_2 - \tfrac{1}{2}x_2^2 D_1,\, q_3 := -\tfrac{1}{2}x_2 D_1 D_2,\, q_4 := -\tfrac{1}{4}x_1 D_2.$$

Step 5. $lm(\boldsymbol{h}) = D_1 D_2^2 \boldsymbol{e}_1$ and $lm(f_j) \mid lm(h)$ for $j = 1, 2$. So,

$$
\begin{aligned}
&(\alpha_{11}, \alpha_{12}) + (1, 2) = (1, 2) \Rightarrow \alpha_{11} = 0, \alpha_{12} = 0, \\
&D_1 D_1 D_2^2 = D_1^2 D_2^2 \Rightarrow c_{\alpha_1,\beta_1} = 1, \\
&(\alpha_{21}, \alpha_{22}) + (1, 1) = (1, 2) \Rightarrow \alpha_{21} = 0, \alpha_{22} = 1, \\
&D_2 D_1 D_2 = 2D_1 D_2^2 + x_2 D_1 D_2 - x_1 D_2^2 \Rightarrow c_{\alpha_2,\beta_2} = 2.
\end{aligned}
$$

Now, we solve the equation

$$lc(\boldsymbol{h}) = \tfrac{1}{2}x_1^2 x_2 = r_1 + 2r_2 x_1 \Rightarrow r_1 = \tfrac{1}{2}x_1^2 x_2,\; r_2 = 0,$$

and with the relations defining $\mathcal{A}$, we compute

$$
\begin{aligned}
\boldsymbol{h} =& \boldsymbol{h} - \frac{1}{2}x_1^2 x_2 \boldsymbol{f}_1 \\
=& \frac{1}{2}x_2^2 D_1 D_2^2 \boldsymbol{e}_2 - D_2^3 \boldsymbol{e}_2 + \frac{3}{4}x_1 x_2 D_1^2 \boldsymbol{e}_2 + \big(-\frac{1}{2}x_1^3 x_2 + \frac{1}{2}x_2^2 - x_1^2\big) D_1 D_2 \boldsymbol{e}_2 + \frac{1}{2}x_2 D_1^2 \boldsymbol{e}_1 \\
& - \frac{1}{2}x_1^2 x_2 D_2^2 \boldsymbol{e}_2 - \frac{1}{4}x_1^2 x_2 D_1 \boldsymbol{e}_2 + \frac{1}{4}x_1^3 D_2 \boldsymbol{e}_2.
\end{aligned}
$$

Further,

$$q_1 := D_1,\ q_2 := \tfrac{1}{2}x_2D_1D_2 - \tfrac{1}{2}x_2^2D_1 - \tfrac{1}{2}x_1^2x_2,\ q_3 := -\tfrac{1}{2}x_2D_1D_2,\ q_4 := -\tfrac{1}{4}x_1D_2.$$

Step 6. $lm(\boldsymbol{h}) = D_1D_2^2\boldsymbol{e}_2$ and $lm(\boldsymbol{f}_j) \mid lm(\boldsymbol{h})$ for $j = 3$. We have,

$$\begin{aligned}&(\alpha_{31}, \alpha_{32}) + (0, 2) = (1, 2) \Rightarrow \alpha_{31} = 1, \alpha_{32} = 0,\\ &D_1D_2^2 = D_1D_2^2 \Rightarrow c_{\alpha_3,\beta_3} = 1,\end{aligned}$$

and $r_3 = \frac{1}{2}x_2^2$. Hence,

$$\begin{aligned}\boldsymbol{h} =& \boldsymbol{h} - \frac{1}{2}x_2^2D_1\boldsymbol{f}_3\\ =& -D_2^3\boldsymbol{e}_2 - \frac{1}{2}x_1x_2^2D_1^2\boldsymbol{e}_1 + \frac{3}{4}x_1x_2D_1^2\boldsymbol{e}_2 + \big(-\frac{1}{2}x_1^3x_2 + \frac{1}{2}x_2^2 - x_1^2\big)D_1D_2\boldsymbol{e}_2 + \frac{1}{2}x_2D_1^2\boldsymbol{e}_1\\ &- \frac{1}{2}x_1x_2D_2^2\boldsymbol{e}_2 - \frac{1}{2}(x_2^3 + \frac{1}{2}x_1^2x_2)D_1\boldsymbol{e}_2 + \frac{1}{4}x_1^3D_2\boldsymbol{e}_2.\end{aligned}$$

Moreover,

$$q_1 := D_1,\ q_2 := \tfrac{1}{2}x_2D_1D_2 - \tfrac{1}{2}x_2^2D_1 - \tfrac{1}{2}x_1^2x_2,\ q_3 := -\tfrac{1}{2}x_2D_1D_2 + \tfrac{1}{2}x_2^2D_1,\ q_4 := -\tfrac{1}{4}x_1D_2.$$

Step 7. $lm(\boldsymbol{h}) = D_2^3\boldsymbol{e}_2$ and $lm(\boldsymbol{f}_j) \mid lm(\boldsymbol{h})$ for $j = 3$. We have,

$$\begin{aligned}&(\alpha_{31}, \alpha_{32}) + (0, 2) = (0, 3) \Rightarrow \alpha_{31} = 0, \alpha_{32} = 1,\\ &D_2D_2^2 = D_2^3 \Rightarrow c_{\alpha_3,\beta_3} = 1,\end{aligned}$$

and $r_3 = -1$. Hence,

$$\begin{aligned}\boldsymbol{h} =& \boldsymbol{h} + D_2\boldsymbol{f}_3\\ =& -\frac{1}{2}x_1x_2^2D_1^2\boldsymbol{e}_1 + 2x_1D_1D_2\boldsymbol{e}_1 + \frac{3}{4}x_1x_2D_1^2\boldsymbol{e}_2 + \big(-\frac{1}{2}x_1^3x_2 + \frac{1}{2}x_2^2 - x_1^2\big)D_1D_2\boldsymbol{e}_2 + \frac{1}{2}x_2D_1^2\boldsymbol{e}_1\\ &- \frac{1}{2}x_1x_2D_2^2\boldsymbol{e}_2 + x_2D_1\boldsymbol{e}_1 - \frac{1}{2}(x_2^3 + \frac{1}{2}x_1^2x_2)D_1\boldsymbol{e}_2 - x_1D_2\boldsymbol{e}_1 + (\frac{1}{4}x_1^3 + x_2)D_2\boldsymbol{e}_2,\end{aligned}$$

and

$$\begin{gathered}q_1 := D_1,\ q_2 := \tfrac{1}{2}x_2D_1D_2 - \tfrac{1}{2}x_2^2D_1 - \tfrac{1}{2}x_1^2x_2,\ q_3 := -\tfrac{1}{2}x_2D_1D_2 + \tfrac{1}{2}x_2^2D_1 - D_2,\\ q_4 := -\tfrac{1}{4}x_1D_2.\end{gathered}$$

Step 8. Finally, note that $lm(h) = D_1^2\boldsymbol{e}_1$ is not divisible by any $lm(f_i)$, $i = 1, 2, 3, 4$. Thus, we have that

$$f = D_1f_1 + (\tfrac{1}{2}x_2D_1D_2 - \tfrac{1}{2}x_2^2D_1 - \tfrac{1}{2}x_1^2x_2)f_2 + (-\tfrac{1}{2}x_2D_1D_2 + \tfrac{1}{2}x_2^2D_1 - D_2)f_3 + (-\tfrac{1}{4}x_1D_2)f_4 + h.$$

We also see that,

$$\begin{gathered}\max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), lm(lm(q_2)lm(\boldsymbol{f}_2)), lm(lm(q_3)lm(\boldsymbol{f}_3)), lm(lm(q_4)lm(\boldsymbol{f}_4))\}\\ = \max\{D_1^2D_2^2e_1, D_1^2D_2^2e_1, D_1D_2^3e_2, D_1^2D_2e_2\} = D_1^2D_2^2e_1 = lm(\boldsymbol{f}).\end{gathered}$$

5.5.3 Gröbner bases for submodules of A^m

Our next purpose is to define Gröbner bases for submodules of A^m.

Definition 5.5.12. *Let $M \neq 0$ be a submodule of A^m and let G be a non empty finite subset of non-zero vectors of M, we say that G is a Gröbner basis for M if each element $0 \neq \boldsymbol{f} \in M$ is reducible w.r.t. G.*

We will say that $\{\boldsymbol{0}\}$ is a Gröbner basis for $M = 0$.

Theorem 5.5.13. *Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero vectors of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for M.*

(ii) *For any vector $\boldsymbol{f} \in A^m$,*

$$\boldsymbol{f} \in M \text{ if and only if } \boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{0}.$$

(iii) *For any $\boldsymbol{0} \neq \boldsymbol{f} \in M$ there exist $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ such that $lm(\boldsymbol{g}_j) | lm(\boldsymbol{f})$, $1 \leq j \leq t$, (i.e., $ind(lm(\boldsymbol{g}_j)) = ind(lm(\boldsymbol{f}))$ and there exist $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(lm(\boldsymbol{g}_j)) = \exp(lm(\boldsymbol{f}))$) and*

$$lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1, \boldsymbol{g}_1}, \dots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t, \boldsymbol{g}_t}\}.$$

(iv) *For $\alpha \in \mathbb{N}^n$ and $1 \leq u \leq m$, let $\langle \alpha, M\}_u$ be the left ideal of R defined by*

$$\langle \alpha, M\}_u := \langle lc(\boldsymbol{f}) | \boldsymbol{f} \in M, ind(lm(\boldsymbol{f})) = u, \exp(lm(\boldsymbol{f})) = \alpha\}.$$

Then, $\langle \alpha, M\}_u = J_u$, with

$$J_u := \langle \sigma^{\beta}(lc(\boldsymbol{g}))c_{\beta, \boldsymbol{g}} | \boldsymbol{g} \in G, ind(lm(\boldsymbol{g})) = u \text{ and } \beta + \exp(lm(\boldsymbol{g})) = \alpha\}.$$

Proof. See [58], Theorem 26. □

From this theorem we get the following consequences.

Corollary 5.5.14. *Let $M \neq 0$ be a submodule of A^m. Then,*

(i) *If G is a Gröbner basis for M, then $M = \langle G \rangle$.*

(ii) *Let G be a Gröbner basis for M, if $\boldsymbol{f} \in M$ and $\boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{h}$, with $\boldsymbol{h}$ reduced, then $\boldsymbol{h} = \boldsymbol{0}$.*

(iii) *Let $G = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$ be a set of non-zero vectors of M with $lc(\boldsymbol{g}_i) \in R^*$ for each $1 \leq i \leq t$. Then, G is a Gröbner basis of M if and only if given $0 \neq \boldsymbol{r} \in M$ there exists i such that $lm(\boldsymbol{g}_i)$ divides $lm(\boldsymbol{r})$.*

Proof. (i): this is a direct consequence of Theorem 5.5.13.

(ii): let $\boldsymbol{f} \in M$ and $f \xrightarrow{G}_+ \boldsymbol{h}$, with $\boldsymbol{h}$ reduced; since $\boldsymbol{f} - \boldsymbol{h} \in \langle G \rangle = M$, then $\boldsymbol{h} \in M$; if $\boldsymbol{h} \neq \boldsymbol{0}$ then $\boldsymbol{h}$ can be reduced by G, but this is not possible since $\boldsymbol{h}$ is reduced.

(iii): if G is a Gröbner basis of M, then given $0 \neq \boldsymbol{r} \in M$, $\boldsymbol{r}$ is reducible w.r.t. G, hence there exists i such that $lm(\boldsymbol{g}_i)$ divides $lm(\boldsymbol{r})$. Conversely, if this condition holds for some i, then $\boldsymbol{r}$ is reducible w.r.t. G since the equation $lc(\boldsymbol{r}) = r_1\sigma^{\alpha_i}(lc(\boldsymbol{g}_i)c_{\alpha_i,\boldsymbol{g}_i}$, with $\alpha_i + \exp(lm(\boldsymbol{g}_i)) = \exp(lm(\boldsymbol{r}))$, is soluble with solution $r_1 = lc(\boldsymbol{r})c^{-1}_{\alpha_i,\boldsymbol{g}_i}(\sigma^{\alpha_i}(lc(\boldsymbol{g}_i)))^{-1}$. □

Note that the remainder of $\boldsymbol{f} \in A^m$ with respect to a Gröbner basis is not unique. Moreover, changing the term order, a Gröbner basis could not be again a Gröbner basis. In fact, a counterexample was given in [75] for the trivial case when $A = R[x_1, \dots, x_n]$ is the commutative polynomial ring.

Of course, there exists a version of Corollary 5.3.4 for the module case.

Corollary 5.5.15. *Let G be a Gröbner basis for a left A-module M. Given $\boldsymbol{g} \in G$, if $\boldsymbol{g}$ is reducible w.r.t. $G' = G - \{\boldsymbol{g}\}$, then G' is a Gröbner basis for M.*

Proof. According to Theorem 5.5.13, is enough to show that all $\boldsymbol{f} \in M$ is reducible w.r.t G'. Let $\boldsymbol{f}$ be a nonzero vector in M; since G is a Gröbner basis for M, $\boldsymbol{f}$ is reducible w.r.t G and there exist elements $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ satisfying the conditions (i), (ii) and (iii) in the Definition 5.5.7. If $\boldsymbol{g} \neq \boldsymbol{g}_i$ for each $1 \leq i \leq t$, then we finished. Suppose that $\boldsymbol{g} = \boldsymbol{g}_j$ for some $j \in \{1, \dots, t\}$ and let $\beta_i = \exp(\boldsymbol{g}_i)$ for $i \neq j$, $\beta = \exp(\boldsymbol{g})$, and $\alpha_i, \alpha \in \mathbb{N}^n$ such that $\alpha_i + \beta_i = \exp(\boldsymbol{f}) = \alpha + \beta$. Thus,

$$lc(\boldsymbol{f}) = r_1\sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\beta_1} + \cdots + r_j\sigma^{\alpha}(lc(\boldsymbol{g}))c_{\alpha,\beta} + \cdots + r_t\sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\beta_t}.$$

On the other hand, since $\boldsymbol{g}$ is reducible w.r.t. G', there exist $\boldsymbol{g}'_1, \dots, \boldsymbol{g}'_s \in G'$ such that $lm(\boldsymbol{g}'_l) \mid lm(\boldsymbol{g})$ and $lc(\boldsymbol{g}) = \sum_{l=1}^{s} r'_l\sigma^{\alpha'_l}(lc(\boldsymbol{g}'_l))c_{\alpha'_l,\beta'_l}$, where $\beta'_l = \exp(\boldsymbol{g}'_l)$, $\alpha'_l \in \mathbb{N}^n$ and $\alpha'_l + \beta'_l = \exp(\boldsymbol{g}) = \beta$. So, $lm(\boldsymbol{g}'_l) \mid lm(\boldsymbol{f})$ for $1 \leq i \leq s$; moreover, using the identities of Remark 1.2.3, we have that

$$\begin{aligned}
\sigma^{\alpha}(lc(\boldsymbol{g}))c_{\alpha,\beta} =& \sigma^{\alpha}(\sum_{l=1}^{s} r'_l\sigma^{\alpha'_l}(lc(\boldsymbol{g}'_l))c_{\alpha'_l,\beta'_l})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)\sigma^{\alpha}\sigma^{\alpha'_1}(lc(\boldsymbol{g}'_1))\sigma^{\alpha}(c_{\alpha'_1,\beta'_1})c_{\alpha,\beta} + \cdots + \sigma^{\alpha}(r'_s)\sigma^{\alpha}\sigma^{\alpha'_s}(lc(\boldsymbol{g}'_s))\sigma^{\alpha}(c_{\alpha'_s,\beta'_s})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)c_{\alpha,\alpha'_1}\sigma^{\alpha+\alpha'_1}(lc(\boldsymbol{g}'_1))c^{-1}_{\alpha,\alpha'_1}\sigma^{\alpha}(c_{\alpha'_1,\beta'_1})c_{\alpha,\beta} + \cdots + \\
& \sigma^{\alpha}(r'_s)c_{\alpha,\alpha'_s}\sigma^{\alpha+\alpha'_s}(lc(\boldsymbol{g}'_s))c^{-1}_{\alpha,\alpha'_s}\sigma^{\alpha}(c_{\alpha'_s,\beta'_s})c_{\alpha,\beta} \\
=& \sigma^{\alpha}(r'_1)c_{\alpha,\alpha'_1}\sigma^{\alpha+\alpha'_1}(lc(\boldsymbol{g}'_1))c_{\alpha+\alpha'_1,\beta'_1} + \cdots + \sigma^{\alpha}(r'_s)c_{\alpha,\alpha'_s}\sigma^{\alpha+\alpha'_s}(lc(\boldsymbol{g}'_s))c_{\alpha+\alpha'_s,\beta'_s}.
\end{aligned}$$

Since $\alpha + \beta = \exp(\boldsymbol{f})$, then $\alpha + \alpha'_l + \beta'_l = \exp(\boldsymbol{f})$. Further, if exists $\boldsymbol{g}_k \in \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$ such that $\boldsymbol{g}_k = \boldsymbol{g}'_l$ for some $l \in \{1, \dots, s\}$, then $\beta'_l = \beta_k$ and $\alpha + \alpha'_l = \alpha_k$; therefore, in the representation of $lc(\boldsymbol{f})$ would appear the term $(r_k + r_j\sigma^{\alpha}(r'_l)c_{\alpha,\alpha'_l})\sigma^{\alpha_k}(lc(\boldsymbol{g}_k))c_{\alpha_k,\beta_k}$. From above it follows that $\boldsymbol{f}$ is reducible w.r.t. G' and, hence, G' is a Gröbner basis for M. □

5.5.4 Buchberger's algorithm for modules

Recall that we are assuming that A is a bijective skew PBW extension. We will prove in the current section that every submodule M of A^m has a Gröbner basis, and also we will construct the Buchberger's algorithm for computing such bases. The results obtained here improve those of [58] and [57], and generalize the results obtained in Section 5.4 for left ideals.

We start fixing some notation and proving a preliminary general result.

Definition 5.5.16. *Let* $F := \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_s\} \subseteq A^m$ *such that the least common multiple of* $\{lm(\boldsymbol{g}_1), \dots, lm(\boldsymbol{g}_s)\}$*, denoted by* $\boldsymbol{X}_F$*, is non-zero. Let* $\theta \in \mathbb{N}^n$*,* $\beta_i := \exp(lm(\boldsymbol{g}_i))$ *and* $\gamma_i \in \mathbb{N}^n$ *such that* $\gamma_i + \beta_i = \exp(\boldsymbol{X}_F)$*,* $1 \le i \le s$*.* $B_{F,\theta}$ *will denote a finite set of generators of*

$$S_{F,\theta} := Syz_R[\sigma^{\gamma_1+\theta}(lc(\boldsymbol{g}_1))c_{\gamma_1+\theta,\beta_1} \ \cdots \ \sigma^{\gamma_s+\theta}(lc(\boldsymbol{g}_s))c_{\gamma_s+\theta,\beta_s})].$$

For $\theta = \mathbf{0} := (0, \dots, 0)$*,* $S_{F,\theta}$ *will be denoted by* S_F *and* $B_{F,\theta}$ *by* B_F*.*

Lemma 5.5.17. *Let* $\boldsymbol{g}_1, \dots, \boldsymbol{g}_s \in A^m$ *,* $c_1, \dots, c_s \in R - \{0\}$ *and* $\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$ *be such that* $lm(x^{\alpha_i} lm(\boldsymbol{g}_i)) = \boldsymbol{X}_\delta$*. If* $lm(\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$*, then there exist* $r_1, \dots, r_k \in R$ *and* $l_1, \dots, l_s \in A$ *such that*

$$\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i = \sum_{j=1}^k r_j x^{\delta-\exp(\boldsymbol{X}_F)} \left(\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i \right) + \sum_{i=1}^s l_i \boldsymbol{g}_i,$$

where $\boldsymbol{X}_F$ *is the least common multiple of* $lm(\boldsymbol{g}_1), \dots, lm(\boldsymbol{g}_s)$*,* $\gamma_i \in \mathbb{N}^n$ *is such that* $\gamma_i + \exp(\boldsymbol{g}_i) = \exp(\boldsymbol{X}_F)$*,* $1 \le i \le s$*, and* $(b_{j1}, \dots, b_{js}) \in B_F$*. Moreover, we have that* $lm(x^{\delta-\exp(\boldsymbol{X}_F)} \sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$ *and* $lm(lm(l_i) lm(\boldsymbol{g}_i)) \prec \boldsymbol{X}_\delta$*.*

Proof. Let $\beta_i := \exp(lm(\boldsymbol{g}_i))$ for $1 \le i \le s$; since $\boldsymbol{X}_\delta = lm(x^{\alpha_i} lm(\boldsymbol{g}_i))$, then $lm(\boldsymbol{g}_i) \mid \boldsymbol{X}_\delta$ and hence $\boldsymbol{X}_F \mid \boldsymbol{X}_\delta$, so there exists $\theta \in \mathbb{N}^n$ such that $\exp(\boldsymbol{X}_F) + \theta = \delta$, with $\delta := \exp(\boldsymbol{X}_\delta)$. On the other hand, $\gamma_i + \beta_i = \exp(\boldsymbol{X}_F)$ and $\alpha_i + \beta_i = \delta$, so $\alpha_i = \gamma_i + \theta$ for every $1 \le i \le s$. Now, $lm(\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$ implies that $\sum_{i=1}^s c_i \sigma^{\alpha_i}(lc(\boldsymbol{g}_i)) c_{\alpha_i,\beta_i} = 0$. So we have $\sum_{i=1}^s c_i \sigma^{\theta+\gamma_i}(lc(\boldsymbol{g}_i)) c_{\theta+\gamma_i,\beta_i} = 0$. Hence, we have that $(c_1, \dots, c_s) \in S_{F,\theta}$; from Remark 5.4.2 we know that exist $(c_1', \dots, c_s') \in S_F$ such that $c_i = \sigma^\theta(c_i') c_{\theta,\gamma_i}$. Then,

$$\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i = \sum_{i=1}^s \sigma^\theta(c_i') c_{\theta,\gamma_i} x^{\alpha_i} \boldsymbol{g}_i.$$

Now,

$$\begin{aligned}
x^\theta c_i' x^{\gamma_i} &= (\sigma^\theta(c_i') x^\theta + p_{c_i',\theta}) x^{\gamma_i} \\
&= \sigma^\theta(c_i') x^\theta x^{\gamma_i} + p_{c_i',\theta} x^{\gamma_i} \\
&= \sigma^\theta(c_i') c_{\theta,\gamma_i} x^{\theta+\gamma_i} + \sigma^\theta(c_i') p_{\theta,\gamma_i} + p_{c_i',\theta} x^{\gamma_i} \\
&= \sigma^\theta(c_i') c_{\theta,\gamma_i} x^{\theta+\gamma_i} + p_i'
\end{aligned}$$

where $p'_i := \sigma^\theta(c'_i)p_{\theta,\gamma_i} + p_{c'_i,\theta}x^{\gamma_i}$; note that $p'_i = 0$ or $lm(p'_i) \prec x^{\theta+\gamma_i}$ for each $1 \leq i \leq s$. Thus, $\sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} = x^\theta c'_i x^{\gamma_i} + p_i$, with $p_i = 0$ or $lm(p_i) \prec x^{\theta+\gamma_i}$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{s} c_i x^{\alpha_i}\boldsymbol{g}_i &= \sum_{i=1}^{s} \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\alpha_i}\boldsymbol{g}_i \\
&= \sum_{i=1}^{s}(x^\theta c'_i x^{\gamma_i} + p_i)\boldsymbol{g}_i \\
&= \sum_{i=1}^{s} x^\theta c'_i x^{\gamma_i}\boldsymbol{g}_i + \sum_{i=1}^{s} p_i\boldsymbol{g}_i,
\end{aligned}$$

with $p_i\boldsymbol{g}_i = 0$ or $lm(lm(p_i)lm(\boldsymbol{g}_i)) \prec x^{\theta+\gamma_i+\beta_i} = x^\delta$. On the other hand, let $B_F := \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_k\} := \{(b_{11}, \dots, b_{1s}),$
$\dots, (b_{k1}, \dots, b_{ks})\}$ be a set of generators of S_F; as $(c'_1, \dots, c'_s) \in S_F$, then there exist $r'_1, \dots, r'_k \in R$ such that $(c'_1, \dots, c'_s) = r'_1\boldsymbol{b}_1 + \cdots + r'_k\boldsymbol{b}_k = r'_1(b_{11}, \dots, b_{1s}) + \cdots + r'_k(b_{k1}, \dots, b_{ks})$, thus $c'_i = \sum_{j=1}^{k} r'_j b_{ji}$. Using this, we have

$$\begin{aligned}
\sum_{i=1}^{s} x^\theta c'_i x^{\gamma_i}\boldsymbol{g}_i &= \sum_{i=1}^{s} x^\theta\Big(\sum_{j=1}^{k} r'_j b_{ji}\Big)x^{\gamma_i}\boldsymbol{g}_i \\
&= \sum_{i=1}^{s}\Big(\sum_{j=1}^{k} x^\theta r'_j b_{ji}\Big)x^{\gamma_i}\boldsymbol{g}_i \\
&= \sum_{i=1}^{s}\Big(\sum_{j=1}^{k}(\sigma^\theta(r'_j)x^\theta + p_{r'_j,\theta})b_{ji}\Big)x^{\gamma_i}\boldsymbol{g}_i \\
&= \sum_{i=1}^{s}\Big(\sum_{j=1}^{k} \sigma^\theta(r'_j)x^\theta b_{ji}x^{\gamma_i}\boldsymbol{g}_i + \sum_{j=1}^{k} p_{r'_j,\theta}b_{ji}x^{\gamma_i}\boldsymbol{g}_i\Big) \\
&= \sum_{j=1}^{k}\sum_{i=1}^{s} \sigma^\theta(r'_j)x^\theta b_{ji}x^{\gamma_i}\boldsymbol{g}_i + \sum_{i=1}^{s}\sum_{j=1}^{k} p_{r'_j,\theta}b_{ji}x^{\gamma_i}\boldsymbol{g}_i \\
&= \sum_{j=1}^{k} \sigma^\theta(r'_j)x^\theta \sum_{i=1}^{s} b_{ji}x^{\gamma_i}\boldsymbol{g}_i + \sum_{i=1}^{s} q_i\boldsymbol{g}_i,
\end{aligned}$$

where $q_i := \sum_{j=1}^{k} p_{r'_j,\theta}b_{ji}x^{\gamma_i} = 0$ or $lm(q_i) \prec x^{\theta+\gamma_i}$. So,

$$\sum_{i=1}^{s} c_i x^{\alpha_i}\boldsymbol{g}_i = \sum_{j=1}^{k} r_j x^\theta \sum_{i=1}^{s} b_{ji}x^{\gamma_i}\boldsymbol{g}_i + \sum_{i=1}^{s} l_i\boldsymbol{g}_i,$$

with $l_i := p_i + q_i$ for $1 \leq i \leq s$ and $r_j := \sigma^\theta(r'_j)$ for $1 \leq j \leq k$. Finally, is easy to see $lm(x^\theta(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}\boldsymbol{g}_i)) \prec \boldsymbol{X}_\delta$ since that $lm(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}\boldsymbol{g}_i) \prec lm(x^{\gamma_i}lm(\boldsymbol{g}_i))$. Moreover, $lm(lm(l_i)lm(\boldsymbol{g}_i)) \preceq \max\{lm(lm(p_i)lm(\boldsymbol{g}_i)), lm(lm(q_i)lm(\boldsymbol{g}_i))\} \prec \boldsymbol{X}_\delta$. $\square$

Using the above result, we can establish Buchberger's algorithm for modules:

Theorem 5.5.18. *Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero generators of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of M.*

(ii) *For all $F := \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_s\} \subseteq G$, with $\boldsymbol{X}_F \neq \boldsymbol{0}$, and for any $(b_1, \dots, b_s) \in B_F$,*

$$\textstyle\sum_{i=1}^s b_i x^{\gamma_i} \boldsymbol{g}_i \xrightarrow{G}_+ 0.$$

Proof. (i) $\Rightarrow$ (ii): we observe that $\boldsymbol{f} := \sum_{i=1}^s b_i x^{\gamma_i+\theta} \boldsymbol{g}_i \in M$, so by Theorem 5.5.13 $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{0}$.
(ii) $\Rightarrow$ (i): let $\boldsymbol{0} \neq \boldsymbol{f} \in M$, we will prove that the condition (iii) of Theorem 5.5.13 holds. Let $G := \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$, then there exist $h_1, \dots, h_t \in A$ such that $f = h_1 \boldsymbol{g}_1 + \cdots + h_t \boldsymbol{g}_t$, we can choose $\{h_i\}_{i=1}^t$ such that $\boldsymbol{X}_\delta := \max\{lm(lm(h_i)lm(\boldsymbol{g}_i))\}_{i=1}^t$ is minimal. Let $lm(h_i) := x^{\alpha_i}$, $c_i := lc(h_i)$, $\exp(lm(\boldsymbol{g}_i)) = \beta_i$ for $1 \leq i \leq t$ and $F := \{\boldsymbol{g}_i \in G \mid lm(lm(h_i)lm(\boldsymbol{g}_i)) = \boldsymbol{X}_\delta\}$; renumbering the elements of G we can assume that $F = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_s\}$. We will consider two possible cases.

Case 1: $lm(\boldsymbol{f}) = \boldsymbol{X}_\delta$. Then $lm(\boldsymbol{g}_i) \mid lm(\boldsymbol{f})$ for $1 \leq i \leq s$ and

$$lc(\boldsymbol{f}) = c_1 \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\beta_1} + \cdots + c_s \sigma^{\alpha_s}(lc(\boldsymbol{g}_s))c_{\alpha_s,\beta_s},$$

i.e., the condition (iii) of Theorem 5.5.13 holds.

Case 2: $lm(\boldsymbol{f}) \prec \boldsymbol{X}_\delta$. We will prove that this produces a contradiction. To begin, note that $\boldsymbol{f}$ can be written as

$$\boldsymbol{f} = \sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i + \sum_{i=1}^s (h_i - c_i x^{\alpha_i}) \boldsymbol{g}_i + \sum_{i=s+1}^t h_i \boldsymbol{g}_i; \tag{5.5.1}$$

we see that $lm(\sum_{i=1}^s (h_i - c_i x^{\alpha_i}) \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$ and $lm(\sum_{i=s+1}^t h_i \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$, therefore $lm(\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$; by lemma 5.5.17, we have

$$\sum_{i=1}^s c_i x^{\alpha_i} \boldsymbol{g}_i = \sum_{j=1}^k r_j x^{\delta - \exp(\boldsymbol{X}_F)} \big(\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i\big) + \sum_{i=1}^s l_i \boldsymbol{g}_i, \tag{5.5.2}$$

where $lm(l_i \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$ for $1 \leq i \leq s$. By hypothesis, $\sum_{i=1}^s b_{ji} x^{\gamma_i+\theta} \boldsymbol{g}_i \xrightarrow{G}_+ 0$, and according to Theorem 5.5.10, there exist $q_1, \dots, q_t \in A$ such that $\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i = \sum_{i=1}^t q_i \boldsymbol{g}_i$, with $lm(\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i) = \max\{lm(lm(q_i)lm(\boldsymbol{g}_i))\}_{i=1}^t$, but $(b_{j1}, \dots, b_{js}) \in S_F$, so $lm(\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i) \prec \boldsymbol{X}_F$ and hence $lm(lm(q_i)lm(\boldsymbol{g}_i)) \prec \boldsymbol{X}_F$ for every $1 \leq i \leq t$. Thus,

$$\begin{aligned}
\sum_{j=1}^k r_j x^{\delta - \exp(\boldsymbol{X}_F)} \big(\sum_{i=1}^s b_{ji} x^{\gamma_i} \boldsymbol{g}_i\big) &= \sum_{j=1}^k r_j x^{\delta - \exp(\boldsymbol{X}_F)} \big(\sum_{i=1}^t q_i \boldsymbol{g}_i\big) \\
&= \sum_{i=1}^t \sum_{j=1}^k r_j x^{\delta - \exp(\boldsymbol{X}_F)} q_i \boldsymbol{g}_i = \sum_{i=1}^t \widetilde{q}_i \boldsymbol{g}_i,
\end{aligned}$$

with $\widetilde{q}_i := \sum_{j=1}^{k} r_j x^{\delta-\exp(\boldsymbol{X}_F)} q_i$ and $lm(\widetilde{q}_i \boldsymbol{g}_i) \prec \boldsymbol{X}_\delta$. Substituting $\sum_{i=1}^{s} c_i x^{\alpha_i} \boldsymbol{g}_i = \sum_{i=1}^{t} \widetilde{q}_i \boldsymbol{g}_i$ into equation 5.5.1, we obtain

$$\boldsymbol{f} = \sum_{i=1}^{t} \widetilde{q}_i \boldsymbol{g}_i + \sum_{i=1}^{s} (h_i - c_i x^{\alpha_i}) \boldsymbol{g}_i + \sum_{i=1}^{s} l_i \boldsymbol{g}_i + \sum_{i=s+1}^{t} h_i \boldsymbol{g}_i,$$

and so we have expressed $\boldsymbol{f}$ as a combination of the vectors $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t$, where every term has leading monomial $\prec \boldsymbol{X}_\delta$. This contradicts the minimality of $\boldsymbol{X}_\delta$ and we finish the proof. □

Corollary 5.5.19. *Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ be a set of non-zero vectors of A^m. The algorithm below produces a Gröbner basis for the submodule $\langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s \rangle$ ($P(X)$ denotes the set of subsets of the set X):*

Buchberger's algorithm for modules
over bijective skew PBW extensions

INPUT: $F := \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\} \subseteq A^m, \boldsymbol{f}_i \neq \boldsymbol{0}, 1 \leq i \leq s$

OUTPUT: $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ *a Gröbner basis for* $\langle F \rangle$

INITIALIZATION: $G := \emptyset, G' := F$

WHILE $G' \neq G$ **DO**

> $D := P(G') - P(G)$
>
> $G := G'$
>
> **FOR** *each* $S := \{\boldsymbol{g}_{i_1}, \ldots, \boldsymbol{g}_{i_k}\} \in D$, *with* $\boldsymbol{X}_S \neq \boldsymbol{0}$, **DO**
>
> > *Compute* B_S
> >
> > **FOR** *each* $\boldsymbol{b} = (b_1, \ldots, b_k) \in B_S$ **DO**
> >
> > > *Reduce* $\sum_{j=1}^{k} b_j x^{\gamma_j} \boldsymbol{g}_{i_j} \xrightarrow{G'}_{+} \boldsymbol{r}$, *with* $\boldsymbol{r}$ *reduced with respect to* G' *and* γ_j *defined as in Definition 5.5.16*
> > >
> > > **IF** $\boldsymbol{r} \neq \boldsymbol{0}$ **THEN**
> > >
> > > $G' := G' \cup \{\boldsymbol{r}\}$

From Theorem 1.2.9 and the previous corollary we get the following direct conclusion.

