UNIVERSIDAD NACIONAL DE COLOMBIA
ESCUELA DE ESTADÍSTICA

Optimal Retention Point Estimation in High Cost Diseases Reinsurance

A Master's Degree Project Submitted by

Jorge Iván Pérez García

In Partial Fulfillment of the Requirements for the Degree of:
Magister en Ciencias Estadística

Advisor:
Norman Diego Giraldo Gómez
M.Sc., Associate Professor, with Tenure
Escuela de Estadística
Universidad Nacional de Colombia

Universidad Nacional de Colombia
Facultad de Ciencias, Escuela de Estadística
Medellín, Colombia
2019

UNIVERSIDAD NACIONAL DE COLOMBIA
ESCUELA DE ESTADÍSTICA

Estimación del Punto de Retención Óptimo en el Caso del Reaseguro para Enfermedades de Alto Costo

Trabajo Final de Maestría Presentado por

Jorge Iván Pérez García

Como Requisito Parcial para Optar el Título de:
Magister en Ciencias Estadística

Asesor:
Norman Diego Giraldo Gómez
M.Sc., Profesor Asociado, con titularidad en
Escuela de Estadística
Universidad Nacional de Colombia

Universidad Nacional de Colombia
Facultad de Ciencias, Escuela de Estadística
Medellín, Colombia
2019

Acknowledgements

My very profound gratitude to my brother Sergio and to my girlfriend Luisa for support me morally and emotionally, because without them it is probable I would not have finished this Master's degree. Special thanks also to my girlfriend for help me in correcting the English wording of this document.

Likewise, I would like to acknowledge to Professor Norman Giraldo and Professor Freddy Hernández, for their ideas, advice, guidance and useful critiques, which helped to keep the progress of this research work.

Finally, I must express my deep gratitude to my parents and to my brother for providing me with unfailing support and continuous encouragement throughout my years of study, because without them it would not have been possible to have reached this accomplishment. Thanks.

Jorge Pérez.

Contents

List of Tables

List of Figures

Abstract

In this Master's degree project, is develops the actuarial valuation methodology of Stop-Loss reinsurance, in order to provide a solution to the problem of calculating the optimal retention point and reinsurance premium, which must have the prepaid medicine companies to cover those high cost procedures that affect their financial health. It should be noted that, for the application of this methodology, we only use the adjustment of the number and claims size, in consequence, no considerations are made about the solvency or probability of ruin of the health-services provider companies.

Despite that reinsurance in prepaid medicine is contemplated in Law 100/93, there are relatively few studies that have been carried out on this subject, and even fewer, the studies carried out from the statistical point of view. For this reason, we implement the methodology Single Loss Approximation (SLA) to estimate the optimal retention point and reinsurance premium, for different levels of relative safety factor, seeking to incorporate recent developments and some alternatives in the adjustment processes.

Keywords: Heavy Tail Distributions, Poisson Mixtures, Spliced Distributions, GAMLSS Distributions, High Cost Diseases, Prepaid Medicine, Stop-Loss Reinsurance, Extreme Values, Optimal Retention Point.

Resumen

En este trabajo final de Maestría, se desarrolla de la metodología de valoración actuarial de reaseguro de excedente de pérdida (Stop-Loss), con el fin de darle solución al problema del cálculo del punto de retención óptimo y prima de reaseguro, que deben tener las compañías de medicina prepagada para cubrir aquellos procedimientos alto costo que afecten su salud financiera. Es de anotar que, para la aplicación de esta metodología, se emplea únicamente el ajuste del número y tamaño de las reclamaciones, y en consecuencia, no se hacen consideraciones sobre la solvencia o probabilidad de ruina de las empresas prestadoras de servicios de salud.

A pesar de que los reaseguros en medicina prepagada están contemplados en la Ley 100/93, son relativamente pocos los estudios que se han realizado sobre este tema, y aún más pocos, los estudios realizados desde el punto de vista estadístico. Por este motivo, se implementa la metodología Single Loss Approximation (SLA) para la estimación del punto de retención óptimo y la prima de reaseguro, para diferentes niveles de factor de seguridad relativa, buscando incorporar desarrollos recientes y algunas alternativas en los procesos de ajuste.

Palabras Clave: Distribuciones de Cola Pesada, Mezclas Poisson, Distribuciones Spliced, Distribuciones GAMLSS, Enfermedades de Alto Costo, Medicina Prepagada, Reaseguro de Excedente de Pérdida, Valores Extremos, Punto de Retención Óptimo.

Introduction

The diseases or procedures known as high-cost or catastrophic (*HCD* onward) are those that given their nature present one or all of the following characteristics: high complexity degree, are long-lasting, significantly deteriorate people's health, have low cost-effectiveness when treated and above all, these generate high costs for insurance companies that must ensure their timely treatment.

The *HCD* can represent great risks to insurers financial health, up to the point of threatening their survival in the market. Due of this, the insurers should look for alternative sources of financing to guarantee the necessary resources to face the high costs generated by the *HCD* and support all the necessary procedures for the proper attention of its users. One of the main alternatives of financing to which the insurers resort is to reinsurance, which is contemplated in Article 19 of Law 1122 of 2007 of Colombia.

Reinsurance contracts are agreements between reinsurers and insurers to avoid financial imbalances because of the *HCD* and thus ensure the financial health of the insurer through risk reducing given a possible loss. In this agreement, the reinsurer decides to accept the transfer part of the risk assumed by the insurer, and covers them once certain threshold or deductible has been overcome, and depending on the agreement, until a certain coverage limit has been exceeded.

In this way, it is possible offer to the insurer better risk management conditions, facilitating the control of problematic risks, such as the individual policies with high severity and low occurrence, managing to protect it against possible large accumulations of individual losses or the possible cumulative loss of a single event, through the fragmentation of such cases in different portions assumed by different reinsurance companies.

In return for the received protection, the insurer pays to the reinsurer(s) an equivalent premium greater than the expected value of the transferred risk. The exchange between the part of the risk retained and the premium paid for the remaining risk, makes of great importance the determination of the optimal retention point.

Another aspect of interest in this work, focuses on Prepaid Medicine Companies (*PMC* onward), which are described in Article 1 of Decree 1486 of 1994 of Colombia, as an "*organized system (...) for the management of medical attention and health services provision (...) by charging a regular price that was previously agreed (...)*". In other words, the *PMC* are entities covered by the law that give users who decide to acquire prepaid medical plans, the assistance and medical procedures that they require, in exchange for a compensation for the services received.

In this sense, the interest of this work is to provide a detailed methodological guide that allows the *PMC* evaluate different aspects of a reinsurance before hiring it, as well established in the Paragraph 4 of Article 162 of Law 100 of 1993 and in the 19 of Law 1122 of 2007 of Colombia, *PMC* have the obligation to contract reinsurance to respond individually or collectively for the risks generated by the *HCD*. See Appendix A for a summary description of the *HCD* legal framework.

Another motivation for this study is because in Colombia, is not wide spread the issue of reinsurance for *HCD*, and this is confirmed by the small number of studies which has been carry out both in academic institutions and in the private sector, where is only possible to highlight the works of Chicaíza and Cabedo (2009) and Girón and Herrera (2015), which despite presenting the estimation of the reinsurance pure premium, they leave a gap in the literature by not developing the issue of optimal retention point.

Given the above and as a summary, the objective of this work will be to find the optimal retention point that an insurer must have to face the risks generated by users suffering from *HCD*, the threshold from which the risk must be transferred to a third party and the premium that must be paid for such protection. This will be done through the employment of a Stop-Loss reinsurance methodology, calculated from the adjustment of *spliced* and GAMLSS distributions.

The document presented here is structured as follows. In the first chapter is made a brief description of the dataset with which the work is carried out. The second chapter describes and performs the adjustment process for the claims number that occur for each *HCD*. In the third chapter the description and adjustment is made for the set of individual costs for each of the *HCD*, through the use of *spliced distributions*.

Subsequently in the fourth chapter is described the aggregate loss distributions theory, the Single Loss Approximation methodology, and the reinsurance theory. Additionally, is carried out the calculation of the risk measures, namely, Value at Risk, Tail Value at Risk, and Expected Shorfall, and is performed the calculation of the optimal retention point for each *HCD*.

Moreover in the fifth chapter, is performed the individual costs adjustment for each *HCD* by means of GAMLSS distributions, and are presented the associated risk measures and optimal retention points for each *HCD*. In the sixth chapter, is present a comparison between the results obtained through the use of spliced and

GAMLSS distributions. Finally, we present the conclusions associated with the results obtained in the work.

As additional topics that may be of interest to readers, we present the Appendix A which introduces the legal framework associated with *HCD*, where it is explained which are the *HCD*, the included treatments and the high-cost services covered by the law. In Appendix B we present all the codes used in this work, which can be found next to the databases, in the following URL: `https://github.com/jiperezga/Masters-degree-project`

Chapter 1

Data Analysis

Table of Contents

In this work, are used two *HCD* datasets dating from 1988 to 1993, namely, hospitalization and general surgery services, where, for both datasets, we have the record of the year and the total cost of each given intervention, in millions pesos, as shown in the Table 1.1. This table is made using the `kable` function of the library `kableExtra`(2018). See Code 1 in Appendix B.

Table 1.1: Header of dataset

Hospitalization		General Surgery	
year	cost	year	cost
1988	19.433239	1988	3.0887698
1988	6.456504	1988	13.0053467
1988	83.058647	1988	8.7786090
1988	5.329266	1988	8.7786090
1988	7.992761	1988	2.1133688
1988	2.043465	1988	0.6502673

For hospitalization services there are a total of 2328 records, while for general surgery services there is a total of 454 records. These records, represent the number and claims size of each medical service between the studies period and are represented in graphic form in the Figures 1.1 and 1.2 by means of the `pareto.chart` function of the library `qcc` (2004) and later in the Tables 1.2 and 1.3 by means of the `kable` function of the library `kableExtra` (2018). See Code 2 and Code 3 in Appendix B.

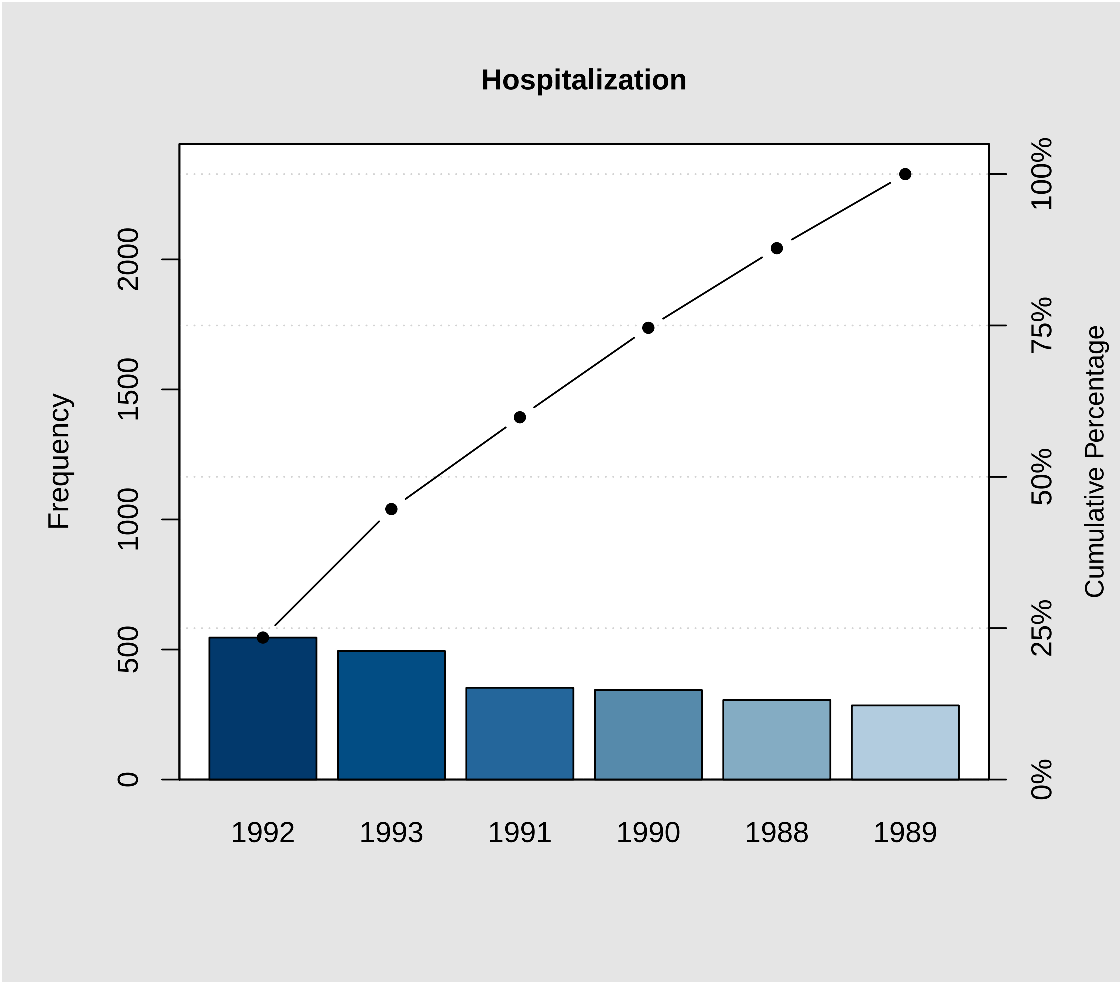

Figure 1.1: Pareto chart for claims distribution in hospitalization

Table 1.2: Distribution records hospitalization per year

	Frequency	Cum.Freq.	Percentage	Cum.Percent.
1992	546	546	23.45361	23.45361
1993	494	1040	21.21993	44.67354
1991	353	1393	15.16323	59.83677
1990	344	1737	14.77663	74.61340
1988	306	2043	13.14433	87.75773
1989	285	2328	12.24227	100.00000

In the Figure 1.1 and Table 1.2 we appreciate that in 1992 and 1993 are the years in which the highest claims number occur with a 23.45% and 21.21%, respectively. While the smallest claims number is observed in the years 1988 and 1989, with a 13.14% and 12.24%, respectively.

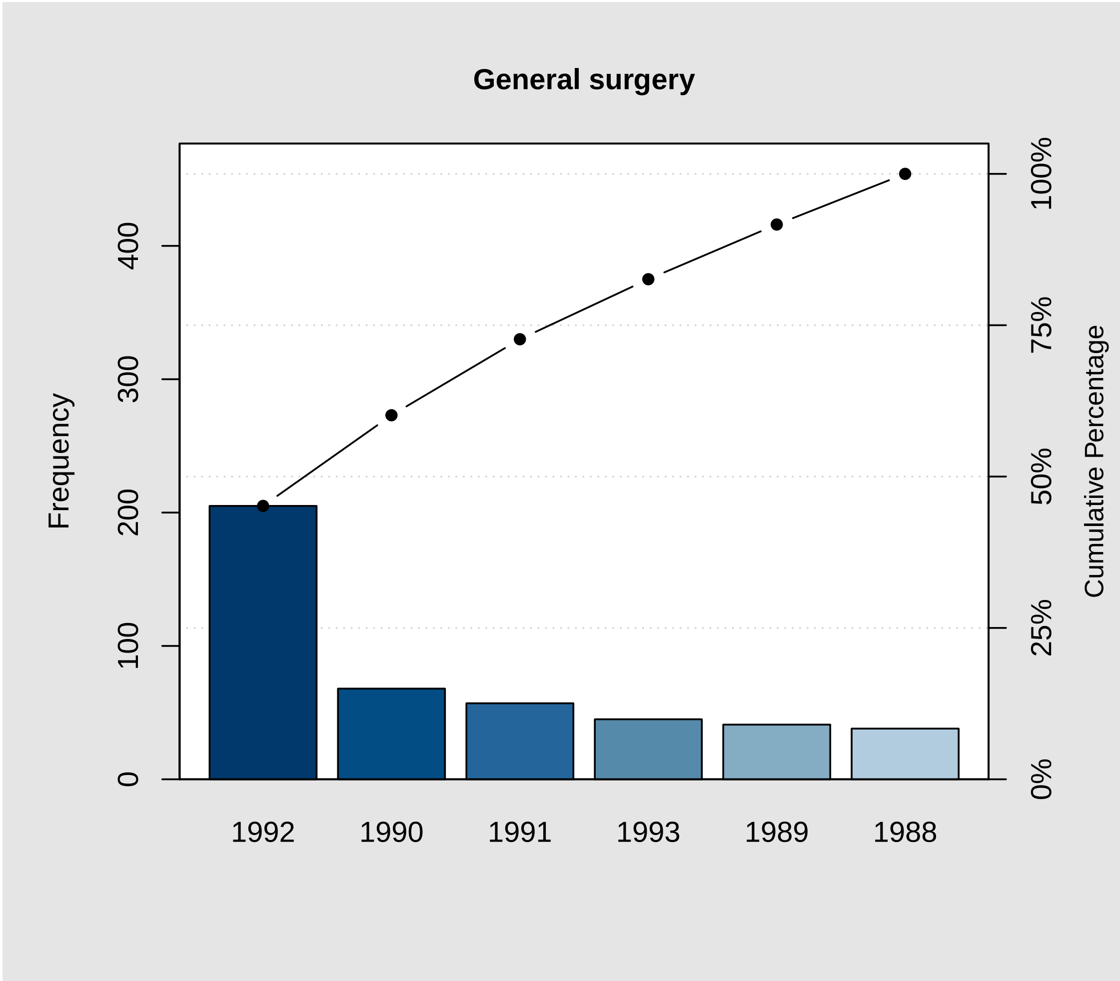

Figure 1.2: Pareto chart for claims distribution in general surgery

Table 1.3: Distribution records general surgery per year

	Frequency	Cum.Freq.	Percentage	Cum.Percent.
1992	205	205	45.154185	45.15418
1990	68	273	14.977974	60.13216
1991	57	330	12.555066	72.68722
1993	45	375	9.911894	82.59912
1989	41	416	9.030837	91.62996
1988	38	454	8.370044	100.00000

In the same way, in the Figure 1.2 and in the Table 1.3 is appreciated that the year in which occurs the largest claims number for general surgery services is 1992, with a percentage of 45.15% of the total data, while the years in which happen the lowest claims number are 1988 and 1989, each with less than 10% of the total frequencies.

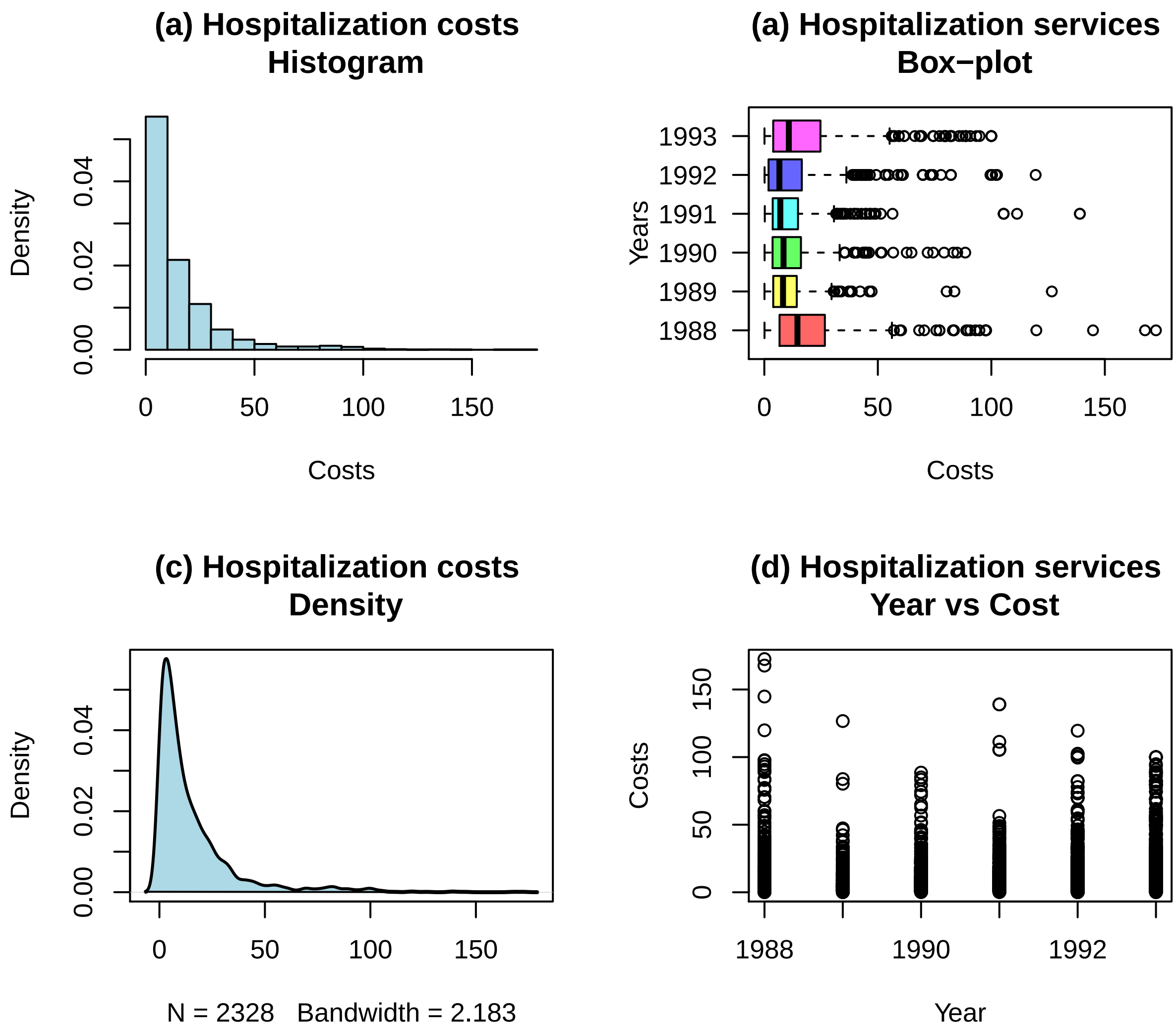

Figure 1.3: Histogram, Box-plot, Density and Scatterplot for hospitalization costs

To obtain a preliminary view of the individual costs of the two medical services, it is presented in graphic form the individual costs in million pesos, in order to have a notion of their distributions, the tails severity level and the amount of extreme observations that each one possesses. See Code 4 and Code 5 in Appendix B.

In the Figures 1.3.a and 1.3.c for hospitalization services, and 1.4.a and 1.4.c for general surgery services, it is observed that the distribution of the individual costs for both, has a positive skewness behavior with a slow decay in the tail area, which is an interesting result from the extreme values theory, hence, this issue is addressed later.

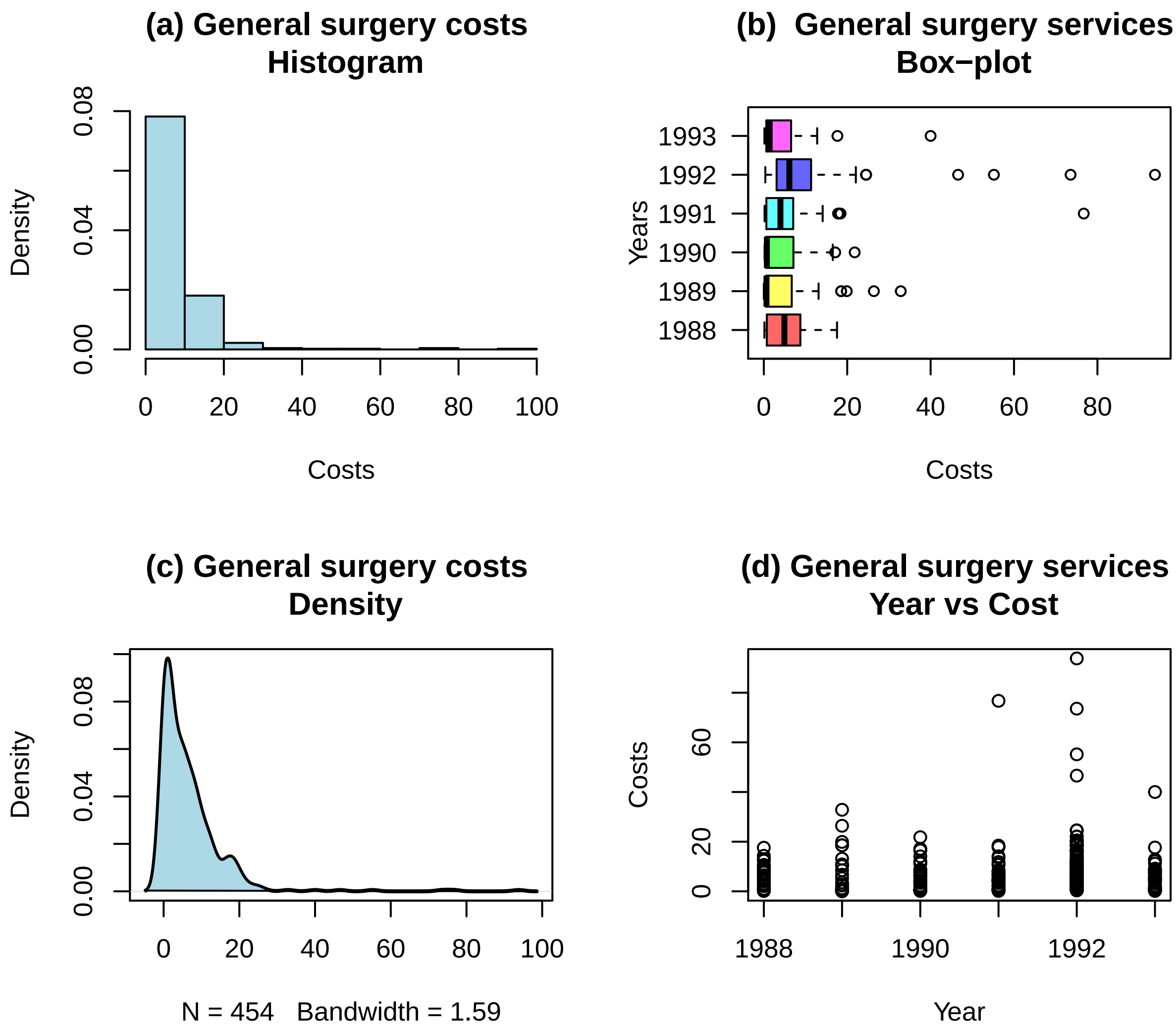

Figure 1.4: Histogram, Box-plot, Density and Scatterplot for general surgery costs

In the Figures 1.3.b and 1.4.b is presented a box-plot for each medical service to show if there is any trend in costs over the years, and also, observe the number of points that are outside the right whisker of the individual costs distribution, since the greater the number of observations in this area, will be obtained more information about the severity level of the distribution tail.

The Figures 1.3.d and 1.4.d show the scatterplot of the costs of each medical service per year, where it is noted that in the hospitalization case, the most expensive interventions took place in 1988, which exceeded 150 million of pesos, while in the general surgery case, the most expensive interventions happened in 1992 with a cost of 93.79 million and in 1991 with 76.71 million.

Chapter 2

Frequency Model Estimation

Table of Contents

2.1 Introduction

In order to perform the reinsurance calculation, it is necessary to select parametric models that allow capturing in the best possible way, the behavior of the observed frequencies $n_{k,i}$ and the observed individual costs $x_{j,i}^{(k)}$ for the k-th medical services derived from a *HCD*, where the subscripts i and j refer to the i-th individual and j-th cost generated.

For the distribution of the observed frequencies $n_{k,i}$, assume a random variable N_k representing the number of patients belonging to the k-th medical services generated by a *HCD* during the period of 1988-1993 and which has a mass function given by $p_{n_k}(x) = \mathbb{P}(N_k = x)$. Deelstra and Plantin (2014, pp. 17–20) point out that to make the adjustment, there are three laws of frequencies commonly used in the practice, which depend on the expected value and the variance of the frequencies observed, namely, the Binomial Law, the Poisson Law (also called Poisson Process) and the Mixed Poisson Law.

The Binomial law is applied in cases in which the variance of the random variable N_k is significantly smaller than its expected value. In this case, it is assumed that the mass function of N_k is given by

$$\mathbb{P}(N_k = x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x} \tag{2.1}$$

with mean and variance given by

$$\mathbb{E}(N_k) = np \quad \text{and} \quad Var(N_k) = np(1-p) \tag{2.2}$$

where n represents the total claims number that occurred in the k-th medical service, $x = 0, \ldots, n$ represents the number of times a claim occurs and $0 < p < 1$ is the adjustment parameter of the distribution and represents the probability of a claim occurring.

The Poisson law is applied in situations in which the variance and the expected value of the random variable N_k are similar. In this case, it is assumed that the mass function of N_k is given by

$$\mathbb{P}(N_k = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \tag{2.3}$$

with mean and variance given by

$$\mathbb{E}(N_k) = Var(N_k) = \lambda t \tag{2.4}$$

where $x = 0, 1, 2, \ldots, \infty$ represents the number of times a claim occurs, $t > 0$ represents the time variable (commonly set $t = 1$) and $\lambda > 0$ is the adjustment parameter of the distribution and represents the intensity parameter.

The Mixed Poisson Law is applied when the variance of the random variable N_k is significantly greater than its expected value. In this case, it is assumed that the mass function of N_k, follows a Poisson Law with random parameter λ (replaced by θ), and *mixing function* F_λ (also called *risk structure function*), such that

$$\mathbb{P}(N_k = x) = \int_0^\infty \frac{e^{-\theta t}(\theta t)^x}{x!} dF_\lambda(\theta) \tag{2.5}$$

with mean and variance given by

$$\mathbb{E}(N_k) = t\mathbb{E}(\lambda) \quad \text{and} \quad Var(N_k) = t\mathbb{E}(\lambda) + t^2 Var(\lambda) \tag{2.6}$$

where $\mathbb{E}(\lambda)$ and $Var(\lambda)$ are respectively the mean and variance of the random parameter λ . (See Albrecher, Beirlant, and Teugels (2017, pp. 146–149) to observe some examples of Mixed Poisson Law)

2.2 Frequency model estimation for hospitalization services

Based on the frequency laws presented by Deelstra and Plantin (2014), we estimate the mean and variance of the claims number for hospitalization services, in order to observe if the empirical variance is greater, less or equal to the empirical mean, to select the most appropriate frequency law. Once the estimate is made, we observe that the expected value is 388 and the variance of 11338.8, which indicates that the variance is significantly greater than the expected value of the claims number, consequently, it is concluded that the claims number in hospitalization services follows a Mixed Poisson Law.

Given that within the category of distributions belonging to the Poisson Mixed Law, a large number of mixtures can occur, such as the Negative Binomial, Generalized Poisson, Poisson-Inverse Gaussian, Sichel, among others, it is necessary to use a statistical software that allows evaluating different distributions of Poisson mixtures and selecting those that best fit the dataset.

In order to find the distribution that presents the best adjustment to the claims number for hospitalization services, we use the functions `fitDist` and `gamlssML` of the library `gamlss`(2005). The `fitDist` function is used to adjust the distribution that presents the best fit, while the `gamlssML` function is employed to adjust the second and third distribution that presents the best fit. See Code 6 in Appendix B.

Table 2.1: Better fit for frequencies of hospitalization services

PIG	GPO	NBI	NBII	DEL
75.0654	75.15057	75.32004	75.32004	75.33117

In Table 2.1 it is observed that the function `fitDist` suggests through the Akaike information criterion (*AIC* onwards), that the distribution that grants the best adjustment to the number of hospitalization claims is the Poisson-Inverse Gaussian (*PIG* onwards) with a value of 75.0654, followed by the Generalized Poisson (*GPO* onwards) with a value of 75.15057 and the Negative Binomial type I (*NBI* onwards) with a value of 75.32004. The description and presentation of the main statistics of the distributions *PIG*, *GPO* and *NBI* are presented in Appendix C.

Due to how tedious it could be to program the mean, variance, skewness and excess kurtosis for these or other distributions, it is decided to use the functions `moments`, `skew` and `kurt` of the library `DistMom` (2018) to carry out the calculation of these values, where it can be proved that the same result is reached using the formulation proposed in (7.11), (7.5) and (7.8). In addition, The empirical skewness and excess kurtosis are calculated with the functions `skewness` and `kurtosis` of the library `e1071` (2018). See Code 7 in Appendix B.

Table 2.2: Statistical measurements of hospitalization frequencies

Dist	Mean	Variance	Skewness	Excess kurtosis
Empirical	388	11338.800	0.6102662	-1.2966996
PIG	388	8913.969	0.6986897	0.8203684
GPO	388	8931.442	0.6291843	0.6580699
BNI	388	8859.088	0.4745443	0.3377319

In the Table 2.2 it is observed that there is no difference between the calculated value of the theoretical mean for the three adjusted distributions. In the same way, it is appreciated that for all cases the value of the theoretical variance is much lower than the value of the observed variance of 11338.8, being the *GPO* distribution the adjusted distribution that has the highest variance with a value of 8931.442. It is also observed that the theoretical value of the skewness is very similar to that adjusted by the distributions *PIG* and *GPO*, but in the excess kurtosis case, none of the distributions coincide with the empirical value of -1.2967, which suggests that the empirical distribution has a platicurtic behavior.

In addition, the Table 2.2 shows that for all the distributions the theoretical value of the skewness and excess kurtosis are positive but relatively close to zero, consequently, if the density is plotted, the claims number for hospitalization services will be in bell-shaped (similar to a normal distribution) with a slight right skewed. We proceed to observe the graphical adjustment by contrasting the empirical cumulative distribution versus theoretical cumulative distributions. See Code 8 in Appendix B.

The Figure 2.1 shows that the behavior of the three adjusted cumulative distributions is very similar, much that each curve overlapped the others. It is also noted that none of the three distributions captures well the behavior of the observed frequency of the claims number in hospitalization service, largely because of the category that is between 353 and 494 claims.

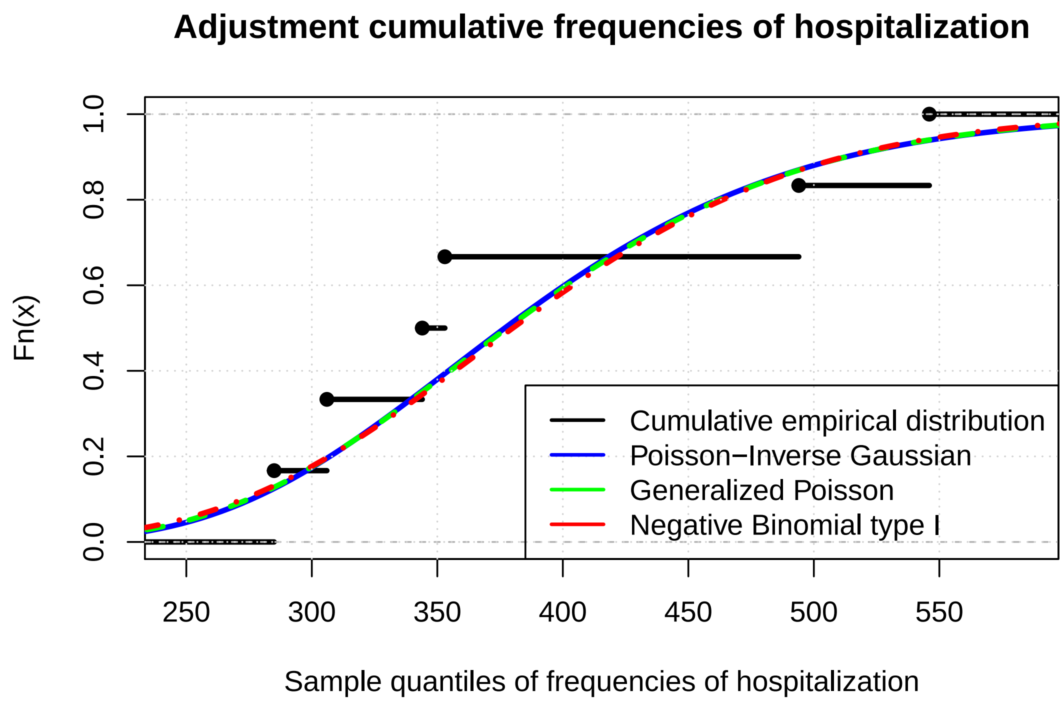

Figure 2.1: Adjustment of observed frequency of the claims number in hospitalization service

Due to the above, it is very likely that the goodness of fit tests of Kolmogorov-Smirnov (`ks.test`), Cramer-von Mises (`w2.test`) y Kuiper (`v.test`) made with the library `truncgof` (2012) throw *P-values* below 0.05, leading that the hypothesis contrast proposed in the equation (2.7) be rejected.

The hypothesis contrast is given by

$$\begin{aligned} H_0 &: F_{N_{hosp}}(x|\theta) \in \hat{F}_{N_{hosp}}(x|\hat{\theta}) \\ H_1 &: F_{N_{hosp}}(x|\theta) \notin \hat{F}_{N_{hosp}}(x|\hat{\theta}) \end{aligned} \tag{2.7}$$

with $F_{N_{hosp}}(x|\theta)$ the distribution function of the claims number for hospitalization services with parameter vector θ and $\hat{F}_{N_{hosp}}(x|\hat{\theta})$ the distribution function adjusted to the claims number for hospitalization services, with vector of estimated parameters $\hat{\theta}$, where $\hat{F}_{N_{hosp}}$ refers to any of the three fitted distributions. See Code 9 in Appendix B.

Table 2.3: Goodness-of-fit tests hospitalization frequencies

Dist	ks.test	w2.test	v.test
PIG	0.04	0.03	0.06
GPO	0.09	0.13	0.08
NBI	0.03	0.04	0.05

Contrary to the expected, the Table 2.3 shows that the Kolmogorov-Smirnov test (`ks.test`) and the Cramer-von Mises test (`w2.test`), only the *GPO* distribution obtained a *P-value* greater than 5%. Additionally, shows that in the Kuiper test (`v.test`) the three adjusted distributions obtained a *P-value* greater than or equal to 5%. In these cases it is concluded that there is not enough empirical evidence to reject the null hypothesis, consequently, it is concluded that the claims number in hospitalization services is distributed as the specific distribution.

From the results obtained in this section, it is assumed in the rest of this work, that the claims number in hospitalization services is distributed *GPO* with estimated parameter $\hat{\mu} = 388$ and $\hat{\sigma} = 0.05663$, due it was found, first, that the difference of *AIC* associated with the adjustment of the distributions *PIG* and *GPO* is small, second, than the graphical adjustment shown in the Figure 2.1 does not present significant differences, and third, that the *GPO* distribution was the only one that obtained a *P-value* greater than 5% in the three association tests presented in the Table 2.3.

2.3 Frequency model estimation for general surgery services

Similar to hospitalization services, is carried out the estimation and comparison of the empirical mean and variance of the claims number for general surgery services, in order to classify this service into one of the frequency laws presented in Deelstra and Plantin (2014). When calculating these measures, it is evident that the expected value is much lower than the value of the variance, being equal to 75.667 and 4139.067, respectively. Therefore, it can be deduced that the claims number in general surgery services belongs to the Mixed Poisson Law.

Given the wide range of mixtures belonging to the Poisson Mixed Law, we should look for the distribution that best fits the claims number for general surgery services, therefore, it is decided to use the `fitDist` function of the library `gamlss` (Rigby and Stasinopoulos, 2005) because it contains a great variety of Poisson mixtures. See Code 10 in Appendix B.

Table 2.4: Better fit for frequencies of general surgery services

DEL	PIG	GPO	SICHEL	SI
61.16453	63.84168	64.34709	64.82573	64.82573

The Table 2.4 shows that the distributions that presented the best adjustment to the claims number for general surgery services are the Delaporte (*DEL* onwards) with an *AIC* of 61.16453, the *PIG* with an *AIC* of 63.84168 and the *GPO* with an *AIC* of 64.34709. It should be noted that the *AIC* of the three distributions does not differ much from the others, thus, the adjustment made by the distributions is expected to be similar. The description and presentation of the main statistics of the distributions *DEL*, *PIG* and *GPO* are presented in Appendix C.

To carry out the calculation of the statistics presented in the equations (7.3), (7.11) and (7.5), is used the library `DistMom` (2018), while, for the calculation of the empirical skewness and excess kurtosis, is used the library `e1071` (2018). See Code 11 in Appendix B.

Table 2.5: Statistical measurements of general surgery frequencies

Dist	Mean	Variance	Skewness	Excess kurtosis
Empirical	75.66667	4139.067	1.672216	0.9822301
DEL	75.66669	4909.595	4.092893	25.3435239
PIG	75.66667	2310.444	1.844012	5.7031503
BNI	75.66666	2331.757	1.684608	4.7209981

The Table 2.5 shows that there is no significant difference between the calculated value of the theoretical mean for the three adjusted distributions and the empirical mean of general surgery services. Similarly, it shows that of the adjusted distributions, the *DEL* is the only one that has a variance close to the empirical variance, the *PIG* and *BNI* values similar to the empirical skewness, and none of the adjusted distributions is close to the empirical value of excess kurtosis, being the *DEL* the furthest with a value of 25.3435.

Below is presented the graphical adjustment of the three theoretical cumulative distribution versus the empirical cumulative distribution. See Code 12 in Appendix B.

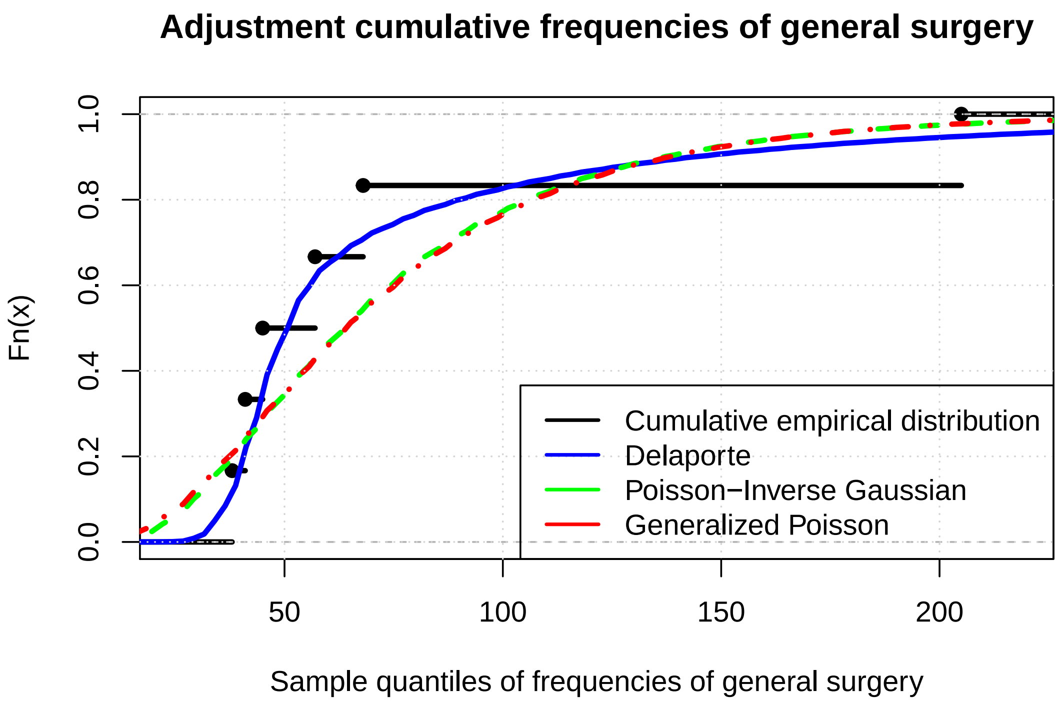

Figure 2.2: Adjustment of observed frequency of the claims number in general surgery service

The Figure 2.2 shows that the adjustment level of the *DEL* is much better than that offered by the *PIG* or the *GPO*, since it captures the behavior of the initial and central probabilities of empirical cumulative distribution. In addition, it is observed that unlike the *PIG* and *GPO*, the distribution *DEL* gives a probability of very low occurrence to values under 38 and grants possibilities of occurrence to values over 205, being those values the minimum and maximum claims number of general surgery services.

As in the hospitalization case, are calculated the goodness of fit tests of Kolmogorov-Smirnov (`ks.test`), Cramer-von Mises (`w2.test`) and Kuiper (`v.test`), in order to obtain a measure of goodness that warns us if the set of observations is well adjusted by a theoretical distribution. For this, we employ the following hypothesis contrast

$$\begin{aligned} H_0 &: F_{N_{surg}}(x|\theta) \in \hat{F}_{N_{surg}}(x|\hat{\theta}) \\ H_1 &: F_{N_{surg}}(x|\theta) \notin \hat{F}_{N_{surg}}(x|\hat{\theta}) \end{aligned} \tag{2.8}$$

with $F_{N_{hosp}}(x|\theta)$ the distribution function of the claims number for general surgery services, with parameter vector θ, and $\hat{F}_{N_{hosp}}(x|\hat{\theta})$ the distribution function adjusted *DEL*, *PIG* and *GPO*, with vector of estimated parameters $\hat{\theta}$. See Code 13 in Appendix

B.

Table 2.6: Goodness-of-fit tests general surgery frequencies

Dist	ks.test	w2.test	v.test
DEL	0.23	0.20	0.58
PIG	0.02	0.01	0.11
GPO	0.00	0.00	0.05

As expected from the good fit obtained by the distribution *DEL* in the graphical analysis, in the Table 2.6 it is observed that for each of the three association tests, is obtained a *P-value* greater than 5%, which concludes that the null hypothesis raised in the equation (2.8) is not rejected in any of them. On the other hand, the Table 2.6 shows that the distributions *PIG* and *GPO* do not present a good fit, since is rejected the hypothesis null in the tests Kolmogorov-Smirnov and Cramer-von Mises, due in these the *P-value* is less than 5%.

Given that in all the obtained results, the distribution *DEL* had a better performance than the other distributions, hence, it is assumed that the claims number for general surgery services is distributed as a *DEL* with estimated parameter $\hat{\mu} = 75.66669$, $\hat{\sigma} = 4.29732$ and $\hat{\nu} = 0.55675$

Chapter 3

Severity Model Estimation

Table of Contents

3.1 Introduction

For the distribution of the observed individual costs $x_{i,j}^{(k)}$, assume a random variable X_k that represents the individual costs of the patients belonging to the k-th medical services generated by a *HCD* during the period of 1988-1993 and which has a cumulative distribution function given by $F_{X_k}(x) = \mathbb{P}(X_k < x)$.

Before examining which distributions adjust to the individual costs of each medical service, It is necessary to perform an exploratory statistical analysis of the data set and introduce some functions that allow obtaining as much information as possible to classify the tail of the empirical distributions according to their degree of severity. The information collected will help to decide a reasonable model for each of the services and the most appropriate method in each situation.

3.2 Mean residual life analysis

In some reinsurance treaties, it is usually useful to condition a random variable X_k to a certain threshold u, such that $X_k > u$, because this allows to identify in a certain way the severity of the right tail of a distribution and can be useful to decide the reinsurance premium.

To identify the severity degree of the right tail distribution for the individual costs $\mathbb{P}(X_k > x)$, is defined for each of the medical services a function $e_{X_k}(u)$, known as a *mean excess function* or *mean residual life function* over the threshold u (*MRL* onwards), such that

$$e_{X_k}(u) = \mathbb{E}(X_k - u | X_k > u), \ u \geq 0 \tag{3.1}$$

is a function that measures the average costs X_k, assuming $X_k > u$. In other words, the equation (3.1) measures the expected individual cost of the k-th medical service once they exceed a threshold u. If $\overline{F}_{X_k}(u) = \mathbb{P}(X_k > u)$ is defined as the survival function of the random variable X_k evaluated in u, then using the definition of conditional expectation in (3.1), it is possible to rewrite this as (Mikosch, 2009, p. 91)

$$e_{X_k}(u) = \frac{\mathbb{E}(X_k - u)_+}{\overline{F}_{X_k}(u)} = \frac{\mathbb{E}((X_k - u)I(X_k > u))}{\overline{F}_{X_k}(u)} \tag{3.2}$$

where $I(X_k > u)$ is an indicator function such that

$$I(X_k > u) = \begin{cases} 1 & \text{if } X_k > u \\ 0 & \text{if } X_k \leq u \end{cases} \tag{3.3}$$

From this result, the equation (3.2) can be rewritten again as (Giraldo, 2018, p. 34)

$$\begin{aligned} e_{X_k}(u) &= \frac{\mathbb{E}((X_k - u)I(X_k > u))}{\overline{F}_{X_k}(u)} \\ &= \frac{\mathbb{E}(X_k I(X_k > u) - uI(X_k > u))}{\overline{F}_{X_k}(u)} \\ &= \frac{\mathbb{E}(X_k I(X_k > u)) - u\mathbb{P}(X_k > u)}{\overline{F}_{X_k}(u)} \\ &= \frac{\mathbb{E}(X_k(1 - I(X_k \leq u))) - u\mathbb{P}(X_k > u)}{\overline{F}_{X_k}(u)} \\ e_{X_k}(u) &= \frac{\mathbb{E}(X_k) - \mathbb{E}(X_k I(X_k \leq u)) - u\mathbb{P}(X_k > u)}{\overline{F}_{X_k}(u)} \end{aligned} \tag{3.4}$$

where $\mathbb{E}(X_k I(X_k \leq u)) - u\mathbb{P}(X_k > u) = \mathbb{E}(X_k \wedge u)$ is called the **limit expected value** (Klugman, Panjer, and Willmot, 2012, p. 25), in consequence, from the equation (3.4) we obtain

$$e_{X_k}(u) = \frac{\mathbb{E}(X_k) - \mathbb{E}(X_k \wedge u)}{\overline{F}_{X_k}(u)} \tag{3.5}$$

where $X_k \wedge u = \min(X_k, u) \in [0, u]$. Therefore, when $X_k > u$, the value of $\mathbb{E}(X_k) - \mathbb{E}(X_k \wedge u) = \mathbb{E}(X_k - u)$, otherwise $\mathbb{E}(X_k) - \mathbb{E}(X_k \wedge u) = 0$. In Beirlant, Goegebeur, Segers, and Teugels (2004, p. 15), the authors show that it is possible to apply Fubini's Theorem to the expression $\mathbb{E}(X_k - u)$, as follows

$$\begin{aligned}\mathbb{E}(X_k - u) &= \int_u^\infty (x_k - u)\ \mathrm{d}F_{X_k}(x_k) \\ &= \int_u^\infty \int_u^{x_k} \mathrm{d}y\ \mathrm{d}F_{X_k}(x_k) \\ &= \int_u^\infty \mathrm{d}y \int_y^\infty\ \mathrm{d}F_{X_k}(x_k) \\ &= \int_u^\infty (1 - F_{X_k}(y))\ \mathrm{d}y \\ \mathbb{E}(X_k - u) &= \int_u^\infty \overline{F}_{X_k}(y)\ \mathrm{d}y\end{aligned} \tag{3.6}$$

The equation (3.6) allows to express the *MRL* in a simpler way, such that

$$e_{X_k}(u) = \frac{\int_u^\infty \overline{F}_{X_k}(y)\ \mathrm{d}y}{\overline{F}_{X_k}(u)} \tag{3.7}$$

where it is observed that there exists a relation between $e_{X_k}(u)$ and the behavior of the survival function $\overline{F}_{X_k}(u)$ of the random variable X_k, evaluated in the threshold u, when $u \to \infty$.

It should be noted that in most situations, the theoretical survival function $\overline{F}_{X_k}(u)$ is not known, due we only have sample data, hence, it is necessary to use the empirical survival function $\overline{F}_{n_k}$, defined for a random sample $X_{k_1}, X_{k_2}, \ldots, X_{k_n}$ as

$$\overline{F}_{n_k}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_{k_i} > x), \quad x \in \mathbb{R} \tag{3.8}$$

for which it is not difficult to see that if $X_{k_{(1,n)}}, X_{k_{(2,n)}}, \ldots, X_{k_{(n,n)}}$ are the order statistics of a random sample, organized in ascending order such that $X_{k_{(1,n)}} \leq X_{k_{(2,n)}} \leq \ldots \leq X_{k_{(n,n)}}$, then, $\overline{F}_{n_k}(X_{k_{(n-m,n)}})$ can be defined as

$$\overline{F}_{n_k}(X_{k_{(n-m,n)}}) = \frac{m}{n}, \quad m = 1, 2, \ldots, n \tag{3.9}$$

Based on these results and that the support of $\overline{F}_{n_k}$ is limited, Mikosch (2009, p. 91) considers the empirical function of *MRL* e_{n_k} as

$$\begin{aligned}e_{n_k}(u) &= \mathbb{E}_{n_k}(Y - u|Y > u) \\ &= \frac{\mathbb{E}_{n_k}(Y - u)_+}{\overline{F}_{n_k}(u)} \\ &= \frac{1}{n}\frac{\sum_{i=1}^n (X_{k_{(i,n)}} - u)_+}{\overline{F}_{n_k}(u)} \\ &= \frac{\sum_{i=1}^n (X_{k_{(i,n)}} - u)_+}{\sum_{i=1}^n I(X_{k_i} > u)}\end{aligned} \tag{3.10}$$

for $u \in \left[X_{k_{(1,n)}}, X_{k_{(n,n)}}\right)$, and

$$(X_{k_{(i,n)}} - u)_+ = \begin{cases} 1, & \text{if } X_{k_{(i,n)}} > u \\ 0, & \text{if } X_{k_{(i,n)}} \leq u \end{cases} \tag{3.11}$$

for $i = 1, 2, \ldots, n$. Also, by the strong law of large numbers, we have that if $E(X_k) < \infty$, then $e_{n_k}(u) \xrightarrow{a.s} e_{X_k}(u)$ for any $u > 0$ when $n \to \infty$. The term $\xrightarrow{a.s}$ means "converge almost sure, with probability 1 to".

In Beirlant et al. (2004, p. 16) and Mikosch (2009, p. 89) presents an important relation of the *MRL* with the exponential distribution, due to its *memoryless property*, where it is observed that the *MRL* of this distribution is a constant and equal to its shape parameter. Then, Based on the constant behavior of the *MRL* of the exponential distribution, it is possible to classify the severity of a random variable X_k, in one of three categories, depending on whether its behavior is increasing, decreasing or constant, such that (Giraldo, 2018)

$$\text{Severity} = \begin{cases} \text{Low = Light-tailed: } e_{X_k}(u) \searrow, \text{ when } u \to \infty \\ \text{Mid = Medium-tailed: } e_{X_k}(u) \to \text{cst, when } u \to \infty \\ \text{High = Heavy-tailed: } e_{X_k}(u) \nearrow, \text{ when } u \to \infty \end{cases}$$

In Moscadelli (2004, p. 22) it is recommended to adjust the Weibull distribution when the data are low severity, the Gamma, Exponential, Gumbel and LogNormal distributions if they are mid severity, and the Pareto distribution when the severity is high. On the other hand, Mora (2010, p. 73) recommends adjusting the Pareto, double exponential, t, mixed model and distributions with Pareto tails, when the dataset has high severity, since these are distributions that have a high kurtosis.

3.2.1 Mean residual life analysis for hospitalization services

To perform the calculation of the *MRL* for the individual costs of hospitalization services and observe graphically if the behavior is increasing, decreasing or constant, is used the function `mrlplot` of the library `evmix` (2018), in order to facilitate the identification of the severity level of the dataset. See Code 14 in Appendix B.

