S-Coherent measures pairs and Orthogonal polynomials with respect to Sobolev products. Applications. (Pares de Medidas s-Coherentes y Polinomios Ortogonales Respecto a Productos de Sobolev. Aplicaciones).

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A mis padres y a Liliana B.

Los verdaderos logros no son consecuencia del azar ni de un esfuerzo individual, un legítimo y desinteresado estímulo de alguien más te ha impulsado por el camino correcto, la gratitud es lo único que te mantendrá por esa senda.

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Abstract

In this dissertation, the concept of $s-$*coherence*, or *symmetric* $(1,1)-$*coherence*, of pairs of quasi-definite linear functionals, and the polynomials orthogonal with respect to certain Sobolev inner product type play a preponderant role. The concept of symmetric $(1,1)-$ coherent pair is defined as follows. Let u and v denote two symmetric quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ will denote their respective sequences of monic orthogonal polynomials, (SMOP in short). Suppose that there exist sequences of non-zero real numbers $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, with $a_n b_n \neq 0$, such that

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0, \tag{1}$$

holds. Then the pair $\{u, v\}$ is said to be a *Symmetric* $(1,1)-$*Coherent Pair*. This concept is introduced in [34] as a natural extension of the concept of symmetric coherent pairs of quasi-definite linear functionals studied in [55].

The structure of this work is as follows. First, a classification of symmetric $(1,1)-$coherent pairs is stated by using a symmetrization process. In addition, we study how from (1), and using the symmetrization process, we can arrive to a non-coherence algebraic relation. Then, the corresponding inverse problem is analyzed exhaustively. After this, we consider the Sobolev inner product

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0, \tag{2}$$

where we assume that u and v are positive-definite, with μ_0 and μ_1 as the respective positive Borel measures and $\left\{S_n^\lambda\right\}_{n\geq 0}$ as the Sobolev orthogonal polynomials associated with (2). So, the algebraic relation

$$S_{n+3}^\lambda(x) + \eta_n(\lambda) S_{n+1}^\lambda(x) = P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x), \tag{3}$$

is considered, where special attention is placed on the so called *Sobolev coefficients* $\{\eta_n(\lambda)\}_{n\geq 0}$. Then, their recurrence properties as well as those of the corresponding Sobolev norms $\left\{\left\|S_n^\lambda\right\|_S^2\right\}_{n\geq 0}$ are studied. On the other hand, the particular symmetric $(1,1)-$coherent pair $\{\mu_0, \mu_1\}$, $d\mu_0 = e^{-x^2}dx$, $d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx$, is taken into account. In this way, limit behavior of Soboleb coefficients and the asymptotic properties of Sobolev polynomials are deeply studied. Finally, we exhibit an algorithm to compute Fourier coefficients in expansions of functions that belong to the Sobolev space $W_2^1[\mathbb{R}, \mu_0, \mu_1]$ by using Sobolev polynomials. In order to do this, we follow the ideas presented in [55].

Keywords: Orthogonal Polynomials, Symmetric $(1,1)-$coherent pairs, Sobolev Orthogonal Polynomials.

Resumen

En esta disertación, el concepto de $s-$*coherencia*, o $(1,1)-$*coherencia simétrica*, de pares de funcionales lineales regulares y los polinomios ortogonales con respecto a cierto producto interno de tipo Sobolev, juegan un papel preponderante. El concepto de par simétrico $(1,1)-$ coherente es definido de la siguiente forma. Sean u y v dos funcionales lineales, simétricos y regulares, donde $\{P_n\}_{n\geq 0}$ y

$\{R_n\}_{n\geq 0}$ representan sus respectivas sucesiones de polinomios ortogonales mónicos, (para ser breves escribiremos SPOM). Supongamos que existen sucesiones no nulas de números reales $\{a_n\}_{n\geq 0}$ y $\{b_n\}_{n\geq 0}$, with $a_n b_n \neq 0$, tales que

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0,$$

es satisfecha. Entonces el par $\{u, v\}$ se denomina *par simétrico* $(1,1)-$*Coherente*. Este concepto es introducido en [34] como una extensión natural del concepto de *par simétrico coherente* estudiado en [55]. La estructura de este trabajo es la siguiente. Primero, una clasificación de pares simétricos $(1,1)-$coherentes es establecida usando cierto proceso de simetrización. Adicionalmente estudiamos cómo de (1), y usando el proceso de simetrización, podemos llegar a una interesante relación algebraica no coherente. El problema inverso asociado a esta relación es analizado exhaustivamente. Luego, consideramos el producto interno de tipo Sobolev

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0,$$

donde asumimos que u y v son definidos positivos con μ_0 y μ_1 como las respectivas medidas de Borel y $\{S_n^\lambda\}_{n\geq 0}$ como la SPOM asociada con (2). Entonces la relación algebraica

$$S_{n+3}^\lambda(x) + \eta_n(\lambda) S_{n+1}^\lambda(x) = P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x),$$

es considerada, donde especial atención es puesta en los llamados *coeficientes de Sobolev* $\{\eta_n(\lambda)\}_{n\geq 0}$. Entonces, sus propiedades de recurrencia como las de las respectivas normas de Sobolev $\left\{\left\|S_n^\lambda\right\|_S^2\right\}_{n\geq 0}$ son estudiadas. De otro lado, el caso particular del par simétrico $(1,1)-$coherente $\{\mu_0, \mu_1\}$, $d\mu_0 = e^{-x^2}dx$, $d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx$, es tenido en cuenta. Así, el comportamiento límite de los coeficientes de Sobolev y propiedades asintóticas de los polinomios de Sobolev son estudiados exhaustivamente. Finalmente exhibimos un algoritmo para calcular los coeficientes de Fourier en expansiones de funciones en el espacio de Sobolev $W_2^1[\mathbb{R}, \mu_0, \mu_1]$ a través de polinomios de Sobolev. Para este fin seguimos las ideas planteadas en [55].

Palabras Clave: Polinomios Ortogonales, Pares Simétricos $(1,1)-$coherentes, Polinomios Ortogonales de Sobolev.

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Introduction

0.1 Historical introduction

Since the so-called *Sobolev Orthogonality* is a recurrent topic in this dissertation, we want to make a summary of both, the historical progress of this concept and the main contributions, as the best way to highlight their influence in the development of our work.

In last decades special attention has been paid to the so-called *Sobolev Orthogonality* defined by the inner product

$$\langle f, g \rangle_S = \sum_{i=0}^{m} \int_{\mathbb{R}} f^{(i)}(x) g^{(i)}(x) d\mu_i(x), \tag{4}$$

where every μ_j represents a positive Borel measure supported on an infinite subset of the real line. Such an inner product is known in the literature as a *Sobolev inner product.*

In 1947 the foundations of the theory of Sobolev Orthogonality are instituted with the pioneering work [67] of D. C. Lewis. He proposed the following minimum problem:

Problem 1. *Given* $\alpha_i(x)$, $i = 0, 1, 2, \cdots, k-1$, *non-decreasing monotonic functions on the interval* $[a, b]$, *and a function* f *that satisfies certain regularity conditions, determine the polynomial* Q_n *with* $\deg Q_n \leq n$ *such that*

$$\sum_{i=0}^{p} \int_a^b \left[f^{(i)}(x) - Q_n^{(i)}(x) \right]^2 d\alpha_i(x),$$

is minimal.

In such a paper, the least squares problem has been proposed in terms of Stieltjes integrals and the solution contains an integral representation for the remainder given in terms of the so-called Peano Kernel. Of course, Lewis did not use Sobolev orthogonal polynomials explicitly. Notice that the Sobolev orthogonality can be defined in terms of Stieltjes integrals. More precisely, a Sobolev inner product has the form

$$\langle f, g \rangle_S = \sum_{i=0}^{m} \int_a^b f^{(i)}(x) g^{(i)}(x) d\alpha_i(x),$$

where (a, b) is a finite or infinite interval and every α_j represents a non-decreasing and bounded function on $[a, b]$, such that for every j the set

$$\vartheta(\alpha_j) = \{ x | \, \alpha_j(x+\epsilon) - \alpha_j(x-\epsilon) > 0, \text{ for every } \epsilon > 0 \},$$

is infinite and

$$\mu_k^{(j)} = \int_a^b x^k d\alpha_j < \infty, \text{ for } k \in \mathbb{N}.$$

In early 60s, the German mathematician P. Althammer presented his first work, (see [12]) based on the seminal paper of Lewis and reformulated the Lewis's problem as follows: given the inner product

$$\langle f,g\rangle_S=\sum_{i=0}^{m}\int_a^b f^{(i)}(x)g^{(i)}(x)w_i(x)dx, \tag{5}$$

where the $w_i's$ are weight functions in $[a,b]$, and a function f defined in $[a,b]$, to determine

$$\min_{Q\in\mathbf{P}_n}\|f-Q\|_S\,.$$

where $\mathbf{P}_n$ represents the linear space of polynomials with degree less than or equal to n and $\|.\|_S$ is the norm induced by $\langle,\rangle_S$. Then, if $\{S_n\}_{n\geq 0}$ is the sequence of polynomials orthogonal with respect to (5), the polynomial Q^*, where the minimum is achieved, will be a linear combination of Sobolev orthogonal polynomials, namely,

$$Q^*(x)=\sum_{k=0}^{n}a_kS_k(x),\ \text{with } a_k=\langle f,S_k\rangle_S\,.$$

Althammer showed also a distinctive feature of the Sobolev orthogonality with respect to localization of the zeros of the corresponding orthogonal polynomials. Indeed, taking into account the inner product (5) with $m=1$, $a=-1$, $b=1$, $w_0(x)=1$, and

$$w_1(x)=\begin{cases}10, & -1\leq x<0,\\ 1, & 0\leq x\leq 1,\end{cases}$$

he proved that one zero of S_2 is located outside of $[-1,1]$. It is well known that this fact does not occur for standard orthogonality with respect to a probability Borel measure supported on the real line. This "incident" led Althammer to work with the Sobolev inner product

$$\langle f,g\rangle_S=\int_{-1}^{1}f(x)g(x)dx+\lambda\int_{-1}^{1}f'(x)g'(x)dx,\qquad \lambda>0. \tag{6}$$

In this case, the respective polynomials S_n, $n\geq 0$, are a generalization of the *Legendre polynomials*, (a particular case of *classical Jacobi polynomials*).

Thus, from the works of Lewis and Althammer, the Sobolev orthogonality emerges turning the study of zeros into a recurrent topic in later contributions. In the next decade, the papers [24], [32] and [51] stand out. In particular, in [24] a generalization of the classical Laguerre polynomials orthogonal with respect to the inner product

$$\langle f,g\rangle_S=\int_0^{\infty}f(x)g(x)e^{-x}dx+\lambda\int_0^{\infty}f'(x)g'(x)e^{-x}dx,\qquad \lambda>0,$$

is studied.

An interesting general approach to the Sobolev orthogonality is given in [107] with the inner product

$$\langle f,g\rangle_S=\sum_{\upsilon,\mu=0}^{\infty}\int_a^b v_{\upsilon\mu}(x)p^{(\upsilon)}(x)q^{(\mu)}(x)d\eta(x), \tag{7}$$

where η is a classical measure, (Hermite, Laguerre or Jacobi), and $\upsilon,\mu\in\mathbb{N}\cup\{0\}$. $v_{\upsilon\mu}$ are polynomials that, under certain conditions and the application of integration by parts, allow turn the inner product (7) into a standard inner product. Besides, (7) generalizes the inner products studied by Althammer and Brenner. However, it took two decades before the interest in the Sobolev orthogonality wake up again. The Sobolev orthogonal polynomials revived in early 90's thanks to the influential work of A. Iserles, P. E. Koch, S. P. Norsett and J. M. Sanz-Serna and their paper [55], where the polynomials orthogonal with respect to the Sobolev inner product

$$\langle f,g\rangle_S=\int_{\mathbb{R}} f(x)g(x)d\mu_0(x)+\lambda\int_{\mathbb{R}} f'(x)g'(x)d\mu_1(x),\quad \lambda\geq 0,$$

were studied. Here μ_0 and μ_1 are positive Borel measures on the real line. The corner stone in [55] is the presentation of the fertile concept of **Coherent Pair** of measures. To be more precise, a pair $\{\mu_0,\mu_1\}$ of positive Borel measures supported on infinite subsets of the real line is a coherent pair if there exist nonzero constants a_n such that $Q_n(x)=\dfrac{P'_{n+1}(x)}{n+1}+a_n\dfrac{P'_n(x)}{n}$, $n>0$, where $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are the sequences of monic polynomials orthogonal with respect to μ_0 and μ_1, respectively. In an analogous way the concept of *Symmetric Coherent Pairs* can be defined when μ_0 and μ_1 are symmetric, i. e. invariant under the transformation $x\to -x$. Among others, interesting algebraic connections between Sobolev polynomials and polynomials orthogonal with respect to μ_0 are obtained, and an efficient algorithm to compute Fourier-Sobolev coefficients in expansions with Sobolev polynomials is presented. A problem that arises immediately is to identify all coherent pairs as well as the symmetric coherent pairs. It should also be pointed out that the work [81] constitutes the foundation for the solution of this problem, task that, later, will be completed by H. G. Meijer in [96], where it was proved that one of the two measures must be classical and its companion is a rational perturbation of it. On Sobolev orthogonal polynomials and coherent pairs, (and *symmetric coherent pairs*), a large and interesting number of papers enrich the literature nowadays. We highlight [2], [9], [10], [13], [28], [29], [30], [74], [76], [78], [79], [85], [91], [93], [97], [98], [104], [105], among others.

At the end of 20th century a wide range of investigations are carried out which are related with the so-called discrete Sobolev inner product or Sobolev inner product of the type II, i. e. when in (4), μ_0 has continuous support and μ_i, for $i=1,...,m$ are supported on finite subsets. In [19] the inner product

$$\langle p,q\rangle_S=\int_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha}dx+\sum_{k=0}^{1}M_k\left(p^{(k)}(-1)q^{(k)}(-1)+p^{(k)}(1)q^{(k)}(1)\right),$$

with $M_k\geq 0$, $\alpha>-1$, is considered. Notice that the respective Sobolev orthogonal polynomials are a generalization of the classical symmetric Gegenbauer polynomials. An expression for every Sobolev polynomial in terms of differential operators is obtained as well as some properties of their zeros and their representation in terms of a hypergeometric series, (*Hypergeometric Representation*). A similar study for polynomials orthogonal with respect to the *Sobolev-Laguerre inner product*

$$\langle p,q\rangle_S=\int_{0}^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx+\sum_{v=0}^{N}M_v p^{(v)}(0)q^{(v)}(0),\ M_v\geq 0,\ \alpha>-1,$$

is presented in [61], where a $(2N+3)-$term recurrence relation for the Sobolev polynomials is obtained. Analytic properties of Sobolev polynomials orthogonal with respect to the above inner product are also studied in [38],($N=1$, $M_0=0$.), [62], [63] and [92]. In such a framework, the authors find, among others, a *Holonomic Equation*, (i.e., a second order ODE with polynomial coefficients), satisfied by the Sobolev polynomials and, as a consequence, the electrostatic interpretation of their zeros is deduced. Asymptotics for Sobolev polynomials orthogonal with respect to

$$\langle p,q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Mp^{(j)}(0)q^{(j)}(0), \quad M>0, \tag{8}$$

are analyzed in [37]. The case

$$\langle p,q\rangle_S = \int_{\mathbb{R}} p(x)q(x)w(x)dx + \lambda p^{(v)}(c)q^{(v)}(c),$$

is analyzed in [82] when $\lambda \geq 0$, $c \in \mathbb{R}$ and $v \in \mathbb{N}\cup\{0\}$. If the weight w is semiclassical, the authors found a second order differential equation satisfied by the Sobolev polynomials, as well as a $(2v+3)-$term recurrence relation is deduced for these polynomials. This is a pioneering work since the value c can be located outside the orthogonality interval. In [77] a thorough study of some properties of zeros is presented as well as their behavior with respect to λ.

The idea of considering discrete measures supported on points outside the support of the measure $d\mu$ is followed in the work [2], where the inner product

$$\langle p,q\rangle_S = \int_E p(x)q(x)d\mu + \sum_{v=0}^{N} M_v p^{(v)}(\zeta)q^{(v)}(\zeta), \; M_v \geq 0, \tag{9}$$

is studied with $\zeta \notin E$. The authors deduced that the number of zeros outside the convex hull of the support of the measure depends on N instead of the order of derivatives. Besides, they obtain interlacing properties of zeros. When $d\mu = x^\alpha e^{-x}dx$, (classical Laguerre measure), analytic properties such as connection formula, hypergeometric representation and Holonomic Equation are studied in [43] for the polynomials orthogonal with respect to (9) with $\zeta < 0$ and $N=0$. The case $N=1$ is studied in [100] where an electrostatic model for the zeros of Sobolev polynomials is deduced. Also, in the Laguerre setting, and for $\zeta<0$, $M_0=0$ and $N=1$, in (9), asymptotic properties such as relative outer asymptotics and Mehler-Heine and Plancherel-Rotach formulas for rescaled Sobolev polynomials are given. In the above context, in [99] similar asymptotic properties are studied when $M_v=0$ for $v=0,1,...,N-1$. A very interesting variant of the discrete Laguerre-Sobolev inner products is given in [70] through the inner product

$$\langle p,q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + M_n p^{(j)}(0)q^{(j)}(0), \; \alpha > -1,$$

where $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers such that

$$\lim_{n\to\infty} M_n n^\beta = M, \; \beta \in \mathbb{R}.$$

This constitutes another generalization of the inner product (8). A Mehler-Heine type formula is deduced and, consequently, also the asymptotic behavior of the zeros. The authors made a similar study for the Jacobi-Sobolev case in [71], and they present a more general approach for any positive Borel measure on the real line in [72].

Other useful references regarding analytical and asymptotic properties of Sobolev type orthogonal polynomials in the discrete case are [7], [13] [27],[45], [68], [85]. In order to conclude this brief introduction the following surveys, sorted according to their appearance in the literature, are especially recommended: [8], [86], [89], [90], [95].

0.2 Structure of the manuscript and original contributions

In the previous section we have highlighted the way as, after two decades, the concept of Coherent pair of measures constitutes a starting point in order to emerge the interest in the Sobolev Orthogonality in early 90s. In this work we are interested in studying some analytic and asymptotic aspects of a natural generalization known in the literature as Symmetric (1,1)-Coherence or S-coherence, which is introduced in [34]. Also we want to present the state of the art around this topic. On the other hand, and inspired by the contributions that helped to solve the problem of the determination of all Coherent Pairs and Symmetric Coherent Pairs, we will deduce a classification of symmetric $(1,1)-$coherent pairs for which the Symmetrization Process plays a central role. In order to this, we will show how to get in a natural way a non-coherent algebraic relation which we will study extensively.

Finally, given the relation between Sobolev type orthogonal polynomials and the symmetric $(1,1)-$ coherence, we will study the asymptotic behavior of the Sobolev orthogonal polynomials associated with a particular symmetric $(1,1)-$coherent pair. In this way, the structure of this dissertation is as follows.

In Chapter 1, the basic background about moment functionals and orthogonal polynomials on the real line is presented. A special emphasis on semiclassical linear functionals is given and the symmetrization process for linear functionals is deeply analyzed. Finally, the main results of asymptotic behavior for classical Laguerre and Hermite polynomials are presented.

In Chapter 2, we define formally the Sobolev type inner product and the concepts of Coherent Pair and Symmetric Coherent pair as a necessary background. We exhibit the main analytic and asymptotic aspects from these concepts.

In Chapter 3, we describe the main results about a natural generalization of the concept of coherent pairs that in the literature is known as $(1,1)-$coherent pair introduced in [33]. We summarize the classification of all $(1,1)-$coherent pairs. A detailed discussion of the subsequent generalizations is presented at the end of the chapter.

In Chapter 4, we study exhaustively the concept of $(1,1)-$symmetric coherent pair or $s-$coherent pair. We propose a classification of $(1,1)-$symmetric coherent pairs through the symmetrization process. In this way we propose and study rigorously an inverse problem associated with non-coherent pairs. The results of this chapter appear in the references [40] and [41].

In Chapter 5, we consider the symmetric $(1,1)-$coherent pair

$$d\mu_0 = e^{-x^2}dx, \qquad d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx, \quad a, b \neq 0, \ a, b \in \mathbb{R}_+.$$

We analyze the relative asymptotics of the polynomials orthogonal with respect to Sobolev inner product

$$\langle p(x), q(x) \rangle_S := \int p(x)q(x)d\mu_0(x) + \lambda \int p'(x)q'(x)d\mu_1(x), \quad \lambda > 0,$$

on the space of polynomials with real coefficients and we obtain Mehler-Heine type formulas. The results of this chapter yield the publication [39].

In Chapter 6 we present applications of some results and ideas exposed throughout the manuscript. In this way we present an algorithm that arises in a natural way to compute the Fourier coefficients of expansions of Sobolev polynomials for functions in certain Sobolev spaces. The contents of this chapter are part of the reference [42]. Finally in Chapter 7 the conclusions of this dissertation and some open problems arising from its elaboration are presented.

Chapter 1

Orthogonal Polynomials on the real Line

This Chapter is devoted to summarize the elements of the general theory of orthogonal polynomials on the real line, as well as basic ideas and tools that will be needed to build the main results of this dissertation. We call it "standard orthogonality". Also we will establish the necessary notation that we will follow in the sequel. For the reader, the only background are first undergraduate courses in real and complex analysis, functional analysis and measure theory. The structure of this chapter is as follows. In Section 1.1 we will deal with moment functionals and we will present some examples including the *quasi-definite linear functionals*. In Section 1.2 we will focus our attention on sequences of polynomials orthogonal with respect to quasi-definite linear functionals. Section 1.3 will be dedicated to *semiclassical linear functionals*, some characterizations and, of course, we will emphasize on the *classical case*. Besides, we will exhibit the classification of semiclassical functionals of class 1 and 2, respectively. In Section 1.4 we will discuss the *symmetrization process* and we will deal with the semiclassical character of *symmetric linear functionals*. In addition we will present the classification of semiclassical symmetric linear functionals of class 1 and 2. Finally, in Section 1.5 we will approach the *Relative Asymptotics* of orthogonal polynomials as well as classical asymptotic results of Laguerre and Hermite polynomials.