Corollary 5.5.20. *Every submodule of A^m has a Gröbner basis.*

Example 5.5.21. For this first example, we consider the ring $\mathcal{R}$ given in the Example 1.3.6. Once again, for computational reasons, we rewrite the generators and relations for this algebra in the following way:

$$x := b, \quad y := a, \quad z := c, \quad w := d,$$

and

$$
\begin{aligned}
&yx = q^{-1}xy, \quad wx = qxw, \quad zy = qyz, \quad wz = qzw\\
&zx = \mu^{-1}xz, \quad wy = yw + (q - q^{-1})xz,
\end{aligned}
$$

and, therefore, $\mathcal{R} \cong \sigma(\Bbbk[x])\langle y, z, w\rangle$. On $Mon(\mathcal{R})$ we use the order deglex with $y \succ z \succ w$ and in $Mon(A^2)$ the TOPREV order, whence $\boldsymbol{e}_1 > \boldsymbol{e}_2$.
Further, we will take $k = \mathbb{Q}$, $\mu = \frac{1}{2}$ and $q = \frac{2}{3}$. From above relations, we obtain that $\sigma_1(x) = \frac{3}{2}x$, $\sigma_2(x) = 2x$ and $\sigma_3(x) = \frac{2}{3}x$. Let $\boldsymbol{f}_1 = xyw\boldsymbol{e}_1 + w\boldsymbol{e}_2$ and $\boldsymbol{f}_2 = x^2zw\boldsymbol{e}_1 + xy\boldsymbol{e}_2$. We will construct a Gröbner basis for the modules $M := {}_{\mathcal{R}}\langle \boldsymbol{f}_1, \boldsymbol{f}_2\rangle$.
Step 1. we start with $G := \varnothing$, $G' := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. Since $G' \neq G$, we make $D := P(G') - P(G)$, i.e., $D := \{S_1, S_2, S_{1,2}\}$, where $S_1 := \{\boldsymbol{f}_1\}$, $S_2 := \{\boldsymbol{f}_2\}$, $S_{1,2} := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. We also make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we compute B_S:
. For S_1 we have $Syz_{\mathbb{Q}[x]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}]$, where $\beta_1 = \exp(lm(\boldsymbol{f}_1)) = (1,0,1)$, $\gamma_1 = (0,0,0)$ and $c_{\gamma_1,\beta_1} = 1$; thus $B_{S_1} = \{0\}$ and we do not add any vector to G'.
. For S_2 we have an identical situation.
. For $S_{1,2}$ we have $X_{1,2} = lcm\{lm(f_1), lm(f_2)\} = yzw\boldsymbol{e}_1$, thus $\gamma_1 = (0,1,0)$ and $\gamma_2 = (1,0,0)$. Since $zyw = \frac{2}{3}yzw$, then $c_{\gamma_1,\beta_1} = \frac{2}{3}$ and $\sigma^{\gamma_1}(lc(f_1)) = \sigma_2(x) = 2x$. Analogously, $c_{\gamma_2,\beta_2} = 1$ and $\sigma^{\gamma_2}(lc(f_2)) = \sigma_1(x^2) = \frac{9}{4}x^2$. Hence, we must calculate a system of generators for $Syz_{\mathbb{Q}[x]}[\frac{4}{3}x, \frac{9}{4}x^2]$. Such generator set can be $B_{S_{1,2}} = \{(\frac{3}{4}x, -\frac{4}{9})\}$. From this, we get

$$
\begin{aligned}
\frac{3}{4}xz\boldsymbol{f}_1 - \frac{4}{9}y\boldsymbol{f}_2 &= \frac{3}{4}xz(xyw\boldsymbol{e}_1 + w\boldsymbol{e}_2) - \frac{4}{9}y(x^2zw\boldsymbol{e}_1 + xy\boldsymbol{e}_2)\\
&= x^2zyw\boldsymbol{e}_1 + \frac{3}{4}xzw\boldsymbol{e}_2 - x^2yzw\boldsymbol{e}_1 - \frac{2}{3}xy^2\boldsymbol{e}_2\\
&= -\frac{2}{3}xy^2\boldsymbol{e}_2 + \frac{3}{4}xzw\boldsymbol{e}_2 := \boldsymbol{f}_3,
\end{aligned}
$$

Observe that $\boldsymbol{f}_3$ is reduced with respect to G'. We make $G' := \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$.

Step 2: since $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2\} \neq G' = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$, we make $D := \mathcal{P}(G') - \mathcal{P}(G)$, i.e., $D := \{S_3, S_{1,3}, S_{2,3}, S_{1,2,3}\}$, where $S_1 := \{\boldsymbol{f}_1\}, S_{1,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_3\}, S_{2,3} := \{\boldsymbol{f}_2, \boldsymbol{f}_3\}, S_{1,2,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$. We make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we must compute B_S. Since $\boldsymbol{X}_{S_{1,3}} = \boldsymbol{X}_{S_{2,3}} = \boldsymbol{X}_{S_{1,2,3}} = \boldsymbol{0}$, we only need to consider S_3.
. We compute

$$
Syz_{\mathbb{Q}[x]}[\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3}],
$$

where $\beta_3 = \exp(lm(\boldsymbol{f}_3)) = (2,0,0)$; $\boldsymbol{X}_{S_3} = lcm\{lm(\boldsymbol{f}_3)\} = lm(\boldsymbol{f}_3) = y^2\boldsymbol{e}_2$; $\exp(\boldsymbol{X}_{S_3}) = (0,2,0)$; $\gamma_3 = \exp(\boldsymbol{X}_{S_3}) - \beta_3$ = (0,0, 0); $x^{\gamma_3}x^{\beta_3} = y^2$, so $c_{\gamma_3,\beta_3} = 1$. Hence

$$
\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3} = \sigma^{\gamma_3}(-\frac{2}{3}x)1 = \sigma_2^0\sigma_3^0(-\frac{2}{3}x) = -\frac{2}{3}x,
$$

and $Syz_{\mathbb{Q}[x]}[-\frac{2}{3}x] = \{0\}$, i.e., $B_{S_3} = \{0\}$. This means that we not add any vector to G' and hence $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a Gröbner basis for M.

Example 5.5.22. For this other example, we employ the *additive analogue of algebra de Weyl*, $A_n(q_1, \ldots, q_n)$ (see Example 1.1.5, (iv)). We will take $n = 2$, $\Bbbk = \mathbb{Q}$, $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{3}$ and $A = A_2(\frac{1}{2}, \frac{1}{3})$. On $Mon(A)$, we use the order deglex with $y_1 \succ y_2$ and in $Mon(A^2)$ the

TOPREV order with $\boldsymbol{e}_1 > \boldsymbol{e}_2$.
Let $\boldsymbol{f}_1 = x_1y_1^2\boldsymbol{e}_1 + x_2y_2\boldsymbol{e}_2$ and $\boldsymbol{f}_2 = x_2y_2^2\boldsymbol{e}_1 + x_1y_1\boldsymbol{e}_2$. We will construct a Gröbner basis for the module $M := {}_A\langle \boldsymbol{f}_1, \boldsymbol{f}_2\rangle$.
Step 1. we start with $G := \varnothing$, $G' := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. Since $G' \neq G$, we make $D := P(G') - P(G)$, i.e., $D := \{S_1, S_2, S_{1,2}\}$, where $S_1 := \{\boldsymbol{f}_1\}$, $S_2 := \{\boldsymbol{f}_2\}$, $S_{1,2} := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. We also make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we compute B_S:
. For S_1 we have $Syz_{\mathbb{Q}[x_1,x_2]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}]$, where $\beta_1 = \exp(lm(\boldsymbol{f}_1)) = (2,0)$, $\gamma_1 = (0,0)$ and $c_{\gamma_1,\beta_1} = 1$; thus $B_{S_1} = \{0\}$ and we do not add any vector to G'.
. For S_2 we have an identical situation.
. For $S_{1,2}$ we compute

$$Syz_{\mathbb{Q}[x_1,x_2]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_1}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2}],$$

where $\beta_1 = \exp(lm(f_1)) = (2,0)$, $\beta_2 = \exp(lm(f_2)) = (0,2)$; we have $X_{1,2} = lcm\{lm(f_1), lm(f_2)\} = y_1^2y_2^2\boldsymbol{e}_1$; $\gamma_1 = (0,2)$; $y^{\gamma_1}y^{\beta_1} = y_1^2y_2^2$, so $c_{\gamma_1,\beta_1} = 1$ and $\sigma^{\gamma_1}(lc(\boldsymbol{f}_1)) = x_1$; analogously, $\sigma_2 = (2,0)$, $c_{\gamma_2,\beta_2} = 1$ and $\sigma^{\gamma_2}(lc(\boldsymbol{f}_2)) = x_2$. Hence, $Syx_{\mathbb{Q}[x_1,x_2]}[x_1,x_2] = \langle (x_2, -x_1)\rangle$ and $B_{S_{1,2}} = \{(x_2, -x_1)\}$. From this we get

$$\begin{aligned} x_2y^{\gamma_1}\boldsymbol{f}_1 - x_1y_1^2\boldsymbol{f}_2 =& x_2y_2^2(x_1y_1^2\boldsymbol{e}_1 + x_2y_2\boldsymbol{e}_2) - x_1y_1^2(x_2y_2^2\boldsymbol{e}_1 + x_1y_1\boldsymbol{e}_2)\\ =& x_1x_2y_2^2y_1^2y_2^2\boldsymbol{e}_1 + x_2y_2^2x_2y_2\boldsymbol{e}_2 - x_1x_2y_1^2y_1^2y_2\boldsymbol{e}_1 - x_1y_1^2x_1y_1\boldsymbol{e}_2\\ =& -\frac{1}{4}x_1^2y_1^3\boldsymbol{e}_2 + \frac{1}{9}x_2^2y_2^3\boldsymbol{e}_2 - \frac{3}{2}x_1y_1^2\boldsymbol{e}_2 + \frac{4}{3}x_2y_2^2\boldsymbol{e}_2 := \boldsymbol{f}_3, \end{aligned}$$

We observe that f_3 is reduced with respect to G'. We make $G' := \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$.

Step 2: since $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2\} \neq G' = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$, we make $D := \mathcal{P}(G') - \mathcal{P}(G)$, i.e., $D := \{S_3, S_{1,3}, S_{2,3}, S_{1,2,3}\}$, where $S_1 := \{\boldsymbol{f}_1\}, S_{1,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_3\}, S_{2,3} := \{\boldsymbol{f}_2, \boldsymbol{f}_3\}, S_{1,2,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$. We make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we must compute B_S. Since $\boldsymbol{X}_{S_{1,3}} = \boldsymbol{X}_{S_{2,3}} = \boldsymbol{X}_{S_{1,2,3}} = \boldsymbol{0}$, we only need to consider S_3.

. We have to compute

$$Syz_{\mathbb{Q}[x_1,x_2]}[\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3}],$$

where $\beta_3 = \exp(lm(\boldsymbol{f}_3)) = (0,3)$; $\boldsymbol{X}_{S_3} = lcm\{lm(\boldsymbol{f}_3)\} = lm(\boldsymbol{f}_3) = y_1^3\boldsymbol{e}_2$; $\exp(\boldsymbol{X}_{S_3}) = (0,3)$; $\gamma_3 = \exp(\boldsymbol{X}_{S_3}) - \beta_3$ = (0, 0); $x^{\gamma_3}x^{\beta_3} = y_1^3$, so $c_{\gamma_3,\beta_3} = 1$. Hence

$$\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3} = \sigma^{\gamma_3}(-x_1^2)1 = \sigma_2^0\sigma_3^0(-x_1^2) = -x_1^2,$$

and $Syz_{\mathbb{Q}[x_1,x_2]}[-x_1^2] = \{0\}$, i.e., $B_{S_3} = \{0\}$. This means that we not add any vector to G' and hence $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a Gröbner basis for M.

Finally, we get the following direct consequence from Theorem 5.5.18.

Corollary 5.5.23. *Let $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a generator set of a module M. If $ind(\boldsymbol{g}_i) \neq ind(\boldsymbol{g}_j)$ for every $i \neq j$, then G is a Gröbner basis for M.*

Proof. If we have $ind(\boldsymbol{g}_i) \neq ind(\boldsymbol{g}_j)$ for every $i \neq j$, then $\boldsymbol{X}_F = \boldsymbol{0}$ for each subset F of G. In this way, the condition (ii) in Theorem 5.5.18 trivially holds; thus $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ is a Gröbner basis for M. □

5.6 Right skew PBW extensions and right Gröbner bases

Our definition of a skew PBW extension A of a ring R depends on assumption that A is a free left R-module over the standard monomials $Mon(A)$ (see Definition 1.1.1). However, if A is bijective, then A is a right free R-module with basis $Mon(A)$ (see Proposition 1.2.4).

Definition 5.6.1. *Let A and R be rings with $R \subseteq A$; let $x_1, \dots, x_n$ be finitely many elements of A. We say that A is a ring of right polynomial type over R w.r.t. $\{x_1, \dots, x_n\}$ if A is a right R-free module with basis*

$$Mon(A) := Mon\{x_1, \dots, x_n\} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, we say that A is a ring of polynomial type over R w.r.t. $x_1, \dots, x_n$ if $Mon(A)$ is a basis for A as a left and as a right R-module.

Thus, if A is a ring of polynomial type w.r.t. $x_1, \dots, x_n$, every element $f \in A$ has a standard representation both left and right in the following way:

$$f = \sum_{i=1}^{s} c_i x^{\alpha_i} = \sum_{j=1}^{t} x^{\beta_j} d_j,$$

for some $c_i, d_j \in R$ and $x^{\alpha_i}, x^{\beta_j} \in Mon(A)$, $1 \leq i \leq s$, $1 \leq j \leq t$. Given a monomial order on $Mon(A)$ (e.g., deglex order), we can rewrite f with the property that $x^{\alpha_1} \succ \cdots \succ x^{\alpha_s}$ and $x^{\beta_1} \succ \cdots \succ x^{\beta_t}$. Thus, the left and right leading monomials of f are, respectively, $lm^l(f) := x^{\alpha_1}$ and $lm^r(f) := x^{\beta_1}$.

Since the habitual definition of skew PBW extensions consider left representation (see Definition 1.1.1), we could call them "*left skew PBW extensions*". Thus, using the right polynomial ring notion, we can establish the definition of "*right skew PBW extension*", as follows.

Definition 5.6.2. *Let R and A be rings, we say that A is a right skew PBW extension of R, if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finite elements $x_1, \dots, x_n \in A$ such A is a right R-free module with basis*

$$Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $d_{i,r} \in R - \{0\}$ such that*

$$r x_i - x_i d_{i,r} \in R. \tag{5.6.1}$$

(iv) *For every $1 \leq i, j \leq n$ there exists $d_{i,j} \in R - \{0\}$ such that*

$$x_j x_i - x_i x_j d_{i,j} \in R + x_1 R + \cdots + x_n R. \tag{5.6.2}$$

Under these conditions, we will write $A = \sigma^r(R)\langle x_1, \dots, x_n \rangle$.

The right version of Theorem 1.2.2 is as follows.

Theorem 5.6.3. *Let A be a ring of right polynomial type over R w.r.t. $\{x_1, \ldots, x_n\}$. A is a right skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in Mon(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha \in R - \{0\}$ and $q_{\alpha,r} \in A$ such that*
$$rx^\alpha = x^\alpha r_\alpha + q_{\alpha,r}, \tag{5.6.3}$$
where $q_{\alpha,r} = 0$ or $\deg(q_{\alpha,r}) < |\alpha|$ if $q_{\alpha,r} \neq 0$. Moreover, if r is right invertible, then r_α is right invertible.

(b) *For every $x^\alpha, x^\beta \in Mon(A)$ there exist unique elements $d_{\alpha,\beta} \in R$ and $q_{\alpha,\beta} \in A$ such that*
$$x^\alpha x^\beta = x^{\alpha+\beta} d_{\alpha,\beta} + q_{\alpha,\beta}, \tag{5.6.4}$$
where $d_{\alpha,\beta}$ is right invertible, $q_{\alpha,\beta} = 0$ or $\deg(q_{\alpha,\beta}) < |\alpha + \beta|$ if $q_{\alpha,\beta} \neq 0$.

Remark 5.6.4. (i) All properties mentioned in Sections 1.1 and 1.2 can be established for right skew PBW extensions. For example, the elements $d_{i,j}$ in (5.6.2) are right invertible for $i < j$: indeed, let $i < j$, by (5.6.2) there exist $d_{j,i}, d_{i,j} \in R$ such that $x_ix_j - x_jx_id_{j,i} \in R + x_1R + \cdots + x_nR$ and $x_jx_i - x_ix_jd_{i,j} \in R + x_1R + \cdots + x_nR$. So, $x_ix_j - x_ix_jd_{i,j}d_{j,i} \in R + x_1R + \cdots + x_nR$ and since $Mon(A)$ is a R-basis for A_R, then $1 = d_{i,j}d_{j,i}$, i.e., for every $1 \leq i < j \leq n$, $d_{i,j}$ has a right inverse and $d_{j,i}$ has a left inverse.

(ii) In a similar way as were defined quasi-commutative and bijective left skew PBW extensions, it is also possible to define the same notions in the right case. Hence, if A is a right skew PBW extension of a ring R, then A is bijective if the endomorphisms induced by the elements $d_{i,r}$ in (5.6.1) are automorphism of R, and the coefficients $d_{i,j}$ in (5.6.2) are invertible (compare with Definition 1.1.4).

Lemma 5.6.5. *Let A be a ring of polynomial type over R w.r.t. $x_1, \ldots, x_n$. If A is a left or right skew PBW extension of R, then $lm^l(f) = lm^r(f)$ for every $f \in A$.*

Proof. Suppose that A is a left skew PBW extension of R; if $f = 0$ there is nothing to prove. If $0 \neq f$ with $lm^r(f) = x^{\beta_1}$, then f has a right representation in the form $f = x^{\beta_1}d_1 + \cdots + x^{\beta_t}d_t$ with $x^{\beta_1} \succ \cdots \succ x^{\beta_t}$ and $0 \neq d_i \in R$, for $1 \leq i \leq t$. From Theorem 1.2.2 we obtain that $f = \sigma^{\beta_1}(d_1)x^{\beta_1} + p_{\beta_1,d_1} + \cdots + \sigma^{\beta_t}(d_t)x^{\beta_t} + p_{\beta_t,d_t}$ where $p_{\beta_i,d_i} = 0$ or $\deg(p_{\beta_i,d_i}) < |\beta_i|$ if $p_{\beta_1,d_1} \neq 0$. From this we get that $lm^l(f) = x^{\beta_1}$. A similar proof holds if we suppose that A is a right skew PBW extension of R. □

The following theorem allow us to establish the Gröbner bases theory for right ideals and right modules of bijective left skew PBW extensions.

Theorem 5.6.6. *Let A and R be rings such that $R \subseteq A$, and let $x_1, \ldots, x_n$ be nonzero elements in A. Suppose that $Mon(A)$ is ordered by some monomial order. Consider the following statements:*

(i) *A is a ring of right polynomial type over R w.r.t. $x_1, \ldots, x_n$ and a left skew PBW extension of R.*

(ii) *A is a ring of left polynomial type over R w.r.t. $x_1, \ldots, x_n$ and a right skew PBW extension of R.*

(iii) *A is a bijective left skew PBW extension of R.*

(iv) *A is a bijective right skew PBW extension of R.*

Then, $(i) \Leftrightarrow (ii)$, $(iii) \Leftrightarrow (iv)$ and $(iii) \Rightarrow (i)$. Further, if in (i) we replace the first condition by A is also a right skew PBW extension of R, then $(i) \Rightarrow (iii)$.

Proof. (i) $\Leftrightarrow$ (ii): Since A is a left skew PBW extension of R, then $Mon(A)$ is a basis for ${}_RA$, i.e., A is a ring of left polynomial type over R w.r.t. $x_1, \ldots, x_n$. Now, since A is a ring of right polynomial type over R w.r.t. $x_1, \ldots, x_n$, then A satisfies (ii) in Definition 5.6.2. On the other hand, given $0 \neq r \in R$ and $1 \leq i \leq n$, we have that $rx_i = x_i d_{i,r} + p_{i,r}$ for some $0 \neq d_{i,r} \in R$ and $p_{i,r} \in R$ (see Lemma 5.6.5). Similarly, for $1 \leq i, j \leq n$, we have that $x_j x_i = c_{i,j} x_i x_j + p_{i,j} = x_i x_j d_{i,j} + q_{i,j}$ for some $0 \neq d_{i,j} \in R$ and $q_{i,j} \in R + x_1 R + \cdots + x_n R$. The proof of (ii) $\Rightarrow$ (i) is analogous.

(iii) $\Leftrightarrow$ (iv): From Proposition 1.2.4 we have that A is a right free R-module with basis $Mon(A)$. Only remains to show that there exist elements $d_{i,r}$ and $d_{i,j}$ in R satisfying (iii) and (iv) in Definition 5.6.2, and that with these elements A turns out to be bijective. Since A is bijective, each endomorphism σ_i in Proposition 1.1.3 is an automorphism; thus given $r \in R$ and $1 \leq i \leq n$, $rx_i - x_i\sigma_i^{-1}(r) \in R$, so it is enough to take $d_{i,r} := \sigma_i^{-1}(r)$. We define $\sigma_i' : R \to R$ as $\sigma_i' := \sigma_i^{-1}$. Thus, (iii) in Definition 5.6.2 holds and, of course, each σ_i' is bijective. For $1 \leq i, j \leq n$, we have that $x_j x_i = c_{i,j} x_i x_j + p_{i,j}$, where $c_{i,j}$ is invertible and $p_{i,j} \in R + Rx_1 + \cdots + Rx_n$. Using again Lemma 5.6.5, as in the first part of the proof, $x_j x_i = x_i x_j d_{i,j} + q_{i,j}$ for some $d_{i,j} \neq 0$ and $q_{i,j} \in R + x_1 R + \cdots + x_n R$. So, (iv) in Definition 5.6.2 holds. Moreover, observe that

$$x_i x_j d_{i,j} = x_i[\sigma_j(d_{i,j})x_j + r] = x_i\sigma_j(d_{i,j})x_j + x_i r = [\sigma_i(\sigma_j(d_{i,j}))x_i + s]x_j + x_i r =$$
$$\sigma_i(\sigma_j(d_{i,j}))x_i x_j + s x_j + \sigma_i(r)x_i + u, \text{ with } r, s, u \in R,$$

whence, $c_{i,j} = \sigma_i(\sigma_j(d_{i,j}))$, i.e., $d_{i,j} = \sigma_j^{-1}(\sigma_i^{-1}(c_{i,j}))$ is invertible. We have proved that A is a bijective right skew PBW extension of R. The reverse implication can be proved similarly.

The implication (iii) $\Rightarrow$ (i) is immediate.

Finally, if A is a left and right skew PBW extension of R, then the endomorphism σ_i is bijective for each $1 \leq i \leq n$: In fact, since for $r \in R$ we have $rx_i = x_i\sigma_i'(r) + q_{i,r} = \sigma_i(\sigma_i'(r))x_i + q_{i,r}'$ for certain $q_{i,r}' \in R$. Uniqueness in the standard representation implies that $r = \sigma_i(\sigma_i'(r))$; i.e., $\sigma_i\sigma_i' = i_R$ and hence σ_i is surjective, but according to Proposition 1.1.3, σ_i is injective. So, σ_i is bijective and $\sigma_i' = \sigma_i^{-1}$. Now, as above, $d_{i,j} = \sigma_j^{-1}(\sigma_i^{-1}(c_{i,j}))$ and $d_{i,j}$ is right invertible (see Remark 5.6.4), then $c_{i,j}$ is right invertible, ie.e, $c_{i,j}$ is invertible for $1 \leq i, j \leq n$. □

Remark 5.6.7. The equivalence (iii)$\Leftrightarrow$(iv) in the previous theorem let us to get the following key conclusion: if A is a bijective skew PBW extension of a ring R (we mean left as always in the present work), A is also a bijective right skew PBW extension of ring R, and therefore, we have a left and a right division algorithm. Obviously, if the elements of A are given by their left standard representation, we may have to rewrite them in their right standard representation, in order to be able to perform right divisions. Left and right versions of Buchberger's algorithm are also available. Thus, the theory of Gröbner bases for left ideals and submodules of left free modules developed in this chapter has its counterpart on the right.

CHAPTER 6

Elementary applications of Gröbner theory

There are some classical and elementary applications of Gröbner theory that we will study in this chapter. We will consider the membership problem, we will compute the syzygy module, free resolutions of modules, the intersection and quotient of ideals and submodules, the matrix presentation of a finitely presented module, and the kernel and the image of homomorphism between modules. Recall that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ represents a bijective skew PBW extension of a LGS ring R.

6.1 The membership problem

Let $F = \{f_1, \dots, f_s\} \subset A$ and $I := \langle F\rangle$ be the left ideal generated by F. The membership problem ask whether one may effectively decide if an element $f \in A$ belongs to I. Gröbner theory provides an easy answer to this problem. Indeed, let G be a Gröbner basis of I; making use of the division algorithm (Theorem 5.2.6), it is possible to obtain polynomials $h_1, \dots, h_t, h \in A$, with h reduced w.r.t. G, such that $f \xrightarrow{G}_+ h$ and $f = q_1 f_1 + \cdots + q_t f_t + h$; according to Corollary 5.3.3 if $h \neq 0$, then $f \notin I$; and if $h = 0$, then $f \in I$.

The next theorem complements the answer allowing us to write f as A-linear combination of $f_1, \dots, f_s$ when $f \in I$.

Theorem 6.1.1. *Let $F = \{f_1, \dots, f_s\}$ be a subset of A and $G = \{g_1, \dots, g_t\}$ be a Gröbner basis of $I := \langle F\rangle$. Then, there exist matrices $H = [h_{ij}] \in M_{s\times t}(A)$ and $Q = [q_{ij}] \in M_{t\times s}(A)$ such that*

$$G^T = H^T F^T \textit{ and } F^T = Q^T G^T,$$

where $G := \begin{bmatrix} g_1 & \cdots & g_t\end{bmatrix}$, $F := \begin{bmatrix} f_1 & \cdots & f_s\end{bmatrix}$ and

$$H := \begin{bmatrix} h_{11} & \cdots & h_{1t} \\ \vdots & \ddots & \vdots \\ h_{s1} & \cdots & h_{st}\end{bmatrix}; \textit{ and } Q := \begin{bmatrix} q_{11} & \cdots & q_{1s} \\ \vdots & \ddots & \vdots \\ q_{t1} & \cdots & q_{ts}\end{bmatrix}.$$

Proof. Initially, we show how the Buchberger's algorithm allows us to compute the matrix H. For this, we take

$$\begin{aligned} G_{-1} &:= \varnothing \\ G_0 &:= F \\ G_{i+1} &:= G_i \cup \big\{r \neq 0 \mid \sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} \xrightarrow{G}_+ r, \text{ for } (b_1, \dots, b_k) \in B_S\big\}, \end{aligned}$$

where $S = \{g_{i_1}, \dots, g_{i_k}\} \in P(G_i) - P(G_{i-1})$ and $G_i := \{g_1, \dots, g_{t_i}\}$. Suppose that

$$\begin{bmatrix} g_1 \\ \vdots \\ g_{t_i} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{s1} \\ \vdots & \ddots & \vdots \\ h_{1t_i} & \cdots & h_{st_i} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix}$$

and let g_{t_i+1} be an element in $A - \{0\}$ such that $\sum_{j_1}^{k} b_j x^{\gamma_j} g_{i_j} \xrightarrow{G_i}_+ g_{t_i+1}$; then, $\sum_{j_1}^{k} b_j x^{\gamma_j} g_{i_j} = a_1 g_1 + \cdots + a_{t_i} g_{t_i} + g_{t_i+1}$, and thus

$g_{t_i+1} = \sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} + (-a_1) g_1 + \cdots + (-a_{t_i}) g_{t_i} = (-a_1) g_1 + \cdots + (b_1 x^{\gamma_1} - a_{i_1}) g_{i_1} + \cdots + (b_k x^{\gamma_k} - a_{i_k}) g_{i_k} + \cdots + (-a_{t_i}) g_{t_i} = (-a_1)(h_{11} f_1 + \cdots + h_{s1} f_s) + \cdots + (b_1 x^{\gamma_1} - a_{i_1})(h_{1i_1} f_1 + \cdots + h_{si_1} f_s) + \cdots + (b_k x^{\gamma_k} - a_{i_k})(h_{1i_k} f_1 + \cdots + h_{si_k} f_s) + \cdots + (-a_{t_i})(h_{1t_i} f_1 + \cdots + h_{st_i} f_s) = (-a_1 h_{11} + \cdots + (b_1 x^{\gamma_1} - a_{i_1}) h_{1i_1} + \cdots + (b_k x^{\gamma_k} - a_{i_k}) h_{1i_k} + \cdots - a_{t_i} h_{1t_i}) f_1 + \cdots + (-a_1 h_{s1} + \cdots + (b_1 x^{\gamma_1} - a_{i_1}) h_{si_1} + \cdots + (b_k x^{\gamma_k} - a_{i_k}) h_{si_k} + \cdots - a_{t_i} h_{st_i}) f_s = h_{1t_i+1} f_1 + \cdots + h_{st_i+1} f_s$,

with $h_{rt_i+1} := -a_1 h_{r1} + \cdots + (b_1 x^{\gamma_1} - a_{i_1}) h_{ri_1} + \cdots + (b_k x^{\gamma_k} - a_{i_k}) h_{ri_k} + \cdots - a_{t_i} h_{rt_i}$, for $1 \leq r \leq s$. With this last we have

$$H_{t_k+1} = \begin{bmatrix} h_{11} & \cdots & h_{1t_i+1} \\ \vdots & \ddots & \vdots \\ h_{s1} & \cdots & h_{st_i+1} \end{bmatrix}.$$

Iterating this construction, we will obtain a matrix H with the required properties.