In the first section of the Figure 3.1, which is delimited by the red dashed line, it is observed that for threshold values u less than 35, the *MRL* has an increasing behavior that would suggest that the individual hospitalization costs have high severity.

In the second section of the Figure 3.1, which is delimited by the blue dashed line, it is observed that for threshold values u between 35 and 60, the *MRL* has a constant behavior that would suggest that the individual hospitalization costs have mid severity.

In the third section of the Figure 3.1, which is limited with the value in which 6 excesses ocurred, it is observed that for threshold values u greater than 60, the *MRL* has a very variable behavior, thus, it is preferable to focus attention on the first and second sections of the figure.

Due to the above, it is not possible to assure with certainty whether the severity level of the individual costs is high or not, therefore it is necessary to use another methodology that allows to observe the severity level of the distribution tail, such as goodness-of-fit tests, or specialized tests, such as increasing conditional mean exceedance (Bryson, 1974), among others.

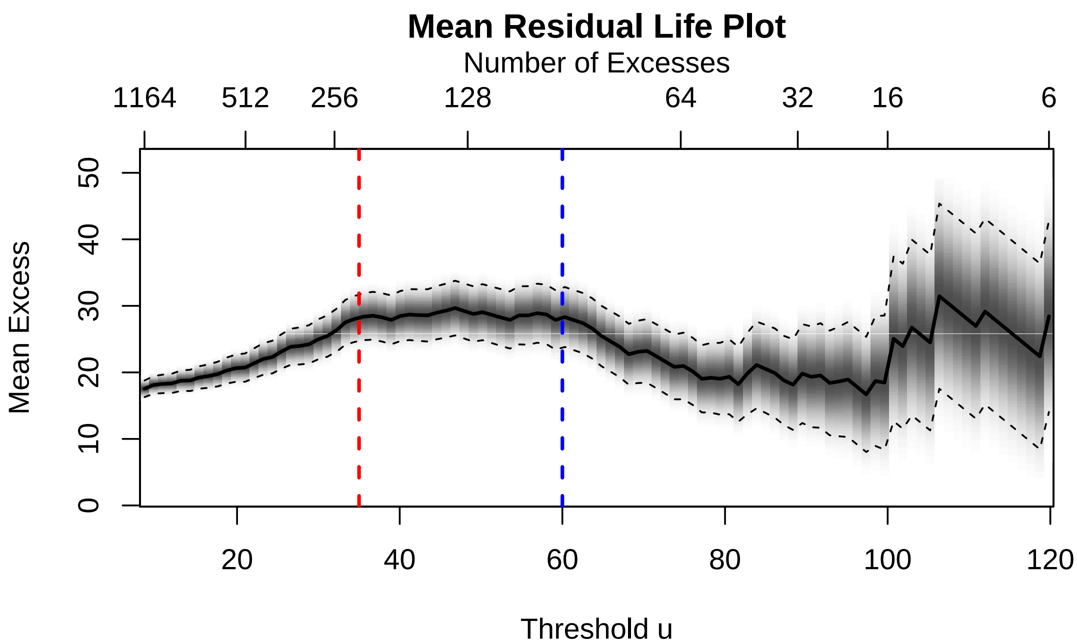

Figure 3.1: Mean residual life for individual costs of hospitalization service

3.2.2 Mean residual life analysis for general surgery services

For general surgery services, is used again the function `mrlplot` of the library `evmix` (2018), in order to show if the graph presents an incresing, decreasing or constant behavior, and for giving a first judgment about the behavior of the distribution tail of individual costs. See Code 15 in Appendix B.

In the first section of the Figure 3.2, which is delimited by the red dashed line, it is appreciated that for threshold values u less than 16, the *MRL* has a weakly increasing behavior, followed by an increasing behavior in the second section of the figure, which is delimited by the value in which 6 excesses occur.

This could be interpreted as possible evidence that the individual costs of general surgery services have a high severity level, but given that the behavior in the first section is weakly increasing, it is necessary to perform other methods of exploratory statistical analysis in order to corroborate whether in fact, the severity is high or not.

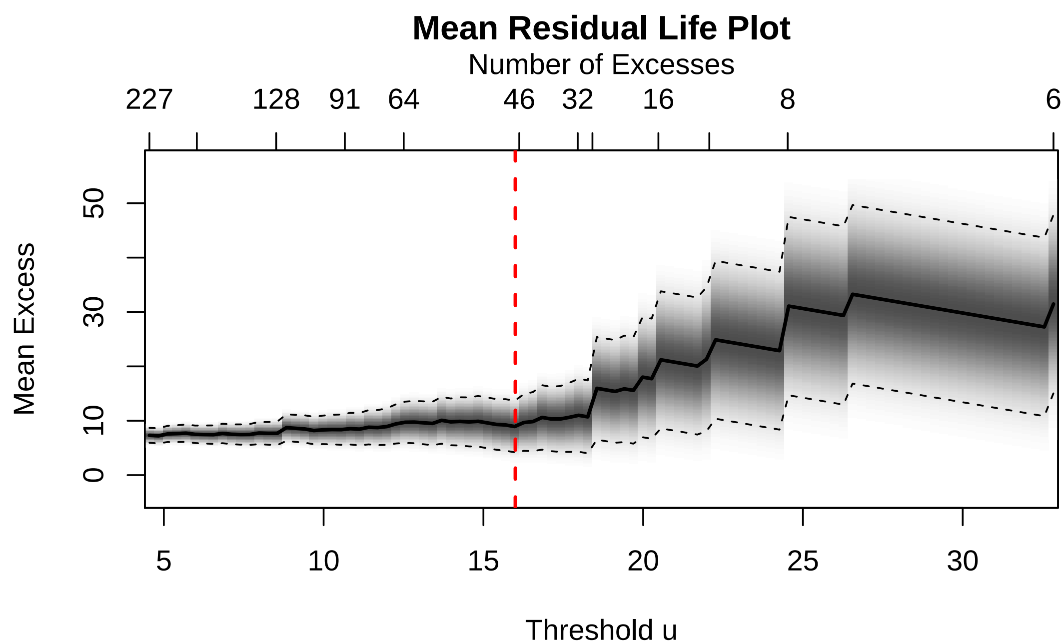

Figure 3.2: Mean residual life for individual costs of general surgery service

3.3 Tail heaviness

In insurance and reinsurance, one of the most important parts in the analysis, is to investigate the tail behavior of the distribution, in order to identify whether or not there are large claims within the individual costs, since its existence allows us to determine which type of asymptotic approximation methodology is the most appropriate to calculate the compound cost distribution.

The Extreme Value Theory (EVT onwards) is a methodology that focuses its attention on the tail of the empirical distribution with the purpose of identifying those distributions that have large claims through the criterion known as the regular variation index. De Haan and Ferreira (2006, Appendix B) define ℓ, as a measurable

Lebesgue function: $\mathbb{R}^+ \to \mathbb{R}$, to be of regular variation to infinity if for some $\alpha \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{\ell(tx)}{\ell(t)} = x^{\alpha}, \quad x > 0 \tag{3.12}$$

which is denoted as $\ell \in RV_\alpha$ and the value α is called the regular variation index. Where $\alpha = 0$, it is said that ℓ, is of slowly varying.

From Ferrari and Fumes (2017, p. 8) and Mikosch (1999, p. 11), consider a non-negative random variable X_k with cumulative distribution F_{X_k} for the k-th medical service, such that $\overline{F}_{X_k} = 1 - F_{X_k}$. Then it is said that the distribution X_k regularly varying with index $\alpha > 0$ if the right tail $\overline{F}_{X_k}$ is regularly varying to infinity with tail index $-\alpha$, such that $\alpha = -1/\xi$ and $\xi > 0$, this is

$$\lim_{t\to\infty} \frac{\overline{F}_{X_k}(tx)}{\overline{F}_{X_k}(t)} = x^{-\frac{1}{\xi}}; \quad x > 0 \tag{3.13}$$

where ξ is defined as the tail index of the distribution. Additionally, the authors point out that when the limit is equal to $x^{-\infty}$, it is said that F_{X_k} is of light right tail with tail index $\xi = 0$. When the limit is equal to 1 then $\overline{F}_{X_k}$ is a slowly varying function and it is said that F_{X_k} is heavy tail with tail index $\xi = \infty$ or equivalently with regular variation index $\alpha = 0$.

Furthermore, from De Haan and Ferreira (2006, p. 17) it is obtained that when $\xi > 0$ and the cumulative distribution F_{X_k} is differentiable and equal to the density function f_{X_k}, then

$$\lim_{t\to\infty} \frac{t f_{X_k}(t)}{\overline{F}_{X_k}(t)} = \frac{1}{\xi} \tag{3.14}$$

An important result can be derived from the equation (3.14), once the adjustment process is done, since it allows to extract the value of the tail index ξ or the regular variation index α from the random variable X_k, through the relation it has with the tail index ξ.

When no distribution has been adjusted to the dataset, different proposals have been developed in order to estimate the value of the tail index ξ, including the simplest but well known, the Hill estimator $\mathrm{H}_{m,n} = \hat{\xi}$, proposed by Hill (1975). In De Haan and Ferreira (2006, pp. 19–20) it is shown that from the equation (3.13), an equivalent expression can be written

$$\lim_{t\to\infty} \frac{\int_t^\infty \frac{1}{x}\overline{F}_{X_k}(x)\ dx}{\overline{F}_{X_k}(t)} = \xi; \quad x > 0 \tag{3.15}$$

which, by partial integration can be rewritten as

$$\lim_{t\to\infty} \frac{\int_t^\infty (\log x - \log t)\ \mathrm{d}F_{X_k}(x)}{\overline{F}_{X_k}(t)} = \xi \tag{3.16}$$

Based on (3.16), by replacing t with $X_{k_{(n-m,n)}}$, the $(n-m)$-th order statistic of a random sample *iid* $X_{k_{(1,n)}}, X_{k_{(2,n)}}, \ldots, X_{k_{(n,n)}}$, with $X_{k_{(1,n)}} \leq X_{k_{(2,n)}} \leq \ldots \leq X_{k_{(n,n)}}$ and replacing F_{X_k} by the empirical distribution function F_{n_k}, we obtain an estimator based on asymptotic results

$$\mathrm{H}_{m,n} = \frac{\int_{X_{k_{(n-m,n)}}}^{\infty} \left(\log x - \log X_{k_{(n-m,n)}}\right) \, \mathrm{d}F_{n_k}(x)}{\overline{F}_{n_k}(X_{k_{(n-m)}})} \tag{3.17}$$

Examining the equation (3.17) it is evident that there is a close relationship between the Hill estimator and the *MRL*, since the divisor of the equation (3.17) is equal to the equation presented in (3.9), while the numerator of the equation (3.17) is similar to that shown in equation (3.6), except that instead of employing $\mathbb{E}(X_k - u)$, is used the expected value of the difference of logarithms, such that

$$\mathbb{E}\left(\log x - \log X_{k_{(n-m,n)}}\right) = \int_{\log X_{k_{(n-m,n)}}}^{\infty} \left(\log x - \log X_{k_{(n-m,n)}}\right) \, \mathrm{d}F_{n_k}(x) \tag{3.18}$$

for $X_{k_{(n-m,n)}} \leq x < \infty$. Due to the relationship of the Hill estimator and the *MRL*, it is possible to use the equation (3.10) and the result of the equation (3.9) to obtain

$$\mathrm{H}_{m,n} = \frac{1}{m} \sum_{i=0}^{m-1} \left(\log X_{(n-i,n)} - \log X_{(n-m,n)}\right) \tag{3.19}$$

where $\mathrm{H}_{m,n}$ is the Hill's tail index estimator. (See Embrechts, Klúppelberg, and Mikosch (1997, Section 6.4), De Haan and Ferreira (2006, Section 3.2) Drees, De Haan, and Resnick (2000) and Hill (1975) for more information on Hill's tail index estimator).

In his work, Hill (1975) establishes that the proposed method depends on the subjective choice of the threshold or value of the order statistic $(n-m)$, and warns that if is selected a very low threshold, there will be many observations, including some that are not extreme, which will cause the estimator to be skewed, while if is selected a very high threshold, there will be few extreme observations, which will cause the estimator to have high variability.

Due to the exchange between bias and variability caused by the arbitrary selection of the threshold, the Hill's basic approach suggests selecting by default an arbitrary threshold close to the order statistic $(n-m)$, corresponding to a percentile between 90% and 95% and then selecting as the tail index of Hill the value where $\mathrm{H}_{m,n}$ is in a stable region (region that is best visualized from the Hill plot). If a stable region is not found in the Hill plot, it is recommended to consult some of the alternatives such as the Smoothed Hill plot, the Alternative Hill plot and the Alternative Smoothed Hill plot. (See Resnick, 2007, Section 4.4.3).

In Resnick and Stărică (1997) it is shown that due to the extreme volatility that can occur in the Hill estimator, finding a stable region can be a very problematic task, therefore recommends applying a smoothing of the statistic to facilitate the location of such a region. Resnick and Stărică (1997, p. 274) explains that the smoothing procedure consists of averaging the values of the Hill estimator for different number of order statistics, namely

$$\mathrm{smooH}_{m,n} = \frac{1}{(r-1)m} \sum_{j=m+1}^{mr} H_{j,n} \tag{3.20}$$

for $r > 1$ (usually 2 or 3). The authors also propose an alternative way to plot the Hill estimator and the Smoothed Hill estimator, where the objective will be to give a larger proportion on the plot to the sectors in which there is a relatively small number of order statistics, and reduce or scale those sectors in which there is a larger number. The alternative way to plot the estimators is given by the relationship

$$\{\theta, \mathrm{H}^{-1}_{\lceil n^{\theta} \rceil, n}\} \tag{3.21}$$

where $0 \leq \theta \leq 1$ and $\lceil y \rceil$ is the ceiling value of y, i.e., is the smallest integer greater or equal to y, with $y \geq 0$. It should be noted that to find the estimated value of ξ in the alternative graphs, one must also look for the region in which the line of the graph is stable.

Once the stable region of the Hill graph has been determined and consequently the value of the Hill's tail index, it is possible the interpretation of this value, due to the relationship that this index has with the shape parameter of the Generalized Pareto distribution (*GPD* onwards), the Generalized Extreme Value distributions (*GEV* onwards) and the tail heaviness of these distributions.

Where, due to the relation $\alpha = \frac{-1}{\varepsilon}$, large values of the form parameter ξ are equivalent to small values of the regular variation index α, hence it means that the distributions tail is heavy. Similarly, small values of the form parameter ξ are equivalent to large values of the regular variation index α, in consequence it means that the distributions tail is light.

Another result derived from the regular variation index, is that if the distribution of claims is distributed as a *GEV* or *GPD*, the number of finite moments they possess can be immediately inferred, because these distributions have the characteristic that their moments depend on the value of the regular variation index, so that $E(X^k) < \infty$ for all $k < \alpha$, i.e., the moments of order equal to or greater than α do not exist.

3.3.1 Tail index with Hill plot for hospitalization services

In order to observe the value of tail index ξ and the value of regular variation index α for the set of individual costs of hospitalization services, are performed the Hill plot, the Alternative Hill plot (*AltHill*), the Smoothed Hill plot (*SmooHill*) and the

Alternative Smoothed Hill plot (*AltSmooHill*). To do this, is used the `hillplot` function from the library `evmix` (2018). See Code 16 in Appendix B.

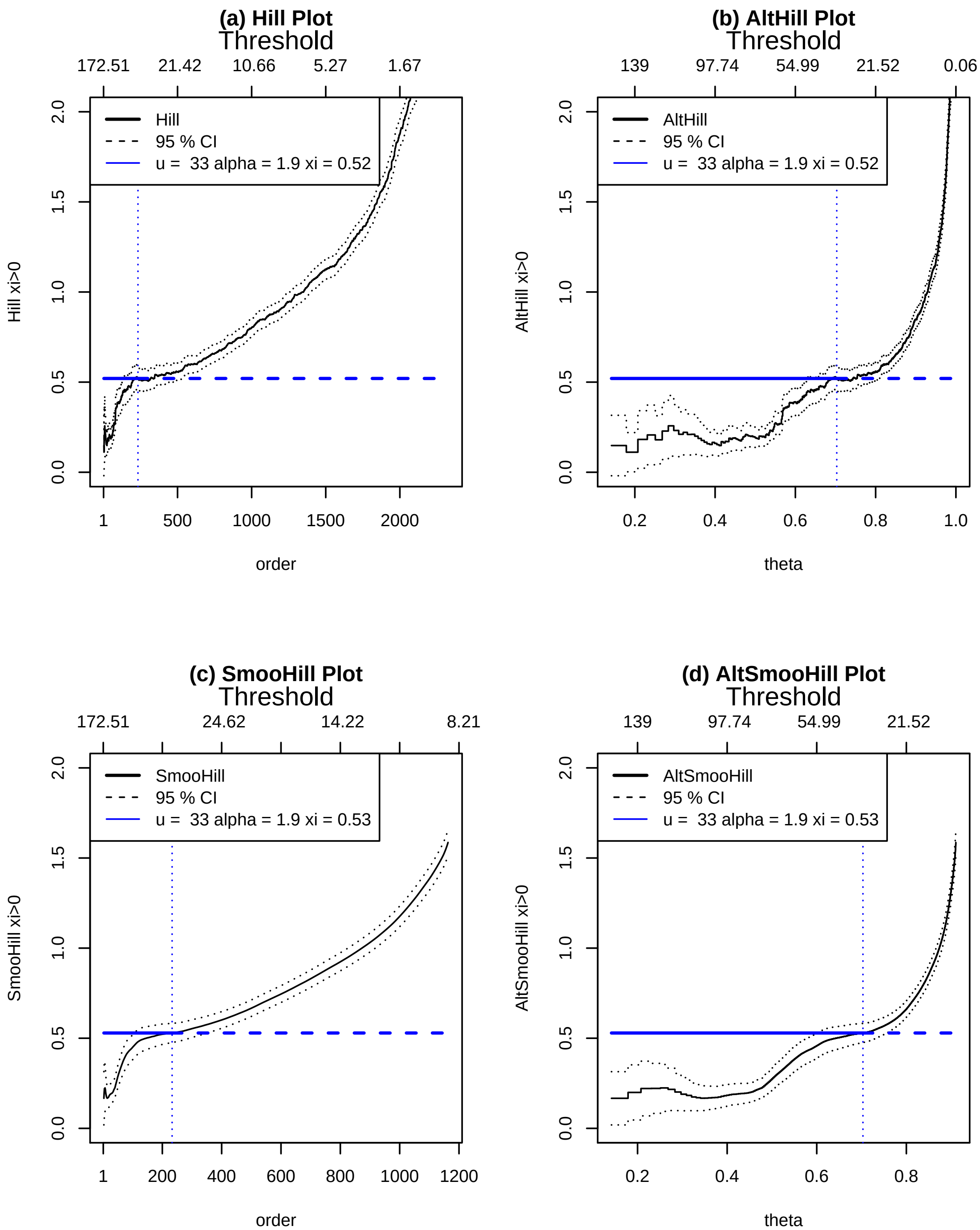

Figure 3.3: Hill, AltHill, SmooHill and AltSmooHill for hospitalization services

In the Figure 3.3 it is observed that in all panels, the value of the tail index ξ is between 0.52 and 0.53, while the value of the regular variation index $\alpha = 1.9$ for a threshold $u = 33.23116$. It should be noted that the value of the threshold u is automatically selected by the library `evmix` (2018), as the percentile that has under its value, 90% of the total of the individual costs of hospitalization services.

Additionally, in the Figures 3.3.a and 3.3.b show that the cut-off point of the horizontal line with the curve, coincides with a slightly stable region for the threshold $u = 33$, but in general it is observed that the curve has a gradual increase, therefore it is not pertinent to suggest that the found values of ξ and α are viable.

On the other hand, and in a contradictory way, the Figures 3.3.c and 3.3.d instead of facilitating the visualization of the stable region, they suggest that in reality there is no region that can be considered as stable in the graphs, due to the increasing and almost constant tendency of the curves, corroborating the results obtained in the Figures 3.3.a and 3.3.b.

Due to the above, it is not possible to affirm with certainty that the value of the tail index for individual hospitalization costs is between 0.52 and 0.53, i.e., it is not possible to make affirmations about the tail heaviness of the individual costs distribution. Therefore, a method of automatic adjustment of distributions will be used later, in order to appreciate which are the distributions that best fit the dataset, and verify if these have characteristics that allow classifying the data set at a specific severity level, with a specific tail index.

3.3.2 Tail index with Hill plot for general surgery services

Like hospitalization services, are used for general surgery services the Hill plot, the Alternative Hill plot (*AltHill*), the Smoothed Hill plot (*SmooHill*) and the Alternative Smoothed Hill plot (*AltSmooHill*), in order to locate stable regions that allow the appropriate selection of the tail index for the dataset. To do this, is employed the `hillplot` function from the library `evmix` (2018). See Code 17 in Appendix B.

The Figures 3.4.a and 3.4.b, show that in the threshold $u = 16$ there is a stable region in the graphs, which corresponds to tail index $\xi = 0.34$ or to regular variation index $\alpha = 2.9$. It should be noted that for these two graphs, the threshold value was automatically selected by the library `evmix`(2018), as the percentile that has under its value, 90% of the total order statistics of the individual costs of general surgery services.

In the Figures 3.4.c and 3.4.d it is shown that for the threshold $u = 16$ (green line) automatically selected by the library `evmix`(2018), the stable region that the graphs possess is not captured, since such a stable region is to the left of the selected threshold. Due to this, it is decided to manually select the threshold value by means of the 95% percentile, and it is observed that for the threshold $u = 18$, the blue line cuts the curve of the graphs in the stable region, showing that the tail index is $\xi = 0.35$ and the regular variation index is $\alpha = 2.8$.

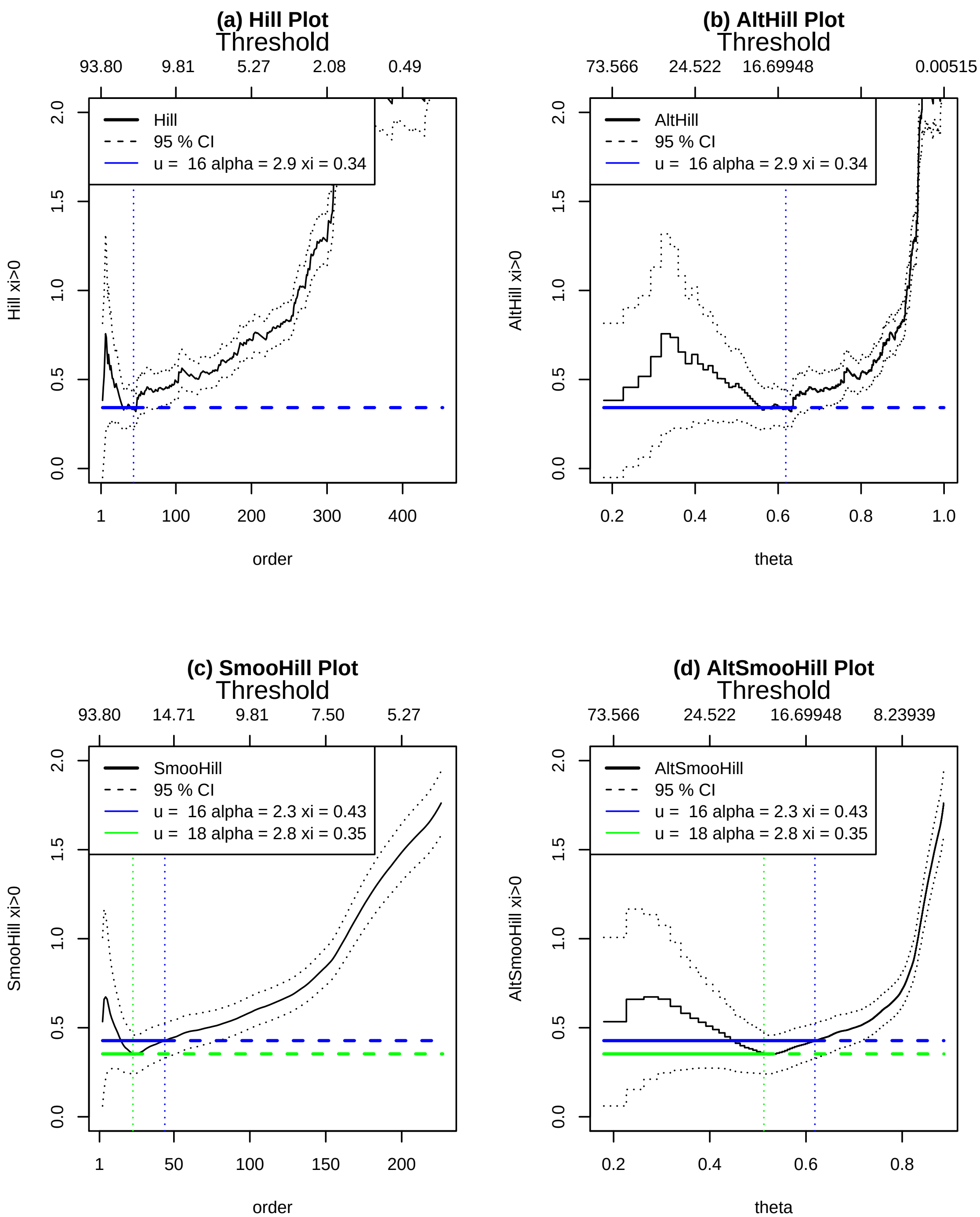

Figure 3.4: Hill, AltHill, SmooHill and AltSmooHill for general surgery services

Due in all the panels of the Figure 3.4 it is observed that exist a stable region in which are obtained similar values for ξ and α, it is possible to affirm that the tail index value for the individual costs of general surgery services is between 0.34 and 0.35, and that the regular variation index value it is between 2.8 and 2.9. This means that if the adjusted distribution to the dataset belongs to the Pareto family, there would be certainty of the existence of its first two moments, but it will not be certain that there are more of these, this being a sign that the individual costs of general surgery services have a heavy tail.

3.4 Spliced distributions

Sometimes, there are datasets that show different statistical behaviors in some of their intervals, which makes the conventional adjustment of a distribution inefficient, generates loss information and causes possible errors in the inference, even in situations in which attention is focused on a single part of the dataset. An example of this is seen in the *EVT*, where interest is focuses on modeling the tail of the dataset, without paying attention to its central part, which is defined as the values that are under a certain threshold.

In order to give a solution to this problem, is proposed the use of mixtures or *spliced distributions*. Albrecher et al. (2017, p. 50) defines a *m-component spliced distribution* with a probability density function as

$$f_{X_k}(x) = \begin{cases} \pi_{k_1} \frac{f_{k_1}(x)}{F_{k_1}(c_{k_1}) - F_{k_1}(c_{k_0})} & c_{k_0} < x \leq c_{k_1} \\ \pi_{k_2} \frac{f_{k_2}(x)}{F_{k_2}(c_{k_2}) - F_{k_2}(c_{k_1})} & c_{k_1} < x \leq c_{k_2} \\ \vdots & \vdots \\ \pi_{k_m} \frac{f_{k_m}(x)}{F_{k_m}(c_{k_m}) - F_{k_m}(c_{k_{m-1}})} & c_{k_{m-1}} < x \leq c_{k_m} \end{cases} \tag{3.22}$$

where $f_{k_i}(x)$ and $F_{k_i}(x)$ are the probability density function and the cumulative distribution function of the i-th interval of a random variable X_k, respectively. $\pi_{k_i} > 0$ is the proportion or weight of each of the categories of the k-th medical service, with $\sum_{i=1}^{m} \pi_{k_i} = 1$. c_{k_i} are the intervals for which the random variable X_k is defined in each category or also called union points. All the above variables defined for $i = 0, 1, \ldots, m$.

A current use given to *spliced distribution* arises from *EVT*, in which the dataset is divided into two parts ($m = 2$), i.e., the dataset is separated in those observations that are under and over a threshold u, to then adjust in the lower part of u a light tail distribution and in the upper part a *GPD*.

Behrens, Lopes, and Gamerman (2004) proposes a parametric form for a *spliced distribution*, from a set of observations $X_{k_1}, X_{k_2}, \ldots, X_{k_n}$ *iid* and a threshold u, which is considered as an additional parameter implicit within the model, such that the observations above u will be distributed as a *GPD*, i.e., $(X_{k_i}|X_{k_i} \geq u) \sim G(\cdot|u, \sigma, \xi)$, being $G(\cdot|u, \sigma, \xi)$ the cumulative distribution function of *GPD* and defined as

$$G(x|u, \sigma_u, \xi) = \mathbb{P}(X_k \leq x|X_k > u) = \begin{cases} 1 - \left[1 + \xi\left(\frac{x-u}{\sigma_u}\right)\right]^{-1/\xi}; & \xi \neq 0 \\ 1 - \exp\left[-\left(\frac{x-u}{\sigma_u}\right)\right]; & \xi = 0 \end{cases} \tag{3.23}$$

where $\sigma_u > 0$ is the form parameter, ξ is the scale parameter and u sometimes described as the location parameter. Using u as an implicit parameter within the model, allows the calculation of the unconditional survival probability of *GPD*, such that

$$\mathbb{P}(X_k > x) = \phi_u[1 - \mathbb{P}(X_k \leq x|X_k > u)] = \phi_u[1 - G(x|u, \sigma_u, \xi)] \tag{3.24}$$

where $\phi_u = \mathbb{P}(X > u)$ is defined as the proportion that exceeds u, such that $0 < \phi < 1$. Furthermore, when $\xi < 0$ the tail of $G(\cdot|u, \sigma, \xi)$ will be light, with $u \leq x \leq u - \sigma_u/\xi$, when $\xi = 0$ the tail of $G(\cdot|u, \sigma, \xi)$ will be exponential type, with $x \geq u$ and when $\xi > 0$ the tail of $G(\cdot|u, \sigma, \xi)$ will be heavy, with $x \geq u$ (MacDonald et al. 2011, p. 2138).

Behrens et al. (2004, p. 229) states that observations below u will be distributed as a $H(\cdot|\eta)$ with parameter vector η, i.e., $(X_{k_i}|X_{k_i} < u) \sim H(\cdot|\eta)$, where $H(\cdot|\eta)$ can be estimated parametrically by distributions such as Gamma, Normal, Weibull, or non-parametrically by smooth kernel density estimator.

Then, the cumulative distribution function of the mixture between $H(\cdot|\eta)$ and $G(\cdot|u, \sigma, \xi)$ for any value of the random variable X_k, can be written as

$$F_{X_k}(x|\eta, u, \sigma_u, \xi) = \begin{cases} H(x|\eta), & x \leq u \\ H(u|\eta) + [1 - H(u|\eta)]G(x|u, \sigma_u, \xi), & x > u \end{cases} \tag{3.25}$$

In order to obtain a more general form of the equation (3.25), Hu and Scarrott (2018) and MacDonald et al. (2011), use the definition obtained in the equation (3.24) and rewrite $1 - H(u|\eta)$ as ϕ_u, getting by reordering terms

$$F_{X_k}(x|\eta, u, \sigma_u, \xi) = \begin{cases} (1 - \phi_u)\frac{H(x|\eta)}{H(u|\eta)}, & x \leq u \\ (1 - \phi_u) + \phi_u G(x|u, \sigma_u, \xi), & x > u \end{cases} \tag{3.26}$$

where it is observed that $(1 - \phi_u) + \phi_u G(x|u, \sigma_u, \xi)$ is the unconditional cumulative distribution function of the *GPD*, obtained from the equation (3.24). From the equation (3.26), it is possible to express the density function of the *spliced distribution* like in the equation (3.22) with $m = 2$, such that

$$f_{X_k}(x) = \pi_k \frac{h(x|\eta)}{H(u|\eta)} I(x \leq u) + (1 - \pi_k) g(x|u, \sigma_u, \xi) I(x > u) \tag{3.27}$$

where $\pi_k = (1 - \phi_u)$ represents the weight of the category, with $0 < \pi_k < 1$, $I(\cdot)$ is an indicator variable, $h(\cdot)$ and $H(\cdot)$ are the parametric density and cumulative distribution function (e.g. Gamma, Normal or Weibull) or non-parametric (e.g. smooth kernel density), and $g(x|u, \sigma_u, \xi)$ is the density function of *GPD*.

3.4.1 Spliced distributions for hospitalization services

To adjust the individual costs of hospitalization services through *spliced distributions*, are used the functions `fgammagpd`, `fnormgpd`, `fweibullgpd` of the library `evmix` (2018) to adjust the mixtures Gamma-Generalized Pareto (*G-GP* onwards), Normal-Generalized Pareto (*N-GP* onwards) and Weibull-Generalized Pareto (*W-GP* onwards), respectively. See Code 18 in Appendix B.

From the adjustment made, it is highlighted that of the three mixtures, the one closest to the threshold 33.23116 proposed as a stable area in the four panels of the Figure 3.3, is the *G-GP* with a value of 33.8329, followed by the *W-GP* with 35.5702 and the *N-GP* with a threshold of 23.03631.

Due the parametric form of the mean, variance, skewness and excess kurtosis of the adjusted *spliced distributions* is unknown, it is decided to use the functions `moments`, `skew` and `kurt` from the `DistMom` (2018) library, to calculate the value of the statistics for the adjusted distributions. To calculate the empirical value of the skewness and excess kurtosis of the individual costs of hospitalization services, is used the library `e1071` (2018). See Code 19 in Appendix B.

Table 3.1: Statistical measurements of spliced distributions for hospitalization services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	14.93863	362.5129	2.982862	11.97454
gammagdp	14.82290	333.8197	3.084240	13.39314
normgdp	14.67612	268.4622	3.370531	18.18369
weibullgdp	14.83413	339.0651	3.043860	12.90941

When comparing the empirical statistics of the individual costs with respect to those presented in the Table 3.1, it is shown that the *W-GP* mixture presents the values closest to all the empirical statistics, followed very closely by the *G-GP* mixture.

On the other hand, the *N-GP* mixture presents the measures furthest away from the empirical values, which could be due to the domain that has the normal distribution $x \in [-\infty, \infty]$. When taking into account these results together with the adjusted threshold by the three mixtures, it is expected that the *spliced distribution W-GP* is the one that presents the best graphical adjustment of the three alternatives.

To observe the adjustment of the mixtures we made the graphs 3.5, 3.6 and 3.7. In the first we present the cumulative distribution of the individual costs of hospitalization versus the cumulative distribution of the adjusted *spliced distributions*. In the second we present the natural logarithm of the survival distribution of the individual costs versus the natural logarithm of the survival functions of the three adjusted mixtures. In the third, we present three panels containing the Q-Q plot of the three adjusted *spliced distributions* to the individual costs of hospitalization services. See Code 20, Code 21, Code 22 in Appendix B.

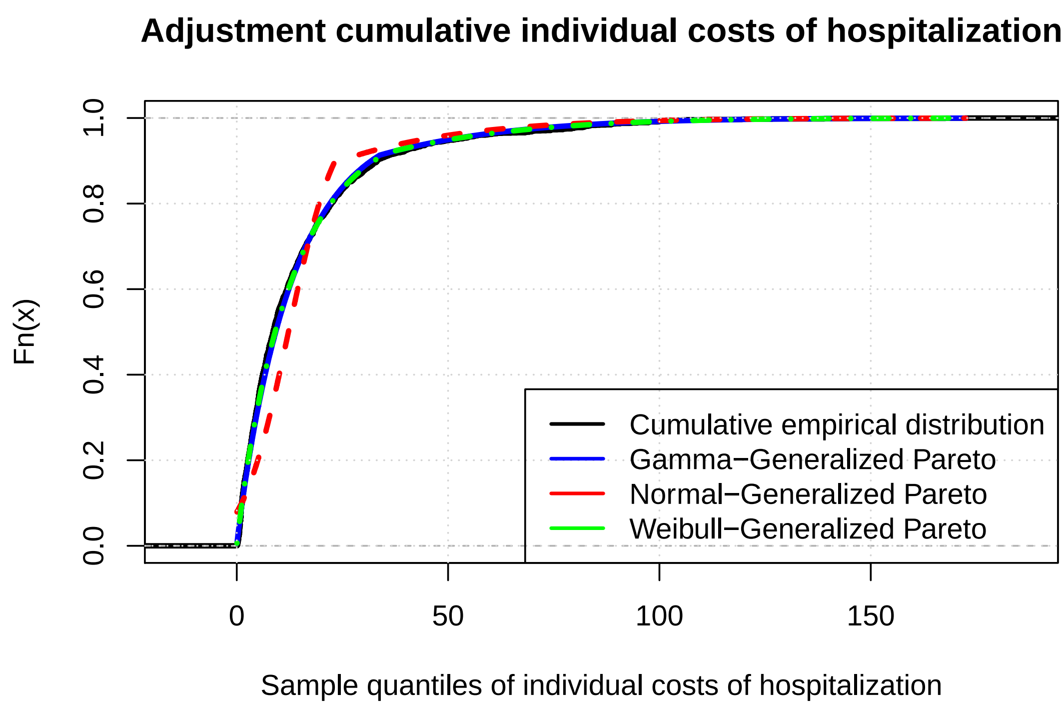

Figure 3.5: Adjustment of the individual costs of hospitalization service with spliced distributions

The Figure 3.5 shows that of the adjusted mixtures, *G-GP* and *W-GP* present very good adjustment to individual costs for hospitalization services, where it is observed that the curves of these mixtures are superimposed on the cumulative empirical distribution.

Furthermore, it is observed that as expected after the results presented in the Table 3.1, the *N-GP* mixture shows a bad adjustment in the initial and central part of the cumulative empirical distribution, due largely to the normal distribution domain.

For its part, the Figure 3.6 shows in more detail the adjustment made by the mixtures to the tail of the individual costs of hospitalization services, allows to see more closely the union area that have the mixtures and allows to observe in which sectors of the adjustment may be lost information.

Given the above, the Figure 3.6 shows that of the three mixtures, the *G-GP* and *W-GP* have a smoother behavior in their union, than the *N-GP*. In addition, it is observed that the *G-GP* and *W-GP* mixtures capture almost perfectly the form of the empirical distribution before and after the threshold u.

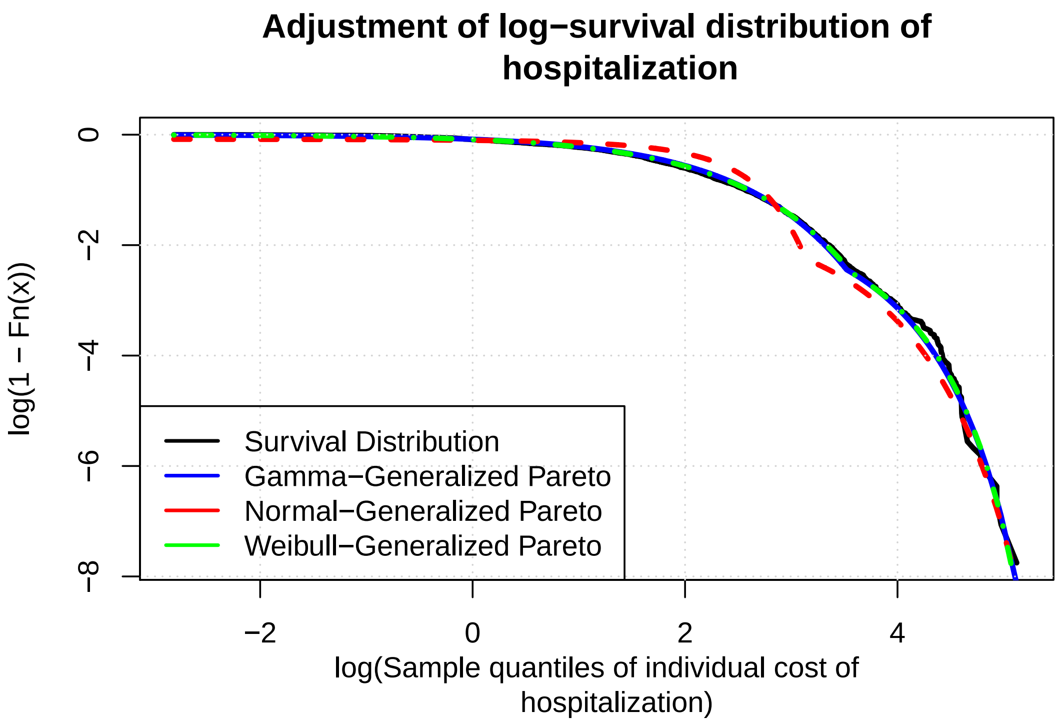

Figure 3.6: Adjustment of log-survival distribution of the individual costs of hospitalization service for spliced distributions

In the Figures 3.7.a and 3.7.c is appreciated that most observations are within the confidence bands, except for a group of values between 50 and 100, which are very close to the outside of the confidence bands. Additionally, it is shown that for values over 100, there are several observations that move away from the diagonal line, but without being outside the confidence bands. Hence, it is concluded that since there are no significant observations outside the confidence bands, it is not possible to say that the individual costs of hospitalization services do not have a *G-GP* or *W-GP spliced distribution*.

For the Figure 3.7.b, it is observed that the *N-GP* mixture has values below 0 that are outside the diagonal line and confidence bands. Furthermore, it is observed that this same behavior for values that are around 100, therefore, it is ruled out that the individual costs of hospitalization services have a *N-GP spliced distribution*.

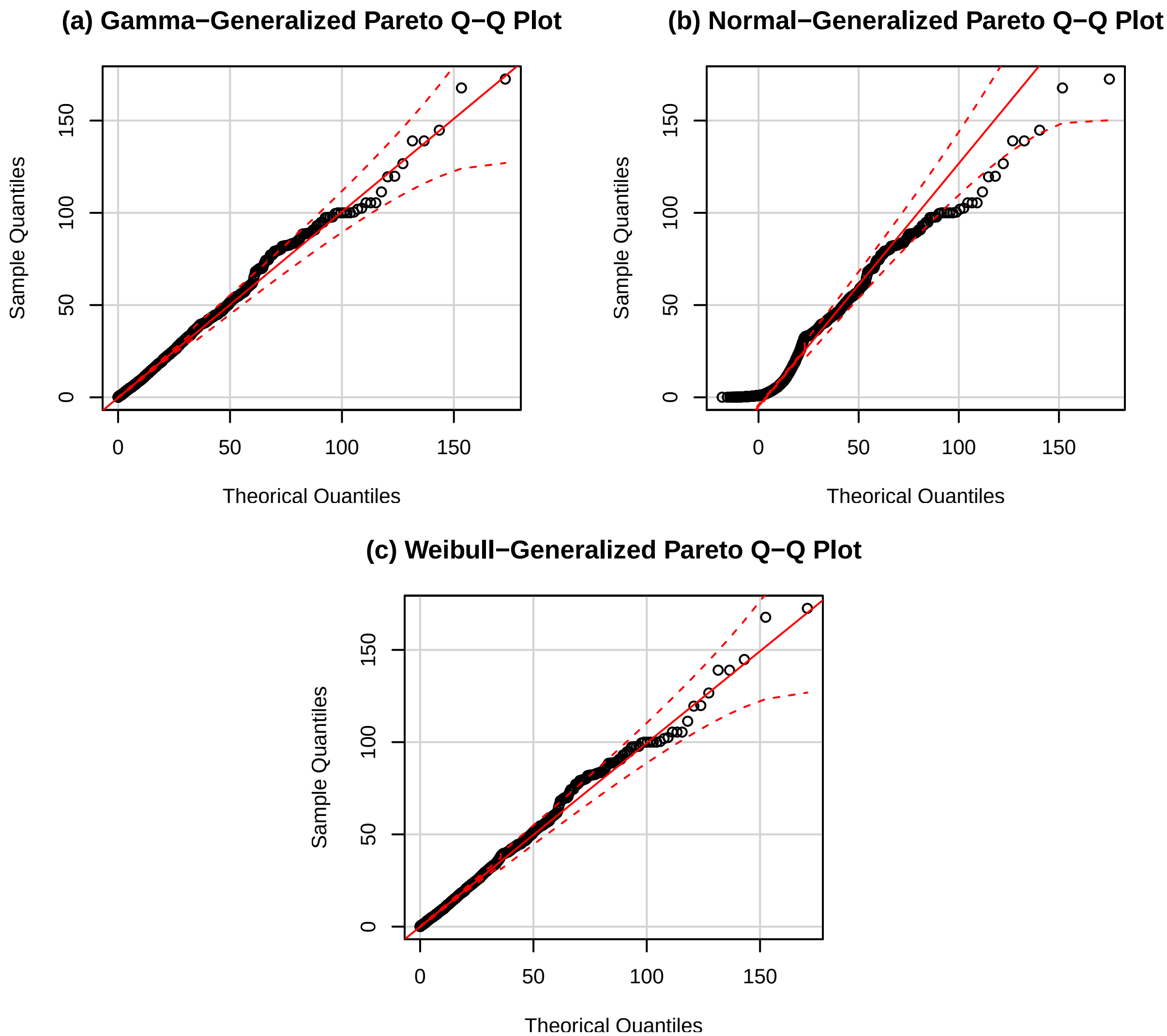

Figure 3.7: Q-Q plot spliced distribution for hospitalization

To perform the statistical test and evidence which mixture is the one with better fit to the individual costs, we use the goodness of fit tests Kolmogorov-Smirnov, Cramer-von Mises and Kuiper. Also, we use specialized goodness of fit tests for heavy tail distributions, namely, the Quadratic Class Upper Tail Anderson-Darling test and Supremum Class Upper Tail Anderson-Darling test (Chernobai, Rachev, and Fabozzi, 2015, pp. 584–585).

The hypothesis contrast is given by

$$\begin{aligned} H_0 &: F_{X_{hosp}}(x|\eta, u, \sigma_u, \xi) \in \hat{F}_{X_{hosp}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi}) \\ H_1 &: F_{X_{hosp}}(x|\eta, u, \sigma_u, \xi) \notin \hat{F}_{X_{hosp}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi}) \end{aligned} \tag{3.28}$$

with $F_{X_{hosp}}(x|\eta, u, \sigma_u, \xi)$ the distribution function of the individual costs with parameters η, u, σ_u, ξ and $\hat{F}_{X_{hosp}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi})$ the distribution function of the adjusted mixtures with estimated parameters $\hat{\eta}$, $\hat{u}$, $\hat{\sigma}_u$, $\hat{\xi}$).

To perform the goodness of fit tests, are used the functions `ks.test` for the Kolmogorov-Smirnov test, `w2.test` for the Cramer-von Mises test, `v.test` for the Kuiper test, `ad2up.test` for the Quadratic Class Upper Tail Anderson-Darling test and `adup.test` for the Supremum Class Upper Tail Anderson-Darling test, from the library `truncgof` (2012). See Code 23 in Appendix B.

Table 3.2: Goodness-of-fit tests hospitalization services for spliced distributions

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
gammagpd	0.87	0.89	0.79	0.04	0.01
normgpd	0.52	0.44	0.58	0.12	0.00
weibullgpd	0.91	0.91	0.87	0.08	0.00

In relation to the *G-GP* mixture, the Table 3.2 shows that for the goodness of fit tests Kolmogorov-Smirnov, Cramer-von Mises and Kuiper, the null hypothesis is not rejected, because are obtained *P-values* higher than the confidence level of 5%. In contrast, for specialized goodness of fit tests for heavy tail distributions, the Table 3.2 shows that are obtained *P-values* of 4% and 1% for the tests Supremum Class Upper Tail Anderson-Darling and Quadratic Class Upper Tail Anderson-Darling, respectively, indicating that the *G-GP* mixture does not offer a good fit in the tail of the individual costs distribution for hospitalization services.

With reference to the *N-GP* mixture, it is observed that despite not showing close values to the empirical statictics in the Table 3.1 or presenting good adjustments in the Figures 3.5, 3.6 and 3.7, we have that in the Table 3.2 only in the Quadratic Class Upper Tail Anderson-Darling test is rejected the adjustment of the mixture to individual costs for hospitalization services. Furthermore, it can be noted that unlike the *G-GP* mixture, the *N-GP* obtains a *P-value* higher than 5% for the Supremum Class Upper Tail Anderson-Darling test, indicating that this mixture presents a good fit in the right tail of the individual costs distribution.

With respect to the *W-GP* mixture, the Table 3.2 shows that, as for the *G-GP* and *N-GP* mixtures, the hypothesis (3.28) is not rejected for the Kolmogorov-Smirnov, Cramer-von Mises and Kuiper tests. Also, unlike the *G-GP* mixture, it is not rejected for the Supremum Class Upper Tail Anderson-Darling test, which as mentioned before, is a specialized test for heavy tail distributions.

From all the previous results it is observed that in the Figures 3.5, 3.6 and 3.7, no differences are found between the *G-GP* and *W-GP* mixture, therefore, the selection of the *spliced distribution* that best fits the individual costs of hospitalization services is based on the Tables 3.1 and 3.2, which show that it also does not reject the Supremum Class Upper Tail Anderson-Darling test, the *W-GP* mixture presented closer statistics to the empirical than those presented by the *G-GP* mixture.

Hence, it is assumed that the individual costs of hospitalization services have a *W-GP spliced distribution*, with parameters $\hat{W}_{shape} = 0.9417177$, $\hat{W}_{scale} = 13.39916$, $\hat{u} = 35.5702$, $\hat{\sigma}_u = 31.77728$, $\hat{\xi} = -0.1190043$, $\hat{\phi}_u = 0.1017503$. Being $\hat{\eta} = (\hat{W}_{shape}, \hat{W}_{scale})$ the parameter vector of the Weibull distribution and $\hat{u}, \hat{\sigma}_u, \hat{\xi}$ and $\hat{\phi}$ the parameter vector of the Generalized Pareto distribution.

3.4.2 Spliced distributions for general surgery services

Like for hospitalization services, are adjusted the mixtures *G-GP*, *N-GP*, and *W-GP* to the individual costs of general surgery services through the functions `fgammagpd`, `fnormgpd`, `fweibullgpd` from the library `evmix`(2018). See Code 24 in Appendix B.

The adjustment shows that *G-GP* and *W-GP* mixtures have thresholds of 16.12222 and 16.12227, respectively, which are very close to the threshold $u = 16.12321$ proposed by the Hill plots presented in the panels (a) and (b) of the Figure 3.4. It also shows that the *N-GP* mixture presents a threshold of 6.324662, which is very far from the threshold proposed in the Hill plots.

Again, are used the libraries `e1071` (2018) and `DistMom` (2018) to calculate the value of the mean, variance, skewness and excess kurtosis of the individual costs for general surgery services and the adjusted *spliced distributions*. See Code 25 in Appendix B.

Table 3.3: Statistical measurements of spliced distributions for general surgery services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	6.680426	82.89872	4.671353	33.36082
gammagdp	7.379634	does not exist	does not exist	does not exist
normgdp	7.142194	66.2263	2.819053	15.91668
weibullgdp	7.731212	does not exist	does not exist	does not exist

The Table 3.3 shows a very interesting result, since with respect to the *G-GP* and *W-GP* mixtures, no moments are found after the first, hence when applying the function `moments` with $k = 2$, and the functions `skew` and `kurt` from the library `DistMom` (2018) results in the message "*The asymptotic method does not converge, the value of the moment is very large or the moment of the distribution does not exist*". Therefore, because the tail of the mixtures is given by a Generalized Pareto distribution, it can be suggested from the *EVT*, that the individual costs for general surgery services have a heavy tail, with a regular variation index $1 < \alpha < 2$.

Another result of the Table 3.3, is that none of the three fitted mixtures has a mean close to the empirical value of the individual costs, being the closest the *N-GP* mixture. Moreover, it is evident that despite the *N-GP* mixture is the closest to the empirical value of the mean, it is observed that for the other statistics, its value differs considerably from the empirical values.

To show the adjustment of the *spliced distributions* are made three graphs, namely, the Figures 3.8, 3.9 and 3.10. In the first are presented the cumulative distributions, in the second, the natural logarithm of the survival distributions, and in the third, the Q-Q plot for the three adjustments. See Code 26, Code 27 and Code 28 in Appendix B.

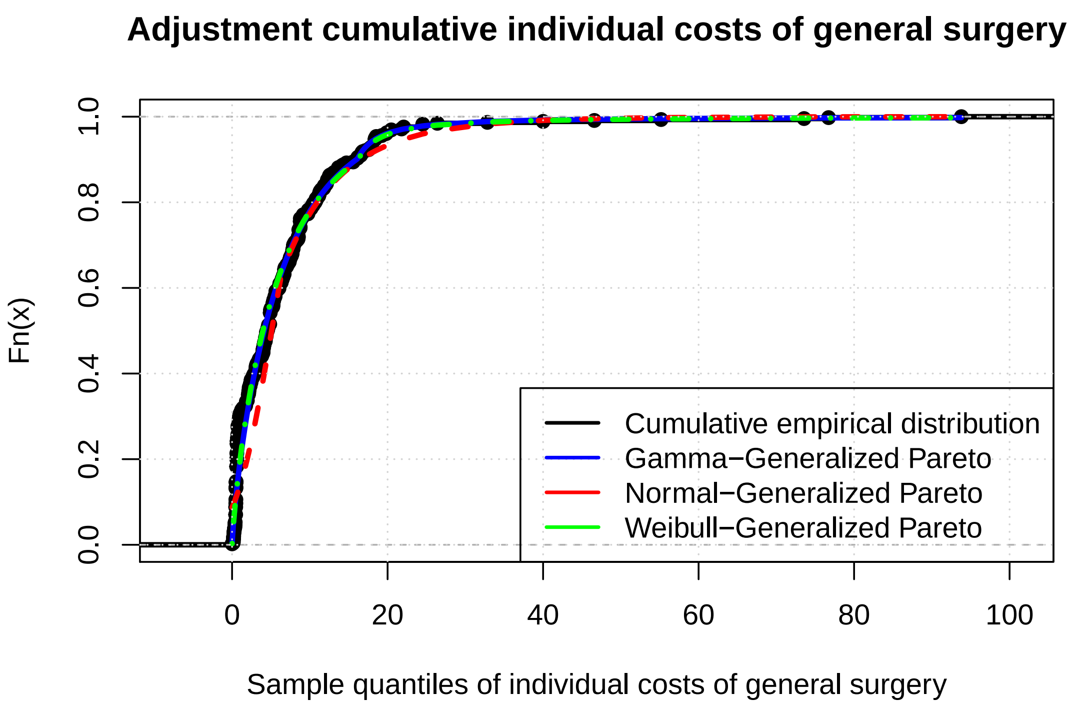

Figure 3.8: Adjustment of the individual costs of general surgery service for spliced distributions

The Figure 3.8 shows that the three adjusted mixtures present problems to capture the initial behavior of the empirical cumulative distribution, being the most notable difference the one presented by the *N-GP* mixture. Additionally, in the union area of the *G-GP* and *W-GP*, the Figure 3.8 shows again that the mixtures present problems to capture the behavior of the empirical cumulative distribution, where it is observed that the empirical curve are above the adjusted curves.

It should be noted that similar to the hospitalization case, in the Figure 3.8 it is evident that the mixture that presents the worst adjustment is the *N-GP*, and it is observed that there are no significant differences between the adjustment of the *G-GP* and *W-GP* mixtures.