1.1 Moment functionals

Let $\mathbf{P}$ be the linear space of polynomials with complex coefficients. $\mathbf{P}_n$ will denote the linear subspace of polynomials of degree at most n. Let u be a linear functional in the algebraic dual space of $\mathbf{P}$. It will be denoted $\mathbf{P}'$. $\langle u, p\rangle$ is the action of the linear functional u on the polynomial $p \in \mathbf{P}$. Let $\{u_n\}_{n\geq 0}$ be a sequence of complex numbers. u is a *moment functional* associated with the *moment sequence* $\{u_n\}_{n\geq 0}$ if u is linear and $u_n = \langle u, x^n\rangle$. A sequence of polynomials $\{P_n\}_{n\geq 0}$, with $\deg P_n = n$, determines an unique sequence of linear functionals $\{\mathbf{p}_n\}_{n\geq 0}$, called *dual basis* associated with $\{P_n\}_{n\geq 0}$, in such a way that $\langle \mathbf{p}_n, P_m\rangle = \delta_{n,m}$, where $\delta_{n,m}$ denotes the Kronecker delta function. In this way every $u \in \mathbf{P}'$ can be expressed in terms of the basis $\{\mathbf{p}_n\}_{n\geq 0}$ as follows

$$u = \sum_{k\geq 0} \langle u, P_k\rangle \, \mathbf{p}_k.$$

On the other hand, if $q \in \mathbf{P}$ and $u \in \mathbf{P}'$, then we define $qu \in \mathbf{P}'$, the *left multiplication*, as

$$\langle qu, p\rangle := \langle u, qp\rangle , \quad p \in \mathbf{P}. \tag{1.1}$$

If $u \in \mathbf{P}'$ and $a, b \in \mathbb{C}$, $b \neq 0$, an affine transformation of u, denoted by $(\tau_a \circ h_b)\, u$, is defined as follows

$$\langle (\tau_a \circ h_b)\, u, p(x)\rangle = \langle u, (h_b \circ \tau_{-a})\, p(x)\rangle = \langle u, p(bx+a)\rangle , \tag{1.2}$$

for every $p \in \mathbf{P}$.

The linear functional $\delta(x-c)$ such that $\langle \delta(x-c), p\rangle := p(c)$, $p \in \mathbf{P}$, $c \in \mathbb{C}$, is said to be the *Dirac delta* linear functional at c.

Given $u \in \mathbf{P}'$, let $\sigma \in \mathbf{P}$ be a polynomial of degree n and denote by $x_i \in \mathbb{C}$, $1 \leq i \leq r$, their zeros with multiplicities n_k, respectively, i.e. $\sum_{k=1}^{r} n_k = n$. Then for every $p \in \mathbf{P}$, we define $\sigma^{-1}(x)u \in \mathbf{P}'$ as follows,

$$\left\langle \sigma^{-1}(x)u, p(x)\right\rangle := \left\langle u, \frac{p(x) - L_\sigma(x;p)}{\sigma(x)} \right\rangle , \tag{1.3}$$

where $L_\sigma(x;p)$ is the interpolatory polynomial

$$L_\sigma(x;p) = \sum_{i=1}^{r} \sum_{j=0}^{n_i - 1} p^{(j)}(x_i) L_{i,j}(x), \tag{1.4}$$

and $L_{i,j}(x)$ is the polynomial of degree at most $n-1$ such that $L_{i,j}^{(k)}(x_l) = \delta_{i,l}\delta_{k,j}$, $i, l = 1, \cdots, r$, and $0 \leq k, j \leq n_i - 1$. If $q \in \mathbf{P}$ and $\sigma(x) = x^2 - \zeta$, $\zeta > 0$, we get

$$L_{x^2-\zeta}(x;q) = \sum_{i=1}^{2} q(x_i) \frac{x^2-\zeta}{(x-x_i)2x_i} = \frac{x^2-\zeta}{2\sqrt{\zeta}} \left(\frac{q(\sqrt{\zeta})}{(x-\sqrt{\zeta})} - \frac{q(-\sqrt{\zeta})}{(x+\sqrt{\zeta})} \right).$$

On the other hand, if $\sigma(x) = (x-\xi)^n$, i.e. σ has a zero of multiplicity n, then for any linear functional u

$$\left\langle (x-\xi)^{-n}u, p(x)\right\rangle = \left\langle u, \frac{p(x) - T_{n-1}^{\xi}(p)(x)}{(x-\xi)^n} \right\rangle , \tag{1.5}$$

where $T_{n-1}^{\xi}(p)$ denotes the Taylor polynomial of degree $n-1$ of the polynomial p around $x = \xi$. When $\xi = 0$, we will write $T_{n-1}(p)$.

Definition 2 (Pochhammer Symbol). *Given $a \in \mathbb{C}$ the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)(a+2)\dots(a+n-1)$, $n \geq 1$, and $(a)_0 = 1$.*

Lemma 3. *Let $p \in \mathbf{P}$ and $q(x) = p\left((x-\xi)^2\right)$. Then for $n \geq 0$ we get*

$$T_n(p)((x-\xi)^2) = T_{2n}^{\xi}(q)(x). \tag{1.6}$$

Proof. The powers of $(x-\xi)$ in q are even non-negative integer numbers, and so the odd powers of $(x-\xi)$ in $T_t^{\xi}(q)(x)$ are zero. Thus

$$T_{2n}^{\xi}(q)(x) = \sum_{k=0}^{2n} \frac{q^{(k)}(\xi)}{k!}(x-\xi)^k = \sum_{k=0}^{n} \frac{q^{(2k)}(\xi)}{(2k)!}(x-\xi)^{2k}.$$

If $p(x)=\sum_{j=0}^{m}a_jx^j$, then $q(x)=\sum_{j=0}^{m}a_j(x-x_j)^{2j}$. Besides $p^{(k)}(x)=\sum_{j=k}^{m}a_j(j-k+1)_kx^{j-k}$ and $q^{(2k)}(x)=\sum_{j=k}^{m}a_j(2j-2k+1)_{2k}(x-\xi)^{2j-2k}$, and, as a consequence, $p^{(k)}(0)=a_kk!$ and $q^{(2k)}(\xi)=a_k(2k)!$.Thus

$$T_{2n}^{\xi}(q)(x)=\sum_{k=0}^{n}\frac{p^{(k)}(0)}{k!}(x-\xi)^{2k}.$$

□

Remark 4. *Since* $T_{2n+1}^{\xi}(q)(x)=T_{2n}^{\xi}(q)(x)$, *then* $T_n(p)((x-\xi)^2)=T_{2n+1}^{\xi}(q)(x)$.

Definition 5. *For any polynomial* q *and* $a\in\mathbb{C}$, *we define the linear operator* $\theta_a:\mathbf{P}\to\mathbf{P}$ *as follows*

$$(\theta_a q)(x)=\frac{q(x)-q(a)}{x-a}. \tag{1.7}$$

Also, $uq\in\mathcal{P}$ denotes the *right-multiplication* of $u\in\mathbf{P}'$ by $q\in\mathbf{P}$, and it is the polynomial defined by

$$(uq)(t):=\left\langle u,\frac{tq(t)-xq(x)}{t-x}\right\rangle,$$

where u acts on the variable x.

Given $p\in\mathcal{P}$, let $p(x)=\sum_{n=0}^{h}g_nx^n$ and $u\in\mathcal{P}'$. After straightforward computations we get

$$(u\theta_0 p)(z):=(u(\theta_0 p))(z)=\sum_{n=0}^{h-1}g_{n+1}\sum_{p=0}^{n}u_pz^{n-p}=\sum_{n=0}^{h-1}\sum_{p=n}^{h-1}(g_{p+1}u_{p-n})\,z^n, \tag{1.8}$$

and, thus, $u\theta_0p$ is a polynomial in z

The $p-th$ derivative of the functional u, $p\in\mathbb{Z}^+\cup\{0\}$, noted by D^pu, is a linear functional such that

$$\langle D^pu,q(x)\rangle:=(-1)^p\left\langle u,q^{(p)}(x)\right\rangle,\quad q\in\mathcal{P}. \tag{1.9}$$

1.1.1 Quasi-definite and Positive-definite linear functionals

Let u be a moment functional and $\{u_n\}_{n\geq 0}$ be the corresponding moment sequence. We define the *Hankel determinant of order* $n+1$

$$\Delta_n^u=\begin{vmatrix} u_0 & u_1 & \cdots & u_n\\ u_1 & u_2 & \cdots & u_{n+1}\\ \vdots & \vdots & \ddots & \vdots\\ u_n & u_{n+1} & \cdots & u_{2n}\end{vmatrix},\quad n\geq 0. \tag{1.10}$$

u is said to be **quasi-definite** or **regular** if the leading principal submatrices of the *Hankel matrix* $(u_{i+j})_{i,j=0}^{\infty}$ are non-singular, i.e. $\Delta_n^u\neq 0$ for $n\geq 0$. u is called **positive-definite** if $\langle u,\pi(x)\rangle>0$ for every nonzero and non-negative real polynomial π. When there is not risk of confusion we will write Δ_n instead of Δ_n^u.

The positive definiteness of a moment functional can be characterized through the associated moment sequence. Namely,

Theorem 6. *([31]). u is positive definite if and only if their moments are real and $\Delta_n > 0$ for $n \geq 0$.*

If u is positive-definite, then there exists a positive Borel measure μ supported on an infinite set $E \subseteq \mathbb{R}$ such that u has an integral representation

$$\langle u, p \rangle = \int_E p(x) d\mu(x), \quad p \in \mathbf{P}.$$

Given a quasi-definite linear functional u on the space $\mathbf{P}(\mathbb{R})$ of polynomials on the real line, a bilinear form $\langle , \rangle_u : \mathbf{P}(\mathbb{R}) \times \mathbf{P}(\mathbb{R}) \to \mathbb{R}$ is defined as $\langle p, q \rangle_u := \langle u, pq \rangle$. If u is positive definite then the bilinear form is an inner product on $\mathbf{P}(\mathbb{R})$ and, as usual, the induced norm will be represented as

$$\|p\|_u = \langle p, p \rangle_u^{1/2} = \left\langle u, p^2 \right\rangle^{1/2} = \left(\int_E p^2(x) d\mu(x) \right)^{1/2},$$

where μ is the positive Borel measure, supported on E, and corresponding to u.

1.2 Orthogonal polynomials

Definition 7. *A sequence $\{P_n\}_{n\geq 0}$ is called an orthogonal polynomial sequence, (OPS in short), with respect to a moment functional u if for $n, m \geq 0$*
i). P_n is a polynomial of degree n.
ii). $\langle u, P_n P_m \rangle = 0$, for $n \neq m$.
iii). $\left\langle u, P_n^2 \right\rangle \neq 0$.

If the leading coefficient of P_n is 1 for every $n \geq 0$, then $\{P_n\}_{n\geq 0}$ is said to be a *monic orthogonal polynomial sequence, (SMOP in short).* The next result gives us conditions that guarantee the existence of an OPS for a given moment functional.

Proposition 8. *([31]). Let u be a moment functional. u is quasi-definite if and only if there exists an OPS $\{P_n\}_{n\geq 0}$ with respect to the functional u.*

Under the conditions of above proposition, if $\{u_n\}_{n\geq 0}$ is the moment sequence associated with u then every monic polynomial P_n can be written as

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} u_0 & u_1 & \cdots & u_n \\ u_1 & u_2 & \cdots & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_n & \cdots & u_{2n-1} \\ 1 & x & & x^n \end{vmatrix}, \quad n \geq 1, \; P_0(x) = 1.$$

Given a quasi-definite moment functional there exist an infinite OPS associated with u. Indeed, if $\{P_n\}_{n\geq 0}$ is an OPS associated with u, then $\{k_n P_n\}_{n\geq 0}$ is also an OPS associated with u for non-zero constants k_n. Then $\{P_n\}_{n\geq 0}$ is uniquely determined if the leading coefficients are fixed. Conversely, if $\{P_n\}_{n\geq 0}$ is an OPS associated with u, for any $k \neq 0$, then $\{P_n\}_{n\geq 0}$ is also an OPS associated with ku. In order to both the quasi-definite moment functional and the OPS are uniquely determined, their normalization will be necessary. In this way, in the sequel we will assume that $\langle u, 1 \rangle = 1$ as well as the respective OPS is monic, unless stated otherwise.

The following result exhibits the existence of a close algebraic connection between the SMOP, its dual basis and its respective linear functional.

Theorem 9. *Let* $\{P_n\}_{n\geq 0}$ *be the SMOP with respect to the quasi-definite linear functional* u *and let* $\{\mathbf{p}_n\}_{n\geq 0}$ *be the respective dual basis. If* $\left\{\mathbf{p}_n^{[1]}\right\}_{n\geq 0}$ *is the dual basis associated to the sequence* $\left\{\dfrac{P'_{n+1}}{n+1}\right\}_{n\geq 0}$, *then*

$$D\mathbf{p}_n^{[1]} = -\frac{(n+1)P_{n+1}(x)}{\|P_{n+1}\|_u^2}u. \tag{1.11}$$

Besides, for every n, $\mathbf{p}_n$ *can be written as*

$$\mathbf{p}_n = \frac{P_n(x)}{\|P_n\|_u^2}u. \tag{1.12}$$

The next theorem describes an important characterization of the orthogonality of a sequence of monic polynomials in terms of a recurrence relation satisfied by their terms. In the literature it is partially accepted that the original version of this result is due to J. Favard with his work [46].

Theorem 10 (Favard's theorem)**.** *([31]). Let* $\{P_n\}_{n\geq 0}$ *be a sequence of monic polynomials.* $\{P_n\}_{n\geq 0}$ *is a MOPS with respect to a quasi-definite linear functional* u *if and only if there exist sequences of numbers* $\{\beta_n\}_{n\geq 1}$ *and* $\{\gamma_n\}_{n\geq 1}$, *with* $\gamma_n \neq 0$ *for* $n \geq 1$, *such that*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1, \tag{1.13}$$

$$P_0(x) = 1, \;\; P_1(x) = x - \beta_0.$$

On the other hand,

$$\beta_n = \frac{\langle u, xP_n^2\rangle}{\langle u, P_n^2\rangle}, \;\; n\geq 0, \quad \gamma_n = \frac{\langle u, xP_nP_{n-1}\rangle}{\langle u, P_{n-1}^2\rangle} = \frac{\langle u, P_n^2\rangle}{\langle u, P_{n-1}^2\rangle}, \;\; n \geq 1.$$

The relation (1.13) is the so-called Three-Term Recurrence Relation, (TTRR in short). A nice survey about the Favard's theorem, its origins and history and its extensions can be seen in [73]. The TTRR is equivalent to the well known and useful *Christoffel-Darboux Identity.*

Theorem 11. *([25], [31]). A SMOP* $\{P_n\}_{n\geq 0}$ *associated with a quasi-definite linear functional* u *satisfies (1.13) if and only if*

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle u, P_k^2\rangle} = \frac{1}{\langle u, P_n^2\rangle}\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y}.$$

Notice that the TTRR can be written in the matrix form $x\mathbf{P}_{n-1} = J_n\mathbf{P}_{n-1} + P_n(x)\mathbf{e}_n$, for $n \geq 1$, where $\mathbf{P}_{n-1} = [P_0, P_1, ..., P_{n-1}]^T$, $\mathbf{e}_n$ is the $n-th$ coordinate vector and J_n is the tridiagonal matrix

$$J_n = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & \cdots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & \cdots & 0 & 0 \\ 0 & 0 & \gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-2} & 1 \\ 0 & 0 & 0 & \cdots & \gamma_{n-1} & \beta_{n-1} \end{bmatrix}.$$

If $\{x_{n,j}\}_{j=1}^{n}$ are the zeros of the polynomial P_n, then it is clear that each $x_{n,j}$ is an eigenvalue of J_n and $[P_0(x_{n,j}), P_1(x_{n,j}), ..., P_{n-1}(x_{n,j})]^T$ is the corresponding eigenvector.

The zeros of orthogonal polynomials have very interesting and nice properties. Next we describe some of them.

Theorem 12. *([31]). Let I be the support of a positive-definite linear functional u and $\{P_n\}_{n\geq 0}$ the respective SMOP. Then*

i). The zeros of P_n are real, simple and located in the interior of the convex hull of I.

ii). (Interlacing property). The zeros of P_n and P_{n+1} mutually separate each other, i.e. if $\{x_{n,j}\}_{j=1}^{n}$ are the n zeros of the polynomial P_n then

$$x_{n+1,j} < x_{n,j} < x_{n+1,j+1}, \quad 1 \leq j \leq n.$$

If u is a positive-definite linear functional, the respective Borel measure μ is supported on $[a, b]$ and if $\zeta \notin (a, b)$, the measure $d\mu_\zeta = |x - \zeta|\, d\mu$ is called a canonical *Christoffel Perturbation*, (see [108], [111] and [112]). In particular, the following result establishes under what conditions the linear functional xu is also quasi-definite.

Proposition 13. *([31]). Let $u \in \mathbf{P}'$ be a quasi-definite linear functional and let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP. If κ is not a zero of P_n, for every $n \geq 1$, $(x-\kappa)u \in \mathbf{P}'$ is quasi-definite and $\left\{\widetilde{P}_n\right\}_{n\geq 0}$, its corresponding SMOP, satisfies*

$$\widetilde{P}_n(x) = (x-\kappa)^{-1}\left(P_{n+1}(x) - \frac{P_{n+1}(\kappa)}{P_n(\kappa)}P_n(x)\right).$$

*Moreover, if u is positive-definite in $[a, b]$ then $(x - \kappa)u$ is also positive-definite on $[a, b]$ if and only if $a \geq \kappa$. The polynomial $\widetilde{P}_n$ is called the $n-th$ monic **Kernel polynomial** corresponding to U whit $K-$parameter κ.*

The above proposition defines a mapping where the starting set and the arriving set are the space of quasi-definite linear functionals. A natural question is, does the application is $1-1$? The answer is no. It is well known that there exist infinitely many SMOP associated with a sequence of Kernel polynomials of $K-$parameter κ.

Let μ be a Borel positive measure supported on an infinite subset of the real line and $M > 0$, $\zeta \in \mathbb{C}$. The measure $d\mu^* := d\mu + M\delta(x - \zeta)$ is called a **Uvarov Perturbation,** (or **Transform)** of μ. This kind of perturbation is introduced in [109].

Theorem 14. *([66]). Let $u \in \mathbf{P}'$ a quasi-definite functional and $\{P_n\}_{n\geq 0}$ its corresponding SMOP. Let $v \in \mathbf{P}'$ defined as $v := u + M\delta(x - a)$, with $M \in \mathbb{C}$, $a \in \mathbb{R}$. Then v is quasi-definite if and only if $d_n := 1 + MK_n(a, a) \neq 0$, $n \geq 0$, where $K_n(x, y)$ is the $n-th$ Kernel polynomial associated to u. Besides $\{R_n\}_{n\geq 0}$, the MOPS associated to v, is given by*

$$R_n(x) = P_n(x) - M\frac{P_n(a)}{d_{n-1}}K_{n-1}(x, a), \quad n \geq 1.$$

Lemma 15. *If $u, v \in \mathbf{P}'$ are related by $xv = u + M\delta(x - a) + N\delta(x)$, $M, a \neq 0$. Then $v = x^{-1}u + \frac{M}{a}\delta(x - a) + \left(\langle v, 1\rangle - \frac{M}{a}\right)\delta(x) - N\delta'(x)$.*

Proof. For any polynomial p, it is enough to consider the action of the linear functional xv, defined as above, on $q(x) := \frac{p(x)-p(0)}{x}$. □

1.3 Semiclassical linear functionals

Let $\widetilde{\phi}$ and $\widetilde{\psi}$ be two nonzero polynomials such that $\deg(\widetilde{\phi}) = m \geq 0$ and $\deg(\widetilde{\psi}) = n \geq 1$ with leading coefficients a_m and b_n, respectively. $(\widetilde{\phi}, \widetilde{\psi})$ is said to be an *admissible pair* if either $m-1 \neq n$ or if $m-1 = n$, then $-ka_{n+1} + b_n \neq 0$ for every $k \in \mathbb{N}$. $u \in \mathbf{P}'$ is said to be *semiclassical* if there exists an admissible pair $(\widetilde{\phi}\ , \widetilde{\psi})$, where $\widetilde{\phi}$ is monic, such that the following differential relation holds,

$$D\left(\widetilde{\phi}u\right) + \widetilde{\psi}u = 0, \qquad \text{(Pearson equation)}. \tag{1.14}$$

A semiclassical linear functional u is said of class s when

$$s = \min_{\Phi} \max\left(\deg\widetilde{\phi} - 2, \deg\widetilde{\psi} - 1\right),$$

where Φ denotes the set of all admissible pairs of nonzero polynomials $\left(\widetilde{\phi}, \widetilde{\psi}\right)$ such that (1.14) holds. With respect to the class of u, we get the next result.

Proposition 16. *([52], [88]). The class of u is the non-negative real number*

$$s = \max\left\{deg(\widetilde{\phi}) - 2, deg(\widetilde{\psi}) - 1\right\},$$

if and only the condition

$$\prod_c \left(\left|\widetilde{\phi}'(c) + \widetilde{\psi}(c)\right| + \left|\left\langle \widetilde{u}, \theta_c\widetilde{\psi} + \theta_c^2\widetilde{\phi}\right\rangle\right|\right) > 0, \tag{1.15}$$

is satisfied, where c is any zero of $\widetilde{\phi}$.

Theorem 17. *(see [88]). If two regular linear functionals u and v are related by an expression of rational type as*

$$p(x)u = r(x)v,$$

where p and r are nonzero polynomials, then u is a semiclassical linear functional if and only if so is v. Moreover, if the class of u is s, then the class of v is at most $s + \deg(p) + \deg(r)$.

Next, some characterizations of semiclassical linear functionals.

Theorem 18. *(see [88]). Let u be a quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ the corresponding SMOP. u is semiclassical of class s if and only if one of the equivalent conditions is satisfied:*

***A)**. There exists a polynomial $\widetilde{\phi}$, with $\deg(\widetilde{\phi}) = t \leq s+2$, such that $\{P_n\}_{n\geq 0}$ satisfies*

$$\widetilde{\phi}(x)\frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} a_{n,k}P_k(x), \quad n \geq s, \tag{1.16}$$

with $a_{n,n-s} \neq 0$ and $n \geq s+1$.