In order to obtain matrix Q, it is enough to remember that if $G = \{g_1, \dots, g_t\}$ is a Gröbner basis for $\langle F \rangle$ then $f_i \xrightarrow{G}_+ 0$ for any$1 \leq i \leq s$; the division algorithm implies that $f_i = q_{1i} g_1 + \cdots + q_{ti} g_t$ for all $1 \leq i \leq s$, and thus the matrix

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1s} \\ \vdots & \ddots & \vdots \\ q_{t1} & \cdots & q_{ts} \end{bmatrix}$$

satisfies the assertion. □

Example 6.1.2. As in the Example 5.4.7, let A be the diffusion algebra. We want to know if the polynomial $f = x_1^2 x_2 D_1 D_2^2 + \frac{3}{2} x_1^2 x_2^2 D_1 D_2 - x_1^2 x_2^3 D_1 + \frac{1}{2} x_1 x_2^2 D_2$ is in the left ideal $I := \langle f_1, f_2 \rangle$, where $f_1 = x_1 D_1 D_2 + x_2$, $f_2 = x_2 D_2^2$. For this task, we calculate a Gröbner basis for I and we check if f can be reduced to 0 with respect to $\{f_1, f_2\}$. We consider the order deglex on $Mon(A)$, with $D_1 \succ D_2$.

We start taking $G := \varnothing$ and $G' := \{f_1, f_2\}$.
Step 1. Since $G' \neq G$, we have $D = \{S_1, S_2, S_{1,2}\}$.
We make $G = G'$.
Since R has not zero divisors, S_1 and S_2 do not add any polynomial to G'. For $S_{1,2}$, we compute $B_{S_{1,2}}$, a generator set of $Syz_R[\sigma^{\gamma_1}(lc(f_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_2}(lc(f_2))c_{\gamma_2,\beta_2}]$: $X_{1,2} = lcm\{D_1D_2, D_2\} = D_1D_2^2$, so $\gamma_1 = (0,1)$, $D_2(D_1D_2) = 2D_1D_2^2 + x_2D_1D_2 - x_1D_2^2$, and whence, $c_{\gamma_1,\beta_1} = 2$; in a similar way, $\gamma_2 = (1,0)$ and $c_{\gamma_2,\beta_2} = 1$. Therefore, $B_{S_{1,2}} = \{(\frac{1}{2}x_2, -x_1)\}$ and we have

$$\tfrac{1}{2}x_2D_2f_1 - x_1D_1f_2 = \tfrac{1}{2}x_1x_2^2D_1D_2 - \tfrac{1}{2}x_1^2x_2D_2^2 + \tfrac{1}{2}x_2^2D_2.$$

Since that

$$\tfrac{1}{2}x_1x_2^2D_1D_2 - \tfrac{1}{2}x_1^2x_2D_2^2 + \tfrac{1}{2}x_2^2D_2 \xrightarrow{G}_+ \tfrac{1}{2}x_2^2D_2 - \tfrac{1}{2}x_2^3 =: f_3$$

and f_3 is reduced with respect to G, we add the polynomial f_3 and we make $G' := \{f_1, f_2, f_3\}$.
Step 2. Since $G' \neq G$, we compute $D = P(G') - P(G)$ and we make $G = G'$. In D we only need to consider three subsets:

$$S_{1,3} = \{f_1, f_3\},\ S_{2,3} = \{f_2, f_3\},\ S_{1,2,3} = \{f_1, f_2, f_3\}.$$

For $S_{1,3}$ we have $X_{1,3} = D_1D_2$ and, hence, $\gamma_1 = (0,0)$ and $\gamma_3 = (1,0)$. From this it follows that $B_{S_{1,3}} = \{(x_2^2, -2x_1)\}$, and we obtain

$$x_2^2f_1 - 2x_1D_1f_3 = x_1x_2^3D_1 + x_2^3 =: f_4$$

and f_4 is reduced with respect to G, we add the polynomial f_4 and we make $G' := \{f_1, f_2, f_3, f_4\}$.
For $S_{2,3}$, $X_{S_{2,3}} = D_2^2$, so $\gamma_2 = (0,0)$ and $c_{\gamma_2,\beta_2} = 1$; in the same way, $\gamma_3 = (0,1)$ and $c_{\gamma_3,\beta_3} = 1$. Thus $B_{S_{2,3}} = \{(x_2, -2)\}$, and

$$x_2f_2 - 2D_2f_3 = x_2^3D_2 \xrightarrow{G} x_2^4 =: f_5.$$

Since f_5 is reduced with respect to G, we add f_5 and we make $G' := \{f_1, f_2, f_3, f_4, f_5\}$.
For $S_{1,2,3}$ we have that $\gamma_1 = (0,1)$, $\gamma_2 = (1,0)$, $\gamma_3 = (1,1)$, and hence, $B_{S_{1,2,3}} = \{(0, x_2, -2), (\frac{1}{2}x_2, -x_1, 0$ for the first generator we obtain a polynomial that can be reduced to 0 by f_1, f_2 and f_3. The same applies for the second generator. Therefore, we do not add any polynomial to G'.
Step 3. Again, $G \neq G'$. Thus, we compute $D = P(G') - P(G)$ and we make $G = G'$. In this case, we need to consider 14 sets in D. For these subsets we obtain polynomials that are reducible to 0 by $G = \{f_1, f_2, f_3, f_4, f_5\}$. Thus, G is a Gröbner basis for $I := \langle f_1, f_2\rangle$.
Finally, applying the division algorithm, f reduces to 0 with respect to $\{f_1, f_2, f_3, f_4, f_5\}$.
Moreover, we have that

$$f = (\frac{1}{2}x_1x_2D_2 + x_1x_2^2)f_1 + \frac{1}{2}x_1^3f_2 - x_1f_3.$$

The membership problem can be extended for modules: let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ be a set of non-zero vectors in A^m and $M := \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s \rangle$ the A-submodule of A^m generated by $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s$; let $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis for M and $\boldsymbol{f} \in A^m$, applying the division algorithm we find $l_1, \ldots, l_t, \in A$ and a reduced vector $\boldsymbol{h} \in A^m$ w.r.t. F such that $\boldsymbol{f} = l_1\boldsymbol{g}_1 + \cdots + l_t\boldsymbol{g}_t + \boldsymbol{h}$; then, $\boldsymbol{f} \in M$ if and only if $\boldsymbol{h} = \boldsymbol{0}$. In addition, Theorem 6.1.1 can be formulated and proved for modules.

Theorem 6.1.3. *Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ be a subset of nonzero vectors of A^m, and $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis of $M := {}_A\langle F \rangle$. Then, there exist matrices $H = [h_{ij}] \in M_{s\times t}(A)$ and $Q = [q_{ij}] \in M_{t\times s}(A)$ such that*

$$G^T = H^T F^T \text{ and } F^T = Q^T G^T, \tag{6.1.1}$$

where $G := \begin{bmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix}$, $F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix}$ and

$$H := \begin{bmatrix} h_{11} & \cdots & h_{1t} \\ \vdots & \ddots & \vdots \\ h_{s1} & \cdots & h_{st} \end{bmatrix}; \text{ and } Q := \begin{bmatrix} q_{11} & \cdots & q_{1s} \\ \vdots & \ddots & \vdots \\ q_{t1} & \cdots & q_{ts} \end{bmatrix}.$$

Therefore, 6.1.1 allow us to write $\boldsymbol{f}$ as A-linear combination of $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s$ when $\boldsymbol{f} \in M$.

As application of the membership problem, given two ideals I and J of A generated by $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_n\}$ respectively, we can effectively decide whether $I = J$: it is enough to check if $f_i \in J$ for all $i \leq i \leq m$, and if $g_j \in I$ for all $1 \leq j \leq n$. A similar remark can be done for modules.

Remark 6.1.4. Of course, Theorems 6.1.1 and 6.1.3 have their right version (see Remark 2.1.2): Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ be a subset of A^m and $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis of $M := \langle F \rangle_A$. Then, there exist matrices $H = [h_{ij}] \in M_{s\times t}(A)$ and $Q = [q_{ij}] \in M_{t\times s}(A)$ such that

$$G = FH \text{ and } F = GQ,$$

where $G := \begin{bmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix}$ and $F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix}$.

6.2 Computing syzygies

Now, we will compute the syzygy module of a finite set of polynomials of A, and more generally, of a finite set of elements of A^m.

Let A^m be the left A-module of column vectors of length $m \geq 1$. Given I a left ideal of A, with $I = \langle f_1, \ldots, f_s \}$, we may define the following A-homomorphism:

$$\phi : A^s \to I; \quad (h_1, \ldots, h_s)^T \mapsto \sum_{i=1}^{s} h_i f_i;$$

Note that ϕ is surjective and, therefore, $I \cong A^s / \ker(\phi)$.

Definition 6.2.1. *The kernel of the homomorphism ϕ is called the **syzygy module** of the matrix $\begin{bmatrix} f_1 & \cdots & f_s \end{bmatrix}$. It is denoted by $Syz(f_1, \ldots, f_s)$. An element $(h_1, \ldots, h_s)^T \in Syz(f_1, \ldots, f_s)$ is called a **syzygy** of $\begin{bmatrix} f_1 & \cdots & f_s \end{bmatrix}$ and satisfies*

$$h_1 f_1 + \cdots + h_s f_s = 0.$$

Note that ϕ can be viewed as the matrix multiplication:

$$\phi(h_1,\dots,h_s) = \begin{bmatrix} h_1 & \cdots & h_s \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix};$$

and $Syz(f_1,\dots f_s)$ as the set of all solutions $(h_1,\dots,h_s)^T \in A^s$ of the linear equation

$$\begin{bmatrix} h_1 & \cdots & h_s \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} = 0.$$

Since A is a left Noetherian ring, then $Syz(f_1,\dots,f_s)$ is a finitely generated left A-module. We will compute a system of generators for $Syz(f_1,\dots,f_s)$ for any $f_1,\dots,f_s \in A$. For this, we first compute a Gröbner basis $G = \{g_1,\dots,g_t\}$ for $I = \langle f_1,\dots,f_s\}$. Next, we obtain a set of generators for $Syz(g_1,\dots,g_t)$ and, finally, we will obtain a system of generators for $Syz(f_1,\dots,f_s)$ from one of $Syz(g_1,\dots,g_t)$.
So, let $G = \{g_1,\dots,g_t\}$ be a Gröbner basis for I, $S = \{g_{i_1},\dots,g_{i_k}\} \subseteq G$ and $\boldsymbol{b} = (b_1,\dots,b_k) \in B_S$ (recall that B_S is a set of generators of $Syz_R(\sigma^{\gamma_j}(lc(g_{i_j}))c_{\gamma_j,\exp(g_{i_j})} \mid 1 \leq j \leq k))$; we know that $\sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} \xrightarrow{G}_+ 0$ and hence there exist $h_1,\dots,h_t \in A$ such that $\sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} = \sum_{i=1}^{t} h_i g_i$. For each $\boldsymbol{b} \in B_S$, we define

$$\boldsymbol{s}_{\boldsymbol{b}S} := \sum_{j=1}^{k} b_j x^{\gamma_j} \boldsymbol{e}_{i_j} - (h_1,\dots,h_t) \in A^t;$$

then $\boldsymbol{s}_{\boldsymbol{b}S} \in Syz(g_1,\dots,g_t)$: in fact,

$$\begin{aligned} \boldsymbol{s}_{\boldsymbol{b}S} \begin{bmatrix} g_1 \\ \vdots \\ g_t \end{bmatrix} &= [\sum_{j=1}^{k} b_j x^{\gamma_j} \boldsymbol{e}_{i_j} - (h_1,\dots,h_t)] \begin{bmatrix} g_1 \\ \vdots \\ g_t \end{bmatrix} \\ &= \sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} - \sum_{i=1}^{t} h_i g_i = 0. \end{aligned}$$

One natural question that aries here is: must we calculate all vectors $\boldsymbol{s}_{\boldsymbol{b}S}$ for each subset of G? The answer is negative; we just need certain particular subsets.

Definition 6.2.2. *Let $X_1,\dots,X_t \in Mon(A)$ and $J \subseteq \{1,\dots,t\}$. Let*

$$X_J := lcm\{X_j \mid j \in J\}.$$

We say that J is saturated with respect to $\{X_1,\dots,X_t\}$, if

$$X_j \mid X_J \Rightarrow j \in J,$$

for any $j \in \{1,\dots,t\}$. The saturation J' of J consists of all $j \in \{1,\dots,t\}$ such that $X_j \mid X_J$.

Theorem 6.2.3. *With the above notations, a generating set for $Syz(g_1,\dots,g_t)$ is*

$$S := \{\boldsymbol{s}_v^J \mid J \subseteq \{1,\dots,t\} \text{ is saturated w.r.t.} \{lm(g_1),\dots,lm(g_t)\}, 1 \leq v \leq l_J\},$$

where

$$\boldsymbol{s}_v^J := \sum_{j\in J} b_{vj}^J x^{\gamma_j}\boldsymbol{e}_j - (h_1^v,\dots,h_t^v),$$

with $\gamma_j \in \mathbb{N}^n$ such that $\gamma_j + \beta_j = \exp(X_J)$, $\beta_j = \exp(g_j)$ for $j \in J$, $B_J := \{\boldsymbol{b}_1^J,\dots,\boldsymbol{b}_{l_J}^J\}$ a system of generators for $S_J := Syz_R[\sigma^{\gamma_j}(lc(g_j))c_{\gamma_j,\beta_j} \mid j \in J]$, and $\boldsymbol{b}_v^J := (b_{vj}^J)_{j\in J}$ for $1 \leq v \leq l_J$.

Proof. We have already seen that ${}_A\langle S\rangle \subseteq Syz(g_1,\dots,g_t)$. Suppose that there exists $\boldsymbol{u} = (u_1,\dots,u_t) \in Syz(g_1,\dots,g_t) - \langle S\rangle$. We can choose $\boldsymbol{u}$ such that

$$x^\delta := \max_{1\leq i\leq t}\{lm(lm(u_i)lm(g_i))\}$$

is minimal with respect to $\preceq$. Let

$$J := \{j \in \{1,\dots,t\} \mid lm(lm(u_j)lm(g_j)) = x^\delta\}.$$

Since $\sum_{i=1}^t u_ig_i = 0$, we have $\sum_{j\in J} lc(u_j)\sigma^{\alpha_j}(lc(g_j))c_{\alpha_j,\beta_j} = 0$, where $\alpha_i := \exp(u_i)$ for $1 \leq i \leq t$. If $X_J := lcm\{lm(g_j) \mid j \in J\}$, then $X_J \mid x^\delta$ and there is $\theta \in \mathbb{N}^n$ with $\theta + \exp(X_J) = \delta$. But $\alpha_j + \beta_j = \delta$ and $\gamma_j + \beta_j = \exp(X_J)$ for all $j \in J$, then $\theta + \gamma_j + \beta_j = \alpha_j + \beta_j$, i.e., $\theta + \gamma_j = \alpha_j$. Thus, $(lc(u_j))_{j\in J} \in S_{J,\theta} := Syz_R[\sigma^{\theta+\gamma_j}(lc(g_j))c_{\theta+\gamma_j,\beta_j} \mid j \in J]$. If J' is the saturation of J, then $X_J = X_{J'}$ and $\boldsymbol{w} = (w_j)_{j\in J'}$ given by

$$w_j = \begin{cases} lc(u_j), & \text{if } j \in J,\\ 0, & \text{if } j \in J' - J\end{cases}$$

is an element of $S_{J',\theta}$. According to Remark 5.4.2, there exists $(b_j)_{j\in J'} \in S_{J'} := Syz_R[\sigma^{\gamma_j}(lc(g_j))c_{\gamma_j,\beta_j} \mid j \in J']$ such that $w_j = \sigma^\theta(b_j)c_{\theta,\gamma_j}$ for $j \in J'$. This implies that $b_j = 0$ for $j \in J' - J$. Now, $(b_j)_{j\in J'} = \sum_{v=1}^{l_{J'}} r'_v\boldsymbol{b}_v^{J'}$, with $B_{J'} := \{\boldsymbol{b}_v^{J'} \mid 1 \leq v \leq l_{J'}\}$ a system of generators for $S_{J'}$ and $r'_v \in R$ for $1 \leq v \leq l_{J'}$. Hence, $b_j = \sum_{v=1}^{l_{J'}} r'_vb_{vj}^{J'}$ and thus $w_j = \sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}$ for all $j \in J'$. Define $\boldsymbol{u}' := \boldsymbol{u} - \sum_{v=1}^{l_{J'}} r_vx^\theta\boldsymbol{s}_v^{J'}$, with $r_v := \sigma^\theta(r'_v)$ for $1 \leq v \leq l_{J'}$; then $\boldsymbol{u}' \in Syz(g_1,\dots,g_t)$ since $\sum_{v=1}^{l_{J'}} r_vx^\theta\boldsymbol{s}_v^{J'} \in {}_A\langle S\rangle$. Note that

$$\begin{aligned}
\sum_{v=1}^{l_{J'}} r_vx^\theta\boldsymbol{s}_v^{J'} &= r_1x^\theta\boldsymbol{s}_1^{J'} + \cdots + r_{l_{J'}}x^\theta\boldsymbol{s}_{l_{J'}}^{J'}\\
&= r_1x^\theta[\sum_{j\in J'} b_{1j}^{J'}x^{\gamma_j}\boldsymbol{e}_j - (h_1^1,\dots,h_t^1)] + \cdots +\\
&\quad r_{l_{J'}}x^\theta[\sum_{j\in J'} b_{l_{J'}j}^{J'}x^{\gamma_j}\boldsymbol{e}_j - (h_1^{l_{J'}},\dots,h_t^{l_{J'}})]\\
&= r_1[\sum_{j\in J'}(\sigma^\theta(b_{1j}^{J'})c_{\theta,\gamma_j}x^{\theta+\gamma_j} + p_j^1)\boldsymbol{e}_j - (h_1^1,\dots,h_t^1)] + \cdots +\\
&\quad r_{l_{J'}}[\sum_{j\in J'}(\sigma^\theta(b_{l_{J'}j}^{J'})c_{\theta,\gamma_j}x^{\theta+\gamma_j} + p_j^{l_{J'}})\boldsymbol{e}_j - (h_1^{l_{J'}},\dots,h_t^{l_{J'}})]
\end{aligned}$$

Thus, for $j \in J$ we have that

$$\begin{aligned} u'_j &= u_j - [\sum_{v=1}^{l_{J'}} r_v\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}x^{\theta+\gamma_j} + \sum_{v=1}^{l_{J'}} p_j^v - \sum_{v=1}^{l_{J'}} h_j^v] \\ &= u_j - [\sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}x^{\alpha_j} + \sum_{v=1}^{l_{J'}} p_j^v - \sum_{v=1}^{l_{J'}} h_j^v] \\ &= u_j - lc(u_j)x^{\alpha_j} - \sum_{v=1}^{l_{J'}} p_j^v + \sum_{v=1}^{l_{J'}} h_j^v \end{aligned}$$

since $j \in J$, $\gamma_j + \theta = \alpha_j$ and $w_j = lc(u_j) = \sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}$. Here $p_j^v = 0$ or $\deg(p_j^v) < |\theta + \gamma_j|$ for every $1 \le v \le l_{J'}$. Then,

$$lm(lm(u_j - lc(u_j)x^{\alpha_j})lm(g_j)) \prec lm(lm(u_j)lm(g_j)) = x^\delta, lm(lm(p_j^v)lm(g_j)) \prec x^{\theta+\gamma_j+\beta_j} = x^\delta,$$

and

$$lm(lm(h_j^v)lm(g_j)) \preceq lm(\sum_{j\in J'} b_{vj}^{J'}x^{\gamma_j}g_j) \prec X_{J'} = X_J \preceq x^\delta,$$

so $lm(lm(u'_j)lm(g_j)) \prec x^\delta$. Now, if $j \in J' - J$, then $w_j = \sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j} = 0$ and $lm(lm(u_j)lm(g_j)) \prec x^\delta$, thus $lm(lm(u'_j)lm(g_j)) \prec x^\delta$. Finally, if $j \notin J'$, then $u'_j = u_j + \sum_{v=1}^{l_{J'}} h_j^v$ and $lm(lm(u'_j)lm(g_j)) \prec x^\delta$. So, $lm(lm(u'_i)lm(g_i)) \prec x^\delta$ for every $1 \le i \le t$ and, by minimality of $\boldsymbol{u}$, we have that $\boldsymbol{u}' \in {}_A\langle S\rangle$ and hence, $\boldsymbol{u} \in {}_A\langle S\rangle$, a contradiction. Therefore, ${}_A\langle S\rangle = Syz(g_1, \ldots, g_t)$. □

Now, we return to the initial problem of calculating a system of generators for $Syz(f_1, \ldots, f_s)$, where $\{f_1, \ldots, f_s\}$ is a collection of nonzero polynomials, which no necessarily form a Gröbner basis for $I = \langle f_1, \ldots, f_s\rangle$. As we saw in Theorem 6.1.1, there exist $H \in M_{s\times t}(A)$ and $Q \in M_{t\times s}(A)$ such that $G^T = H^T F^T$ and $F^T = Q^T G^T$, where $G := \begin{bmatrix} g_1 & \cdots & g_t \end{bmatrix}$, $F := \begin{bmatrix} f_1 & \cdots & f_s \end{bmatrix}$ and G is a Gröbner basis for I. By Theorem 6.2.3, we may compute a set of generators $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_l\}$ for $Syz(g_1, \ldots, g_t)$. Thus, for each $1 \le i \le l$ we have that

$$\boldsymbol{s}_i H^T F^T = \boldsymbol{s}_i G^T = 0,$$

and therefore, $\langle \boldsymbol{s}_i H^T \mid 1 \le i \le l\rangle \subseteq Syz(f_1, \ldots, f_s)$. Further,

$$\begin{bmatrix} I_s - Q^T H^T \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} - Q^T H^T \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

and thereby the rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s$ of $I_s - Q^T H^T$ also belong to $Syz(f_1, \ldots, f_s)$.

Theorem 6.2.4. *With the above notation, we have*

$$Syz(f_1, \ldots, f_s) = \langle \boldsymbol{s}_1 H^T, \ldots, \boldsymbol{s}_l H^T, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_s\rangle \le A^s.$$

Proof. Let $\boldsymbol{s} = (a_1, \ldots, a_s)^T$ be an element in $Syz(f_1, \ldots, f_s)$, then

$$0 = \boldsymbol{s}^T F^T = \boldsymbol{s}^T Q^T G^T,$$

and therefore $\boldsymbol{s}^T Q^T \in Syz(g_1, \ldots, g_t)$. Thus, $\boldsymbol{s}^T Q^T = \sum_{i=1}^{l} p_i \boldsymbol{s}_i$ for some $p_i \in A$. Thereby, $\boldsymbol{s}^T Q^T H^T = \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T)$ and

$$\begin{aligned}
\boldsymbol{s}^T &= \boldsymbol{s}^T - \boldsymbol{s}^T Q^T H^T + \boldsymbol{s} Q^T H^T \\
&= \boldsymbol{s}^T (I_s - Q^T H^T) + \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T) \\
&= \sum_{i=1}^{s} a_i \boldsymbol{r}_i + \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T);
\end{aligned}$$

thus, $\boldsymbol{s}^T \in \langle \boldsymbol{s}_1 H^T, \ldots, \boldsymbol{s}_l H^T, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_s \rangle$ and we obtain the required equality. $\square$

Remark 6.2.5. Note that if G is a Gröbner basis obtained through the Corollary 5.4.5, the matrices Q and H in the Theorem 6.1.1 satisfies that $Q^T H^T = I_s$. In such case, a generator set for $Syz_A(F)$ is given by $\{\boldsymbol{s}_1 H^T, \ldots, \boldsymbol{s}_l H^T\}$, where $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_l\}$ is a system of generators for $Syz_A(G)$.

Example 6.2.6. We continue to work with the Example 5.4.7, where $\mathcal{A}$ is the *diffusion algebra* described in Example 1.3.3, with $n = 2$, $\Bbbk = \mathbb{Q}$, $d_{12} = -2$ and $d_{21} = -1$. In this ring, we have $D_2 D_1 = 2D_1 D_2 + x_2 D_1 - x_1 D_2$ and the automorphisms σ_1 and σ_2 are the identity. We consider the order deglex with $D_1 \succ D_2$ and the polynomials $f_1 = x_1^2 x_2 D_1^2 D_2$, $f_2 = x_2^2 D_1 D_2^2$. As we saw, $G = \{f_1, f_2, f_3, f_4\}$ is a Gröbner basis for $I := {}_A\langle f_1, f_2\}$, where $f_3 = -\frac{1}{4} x_1^3 x_2^3 D_1 D_2 + \frac{1}{4} x_1^4 x_2^2 D_2^2$ $f_4 = x_1^3 x_2 f_2 + 2D_2 f_3 = \frac{1}{2} x_1^4 x_2^2 D_2^3 - \frac{1}{2} x_1^3 x_2^4 D_1 D_2 + \frac{1}{2} x_1^4 x_2^3 D_2^2$. We will use this for computing a system of generators for $Syz_A\{f_1, f_2\}$.
Now, according to Theorem 6.2.3, we must consider the saturated subsets of $\{1, 2, 3, 4\}$ w.r.t. $\{lm(f_i)\}_{i=1}^4$; these sets are: $J_3 = \{3\}$, $J_4 = \{4\}$, $J_{1,3} = \{1, 3\}$, $J_{2,3} = \{2, 3\}$, $J_{1,2,3} = \{1, 2, 3\}$, $J_{2,3,4} = \{2, 3, 4\}$ and $J_{1,2,3,4} = \{1, 2, 3, 4\}$. We have:
. For $J_3 = \{1\}$ we compute a system B_{J_3} of generators of $Syz_R[\sigma^{\gamma_1}(lc(\boldsymbol{f}_3))]c_{\gamma_3,\beta_3}$, where $\gamma_1 = \boldsymbol{X}_{J_3} - \beta_3 = (0, 0)$. Then $B_{J_3} = \{0\}$, and hence we have only one generator $\boldsymbol{b}_1^{J_3} = (b_{11}^{J_3}) = 0$ and $s_1^{J_3} = b_{11}^{J_3} x^{\gamma_3} \widetilde{\boldsymbol{e}}_3 - (0, 0, 0, 0) = (0, 0, 0, 0)$, con $\widetilde{\boldsymbol{e}}_1 = (0, 0, 0, 0)^T$.
. For $J_4 = \{4\}$ the situation is similar.
. For $J_{1,3}$: $X_{J_{1,3}} = D_1^2 D_2$ and $\gamma_1 = (0, 0)$, $\gamma_3 = (1, 0)$; thus, $c_{\gamma_1,\beta_1} = 1$ and $c_{\gamma_3,\beta_3} = 1$. A system of generators of

$$Syz_R[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3}] = Syz_R[x_1^2 x_2, -\tfrac{1}{4} x_1^3 x_2]$$

is $B_{J_{1,3}} = \{(x_1 x_2^2, 4)\}$.
Thus, we only have one generator $b_1^{J_{1,3}} = (x_1 x_2^2, 4)$.
Since that

$$x_1 x_2^2 f_1 + 4D_1 f_3 = x_1^4 f_2,$$

then

$$\begin{aligned} \boldsymbol{s}_1^{J_{1,3}} &= x_1x_2^2\widetilde{\boldsymbol{e}}_1 + 4D_1\widetilde{\boldsymbol{e}}_3 - (0, x_1^4, 0, 0) \\ &= \begin{bmatrix} x_1x_2^2 \\ -x_1^4 \\ 4D_1 \\ 0 \end{bmatrix}. \end{aligned}$$

. For $J_{2,3}$: $X_{J_{2,3}} = D_1D_2^2$ and $\gamma_2 = (0,0)$, $\gamma_3 = (0,1)$; thus, $c_{\gamma_2,\beta_2} = 1$. Since $D_2(D_1D_2) = 2D_1D_2^2 + x_2D_1D_2 - x_1D_2^2$, then $c_{\gamma_3,\beta_3} = 2$. A system of generators of

$$Syz_R[\sigma^{\gamma_2}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2}, \sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3}] = Syz_R[x_2, -\tfrac{1}{2}x_1^3x_2^3]$$

is $B_{J_{2,3}} = \{(x_1^3x_2, 2)\}$.
Therefore,

$$x_1^3x_2f_2 + 2D_2f_3 = f_4,$$

and

$$\begin{aligned} \boldsymbol{s}_1^{J_{2,3}} &= x_1^3x_2\widetilde{\boldsymbol{e}}_2 + 2D_2\widetilde{\boldsymbol{e}}_3 - (0, 0, 0, 1) \\ &= \begin{bmatrix} 0 \\ x_1^3x_2 \\ 2D_2 \\ -1 \end{bmatrix}. \end{aligned}$$

. For $J_{1,2,3}$: $X_{J_{1,2,3}} = D_1^2D_2^2$ and $\gamma_1 = (0,1)$, $\gamma_2 = (1,0)$ and $\gamma_3 = (1,1)$. Now, since

$$\begin{aligned} D_2D_1^2D_2 &= 4D_1^2D_2^2 + 3x_2D_1^2D_2 - 4x_1D_1D_2^2 - x_1x_2D_1D_2 + x_1^2D_2^2, \\ D_1D_2D_1D_2 &= 2D_1^2D_2^2 + x_2D_1^2D_2 - x_1D_1D_2^2, \end{aligned}$$

then $c_{\gamma_1,\beta_1} = 4$, $c_{\gamma_2,\beta_2} = 1$ and $c_{\gamma_3,\beta_3} = 2$. We have that,

$$Syz_R[4x_1^2x_2, x_2^2, -\tfrac{1}{2}x_1^3x_2^3] = \langle (\tfrac{1}{4}x_2, -x_1^2, 0), (\tfrac{1}{4}x_1x_2^2, 0, 2) \rangle.$$

For $b_1^{J_{1,2,3}} = (\frac{1}{4}x_2, -x_1^2, 0)$, is obtained

$$\tfrac{1}{4}x_2D_2f_1 - x_1^2D_1f_2 = \tfrac{3}{4}x_2^2f_1 - x_1^3f_2 + f_3$$

and

$$\begin{aligned} \boldsymbol{s}_1^{J_{1,2,3}} &= \frac{1}{4}x_2D_2\widetilde{\boldsymbol{e}}_1 - x_1^2D_1\widetilde{\boldsymbol{e}}_2 - (\frac{3}{4}x_2^2, -x_1^3, 1, 0) \\ &= \begin{bmatrix} \frac{1}{4}x_2D_2 - \frac{3}{4}x_2^2 \\ -x_1^2D_1 + x_1^3 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

For $b_2^{J_{1,2,3}} = (\frac{1}{4}x_1x_2^2, 0, 2)$, is obtained

$$\tfrac{1}{4}x_1x_2^2D_2f_1 + 2D_1D_2f_3 = \tfrac{3}{4}x_1x_2^3f_1 - x_1^4x_2f_2 + x_1x_2f_3 + D_1f_4$$

and

$$\begin{aligned} \boldsymbol{s}_2^{J_{1,2,3}} &= \frac{1}{4}x_1x_2^2D_2\widetilde{\boldsymbol{e}}_1 + 2D_1D_2\widetilde{\boldsymbol{e}}_2 - (\frac{3}{4}x_1x_2^3, -x_1^4x_2, x_1x_2, D_1) \\ &= \begin{bmatrix} \frac{1}{4}x_1x_2^2D_2 - \frac{3}{4}x_1x_2^3 \\ x_1^4x_2 \\ 2D_1D_2 - x_1x_2 \\ -D_1 \end{bmatrix}. \end{aligned}$$

- For $J_{2,3,4}$: $X_{J_{2,3,4}} = D_1D_2^3$, so $\gamma_2 = (0,1)$, $\gamma_3 = (0,2)$ and $\gamma_4 = (1,0)$. Now, since

$$\begin{gathered} D_2D_1D_2^2 = 2D_1D_2^3 + x_2D_1D_2^2 - x_1D_2^3, \\ D_2^2D_1D_2 = 4D_1D_2^3 + 4x_2D_1D_2^2 - 3x_1D_2^3 + x_2^2D_1D_2 - x_1x_2D_2^2, \end{gathered}$$

then $c_{\gamma_2,\beta_2} = 2$, $c_{\gamma_3,\beta_3} = 4$ and $c_{\gamma_4,\beta_4} = 1$. We have that

$$Syz_R[2x_2^2, -x_1^3x_2^3, \tfrac{1}{2}x_1^4x_2^2] = \langle (\tfrac{1}{2}x_1^3x_2, 1, 0), (\tfrac{1}{2}x_1^4, 0, -2)\rangle.$$

For $b_1^{J_{2,3,4}} = (\frac{1}{2}x_1^3x_2, 1, 0)$, the following equality holds

$$\tfrac{1}{2}x_1^3x_2D_2f_2 + D_2^2f_3 = \tfrac{1}{2}D_2f_4$$

and

$$\begin{aligned} \boldsymbol{s}_1^{J_{2,3,4}} &= \frac{1}{2}x_1^3x_2D_2\widetilde{\boldsymbol{e}}_2 + D_2^2\widetilde{\boldsymbol{e}}_3 - (0,0,0,\frac{1}{2}D_2) \\ &= \begin{bmatrix} 0 \\ \frac{1}{2}x_1^2x_2 \\ D_2^2 \\ \frac{1}{2}D_2 \end{bmatrix}. \end{aligned}$$