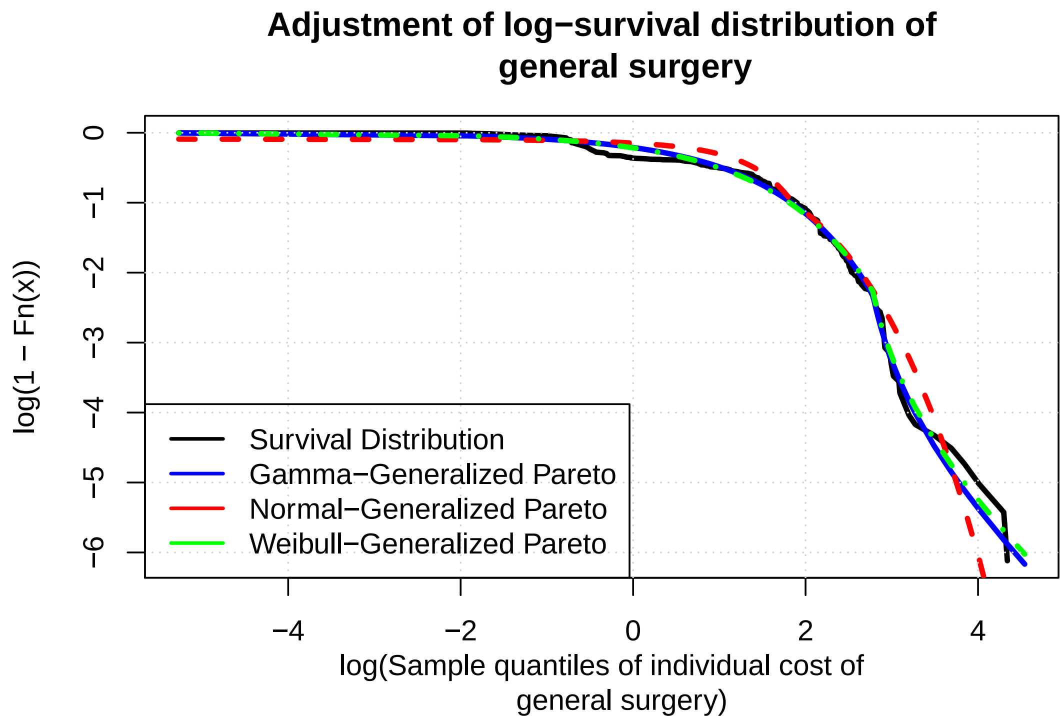

Figure 3.9: Adjustment of log-survival distribution of the individual cost in general surgery service for spliced distributions

The Figure 3.9 shows in more detail the tail behavior of the empirical distribution, where it is observed that both the empirical survival distribution and the *G-GP* and *W-GP* mixtures seem to have a slow decay in the tail area, more precisely, after the value 2.78026, which is the natural logarithm of the threshold value $u = 16.12321$.

It should be noted that in the final part of the Figure 3.9 it can be seen that the curve associated with the *W-GP* mixture decays more slowly than the curve representing the *G-GP* mixture, indicating that the *W-GP* has a heavier tail than the *G-GP*.

Additionally, the Figure 3.9 shows the bad adjustment made by the *N-GP* mixture, where it is evident that around 0, is well above the empirical value, later around 3, it is very far to the right of the empirical value, and finally around 4, it presents a faster decay to zero than the log-survival distribution.

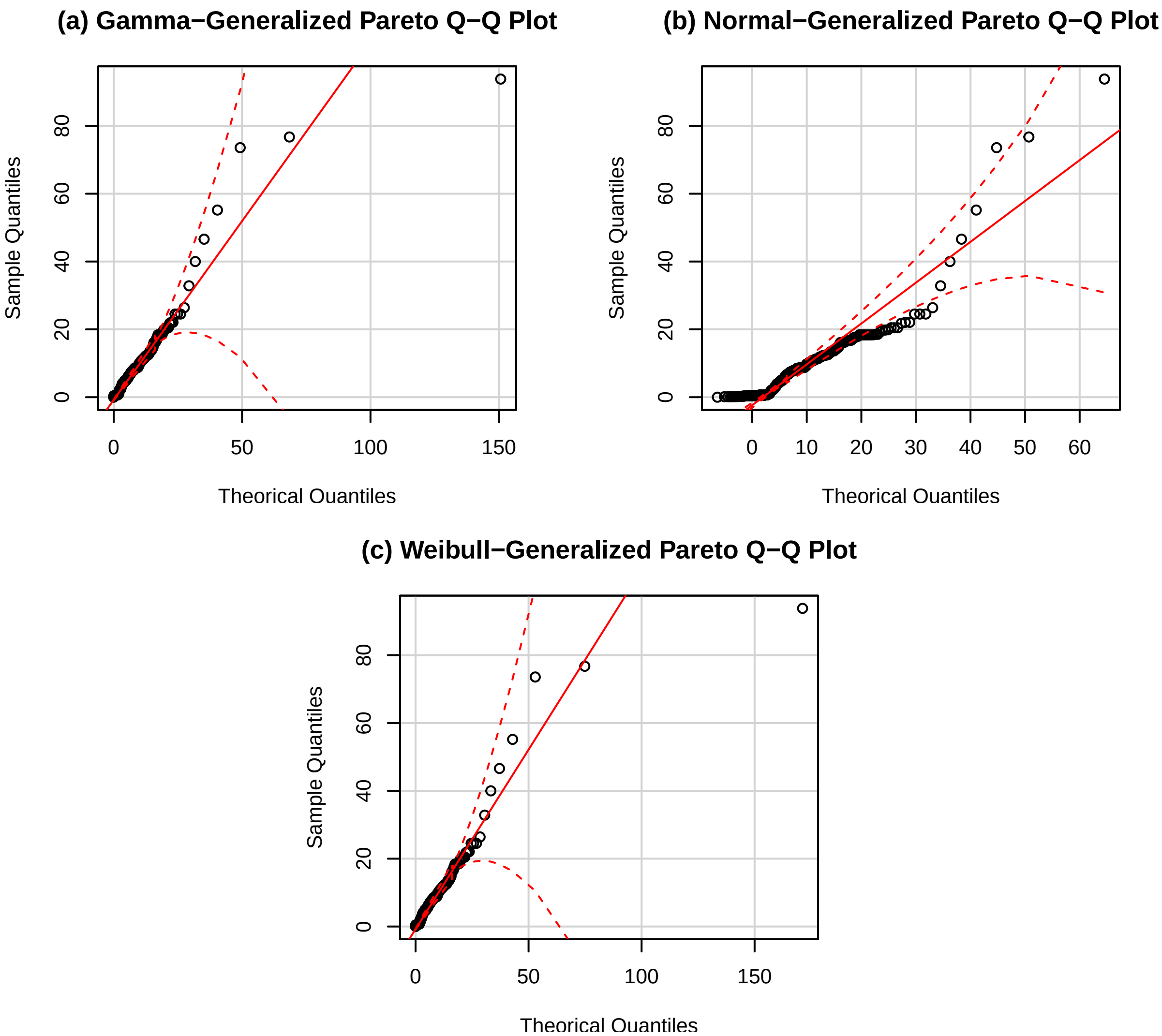

Figure 3.10: Q-Q plot spliced distribution for hospitalization

The Figures 3.10.a and 3.10.c show unusual behavior in their confidence bands, which could be related to the result obtained in the Table 3.3 on which the variance, skewness and excess kurtosis for these mixtures do not exist. Moreover, in both panels, there are several observations that despite being within the confidence bands are very far from the diagonal line.

For its part, the Figure 3.10.b shows a behavior similar to that presented in the hospitalization case, where it is evident that there is a group of observations that are below zero due to the domain that has the normal distribution. In addition, between the values 20 and 35, it is observed that a group of points appears that move away from the diagonal line, leaving outside the confidence bands.

To perform the statistical contrast, are used the conventional goodness-of-fit tests and specialized goodness-of-fit tests for heavy tail distributions, namely, Kolmogorov-Smirnov, Cramer-von Mises, Kuiper, Supremum Class Upper Tail Anderson-Darling and Quadratic Class Upper Tail Anderson-Darling.

For this purpose, is established the following hypothetical contrast

$$\begin{aligned} H_0 &: F_{X_{surg}}(x|\eta, u, \sigma_u, \xi) \in \hat{F}_{X_{surg}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi}) \\ H_1 &: F_{X_{surg}}(x|\eta, u, \sigma_u, \xi) \notin \hat{F}_{X_{surg}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi}) \end{aligned} \tag{3.29}$$

with $F_{X_{surg}}(x|\eta, u, \sigma_u, \xi)$ the distribution function of the individual costs for general surgery services with parameters η, u, σ_u, ξ and $\hat{F}_{X_{hosp}}(x|\hat{\eta}, \hat{u}, \hat{\sigma}_u, \hat{\xi})$ the distribution function of the adjusted mixtures with estimated parameters $\hat{\eta}$, $\hat{u}$, $\hat{\sigma}_u$, $\hat{\xi}$.

To perform the conventional goodness of fit tests, are employed the functions `ks.test`, `w2.test` and `v.test`, while for specialized goodness of fit tests for heavy tail distributions, are used the functions `adup.test` and `ad2up.test`, all belonging to the library `truncgof` (2012). See Code 29 in Appendix B.

Table 3.4: Goodness-of-fit tests general surgery services for spliced distributions

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
gammagpd	0.31	0.23	0.17	0.21	0.06
normgpd	0.62	0.65	0.78	0.02	0.00
weibullgpd	0.26	0.39	0.27	0.26	0.07

The Table 3.4 shows that the *G-GP* and *W-GP* mixtures have a similar behavior, in the sense that in none of the five goodness of fit tests reject the hypothesis (3.29). Moreover, it is appreciated that the *N-GP* mixture, reject the null hypothesis for the specialized tests in distributions of heavy tail, because are obtained *P-values* of 2% and 0%, respectively.

From the results obtained in this section, for the adjustment of the individual costs of general surgery services by means of *spliced distributions*, it is observed that no large differences were found between the *G-GP* and *W-GP* mixtures, since it is only possible to point out that in the Figure 3.9 the *W-GP* mixture has a slightly heavier tail than the *G-GP* and that in the Table 3.4 the *W-GP* mixture obtained slightly larger values in almost all the statistics than the *G-GP*.

Taking into consideration that no other test is available to select which of the two mixtures is the one that presents the best fit, it is assumed that the individual costs of general surgery services have a *W-GP spliced distribution*, with parameters $\hat{W}_{shape} = 0.8458848$, $\hat{W}_{scale} = 6.121305$, $\hat{u} = 16.12227$, $\hat{\sigma}_u = 2.802543$, $\hat{\xi} = 0.853189$ and $\hat{\phi}_u = 0.1034537$, where $\hat{\eta} = (\hat{W}_{shape}, \hat{W}_{scale})$ are the parameter vector of the Weibull distribution and $\hat{u}, \hat{\sigma}_u, \hat{\xi}$ and $\hat{\phi}$ are the parameter vector of the Generalized Pareto distribution.

Chapter 4

Aggregate Loss Distribution

Table of Contents

4.1 Introduction

After selecting the frequency distribution N_k for the claims number that occur during a one year period and the severity distribution X_k for the individual costs of medical services, for each portfolio[1], we proceed to calculate the aggregate loss distribution S_k, for the annual cost of medical services, for each of the k portfolios.

For this, we start with Embrechts et al. (1997, p. 24), which defines the total amount of claims S_k as

$$S_k = \begin{cases} \sum_{i=1}^{N_k} X_{k,i} & N_k > 0 \\ 0 & N_k = 0 \end{cases} \tag{4.1}$$

where, $X_{k,1}, X_{k,2}, \ldots, X_{k,N_k}$ is a succession of non-negative random variables *iid* $X_{k,i} > 0$, with cumulative distribution $F_{X_k}(x) = \mathbb{P}(X_k \leq x)$ with support $[0, \infty)$,

[1]The word "portfolio" refers to each of the medical services of interest.

such that $F_{X_k}(x) < 1, \forall x > 0$, $F_{X_k}(0_+) = 0$, and with $\mathbb{E}(X_{k,i}) < \infty$. Additionally, N_k and $X_{k,i}$ are taken as mutually independent random variables $\forall i = 1, 2, \ldots$ (Bowers et al., 1997, p. 367).

The compound distribution S_k is obtained through the weighted infinite sum of succesive N_k-convolutions of F_k with itself, where, depending on the distribution of N_k, it will have the distribution type of S_k, e.g., if N_k is distributed Poisson, then S_k will have a compound Poisson distribution or if N_k is distributed (Negative) Binomial, then S_k will have a compound (Negative) Binomial distribution (Kaas, Goovaerts, Dhaene, and Denuit, 2008, p. 41). Moreover, the cumulative distribution of the variable S_k is denoted by $F_{S_k}(x)$, and is obtained by the Total Probability Theorem

$$\begin{aligned} F_{S_k}(x) &= \mathbb{P}\left(\sum_{i=0}^{N_k} X_{k,i} \leq x\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(N_k = n)\mathbb{P}\left(\sum_{i=0}^{N_k} X_{k,i} \leq x | N_k = n\right) \\ &= \sum_{n=0}^{\infty} p_{n_k}\mathbb{P}(X_{k,1} + X_{k,2} + \ldots + X_{k,n} \leq x) \\ F_{S_k}(x) &= \sum_{n=0}^{\infty} p_{n_k} F_{X_k}^{*n}(x) \end{aligned} \tag{4.2}$$

where $p_{n_k} = \mathbb{P}(N_k = n)$ is the probability distribution of N_k evaluated in n and $F_{X_k}^{*n}(x) = \mathbb{P}(X_{k,1} + X_{k,2} + \ldots + X_{k,n} \leq x)$ is the n-th convolution of F_{X_k} with itself (Feller, 1978).

Since the compound distribution of S_k is an infinite weigthed sum of distributions, obtaining a closed form for F_{S_k} is often difficult to find analytically, which renders impossible to calculate probability or quantiles for S_k. Therefore, in order to overcome this difficulty, some authors have proposed functions and algorithms that allow to approximate the extreme quantiles for F_{S_k}.

These approaches seek to determine the value of the extreme quantiles in the right part of the aggregate loss distribution, in order to quantify the risk associated with high losses that may affect the company's financial health. Some of these approximations are described in Kaas et al. (2008, Chapters 2–3), Beard, Pentikáinen, and Pesonen (1984, Chapter 3) and Albrecher et al. (2017, Chapter 6).

4.2 Single Loss Approximation

Although there are different approximations, the implementation of each of them is beyond the scope of this work, therefore we focus on a single approach known as Single Loss Approximation (*SLA* onwards) which was introduced by Böcker and Klüppelberg (2005), in the case of heavy or regular variation distributions. The reason for this is that the *SLA* method provides a closed approximation asymptotic formula that does not require algorithms for its calculation, and also, it has been demonstrated in the literature that it offers more accurate results under different conditions than other methods, e.g., see Hess (2011), Opdyke (2014), Peters, Targino, and Shevchenko (2013).

To introduce the SLA method, we start from Embrechts et al. (1997, Theorem 1.3.9, p. 45) and Albrecher et al.(2017, p. 36), which assume that the severities $X_{k,i}$ have a distribution F_{X_k}, that is classified within subexponential class distributions, such that for any non-negative integer n, when $x \to \infty$ we have to

$$\lim_{x\to\infty} \frac{1 - F^{*n}_{X_k}(x)}{1 - F_{X_k}(x)} = \lim_{x\to\infty} \frac{\bar{F}^{*n}_{X_k}(x)}{\bar{F}_{X_k}(x)} = n \tag{4.3}$$

where, $\bar{F}_{X_k}(\cdot) = 1 - F_{X_k}(\cdot)$ is the tail distribution of the individual costs and $\bar{F}^{*n}_{X_k}(\cdot) = 1 - F^{*n}_{X_k}(\cdot)$ is the tail distribution of the convolution of n individual costs. Then, from Embrechts et al. (1997, Theorem 1.3.9, p. 45) we have for a fixed time $t > 0$ and assuming that p_{n_k} satisfies

$$\sum_{n=0}^{\infty} (1+\varepsilon)^n p_{n_k} < \infty \tag{4.4}$$

for some $\varepsilon > 0$, it is said that F_{S_k} is classify within the subexponential class, when its tail behavior is given by

$$\bar{F}_{S_k}(x) \sim \mathbb{E}(N_k)\bar{F}_{X_k}(x); \qquad x \to \infty \tag{4.5}$$

where, $\bar{F}_{S_k}(\cdot) = 1 - F_{S_k}(\cdot)$ is the tail distribution of the aggregate loss and $\mathbb{E}(N_k)$ is the expected value of the frequency distribution N_k. Furthermore, the symbol $\sim$ is equivalent to say that for every fixed time $t > 0$

$$\lim_{x\to\infty} \frac{\bar{F}_{S_k}(x)}{\mathbb{E}(N_k)\bar{F}_{X_k}(x)} = 1 \tag{4.6}$$

or equivalently

$$\lim_{x\to\infty} \frac{\bar{F}_{S_k}(x)}{\bar{F}_{X_k}(x)} = \mathbb{E}(N_k) \tag{4.7}$$

The equation (4.5), will then be known as the Böcker-Klüppelberg formula, and from this, is developed the whole mathematical procedure necessary to obtain the approximation formula of the method *SLA* for the Value at Risk, Expected Shortfall and Stop-Loss Premium, which are the most commonly used risk measures in practice[2].

[2]For other risk measures, see Denuit, Dhaene, Goovaerts, and Kaas (2005, Chapter 2) and Kaas et al. (2008, Chapter 5)

4.2.1 Value at Risk (VaR)

Bockër and Klüppelberg Approximation

To obtain the equality between the right and left terms of the equation (4.5), in Albrecher, Hipp, and Kortschak (2010, p. 106) is presented the first-order asymptotic approximation for compound sums, such that

$$\bar{F}_{S_k}(x) = \mathbb{E}(N_k)\bar{F}_{X_k}(x)(1 + o(1)); \qquad x \to \infty \tag{4.8}$$

where $o(1) \to 0$ when $x \to \infty$. It should be noted that, the equations (4.5) and (4.8) are related, being the term $(1 + o(1))$ the value that generates the equality between the left and right terms of the equation (4.5).

Since $\bar{F}_{S_k}(x) = 1 - F_{S_k}(x)$, then, by clearing the term $F_{S_k}(x)$, from the equation (4.8) we get

$$F_{S_k}(x) = 1 - \mathbb{E}(N_k)\bar{F}_{X_k}(x)(1 + o(1)); \qquad x \to \infty \tag{4.9}$$

Böcker and Klüppelberg (2005, p. 91) equals the right member of the equation (4.9) to a value κ, with $\kappa \to 1$, in order to obtain an asymptotic solution for $F_{X_k}(x)$, such that

$$F_{X_k}(x) = 1 - \frac{1-\kappa}{\mathbb{E}(N_k)}(1 + o(1)); \qquad x \to \infty \tag{4.10}$$

Applying the inverse transformation $F^{-1}_{X_k}$ in both sides of the equation (4.10), we obtain that the value x is given by

$$x = F^{-1}_{X_k}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}(1 + o(1))\right); \qquad \kappa \to 1 \tag{4.11}$$

Similarly, Böcker and Klüppelberg (2005, p. 91) equals the left side of the equation (4.9) to the same value κ, and applies the inverse function $F^{-1}_{S_k}$ on both sides of the equation, to obtain the next value for x

$$x = F^{-1}_{S_k}(\kappa); \qquad \kappa \to 1 \tag{4.12}$$

Finally, by equating (4.11) and (4.12), the authors obtain a closed expression for the calculation of the value x associated with an extreme quantiles of the aggregate loss function S_k, such that

$$x = F^{-1}_{S_k}(\kappa) = F^{-1}_{X_k}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}(1 + o(1))\right); \qquad \kappa \to 1 \tag{4.13}$$

From the equation (4.13), Böcker and Klüppelberg (2005, p. 91), present the approximation of the *VaR* by means of the *SLA* method, for a level κ, with $0 < \kappa < 1$ and defined as the κ-quantile of the aggregate loss distribution

$$VaR_{S_k}(\kappa) = F^{-1}_{X_k}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}(1 + o(1))\right); \qquad \kappa \to 1 \tag{4.14}$$

or equivalently

$$VaR_{S_k}(\kappa) \sim F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right); \qquad \kappa \to 1 \tag{4.15}$$

Bockër and Sprittulla Approximation

Although the equation (4.15) has a very attractive structure, in Böcker and Sprittulla (2006, p. 96), the authors point out that this equation must be used carefully because this approximation underestimate the *VaR*, therefore, its estimation error could be considerably large. The underestimation of the *VaR*, is because the aproximation does not take into account all loss events $X_{k,i}$, which contribute to the aggregate loss S_k.

Given the above and in order to refine the equation (4.13), Böcker and Sprittulla (2006, p. 97) start from one of the properties presented in Embrechts et al. (1997, p. 41) for subexponential distributions

$$\lim_{x\to\infty} \frac{\bar{F}_{X_k}(x-y)}{\bar{F}_{X_k}(x)} = 1; \qquad y \in \mathbb{R}^+ \tag{4.16}$$

this property allows authors to present the following two relationships for $F_{S_k}(x)$ and $F_{X_k}(x)$, assuming that the individual losses distribution has a finite mean ($\mathbb{E}(X_{k,i} < \infty)$)

$$\begin{aligned} \bar{F}_{S_k}(x) &\sim \bar{F}_{S_k}(x + \mathbb{E}(N_k)\mathbb{E}(X_{k,i})); \qquad x \to \infty \\ \bar{F}_{X_k}(x) &\sim \bar{F}_{X_k}(x + E(X_{k,i})); \qquad x \to \infty \end{aligned} \tag{4.17}$$

Then, by replacing the equation (4.17) in (4.5), is obtained

$$\bar{F}_{S_k}(x + \mathbb{E}(N_k)\mathbb{E}(X_{k,i})) \sim \mathbb{E}(N_k)\bar{F}_{X_k}(x + E(X_{k,i})); \qquad x \to \infty \tag{4.18}$$

From this result, Böcker and Sprittulla (2006, p. 97) present an improved approximation for the *VaR*, which seeks to correct the underestimation of *VaR*, by adding the correction constant given by $C = (\mathbb{E}(N_k) - 1)\mathbb{E}(X_{k,i})$. Given the above, the authors define the improved approximation for the *VaR* as

$$VaR_{S_k}(\kappa) = F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}(1 + o(1))\right) + (\mathbb{E}(N_k) - 1)\mathbb{E}(X_{k,i}); \qquad \kappa \to 1 \tag{4.19}$$

where it is evident that the $VaR_{S_k}(\kappa)$ obtained in this equation has a similar structure to the one presented in the equation (4.14), but differently in the equation (4.19) there is a second term that does not depend on the confidence level κ, and represents a constant value associated with the expected loss size.

This value is denoted by Böcker and Sprittulla (2006, p. 97) as a mean correction factor for the VaR and point out that the value $(\mathbb{E}(N_k)-1)\mathbb{E}(X_{k,i})$ could be replaced by any other constant value, because independently of the assumed constant, the limit behavior of the equation (4.17) is preserved. They also point out that the constant selected by them is not the one that produces the most accurate approximation of VaR, but it is selected, due the stronger convergence properties hold when using centered random variables.

Degen Approximation

After the proposal of Böcker and Sprittulla (2006), Degen (2010, p. 7), points out that the constant mean correction factor proposed by Böcker and Sprittulla (2006) lacks a mathematical justification and shows that this can be replaced by a non-constant factor of the order $o(F^{-1}_{X_k}(\kappa))$, with $\kappa \to 1$, such that

$$F^{-1}_{S_k}(\kappa) = F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}(1+o(1))\right)\left(1+\underbrace{\frac{(\mathbb{E}(N_k)-1)\mathbb{E}(X_{k,i})}{F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}(1+o(1))\right)}}_{=o(1)}\right); \qquad \kappa \to 1 \tag{4.20}$$

In order to give analytical support to the SLA method, Degen (2010) proposes an estimation framework for the extreme quantiles calculation for the aggregate loss distribution, which can be applied with a wide range of severity distributions, regardless of whether or not they have finite mean.

To carry out his proposal, Degen (2010, p. 5) began with the second order subexponential theory given by the expansion

$$\bar{F}_{S_k}(x) = \mathbb{E}(N_k)\bar{F}_{X_k}(x)(1+cb(x)+o(b(x))) \qquad x \to \infty \tag{4.21}$$

with

$$b(x) = \begin{cases} \frac{f_{X_k}(x)}{\bar{F}_{X_k}(x)} & \text{if } \mathbb{E}[X_k] < \infty \\ \frac{\mu_{F_{X_k}}(x) f_{X_k}(x)}{\bar{F}(x)} & \text{if } \mathbb{E}[X_k] = \infty \end{cases} \tag{4.22}$$

$$c = \begin{cases} \frac{\mathbb{E}[X_k]\mathbb{E}[(N_k-1)N_k]}{N_k} & \text{if } \mathbb{E}[X_k] < \infty \\ \frac{c_\xi \mathbb{E}[(N_k-1)N_k]}{\mathbb{E}[N_k]} & \text{if } \mathbb{E}[X_k] = \infty \end{cases} \tag{4.23}$$

and with $\mu_{F_{X_k}}(x)$ and c_ξ given by (Degen, 2010, p. 15)

$$\mu_{F_{X_k}}(x) = \int_0^x F_(s)\mathrm{d}s \tag{4.24}$$

$$c_\xi = \begin{cases} 1 & \text{if } \xi = 1 \\ (1-\xi)\frac{\Gamma^2(1-1/\xi)}{2\gamma(1-2/\xi)} & \text{if } 1 < \xi < \infty \end{cases} \tag{4.25}$$

When calculating the inverse relationship of the equation (4.21), through the procedure described between the equations (4.8) to (4.14), we obtain

$$F_{S_k}^{-1}(\kappa) = F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\frac{1}{1+cb(F_{X_k}^{-1}(\tilde{\kappa})) + o(b(F_{X_k}^{-1}(\tilde{\kappa})))}\right); \qquad \kappa \to 1 \tag{4.26}$$

with $\tilde{\kappa} = 1 - (1-\kappa)/\mathbb{E}(N_k)$.

Additionally, the author proposes an equation that allows to measure the approximation error that exists between the quantile function of the aggregate losses $F_{S_k}^{-1}(\cdot)$ and the quantile function of the individual losses $F_{X_k}^{-1}(\cdot)$, for different levels of κ, such that

$$e(\kappa) = \frac{F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right)}{F_{S_k}^{-1}(\kappa)} - 1; \qquad \kappa \in (0,1) \tag{4.27}$$

By clearing the term $F_{S_k}^{-1}(\kappa)$ from the equation (4.27), Degen (2010, p. 8) gets an improved approximation equation for the quantiles calculation of the aggregate loss distributions

$$F_{S_k}^{-1}(\kappa) = \frac{F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right)}{1+e(\kappa)}; \qquad \kappa \to 1 \tag{4.28}$$

where it is shown that, for the particular case of heavy tail distributions with tail index $\xi > 0$, and when $\kappa \to 1$, the term $\frac{1}{1+e(\kappa)}$ is equal to

$$\frac{1}{1+e(\kappa)} = 1 + \xi cb(F_{X_k}^{-1}(\tilde{\kappa})) + o(b(F_{X_k}^{-1}(\tilde{\kappa}))); \qquad \kappa \in (0,1) \tag{4.29}$$

Finally, from the equations (4.26) and (4.29), Degen (2010) derives three approximations for the extreme quantiles calculation for the aggregate loss distribution. Namely, the first is used when it has severities distribution with heavy tail and finite mean, the second is employed when the severities distribution has heavy tail and infinite mean, and the third is used when the tail of the severities distribution is semi-heavy.

Heavy-tailed, finite mean: It arises when the distribution of the individual costs X_k is classified within the subexponential class of distributions with a tail index $0 < \xi < 1$. In such situation, we have $b(x) \sim 1/(\xi x)$, with $x \to \infty$, which leads to the improved *SLA* approximation proposed by Degen (2010, p. 10) of

$$F_{S_k}^{-1}(\kappa) \approx F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) + \mathbb{E}(X_k)(\mathbb{E}(N_k) + d - 1) \tag{4.30}$$

where d is a dispersion factor defined as $d = Var(N_k)/\mathbb{E}(N_k))$. An alternative structure of the equation (4.30) derived from by Albrecher et al. (2017, p. 199), which replaces the $Var(N_k)$ of the term d by $\mathbb{E}(N_k^2) - \mathbb{E}(N_k)^2$ obtaining

$$F_{S_k}^{-1}(\kappa) \approx F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) + \mathbb{E}(X_k)\left(\mathbb{E}(N_k) + \frac{\mathbb{E}(N_k^2) - \mathbb{E}(N_k)^2}{\mathbb{E}(N_k)} - 1\right) \tag{4.31}$$

or equivalently

$$F_{S_k}^{-1}(\kappa) \approx F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) + \mathbb{E}(X_k)\left(\frac{\mathbb{E}(N_k^2)}{\mathbb{E}(N_k)} - 1\right) \tag{4.32}$$

Heavy-tailed, infinite mean: The heavy tail distributions with infinite mean, are presented by two different approximations, namely, when the tail index ξ is equal to 1 or when it is greater than 1. If $\xi > 1$, Degen (2010, p. 10) points out that $b(x) = \mu_{F_{X_k}} f_{X_k}/F_{X_k}$, which guarantees by regular variation theory that

$$b(x) \sim \frac{1}{\xi - 1}\bar{F}(X_k); \qquad x \to \infty \tag{4.33}$$

From this and the equation (4.29), is given place the following approximation when the tail index $\xi > 1$

$$F_{S_k}^{-1}(\kappa) \approx F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) - (1-\kappa)F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right)\frac{c_\xi}{1-(1/\xi)}\left(1 + \frac{d-1}{\mathbb{E}(N_k)}\right) \tag{4.34}$$

Similarly, we have the special event of $\xi = 1$, Degen (2010, p. 10) states that $b(x) = \mu_{F_{X_k}}/F_{X_k}$, when $x \to \infty$, and presents the following approximation for the *SLA* method

$$F_{S_k}^{-1}(\kappa) \approx F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) + \left(\mathbb{E}(N_k) + d - 1\right)\mu_{F_{X_k}}\left(F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right)\right) \tag{4.35}$$

Semi-heavy tailed: This occurs when the tail index $\xi = 0$, which does not make possible to directly use the equation (4.29) to obtain the *SLA* approximation for the extreme quantiles calculation of the aggregate loss distribution. Due to this, and in order to obtain an approximation for the *SLA* method, Degen (2010, p. 12) derives the following equation

$$\frac{1}{1 + e(\kappa)} = 1 + \frac{c}{F_{X_k}^{-1}(\tilde{\kappa})} \tag{4.36}$$

where c is defined in the equation (4.25). Moreover, by replacing the equation (4.36) in (4.28), Degen (2010, p. 12) obtains that the approximation *SLA* has an identical structure to the equation (4.30) given by

$$F_{S_k}^{-1}(\kappa) = F_{X_k}^{-1}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_k)}\right) + \mathbb{E}(X_k)(\mathbb{E}(N_k) + d - 1) \tag{4.37}$$

4.2.2 Expected Shortfall (ES/TVaR/AVaR/CVaR)

With the purpose of providing a closed equation for the *ES* calculation, Biagini and Ulmer (2009) employ the results obtained by Böcker and Klüppelberg (2005), and find an expression that associates *VaR* and *ES*, such that

$$ES_{S_k}(\kappa) = \frac{\mathbb{E}(N_k)}{1-\kappa}\frac{\alpha}{\alpha-1}VaR_{S_k}(\kappa)\bar{F}_{X_k}(VaR_{S_k}(\kappa)) \tag{4.38}$$

where $\bar{F}_{X_k} \in RV_{-\alpha}$, with α the regular variation index. In his work, Biagini and Ulmer (2009) assumes that the $VaR_{S_k}(\kappa)$ is given by the equation (4.15), therefore, when applying the Asymptotic Investment Theorem presented in Bingham, Goldie, and Teugels (1989, Theorem 1.5.12, p. 28), it is possible to replace $\bar{F}_{X_k}(VaR_{S_k}(\kappa))$ by

$$\bar{F}_{X_k}\left(F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)\right) = \frac{1-\kappa}{\mathbb{E}(N_k)} \tag{4.39}$$

Then, by replacing the equation (4.39) in (4.38) we get

$$ES_{S_k}(\kappa) \sim \frac{\alpha}{\alpha-1}VaR_{S_k}(\kappa); \qquad \kappa \to 1 \tag{4.40}$$

It is important to note that the result shown on (4.40) will only be fulfilled if the $VaR_{S_k}(\kappa)$ is given by the equation (4.15), therefore, the procedure described in Biagini and Ulmer (2009, p. 8) could only be applied when is used the *VaR* proposed by Böcker and Klüppelberg (2005), which as it was previously mention, has the problem of not taking into consideration all loss events $X_{k,i}$, that contribute to the aggregate loss S_k, generating an underestimation of *VaR*.

Due to the above, if is replaced the *VaR* proposed by Böcker and Klüppelberg (2005) in the equation (4.40), by a corrected *VaR* such as those proposed in Böcker and Sprittulla (2006) or Degen (2010), with the structure

$$VaR_{S_k}(\kappa) = F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right) + C \tag{4.41}$$

it would be obtained by following the procedure described in Biagini and Ulmer (2009, p. 8) an approximation for *ES* with the structure

$$ES_{S_k}(\kappa) = \frac{\mathbb{E}(N_k)}{1-\kappa}\frac{\alpha}{\alpha-1}\left[F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)+C\right]\bar{F}_{X_k}\left(F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)+C\right) \tag{4.42}$$

At this point, it is not possible to conclude the procedure described in Biagini and Ulmer (2009, p. 8), since it would not be mathematically correct to replace the equation (4.41) in the equation (4.40), because the correction factor C, therefore, it is not possible to conclude the following equalities

$$\bar{F}_{X_k}\left(F^{-1}_{X_k}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)+C\right) = \frac{1-\kappa}{\mathbb{E}(N_k)}+C \tag{4.43}$$

or

$$\bar{F}_{X_k}\left(F_{X_k}^{-1}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)+C\right)=\frac{1-\kappa}{\mathbb{E}(N_k)}+F_{X_k}^{-1}\left(C\right) \tag{4.44}$$

when C is a variable that represents those constant and non-constant values found to the right of $F_{X_k}^{-1}\left(1-\frac{1-\kappa}{\mathbb{E}(N_k)}\right)$ in the equations (4.19), (4.30), (4.34), (4.35) and (4.37).

Then, we decided to use the basic definition of *ES* presented in Kaas et al. (2008, p. 129) and Denuit et al. (2005, p. 72) as

$$ES_{S_k}(\kappa)=\frac{1}{1-\kappa}\int_{\kappa}^{1}VaR_{S_k}(\theta)\mathrm{d}\theta \tag{4.45}$$

which despite not having a closed form as proposed by Biagini and Ulmer (2009), it is simple to numerically calculate and takes into account those constant and non-constant C corrections proposed by Böcker and Sprittulla (2006) and Degen (2010). If we replaced the equation (4.41) in (4.45) we will have that $ES_{S_k}(\kappa)$ would be determined by

$$ES_{S_k}(\kappa)=\frac{1}{1-\kappa}\left[\int_{\kappa}^{1}\left(F_{X_k}^{-1}\left(1-\frac{1-\theta}{\mathbb{E}(N_k)}\right)+C\right)\mathrm{d}\theta\right] \tag{4.46}$$

In the particular case in which the term C is constant, i.e, it does not depend on θ, the $ES_{S_k}(\kappa)$ can be rewritten as

$$ES_{S_k}(\kappa)=\frac{1}{1-\kappa}\left[\int_{\kappa}^{1}F_{X_k}^{-1}\left(1-\frac{1-\theta}{\mathbb{E}(N_k)}\right)\mathrm{d}\theta\right]+C \tag{4.47}$$

4.2.3 Stop-Loss Premium (SLP)

To obtain an expression for the *SLP* calculation, we use the definition presented by Denuit et al. (2005, p. 73), in which the authors propose the relationship that *SLP* has with respect to *VaR* and *ES*. To find this relationship, the authors start from the basic definition of *SLP* given by

$$SLP_{S_k}(\kappa)=\mathbb{E}[(S_k-VaR_{S_k}(\kappa))_+] \tag{4.48}$$

when applying the definition of the expected value, the equation (4.48) can be rewritten as

$$SLP_{S_k}(\kappa)=\int_{0}^{1}\left(VaR_{S_k}(\theta)-VaR_{S_k}(\kappa)\right)_+\mathrm{d}\theta \tag{4.49}$$

$$=\int_{\kappa}^{1}VaR_{S_k}(\theta)\mathrm{d}\theta-VaR_{S_k}(\kappa)(1-\kappa) \tag{4.50}$$

Now, from the definition of ES_{S_k} given in (4.45), the equation (4.49) can be rewritten as

$$SLP_{S_k}(\kappa)=(1-\kappa)\left[ES_{S_k}(\kappa)-VaR_{S_k}(\kappa)\right] \tag{4.51}$$

which gives a simple expression for the *SLP* calculation, based on the values obtained from the *VaR* and *ES* calculation.

4.3 Risk measures estimation for hospitalization services

To estimate the risk measures associated with the aggregate loss distribution for hospitalization services, it is necessary to identify the the tail index value ξ associated with the adjusted distribution to the individual costs of hospitalization services. For this end, we use the equation proposed by De Haan and Ferreira (2006, p. 17) defined in (3.14) as

$$\lim_{x\to\infty} \frac{x f_{X_k}(x)}{\overline{F}_{X_k}(x)} = \frac{1}{\xi} \tag{4.52}$$

Then, by inverting the place of the numerator and the denominator, we have an equation that allows us to obtain the tail index value ξ as

$$\lim_{x\to\infty} \frac{\overline{F}_{X_k}(x)}{x f_{X_k}(x)} = \xi \tag{4.53}$$

It should be noted that sometimes, the value $\overline{F}_{X_k}(x)$ can get faster to 0 than $f_{X_k}(x)$ when $x \to \infty$, or vice versa, therefore, is used a graphical representation of the behavior of the equation (4.53), with the objective of looking for stable region in the graph, since this stable region is associated to the value of tail index ξ. See Code 30 in Appendix B.

In the Figure 4.1 it is observed that as x increases, the limit value presented in the equation (4.53) decreases until it reaches zero, which means that the tail index value for the individual costs distribution of hospitalization services is equal to zero, i.e, $\xi = 0$.

Given the previous result, the individual costs distribution has to be found in the semi-heavy tail situation proposed by Degen (2010, p. 12), hence, to carry out the estimation of the *VaR*, is used the equation of (4.37) given by

$$VaR_{S_{hosp}}(\kappa) = F^{-1}_{X_{hosp}}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_{hosp})}\right) + \mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp}) + d - 1) \tag{4.54}$$

where $d = Var(N_{hosp})/\mathbb{E}(N_{hosp})$. From the equation (4.54) it is observed that the $VaR_{S_{hosp}}$ depends on a value κ, which represents the κ-quantile of the aggregate costs distribution, of the quantile function $F^{-1}_{X_{hosp}}$ and the expected value $\mathbb{E}(X_{hosp})$ of the individual costs, and of the expected value $\mathbb{E}(N_{hosp})$ and variance $Var(N_{hosp})$ of the claims number for hospitalization services.

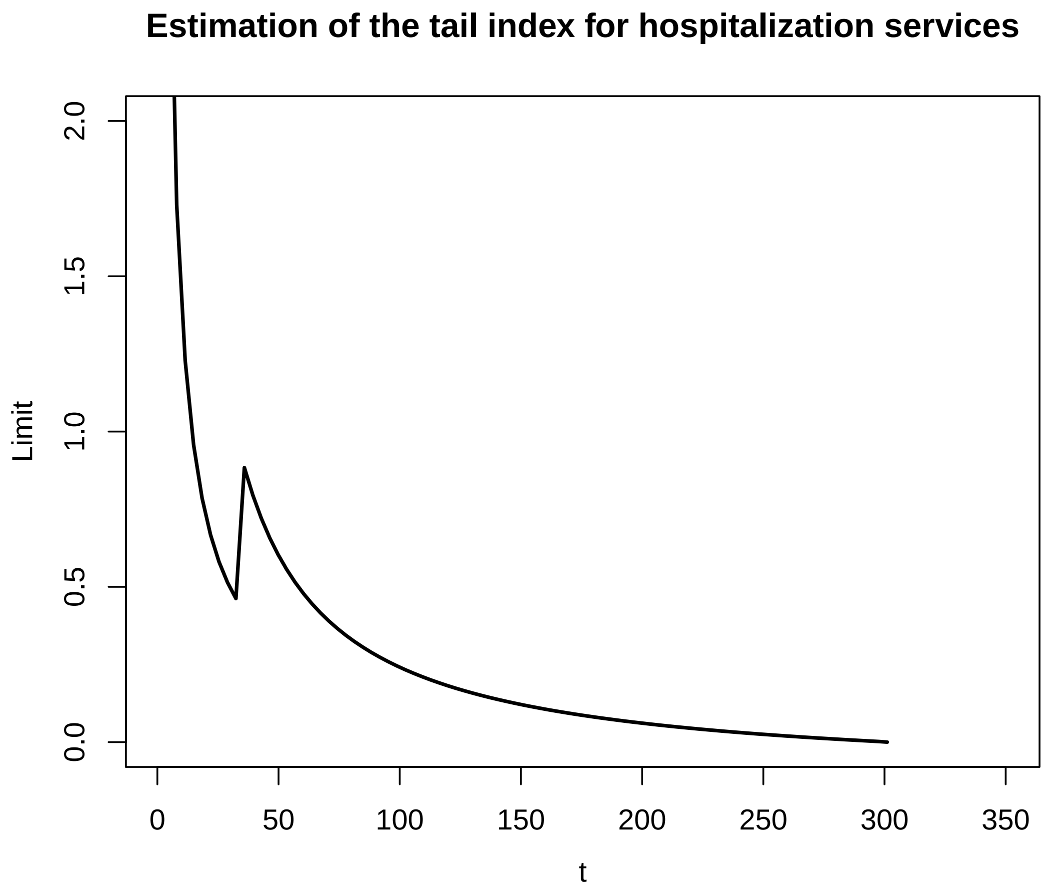

Figure 4.1: Tail index estimation for hospitalization services

Since the interest of the $VaR_{S_{hosp}}(\kappa)$ is centered for values of κ close to 1, is shown in the Figure 4.2 the value of the $VaR_{S_{hosp}}(\kappa)$ for values of κ between 0.9 and 0.999. Additionally, to perform the calculation of the quantile function $F^{-1}_{X_{hosp}}$, is used the function `qweibullgpd` of the library `evmix`(2018), with the parameters obtained in the adjustment process.

Similarly, for the calculation of the expected values $\mathbb{E}(X_{hosp})$, $\mathbb{E}(N_{hosp})$ and variance $Var(N_{hosp})$, we use the values presented in the Tables 3.1 and 2.2, which were obtained by the function `moments` of the library `DistMom`(2018). See Code 31 in Appendix B.

From the Figure 4.2 it is evident that the *VaR* for the aggregate costs of hospitalization services, when $0.9 < \kappa < 0.999$, it is between 6250.264 and 6307.058 million of pesos, where as specific cases, the $VaR_{S_{hosp}}(\kappa) = 6260.922$ when $\kappa = 0.95$ and $VaR_{S_{hosp}}(\kappa) = 6282.528$ when $\kappa = 0.99$.

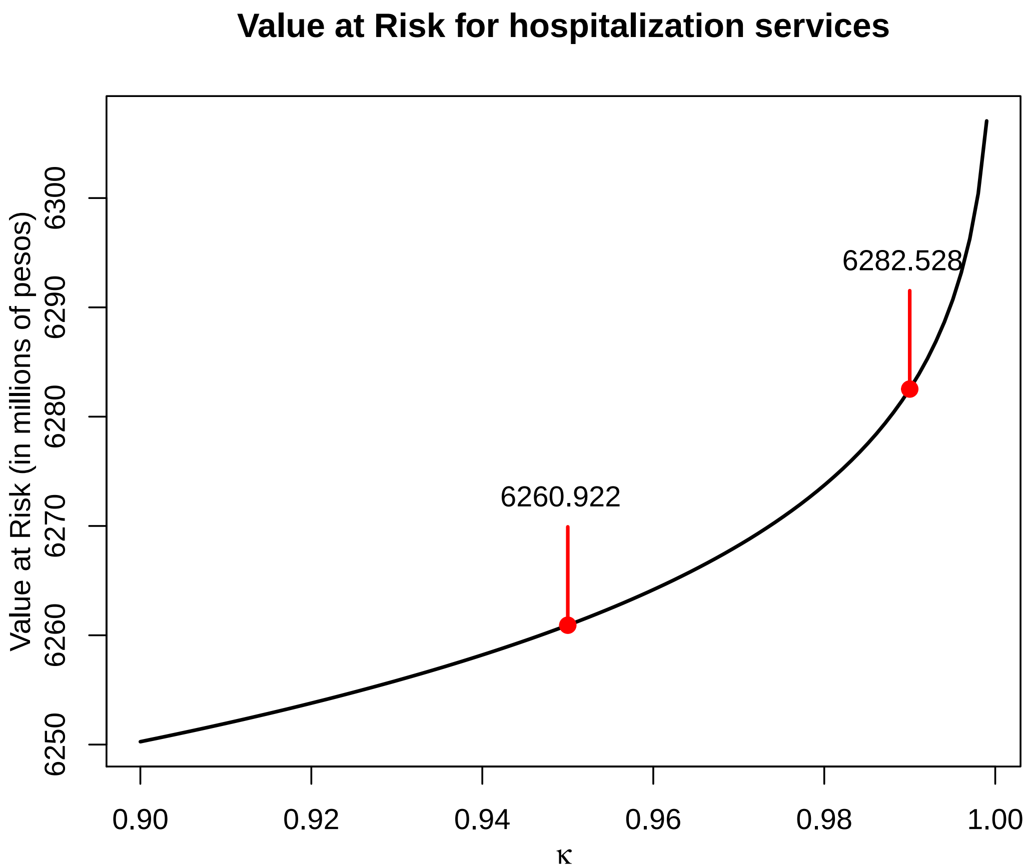

Figure 4.2: Value at Risk for hospitalization services

To perform the *ES* calculation, we start from the equation (4.47), which takes into account those constant and non-constant C values proposed to correct the *VaR*. In this case, the individual costs distribution has a semi-heavy tail, therefore, the C value of the equation (4.47) is replaced by the correction constant $\mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp})+d-1)$ raised by Degen (2010, p. 12) and presented in the equation (4.37), obtaining that the *ES* for the aggregate costs of hospitalization services is given by

$$ES_{S_{hosp}}(\kappa) = \frac{1}{1-\kappa}\left[\int_{\kappa}^{1} F^{-1}_{X_{hosp}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{hosp})}\right)\mathrm{d}\theta\right] + \mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp}) + d - 1) \tag{4.55}$$

from the equation (4.55) it is observed that the $ES_{S_{hosp}}(\kappa)$, depends on κ, $F^{-1}_{X_{hosp}}$, $\mathbb{E}(X_{hosp})$, $\mathbb{E}(N_{hosp})$ and $Var(N_{hosp})$, as in the $VaR_{S_{hosp}}(\kappa)$ case, but unlike this one, the $ES_{S_{hosp}}(\kappa)$ has an integral. To solve this integral, is used the `integrate` function of the library `stats` (2018). See Code 32 in Appendix B.

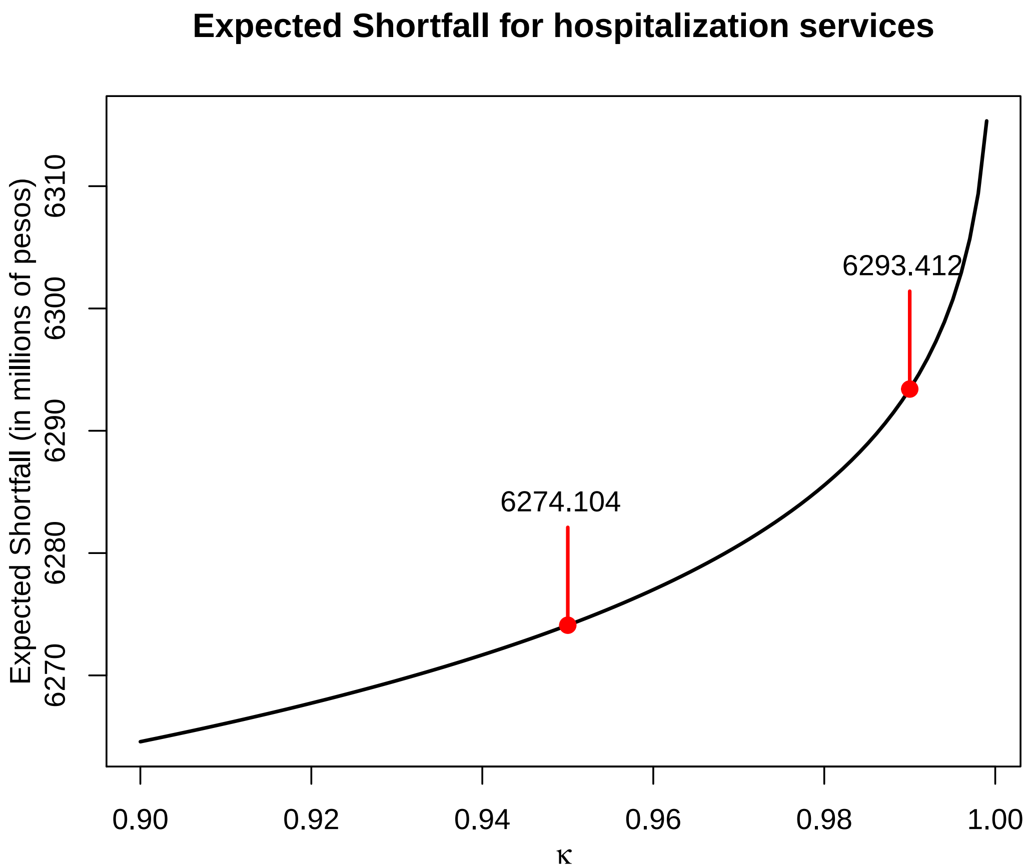

Figure 4.3: Expected Shortfall for hospitalization services

From the Figure 4.3 it can be seen that the ES is between the 6264.579 and 6315.334 millions pesos for values of κ between 0.90 and 0.999, where it is noted that, for specific values of κ, we have that $ES_{S_{hosp}}(0.95) = 6274.104$ and $ES_{S_{hosp}}(0.99) = 6293.412$, where said values are greater than those found for the VaR with the same quantiles.

Finally, to perform the SLP calculation, we employed the equation (4.51), which suggests that, once are calculated the values of the $VaR_{S_{hosp}}(\kappa)$ and $ES_{S_{hosp}}(\kappa)$, then the $SLP_{S_{hosp}}(\kappa)$, can be calculated by replacing the equations (4.54) and (4.55) in (4.51), obtaining

$$\begin{aligned} SLP_{S_{hosp}}(\kappa) =&(1-\kappa)\left(\frac{1}{1-\kappa}\left[\int_{\kappa}^{1} F_{X_{hosp}}^{-1}\left(1-\frac{1-\theta}{\mathbb{E}(N_{hosp})}\right)\mathrm{d}\theta\right]+\right. \\ &\mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp})+d-1) - F_{X_{hosp}}^{-1}\left(1-\frac{1-\kappa}{\mathbb{E}(N_{hosp})}\right) - \\ &\left.\mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp})+d-1)\right) \end{aligned} \tag{4.56}$$

where it is noted that the term $\mathbb{E}(X_{hosp})(\mathbb{E}(N_{hosp})+d-1)$ disappears from the equation (4.56), finally obtaining that SLP for hospitalization services is given by

$$SLP_{S_{hosp}}(\kappa) = \left[\int_{\kappa}^{1} F^{-1}_{X_{hosp}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{hosp})}\right) \mathrm{d}\theta\right] - (1-\kappa)F^{-1}_{X_{hosp}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{hosp})}\right) \tag{4.57}$$

this indicates that, if constant values are used to correct the VaR, the SLP calculated value would not be affected. Below it is graphically present the behavior of the SLP, for values of κ between 0.90 and 0.999. See Code 33 in Appendix B.

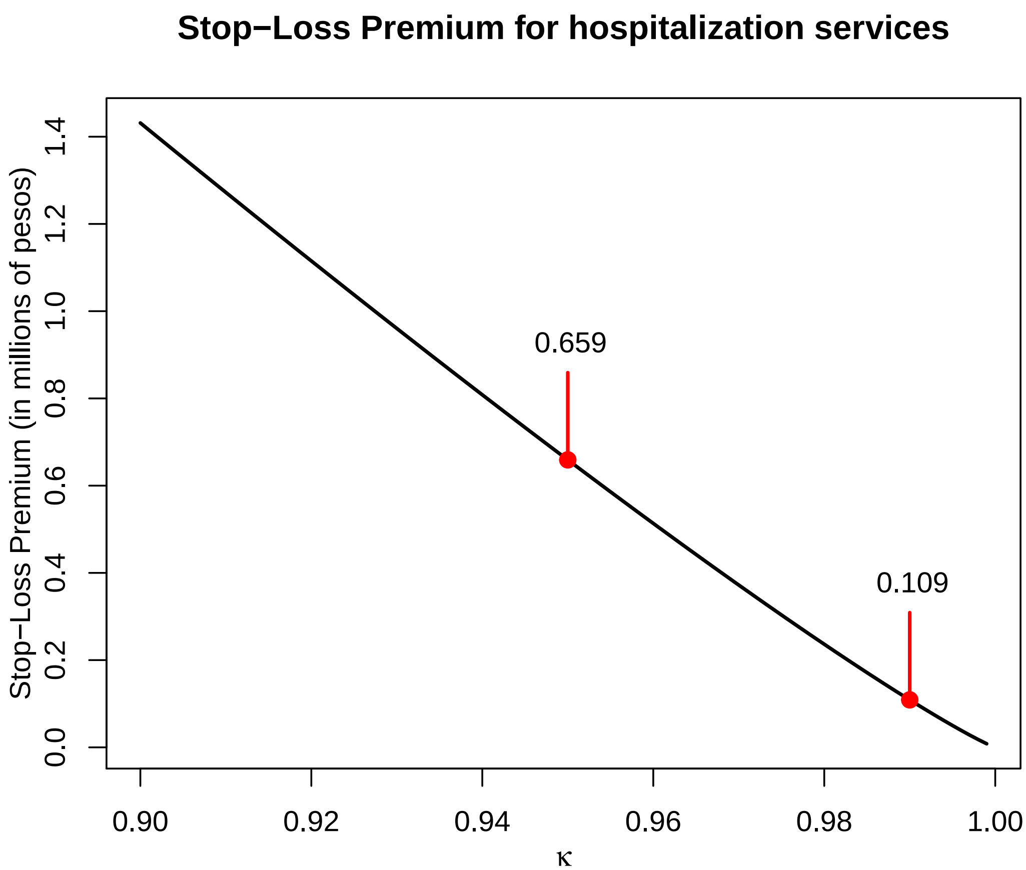

Figure 4.4: Stop-Loss Premium for hospitalization services

From the Figure 4.4 it is observed that the SLP has a decreasing behavior that ranges from 1.432 to 0.008 million of pesos as the value of κ increases from 0.9 to 0.999. Particularly when κ is equal to 0.95 and 0.99, the $SLP_{S_k}(\kappa)$ is equal to 0.659 and 0.109 million of pesos, respectively.