***B)**. There exists a monic polynomial $\widetilde{\phi}$ such that the sequence $\left\{\frac{P'_{n+1}}{n+1}\right\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to $\widetilde{\phi}u$, i.e.*

$$\left\langle \widetilde{\phi}u, x^k\frac{P'_{n+1}(x)}{n+1}\right\rangle = 0, \quad k \leq n-s-1,$$

and

$$\left\langle \widetilde{\phi}u, x^{n-s}\frac{P'_{n+1}(x)}{n+1}\right\rangle \neq 0.$$

Notice that the full classification of quasi-definite linear functionals of class $s = 1$ is given in [20]. In order to obtain them, the method consists in the construction of canonical forms for the functional equations satisfied by linear functionals of class one. After this, by rescaling parameters, to establish irreducible canonical functional equations of class one, and, finally, to study their integral representations. In this way, the semiclassical linear functionals of class 2 are obtained in [83]. In tables 1.1, 1.2, 1.5 and 1.6 we exhibit such a classification in the positive-definite framework, and assuming that $\widetilde{u}$ is semiclassical of class 1 and 2, and satisfying

$$D\left(\widetilde{\phi}\widetilde{u}\right) + \widetilde{\psi}\widetilde{u} = 0. \tag{1.17}$$

$\widetilde{\phi}$	*Weight Function*	*Irreducibility Conditions*
$A_1.$ $\widetilde{\phi}(x) = x(x-1)(x-\zeta)$	$\widetilde{\omega}(x) = (1-x)^{\alpha}\, x^{\beta}\, \lvert x-\zeta\rvert^{\gamma}$	$x \in [0,1]$, $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > -1$, $\zeta \in (0,1)$.
$A_2.$ $\widetilde{\phi}(x) = x^2(x-1)$	$\widetilde{\omega}(x) = (1-x)^{\alpha}\, x^{\beta} e^{-\frac{\gamma}{x}}$	$x \in [0,1]$, $\alpha\gamma \neq 0$, $\gamma > 0$, $\alpha > -1$.

Table 1.1: $\widetilde{u}$ semiclassical of class 1 with $\deg \widetilde{\phi}(x) = 3$, $1 \leq \deg \widetilde{\psi}(x) \leq 2$

$\widetilde{\phi}$	*Weight Function*	*Irreducibility Conditions*
$B_1.$ $\widetilde{\phi}(x) = x(x-1)$	$\widetilde{\omega}(x) = (1-x)^{\alpha+1}\, x^{\beta+1} e^{-\lambda x}$	$x \in [0,1]$, $\alpha\beta \neq 0$, $\alpha, \beta > -1$,
$B_2.$ $\widetilde{\phi}(x) = x^2$	$\widetilde{\omega}(x) = x^{\alpha} e^{-x+\frac{\beta}{x}}$	$x \in [0,\infty)$, $\beta < 0$, $\alpha, \beta > -1$.
$B_3.$ $\widetilde{\phi}(x) = 1$	$\widetilde{\omega}(x) = x^{2\mu} e^{-x^2-\lambda x}$	$x \in [0,\infty)$, $\mu > -1/2$, $\lambda \in \mathbb{R}$.

Table 1.2: $\widetilde{u}$ semiclassical of class 1, with $\deg \widetilde{\phi}(x) < 3$, $\deg \widetilde{\psi}(x) = 2$

1.3.1 Classical orthogonal polynomials

$u \in \mathbf{P}'$ is said to be *classical* if its class is $s = 0$, i.e. if there exist non zero polynomials $\widetilde{\phi}$ and $\widetilde{\psi}$, with $\deg(\widetilde{\phi}) \leq 2$ and $\deg(\widetilde{\psi}) = 1$, such that (1.14) holds. In this case, the SMOP associated with u is called a *classical* SMOP. Up to affine transformations of the variable, the Hermite, Laguerre, Bessel and Jacobi polynomials are the classical SMOP, (see Table 1.3). Besides, except the Bessel polynomials, if u is classical, under certain restrictions on the parameters, then it is positive-definite and it has a integral representation with respect to a weight function ω on an interval (a,b) as described in Table 1.4.

Remark 19. *Actually the Bessel functional $\mathcal{B}^{(\alpha)}$ has an integral representation, being the path of integration the unit circle in the complex plane instead of some interval in real line. Indeed*

$$\left\langle \mathcal{B}^{(\alpha)}, p(z) \right\rangle = \int_{\mathcal{U}} p(z) e^{-2/z} z^{\alpha} dz, \quad \alpha \neq -n, \quad n \geq 2,$$

where $\mathcal{U} = \{z \in \mathbb{C} |\, |z| = 1\}$. For the interested reader, the paper [64] contains the first systematic study of the Bessel polynomials.

Linear Functional	$\widetilde{\phi}; \widetilde{\psi}$	*Parameter Restriction*
$\mathcal{H}$, *(Hermite)*	$1;\, 2x$	$--$
$\mathcal{L}^{(\alpha)}$, *(Laguerre)*	$x;\, x-(\alpha+1),$	$-\alpha \notin \mathbb{N},$
$\mathcal{B}^{(\alpha)}$, *(Bessel)*	$x^2;\, -2(\alpha x+1)$	$-2\alpha \notin \mathbb{N}$
$\mathcal{J}^{(\alpha,\beta)}$, *(Jacobi)*	$x^2-1;$ $-(\alpha+\beta+2)x+\beta-\alpha$	$-\alpha, -\beta \notin \mathbb{N},$ $-\alpha-\beta \notin \mathbb{N}\setminus\{1\}$

Table 1.3: Quasi-definite Classical Orthogonal Polynomials

Linear Functional	(a,b)	$\omega(x)$	*Parameters Restriction*
$\mathcal{H}$	$(-\infty,\infty)$	e^{-x^2}	$--$
$\mathcal{L}^{(\alpha)}$	$[0,\infty)$	$x^{\alpha}e^{-x},$	$\alpha > -1$
$\mathcal{J}^{(\alpha,\beta)}$	$[-1,1]$	$(1-x)^{\alpha}(1+x)^{\beta},$	$\alpha, \beta > -1$

Table 1.4: Classical Orthogonal Polynomials, positive-definite case

The shifted Jacobi functional on a finite interval $[a,b]$ will be denoted by $\mathcal{J}_{[a,b]}^{(\alpha,\beta)}$, and $\mathcal{J}_{[-1,1]}^{(\alpha,\beta)} := \mathcal{J}^{(\alpha,\beta)}$. Also, the shifted Laguerre Functional on $[a,\infty)$ will be denoted by $\mathcal{L}_{[a,\infty)}^{(\alpha)}$, and $\mathcal{L}_{[0,\infty)}^{(\alpha)} := \mathcal{L}^{(\alpha)}$. In this way, the Jacobi functional $\mathcal{J}_{[a,b]}^{(\alpha,\beta)}$ satisfies

$$D\left[(x-a)(x-b))\mathcal{J}_{[a,b]}^{(\alpha,\beta)}\right] = ((\alpha+\beta+2)\,x - [a\,(\alpha+1)+b\,(\beta+1)])\,\mathcal{J}_{[a,b]}^{(\alpha,\beta)},$$

and

$$\left\langle \mathcal{J}_{[a,b]}^{(\alpha,\beta)}, p(x) \right\rangle = \int_a^b p(x)\,(b-x)^{\alpha}\,(x-a)^{\beta}\,dx, \qquad p \in \mathbf{P}.$$

The Laguerre functional $\mathcal{L}_{[a,\infty)}^{(\alpha)}$ satisfies

$$D\left((x-a)\mathcal{L}_{[a,\infty)}^{(\alpha)}\right) = (-x+\alpha+a+1)\,\mathcal{L}_{[a,\infty)}^{(\alpha)},$$

and

$$\left\langle \mathcal{L}^{(\alpha)}_{[a,\infty)}, p(x) \right\rangle = \int_a^\infty p(x) e^{-x} (x-a)^\alpha dx, \qquad p \in \mathbf{P}.$$

$\widetilde{\phi}$	*Weight Function*	*Irreducibility Conditions*
$A_{1.}$ $\widetilde{\phi}(x) = x(x-1)(x-\zeta_1)(x-\zeta_1)$	$\widetilde{\omega}(x) = x^\alpha (1-x)^\beta \lvert x-\zeta_1\rvert^\gamma \lvert\zeta_2 - x\rvert^\rho$	$x \in [0,1]$, $\zeta_1, \zeta_2 \in (0,1)$, $\zeta_1, < \zeta_2$ $\alpha\beta\gamma\rho \neq 0$, $\alpha, \beta, \lambda, \rho > -1$
$A_{2.}$ $\widetilde{\phi}(x) = x^2(x-1)(x-\zeta)$	$\widetilde{\omega}(x) = (1-x)^\alpha (\zeta - x)^\rho x^{2\beta+1} e^{\gamma/x}$	$x \in [0,1]$, $\alpha\gamma\rho \neq 0$, $\alpha, \beta, \rho > -1$, $\zeta > 1$, $\gamma < 0$
$A_{3.1.}$ $\widetilde{\phi}(x) = x^2(x-1)^2$	$\widetilde{\omega}(x) = (1-x)^\alpha x^\beta e^{-\frac{\gamma}{1-x} + \frac{\rho}{x}}$	$x \in [0,1]$, $\gamma\rho \neq 0$, $\alpha, \beta > -1$
$A_{3.2.}$ $\widetilde{\phi}(x) = x^3(x-1)$	$\widetilde{\omega}(x) = x^{\beta-2} (1-x)^\alpha e^{\frac{\lambda}{x^2} + \frac{\gamma}{x}}$	$x \in [0,1]$, $\alpha\lambda \neq 0$, $\alpha, \beta > -1$, $\lambda, \alpha < 0$

Table 1.5: $\widetilde{u}$ semiclassical of class 2, with $\deg \widetilde{\phi}(x) = 4$, $\deg \widetilde{\psi}(x) \leq 3$

$\widetilde{\phi}$	*Weight Function*	*Irreducibility Conditions*
$B_{1.}$ $\widetilde{\phi}(x) = x(x-1)(x-\zeta)$	$\widetilde{\omega}(x) = \lvert 1-x\rvert^\alpha x^\beta \lvert x-\zeta\rvert^\gamma e^{-\lambda x}$	$x \in (0,\infty)$, $\alpha\beta\gamma \neq 0$, $\alpha, \beta > -1$, $\zeta \in (-1,1)$, $\lambda > 0$.
$B_{2.}$ $\widetilde{\phi}(x) = x^2(x-1)$	$\widetilde{\omega}(x) = \lvert x\rvert^{\beta-1} (1-x)^\alpha e^{-\lambda x + \frac{\gamma}{x}}$	$x \in (-\infty, 0)$, $\alpha\gamma \neq 0$, $\alpha > -1$, $\beta > 0$, $\lambda, \gamma < 0$
$B_{3.}$ $\widetilde{\phi}(x) = x^3$	$\widetilde{\omega}(x) = x^{\alpha-1} e^{-x + \frac{\beta}{x} + \frac{\gamma}{x^2}}$,	$x \in [0,\infty)$, $\alpha > 0$, $\beta, \gamma < 0$
$B_{4,1.}$ $\widetilde{\phi}(x) = (x^2-1)$	$\widetilde{\omega}(x) = \lvert x-1\rvert^\beta (x+1)^\gamma e^{-\lambda x^2 - \alpha x}$	$x \in (1,\infty)$, $\beta\gamma \neq 0$, $\lambda > 0$ $\beta, \gamma > -1$, $\alpha \in \mathbb{R}$
$B_{4.2.}$ $\widetilde{\phi}(x) = x^2$	$\widetilde{\omega}(x) = x^{2\alpha+1} e^{-x^2 + 2\lambda x + \frac{\beta}{x}}$	$x \in [0,\infty)$, $\alpha > -1$, $\beta > 0$
$B_{5.}$ $\widetilde{\phi}(x) = x$	$\widetilde{\omega}(x) = x^\gamma e^{-\lambda x^3 - \beta x^2 - \alpha x}$	$x \in [0,\infty)$, $\gamma \neq 0$, $\gamma > -1$, $\lambda > 0$
$B_{6.}$ $\widetilde{\phi}(x) = 1$	$\widetilde{\omega}(x) = e^{-x^4 - \alpha x^3 - \lambda x^2 - \beta x}$	$x \in \mathbb{R}$

Table 1.6: $\widetilde{u}$ semiclassical of class 2, with $\deg \widetilde{\psi} = 3$ and $0 \leq \deg \widetilde{\phi} \leq 3$.

1.3.2 Stieltjes Series

Definition 20. *Given a quasi-definite linear functional u, the formal **Stieltjes Series** S_u associated with u is defined by*

$$S_u(z) := -\sum_{n\geq 0} \frac{u_n}{z^{n+1}}.$$

Theorem 21. *([88]). A regular linear functional u is semiclassical if and only if there exist polynomials σ, C, and D such that its formal Stieltjes series S_u is a (formal) solution of the following non-homogeneous first order linear differential equation*

$$\sigma(z)S_u'(z) = C(z)S_u(z) + D(z).$$

Moreover, if the polynomials σ, C, and D are mutually coprime, then the class of u is given by $s = max\{\deg(C)-1, \deg(D)\}$. Notice that $D(z)$ is related to σ and C.

By using well known techniques, (see [58] or [59]), it is not difficult to prove the next

Theorem 22. *Let u and v be quasi-definite linear functional and let Φ_{M+n} and Ψ_{N+p+n} be polynomials such that $\deg \Phi_{M+n} = M+n$ and $\deg \Psi_{N+p+n} = N+p+n$, with $M, N, p \in \mathbb{Z}^+ \cup \{0\}$ and $n \geq 0$. Assume that*

$$D^p(\Phi_{M+n}(x)u) = \Psi_{N+p+n}(x)v,$$

holds. Then the formal Stieltjes series associated with u and v satisfy

$$\Psi_{N+p+n}(z)S_v(z) - (\Phi_{M+n}(z)S_u(z))^{(p)} = A_n(z),$$

where, according to (1.8), A_n is the polynomial

$$A_n(z) = \left(u\theta_0\Phi_{M+n}\right)^{(p)}(z) - \left(v\theta_0\Psi_{N+p+n}\right)(z).$$

Besides S_u satisfies the following non-homogeneous ordinary differential equation of order p

$$\sum_{j=0}^{p} B_j(z)S_u^{(j)}(z) = C(z),$$

where

$$B_j(z) = \binom{p}{j}\left(\Psi_{N+p}(z)\Phi_{M+1}^{(p-j)}(z) - \Psi_{N+p+1}(z)\Phi_{M}^{(p-j)}(z)\right),$$

and

$$C(z) = \Psi_{N+p+1}(z)A_0(z) - \Psi_{N+p}(z)A_1(z).$$

1.4 Symmetric Linear Functionals

A linear functional $u \in \mathbf{P}'$ is called symmetric if $u_{2n+1} = \left\langle u, x^{2n+1}\right\rangle = 0$, for every $n \in \mathbb{N}$. (See [31] for more characterizations of symmetric regular linear functionals). If $u \in \mathbf{P}'$ is symmetric and quasi-definite, and $\{P_n\}_{n\geq 0}$ is its corresponding SMOP, we can define $\widetilde{u} \in \mathbf{P}'$ by

$$\langle \widetilde{u}, x^n \rangle = \left\langle u, x^{2n}\right\rangle, \quad n \in \mathbb{N}, \tag{1.18}$$

and the monic polynomials A_n and $\widetilde{A}_n$, $n \geq 0$, by

$$P_{2n}(x) = A_n(x^2) \quad \text{and} \quad P_{2n+1}(x) = x\widetilde{A}_n(x^2). \tag{1.19}$$

As a consequence of the above definition, if u is both positive-definite and symmetric and it has an integral representation in terms of the weight even function w on $[-\zeta, \zeta]$, then we get

$$\langle u, p(x) \rangle = \int_{-\zeta}^{\zeta} p(x)w(x)dx,$$

and

$$\langle \widetilde{u}, p(x) \rangle = \int_{0}^{\zeta^2} p(x)x^{-1/2}w(x^{1/2})dx,$$

assuming the convergence of the integrals.

Example 23. *The classical Hermite functional $u = \mathcal{H}$ is the typical example of a symmetric positive-definite linear functional. Notice that in this case we get $\widetilde{u} = \mathcal{L}^{(-1/2)}$ and the identities*

$$H_{2n}(x) = L_n^{-1/2}(x^2), \text{ and } H_{2n+1}(x) = xL_n^{1/2}(x^2). \tag{1.20}$$

Here $\{H_n\}_{n\geq 0}$ and $\{L_n^{\alpha}\}_{n\geq 0}$ denote the monic Hermite and Laguerre polynomials, respectively.

Example 24. *As an relevant particular case of the classical Jacobi polynomials, if $\alpha = \beta = \lambda - 1/2$ we obtain the well known symmetric Gegenbauer linear functional $\mathcal{G}^{(\lambda)}$, whose integral representation is*

$$\left\langle \mathcal{G}^{(\lambda)}, p(x) \right\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\lambda - 1/2}dx, \quad \lambda > -1/2. \tag{1.21}$$

Theorem 25. *([31]). If $u \in \mathbf{P}'$ is a symmetric and quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ is its corresponding SMOP, then $\widetilde{u}$, defined by (1.18), is quasi-definite. Besides, $\{A_n\}_{n\geq 0}$ and $\left\{\widetilde{A}_n\right\}_{n\geq 0}$ defined by (1.19) are the MOPS related with $\widetilde{u}$ and $x\widetilde{u}$, respectively.*

Conversely, if $\widetilde{u} \in \mathbf{P}'$ is quasi-definite, we can define the symmetric linear functional $u \in \mathbf{P}'$ such that

$$\left\langle u, x^{2n} \right\rangle = \langle \widetilde{u}, x^n \rangle \text{ and } \left\langle u, x^{2n+1} \right\rangle = 0, \quad n \geq 0. \tag{1.22}$$

Theorem 26. *([31]). If $\widetilde{u}$ and $x\widetilde{u}$ are quasi-definite linear functionals on $\mathbf{P}'$ and $\{A_n\}_{n\geq 0}$ and $\left\{\widetilde{A}_n\right\}_{n\geq 0}$ are their corresponding SMOP, then the symmetric linear functional $u \in \mathbf{P}'$, defined by (1.22), is quasi-definite and its SMOP $\{P_n\}_{n\geq 0}$ is given by (1.19).*

Remark 27. *Notice that $\left\{\widetilde{A}_n\right\}_{n\geq 0}$ are the Kernel polynomials with $\kappa-$parameter 0 associated with $\widetilde{u}$. Besides, u is called the symmetrized linear functional of $\widetilde{u}$.*

Theorem 28. *([31]). With $b > 0$, u is positive definite on $\left[-\sqrt{b}, \sqrt{b}\right]$ if and only if $\widetilde{u}$ and $x\widetilde{u}$ are positive definite on $[0, b]$.*

Now, we deduce some interesting consequences of (1.3).

Lemma 29. *Let u be the symmetrization of $\widetilde{u} \in \mathbf{P}'$. Let σ be a polynomial with nonzero simple zeros. Then*

$$\left\langle \sigma^{-1}(x^2)u, q(x^2) \right\rangle = \left\langle \sigma^{-1}(x)\widetilde{u}, q(x) \right\rangle, \tag{1.23}$$

for any polynomial q.

Proof. If $\sigma(x) = \prod_{i=1}^{k}(x - x_i)$, let denote $\overline{\sigma}(x) = \sigma(x^2) = \prod_{i=1}^{2k}(x - y_i)$, where $y_{2j} = \sqrt{x_j}$ and $y_{2j-1} = -\sqrt{x_j}$ for $j = 1, ..., k$. Then from (1.3)

$$\left\langle \sigma^{-1}(x^2)u, q(x^2) \right\rangle = \left\langle \overline{\sigma}^{-1}(x)u, q(x^2) \right\rangle = \left\langle u, \frac{q(x^2) - L_{\overline{\sigma}}(x; q(x^2))}{\overline{\sigma}(x)} \right\rangle,$$

and from (1.4)

$$\begin{aligned} L_{\overline{\sigma}}(x; q(x^2)) &= \sum_{i=1}^{2k} q(y_i^2) \frac{\overline{\sigma}(x)}{(x - y_i)\overline{\sigma}'(y_i)} \\ &= \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)2y_i} \left[\frac{1}{(x - \sqrt{x_i})} - \frac{1}{(x + \sqrt{x_i})} \right] \\ &= \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}. \end{aligned}$$

Then

$$\left\langle \sigma^{-1}(x^2)u, q(x^2) \right\rangle = \left\langle u, \frac{q(x^2) - \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}}{\sigma(x^2)} \right\rangle.$$

Besides, since u is the symmetrization of $\widetilde{u}$, then for any polynomial p, $\left\langle u, p(x^2) \right\rangle = \langle \widetilde{u}, p(x) \rangle$ and, as a consequence,

$$\begin{aligned} \left\langle \sigma^{-1}(x^2)u, q(x^2) \right\rangle &= \left\langle \widetilde{u}, \frac{q(x) - \sum_{i=1}^{k} \frac{q(x_i)\sigma(x)}{\sigma'(x_i)(x - x_i)}}{\sigma(x)} \right\rangle \\ &= \left\langle \widetilde{u}, \frac{q(x) - L_{\sigma}(x; q(x))}{\sigma(x)} \right\rangle \\ &= \left\langle \sigma^{-1}(x)\widetilde{u}, q(x) \right\rangle. \end{aligned}$$

□

Given a semiclassical quasi-definite linear functional $\widetilde{u}$, the semiclassical character of the symmetrized linear functional of $\widetilde{u}$, its class and the respective Pearson equation are described in the next theorem.