For $b_2^{J_{2,3,4}} = (\frac{1}{2}x_1^4, 0, -2)$,

$$\tfrac{1}{2}x_1^4D_2f_2 - 2D_1f_4 = x_1x_2^3f_1 - \tfrac{1}{2}x_1^4x_2f_2 - 2x_1x_2f_3 - x_1f_4$$

and hence

$$\begin{aligned} \boldsymbol{s}_2^{J_{2,3,4}} &= \frac{1}{2}x_1^4D_2\widetilde{\boldsymbol{e}}_2 - 2D_1\widetilde{\boldsymbol{e}}_4 - (x_1x_2^3, -\frac{1}{2}x_1^4x_2, -2x_1x_2, -x_1) \\ &= \begin{bmatrix} -x_1x_2^3 \\ \frac{1}{2}x_1^4 + \frac{1}{2}x_1^4x_2 \\ 2x_1x_2 \\ -2D_1 + x_1 \end{bmatrix}. \end{aligned}$$

- For $J_{1,2,3,4}$: $X_{J_{1,2,3,4}} = D_1^2D_2^3$, so $\gamma_1 = (0,2)$, $\gamma_2 = (1,1)$, $\gamma_3 = (1,2)$ and $\gamma_4 = (2,0)$. In this case, $c_{\gamma_1,\beta_1} = 16$, $c_{\gamma_2,\beta_2} = 2$, $c_{\gamma_3,\beta_3} = 4$ and $c_{\gamma_4,\beta_4} = 1$. We have that

$Syz_R[16x_1^2x_2, 2x_2^2, -x_1^3x_2^3, \frac{1}{2}x_1^4x_2^2] = \langle (\frac{1}{16}x_2, -\frac{1}{2}x_1^2, 0, 0), (\frac{1}{16}x_1x_2^2, 0, 1, 0), (\frac{1}{16}x_1^2x_2, 0, 0, -2)\rangle.$

For $b_1^{J_{1,2,3,4}} = (\frac{1}{16}x_2, -\frac{1}{2}x_1^2, 0, 0)$ we obtain

$$\tfrac{1}{16}x_2D_2^2f_1 - \tfrac{1}{2}x_1^2D_1D_2f_2 = \tfrac{9}{16}x_2^3f_1 + (x_1^2x_2D_1 - \tfrac{1}{2}x_1^3D_2 - \tfrac{17}{8}x_1^3x_2)f_2 + \tfrac{21}{4}x_2f_3 + \tfrac{17}{8}f_4,$$

thereby

$$\begin{aligned}\boldsymbol{s}_1^{J_{1,2,3,4}} &= \frac{1}{16}x_2D_2^2\widetilde{\boldsymbol{e}}_1 - \frac{1}{2}x_1^2D_1D_2\widetilde{\boldsymbol{e}}_2 - (\frac{9}{16}x_2^3, x_1^2x_2D_1 - \frac{1}{2}x_1^3D_2 - \frac{17}{8}x_1^3x_2, \frac{21}{4}x_2, \frac{17}{8}) \\ &= \begin{bmatrix} \frac{1}{16}x_2D_2^2 - \frac{9}{16}x_2^3 \\ -\frac{1}{2}x_1^2D_1D_2 - x_1^2x_2D_1 + \frac{1}{2}x_1^3D_2 + \frac{17}{8}x_1^3x_2 \\ -\frac{21}{4}x_2 \\ -\frac{17}{8}\end{bmatrix}.\end{aligned}$$

For $b_2^{J_{1,2,3,4}} = (\frac{1}{16}x_1x_2^2, 0, 1, 0)$,

$$\tfrac{1}{16}x_1x_2^2D_2^2f_1 + D_1D_2^2f_3 = \tfrac{9}{16}x_1x_2^4f_1 - \tfrac{13}{8}x_1^4x_2^2f_2 + \tfrac{13}{4}x_1x_2^2f_3 + (\tfrac{1}{2}D_1D_2 - x_2D_1 + \tfrac{9}{8}x_1x_2)f_4$$

and

$$\begin{aligned}\boldsymbol{s}_2^{J_{1,2,3,4}} &= \frac{1}{16}x_1x_2^2D_2^2\widetilde{\boldsymbol{e}}_1 + D_1D_2^2\widetilde{\boldsymbol{e}}_3 - (\frac{9}{16}x_1x_2^4, -\frac{13}{8}x_1^4x_2^2, \frac{13}{4}x_1x_2^2, \frac{1}{2}D_1D_2 - x_2D_1 + \frac{9}{8}x_1x_2) \\ &= \begin{bmatrix} \frac{1}{16}x_1x_2^2D_2^2 - \frac{9}{16}x_1x_2^4 \\ \frac{13}{8}x_1^4x_2^2 \\ D_1D_2^2 - \frac{13}{4}x_1x_2^2 \\ -\frac{1}{2}D_1D_2 + x_2D_1 - \frac{9}{8}x_1x_2\end{bmatrix}.\end{aligned}$$

For $b_3^{J_{1,2,3,4}} = (\frac{1}{16}x_1^2x_2, 0, 0, -2)$,

$$\begin{gathered}\tfrac{1}{16}x_1^2x_2D_2^2f_1 - 2D_1^2f_4 = \\ (x_1x_2^3D_1 + \tfrac{33}{16}x_1^2x_2^3)f_1 + (\tfrac{1}{2}x_1^4x_2D_1 - \tfrac{17}{8}x_1^5x_2)f_2 + \tfrac{11}{2}x_1^2x_2f_3 + (-3x_1D_1 + \tfrac{9}{8}x_1^2)f_4\end{gathered}$$

and

$$\begin{aligned}\boldsymbol{s}_3^{J_{1,2,3,4}} &= \frac{1}{16}x_1^2x_2D_2^2\widetilde{\boldsymbol{e}}_1 - 2D_1^2\widetilde{\boldsymbol{e}}_4 - (x_1x_2^3D_1 + \frac{33}{16}x_1^2x_2^3, \frac{1}{2}x_1^4x_2D_1 - \frac{17}{8}x_1^5x_2, \frac{11}{2}x_1^2x_2, -3x_1D_1 + \frac{9}{8}x_1^2) \\ &= \begin{bmatrix} \frac{1}{16}x_1^2x_2D_2^2 - x_1x_2^3D_1 - \frac{33}{16}x_1^2x_2^3 \\ -\frac{1}{2}x_1^4x_2D_1 + \frac{17}{8}x_1^5x_2 \\ -\frac{11}{2}x_1^2x_2 \\ -2D_1^2 + 3x_1D_1 - \frac{9}{8}x_1^2\end{bmatrix}.\end{aligned}$$

In consequence, $S = \{s_1^{J_{1,3}}, s_1^{J_{2,3}}, s_1^{J_{1,2,3}}, s_2^{J_{1,2,3}}, s_1^{J_{2,3,4}}, s_2^{J_{2,3,4}}, s_1^{J_{1,2,3,4}}, s_2^{J_{1,2,3,4}}, s_3^{J_{1,2,3,4}}\}$ is a set of generators for $Syz_A(G)$. For computing a generator set for $Syz_A(M)$, we use the Theorem 6.2.4: in this case the matrices H and Q in Theorem 6.1.3 are:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0\end{bmatrix}; H = \begin{bmatrix} 1 & 0 & \frac{1}{4}x_2D_2 - \frac{3}{4}x_2^2 & \frac{1}{2}x_2D_2^2 - \frac{3}{2}x_2^2D_2 \\ 0 & 1 & -x_1^2D_1 + x_1^3 & -4x_1^2D_1D_2 - 2x_1^2x_2D_1 + 4x_1^3D_2 + x_1^3x_2\end{bmatrix}.$$

Since $I_2 - Q^T H^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then the generators for $Syz_A(f_1, f_2)$ are given by $\boldsymbol{s}H^T$ for each $\boldsymbol{s} \in S$. Therefore:

- $\boldsymbol{s}_1 := \boldsymbol{s}_1^{J_{1,3}} H^T = \begin{bmatrix} x_2 D_1 D_2 - 3x_2^2 D_1 + x_1 x_2^2 \\ -4x_1^2 D_1^2 + 4x_1^3 D_1 - x_1^4 \end{bmatrix}$
- $\boldsymbol{s}_1^{J_{2,3}} H^T = \boldsymbol{s}_1^{J_{1,2,3}} H^T = \boldsymbol{s}_2^{J_{1,2,3}} H^T = \boldsymbol{0}$
- $\boldsymbol{s}_2 := \boldsymbol{s}_1^{J_{2,3,4}} H^T = \begin{bmatrix} \frac{1}{2}x_2 D_2^3 - \frac{3}{2}x_2^2 D_2^2 \\ -8x_1^2 D_1 D_2^2 - 8x_1^2 x_2 D_1 D_2 + 7x_1^3 D_2^2 - 2x_1^2 x_2^2 D_1 + \frac{5}{2}x_1^3 x_2 D_2 + \frac{1}{2}x_1^2 x_2 \end{bmatrix}$
- $\boldsymbol{s}_3 := \boldsymbol{s}_2^{J_{2,3,4}} H^T = \begin{bmatrix} -x_2 D_1 D_2^2 + 3x_2^2 D_1 D_2 + \frac{1}{2}x_1 x_2 D_1^2 - x_1 x_2^2 D_2 - \frac{5}{2}x_1 x_2^3 \\ 8x_1^2 D_1^2 D_2 - 12x_1^3 D_1 D_2 + 4x_1^2 x_2 D_1^2 - 6x_1^3 x_2 D_1 + 4x_1^4 D_2 + \frac{5}{2}x_1^4 x_2 + \frac{1}{2}x_1^4 \end{bmatrix}$
- $\boldsymbol{s}_4 := \boldsymbol{s}_1^{J_{1,2,3,4}} H^T = \begin{bmatrix} -x_2 D_2^2 + \frac{15}{8}x^2 D_2 + \frac{27}{8}x_2^3 \\ 8x_1^2 D_1 D_2 + \frac{17}{2}x_1^2 x_2 D_1 - 8x_1^3 D_2 - \frac{21}{4}x_1^3 x_2 \end{bmatrix}$
- $\boldsymbol{s}_5 := \boldsymbol{s}_2^{J_{1,2,3,4}} H^T = \begin{bmatrix} \frac{1}{2}x_2^2 D_1 D_2^2 - \frac{3}{2}x_2^3 D_1 D_2 - \frac{1}{2}x_1 x_2^2 D_2^2 + \frac{7}{8}x_1 x_2^3 D_2 + \frac{15}{8}x_1 x_2^4 \\ -4x_1^2 x_2 D_1^2 D_2 + 8x_1^3 x_2 D_1 D_2 - 2x_1^2 x_2^2 D_1^2 - \frac{9}{2}x_1^4 x_2 D_2 - \frac{11}{4}x_1^4 x_2^2 \end{bmatrix}$
- $\boldsymbol{s}_6 := \boldsymbol{s}_3^{J_{1,2,3,4}} H^T = \begin{bmatrix} -x_2 D_1^2 D_2^2 + 3x_2^2 D_1^2 D_2 + \frac{3}{2}x_1 x_2 D_1 D_2^2 - \frac{9}{2}x_1 x_2^2 D_1 D_2 - \frac{1}{2}x_1^2 x_2 D_2^2 - x_1 x_2^3 D_1 + \frac{5}{16}x_1^2 x_2^2 D_2 + \frac{33}{16}x_1^2 x_2^3 \\ 8x_1^2 D_1^3 D_2 - 20x_1^3 D_1^2 D_2 + 4x_1^2 x_2 D_1^3 - 8x_1^3 x_2 D_1^2 + \frac{33}{2}x_1^4 D_1 D_2 + \frac{41}{4}x_1^4 x_2 D_1 - \frac{9}{2}x_1^5 D_2 - \frac{9}{2}x_1^5 x_2 \end{bmatrix}$.

Hence, $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4, \boldsymbol{s}_5, \boldsymbol{s}_6\}$ is a generator set for $Syz_A(f_1, f_2)$.

The above allow us to establish the following remarkable fact about the behaviour of Gröbner soluble property on bijective skew PBW extensions.

Corollary 6.2.7. *Let R be a LGS ring. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension of R, then A is LGS.*

Proof. This follows from Hilbert Basis Theorem (Theorem 1.2.9), the discussion at the beginning of previous section, Theorem 6.1.1, and from Theorem 3.2.4. □

Remark 6.2.8. (a) Adapting the conditions (i), (ii) and (iii) in Definition 5.2.1 we can define the notion of *right Gröbner soluble rings (RGS)*.

(b) From Theorems 1.2.9 and 5.6.6 is immediate that Hilbert basis theorem holds for bijective right skew PBW extensions. Moreover, the applications established in this chapter for left ideals and submodules of left free modules, have also their right version. Therefore, we have a natural right counterpart of the Corollary 6.2.7.

Corollary 6.2.9. *Let R be a RGS ring. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective right skew PBW extension of R, then A is RGS.*

Now, we can generalize the method described above for computing the syzygy module of a submodule $M = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s\rangle$ of A^m. Let $F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix}$, we recall that $Syz(M) := Syz(F)$ consists of column vectors $\boldsymbol{h} = \begin{bmatrix} h_1 & \cdots & h_s \end{bmatrix}^T \in A^s$ such that

$$h_1 \boldsymbol{f}_1 + \cdots + h_s \boldsymbol{f}_s = \mathbf{0},$$

i.e., $\boldsymbol{h}^T F^T = \mathbf{0}$. We note that $Syz(F)$ is a submodule of A^s and we can set a matrix with its generators, so sometimes we will refer to $Syz(F)$ as a matrix. We also will write

$$Syz(M) = Syz(F) = Syz(\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}). \tag{6.2.1}$$

The computation of $Syz(F)$ is done in two steps. First, we consider a Gröbner basis $G = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$ for M and we compute $Syz(G) := Syz(\{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}) \le A^t$, and then, we obtain a system of generators for $Syz(F)$ from one for $Syz(G)$. For $S = \{\boldsymbol{g}_{i_1}, \ldots, \boldsymbol{g}_{i_k}\} \subseteq G$ and $(b_1, \ldots, b_k) \in B_S$, with B_S a set of generators of $Syz_R(\sigma^{\gamma_j}(lc(\boldsymbol{g}_{i_j}))c_{\gamma_j, \exp(\boldsymbol{g}_{i_j})} \mid 1 \leq j \leq k)$, we have that $\sum_{j=1}^{k} b_j x^{\gamma_j}\boldsymbol{g}_{i_j} \xrightarrow{G}_+ 0$, and hence, there exist $h_1, \ldots, h_s \in A$ such that $\sum_{j=1}^{k} b_j x^{\gamma_j}\boldsymbol{g}_{i_j} = \sum_{i=1}^{t} h_i\boldsymbol{g}_i$. For each $\boldsymbol{b} \in B_S$, we define

$$\boldsymbol{s}_{\boldsymbol{b}S} := \sum_{j=1}^{k} b_j x^{\gamma_j}\boldsymbol{e}_{i_j} - (h_1, \ldots, h_t) \in A^t;$$

then $\boldsymbol{s}_{\boldsymbol{b}S} \in Syz(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t)$: in fact,

$$\begin{aligned}\boldsymbol{s}_{\boldsymbol{b}S}\begin{bmatrix}\boldsymbol{g}_1\\ \vdots\\ \boldsymbol{g}_t\end{bmatrix} &= [\sum_{j=1}^{k} b_j x^{\gamma_j}\boldsymbol{e}_{i_j} - (h_1, \ldots, h_t)]\begin{bmatrix}\boldsymbol{g}_1\\ \vdots\\ \boldsymbol{g}_t\end{bmatrix}\\ &= \sum_{j=1}^{k} b_j x^{\gamma_j}\boldsymbol{g}_{i_j} - \sum_{i=1}^{t} h_i\boldsymbol{g}_i = 0.\end{aligned}$$

Definition 6.2.10. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t \in Mon(A^m)$ and $J \subseteq \{1, \ldots, t\}$. Let*

$$\boldsymbol{X}_J := lcm\{\boldsymbol{X}_j \mid j \in J\}.$$

We say that J is saturated with respect to $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t\}$, if

$$\boldsymbol{X}_j \mid \boldsymbol{X}_J \Rightarrow j \in J,$$

for any $j \in \{1, \ldots, t\}$. The saturation J' of J consists of all $j \in \{1, \ldots, t\}$ such that $\boldsymbol{X}_j \mid \boldsymbol{X}_J$.

Theorem 6.2.11. *With the above notations, a generating set for $Syz(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t)$ is*

$$S := \{\boldsymbol{s}_v^J \mid J \subseteq \{1, \ldots, t\} \text{ is saturated w.r.t.}\{lm(\boldsymbol{g}_1), \ldots, lm(\boldsymbol{g}_t)\}, 1 \leq v \leq l_J\},$$

where

$$\boldsymbol{s}_v^J := \sum_{j \in J} b_{vj}^J x^{\gamma_j}\boldsymbol{e}_j - (h_1^v, \ldots, h_t^v),$$

with $\gamma_j \in \mathbb{N}^n$ such that $\gamma_j + \beta_j = \exp(\boldsymbol{X}_J)$, $\beta_j = \exp(\boldsymbol{g}_j)$, $j \in J$, $B^J := \{\boldsymbol{b}_1^J, \ldots, \boldsymbol{b}_{l_J}^J\}$ is a system of generators for $S^J := Syz_R[\sigma^{\gamma_j}(lc(\boldsymbol{g}_j))c_{\gamma_j,\beta_j} \mid j \in J]$, and $\boldsymbol{b}_v^J := (b_{vj}^J)_{j \in J}$.

Proof. We have already seen that ${}_A\langle S\rangle \subseteq Syz(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t)$. Suppose that there exists $\boldsymbol{u} = (u_1, \ldots, u_t) \in Syz(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t) - \langle S\rangle$. We can choose $\boldsymbol{u}$ with $\boldsymbol{X}_\delta := \max_{1 \leq i \leq t}\{lm(lm(u_i)lm(\boldsymbol{g}_i))\}$ minimal with respect to $\preceq$. Let

$$J := \{j \in \{1, \ldots, t\} \mid lm(lm(u_j)lm(\boldsymbol{g}_j)) = \boldsymbol{X}_\delta\}.$$

Since $\sum_{i=1}^{t} u_i\boldsymbol{g}_i = 0$, in particular we have $\sum_{j \in J} lc(u_j)\sigma^{\alpha_j}(lc(\boldsymbol{g}_j))c_{\alpha_j,\beta_j} = 0$, where $\alpha_i := \exp(u_i)$ for $1 \leq i \leq t$. If $\boldsymbol{X}_J := lcm\{lm(\boldsymbol{g}_j) \mid j \in J\}$, then $\boldsymbol{X}_J \mid \boldsymbol{X}_\delta$ therefore there is $\theta \in \mathbb{N}^n$ with with property $\theta + \exp(\boldsymbol{X}_J) = \delta$. But $\alpha_j + \beta_j = \delta$ and $\gamma_j + \beta_j = \exp(\boldsymbol{X}_J)$ for all $j \in J$, then $\theta + \gamma_j + \beta_j = \alpha_j + \beta_j$, i.e., $\theta + \gamma_j = \alpha_j$. Thus, $(lc(u_j))_{j \in J} \in S_\theta^J :=$

$Syz_R[\sigma^{\theta+\gamma_j}(lc(\boldsymbol{g}_j))c_{\theta+\gamma_j,\beta_j} \mid j \in J]$. If J' is the saturation of J, then $\boldsymbol{X}_J = \boldsymbol{X}_{J'}$ and $\boldsymbol{w} = (w_j)_{j\in J'}$ given by

$$w_j = \begin{cases} lc(u_j), & \text{if } j \in J, \\ 0, & \text{if } j \in J' - J \end{cases}$$

is an element of $S_{J',\theta}$. According to Remark 5.4.2, there exists

$$(b_j)_{j\in J'} \in S_{J'} := Syz_R[\sigma^{\gamma_j}(lc(g_j))c_{\gamma_j,\beta_j} \mid j \in J']$$

such that $w_j = \sigma^\theta(b_j)c_{\theta,\gamma_j}$ for $j \in J'$. This implies that $b_j = 0$ for $j \in J' - J$. Now, $(b_j)_{j\in J'} = \sum_{v=1}^{l_{J'}} r'_v \boldsymbol{b}_v^{J'}$, with $\{\boldsymbol{b}_v^{J'} \mid 1 \leq v \leq l_{J'}\}$ a system of generators for $S^{J'}$ and $r'_v \in R$ for $1 \leq v \leq l_{J'}$. Hence, $b_j = \sum_{v=1}^{l_{J'}} r'_v b_{vj}^{J'}$ and thus $w_j = \sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}$ for all $j \in J'$. Define $\boldsymbol{u}' := \boldsymbol{u} - \sum_{v=1}^{l_{J'}} r_v x^\theta \boldsymbol{s}_v^{J'}$, with $r_v := \sigma^\theta(r'_v)$ for $1 \leq v \leq l_{J'}$; then $\boldsymbol{u}' \in Syz(G)$ since $\sum_{v=1}^{l_{J'}} r_v x^\theta \boldsymbol{s}_v^{J'} \in \langle S \rangle$. Note that

$$\begin{aligned}
\sum_{v=1}^{l_{J'}} r_v x^\theta \boldsymbol{s}_v^{J'} =& r_1 x^\theta \boldsymbol{s}_1^{J'} + \cdots + r_{l_{J'}} x^\theta \boldsymbol{s}_{l_{J'}}^{J'} \\
=& r_1 x^\theta [\sum_{j\in J'} b_{1j}^{J'} x^{\gamma_j} \boldsymbol{e}_j - (h_1^1, \ldots, h_t^1)] + \cdots + \\
& r_{l_{J'}} x^\theta [\sum_{j\in J'} b_{l_{J'}j}^{J'} x^{\gamma_j} \boldsymbol{e}_j - (h_1^{lJ'}, \ldots, h_t^{l_{J'}})] \\
=& r_1 [\sum_{j\in J'} \sigma^\theta(b_{1j}^{J'}) c_{\theta,\gamma_j} x^{\theta+\gamma_j} + p_j^1 \boldsymbol{e}_j - (h_1^1, \ldots, h_t^1)] + \cdots + \\
& r_{l_{J'}} [\sum_{j\in J'} \sigma^\theta(b_{l_{J'}j}^{J'}) c_{\theta,\gamma_j} x^{\theta+\gamma_j} + p_j^{l_{J'}} \boldsymbol{e}_j - (h_1^{l_{J'}}, \ldots, h_t^{l_{J'}})]
\end{aligned}$$

Thus, for $j \in J$ we have that

$$\begin{aligned}
u'_j &= u_j - [\sum_{v=1}^{l_{J'}} r_v \sigma^\theta(b_{vj}^{J'}) c_{\theta,\gamma_j} x^{\theta+\gamma_j} + \sum_{v=1}^{l_{J'}} p_j^v - \sum_{v=1}^{l_{J'}} h_j^v] \\
&= u_j - [\sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'}) c_{\theta,\gamma_j} x^{\alpha_j} + \sum_{v=1}^{l_{J'}} p_j^v - \sum_{v=1}^{l_{J'}} h_j^v] \\
&= u_j - lc(u_j)x^{\alpha_j} - \sum_{v=1}^{l_{J'}} p_j^v + \sum_{v=1}^{l_{J'}} h_j^v
\end{aligned}$$

since for $j \in J$, $\gamma_j + \theta = \alpha_j$ and $w_j = lc(u_j) = \sum_{v=1}^{l_{J'}} \sigma^\theta(r'_v)\sigma^\theta(b_{vj}^{J'})c_{\theta,\gamma_j}$. Here $p_j^v = 0$ or $\deg(p_j^v) < |\theta + \gamma_j|$ for every $1 \leq v \leq l_{J'}$. Then $lm(lm(u_j - lc(u_j)x^{\alpha_j})lm(\boldsymbol{g}_j)) \prec lm(lm(u_j)lm(\boldsymbol{g}_j)) = \boldsymbol{X}_\delta$, $lm(p_j^v \boldsymbol{g}_j) \prec x^{\theta+\gamma_j+\beta_j} = \boldsymbol{X}_\delta$, and

$$lm(lm(h_j^v)lm(\boldsymbol{g}_j)) \preceq lm(\textstyle\sum_{j\in J'} b_{vj}^{J'} x^{\gamma_j} \boldsymbol{g}_j) \prec \boldsymbol{X}_{J'} = \boldsymbol{X}_J \preceq \boldsymbol{X}_\delta$$

and therefore $lm(lm(u'_j)lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}_\delta$. Now, if $j \in J' - J$, then

$$w_j = \sum_{v=1}^{l_{J'}} \sigma^{\theta}(r'_v)\sigma^{\theta}(b^{J'}_{vj})c_{\theta,\gamma_j} = 0,$$

and $lm(lm(u_j)lm(g_j)) \prec \boldsymbol{X}_\delta$, and thus $lm(lm(u'_j)lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}_\delta$. Finally, if $j \notin J'$, then $u'_j = u_j + \sum_{v=1}^{l_{J'}} h^v_j$ and $lm(lm(u'_j)lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}_\delta$. So, $lm(lm(u'_i)lm(\boldsymbol{g}_i)) \prec \boldsymbol{X}_\delta$ for every $1 \le i \le t$ and, by minimality of $\boldsymbol{u}$, we have that $\boldsymbol{u}' \in \_A\langle S\rangle$ and hence, $\boldsymbol{u} \in {}_A\langle S\rangle$, a contradiction. Thus ${}_A\langle S\rangle = Syz(\boldsymbol{g}_1,\dots,\boldsymbol{g}_t)$. □

We return to the task of calculating a system of generators for $Syz(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$, where $\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_s\}$ is a collection of nonzero vectors, which non necessarily form a Gröbner basis for $M = \langle \boldsymbol{f}_1,\dots,\boldsymbol{f}_s\rangle$. From Theorem 6.1.3, there exist $H \in M_{s\times t}(A)$ and $Q \in M_{t\times s}(A)$ such that $G^T = H^T F^T$ and $F^T = Q^T G^T$, where $G := \begin{bmatrix}\boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t\end{bmatrix}$, $F := \begin{bmatrix}\boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s\end{bmatrix}$ and G is a Gröbner basis for $\langle \boldsymbol{f}_1,\dots,\boldsymbol{f}_s\rangle$. By Theorem 6.2.11, we compute a set of generators $\{\boldsymbol{s}_1,\dots,\boldsymbol{s}_l\}$ for $Syz(\boldsymbol{g}_1,\dots,\boldsymbol{g}_t)$. Thus, for each $1 \le i \le l$ we have

$$\boldsymbol{s}_i H^T F^T = \boldsymbol{s}_i G^T = 0,$$

and therefore, $\langle \boldsymbol{s}_i H^T \mid 1 \le i \le l\rangle \subseteq Syz(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$. If $Syz(G) := Z(G) := \begin{bmatrix}\boldsymbol{s}_1 & \cdots & \boldsymbol{s}_l\end{bmatrix}$, then $Syz(\boldsymbol{g}_1,\dots,\boldsymbol{g}_t)$ is the module generated by columns of $Z(G)$ and this last equation may be written as

$$Z(G)^T H^T F^T = Z(G)^T G^T = 0. \tag{6.2.2}$$

Further,

$$[I_s - Q^T H^T]\begin{bmatrix}\boldsymbol{f}_1\\ \vdots \\ \boldsymbol{f}_s\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}_1\\ \vdots \\ \boldsymbol{f}_s\end{bmatrix} - Q^T H^T \begin{bmatrix}\boldsymbol{f}_1\\ \vdots \\ \boldsymbol{f}_s\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix},$$

and thereby the rows $\boldsymbol{r}_1,\dots,\boldsymbol{r}_s$ of $I_s - Q^T H^T$ also belong to $Syz(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$.

Theorem 6.2.12. *With the above notation, we have*

$$Syz(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s) = \langle \boldsymbol{s}_1 H^T,\dots,\boldsymbol{s}_l H^T, \boldsymbol{r}_1,\dots,\boldsymbol{r}_s\rangle \le A^s.$$

In a matrix notation, $Syz(F)$ coincides with the column module of the extended matrix $\big[(Z(G)^T H^T)^T \quad I_s - ($ i.e.,

$$Syz(F) = \begin{bmatrix}(Z(G)^T H^T)^T & I_s - (Q^T H^T)^T\end{bmatrix} \tag{6.2.3}$$

Proof. Let $\boldsymbol{s}^T = (a_1,\dots,a_s)$ be an element in $Syz(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$, then

$$0 = \boldsymbol{s}^T F^T = \boldsymbol{s}^T Q^T G^T,$$

and therefore $\boldsymbol{s}^T Q^T \in Syz(\boldsymbol{g}_1,\dots,\boldsymbol{g}_t)$. Thus, $\boldsymbol{s}^T Q^T = \sum_{i=1}^{l} p_i \boldsymbol{s}_i$ for some $p_i \in A$. Thereby, $\boldsymbol{s}^T Q^T H^T = \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T)$, and thus

$$\begin{aligned}\boldsymbol{s}^T &= \boldsymbol{s}^T - \boldsymbol{s}^T Q^T H^T + \boldsymbol{s}^T Q^T H^T\\ &= \boldsymbol{s}^T(I_s - Q^T H^T) + \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T)\\ &= \sum_{i=1}^{s} a_i \boldsymbol{r}_i + \sum_{i=1}^{l} p_i(\boldsymbol{s}_i H^T);\end{aligned}$$

whence, $\boldsymbol{s}^T \in \langle \boldsymbol{s}_1 H^T,\dots,\boldsymbol{s}_l H^T, \boldsymbol{r}_1,\dots,\boldsymbol{r}_s\rangle$ and we obtain the required equality. □

Remark 6.2.13. When the homomorphisms are disposed by rows and homomorphisms acts from left to right (compare with [78] and see Remark 2.1.2), we have

$$Syz(F) = \begin{bmatrix} HZ(G) & I_s - HQ \end{bmatrix}.$$

Example 6.2.14. Once more, we consider the *additive analogue of the Weyl algebra* $A = A_2(\frac{1}{2}, \frac{1}{3})$, used in the Example 5.5.22, with the same monomial order on $Mon(A)$ and on $Mon(A^2)$. For this example, we want to find a finite set of generators for $Syz_A[\boldsymbol{f}_1, \boldsymbol{f}_2]$, where $\boldsymbol{f}_1 = x_1y_1^2\boldsymbol{e}_1 + x_2y_2\boldsymbol{e}_2$ and $\boldsymbol{f}_2 = x_2y_2^2\boldsymbol{e}_1 + x_1y_1\boldsymbol{e}_2$. As we saw in the Example 5.5.22, $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$, with $\boldsymbol{f}_3 = -\frac{1}{4}x_1^2y_1^3\boldsymbol{e}_2 + \frac{1}{9}x_2^2y_2^3\boldsymbol{e}_2 - \frac{3}{2}x_1y_1^2\boldsymbol{e}_2 + \frac{4}{3}x_2y_2^2\boldsymbol{e}_2$ is a Gröbner basis for M.

Now, according to the Theorem 6.2.11, to compute a system of generators for $Syz(G) = Syz_A[\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3]$, we must compute the saturated subsets J of $\{1, 2, 3\}$ with respect to $\{y_1^2\boldsymbol{e}_1, y_2^2\boldsymbol{e}_1, y_1^3\boldsymbol{e}_2\}$. We have:

- For $J_1 = \{1\}$ we compute a system B_{J_1} of generators of $Syz_R[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))]c_{\gamma_1,\beta_1}$, where $\beta_1 := \exp(lm(\boldsymbol{f}_1))$ and $\gamma_1 = \boldsymbol{X}_{J_1} - \beta_1 = (0, 0)$. Then $B_{J_1} = \{0\}$, and hence we have only one generator $\boldsymbol{b}_1^{J_1} = (b_{11}^{J_1}) = 0$ and $s_1^{J_1} = b_{11}^{J_1}x^{\gamma_1}\widetilde{\boldsymbol{e}}_1 - (0, 0, 0) = (0, 0, 0)$, con $\widetilde{\boldsymbol{e}}_1 = (0, 0, 0)^T$.
- For $J_2 = \{2\}$ and $J_3 = \{3\}$ the situation is similar.
- For $J_{1,2} = \{1, 2\}$, a system of generators of

$$Syz_R[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}, \sigma^{\gamma_1}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2}],$$

where $\beta_1 = \exp(lm(\boldsymbol{f}_1))$, $\beta_2 = \exp(lm(\boldsymbol{f}_2))$, $\gamma_1 = (0, 2)$, $\gamma_2 = (2, 0)$, $c_{\gamma_1,\beta_1} = 1$ and $c_{\gamma_2,\beta_2} = 1$, is $B_{J_{1,2}} = \{(x_2, -x_1)\}$.