4.4 Risk measures estimation for general surgery services

Similar to hospitalization services, we use the equation (4.53) to find the tail index value ξ for the adjusted distribution to the individual costs of general surgery, where as previously indicated, should be located that section of the plot where is evidenced a stable region. See Code 34 in Appendix B.

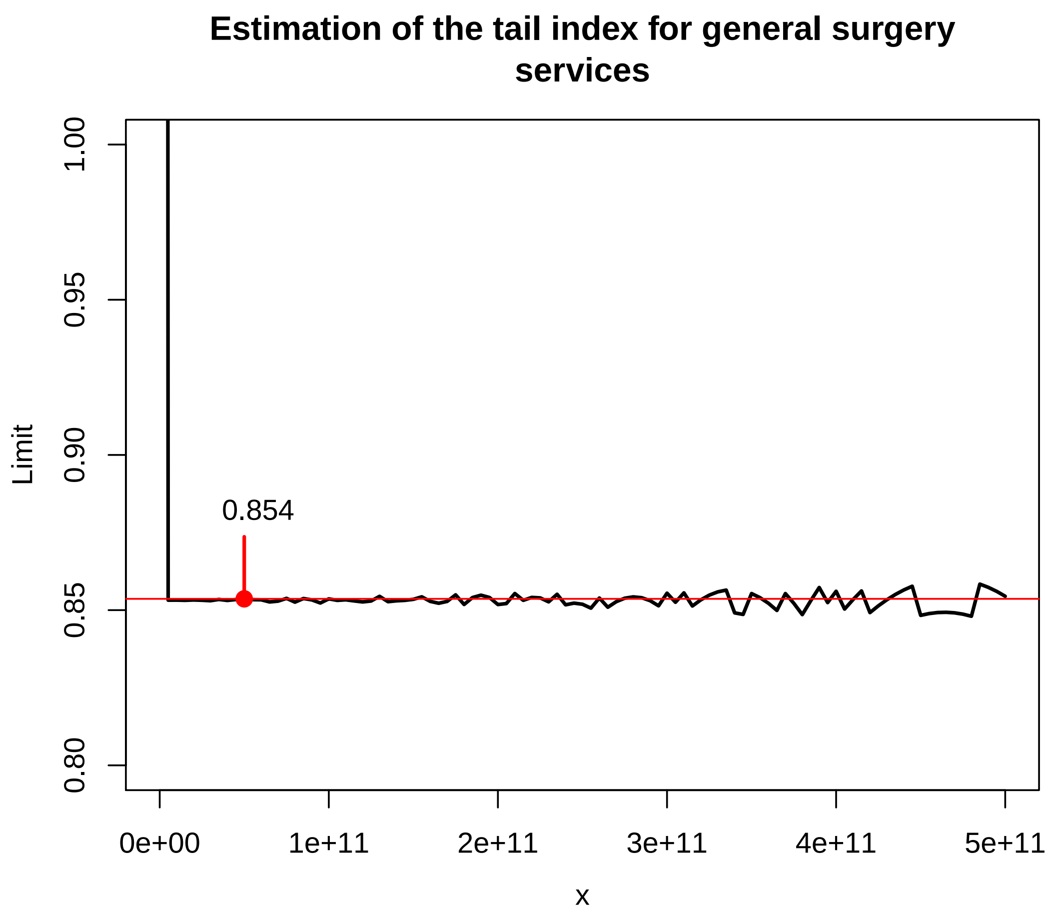

Figure 4.5: Tail index estimation for general surgery services

The Figure 4.5 shows that the limit proposed in the equation (4.53) has a stable region around 0.84, consequently, it is concluded that the tail index for the individual costs distribution of general surgery services is approximately 0.84. As previously mentioned, the value ξ is related to the tail heaviness of the distribution, therefore, the $\xi \approx 0.84$ value is the reason why the mixture *W-GP* had not defined a value for any of the moments that were above the first.

From this result, it is possible to classify the individual costs distribution in the scenario of heavy tail with finite mean proposed by Degen (2010, p. 10), which calculated the *VaR* given by (4.30) as

$$VaR_{S_{surg}}(\kappa) = F^{-1}_{X_{surg}}\left(1 - \frac{1-\kappa}{\mathbb{E}(N_{surg})}\right) + \mathbb{E}(X_{surg})(\mathbb{E}(N_{surg}) + d - 1) \tag{4.58}$$

Below is graphically shown the behavior of *VaR* for general surgery, for values of κ between 0.9 and 0.999. See Code 35 in Appendix B.

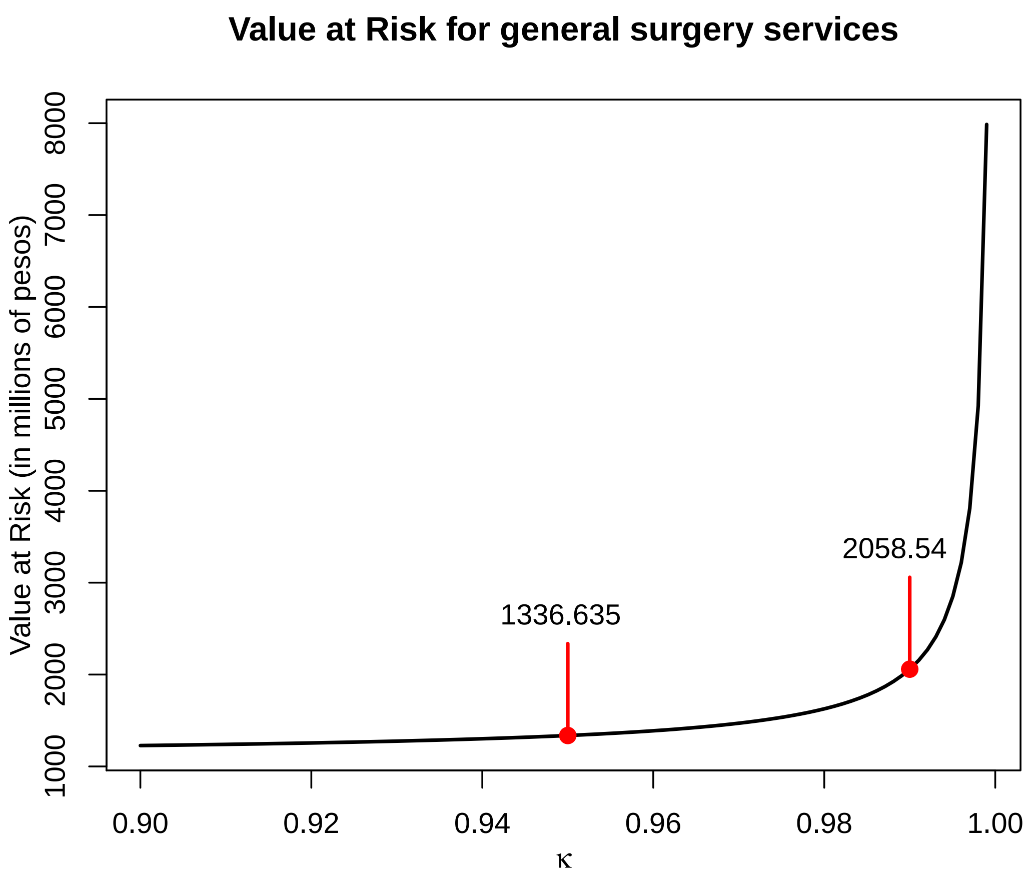

Figure 4.6: Value at Risk for general surgery services

In the Figure 4.6 it is shown that the *VaR* of aggregate costs for general surgery services is between 1227.303 and 7986.619 million of pesos, when κ is between 0.9 and 0.999, since when $\kappa \to 1$, the value of $VaR_{S_{surg}}(\kappa) \to \infty$.

This difference between the quantiles 0.9 and 0.999 is due to the behavior of the aggregate costs distribution for general surgery services, which has a heavy tail, because to this, it is expected a significant change in the values *VaR* when κ approaches 1. Additionally, as specific cases we have that for $\kappa = 0.95$, the

$VaR_{S_{surg}}(\kappa) = 1336.635$ and for $\kappa = 0.99$, the $VaR_{S_{surg}}(\kappa) = 2058.540$.

In order to calculate the *ES*, we start from the fact that the equation (4.58) for the calculation of *VaR*, has the same structure of the equation (4.54), such that

$$ES_{S_{surg}}(\kappa) = \frac{1}{1-\kappa}\left[\int_{\kappa}^{1} F^{-1}_{X_{surg}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{surg})}\right) \mathrm{d}\theta\right] + \mathbb{E}(X_{surg})(\mathbb{E}(N_{surg}) + d - 1) \tag{4.59}$$

From the equation (4.59) is made the *ES* calculation and is presented its behavior graphically for values of κ from 0.9 to 0.999 since when $\kappa \to 1$, the value of $ES_{S_{surg}}(\kappa) \to \infty$. See Code 36 in Appendix B.

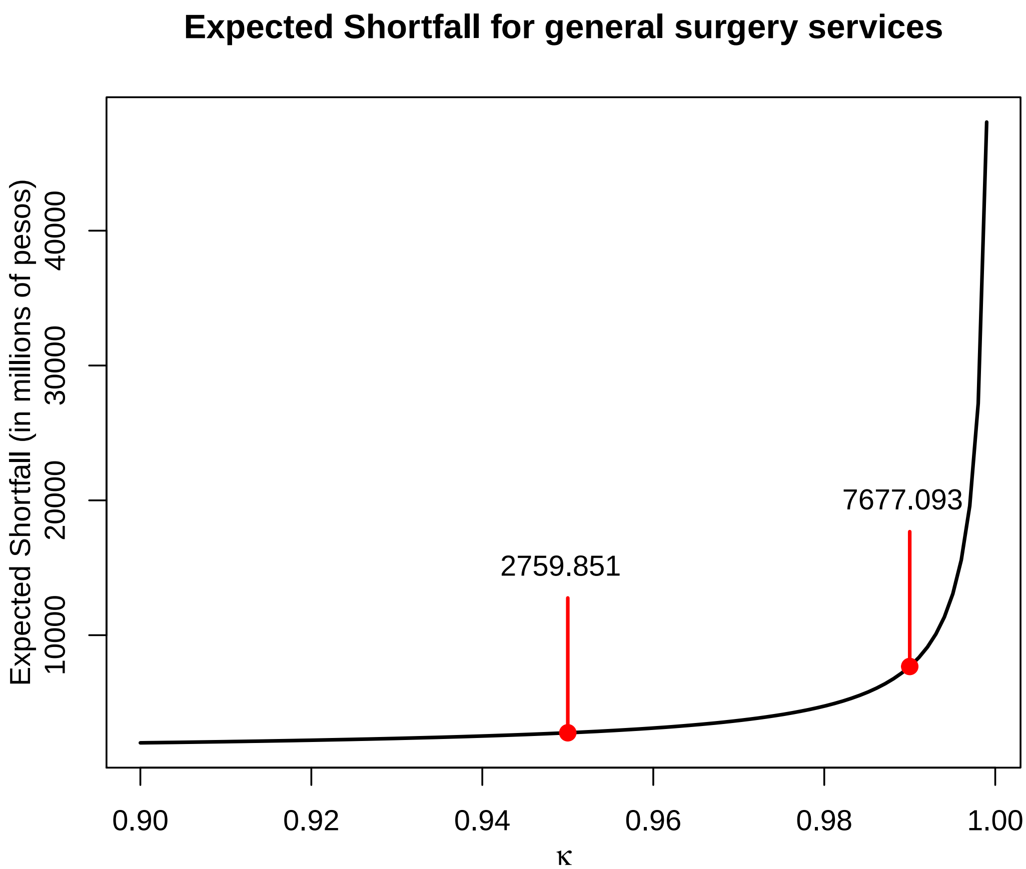

Figure 4.7: Expected Shortfall for general surgery services

The Figure 4.7 shows an increasing behavior, where as the value κ approaches 1 the value of *ES* increases to infinity. Additionally it is noted that for values of $\kappa = 0.95$ and $\kappa = 0.99$, the $ES_{S_{surg}}(\kappa)$ has values of 2759.851 and 7677.093 million pesos, respectively.

Finally, the *SLP* calculation is implemented for general surgery services, where it is employed a similar equation to the one used for hospitalization, because the *VaR* in both scenarios have the same structure, hence, $SLP_{S_{surg}}(\kappa)$ is given by

$$SLP_{S_{surg}}(\kappa) = \left[\int_{\kappa}^{1} F^{-1}_{X_{surg}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{surg})}\right) \mathrm{d}\theta\right] - (1-\kappa) F^{-1}_{X_{surg}}\left(1 - \frac{1-\theta}{\mathbb{E}(N_{surg})}\right) \tag{4.60}$$

In the Figure 4.8 it is shown that the *SLP* for general surgery services, has a decreasing behavior ranging from 78.784 to 40.069 million of pesos, when κ goes from 0.90 to 0.999. Additionally, it is evidenced that for point quantiles of $\kappa = 0.95$ and $\kappa = 0.99$, the *SLP* is 71.161 and 56.186 million of pesos. See Code 37 in Appendix B.

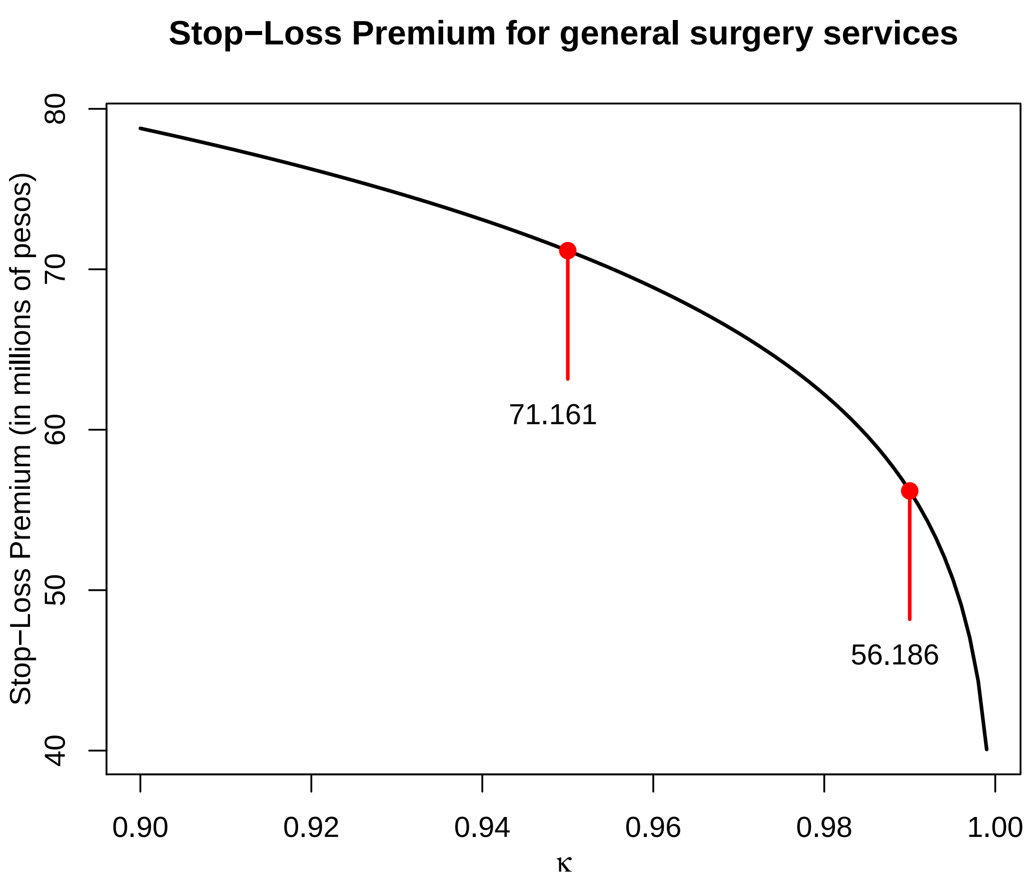

Figure 4.8: Stop-Loss Premium for general surgery services

4.5 Reinsurance

An insurance contract is an agreement where in exchange for a premium, an insurance company agrees to indemnify in part or in all of the agreed risks, once occurs the agreed specific event within the contract coverage limits. At the same time, the insurance company may reach an agreement with a reinsurer, in order to transfer some or all of the assumed risks in the agreed contracts, to minimize the loss risk.

In the same way, a reinsurance contract fulfills the same characteristics of an insurance contract, with the difference that reinsurance is an insurance agreed between insurance companies. In consecuense, the reinsurer will never have any direct link with the insurance policies buyers, if not only with the reinsured company. Furthermore, the coverage level provided by the reinsurer depends on the type and form of the reinsurance contract agreed after the negotiation between entities, since the decision of which contract is more convenient for the insurer depends on the severity of the policies, their reserves and their needs.

Before addressing the types of reinsurance, consider a variable $k \geq 1$ that represents the number of *HCD* portfolios of an insurer. For example, chemotherapy and radiotherapy treatment for cancer, AIDS treatment and its complications, intensive care unit treatment for more than five days, among others. If we define the aggregate costs associated with the k-th portfolio as in the equation (4.1), for the validity period of a policy (usually one year). Then a reinsurance contract for the k-th *HCD* can be defined as

$$S_k = D_k + R_k \tag{4.61}$$

where D_k represents the amount deductible or retained by the insurer and R_k represents the amount paid by the reinsurer (Albrecher, et al., 2017, p. 19).

To calculate the retained part by the insurer and the ceded part to the reinsurer, there are different reinsurance types, such as Quota-Share, Surplus, Excess of Loss, Stop-Loss, etc., but the implementation of each of them is beyond the scope of this work, therefore, in order to provide an approximation to the calculation of these values, is used a Stop-Loss reinsurance, given that this is the reinsurance method that provides less variation of the risk retained by the insurer (Cai and Tan, 2007, p. 94).

In Stop-Loss reinsurance, are reinsured only those costs $X_{k,i}$ that exceed a fixed amount of retention $M_k > 0$, which may or may not be restricted to a higher limit L_k, defined as the maximum coverage limit amount of the reinsurance contract. The above means that, if is agreed a maximum coverage limit of L_k, then the reinsurer will cover those costs between M_k and the limit L_k, and costs that are below M_k or those that exceed L_k, are covered by the insurer.

If we take the situation where there is no reinsurance limit, i.e, $L_k = \infty$, then the Stop-Loss reinsurance for the k-th portfolio is defined as

$$D_k = (S_k \wedge M_k), \quad R_k = (S_k - M_k)_+ \tag{4.62}$$

where the notation $(a \wedge b)$ represents

$$(a \wedge b) = \min(a, b) \tag{4.63}$$

and the notation $(a - b)_+$ represents

$$(a-b)_+ = \begin{cases} 0 & (a-b) \leq 0 \\ (a-b) & (a-b) > 0 \end{cases} \tag{4.64}$$

Otherwise, if $L_k \leq \infty$, reinsurance is defined as

$$D_k = \Big[(S_k \wedge M_k) + (S_k - L_k)_+\Big], \quad R_k = \Big[(S_k - M_k)_+ \wedge (L_k - M_k)\Big] \tag{4.65}$$

In exchange for the coverage offered by the reinsurer for a particular risk, the insurer pays compensation, which we assume by the expected value principle[3], defined as

$$\delta(M_k) = (1+\rho)\pi(M_k) \tag{4.66}$$

where $\delta(M_k)$ is the reinsurance premium, $\rho > 0$ is defined as the reinsurer's relative safety load factor, which can be interpreted as a risk premium rate, and $\pi(M_k)$ it is known as the net Stop-Loss premium, defined in Klugman et al. (2012, p. 146) as

$$\pi(M_k) = \mathbb{E}[(S_k - M_k)_+] = \int_{M_k}^{\infty} \bar{F}_{S_k}(x)dx \tag{4.67}$$

From the equation (4.67), it is evidenced that there is a direct relationship between this measure and the SLP defined in the equation (4.48), when the retention level M_k is equal to $VaR_{S_k}(\kappa)$. The above is stated in Denuit et al. (2005, p. 73), where the authors point out that an important aspect of the Stop-Loss reinsurance contract is that for a retention point M_k equal to $VaR_{S_k}(\kappa)$, the value of the net premium $\pi(M_k)$ is given by $SLP_{S_k}(\kappa)$.

If we define T_k as the total costs paid by the insurer in a Stop-Loss contract for the k-th medical service, then we can write T_k as

$$T_k = D_k + \delta(M_k) \tag{4.68}$$

From the equation (4.68) is observed an exchange between the amount deductible or retained by the insurer and the premium paid for the coverage of the reinsurance, where, Cai and Tan (2007, p. 95) indicate that, if is selected a low for the retention level M_k, then the reinsurance premium will be high, because the insurer will be

[3]Other principles for the premium calculation of reinsurance can be consulted in Centeno and Simões (2009, p. 389)

transferring the greater amount of risk to the reinsurer, while, if is selected a low level for the reinsurance premium $\delta(M_k)$, it is assumed that the insurer will be transferring a small amount of risk to the reinsurer, therefore, it will be exposed to a potentially dangerous liability for its financial health.

For this reason, it is necessary to find the optimal retention level that an insurer must have, given a relative security charge level ρ, in such a way that the retained risk is minimized and/or the insurer's utility is maximized when is used a Stop-Loss reinsurance method. In other words, the objective is to ensure that the risk measure associated with T_k is as small as possible (Cai and Tan, 2007, p. 96).

To achieve this objective, we use the *VaR* minimization criterion of the total costs T_k, where to find that expression, it is necessary to start from the definition of the *VaR* of the retained loss, which is presented by Cai and Tan (2007, p. 98) as

$$VaR_{D_k}(M_k,\kappa) = \begin{cases} M_k; & 0 < M_k < VaR_{S_k}(\kappa) \\ VaR_{S_k}(\kappa); & M_k > VaR_{S_k}(\kappa) \end{cases} \tag{4.69}$$

From the equations (4.69) and (4.68) it follows that the *VaR* of the total costs T_k is equal to

$$VaR_{T_k}(M_k,\kappa) = VaR_{D_k}(M_k,\kappa) + \delta(M_k) \tag{4.70}$$

or equivalently to

$$VaR_{T_k}(M_k,\kappa) = \begin{cases} M_k + \delta(M_k); & 0 < M_k < VaR_{S_k}(\kappa) \\ VaR_{S_k}(\kappa) + \delta(M_k); & M_k > VaR_{S_k}(\kappa) \end{cases} \tag{4.71}$$

If we define M_k^* as a random variable that represents the optimal retention level for a Stop-Loss reinsurance, then using the equation (4.71), the objective is to find the value M_k^* that minimizes the function $VaR_{T_k}(M_k,\kappa)$, such that

$$VaR_{T_k}(M_k^*,\kappa) = \min_{M_k>0}\left(VaR_{T_k}(M_k,\kappa)\right) \tag{4.72}$$

To find the optimal point M_k^*, in Cai and Tan (2007, pp. 98–100) are established the sufficient and necessary conditions for the existence of the optimal retention point for the $VaR_{T_k}(M_k^*,\kappa)$, where, the authors indicate that the optimal point M_k^* is equal to

$$M_k^* = VaR_{S_k}(\rho^*) \tag{4.73}$$

if and only if, there is a term $\rho^* = \frac{1}{1+\rho}$, so that the following two conditions are met

$$1-\kappa < \rho^* < \bar{F}_{S_k}(0) \tag{4.74}$$

and

$$VaR_{S_k}(\kappa) \geq VaR_{S_k}(\rho^*) + \delta(VaR_{S_k}(\rho^*)) \tag{4.75}$$

if both conditions are fulfilled, the *VaR* of the total costs T_k, paid by the insurer in a Stop-Loss contract for the k-th portfolio is given by

$$VaR_{T_k}(M_k^*, \kappa) = M_k^* + \delta(M_k^*) \tag{4.76}$$

where it is noted that for the optimal retention point M_k^* calculation, it is only necessary to know the aggregate loss distribution of S_k, the VaR of S_k and the reinsurer's relative safety load factor ρ.

4.6 Optimum retention point estimation for hospitalization services

To obtain the optimal retention point for the aggregate costs of hospitalization services, we decided to construct a results table, which presents for different levels of ρ, the value of the optimal quantile $\kappa_{\rho^*} = 1 - \rho^*$, the optimal retention point $M^*_{hosp} = VaR_{S_{hosp}}(\kappa_{\rho^*})$, the reinsurance premium $\delta(M^*_{hosp}) = (1+\rho)SLP_{S_{hosp}}(\kappa_{\rho^*})$, and the *VaR* of the total costs T_{hosp}, $VaR_{T_{hosp}}(\kappa_{\rho^*})$. See Code 38 in Appendix B.

Table 4.1: Optimum retention point estimation for hospitalization services

ρ	κ_{ρ^*}	M^*_{hosp}	$\delta(M^*_{hosp})$	$VaR_{T_{hosp}}(\kappa_{\rho^*})$
0.1	0.090909	6209.826	18.616	6228.442
0.2	0.166667	6211.630	18.424	6230.054
0.3	0.230769	6213.272	18.249	6231.521
0.5	0.333333	6216.170	17.941	6234.111
0.8	0.444444	6219.791	17.556	6237.346
1.0	0.500000	6221.848	17.337	6239.185
1.2	0.545455	6223.686	17.142	6240.828
1.5	0.600000	6226.120	16.883	6243.003
2.0	0.666667	6229.527	16.520	6246.048
3.0	0.750000	6234.756	15.964	6250.720
4.0	0.800000	6238.689	15.546	6254.236
5.0	0.833333	6241.827	15.212	6257.039
7.0	0.875000	6246.642	14.700	6261.342
10.0	0.909091	6251.782	14.154	6265.936
20.0	0.952381	6261.640	13.105	6274.745
50.0	0.980392	6273.989	11.792	6285.781

In the Table 4.1 it is shown that as the reinsurer's relative safety load factor increases, the value of the optimal quantile κ_{ρ^*} rises until it is close to 99% when $\rho = 50$. It is also observed that as the retention level increases, the premium paid by the insurer declines, this could be explained by the almost constant decreasing behavior of $SLP_{S_{hosp}}(\kappa)$ and the tendency to infinite of $VaR_{S_{hosp}}(\kappa)$ when $\kappa = 1$. Additionally, it is evident that there are no sudden changes in the increments of the variables M^*_{hosp} or $VaR_{T_{hosp}}(\kappa_{\rho^*})$.

It should be noted that, if is used the `optim` function of the library `stats` (2018) to obtain the optimal retention points, then, by optimizing the *VaR* function for the total costs T_{hosp}, we arrive at the same results as using the value of the optimal quantile $\kappa_{\rho^*} = 1 - \rho^*$.

4.7 Optimum retention point estimation for general surgery services

Similar to hospitalization, we build the Table 4.2 to summarize the results obtained for general surgery services. In this table, we could appreciate that the reinsurance premium has a different behavior from the one presented in the hospitalization case, since as the optimal retention level of the insurer increases, the premium paid for coverage increases too in a greater proportion than the retention level, to the point where the premium paid is greater than the retention level when $\rho \geq 20$. See Code 39 in Appendix B.

Table 4.2: Optimum retention point estimation for general surgery services

ρ	κ_{ρ^*}	M^*_{surg}	$\delta(M^*_{surg})$	$VaR_{T_{surg}}(\kappa_{\rho^*})$
0.1	0.090909	1112.357	119.829	1232.186
0.2	0.166667	1113.946	129.063	1243.009
0.3	0.230769	1115.515	138.185	1253.701
0.5	0.333333	1118.603	156.130	1274.733
0.8	0.444444	1123.125	182.407	1305.532
1.0	0.500000	1126.077	199.564	1325.641
1.2	0.545455	1128.986	216.470	1345.456
1.5	0.600000	1133.278	241.415	1374.694
2.0	0.666667	1140.270	282.047	1422.317
3.0	0.750000	1153.771	360.510	1514.282
4.0	0.800000	1166.781	436.114	1602.895
5.0	0.833333	1179.411	509.515	1688.926
7.0	0.875000	1203.801	651.258	1855.059
10.0	0.909091	1238.787	854.577	2093.364
20.0	0.952381	1347.044	1483.711	2830.756
50.0	0.980392	1636.039	3163.199	4799.239

The above can be explained by the $SLP_{S_{surg}}(\kappa)$ when evaluating values of κ close to 1, since it is observed that these are not close to 0. Additionally, it can be explained by the value of the $VaR_{S_{surg}}(\kappa)$ when $\kappa \to 1$, which indicates that if the insurer only wants to cover these extreme values, he should pay a higher premium.

Chapter 5

GAMLSS distributions an alternative to adjust severity distributions

Table of Contents

5.1 Introduction

The generalized additive models for location, scale and shape (GAMLSS onwards) proposed by Rigby and Stasinopoulos (2005), are a general class of semi-parametric regression models composed of a response variable and one or more explanatory variables, where the response variable requires a parametric structure, which can be selected from a more general family of distributions than the exponential family, while the explanatory variables can have fixed effects, random effects or smoothed nonparametric functions.

The main advantage of the GAMLSS compared to conventional regression models is that they allow to adjust a great collection of discrete or continuous distributions to the response variable, which can have very asymmetric shapes, with positive or negative biases, and/or shapes with large kurtosis. Among the distributions that can be adjusted to the response variable, is highlighted a large amount of Poisson mixtures for the discrete case, along with distributions belonging to the Pareto and Power Exponential families for the continuous case.

To conceptually introduce the GAMLSS, we start from Stasinopoulos and Rigby (2007, p. 2) which present the GAMLSS as models that assume a set of independent observations $y_1, y_2, \ldots, y_n$, with probability function $f(y_i|\theta_i)$ conditioned to θ_i, a vector of up to four parameters, with structure $\theta_i = \mu_i$ or $\theta_i = (\mu_i, \sigma_i)$ or $\theta_i = (\mu_i, \sigma_i, \nu_i)$ or $\theta_i = (\mu_i, \sigma_i, \nu_i, \tau_i)$, which can be taken as functions of the explanatory variables, and in turn are considered as the parameters of the distribution of y_i.

In addition, given a response vector $\mathbf{y}^T = (y_1, y_2, \ldots, y_n)$, Rigby and Stasinopoulos (2005, p. 509), propose an equation that allows to express mathematically the GAMLSS, as

$$g_k(\boldsymbol{\theta}_k) = \boldsymbol{\eta}_k = \mathbf{X}_k\boldsymbol{\beta}_k + \sum_{j=1}^{J_k} \mathbf{Z}_{jk}\boldsymbol{\gamma}_{jk} \tag{5.1}$$

with $g_k(\cdot)$ a monotone link function that relates the parameter vector $\boldsymbol{\theta}_k$ with the explanatory variables, being $\boldsymbol{\theta}_k^T = (\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\nu}, \boldsymbol{\tau})$ for $k = 1, 2, 3, 4$, respectively. Furthermore, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ and $\boldsymbol{\eta}_k$ are vectors of size n, $\boldsymbol{\beta}_k^T = (\beta_{1k}, \beta_{2k}, \ldots, \beta_{J'_k k})$ is a parameters vector of size J'_k, $\mathbf{Z}_{jk}$ is a fixed known design matrix of size $n \times q_{jk}$ and γ_{jk} is a random variable of dimension q_{jk}.

It should be noted that J'_k represents the number of covariates used in the fixed effects in $\boldsymbol{\eta}_k$, and J_k represents the number of random effects in $\boldsymbol{\eta}_k$. Rigby and Stasinopoulos (2005, p. 509) points out that if $J_k = 0$ for $k = 1, 2, 3, 4$, then the model presented in the equation (5.1) is reduced to a completely parametric structure, given by

$$g_k(\boldsymbol{\theta}_k) = \boldsymbol{\eta}_k = \mathbf{X}_k\boldsymbol{\beta}_k \tag{5.2}$$

whereas, if $\mathbf{Z}_{jk} = \mathbf{I}_n$ is an identity matrix $n \times n$, and $\boldsymbol{\gamma}_{jk} = h_{jk}(\mathbf{x}_{jk})$ for all combinations of $j = 1, 2, \ldots, J_k$ and $k = 1, 2, 3, 4$, then the model presented in the equation (5.1) can be rewritten as

$$g_k(\boldsymbol{\theta}_k) = \boldsymbol{\eta}_k = \mathbf{X}_k\boldsymbol{\beta}_k + \sum_{j=1}^{J_k} h_{jk}(\mathbf{x}_{jk}) \tag{5.3}$$

with h_{jk} an unknown function of explanatory variables X_{jk}, and $\mathbf{x}_{jk}$ a known vector of explanatory variables of size n.

To measure the tail heaviness of the continuous distributions $f_Y(y)$, Rigby, Stasinopoulos, Heller, and Voudouri (2014, Chapter 12), advise focus the attention on the logarithm of the distribution, because this exaggerates the distribution tail behavior and allows to evidence more easily the true heaviness that it has.

Thus to define if one distribution has a heavier tail than another, the authors define two continuous random variables Y_1 and Y_2 with probability densities $f_{Y_1}(y)$ and $f_{Y_2}(y)$ for which it is satisfied $\lim_{y\to\infty} f_{Y_1}(y) = \lim_{y\to\infty} f_{Y_2}(y) = 0$, then

$$Y_2 \text{ has a heavier right tail than } Y_1 \Leftrightarrow \lim_{y\to\infty} [\log\, f_{Y_2}(y) - \log\, f_{Y_1}(y)] = \infty \tag{5.4}$$

where, if the equation (5.4) is fulfilled, then it must be satisfied that (Rigby et al. 2014, Appendix 12.5.1)

$$Y_2 \text{ has a heavier right tail than } Y_1 \Leftrightarrow f_{Y_1}(y) = o\left(f_{Y_2}(y)\right) \text{ as } y \to \infty \tag{5.5}$$

with $o(\cdot)$ the term known as *little-o*, which refers to that $f_{Y_2}(y)$ grows much faster than $f_{Y_1}(y)$, for sufficiently large values of y, when $f_{Y_1}(y) = o\left(f_{Y_2}(y)\right)$. Additionally, if the equation (5.5) is fulfilled, then it must also be satisfied that (Rigby et al. 2014, Appendix 12.5.2)

$$Y_2 \text{ has a heavier right tail than } Y_1 \Leftrightarrow \overline{F}_{Y_1}(y) = o\left(\overline{F}_{Y_2}(y)\right) \text{ as } y \to \infty \tag{5.6}$$

with $\overline{F}_{Y_i}(y) = 1 - F_{Y_i}(y)$, being $F_{Y_i}(y)$ the cumulative distribution function of Y_i for $i = 1, 2$.

From the previous results, Rigby et al. (2014, p. 200) classify the asymptotic behavior of the logarithm of the distributions, $\log\, f_Y(y)$, in three of its main forms when $y \to \infty$, being $\log\, f_Y(y) \sim$

Type I: $-\, k_2 \left(\log\, |y|\right)^{k_1}$,

Type II: $-\, k_4 \left|y\right|^{k_3}$,

Type III: $-\, k_6 e^{k_5 |y|}$

where the tail **Type I** is heavier than the tails **Type II** and **Type III**, while the tail **Type II** is heavier than the tail **Type III**. In order to understand the difference among these, we present the Figure 5.1, in which it is shown the tail behavior of the three different types for k_1, k_3, $k_5 = 1$, 2, and k_2, k_4, $k_6 = 1$, 2. See Code 40 in Appendix B.

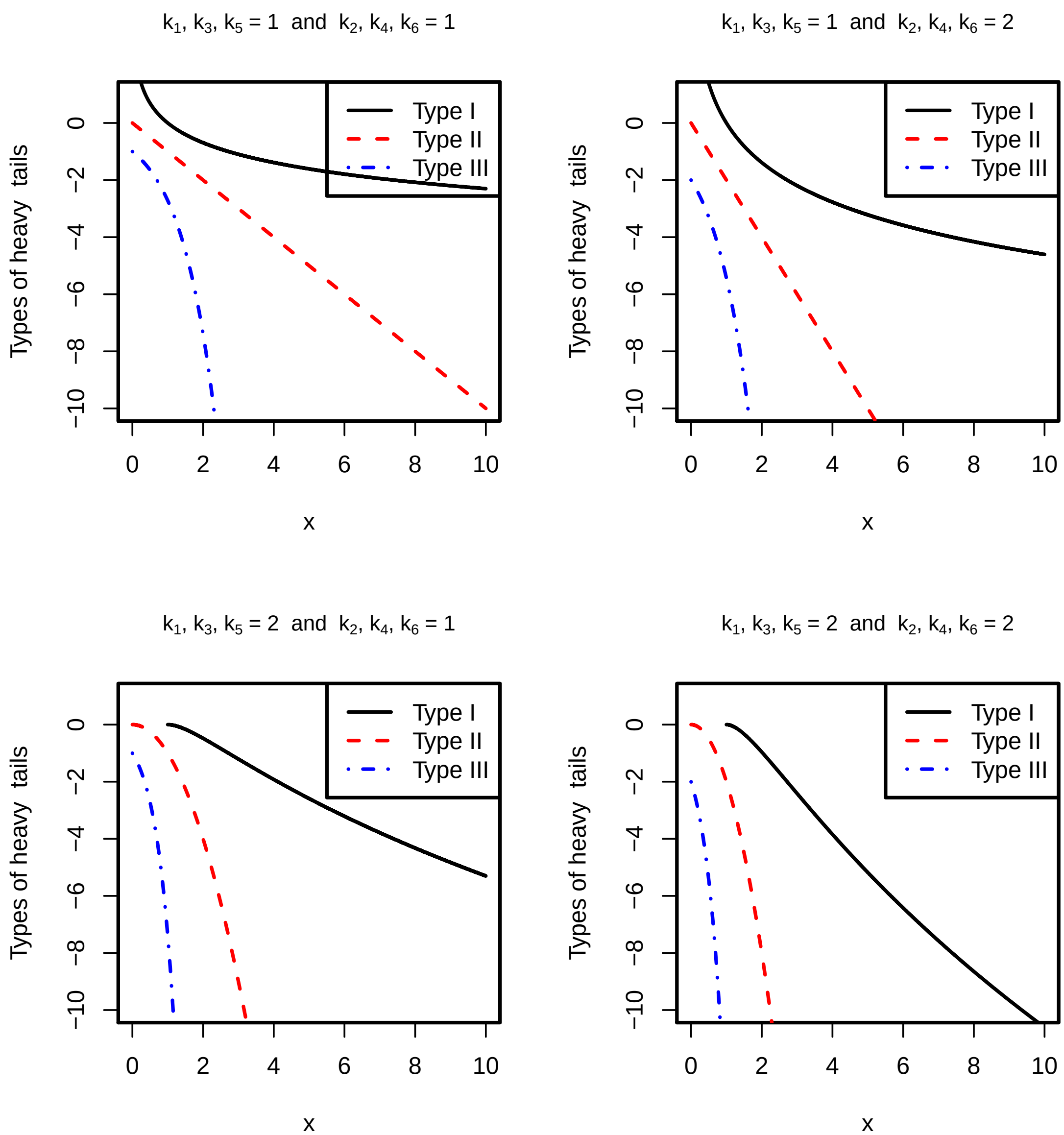

Figure 5.1: Figure shows the tail shape for diferent types of GAMLSS distributions for $k_1, k_3, k_5 = 1, 2$, and $k_2, k_4, k_6 = 1, 2$. Smaller values i the k's result heavier tails. Rigby et al. (2014)

From these results, Rigby et al. (2014, p. 204) classify the tail heaviness of the distributions in four categories of the seven proposed by Mandelbrot (1997, Chapter E5), namely, mild, slow, wild and extreme randomness. On the other hand, Ferrari and Fumes (2017, pp. 9–10) based on the equations (3.13) and using the Maple 13 software, complements the classification made by Rigby et al. (2014, p. 204), by adding the tail index ξ corresponding to each situation

- Non-heavy tail (mild): $k_3 \geq 1$ or $0 < k_5 < \infty$, with tail index $\xi = 0$

- Heavy tail (slow): $k_1 > 1$ and $0 < k_3 < 1$, with tail index $\xi = 0$
- Paretian type tail (wild): $k_1 = 1$ and $k_2 > 1$, with tail index $\xi = 1/(k_2 - 1)$
- Heavier than any Paretian type tail (extreme): $k_1 = 1$ and $k_2 = 1$, with tail index $\xi = \infty$

where the "heavy tail" category means that the distribution has a heavier tail than any exponential distribution but lighter than any Parentian distribution.

Additionally in Rigby et al. (2014, p. 204), the authors create a sub-classification for the tails **Type II**, from the value that has k_3, where

- if $0 < k_3 < 1$, then the distribution has a heavier tail than the Laplace.
- if $1 < k_3 < 2$, then the distribution has a lighter tail than the Laplace but heavier than the normal.
- if $k_3 > 2$, then the distribution has a lighter tail than the normal.

5.2 Adjustment of GAMLSS distributions for hospitalization services severities

To adjust the individual costs of hospitalization services through GAMLSS distributions, is used the function `fitDist` of the library `gamlss` (2005) to find the distribution that offers a better fit. Additionally, since the `fitDist` function only stores the parameters of the distribution that provide the best fit, is employed the `gamlssML` function of the library `gamlss` (2005) to adjust the second and third distribution that present the best adjustment. See Code 41 in Appendix B.

Table 5.1: Better fit for individual cost of hospitalization services with GAMLSS distributions

GG	GB2	BCPE	BCPEo	BCCG
17102.02	17104.02	17104.86	17104.86	17105.39

The Table 5.1, shows that the distributions that provide the best fit with the library `gamlss` (2005), are the Generalized Gamma (*GG* onwards), followed by the Generalized Beta type 2 (*GB2* onwards) and the Box-Cox Power Exponential (*BCPE* onwards), with an *AIC* of 17102.02, 17104.02 and 17104.86, respectively. Due the *AIC* of the three fitted distributions does not differ by much, it is expected that the adjustment presented by the three will be similar. The description and presentation of the main statistics of the distributions *GG*, *GB2* and *BCPE* are presented in Appendix C.

From the equation (7.21), it is observed that the existence of the *GG* moments are conditioned to the value taken by ν and $\sigma^2|\nu|$, where, as evidenced in Rigby et al. (2014, p. 203), the tail heaviness of the *GG* distribution depends on these same values, showing that in case of having a Parentian tail, there is a direct relationship between the existence of the moments and the value of $\sigma^2|\nu|$.

Namely, when $\nu < 0$, the value $k_1 = 1$ and $k_2 = (\sigma^2|\nu|)^{-1} + 1$, then, if $(\sigma^2|\nu|) \to \infty$ the *GG* right tail is considered heavier than any Parentian type tail because $k_2 \to 1$, otherwise, the distribution tail is regarded Parentian type. Moreover, when $\nu > 0$, the value $k_3 = \nu$ and $k_4 = (\mu^\nu \sigma^2 \nu^2)^{-1}$, therefore, if $\nu < 1$ the *GG* right tail would be heavier than the Laplace or any exponential distribution tail, but lighter than any Parentian type tail, otherwise, the distribution tail would have a light tail. In this case, k_4 makes the tail a little heavier (small k_4) or lighter (large k_4).

The adjustment shows that $\hat{\sigma} = 1.209628$ and $\hat{\nu} = 0.4395007 > 0$, i.e., $k_3 = 0.4395007$ and $k_4 = 1.258322$, giving as result that the adjusted *GG* distribution tail is heavier than the Laplace or any exponential distribution tail, but lighter than any Parentian type tail, with a tail index $\xi = 0$.

The equation (7.18) shows that the existence of the moments of the *GB2* distribution are conditioned to the values that take τ and σ, where we have that the four moments exist if $\tau > 4\sigma^{-1}$. Additionally, Rigby et al. (2014, p. 203) shows that the *GB2* distribution has a Paretian type tail, because independently of the values of the parameters σ and τ, the value of k_1 will always be equal to 1, but k_2 always has a value greater than 1, given that $k_2 = \sigma\tau + 1$.

The obtained adjustment shows that $\hat{\sigma} = 0.4392977$ and $\hat{\tau} = 14972.83$, as a consequence the adjusted *GB2* distribution has a Paretian type tail with values $k_1 = 1$ and $k_2 = 6578.53$, and tail index $\xi = 1/(k_2 - 1) \approx 0$.

From the tables presented in Rigby et al. (2014, pp. 202–203) it is observed that the *BCPE* distribution can possess any of the four tail types presented in the previous section, hence is constructed the Table 5.2 as a guide for the classification of the *BCPE* distribution tail.

Table 5.2: Classification table for tail heaviness of the BCPE distribution

Condition	Value of $k_1 - k_6$	Tail Heavier
$\nu < 0$	$k_1 = 1; k_2 = \|\nu\| + 1$	Paretian type tail
$\nu = 0; \tau < 1$	$k_1 = 1; k_2 = 1$	Heavier than any Parentian type tail
$\nu = 0; \tau = 1$	$k_1 = 1; k_2 = 1 + (\Gamma(1/\nu)^{1/2}\Gamma(3/\nu)^{-1/2}\sigma)^{-\tau}$	Paretian type tail
$\nu = 0; \tau > 1$	$k_1 = \tau; k_2 = (\Gamma(1/\nu)^{1/2}\Gamma(3/\nu)^{-1/2}\sigma)^{-\tau}$	Heavy tail
$\nu > 0$	$k_3 = \nu\tau; k_4 = (\Gamma(1/\nu)^{1/2}\Gamma(3/\nu)^{-1/2}\mu^{\nu}\sigma\nu)^{-\tau}$	Heavy tail if $0 < k_3 < 1$ Non heavy-tail if $k_3 \geq 1$

From the obtained adjustment and based on the Table 5.2 it is observed that $\hat{\mu} = 8.366066$, $\hat{\sigma} = 1.250054$, $\hat{\nu} = 0.1361926$ and $\hat{\tau} = 2.165023$, hence, given that $\nu > 0$ the adjusted *BCPE* distribution has a heavy tail with values $k_3 = 0.2948602$ and $k_4 = 2.334661 \times 10^{19}$, and tail index $\xi = 0$. Besides, since $0 < k_3 < 1$, the adjusted tail is heavier than the Laplace distribution.

To perform the calculation of the theorical and empirical mean, variance, skewness and excess kurtosis of the distributions that presented better fit for the individual costs for hospitalization services, we use the functions `moments`, `skew`, `kurt`, `skewness` and `kurtosis` of the libraries `DistMom` (2018) and `e1071` (2018). See Code 42 in Appendix B.

Table 5.3: Statistical measurements of GAMLSS distributions for hospitalization services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	14.93863	362.5129	2.982862	11.97454
GG	14.97395	387.1405	3.820502	27.59570
GB2	14.96473	386.7492	3.822866	27.64090
BCPE	14.94636	377.6279	3.642886	24.80356

The Table 5.3 shows that the three adjusted distributions have values similar to the empirical mean of 14.93863, being the *BCPE* distribution the one that most closely approximates with a value of 14.94636. Similarly for the other three statistics, it is observed that despite not presenting values similar to the empirical, of the three adjusted distributions, the *BCPE* is the one that presents statistics closest to the empiricals.

Once the estimations are made, we proceed to present three graphs, in the first we plot the theoricals vs the empirical cumulative distribution, in the second we show the theoricals vs the empirical natural logarithm of the survival distribution, and in the third we display the Q-Q plot of the three adjusted distributions. See Code 43, Code 44 and Code 45 in Appendix B.

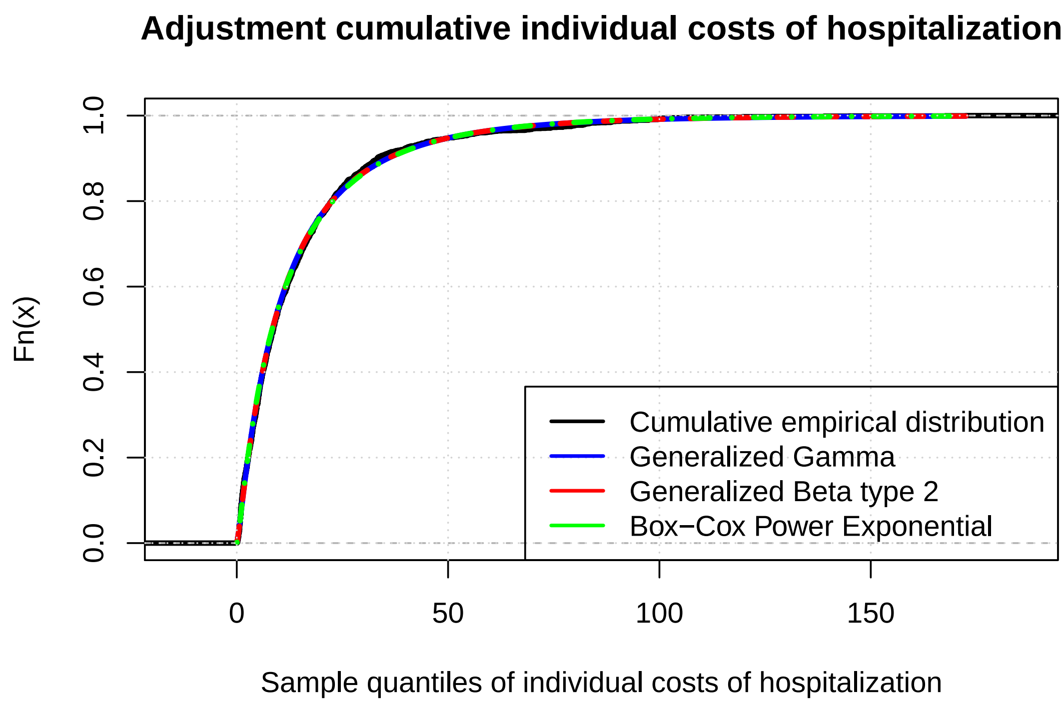

Figure 5.2: Adjustment of the individual costs of general surgery service with GAMLSS distributions

In the Figure 5.2 no differences are observed among the three adjusted distributions, since each curve is overlapped on the others. It is also evidenced that the three distributions capture well the behavior of the individual costs of hospitalization service, being only possible to note the curve associated to the cumulative empirical distribution shortly before the value 50 and between 50 and 100.

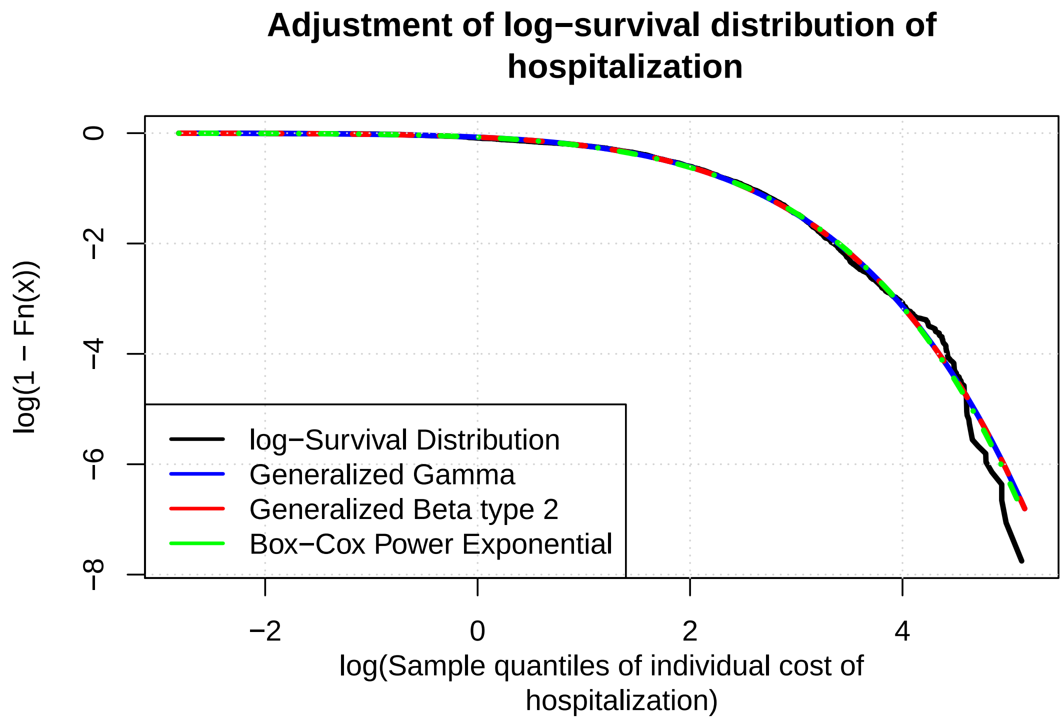

Figure 5.3: Adjustment of log-survival distribution of the individual cost in general surgery service for GAMLSS distributions

To improve the visualization in the tail area of the adjusted distributions by `gamlss` (2005), we build the Figure 5.3 and it is observed that there is no difference between the adjustment made by the distributions *GG* and *GB2* because their curves are overlapped in the whole of the plot. Furthermore, the Figure 5.3 shows that although the *BCPE* distribution has a behavior very similar to the other distributions, this exhibits a little faster decay than the distributions *GG* and *GB2* in the final area of the curve, which makes it look more similar to the empirical log-survival distribution curve.

In panels (a), (b) and (c) of the Figure 5.4 it is appreciated that between the values 50 and 100, there is a set of points that are found outside of the confidence bands, which could generate that goodness of fit tests reject the adjustment made by any of these distributions to the individual costs of hospitalization service. It is also evidenced that for values higher than 150 the observations move away from the diagonal line, which means that the adjustment of the distributions *GG*, *GB2* and *BCPE* are not very good to capture the behavior of the higher costs of this medical service. It should be noted that this behavior can also be seen in the Figure 5.3 for values higher than 5.