Theorem 30 (Symmetrization Theorem)**.** *([17]). If $\widetilde{u}$ is a semiclassical linear functional of class $\widetilde{s}$ satisfying the Pearson equation*

$$D\left[\widetilde{\phi}(x)\widetilde{u}\right] + \widetilde{\psi}(x)\widetilde{u} = 0, \tag{1.24}$$

and $x\widetilde{u}$ is a quasi-definite linear functional, then u, the symmetrization of $\widetilde{u}$, is semiclassical of class s satisfying the Pearson equation

$$D\left[\phi u\right]+\psi(x)u=0, \tag{1.25}$$

where the number s and the polynomials ϕ and ψ are defined according with next cases:
***i)**. If*

$$\widetilde{\phi}(0)=0 \quad \text{and} \quad \widetilde{\phi}'(0)+2\widetilde{\psi}(0)=0, \tag{1.26}$$

then

$$\phi(x)=(\theta_0\widetilde{\phi})(x^2), \quad \psi(x)=x\left[2(\theta_0\widetilde{\psi})(x^2)+(\theta_0^2\widetilde{\phi})(x^2)\right], \tag{1.27}$$

and $s=2\widetilde{s}$.
***ii)**. If $\widetilde{\phi}(0)=0$ and $\widetilde{\phi}'(0)+2\widetilde{\psi}(0)\neq 0$ then*

$$\phi(x)=x(\theta_0\widetilde{\phi})(x^2), \;\; \psi(x)=2\widetilde{\psi}(x^2),$$

and $s=2\widetilde{s}+1$.
***iii)**. If $\widetilde{\phi}(0)\neq 0$ then*

$$\phi(x)=x\widetilde{\phi}(x^2), \quad \psi(x)=2\left[x^2\widetilde{\psi}(x^2)-\widetilde{\phi}(x^2)\right],$$

and $s=2\widetilde{s}+3$.

Corollary 31. *If s is odd, the polynomials ϕ and ψ in (1.25) are, respectively, odd and even functions. If s is even, the polynomials ϕ and ψ in (1.25) are, respectively, even and odd functions.*

In the cases $\mathcal{J}_{[0,1]}^{(\alpha,\beta)}$ and $\mathcal{L}^{(\alpha)}$, where the weight functions are $\omega(x)=(1-x)^\alpha x^\beta$ on $[0,1]$ and $\omega(x)=e^{-x}x^\alpha$ on $[0,\infty)$, respectively, the new weight functions associated with the symmetrized linear functionals $\overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$ and $\overline{\mathcal{L}}^{(\alpha)}$ are $\omega(x)=(1-x^2)\,|x|^{2\beta+1}$ on $[-1,1]$ and $\omega(x)=e^{-x^2}\,|x|^{2\alpha+1}$ on $\mathbb{R}$, respectively. For see that, it is enough and easy to prove that $\left\langle \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)},p(x^2)\right\rangle=\left\langle \mathcal{J}_{[0,1]}^{(\alpha,\beta)},p(x)\right\rangle$ and $\left\langle \overline{\mathcal{L}}^{(\alpha)},p(x^2)\right\rangle=\left\langle \mathcal{L}^{(\alpha)},p(x)\right\rangle$ for any polynomial p. On purpose, the symmetric linear functionals of class 1, described in [17], are the symmetrization of the classical linear functionals. In [36] the symmetric linear functionals of class 2 are obtained by using the Symmetrization Theorem. This allows to take the search of these functionals to class 1 framework. The linear functionals of class 3 ones are studied in [84]. Next we describe the classification of the semiclassical symmetric linear functional of class 1 and 2. We assume that u is positive-definite and that it satisfies

$$D(\phi u)+\psi u=0.$$

In tables 1.7, 1.8 and 1.9, the symmetric semiclassical linear functionals of class 1 and 2 are described in the positive-definite framework

ϕ	*Weight Function*	*Irreducibility Conditions*
Generalized Gegenbauer $\phi(x) = x(x^2 - 1)$	$\omega(x) = \lvert x\rvert^{\alpha} \left(1 - x^2\right)^{\lambda - 1/2}$	$x \in [-1, 1]$, $\alpha \neq 0$, $\alpha > -1$ $\lambda > -1/2$,
$Hermite - Chihara$ $\phi(x) = x$	$\omega(x) = \lvert x\rvert^{\alpha} e^{-x^2}$	$x \in \mathbb{R}$, $\alpha \neq 0$, $\alpha > -1$.

Table 1.7: u symmetric semiclassical of class 1

ϕ	*Weight Function*	*Irreducibility Conditions*
$\phi(x) = (x^2 - 1)(x^2 - \lambda)$	$\omega(x) = \left\lvert \lambda - x^2 \right\rvert^{\gamma} (1 - x^2)^{\beta}$	$x \in [-1, 1]$, $\lvert\lambda\rvert > 1$, $\gamma \neq 0$, $\beta > -1$
$\phi(x) = x^2(x^2 - 1)$	$d\mu = \lvert x\rvert^{2\beta+1} (1 - x^2)^{\alpha} dx + \lambda\delta(x)$	$x \in [-1, 1]$, $\beta \neq 0$, $\alpha, \beta > -1$.
$\phi(x) = (x^2 - 1)^2$	$\omega(x) = \left(1 - x^2\right)^{\beta} e^{-\frac{\gamma}{1-x^2}}$	$x \in [-1, 1]$, $\beta > -1$, $\gamma > 0$

Table 1.8: u symmetric semiclassical of class 2, $\deg \phi = 4$ and $1 \leq \deg \psi \leq 3$.

ϕ	*Weight Function*	*Irreducibility Conditions*
$\phi(x) = x^2 - 1$	$\omega(x) = (1 - x^2)^{\beta} e^{-\lambda x^2}$	$x \in [-1, 1]$, $\beta > -1$.
$\phi(x) = x^2$	$d\mu = \lvert x\rvert^{2\alpha+1} e^{-x^2} dx + \lambda\delta(x)$	$x \in \mathbb{R}$, $\alpha > -1$.
$\phi(x) = 1$	$\omega(x) = e^{-x^4 - \lambda x^2}$	$x \in \mathbb{R}$

Table 1.9: u symmetric semiclassical of class 2, $\deg \psi = 3$ and $0 \leq \deg \phi \leq 2$.

1.5 Asymptotics for Orthogonal Polynomials

Let μ be a positive Borel measure supported on both an infinite and compact subset of the real line denoted by $supp\,(\mu)$. It is well known that through Gram-Schmidt orthogonalization process applied to $\{x^n\}_{n\geq 0}$ and in $L^2(\mu)$, it is possible to build an orthonormal sequence of polynomials $\{p_n(.;\mu)\}_{n\geq 0}$ The asymptotic behavior is a widely and rich branch of the theory of orthogonal polynomials and it is related to with the study of the behavior of $\{p_n(.;\mu)\}_{n\geq 0}$ when n tends to infinity. The importance of asymptotics lies in its applications: linear predictors in the theory of stochastic processes, random matrix theory, Fisher–Hartwig conjectures and Ising models, study of algorithms, entropy, among others, (see the nice survey [69]). Usually there are three different asymptotic behavior of an OPS:

i). *Root Asymptotics.* It deals with $\lim_{n\to\infty} \lvert p_n(x;\mu)\rvert^{1/n}$.

ii). *Outer Ratio Asymptotics.* It has to do with $\lim_{n\to\infty} \dfrac{p_n(x;\mu)}{p_{n-1}(x;\mu)}$.

iii). *Szegő Asymptotics.* It is related to $\lim_{n\to\infty} \frac{p_n(x;\mu)}{\varphi^n(x)}$ where φ is analytic on $\mathbb{C}\backslash\overline{supp}(\mu)$ and $\overline{supp}(\mu)$ denotes the convex hull of $supp(\mu)$.

It is also well known that i)⇒ii)⇒iii). but, in general, the converse statements do not hold. In this dissertation we are interested in the study of *Relative Asymptotics* for Orthogonal polynomials in the real line as well as certain types of the so called *Mehler-Heine* and *Plancherel-Rotach* formulas. Let μ and ν be positive Borel measures supported on compact subsets such that $d\nu = \frac{1}{x-c}d\mu + \lambda\delta(x-c)$, with $\lambda > 0$ and $c \notin supp(\mu)$. ν is called a *Geronimus Perturbation* of μ and it was introduced by Geronimus in his pioneer works [49] and [50]. Then it is reasonable to think that there should be a relation between the respective SMOP $\{p_n(.;\mu)\}_{n\geq 0}$ and $\{p_n(.;\nu)\}_{n\geq 0}$. This argument gives sense to the study of $\lim_{n\to\infty} \frac{p_n(x;\mu)}{p_n(x;\nu)}$ outside $supp(\mu)$. Such a limit is known as Outer Relative asymptotics. In other words allows us to study the asymptotic behavior of the SMOP associated with one measure when we know the behavior of the other one.

With the purpose that this memory is as self-contained as possible, next we present the various kinds of asymptotics for Laguerre and Hermite polynomials.

Theorem 32. *Let* $\left\{\widehat{L}_n^{\alpha}\right\}_{n\geq 0}$ *be the sequence of classical Laguerre polynomials with leading coefficient* $\frac{(-1)^n}{n!}$. *Then the following statements hold.*

i). (Perron's asymptotics formula on $\mathbb{C}\backslash\mathbb{R}$, *[108])*

$$\widehat{L}_n^{\alpha}(x) = 2^{-1}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}e^{2(-nx)^{1/2}}\left(1+\mathcal{O}(n^{-1/2})\right).$$

This relation holds for x *in the complex plane cut along the positive real semiaxis; both* $(-x)^{-\alpha/2-1/4}$ *and* $(-x)^{-1/2}$ *must be taken real and positive if* $x<0$. *The bound of the remainder holds uniformly in every closed domain which does not overlap the positive real semiaxis*

ii). (Mehler-Heine, [108])

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(x/(n+k))}{n^{\alpha}} = x^{-\alpha/2}J_{\alpha}(2\sqrt{x}), \tag{1.28}$$

uniformly on compact subsets of $\mathbb{C}$ *and uniformly on* $k\in\mathbb{N}\cup\{0\}$. *Besides* J_{α} *represents the Bessel's function of the first kind defined by*

$$J_{\alpha}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\alpha+1)}\left(\frac{x}{2}\right)^{2j+\alpha}.$$

iii). ([108])

$$\frac{\widehat{L}_n^{\alpha}(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2} + J_{\alpha}(2\sqrt{nx}) + \mathcal{O}(n^{-3/4}),$$

uniformly on compact subsets of $\mathbb{R}_+$

iv). ([44]). For $x\in\mathbb{C}\backslash\mathbb{R}_+$

$$\frac{\widehat{L}_{n+j}^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}}j + \left(\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$

where $\sqrt{-x}$ *must be taken real and positive if* $x<0$.

v). ([13])

$$\lim_{n\to\infty} n^{(l-j)/2}\frac{\widehat{L}_{n+k}^{(\alpha+j)}(x)}{\widehat{L}_{n+h}^{(\alpha+l)}(x)} = (-x)^{(l-j)/2}, \quad j,l\in\mathbb{R},\ k,h\in\mathbb{Z},$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

vi). (Plancherel-Rotach type formula, [10]). Let

$$\varphi(z) = z+\sqrt{z^2-1}, \tag{1.29}$$

denote the conformal mapping of $\mathbb{C}\backslash[-1,1]$ *onto the exterior of the closed unit disk. Then*

$$\lim_{n\to\infty}\frac{\widehat{L}_{n-1}^{\alpha}((n+j)x)}{\widehat{L}_{n}^{\alpha}((n+j)x)} = -\frac{1}{\varphi((x-2)/2)},$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$, *and uniformly on* $j\in\mathbb{N}\cup\{0\}$.

vii). (Scaled asymptotics, [48]).

$$\begin{aligned}\lim_{n\to\infty}&(-1)^n 2^n\sqrt{2\pi n}\left(x-2+\sqrt{x^2-4x}\right)^{-n}\exp\left(\frac{-2nx}{x+\sqrt{x^2-4x}}\right)\widehat{L}_n^{\alpha}(nx)\\ =\;& 2^{-\alpha-1/2}x^{\alpha}\left(x-2+\sqrt{x^2-4x}\right)^{1/2}\left(x+\sqrt{x^2-4x}\right)^{\alpha}\left(x^2-4x\right)^{-1/4},\end{aligned} \tag{1.30}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$, *taking into account that the square roots in (1.30) are negative if x is negative.*

Theorem 33. *Let* $\{H_n\}_{n\geq 0}$ *be the sequence of classical monic Hermite polynomials. Then*

i). ([108])

$$\lim_{n\to\infty}\frac{nH_n(x)}{2H_{n+2}(x)} = -1, \tag{1.31}$$

uniformly on compact sets of $\mathbb{C}\backslash\mathbb{R}$.

ii).([1]). For $j\in\mathbb{Z}$ *fixed*

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{n!}H_{2n}\left(\frac{x}{2\sqrt{n+j}}\right) = \left(\frac{x}{2}\right)^{1/2}J_{-1/2}(x) \tag{1.32}$$

and

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}H_{2n+1}\left(\frac{x}{2\sqrt{n+j}}\right) = \left(\frac{x}{2}\right)^{1/2}J_{1/2}(x), \tag{1.33}$$

uniformly on compact sets of $\mathbb{C}$.

iii). ([108])

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n+j}}{n!}H_{2n}\left(\frac{x}{2\sqrt{n+j}}\right) = \frac{1}{\sqrt{\pi}}\cos(x), \tag{1.34}$$

and

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}H_{2n+1}\left(\frac{x}{2\sqrt{n+j}}\right) = \frac{1}{\sqrt{\pi}}\sin(x), \tag{1.35}$$

uniformly on compact subsets of $\mathbb{C}$ *and uniformly on* $j\in\mathbb{N}\cup\{0\}$.

iv). ([102])

$$\lim_{n\to\infty}\sqrt{\left[\frac{n}{2}\right]}\frac{H_n(x)}{H_{n+1}(x)} = \begin{cases} -ix, & x\in\mathbb{C}_+\\ ix, & x\in\mathbb{C}_-,\end{cases} \tag{1.36}$$

uniformly on compact subsets of half planes $\mathbb{C}_+$ *and* $\mathbb{C}_-$*, respectively*

v). ([110]). For $j \in \mathbb{Z}$ *fixed*

$$\lim_{n\to\infty} \sqrt{n} \frac{H_{n-1}\left(\sqrt{n+j}z\right)}{H_n\left(\sqrt{n+j}z\right)} = \frac{\sqrt{2}}{\varphi\left(z/\sqrt{2}\right)}, \qquad j \in \mathbb{Z} \tag{1.37}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus\left[-\sqrt{2},\sqrt{2}\right]$.

Corollary 34. *vi). (Consequence of (1.37)). For* $j \in \mathbb{Z}$ *fixed, and non-negative integers* p *and* q *such that* $n > p-1$

$$\lim_{n\to\infty} \frac{(\sqrt{n})^{p+q}\, H_{n-p}\left(\sqrt{n+j}z\right)}{H_{n+q}\left(\sqrt{n+j}z\right)} = \left(\frac{\sqrt{2}}{\varphi\left(z/\sqrt{2}\right)}\right)^{p+q}, \tag{1.38}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus\left[-\sqrt{2},\sqrt{2}\right]$

Chapter 2

Coherent Pairs and Sobolev type Orthogonal Polynomials

2.1 Sobolev orthogonality

In the framework of the so-called nonstandard orthogonality we have the *Sobolev type orthogonality.* Let $\mu_0, \mu_1, \ldots, \mu_m$ be $m+1$ positive Borel measures supported on compact subsets of $\mathbb{C}$, such that $\left\{x^k\right\}_{k\geq 0} \subseteq L^2(\mu_i)$ for $i = 0, \ldots, m$. We also assume that $supp\,(\mu_0)$ is infinite and that μ_m is not the trivial measure. Besides $\langle , \rangle_{\mu_i}$ and $\|.\|_{\mu_i}$ will denote the inner product and the induced norm in $L^2(\mu_i)$, respectively. Then the inner product on the linear space of polynomials with complex coefficients

$$\langle p(x), q(x)\rangle_S := \sum_{k=0}^{m} \int p^{(k)}(x)\overline{q^{(k)}(x)}d\mu_k(x) = \sum_{k=0}^{m} \left\langle p^{(k)}, q^{(k)} \right\rangle_{\mu_k}, \tag{2.1}$$

is called a *Sobolev Inner Product* associated with the measures μ_k, $0 \leq k \leq m$, and the respective induced norm $\|p(x)\|_S = \langle p, p\rangle_S^{1/2} := \left(\sum\limits_{k=0}^{m} \left\|p^{(k)}\right\|_{\mu_k}^2 \right)^{1/2}$. A sequence of polynomials $\{S_n\}_{n\geq 0}$, with $\deg S_n = n$ for $n \geq 0$, is orthogonal with respect to the inner product (2.1) if the following statements hold.

i). $\langle S_n(x), S_m(x)\rangle_S = 0$ if $n \neq m$, and

ii). $\langle S_n(x), S_n(x)\rangle_S \neq 0$, $n \geq 0$.

Then S_n is said to be $n-th$ Sobolev polynomial associated with (2.1), and $\{S_n\}_{n\geq 0}$ is called a sequence of *Sobolev orthogonal polynomials.*

In the sequel we assume that $\mathbf{P}$ is the linear space of polynomials with real coefficients on the real line. It should not be a surprise the above orthogonality definition in terms of an inner product. Indeed, the orthogonality can be defined alternatively in terms of inner products on $\mathbf{P}$.

Definition 35. *A sequence $\{P_n\}_{n\geq 0}$ is called an orthogonal polynomials sequence with respect to an inner product $\langle , \rangle$ on $\mathbf{P}$ if for $n, m \geq 0$,*

i). P_n is a polynomial of degree n

ii). $\langle P_n, P_m\rangle = 0$, for $n \neq m$.

iii). $\langle P_n, P_n\rangle \neq 0$.

Notice that it is possible to develop the theory of orthogonal polynomials in a more general framework. For instance, the determinants Δ_n, defined in (1.10) can be expressed as

$$\Delta_n = \begin{vmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \cdots & \langle 1,x^n\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \cdots & \langle x,x^n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x^n,1\rangle & \langle x^n,x\rangle & \cdots & \langle x^n,x^n\rangle \end{vmatrix}, \quad n \geq 0.$$

Having said that, now we want to discuss the difference between the standard and non standard orthogonality.

Definition 36. *Let X be the vector space of complex-valued functions with inner product $\langle , \rangle$. The inner product is said to be standard if*

$$\langle xf(x), g(x)\rangle = \langle f(x), \overline{x}g(x)\rangle ,$$

for every $f, g \in X$.

In other words, the inner product $\langle , \rangle$ is said to be standard if the *multiplication operator* $\mathcal{M}_x : X \to X$, $M_x(f) = xf$ is symmetric. When $X = \mathbf{P}$, it is possible to establish a significant connection with the Favard's theorem.

Lemma 37. *A sequence of monic polynomials orthogonal with respect to an inner product satisfies a TTRR if and only if the operator*

$$\begin{array}{rccc} \mathcal{M}_x : & \mathbf{P} & \to & \mathbf{P} \\ & p & \mapsto & \mathcal{M}_x[p] := xp, \end{array}$$

is symmetric.

When the operator $\mathcal{M}_x$ is symmetric, the orthogonality is called *standard.* In general, notice that the orthogonality defined through a Sobolev inner product (2.1) is not standard and, consequently, we can not expect that the corresponding Sobolev orthogonal polynomials satisfy a TTRR. On the other hand, it is well known that in the standard orthogonality the zeros of each orthogonal polynomial are real, simple and lie in the convex hull of the support of the measure. In the pioneer work [12] the next Sobolev inner product is considered

$$\langle p(x), q(x)\rangle_S := \int_{-1}^{1} p(x)q(x)dx + \lambda \int_{-1}^{1} p'(x)q'(x)dx, \ \lambda > 0.$$

In such paper Althammer showed that if the inner product is written as

$$\langle p(x), q(x)\rangle_S := \int_{-1}^{1} p(x)q(x)dx + \lambda \int_{-1}^{1} p'(x)q'(x)w(x)dx,$$

with

$$w(x) = \begin{cases} 10, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1, \end{cases}$$

then the respective Sobolev polynomial $S_2^{\lambda}(x) = x^2 + \frac{27}{35}x - \frac{1}{3}$ has a real zero outside of $(-1, 1)$. Even more so, if we consider the Sobolev inner product

$$\langle p(x), q(x)\rangle_S := \int p(x)q(x)d\mu_0(x) + \lambda \int p'(x)q'(x)d\mu_1(x), \quad \lambda > 0,$$

(studied in [93]) where $\{\mu_0, \mu_1\}$ is a so-called *symmetric coherent pair*, (this concept will be defined in the next section), then the Sobolev polynomials S_{2n}^{λ} can have complex zeros. Maybe these differences with the standard orthogonality have contributed to make the Sobolev orthogonality so attractive and interesting for researchers in the last two decades. Of course, there exist many other powerful reasons that justify the study of Sobolev orthogonality: coherent pairs, spectral theory for ordinary differential equations, search of efficient algorithms for computing coefficients in Sobolev-Fourier expansions, extension of Gauss quadrature formulas, among others. Notice that in the introduction of this manuscript we have made a brief account of both historical development and the state of the art Sobolev orthogonality.

The first result about asymptotics of Sobolev orthogonal polynomials was given in [85] for the Sobolev inner product

$$\langle p(x), q(x)\rangle_S := \int p(x)q(x)d\mu(x) + Mp'(\zeta)q'(\zeta),\ \zeta \in \mathbb{R},\ M \geq 0, \tag{2.2}$$

with μ in the *Nevai class* $M(0,1)$, (see [103]), which consists of all positive Borel measures μ for which the corresponding OPS $\{p_n(x;\mu)\}_{n\geq 0}$ satisfies the TTRR

$$xp_n(x;\mu) = a_{n+1}p_{n+1}(x;\mu) + b_n p_n(x;\mu) + a_n p_{n-1}(x;\mu),$$

where $a_n \to 1/2$ and $b_n \to 0$ when $n \to \infty$. Indeed, in [85] is proved the next result that deals with outer relative asymptotics when $\mu \in M(0,1)$.

Theorem 38.

$$\lim_{n\to\infty} \frac{S_n^{\lambda}(x)}{L_n(x)} = \frac{(\varphi(x) - \varphi(\zeta))^2}{2\varphi(x)(x-\zeta)},$$

uniformly on every compact subset of $\overline{\mathbb{C}}\backslash supp\,(\mu)$, *where* $\{L_n\}_{n\geq 0}$ *is the SMOP associated with* μ, $\{S_n^{\lambda}\}_{n\geq 0}$ *is orthogonal with respect to the Sobolev inner product*

$$\langle p, q\rangle_S = \int pqd\mu + \lambda p'(\zeta)q'(\zeta),\ \lambda > 0,\ \zeta \in \mathbb{R}\backslash supp\,(\mu)\,,$$

and φ *is the conformal mapping* $\varphi(x) = x + \sqrt{x^2-1}$, *with the convention* $\sqrt{x^2-1} > 0$ *for* $x > 1$.