Thus, we only have one generator $b_1^{J_{1,2}} = (x_2, -x_1)$.

Since that

$$x_2y_2^2\boldsymbol{f}_1 - x_1y_1^2\boldsymbol{f}_2 = \boldsymbol{f}_3,$$

then

$$\begin{aligned} \boldsymbol{s}_1^{J_{1,2}} &= x_2y_2^2\widetilde{\boldsymbol{e}}_1 - x_1y_1^2\widetilde{\boldsymbol{e}}_2 - (0, 0, 1) \\ &= \begin{bmatrix} x_2y_2^2 \\ -x_1y_1^2 \\ -1 \end{bmatrix}. \end{aligned}$$

- For $J_{1,3} = \{1, 3\}$ and $J_{2,3} = \{2, 3\}$, we have $\boldsymbol{X}_{J_{1,3}} = \boldsymbol{X}_{J_{2,3}} = \boldsymbol{0}$.

Hence,

$$Syz(G) = \left\langle \begin{bmatrix} x_2y_2^2 \\ -x_1y_1^2 \\ -1 \end{bmatrix} \right\rangle$$

Finally, we compute a generator set for $Syz_A(M)$: let $\boldsymbol{s} = \begin{bmatrix} x_2y_2^2 & -x_1y_1^2 & -1 \end{bmatrix}^T$; from Theorem 6.1.3 there exist matrices H and Q such that $G^T = H^TF^T$ and $F^T = Q^TG^T$; in this case,

$$H = \begin{bmatrix} 1 & 0 & x_2y_2^2 \\ 0 & 1 & -x_1y_1^2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, $\boldsymbol{s}^T H^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $I_2 - Q^T H^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $Syz_A(\boldsymbol{f}_1, \boldsymbol{f}_2) = \mathbf{0}$ and therefore, M is a free left module of rank two.

6.3 Intersections

Using syzygies, we will compute in this section the intersection of left ideals of A and submodules of A^m. For this, let $I = \langle f_1, \ldots, f_s\}$ and $J = \langle g_1, \ldots, g_t\}$ be left ideals of A; for $h \in I \cap J$ there exist some $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ elements in A such that

$$h = a_1 f_1 + \cdots + a_s f_s = b_1 g_1 + \cdots + b_t g_t.$$

The above can be reformulated saying that

$$\begin{bmatrix} -h & a_1 & \ldots & a_s \end{bmatrix} \begin{bmatrix} 1 \\ f_1 \\ \vdots \\ f_s \end{bmatrix} = 0 \text{ and } \begin{bmatrix} -h & b_1 & \ldots & b_t \end{bmatrix} \begin{bmatrix} 1 \\ g_1 \\ \vdots \\ g_t \end{bmatrix} = 0,$$

i.e., $(-h, a_1, \ldots, a_s)^T \in Syz(1, f_1, \ldots, f_s)$ and $(-h, b_1, \ldots, b_t)^T \in Syz(1, g_1, \ldots, g_t)$. Setting $\boldsymbol{i} := (1,1), \boldsymbol{f}_1 := (f_1, 0), \ldots, \boldsymbol{f}_s := (f_s, 0), \boldsymbol{g}_1 := (0, g_1), \ldots, \boldsymbol{g}_t := (0, g_t)$, these two conditions may be rewritten as the following single condition: there exist polynomials $a_1, \ldots, a_s, b_1, \ldots, b_t \in A$ such that the vector $(-h, a_1, \ldots, a_s, b_1, \ldots, b_t)^T$ is a syzygy of L, where $L = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s & \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix}$. Since $h \in I \cap J$ if and only if $-h \in I \cap J$, we may rephrase the above by the more natural condition that $(h, a_1, \ldots, a_s, b_1, \ldots, b_t)^T$ be a syzygy of L. Thus, we have proved the following result.

Theorem 6.3.1. *The elements in $I \cap J$ are polynomials $h \in A$ with the property that there exist $a_1, \ldots, a_s, b_1, \ldots, b_t \in A$ such that $(h, a_1, \ldots, a_s, b_1, \ldots, b_t)^T \in Syz(L)$.*

A system of generators for the intersection is given in the following corollary.

Corollary 6.3.2. *Let $\{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l\}$ be a generating set for $Syz(L)$. If h_{1j} is the first coordinate of $\boldsymbol{h}_j$, for $1 \le j \le l$, then $L = \{h_{11}, \ldots, h_{1l}\}$ generates $I \cap J$.*

Proof. Let $h \in I \cap J$, then there exist $a_1, \ldots, a_s, b_1, \ldots, b_t \in A$ such that $h = a_1 f_1 + \cdots + a_s f_s = b_1 g_1 + \cdots + b_t g_t$; thus, $(h, a_1, \ldots, a_s, b_1, \ldots, b_t)^T \in Syz(L)$, and hence $(h, a_1, \ldots, a_s, b_1, \ldots, b_t)^T = \sum_{j=1}^{l} r_j \boldsymbol{h}_j$ for certain $r_1, \ldots, r_l \in A$. From this we get that $h = \sum_{j=1}^{l} r_j h_{1j}$, i.e., $I \cap J \subseteq \langle L\}$. The other inclusion follows from the definition of $Syz(L)$. □

Example 6.3.3. Let $A = \sigma(\mathbb{Q})\langle x, y\rangle$ defined through the relation $yx = -xy + 1$. Over $Mon(A)$ we consider the deglex order, with $x \succ y$. Let $I = {}_A\langle xy, y^2\}$ and and $J = {}_A\langle y\}$ be left ideals of A. We will compute a system of generators of $I \cap J$. In this case

$$L = \begin{bmatrix} 1 & xy & x^2 & 0 \\ 1 & 0 & 0 & y \end{bmatrix}.$$

Employing the TOPREV order on $Mon(A^2)$, with $e_1 < e_2$, and using the method described above for computing syzygies, we have the following generator set for $Syz_A(L)$: $\{(xy, -1, 0, -x), (0, -x, y, 0), (-x^2y, 0, y, x^2)\}$. Hence, $I \cap J = {}_A\langle xy, x^2y\} = {}_A\langle xy\}$.

Now, we consider the intersection of a arbitrary finite family of left ideals of A, $I_j = \langle f_{1j}, \ldots, f_{t_jj}\}$, $1 \leq j \leq r$. We define

$$\boldsymbol{i} := (1, 1, \ldots, 1), \boldsymbol{f}_{11} = (f_{11}, 0, \ldots, 0), \boldsymbol{f}_{21} = (f_{21}, 0, \ldots, 0) \ldots, \boldsymbol{f}_{t_11} = (f_{t_11}, 0, \ldots, 0), \ldots, \boldsymbol{f}_{1r} = (0, \ldots, 0, f_{1r}), \ldots, \boldsymbol{f}_{t_rr} = (0, \ldots, 0, f_{t_rr}),$$

and

$$L = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{f}_{11} & \boldsymbol{f}_{21} & \cdots & \boldsymbol{f}_{t_11} & \cdots & \boldsymbol{f}_{1r} & \boldsymbol{f}_{2r} & \cdots & \boldsymbol{f}_{t_rr} \end{bmatrix} \in M_{r \times l}(A),$$

where $l = 1 + \sum_{j=1}^{r} t_j$. Thus, if $\boldsymbol{s} \in Syz(L)$, then $\boldsymbol{s}^T L^T = 0$. As we observed above, the first coordinates of a generating set for $Syz(L)$ turn out to be a generating set for $I_1 \cap \cdots \cap I_r$.

We can extend the previous results to compute the intersection of submodules. For this, let M and N be two submodules of A^m, with $m \geq 1$. Suppose that $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\rangle$ and $N = \langle \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r\rangle$. Thus, $\boldsymbol{h} \in M \cap N$ if and only if there exist $a_1, \ldots, a_s, b_1, \ldots, b_t \in A$ such that

$$\boldsymbol{h} = a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s = b_1\boldsymbol{g}_1 + \cdots + b_t\boldsymbol{g}_t.$$

If $h = \begin{bmatrix} h_1 & \cdots & h_m \end{bmatrix}^T$, then

$$\begin{bmatrix} -h_1 & \cdots & -h_m & a_1 & \cdots & a_s \end{bmatrix}^T \text{ and } \begin{bmatrix} -h_1 & \cdots & -h_m & b_1 & \cdots & b_t \end{bmatrix}^T$$

are a syzygies of the matrices

$$\begin{bmatrix} I_m & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix} \text{ and } \begin{bmatrix} I_m & \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix},$$

respectively, where I_m is the identity matrix of order m. Mimicking the reasoning for the ideal case, we define the matrix L, given by

$$L = \begin{bmatrix} I_m & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s & 0 & \cdots & 0 \\ I_m & 0 & \cdots & 0 & \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix},$$

and it is easy to prove the following result.

Proposition 6.3.4. *With the above notation, $M \cap N$ consists exactly of vectors $\boldsymbol{h}$ whose coordinates are precisely the first m elements of vectors of $Syz(L)$. Moreover, the set of vectors which consisting of the firsts m coordinates of each element of a set of generators for $Syz(L)$ is system of generators for $M \cap N$.*

The previous result can be extended to a finite set of modules: let $M_1, \ldots, M_r$ be submodules of A^m, with $r \geq 3$. Suppose that each M_i is generated by the columns of some matrix $\boldsymbol{F}_i \in M_{m \times t_i}(A)$, and define

$$L = \begin{bmatrix} I_m & \boldsymbol{F}_1 & 0 & \cdots & 0 \\ I_m & 0 & \boldsymbol{F}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_m & 0 & 0 & \cdots & \boldsymbol{F}_r \end{bmatrix}.$$

Proposition 6.3.5. *With the previous notation, the intersection $\bigcap_{i=1}^{r} M_i$ is the set of all vectors $\boldsymbol{h}$ which are the first m coordinates of vectors in $Syz(L)$. Furthermore, the set of vectors that consist of the first m entries of each of vectors of a generator set for $Syz(L)$ is a system of generators for the intersection.*

Example 6.3.6. We consider the Example 6.2.5 in [19] and we verify the calculations developed there, using our algorithms. Let $A = \sigma(\mathbb{Q})\langle x, y\rangle$, with $yx = -xy$ and the deglex order on $Mon(A)$. Let M, N be submodules of A^2, where $M = {}_A\langle (x,x),(y,0)\rangle$ and $N = {}_A\langle (0,y^2),(y,x)\rangle$. In this case, the matrix L is given by

$$L = \begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & y \\ 0 & 1 & 0 & 0 & y^2 & x \end{bmatrix}.$$

So, if we consider the TOP order on $Mon(A^4)$, with $\boldsymbol{e}_4 > \boldsymbol{e}_3 > \boldsymbol{e}_2 > \boldsymbol{e}_1$, then a Gröbner basis for the left A-module generated by the columns of L is $G = \{\boldsymbol{f}_i\}_{i=1}^{8}$, where $\boldsymbol{f}_i$ is the i-th column of L for $1 \leq i \leq 6$, $\boldsymbol{f}_7 = y^2\boldsymbol{e}_2$ and $\boldsymbol{f}_8 = -x\boldsymbol{e}_1 - y\boldsymbol{e}_3$. A set of generators for $Syz_A(G)$ is

$$\{y^2\boldsymbol{e}_2 - \boldsymbol{e}_5 - \boldsymbol{e}_7, x\boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_6 - \boldsymbol{e}_8, y^2\boldsymbol{e}_3 - xy\boldsymbol{e}_4 - x\boldsymbol{e}_7, -y^2\boldsymbol{e}_1 + (x+y)\boldsymbol{e}_4 - y\boldsymbol{e}_8, xy^2\boldsymbol{e}_2 - x\boldsymbol{e}_5 - x\boldsymbol{e}_7, y^3\boldsymbol{e}_1 + xy^2\boldsymbol{e}_2 - y^2\boldsymbol{e}_4 - y^2\boldsymbol{e}_6 - x\boldsymbol{e}_7\}.$$

Computing the corresponding matrix H in Theorem 6.1.3, we have that

$$Syz_A(L) = {}_A\langle (0, -xy^2, y^2 - xy, x, 0), (-y^2, xy, y, x+y, 0, y), (y^3, 0, 0, -y^2, x, -y^2)\rangle.$$

Thus, $M \cap N$ is generated by $(0, -xy^2)$, $(-y^2, xy)$, $(y^3, 0)$; but $(y^3, 0) = -y(-y^2, xy) + (0, -xy^2)$, hence $M \cap N = {}_A\langle (0, xy^2), (-y^2, xy)\rangle$.

6.4 Quotients

We can use syzygies to compute a set of generators for the quotient of left ideals and modules. For this, let I be a finitely generated left ideal of A, say $I = \langle f_1, \ldots, f_s\}$, and let G be an arbitrary subset of A. Recall that $(I : G)$ consist of elements $h \in A$ such that $hg \in I$ for all $g \in G$, in other words, for every $g \in G$ there exist $a_{1g}, \ldots, a_{sg} \in A$ with property $hg = \sum_{i=1}^{s} a_{ig} f_i$. It is straightforward to show that $(I : G)$ is a left ideal of A. Furthermore,

$$(I : G) = \bigcap_{g \in G} (I : g).$$

So, if $G = \{g_1 \ldots, g_t\}$, then

$$(I : G) = \bigcap_{i=1}^{t} (I : g).$$

Note that, given a polynomial $g \in A$, $h \in (I : g)$ if, and only if, $(-h, h_1, \ldots, h_s) \in Syz_A(g, f_1, \ldots, f_s)$ where $h_1, \ldots, h_s \in A$ are elements such that $hg = h_1 f_1 + \cdots + h_s f_s$.

But, $h \in (I : g)$ if, and only if, $-h \in (I : g)$, thus for computing a system of generators of $(I : G)$, with $G = \{g_1 \ldots, g_t\}$, we will consider the matrix L given by

$$L = \begin{bmatrix} g_1 & f_1 & \cdots & f_s & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \cdots & \vdots \\ g_t & 0 & \cdots & 0 & f_1 & \cdots & f_s \end{bmatrix}.$$

In consequence, $(I : G)$ is the set of all elements in A that are the first coordinates of vectors in $Syz(L)$, and a generator set is given by the first coordinates of the vectors in a generator system for $Syz(L)$.

Example 6.4.1. Let A be the ring $\sigma(\mathbb{Q})\langle x, y\rangle$, where $yx = xy + x$. Given $I = {}_A\langle x^2y, xy\}$ and $G = \{x^2, y\}$, we will compute a generator set for $(I : G)$. For this, we consider the following matrix

$$\begin{bmatrix} x^2 & x^2y & xy & 0 & 0 \\ y & 0 & 0 & x^2y & xy \end{bmatrix}$$

Now, if $Mon(A)$ is ordered by deglex order, with $x \succ y$, and $Mon(A^2)$ is ordered by TOPREV order, with $\boldsymbol{e}_1 > \boldsymbol{e}_2$, then a Gröbner basis for the left A-module generated by columns of L is $G = \{\boldsymbol{f}_i\}_{i=1}^6$, where $\boldsymbol{f}_i$ is the i-th column of L and $\boldsymbol{f}_6 = y^2\boldsymbol{e}_2 - 2y\boldsymbol{e}_2$. Further,

$$Syz(G) = {}_A\langle (y-2)\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_6, (y-2)\boldsymbol{e}_1 - x\boldsymbol{e}_3 - \boldsymbol{e}_6, \boldsymbol{e}_4 - x\boldsymbol{e}_5, (y-3)\boldsymbol{e}_5 - x\boldsymbol{e}_6, (y-1)\boldsymbol{e}_4 - xy\boldsymbol{e}_5, -3\boldsymbol{e}_4, xy\boldsymbol{e}_5 - x^2\boldsymbol{e}_6\rangle.$$

From this it follows that a system of generators for $Syz_A(L)$ is:

$$\{(0,1,-x,0,0), (0,0,0,1,-x), (-xy+2x, x, 0, 0, y-3), (0,0,0,y-1,-xy), (-x^2y + 2x^2, x^2, 0, -3, xy)\}.$$

In consequence, $(I : G) = {}_A\langle -xy + 2x\rangle$.

6.5 Presentation of a module

Let $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\rangle$ be a submodule of A^m, there exists a natural surjective homomorphism $\pi_M : A^s \longrightarrow M$ defined by $\pi_M(\boldsymbol{e}_i) := \boldsymbol{f}_i$, where $\{\boldsymbol{e}_i\}_{1\leq i\leq s}$ is the canonical basis of A^s. We have the isomorphism $\overline{\pi_M} : A^s/\ker(\pi_M) \cong M$, defined by $\overline{\pi_M}(\overline{\boldsymbol{e}_i}) := \boldsymbol{f}_i$, where $\overline{\boldsymbol{e}_i} := \boldsymbol{e}_i + \ker(\pi_M)$. We note that $\ker(\pi_M)$ is also a finitely generated module, $\ker(\pi_M) := \langle \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{s_1}\rangle$, and hence, we have the exact sequence

$$A^{s_1} \xrightarrow{\delta_M} A^s \xrightarrow{\pi_M} M \longrightarrow 0, \tag{6.5.1}$$

with $\delta_M := l_M \circ \pi'_M$, where l_M is the inclusion of $\ker(\pi_M)$ in A^s and π'_M is the natural surjective homomorphism from A^{s_1} to $\ker(\pi_M)$. We note that $\ker(\pi_M) = Syz(M) = Syz(F)$, where $F = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_s] \in M_{m\times s}(A)$

Definition 6.5.1. *It says that $A^s/Syz(M)$ is a presentation of M. It says also that the sequence (6.5.1) is a finite presentation of M, and M is a finitely presented module.*

Theorem 6.2.11 gives a method for computing a presentation of M when A is a bijective skew PBW extension. Moreover, let Δ_M be the matrix of δ_M in the canonical bases of A^{s_1} and A^s; since $Im(\delta_M) = \ker(\pi_M)$, then

$$\Delta_M = \begin{bmatrix} \boldsymbol{h}_1 & \cdots & \boldsymbol{h}_{s_1} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1s_1} \\ \vdots & & \vdots \\ h_{s1} & \cdots & h_{ss_1} \end{bmatrix} \in M_{s\times s_1}(A),$$

and hence, the columns of Δ_M are the generators of $Syz(F)$. With the notation of Section 6.2, $\Delta_M = Z(F)$.

Definition 6.5.2. *With the previous notation, it says that Δ_M is a matrix presentation of M.*

As we just saw, Δ_M is computable when A is a bijective skew PBW extension. We can also compute presentations of quotient modules. Indeed, let $N \subseteq M$ be submodules of A^m, where $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\rangle$, $N = \langle \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\rangle$ and $M/N = \langle \overline{\boldsymbol{f}_1}, \ldots, \overline{\boldsymbol{f}_s}\rangle$, then we have a canonical surjective homomorphism $A^s \longrightarrow M/N$ such that a presentation of M/N is given by $M/N \cong A^s/Syz(M/N)$. But $Syz(M/N)$ can be computed in the following way: $\boldsymbol{h} = (h_1, \ldots, h_s)^T \in Syz(M/N)$ if and only if $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s \in \langle \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\rangle$ if and only if there exist $h_{s+1}, \ldots, h_{s+t} \in A$ such that $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s + h_{s+1}\boldsymbol{g}_1 + \cdots + h_{s+t}\boldsymbol{g}_t = \boldsymbol{0}$ if and only if $(h_1, \ldots, h_s, h_{s+1}, \ldots, h_{s+t}) \in Syz(H)$, where

$$H := [\boldsymbol{f}_1 \ \cdots \ \boldsymbol{f}_s \ \boldsymbol{g}_1 \ \cdots \ \boldsymbol{g}_t].$$

Theorem 6.5.3. *With the notation above, a presentation of M/N is given by $A^s/Syz(M/N)$, where a set of generators of $Syz(M/N)$ are the first s coordinates of generators of $Syz(H)$. Thus, a finite presentation of M/N is effective computable.*

Example 6.5.4. Again, let A be the ring $\sigma(\mathbb{Q})\langle x, y\rangle$, where $yx = xy + x$. Given $M = {}_A\langle (1,1), (xy, 0), (y^2, 0), (0, x)\rangle$, we will compute a finite presentation for M. For this, use the deglex order on $Mon(A)$, with $x \succ y$, and the TOP order over $Mon(A^2)$, with $\boldsymbol{e}_2 > \boldsymbol{e}_1$. A straightforward calculation shows that

$$G = \{(1,1), (xy, 0), (y^2, 0), (0, x), (x, 0)\}$$

is a Gröbner basis for M. Moreover, a set of generators for $Syz_A(G)$ is given by

$$\{(x, 0, 0, -1, -1), (0, 1, 0, 0, -y+1), (0, -y+1, x, 0, 0), (0, -y-1, 0, 0, y^2-1)\}$$

and, therefore, $Syz_A(M) = {}_A\langle \boldsymbol{s}_1 = (0, -y+1, x, 0), \boldsymbol{s}_2 = (-xy, 1, 0, y-1), \boldsymbol{s}_3 = (xy^2 + 2xy, -y-1, 0, 1-y^2)\rangle$. Thus, we have obtained the following presentation for M:

$$M \cong A^4/\langle \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\rangle.$$

6.6 Computing free resolutions

In this section, we will compute free resolutions for left submodules of A^m. Let M be a submodule of A^m, we recall that a free resolution of M is an exact sequence of free modules

$$\cdots \xrightarrow{F_{r+2}} A^{s_r} \xrightarrow{F_r} A^{s_{r-1}} \xrightarrow{F_{r-1}} \cdots \xrightarrow{F_2} A^{s_1} \xrightarrow{F_1} A^{s_0} \xrightarrow{F_0} M \longrightarrow 0,$$

with $s_i \geq 0$ for each $i \geq 0$. We assume that $A^0 = 0$. r is the length of this sequence if $s_r \neq 0$ and $s_i = 0$ for $i \geq r+1$. The following proposition describes a simple procedure for constructing a free resolution of M.

Theorem 6.6.1. *Let $M = \langle \boldsymbol{f}_1^{(0)}, \ldots, \boldsymbol{f}_{s_0}^{(0)} \rangle$ be a submodule of the free left module A^m. Let F_0 be the matrix whose columns are $\boldsymbol{f}_1^{(0)}, \ldots, \boldsymbol{f}_{s_0}^{(0)}$, and for $i \geq 1$ let*

$$F_i := Syz(F_{i-1}) = \begin{bmatrix} \boldsymbol{f}_1^{(i)} & \cdots & \boldsymbol{f}_{s_i}^{(i)} \end{bmatrix}.$$

Then,

$$\cdots \xrightarrow{f_{r+2}} A^{s_r} \xrightarrow{f_r} A^{s_{r-1}} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} A^{s_1} \xrightarrow{f_1} A^{s_0} \xrightarrow{f_0} M \longrightarrow 0,$$

is a free resolution of M, where

$$f_i(\boldsymbol{e}_{j_i}^{(i)}) = [(\boldsymbol{e}_{j_i}^{(i)})^T F_i^T]^T = \boldsymbol{f}_{j_i}^{(i)}$$

and $\{\boldsymbol{e}_{j_i}^{(i)}\}_{1 \leq j_i \leq s_i}$ is the canonical basis of A^{s_i}.

Proof. Each homomorphism f_i is represented by a matrix, and hence, a resolution of M is described as a sequence of matrices $\{F_i\}_{i \geq 0}$, where the columns of F_i are the generators of $Syz(F_{i-1})$, $i \geq 1$. The columns of F_0 are the generators of M. Thus, by definition of matrices F_i, we have that $Im(f_i) = Syz(F_{i-1}) = \ker(f_{i-1})$ for each $i \geq 1$, and that F_0 is a surjective homomorphism. □

We can illustrate this procedure in the following example.

Example 6.6.2. Let A be the ring $\sigma(\mathbb{Q})\langle x, y \rangle$, where $yx = xy + x$. We will calculate a free resolution for the left module $M := {}_A\langle (1,1), (xy, 0), (y^2, 0), (0, x) \rangle$ given in the Example 6.5.4. There we saw that $M \cong A^4/\langle \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3 \rangle$, where $\boldsymbol{s}_1 = (0, -y+1, x, 0), \boldsymbol{s}_2 = (-xy, 1, 0, y-1), \boldsymbol{s}_3 = (xy^2 + 2xy, -y-1, 0, 1-y^2)$. Now, we must compute a generator set for $Syz_A(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3)$. For such task, we consider the deglex order on $Mon(A)$, with $x \succ y$, and the TOP order over $Mon(A^2)$, with $\boldsymbol{e}_2 > \boldsymbol{e}_1$. Is not difficult to see that $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$ is a Gröbner basis; so, $Syz_A(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3) = {}_A\langle (0, y+1, 1) \rangle$. Finally, $Syz_A(\boldsymbol{s}) = \boldsymbol{0}$, where $\boldsymbol{s} = (0, y+1, 1)$. In consequence,

$$F_0 = \begin{bmatrix} 1 & xy & y^2 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}, F_1 = \begin{bmatrix} 0 & -xy & xy^2 + 2xy \\ -y+1 & 1 & -y-1 \\ x & 0 & 0 \\ 0 & y-1 & 1-y^2 \end{bmatrix}, F_2 = \begin{bmatrix} 0 \\ y+1 \\ 1 \end{bmatrix}$$

and a free resolution for M is given by

$$0 \longrightarrow A \xrightarrow{F_2} A^3 \xrightarrow{F_1} A^4 \xrightarrow{F_0} M \longrightarrow 0$$

6.7 Kernel and image of an homomorphism

Let $M \subseteq A^m$ and $N \subseteq A^l$ be modules, with $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s \rangle$, $N = \langle \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t \rangle$, and let $\phi : M \longrightarrow N$ be a homomorphism. Then, there exists a matrix $\Phi = [\phi_{ji}]$ of size $t \times s$ with entries in A given by

$$\phi(\boldsymbol{f}_i) = \phi_{1i}\boldsymbol{g}_1 + \cdots + \phi_{ti}\boldsymbol{g}_t,$$

for each $1 \leq i \leq s$. In this section, we will calculate a system of generators and presentations for $\ker \phi$ and $Im(\phi)$ by using the matrix Φ induced by the homomorphism ϕ. Let $A^s/Syz(M)$ and $A^t/Syz(N)$ be presentations of M and N respectively. We consider the canonical isomorphisms

$$\overline{\pi_M} : A^s/Syz(M) \longrightarrow M, \overline{\pi_N} : A^t/Syz(N) \longrightarrow N$$

defined by $\overline{\pi_M}(\overline{\boldsymbol{e}_i}) = \boldsymbol{f}_i$, for $1 \leq i \leq s$, and $\overline{\pi_N}(\overline{\boldsymbol{e}'_j}) = \boldsymbol{g}_j$, for $1 \leq j \leq t$, where $\{\boldsymbol{e}_i\}_{1 \leq i \leq s}$ is the canonical basis of A^s and $\{\boldsymbol{e}'_j\}_{1 \leq j \leq t}$ is the canonical basis of A^t. Thus, we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ A^s/Syz(M) & \xrightarrow[\overline{\phi}]{} & A^t/Syz(N) \end{array} \tag{6.7.1}$$

where the vertical arrows are the isomorphisms $(\overline{\pi_M})^{-1}$ and $(\overline{\pi_N})^{-1}$. Hence, $\overline{\phi}(\overline{\boldsymbol{e}_i}) = (\overline{\pi_N})^{-1} \circ \phi \circ \overline{\pi_M}(\overline{\boldsymbol{e}_i}) = \phi_{1i}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{ti}\overline{\boldsymbol{e}'_t}$, for each $1 \leq i \leq s$. Note that $\ker(\phi) \cong \ker(\overline{\phi})$ and $Im(\phi) \cong Im(\overline{\phi})$: in fact, is enough to see that $(\overline{\pi_M})^{-1}$ restricted to $\ker(\phi)$ is an isomorphism between $\ker(\phi)$ and $\ker(\overline{\phi})$; analogously for $Im(\phi)$ and $Im(\overline{\phi})$. Let $m \in \ker(\phi)$, then $m = a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s$ and thus, $(\overline{\pi_N})^{-1}(\phi(h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s)) = \overline{\boldsymbol{0}} = \overline{\phi}((\overline{\pi_M})^{-1}(h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s)) = \overline{\phi}(h_1\overline{\boldsymbol{e}_1} + \cdots + h_s\overline{\boldsymbol{e}_s}) = h_1\overline{\phi}(\overline{\boldsymbol{e}_1}) + \cdots + h_s\overline{\phi}(\overline{\boldsymbol{e}_s}) = h_1(\phi_{11}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{t1}\overline{\boldsymbol{e}'_t}) + \cdots + h_s(\phi_{1s}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{ts}\overline{\boldsymbol{e}'_t}) = (h_1\phi_{11} + \cdots + h_s\phi_{1s})\overline{\boldsymbol{e}'_1} + \cdots + (h_1\phi_{t1} + \cdots + h_s\phi_{ts})\overline{\boldsymbol{e}'_t}$. This implies that $(h_1\phi_{11} + \cdots + h_s\phi_{1s})\boldsymbol{e}'_1 + \cdots + (h_1\phi_{t1} + \cdots + h_s\phi_{ts})\boldsymbol{e}'_t \in Syz(N)$. By Theorem 6.2.11, we can compute a system of generators for $Syz(N) = \langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{t_1} \rangle \subseteq A^t$. Hence, there exist $a_{s+1}, \ldots, a_{s+t_1} \in A$ such that

$$a_1 \begin{bmatrix} \phi_{11} \\ \vdots \\ \phi_{t1} \end{bmatrix} + \cdots + a_s \begin{bmatrix} \phi_{1s} \\ \vdots \\ \phi_{ts} \end{bmatrix} + a_{s+1}\boldsymbol{s}_1 + \cdots + a_{s+t_1}\boldsymbol{s}_{t_1} = \mathbf{0}.$$

Conversely, if $\overline{(a_1, \ldots, a_s)} \in \ker(\overline{\phi})$, the above calculations allow us conclude that $a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s \in \ker(\phi)$; thus, we have obtained that

$$a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s \in \ker(\phi) \Leftrightarrow \overline{(a_1, \ldots, a_s)} \in \ker(\overline{\phi}).$$

We have proved the following theorem.

Theorem 6.7.1. *With the above notation, let*

$$H = \begin{bmatrix} \Phi_1 & \cdots & \Phi_s & \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{t_1} \end{bmatrix},$$

where Φ_i is the $i-th$ column of the matrix Φ, for $1 \leq i \leq s$. Then,

$$(a_1, \ldots, a_s, a_{s+1}, \ldots, a_{s+t_1}) \in Syz(H) \Leftrightarrow a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s \in \ker(\phi).$$

Thus, if $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_v\} \subset A^{s+t_1}$ is a system of generators of $Syz(H)$, let $\boldsymbol{z}'_k \in A^s$ be the vector obtained from $\boldsymbol{z}_k$ when omitting the last t_1 components, $1 \leq k \leq v$, then $\{\overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v}\}$ is a system of generators for $\ker(\overline{\phi})$. Moreover, if

$$\boldsymbol{z}'_1 = (h_{11}, \ldots, h_{1s}), \ldots, \boldsymbol{z}'_v = (h_{v1}, \ldots, h_{vs}),$$

then $\{h_{11}\boldsymbol{f}_1 + \cdots + h_{1s}\boldsymbol{f}_s, \ldots, h_{v1}\boldsymbol{f}_1 + \cdots + h_{vs}\boldsymbol{f}_s\}$ is a system of generators for $\ker(\phi)$.

A presentation of $\ker(\phi)$ is given in the following way.

Corollary 6.7.2. *With the notation of this section, a presentation of $\ker(\phi)$ is given by A^v/K, where*

$$K = Syz(\ker(\phi)) = Syz\begin{bmatrix} h_{11}\boldsymbol{f}_1 + \cdots + h_{1s}\boldsymbol{f}_s & \cdots & h_{v1}\boldsymbol{f}_1 + \cdots + h_{vs}\boldsymbol{f}_s \end{bmatrix}.$$

Now we also want to compute also an explicit presentation for $\ker(\overline{\phi})$. We assume that we have computed a system of generators for $Syz(M) = \langle \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{s_1} \rangle \subseteq A^s$. We know that a presentation of $\ker(\overline{\phi})$ is given by $\ker(\overline{\phi}) \cong A^v/K'$, where $K' = Syz(\ker(\overline{\phi})) = Syz(\langle \overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v} \rangle)$. But, $(l_1, \ldots, l_v) \in Syz(\langle \overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v} \rangle)$ if and only if there exist $l_{v+1}, \ldots, l_{v+s_1} \in A$ such that $l_1\boldsymbol{z}'_1 + \cdots + l_v\boldsymbol{z}'_v + l_{v+1}\boldsymbol{w}_1 + \cdots + l_{v+s_1}\boldsymbol{w}_{s_1} = \mathbf{0}$. Thus, we have proved the following corollary.