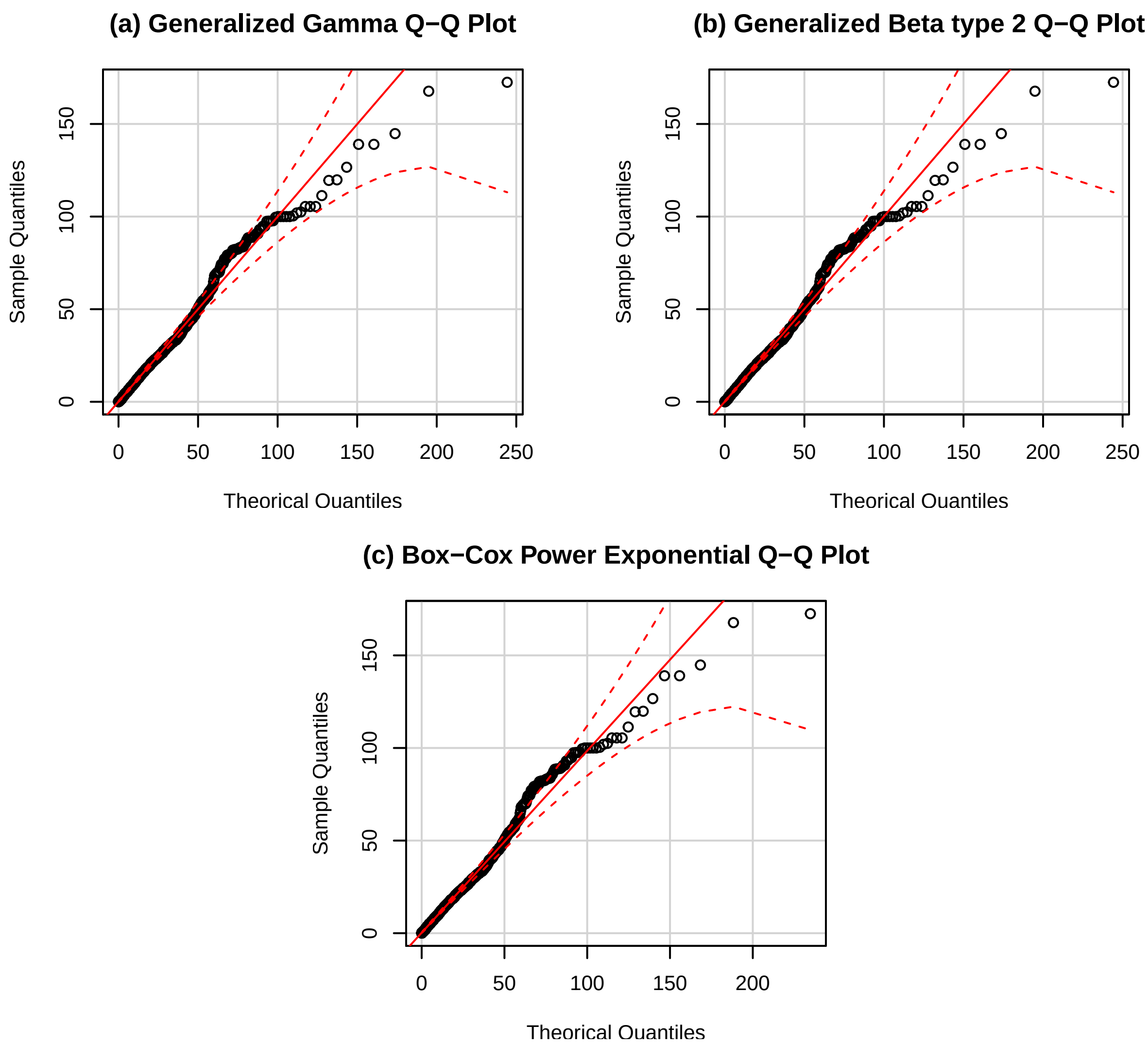

Figure 5.4: Q-Q plot GAMLSS distribution for hospitalization

To perform the hypothesis test, we employ the same goodness of fit tests used in the *spliced distribution* case, namely the tests Kolmogorov-Smirnov, Cramer-von Mises, Kuiper, Supremum class Upper Tail Anderson-Darling and Quadratic Class Upper Tail Anderson-Darling. Being the hypothesis contrast given by

$$\begin{aligned} H_0 &: F_{X_{hosp}}(x|\theta) \in \hat{F}_{X_{hosp}}(x|\hat{\theta}) \\ H_1 &: F_{X_{hosp}}(x|\theta) \notin \hat{F}_{X_{hosp}}(x|\hat{\theta}) \end{aligned} \tag{5.7}$$

with $F_{X_{hosp}}(x|\theta)$ the distribution function of individual costs for hospitalization services with parameter $\theta = (\mu, \sigma, \nu)$ or (μ, σ, ν, τ), and $\hat{F}_{X_{hosp}}(x|\hat{\theta})$ the adjusted distribution function by the library `gamlss`(2005) with estimated parameter $\hat{\theta} = (\hat{\mu}, \hat{\sigma}, \hat{\nu})$ or $(\hat{\mu}, \hat{\sigma}, \hat{\nu}, \hat{\tau})$ depending on whether the adjusted distribution is GG or are *GB2*, *BCPE*, respectively.

To perform the goodness of fit tests and prove whether the hypothesis presented in the equation (5.7) is rejected or not, we use the functions `ks.test`, `w2.test`, `v.test`, `adup.test` and `ad2up.test` from the library `truncgof`(2012). See Code 46 in Appendix B.

Table 5.4: Goodness-of-fit tests for hospitalization services with GAMLSS distributions

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
GG	0.02	0.04	0.03	0.47	0.09
GB2	0.98	0.99	1.00	0.00	0.97
BCPE	0.00	0.01	0.01	0.49	0.04

For the *GG* distribution case it is shown that the three conventional tests of goodness of fit, Kolmogorov-Smirnov, Cramer-von Mises and Kuiper, are rejected with *P-values* of 2%, 4% and 3%, respectively, while in the specialized goodness tests in heavy tail distributions, it is appreciated that neither of the two tests rejects the null hypothesis (5.7), because the *P-values* are 47% and 9% for the Supremum and Quadratic Class Upper Tail Anderson-Darling, respectively.

For the GB2 distribution it is observed that four of the five tests present values close to 100%, being the hypothesis (5.7) only rejected by the Supremum Class Upper Tail Anderson-Darling test with a *P-value* of 0%. On the contrary, for the distribution BCPE it is noted that only one of the five tests presents a value higher than 5%, being the Supremum Class Upper Tail Anderson-Darling test not rejected because its *P-value* is 49%.

Since in the Table 5.3 and in the Figures 5.2, 5.3 and 5.4 there are not significant differences between the adjustment distributions *GG*, *GB2*, *BCPE*, the selection of the best fit for the individual costs of hospitalization services is completely based on the result obtained in the Table 5.4, in which it is observed that the *GB2* distribution does not reject four out of five tests of goodness of fit. Therefore, it is assumed that the individual costs of hospitalization services have a *GB2* distribution, with estimated parameters $\hat{\mu} = 1884308708$, $\hat{\sigma} = 0.4392977$, $\hat{\nu} = 3.542926$ and $\hat{\tau} = 14972.83$.

5.3 Risk measures estimation for hospitalization services with GAMLSS

In the previous section it was mentioned that the adjusted *GB2* distribution had a Parentian type tail with a value $k_2 = 6578.53$, which indicated that the tail index $\xi = 1/(k_2 - 1) \approx 0$. In order to corroborate this, we use a graphic representation of the equation (4.53), where the objective is to look for a stable region in the graph, because this represents the value of the tail index of the distribution. See Code 47 in Appendix B.

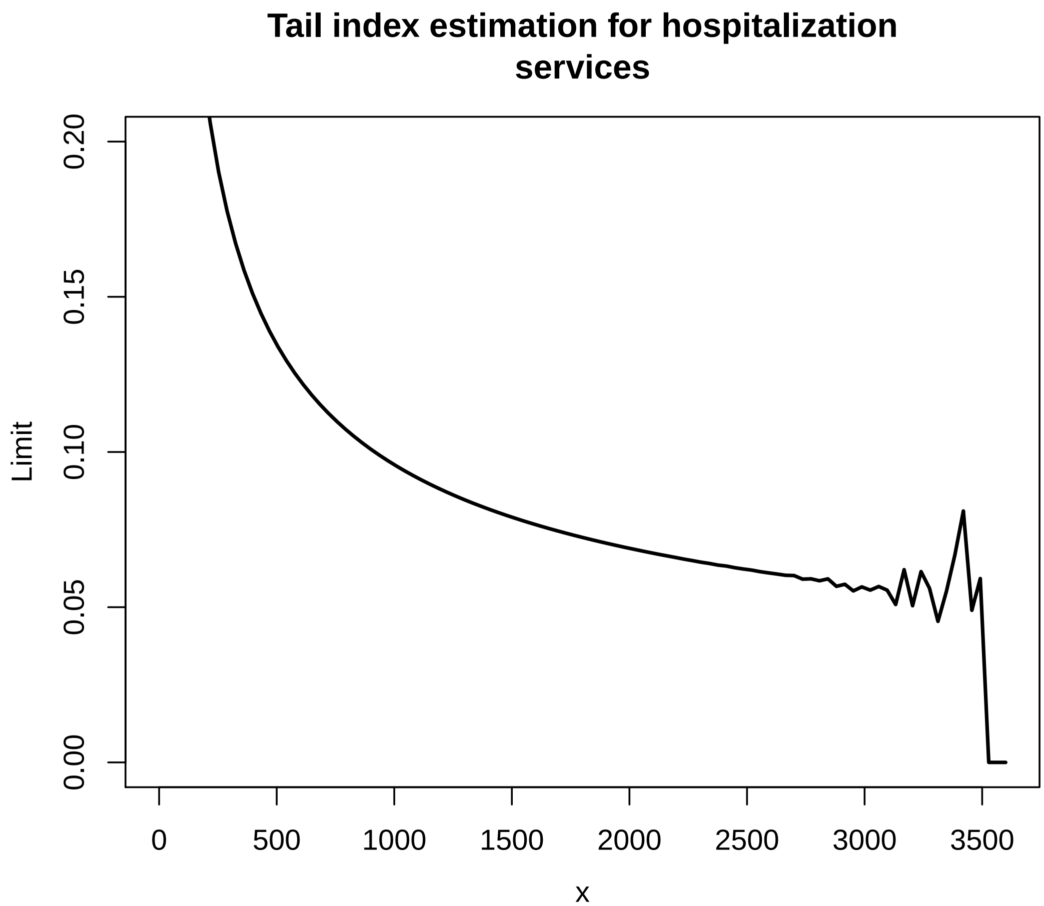

Figure 5.5: Tail index estimation for hospitalization services with GAMLSS

The Figure 5.5 shows a constantly decreasing behavior, which does not ensure the existence of a stable region in the graph, suggesting that the tail index of the *GB2* distribution is $\xi = 0$, which corroborated the previously raised by the value $\xi = 1/(k_2 - 1)$. The above is clear evidence that the adjusted distribution to the individual costs for hospitalization services has a semi-heavy tail as proposed in Degen (2010, p. 12).

Accordingly, to perform the estimation of *VaR*, *ES* and *SLP* when the distribution has a semi-heavy tail, we employ the equations (4.54), (4.55) and (4.57), respectively, to present a graphical behavior representation of each measurements. See Code 48 in Appendix B.

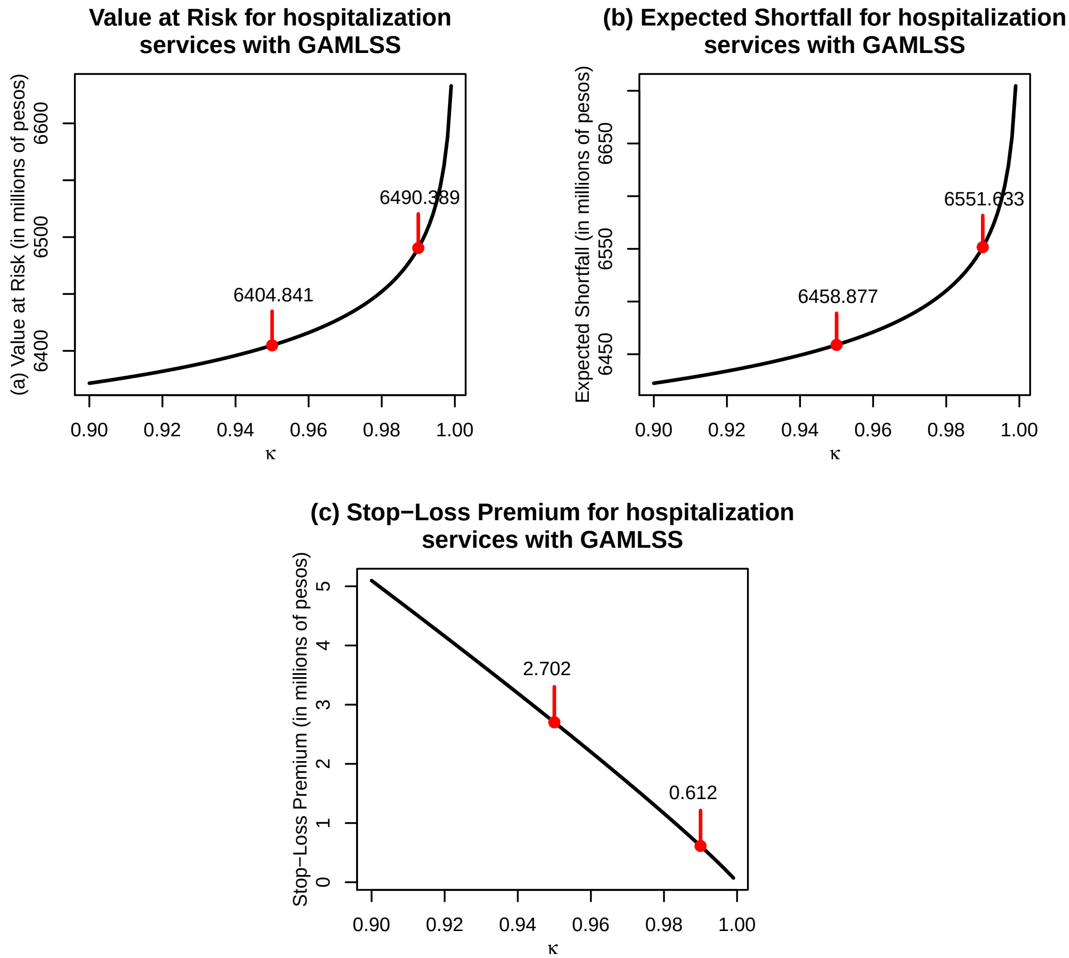

Figure 5.6: Risk measures for hospitalization services with GAMLSS

In panels (a) and (b) of the Figure 5.6 it is shown that both *VaR* and *ES* have an increasing behavior until infinity, where as to be expected, the values of *ES* are always above the values of *VaR*. This can be better seen, by observing specific values for κ, where it is appreciated that when $\kappa = 0.95$, the *VaR* is equal to 6404.842, while the *ES* is equal to 6458.878, and when $\kappa = 0.99$, the *VaR* is equal to 6490.391, while the *ES* is equal to 6551.634.

In the other hand, panels (c) of the Figure 5.6 show that *SLP* has a decreasing behavior that goes from a value close to 5 million pesos when $\kappa = 0.90$ to 0 when $\kappa \approx 1$. In addition, for specific values of $\kappa = 0.95, 0.99$, are obtained values of *SLP* of 2.702 and 0.612, respectively.

5.4 Optimum retention point estimation for hospitalization service with GAMLSS

In the following table, we present for different levels of the relative safety load factor ρ the optimal retention point that an insurer should have when the individual hospitalization costs are adjusted through a *GB2* distribution. See Code 49 in Appendix B.

Table 5.5: Optimum retention point estimation for hospitalization services with GAMLSS

ρ	κ_{ρ^*}	M^*_{hosp}	$\delta(M^*_{hosp})$	$VaR_{T_{hosp}}(\kappa_{\rho^*})$
0.1	0.090909	6279.488	41.256	6320.744
0.2	0.166667	6282.714	41.636	6324.350
0.3	0.230769	6285.711	41.985	6327.697
0.5	0.333333	6291.139	42.611	6333.750
0.8	0.444444	6298.182	43.409	6341.591
1.0	0.500000	6302.319	43.870	6346.189
1.2	0.545455	6306.102	44.288	6350.390
1.5	0.600000	6311.239	44.848	6356.087
2.0	0.666667	6318.688	45.649	6364.337
3.0	0.750000	6330.738	46.914	6377.651
4.0	0.800000	6340.333	47.896	6388.230
5.0	0.833333	6348.335	48.700	6397.035
7.0	0.875000	6361.257	49.971	6411.228
10.0	0.909091	6375.985	51.381	6427.366
20.0	0.952381	6407.265	54.253	6461.518
50.0	0.980392	6453.197	58.217	6511.414

The Table 5.5 shows the different levels of optimal retention, M^*_{hosp}, associated with the optimal reinsurance premiums $\delta(M^*_{hosp})$ and the optimal *VaR* of the total costs $VaR_{T_{hosp}}(\kappa_{\rho^*})$, depending on the relative security load levels ρ selected by the reinsurer. Besides it is appreciated that the variables M^*_{hosp}, $\delta(M^*_{hosp})$ and $VaR_{T_{hosp}}(\kappa_{\rho^*})$ have an increasing behavior as ρ increases.

The above is evidenced, when the optimal retention point increases from 6279.488 million pesos when $\rho = 0.1$, up to 6453.197 million pesos when $\rho = 50$. Similarly, it is observed that the optimal reinsurance premium increases from 41.256, up to 58.217 million pesos, and the total *VaR* increases from 6320.744 to 6511.414 millions pesos when the security load factor level goes from 0.1 to 50.

5.5 Adjustment of GAMLSS distributions for severities of general surgery services

To find the GAMLSS distribution that present a better fit to the individual costs of general surgery services, we employ the functions `fitDist` and `gamlssML` of the library `gamlss` (2005) to adjust the parameters of the three distributions that present the best fit to the data set. See Code 50 in Appendix B.

Table 5.6: Better fit for individual cost of general surgery services with GAMLSS distributions

BCPEo	BCPE	GG	GB2	BCCGo
2535.645	2585.225	2604.319	2606.321	2606.974

The Table 5.6, shows that the Box-Cox Power Exponential - original (*BCPEo* onwards) is the one that presents the best adjustment to the individual costs of general surgery services with a *AIC* of 2535.645, followed by the *BCPE* distribution with a *AIC* of 2585.225 and the *GG* distribution with a *AIC* of 2604.319. Unlike the hospitalization services, in this case it is observed that the *AIC* of the *BCPEo* distribution differs from the other distributions by more than 60 units, thus, it is expected that significant differences will be found in the calculated statistics, in the graphic adjustment and in the goodness of fit tests. The description and presentation of the main statistics of the distributions *BCPEo*, *BCPE* and *GG* are presented in Appendix C.

For the *BCPEo* distribution, is found that the adjustment parameters are $\hat{\mu} = 0.4877005$, $\hat{\sigma} = 116.0758$, $\hat{\nu} = 1.370003$ and $\hat{\tau} = 0.2788162$, then based on the Table 5.2 it is seen that the parameter $\nu > 0$, and because the value $k_3 = 0.3819789$ we have that the adjusted *BCPEo* distribution has a heavier tail that the any exponential distribution or Laplace, but lighter than a Parentian type tail, with tail index $\xi = 0$.

On the other hand, the parameters adjusted by the *BCPE* distribution are $\hat{\mu} = 3.271356$, $\hat{\sigma} = 1.343855$, $\hat{\nu} = 0.1164034$ and $\hat{\tau} = 3.126172$, and as a result we get that the adjusted *BCPE* distribution has a lighter tail than the Parentian type tail, but has a heavier tail than any exponential or Laplace, because $\nu > 0$ and its value $k_3 = 0.363897$.

From the parameters adjusted by the *GG* distribution, we have that the value of $\hat{\sigma}$ is 1.270459 and $\hat{\nu}$ is 0.5135504, so that the *GG* distribution it is within the category of heavy tails, i.e, it has a lighter tail than Parentian type tails because $\nu > 0$, but it has a heavier tail than any exponential or Laplace distribution because $k_3 = 0.5135504$.

Once is known the distributions tail heaviness, we proceed to calculate the mean, variance, skewness and excess kurtosis of the distributions that best adjusted the individual costs for general surgery services. For this, we use the libraries `DistMom` (2018) and `e1071` (2018). See Code 51 in Appendix B.

Table 5.7: Statistical measurements of GAMLSS distributions for general surgery services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	6.680426	82.89872	4.671353	33.36082
BCPEo	6.787825	122.55566	5.179333	53.99778
BCPE	6.369685	65.78484	2.648852	10.67166
GG	6.721370	82.68801	3.710635	25.10128

The Table 5.7 shows that the distribution that has mean, variance and excess kurtosis closest to the empirical values is *GG*, while the distribution that has skewness closer to the empirical value is the *BCPEo*. The previous result seems contradictory to what is found in the Table 5.6, since the *GG* distribution could be expected to better adjust the individual costs of general surgery services than the *BCPEo* distribution.

To test whether this result is consistent with the graphical analysis, we plot the cumulative distribution of the individual costs of general surgery versus the adjusted cumulative distribution functions, the natural logarithm of the empirical survival distribution versus the natural logarithm of the adjusted survival functions, and the Q-Q plots of the adjusted distributions. See Code 52, Code 53 and Code 54 in Appendix B.

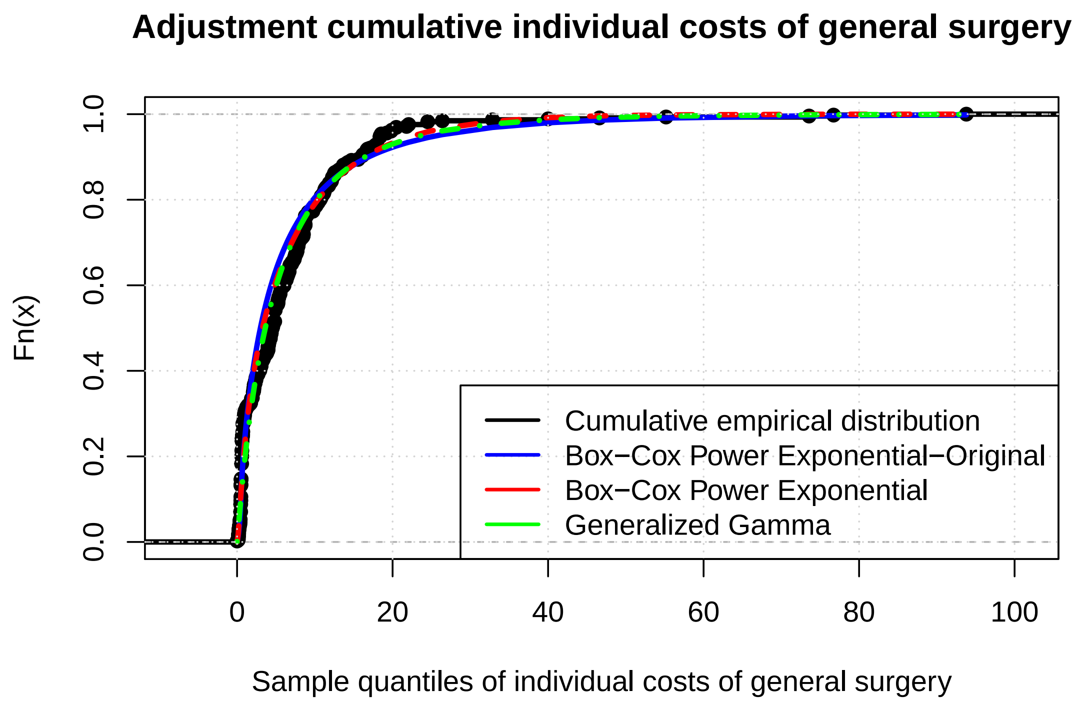

Figure 5.7: Adjustment of the individual costs of general surgery service with GAMLSS distributions

The Figure 5.7 shows that the adjusted curves do not have the shape of the cumulative empirical curve, where it is noted that between the values 0 and 20 the three curves are to the left of the cumulative empirical curve, and after 20 these are below the cumulative empirical curve. In addition, it is possible to notice that the *BCPEo* distribution is below the other distributions, i.e, its cumulative distribution function takes longer to reach 1, than the other distributions.

The Figure 5.8 presents better the behavior and the adjustment made by the distributions *BCPEo*, *BCPE* and *GG* to the individual costs of general surgery services, and it is observed that none of the three curves are able to capture the individual costs behavior, because in spite of trying to have a good adjustment in the tail area of the distribution, they do not capture the set of observations that are around $3 \approx \log(20)$.

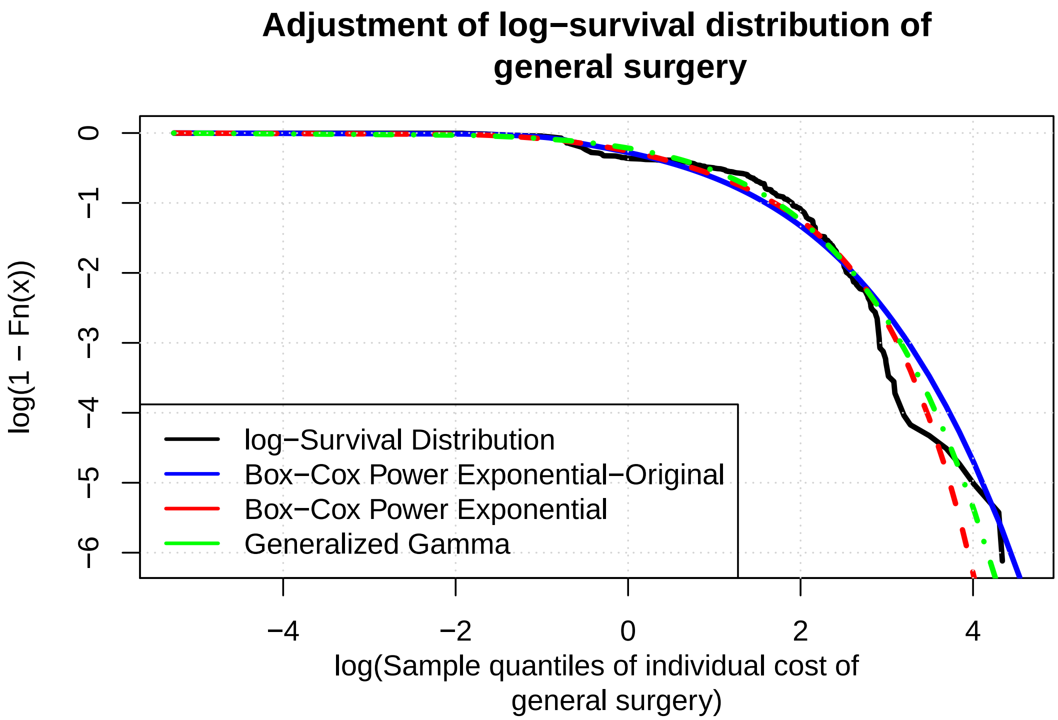

Figure 5.8: Adjustment of log-survival distribution of the individual cost in general surgery service for GAMLSS distributions

The Figure 5.9.a shows that despite capturing those high individual costs, it does not manage to adequately capture the behavior of those central costs, which are between 20 and 50, since there is a large number of points that are outside the confidence bands. This could be a clear sign that the *BCPEo* distribution is not an adequate distribution to model the individual costs of hospitalization services.

The Figure 5.9.b, displays a situation similar to the one presented in the Figure 5.9.a, because are observed groups of points that are outside of the confidence bands, which can be appreciated between 20 and 35, and between 40 and 50. This could be taken as a sign that the *BCPE* distribution does not adequately capture the behavior of the individual costs of hospitalization services, since in this case, the distribution is not capable of modeling the individual costs associated with mid or extreme values.

The Figure 5.9.c has a very similar shape to the Figure 5.8.a, since it is capable of capturing the highest individual costs, but not the individual costs that are between 20 and 40. The most notorious difference between these two figures is that the set of observations that are outside the confidence bands are greater that in the *BCPEo* distribution case, and that the observations associated with highest individual costs are closer to the diagonal line in the *GG* distribution case.

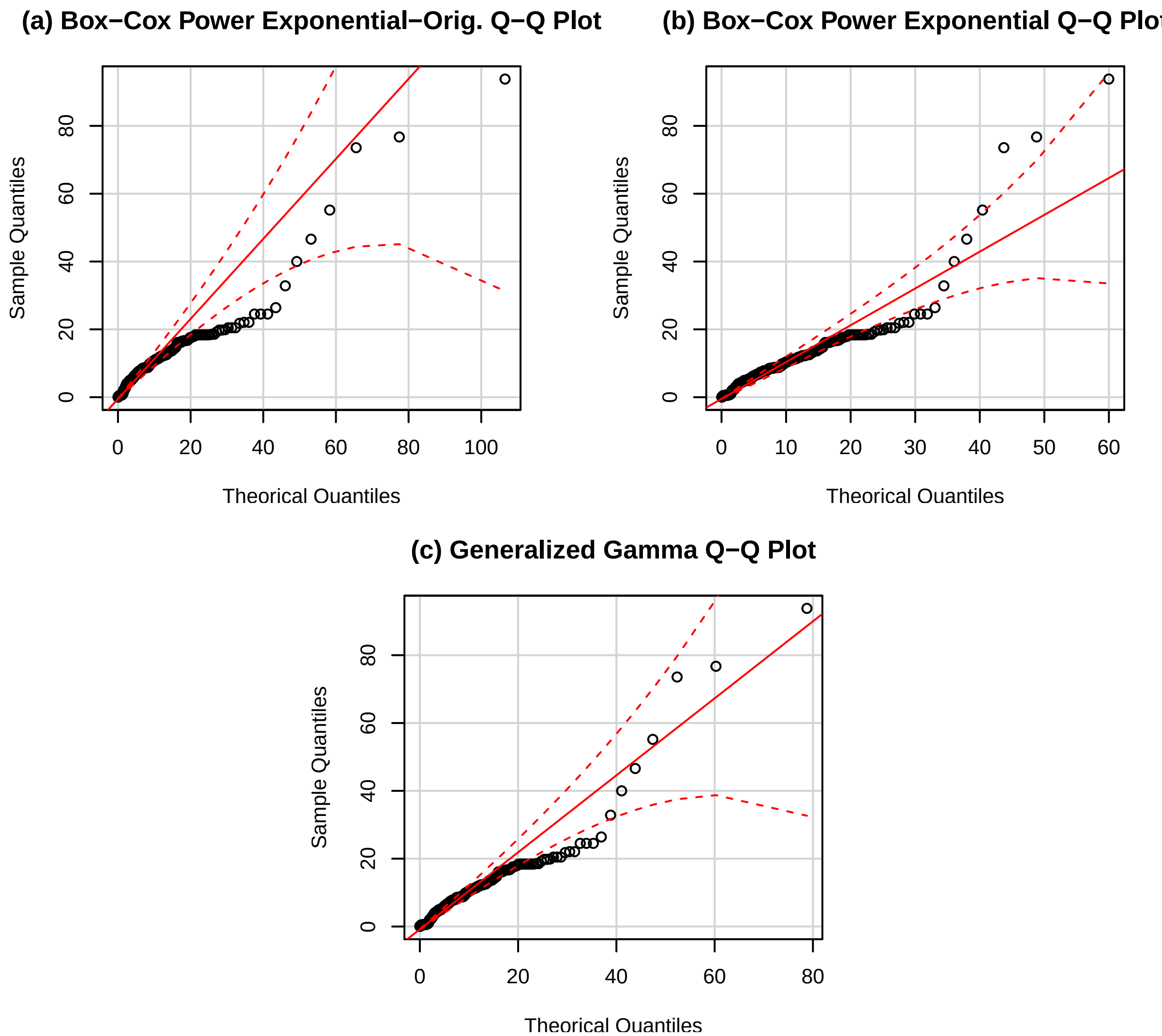

Figure 5.9: Q-Q plot GAMLSS distribution for general surgery

Finally, as in the other subsections, we present the hypothesis set of the adjusted distributions, in order to observe whether when performing the goodness of fit tests of the library `truncgof` (2012), the null hypothesis (5.8) is not rejected and it is possible to conclude that there is insufficient evidence against the individual costs of general surgery services being distributed as one of the three adjusted distributions. See Code 55 in Appendix B.

$$
\begin{aligned}
H_0 &: F_{X_{surg}}(x|\theta) \in \hat{F}_{X_{surg}}(x|\hat{\theta}) \\
H_1 &: F_{X_{surg}}(x|\theta) \notin \hat{F}_{X_{surg}}(x|\hat{\theta})
\end{aligned}
\tag{5.8}
$$

Table 5.8: Goodness-of-fit tests for general surgery services with GAMLSS distributions

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
BCPEo	0.35	0.34	0.29	0.37	0.27
BCPE	0.01	0.02	0.00	0.00	0.00
GG	0.03	0.00	0.00	0.15	0.01

The Table 5.8 shows that the distribution BCPEo gets *P-values* greater than 5% in each of the tests, suggesting that the individual costs of hospitalization have a distribution BCPEo. Also note that for the BCPE all tests are rejected with *P-values* less than 2%. While for the *GG* distribution it is appreciated that are rejected four of the five tests performed, only obtaining a *P-value* greater than 5% for the Supremum Class Upper Tail Anderson-Darling test.

Taking into consideration the previous results, it is assumed that the individual costs of general surgery services have a *BCPEo* distribution with parameters $\hat{\mu} = 0.4877005$, $\hat{\sigma} = 116.0758$, $\hat{\nu} = 1.370003$ and $\hat{\tau} = 0.2788162$, because, in first place, the Table 5.6 shows that the *BCPEo* distribution reports a lower AIC than the others, in second place, the Table 5.7 exhibits that the *BCPEo* has statistics closer to the empirical values, and in third place, the *BCPEo* is the only distribution of the three proposals, which does not reject any of the five goodness-of-fit tests proposed in the table 5.8).

5.6 Risk measures estimation for general surgery services with GAMLSS

To observe the tail type that has the distribution adjusted to the individual costs of general surgery services, we use the equation (4.53), where it is expected to find that the tail index is consistent with that presented in the previous section. See Code 56 in Appendix B.

In the figure 5.10 it is evident that the curve does not present a stable region, due the curve decreases constantly. It should be noted that although there seems to be a stable region between 2.500 and 3.000, the behavior in this area is unstable and it would not be appropriate to establish a stable region in this area. Consequently, it is concluded that the tail index of the adjusted distribution is $\xi = 0$, which is consistent with the previous section, where it was presented that the parameter ν of the *BCPEo* distribution was greater than zero, thus the tail index would be equal to zero.

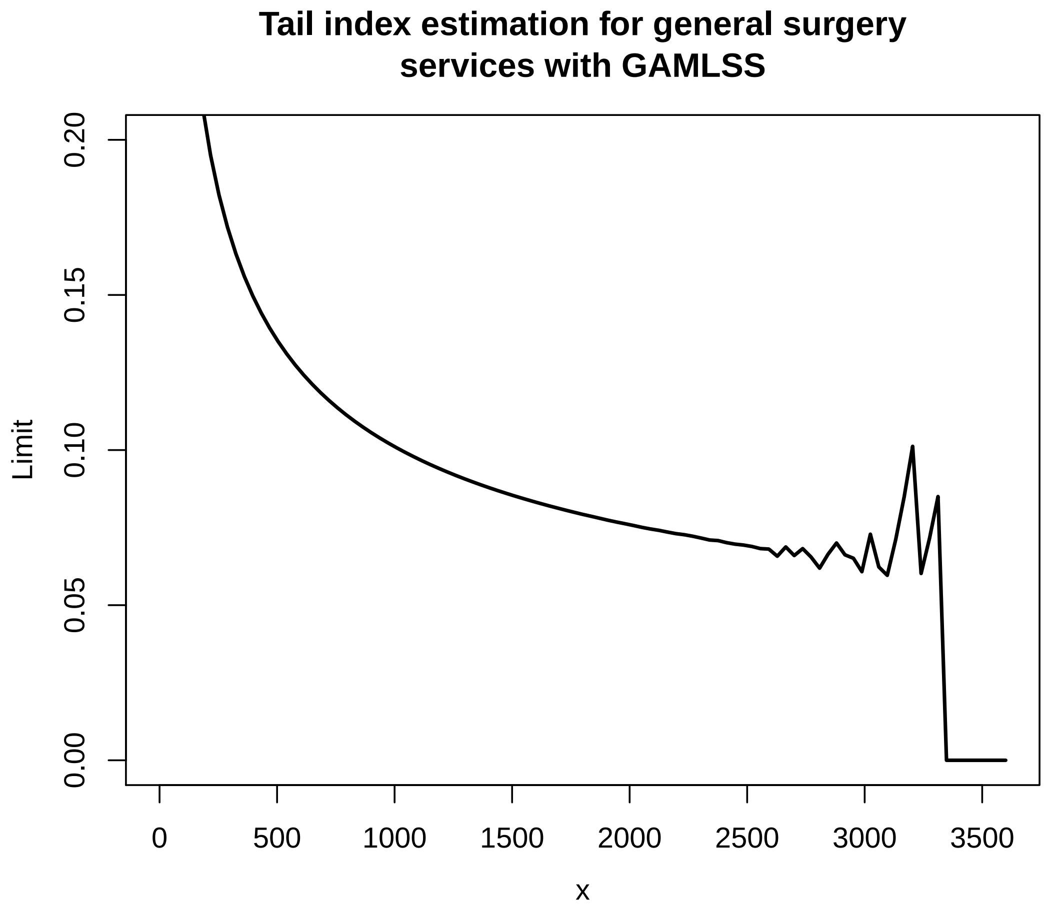

Figure 5.10: Tail index estimation for general surgery services with GAMLSS

The figure 5.11 presents different risk measures associated with the aggregate costs distribution, when we use a *BCPEo* distribution to adjust the individual costs of general surgery services. In this case, given that $\xi = 0$, the *BCPEo* distribution tail is classified within the class of semi-heavy tail distributions, consequently the equations (4.58), (4.59) and (4.60) are used for the graphic representation of *VaR*, *ES* y *SLP*, respectively. See Code 57 in Appendix B.

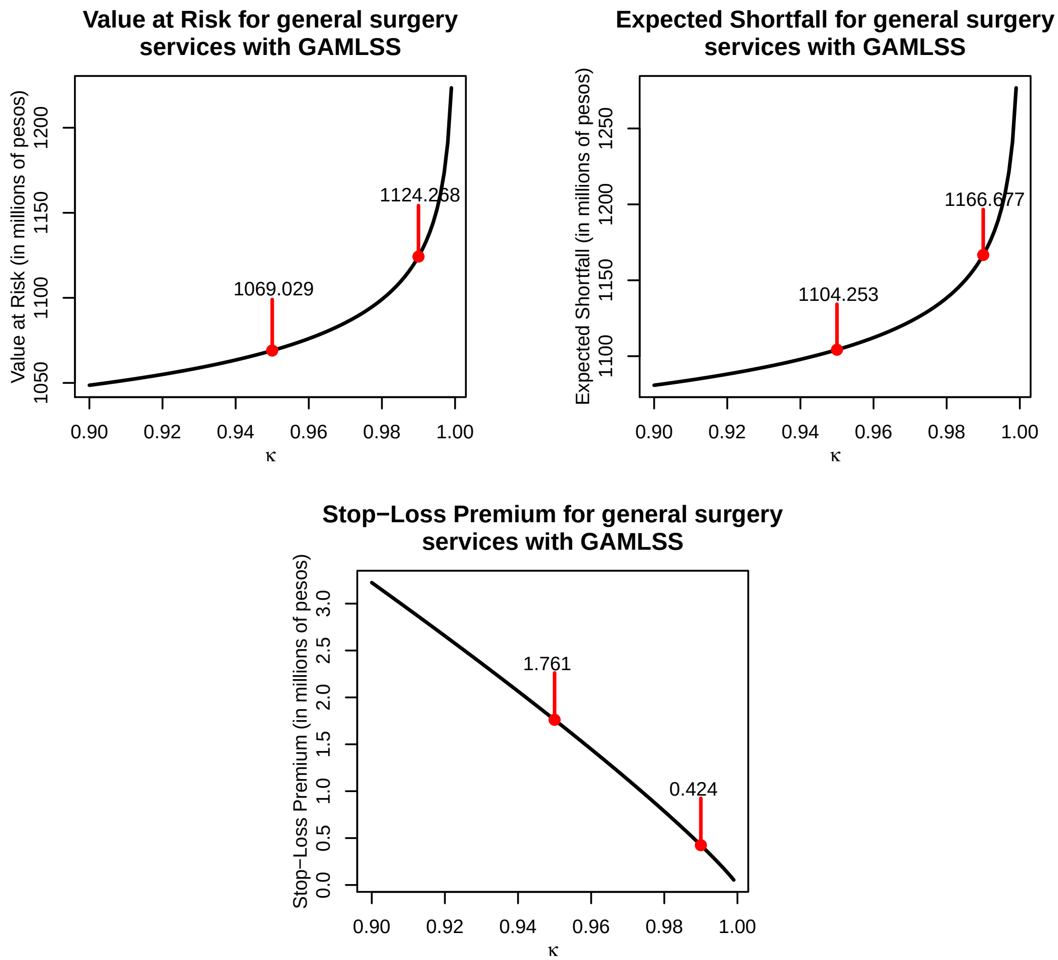

Figure 5.11: Risk measures for general surgery services with GAMLSS

In the Figure 5.11.a it is evident that the *VaR* has an increasing form, that goes from 1048.636 when $\kappa = 0.90$ up to 1223.431 when $\kappa = 0.999$, where, the upper value of $\kappa = 0.999$ is established, because in $\kappa = 1$ the *VaR* is equal to infinity. In addition, as particular values $\kappa = 0.95, 0.99$ the *VaR* takes values of 1069.029 and 1124.268 million pesos, respectively.

Similarly, in the Figure 5.11.b it is evident that the *ES* also has an increasing behavior, that goes from 1104.253 to 1166.677, when κ ranging from 0.90 to 0.999, where, on this occasion the upper value of $\kappa = 0.999$ is established because the integral presented in the equation (4.59) is singular when $\theta = 1$. It should be noted that the values of *ES* when $\kappa = 0.95$ and 0.99 are 1104.253 and 1166.677 million pesos, respectively.

In the Figure 5.11.c it is also shown the behavior of the curve for values of κ between 0.9 and 0.999, and it is appreciated that it has a decreasing behavior ranging from 3.224 to 0.053 million pesos, where, the *SLP* has a value of 1.761 when $\kappa = 0.95$

and 0.424 when $\kappa = 0.99$.

5.7 Optimum retention point estimation for general surgery service with GAMLSS

Finally, we displayed the Table 5.9 to show the behavior of the optimal retention point and the reinsurance premium that an insurer should pay when using different levels of the relative safety load factor ρ and the *BCPEo* distribution to adjust the individual costs of general surgery services. See Code 58 in Appendix B.

Table 5.9: Optimum retention point estimation for general surgery services with GAMLSS

ρ	κ_{ρ^*}	M^*_{surg}	$\delta(M^*_{surg})$	$VaR_{T_{surg}}(\kappa_{\rho^*})$
0.1	0.090909	996.633	23.193	1019.826
0.2	0.166667	998.322	23.537	1021.859
0.3	0.230769	999.902	23.854	1023.756
0.5	0.333333	1002.787	24.422	1027.210
0.8	0.444444	1006.577	25.151	1031.729
1.0	0.500000	1008.826	25.575	1034.401
1.2	0.545455	1010.898	25.959	1036.857
1.5	0.600000	1013.732	26.477	1040.208
2.0	0.666667	1017.884	27.219	1045.103
3.0	0.750000	1024.703	28.400	1053.103
4.0	0.800000	1030.220	29.324	1059.543
5.0	0.833333	1034.875	30.084	1064.959
7.0	0.875000	1042.494	31.293	1073.787
10.0	0.909091	1051.322	32.646	1083.968
20.0	0.952381	1070.541	35.436	1105.976
50.0	0.980392	1099.792	39.359	1139.151

From the Table 5.9 it is observed that the optimal retention point, the reinsurance premium and the *VaR* of the total costs have an increasing behavior when the relative safety level increases. In addition, it is evident that the optimum retention point and the reinsurance premium amount to 1051.322 and 32.646 million pesos, when $\rho = 10$, and they amount to 1099.792 and 39.359 million pesos, when $\rho = 50$, respectively.

Chapter 6

Adjustment comparison between the spliced and GAMLSS distribution

Table of Contents

6.1 Introduction

Because two alternatives were proposed to adjust the individual costs of medical services, the objective in this section is to perform a comprehensive comparative analysis using tables and graphs of the results obtained through the adjustment presented by spliced distributions Section 3.4 and GAMLSS distributions Chapter 5, to observe if there are significant differences between the optimal retention points and reinsurance premiums obtained in each case.

6.2 Adjustment comparison between the spliced and GAMLSS distribution for hospitalization services

To observe if there are significant differences between the *W-GP* mixture with estimated parameters $\hat{W}_{shape} = 0.9417177$, $\hat{W}_{scale} = 13.39916$, $\hat{u} = 35.5702$, $\hat{\sigma}_u = 31.77728$, $\hat{\xi} = -0.1190043$, $\hat{\phi}_u = 0.101750$ and the *GB2* distribution with estimated parameters $\hat{\mu} = 1884308708$, $\hat{\sigma} = 0.4392977$, $\hat{\nu} = 3.542926$ and $\hat{\tau} = 14972.83$, regarding the adjustment of the individual costs of hospitalization services, is presented the Table 6.1 in order to show which of the two distributions presents statistics closer to the mean, variance, skewness and excess kurtosis of the individual costs of hospitalization services. See Code 59 in Appendix B.

Table 6.1: Statistical measurements comparison for hospitalization services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	14.93863	362.5129	2.982862	11.97454
weibullgpd	14.83413	339.0651	3.043860	12.90941
GB2	14.96473	386.7492	3.822866	27.64090

The Table 6.1 shows that of the four statistics, the *GB2* distribution has a value of mean closer than the *W-GP* mixture, but it has an value of excess kurtosis of more than double the empirical value. For its part, the *W-GP* mixture presents a variance, skewness and excess kurtosis more similar to the empirical values, which could be a signal in favor of the *W-GP* mixture as the distribution that best fits the individual costs of hospitalization services.

To observe the graphical adjustment, the graphs of the cumulative distributions, the natural logarithm of the survival distributions and the Q-Q plots of the individual costs of hospitalization services versus the *W-GP* mixture and *GB2* distribution are presented below. See Code 60, Code 61 and Code 62 in Appendix B.

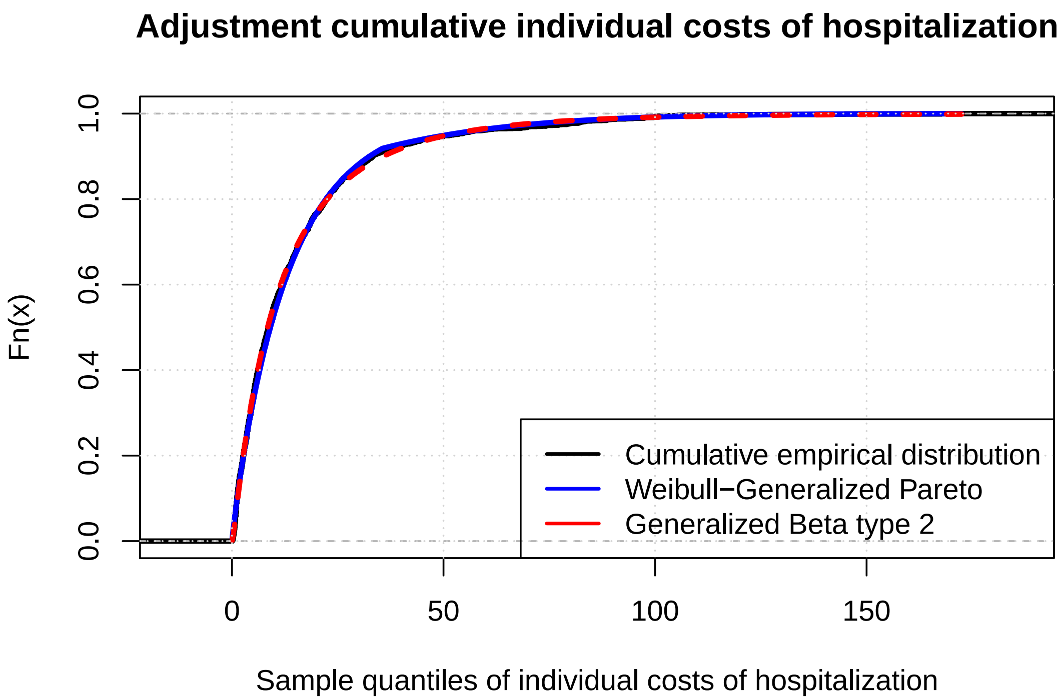

Figure 6.1: Adjustment comparison for cumulative individual costs of hospitalization services

In the Figure 6.1 it is observed that the adjustment of the Weibull-Pareto Generalized mixture is very similar to that of the *GB2* distribution, except for values between 25 and 50, where we can see that the *GB2* distribution is below the cumulative empirical distribution and the *W-GP* mixture. Even though we see in the Figure 6.1 some difference in the adjustment of the curves between the values 25 and 50, this difference is not very informative, thus it will not be taken into account for the decision on which of the two adjusted distributions presents a better fit.

The Figure 6.2 presents in more detail the adjustment made by the adjusted distributions in the tail area, being the *W-GP* mixture the one that captures better the behavior of the individual costs for services of hospitalization in the area of the tail, since it can be seen that the fall of the curve associated with the *W-GP* mixture has a similar shape to the empirical curve, while the curve associated with the *GB2* distribution has a fall that is to the right of the empirical curve, i.e., a slower fall or a heavier tail.

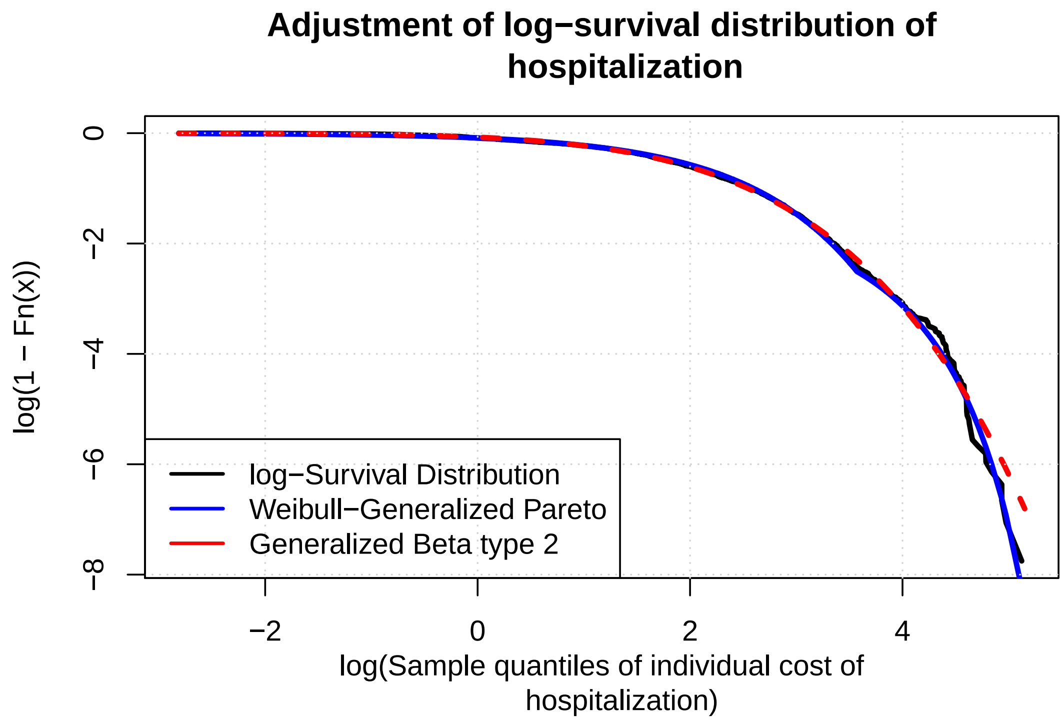

Figure 6.2: Adjustment comparison for log-survival costs of hospitalization services

In panels (a) and (b) of the Figure 6.3 are shown the Q-Q plots for the *W-GP* mixture and the *GB2* distribution, respectively, showing that the set of points associated with the *W-GP* mixture oscillate around the diagonal line very closely, while the set of points associated with the *GB2* distribution are farther from the diagonal line, in special in the upper right area.

This supports the adjustment observed in the Figure 6.2, where the *W-GP* mixture presents a similar adjustment to the *GB2* distribution, when the individual costs are low or medium, but presents a better fit than the *GB2* distribution when the individual costs are high or extreme, this being a point in favor of the *W-GP* mixture, in the decision making.

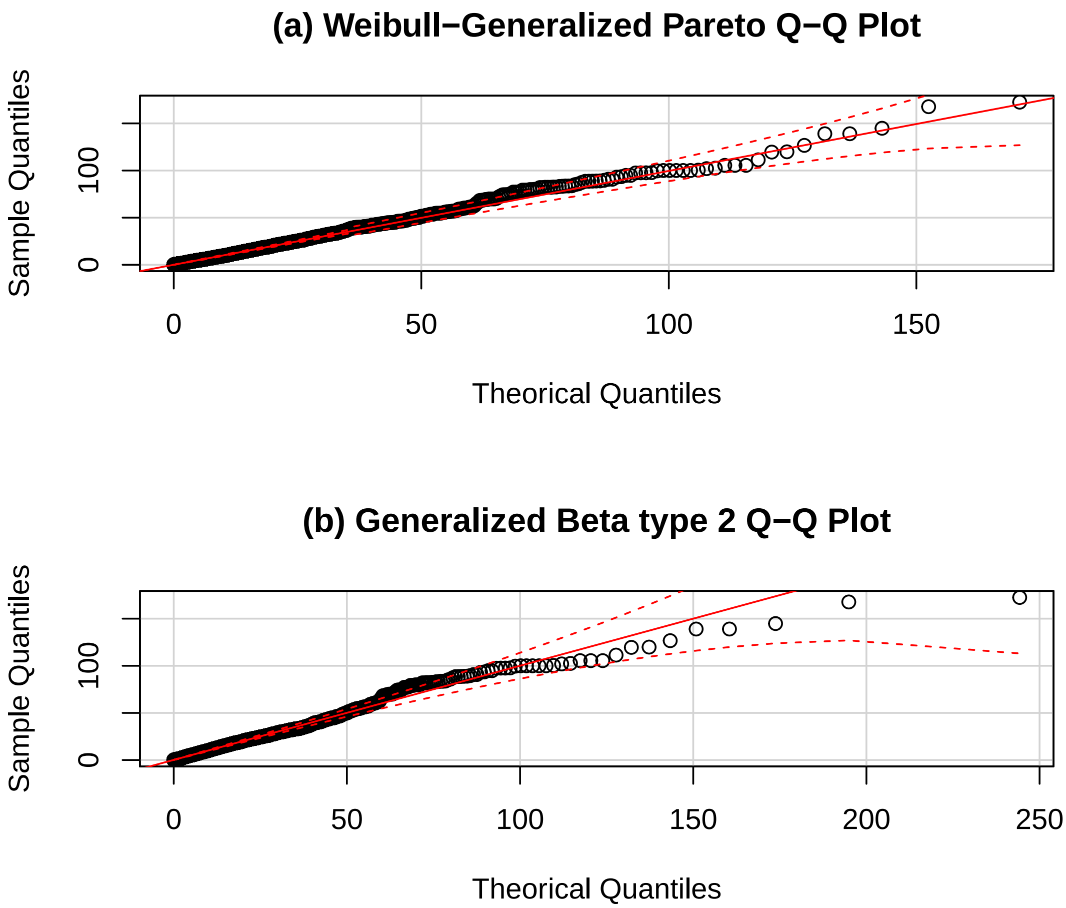

Figure 6.3: Q-Q plot comparison for hospital services

To perform the hypothesis testing for the goodness of fit tests of the spliced and GAMLSS distributions, we employed the equations (3.28) and (5.7), respectively. In addition, we used the values already obtained in the Tables 3.2 and 5.8 to construct a new table that facilitates the appreciation of the results obtained in these tests. See Code 63 in Appendix B.