Notice that the product (2.2) is a particular case of the very special case so-called *Discrete Sobolev inner products* or *Sobolev inner products of the second type.* Namely, it is obtained from (2.1) when μ_k, $k = 1, 2, ..., m$, is an atomic measure, i. e. such measures are supported on finite subsets of the real line. This kind of Sobolev inner product can be written as

$$\langle p(x), q(x)\rangle_S := \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \sum_{k=1}^{m} \int_{\mathbb{R}} p^{(k)}(x)q^{(k)}(x)d\mu_k(x),$$

where often, for $k = 1, ..., m$, $d\mu_k(x) = A_k\delta(x-c)$, $c \in \mathbb{R}$ and $A_k \geq 0$. Contributions on particular cases when μ_0 has unbounded support can be seen in [38] and [53]. The references [8] and [86] are highly recommendable surveys on analytic and asymptotics properties of Sobolev orthogonal polynomials in both discrete and continuous cases.

2.2 Coherent Pairs and Symmetric Coherent Pairs

In [55] the concepts of *coherent pair* and *symmetric coherent pair* are introduced in the Sobolev orthogonal polynomials framework.

Definition 39. *Let $\{\mu_0, \mu_1\}$ be a pair of positive Borel measures and let $\{P_n(.;\mu_0)\}_{n\geq 0}$ and $\{P_n(.;\mu_1)\}_{n\geq 0}$ be the corresponding SMOP. The pair $\{\mu_0, \mu_1\}$ is called a coherent pair if there exist non-zero constants σ_n such that*

$$P_n(x;\mu_1) = \frac{P'_{n+1}(x;\mu_0)}{n+1} + \sigma_n \frac{P'_n(x;\mu_0)}{n}, \qquad n \geq 1. \tag{2.3}$$

If μ_0 and μ_1 are symmetric then the pair $\{\mu_0, \mu_1\}$ is called a symmetric coherent pair if there exist non-zero constants γ_n such that

$$P_{n+1}(x;\mu_1) = \frac{P'_{n+2}(x;\mu_0)}{n+2} + \gamma_n \frac{P'_n(x;\mu_0)}{n}, \qquad n \geq 1. \tag{2.4}$$

Remark 40. *The above definitions can be formulated equivalently in terms of quasi-definite linear functionals instead of positive Borel measures. In fact, if $\{u, v\}$ is a (symmetric) coherent pair, then the corresponding SMOP will be denoted as $\{P_n(.;u)\}_{n\geq 0}$ and $\{P_n(;v)\}_{n\geq 0}$, respectively.*

A (symmetric) coherent pair of measures $\{\mu_0, \mu_1\}$ determines, in a natural way, a Sobolev inner product on $\mathbf{P}$. Indeed, for real polynomials p and q we get

$$\langle p(x), q(x)\rangle_S := \int p(x)q(x)d\mu_0(x) + \lambda \int p'(x)q'(x)d\mu_1(x), \quad \lambda > 0, \tag{2.5}$$

and $\{S_n^\lambda\}_{n\geq 0}$ denotes the SMOP associated with (2.5).

In [55], the algebraic connection between the Sobolev polynomials and orthogonal polynomials with respect to one of the two measures in the coherent pair is studied. More precisely,

Theorem 41. *If $\{\mu_0, \mu_1\}$ is a coherent pair then there exists a sequence $\{\eta_n(\lambda)\}_{n\geq 0}$ such that*

$$S_{n+2}^\lambda(x) + \eta_n(\lambda) S_{n+1}^\lambda(x) = P_{n+2}(x;\mu_0) + \sigma_n P_{n+1}(x;\mu_0), \quad n \geq 0. \tag{2.6}$$

Notice that the converse of the above result is not true, i.e., if (2.6) holds, then the pair $\{\mu_0, \mu_1\}$ is not necessarily a coherent pair. However, under the assumption (2.6), it is possible to obtain an algebraic relation between $\{P_n(.;\mu_0)\}_{n\geq 0}$ and $\{P_n(.;\mu_1)\}_{n\geq 0}$. Such a relation will generalize the concept of coherent pair in order to define the so-called $(1,1)-$*coherent pairs*, which will be discussed in the next chapter.

Under a suitable non-monic normalization of Sobolev polynomials orthogonal with respect to 2.5, ($\left\{\widetilde{S}_n^\lambda\right\}_{n\geq 0}$ denotes such a normalized sequence), in [55] a natural and special algebraic connection with the sequence $\{P_n(.;\mu_0)\}_{n\geq 0}$ is obtained. Namely,

Theorem 42. *If $\{\mu_0, \mu_1\}$ is a coherent pair such that (2.3) holds, then*

$$\widetilde{S}_n^\lambda(x) = \gamma_n(\lambda) P_n(x;\mu_0) + \sum_{k=0}^{n-1} \alpha_k(\lambda) P_k(x;\mu_0),$$

where, for $0 \leq k \leq n-1$, $\alpha_k(\lambda)$ does not depends on n. In addition, the sequence $\{\alpha_k(\lambda)/\lambda\}_{k\geq 1}$ satisfies a TTRR in the standard sense.

On the other hand, in [55] the relation (2.6) is fundamental for the implementation of an algorithm that allows to evaluate efficiently expansions of functions on the *Sobolev Space*

$$W^{1,2}\left[E,\mu_0,\mu_1\right]=\left\{f:E\to\mathbb{R}|f\in L^2(E;\mu_0),\ f'\in L^2(E;\mu_1)\right\},$$

in terms of the Sobolev polynomials orthogonal with respect to (2.5), without the explicit knowledge of the Sobolev polynomials.

Since Sobolev orthogonality is nonstandard, in general, the corresponding Sobolev orthogonal polynomials will not satisfy a TTRR. However, the polynomials $\left\{S_n^\lambda\right\}_{n\geq 0}$ can still be obtained recursively.

Proposition 43. *([29]). If $\{\mu_0,\mu_1\}$ is a coherent pair, the Sobolev polynomials $\left\{S_n^\lambda\right\}_{n\geq 0}$, orthogonal with respect to (2.5), satisfy a four term recurrence relation.*

With respect to the nature and location of zeros of Sobolev polynomials, in [93] is proved that if $\{\mu_0,\mu_1\}$ is a coherent pair, λ is sufficiently large and $n>1$, then the zeros of S_n^λ are real and simple, and they are interlaced with the zeros of $P_{n-1}\left(x;\mu_0\right)$ and with those of $P_{n-1}\left(x;\mu_1\right)$.

On the other hand, in order to identify all coherent pairs, in [81] the coherence is studied in a more general framework. In that work the functionals u and v are quasi-definite and the corresponding MOPS must satisfy

$$P_{n+1}\left(x;v\right)=\frac{P'_{n+2}(x;u)}{n+2}+\gamma_n\frac{P'_{n+1}(x;u)}{n+1}. \tag{2.7}$$

Then, with the assumption that one of the two sequences of polynomials is fixed, for instance $\left\{P_n\left(.;u\right)\right\}_{n\geq 0}$, the problem of determine all possible sequences $\left\{P_n\left(.;v\right)\right\}_{n\geq 0}$ and all the sequences of compatible parameters $\left\{\gamma_n\right\}_{n\geq 0}$ that satisfy (2.7) is solved. From that general perspective, in [78] is proved that all coherent pairs can be described in terms of semiclassical orthogonal polynomials, i.e., both polynomials are semiclassical and the quasi-definite linear functionals satisfy a rational relation. Namely,

Theorem 44. *([78]). If $\{u,v\}$ is a coherent pair, then u and v satisfy*

$$D\left(\pi_n v\right)=n\frac{P_n\left(x;u\right)}{\left\|P_n\left(\cdot;u\right)\right\|_u^2}u,\quad n\geq 1,$$

where

$$\pi_n(x)=\gamma_n\frac{P_n\left(x;v\right)}{\left\langle v,P_n^2\left(x;v\right)\right\rangle}-\frac{P_{n-1}\left(x;v\right)}{\left\langle v,P_{n-1}^2\left(x;v\right)\right\rangle}.$$

Finally, in [96] the classification of all the coherent pairs and symmetric coherent pairs is given, being always classical one of the two measures. The next results summarize and formalize this achievement.

Theorem 45. *([96]). Let $\{u,v\}$ denote a coherent pair. Then*

$$\varphi Dv=\pi v,\quad \varphi u=Bv,\ \ \pi u=BDv, \tag{2.8}$$

where $\varphi,\pi,$ and B are polynomials such that $\deg\varphi\leq 3$, $\deg\pi\leq 2$ *and* $\deg B=2$.

Theorem 46. *([96]). If the polynomial B has a double zero ξ, then u is classical and*

$$\widetilde{\varphi}u = (x-\xi)\,v, \quad \deg\widetilde{\varphi} \leq 2.$$

The corresponding coherent pairs $\{u,v\}$ are exhibited in Table 2.1.

Coherent Pair $\{u,v\}$	*Restrictions*
$CP1$ $u=\mathcal{L}^{(\alpha)}$ $v=\dfrac{1}{x-\xi}\mathcal{L}^{(\alpha+1)}+M\delta(x-\xi)$	$x\in[0,\infty]$, $\alpha>-1$, $\xi\leq 0$, $M\geq 0$.
$CP2$ $u=\mathcal{J}^{(\alpha,\beta)}$ $v=\dfrac{1}{\lvert x-\xi\rvert}\mathcal{J}^{(\alpha+1,\beta+1)}+M\delta(x-\xi)$	$x\in[-1,1]$, $\alpha,\beta>-1$, $\lvert\xi\rvert\geq 1$, $M\geq 0$.

Table 2.1: Coherent pairs, u classical

Theorem 47. *([96]). If the polynomial B has two different zeros then v is classical, and there exists ζ such that*

$$\widetilde{\varphi}u = (x-\zeta)\,v, \quad \deg\widetilde{\varphi} \leq 2.$$

The corresponding coherent pairs (u,v) are exhibited in Table 2.2.

Coherent Pair $\{u,v\}$	*Restrictions*
$CP3$ $u=(x-\zeta)\,\mathcal{L}^{(\alpha-1)}$ $v=\mathcal{L}^{(\alpha)}$	$x\in[0,\infty]$, $\alpha>0$, $\zeta<0$, $M\geq 0$
$CP4$ $u=\mathcal{L}^{(0)}+M\delta(x)$ $v=\mathcal{L}^{(0)}$	$x\in[0,\infty]$.
$CP5$ $u=\lvert x-\zeta\rvert\,\mathcal{J}^{(\alpha-1,\beta-1)}$ $v=\mathcal{J}^{(\alpha,\beta)}$	$x\in[-1,1]$, $\alpha,\beta>0$, $\lvert\zeta\rvert\geq 1$.
$CP6$ $u=\mathcal{J}^{(0,\beta-1)}+M\delta(x-1)$ $v=\mathcal{J}^{(0,\beta)}$	$x\in[-1,1]$, $\beta>0$, $M\geq 0$.
$CP7$ $u=\mathcal{J}^{(\alpha-1,0)}+M\delta(x+1)$ $v=\mathcal{J}^{(\alpha,0)}$	$x\in[-1,1]$, $\alpha>0$, $M\geq 0$.

Table 2.2: Coherent pairs, v classical

Analogously, in the case of symmetric coherent pairs we get the next results

Theorem 48. *([96]). Let $\{u,v\}$ denote a symmetric coherent pair. Then*

$$\varphi Dv = \pi v, \quad \varphi u = Bv, \;\; \pi u = BDv, \tag{2.9}$$

where φ, π *and* B *are polynomials such that* $\deg \varphi \leq 4$, $\deg \pi \leq 3$ *and* $\deg B = 4$.

Theorem 49. *([96]). Let* B *be of the form* $B(x) = k\left(x^2 - \xi^2\right)^2$. *Then* u *is classical and*

$$\widetilde{\varphi} u = \left(x^2 - \xi^2\right) v, \quad \deg \widetilde{\varphi} \leq 2.$$

The corresponding symmetric coherent pairs $\{u, v\}$ *are exhibited in 2.3.*

Symmetric *Coherent Pair* $\{u, v\}$	*Restrictions*
$SCP1. \quad u = \mathcal{H}, \;\; v = \dfrac{1}{x^2+\xi^2}\mathcal{H}$	$x \in \mathbb{R}$, $\xi^2 \neq 0$.
$SCP2. \quad u = \mathcal{G}^{(\lambda-1)}, \quad v = \dfrac{1}{x^2+\xi^2}\mathcal{G}^{(\lambda)}$	$x \in [-1, 1]$, $\xi^2 \neq 0$, $\lambda > 1/2$.
$SCP3$ $u = \mathcal{G}^{(\lambda-1)},$ $v = \dfrac{1}{\xi^2 - x^2}\mathcal{G}^{(\lambda)} + \dfrac{\eta}{2}\left(\delta(x - \lvert\xi\rvert) + \delta(x + \lvert\xi\rvert)\right)$	$x \in [-1, 1]$, $\lvert\xi\rvert > 1$, $\lambda > 1/2$, $\eta \geq 0$.

Table 2.3: Symmetric coherent pairs, u classical

Theorem 50. *([96]). If the polynomial* B *has two different pairs of zeros then* v *is classical, and there exists* ζ *such that*

$$\widetilde{\varphi} u = \left(x^2 - \zeta^2\right) v, \quad \deg \widetilde{\varphi} \leq 2.$$

The corresponding symmetric coherent pairs $\{u, v\}$ *are exhibited in Table 2.4.*

Symmetric *Coherent Pair* $\{u, v\}$	*Restrictions*
$SCP4. \quad u = \left(x^2+\zeta^2\right)\mathcal{H}, \;\; v = \mathcal{H}$	$x \in \mathbb{R}$.
$SCP5. \quad u = \left(x^2+\zeta^2\right)\mathcal{G}^{(\lambda-1)}, \;\; v = \mathcal{G}^{(\lambda)}$	$x \in [-1, 1]$, $\lambda > 1/2$.
$SCP6. \quad u = \left(\zeta^2 - x^2\right)\mathcal{G}^{(\lambda-1)}, \;\; v = \mathcal{G}^{(\lambda)}$	$x \in [-1, 1]$, $\lambda > 1/2$, $\lvert\zeta\rvert \geq 1$.
$SCP7$ $u = \mathcal{G}^{(1/2)} + \dfrac{\eta}{2}\left(\delta(x-1) + \delta(x+1)\right)$ $v = \mathcal{G}^{(1/2)}$	$x \in [-1, 1]$, $\eta \geq 0$.

Table 2.4: Symmetric coherent pairs, v classical

In order to see additional results on algebraic properties of the Sobolev orthogonal polynomials with respect to coherent and symmetric coherent pairs, the reader can also refer to [80].

2.3 Asymptotics and coherent pairs

As soon as the paper [55] appeared, the study of asymptotic results for coherent pairs and symmetric coherent pairs was almost completely solved throughout the next decade. In this section we will describe the main achievements in this direction. The first result about outer relative asymptotics for coherent pairs is given in [91], where the coherent pairs $\{\mu_0, \mu_1\}$ with *supp* $(\mu_0) = [-1, 1]$ are investigated.

Theorem 51. *([91]). Let $\{\mu_0,\mu_1\}$ be a coherent pair of measures, with supp $(\mu_0)=[-1,1]$. Then*

$$\lim_{n\to\infty}\frac{S_n^{\lambda}(x)}{P_n(x;\mu_1)}=\frac{2}{\varphi'(x)},$$

uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$. φ is defined in (1.29).

In [74] the authors extend the validity of above result to the symmetric coherent pair setting, when *supp* $(\mu_0)=[-1,1]$, (Gegenbauer case). In the same way, in [98], additional relative and strong asymptotics are deduced in both $\mathbb{C}\backslash[-1,1]$ and $(-1,1)$ respectively. In the next theorem we summarize some of these results.

Theorem 52. *([98]). Let $\{\mu_0,\mu_1\}$ denote a coherent pair and let $\{S_n^{\lambda}\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to (2.5). Also $\left\{P_n^{(\alpha,\beta)}\right\}_{n\geq 0}$ will denote the classical monic Jacobi polynomials*

i). If $d\mu_1=(1-x)^{\alpha}(1+x)^{\beta}dx$ on $(-1,1)$, then

$$\lim_{n\to\infty}\frac{S_n^{\lambda}(x)}{P_n^{(\alpha-1,\beta-1)}(x)}=1,$$

uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$, moreover, if $0<\theta<\pi$, then

$$S_n^{\lambda}(\cos\theta)=P_n^{(\alpha-1,\beta-1)}(\cos\theta)+\mathcal{O}\left(n^{-5/2}\right),$$

uniformly on compact subsets of $(0,\pi)$.

ii). If $d\mu_0=(1-x)^{\alpha}(1+x)^{\beta-1}dx$ on $(-1,1)$, (as a consequence $d\mu_1=\dfrac{1}{|x-\xi|}(1-x)^{\alpha+1}(1+x)^{\beta}dx+M\delta(x-\xi)$), then

$$\lim_{n\to\infty}\frac{S_n^{\lambda}(x)}{P_n^{(\alpha,\beta-1)}(x)}=1,$$

uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$. In addition, if $0<\theta<\pi$, then

$$\begin{aligned}&S_n^{\lambda}(\cos\theta)\\ =\;&\frac{n+\alpha+\beta-1}{2n+\alpha+\beta}P_n^{(\alpha,\beta-1)}(\cos\theta)+\frac{n+\beta-1}{2n+\alpha+\beta}c_{n-1}P_{n-1}^{(\alpha,\beta-1)}(\cos\theta)+\mathcal{O}\left(n^{-5/2}\right),\end{aligned}$$

uniformly on compact subsets of $(0,\pi)$. Here

$$c_n=\begin{cases}1/\varphi(\xi)+\mathcal{O}\left(n^{-1}\right), & \text{if } M=0,\\ \varphi(\xi)+\mathcal{O}\left(n^{-1}\right), & \text{if } M\geq 0.\end{cases}$$

Complementary results on asymptotics in the Jacobi case can be found in [104]. Finally, in the Gegenbauer case, asymptotic results are analyzed in [14] where $d\mu_0$ is the Gegenbauer measure and

$$d\mu_1=\left(\kappa_1+\frac{\kappa_2}{1+qx^2}\right)\left(1-x^2\right)d\mu_0+\kappa_2M_q\left(\delta'\left(x+\frac{1}{\sqrt{-q}}\right)+\delta'\left(x-\frac{1}{\sqrt{-q}}\right)\right),$$

holds, besides $\kappa_1,\kappa_2\geq 0$, $q\geq -1$, $M_q>0$ if $-1\leq q<0$ and $M_q=0$ if $q\geq 0$. This case should be considered as an extension of symmetric coherence because when $\kappa_1=0$, $\kappa_2>0$, and $q\neq 0$ we get that the pair $\{\mu_0,\mu_1\}$ is a symmetric coherent pair.

We now describe some important asymptotic results in the case of unbounded support measures. With respect to coherent pair CP3, (Type I in the literature), in [105], by using potential theory, both the root asymptotics of the Sobolev polynomials and the zero distribution with respect to the zeros of Laguerre polynomials are studied. *Asymptotic expansions,* (see [26]), of Sobolev polynomials for pairs CP1 and CP3 are studied in [97]. With respect to coherent pair $CP1$, and the outer relative asymptotics for the corresponding Sobolev orthogonal polynomials we get the next result.

Theorem 53. *(See [97]). Let* $\{S_n^\lambda\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to (2.5) with* $d\mu_0 = x^\alpha e^{-x}dx$ *and* $d\mu_1 = \dfrac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta(x-\xi)$*, and the leading coefficient of* S_n^λ *equal to the leading coefficient of* L_n^α*. If* $\eta := \dfrac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}$ *then*

$$\lim_{n\to\infty}\frac{S_n^\lambda(x)}{L_n^{(\alpha-1)}(x)} = \begin{cases} \dfrac{1}{1-\eta}\left(1+\sqrt{\dfrac{\xi}{x}}\right), & \text{if } M=0, \\ \dfrac{1}{1-\eta}\left(1-\sqrt{\dfrac{\xi}{x}}\right), & \text{if } M>0, \end{cases}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

The Mehler–Heine and Plancherel–Rotach type formulas for Sobolev polynomials in the case CP3 are obtained in [10]. More precisely,

Theorem 54. *(See [10]). Let* $\{S_n^\lambda\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to (2.5) with* $d\mu_0 = x^\alpha e^{-x}dx$ *and* $d\mu_1 = \dfrac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta(x-\xi)$ *where the leading coefficient of* S_n^λ *is* $(-1)^n/n!$*. Besides, we define*

$$d(\xi) = \begin{cases} -\sqrt{-\xi}, & \text{if } M=0, \\ \sqrt{-\xi}, & \text{if } M>0, \end{cases}$$

and

$$s(x) = x^{-(\alpha-1)/2}J_{\alpha-1}\left(2\sqrt{x}\right) - (\alpha+1)\,x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right).$$

Then,

a)*. (Mehler–Heine). It holds*

$$\lim_{n\to\infty}\frac{S_n^\lambda(x/n)}{n^{\alpha-1/2}} = -\frac{d(\xi)}{1-\dfrac{1}{\varphi\left(\frac{\lambda+2}{2}\right)}}x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right), \quad \xi<0,$$

$$\lim_{n\to\infty}\frac{S_n^\lambda(x/n)}{n^{\alpha-1}} = \frac{1}{1-\dfrac{1}{\varphi\left(\frac{\lambda+2}{2}\right)}}s(x), \quad \xi=0,\ M>0,$$

and

$$\lim_{n\to\infty}\frac{S_n^\lambda(x/n)}{n^{\alpha-1}} = \frac{1}{1-\dfrac{1}{\varphi\left(\frac{\lambda+2}{2}\right)}}x^{-(\alpha-1)/2}J_{\alpha-1}\left(2\sqrt{x}\right), \quad \xi=M=0,$$

all uniformly on compact subsets of $\mathbb{C}$.

b). *(Plancherel–Rotach).*

$$\lim_{n\to\infty}\frac{S_n^\lambda(nx)}{L_n^\alpha(nx)}=\frac{\varphi\left(\frac{x-2}{2}\right)+1}{\varphi\left(\frac{x-2}{2}\right)+\dfrac{1}{\varphi\left(\frac{\lambda+2}{2}\right)}},$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$.