Corollary 6.7.3. *With the above notation, let*

$$L = \begin{bmatrix} \boldsymbol{z}'_1 & \cdots & \boldsymbol{z}'_v & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{s_1} \end{bmatrix}.$$

If $\{\boldsymbol{l}_1, \ldots, \boldsymbol{l}_q\} \subseteq A^{v+s_1}$ is a system of generators of $Syz(L)$, let $\boldsymbol{l}'_k \in A^v$ be the vector obtained from $\boldsymbol{l}_k$ when omitting the last s_1 components, $1 \leq k \leq q$, then $\{\boldsymbol{l}'_1, \ldots, \boldsymbol{l}'_q\}$ is a system of generators for K', and hence, a presentation of $\ker(\overline{\phi})$ is given by A^v/K'.

We consider now the image of homomorphism $\phi : M \longrightarrow N$ in (6.7.1). Then the following result is clear from the above discussion.

Corollary 6.7.4. *A system of generators for $Im(\phi)$ is given by*

$$Im(\phi) = \langle \phi_{11}\boldsymbol{g}_1 + \cdots + \phi_{t1}\boldsymbol{g}_t, \ldots, \phi_{1s}\boldsymbol{g}_1 + \cdots + \phi_{ts}\boldsymbol{g}_t \rangle.$$

A presentation of $Im(\phi)$ is A^s/I, where

$$I = Syz\begin{bmatrix}\phi_{11}\boldsymbol{g}_1 + \cdots + \phi_{t1}\boldsymbol{g}_t & \cdots & \phi_{1s}\boldsymbol{g}_1 + \cdots + \phi_{ts}\boldsymbol{g}_t\end{bmatrix}.$$

Many of the theoretical results of the present chapter will be illustrated with other concrete examples in the last chapter.

We conclude this section by showing an explicit presentation of $Im(\overline{\phi})$. We know that $Im(\overline{\phi}) = \langle \phi_{11}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{t1}\overline{\boldsymbol{e}'_t}, \ldots, \phi_{1s}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{ts}\overline{\boldsymbol{e}'_t}\rangle$, thus a presentation of $Im(\overline{\phi})$ is given by $Im(\overline{\phi}) \cong A^s/Syz(Im(\overline{\phi}))$. Let $(h_1, \ldots, h_s) \in Syz(Im(\overline{\phi}))$, then there exist $h_{s+1}, \ldots, h_{s+t_1} \in A$ such that

$$h_1\begin{bmatrix}\phi_{11}\\ \vdots \\ \phi_{t1}\end{bmatrix} + \cdots + h_s\begin{bmatrix}\phi_{1s}\\ \vdots \\ \phi_{ts}\end{bmatrix} + h_{s+1}\boldsymbol{u}_1 + \cdots + h_{s+t_1}\boldsymbol{u}_{t_1} = \mathbf{0}.$$

Thus, we have proved the following corollary.

Corollary 6.7.5. *Let H be the matrix in Theorem 6.7.1. If $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_v\} \subseteq A^{s+t_1}$ is a system of generators of $Syz(H)$, let $\boldsymbol{z}'_k \in A^s$ be the vector obtained from $\boldsymbol{z}_k$ when omitting the last t_1 components, $1 \leq k \leq v$. Then, $\{\boldsymbol{z}'_1, \ldots, \boldsymbol{z}'_v\}$ is a system of generators for $Syz(Im(\overline{\phi}))$ and $A^s/Syz(Im(\overline{\phi}))$ is a presentation of $Im(\overline{\phi})$.*

Example 6.7.6. Let $A := \sigma(\mathbb{Q}[x_1])\langle x_2, x_3\rangle = \mathcal{O}_3\left(2, \frac{1}{2}, 3\right)$. Let $M := \langle \boldsymbol{f}_1, \boldsymbol{f}_2\rangle \subseteq A^2$, where $\boldsymbol{f}_1 = x_1^2x_2^2\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2$ and $\boldsymbol{f}_2 = 2x_1x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$. In a similar way as was done in Example 6.2.14, we can prove that $Syz(M) = 0$ and hence M is free with basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. Let $N := \langle \boldsymbol{g}_1, \boldsymbol{g}_2\rangle \subseteq A^2$, where $\boldsymbol{g}_1 = (2x_1 + 1)x_2^2\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2$ and $\boldsymbol{g}_2 = (4x_1^2 + x_1)\boldsymbol{e}_1 + x_1x_2^2x_3\boldsymbol{e}_2$. We consider the homomorphism $\phi : M \longrightarrow N$ given by

$$\phi(\boldsymbol{f}_1) := \boldsymbol{g}_1 + 2\boldsymbol{g}_2$$

$$\phi(\boldsymbol{f}_2) := x_1\boldsymbol{g}_1 + \boldsymbol{g}_2.$$

The matrix Φ induced by ϕ is

$$\Phi = \begin{bmatrix}1 & x_1\\ 2 & 1\end{bmatrix}$$

Using the results of Section 6.2 we verify that

$$Syz(N) = \left\langle \begin{pmatrix}x_1x_2\\ -1\end{pmatrix}\right\rangle,$$

so the matrix H of Theorem 6.7.1 is

$$H = \begin{bmatrix}1 & x_1 & x_1x_2\\ 2 & 1 & -1\end{bmatrix}.$$

Once more, by the results of Section 6.2, a system of generators of $Syz(H)$ is

$$\left\{ \begin{pmatrix} 2x_1^2 - \frac{1}{2}x_1 + x_1^2x_2 - \frac{1}{2}x_1x_2 \\ -2x_1 + \frac{1}{2} - 2x_1^2x_2 + x_1x_2 \\ 4x_1^2 - 3x_1 + \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2}x_1 + 2x_1^2x_2 + \frac{1}{2}x_1x_2 + x_1^2x_2^2 \\ -\frac{1}{2} - 2x_1^2x_2^2 - \frac{3}{2}x_1x_2 \\ 4x_1^2x_2 - \frac{1}{2}x_1x_2 + x_1 - \frac{1}{2} \end{pmatrix} \right\} \in A^3,$$

and by Theorem 6.7.1, a system of generators of $\ker(\phi)$ is

$$\begin{gathered} \{\left(2x_1^2 - \tfrac{1}{2}x_1 + x_1^2x_2 - \tfrac{1}{2}x_1x_2\right)\boldsymbol{f}_1 + (-2x_1 + \tfrac{1}{2} - 2x_1^2x_2 + x_1x_2)\boldsymbol{f}_2, \\ (\tfrac{1}{2}x_1 + 2x_1^2x_2 + \tfrac{1}{2}x_1x_2 + x_1^2x_2^2)\boldsymbol{f}_1 + (-\tfrac{1}{2} - 2x_1^2x_2^2 - \tfrac{3}{2}x_1x_2)\boldsymbol{f}_2\}, \end{gathered}$$

and a system of generators of $Im(\phi)$ is $\{\phi(\boldsymbol{f}_1), \phi(\boldsymbol{f}_2)\} = \{\boldsymbol{g}_1 + 2\boldsymbol{g}_2, x_1\boldsymbol{g}_1 + \boldsymbol{g}_2\}$.

CHAPTER 7

Matrix computations on projective modules using Gröbner bases

In this last chapter, we will use the constructive proofs developed in the former part of this thesis and the Gröbner basis theory, in the order of establishing several algorithms that will allow us to carry out effective calculations as projective dimension, testing stably freeness, constructing minimal presentations and obtaining bases for free modules.

7.1 Computing the inverse of a matrix

We will present an algorithm that determines whether a given rectangular matrix over a bijective skew PBW extension is left invertible and, in such a case, this computes one of its left inverses. A similar algorithm will be constructed for the right side case. We start with the following elementary fact about left invertible matrices.

Proposition 7.1.1. *Let F be a rectangular matrix of size $r \times s$ with entries in a ring S. If F has left inverse, then $r \geq s$. Moreover, F has a left inverse if and only if the left module generated by the rows of F coincides with S^s.*

Proof. First statement follows from the fact that we are assuming S satisfying the $\mathcal{RC}$ condition (see Proposition 2.1.4 and Remark 2.1.9). Now, suppose that F has a left inverse $L \in M_{s\times r}(S)$, i.e., $LF = I_s$. Define the following S-homomorphisms

$$\begin{aligned} f^t : S^r &\to S^s & \qquad l^t : S^s &\to S^r \\ \boldsymbol{a} &\mapsto (\boldsymbol{a}^T F)^T & \boldsymbol{b} &\mapsto (\boldsymbol{b}^T L)^T, \end{aligned}$$

then $m(f^t) = F^T$ and $m(l^t) = L^T$ (for the notation, see Chapter 1). Whence, $m(f^t \circ l^t) = (LF)^T = I_s^T = I_s$, i.e, f^t is an epimorphism. Hence, $Im(f^t) = S^s$, i.e., the left submodule generated by the rows of F coincides with the free module S^s. Conversely, suppose that the module generated by the rows of F coincides wit S^s, then for f^t defined as above, there exist $\boldsymbol{a}_1 \ldots, \boldsymbol{a}_s \in S^r$ such that $f^t(\boldsymbol{a}_i) = \boldsymbol{e}_i$ for each $1 \leq i \leq s$, and where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s$

denote the canonical vectors of S^s. Thus, if $\boldsymbol{a}_i = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ri} \end{bmatrix}^T$, we have

$$\boldsymbol{a}_i^T F = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ri} \end{bmatrix} F = a_{1i}F_{(1)} + \cdots + a_{ri}F_{(r)} = \boldsymbol{e}_i,$$

where $F_{(j)}$ denotes the j-th row of F, $1 \leq j \leq r$. Therefore, if L is the matrix whose rows are the vectors $\boldsymbol{a}_i^T$, then $LF = I_s$, i.e., F has a left inverse. □

Corollary 7.1.2. *Let A be a bijective skew PBW extension and let $F \in M_{r\times s}(A)$ be a rectangular matrix over A. The algorithm below determines whether F is left invertible, and in the positive case, it computes a left inverse of F:*

Algorithm for the left inverse of a matrix

INPUT: A rectangular matrix $F \in M_{r\times s}(A)$

OUTPUT: A matrix $L \in M_{s\times r}(A)$ satisfying $LF = I_s$ in case that it exists, and 0 in other case

INITIALIZATION:

IF $r < s$

RETURN 0

IF $r \geq s$, let $G := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis for the left submodule generated by rows of F and let $\{\boldsymbol{e}_i\}_{i=1}^s$ be the canonical basis of A^s. Use the division algorithm to verify whether $\boldsymbol{e}_i \in {}_A\langle G\rangle$ for each $1 \leq i \leq s$.

IF there exists some $\boldsymbol{e}_i$ such that $\boldsymbol{e}_i \notin \langle G\rangle$,

RETURN 0

IF $\langle G\rangle = A^s$, let $H \in M_{r\times t}(A)$ with the property $G^T = H^T F$, and consider $K := [k_{ij}] \in M_{t\times s}$, where the k_{ij}'s are such that $\boldsymbol{e}_i = k_{1i}\boldsymbol{g}_1 + k_{2i}\boldsymbol{g}_2 + \cdots + k_{ti}\boldsymbol{g}_t$ for $1 \leq i \leq s$. Thus, $I_s = K^T G^T$

RETURN $L := K^T H^T$

Example 7.1.3. Let $A = \sigma(\mathbb{Q})\langle x, y\rangle$ defined through the relation $yx = -xy + 1$. Given the matrix

$$F = \begin{bmatrix} 1 & 1 \\ xy & 0 \\ x^2 & 0 \\ 1 & y \end{bmatrix},$$

we apply the above algorithm in order to verify whether F has a left inverse. For this, we compute a Gröbner basis of the left module generated by the rows of F. Considering the deglex order on $Mon(A)$, with $x \succ y$, and the TOPREV order on $Mon(A^2)$, with $\boldsymbol{e}_1 > \boldsymbol{e}_2$, a Gröbner basis for ${}_A\langle F^T\rangle$ is $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ (here, we also used the Corollary 5.3.4). In consequence, F has a left inverse and, from calculations obtained during the process of Buchberger's algorithm, we have that

$$L = \begin{bmatrix} xy^2 - y & y+1 & 0 & -xy+1 \\ -xy^2 + y + 1 & -y-1 & 0 & xy-1 \end{bmatrix}$$

is a left inverse for F.

Corollary 7.1.4. *Let F be a square matrix of size $r \times r$ with entries in a ring S. Then, F is invertible if and only if the rows of F shape a basis of S^s.*

Proof. Let $L \in M_r(A)$ such that $LF = I_r = FL$. From $LF = I_r$ it follows that the rows of F generate S^r. Let f^t and l^t be like in the proof of Proposition 7.1.1; since $FL = I_r$, then $l^t \circ f^t = i_{S^r}$ and, therefore, f^t is a monomorphism, i.e., $Syz(F^T) = 0$. Thus, the rows of F are linearly independent, and this complete the first implication. Conversely, since the rows of F generate S^r, by Proposition 7.1.1, F has a left inverse. Let L be a such inverse, then $LF = I_r$. We have $FLF = F$, this implies that $(FL - I_r)F = 0_r$, but $Syz(F^T) = 0$, then $FL = I_r$, i.e., $F^{-1} = L$. □

Corollary 7.1.5. *Let A be a bijective skew PBW extension and $F \in M_r(A)$ a square matrix over A. The algorithm below determines whether F is invertible, and in the positive case, it computes the inverse of F:*

Algorithm for the inverse of a square matrix

INPUT: *A square matrix $F \in M_r(A)$*

OUTPUT: *A matrix $L \in M_r(A)$ satisfying $LF = I_r = FL$ in case that it exists, and 0 in other case*

INITIALIZATION:

Use the algorithm in Corollary 7.1.2 to determine whether F is left invertible

IF *F is not left invertible*
RETURN 0

ELSE *Compute $Syz(F^T)$*

IF $Syz(F^T) \neq 0$

RETURN 0

ELSE *Compute the matrices H and K in the algorithm of Corollary 7.1.2*

RETURN $L := K^T H^T$

Example 7.1.6. Once more, we consider the *additive analogue of the Weyl algebra* $A = A_2(\frac{1}{2}, \frac{1}{3})$, used in Example 5.5.22, with the same monomial order on $Mon(A)$ and on $Mon(A^2)$. For this example, let F be the following matrix

$$F = \begin{bmatrix} x_1y_1^2 & x_2y_2^2 \\ x_2y_2 & x_1y_1 \end{bmatrix}.$$

We want to check whether columns of F conform a basis for A^2. From Section 2.1, we know that this is true if and only if F^T is invertible. Using the above algorithm, we start verifying that F^T has a left inverse; for this purpose, we compute a Gröber basis of the left A-module generated by the rows of F^T, i.e., of the left A-module $Im(F)$. As we saw, (see Example 5.5.22) $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a Gröbner basis for this module, where $\boldsymbol{f}_1 = x_1y_1^2\boldsymbol{e}_1 + x_2y_2\boldsymbol{e}_2$, $\boldsymbol{f}_2 = x_2y_2^2\boldsymbol{e}_1 + x_1y_1\boldsymbol{e}_2$ and $\boldsymbol{f}_3 = -\frac{1}{4}x_1^2y_1^3\boldsymbol{e}_2 + \frac{1}{9}x_2^2y_2^3\boldsymbol{e}_2 - \frac{3}{2}x_1y_1^2\boldsymbol{e}_2 + \frac{4}{3}x_2y_2^2\boldsymbol{e}_2$. Using the division algorithm we can check that $\boldsymbol{e}_1 \notin \langle G\rangle$, whereby ${}_A\langle G\rangle \neq A^2$. Thus F^T has no a left inverse and, hence, the columns of F are not a basis for A^2.

Remark 7.1.7. If S is a left (or right) Noetherian ring, then every epimorphism $\alpha : S^r \to S^r$ is an automorphism (see Proposition 1.14 in [63]). In terms of the Remark 2.1.9, we have that every left (or right) Noetherian ring is $\mathcal{WF}$. Therefore, to test that $F \in M_r(S)$ is invertible, it is enough to show that F has a right or a left inverse. So, in the above algorithm, when A is a bijective PBW extension of a LGS ring, it is not necessary the computation of $Syz_S(F^T)$ to test whether the matrix is invertible, it would be sufficient to apply the algorithm for the left inverse given in Corollary 7.1.2.

Now we will consider the right inverse of a rectangular matrix. We start with the following theoretical result.

Proposition 7.1.8. *Let F be a rectangular matrix of size $r \times s$ with entries in the ring S. If F has right inverse, then $s \geq r$ and the module of syzygies of the submodule generated by the rows of F is zero, i.e., $Syz(F^T) = 0$. In other words, if F has a right inverse then the rows of F are linearly independent.*

Proof. $s \geq r$ since we are assuming that S is $\mathcal{RC}$ (Proposition 2.1.4 and Remark 2.1.9). Let $L \in M_{s\times r}(S)$ such that $FL = I_r$. Consider the homomorphisms f^t and l^t as in Proposition 7.1.1, then f^t is a monomorphism. Hence, $\ker(f^t) = 0$, i.e., $Syz(F^T) = 0$. □

Proposition 7.1.9. *Let F be a rectangular matrix of size $r \times s$ with entries in the ring S. If F has right inverse, then $s \geq r$. Moreover, F has a right inverse if and only if $Syz(F^T) = \boldsymbol{0}$ and $Im(F^T)$ is a summand direct of S^s, where $Im(F^T)$ denotes the module generated by the columns of F^T i.e., the module generated by the rows of F.*

Proof. To begin, $s \geq r$ since we are assuming that S is $\mathcal{RC}$ (Proposition 2.1.4 and Remark 2.1.9). Now, let $L \in M_{s\times r}(S)$ such that $FL = I_r$. Consider the homomorphisms f^t and l^t as in Proposition 7.1.1, then $l^t \circ f^t = i_{S^r}$, i.e, f^t is a split monomorphism. Thus, $S^s = Im(f^t) \oplus \ker(l^t)$, and $Im(f^t)$ is a direct summand of S^s. Conversely, let M be a submodule of S^s such that $S^s = Im(f^t) \oplus M$. So, given $\boldsymbol{f} \in S^s$ there exist unique elements $\boldsymbol{f}_1 \in Im(f^t)$ and $\boldsymbol{f}_2 \in M$ such that $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$. Define the homomorphism $l^t : S^s \to S^r$ as $l^t(\boldsymbol{f}) := \boldsymbol{h_f}$, where $\boldsymbol{h_f} \in S^r$ is such that $f^t(\boldsymbol{h_f}) = \boldsymbol{f}_1$. By hypothesis $Syz(F^T) = \boldsymbol{0}$, so f^t is injective and we get that l^t is well defined. It is not difficult to show that l^t is a S-homomorphism. Consequently, $l^t \circ f^t = i_{S^r}$ and if $L^T := m(l^t)$, then $FL = I_r$, i.e., F has a right inverse. □

Remark 7.1.10. If we had a computational tool for to check when a submodule of a free module is a summand direct, then Proposition 7.1.9 would establish an algorithm to check the existence of a right inverse.

Following [23] and [105], consider a matrix $F := [f_{ij}] \in M_{r\times s}(A)$, with $s \geq r$, where A is a bijective skew PBW extension endowed with an involution θ, i.e., a function $\theta : S \to S$ such that $\theta(a+b) = \theta(a) + \theta(b), \theta(ab) = \theta(b)\theta(a)$ and $\theta^2 = i_S$, for all $a, b \in S$. Note that $\theta(1) = 1$, and hence, θ is an anti-isomorphism of S. We define $\theta(F) := [\theta(f_{ij})]$. Observe that if $K \in M_{s\times r}(A)$, then

$$\theta(FK)^T = \theta(K)^T\theta(F)^T. \tag{7.1.1}$$

Proposition 7.1.11. *Let A be a bijective skew PBW extension endowed with an involution θ and let $F := [f_{ij}] \in M_{r\times s}(A)$, with $s \geq r$. Then, F has a right inverse if and only if for each $1 \leq j \leq r$, $\boldsymbol{e}_j \xrightarrow{G'}_+ \mathbf{0}$, where G' is a Gröbner basis of the left A-module generated by the columns of $\theta(F)$ and $\{\boldsymbol{e}_j\}_{j=1}^r$ is the canonical basis of A^r.*

Proof. $G := [g_{ij}] \in M_{s\times r}(A)$ is a right inverse of F if and only if $FG = I_r$, and this is equivalent to say that

$$\boldsymbol{e}_j = \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{r1} \end{bmatrix} \cdot g_{1j} + \cdots + \begin{bmatrix} f_{1s} \\ f_{2s} \\ \vdots \\ f_{rs} \end{bmatrix} \cdot g_{sj}, 1 \leq j \leq r;$$

applying θ we obtain

$$\boldsymbol{e}_j = \theta(g_{1j}) \cdot \begin{bmatrix} \theta(f_{11}) \\ \theta(f_{21}) \\ \vdots \\ \theta(f_{r1}) \end{bmatrix} + \cdots + \theta(g_{sj}) \cdot \begin{bmatrix} \theta(f_{1s}) \\ \theta(f_{2s}) \\ \vdots \\ \theta(f_{rs}) \end{bmatrix}.$$

Thus, G is a right inverse of F if and only if the canonical vectors of A^r belong to the left A-module generated by the columns of $\theta(F)$, i.e., $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r \in \langle\theta(F)\rangle$. Let G' be a Gröbner basis of $\langle\theta(F)\rangle$, then by Theorem 5.5.13, G is a right inverse of F if and only if for each j, $\boldsymbol{e}_j \xrightarrow{G'}_+ \mathbf{0}$. □

Corollary 7.1.12. *Let A be a bijective skew PBW extension with an involution θ, and $F \in M_{r\times s}(A)$ be a rectangular matrix over A. The algorithm below determines whether F is right invertible, and in the positive case, it computes the right inverse of F:*

Algorithm 1 for the right inverse of a matrix

INPUT: An involution θ of A; a rectangular matrix $F \in M_{r\times s}(A)$

OUTPUT: A matrix $H \in M_{s\times r}(A)$ satisfying $FH = I_r$ if it exists, and 0 in other case

INITIALIZATION:

IF $s < r$

RETURN 0

IF $s \geq r$, let $G' := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis for the left submodule generated by columns of $\theta(F)$ and let $\{\boldsymbol{e}_j\}_{j=1}^{r}$ be the canonical basis of A^r. Use the division algorithm to verify if $\boldsymbol{e}_j \in \langle G' \rangle$ for each $1 \leq j \leq r$.

IF there exists some $\boldsymbol{e}_j$ such that $\boldsymbol{e}_j \notin \langle G' \rangle$,

RETURN 0

IF $\langle G' \rangle = A^r$, let $J \in M_{s\times t}(A)$ with the property $G'^T = J^T\theta(F)^T$, and consider $K := [k_{ij}] \in M_{t\times r}$, where the k_{ij}'s are such that $\boldsymbol{e}_j = k_{1j}\boldsymbol{g}_1 + k_{2j}\boldsymbol{g}_2 + \cdots + k_{tj}\boldsymbol{g}_t$ for $1 \leq j \leq r$. Thus, $I_r = K^T G'^T$

RETURN $H := \theta(J)\theta(K)$

Proof. Applying (7.1.1) we get

$$I_r = K^T G'^T = K^T J^T \theta(F)^T = \theta(\theta(K))^T \theta(\theta(J))^T \theta(F)^T = \theta(\theta(J)\theta(K))^T \theta(F)^T = \theta(F\theta(J)\theta(K))^T,$$

so $I_r = \theta(F\theta(J)\theta(K)) = \theta(I_r)$, and from this we get $I_r = F\theta(J)\theta(K)$. □

Example 7.1.13. Let us consider the ring $A = \sigma(\mathbb{Q})\langle x, y\rangle$, with $yx = -xy + 1$. Using the above algorithm, we will compute a right inverse for

$$F = \begin{bmatrix} x & 0 & 1 \\ y-1 & x-1 & x-y \end{bmatrix}$$

provided that it exists. For this, we consider the involution θ on A given by $\theta(x) = -x$ and $\theta(y) = -y$. With this involution, we have that $\theta(xy) = -xy + 1$. Thus,

$$\theta(F) = \begin{bmatrix} -x & 0 & 1 \\ -y-1 & -x-1 & -x+y \end{bmatrix}$$

We start computing a Gröbner basis for the left module generated by the columns of $\theta(F)$. From Corollaries 5.3.4 and 5.4.5, we get $G' = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a Gröbner basis for ${}_A\langle\theta(F)\rangle$. In this case, F has a right inverse and

$$J = \begin{bmatrix} -x+y & -1 \\ x^2+2xy-y^2-x+y-1 & x+y-1 \\ -x^2-xy+2 & -x \end{bmatrix} \text{ is such that } G'^T = J^T\theta(F)^T.$$

Since $G'^T = I_2$, then $K = I_2$ and $L := \theta(J)$ is a right inverse for F, where

$$\theta(J) = \begin{bmatrix} x-y & -1 \\ x^2-2xy-y^2+x-y+1 & -x-y-1 \\ -x^2+xy+1 & x \end{bmatrix}.$$

To find involutions of skew PBW extensions it is a difficult task, so the above algorithm is not practical. A second algorithm for testing the existence and computing a right inverse of a matrix uses the theory of Gröbner bases for right modules developed in Section 5.6. For this we will make a simple adaptation of Proposition 7.1.1 and Corollary 7.1.2 for right submodules, using the right notation in Remark 2.1.2.

Proposition 7.1.14. *Let F be a rectangular matrix of size $r \times s$ with entries in a ring S. If F has right inverse, then $s \geq r$. Moreover, F has a right inverse if and only if the right module generated by the columns of F coincides with S^r.*

Proof. The first statement follows from Proposition 2.1.4 and Remark 2.1.9. Now, suppose that F has a right inverse and let L be a matrix such that $FL = I_r$. Define the following homomorphism of right free S-modules:

$$\begin{aligned} f : S^s &\to S^r & \qquad l : S^r &\to S^s \\ \boldsymbol{a} &\mapsto F\boldsymbol{a} & \boldsymbol{b} &\mapsto L\boldsymbol{b}, \end{aligned}$$

then $m(f) = F$ and $m(l) = L$. Whence, $m(f \circ l) = FL = I_r$, i.e, f is an epimorphism. Therefore, $Im(f) = S^r$, i.e., the right submodule generates by columns of F coincides with the free module S^r. Conversely, if $Im(F) = S^r$, then for f defined as above, there exist $\boldsymbol{a}_1 \ldots, \boldsymbol{a}_s \in S^s$ such that $f(\boldsymbol{a}_i) = \boldsymbol{e}_i$ for each $1 \leq i \leq s$, and where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s$ denote the canonical vectors of S^s. Thus, if $\boldsymbol{a}_j = \begin{bmatrix} a_{1j} & a_{2j} & \cdots & a_{rj} \end{bmatrix}^T$, we have

$$F\boldsymbol{a}_j = F\begin{bmatrix} a_{1j} & a_{2j} & \cdots & a_{rj} \end{bmatrix} = F^{(1)}a_{1j} + \cdots + F^{(r)}a_{rj} = \boldsymbol{e}_j,$$

where $F^{(j)}$ denotes the j-th column of F, $1 \leq j \leq r$. So, if L is the matrix whose columns are the vectors $\boldsymbol{a}_j^T$, then $FL = I_r$, i.e., F has a right inverse. □

Thus, considering the results of Section 5.6, we have the following alternative algorithm for testing the existence of a right inverse.

Corollary 7.1.15. *Let A be a bijective skew PBW extension and $F \in M_{r\times s}(A)$ be a rectangular matrix over A. The algorithm below determines whether F is right invertible, and in the positive case, it computes a right inverse of F:*

Algorithm 2 for the right inverse of a matrix

INPUT: A rectangular matrix $F \in M_{r\times s}(A)$

OUTPUT: A matrix $L \in M_{s\times r}(A)$ satisfying $FL = I_r$ when it exists, and 0 in other case

INITIALIZATION:

IF $s < r$

RETURN 0

IF $s \geq r$, let $G := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a right Gröbner basis for the right submodule generated by columns of F and let $\{\boldsymbol{e}_j\}_{j=1}^{r}$ be the canonical basis of A^r_A. Use right version of division algorithm to verify if $\boldsymbol{e}_i \in \langle G\rangle_A$ for each $1 \leq i \leq r$.

IF there exists some $\boldsymbol{e}_j$ such that $\boldsymbol{e}_j \notin \langle G\rangle_A$,

RETURN 0

IF $\langle G\rangle_A = A^r$, let $H \in M_{s\times t}(A)$ with the property $G = FH$ (see Remark 6.1.4), and consider $K := [k_{ij}] \in M_{t\times s}$, where the k_{ij}'s are such that $\boldsymbol{e}_j = \boldsymbol{g}_1 k_{1j} + \boldsymbol{g}_2 k_{2j} + \cdots + \boldsymbol{g}_t k_{tj}$ for $1 \leq i \leq r$. Thus, $I_r = GK$

RETURN $L := HK$

Example 7.1.16. Consider the ring $A = \sigma(\mathbb{Q})\langle x, y\rangle$, with $yx = -xy + 1$, and let F be the matrix given by

$$F = \begin{bmatrix} y^2 & -xy & y \\ xy-1 & x^2 & x \end{bmatrix}.$$

Applying the right versions of Buchberger's algorithm and Corollary 5.5.15, we have that a Gröbner basis for the right module generated by the columns of F is $G = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. From Corollary 7.1.15 we can show that F has a right inverse; moreover, one right inverse for F is given by

$$L = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ x & y \end{bmatrix}.$$

7.2 Computing projective dimension

Theorem 2.4.2 holds for any projective resolution of M, thus we can consider a free resolution $\{f_i\}_{i\geq 0}$ computed using the results of Section 6.6. Hence, by Theorem 2.4.3 we obtain the following algorithm which computes the projective dimension of a module $M \subseteq A^m$ given by a finite set of generators, where A is a bijective skew PBW extension of a LGS ring R with finite left global dimension. Note that A is left Noetherian (Theorem 1.2.9) and $\mathrm{lgld}(A) < \infty$ (see [83]).

Projective dimension of a module
over a bijective skew PBW extension
Algorithm 1

INPUT: lgld$(A) < \infty, M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s \rangle \subseteq A^m$, with $\boldsymbol{f}_k \neq 0$, $1 \leq k \leq s$

OUTPUT: pd(M)

INITIALIZATION: Compute a free resolution $\{f_i\}_{i \geq 0}$ of M

$i := 0$

WHILE $i \leq$ lgld(A) **DO**

IF $Im(f_i)$ is projective **THEN** pd$(M) = i$

ELSE $i = i + 1$

Observe that, in the previous algorithm, we no need to compute finite free resolutions of M; any free resolution computed using syzygies is enough.

Next, we present another algorithm for computing the left projective dimension of a module $M \subseteq A^m$ given by a finite free resolution:

$$0 \to A^{s_m} \xrightarrow{f_m} A^{s_{m-1}} \xrightarrow{f_{m-1}} A^{s_{m-2}} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_2} A^{s_1} \xrightarrow{f_1} A^{s_0} \xrightarrow{f_0} M \longrightarrow 0. \tag{7.2.1}$$

This algorithm is supported by Corollary 2.4.4 and Theorem 2.4.5.