Table 6.2: Goodness-of-fit tests comparison for hospitalization services

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
weibullgpd	0.91	0.91	0.87	0.08	0.00
GB2	0.98	0.99	1.00	0.00	0.97

The Table 6.2 shows that neither of the two distributions rejects the null hypothesis for the Kolmogorov-Smirnov, Cramer-von Mises and Kuiper tests, since they get *P-values* higher than 5% in each of these tests. Additionally, it is observed that the *W-GP* mixture presents a *P-value* greater than 5% in the Supremum class Upper Tail Anderson-Darling test but a *P-value* less than 5% in the Quadratic Class Upper Tail Anderson-Darling test, while the *GB2* distribution presents a *P-value* less than 5% in the Supremum class Upper Tail Anderson-Darling test and a *P-value* more than 5% in the Quadratic Class Upper Tail Anderson-Darling test.

From the results presented in this subsection, it is observed that the *W-GP* mixture presents statistics closer to the empirical values of the individual costs, than the *GB2* distribution. Also graphically, it is observed that in the Figures 6.2 and 6.3, the *W-GP* mixture better fits the shape of the tail of the the individual costs, showing that their estimated values are closer to the empirical values, than the values estimated by adjusting the *GB2* distribution.

Therefore, for the calculation of the risk measures, the optimal retention points and the reinsurance premiums are taken as the reference values, those adjusted by means of the *W-GP* mixture.

6.3 Risk measures comparison for hospitalization services

In order to make a comparison between risk measures when a *W-GP* mixture or *GB2* distribution is adjusted to the individual costs of hospitalization services, is made a graphical representation of the behavior of *VaR*, *ES* and *SLP* for each adjustment. See Code 64 in Appendix B.

The Figure 6.4 shows that risk measures associated with the *GB2* distribution are above those associated with the *W-GP* mixture, and based on what was mentioned in the previous section, about the mixture *W-GP* is the one that offers the best adjustment to the set of individual costs of hospitalization services, and consequently, it is the situation that is taken as a point of reference, then, it is concluded that the *GB2* distribution overestimates the value of the risk measures.

Additionally, in the Figure 6.4.a it is observed that when is used a percentile $\kappa = 0.9$, the *GB2* adjustment overestimates the *VaR*, for a little less than 150 million pesos, while, when the value of κ approaches 0.999, the *GB2* adjustment increases the overestimation to close to 326 million pesos.

In the same way, it is observed in the Figure 6.4.b that the overestimation of *ES* goes from 157.910, to 389.333 millions of pesos, when κ increases from 0.90 to 0.999. While, in the Figure 6.4.c the difference of the *SLP* decreases between adjustments, as the value of κ increases, being the difference of little more than 3.5 millon pesos when $\kappa = 0.9$, to be less of 1 million pesos when $\kappa = 0.999$.

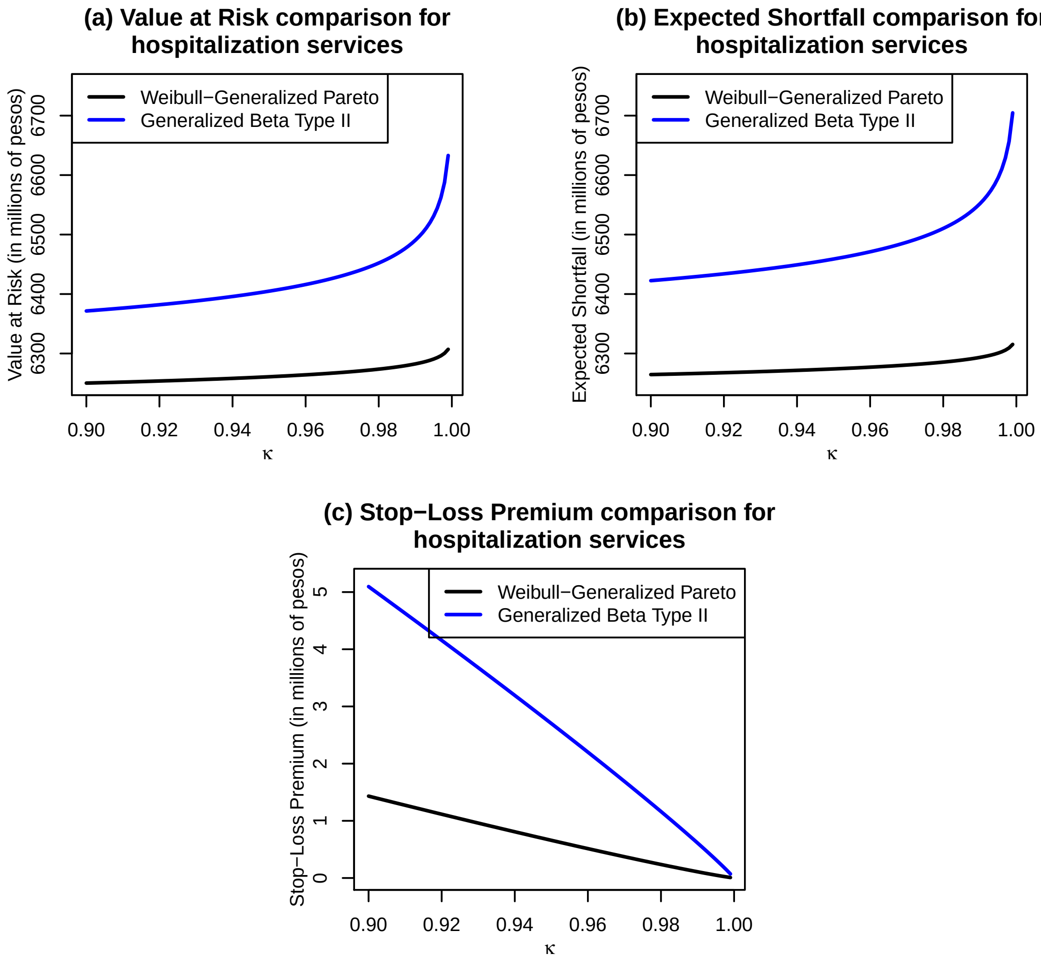

Figure 6.4: Risk measures comparison for hospitalization services

6.4 Optimum retention point comparison for hospitalization services

To observe the difference between the optimal retention points and the reinsurance premiums obtained by adjusting the *W-GP* mixture and the *GB2* distribution to the individual costs of hospitalization services, we present the Table 6.3, which shows for different safety load factor levels, the optimal retention point and the reinsurance premium that an insurer must have depending on the adjusted distribution. See Code 65 in Appendix B.

The Table 6.3 shows that both the optimal retention point and the reinsurance premium are greater for all relative safety factor levels, when is used the adjustment of the *GB2* distribution to fit the individual costs, than when is employ the *W-GP* mixture.

Table 6.3: Optimum retention point comparison for hospitalization services

ρ	κ_{ρ^*}	$M^*_{hospW-GP}$	$M^*_{hospGB2}$	$\delta(M^*_{hospW-GP})$	$\delta(M^*_{hospGB2})$
0.1	0.090909	6209.826	6279.488	18.616	41.256
0.2	0.166667	6211.630	6282.714	18.424	41.636
0.3	0.230769	6213.272	6285.711	18.249	41.985
0.5	0.333333	6216.170	6291.139	17.941	42.611
0.8	0.444444	6219.791	6298.182	17.556	43.409
1.0	0.500000	6221.848	6302.319	17.337	43.870
1.2	0.545455	6223.686	6306.102	17.142	44.288
1.5	0.600000	6226.120	6311.239	16.883	44.848
2.0	0.666667	6229.527	6318.688	16.520	45.649
3.0	0.750000	6234.756	6330.738	15.964	46.914
4.0	0.800000	6238.689	6340.333	15.546	47.896
5.0	0.833333	6241.827	6348.335	15.212	48.700
7.0	0.875000	6246.642	6361.257	14.700	49.971
10.0	0.909091	6251.782	6375.985	14.154	51.381
20.0	0.952381	6261.640	6407.265	13.105	54.253
50.0	0.980392	6273.989	6453.197	11.792	58.217

Consequently, it is concluded that in this case, the use of the *GB2* distribution leads to an overestimation of the reinsurance premium in an amount exceeding 20 million pesos, and overestimates the optimal retention point for a value exceeding 65 million pesos, leading the insurer to pay a greater amount to cover the risk associated to hospitalization services.

6.5 Adjustment comparison between the spliced and GAMLSS distribution for general surgery services

Similar to the hospitalization case, the objective of this subsection is to decide which of the two proposed alternatives is the distribution that best fits the individual costs of general surgery services. Namely, the first alternative was obtained by adjusting *spliced distributions* with a *W-GP* mixture with estimated parameters $\hat{W}_{shape} = 0.8458848$, $\hat{W}_{scale} = 6.121305$, $\hat{u} = 16.12227$, $\hat{\sigma}_u = 2.802543$, $\hat{\xi} = 0.853189$ and $\hat{\phi}_u = 0.1034537$, and the second alternative was obtained by GAMLSS distributions with a *BCPEo* distribution with estimated parameters $\hat{\mu} = 0.4877005$, $\hat{\sigma} = 116.0758$, $\hat{\nu} = 1.370003$ and $\hat{\tau} = 0.2788162$.

To make such a decision, we initially present the Table 6.4, which shows the values of the mean, variance, skewness and excess kurtosis of the individual costs of general surgery and the estimated statistics associated to the *W-GP* mixture and the *BCPEo* distribution. See Code 66 in Appendix B.

Table 6.4: Statistical measurements comparison for general surgery services

Dist	Mean	Variance	Skewness	Excess Kurtosis
Empirical	6.680426	82.89872	4.671353	33.36082
weibullgdp	7.731212	does not exist	does not exist	does not exist
BCPEo	6.787825	122.55566	5.179333	53.99778

The Table 6.4 is not very useful to compare the fitted distributions, because only the first moment is defined for the *W-GP* mixture, consequently, it is not possible to make a comparison of the similarity between the empirical and estimated statistics by this distribution. This does not mean that the *W-GP* mixture can not make a good fit to the data set, since the values that present the individual costs for variance, skewness and excess kurtosis are calculated from observed data and not from an infinite set of identically distributed observations.

Additionally, the Table 6.4 shows that unlike the *W-GP* mixture, the *BCPEo* distribution presents values for each of the statistics, among which we highlight the value of the estimated mean and skewness, due to the closeness they have with respect to the empirical mean and skewness.

To observe the adjustment of the fitted distributions, we build three plots. In the first we present the cumulative distribution of the individual costs of general surgery versus the adjustment of the *W-GP* mixture and *BCPEo* distribution. In the second we present the natural logarithm of the survival distribution of the individual costs versus the natural logarithm of the survival functions of the fitted distributions. In the third, two panels are presented that contain the Q-Q plot of each adjusted distribution to the individual costs of general surgery services. See Code 67, Code 68 and Code 69 in Appendix B.

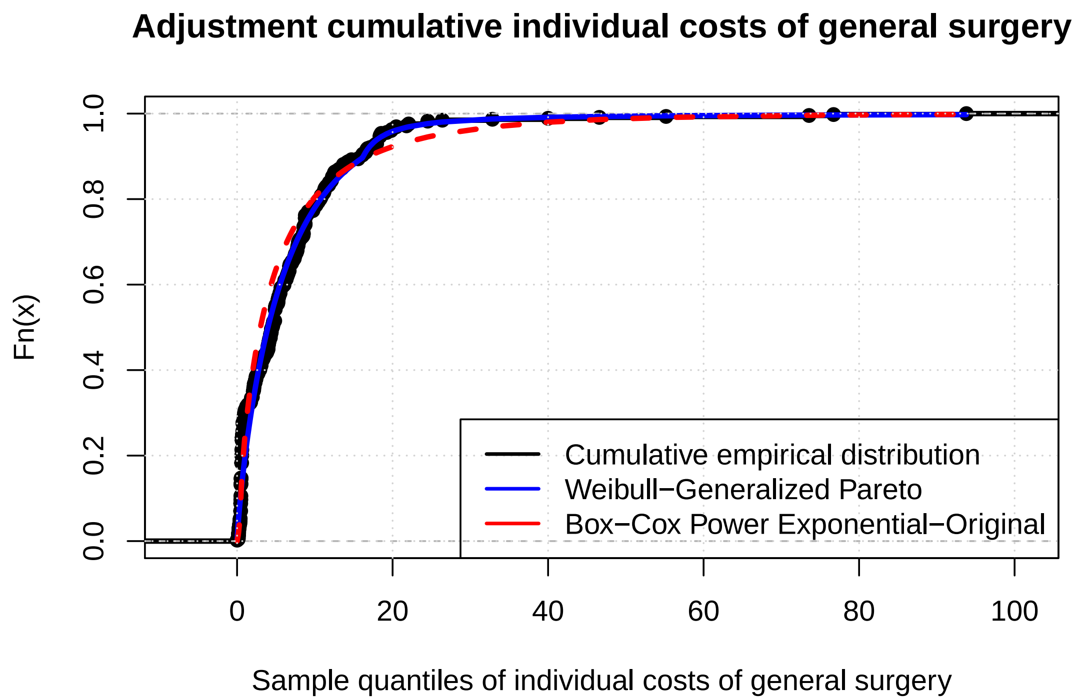

Figure 6.5: Adjustment comparison for cumulative individual costs of general surgery services

The Figure 6.5 shows that the curve associated to the *BCPEo* distribution does not have the shape of the empirical curve, since it is on the left in the initial part, and below in the middle part of the cumulative empirical curve. On the contrary, the curve associated to the *W-GP* mixture manages to adequately capture the behavior of the empirical curve, being the threshold $u = 16.12227$, the place where the greatest difference between curves can be seen, being this the continuity point between the Weibull and Generalized Pareto distribution.

In the Figure 6.6 we can see that the curve associated to the natural logarithm of the survival distribution of the individual costs has a concave behavior, followed by a convex behavior from a value close to 2.5. Due to this, we observe as the use of a distribution by parts such as the *W-GP* mixture, can provide a better fit than a complete distribution such as the *BCPEo*.

To illustrate the above, the Figure 6.6 shows how the curve associated with the *W-GP* mixture presents a good fit both in the Weibull and Generalized Pareto part, being the value 2.780202 the continuity point in the mixture, such that this value is simply the natural logarithm of the threshold $u = 16.12227$.

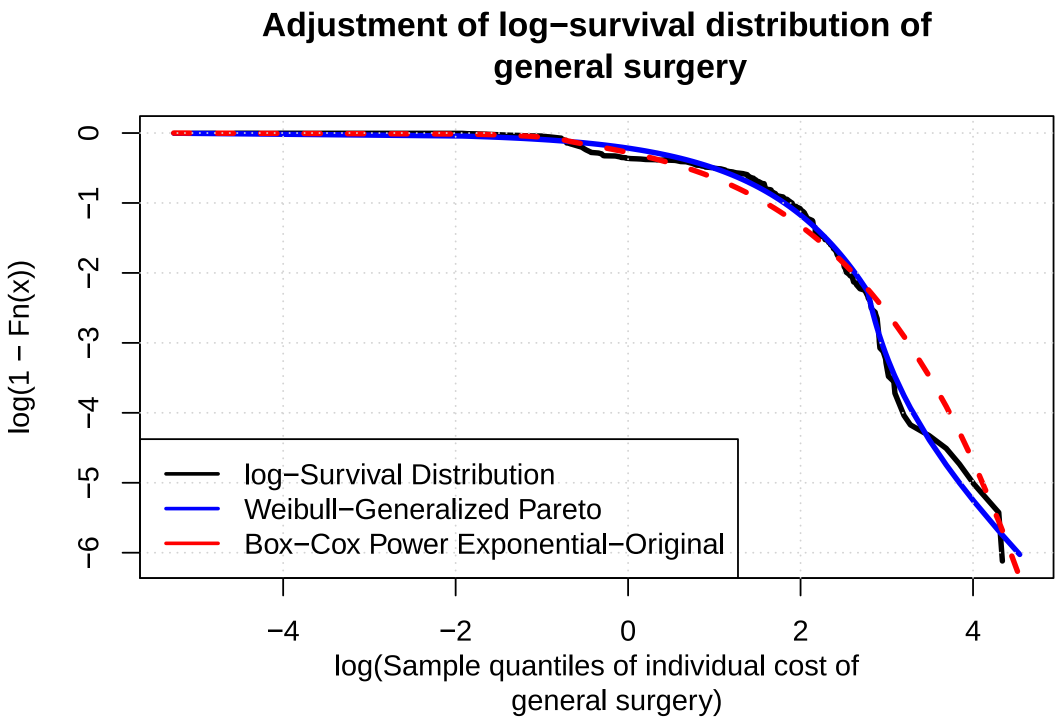

Figure 6.6: Adjustment comparison for log-survival costs of general surgery services

In addition, the Figure 6.6 shows as the curve associated with the *BCPEo* distribution, presents a good fit in the initial part while the empirical curve has a concave shape, but once the empirical curve takes convex shape, the Figure 6.6 displays as the curve associated with the *BCPEo* distribution, begins to present problems in the adjustment, since it is always to the right of the empirical curve.

The Q-Q plots for the *W-GP* mixture and the *BCPEo* distribution are exhibited respectively, in the panels (a) and (b) of the Figure 6.7. In the panel (a) it is evidenced that the confidence bands indefinitely open, covering all the points of the graph, where, this behavior can be explained by the fact that the adjusted distribution only has defined its first moment.

For its part, in the panel (b) presents confidence bands defined along the graph, and it is shown that there is a set of points that are below and above the confidence bands, which can be taken as a sign that the adjusted distribution is not the one indicated for modeling the set of individual costs of general surgery services.

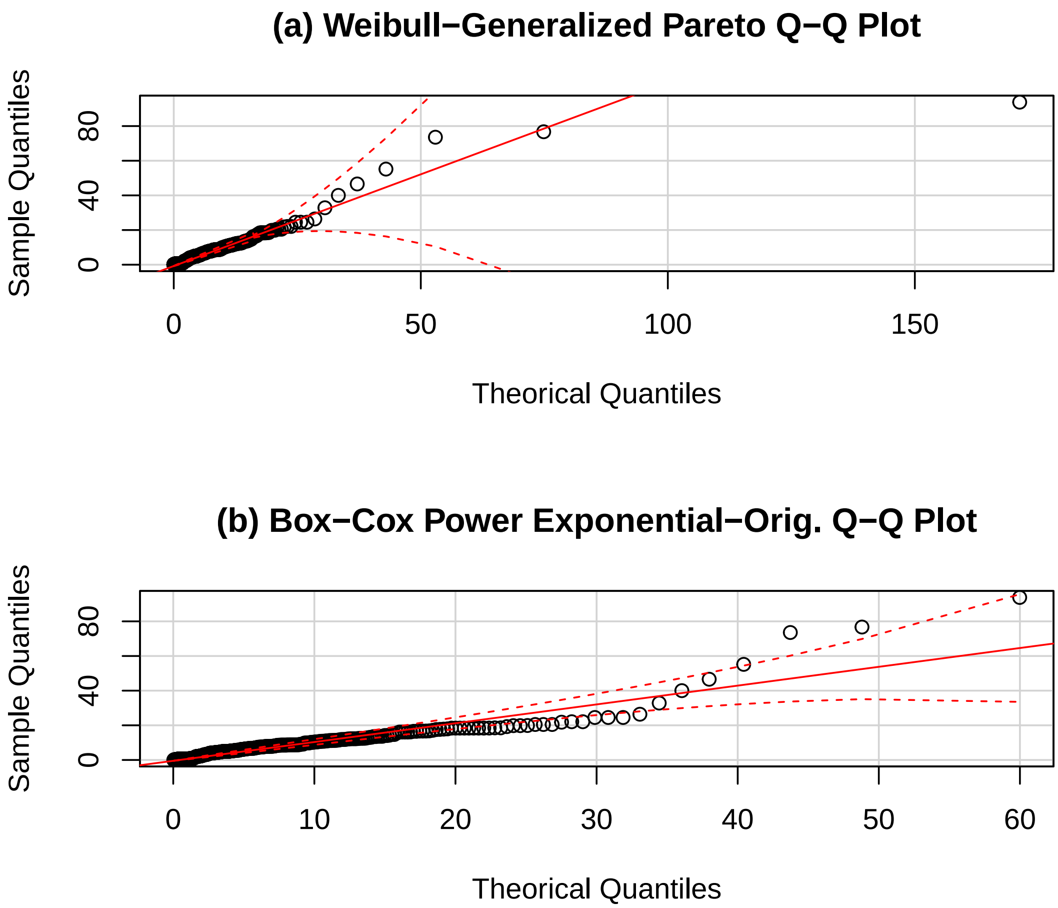

Figure 6.7: Q-Q plot comparison for general surgery services

For the hypothesis testing of the spliced and the GAMLSS distributions, we use the equations (3.29) and (5.8), with the values already reported in the Tables 3.4 and 5.8, in order to build a table that facilitates the comparison of the *P-Value* obtained in the goodness of fit tests, with the *W-GP* mixture and the *BCPEo* distribution. See Code 70 in Appendix B.

Table 6.5: Goodness-of-fit tests comparison for general surgery services

Dist	ks.test	w2.test	v.test	adup.test	ad2up.test
weibullgpd	0.26	0.39	0.27	0.26	0.07
BCPEo	0.35	0.34	0.29	0.37	0.27

The Table 6.5 shows that for all tests we get *P-values* greater than 5%, therefore, none of the tests used rejects the null hypothesis, thus it is concluded that there is not enough evidence to say that the individual costs for general surgery services are not distributed as a *W-GP* mixture or as a *BCPEo* distribution.

Considering that none of the tables presented in this subsection were informative to decide which distribution best fits the individual costs of general surgery services, the decision is made based on the graphic evidence obtained in the Figures 6.5, 6.6 and 6.7.

Hence, it is concluded that the distribution that presents a better adjustment to the individual costs of general surgery services is the mixture *W-GP*, consequently, it is the distribution that is taken as a reference for the calculation of risk measures, optimal retention points and reinsurance premiums.

6.6 Risk measures comparison for general surgery services

To observe the difference between the risk measures associated with the adjustment of the *W-GP* mixture and the *GB2* distribution to the individual costs of hospitalization services, we presented graphically the behavior of the *VaR*, *ES* and *SLP* for each adjustment. See Code 71 in Appendix B.

The Figure 6.8 shows that there are significant differences in all the risk measures associated with the *BCPEo* distribution and the *W-GP* mixture, where, it is evidenced that the curves associated with the adjustment *W-GP* are above those associated with the adjustment *BCPEo* in all the panels.

Because the mixture *W-GP* presented better adjustment to the individual costs of general surgery services and it was decided to use it as the reference distribution, then from the results obtained in the Figure 6.8, it is concluded that the adjustment of the distribution *BCPEo* underestimates the value of all the risk measures.

In the panel (a) is noted that the underestimation of *VaR* by the adjustment *BCPEo* is 1169.887 when $\kappa = 0.90$ and 7754.409 when $\kappa = 0.999$, similarly, the panel (b) shows that the underestimation of the *ES* amounts from 1925.483 to 47770.578 when κ rises from 0.90 to 0.999, while, in the panel (c) it is evident that the underestimation of *SLP* drops from 75.560 when $\kappa = 0.90$ to 40.016 when $\kappa = 0.999$.

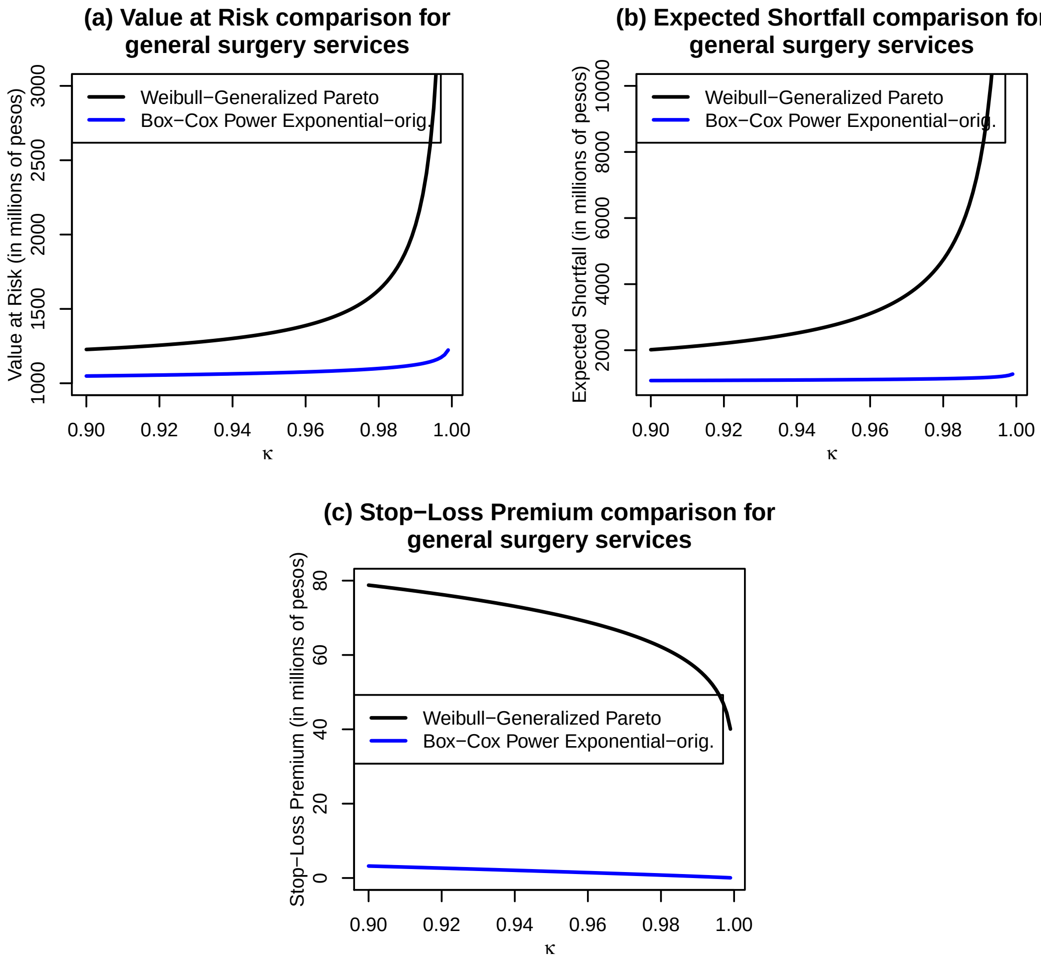

Figure 6.8: Risk measures comparison for general surgery services

6.7 Optimum retention point comparison for general surgery services

In order to observe the difference between the optimal retention points and the reinsurance premiums that an insurer must pay for different levels of the security load factor, when using the adjustment *BCPEo* and *W-GP* to the individual costs of general surgery services, we presented the Table 6.6. See Code 72 in Appendix B.

In this table it is appreciated that there are significant differences between the points of optimal retention and the reinsurance premiums, when is used the adjustment of the *BCPEo* distribution or the *W-GP* mixture for the individual costs of general surgery services.

Table 6.6: Optimum retention point comparison for general surgery services

ρ	κ_{ρ^*}	$M^*_{surgW-GP}$	$M^*_{surgBCPEo}$	$\delta(M^*_{surgW-GP})$	$\delta(M^*_{surgBCPEo})$
0.1	0.090909	2103.577	996.633	119.829	23.193
0.2	0.166667	2105.166	998.322	129.063	23.537
0.3	0.230769	2106.736	999.902	138.185	23.854
0.5	0.333333	2109.824	1002.787	156.130	24.422
0.8	0.444444	2114.345	1006.577	182.407	25.151
1.0	0.500000	2117.298	1008.826	199.564	25.575
1.2	0.545455	2120.207	1010.898	216.470	25.959
1.5	0.600000	2124.499	1013.732	241.415	26.477
2.0	0.666667	2131.491	1017.884	282.047	27.219
3.0	0.750000	2144.992	1024.703	360.510	28.400
4.0	0.800000	2158.002	1030.220	436.114	29.324
5.0	0.833333	2170.632	1034.875	509.515	30.084
7.0	0.875000	2195.022	1042.494	651.258	31.293
10.0	0.909091	2230.008	1051.322	854.577	32.646
20.0	0.952381	2338.265	1070.541	1483.711	35.436
50.0	0.980392	2627.260	1099.792	3163.199	39.359

Since it is evident that as the reinsurer's relative safety factor increases, the underestimation by the *BCPEo* distribution of retention points and reinsurance premiums increases too from 1106.944 and 96.636 million pesos when $\rho = 0.1$, to 1527.468 and 3123.84 million pesos when $\rho = 50$, respectively.

The biggest difference between the value of the reinsurance premium for the adjustment *BCPEo* and *W-GP* when $\kappa_{p^*} \to 1$, may be due to the adjustment in the tail area presented by the distributions in the Figures 6.6 and 6.7, where it is observed that unlike the *W-GP* mixture, the *BCPEo* distribution does not capture the tail area behavior of the individual costs, which also leads to a bad adjustment for the reinsurance premium.

Chapter 7

Conclusions

An objective of this work was to propose the GAMLSS distributions as an alternative to the distributions usually employed in practice for adjusting the number and claims size, since are generally employed distributions such as Poisson, Binomial or Negative Binomial to adjust the claims number, and distributions such as Weibull, Log-Normal, Gumbel, the Pareto family distributions, or *spliced distributions* to adjust the claims size.

For the claims number, stand out the fact that there is a wide variety of distributions and mixtures that can be employed for frequency adjustment, because in practice, when is applied the actuarial method, is usually assumed that the claims number is distributed Poisson, Binomial or Negative Binomial, when there really are other distributions or mixtures that can offer a better fit.

A sample of this is presented in the Figure 2.2, where the adjustment process is done through the wide list of distributions contained in GAMLSS, where, it is observed that the Delaporte distribution is the one that offers a better adjustment to the claims number for general surgery services.

For the claims size, are compared the GAMLSS distributions adjustments performance against *spliced distributions*, obtaining that in this case, as shown in the figures 6.2 and 6.6, the *spliced distributions* offer a better adjustment, in consecuense, these distributions should be the ones used for the calculation of optimal retention points and reinsurance premiums, since the use of distributions that do not adjust correctly the claims size, may lead to underestimations or overestimates of these measurements.

Because of this, it is recommended not to make assumptions about the distribution of the number or claims size, since assuming or using a distribution that does not offer a good adjustment, can generate overestimates or underestimates both in the calculation of the reinsurance premium, and in the optimum retention point that an insurer must have to cover a certain risk associated with a *HCD*, as shown in the tables 6.3 and 6.6.

Evidence of this is presented in Chapter 6, where it is shown that adjusting a *GB2* distribution to the individual costs of hospitalization services, overestimates retention points and optimal reinsurance premiums, while adjusting a *BCPEo* distribution to the individual costs of general surgery services, significantly underestimates these measures.

Therefore, to avoid overestimation and underestimation of retention and reinsurance premium, it is recommended to test the adjustment of as many distributions and mixtures as possible, since the better the adjustment to the number and claims size, the more accurate are the optimal calculated values, taking into consideration the real risk of the claims.

Additionally, it is recommended to use data sets that have a tail index $\xi = 1$ and $\xi > 1$, in order to test the behavior of the reinsurance premium under the two scenarios outlined in the section 4.2.1, which could not be covered in this Master's degree project, due to the limitations of time and information, since in this work only could be covered the case of heavy tail distributions with finite mean and semi-heavy tail distributions.

Another recommendation is not to indiscriminately use the equations and approximations proposed in papers, and we invite you to carefully review this before using them, because, as shown in section 4.2.2, although the proposal for the calculation of *TVaR* made by Biagini and Ulmer (2009) and presented in the equation (4.40) is well-founded under certain conditions, this can not be applied when changing the approximation of *VaR* proposed by Böcker and Klüppelberg (2005), for a corrected approximation as the one proposed by Böcker and Sprittulla (2006) or Degen (2010).

Finally, it is expected that the methodology presented here for the calculation of the optimum retention point that an insurer must have for a *HCD*, is useful for prepaid medicine companies, especially, for those people who are interested in the subject of reinsurance, because an effort was made to explain in the best possible way, both the theoretical and practical part necessary for the application of the methodology.

Appendix A: Colombia's regulation of critical diseases in medical services

In Colombia, as well defined by Article 16 of Resolution 5261 of 1994, the *HCD* are those that due to their nature, have a low frequency, have a high technical complexity in their management, represent a low cost-effectiveness in their treatment and they generate high costs. In addition, in Article 17 of the same resolution, are described the treatments that are included or accepted for the management of *HCD*:

a) Treatment with radiotherapy and chemotherapy for cancer.
b) Dialysis for chronic renal failure, kidney transplantation, heart, bone marrow and cornea.
c) Treatment for AIDS and its complications.
d) Surgical treatment for heart diseases and central nervous system.
e) Surgical treatment for genetic origin or congenital diseases.
f) Surgical medical treatment for major trauma.
g) Intensive care unit therapy.
h) Joint replacements.

Due to the *HCD* nature, it is established in Paragraph 4 of Article 162 of Law 100 of 1993 that every health promoting entity will reinsure the risks derived from the care of diseases qualified by the National Council of Social Security as high cost.

Additionally, the Article 19 of Law 1122 of 2007 establishes that for the care of *HCD*, health promoting entities will contract reinsurance or respond, directly or collectively for said risk, in accordance with the regulations issued by the National Government.

Furthermore, it is decided to include other diseases among the *HCD*, and they are described in the Article 1 of Resolution 3974 of 2009 as:

1. Cervical cancer
2. Breast cancer
3. Stomach cancer
4. Colon and rectum cancer
5. Prostate cancer

6. Acute lymphocytic leukemia
7. Acute myeloid leukemia
8. Hodgkin lymphoma
9. Non-hodgkin lymphoma
10. Epilepsy
11. Rheumatoid arthritis
12. Infection by the Human Immunodeficiency Virus (HIV) and Acquired Immunodeficiency Syndrome (AIDS).

To complete the definition of high-cost events and services, in the Article 126 of Resolution 5521 of 2013 defines high-cost events and services as:

1. Kidney transplant, heart, liver, bone marrow y córnea.
2. Peritoneal dialysis and hemodialysis.
3. Surgical management for heart diseases.
4. Surgical management for diseases of the central nervous system.
5. Joint replacements.
6. Medical surgical management of the large burned patient.
7. Major trauma management.
8. Diagnosis and management of the HIV-infected patient.
9. Chemotherapy and radiotherapy for cancer.
10. Management of patients in the Intensive Care Unit.
11. Surgical management of congenital diseases.

In addition, in the Article 127 of Resolution 5521 of 2013 defines the types of injuries that a patient must have to be considered as a large burn, therefore, be considered as a patient that requires a high-cost service or event.

1. Burns of 2° and 3° degree in more than 20% of body surface.
2. Burns of total or deep thickness, in any extension, affecting hands, face, eyes, ears, feet and perineum or genital anus.
3. Burns complicated by aspiration injury.
4. Deep and mucous burns, electrical and/or chemical.
5. Complicated burns with fractures and other major trauma.
6. Burns in patients at high risk for being younger than 5 years and older than 60 years or complicated by intercurrent diseases, severe or previous critical condition.

Finally, in the Article 128 of Resolution 5521 of 2013 states that in order to consider a patient as having greater trauma, it is required that this has one or more serious injuries caused by external violence, which require for medical-surgical management the performance of multiple therapeutic procedures or interventions, which require for medical-surgical management the performance of multiple therapeutic procedures or interventions, which is carried out under a service of high complexity.

Appendix B: R language scripts

Data analysis

Code 1: Load dataset and header

```
if(!require("kableExtra")) install.packages("kableExtra")
require(kableExtra)
### Loading data
hospita <- read.table("data/hospitalization.dat", header = T)
hospita$cost <- hospita$cost/1000000
surgery <- read.table("data/surgery.dat", header = T)
surgery$cost <- surgery$cost/1000000
### Header
kable(cbind(head(hospita), round(head(surgery), 3)),
      caption = "Header of dataset
      \\label{tab:header}",
      "latex", booktabs = T) %>%
      add_header_above(c("Hospitalization" = 2,
                         "General Surgery" = 2)) %>%
      column_spec(column = 2, border_right = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 2: Pareto chart and table for hospitalization frequencies

```
if(!require("qcc")) install.packages("qcc")
require(qcc)
### Pareto chart hospitalization
parhosp <- pareto.chart(table(hospita$year),
           main = "Hospitalization", las = 1)
### Table
kable(parhosp,
      caption = "Distribution records hospitalization per year
      \\label{tab:frech}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 3: Pareto chart and table for general surgery frequencies

```r
### Pareto chart
parsurg <- pareto.chart(table(surgery$year),
           main = "General surgery", las = 1)
### Table
kable(parsurg,
      caption = "Distribution records general surgery per year
      \\label{tab:frecs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 4: Histogram, Box-plot, Density and Scatterplot for hospitalization costs

```r
par(mfrow=c(2,2))
### Histogram
hist(hospita$cost, freq = F, xlab = "Costs",
     main = "(a) Hospitalization costs \n Histogram",
     col = "lightblue")
### Box-plot
boxplot(hospita$cost~factor(hospita$year), col = rainbow(6, s = 0.6),
        main = "(a) Hospitalization services \n Box-plot",
        xlab = "Costs", ylab = "Years", horizontal = T, las=1)
### Density
plot(density(hospita$cost), lwd = 2,
     main = "(c) Hospitalization costs \n Density")
polygon(density(hospita$cost), col = "lightblue")
### Scatterplot
plot(hospita$year,hospita$cost, ylab="Costs", xlab = "Year",
     main = "(d) Hospitalization services \n Year vs Cost")
```

Code 5: Histogram, Box-plot, Density and Scatterplot general surgery costs

```r
par(mfrow=c(2,2))
### Histogram
hist(surgery$cost, freq = F, xlab = "Costs",
     main = "(a) General surgery costs \n Histogram",
     col = "lightblue")
### Box-plot
boxplot(surgery$cost~factor(surgery$year), col = rainbow(6, s = 0.6),
        main = "(b)  General surgery services \n Box-plot",
        xlab = "Costs", ylab = "Years", horizontal = T, las=1)
### Density
plot(density(surgery$cost), lwd = 2,
     main = "(c) General surgery costs \n Density")
polygon(density(surgery$cost), col = "lightblue")
```

```
### Scatterplot
plot(surgery$year,surgery$cost, ylab="Costs", xlab = "Year",
     main = "(d) General surgery services \n Year vs Cost")
```

Frequency model estimation for hospitalization

Code 6: Best fit with GAMLSS for frequencies of hospitalization services

```
if(!require("gamlss")) install.packages("gamlss")
require(gamlss)
### The adjustment is made
FitN_hosp1 <- fitDist(y = table(hospita$year), type = "counts")
### Estimation of second and third distribution with best fit
FitN_hosp2 <- gamlssML(formula = table(hospita$year), family = GPO)
FitN_hosp3 <- gamlssML(formula = table(hospita$year), family = NBI)
### The five distributions that present the best fit are
kable(rbind(FitN_hosp1$fits[1:5]),
      caption = "Better fit for frequencies of hospitalization services
      \\label{tab:fitfhosp}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Once the estimates have been made, it is possible to extract the value of the parameters μ, σ, ν and τ, depending on whether the estimated distribution has one, two, three or four parameters, respectively. To do this, use the command `FitN_hosp1$mu` to extract the parameter μ, from the *PIG* distribution, or the command `FitN_hospi3$sigma` to extract the parameter σ, of the *NBI* distribution.

Code 7: Statistical measurements of hospitalization frequencies

```
if(!require("devtools")) install.packages("devtools")
require(devtools)
if(!require("DistMom")) install_github("jiperezga/DistMom")
require(DistMom)
if(!require("e1071")) install.packages("e1071")
require(e1071)
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanNhEmp <- mean(table(hospita$year))
MeanNhPIG <- moments(k = 1, dist = "PIG", domain = "counts",
               param = c(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
MeanNhGPO <- moments(k = 1, dist = "GPO", domain = "counts",
               param = c(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
```

```
MeanNhNBI <- moments(k = 1, dist = "NBI", domain = "counts",
             param = c(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))
#### Variance
VariNhEmp <- var(table(hospita$year))
VariNhPIG <- moments(k = 2, dist = "PIG", domain = "counts",
             param = c(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma),
             central = TRUE)
VariNhGPO <- moments(k = 2, dist = "GPO", domain = "counts",
             param = c(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma),
             central = TRUE)
VariNhNBI <- moments(k = 2, dist = "NBI", domain = "counts",
             param = c(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma),
             central = TRUE)
### Skewness
SkewNhEmp <- skewness(table(hospita$year), type = 1)
SkewNhPIG <- skew(dist = "PIG", domain = "counts",
             param = c(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
SkewNhGPO <- skew(dist = "GPO", domain = "counts",
             param = c(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
SkewNhNBI <- skew(dist = "NBI", domain = "counts",
             param = c(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))
### Excess Kurtosis
KurtNhEmp <- kurtosis(table(hospita$year), type = 1)
KurtNhPIG <- kurt(dist = "PIG", domain = "counts", excess = TRUE,
             param = c(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
KurtNhGPO <- kurt(dist = "GPO", domain = "counts", excess = TRUE,
             param = c(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
KurtNhNBI <- kurt(dist = "NBI", domain = "counts", excess = TRUE,
             param = c(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))

kable(cbind(data.frame(Dist = c("Empirical", "PIG", "GPO", "BNI")),
            "Mean" = c(MeanNhEmp, unname(MeanNhPIG), unname(MeanNhGPO),
            unname(MeanNhNBI)), "Variance" = c(VariNhEmp,
            unname(VariNhPIG), unname(VariNhGPO), unname(VariNhNBI)),
            "Skewness" = c(SkewNhEmp, unname(SkewNhPIG),
            unname(SkewNhGPO), unname(SkewNhNBI)), "Excess kurtosis" =
            c(KurtNhEmp, unname(KurtNhPIG), unname(KurtNhGPO),
            unname(KurtNhNBI))),
      caption = "Statistical measurements of hospitalization
      frequencies \\label{tab:StatisticsNh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 8: Adjustment cumulative frequencies of hospitalization

```r
### Empirical vs Theorical cumulative distribution function
plot(ecdf(table(hospita$year)), lwd = 3,
     xlab = "Sample quantiles of frequencies of hospitalization",
     main = "Adjustment cumulative frequencies of hospitalization")
curve(pPIG(x, mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma),
      add = T, from = 200, to = 600, lwd = 3, col = "blue", lty = 1)
curve(pGPO(x, mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma),
      add = T, from = 200, to = 600, lwd = 3, col = "green", lty = 2)
curve(pNBI(x, mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma),
      add = T, from = 200, to = 600, lwd = 3, col = "red", lty = 4)
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "green",
       "red"), legend = c("Cumulative empirical distribution",
       "Poisson-Inverse Gaussian", "Generalized Poisson",
       "Negative Binomial type I"), lwd = 2)
```

Code 9: Goodness-of-fit tests for hospitalization frequencies

```r
# A seed is established so that the results can be replicated
set.seed(1248)
if(!require("truncgof")) install.packages("truncgof")
require(truncgof)
freqhosp <- table(hospita$year)
### Kolmogorov-Smirnov test
kshPIG <- ks.test(x = freqhosp, distn =  "pPIG", H = min(freqhosp),
          fit = list(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
kshGPO <- ks.test(x = freqhosp, distn =  "pGPO", H = min(freqhosp),
          fit = list(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
kshNBI <- ks.test(x = freqhosp, distn =  "pNBI", H = min(freqhosp),
          fit = list(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))
### Cramer-von Mises test
cvmhPIG <- w2.test(x = freqhosp, distn =  "pPIG", H = min(freqhosp),
           fit = list(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
cvmhGPO <- w2.test(x = freqhosp, distn =  "pGPO", H = min(freqhosp),
           fit = list(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
cvmhNBI <- w2.test(x = freqhosp, distn =  "pNBI", H = min(freqhosp),
           fit = list(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))
### Kuiper test
kuhPIG <- v.test(x = freqhosp, distn =  "pPIG", H = min(freqhosp),
          fit = list(mu = FitN_hosp1$mu, sigma = FitN_hosp1$sigma))
kuhGPO <- v.test(x = freqhosp, distn =  "pGPO", H = min(freqhosp),
          fit = list(mu = FitN_hosp2$mu, sigma = FitN_hosp2$sigma))
kuhNBI <- v.test(x = freqhosp, distn =  "pNBI", H = min(freqhosp),
          fit = list(mu = FitN_hosp3$mu, sigma = FitN_hosp3$sigma))
```

```
kable(cbind(data.frame(Dist = c("PIG", "GPO", "BNI")),
            "ks.test" = c(kshPIG$p.value, kshGPO$p.value,
            kshNBI$p.value), "w2.test" = c(cvmhPIG$p.value,
            cvmhGPO$p.value, cvmhNBI$p.value), "v.test" = c(
            kuhPIG$p.value, kuhGPO$p.value, kuhNBI$p.value)),
      caption = "Goodness-of-fit tests hospitalization frequencies
      \\label{tab:gftNh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Frequency model estimation for general surgery

Code 10: Best fit with gamlss for frequencies of general surgery services

```
### The adjustment is made
FitN_surg1 <- fitDist(y = table(surgery$year), type = "counts")
### Estimation of second and third distribution with best fit
FitN_surg2 <- gamlssML(formula = table(surgery$year), family = PIG)
FitN_surg3 <- gamlssML(formula = table(surgery$year), family = GPO)
### The five distributions that present the best fit are
kable(rbind(FitN_surg1$fits[1:5]),
      caption = "Better fit for frequencies of general surgery
      services \\label{tab:fitfhosp}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 11: Statistical measurements of general surgery frequencies

```
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanNsEmp <- mean(table(surgery$year))
MeanNsDEL <- moments(k = 1, dist = "DEL", domain = "counts",
             param = c(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
                       nu = FitN_surg1$nu))
MeanNsPIG <- moments(k = 1, dist = "PIG", domain = "counts",
             param = c(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
MeanNsGPO <- moments(k = 1, dist = "GPO", domain = "counts",
             param = c(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))
#### Variance
VariNsEmp <- var(table(surgery$year))
VariNsDEL <- moments(k = 2, dist = "DEL", domain = "counts",
             param = c(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
                       nu = FitN_surg1$nu), central = TRUE)
```

```
VariNsPIG <- moments(k = 2, dist = "PIG", domain = "counts",
             param = c(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma),
             central = TRUE)
VariNsGPO <- moments(k = 2, dist = "GPO", domain = "counts",
             param = c(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma),
             central = TRUE)
### Skewness
SkewNsEmp <- skewness(table(surgery$year), type = 1)
SkewNsDEL <- skew(dist = "DEL", domain = "counts",
             param = c(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
                  nu = FitN_surg1$nu))
SkewNsPIG <- skew(dist = "PIG", domain = "counts",
              param = c(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
SkewNsGPO <- skew(dist = "GPO", domain = "counts",
              param = c(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))
### Excess Kurtosis
KurtNsEmp <- kurtosis(table(surgery$year), type = 1)
KurtNsDEL <- kurt(dist = "DEL", domain = "counts", excess = TRUE,
             param = c(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
                  nu = FitN_surg1$nu))
KurtNsPIG <- kurt(dist = "PIG", domain = "counts", excess = TRUE,
             param = c(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
KurtNsGPO <- kurt(dist = "GPO", domain = "counts", excess = TRUE,
             param = c(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))

kable(cbind(data.frame(Dist = c("Empirical", "DEL", "PIG", "BNI")),
            "Mean" = c(MeanNsEmp, unname(MeanNsDEL), unname(MeanNsPIG),
            unname(MeanNsGPO)), "Variance" = c(VariNsEmp,
            unname(VariNsDEL), unname(VariNsPIG), unname(VariNsGPO)),
            "Skewness" = c(SkewNsEmp, unname(SkewNsDEL),
            unname(SkewNsPIG), unname(SkewNsGPO)), "Excess kurtosis" =
            c(KurtNsEmp, unname(KurtNsDEL), unname(KurtNsPIG),
            unname(KurtNsGPO))),
      caption = "Statistical measurements of general surgery
                 frequencies \\label{tab:StatisticsNs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 12: Adjustment cumulative frequencies of general surgery

```
### Empirical vs Theorical cumulative distribution function
plot(ecdf(table(surgery$year)), lwd = 3,
     xlab = "Sample quantiles of frequencies of general surgery",
     main = "Adjustment cumulative frequencies of general surgery")
```

```
curve(pDEL(x, mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
      nu = FitN_surg1$nu), add = T, from = 10, to = 250, lwd = 3,
      col = "blue", lty = 1)
curve(pPIG(x, mu = FitN_surg2$mu, sigma = FitN_surg2$sigma),
      add = T, from = 10, to = 250, lwd = 3, col = "green", lty = 2)
curve(pGPO(x, mu = FitN_surg3$mu, sigma = FitN_surg3$sigma),
      add = T, from = 10, to = 250, lwd = 3, col = "red", lty = 4)
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "green",
       "red"), legend = c("Cumulative empirical distribution",
       "Poisson-Inverse Gaussian", "Generalized Poisson",
       "Negative Binomial type I"), lwd = 2)
```

Code 13: Goodness-of-fit for tests general surgery frequencies

```
# A seed is established so that the results can be replicated
set.seed(1248)
freqsurg <- table(surgery$year)
### Kolmogorov-Smirnov test
kshDEL <- ks.test(x = freqsurg, distn =  "pDEL", H = min(freqsurg),
          fit = list(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
          nu = FitN_surg1$nu))
kshPIG <- ks.test(x = freqsurg, distn =  "pPIG", H = min(freqsurg),
          fit = list(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
kshGPO <- ks.test(x = freqsurg, distn =  "pGPO", H = min(freqsurg),
          fit = list(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))
### Cramer-von Mises test
cvmhDEL <- w2.test(x = freqsurg, distn =  "pDEL", H = min(freqsurg),
           fit = list(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
           nu = FitN_surg1$nu))
cvmhPIG <- w2.test(x = freqsurg, distn =  "pPIG", H = min(freqsurg),
           fit = list(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
cvmhGPO <- w2.test(x = freqsurg, distn =  "pGPO", H = min(freqsurg),
           fit = list(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))
### Kuiper test
kuhDEL <- v.test(x = freqsurg, distn =  "pDEL", H = min(freqsurg),
          fit = list(mu = FitN_surg1$mu, sigma = FitN_surg1$sigma,
          nu = FitN_surg1$nu))
kuhPIG <- v.test(x = freqsurg, distn =  "pPIG", H = min(freqsurg),
          fit = list(mu = FitN_surg2$mu, sigma = FitN_surg2$sigma))
kuhGPO <- v.test(x = freqsurg, distn =  "pGPO", H = min(freqsurg),
          fit = list(mu = FitN_surg3$mu, sigma = FitN_surg3$sigma))

kable(cbind(data.frame(Dist = c("DEL", "PIG", "GPO")),
```

```
            "ks.test" = c(kshDEL$p.value, kshPIG$p.value,
            kshGPO$p.value), "w2.test" = c(cvmhDEL$p.value,
            cvmhPIG$p.value, cvmhGPO$p.value), "v.test" = c(
            kuhDEL$p.value, kuhPIG$p.value, kuhGPO$p.value)),
      caption = "Goodness-of-fit tests general surgery frequencies
      \\label{tab:gftNs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Mean residual life analysis for hospitalization

Code 14: Mean residual life hospitalizacion service

```
if(!require("evmix")) install.packages("evmix")
require(evmix)
### Graphic adjustment of mean residual life
mrlplot(hospita$cost, legend.loc = NULL, try.thresh = NULL)
abline(v = 35, col = "red", lwd = 2, lty = 2)
abline(v = 60, col = "blue", lwd = 2, lty = 2)
```

Mean residual life analysis for general surgery

Code 15: Mean residual life hospitalizacion service

```
### Graphic adjustment of mean residual life
mrlplot(surgery$cost, legend.loc = NULL, try.thresh = NULL)
abline(v = 16, col = "red", lwd = 2, lty = 2)
```

Tail index with Hill plot for hospitalization

Code 16: Hill, AltHill, SmooHill and AltSmooHill for hospitalizacion service

```
par(mfrow = c(2, 2), cex = 0.7)
### Hill Plot
hillplot(data = hospita$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "Hill", ylim = c(0, 2), main = "")
title(main = "(a) Hill Plot", line = 3.3)
### AltHill Plot
hillplot(data = hospita$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "Hill", ylim = c(0, 2), x.theta = TRUE,
         main = "")
```

```
title(main = "(b) AltHill Plot", line = 3.3)
### SmooHill Plot
hillplot(data = hospita$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "SmooHill", ylim = c(0, 2), main = "")
title(main = "(c) SmooHill Plot", line = 3.3)
### AltSmooHill plot
hillplot(data = hospita$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "SmooHill", ylim = c(0, 2), x.theta = TRUE,
         main = "")
title(main = "(d) AltSmooHill Plot", line = 3.3)
```

Tail index with Hill plot for general surgery

Code 17: Hill, AltHill, SmooHill and AltSmooHill for general surgery service

```
par(mfrow = c(2, 2), cex = 0.7)
### Hill Plot
hillplot(data = surgery$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "Hill", ylim = c(0, 2), main = "")
title(main = "(a) Hill Plot", line = 3.3)
### AltHill Plot
hillplot(data = surgery$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "Hill", ylim = c(0, 2), x.theta = TRUE,
         main = "")
title(main = "(b) AltHill Plot", line = 3.3)
### SmooHill Plot
hillplot(data = surgery$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "SmooHill", ylim = c(0, 2), main = "",
         try.thresh = quantile(surgery$cost, c(0.9, 0.95),
                               na.rm = TRUE))
title(main = "(c) SmooHill Plot", line = 3.3)
### AltSmooHill plot
hillplot(data = surgery$cost, alpha = 0.05, legend.loc = "topleft",
         hill.type = "SmooHill", ylim = c(0, 2), x.theta = TRUE,
         main = "", try.thresh = quantile(surgery$cost,
         c(0.9, 0.95), na.rm = TRUE))
title(main = "(d) AltSmooHill Plot", line = 3.3)
```

Adjustment with spliced distributions for hospitalization

Code 18: Adjustment with spliced distributions for hospitalization service

```
### The adjustment with evmix is made
SpHfit1 <- fgammagpd(x = hospita$cost)
SpHfit2 <- fnormgpd(x = hospita$cost)
SpHfit3 <- fweibullgpd(x = hospita$cost)
```

Code 19: Statistical measurements of spliced distributions for hospitalization services

```
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanXhSpE <- mean(hospita$cost)
MeanXhSpG <- moments(k = 1, dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpHfit1$phiu, gshape = SpHfit1$gshape,
             gscale = SpHfit1$gscale, u = SpHfit1$u, xi = SpHfit1$xi,
             sigmau = SpHfit1$sigmau))
MeanXhSpN <- moments(k = 1, dist = "normgpd", domain = "realline",
             param = c(phiu = SpHfit2$phiu, nmean = SpHfit2$nmean,
             nsd = SpHfit2$nsd, u = SpHfit2$u, xi = SpHfit2$xi,
             sigmau = SpHfit2$sigmau))
MeanXhSpW <- moments(k = 1, dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpHfit3$phiu, wshape = SpHfit3$wshape,
             wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
             sigmau = SpHfit3$sigmau))
#### Variance
VariXhSpE <- var(hospita$cost)
VariXhSpG <- moments(k = 2, dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpHfit1$phiu, gshape = SpHfit1$gshape,
             gscale = SpHfit1$gscale, u = SpHfit1$u, xi = SpHfit1$xi,
             sigmau = SpHfit1$sigmau), central = TRUE)
VariXhSpN <- moments(k = 2, dist = "normgpd", domain = "realline",
             param = c(phiu = SpHfit2$phiu, nmean = SpHfit2$nmean,
             nsd = SpHfit2$nsd, u = SpHfit2$u, xi = SpHfit2$xi,
             sigmau = SpHfit2$sigmau), central = TRUE)
VariXhSpW <- moments(k = 2, dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpHfit3$phiu, wshape = SpHfit3$wshape,
             wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
             sigmau = SpHfit3$sigmau), central = TRUE)
### Skewness
SkewXhSpE <- skewness(hospita$cost, type = 1)
SkewXhSpG <- skew(dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpHfit1$phiu, gshape = SpHfit1$gshape,
             gscale = SpHfit1$gscale, u = SpHfit1$u, xi = SpHfit1$xi,
             sigmau = SpHfit1$sigmau))
SkewXhSpN <- skew(dist = "normgpd", domain = "realline",
             param = c(phiu = SpHfit2$phiu, nmean = SpHfit2$nmean,
             nsd = SpHfit2$nsd, u = SpHfit2$u, xi = SpHfit2$xi,
```