The paper [9] deals with asymptotics for symmetric coherent pairs when one of the two functionals is the classical Hermite functional, in particular the cases SCP1 and SPC4 in tables 12 and 13. The authors provided relative outer asymptotics for the respective Sobolev polynomials. A Plancherel-Rotach type formula for scaled polynomials is deduced. Indeed, for the pair SCP4 we get the next result.

Theorem 55. *([9]). Let* $\{u,v\}$ *be the symmetric coherent pair given by* $u=\left(x^2+\zeta^2\right)\mathcal{H}$ *and* $v=\mathcal{H}$ *and let* $\left\{S_n^\lambda\right\}_{n\geq 0}$ *be the Sobolev polynomials orthogonal with respect to the inner product*

$$\langle p(x),q(x)\rangle_S:=\int_{\mathbb{R}}p(x)q(x)\left(x^2+\zeta^2\right)e^{-x^2}dx+\lambda\int_{\mathbb{R}}p'(x)q'(x)e^{-x^2}dx.$$

Assume that the leading coefficient of H_n *and* S_n^λ *is* 2^n. *Then*

i). *(Outer Relative asymptotics).*

$$\lim_{n\to\infty}\frac{S_n^\lambda(x)}{H_n(x)}=\frac{\varphi(2\lambda+1)}{\varphi(2\lambda+1)-1},$$

holds uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

ii). *(Outer Relative asymptotics for scaled polynomials).*

$$\lim_{n\to\infty}\frac{S_n^\lambda(\sqrt{n}x)}{H_n(\sqrt{n}x)}=\frac{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1},$$

holds uniformly on compact sets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$.

iii). *(Plancherel-Rotach type formula).*

$$\lim_{n\to\infty}\left(\frac{2^n}{n!\sqrt{\pi}}\right)^{1/2}\frac{S_n^\lambda(\sqrt{n}x)}{\prod_{k=1}^{n}\varphi\left(\sqrt{\dfrac{n}{2k}}x\right)}=\left(\frac{x^2-2}{x^2}\right)^{-1/4}\frac{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1},$$

uniformly on compact sets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$.

In order to complete the description of the asymptotics for the symmetric coherent pair $\left\{\left(x^2+\zeta^2\right)\mathcal{H},\mathcal{H}\right\}$, the next result gives Mehler-Heine type formulas.

Theorem 56. *([30]). (Mehler-Heine type formulas).*

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}S_{2n}^\lambda(x/(2\sqrt{n}))}{2^n n!}=\frac{\varphi(2\lambda+1)}{\varphi(2\lambda+1)-1}\frac{\cos(x)}{\sqrt{\pi}},$$

and

$$\lim_{n\to\infty}\frac{(-1)^n S_{2n+1}^\lambda(x/(2\sqrt{n}))}{2^{n+1}n!}=\frac{\varphi(2\lambda+1)}{\varphi(2\lambda+1)-1}\frac{\sin(x)}{\sqrt{\pi}},$$

both uniformly on compact subsets of $\mathbb{C}$.

The references [28] and [76] are highly recommended surveys on asymptotics for Sobolev Polynomials in the unbounded support setting, (Hermite and Laguerre), as well as for properties and asymptotic behavior of the corresponding zeros.

Chapter 3

(1,1)-Coherent Pairs

The concept of Coherent Pair admits a natural generalization as we will see now. Assume that $\{\mu_0, \mu_1\}$ is a coherent pair of measures, i.e., the corresponding SMOP satisfy

$$P_n\left(x;\mu_1\right) = \frac{P'_{n+1}(x;\mu_0)}{n+1} + \sigma_n \frac{P'_n(x;\mu_0)}{n}, \quad \sigma_n \neq 0, \quad n \geq 1,$$

and consider the Sobolev inner product

$$\langle p(x), q(x)\rangle_S := \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1, \ \lambda > 0, \tag{3.1}$$

$p, q \in \mathbf{P}$. Then, as it was described in the previous chapter, it is possible to prove that the SMOP $\left\{S_n^{\lambda}\right\}_{n\geq 0}$ associated with (3.1) and $\left\{P_n\left(.;\mu_0\right)\right\}_{n\geq 0}$ are related through the algebraic relation

$$S_{n+2}^{\lambda}(x) + \eta_n(\lambda) S_{n+1}^{\lambda}(x) = P_{n+2}(x;\mu_0) + \sigma_n P_{n+1}(x;\mu_0), \quad n \geq 0. \tag{3.2}$$

The proof is based on the fact that, for every n, the coefficients of S_n^{λ} are rational functions in the variable λ where the degree of the numerator it is not greater than the degree of the denominator and, as a consequence, it is possible to define the sequence of polynomials $\left\{W_n\right\}_{n\geq 0}$, where

$$W_n(x) = \lim_{\lambda\to\infty} S_n^{\lambda}(x), \quad n \geq 0.$$

In addition, for every n, $W'_{n+1}(x) = (n+1)P_n(x;\mu_1)$. Then we get the question that arises in natural way, is the converse statement true?, i.e. (3.2) implies the coherence? The answer is no. Indeed, if (3.2) holds, then it is not difficult to prove that there exist constants δ_n not necessarily zero such that the sequences $\left\{P_n\left(.;\mu_0\right)\right\}_{n\geq 0}$ and $\left\{P_n(.;\mu_1)\right\}_{n\geq 0}$ satisfy

$$\frac{P'_{n+2}(x;\mu_0)}{n+2} + \sigma_n \frac{P'_{n+1}(x;\mu_0)}{n+1} = P_{n+1}(x;\mu_1) + \delta_n P_n(x;\mu_1), \quad n \geq 0, \tag{3.3}$$

which is not a coherence relation unless $\delta_n = 0$ for every n. This result gives rise to a natural generalization of the concept of coherent pair, generalization introduced in [33] and that motivates the contents of this chapter. In section 3.1 we formalize this topic and we present the main results. We also exhibit the full classification of all pairs $\{\mu_0, \mu_1\}$ whose SMOP satisfy (3.3). The end of the chapter will be dedicated to show the extensions of the concept and the main contributions.

3.1 $(1,1)-$Coherent Pairs and Classification

Definition 57. *Let $\{u,v\}$ be a pair of quasi-definite linear functionals with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ as the corresponding SMOP. $\{u,v\}$ is said to be a $(1,1)-$**coherent pair** if there exist sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ such that*

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_{n+1}(x)}{n+1} = Q_{n+1}(x) + a_n Q_n(x), \qquad n \geq 0, \tag{3.4}$$

holds, with $b_n \neq 0$.

This concept was introduced in [33]. There, the associated *inverse problem* is solved. This means that assuming the $(1,1)-$ coherence relation (3.4), then an algebraic relation between the linear functionals is found. To be more precise,

Theorem 58. *([33]). If $\{u,v\}$ is a $(1,1)-$coherent pair given by (3.4), such that $a_0 \neq b_0$, (or equivalently $Q_n \neq \frac{P'_{n+1}}{n+1}$, for $n \geq 1$), then*

***i)**. Either u is a semiclassical linear functional of class at most 1, i. e. there exist polynomials $\widetilde{\beta}$ and φ with $\deg(\widetilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that*

$$D[\widetilde{\beta} u] = -\varphi u.$$

***ii)**. Or v is a semiclassical linear functional of class at most 1, i. e. there exist polynomials $\widetilde{\beta}$ and φ with $\deg(\widetilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that*

$$D[\widetilde{\beta} v] = -\varphi v.$$

Besides there exists a constant ζ such that the pair $\{u,v\}$ satisfies

$$(x-\zeta)v = \widetilde{\beta} u. \tag{3.5}$$

Furthermore, in [33] all $(1,1)-$ coherent pairs of linear functionals were determined. In each case **i)**. and **ii)**. of the above theorem the pair $\{u,v\}$ is called either *type I* or *type II*, respectively. In Table 3.1 the $(1,1)-$coherent pairs of type I are described when u is classical. In Table 3.2 the $(1,1)-$coherent pairs of type I are described when u is semiclassical of class 1. Here and in the sequel, μ_0 and μ_1 are the measures associated with u and v, respectively. In Table 3.3 the $(1,1)-$coherent pairs of type II are described when v is classical. Finally, in Table 3.4 we write the $(1,1)-$coherent pairs when v is semiclassical of class 1, according to [20] where the full classification of semiclassical functionals of class 1 is given as well as their integral representation in the positive-definite framework.

$\{u, v\}$	Restriction and support of u	$\widetilde{\beta}(x)$ in (3.5)
$J_{1,1}$ $u = \mathcal{J}^{(\alpha,\beta)}_{[0,1]}$ $v = \frac{\lvert x-\xi\rvert}{\lvert x-\zeta\rvert}\mathcal{J}^{(\alpha+1,\beta+1)}_{[0,1]} + M\delta(x-\zeta)$	$\alpha, \beta > -1$ $\lvert\zeta - 1/2\rvert > 1/2, \quad \lvert\xi - 1/2\rvert > 1/2,$ If $\zeta = \xi$, then $0 < \zeta < 1, x \in [0, 1]$	$x(x-\xi)(x-1)$
$B_{1,1}$ $u = \mathcal{B}^{(\alpha)}, v = \frac{x-\xi}{x-\zeta}\mathcal{B}^{(\alpha+2)} + M\delta(x-\zeta)$	No positive- definite	$(x-\xi)x^2$
$L_{1,1}$ $u = \mathcal{L}^{(\alpha)}, v = \frac{x-\xi}{x-\zeta}\mathcal{L}^{(\alpha+1)} + M\delta(x-\zeta)$	$\alpha > -1, \zeta, \xi < 0,$ If $\zeta = \xi$ then $\zeta > 0, x \in [0, \infty)$.	$(x-\xi)x$
$J_{1,2}$ $u = \mathcal{J}^{(\alpha,\beta)}_{[0,1]}, v = \frac{\mathcal{J}^{(\alpha+1,\beta+1)}_{[0,1]}}{\lvert x-\zeta\rvert} + M\delta(x-\zeta)$	$\alpha, \beta > -1$ $\lvert\zeta - 1/2\rvert > 1/2$ $x \in [0, 1]$	$x(x-1)$
$B_{1,2}$ $u = \mathcal{B}^{(\alpha)}, v = \frac{1}{x-\zeta}\mathcal{B}^{(\alpha+2)} + M\delta(x-\zeta)$	No positive-definite	x^2
$L_{1,2}$ $u = \mathcal{L}^{(\alpha)}, v = \frac{1}{x-\zeta}\mathcal{L}^{(\alpha+1)} + M\delta(x-\zeta)$	$\alpha > -1, \quad \zeta < 0,$ $x \in [0, \infty)$	x

Table 3.1: (1,1)-Coherent pairs of Type I, u classical

$\{\mu_0, \mu_1\}$	Restriction and support support of μ_0	$\widetilde{\beta}(x)$ in (3.5)
$S_{1,1}$ $d\mu_0 = (1-x)^\alpha x^\beta \lvert x-\lambda\rvert^\gamma \, dx$ $d\mu_1 = \frac{(1-x)^{\alpha+1}}{\lvert x-\zeta\rvert} x^{\beta+1} \lvert x-\lambda\rvert^{\gamma+1} \, dx + M\delta(x-\zeta)$	$\alpha, \beta, \gamma > -1, \quad \left\lvert \lambda - \frac{1}{2}\right\rvert > \frac{1}{2},$ $\alpha\beta\gamma \neq 0, \quad \left\lvert \zeta - \frac{1}{2}\right\rvert > \frac{1}{2}$ $x \in [0,1]$	$x(x-1)(x-\lambda)$
$S_{1,2}$ $d\mu_0 = (1-x)^\alpha x^\beta e^{-\frac{\gamma}{x}} dx$ $d\mu_1 = \frac{(1-x)^{\alpha+1}}{\lvert x-\zeta\rvert} x^{\beta+2} e^{-\frac{\gamma}{x}} dx + M\delta(x-\zeta\ )$	$\alpha\beta \neq 0, \ \gamma > 0, \ \alpha > -1$ $\left\lvert \zeta - \frac{1}{2}\right\rvert > \frac{1}{2}$ $x \in [0,1]$	$x^2(x-1)$
$S_{1,3}$ $d\mu_0 = x^\alpha e^{\frac{\beta}{x} - \frac{2}{x^2}} dx$ $d\mu_1 = \frac{1}{\lvert x-\zeta\rvert} x^{\alpha+3} e^{\frac{\beta}{x} - \frac{2}{x^2}} dx + M\delta(x-\zeta)$	No supported on the the real line.	x^3
$S_{1,4}$ $d\mu_0 = (1-x)^\alpha x^\beta e^{-\lambda x} dx$ $d\mu_1 = \frac{(1-x)^{\alpha+1} x^{\beta+1}}{\lvert x-\zeta\rvert} e^{-\lambda x} dx + M\delta(x-\zeta) + N\delta(x)$	$\alpha, \beta > -1,$ $\alpha\beta\lambda \neq 0, \quad \left\lvert \zeta - \frac{1}{2}\right\rvert > \frac{1}{2}$ $x \in [0,1]$	$x(x-1)$
$S_{1,5}$ $d\mu_0 = x^\alpha e^{-x+\frac{\beta}{x}} dx,$ $d\mu_1 = \frac{1}{x-\zeta} x^{\alpha+2} e^{-x+\frac{\beta}{x}} dx + M\delta(x-\zeta)$	$\beta, \ \zeta < 0, \alpha > -1$ $x \in [0,\infty)$	x^2
$S_{1,6}$ $d\mu_0 = x^{2\mu} e^{-x^2-\lambda x} dx$ $d\mu_1 = \frac{1}{x-\zeta} x^{2\mu+1} e^{-x^2-\lambda x} dx + M\delta(x-\zeta)$	$\zeta < 0, \quad \mu > -1/2, \quad \mu \neq 0$ $x \in [0,\infty)$	x

Table 3.2: (1,1)-Coherent pairs of Type I, u semiclassical of class 1

$\{u,v\}$	Restriction and support of v	$\widetilde{\beta}(x)$ in (3.5)
$J_{2,1}$ $v=\mathcal{J}_{[0,1]}^{(\alpha+1,\beta+1)}$ $u=\frac{\lvert x-\zeta\rvert}{\lvert x-\xi\rvert}\mathcal{J}_{[0,1]}^{(\alpha,\beta)}+M\delta(x-\xi)$	$\alpha,\beta>-1$ $\lvert\zeta-\frac{1}{2}\rvert>\frac{1}{2},\lvert\xi-\frac{1}{2}\rvert>\frac{1}{2}$ or $\zeta=\xi\in[0,1],x\in[0,1]$	$x(x-\xi)(x-1)$
$J_{2,2}$ $v=\mathcal{J}_{[0,1]}^{(0,\beta+1)}$ $u=\frac{1}{\lvert x-\xi\rvert}\mathcal{J}_{[0,1]}^{(0,\beta)}+M\delta(x-\xi)+N\delta(x-1)$	$\beta>-1$ $\lvert\xi-\frac{1}{2}\rvert>\frac{1}{2}$ $x\in[0,1]$	$x(x-\xi)(x-1)$
$B_{2,1}$ $v=\mathcal{B}^{(\alpha)}$ $u=\frac{(x-\zeta)}{x^2(x-\xi)}\mathcal{B}^{(\alpha)}+M_0\delta(x)+M_1\delta'(x)$ $+M_2\delta_{0,\xi}\delta''(x)+N\delta(x-\xi)\mathbf{1}_{\mathbb{R}-\{0\}}(\xi)$	No positive-definite	$(x-\xi)x^2$
$L_{2,1}$ $v=\mathcal{L}^{(\alpha+1)},u=\frac{x-\zeta}{x-\xi}\mathcal{L}^{(\alpha)}+M\delta(x-\xi)$	$\alpha>-1,\quad\zeta,\xi<0$ $x\in[0,\infty)$	$(x-\xi)x$
$L_{2,2}$ $v=\mathcal{L}^{(0)}$ $u=\frac{1}{x-\xi}\mathcal{L}^{(0)}+M\delta(x-\xi)+N\delta(x)$	$\zeta=0,\xi<0$ $x\in[0,\infty)$	$(x-\xi)x$
$J_{2,3}$ $v=\mathcal{J}_{[0,1]}^{(\alpha+1,\beta+1)},u=\lvert x-\zeta\rvert\,\mathcal{J}_{[0,1]}^{(\alpha,\beta)}$	$\alpha,\beta>-1$ $\lvert\zeta-\frac{1}{2}\rvert>\frac{1}{2},x\in[0,1]$	$x(x-1)$
$B_{2,2}$ $v=\mathcal{B}^{(\alpha)}$ $u=\frac{(x-\zeta)}{x^2}\mathcal{B}^{(\alpha)}+M_0\delta(x)+M_1\delta'(x)$	No positive-definite	x^2
$L_{2,3}$ $v=\mathcal{L}^{(\alpha+1)},u=(x-\zeta)\,\mathcal{L}^{(\alpha)}$	$\alpha>-1,\quad\zeta<0$ $x\in[0,\infty)$	x
$L_{2,4}$ $v=\mathcal{L}^{(0)},u=\mathcal{L}^{(0)}+M\delta(x)$	$\zeta<0$ $x\in[0,\infty)$	x

Table 3.3: (1,1)-Coherent pairs of Type II, v classical

$\{\mu_0, \mu_1\}$	Restriction and Support of μ_1	$\widetilde{\beta}(x)$ in (3.5)
$S_{2,1}$ $d\mu_0 = \lvert x-\zeta\rvert\,(1-x)^{\alpha}x^{\beta}\,\lvert x-\lambda\rvert^{\gamma}\,dx$ $d\mu_1 = (1-x)^{\alpha+1}x^{\beta+1}\,\lvert x-\lambda\rvert^{\gamma+1}\,dx$	$\alpha, \beta, \gamma > -1, \quad \alpha\beta\gamma \neq 0$ $0 < \lambda < 1, \ \left\lvert \zeta - \frac{1}{2} \right\rvert > \frac{1}{2}$ $x \in [0,1]$	$x(x-1)(x-\lambda)$
$S_{2,2}$ $d\mu_0 = \lvert x-\zeta\rvert\,(1-x)^{\alpha}x^{\beta}e^{-\frac{\gamma}{x}}dx$ $d\mu_1 = (1-x)^{\alpha+1}x^{\beta+2}e^{-\frac{\gamma}{x}}dx$	$\alpha, \beta > -1$ $\alpha\beta \neq 0, \ \left\lvert \zeta - \frac{1}{2} \right\rvert > \frac{1}{2}$ $\gamma > 0, \quad x \in [0,1]$	$x^2(x-1)$
$S_{2,3}$ $d\mu_0 = x^{\beta}e^{-\frac{\gamma}{x}}dx + M\delta(x-1)$ $d\mu_1 = x^{\beta+2}e^{-\frac{\gamma}{x}}dx$	$\beta > -1, \ \beta \neq 0$ $\gamma > 0, \quad x \in [0,1]$	$x^2(x-1)$
$S_{2,4}$ $d\mu_0 = (x-\zeta)x^{\alpha}e^{\frac{\beta}{x}-\frac{2}{x^2}}dx$ $d\mu_1 = x^{\alpha+3}e^{\frac{\beta}{x}-\frac{2}{x^2}dx}$	No positive-definite	x^3
$S_{2,5}$ $d\mu_0 = (x-\zeta)x^{\alpha}(1+x)^{\beta}e^{-\lambda x}dx,$ $d\mu_1 = x^{\alpha+1}(1+x)^{\beta+1}e^{-\lambda x}dx$	$\alpha, \beta > -1$ $\alpha\beta \neq 0$ $\lambda > 0, \ \zeta < 0$ $x \in [0,\infty)$	$x\,(x+1)$
$S_{2,6}$ $d\mu_0 = x^{\alpha}e^{-\lambda x}dx + M\delta(x+1)$ $d\mu_1 = x^{\alpha+1}e^{-\lambda x}dx$	$\alpha > -1, \ \alpha \neq 0$ $\lambda > 0$ $\zeta < -1$ o $\zeta < 0.$ $x \in [0,\infty)$	$x\,(x+1)$
$S_{2,7}$ $d\mu_0 = x^{\beta}e^{-\lambda x}dx + M\delta(x-1)$ $d\mu_1 = x^{\beta+1}e^{-\lambda x}dx$	$\beta > -1, \ \beta \neq 0$ $\lambda > 0$ $\zeta < -1$ o $\zeta < 0.$ $x \in [0,\infty)$	$x\,(x-1)$
$S_{2,8}$ $d\mu_0 = (x-\zeta)x^{\alpha}e^{-x+\frac{\beta}{x}}dx$ $d\mu_1 = x^{\alpha+2}e^{-x+\frac{\beta}{x}}dx$	$\beta < 0, \alpha > -1, \ \zeta < 0$ $x \in [0,\infty)$	x^2
$S_{2,9}$ $d\mu_0 = e^{-x+\frac{\beta}{x}}dx + M\delta(x)$ $d\mu_1 = xe^{-x+\frac{\beta}{x}}dx$	$\beta < 0,$ $x \in [0,\infty)$	x^2
$S_{2,10}$ $d\mu_0 = (x-\zeta)x^{2\mu}e^{-x^2-\lambda x}dx$ $d\mu_1 = x^{2\mu+1}e^{-x^2-\lambda x}dx$	$\mu > -1/2, \ \zeta < 0$ $x \in [0,\infty)$	x
$S_{2,11}$ $d\mu_0 = e^{-x^2-\lambda x}dx + M\delta(x)$ $d\mu_1 = e^{-x^2-\lambda x}dx$	$\zeta < 0$ $x \in [0,\infty)$	x

Table 3.4: (1,1)-Coherent pairs of Type II, v semiclassical of class 1

3.2 $(M,N)-$coherent pairs of order (m,k)

The work of A. Iserles et al, (see [55]), where the concept of coherent pair is introduced in connection with Sobolev Orthogonality, was the reagent of an important series of investigations that have not only contributed to solve problems associated with the concept itself, as the asymptotic behavior or the classification of all coherent pairs, but have contributed to extend and generalize the concept. We are going to start from the end. Let $\{u,v\}$ be a pair of quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ the corresponding SMOP.