Projective dimension of a module
over a bijective skew PBW extension
Algorithm 2

INPUT: An A-module M defined by a finite free resolution (7.2.1)

OUTPUT: $\mathrm{pd}(M)$

INITIALIZATION: Set $j := m$ and $H_j := F_m$, with F_m the matrix of f_m in the canonical bases

WHILE $j \leq m$ **DO**

Step 1. Check whether or not H_j^T admits a right inverse G_j^T:

(a) If no right inverse of H_j^T exists, then $\mathrm{pd}(M) = j$

(b) If there exists a right inverse G_j^T of H_j^T and

(i) If $j = 1$, then $\mathrm{pd}(M) = 0$
(ii) If $j = 2$, then compute (2.4.6)
(iii) If $j \geq 3$, then compute (2.4.5)

Step 2. $j := j - 1$

Example 7.2.1. Let A be the ring $\sigma(\mathbb{Q})\langle x, y\rangle$, where $yx = xy + x$. We will calculate the projective dimension of the left module $M = {}_A\langle(1,1),(xy,0),(y^2,0),(0,x)\rangle$ given in the Example 6.5.4. As we saw in the Example 6.6.2, a free resolution for M is given by:

$$0 \longrightarrow A \xrightarrow{F_2} A^3 \xrightarrow{F_1} A^4 \xrightarrow{F_0} M \longrightarrow 0$$

where,

$$F_0 = \begin{bmatrix} 1 & xy & y^2 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}, F_1 = \begin{bmatrix} 0 & -xy & xy^2+2xy \\ -y+1 & 1 & -y-1 \\ x & 0 & 0 \\ 0 & y-1 & 1-y^2 \end{bmatrix}, F_2 = \begin{bmatrix} 0 \\ y+1 \\ 1 \end{bmatrix}.$$

In order to apply the above algorithm, we start checking whether $F_2 = \begin{bmatrix} 0 & y+1 & 1 \end{bmatrix}^T$ has a right inverse. A straightforward calculation shows that a right inverse for F_2 is $G_2 = \begin{bmatrix} 0 & 1 & -y \end{bmatrix}^T$, so we compute (2.4.6):

$$0 \longrightarrow A^3 \xrightarrow{H_1} A^5 \xrightarrow{H_0} M \longrightarrow 0 \tag{7.2.2}$$

where

$$H_1 := \begin{bmatrix} 0 & -xy & xy^2+2xy \\ -y+1 & 1 & -y-1 \\ x & 0 & 0 \\ 0 & y-1 & 1-y^2 \\ 0 & 1 & -y \end{bmatrix} \text{ and } H_0 := \begin{bmatrix} 1 & xy & y^2 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}.$$

To verify whether H_1^T has a right inverse, we must calculate a Gröbner basis for the right module generated by the columns of H_1^T. Since the ring A considered is a bijective skew PBW extension, we can use the right version of Buchberger's algorithm. For this, we consider the deglex order on $Mon(A)$, with $x \succ y$, and the TOP order over $Mon(A^3)$, with $\boldsymbol{e}_1 < \boldsymbol{e}_2 < \boldsymbol{e}_3$. Applying this algorithm, along with Corollary 5.5.15, we obtain the following Gröbner basis for $\langle H_1^T \rangle_A$, $G = \{(x, 0, 0), (1-y, 0, -1), (0, -1, 1), (0, -x, 0), (0, y-1, 0)\}$. Note that $\boldsymbol{e}_1$ is not reducible by G, thus $\langle G \rangle_A \neq A^3$ and hence H_1^T does not have a right inverse. Therefore, $\mathrm{pd}(M) = 1$.

Remark 7.2.2. The above algorithms can be used for testing whether a given module M is projective: we can compute its projective dimension, and hence, M es projective if and only if $\mathrm{pd}(M) = 0$.

7.3 Test for stably-freeness

Theorem 2.2.15 gives a procedure for testing stably-freeness for a module $M \subseteq A^m$ given by an exact sequence

$$0 \to A^s \xrightarrow{f_1} A^r \xrightarrow{f_0} M \to 0,$$

where A is a bijective skew PBW extension.

Test for stably-freeness
Algorithm 1

INPUT: M an A-module with exact sequence

$$0 \to A^s \xrightarrow{f_1} A^r \xrightarrow{f_0} M \to 0$$

OUTPUT: TRUE in case that M is stably free, FALSE otherwise

INITIALIZATION: Compute the matrix F_1 of f_1

IF F_1^T has right inverse **THEN**

  RETURN TRUE

ELSE

  RETURN FALSE

Example 7.3.1. Let $A = \sigma(\mathbb{Q})\langle x, y \rangle$, with $yx = -xy$. We want to know whether the left A-module M given by

$$M = {}_A\langle \boldsymbol{e}_3 + \boldsymbol{e}_1, \boldsymbol{e}_4 + \boldsymbol{e}_2, x\boldsymbol{e}_2 + x\boldsymbol{e}_1, y\boldsymbol{e}_1, y^2\boldsymbol{e}_4, x\boldsymbol{e}_4 + y\boldsymbol{e}_3 \rangle$$

is stably free or not. To answer this question, we start computing a finite presentation of M. Considering the deglex order on $Mon(A)$ with $x \succ y$, the TOP order on $Mon(A^4)$ with $\boldsymbol{e}_4 > \boldsymbol{e}_3 > \boldsymbol{e}_2 > \boldsymbol{e}_1$, and using the methods established in the previous sections, we have that a system of generators for $Syz(M)$ is given by

$$S = \{(0, -xy^2, y^2, -xy, x, 0), (-y^2, xy, y, x+y, 0, y), (y^3, 0, 0, -y^2, x, -y^2)\}.$$

Therefore, we get a following finite presentation for M:

$$A^3 \xrightarrow{F_1} A^6 \xrightarrow{F_0} M \longrightarrow 0 \tag{7.3.1}$$

where,

$$F_1 := \begin{bmatrix} 0 & -y^2 & y^3 \\ -xy^2 & xy & 0 \\ y^2 & y & 0 \\ -xy & x+y & -y^2 \\ x & 0 & x \\ 0 & y & -y^2 \end{bmatrix} \text{ and } F_0 := \begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & y \\ 0 & 1 & 0 & 0 & y^2 & x \end{bmatrix}.$$

Applying the method for computing the syzygy module, we have that $Syz_A(F_1) = 0$, so the presentation obtained in 7.3.1 becomes

$$0 \longrightarrow A^3 \xrightarrow{F_1} A^6 \xrightarrow{F_0} M \longrightarrow 0$$

Finally, we must to test whether F_1^T has a right inverse. For this, we calculate a Gröbner basis for the right module generated by the columns of F_1^T. Using the TOP order on $Mon(A^3)$, with $\boldsymbol{e}_3 > \boldsymbol{e}_2 > \boldsymbol{e}_1$, a Gröbner basis for $\langle F_1^T \rangle_A$ is given by $G = \{\boldsymbol{f}_i\}_{i=1}^7$, where $\boldsymbol{f}_i$ is the i-th column of F_1^T for $1 \leq i \leq 6$, and $\boldsymbol{f}_7 = -\boldsymbol{e}_2 xy^2 + \boldsymbol{e}_1 xy^2$. Note that, for example, $\boldsymbol{e}_1 \notin \langle G \rangle_A$ so that $A^6 \neq \langle G \rangle_A$. Thus, F_1^T has not right inverse and hence M is not stably free.

Remark 7.3.2. From Theorem 2.2.15, if M is a left A-module with exact sequence $0 \to A^s \xrightarrow{f_1} A^r \xrightarrow{f_0} M \to 0$, then $M^T \cong Ext_A^1(M, A)$, where $M^T = S^s/Im(f_1^T)$ and $f_1^T : S^r \to S^s$ is the homomorphism of right free S-modules induced by the matrix F_1^T. Thus, for testing stably freeness of M, we can use the results in the Section 5.6 and computing a Gröbner basis for the right module generated by columns of F_1^T. Using the right version of the division algorithm, is possible to check whether $S^s = Im(F_1^T)$. If this last equality holds, then $M^T = 0$ and M is stably free.

Corollary 2.4.4 gives another procedure for testing stably-freeness for a module $M \subseteq A^m$ given by a finite free resolution (2.4.4) with $S = A$: Indeed, if $m \geq 3$ and f_m has not left inverse, then M is non stably free; if f_m has a left inverse, we compute the new finite free resolution (2.4.5) and we check whether h_{m-1} has a left inverse. We can repeat this procedure until (2.4.6); if h_1 has not left inverse, then M is non stably free. If h_1 has a left inverse, then M is stably free.

Example 7.3.3. Let A be the ring $\sigma(\mathbb{Q})\langle x, y\rangle$, where $yx = xy + x$ and consider the left module $M = {}_A\langle(1,1),(xy,0),(y^2,0),(0,x)\rangle$ given in the Example 6.5.4. As we saw in the Example 7.2.1, a finite presentation for M is given by:

$$0 \longrightarrow A^3 \xrightarrow{H_1} A^5 \xrightarrow{H_0} M \longrightarrow 0 \tag{7.3.2}$$

where

$$H_1 := \begin{bmatrix} 0 & -xy & xy^2+2xy \\ -y+1 & 1 & -y-1 \\ x & 0 & 0 \\ 0 & y-1 & 1-y^2 \\ 0 & 1 & -y \end{bmatrix} \text{ and } H_0 := \begin{bmatrix} 1 & xy & y^2 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}.$$

In such example, we showed that H_1^T has not a right inverse, hence M is not a stably free module.

7.4 Computing minimal presentations

If $M \subseteq A^m$ is a stably free module given by the finite free resolution (2.4.4) with $S = A$, then the Corollary 2.4.4 gives a procedure for computing a minimal presentation of M. In fact, if $m \geq 3$, then f_m has a left inverse (if not, $\mathrm{pd}(M) = m$, but this is impossible by Theorem 2.4.5 since M is projective). Hence, we compute the new finite presentation (2.4.5) and we will repeat the procedure until we get a finite presentation as in (2.4.6), which is a minimal presentation of M.

Example 7.4.1. Let us consider again the ring $A = \sigma(\mathbb{Q})\langle x, y\rangle$, with $yx = -xy + 1$. Let M be the left A-module given by presentation $A^2/Im(F_1)$, where

$$F_1 = \begin{bmatrix} y^2 & xy-1 \\ -xy & x^2 \end{bmatrix}.$$

Regarding the deglex order on $Mon(A)$, with $y \succ x$, and the TOP order over $Mon(A^2)$ with $\boldsymbol{e}_2 > \boldsymbol{e}_1$, we have that $Syz_A(F_1)$ is generated by (x, y). So, the following exact sequence is obtained:

$$0 \longrightarrow A \xrightarrow{F_2} A^2 \xrightarrow{F_1} A^2 \xrightarrow{\pi} M \longrightarrow 0$$

where $F_2 := \begin{bmatrix} x & y \end{bmatrix}^T$. Note that F_2^T has a right inverse: $G_2^T = \begin{bmatrix} y \\ x \end{bmatrix}$; thus, from Corollary 2.4.4 we get the following finite presentation for M:

$$0 \longrightarrow A^2 \xrightarrow{h_1} A^3 \xrightarrow{h_0} M \longrightarrow 0 \tag{7.4.1}$$

with $H_1^T = \begin{bmatrix} F_1^T & G_2^T \end{bmatrix}$ and $h_0 = \begin{bmatrix} f_0 & 0 \end{bmatrix}^T$. In the Example 7.1.16, we showed that H_1^T has a right inverse; moreover, one right inverse for H_1^T is

$$L_1^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ x & y \end{bmatrix}.$$

In consequence, (7.4.1) is a minimal presentation for M, and M turns out to be a stably free module.

7.5 Computing free bases

In the Section 2.3, it was showed that if M is a stably free module with $\mathrm{rank}(M) \geq \mathrm{sr}(S)$, then M is free with dimension equals to $\mathrm{rank}(\mathrm{M})$. For computing a basis of M, we start establishing an algorithm for to calculate the elementary matrix U in the Proposition 2.3.3:

Algorithm for computing U in Proposition 2.3.3

INPUT: An unimodular stable column vector $\boldsymbol{v} = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}^T$ over S.

OUTPUT: An elementary matrix $U \in M_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$.

DO:

1. Compute $a_1, \ldots, a_{r-1} \in S$ such that (2.3.1) holds.
2. Compute the matrix E_1 given in (2.3.2).
3. Calculate the elements $b_1, \ldots, b_{r-1} \in S$ with the property that $\sum_{i=1}^{r-1} b_i v_i' = 1$, with $v_i' = v_i + a_i v_r$ for $1 \leq i \leq r-1$.
4. Define $v_i'' := (v_i' - 1 - v_r)b_i$ for $1 \leq i \leq r-1$, and compute the matrices E_2, E_3 and E_4 given in (2.3.3)-(2.3.5).

RETURN: $U := E_4E_3E_2E_1$.

We will illustrate below this algorithm.

Example 7.5.1. Consider the *Quantum Weyl Algebra* $A_2(J_{a,b})$, described in the Example 1.3.1, with $\Bbbk = \mathbb{Q}$, $a = 0$ and $b = -1$. Thus, the relations in this ring are given by:

$$\begin{aligned}
x_1x_2 &= x_2x_1 \\
\partial_2\partial_1 &= \partial_1\partial_2 - \partial_2^2 \\
\partial_1 x_1 &= 1 + x_1\partial_1 \\
\partial_1 x_2 &= x_2\partial_1 - x_2\partial_2 \\
\partial_2 x_1 &= x_1\partial_2 \\
\partial_2 x_2 &= 1 + x_1\partial_2 + x_2\partial_2.
\end{aligned}$$

$E_4(A_2(J_{0,-1}))$ it will denote the group generated by all elementary matrices of size 4×4 over $A_2(J_{0,-1})$. Let $\boldsymbol{v} = \begin{bmatrix} \partial_2 + x_1 & \partial_2 + \partial_1 & x_2 & \partial_1 \end{bmatrix}^T$, then $\boldsymbol{u} = \begin{bmatrix} \partial_1 & -\partial_2 & 0 & -x_1 \end{bmatrix}$ is such that $\boldsymbol{uv} = 1$, whereby $\boldsymbol{v} \in Um_c(4, A_2(J_{0,-1}))$. Moreover, the column vector $\boldsymbol{v}' = \begin{bmatrix} \partial_2 + x_1 & \partial_2 & x_2 \end{bmatrix}^T$ has a left inverse $\boldsymbol{u}' = \begin{bmatrix} 0 & x_2 - x_1 & \partial_2 \end{bmatrix}$, so $\boldsymbol{v}$ is a stable unimodular column. In this case, $a_1 = 0$, $a_2 = -1$, $a_3 = 0$ and the matrix E_1 is given by

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With this elementary matrix we get $E_1\boldsymbol{v} = \begin{bmatrix} \partial_2 + x_1 & \partial_2 & x_2 & \partial_1 \end{bmatrix}^T$. If we define $v_1'' := 0$, $v_2'' := (\partial_2 + x_1 - 1 - \partial_1)(x_2 - x_1)$, $v_3'' = (\partial_2 + x_1 - 1 - \partial_1)\partial_2$ and

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & v_2'' & v_3'' & 1 \end{bmatrix},$$

we obtain $E_2E_1\boldsymbol{v} = \begin{bmatrix} \partial_2 + x_1 & \partial_2 & x_2 & \partial_2 + x_1 - 1 \end{bmatrix}^T$. Finally, if we define

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in E_4(A_2(J_{0,-1})),\ E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\partial_2 & 1 & 0 & 0 \\ -x_2 & 0 & 1 & 0 \\ -\partial_2 - x_1 + 1 & 0 & 0 & 1 \end{bmatrix} \in E_4(A_2(J_{0,-1}))$$

and $U := E_4E_3E_2E_1 \in E_4(A_2(J_{0,-1}))$, then we have $U\boldsymbol{v} = \boldsymbol{e}_1$.

The proof of Theorem 2.3.6 allows us to establish an algorithm to compute a basis for M, when M is a stably free module given by a minimal presentation

$$0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0, \tag{7.5.1}$$

with $g_1 : S^r \to S^s$ such that $g_1 \circ f_1 = i_{S^s}$, and $\operatorname{rank}(M) = r - s \geq \operatorname{sr}(S)$.

Algorithm for computing bases

INPUT: $F_1 = m(f_1)$ such that $F_1^T \in M_{s\times r}(S)$ has a right inverse $G_1^T \in M_{r\times s}(S)$, and satisfies $r - s \geq \mathrm{sr}(S)$.

OUTPUT: A matrix $U \in M_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s & 0 \end{bmatrix}^T$; by Lemma 2.3.5 the set $\{(U^T)^{(s+1)}, \ldots, (U^T)^{(r)}\}$ is a basis for M, where $(U^T)^{(j)}$ denotes the j-th column of U^T for $s+1 \leq j \leq r$.

INITIALIZATION: $i = 1$, $V = I_r$.

WHILE $i < r$ **DO**:

1. Denote by $\boldsymbol{v}_i \in S^{r-i+1}$ the column vector given by taking the last $r - i + 1$ entries of the i-th column of VG_1^T.
2. Apply the previous algorithm to compute $L_i \in E_{r-i+1}(S)$ such that $L_i\boldsymbol{v}_i = \boldsymbol{e}_1$.
3. Define the matrix $U_i := \begin{bmatrix} I_{i-1} & 0 \\ 0 & L_i \end{bmatrix} \in E_r(S)$ for $i > 1$, and $U_1 := L_1$.
4. $i = i + 1$

RETURN $U := PU_sV$, where P is an adequate elementary matrix.

Example 7.5.2. Let A be the *Quantum Weyl Algebra* $A_2(J_{a,b})$ considered in Example 7.5.1, with $\Bbbk = \mathbb{Q}$, $a = 0$ and $b = -1$. In order to illustrate the previous algorithm, take $M = A^6/Im(F_1)$, where

$$F_1 = \begin{bmatrix} 0 & \partial_1 \\ x_2 & \partial_2 \\ 0 & -x_1 \\ \partial_1 & 0 \\ x_1 & 1 \\ \partial_2 & -1 \end{bmatrix}.$$

Using the algorithm described in Corollary 7.1.15, the deglex order over $Mon(A)$, with $x_2 > \partial_1$, and the TOPREV order on $Mon(A^6)$, with $\boldsymbol{e}_1 > \boldsymbol{e}_2$, it is possible to show that F_1^T has a right inverse given by:

$$G_1^T = \begin{bmatrix} x_1\partial_1 & x_1 \\ 0 & 0 \\ \partial_1^2 & \partial_1 \\ x_1 & 0 \\ -\partial_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we have the following minimal presentation for M:

$$0 \to A^2 \xrightarrow{F_1} A^6 \xrightarrow{\pi} M \to 0, \tag{7.5.2}$$

where π is the canonical projection. Thus, M is a stably free A-module with $\operatorname{rank}(\mathrm{M}) = 4$. Since lKdim$(A) = 3$ (see [38], Theorem 2.2), then $\operatorname{sr}(A) \leq 4$ and by the Theorem 2.3.6, M is free with dimension equals to $\operatorname{rank}(M)$. We will use the previous algorithm for computing a basis of M.

. *Step 1.* Let $V = I_6$ and $\boldsymbol{v}_1$ the first column of VG_1^T, i.e.,

$$\boldsymbol{v}_1 = \begin{bmatrix} x_1\partial_1 & 0 & \partial_1^2 & x_1 & -\partial_1 & 0 \end{bmatrix}^T,$$

then $\boldsymbol{v}_1 \in Um_c(6, A)$ and $\boldsymbol{u}_1 = \begin{bmatrix} 0 & x_2 & 0 & \partial_1 & x_1 & -\partial_1 \end{bmatrix}$ is such that $\boldsymbol{u}_1\boldsymbol{v}_1 = 1$. Note that $\boldsymbol{v}_1' = \begin{bmatrix} x_1\partial_1 & 0 & \partial_1^2 & x_1 & -\partial_1 \end{bmatrix}^T$ is trivially unimodular. Applying to $\boldsymbol{v}_1$ the first algorithm of the current section, we have that $E_1 = I_6$,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & (x_1\partial_1 - 1)x_2 & 0 & (x_1\partial_1 - 1)\partial_1 & (x_1\partial_1 - 1)x_1 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and, } E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\partial_1^2 & 0 & 1 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 1 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 & 1 & 0 \\ -x_1\partial_1 + 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can check that

$$U_1 := E_4E_3E_2E_1 =$$
$$\begin{bmatrix} 1 & -(x_1\partial_1 - 1)x_2 & 0 & -(x_1\partial_1 - 1)\partial_1 & -(x_1\partial_1 - 1)x_1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\partial_1^2 & \partial_1^2(x_1\partial_1 - 1)x_2 & 1 & \partial_1^2(x_1\partial_1 - 1)\partial_1 & \partial_1^2(x_1\partial_1 - 1)x_1 & \partial_1^2 \\ -x_1 & x_1(x_1\partial_1 - 1)x_2 & 0 & x_1(x_1\partial_1 - 1)\partial_1 + 1 & x_1(x_1\partial_1 - 1)x_1 & x_1 \\ \partial_1 & -\partial_1(x_1\partial_1 - 1)x_2 & 0 & -\partial_1(x_1\partial_1 - 1)\partial_1 & -\partial_1(x_1\partial_1 - 1)x_1 + 1 & -\partial_1 \\ -x_1\partial_1 + 1 & x_1\partial_1(x_1\partial_1 - 1)x_2 & 0 & x_1\partial_1(x_1\partial_1 - 1)\partial_1 & x_1\partial_1(x_1\partial_1 - 1)x_1 & x_1\partial_1 \end{bmatrix} \in E_6(A)$$

and

$$U_1G_1^T = \begin{bmatrix} 1 & x_1 \\ 0 & 0 \\ 0 & -x_1\partial_1^2 - \partial_1 \\ 0 & -x_1^2 \\ 0 & x_1\partial_1 + 1 \\ 0 & -x_1^2\partial_1 \end{bmatrix}.$$

. *Step 2.* Make $V := U_1$ and let $\boldsymbol{v}_2$ be the column vector given by taking the last five entries of the 2-th column of VG_1^T; i.e., $\boldsymbol{v}_2 = \begin{bmatrix} 0 & -x_1\partial_1^2 - \partial_1 & -x_1^2 & x_1\partial_1 + 1 & -x_1^2\partial_1 \end{bmatrix}^T$. Note that $\boldsymbol{u}_2 = \begin{bmatrix} 0 & -x_1 & \partial_1^2 & 3 & 0 \end{bmatrix}$ satisfies $\boldsymbol{u}_2\boldsymbol{v}_2 = 1$, thus $v_2 \in Um_c(5, A)$. Moreover, $\boldsymbol{v}_2' = \begin{bmatrix} 0 & -x_1\partial_1^2 - \partial_1 & -x_1^2 & x_1\partial_1 + 1 \end{bmatrix}$ is unimodular with $\boldsymbol{u}_2' = \begin{bmatrix} 0 & -x_1 & \partial_1^2 & 3 \end{bmatrix}$ such that $\boldsymbol{u}_2'\boldsymbol{v}_2' = 1$, and hence $\boldsymbol{v}_2$ is stable. Using the algorithm at the beginning of this section, we have that $E_1 = I_5$,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -(-1 + x_1^2\partial_1)x_1 & (-1 + x_1^2\partial_1)\partial_1^2 & 3(-1 + x_1^2\partial_1) & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{and, } E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1\partial_1^2 + \partial_1 & 1 & 0 & 0 & 0 \\ x_1^2 & 0 & 1 & 0 & 0 \\ -x_1\partial_1 - 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Making the respective calculations, we have that

$$L_2 := E_4E_3E_2E_1 =$$
$$\begin{bmatrix} 1 & (-1 + x_1^2\partial_1)x_1 & -(-1 + x_1^2\partial_1)\partial_1^2 & -3(-1 + x_1^2\partial_1) & -1 \\ x_1\partial_1^2 + \partial_1 & 1 + (x_1\partial_1^2 + \partial_1)(-1 + x_1^2\partial_1)x_1 & -(x_1\partial_1^2 + \partial_1)(-1 + x_1^2\partial_1)\partial_1^2 & -3(x_1\partial_1^2 + \partial_1)(-1 + x_1^2\partial_1) & -(x_1\partial_1^2 + \partial_1) \\ x_1^2 & x_1^2(-1 + x_1^2\partial_1)x_1 & 1 - x_1^2(-1 + x_1^2\partial_1)\partial_1^2 & -3x_1^2(-1 + x_1^2\partial_1) & -x_1^2 \\ -(x_1\partial_1 + 1) & -(x_1\partial_1 + 1)(-1 + x_1^2\partial_1)x_1 & (x_1\partial_1 + 1)(-1 + x_1^2\partial_1)\partial_1^2 & 1 + 3(x_1\partial_1 + 1)(-1 + x_1^2\partial_1) & x_1\partial_1 + 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

and $L_2\boldsymbol{v}_2 = \boldsymbol{e}_1 \in A^5$. Define $U_2 := \begin{bmatrix} 1 & 0 \\ 0 & L_2 \end{bmatrix}$; then

$$U_2U_1G_1^T = \begin{bmatrix} 1 & x_1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, if

$$P_1 := \begin{bmatrix} 1 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ then } UG_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where $U := P_1U_2U_1$. Thus, a basis for M is given by $\{\pi(U_{(3)}), \pi(U_{(4)}), \pi(U_{(5)}), \pi(U_{(6)})\}$, with $U_{(i)}^T$ denoting the transpose of i-th row of the matrix U, for $i = 3, 4, 5, 6$; i.e.,

$$U_{(3)}^T = \begin{bmatrix} -x_1^3\partial_1^2 + x_1\partial_1^3 - 4x_1^2\partial_1 - 2x_1 \\ (x_1\partial_1^2 + \partial_1)(1 - x_1\partial_1^2 x_2 + x_1^3\partial_1^3 x_2 + \partial_1 x_2) \\ 1 + (x_1\partial_1^2 + \partial_1)(-1 + x_1^2\partial_1)x_1 \\ (x_1\partial_1^2 + \partial_1)(x_1^3\partial_1^4 - x_1\partial_1^3 + 2\partial_1^2 - x_1\partial_1^3) \\ (x_1\partial_1^2 + \partial_1)(\partial_1 x_1 - x_1\partial_1^2 x_1 + x_1^3\partial_1^3 x_1 - 3x_1^2\partial_1 + 3) \\ (x_1\partial_1^2 + \partial_1)(-\partial_1 + x_1^2\partial_1^2 - x_1\partial_1) + \partial_1^2 \end{bmatrix},$$

$$U_{(4)}^T = \begin{bmatrix} x_1^2\partial_1 - x_1^4\partial_1^2 + x_1^3\partial_1 - x_1^2 - x_1 \\ x_1^2 + (-x_1^2\partial_1 + x_1^4\partial_1^2 - x_1^3\partial_1 + x_1)(x_1\partial_1 - 1)x_2 \\ -x_1^3 + x_1^5\partial_1 + x_1^4 \\ -x_1^3\partial_1^3 + x_1^5\partial_1^4 + 2x_1^2\partial_1^2 - x_1\partial_1 - x_1^4\partial_1^3 + 1 \\ -x_1^4\partial_1^2 - x_1^3\partial_1 + x_1^6\partial_1^3 + 3x_1^5\partial_1^2 - 3x_1^4\partial_1 + 3x_1^2 \\ -x_1^2\partial_1 + x_1^4\partial_1^2 - x_1^3\partial_1 + x_1 \end{bmatrix},$$

$$U_{(5)}^T = \begin{bmatrix} -x_1\partial_1^2 + x_1^3\partial_1^3 + 2x_1^2\partial_1^2 - x_1\partial_1 + 1 \\ x_1\partial_1(-1 + x_1\partial_1^2 x_2 - x_1^3\partial_1^3 x_2) - x_1^3\partial_1^3 x_2 - 1 \\ -(x_1\partial_1 + 1)(-1 + x_1^2\partial_1)x_1 \\ (x_1\partial_1 + 1)(x_1\partial_1^3 - x_1^3\partial_1^4 + x_1^2\partial_1^3 - \partial_1^2) \\ (x_1\partial_1 + 1)(x_1\partial_1^2 x_1 - x_1^3\partial_1^3 + 3x_1^2\partial_1 - 3) - x_1^2\partial_1^2 + 2x_1\partial_1 + 1 \\ -(x_1\partial_1 + 1)(-\partial_1 + x_1^2\partial_1^2 - x_1\partial_1) - \partial_1 \end{bmatrix}, \quad U_{(6)}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

APPENDIX A

Filtered-graded transfer of Gröbner bases

In [84] it was shown that if $A = \Bbbk[a_i]_{i\in\Lambda}$ is a $\Bbbk$-algebra generated by $\{a_i\}_{i\in\Lambda}$ over the field $\Bbbk$, and I a left ideal of A, then a nonempty subset G of I is a Gröbner basis for I if, and only if, $\overline{G}$ is a Gröbner basis of $Gr(I)$, where $\overline{G}$ denotes the image of G in $Gr(A)$ and $Gr(I)$ is the left ideal associated to I in $Gr(A)$. A similar fact is proved in [19] for the case of PBW algebras. We will present an analogous result for skew PBW extensions, specifically for those of bijective type.

A.1 For left ideals

In [83] was showed that if A is a skew PBW extension, then its associated graded ring $Gr(A)$ is a quasi-commutative skew PBW extension (see Theorem 1.2.5). In this section we will prove this fact using a different technique. Furthermore, we establish the transfer of Gröbner bases between A and $Gr(A)$.

By (1.2.7), given A a skew PBW extension of the ring R, the collection of subsets $\{F_p(A)\}_{p\in\mathbb{Z}}$ of A defined by

$$F_p(A) := \begin{cases} 0, & \text{if } p \leq -1, \\ R, & \text{if } p = 0, \\ \{f \in A \mid \deg(lm(f)) \leq p\}, & \text{if } p \geq 1. \end{cases}$$

is a filtration for the ring A, named *standard filtration*.
Now, notice that

$$F_p(A) = \big\{\sum c_\alpha x^\alpha \mid c_\alpha \in R \setminus \{0\},\, x^\alpha \in Mon(A),\, \deg(x^\alpha) \leq p\big\};$$

in this case, we say that this filtration is the *filtration $Mon(A)$-standard on A*. Moreover,

$$Mon(A) = \bigcup_{p\geq 0} Mon(A)_p,$$

where $Mon(A)_p := \{x^\alpha \in Mon(A) \mid \deg(x^\alpha) \leq p\}$, and if $|\alpha| = p$, then $x^\alpha \notin Mon(A)_{p-1}$. In this case, it says that $Mon(A)$ is a *strictly filtered basis*.

It can be noted that any filtration $\{F_p(A)\}_{p\in\mathbb{Z}}$ on A defines an order function $v : A \to \mathbb{Z}$ in the following way:

$$v(f) := \begin{cases} p, & \text{if } f \in F_p(A) - F_{p-1}(A), \\ -\infty, & \text{if } f \in \cap_{p\in\mathbb{Z}} F_p(A). \end{cases}$$

Definition A.1.1. *Let $Gr(A)$ be the graded ring associated to the filtered ring A, and let $f \in A$ with $f = \sum_{|\alpha|\leq p} c_\alpha x^\alpha$, where p =deg(f), $c_\alpha \in R \setminus \{0\}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. In what follows, $\eta(f)$ will denote the image (or principal symbol) of f in $Gr(A)$, i.e.,*

$$\eta(f) := \sum_{|\alpha|=p} c_\alpha x^\alpha + F_{p-1}(A) \in F_p(A)/F_{p-1}(A).$$

Lemma A.1.2. *Let A, $Mon(A)$ and $\{F_p(A)\}_p$ as above, then:*

(i) *For each $f \in A$, deg(f) $= v(f)$.*

(ii) *For each $p \in \mathbb{N}$, $Mon(A)_p$ is a R-basis for $F_p(A)$.*

(iii) *For x^α, $x^\beta \in Mon(A)$, $\eta(x^\alpha) = \eta(x^\beta)$ if and only if $x^\alpha = x^\beta$.*

Proof. (i) From definition of $\{F_p(A)\}_{p\in\mathbb{Z}}$ it follows that if $0 \neq f \in A$, then there exists $p \in \mathbb{N}$ such that $f \in F_p(A) - F_{p-1}(A)$ and, therefore, $v(f) = p$. But, if $f \in F_p(A) - F_{p-1}(A)$, then $\deg(f) = p$ and we obtain the equality.
(ii) Let $f \in F_p(A)$, then $f = \sum_{|\alpha|\leq p} c_\alpha x^\alpha$, and hence, $f \in {}_R\langle Mon(A)_p\}$. The linear independence of $Mon(A)_p$ it follows from fact that $Mon(A)_p \subseteq Mon(A)$ and $Mon(A)$ is linearly independent.
(iii) Let x^α, $x^\beta \in Mon(A)$ such that $0 \neq \eta(x^\alpha) = \eta(x^\beta) \in Gr(A)_p = F_p(A)/F_{p-1}(A)$; this last implies that $x^\alpha - x^\beta \in F_{p-1}(A)$, i.e., $x^\alpha - x^\beta \in {}_R\langle Mon(A)_{p-1}\}$. Now, since x^α, $x^\beta \notin F_{p-1}(A)$, we have that $x^\alpha - x^\beta = 0$, namely $x^\alpha = x^\beta$. The other implication is a straightforward reasoning. □

Lemma A.1.3. *If x^α, $x^\beta \in Mon(A)$, with deg(x^α) $= p$ and deg(x^β) $= q$, then $\eta(x^\alpha x^\beta) = \eta(x^\alpha)\eta(x^\beta)$. In particular, if $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in F_p(A) - F_{p-1}(A)$, necessarily $\eta(x^\alpha) \neq 0$ and $\eta(x^\alpha) = \eta(x_1)^{\alpha_1} \cdots \eta(x_n)^{\alpha_n} \in Gr(A)_p$.*

Proof. In fact, $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta} \in R$ is left invertible and $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta| = p + q$ (see Theorem 1.2.2), whence $0 \neq \eta(x^\alpha x^\beta) = \overline{c_{\alpha,\beta} x^{\alpha+\beta}} = c_{\alpha,\beta}\overline{x^{\alpha+\beta}} \in F_{p+q}(A)/F_{p+q-1}(A)$. Furthermore, $0 \neq \eta(x^\alpha)\eta(x^\beta) = \overline{x^\alpha}\,\overline{x^\beta} = \overline{x^\alpha x^\beta} \in F_{p+q}(A)/F_{p+q-1}(A)$; but $x^\alpha x^\beta - c_{\alpha,\beta} x^{\alpha+\beta} = p_{\alpha,\beta} \in F_{p+q-1}(A)$, then $\overline{x^\alpha x^\beta} = \overline{c_{\alpha,\beta} x^{\alpha+\beta}}$, i.e., $\eta(x^\alpha x^\beta) = \eta(x^\alpha)\eta(x^\beta)$. □

Proposition A.1.4. *Let A, $Mon(A)$ and $\{F_p(A)\}$ as before, then $\eta(Mon(A)_p) := \{\eta(x^\alpha) \mid x^\alpha \in Mon(A)_p\}$, forms a R-basis of $Gr(A)_p$ for each $p \in \mathbb{N}$. Moreover, $\eta(Mon(A)) := \{\eta(x^\alpha) \mid x^\alpha \in Mon(A)\}$ is a R-basis for $Gr(A)$.*

Proof. Let $f \in F_p(A) \setminus F_{p-1}(A)$, then $f = \sum_{|\alpha| \leq p} c_\alpha x^\alpha$ with $c_\alpha \in R \setminus \{0\}$ y $\eta(f) = \sum_{|\alpha|=p} c_\alpha \eta(x^\alpha) \neq 0$. By Lemma A.1.3, $\eta(x^\alpha) \in Gr(A)_p$ for every α with $|\alpha| = p$, thus $\eta(Mon(A)_p)$ is a generating set for the left R-module $Gr(A)_p$. Now, suppose that there are $\lambda_i \in R$ such that $0 = \sum \lambda_i \eta(x^{\alpha_i}) \in Gr(A)_p$ for certain $x^{\alpha_i} \in Mon(A)_p$, then $\sum \lambda_i x^{\alpha_i} \in F_{p-1}(A)$; but $\deg(x^{\alpha_i}) = p$ for each i and $Mon(A)$ is a R-basis filtered strictly, hence $\lambda_i = 0$ for every i. □

The above preliminaries enable us to establish one of the main theorems of this section.