```
             sigmau = SpHfit2$sigmau))
SkewXhSpW <- skew(dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpHfit3$phiu, wshape = SpHfit3$wshape,
             wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
             sigmau = SpHfit3$sigmau))
### Excess Kurtosis
KurtXhSpE <- kurtosis(hospita$cost, type = 1)
KurtXhSpG <- kurt(dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpHfit1$phiu, gshape = SpHfit1$gshape,
             gscale = SpHfit1$gscale, u = SpHfit1$u, xi = SpHfit1$xi,
             sigmau = SpHfit1$sigmau), excess = TRUE)
KurtXhSpN <- kurt(dist = "normgpd", domain = "realline",
             param = c(phiu = SpHfit2$phiu, nmean = SpHfit2$nmean,
             nsd = SpHfit2$nsd, u = SpHfit2$u, xi = SpHfit2$xi,
             sigmau = SpHfit2$sigmau), excess = TRUE)
KurtXhSpW <- kurt(dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpHfit3$phiu, wshape = SpHfit3$wshape,
             wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
             sigmau = SpHfit3$sigmau), excess = TRUE)

kable(cbind(data.frame(Dist = c("Empirical", "gammagdp", "normgdp",
            "weibullgdp")), "Mean" = c(MeanXhSpE, unname(MeanXhSpG),
            unname(MeanXhSpN), unname(MeanXhSpW)), "Variance" = c(
            VariXhSpE, unname(VariXhSpG), unname(VariXhSpN),
            unname(VariXhSpW)), "Skewness" = c(SkewXhSpE,
            unname(SkewXhSpG), unname(SkewXhSpN), unname(SkewXhSpW)),
            "Excess Kurtosis" = c(KurtXhSpE, unname(KurtXhSpG),
            unname(KurtXhSpN), unname(KurtXhSpW))),
      caption = "Statistical measurements of spliced distributions for
      hospitalization services \\label{tab:StatisticsSpXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 20: Adjustment of cumulative individual costs of hospitalization services with spliced distributions

```
FnXh <- ecdf(hospita$cost)
sortXh <- sort(hospita$cost)
### Empirical vs Theorical cumulative distribution function
plot(FnXh, lwd = 3,
    xlab = "Sample quantiles of individual costs of hospitalization",
    main = "Adjustment cumulative individual costs of hospitalization")
fitXhG <- pgammagpd(q = sortXh, phiu = SpHfit1$phiu,
          gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
```

```
          u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau)
lines(sortXh, fitXhG, lwd = 3, lty = 1, col = "blue")
fitXhN <- pnormgpd(q = sortXh, phiu = SpHfit2$phiu,
          nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
          xi = SpHfit2$xi, sigmau = SpHfit2$sigmau)
lines(sortXh, fitXhN, lwd = 3, lty = 2, col = "red")
fitXhW <- pweibullgpd(q = sortXh, phiu = SpHfit3$phiu,
          wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
          u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau)
lines(sortXh, fitXhW, lwd = 3, lty = 4, col = "green")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red",
       "green"), legend = c("Cumulative empirical distribution",
       "Gamma-Generalized Pareto", "Normal-Generalized Pareto",
       "Weibull-Generalized Pareto"), lwd = 2)
```

Code 21: Adjustment of log-survival distribution of hospitalization with spliced distributions

```
### Empirical vs Theorical log-survival distribution function
survXh <- 1 - FnXh(sortXh)
plot(x = log(sortXh), y = log(survXh), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
     hospitalization)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     hospitalization", type = "l")
survXhG <- 1 - fitXhG
lines(log(sortXh), log(survXhG), lwd = 3, col = "blue")
survXhN <- 1 - fitXhN
lines(log(sortXh), log(survXhN), lwd = 3, col = "red", lty = 2)
survXhW <- 1 - fitXhW
lines(log(sortXh), log(survXhW), lwd = 3, col = "green", lty = 4)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red", "green"),
       legend = c("log-Survival Distribution",
        "Gamma-Generalized Pareto", "Normal-Generalized Pareto",
        "Weibull-Generalized Pareto"), lwd = 2)
```

Code 22: Q-Q plot spliced distribution for hospitalization

```
if(!require("car")) install.packages("car")
require(car)
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
```

```
### QQ-plot
qqPlot(x = hospita$cost, lwd = 1, distribution = "gammagpd",
       phiu = SpHfit1$phiu, gshape = SpHfit1$gshape,
       gscale = SpHfit1$gscale, u = SpHfit1$u, xi = SpHfit1$xi,
       sigmau = SpHfit1$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(a) Gamma-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = hospita$cost, lwd = 1, distribution = "normgpd",
       phiu = SpHfit2$phiu, nmean = SpHfit2$nmean,
       nsd = SpHfit2$nsd, u = SpHfit2$u, xi = SpHfit2$xi,
       sigmau = SpHfit2$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Normal-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = hospita$cost, lwd = 1, distribution = "weibullgpd",
       phiu = SpHfit3$phi, wshape = SpHfit3$wshape,
       wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
       sigmau = SpHfit3$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(c) Weibull-Generalized Pareto Q-Q Plot", id = FALSE)
```

Code 23: Goodness-of-fit tests for hospitalization services for spliced distributions

```
# A seed is established so that the results can be replicated
set.seed(1248)
### Kolmogorov-Smirnov test
kolmSpXhG <- ks.test(x = hospita$cost, distn =  "pgammagpd",
             H = min(hospita$cost), fit = list(phiu = SpHfit1$phiu,
             gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
             u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau))
kolmSpXhN <- ks.test(x = hospita$cost, distn =  "pnormgpd",
             H = min(hospita$cost), fit = list(phiu = SpHfit2$phiu,
             nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
             xi = SpHfit2$xi, sigmau = SpHfit2$sigmau))
kolmSpXhW <- ks.test(x = hospita$cost, distn =  "pweibullgpd",
             H = min(hospita$cost), fit = list(phiu = SpHfit3$phiu,
             wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
             u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau))
### Cramer-von Mises test
cramSpXhG <- w2.test(x = hospita$cost, distn =  "pgammagpd",
             H = min(hospita$cost), fit = list(phiu = SpHfit1$phiu,
             gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
             u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau))
cramSpXhN <- w2.test(x = hospita$cost, distn =  "pnormgpd",
             H = min(hospita$cost), fit = list(phiu = SpHfit2$phiu,
             nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
```

```
               xi = SpHfit2$xi, sigmau = SpHfit2$sigmau))
cramSpXhW <- w2.test(x = hospita$cost, distn =  "pweibullgpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit3$phiu,
               wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
               u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau))
### Kuiper test
kuipSpXhG <- v.test(x = hospita$cost, distn =  "pgammagpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit1$phiu,
               gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
               u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau))
kuipSpXhN <- v.test(x = hospita$cost, distn =  "pnormgpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit2$phiu,
               nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
               xi = SpHfit2$xi, sigmau = SpHfit2$sigmau))
kuipSpXhW <- v.test(x = hospita$cost, distn =  "pweibullgpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit3$phiu,
               wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
               u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau))
### Supremum class Upper Tail Anderson-Darling test
adupSpXhG <- adup.test(x = hospita$cost, distn =  "pgammagpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit1$phiu,
               gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
               u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau))
adupSpXhN <- adup.test(x = hospita$cost, distn =  "pnormgpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit2$phiu,
               nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
               xi = SpHfit2$xi, sigmau = SpHfit2$sigmau))
adupSpXhW <- adup.test(x = hospita$cost, distn =  "pweibullgpd",
               H = min(hospita$cost), fit = list(phiu = SpHfit3$phiu,
               wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
               u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau))
### Quadratic Class Upper Tail Anderson-Darling test
ad2upSpXhG <- ad2up.test(x = hospita$cost, distn =  "pgammagpd",
                H = min(hospita$cost), fit = list(phiu = SpHfit1$phiu,
                gshape = SpHfit1$gshape, gscale = SpHfit1$gscale,
                u = SpHfit1$u, xi = SpHfit1$xi, sigmau = SpHfit1$sigmau))
ad2upSpXhN <- ad2up.test(x = hospita$cost, distn =  "pnormgpd",
                H = min(hospita$cost), fit = list(phiu = SpHfit2$phiu,
                nmean = SpHfit2$nmean, nsd = SpHfit2$nsd, u = SpHfit2$u,
                xi = SpHfit2$xi, sigmau = SpHfit2$sigmau))
ad2upSpXhW <- ad2up.test(x = hospita$cost, distn =  "pweibullgpd",
                H = min(hospita$cost), fit = list(phiu = SpHfit3$phiu,
                wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
                u = SpHfit3$u, xi = SpHfit3$xi, sigmau = SpHfit3$sigmau))
```

```
### Results table
kable(cbind(data.frame(Dist = c("gammagpd", "normgpd", "weibullgpd")),
            "ks.test" = c(kolmSpXhG$p.value, kolmSpXhN$p.value,
            kolmSpXhW$p.value), "w2.test" = c(cramSpXhG$p.value,
            cramSpXhN$p.value, cramSpXhW$p.value), "v.test" = c(
            kuipSpXhG$p.value, kuipSpXhN$p.value, kuipSpXhW$p.value),
            "adup.test" = c(adupSpXhG$p.value, adupSpXhN$p.value,
            adupSpXhW$p.value), "ad2up.test" = c(ad2upSpXhG$p.value,
            ad2upSpXhN$p.value, ad2upSpXhW$p.value)),
      caption = "Goodness-of-fit tests hospitalization services for
      spliced distributions \\label{tab:gftSpXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Adjustment with spliced distributions for general surgery

Code 24: Adjustment with spliced distributions for general surgery service

```
### The adjustment with evmix is made
SpSfit1 <- fgammagpd(x = surgery$cost)
SpSfit2 <- fnormgpd(x = surgery$cost)
SpSfit3 <- fweibullgpd(x = surgery$cost)
```

Code 25: Statistical measurements of spliced distributions for general surgery services

```
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanXsSpE <- mean(surgery$cost)
MeanXsSpG <- moments(k = 1, dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpSfit1$phiu, gshape = SpSfit1$gshape,
             gscale = SpSfit1$gscale, u = SpSfit1$u, xi = SpSfit1$xi,
             sigmau = SpSfit1$sigmau))
MeanXsSpN <- moments(k = 1, dist = "normgpd", domain = "realline",
             param = c(phiu = SpSfit2$phiu, nmean = SpSfit2$nmean,
             nsd = SpSfit2$nsd, u = SpSfit2$u, xi = SpSfit2$xi,
             sigmau = SpSfit2$sigmau))
MeanXsSpW <- moments(k = 1, dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpSfit3$phiu, wshape = SpSfit3$wshape,
             wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
             sigmau = SpSfit3$sigmau))
#### Variance
VariXsSpE <- var(surgery$cost)
```

```r
VariXsSpG <- moments(k = 2, dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpSfit1$phiu, gshape = SpSfit1$gshape,
             gscale = SpSfit1$gscale, u = SpSfit1$u, xi = SpSfit1$xi,
             sigmau = SpSfit1$sigmau), central = TRUE)
VariXsSpN <- moments(k = 2, dist = "normgpd", domain = "realline",
             param = c(phiu = SpSfit2$phiu, nmean = SpSfit2$nmean,
             nsd = SpSfit2$nsd, u = SpSfit2$u, xi = SpSfit2$xi,
             sigmau = SpSfit2$sigmau), central = TRUE)
VariXsSpW <- moments(k = 2, dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpSfit3$phiu, wshape = SpSfit3$wshape,
             wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
             sigmau = SpSfit3$sigmau), central = TRUE)
### Skewness
SkewXsSpE <- skewness(surgery$cost, type = 1)
SkewXsSpG <- skew(dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpSfit1$phiu, gshape = SpSfit1$gshape,
             gscale = SpSfit1$gscale, u = SpSfit1$u, xi = SpSfit1$xi,
             sigmau = SpSfit1$sigmau))
SkewXsSpN <- skew(dist = "normgpd", domain = "realline",
             param = c(phiu = SpSfit2$phiu, nmean = SpSfit2$nmean,
             nsd = SpSfit2$nsd, u = SpSfit2$u, xi = SpSfit2$xi,
             sigmau = SpSfit2$sigmau))
SkewXsSpW <- skew(dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpSfit3$phiu, wshape = SpSfit3$wshape,
             wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
             sigmau = SpSfit3$sigmau))
### Excess Kurtosis
KurtXsSpE <- kurtosis(surgery$cost, type = 1)
KurtXsSpG <- kurt(dist = "gammagpd", domain = "realplus",
             param = c(phiu = SpSfit1$phiu, gshape = SpSfit1$gshape,
             gscale = SpSfit1$gscale, u = SpSfit1$u, xi = SpSfit1$xi,
             sigmau = SpSfit1$sigmau), excess = TRUE)
KurtXsSpN <- kurt(dist = "normgpd", domain = "realline",
             param = c(phiu = SpSfit2$phiu, nmean = SpSfit2$nmean,
             nsd = SpSfit2$nsd, u = SpSfit2$u, xi = SpSfit2$xi,
             sigmau = SpSfit2$sigmau), excess = TRUE)
KurtXsSpW <- kurt(dist = "weibullgpd", domain = "realplus",
             param = c(phiu = SpSfit3$phiu, wshape = SpSfit3$wshape,
             wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
             sigmau = SpSfit3$sigmau), excess = TRUE)

kable(cbind(data.frame(Dist = c("Empirical", "gammagdp", "normgdp",
            "weibullgdp")), "Mean" = c(MeanXsSpE, unname(MeanXsSpG),
            unname(MeanXsSpN), unname(MeanXsSpW)), "Variance" = c(
```

```
          VariXsSpE, "does not exist", unname(VariXsSpN),
          "does not exist"), "Skewness" = c(SkewXsSpE,
          "does not exist", unname(SkewXsSpN), "does not exist"),
          "Excess Kurtosis" = c(KurtXsSpE, "does not exist",
          unname(KurtXsSpN), "does not exist")),
     caption = "Statistical measurements of spliced distributions for
     general surgery services \\label{tab:StatisticsSpXs}",
     "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 26: Adjustment of cumulative individual costs of general surgery services with spliced distributions

```
FnXs <- ecdf(surgery$cost)
sortXs <- sort(surgery$cost)
### Empirical vs Theorical cumulative distribution function
plot(FnXs, lwd = 3,
    xlab = "Sample quantiles of individual costs of general surgery",
    main = "Adjustment cumulative individual costs of general surgery")
fitXsG <- pgammagpd(q = sortXs, phiu = SpSfit1$phiu,
          gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
          u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau)
lines(sortXs, fitXsG, lwd = 3, lty = 1, col = "blue")
fitXsN <- pnormgpd(q = sortXs, phiu = SpSfit2$phiu,
          nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
          xi = SpSfit2$xi, sigmau = SpSfit2$sigmau)
lines(sortXs, fitXsN, lwd = 3, lty = 2, col = "red")
fitXsW <- pweibullgpd(q = sortXs, phiu = SpSfit3$phiu,
          wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
          u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau)
lines(sortXs, fitXsW, lwd = 3, lty = 4, col = "green")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red",
       "green"), legend = c("Cumulative empirical distribution",
       "Gamma-Generalized Pareto", "Normal-Generalized Pareto",
       "Weibull-Generalized Pareto"), lwd = 2)
```

Code 27: Adjustment of log-survival distribution of general surgery with spliced distributions

```
### Empirical vs Theorical log-survival distribution function
survXs <- 1 - FnXs(sortXs)
plot(x = log(sortXs), y = log(survXs), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
```

```
     general surgery)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     general surgery", type = "l")
survXsG <- 1 - fitXsG
lines(log(sortXs), log(survXsG), lwd = 3, col = "blue")
survXsN <- 1 - fitXsN
lines(log(sortXs), log(survXsN), lwd = 3, col = "red", lty = 2)
survXsW <- 1 - fitXsW
lines(log(sortXs), log(survXsW), lwd = 3, col = "green", lty = 4)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red", "green"),
       legend = c("log-Survival Distribution",
       "Gamma-Generalized Pareto", "Normal-Generalized Pareto",
       "Weibull-Generalized Pareto"), lwd = 2)
```

Code 28: Q-Q plot spliced distribution for general surgery

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
### QQ-plot
qqPlot(x = surgery$cost, lwd = 1, distribution = "gammagpd",
       phiu = SpSfit1$phiu, gshape = SpSfit1$gshape,
       gscale = SpSfit1$gscale, u = SpSfit1$u, xi = SpSfit1$xi,
       sigmau = SpSfit1$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(a) Gamma-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = surgery$cost, lwd = 1, distribution = "normgpd",
       phiu = SpSfit2$phiu, nmean = SpSfit2$nmean,
       nsd = SpSfit2$nsd, u = SpSfit2$u, xi = SpSfit2$xi,
       sigmau = SpSfit2$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Normal-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = surgery$cost, lwd = 1, distribution = "weibullgpd",
       phiu = SpSfit3$phi, wshape = SpSfit3$wshape,
       wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
       sigmau = SpSfit3$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(c) Weibull-Generalized Pareto Q-Q Plot", id = FALSE)
```

Code 29: Goodness-of-fit tests for general surgery services for spliced distributions

```
# A seed is established so that the results can be replicated
set.seed(1248)
### Kolmogorov-Smirnov test
```

```
kolmSpXsG <- ks.test(x = surgery$cost, distn =  "pgammagpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit1$phiu,
             gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
             u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau))
kolmSpXsN <- ks.test(x = surgery$cost, distn =  "pnormgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit2$phiu,
             nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
             xi = SpSfit2$xi, sigmau = SpSfit2$sigmau))
kolmSpXsW <- ks.test(x = surgery$cost, distn =  "pweibullgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit3$phiu,
             wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
             u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau))
### Cramer-von Mises test
cramSpXsG <- w2.test(x = surgery$cost, distn =  "pgammagpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit1$phiu,
             gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
             u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau))
cramSpXsN <- w2.test(x = surgery$cost, distn =  "pnormgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit2$phiu,
             nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
             xi = SpSfit2$xi, sigmau = SpSfit2$sigmau))
cramSpXsW <- w2.test(x = surgery$cost, distn =  "pweibullgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit3$phiu,
             wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
             u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau))
### Kuiper test
kuipSpXsG <- v.test(x = surgery$cost, distn =  "pgammagpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit1$phiu,
             gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
             u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau))
kuipSpXsN <- v.test(x = surgery$cost, distn =  "pnormgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit2$phiu,
             nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
             xi = SpSfit2$xi, sigmau = SpSfit2$sigmau))
kuipSpXsW <- v.test(x = surgery$cost, distn =  "pweibullgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit3$phiu,
             wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
             u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau))
### Supremum class Upper Tail Anderson-Darling test
adupSpXsG <- adup.test(x = surgery$cost, distn =  "pgammagpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit1$phiu,
             gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
             u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau))
adupSpXsN <- adup.test(x = surgery$cost, distn =  "pnormgpd",
```

```
            H = min(surgery$cost), fit = list(phiu = SpSfit2$phiu,
            nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
            xi = SpSfit2$xi, sigmau = SpSfit2$sigmau))
adupSpXsW <- adup.test(x = surgery$cost, distn =  "pweibullgpd",
            H = min(surgery$cost), fit = list(phiu = SpSfit3$phiu,
            wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
            u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau))
### Quadratic Class Upper Tail Anderson-Darling test
ad2upSpXsG <- ad2up.test(x = surgery$cost, distn =  "pgammagpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit1$phiu,
             gshape = SpSfit1$gshape, gscale = SpSfit1$gscale,
             u = SpSfit1$u, xi = SpSfit1$xi, sigmau = SpSfit1$sigmau))
ad2upSpXsN <- ad2up.test(x = surgery$cost, distn =  "pnormgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit2$phiu,
             nmean = SpSfit2$nmean, nsd = SpSfit2$nsd, u = SpSfit2$u,
             xi = SpSfit2$xi, sigmau = SpSfit2$sigmau))
ad2upSpXsW <- ad2up.test(x = surgery$cost, distn =  "pweibullgpd",
             H = min(surgery$cost), fit = list(phiu = SpSfit3$phiu,
             wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
             u = SpSfit3$u, xi = SpSfit3$xi, sigmau = SpSfit3$sigmau))
### Results table
kable(cbind(data.frame(Dist = c("gammagpd", "normgpd", "weibullgpd")),
            "ks.test" = c(kolmSpXsG$p.value, kolmSpXsN$p.value,
            kolmSpXsW$p.value), "w2.test" = c(cramSpXsG$p.value,
            cramSpXsN$p.value, cramSpXsW$p.value), "v.test" = c(
            kuipSpXsG$p.value, kuipSpXsN$p.value, kuipSpXsW$p.value),
            "adup.test" = c(adupSpXsG$p.value, adupSpXsN$p.value,
            adupSpXsW$p.value), "ad2up.test" = c(ad2upSpXsG$p.value,
            ad2upSpXsN$p.value, ad2upSpXsW$p.value)),
      caption = "Goodness-of-fit tests general surgery services for
      spliced distributions \\label{tab:gftSpXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Risk measures estimation for hospitalization services

Code 30: Tail index estimation for hospitalization services

```
### Density function
fhosp <- function(x) dweibullgpd(x, wshape = SpHfit3$wshape,
         wscale = SpHfit3$wscale, sigmau = SpHfit3$sigmau,
         u = SpHfit3$u, xi = SpHfit3$xi, phiu = SpHfit3$phiu)
```

```
### Cumulative function
Fhosp <- function(x) pweibullgpd(x, wshape = SpHfit3$wshape,
         wscale = SpHfit3$wscale, sigmau = SpHfit3$sigmau,
         u = SpHfit3$u, xi = SpHfit3$xi, phiu = SpHfit3$phiu)
### Equation to calculate the tail index
tailindexH <- function(x) (1-Fhosp(x))/(x * fhosp(x))
curve(expr = tailindexH, from = 1, to = 320, ylim = c(0,1),
      ylab = "Limit", xlab = "x", mgp=c(2.5,1,0), lwd = 2,
      main = "Estimation of the tail index for hospitalization
services")
```

Code 31: Value at Risk for hospitalization services

```
quantileH <- function(kappa) 1 - ((1 - kappa) / MeanNhGPO)
FquantileH <- function(kappa) qweibullgpd(p = quantileH(kappa),
        wshape = SpHfit3$wshape, wscale = SpHfit3$wscale,
        sigmau = SpHfit3$sigmau, u = SpHfit3$u, xi = SpHfit3$xi,
        phiu = SpHfit3$phiu)
correctionH <- MeanXhSpW * (MeanNhGPO + (VariNhGPO / MeanNhGPO) - 1)
VaRH <- function(kappa) FquantileH(kappa) + correctionH

curve(expr = VaRH, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Value at Risk for hospitalization services")
rect(xleft = 0.95, ybottom = VaRH(0.95) + 9, xright = 0.95,
     ytop = VaRH(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = VaRH(0.99) + 9, xright = 0.99,
     ytop = VaRH(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(VaRH(0.95), VaRH(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = VaRH(0.95) + 15, bty = "n",
       legend = round(VaRH(0.95), 3))
legend(x = 0.98, y = VaRH(0.99) + 15, bty = "n",
       legend = round(VaRH(0.99), 3))
```

Code 32: Expected Shortfall for hospitalization services

```
TVaRH <- function(kappa) (1/(1-kappa)) * as.numeric(integrate(
                         f = VaRH, lower = kappa, upper = 1)$value)
TVaRH <- Vectorize(TVaRH)

curve(expr = TVaRH, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
```

```
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Expected Shortfall for hospitalization services")
rect(xleft = 0.95, ybottom = TVaRH(0.95) + 8, xright = 0.95,
     ytop = TVaRH(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = TVaRH(0.99) + 8, xright = 0.99,
     ytop = TVaRH(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(TVaRH(0.95), TVaRH(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = TVaRH(0.95) + 15, bty = "n",
       legend = round(TVaRH(0.95), 3))
legend(x = 0.98, y = TVaRH(0.99) + 15, bty = "n",
       legend = round(TVaRH(0.99), 3))
```

Code 33: Stop-Loss Premium for hospitalization services

```
ESH <- function(kappa) (1 - kappa)*(TVaRH(kappa) - VaRH(kappa))

curve(expr = ESH, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Stop-Loss Premium for hospitalization services")
rect(xleft = 0.95, ybottom = ESH(0.95) + 0.2, xright = 0.95,
     ytop = ESH(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = ESH(0.99) + 0.2, xright = 0.99,
     ytop = ESH(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(ESH(0.95), ESH(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.942, y = ESH(0.95) + 0.35, bty = "n",
       legend = round(ESH(0.95), 3))
legend(x = 0.982, y = ESH(0.99) + 0.35, bty = "n",
       legend = round(ESH(0.99), 3))
```

Risk measures estimation for general surgery services

Code 34: Tail index estimation for general surgery services

```
### Density function
fsurg <- function(x) dweibullgpd(x, wshape = SpSfit3$wshape,
         wscale = SpSfit3$wscale, sigmau = SpSfit3$sigmau,
         u = SpSfit3$u, xi = SpSfit3$xi, phiu = SpSfit3$phiu)
### Cumulative function
Fsurg <- function(x) pweibullgpd(x, wshape = SpSfit3$wshape,
```

```
        wscale = SpSfit3$wscale, sigmau = SpSfit3$sigmau,
        u = SpSfit3$u, xi = SpSfit3$xi, phiu = SpSfit3$phiu)
### Equation to calculate the tail index
tailindexS <- function(x) (1-Fsurg(x))/(x * fsurg(x))
curve(expr = tailindexS, from = 1, to = 4e11, ylim = c(0.80,0.90),
      ylab = "Limit", xlab = "x", mgp=c(2.5,1,0), lwd = 2,
      main = "Estimation of the tail index for general surgery
services")
abline(h = tailindexS(0.9e11), col = "red")
rect(xleft = 5e10, ybottom = tailindexS(0.9e11), xright = 5e10,
     ytop = tailindexS(0.9e11) + 0.02, lwd = 2, border = "red")
points(x = 5e10, y = tailindexS(0.9e11),
       pch = 19, col = "red", cex = 1.2)
legend(x = 1.2e10, y = tailindexS(0.9e11) + 0.032, bty = "n",
       legend = round(tailindexS(0.9e11), 3))
```

Code 35: Value at Risk for general surgery services

```
quantileS <- function(kappa) 1 - ((1 - kappa) / MeanNsDEL)
FquantileS <- function(kappa) qweibullgpd(p = quantileS(kappa),
        wshape = SpSfit3$wshape, wscale = SpSfit3$wscale,
        sigmau = SpSfit3$sigmau, u = SpSfit3$u, xi = SpSfit3$xi,
        phiu = SpSfit3$phiu)
correctionS <- MeanXsSpW * (MeanNsDEL + (VariNsDEL / MeanNsDEL) - 1)
VaRS <- function(kappa) FquantileS(kappa) + correctionS

curve(expr = VaRS, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Value at Risk for general surgery services")
rect(xleft = 0.95, ybottom = VaRS(0.95) + 1000, xright = 0.95,
     ytop = VaRS(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = VaRS(0.99) + 1000, xright = 0.99,
     ytop = VaRS(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(VaRS(0.95), VaRS(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = VaRS(0.95) + 1700, bty = "n",
       legend = round(VaRS(0.95), 3))
legend(x = 0.98, y = VaRS(0.99) + 1700, bty = "n",
       legend = round(VaRS(0.99), 3))
```

Code 36: Expected Shortfall for general surgery services

```
TVaRS <- function(kappa) (1/(1-kappa)) * as.numeric(integrate(
         f = VaRS, lower = kappa, upper = 1)$value)
TVaRS <- Vectorize(TVaRS)

curve(expr = TVaRS, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Expected Shortfall for general surgery services")
rect(xleft = 0.95, ybottom = TVaRS(0.95) + 10000, xright = 0.95,
     ytop = TVaRS(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = TVaRS(0.99) + 10000, xright = 0.99,
     ytop = TVaRS(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(TVaRS(0.95), TVaRS(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = TVaRS(0.95) + 15000, bty = "n",
       legend = round(TVaRS(0.95), 3))
legend(x = 0.98, y = TVaRS(0.99) + 15000, bty = "n",
       legend = round(TVaRS(0.99), 3))
```

Code 37: Stop-Loss Premium for general surgery services

```
ESS <- function(kappa) (1 - kappa)*(TVaRS(kappa) - VaRS(kappa))

curve(expr = ESS, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Stop-Loss Premium for general surgery services")
rect(xleft = 0.95, ybottom = ESS(0.95) - 8, xright = 0.95,
     ytop = ESS(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = ESS(0.99) - 8, xright = 0.99,
     ytop = ESS(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(ESS(0.95), ESS(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.942, y = ESS(0.95) - 8, bty = "n",
       legend = round(ESS(0.95), 3))
legend(x = 0.982, y = ESS(0.99) - 8, bty = "n",
       legend = round(ESS(0.99), 3))
```

Optimum retention point estimation for hospitalization service

Code 38: Optimum retention point estimation for hospitalization service

```
rho <- c(0.1, 0.2, 0.3, 0.5, 0.8, 1, 1.2, 1.5, 2, 3, 4, 5, 7,
         10, 20, 50)
VaRTH <- function(rho, kappa) VaRH(kappa) + (1 + rho)*ESH(kappa)
ResultH <- function(rho){
  kapparho <- 1 - 1/(1 + rho)
  VaRHrho <- round(VaRH(kapparho), 3)
  DeltaHrho <- round((1+rho)*ESH(kapparho), 3)
  VaRTHrho <- round(VaRTH(rho, kapparho), 3)
  return(c(rho, kapparho, VaRHrho, DeltaHrho, VaRTHrho))
}

tableH <- round(t(sapply(X = rho, FUN = ResultH)), 6)

kable(cbind(data.frame(tableH)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
                    "$M_{hosp}^*$", "$\\delta(M_{hosp}^*)$",
                    "$VaR_{T_{hosp}}(\\kappa_{\\rho^*})$"),
      caption = "Optimum retention point estimation for
      hospitalization services \\label{tab:retentionH}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

Optimum retention point estimation for general surgery services

Code 39: Optimum retention point estimation for hospitalization service

```
VaRTS <- function(rho, kappa) VaRS(kappa) + (1 + rho)*ESS(kappa)
ResultS <- function(rho){
  kapparho <- 1 - 1/(1 + rho)
  VaRSrho <- round(VaRS(kapparho), 3)
  DeltaSrho <- round((1+rho)*ESS(kapparho), 3)
  VaRTSrho <- round(VaRTS(rho, kapparho), 3)
  return(c(rho, kapparho, VaRSrho, DeltaSrho, VaRTSrho))
}

tableS <- round(t(sapply(X = rho, FUN = ResultS)), 6)

kable(cbind(data.frame(tableS)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
                    "$M_{surg}^*$", "$\\delta(M_{surg}^*)$",
                    "$VaR_{T_{surg}}(\\kappa_{\\rho^*})$"),
```

```
      caption = "Optimum retention point estimation for
      general surgery services \\label{tab:retentionS}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

GAMLSS distributions to adjust severity distributions

Code 40: Figure shows the tail shape for diferent types of GAMLSS distributions for $k_1, k_3, k_5 = 1, 2$, and $k_2, k_4, k_6 = 1, 2$. Smaller values i the k's result heavier tails. Rigby et al. (2014)

```
### Types of distributions
TypeI <- function(x, k1, k2) - k2 * log(abs(x))^k1
TypeII <- function(x, k3, k4) - k4 * abs(x)^k3
TypeIII <- function(x, k5, k6) - k6 * exp(k5 * abs(x))
x <- seq(0, 10, 0.001)
x2 <- seq(1, 10, 0.001)
par(mfrow = c(2, 2), cex.main = 0.9, lwd = 2)

plot(x = x, y = TypeIII(x = x, k5 = 1, k6 = 1), type = "l",
     main = expression(paste(k[1], ", ", k[3], ", ", k[5], " = ", 1,
     "  and  ", k[2], ", ", k[4], ", ", k[6], " = ", 1)),  lty = 4,
     ylab = "Types of heavy  tails", col = "blue", ylim = c(-10, 1))
legend("topright", legend = c("Type I", "Type II", "Type III"),
       col = c("black", "red", "blue"), lty = c(1,2,4))
lines(x = x, y = TypeII(x = x, k3 = 1, k4 = 1), type = "l",
      col = "red", lwd = 2, lty = 2)
lines(x = x, y = TypeI(x = x, k1 = 1, k2 = 1), type = "l",
      col = "black", lwd = 2, lty = 1)

plot(x = x, y = TypeIII(x = x, k5 = 1, k6 = 2), type = "l",
     main = expression(paste(k[1], ", ", k[3], ", ", k[5], " = ", 1,
     "  and  ", k[2], ", ", k[4], ", ", k[6], " = ", 2)), lty = 4,
     ylab = "Types of heavy  tails", col = "blue", ylim = c(-10, 1))
legend("topright", legend = c("Type I", "Type II", "Type III"),
       col = c("black", "red", "blue"), lty = c(1,2,4))
lines(x = x, y = TypeII(x = x, k3 = 1, k4 = 2), type = "l",
      col = "red", lwd = 2, lty = 2)
lines(x = x, y = TypeI(x = x, k1 = 1, k2 = 2), type = "l",
      col = "black", lwd = 2, lty = 1)
```

```
plot(x = x, y = TypeIII(x = x, k5 = 2, k6 = 1), type = "l",
     main = expression(paste(k[1], ", ", k[3], ", ", k[5], " = ", 2,
     "  and  ", k[2], ", ", k[4], ", ", k[6], " = ", 1)), lty = 4,
     ylab = "Types of heavy  tails", col = "blue", ylim = c(-10, 1))
legend("topright", legend = c("Type I", "Type II", "Type III"),
       col = c("black", "red", "blue"), lty = c(1,2,4))
lines(x = x, y = TypeII(x = x, k3 = 2, k4 = 1), type = "l",
      col = "red", lwd = 2, lty = 2)
lines(x = x2, y = TypeI(x = x2, k1 = 2, k2 = 1), type = "l",
      col = "black", lwd = 2, lty = 1)

plot(x = x, y = TypeIII(x = x, k5 = 2, k6 = 2), type = "l",
     main = expression(paste(k[1], ", ", k[3], ", ", k[5], " = ", 2,
     "  and  ", k[2], ", ", k[4], ", ", k[6], " = ", 2)), lty = 4,
     ylab = "Types of heavy  tails", col = "blue", ylim = c(-10, 1))
legend("topright", legend = c("Type I", "Type II", "Type III"),
       col = c("black", "red", "blue"), lty = c(1,2,4))
lines(x = x, y = TypeII(x = x, k3 = 2, k4 = 2), type = "l",
      col = "red", lwd = 2, lty = 2)
lines(x = x2, y = TypeI(x = x2, k1 = 2, k2 = 2), type = "l",
      col = "black", lwd = 2, lty = 1)
```

Adjustment of GAMLSS distributions for hospitalization services severities

Code 41: Best fit with GAMLSS distributions for individual cost of hospitalization services

```
### The adjustment is made
GdHfit1 <- fitDist(y = hospita$cost, type = "realplus")
### Estimation of second and third distribution with best fit
GdHfit2 <- gamlssML(formula = hospita$cost, family = GB2)
GdHfit3 <- gamlssML(formula = hospita$cost, family = BCPE)
### The five distributions that present the best fit are
kable(rbind(GdHfit1$fits[1:5]),
      caption = "Better fit for individual cost of hospitalization
      services with GAMLSS distributions \\label{tab:fitfhosp}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 42: Statistical measurements of GAMLSS distributions for hospitalization services

```r
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanXhGdE <- mean(hospita$cost)
MeanXhGdGG <- moments(k = 1, dist = "GG", domain = "realplus",
              param = c(mu = GdHfit1$mu, sigma = GdHfit1$sigma,
              nu = GdHfit1$nu))
MeanXhGdGB2 <- moments(k = 1, dist = "GB2", domain = "realplus",
               param = c(mu = GdHfit2$mu, sigma = GdHfit2$sigma,
               nu = GdHfit2$nu, tau = GdHfit2$tau))
MeanXhGdBCPE <- moments(k = 1, dist = "BCPE", domain = "realplus",
                param = c(mu = GdHfit3$mu, sigma = GdHfit3$sigma,
                nu = GdHfit3$nu, tau = GdHfit3$tau))
#### Variance
VariXhGdE <- var(hospita$cost)
VariXhGdGG <- moments(k = 2, dist = "GG", domain = "realplus",
              param = c(mu = GdHfit1$mu, sigma = GdHfit1$sigma,
              nu = GdHfit1$nu), central = TRUE)
VariXhGdGB2 <- moments(k = 2, dist = "GB2", domain = "realplus",
               param = c(mu = GdHfit2$mu, sigma = GdHfit2$sigma,
               nu = GdHfit2$nu, tau = GdHfit2$tau), central = TRUE)
VariXhGdBCPE <- moments(k = 2, dist = "BCPE", domain = "realplus",
                param = c(mu = GdHfit3$mu, sigma = GdHfit3$sigma,
                nu = GdHfit3$nu, tau = GdHfit3$tau), central = TRUE)
### Skewness
SkewXhGdE <- skewness(hospita$cost, type = 1)
SkewXhGdGG <- skew(dist = "GG", domain = "realplus",
              param = c(mu = GdHfit1$mu, sigma = GdHfit1$sigma,
              nu = GdHfit1$nu))
SkewXhGdGB2 <- skew(dist = "GB2", domain = "realplus",
               param = c(mu = GdHfit2$mu, sigma = GdHfit2$sigma,
               nu = GdHfit2$nu, tau = GdHfit2$tau))
SkewXhGdBCPE <- skew(dist = "BCPE", domain = "realplus",
                param = c(mu = GdHfit3$mu, sigma = GdHfit3$sigma,
                nu = GdHfit3$nu, tau = GdHfit3$tau))
### Excess Kurtosis
KurtXhGdE <- kurtosis(hospita$cost, type = 1)
KurtXhGdGG <- kurt(dist = "GG", domain = "realplus",
              param = c(mu = GdHfit1$mu, sigma = GdHfit1$sigma,
              nu = GdHfit1$nu), excess = TRUE)
KurtXhGdGB2 <- kurt(dist = "GB2", domain = "realplus",
               param = c(mu = GdHfit2$mu, sigma = GdHfit2$sigma,
               nu = GdHfit2$nu, tau = GdHfit2$tau), excess = TRUE)
KurtXhGdBCPE <- kurt(dist = "BCPE", domain = "realplus",
                param = c(mu = GdHfit3$mu, sigma = GdHfit3$sigma,
```

```
                nu = GdHfit3$nu, tau = GdHfit3$tau), excess = TRUE)

kable(cbind(data.frame(Dist = c("Empirical", "GG", "GB2",
            "BCPE")), "Mean" = c(MeanXhGdE, unname(MeanXhGdGG),
            unname(MeanXhGdGB2), unname(MeanXhGdBCPE)),
            "Variance" = c(VariXhGdE, unname(VariXhGdGG),
            unname(VariXhGdGB2), unname(VariXhGdBCPE)),
            "Skewness" = c(SkewXhGdE, unname(SkewXhGdGG),
            unname(SkewXhGdGB2), unname(SkewXhGdBCPE)),
            "Excess Kurtosis" = c(KurtXhGdE, unname(KurtXhGdGG),
            unname(KurtXhGdGB2), unname(KurtXhGdBCPE))),
      caption = "Statistical measurements of GAMLSS distributions for
      hospitalization services \\label{tab:StatisticsGdXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 43: Adjustment of cumulative individual costs of hospitalization services with GAMLSS distributions

```
FnXh <- ecdf(hospita$cost)
sortXh <- sort(hospita$cost)
### Empirical vs Theorical cumulative distribution function
plot(FnXh, lwd = 3,
    xlab = "Sample quantiles of individual costs of hospitalization",
    main = "Adjustment cumulative individual costs of hospitalization")
fitXhGG <- pGG(q = sortXh, mu = GdHfit1$mu, sigma = GdHfit1$sigma,
           nu = GdHfit1$nu)
lines(sortXh, fitXhGG, lwd = 3, lty = 1, col = "blue")
fitXhGB2 <- pGB2(q = sortXh, mu = GdHfit2$mu, sigma = GdHfit2$sigma,
            nu = GdHfit2$nu, tau = GdHfit2$tau)
lines(sortXh, fitXhGB2, lwd = 3, lty = 2, col = "red")
fitXhBCPE <- pBCPE(q = sortXh, mu = GdHfit3$mu,
             sigma = GdHfit3$sigma, nu = GdHfit3$nu, tau = GdHfit3$tau)
lines(sortXh, fitXhBCPE, lwd = 3, lty = 4, col = "green")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red",
       "green"), legend = c("Cumulative empirical distribution",
       "Generalized Gamma", "Generalized Beta type 2",
       "Box-Cox Power Exponential"), lwd = 2)
```

Code 44: Adjustment of log-survival distribution of hospitalization with GAMLSS distributions

```
### Empirical vs Theorical log-survival distribution function
survXh <- 1 - FnXh(sortXh)
plot(x = log(sortXh), y = log(survXh), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
     hospitalization)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     hospitalization", type = "l")
survXhGG <- 1 - fitXhGG
lines(log(sortXh), log(survXhGG), lwd = 3, col = "blue")
survXhGB2 <- 1 - fitXhGB2
lines(log(sortXh), log(survXhGB2), lwd = 3, col = "red", lty = 2)
survXhBCPE <- 1 - fitXhBCPE
lines(log(sortXh), log(survXhBCPE), lwd = 3, col = "green", lty = 4)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red", "green"),
       legend = c("log-Survival Distribution", "Generalized Gamma",
       "Generalized Beta type 2", "Box-Cox Power Exponential"),
       lwd = 2)
```

Code 45: Q-Q plot GAMLSS distribution for hospitalization services

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
### QQ-plot
qqPlot(x = hospita$cost, lwd = 1, distribution = "GG",
       mu = GdHfit1$mu, sigma = GdHfit1$sigma, nu = GdHfit1$nu,
       cex = 1, col.lines = "red" , xlab = "Theorical Quantiles",
       ylab = "Sample Quantiles", id = FALSE,
       main = "(a) Generalized Gamma Q-Q Plot")
qqPlot(x = hospita$cost, lwd = 1, distribution = "GB2",
       mu = GdHfit2$mu, sigma = GdHfit2$sigma, nu = GdHfit2$nu,
       tau = GdHfit2$tau, cex = 1, col.lines = "red",
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Generalized Beta type 2 Q-Q Plot", id = FALSE)
qqPlot(x = hospita$cost, lwd = 1, distribution = "BCPE",
       mu = GdHfit3$mu, sigma = GdHfit3$sigma, nu = GdHfit3$nu,
       tau = GdHfit3$tau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(c) Box-Cox Power Exponential Q-Q Plot", id = FALSE)
```

Code 46: Goodness-of-fit tests for hospitalization services for GAMLSS

```
# A seed is established so that the results can be replicated
set.seed(1248)
### Kolmogorov-Smirnov test
```

```
kolmGdXhGG <- ks.test(x = hospita$cost, distn =  "pGG",
              H = min(hospita$cost), fit = list(mu = GdHfit1$mu,
              sigma = GdHfit1$sigma, nu = GdHfit1$nu))
kolmGdXhGB2 <- ks.test(x = hospita$cost, distn =  "pGB2",
               H = min(hospita$cost), fit = list(mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau))
kolmGdXhBCPE <- ks.test(x = hospita$cost, distn =  "pBCPE",
                H = min(hospita$cost), fit = list(mu = GdHfit3$mu,
                sigma = GdHfit3$sigma, nu = GdHfit3$nu,
                tau = GdHfit3$tau))
### Cramer-von Mises test
cramGdXhGG <- w2.test(x = hospita$cost, distn =  "pGG",
              H = min(hospita$cost), fit = list(mu = GdHfit1$mu,
              sigma = GdHfit1$sigma, nu = GdHfit1$nu))
cramGdXhGB2 <- w2.test(x = hospita$cost, distn =  "pGB2",
               H = min(hospita$cost), fit = list(mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau))
cramGdXhBCPE <- w2.test(x = hospita$cost, distn =  "pBCPE",
                H = min(hospita$cost), fit = list(mu = GdHfit3$mu,
                sigma = GdHfit3$sigma, nu = GdHfit3$nu,
                tau = GdHfit3$tau))
### Kuiper test
kuipGdXhGG <- v.test(x = hospita$cost, distn =  "pGG",
              H = min(hospita$cost), fit = list(mu = GdHfit1$mu,
              sigma = GdHfit1$sigma, nu = GdHfit1$nu))
kuipGdXhGB2 <- v.test(x = hospita$cost, distn =  "pGB2",
               H = min(hospita$cost), fit = list(mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau))
kuipGdXhBCPE <- v.test(x = hospita$cost, distn =  "pBCPE",
                H = min(hospita$cost), fit = list(mu = GdHfit3$mu,
                sigma = GdHfit3$sigma, nu = GdHfit3$nu,
                tau = GdHfit3$tau))
### Supremum class Upper Tail Anderson-Darling test
adupGdXhGG <- adup.test(x = hospita$cost, distn =  "pGG",
              H = min(hospita$cost), fit = list(mu = GdHfit1$mu,
              sigma = GdHfit1$sigma, nu = GdHfit1$nu))
adupGdXhGB2 <- adup.test(x = hospita$cost, distn =  "pGB2",
               H = min(hospita$cost), fit = list(mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau))
adupGdXhBCPE <- adup.test(x = hospita$cost, distn =  "pBCPE",
```

```
                H = min(hospita$cost), fit = list(mu = GdHfit3$mu,
                sigma = GdHfit3$sigma, nu = GdHfit3$nu,
                tau = GdHfit3$tau))
### Quadratic Class Upper Tail Anderson-Darling test
ad2upGdXhGG <- ad2up.test(x = hospita$cost, distn =  "pGG",
              H = min(hospita$cost), fit = list(mu = GdHfit1$mu,
              sigma = GdHfit1$sigma, nu = GdHfit1$nu))
ad2upGdXhGB2 <- ad2up.test(x = hospita$cost, distn =  "pGB2",
                H = min(hospita$cost), fit = list(mu = GdHfit2$mu,
                sigma = GdHfit2$sigma, nu = GdHfit2$nu,
                tau = GdHfit2$tau))
ad2upGdXhBCPE <- ad2up.test(x = hospita$cost, distn =  "pBCPE",
                 H = min(hospita$cost), fit = list(mu = GdHfit3$mu,
                 sigma = GdHfit3$sigma, nu = GdHfit3$nu,
                 tau = GdHfit3$tau))
### Results table
kable(cbind(data.frame(Dist = c("GG", "GB2", "BCPE")),
            "ks.test" = c(kolmGdXhGG$p.value, kolmGdXhGB2$p.value,
            kolmGdXhBCPE$p.value), "w2.test" = c(cramGdXhGG$p.value,
            cramGdXhGB2$p.value, cramGdXhBCPE$p.value), "v.test" = c(
            kuipGdXhGG$p.value, kuipGdXhGB2$p.value,
            kuipGdXhBCPE$p.value), "adup.test" = c(adupGdXhGG$p.value,
            adupGdXhGB2$p.value, adupGdXhBCPE$p.value),
            "ad2up.test" = c(ad2upGdXhGG$p.value,
            ad2upGdXhGB2$p.value, ad2upGdXhBCPE$p.value)),
      caption = "Goodness-of-fit tests for hospitalization services
      with GAMLSS distributions \\label{tab:gftGdXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Risk measures estimation for hospitalization services with GAMLSS

Code 47: Tail index estimation for hospitalization services with GAMLSS

```
### Density function
fhospG <- function(x) dGB2(x, mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau)
### Cumulative function
FhospG <- function(x) pGB2(x, mu = GdHfit2$mu,
               sigma = GdHfit2$sigma, nu = GdHfit2$nu,
               tau = GdHfit2$tau)
```

```
### Equation to calculate the tail index
tailindexHG <- function(x) (1-FhospG(x))/(x * fhospG(x))
curve(expr = tailindexHG, from = 1, to = 3600, ylim = c(0,0.2),
      ylab = "Limit", xlab = "x", mgp=c(2.5,1,0), lwd = 2,
      main = "Tail index estimation for hospitalization
services")
```

Code 48: Risk measures for hospitalization services with GAMLSS

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))

### Value at Risk
quantileHG <- function(kappa) 1 - ((1 - kappa) / MeanNhGPO)
FquantileHG <- function(kappa) qGB2(quantileHG(kappa),
               mu = GdHfit2$mu, sigma = GdHfit2$sigma,
               nu = GdHfit2$nu, tau = GdHfit2$tau)
correctionHG <- MeanXhGdGB2 * (MeanNhGPO + (VariNhGPO / MeanNhGPO) - 1)
VaRHG <- function(kappa) FquantileHG(kappa) + correctionHG

curve(expr = VaRHG, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(a) Value at Risk for hospitalization
services with GAMLSS")
rect(xleft = 0.95, ybottom = VaRHG(0.95) + 30, xright = 0.95,
     ytop = VaRHG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = VaRHG(0.99) + 30, xright = 0.99,
     ytop = VaRHG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(VaRHG(0.95), VaRHG(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.935, y = VaRHG(0.95) + 65, bty = "n",
       legend = round(VaRHG(0.95), 3))
legend(x = 0.975, y = VaRHG(0.99) + 65, bty = "n",
       legend = round(VaRHG(0.99), 3))

### Expected Shortfall
TVaRHG <- function(kappa)(1/(1-kappa)) * as.numeric(integrate(
                         f = VaRHG, lower = kappa, upper = 1)$value)
TVaRHG <- Vectorize(TVaRHG)

curve(expr = TVaRHG, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(b) Expected Shortfall for hospitalization
```

```
services with GAMLSS")
rect(xleft = 0.95, ybottom = TVaRHG(0.95) + 30, xright = 0.95,
     ytop = TVaRHG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = TVaRHG(0.99) + 30, xright = 0.99,
     ytop = TVaRHG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(TVaRHG(0.95), TVaRHG(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.935, y = TVaRHG(0.95) + 68, bty = "n",
       legend = round(TVaRHG(0.95), 3))
legend(x = 0.975, y = TVaRHG(0.99) + 68, bty = "n",
       legend = round(TVaRHG(0.99), 3))

### Stop-Loss Premium
ESHG <- function(kappa) (1 - kappa)*(TVaRHG(kappa) - VaRHG(kappa))

curve(expr = ESHG, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(c) Stop-Loss Premium for hospitalization
services with GAMLSS")
rect(xleft = 0.95, ybottom = ESHG(0.95) + 0.6, xright = 0.95,
     ytop = ESHG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = ESHG(0.99) + 0.6, xright = 0.99,
     ytop = ESHG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(ESHG(0.95), ESHG(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.938, y = ESHG(0.95) + 1.3, bty = "n",
       legend = round(ESHG(0.95), 3))
legend(x = 0.978, y = ESHG(0.99) + 1.3, bty = "n",
       legend = round(ESHG(0.99), 3))
```

Optimum retention point estimation for hospitalization service with GAMLSS

Code 49: Optimum retention point estimation for hospitalization service with GAMLSS

```
VaRTHG <- function(rho, kappa) VaRHG(kappa) + (1 + rho)*ESHG(kappa)
ResultHG <- function(rho){
  kapparho <- 1 - 1/(1 + rho)
  VaRHGrho <- round(VaRHG(kapparho), 3)
  DeltaHGrho <- round((1+rho)*ESHG(kapparho), 3)
  VaRTHGrho <- round(VaRTHG(rho, kapparho), 3)
  return(c(rho, kapparho, VaRHGrho, DeltaHGrho, VaRTHGrho))
```

```
}

tableHG <- round(t(sapply(X = rho, FUN = ResultHG)), 6)

kable(cbind(data.frame(tableHG)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
                    "$M_{hosp}^*$", "$\\delta(M_{hosp}^*)$",
                    "$VaR_{T_{hosp}}(\\kappa_{\\rho^*})$"),
      caption = "Optimum retention point estimation for
      hospitalization services with GAMLSS \\label{tab:retentionHG}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

Adjustment of GAMLSS distributions for general surgery services severities

Code 50: Best fit with GAMLSS distributions for individual cost of general surgery services

```
### The adjustment is made
GdSfit1 <- fitDist(y = surgery$cost, type = "realplus")
### Estimation of second and third distribution with best fit
GdSfit2 <- gamlssML(formula = surgery$cost, family = BCPE)
GdSfit3 <- gamlssML(formula = surgery$cost, family = GG)
### The five distributions that present the best fit are
kable(rbind(GdSfit1$fits[1:5]),
      caption = "Better fit for individual cost of general surgery
      services with GAMLSS distributions \\label{tab:GdSfit}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 51: Statistical measurements of GAMLSS distributions for general surgery services

```
### Estimation of mean, variance, skewness and excess kurtosis
#### Mean
MeanXsGdE <- mean(surgery$cost)
MeanXsGdBCPEo <- moments(k = 1, dist = "BCPEo", domain = "realplus",
                 param = c(mu = GdSfit1$mu, sigma = GdSfit1$sigma,
                 nu = GdSfit1$nu, tau = GdSfit1$tau))
MeanXsGdBCPE <- moments(k = 1, dist = "BCPE", domain = "realplus",
                param = c(mu = GdSfit2$mu, sigma = GdSfit2$sigma,
```