Definition 59. *$\{u,v\}$ is a $(M,N)-$coherent pair of order (m,k) if the corresponding SMOP satisfies*

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{[m]}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{[k]}(x), \tag{3.6}$$

where $m,k,M,N \in \mathbb{N}\cup\{0\}$, $P_n^{[i]}(x) := \dfrac{P_{n+i}^{(i)}(x)}{(n+1)_i}$, and $\{a_{i,n}\}_{n\geq 0}$, $\{b_{j,n}\}_{n\geq 0}$, $0\leq i\leq M$, $0\leq j\leq N$ are sequences of numbers with $a_{0,n} = b_{0,n} = 1$. If $k=0$, then we will say that $\{u,v\}$ is an $(M,N)-$coherent pair of order m

The notion of "$(M,N)-$coherence" was introduced in [60] for order one, where the natural connection with Sobolev polynomials orthogonal with respect to a Sobolev inner product like 3.1 is presented. Notice that the coherence defined in [55] occurs when $M=1$, $N=0$, $m=1$ and $k=0$, i.e. a coherent pair is a $(1,0)-$coherent pair of order 1, and the $(1,1)-$coherence defined in the above section occurs when $M=N=1$, $m=1$, $k=0$, then in such a case we get a $(1,1)-$coherent pair of order 1. In the context of inverse problems the question is how functionals that form a $(M,N)-$coherent pair of order (m,k) are related. The answer is given in [59] under certain natural conditions imposed on the $M\times N$ sequences $\{a_{i,n}\}_{n\geq 0}$ and $\{b_{j,n}\}_{n\geq 0}$, $i,j\geq 1$. In fact, the authors prove that there exist polynomials ψ_{N+m+i} and ϕ_{M+k+i}, $i=0,1$, with degrees defined by their subscript such that if $k\geq m$, then

$$D^{k-m}\left(\psi_{N+m+i}u\right) = \phi_{M+k+i}v,$$

holds. In such a paper the relation between the formal Stieltjes series associated with the functionals u and v is also determined. In [58] this general approach is also studied. There, when $m>k+1$ it is proved that the functionals u and v are related through a rational relation and they are semiclassical. To be more precise we present the next

Theorem 60. *([58]). Let $\{u,v\}$ be a $(M,N)-$coherent pair of order (m,k) given by (3.6), with $m\geq k$, and let $A_{M+N} = [\alpha_{i,j}]_{i,j=0}^{M+N-1}$ be a square matrix with entries*

$$\alpha_{i,j} = \begin{cases} a_{j-i,j}, & \text{if } 0\leq i\leq N-1 \text{ and } i\leq j\leq M+i, \\ b_{j-i+N,j}, & \text{if } N\leq i\leq M+N-1 \text{ and } i-N\leq j\leq i, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $a_{0,j_1} = b_{0,j_2} = 1$ for $0\leq j_1\leq N-1$ and $0\leq j_2\leq M-1$, besides $|A_{M+N}|\neq 0$. Then there exist polynomials ϕ_{M+k+n} and ψ_{N+m+n} of degrees $M+k+n$ and $N+m+n$, respectively, such that

$$D^{m-k}\left(\phi_{M+k+n}v\right) = \psi_{N+m+n}u, \quad n\geq 0,$$

and each one of the functionals u and v is a rational modification of the other one, i.e., there exist polynomials φ and ρ such that

$$\varphi(x)u = \rho(x)v.$$

Moreover, we have the following.

***(i)**. If $m = k$, then u is a semiclassical linear functional if and only if v is semiclassical.*

***(ii)**. If $m > k$, then u and v are semiclassical linear functionals.*

In addition, and as usual, the inner product

$$\langle p(x), q(x)\rangle_S := \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1, \ \lambda > 0, \tag{3.7}$$

is considered when $k = 0$. In this way, a generalization of (3.2) is obtained.

Theorem 61. *([58]). Let $\{\mu_0, \mu_1\}$ be a $(M, N)-$coherent pair of measures of order m with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ as the respective SMOP, besides, let $\{S_n^\lambda\}_{n\geq 0}$ be the monic Sobolev polynomials orthogonal with respect to (3.7). Then*

$$S_n^\lambda(x) + \sum_{i=m}^{n-1} \frac{\langle T_n, S_i^\lambda(x)\rangle_{\mu_0}}{\|S_i^\lambda\|_S^2} S_i^\lambda(x) = P_n(x) + \sum_{i=m}^{n-1} \frac{\langle T_n, P_i(x)\rangle_{\mu_0}}{\|P_i\|_{\mu_0}^2} P_i(x),$$

for $n \geq m$, and $S_n^\lambda = P_n$ for $n \leq m$, hold, where $T_n(x) := \lim_{\lambda\to\infty} S_n^\lambda(x)$, $n \geq 0$, and $\langle p, q\rangle_{\mu_0} := \int_{\mathbb{R}} p(x)q(x)d\mu_0$, being $\|.\|_{\mu_0}$ the norm induced by this inner product.

Finally, the algorithm proposed in [55] for $(1, 0)-$coherent pairs of order 1 is generalized in a natural way for $(M, N)-$coherent pairs of measures of order m. We are going to see now what are the contributions that allowed to reach the state of generality presented in (3.7). We have already mentioned the importance of the work [55] as the genesis of coherence and we already described the extension so-called $(1, 1)-$coherent pair in [33]. However, before this last work, a generalization of the concept of Coherent Pairs was proposed in [29] where, given a pair $\{\mu_0, \mu_1\}$ of positive Borel measures, the existence of constants A_n, B_n such that

$$P_n(x; \mu_1) = \frac{P'_{n+1}(x; \mu_0)}{n+1} + A_n \frac{P'_n(x; \mu_0)}{n} + B_n \frac{P'_{n-1}(x; \mu_0)}{n-1}, \quad n > 1,$$

holds was assumed. Notice that then $\{\mu_0, \mu_1\}$ is a $(2, 0)-$coherent pair of order 1, if $B_n = 0$ we get the coherence and if $A_n = 0$ and μ_0 and μ_1 are symmetric then we get the symmetric coherence. In order to describe the next extension, in [75] the generalization called $k-$coherence is proposed.

Definition 62 ($k-$coherence)**.** *Let μ_i , $i = 0, 1$, be two positive Borel measures. Then $\{\mu_0, \mu_1\}$ is a $k-$**coherent pair of measures** ($k \geq 0$) if for every $n \in \mathbb{N}$, $n \geq k+1$, there exist $\sigma_{n,\mu_0,\mu_1}^{(n)}, \ldots, \sigma_{n,\mu_0,\mu_1}^{(n-k)} \in \mathbb{R}$, with $\sigma_{n,\mu_0,\mu_1}^{(n-k)} \neq 0$ such that*

$$P_n(x; \mu_1) = \frac{P'_{n+1}(x; \mu_0)}{n+1} + \sum_{j=0}^{k} \sigma_{n,\mu_0,\mu_1}^{(n-j)} \frac{P'_{n-j}(x; \mu_0)}{n-j}.$$

This is a $(k+1, 0)-$coherence of order $(1, 0)$. Of course, the $0-$coherence is the coherence defined in [55] and the $1-$coherence is defined in [29]. With respect to this generalization, the $1-$coherence is studied again in [65], but from a more general approach considering pairs of quasi-definite functionals, and proving that if (u, v) is a $1-$coherent pair, then u and v must be semiclassical, of class at most 6 and 2 respectively. In the paper [3] the next result is obtained for the $(1, 1)-$coherence of order 0.

Theorem 63. *([3]). Let $\{u,v\}$ be a pair of quasi-definite linear functionals with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ as the respective SMOP. Then*

$$P_n(x)+s_nP_{n-1}(x)=Q_n(x)+t_nQ_{n-1}(x), \quad n\geq 1,$$

holds for sequences $\{s_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ such that with $s_1\neq t_1$ and $s_nt_n\neq 0$ for $n\geq 1$, if and only if for every $n\geq 1$, $P_n \neq Q_n$ and there exist complex numbers λ, a, $\widetilde{a}$ such that

$$(x-\widetilde{a})u=\lambda(x-a)v. \tag{3.8}$$

In [4] the corresponding direct problem is studied, i.e. under the assumptions that (3.8) holds for two functionals u and v, and that u is quasi-definite, then the conditions under which v is quasi-definite are deduced.

As for the solution of the inverse problem for the (M,N) coherence of order 0, the natural conjecture that arise is that the degrees of the polynomials in the rational relation that u and v should satisfy, depend on M and N. Indeed in [106] the next theorem is proved.

Theorem 64. *([106]). Let $\{u,v\}$ be a $(M,N)-$coherent pair of order 0, i.e., the respective SMOP $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy*

$$P_n(x)+\sum_{i=1}^{M}r_{i,n}P_{n-i}(x)=Q_n(x)+\sum_{i=1}^{N}s_{i,n}Q_{n-i}(x), n\geq 0,$$

further assume that $r_{M,M+N}s_{N,M+N}\neq 0$, and $|A_{M+N}|\neq 0$ for $A_{M+N}=[\alpha_{i,j}]_{i,j=1}^{M+N-1}$ with entries

$$\alpha_{i,j}=\begin{cases} r_{j-i,j-1}, & \text{if } 0\leq i\leq N \text{ and } i\leq j\leq M+i\\ s_{j-i+N,j-1}, & \text{if } N+1\leq i\leq M+N \text{ and } i-N\leq j\leq i\\ 0, & \text{otherwise.}\end{cases}$$

with the convention $r_{0,k}=s_{0,h}=1$ for $0\leq k\leq N$ and $0\leq h\leq M$. Then there exist polynomials ϕ and ψ of degrees M and N, respectively, such that

$$\phi u=\psi v.$$

Later on, in [5] a direct problem associated with the $(M,0)-$coherence of order 0 is studied. Indeed, given a SMOP $\{P_n\}_{n\geq 0}$ and the sequence of polynomials $\{Q_n\}_{n\geq 0}$ defined by

$$Q_n(x)=P_n(x)+\sum_{i=1}^{M}a_iP_{n-i}(x),\ n>M,\ a_M\neq 0,$$

the authors studied necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n\geq 0}$, and the families $\{Q_n\}_{n\geq 0}$ are characterized explicitly when $M=2$. Notice that this is a particular case of $(M,0)-$coherence of order 0 given that Q_n is a linear combination of polynomials of $\{P_n\}_{n\geq 0}$. Without these particular conditions the author returns to the problem in [6]. Additional studies on $(2,0)-$coherence of order 0 appear in [23]. Finally the inverse problem associated with the $(2,1)-$coherence of order 0

$$P_n(x)+r_nP_{n-1}(x)+s_nP_{n-2}(x)=Q_n(x)+t_nQ_{n-1}(x),$$

is studied in [11], where the interesting topic about when the $(2,1)-$coherence algebraic relation is **non-degenerate** is analyzed, i.e. the conditions that the sequences $\{r_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$, and $\{t_n\}_{n\geq 0}$ must satisfy in such a way that the $(2,1)-$coherence relation can not be reduced to a $(M,N)-$coherence relation with either $M<2$ or $N<1$, are given.

Chapter 4

S-coherence

The concept of symmetric coherence, presented in [55], is extended in [34] through the coherence relation

$$P_{n+2}(x;\mu_1)+\delta_n P_n(x;\mu_1)=\frac{P'_{n+3}(x;\mu_0)}{n+3}+\sigma_n\frac{P'_{n+1}(x;\mu_0)}{n+1},\quad n\geq 0,$$

where μ_0 and μ_1 are positive symmetric Borel measures. Such generalization also arises in a natural way as a necessary and sufficient condition for an algebraic connection with Sobolev polynomials like

$$S^{\lambda}_{n+2}(x)+\eta_n(\lambda)S^{\lambda}_n(x)=P_{n+2}(x;\mu_0)+\sigma_n P_n(x;\mu_0),\ n\geq 0, \tag{4.1}$$

holds. In the sequel, this extension will be called *symmetric* $(1,1)-$*coherence*, (also, it is called $s-$coherence). In [34] the authors deal with the particular case when μ_0 is classical, (Hermite and Gegenbauer), in such a way that the respective inverse problem is studied and the connection between the recurrence and coherence coefficients is analyzed. In addition, the respective companions of μ_0 are given. This chapter begins with the analysis of the inverse problem associated with the Symmetric $(1,1)-$coherence. In section 4.2 a classification of symmetric $(1,1)-$coherent pairs is stated by using the symmetrization process. In section 4.3 we study how through this symmetrization process we can arrive to a non-coherence algebraic relation. Then the corresponding inverse problem is analyzed exhaustively.

4.1 Symmetric $(1,1)-$coherent pairs and the inverse problem

We begin with the definition of Symmetric $(1,1)-$Coherent Pair introduced in [34]. From now on in this manuscript we assume that any moment functional u is normalized by the condition $\langle u,1\rangle=1$

Definition 65. *Let u and v denote two symmetric quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ will denote their respective SMOP. Suppose that there exist sequences of non-zero real numbers $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, with $a_nb_n\neq 0$, such that*

$$\frac{P'_{n+3}(x)}{n+3}+a_n\frac{P'_{n+1}(x)}{n+1}=R_{n+2}(x)+b_nR_n(x),\quad n\geq 0, \tag{4.2}$$

holds. Then the pair $\{u,v\}$ is said to be a Symmetric $(1,1)-$Coherent Pair. Furthermore, if u and v are positive-definite and μ_0 and μ_1 are the respective positive Borel measures then $\{\mu_0,\mu_1\}$ is said to be a Symmetric $(1,1)-$Coherent Pair of measures.

Remark 66. *In connection with the Sobolev inner products, in [21] a particular case of symmetric* $(1,1)-$*coherent pair was studied when* u *is classical. Indeed, a relation of the type 4.1 is obtained, which is a necessary and sufficient condition in order to obtain the respective symmetric* $(1,1)-$*coherence relation.*

With the condition $a_n b_n \neq 0,\ n \geq 0$, we are assuming that the relation (4.2) is non-degenerated. Moreover if $a_i \neq b_i$, $i = 0, 1$, we can avoid dealing with trivial cases. Namely

Proposition 67. *([35]). Let* $\{u, v\}$ *be a Symmetric* $(1,1)-$*coherent pair satisfying (4.2). The following statements are equivalent.*

i). $a_i \neq b_i$, $i = 0, 1$.

ii). $R_n(x) \neq \dfrac{P'_{n+1}(x)}{n+1}$, *for* $n \geq 2$.

We will deduce some tools that will be important later on describing how the symmetrization process allows us to take the current problem to simpler case. For $n \geq 0$ and by using (4.2) inductively, we can obtain

$$\begin{aligned}&\frac{P'_{2n+3}(x)}{2n+3}\\ =\ & R_{2n+2}(x) + \sum_{j=0}^{n}(-1)^j \left(\prod_{k=0}^{j} \widetilde{u}_{2n-2(k-1)}\right)(s_{2n-2j} - u_{2n-2j})\, R_{2n-2j}(x),\end{aligned}$$

where

$$\widetilde{u}_{2n-2(k-1)} = \begin{cases} u_{2n-2(k-1)}, & 1 \leq k \leq j, \\ 1, & k = 0, \end{cases}$$

and for $n \geq 1$

$$\begin{aligned}&\frac{P'_{2n+2}(x)}{2n+2}\\ =\ & R_{2n+1}(x) + \sum_{j=0}^{n-1}(-1)^j \left(\prod_{k=0}^{j} \widetilde{u}_{2n-2k+1}\right)(s_{2n-1-2j} - u_{2n-1-2j})\, R_{2n-1-2j}(x),\end{aligned}$$

where

$$\widetilde{u}_{2n-2k+1} = \begin{cases} u_{2n-2k+1}, & 1 \leq k \leq j, \\ 1, & k = 0. \end{cases} .$$

Let us define $r_{2n+1}(x) := R_{2n+1}(x) + A_{2n+1}x$, $n \geq 1$. Then

$$\begin{aligned} 0 &= \left\langle (R_{2n+1} + A_{2n+1}x)\, v, \frac{P'_{2n+4}}{2n+4} \right\rangle \\ &= (s_{2n+1} - u_{2n+1}) \left\langle v, R^2_{2n+1} \right\rangle + A_{2n+1}(-1)^n \left(\prod_{k=0}^{n-1} u_{2n+1-2k}\right)(s_1 - u_1) \left\langle v, R_1^2 \right\rangle, \end{aligned}$$

if and only if

$$A_{2n+1} = \frac{(s_{2n+1} - u_{2n+1}) \left\langle v, R^2_{2n+1} \right\rangle}{(-1)^{n+1} \left(\prod\limits_{k=0}^{n-1} u_{2n+1-2k}\right)(s_1 - u_1) \left\langle v, R_1^2 \right\rangle}, \qquad n \geq 1,$$

and, inductively, we can prove that $\langle (R_{2n+1} + A_{2n+1}x)\, v, P'_{2n+2k}\rangle = 0$ for $k \geq 2$. On the other hand, for $n \geq 1$, we define $r_{2n}(x) := R_{2n}(x) + A_{2n}$. Then

$$\begin{aligned} & \left\langle r_{2n}v, \frac{P'_{2n+3}}{2n+3} \right\rangle \\ = \; & (s_{2n} - u_{2n}) \langle v, R_{2n}^2 \rangle + A_{2n} \left\langle v, (-1)^n \left(\prod_{k=0}^{n} \widetilde{u}_{2n-2(k-1)} \right) (s_0 - u_0) \right\rangle = 0 \end{aligned}$$

if and only if

$$A_{2n} = \frac{(s_{2n} - u_{2n}) \langle v, R_{2n}^2 \rangle}{(-1)^{n+1} \left(\prod\limits_{k=1}^{n} u_{2n-2(k-1)} \right) (s_0 - u_0)}.$$

Furthermore, $\left\langle r_{2n}v, \frac{P'_k}{k} \right\rangle = 0$, $k \geq 2n+2$. On the other hand, let consider the linear functional $r_{2n+1}v$ and its expansion in terms of the dual basis $\{\widehat{U}_n\}_{n\geq 0}$ associated with $\{\frac{P'_{n+1}}{n+1}\}_{n\geq 0}$. Namely,

$$r_{2n+1}v = \sum_{k=0}^{\infty} \widetilde{\lambda}_{nk} \widehat{U}_k, \qquad \widetilde{\lambda}_{nk} = \left\langle r_{2n+1}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0 \;\;, \quad k \geq 2n+2,$$

where $\widetilde{\lambda}_{nk} = \left\langle r_{2n+1}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k+1$ is odd. If $r_{2n+1}v = \sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} \widehat{U}_{2k+1}$, then we can apply the distributional derivative in both sides and we obtain

$$D\left[r_{2n+1}v\right] = -\sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} (2k+2) U_{2k+2},$$

where $\{U_n\}_{n\geq 0}$ is the dual basis associated with $\{P_n\}_{n\geq 0}$. Since $U_m = \frac{P_m}{\langle u, P_m^2\rangle} u$, then

$$D\left[r_{2n+1}v\right] = -\left(\sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} (2k+2) \frac{P_{2k+2}}{\langle u, P_{2k+1}^2 \rangle} \right) u.$$

In an analogous way, for $n \geq 1$, we consider $r_{2n}v$ and, as above,

$$r_{2n}v = \sum_{k=0}^{\infty} \lambda_{nk} \widehat{U}_k, \qquad \lambda_{nk} = \left\langle r_{2n}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0, \text{ if } k \geq 2n+3,$$

and $\lambda_{nk} = \left\langle r_{2n}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k+1$ is even. Then $D\left[r_{2n}v\right] = -\sum_{k=0}^{n} \lambda_{n,2k}(2k+1) U_{2k+1}$, and

$$D\left[r_{2n}v\right] = -\left(\sum_{k=0}^{n} \lambda_{n,2k} (2k+1) \frac{P_{2k+1}(x)}{\langle u, P_{2k+1}^2 \rangle} \right) u.$$

Next we summarize the above results.

Proposition 68. *For $m \geq 2$ there exist polynomials r_m, ϕ_{m+1} with $\deg r_m = m$, $\deg \phi_{m+1} \leq m+1$, such that*

$$D\left[r_m v\right] = -\phi_{m+1} u, \qquad m \geq 2, \tag{4.3}$$

with

$$r_m(x) := R_m(x) + A_m x^{\left(\frac{1-(-1)^m}{2}\right)},$$

$$A_m = \frac{(s_m - u_m)\left\langle v, R_m^2 \right\rangle}{(-1)^{[m/2]+1}\left(\prod_{k=1}^{n} u_{m+2-2k}\right)\left(s_{\frac{1-(-1)^m}{2}} - u_{\frac{1-(-1)^m}{2}}\right)\left\langle v, R^2_{\frac{1-(-1)^m}{2}}\right\rangle}.$$

Moreover,

$$\phi_{2n+2}(x) = \sum_{k=0}^{n} \frac{\widetilde{\lambda}_{n,2k+1}(2k+2)}{\left\langle u, P_{2k+2}^2\right\rangle} P_{2k+2}(x), \qquad n \geq 1, \tag{4.4}$$

and

$$\phi_{2n+1}(x) = \sum_{k=0}^{n} \frac{\lambda_{n,2k}(2k+1)}{\left\langle u, P_{2k+1}^2\right\rangle} P_{2k+1}(x), \qquad n \geq 1. \tag{4.5}$$

Besides, the associated inverse problem is solved in [35], namely

Theorem 69. *Let $\{u, v\}$ be a symmetric (1,1)-coherent pair. There exist polynomials A, B and C with* $\deg(A) = 4$, $\deg(B) \leq 5$ *and* $\deg(C) \leq 6$, *such that*

$$A(x)Dv = B(x)u, \tag{4.6}$$

$$B(x)v = C(x)Dv, \tag{4.7}$$

$$xC(x)u = xA(x)v, \tag{4.8}$$

where

$$A(x) = \frac{r_4'(x)r_2(x) - r_4(x)r_2'(x)}{x}, \tag{4.9}$$

$$B(x) = \frac{r_2'(x)\phi_5(x) - r_4'(x)\phi_3(x)}{x}, \tag{4.10}$$

$$C(x) = \frac{r_4(x)\phi_3(x) - r_2(x)\phi_5(x)}{x}. \tag{4.11}$$

Depending on the nature of the zeros of A, it is possible to refine the rational relation (4.8). Besides, according to (4.9), A is an even function. In this way, either $A(x) = 2(x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$, or $A(x) = 2\left(x^2 - \xi^2\right)^2$. In the sequel, we will assume that $\xi^2, \xi_1^2, \xi_2^2 \in \mathbb{R}$. Next, we study each case. Before that some technical lemma that describes how the symmetrization process allows us to take the current problem to a simpler case.