Theorem A.1.5. *If $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a (bijective) skew PBW extension of ring R, then $Gr(A)$ is a (bijective) quasi-commutative skew PBW extension of R.*

Proof. We must show that in $Gr(A)$ there exist nonzero elements $y_1, \ldots, y_n$ satisfying the conditions in (a) from Definition 1.1.4. Define $y_i := \eta(x_i)$ for each $1 \leq i \leq n$; by Proposition A.1.4 we have that

$$\eta(Mon(A)) := \{\eta(x^\alpha) = \eta(x_1)^{\alpha_1} \cdots \eta(x_n)^{\alpha_n} \mid x^\alpha \in Mon(A)\}$$

is a R-basis for $Gr(A)$. Now, given $r \in R \setminus \{0\}$, there is $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i = p_{i,r} \in R$; from last equality it follows that $\eta(x_i r) - \eta(c_{i,r} x_i) = \eta(p_{i,r}) = 0$, i.e., $\eta(x_i r) = \eta(c_{i,r} x_i) = c_{i,r} \eta(x_i)$; but $x_i r \neq 0$ for any nonzero $r \in R$ because $Mon(A)$ is a R-basis for the right R-module A_R (see Proposition 1.2.4), thus $\eta(x_i r) = \eta(x_i)\eta(r) = \eta(x_i) r$, and consequently $\eta(x_i) r = c_{i,r} \eta(x_i)$. On the other hand, given $i, j \in \{1, \ldots n\}$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j = p_{i,j} \in R + Rx_1 + \cdots + Rx_n$; hence we have that $\eta(x_j x_i) = \eta(c_{i,j} x_i x_j) = c_{i,j} \eta(x_i)\eta(x_j)$, and by Lemma A.1.3 $\eta(x_j x_i) = \eta(x_j)\eta(x_i)$, therefore $\eta(x_j)\eta(x_i) = c_{i,j} \eta(x_i)\eta(x_j)$. Since the $c_{i,r}$'s and $c_{i,j}$'s that define to $Gr(A)$ as a quasi-commutative skew PBW extension are the same that define A as a skew PBW extension of R, then the bijectivity of A implies the of $Gr(A)$. □

Remark A.1.6. The last theorem will allow us to establish a back and forth between Gröbner bases theory for A and $Gr(A)$. As we will show, the existence of one theory implies the existence of the other.

In the following, the set $\eta(Mon(A))$ will be denoted by $Mon(Gr(A))$. Thus, $Mon(Gr(A))$ is the basis for the left R-module $Gr(A)$ composed by the standard monomials in the variables $\eta(x_1), \ldots, \eta(x_n)$.

Proposition A.1.7. *If $\succeq$ is a monomial order on $Mon(A)$, then relation $\succeq_{gr}$ defined over $Mon(Gr(A))$ by*

$$\eta(x^\alpha) \succeq_{gr} \eta(x^\beta) \Leftrightarrow x^\alpha \succeq x^\beta \tag{A.1.1}$$

is a monomial order for $Mon(Gr(A))$.

Proof. We will show that $\succeq_{gr}$ satisfies the conditions in the Definition 5.1.1: (i) Let $\eta(x^\alpha)$, $\eta(x^\beta)$, $\eta(x^\lambda)$, $\eta(x^\gamma) \in Mon(Gr(A))$ and suppose that $\eta(x^\beta) \succeq_{gr} \eta(x^\alpha)$, then,

$$lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)) \Leftrightarrow lm(\eta(x^\gamma x^\beta x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma x^\alpha x^\lambda)).$$

But, $\eta(lm(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma x^\beta x^\lambda))$ for all γ, β, $\lambda \in \mathbb{N}^n$: indeed, $\eta(x^\gamma x^\beta x^\lambda) = \overline{cx^{\gamma+\beta+\lambda}}$ $= c\eta(x^{\gamma+\beta+\lambda})$, where $c := c_{\gamma,\beta}c_{\gamma+\beta,\lambda}$ (see Remark 1.2.3). Therefore,

$$lm(\eta(x^\gamma x^\beta x^\lambda)) = lm(c\eta(x^{\gamma+\beta+\lambda})) = \eta(x^{\gamma+\beta+\lambda}) = \eta(lm(x^\gamma x^\beta x^\lambda)).$$

Since $\succeq$ is a order monomial on $Mon(A)$, it has $lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda)$, so that $\eta(lm(x^\gamma x^\beta x^\lambda)) \succeq_{gr} \eta(lm(x^\gamma x^\alpha x^\lambda))$, i.e., $lm(\eta(x^\gamma x^\beta x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma x^\alpha x^\lambda))$. In consequence, $lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda))$.
The conditions (ii) y (iii) in Definition 5.1.1 are easily verifiable. □

Lemma A.1.8. *Let A as before, $\succeq$ a monomial order on $Mon(A)$ and $f \in A$ an arbitrary element. Then,*

(i) *$f \in F_p(A)$ if and only if $deg(f) \leq p$. Further, $f \in F_p(A) - F_{p-1}(A)$ if, and only, if $deg(f) = p$.*

(ii) *$\eta(lm(f)) = lm(\eta(f))$.*

Proof. (i) It follows from the definition of $F_p(A)$ and Lemma A.1.2.
(ii) Let f be a nonzero polynomial in A; there exists $p \in \mathbb{N}$ such that $f \in F_p(A) - F_{p-1}(A)$. Let $f = \sum_{i=1}^n \lambda_i x^{\alpha_i}$, with $\lambda_i \in R \setminus \{0\}$ y $x^{\alpha_i} \in Mon(A)_p$, $1 \leq i \leq n$, where $x^{\alpha_1} \succ x^{\alpha_2} \succ \cdots \succ x^{\alpha_n}$. Hence, $lm(f) = x^{\alpha_1}$, $\deg(f) = p$ and $\eta(f) = \sum_{|\alpha_i|=p} \lambda_i \eta(x^{\alpha_i})$. From the definition of $\succeq_{gr}$, we have that $lm(\eta(f)) = \eta(x^{\alpha_1}) = \eta(lm(f))$. □

We will prove that the reciprocal of the Proposition A.1.7 also holds.

Proposition A.1.9. *Let A and $Gr(A)$ as before. If $\succeq_{gr}$ is a monomial order on $Mon(Gr(A))$, then the relation $\succeq$ defined as*

$$x^\alpha \succeq x^\beta \Leftrightarrow \eta(x^\alpha) \succeq_{gr} \eta(x^\beta) \tag{A.1.2}$$

is a monomial order over $Mon(A)$.

Proof. Since $\succeq_{gr}$ is a well order, from (A.1.2) it follows that $\succeq$ is a well order also. Now, we show that $\succeq$ is a monomial order: indeed, let x^α, x^β, x^γ, $x^\lambda \in Mon(A)$ and suppose that $x^\beta \succeq x^\alpha$, so:

$$\begin{cases} \eta(x^\beta) \succeq \eta(x^\alpha) \\ \eta(lm(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \\ \eta(lm(x^\gamma x^\alpha x^\lambda)) = lm(\eta(x^\gamma x^\alpha x^\lambda)) = lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)) \\ lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)), \end{cases}$$

and hence, $lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda)$. Clearly $x^\alpha \succeq 1$ for all $x^\alpha \in Mon(A)$, and $\succeq$ is degree compatible. □

Definition A.1.10. *Let I be a left (right or two side) ideal of A. The graduation of I (or the associated graded ideal to I) is defined as $G(I) := \oplus_p Gr(I)_{p\in\mathbb{N}}$, where $Gr(I)_p := I \cap F_p(A)/I \cap F_{p-1}(A) \cong (I + F_{p-1}(A)) \cap F_p(A)/F_{p-1}(A)$, for each $p \in \mathbb{N}$; (e.g., see [97]).*

The following theorem shows how calculate Gröbner basis for I, if we have one for $Gr(I)$.

Theorem A.1.11. *Let A, $Gr(A)$, $Mon(A)$ and $Mon(Gr(A))$ as before, $\succeq$ a monomial order over $Mon(A)$, and I a left ideal of A. If $\overline{\mathcal{G}} = \{G_j\}_{j\in J}$ is a Gröbner basis for $Gr(I)$, with respect to the monomial order $\succeq_{gr}$, and such basis is formed by homogeneous elements, then $\mathcal{G} := \{g_j\}_{j\in J}$ is a Gröbner basis for I, where $g_j \in I$ is a selected polynomial with property that $\eta(g_j) = G_j$ for each $j \in J$.*

Proof. Let $0 \neq f \in I \cap F_p(A) \setminus F_{p-1}(A)$; we shall show that the condition (iii) in the Theorem 5.3.2 is satisfied: let $\overline{f} := \eta(f)$, then $0 \neq \overline{f} \in G(I)_p$. Since $\overline{\mathcal{G}}$ is a Gröbner basis of $G(I)$, there exist $G_1, \ldots, G_t \in \overline{\mathcal{G}}$ such that $lm(G_j) \mid lm(\overline{f})$ for each $1 \leq j \leq t$ and $lc(\overline{f}) \in \langle \sigma^{\alpha_1}(lc(G_1))c_{\alpha_1,G_1}, \ldots, \sigma^{\alpha_t}(lc(G_t))c_{\alpha_t,G_t}\}$, with $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + exp(lm(G_j)) = exp(lm(\overline{f})) = exp(lm(f)) = p$ and c_{α_j,G_j} is the coefficient determined by the product $\eta(x)^{\alpha_j} lm(G_j)$ in $Gr(A)$, for $1 \leq j \leq t$. From this last it follows that $lm(\eta(x)^{\alpha_j} lm(G_j)) = lm(\overline{f})$; but $lm(\eta(x)^{\alpha_j} lm(G_j)) = lm(\eta(x^{\alpha_j} x^{\beta_j}))$, where $x^{\beta_j} := lm(g_j)$ y $g_j \in I \cap F_p(A)$ is such that $\eta(g_j) = G_j$. From Lemma A.1.8 we get that $lm(\eta(x^{\alpha_j} x^{\beta_j})) = \eta(lm(x^{\alpha_j} x^{\beta_j})) \in F(A)_p/F(A)_{p-1}$, so that $\eta(lm(x^{\alpha_j} x^{\beta_j})) = lm(\overline{f}) = \eta(lm(f))$. The latter implies that $lm(x^{\alpha_j} x^{\beta_j}) - lm(f) \in F_{p-1}(A)$ and, therefore, $lm(x^{\alpha_j} x^{\beta_j}) = lm(f)$, i.e., $lm(g_j) \mid lm(f)$ for each $1 \leq j \leq t$. Further, $lc(h) = lc(\eta(h))$ for all $h \in A$, then $lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \ldots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}$. $\square$

In this way, a Gröbner basis of $Gr(I)$ can be transfer to a Gröbner basis of I. In particular, from a Gröbner basis of $Gr(I)$ we can get a set of generators for I. Reciprocally, when we need obtain a generating set of $Gr(I)$ from one of $I = \langle f_1, \ldots, f_r\}$, we could think that $Gr(I) = \langle \eta(f_1), \ldots, \eta(f_r)\}$. Nevertheless, this affirmation in general is not true: in fact, let $A = A_2(\Bbbk)$, the second Weyl algebra, i.e., $A = \Bbbk[x_1, x_2][y_1, \frac{\partial}{\partial x_1}][y_2, \frac{\partial}{\partial x_2}]$ with its associated standard filtration, and consider the left ideal I generated by $f_1 = x_1 y_1$ and $f_2 = x_2 y_1^2 - y_1$. Note that $x_1 \in I$, since $x_1 = (t_2 x_1^2 - x_1) f_1 - (t_1 x_1 + 2) f_2$, but $\eta(x_1) \notin \langle \eta(f_1), \eta(f_2)\}$, where $\eta(f_1) = \eta(t_1)\eta(x_1) \in Gr(I)_1$ and $\eta(f_2) = \eta(t_2)\eta(x_1)^2 \in Gr(I)_2$ (see [84]). However, if $G = \{f_1, \ldots, f_r\}$ is a Gröbner basis for I, we will show that $\eta(G) = \{\eta(f_1), \ldots, \eta(f_r)\}$ is a Gröbner basis for $Gr(I)$ and, from this we will have a generating set for $Gr(I)$.

Theorem A.1.12. *With notation as above, let $\mathcal{G} = \{g_i\}_{i\in J}$ be a Gröbner basis for a left ideal I of A. Then $\overline{\mathcal{G}} = \{\eta(g_i)\}_{i\in J}$ is a Gröbner basis of $Gr(I)$ consisting of homogeneous elements.*

Proof. Since $Gr(I)$ is a homogeneous ideal, it suffices to show that every nonzero homogeneous element $F \in Gr(I)$ satisfies the condition (iii) in the Theorem 5.3.2. Let $0 \neq F \in Gr(I)_p$, then $F = \eta(f)$ for some $f \in I \cap F_p(A) - I \cap F_{p-1}(A)$ and there exist $g_1, \ldots, g_t \in \mathcal{G}$ with the property that $lm(g_i) \mid lm(f)$ and $lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \ldots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}$, where $\alpha_i \in \mathbb{N}^n$ is such that $\alpha_i + \exp(g_i) = \exp(f)$ for each $1 \leq i \leq t$. By Lemma A.1.8 we have that $\eta(lm(f)) = lm(\eta(f)) = lm(F)$, then $lm(\eta(g_i)) \mid lm(F)$. Further, since $lc(f) = lc(\eta(f)) = lc(F)$, it follows that $lc(F) \in \langle \sigma^{\alpha_1}(lc(\eta(g_1)))c_{\alpha_1,\eta(g_1)}, \ldots, \sigma^{\alpha_t}(lc(\eta(g_t)))c_{\alpha_t,\eta(g_t)}\}$ and, in consequence $\overline{\mathcal{G}}$ is a Gröbner basis for $Gr(I)$. $\square$

A.2 For modules

Similar results to those presented in the previous section can be proved in the case of modules. For this, let M be a submodule of the free module A^m, $m \geq 1$, where A is a skew PBW extension of a ring R. Define the following collection of subsets of M:

$$F_p(M) := \{\boldsymbol{f} \in M \mid \deg(\boldsymbol{f}) \leq p\}. \tag{A.2.1}$$

It is not difficult to show that the collection $\{F_p(M)\}_{p\geq 0}$ given in (A.2.1) is a filtration for M, called the *natural filtration* on M. With this filtration we can define the graded module associated to M, which will be denoted by $Gr(M)$, in the following way: $Gr(M) := \oplus_{p\geq 0} F_p(M)/F_{p-1}(M)$; if $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$, then $\boldsymbol{f}$ is said to have degree p. Thus, we may associate to $\boldsymbol{f}$ its *principal symbol* $\eta(\boldsymbol{f}) := \boldsymbol{f} + F_{p-1}(M) \in G_p(M) = F_p(M)/F_{p-1}(M)$. The $Gr(A)$-structure is given by, via distributive laws, the following multiplication:

$$\eta(r)\eta(\boldsymbol{f}) := \begin{cases} \eta(r\boldsymbol{f}), & \text{if } r\boldsymbol{f} \notin F_{i+j-1}(M), \\ 0, & \text{otherwise} \end{cases} \tag{A.2.2}$$

where $r \in F_i(A) - F_{i-1}(A)$ and $\boldsymbol{f} \in F_j(M) - F_{j-1}(M)$.

Notice that any filtration $\{F_p(M)\}_{p\in\mathbb{Z}}$ on M defines an order function $v : M \to \mathbb{Z}$ in the following way:

$$v(\boldsymbol{f}) := \begin{cases} p, & \text{if } \boldsymbol{f} \in F_p(M) - F_{p-1}(M), \\ -\infty, & \text{if } \boldsymbol{f} \in \cap_{p\in\mathbb{Z}} F_p(M). \end{cases}$$

Lemma A.2.1. *Let A, M and $\{F_p(M)\}_p$ as above. Then for each $\boldsymbol{f} \in M$, deg$(\boldsymbol{f}) = v(\boldsymbol{f})$.*

Proof. From definition of $\{F_p(M)\}_{p\geq 0}$, it follows that if $\mathbf{0} \neq \boldsymbol{f} \in M$, then there exists $p \in \mathbb{N}$ such that $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ and, therefore, $v(\boldsymbol{f}) = p$. But, if $f \in F_p(M) - F_{p-1}(M)$, then $\deg(\boldsymbol{f}) = p$ and we obtain the equality. □

We have a version of the Proposition A.1.7 for module case.

Proposition A.2.2. *If $>$ is a monomial order on $Mon(A^m)$, then relation $>_{gr}$ defined over $Mon(Gr(A)^m)$ by*

$$\eta(\boldsymbol{X}) >_{gr} \eta(\boldsymbol{Y}) \Leftrightarrow \boldsymbol{X} > \boldsymbol{Y} \tag{A.2.3}$$

is a monomial order for $Mon(Gr(A)^m)$.

Proof. We will show that $\succeq_{gr}$ satisfies the conditions in the Definition 5.5.2: to begin, note that $>_{gr}$ is a total order because $>$ it is. Now, to prove (i) we must show that $lm(\eta(x^\beta)\eta(x^\alpha))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i$ for every $\overline{\boldsymbol{X}} = \eta(x^\alpha)\overline{\boldsymbol{e}}_i \in Mon(Gr(A)^m)$ and $\eta(x^\beta) \in Mon(Gr(A))$. It can be noted that,

$$lm(\eta(x^\beta)\eta(x^\alpha))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i \Leftrightarrow \eta(lm(x^\beta x^\alpha))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i.$$

Since $>$ is a monomial order on $Mon(A^m)$, we have that $lm(x^\beta x^\alpha)\boldsymbol{e}_i \geq x^\alpha \boldsymbol{e}_i$ and, from (A.2.3) it follows that $\eta(lm(x^\beta x^\alpha))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i$. So, $lm(\eta(x^\beta)\eta(x^\alpha))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i$.
For (ii), let $\overline{\boldsymbol{Y}} = \eta(x^\beta)\overline{\boldsymbol{e}}_j$ and $\overline{\boldsymbol{X}} = \eta(x^\alpha)\overline{\boldsymbol{e}}_i$ monomials in $Mon(Gr(A)^m)$ such that $\overline{\boldsymbol{Y}} \geq_{gr} \overline{\boldsymbol{X}}$. Given $\eta(x^\gamma) \in Mon(Gr(A))$, we have

$$lm(\eta(x^\gamma)\eta(x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha))\overline{\boldsymbol{e}}_i \Leftrightarrow \eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} \eta(lm(x^\gamma x^\alpha))\overline{\boldsymbol{e}}_i.$$

In $Mon(A)$ we get that $lm(x^\gamma x^\beta)\boldsymbol{e}_j \geq lm(x^\gamma x^\alpha)\boldsymbol{e}_i$ and, once again, from (A.2.3) it follows that $\eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} \eta(lm(x^\gamma x^\alpha))\overline{\boldsymbol{e}}_i$.
Finally is easily verifiable that $\geq_{gr}$ is degree compatible. □

Lemma A.2.3. *Let A, M, $Gr(A)$, $Gr(M)$ and $<$ as before, and consider an arbitrary element $\boldsymbol{f} \in M$. Then,*

(i) *$\boldsymbol{f} \in F_p(M)$ if, and only if, $deg(\boldsymbol{f}) \leq p$. Further, $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ if, and only, if $deg(\boldsymbol{f}) = p$.*

(ii) *$\eta(lm(\boldsymbol{f})) = lm(\eta(\boldsymbol{f}))$.*

Proof. (i) It follows from the definition of $F_p(M)$ and Lemma A.2.1.
(ii) Let $\boldsymbol{f}$ be a nonzero vector in M, then there exists $p \in \mathbb{N}$ such that $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$. Thus, $\boldsymbol{f} = \sum_{i=1}^{l} \lambda_i \boldsymbol{X}_i$ with $\lambda_i \in R \setminus \{0\}$, $\boldsymbol{X}_i \in Mon(A^m)$ where $\deg(\boldsymbol{X}_i) \leq p$ for each $1 \leq i \leq l$, and $\boldsymbol{X}_1 > \boldsymbol{X}_2 > \cdots > \boldsymbol{X}_l$. Whence, $lm(\boldsymbol{f}) = \boldsymbol{X}_1$ and since $\deg(\boldsymbol{f}) = p$ and $\eta(\boldsymbol{f}) = \sum_{|\exp(\boldsymbol{X}_i)|=p} \lambda_i \eta(\boldsymbol{X}_i)$, from the definition given for $\geq_{gr}$, we have that $lm(\eta(\boldsymbol{f})) = \eta(\boldsymbol{X}_1) = \eta(lm(\boldsymbol{f}))$. □

The conversely of Proposition A.2.2 is also true, as will be shown below.

Proposition A.2.4. *With the same notation used so far, if $\geq_{gr}$ a monomial order on $Mon(Gr(A)^m)$, then $\geq$ defined as*

$$\boldsymbol{X} \geq \boldsymbol{Y} \Leftrightarrow \eta(\boldsymbol{X}) \geq_{gr} \eta(\boldsymbol{Y}) \tag{A.2.4}$$

is a monomial order over $Mon(A^m)$.

Proof. Since $\geq_{gr}$ is a total order, from (A.2.4) it follows that $\geq$ is a total order also. Now, we show that $\geq$ is a monomial order: indeed, let $x^\beta \in Mon(A)$ and $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i$ an element in $Mon(A^m)$; we must to show $lm(x^\gamma x^\beta)\boldsymbol{e}_i \geq x^\alpha \boldsymbol{e}_i$ for all $x^\gamma \in Mon(A)$; however

$$\eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i \Leftrightarrow lm(\eta(x^\gamma)\eta(x^\beta))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^\alpha)\overline{\boldsymbol{e}}_i$$

and since $\geq_{gr}$ is a monomial order, this last inequality is true. From (A.2.4) it follows that $lm(x^\gamma x^\beta)\boldsymbol{e}_i \geq x^\alpha \boldsymbol{e}_i$, as we had to show. Now, if $\boldsymbol{Y} = x^\beta \boldsymbol{e}_j$ and $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i$ are monomials in $Mon(A^m)$ such that $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\eta(\boldsymbol{Y}) \geq_{gr} \eta(\boldsymbol{X})$. Thus, given $\eta(x^\gamma) \in Mon(Gr(A))$ we have that

$$lm(\eta(x^\gamma)\eta(x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha))\overline{\boldsymbol{e}}_i$$

i.e.,

$$\eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} \eta(lm(x^\gamma x^\alpha))\overline{\boldsymbol{e}}_i.$$

This implies that $lm(x^\gamma x^\beta)\boldsymbol{e}_j \geq lm(x^\gamma x^\alpha)\boldsymbol{e}_i$. Finally, it is easy to prove that $\geq$ is degree compatible. $\square$

We are ready to prove the main theorem of this last section.

Theorem A.2.5. *Let A, $Gr(A)$, $Mon(A)$ and $Mon(Gr(A))$ be as before, $\geq$ a monomial order over $Mon(A^m)$, and M a nonzero submodule of A^m. The following statements hold:*

(i) *If $\overline{\mathcal{G}} = \{\boldsymbol{G}_j\}_{j\in J}$ is a Gröbner basis for $Gr(M)$, with respect to the monomial order $\geq_{gr}$, and such basis is formed by homogeneous elements, then $\mathcal{G} := \{\boldsymbol{g}_j\}_{j\in J}$ is a Gröbner basis for M, where $\boldsymbol{g}_j \in M$ is a selected vector with the property that $\eta(\boldsymbol{g}_j) = \boldsymbol{G}_j$ for each $j \in J$.*

(ii) *If $\mathcal{G} = \{\boldsymbol{g}_i\}_{i\in J}$ is a Gröbner basis for M, then $\overline{\mathcal{G}} = \{\eta(\boldsymbol{g}_i)\}_{i\in J}$ is a Gröbner basis of $Gr(M)$ consisting of homogeneous elements.*

Proof. (i) Let $\boldsymbol{0} \neq \boldsymbol{f} \in F_p(M) \setminus F_{p-1}(M)$; we shall show that the condition (iii) in Theorem 5.5.13 is satisfied (see also [58], Theorem 26): let $\overline{\boldsymbol{f}} := \eta(\boldsymbol{f})$, then $\boldsymbol{0} \neq \overline{\boldsymbol{f}} \in G(M)_p$. Since $\overline{\mathcal{G}}$ is a Gröbner basis of $G(M)$, there exist $\boldsymbol{G}_1, \dots, \boldsymbol{G}_t \in \overline{\mathcal{G}}$ such that $lm(\boldsymbol{G}_j) \mid lm(\overline{\boldsymbol{f}})$ for each $1 \leq j \leq t$ and $lc(\overline{\boldsymbol{f}}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{G}_1))c_{\alpha_1,\boldsymbol{G}_1}, \dots, \sigma^{\alpha_t}(lc(\boldsymbol{G}_t))c_{\alpha_t,\boldsymbol{G}_t}\}$, with $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(lm(\boldsymbol{G}_j)) = \exp(lm(\overline{\boldsymbol{f}})) = p$ and $c_{\alpha_j,\boldsymbol{G}_j}$ is the coefficient determined by the product $\eta(x)^{\alpha_j} lm(\boldsymbol{G}_j)$ in $Gr(M)$, for $1 \leq j \leq t$. But, $\exp(lm(\overline{\boldsymbol{f}})) = \exp(lm(\boldsymbol{f}))$, thus of the above mentioned follows that $lm(\eta(x^{\alpha_j})lm(\boldsymbol{G}_j)) = lm(\overline{\boldsymbol{f}})$; note that $lm(\eta(x^{\alpha_j})lm(\boldsymbol{G}_j)) = lm(\eta(x^{\alpha_j}\boldsymbol{X}_j))$, where $\boldsymbol{X} := lm(\boldsymbol{g}_j)$ and $\boldsymbol{g}_j \in F_p(M)$ is such that $\eta(\boldsymbol{g}_j) = \boldsymbol{G}_j$. From Lemma A.2.3 we get that $lm(\eta(x^{\alpha_j}\boldsymbol{X})) = \eta(lm(x^{\alpha_j}\boldsymbol{X})) \in F(M)_p/F(M)_{p-1}$, so that $\eta(lm(x^{\alpha_j}\boldsymbol{X})) = lm(\overline{\boldsymbol{f}}) = \eta(lm(\boldsymbol{f}))$. The latter implies that $lm(x^{\alpha_j}\boldsymbol{X}) - lm(\boldsymbol{f}) \in F_{p-1}(M)$ and, therefore, $lm(x^{\alpha_j}\boldsymbol{X}) = lm(\boldsymbol{f})$, i.e., $lm(\boldsymbol{g}_j) \mid lm(\boldsymbol{f})$ for each $1 \leq j \leq t$. Further, $lc(h) = lc(\eta(\boldsymbol{h}))$ for all $\boldsymbol{h} \in A^m$, then $lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\boldsymbol{g}_1}, \dots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\boldsymbol{g}_t}\}$.

(ii) Since $Gr(M)$ is a graded module, it suffices to show that every nonzero homogeneous element $\boldsymbol{F} \in Gr(M)$ satisfies the condition (iii) in the Theorem 5.5.13. Suppose that $\boldsymbol{F} \in Gr(M)_p$; then, $\boldsymbol{F} = \eta(\boldsymbol{f})$ for some $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ and there exist $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in \mathcal{G}$ with the property that $lm(\boldsymbol{g}_i) \mid lm(\boldsymbol{f})$ and $lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\boldsymbol{g}_1}, \dots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\boldsymbol{g}_t}\}$, where $\alpha_i \in \mathbb{N}^n$ is such that $\alpha_i + \exp(\boldsymbol{f}_i) = \exp(\boldsymbol{f})$ for each $1 \leq i \leq t$. By Lemma A.2.3 we have that $lm(\boldsymbol{f}) = lm(\eta(\boldsymbol{f})) = lm(\boldsymbol{F})$, then $lm(\eta(\boldsymbol{g}_i)) \mid lm(\boldsymbol{F})$ and, since $lc(\boldsymbol{f}) = lc(\eta(\boldsymbol{f})) = lc(\boldsymbol{F})$, it follows that $lc(\boldsymbol{F}) \in \langle \sigma^{\alpha_1}(lc(\eta(\boldsymbol{g}_1)))c_{\alpha_1,\eta(\boldsymbol{g}_1)}, \dots, \sigma^{\alpha_t}(lc(\eta(\boldsymbol{g}_t)))c_{\alpha_t,\eta(\boldsymbol{g}_t)}\}$ and, hence, $\overline{\mathcal{G}}$ is a Gröbner basis for $Gr(M)$. $\square$

Future work

Some other tasks closely related to the research of projective modules over skew PBW extensions consist of giving constructive proofs of the following theorems that were established in previous works by using tools of rings and modules and classical homological techniques:

- Serre's theorem about stably free modules: *Let A be a bijective skew PBW extension of a ring R such that R is left (right) Noetherian, left (right) regular and $\mathcal{PSF}$. Then A is $\mathcal{PSF}$.*

  A non-constructive proof of this theorem was given in [83], Corollary 2.8. A constructive proof for the habitual commutative ring of polynomials can be found for example in [78].

- Hilbert's syzygy theorem about the global dimension of bijective skew PBW extensions.

  A non-constructive proof of this theorem was given in [83], Theorem 4.2.

Another problem to be considered is the computation of Ext and Tor for bimodules over bijective skew PBW extensions. In order to do this, it is necessary to construct the theory of two-sided Gröbner bases for bijective skew PBW extensions with some extra conditions. These constructions could be useful for the study of some algebras of recent interest arising in non-commutative algebraic geometry such as Artin-Schelter regular algebras and Calabi-Yau algebras (see [109]).

On the other hand, it would be really important developing a computational package for the calculation of Gröbner bases on bijective skew PBW extensions, besides to be able to perform computations related with the matrix-constructive interpretations of properties as being a projective-free, $\mathcal{PSF}$, Hermite or cancellable ring.

Finally, another field of future investigation is the application in algebraic analysis of theorems, algorithms and Gröbner theory presented in this thesis (see [13] and [25]).

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