```
                nu = GdSfit2$nu, tau = GdSfit2$tau))
MeanXsGdGG <- moments(k = 1, dist = "GG", domain = "realplus",
              param = c(mu = GdSfit3$mu, sigma = GdSfit3$sigma,
              nu = GdSfit3$nu))
#### Variance
VariXsGdE <- var(surgery$cost)
VariXsGdBCPEo <- moments(k = 2, dist = "BCPEo", domain = "realplus",
                 param = c(mu = GdSfit1$mu, sigma = GdSfit1$sigma,
                 nu = GdSfit1$nu, tau = GdSfit1$tau), central = TRUE)
VariXsGdBCPE <- moments(k = 2, dist = "BCPE", domain = "realplus",
                param = c(mu = GdSfit2$mu, sigma = GdSfit2$sigma,
                nu = GdSfit2$nu, tau = GdSfit2$tau), central = TRUE)
VariXsGdGG <- moments(k = 2, dist = "GG", domain = "realplus",
              param = c(mu = GdSfit3$mu, sigma = GdSfit3$sigma,
              nu = GdSfit3$nu), central = TRUE)
### Skewness
SkewXsGdE <- skewness(surgery$cost, type = 1)
SkewXsGdBCPEo <- skew(dist = "BCPEo", domain = "realplus",
                 param = c(mu = GdSfit1$mu, sigma = GdSfit1$sigma,
                 nu = GdSfit1$nu, tau = GdSfit1$tau))
SkewXsGdBCPE <- skew(dist = "BCPE", domain = "realplus",
                param = c(mu = GdSfit2$mu, sigma = GdSfit2$sigma,
                nu = GdSfit2$nu, tau = GdSfit2$tau))
SkewXsGdGG <- skew(dist = "GG", domain = "realplus",
              param = c(mu = GdSfit3$mu, sigma = GdSfit3$sigma,
              nu = GdSfit3$nu))
### Excess Kurtosis
KurtXsGdE <- kurtosis(surgery$cost, type = 1)
KurtXsGdBCPEo <- kurt(dist = "BCPEo", domain = "realplus",
                 param = c(mu = GdSfit1$mu, sigma = GdSfit1$sigma,
                 nu = GdSfit1$nu, tau = GdSfit1$tau), excess = TRUE)
KurtXsGdBCPE <- kurt(dist = "BCPE", domain = "realplus",
                param = c(mu = GdSfit2$mu, sigma = GdSfit2$sigma,
                nu = GdSfit2$nu, tau = GdSfit2$tau), excess = TRUE)
KurtXsGdGG <- kurt(dist = "GG", domain = "realplus",
              param = c(mu = GdSfit3$mu, sigma = GdSfit3$sigma,
              nu = GdSfit3$nu), excess = TRUE)

kable(cbind(data.frame(Dist = c("Empirical", "GG", "BCPEo",
            "BCPE")), "Mean" = c(MeanXsGdE, unname(MeanXsGdBCPEo),
            unname(MeanXsGdBCPE), unname(MeanXsGdGG)),
            "Variance" = c(VariXsGdE, unname(VariXsGdBCPEo),
            unname(VariXsGdBCPE), unname(VariXsGdGG)),
            "Skewness" = c(SkewXsGdE, unname(SkewXsGdBCPEo),
```

```
            unname(SkewXsGdBCPE), unname(SkewXsGdGG)),
            "Excess Kurtosis" = c(KurtXsGdE, unname(KurtXsGdBCPEo),
            unname(KurtXsGdBCPE), unname(KurtXsGdGG))),
      caption = "Statistical measurements of GAMLSS distributions for
      general surgery services \\label{tab:StatisticsGdXs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 52: Adjustment of cumulative individual costs of general surgery services with GAMLSS distributions

```
FnXs <- ecdf(surgery$cost)
sortXs <- sort(surgery$cost)
### Empirical vs Theorical cumulative distribution function
plot(FnXs, lwd = 3,
    xlab = "Sample quantiles of individual costs of general surgery",
    main = "Adjustment cumulative individual costs of general surgery")
fitXsBCPEo <- pBCPEo(q = sortXs, mu = GdSfit1$mu,
              sigma = GdSfit1$sigma, nu = GdSfit1$nu,
              tau = GdSfit1$tau)
lines(sortXs, fitXsBCPEo, lwd = 3, lty = 1, col = "blue")
fitXsBCPE <- pBCPE(q = sortXs, mu = GdSfit2$mu, sigma = GdSfit2$sigma,
             nu = GdSfit2$nu, tau = GdSfit2$tau)
lines(sortXs, fitXsBCPE, lwd = 3, lty = 2, col = "red")
fitXsGG <- pGG(q = sortXs, mu = GdSfit3$mu, sigma = GdSfit3$sigma,
           nu = GdSfit3$nu)
lines(sortXs, fitXsGG, lwd = 3, lty = 4, col = "green")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red",
       "green"), legend = c("Cumulative empirical distribution",
       "Box-Cox Power Exponential-Original",
       "Box-Cox Power Exponential", "Generalized Gamma"), lwd = 2)
```

Code 53: Adjustment of log-survival distribution of general surgery with GAMLSS distributions

```
### Empirical vs Theorical log-survival distribution function
survXs <- 1 - FnXs(sortXs)
plot(x = log(sortXs), y = log(survXs), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
     general surgery)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     general surgery", type = "l")
survXsBCPEo <- 1 - fitXsBCPEo
```

```
lines(log(sortXs), log(survXsBCPEo), lwd = 3, col = "blue")
survXsBCPE <- 1 - fitXsBCPE
lines(log(sortXs), log(survXsBCPE), lwd = 3, col = "red", lty = 2)
survXsGG <- 1 - fitXsGG
lines(log(sortXs), log(survXsGG), lwd = 3, col = "green", lty = 4)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red", "green"),
       legend = c("log-Survival Distribution",
       "Box-Cox Power Exponential-Original",
       "Box-Cox Power Exponential", "Generalized Gamma"),
       lwd = 2)
```

Code 54: Q-Q plot GAMLSS distribution for general surgery

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
### QQ-plot
qqPlot(x = surgery$cost, lwd = 1, distribution = "BCPEo",
       mu = GdSfit1$mu, sigma = GdSfit1$sigma, nu = GdSfit1$nu,
       tau = GdSfit1$tau, cex = 1, col.lines = "red", id = FALSE,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(a) Box-Cox Power Exponential-Orig. Q-Q Plot")
qqPlot(x = surgery$cost, lwd = 1, distribution = "BCPE",
       mu = GdSfit2$mu, sigma = GdSfit2$sigma, nu = GdSfit2$nu,
       tau = GdSfit2$tau, cex = 1, col.lines = "red", id = FALSE,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Box-Cox Power Exponential Q-Q Plot")
qqPlot(x = surgery$cost, lwd = 1, distribution = "GG",
       mu = GdSfit3$mu, sigma = GdSfit3$sigma, nu = GdSfit3$nu,
       cex = 1, col.lines = "red", id = FALSE,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(c) Generalized Gamma Q-Q Plot")
```

Code 55: Goodness-of-fit tests for general surgery services for GAMLSS distributions

```
# A seed is established so that the results can be replicated
set.seed(1248)
### Kolmogorov-Smirnov test
kolmGdXsBCPEo <- ks.test(x = surgery$cost, distn =  "pBCPEo",
                 H = min(surgery$cost), fit = list(mu = GdSfit1$mu,
                 sigma = GdSfit1$sigma,nu = GdSfit1$nu,
                 tau = GdSfit1$tau))
kolmGdXsBCPE <- ks.test(x = surgery$cost, distn =  "pBCPE",
                H = min(surgery$cost), fit = list(mu = GdSfit2$mu,
                sigma = GdSfit2$sigma, nu = GdSfit2$nu,
```

```
                tau = GdSfit2$tau))
kolmGdXsGG <- ks.test(x = surgery$cost, distn =  "pGG",
              H = min(surgery$cost), fit = list(mu = GdSfit3$mu,
              sigma = GdSfit3$sigma, nu = GdSfit3$nu))
### Cramer-von Mises test
cramGdXsBCPEo <- w2.test(x = surgery$cost, distn =  "pBCPEo",
                 H = min(surgery$cost), fit = list(mu = GdSfit1$mu,
                 sigma = GdSfit1$sigma,nu = GdSfit1$nu,
                 tau = GdSfit1$tau))
cramGdXsBCPE <- w2.test(x = surgery$cost, distn =  "pBCPE",
                H = min(surgery$cost), fit = list(mu = GdSfit2$mu,
                sigma = GdSfit2$sigma, nu = GdSfit2$nu,
                tau = GdSfit2$tau))
cramGdXsGG <- w2.test(x = surgery$cost, distn =  "pGG",
               H = min(surgery$cost), fit = list(mu = GdSfit3$mu,
               sigma = GdSfit3$sigma, nu = GdSfit3$nu))
### Kuiper test
kuipGdXsBCPEo <- v.test(x = surgery$cost, distn =  "pBCPEo",
                 H = min(surgery$cost), fit = list(mu = GdSfit1$mu,
                 sigma = GdSfit1$sigma,nu = GdSfit1$nu,
                 tau = GdSfit1$tau))
kuipGdXsBCPE <- v.test(x = surgery$cost, distn =  "pBCPE",
                H = min(surgery$cost), fit = list(mu = GdSfit2$mu,
                sigma = GdSfit2$sigma, nu = GdSfit2$nu,
                tau = GdSfit2$tau))
kuipGdXsGG <- v.test(x = surgery$cost, distn =  "pGG",
              H = min(surgery$cost), fit = list(mu = GdSfit3$mu,
              sigma = GdSfit3$sigma, nu = GdSfit3$nu))
### Supremum class Upper Tail Anderson-Darling test
adupGdXsBCPEo <- adup.test(x = surgery$cost, distn =  "pBCPEo",
                 H = min(surgery$cost), fit = list(mu = GdSfit1$mu,
                 sigma = GdSfit1$sigma,nu = GdSfit1$nu,
                 tau = GdSfit1$tau))
adupGdXsBCPE <- adup.test(x = surgery$cost, distn =  "pBCPE",
                H = min(surgery$cost), fit = list(mu = GdSfit2$mu,
                sigma = GdSfit2$sigma, nu = GdSfit2$nu,
                tau = GdSfit2$tau))
adupGdXsGG <- adup.test(x = surgery$cost, distn =  "pGG",
              H = min(surgery$cost), fit = list(mu = GdSfit3$mu,
              sigma = GdSfit3$sigma, nu = GdSfit3$nu))
### Quadratic Class Upper Tail Anderson-Darling test
ad2upGdXsBCPEo <- ad2up.test(x = surgery$cost, distn =  "pBCPEo",
                  H = min(surgery$cost), fit = list(mu = GdSfit1$mu,
                  sigma = GdSfit1$sigma,nu = GdSfit1$nu,
```

```
                  tau = GdSfit1$tau))
ad2upGdXsBCPE <- ad2up.test(x = surgery$cost, distn =  "pBCPE",
                 H = min(surgery$cost), fit = list(mu = GdSfit2$mu,
                 sigma = GdSfit2$sigma, nu = GdSfit2$nu,
                 tau = GdSfit2$tau))
ad2upGdXsGG <- ad2up.test(x = surgery$cost, distn =  "pGG",
               H = min(surgery$cost), fit = list(mu = GdSfit3$mu,
               sigma = GdSfit3$sigma, nu = GdSfit3$nu))
### Results table
kable(cbind(data.frame(Dist = c("BCPEo", "BCPE", "GG")),
            "ks.test" = c(kolmGdXsBCPEo$p.value, kolmGdXsBCPE$p.value,
            kolmGdXsGG$p.value), "w2.test" = c(cramGdXsBCPEo$p.value,
            cramGdXsBCPE$p.value, cramGdXsGG$p.value), "v.test" = c(
            kuipGdXsBCPEo$p.value, kuipGdXsBCPE$p.value,
            kuipGdXsGG$p.value), "adup.test" = c(adupGdXsBCPEo$p.value,
            adupGdXsBCPE$p.value, adupGdXsGG$p.value), "ad2up.test" =
            c(ad2upGdXsBCPEo$p.value, ad2upGdXsBCPE$p.value,
            ad2upGdXsGG$p.value)),
      caption = "Goodness-of-fit tests for general surgery services
      with GAMLSS distributions \\label{tab:gftGdXs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Risk measures estimation for general surgery services with GAMLSS

Code 56: Tail index estimation for general surgery services with GAMLSS

```
### Density function
fsurgG <- function(x) dBCPEo(x, mu = GdSfit1$mu,
         sigma = GdSfit1$sigma, nu = GdSfit1$nu,
         tau = GdSfit1$tau)
### Cumulative function
FsurgG <- function(x) pBCPEo(x, mu = GdSfit1$mu,
         sigma = GdSfit1$sigma, nu = GdSfit1$nu,
         tau = GdSfit1$tau)

### Equation to calculate the tail index
tailindexSG <- function(x) (1-FsurgG(x))/(x * fsurgG(x))
curve(expr = tailindexSG, from = 1, to = 3600, ylim = c(0,0.2),
      ylab = "Limit", xlab = "x", mgp=c(2.5,1,0), lwd = 2,
      main = "Tail index estimation for general surgery
services with GAMLSS")
```

Code 57: Risk measures for general surgery services with GAMLSS

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))

### Value at Risk
quantileSG <- function(kappa) 1 - ((1 - kappa) / MeanNsDEL)
FquantileSG <- function(kappa) qBCPEo(quantileSG(kappa),
               mu = GdSfit1$mu, sigma = GdSfit1$sigma,
               nu = GdSfit1$nu, tau = GdSfit1$tau)
correctionSG <- MeanXsGdBCPEo * (MeanNsDEL + (VariNsDEL / MeanNsDEL) -
                                 1)
VaRSG <- function(kappa) FquantileSG(kappa) + correctionSG

curve(expr = VaRSG, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Value at Risk for general surgery
services with GAMLSS")
rect(xleft = 0.95, ybottom = VaRSG(0.95) + 30, xright = 0.95,
     ytop = VaRSG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = VaRSG(0.99) + 30, xright = 0.99,
     ytop = VaRSG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(VaRSG(0.95), VaRSG(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = VaRSG(0.95) + 50, bty = "n",
       legend = round(VaRSG(0.95), 3))
legend(x = 0.98, y = VaRSG(0.99) + 50, bty = "n",
       legend = round(VaRSG(0.99), 3))

### Expected Shortfall
TVaRSG <- function(kappa) (1/(1-kappa)) * as.numeric(integrate(
         f = VaRSG, lower = kappa, upper = 1)$value)
TVaRSG <- Vectorize(TVaRSG)

curve(expr = TVaRSG, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Expected Shortfall for general surgery
services with GAMLSS")
rect(xleft = 0.95, ybottom = TVaRSG(0.95) + 30, xright = 0.95,
     ytop = TVaRSG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = TVaRSG(0.99) + 30, xright = 0.99,
     ytop = TVaRSG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(TVaRSG(0.95), TVaRSG(0.99)),
```

```
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.94, y = TVaRSG(0.95) + 52, bty = "n",
       legend = round(TVaRSG(0.95), 3))
legend(x = 0.98, y = TVaRSG(0.99) + 52, bty = "n",
       legend = round(TVaRSG(0.99), 3))

### Stop-Loss Premium
ESSG <- function(kappa) (1 - kappa)*(TVaRSG(kappa) - VaRSG(kappa))

curve(expr = ESSG, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "Stop-Loss Premium for general surgery
services with GAMLSS")
rect(xleft = 0.95, ybottom = ESSG(0.95) + 0.5, xright = 0.95,
     ytop = ESSG(0.95), lwd = 2, border = "red")
rect(xleft = 0.99, ybottom = ESSG(0.99) + 0.5, xright = 0.99,
     ytop = ESSG(0.99), lwd = 2, border = "red")
points(x = c(0.95, 0.99), y = c(ESSG(0.95), ESSG(0.99)),
       pch = 19, col = "red", cex = 1.2)
legend(x = 0.942, y = ESSG(0.95) + 0.85, bty = "n",
       legend = round(ESSG(0.95), 3))
legend(x = 0.982, y = ESSG(0.99) + 0.85, bty = "n",
       legend = round(ESSG(0.99), 3))
```

Optimum retention point estimation for general surgery service with GAMLSS

Code 58: Optimum retention point estimation for general surgery service with GAMLSS

```
VaRTSG <- function(rho, kappa) VaRSG(kappa) + (1 + rho)*ESSG(kappa)
ResultSG <- function(rho){
  kapparho <- 1 - 1/(1 + rho)
  VaRSGrho <- round(VaRSG(kapparho), 3)
  DeltaSGrho <- round((1+rho)*ESSG(kapparho), 3)
  VaRTSGrho <- round(VaRTSG(rho, kapparho), 3)
  return(c(rho, kapparho, VaRSGrho, DeltaSGrho, VaRTSGrho))
}

tableSG <- round(t(sapply(X = rho, FUN = ResultSG)), 6)

kable(cbind(data.frame(tableSG)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
```

```
                    "$M_{surg}^*$", "$\\delta(M_{surg}^*)$",
                    "$VaR_{T_{surg}}(\\kappa_{\\rho^*})$"),
      caption = "Optimum retention point estimation for
      general surgery services with GAMLSS \\label{tab:retentionSG}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

Adjustment comparison between the spliced and GAMLSS distribution for hospitalization services

Code 59: Statistical measurements comparison for hospitalization services

```
kable(cbind(data.frame(Dist = c("Empirical", "weibullgpd", "GB2")),
            "Mean" = c(MeanXhGdE, unname(MeanXhSpW),
            unname(MeanXhGdGB2)), "Variance" = c(VariXhGdE,
            unname(VariXhSpW), unname(VariXhGdGB2)),
            "Skewness" = c(SkewXhGdE, unname(SkewXhSpW),
            unname(SkewXhGdGB2)), "Excess Kurtosis" = c(KurtXhGdE,
            unname(KurtXhSpW), unname(KurtXhGdGB2))),
      caption = "Statistical measurements comparison for
      hospitalization services \\label{tab:StatisticsCpXs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 60: Adjustment comparison for cumulative individual costs of hospitalization services

```
### Empirical vs Theorical cumulative distribution function
plot(FnXh, lwd = 3,
    xlab = "Sample quantiles of individual costs of hospitalization",
    main = "Adjustment cumulative individual costs of hospitalization")
lines(sortXh, fitXhW, lwd = 3, lty = 1, col = "blue")
lines(sortXh, fitXhGB2, lwd = 3, lty = 2, col = "red")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red"),
       lwd = 2, legend = c("Cumulative empirical distribution",
       "Weibull-Generalized Pareto", "Generalized Beta type 2"))
```

Code 61: Adjustment comparison for log-survival costs of hospitalization services

```
### Empirical vs Theorical log-survival distribution function
plot(x = log(sortXh), y = log(survXh), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
     hospitalization)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     hospitalization", type = "l")
lines(log(sortXh), log(survXhW), lwd = 3, col = "blue", lty = 1)
lines(log(sortXh), log(survXhGB2), lwd = 3, col = "red", lty = 2)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red"),
       lwd = 2, legend = c("log-Survival Distribution",
       "Weibull-Generalized Pareto", "Generalized Beta type 2"))
```

Code 62: Q-Q plot comparison for hospitalization services

```
par(mfrow = c(2,1))
### QQ-plot
qqPlot(x = hospita$cost, lwd = 1, distribution = "weibullgpd",
       phiu = SpHfit3$phi, wshape = SpHfit3$wshape,
       wscale = SpHfit3$wscale, u = SpHfit3$u, xi = SpHfit3$xi,
       sigmau = SpHfit3$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(a) Weibull-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = hospita$cost, lwd = 1, distribution = "GB2",
       mu = GdHfit2$mu, sigma = GdHfit2$sigma, nu = GdHfit2$nu,
       tau = GdHfit2$tau, cex = 1, col.lines = "red",
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Generalized Beta type 2 Q-Q Plot", id = FALSE)
```

Code 63: Goodness-of-fit tests comparison for hospitalization services

```
### Results table
kable(cbind(data.frame(Dist = c("weibullgpd", "GB2")),
            "ks.test" = c(kolmSpXhW$p.value, kolmGdXhGB2$p.value),
            "w2.test" = c(cramSpXhW$p.value, cramGdXhGB2$p.value),
            "v.test" = c(kuipSpXhW$p.value, kuipGdXhGB2$p.value),
            "adup.test" = c(adupSpXhW$p.value, adupGdXhGB2$p.value),
            "ad2up.test" = c(ad2upSpXhW$p.value, ad2upGdXhGB2$p.value)),
      caption = "Goodness-of-fit tests comparison for hospitalization
      services
      \\label{tab:gftCpXh}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Risk measures comparison for hospitalization services

Code 64: Risk measures comparison for hospitalization services

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
### Value at Risk
curve(expr = VaRH, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(a) Value at Risk comparison for
hospitalization services", ylim = c(6250, 6750))
curve(expr = VaRHG, from = 0.90, to = 0.999, add = T,
      col = "blue", lwd = 2)

legend("topleft", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
       "Generalized Beta Type II"), lwd = c(2, 2))

### Expected Shortfall
curve(expr = TVaRH, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(b) Expected Shortfall comparison for
hospitalization services", ylim = c(6250, 6750))
curve(expr = TVaRHG, from = 0.90, to = 0.999, add = T,
      col = "blue", lwd = 2)

legend("topleft", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
       "Generalized Beta Type II"), lwd = c(2, 2))

### Stop-Loss Premium
curve(expr = ESH, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(c) Stop-Loss Premium comparison for
hospitalization services", ylim = c(0, 5.2))
curve(expr = ESHG, from = 0.90, to = 0.999, add = T,
      col = "blue", lwd = 2)

legend("topright", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
```

```
       "Generalized Beta Type II"), lwd = c(2, 2))
```

Optimum retention point comparison for hospitalization services

Code 65: Optimum retention point comparison for hospitalization services

```
tableHC <- cbind(tableH[, 1:3], tableHG[, 3], tableH[, 4],
                 tableHG[, 4])

kable(cbind(data.frame(tableHC)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
                    "$M_{hosp_{W-GP}}^*$", "$M_{hosp_{GB2}}^*$",
                    "$\\delta(M_{hosp_{W-GP}}^*)$",
                    "$\\delta(M_{hosp_{GB2}}^*)$"),
      caption = "Optimum retention point comparison for
      hospitalization services \\label{tab:retentionHC}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

Adjustment comparison between the spliced and GAMLSS distribution for general surgery

Code 66: Statistical measurements comparison for general surgery services

```
kable(cbind(data.frame(Dist = c("Empirical", "weibullgpd",
            "BCPEo")), "Mean" = c(MeanXsGdE, unname(MeanXsSpW),
            unname(MeanXsGdBCPEo)), "Variance" = c(round(VariXsGdE,
            5), "does not exist", round(unname(VariXsGdBCPEo), 5)),
            "Skewness" = c(round(SkewXsGdE, 6), "does not exist",
            round(unname(SkewXsGdBCPEo), 6)), "Excess Kurtosis" =
            c(round(KurtXsGdE, 5), "does not exist", round(unname(
            KurtXsGdBCPEo, 5)))),
      caption = "Statistical measurements comparison for
      general surgery services \\label{tab:StatisticsCpXs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Code 67: Adjustment comparison for cumulative individual costs of general surgery services

```
### Empirical vs Theorical cumulative distribution function
plot(FnXs, lwd = 3,
    xlab = "Sample quantiles of individual costs of general surgery",
    main = "Adjustment cumulative individual costs of general surgery")
lines(sortXs, fitXsW, lwd = 3, lty = 1, col = "blue")
lines(sortXs, fitXsBCPEo, lwd = 3, lty = 2, col = "red")
grid()
legend("bottomright", lty = 1, col = c("black", "blue", "red"),
       lwd = 2, legend = c("Cumulative empirical distribution",
       "Weibull-Generalized Pareto",
       "Box-Cox Power Exponential-Original"))
```

Code 68: Adjustment comparison for log-survival costs of general surgery services

```
### Empirical vs Theorical log-survival distribution function
plot(x = log(sortXs), y = log(survXs), lwd = 3,
     xlab = "log(Sample quantiles of individual cost of
     general surgery)", ylab = "log(1 - Fn(x))",
     main = "Adjustment of log-survival distribution of
     general surgery", type = "l")
lines(log(sortXs), log(survXsW), lwd = 3, col = "blue", lty = 1)
lines(log(sortXs), log(survXsBCPEo), lwd = 3, col = "red", lty = 2)
grid()
legend("bottomleft", lty = 1, col = c("black", "blue", "red"),
       lwd = 2, legend = c("log-Survival Distribution",
       "Weibull-Generalized Pareto",
       "Box-Cox Power Exponential-Original"))
```

Code 69: Q-Q plot comparison for general surgery services

```
par(mfrow = c(2,1))
### QQ-plot
qqPlot(x = surgery$cost, lwd = 1, distribution = "weibullgpd",
       phiu = SpSfit3$phi, wshape = SpSfit3$wshape,
       wscale = SpSfit3$wscale, u = SpSfit3$u, xi = SpSfit3$xi,
       sigmau = SpSfit3$sigmau, cex = 1, col.lines = "red" ,
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(a) Weibull-Generalized Pareto Q-Q Plot", id = FALSE)
qqPlot(x = surgery$cost, lwd = 1, distribution = "BCPEo",
       mu = GdSfit2$mu, sigma = GdSfit2$sigma, nu = GdSfit2$nu,
       tau = GdSfit2$tau, cex = 1, col.lines = "red",
       xlab = "Theorical Quantiles", ylab = "Sample Quantiles",
       main = "(b) Box-Cox Power Exponential-Orig. Q-Q Plot",
       id = FALSE)
```

Code 70: Goodness-of-fit tests comparison for general surgery services

```
### Results table
kable(cbind(data.frame(Dist = c("weibullgpd", "BCPEo")),
            "ks.test" = c(kolmSpXsW$p.value, kolmGdXsBCPEo$p.value),
            "w2.test" = c(cramSpXsW$p.value, cramGdXsBCPEo$p.value),
            "v.test" = c(kuipSpXsW$p.value, kuipGdXsBCPEo$p.value),
            "adup.test" = c(adupSpXsW$p.value, adupGdXsBCPEo$p.value),
            "ad2up.test" = c(ad2upSpXsW$p.value,
                             ad2upGdXsBCPEo$p.value)),
      caption = "Goodness-of-fit tests comparison for general surgery
      services
      \\label{tab:gftCpXs}",
      "latex", booktabs = T) %>%
kable_styling(latex_options = c("striped", "hold_position"))
```

Risk measures comparison for general surgery services

Code 71: Risk measures comparison for general surgery services

```
par(mai=rep(0.5, 4))
layout(matrix(c(1,1, 2,2, 0, 3,3, 0), ncol = 4, byrow = TRUE))
### Value at Risk
curve(expr = VaRS, from = 0.90, to = 0.999,
      ylab = "Value at Risk (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(a) Value at Risk comparison for
general surgery services", ylim = c(1000, 3000))
curve(expr = VaRSG, from = 0.90, to = 0.999, add = T,
      col = "blue", lwd = 2)

legend("topleft", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
       "Box-Cox Power Exponential-orig."), lwd = c(2, 2))

### Expected Shortfall
curve(expr = TVaRS, from = 0.90, to = 0.999,
      ylab = "Expected Shortfall (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(b) Expected Shortfall comparison for
general surgery services", ylim = c(1000, 10000))
curve(expr = TVaRSG, from = 0.90, to = 0.999, add = T,
```

```
      col = "blue", lwd = 2)

legend("topleft", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
       "Box-Cox Power Exponential-orig."), lwd = c(2, 2))

### Stop-Loss Premium
curve(expr = ESS, from = 0.90, to = 0.999,
      ylab = "Stop-Loss Premium (in millions of pesos)",
      xlab = expression(kappa), mgp=c(2,1,0), lwd = 2,
      main = "(c) Stop-Loss Premium comparison for
general surgery services", ylim = c(0, 80))
curve(expr = ESSG, from = 0.90, to = 0.999, add = T,
      col = "blue", lwd = 2)

legend("left", col = c("black", "blue"), lty = c(1, 1),
       legend = c("Weibull-Generalized Pareto",
       "Box-Cox Power Exponential-orig."), lwd = c(2, 2))
```

Optimum retention point comparison for general surgery services

Code 72: Optimum retention point comparison for general surgery service

```
tableSC <- cbind(tableS[, 1:3], tableSG[, 3], tableS[, 4],
                 tableSG[, 4])

kable(cbind(data.frame(tableSC)), escape = FALSE,
      col.names = c("$\\rho$", "$\\kappa_{\\rho^*}$",
                    "$M_{surg_{W-GP}}^*$", "$M_{surg{BCPEo}}^*$",
                    "$\\delta(M_{surg_{W-GP}}^*)$",
                    "$\\delta(M_{surg_{BCPEo}}^*)$"),
      caption = "Optimum retention point comparison for
      general surgery services \\label{tab:retentionSC}",
      "latex", booktabs = T) %>%
  kable_styling(latex_options = c("striped", "hold_position"),
                position = "center")
```

Appendix C: Description of GAMLSS distributions

Este apéndice busca realizar una descripción a las distribuciones ajustadas mediante la librería `gamlss` (2005), que se emplearon en los capítulos 2 y 5.

Frequency Model

Delaporte (DEL)

The *DEL* distribution is a Poisson mixtures known in actuarial literature, according to Johnson, Kemp, and Kotz (2005), as the convolution of a Negative Binomial distribution with a Poisson distribution. In Rigby et al. (2008, pp. 385–386), the authors define the probability function of the *DEL* distribution as

$$f_y(y) = \frac{e^{\mu\nu}}{\Gamma(1/\sigma)}[1 + \mu\sigma(1-\nu)]^{-1/\sigma} S \tag{7.1}$$

where

$$S = \sum_{j=0}^{y} \binom{y}{j} \frac{\mu^y \nu^{y-j}}{y!} \left[\mu + \frac{1}{\sigma(1-\nu)}\right]^{-j} \Gamma\left(\frac{1}{\sigma} + j\right) \tag{7.2}$$

for $y = 0, 1, 2, \ldots, \infty$, $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$. Furthermore, in Rigby et al. (2008, p. 393) establishes the mean, variance, skewness and excess kurtosis of the *DEL* distribution as

$$\begin{aligned}
&\mathbb{E}(Y) = \mu \\
&Var(Y) = \mu + \mu^2\sigma(1-\nu)^2 \\
&Skew(Y) = \frac{\mu[1 + 3\mu\sigma(1-\nu)^2 + 2\mu^2\sigma^2(1-\nu)^3]}{(\mu + \mu^2\sigma(1-\nu)^2)^{1.5}} \\
&Kurt(Y) = \frac{\mu[1 + 7\mu\sigma(1-\nu)^2 + 12\mu^2\sigma^2(1-\nu)^3 + 6\mu^3\sigma^3(1-\nu)^4]}{(\mu + \mu^2\sigma(1-\nu)^2)^2}
\end{aligned} \tag{7.3}$$

Generalized Poisson (GPO)

The *GPO* distribution proposed by Consul and Jain (1973) is defined as a generalization of the Poisson distribution with two parameters, obtained in a limit way when approximating the two gamma functions that have the Generalized Negative Binomial distribution through the Stirling's formula. In Rigby et al. (2017, p. 319) is defined the probability function of the *GPO* distribution as

$$f_y(y) = \left(\frac{\mu}{1+\mu\sigma}\right)^y \frac{(1+\mu\sigma)^{y-1}}{y!} \exp\left[-\frac{\mu(1+\sigma y)}{1+\mu\sigma}\right] \tag{7.4}$$

for $y = 0, 1, \ldots, \infty$, where $\mu > 0$, $\sigma > 0$. In addition, Rigby et al. (2017, p. 319) define the mean, variance, skewness and excess kurtosis of Y as

$$\begin{aligned} &\mathbb{E}(Y) = \mu \\ &Var(Y) = \mu(1+\mu\sigma)^2 \\ &Skew(Y) = (1+3\mu\sigma)/\mu^{0.5} \\ &Kurt(Y) = (1+10\mu\sigma+15\mu^2\sigma^2)/\mu \end{aligned} \tag{7.5}$$

Negative Binomial type I (NBI)

The *NBI* distribution is one of the most used distributions within the Mixed Poisson Law, since by assuming that the parameter $\lambda \sim Gamma(\alpha, \beta)$, is obtained

$$dF_\lambda(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0 \tag{7.6}$$

with $\alpha > 0$ and $\beta > 0$. In Rigby et al. (2017, p. 321) the authors denote the probability function of the *NBI* distribution as

$$f_y(y) = \frac{\Gamma(y+(1/\sigma))}{\Gamma(1/\sigma)\Gamma(y+1)} \left(\frac{\mu\sigma}{1+\mu\sigma}\right)^y \left(\frac{1}{1+\mu\sigma}\right)^{1/\sigma} \tag{7.7}$$

for $y = 0, 1, \ldots, \infty$, with $\mu > 0$, $\sigma > 0$. Also, Rigby et al., (2017, p. 321) define the mean, variance, skewness and excess kurtosis of the distribution *NBI* as

$$\begin{aligned} &\mathbb{E}(Y) = \mu \\ &Var(Y) = (\mu+\mu^2\sigma) \\ &Skew(Y) = (1+2\mu\sigma)/(\mu+\mu^2\sigma)^{0.5} \\ &Kurt(Y) = 6\sigma + (1/(\mu+\mu^2\sigma)) \end{aligned} \tag{7.8}$$

Poisson-Inverse Gaussian (PIG)

As pointed out by Willmot (1987), the *PIG* distribution as a Poisson mixture distribution has statistical and mathematical properties very similar to the Negative Binomial distribution. On the other hand, Rigby et al. (2017, p. 325) defines the probability function of the *PIG* distribution as

$$f_y(y) = \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{\mu^y e^{\frac{1}{\sigma}} K_{y-\frac{1}{2}}(\alpha)}{(\alpha\sigma)^y y!} \tag{7.9}$$

where $\mu > 0$, $\sigma > 0$, $\alpha^2 = \frac{1}{\sigma^2} + \frac{2\mu}{\sigma}$, for $y = 0, 1, \ldots, \infty$, and

$$K_\lambda(t) = \frac{1}{2}\int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}t(x+x^{-1})} dx \tag{7.10}$$

is the modified Bessel function of the third type. Furthermore, Rigby et al. (2017, p. 325) establish that the mean, variance, skewness and excess kurtosis of Y are given by

$$\begin{aligned}
&\mathbb{E}(Y) = \mu \\
&Var(Y) = \mu(1 + \mu\sigma) \\
&Skew(Y) = (1 + 3\mu\sigma + 3\mu^2\sigma^2)/\left[\mu^{0.5}(1 + \mu\sigma)^{1.5}\right] \\
&Kurt(Y) = (1 + 7\mu\sigma + 18\mu^2\sigma^2 + 15\mu^3\sigma^3)/\left[\mu(1 + \mu\sigma)^2\right]
\end{aligned} \tag{7.11}$$

Severity model

Box-Cox Power Exponential (BCPE)

The *BCPE* distribution was proposed by Rigby and Stasinopoulos (2004) and is presented by the authors as a generalization of the distributions Power Exponential (when the skewness parameter is zero) and Normal Box-Cox (when the kurtosis parameter is zero), with the advantage of modeling both the skewness and the kurtosis of the dependent variable Y.

In Rigby and Stasinopoulos (2004, p. 3058) and Rigby et al. (2017, p. 291) the probability distribution of the *BCPE* is presented as

$$f_y(y) = \frac{y^{\nu-1} f_T(z)}{\mu^\nu \sigma F_T\left(\frac{1}{\sigma|\nu|}\right)} \tag{7.12}$$

where $y > 0$, $\mu > 0$, $\sigma > 0$, $-\infty < \nu < \infty$, T is distributed as a standard Power Exponential (mean zero and variance one), $f_T(z)$ is the density function of T evaluated in the transformed random variable Z defined as

$$f_T(z) = \frac{\tau e^{-|z/c|^\tau}}{2c\Gamma(1/\tau)} \tag{7.13}$$

where $\tau > 0$, $c^2 = \frac{\Gamma(1/\tau)}{\Gamma(3/\tau)}$, and Z is a transformed random variable that follows a truncated standard normal distribution, with range $Z > -1/(\sigma\nu)$ if $\nu > 0$ or $Z < -1/(\sigma\nu)$ if $\nu < 0$, defined as

$$Z = \begin{cases} \frac{1}{\sigma\nu}\left[\left(\left(\frac{Y}{\mu}\right)^{\nu} - 1\right)\right] & \text{if } \nu \neq 0 \\ \frac{1}{\sigma}\log\left(\frac{Y}{\mu}\right) & \text{if } \nu = 0 \end{cases} \tag{7.14}$$

and $F_T(1/\sigma|\nu|)$ is the cumulative distribution of T evaluated in $\frac{1}{\sigma\nu}$ defined as

$$F_T\left(\frac{1}{\sigma|\nu|}\right) = \frac{1}{2}\left[1 + \frac{\gamma\left(\frac{1}{\tau}, \left|\frac{1}{c\sigma|\nu|}\right|^{\tau}\right)}{\Gamma\left(\frac{1}{\tau}\right)} sign\left(\frac{1}{\sigma|\nu|}\right)\right] \tag{7.15}$$

where $\Gamma(\cdot)$ is the gamma function and $\gamma(\cdot)$ is the incomplete gamma function.

It should be noted that the *BCPE* distribution does not have equations for the calculation of its moments, but in Rigby and Stasinopoulos (2004, p. 3073), the authors present approximations of the mean, coefficient of variation, skewness and kurtosis of Y, when the parameter $\nu > 0$ and $\sigma < 0.2/max(|\nu|, 1)$, due this allows us to ignore the truncation of the random variable Z. (See Rigby and Stasinopoulos (2004, p. 3073) for the approximation equations of the moments of the *BCPE* distribution).

Box-Cox Power Exponential - original (BCPEo)

It is worth noting what is expressed by Rigby et al. (2017, p. 292), with respect to the relationship of the distributions *BCPEo* and *BCPE*, where the authors point out that both distributions have the same probability density function, but they differ in the link function to calculate the μ parameter. The link function of the parameter μ for the *BCPEo* is `log` while for the *BCPE* is `identity`, therefore, to adjust the parameter μ in the *BCPEo*, the optimization algorithm uses the domain $(-\infty, \infty)$, whereas in the *BCPEo*, the optimization algorithm uses the domain $(0, \infty)$.

Additionally, in Stasinopoulos, Rigby, Heller, Voudouris, and De Bastiani (2017, p. 454) it is indicated that the selection of the link function for a parameter does not generate problems, and that the preference for one or another link function (in this case, using one or another distribution) lies in the one that obtains a $GAIC(\omega)$ smaller in the adjustment (In this case $\omega = 2$ thus $GAIC(2)$ it is equivalent to AIC).

Generalized Beta type 2 (GB2)

The *GB2* distribution is defined by McDonald and Xu (1995), as a particular case of the Generalized Beta distribution, when the parameter $c = 1$. Also, in McDonald and Xu (1995, p. 136) the authors point out that the *GB2* has a large number of nested distributions, among which stand out the Generalized Gamma, Burr type 3 and type 12, Lognormal, Weibull, Gamma, Lomax, F statistic, Rayleigh, Chi-square, Half-normal, Half-Student's t, Exponential and Log-logistic.

In Rigby et al. (2017, p. 293) the probability function of the *GB2* is defined as

$$f_y(y) = \frac{|\sigma| y^{\sigma\nu - 1}}{\mu^{\sigma\nu} B(\nu, \tau)[1 + (y/\mu^{\sigma})]^{\sigma+\tau}} \tag{7.16}$$

for $y > 0$, $\mu > 0$, $\sigma > 0$, $\nu > 0$, $\tau > 0$ and where $B(a, b)$ is the beta function defined as

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{7.17}$$

Furthermore, Rigby et al. (2017, p. 293), defines the mean, variance, skewness and excess kurtosis of the *GB2* distribution as

$$\mathbb{E}(Y) = \begin{cases} \frac{\mu B(\nu+\sigma^{-1},\tau-\sigma^{-1})}{B(\nu,\tau)}, & \text{if } \tau > \frac{1}{\sigma} \\ \infty, & \text{if } \tau \leq \frac{1}{\sigma} \end{cases}$$

$$Var(Y) = \begin{cases} \frac{\mu_2}{B(\nu,\tau)^2}, & \text{if } \tau > \frac{2}{\sigma} \\ \infty, & \text{if } \tau \leq \frac{2}{\sigma} \end{cases}$$
$$\text{where } \mu_2 = \left[B(\nu+2\sigma^{-1},\tau-2\sigma^{-1})B(\nu,\tau) - B(\nu+\sigma^{-1},\tau-\sigma^{-1})^2\right]$$

$$\begin{aligned} Skew(Y) &= \begin{cases} \frac{\mu_3}{B(\nu,\tau)^3 Var(Y)^{1.5}}, & \text{if } \tau > \frac{3}{\sigma} \\ \infty, & \text{if } \tau \leq \frac{3}{\sigma} \end{cases} \\ \text{where } \mu_3 &= \mu^3[B(\nu+3\sigma^{-1},\tau-3\sigma^{-1})B(\nu,\tau)^2 \\ &\quad - 3B(\nu+2\sigma^{-1},\tau-2\sigma^{-1})B(\nu+\sigma^{-1},\tau-\sigma^{-1})B(\nu,\tau) \\ &\quad + 2B(\nu+\sigma^{-1},\tau-\sigma^{-1})^3] \end{aligned} \tag{7.18}$$

$$\begin{aligned} Kurt(Y) &= \begin{cases} \frac{\mu_4}{B(\nu,\tau)^4 Var(Y)^2} - 3, & \text{if } \tau > \frac{4}{\sigma} \\ \infty, & \text{if } \tau \leq \frac{4}{\sigma} \end{cases} \\ \text{where } \mu_4 &= \mu^4[B(\nu+4\sigma^{-1},\tau-4\sigma^{-1})B(\nu,\tau)^3 \\ &\quad - 4B(\nu+3\sigma^{-1},\tau-3\sigma^{-1})B(\nu+\sigma^{-1},\tau-\sigma^{-1})B(\nu,\tau)^2 \\ &\quad + 6B(\nu+2\sigma^{-1},\tau-2\sigma^{-1})B(\nu+\sigma^{-1},\tau-\sigma^{-1})^2 B(\nu,\tau) \\ &\quad - 3B(\nu+\sigma^{-1},\tau-\sigma^{-1})^4] \end{aligned}$$

Generalized Gamma

The GG distribution was proposed by Stacy (1962) in its article "*A generalization of the Gamma Distribution*" and is known for being a generalization of the Gamma and Weibull distributions. Additionally, as shown in Crooks (2010), the GG distribution is a particular case of the four-parameter income function proposed by Amoroso (1925, p. 124).

Where, after completing and reparameterizing the Amoroso function so that it was a probability density function, Stacy (1962) equals the location parameter to zero to finally obtain the GG distribution formula. In Rigby et al. (2017, p. 284), the authors present the probability function of the GG distribution as

$$f_y(y) = \frac{|\nu| z^\theta \theta^\theta e^{-z\theta}}{\Gamma(\theta) y} \tag{7.19}$$

where

$$z = \left(\frac{y}{\mu}\right)^\nu \text{ and } \theta = \frac{1}{\sigma^2 \nu^2} \tag{7.20}$$

for $y > 0$, $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$, with $\nu \neq 0$. Furthermore, Rigby et al., (2017, p. 284) point out that the mean, variance, skewness and excess kurtosis of Y are given as

$$
\begin{aligned}
\mathbb{E}(Y) &= \begin{cases} \frac{\mu\Gamma\left(\theta+\frac{1}{\nu}\right)}{\theta^{1/\nu}\Gamma(\theta)}, & \text{if } \{\nu > 0\} \text{ or } \{\nu < 0,\ \sigma^2|\nu| < 1\} \\ \infty, & \text{if } \nu < 0 \text{ and } \sigma^2|\nu| \geq 1 \end{cases} \\
\\
Var(Y) &= \begin{cases} \frac{\mu_2}{\theta^{2/\nu}\Gamma(\theta)^2}, & \text{if } \{\nu > 0\} \text{ or } \left\{\nu < 0,\ \sigma^2|\nu| < \frac{1}{2}\right\} \\ \infty, & \text{if } \nu < 0 \text{ and } \sigma^2|\nu| \geq \frac{1}{2} \end{cases} \\
\text{where } \mu_2 &= \mu^2\left[\Gamma\left(\theta+\frac{2}{\nu}\right)\Gamma(\theta) - \Gamma\left(\theta+\frac{1}{\nu}\right)^2\right] \\
\\
Skew(Y) &= \begin{cases} \frac{\mu_3}{\theta^{3/\nu}\Gamma(\theta)^3 Var(Y)^{1.5}}, & \text{if } \{\nu > 0\} \text{ or } \left\{\nu < 0,\ \sigma^2|\nu| < \frac{1}{3}\right\} \\ \infty, & \text{if } \nu < 0 \text{ and } \sigma^2|\nu| \geq \frac{1}{3} \end{cases} \\
\text{where } \mu_3 &= \mu^3\left[\Gamma\left(\theta+\frac{3}{\nu}\right)\Gamma(\theta)^2 - 3\Gamma\left(\theta+\frac{2}{\nu}\right)\Gamma\left(\theta+\frac{1}{\nu}\right)\Gamma(\theta) + 2\Gamma\left(\theta+\frac{1}{\nu}\right)^3\right] \\
\\
Kurt(Y) &= \begin{cases} \frac{\mu_4}{\theta^{4/\nu}\Gamma(\theta)^4 Var(Y)^2} - 3, & \text{if } \{\nu > 0\} \text{ or } \left\{\nu < 0,\ \sigma^2|\nu| < \frac{1}{4}\right\} \\ \infty, & \text{if } \nu < 0 \text{ and } \sigma^2|\nu| \geq \frac{1}{4} \end{cases} \\
\text{where } \mu_4 &= \mu^4\left[\Gamma\left(\theta+\frac{4}{\nu}\right)\Gamma(\theta)^3 - 4\Gamma\left(\theta+\frac{3}{\nu}\right)\Gamma\left(\theta+\frac{1}{\nu}\right)\Gamma(\theta)^2\right. \\
&\quad \left. + 6\Gamma\left(\theta+\frac{2}{\nu}\right)\Gamma\left(\theta+\frac{1}{\nu}\right)^2\Gamma(\theta) - 3\Gamma\left(\theta+\frac{1}{\nu}\right)^4\right]
\end{aligned} \tag{7.21}
$$

being $\Gamma(\cdot)$ the gamma function.

Bibliography

Albrecher, H., Beirlant, J., and Teugels, J. (2017). *Reinsurance: Actuarial and statistical aspects*. John Wiley & Sons.

Albrecher, H., Hipp, C., and Kortschak, D. (2010). Higher-order expansions for compound distributions and ruin probabilities with subexponential claims. *Scandinavian Actuarial Journal*, *2010*(2), 105–135.

Amoroso, L. (1925). Ricerche intorno alla curva dei redditi. *Annali Di Matematica Pura Ed Applicata*, *2*(1), 123–159.

Beard, R., Pentikáinen, T., and Pesonen, E. (1984). *Risk theory: The stochastic basis of insurance* (Vol. 3). London, Chapman; Hall.

Behrens, C. N., Lopes, H. F., and Gamerman, D. (2004). Bayesian analysis of extreme events with threshold estimation. *Statistical Modelling*, *4*(3), 227–244.

Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. (2004). *Statistics of extremes: Theory and applications*. John Wiley & Sons.

Biagini, F., and Ulmer, S. (2009). Asymptotics for operational risk quantified with expected shortfall. *ASTIN Bulletin*, *39*(2), 735–752.

Bingham, N., Goldie, C., and Teugels, J. (1989). *Regular variation* (Vol. 27). Cambridge University Press.

Bowers, N., Gerber, H., Hickman, J., Jones, D., and Nesbitt, C. (1997). Actuarial mathematics. *Society of Actuaries*, *2*.

Böcker, K., and Klüppelberg, C. (2005). Operational var: A closed-form approximation. *Risk, Operational Risk*, *18*(12), 90–93.

Böcker, K., and Sprittulla, J. (2006). Operational var: Meaningful means. *Risk, Brief Communication*, 96–98.

Bryson, M. C. (1974). Heavy-tailed distributions: Properties and tests. *Technometrics*, *16*(1), 61–68.

Cai, J., and Tan, K. S. (2007). Optimal retention for a stop-loss reinsurance under the var and cte risk measures. *ASTIN Bulletin*, *37*(1), 93–112.

Centeno, M., and Simões, O. (2009). Optimal reinsurance. *RACSAM - Revista de La*

Real Academia de Ciencias Exactas, Fisicas Y Naturales. Serie A. Matematicas, *103*(2), 387–404.

Chernobai, A., Rachev, S., and Fabozzi, F. (2015). Composite goodness-of-fit tests for left-truncated loss samples. In *Handbook of financial econometrics and statistics* (pp. 575–596). Springer.

Chicaíza, L., and Cabedo, D. (2009). Using the Black-Scholes method for estimating high-cost illness insurance premiums in Colombia. *Innovar*, *19*(33), 119–130.

Consul, P. C., and Jain, G. C. (1973). A generalization of the poisson distribution. *Technometrics*, *15*(4), 791–799.

Crooks, G. E. (2010). The amoroso distribution. *arXiv Preprint arXiv:1005.3274*.

Deelstra, G., and Plantin, G. (2014). *Risk theory and reinsurance*. Springer-Verlag London.

Degen, M. (2010). The calculation of minimum regulatory capital using single-loss approximations. *The Journal of Operational Risk*, *5*(4), 3–17.

De Haan, L., and Ferreira, A. (2006). *Extreme value theory: An introduction*. Springer Science & Business Media.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial theory for dependent risks: Measures, orders and models*. John Wiley & Sons.

Drees, H., De Haan, L., and Resnick, S. (2000). How to make a hill plot. *The Annals of Statistics*, *28*(1), 254–274.

Embrechts, P., Klúppelberg, C., and Mikosch, T. (1997). *Modelling extremal events: For insurance and finance* (Vol. 33). Applications of Mathematics – Stochastic Modelling; Applied Probability.

Feller, W. (1978). *Introducción a la teoría de probabilidades y sus aplicaciones* (Vol. 2). Limusa - Wiley, S.A.

Ferrari, S. L., and Fumes, G. (2017). Box–cox symmetric distributions and applications to nutritional data. *AStA Advances in Statistical Analysis*, *101*(3), 321–344.

Giraldo, N. (2018). Gestion cuantitativa de riesgos, con aplicaciones en r. *Notas de Clase Disponibles En Dropbox*, Universidad Nacional de Colombia.

Girón, L., and Herrera, F. (2015). Cálculo y comparación de la prima de un reaseguro de salud usando el modelo de opciones de Black-Scholes y el modelo actuarial. *Revista de Economía Del Rosario. Julio-Diciembre*, *18*(2), 211–248. https://doi.org/10.12804/rev.econ.rosario.18.02.2015.03

Hess, C. (2011). Can the single-loss approximation method compete with the standard monte carlo simulation technique? *The Journal of Operational Risk*, *6*(2), 31–43.

Hill, B. (1975). A simple general approach to inference about the tail of a distribution. *The Annals of Statistics*, 1163–1174.

Hu, Y., and Scarrott, C. (2018). evmix: An R package for extreme value mixture modeling, threshold estimation and boundary corrected kernel density estimation. *Journal of Statistical Software*, *84*(5), 1–27. https://doi.org/10.18637/jss.v084.i05

Johnson, N. L., Kemp, A. W., and Kotz, S. (2005). *Univariate discrete distributions* (Vol. 444). John Wiley & Sons.

Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. (2008). *Modern actuarial risk theory: Using r* (Vol. 2). Springer Science & Business Media.

Klugman, S., Panjer, H., and Willmot, G. (2012). *Loss models: From data to decisions* (Vol. 4). John Wiley & Sons.

MacDonald, A., Scarrott, C. J., Lee, D., Darlow, B., Reale, M., and Russell, G. (2011). A flexible extreme value mixture model. *Computational Statistics & Data Analysis*, *55*(6), 2137–2157.

Mandelbrot, B. B. (1997). *Fractals and scaling in finance: Discontinuity, concentration, risk*. Springer Science & Business Media.

McDonald, J. B., and Xu, Y. J. (1995). A generalization of the beta distribution with applications. *Journal of Econometrics*, *66*(1-2), 133–152.

Meyer, D., Dimitriadou, E., Hornik, K., Weingessel, A., and Leisch, F. (2018). *E1071: Misc functions of the department of statistics, probability theory group (formerly: E1071), tu wien*. Retrieved from https://CRAN.R-project.org/package=e1071

Mikosch, T. (1999). *Regular variation, subexponentiality and their applications in probability theory*. Eindhoven University of Technology.

Mikosch, T. (2009). *Non-life insurance mathematics: An introduction with the poisson process*. Springer Science & Business Media.

Mora, A. (2010). Estimadores del índice de cola y el valor en riesgo. *Cuadernos de Administración, Universidad del Valle, Julio-Diciembre*(44), 71–88.

Moscadelli, M. (2004). *The modelling of operational risk: Experience with the analysis of the data collected by the basel committee* (No. 517). Temi di discussione, Banca D'Italia.

Opdyke, J. (2014). Estimating operational risk capital with greater accuracy, precision and robustness. *The Journal of Operational Risk*, *9*(4), 3–79.

Perez, J. (2018). *DistMom: Moments or cumulants for any distribution*.

Peters, G., Targino, R., and Shevchenko, P. (2013). Understanding operational risk capital approximations: First and second orders. *Journal of Governance and Reg-*

ulation, *2*, 58–78.

R Core Team. (2018). *R: A language and environment for statistical computing.* Vienna, Austria: R Foundation for Statistical Computing. Retrieved from https://www.R-project.org/

Resnick, S. I. (2007). *Heavy-tail phenomena: Probabilistic and statistical modeling.* Springer Science & Business Media.

Resnick, S., and Stărică, C. (1997). Smoothing the hill estimator. *Advances in Applied Probability*, *29*(1), 271–293.

Rigby, B., Stasinopoulos, M., Heller, G., and Voudouris, V. (2014). The distribution toolbox of gamlss. *The GAMLSS Team.*

Rigby, R., and Stasinopoulos, D. (2004). Smooth centile curves for skew and kurtotic data modelled using the box–cox power exponential distribution. *Statistics in Medicine*, *23*(19), 3053–3076.

Rigby, R., and Stasinopoulos, D. (2005). Generalized additive models for location, scale and shape,(with discussion). *Applied Statistics*, *54.3*, 507–554.

Rigby, R., Stasinopoulos, D., and Akantziliotou, C. (2008). A framework for modelling overdispersed count data, including the poisson-shifted generalized inverse gaussian distribution. *Computational Statistics & Data Analysis*, *53*(2), 381–393.

Rigby, R., Stasinopoulos, D., Heller, G., and De Bastiani, F. (2017). Distributions for Modelling Location, Scale, and Shape: Using GAMLSS in R. *URL www. gamlss. org.(last accessed 12 August 2018).*

Scrucca, L. (2004). Qcc: An r package for quality control charting and statistical process control. *R News*, *4/1*, 11–17. Retrieved from https://cran.r-project.org/doc/Rnews/

Stacy, E. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, *33*(3), 1187–1192.

Stasinopoulos, D., and Rigby, R. (2007). Generalized additive models for location scale and shape (gamlss) in r. *Journal of Statistical Software*, *23*(7), 1–46.

Stasinopoulos, M., Rigby, R., Heller, G., Voudouris, V., and De Bastiani, F. (2017). *Flexible regression and smoothing: Using gamlss in r.* Chapman; Hall/CRC.

Willmot, G. (1987). The poisson-inverse gaussian distribution as an alternative to the negative binomial. *Scandinavian Actuarial Journal*, *1987*(3-4), 113–127.

Wolter, T. (2012). *Truncgof: GoF tests allowing for left truncated data.* Retrieved from https://CRAN.R-project.org/package=truncgof

Zhu, H. (2018). *KableExtra: Construct complex table with 'kable' and pipe syntax.* Retrieved from http://haozhu233.github.io/kableExtra/, https://github.com/haozhu233/kableExtr