Definition 70. *Given an even polynomial p of degree n, the polynomial p^E, with of $deg(p^E) = n/2$, is defined as $p^E(x^2) := p(x)$*

Lemma 71. ***i).*** *Let u and v be the symmetrized of $\widetilde{u}$ and $\widetilde{v}$, respectively. If ϕ and ψ are even polynomials such that $\phi u = \psi v$, holds, then $\phi^E \widetilde{u} = \psi^E \widetilde{v}$. Besides the converse also holds.*

ii). *If u and v satisfy $D(x\phi u) = \psi v$, where ϕ and ψ are even polynomials, then $D\left(x\phi^E \widetilde{u}\right) = \frac{1}{2}\psi^E \widetilde{v}$.*

4.2 A classification of symmetric $(1,1)-$coherent pairs

4.2.1 Case $A(x) = 2\left(x^2-\xi^2\right)^2$

If $A(x) = 2\left(x^2-\xi^2\right)^2$, then (4.8) can be written as $xC(x)u = 2x\left(x^2-\xi^2\right)^2 v$ as well as $r_2(x) = x^2 - a^2$ and $r_4(x) = x^4 + fx^2 + g$. As a consequence, $A(x) = 2x^4 - 4x^2\xi^2 + 2\xi^4$, where $\xi^2 = a^2$ and $fa^2 + g = \xi^4$. Thus, $r_2(x) = x^2 - \xi^2$. Since $(xA(x))' = r_4''(x)r_2(x) - r_4(x)r_2''(x)$, we deduce $r_4''(\xi)r_2(\xi) - 2r_4(\xi) = 0$. From this expression, taking into account $r_2(\xi) = 0$, we get $r_4(\xi) = 0$. As a consequence, $r_4(x) = r_2(x)\rho_2(x)$, where $\rho_2(x) := x^2 - \delta$. From (4.11),

$$C(x) = r_4(x)\frac{\phi_3(x)}{x} - r_2(x)\frac{\phi_5(x)}{x} = r_2(x)\left(\rho_2(x)\frac{\phi_3(x)}{x} - \frac{\phi_5(x)}{x}\right) = r_2(x)\sigma_4(x).$$

According to (4.10), $B(x) = 2\phi_5(x) - 2\left(\rho(x) + r_2(x)\right)\phi_3(x)$, then from (4.6) we get

$$r_2^2(x)Dv = \left(\phi_5(x) - \left(x^2 + r_2(x)\right)\phi_3(x)\right)u.$$

For $m = 2$, multiplying (4.3) by $r_2(x)$ we deduce, $r_2^2(x)Dv = -r_2(x)\phi_3(x)u - 2xr_2(x)v$. On the other hand, from the above expressions we get

$$\left(\phi_5(x) - \left(\rho(x) + r_2(x)\right)\phi_3(x)\right)u + r_2(x)\phi_3(x)u + 2xr_2(x)v = 0,$$

i.e.

$$\left(\phi_5(x) - \rho(x)\phi_3(x)\right)u + 2xr_2(x)v = 0.$$

Thus, we get

$$x\sigma_4(x)u = 2xr_2(x)v, \tag{4.12}$$

where

$$\sigma_4(x) = \rho(x)\frac{\phi_3(x)}{x} - \frac{\phi_5(x)}{x}. \tag{4.13}$$

If a symmetric $(1,1)-$coherent pair (u,v) satisfies (4.12), then

$$x^2\sigma_4(x)u = 2x^2r_2(x)v. \tag{4.14}$$

Through the symmetrization process, we can find pairs (u,v) of symmetric linear functionals such that (4.14) holds. Among such pairs, we will identify all the symmetric $(1,1)-$ coherent ones later on.

Lemma 72. ***i).*** *For $m = 2n$, (4.3) implies*

$$xD\left(r_{2n}^E(x)\widetilde{v}\right) = -\frac{1}{2}r_{2n}^E(x)\widetilde{v} - \frac{1}{2}x\widetilde{\phi}_{2n}^E(x)\widetilde{u},$$

where $\phi_{2n+1}(x) := x\widetilde{\phi}_{2n}(x)$.

ii). *For $m = 2n+1$, (4.3) yields*

$$D\left(x\widetilde{r}_{2n}^E(x)\widetilde{v}\right) = -\frac{1}{2}x\phi_{2n+2}^E(x)\widetilde{u},$$

where $r_{2n+1}(x) = x\widetilde{r}_{2n}(x)$.

Proof. We will prove ii). The proof of i) is similar. The Pearson type relation is equivalent to

$$D(x\widetilde{r}_{2n}(x)v) = -\phi_{2n+2}(x)u. \tag{4.15}$$

For every polynomial p,

$$\begin{aligned}\left\langle D\left(x\widetilde{r}_{2n}^{E}(x)\widetilde{v}\right), p(x)\right\rangle &= -\left\langle \widetilde{v}, x\widetilde{r}_{2n}^{E}(x)p'(x)\right\rangle \\ &= -\left\langle v, x^2\widetilde{r}_{2n}(x)p'(x^2)\right\rangle \\ &= \frac{1}{2}\left\langle D\left(x\widetilde{r}_{2n}(x)v\right), p(x^2)\right\rangle,\end{aligned}$$

and from (6.5)

$$\begin{aligned}\left\langle D\left(x\widetilde{r}_{2n}^{E}(x)\widetilde{v}\right), p(x)\right\rangle &= -\frac{1}{2}\left\langle \phi_{2n+2}(x)u, p(x^2)\right\rangle \\ &= -\frac{1}{2}\left\langle \phi_{2n+2}^{E}(x)\widetilde{u}, p(x)\right\rangle.\end{aligned}$$

Thus, our statement follows. □

From the previous lemma, we get that $D(r_2(x)v) = -\phi_3(x)u$ implies

$$xD\left(r_2^E(x)\widetilde{v}\right) = -\frac{1}{2}r_2^E(x)\widetilde{v} - \frac{1}{2}x\widetilde{\phi}_2^E(x)\widetilde{u}. \tag{4.16}$$

On the other hand, $D(r_3(x)v) = -\phi_4(x)u$ is equivalent to

$$D(x\widetilde{r}_2^E(x)\widetilde{v}) = -\frac{1}{2}\phi_4^E(x)\widetilde{u}, \tag{4.17}$$

and $D(r_4(x)v) = -\phi_5(x)u = -x\widetilde{\phi}_4(x)u$ yields

$$xD(r_4^E(x)\widetilde{v}) = -\frac{1}{2}r_4^E(x)\widetilde{v} - \frac{1}{2}x\widetilde{\phi}_4^E(x)\widetilde{u}. \tag{4.18}$$

On the other hand, let u and v be the symmetrizations of $\widetilde{u}$ and $\widetilde{v}$, respectively. Then,

Lemma 73. *u and v satisfy (4.14) if and only if $\widetilde{u}$ and $\widetilde{v}$ satisfy*

$$x\sigma_4^E(x)\widetilde{u} = 2xr_2^E(x)\widetilde{v}. \tag{4.19}$$

Proof. We assume that $x^2\sigma_4(x)u = 2x^2r_2(x)v$. Let p be any polynomial. Then

$$\left\langle x\sigma_4^E(x)\widetilde{u}, p(x)\right\rangle = \left\langle u, x^2p(x)^2\sigma_4^E(x^2)\right\rangle = \left\langle 2xr_2^E(x)\widetilde{v}, p(x)\right\rangle.$$

On the other hand, assume that $x\sigma_4^E(x)\widetilde{u} = 2xr_2^E(x)\widetilde{v}$. If $p(x) = \sum_{k=0}^{n} a_kx^k$, then $p^E(x^2) = \sum_{k=0}^{[n/2]} a_{2k}x^{2k}$. As a consequence, $\left\langle x^2\sigma_4(x)u, p(x)\right\rangle = \left\langle 2x^2r_2(x)v, p(x)\right\rangle$. □

Taking derivatives in both hand sides of 4.19 and by using (4.16) we get

$$D(x\sigma_4^E(x)\widetilde{u}) = 2D(xr_2^E\widetilde{v}) = r_2^E(x)\widetilde{v} - x\widetilde{\phi}_2^E(x)\widetilde{u}.$$

If we multiply by x, then from 4.19

$$xD(x\sigma_4^E(x)\widetilde{u}) = xr_2^E(x)\widetilde{v} - x^2\widetilde{\phi}_2^E(x)\widetilde{u} = \left(\frac{1}{2}x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u}$$

and, equivalently,

$$D(x^2\sigma_4^E(x)\widetilde{u}) = \left(\frac{1}{2}x\sigma_4^E(x) + x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u} = \left(\frac{3}{2}x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u}.$$

Next we summarize the above results.

Proposition 74. *If* $A(x) = 2\left(x^2 - \xi^2\right)^2$ *and* (u,v) *is a symmetric* $(1,1)-$*coherent pair, then* $(\widetilde{u},\widetilde{v})$ *satisfy (4.19) and*

$$D(\widetilde{\phi}(x)\widetilde{u}) + \widetilde{\psi}(x)\widetilde{u} = 0, \tag{4.20}$$

where

$$\widetilde{\phi}(x) = x^2\sigma_4^E(x),$$
$$\sigma_4^E(x) = x\widetilde{\phi}_2^E(x) - \widetilde{\phi}_4^E(x),$$

and

$$\widetilde{\psi}(x) = x^2\widetilde{\phi}_2^E(x) - \frac{3}{2}x\sigma_4^E(x) = -\frac{1}{2}x^2\widetilde{\phi}_2^E(x) + \frac{3}{2}x\widetilde{\phi}_4^E(x).$$

Moreover $\deg\widetilde{\psi} \leq 3$ *and* $\deg\widetilde{\phi} \leq 4$. *As a consequence,* $\widetilde{u}$ *is a semiclassical linear functional of class at most* 2.

In the sequel, given a linear functional $\widetilde{U}$ and its symmetrized U, $\left\{\widetilde{\mu}_n^U\right\}_{n\geq 0}$ and $\left\{\mu_n^U\right\}_{n\geq 0}$ will denote the corresponding moment sequences. From (4.5) we get $\phi_3(x) = \frac{\lambda_{1,0}}{\langle u,P_1^2\rangle}x + \frac{3\lambda_{1,2}}{\langle u,P_3^2\rangle}P_3(x)$, with $\lambda_{1,0} = \langle v, r_2\rangle = \mu_2^v - \xi^2$. After some straightforward calculations, we get

$$\lambda_{1,2} = \mu_4^v - \left(\xi^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \frac{\xi^2}{3}\left(\gamma_1^u + \gamma_2^u\right),$$

where $\{\gamma_n^u\}_{n\geq 1}$ are the coefficients of the three term recurrence relation that the MOPS $\{P_n\}_{n\geq 0}$ satisfies. Then

$$\widetilde{\phi}_2^E(x) = \frac{3\lambda_{1,2}}{\langle u,P_3^2\rangle}x + \left(\frac{\lambda_{1,0}}{\langle u,P_1^2\rangle} - \frac{3\lambda_{1,2}}{\langle u,P_3^2\rangle}\left(\gamma_1^u + \gamma_2^u\right)\right).$$

In particular,

$$\widetilde{\phi}_2^E(0) = \frac{\mu_2^v - \xi^2}{\langle u,P_1^2\rangle} - \frac{3\mu_4^v - 3\left(\xi^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \xi^2\left(\gamma_1^u + \gamma_2^u\right)}{\langle u,P_3^2\rangle}\left(\gamma_1^u + \gamma_2^u\right). \tag{4.21}$$

From (4.19) and taking into account $\widetilde{u}$ is a linear functional of class $s \leq 2$, according to the above classification we can find its companion $\widetilde{v}$. As a consequence, we can deduce all the candidates (u,v) to be symmetric $(1,1)-$coherent pairs. From (4.20) we get

$$x^2\sigma_4^E(x)D(\widetilde{u}) = -\left(\widetilde{\psi}(x) + \left(x^2\sigma_4^E(x)\right)'\right)\widetilde{u}. \tag{4.22}$$

In the sequel we consider $\widetilde{s} \leq 1$. The case $\widetilde{s} = 2$ will not be considered. From the classification of the semiclassical linear functionals of class $\widetilde{s} \leq 1$, we will analyze the semiclassical character of $\widetilde{u}$ taking into account the algebraic structure of $\sigma_4^E(x)$.

$\widetilde{u}$ **of class** $\widetilde{s}=0$

In order to arrive to a classical case, we start the discussion by considering the following situations

i). $\sigma_4^E(x)=x^2$, $\widetilde{\psi}(x)=x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}x\right)$. From (4.20) we get $D(x^4\widetilde{u})=-x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}x\right)\widetilde{u}$ or, equivalently,

$$D(x^2\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{1}{2}x\right)\widetilde{u}+N_1\delta(x)+N_2\delta'(x).$$

It is easy to see that $N_1=\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)+\frac{1}{2}x\right\rangle$, and $N_2=\frac{1}{2}\left\langle\widetilde{u},x^2\right\rangle-\left\langle\widetilde{u},x\widetilde{\phi}_2^{E(x)}\right\rangle$. Thus, if $\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)\right\rangle+\frac{1}{2}\widetilde{\mu}_1^u=\frac{1}{2}\widetilde{\mu}_2^u-\left\langle\widetilde{u},x\widetilde{\phi}_2^E(x)\right\rangle=0$, then $\widetilde{u}$ is the Bessel classical functional since

$$D(x^2\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{1}{2}x\right)\widetilde{u}.$$

ii). $\sigma_4^E(x)=x(x-1)$, $\widetilde{\psi}(x)=x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}(x-1)\right)$. Then $D(x^3(x-1)\widetilde{u})=-x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}(x-1)\right)\widetilde{u}$.

This yields

$$D(x(x-1)\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{1}{2}(x-1)\right)\widetilde{u}+N_1\delta(x)+N_2\delta'(x),$$

where $N_1=\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)+\frac{1}{2}(x-1)\right\rangle$ and $N_2=\frac{1}{2}\left\langle\widetilde{u},x(x-1)\right\rangle-\left\langle\widetilde{u},x\widetilde{\phi}_2^E(x)\right\rangle$. If $\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)\right\rangle+\frac{1}{2}(\widetilde{\mu}_1^u-1)=\frac{1}{2}(\widetilde{\mu}_2^u-\widetilde{\mu}_1^u)-\left\langle\widetilde{u},x\widetilde{\phi}_2^E(x)\right\rangle=0$, then $\widetilde{u}=\mathcal{J}_{[0,1]}^{(\alpha,\beta)}$, i.e. the Jacobi classical functional on $[0,1]$, such that

$$D(x(x-1)\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{1}{2}(x-1)\right)\widetilde{u}.$$

iii). $\sigma_4^E(x)=x$, $\widetilde{\psi}(x)=x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}\right)$. From (4.20) we get $x^3D(\widetilde{u})=-\left(x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}\right)+3x^2\right)\widetilde{u}$

and

$$xD(\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{3}{2}\right)\widetilde{u}+N_1\delta(x)+N_2\delta'(x).$$

Then $\langle xD(\widetilde{u}),1\rangle=-\left\langle\widetilde{u},\left(\widetilde{\phi}_2^E(x)+\frac{3}{2}\right)\right\rangle+N_1$ and $\langle xD(\widetilde{u}),x\rangle=-\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)+\frac{3}{2}x\right\rangle-N_2$. If $N_1=\left\langle\widetilde{u},\widetilde{\phi}_2^E(x)\right\rangle+\frac{1}{2}=0$ and $N_2=-\left\langle\widetilde{u},\widetilde{\phi}_2^{E(}x)\right\rangle+\frac{1}{2}\widetilde{\mu}_1^u=0$ we get $D(x\widetilde{u})=-\left(\widetilde{\phi}_2^E(x)+\frac{1}{2}\right)\widetilde{u}$, i.e. $\widetilde{u}$ is the classical Laguerre linear functional.

Remark 75. *We do not consider* $\sigma_4^E(x)=1$, *since in such a case* $\widetilde{u}$ *is the classical Hermite functional.*

$\widetilde{u}$ **of class** $\widetilde{s}=1$

In order to analyze the semiclassical case when $\widetilde{s}=1$ we will discuss two possible situations in order to reduce the degrees of the polynomials involved in the initial Pearson equation.

a).

i). $\sigma_4^E(x)=x^2$, $\widetilde{\psi}(x)=x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}x\right)$. From (4.20)

$$x^3D(\widetilde{u})=-\left(\widetilde{\psi}(x)+4x^3\right)\widetilde{u}=-x\left(\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}x\right)+4x\right)\widetilde{u}+M\delta(x).$$

If

$$M = \langle x^3 D(\widetilde{u}), 1\rangle + \left\langle x\left(\widetilde{\phi}_2^E(x) + \frac{5}{2}x\right)\widetilde{u}, 1\right\rangle = \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u = 0,$$

then you can reduce the Pearson equation to

$$D(x^3\widetilde{u}) = -x\left(\widetilde{\phi}_2^E(x) + \frac{5}{2}x\right)\widetilde{u} + 3x^2\widetilde{u} = \left(-x\widetilde{\phi}_2^E(x) + \frac{1}{2}x^2\right)\widetilde{u},$$

and you have here $\widetilde{\psi}(x) = x\widetilde{\phi}_2^E(x) - \frac{1}{2}x^2$, $\widetilde{\psi}(0) = 0$ and $\widetilde{\psi}'(0) = \widetilde{\phi}_2^E(0)$. If $\widetilde{\phi}_2^E(0) \neq 0$, then $\widetilde{u}$ corresponds to the case $A_{3,2}$ of the Belmehdi's classification in [20], and, as a consequence, $\widetilde{u} = x^{-1}\mathcal{B}^{(\alpha)} + M\delta(x)$.

ii). $\sigma_4^E(x) = x(x-1)$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)\right)$. In this case

$$x^2(x-1)D(\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) - \frac{3}{2}x^2 + \frac{3}{2}x + 4x^2 - 3x\right)\widetilde{u} + M_1\delta(x).$$

Then

$$\begin{aligned} M_1 &= -3\langle \widetilde{u}, x^2\rangle + 2\langle \widetilde{u}, x\rangle + \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle + \frac{5}{2}\langle \widetilde{u}, x^2\rangle - \frac{3}{2}\langle \widetilde{u}, x\rangle \\ &= \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u + \frac{1}{2}\widetilde{\mu}_1^u. \end{aligned}$$

If $M_1 = 0$, then $D(x^2(x-1)\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) + \frac{1}{2}x - \frac{1}{2}x^2\right)\widetilde{u}$, and, according to the case A_2 in [20], $\widetilde{u}$ has an integral representation with weight function $w(x) = (1-x)^\alpha x^\beta e^{-\frac{\gamma}{x}}$, on $[0,1]$, with $\alpha\gamma \neq 0$, $\gamma > 0$, $\alpha > -1$.

iii). $\sigma_4^E(x) = (x-1)(x-\zeta)$, $\zeta \neq 0, 1$, $\widetilde{\psi}(x) = x\left(x\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)(x-\zeta)\right)$. Then,

$$D(x(x-1)(x-\zeta)\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) - \frac{1}{2}(x-1)(x-\zeta)\right)\widetilde{u} + M\delta(x).$$

If

$$M = \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u + \left(\frac{1}{2}\zeta + \frac{1}{2}\right)\widetilde{\mu}_1^u - \frac{1}{2}\zeta = 0,$$

this corresponds to the case A_1 in [20] with $\widetilde{\omega}(x) = (1-x)^\alpha x^\beta |x-\zeta|^\gamma$ on $[0,1]$ with the conditions $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > -1$, $\zeta \in (0,1)$.

b).

i). $\sigma_4^E(x) = x$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}\right)$. As above, if $M = \left\langle \widetilde{u}, x\widetilde{\phi}_2^E\right\rangle - \frac{1}{2}\widetilde{\mu}_1^u = 0$, then

$$D(x^2\widetilde{u}) = -x\left(\widetilde{\phi}_2^E(x) - \frac{1}{2}\right)\widetilde{u},$$

and, according to the case B_2 in [20], we obtain an integral representation of $\widetilde{u}$ in terms of the weight function

$$w(x) = x^\alpha (1+x)^{\beta+1} e^{-x+\frac{\beta}{x}},$$

on $[0,\infty)$, with $\beta < 0$, $\alpha, \beta > -1$.

ii).$\sigma_4^E(x)=x-1$, $\widetilde{\psi}(x)=x\left(x\widetilde{\phi}_2^E(x)-\frac{3}{2}(x-1)\right)$. Then, $D(x(x-1)\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}(x-1)\right)\widetilde{u}$, when $M=\left\langle\widetilde{u},x\widetilde{\phi}_2^E(x)\right\rangle-\frac{1}{2}\widetilde{\mu}_1^u+\frac{1}{2}=0$. This is the case B_1 in [20] with $\widetilde{\omega}(x)=(1-x)^{\alpha+1}x^{\beta+1}e^{-\lambda x}$ on $[0,1]$ and the conditions $\alpha\beta\neq 0$, $\alpha,\beta>-1$.

iii). $\sigma_4^E(x)=1$, $\widetilde{\psi}(x)=x\left(x\widetilde{\phi}_2^E(x)-\frac{3}{2}\right)$. If $M=\left\langle\widetilde{u},x\widetilde{\phi}_2^E\right\rangle-\frac{1}{2}=0$, then $D(x\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\right)\widetilde{u}$, and, according to the case B_3 in [20], we get that $\widetilde{u}$ is represented in terms of the weight function $w(x)=x^{2\mu}e^{-x^2-\lambda x}$, on $\mathbb{R}^+$, $\mu>-1/2$, $\lambda\in\mathbb{R}$.

As the classical case, it is possible to reduce (4.19). Indeed, the general form of the Pearson equation is

$$D(x\sigma_4^E(x)\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\sigma_4^E(x)\right)\widetilde{u}.$$

Taking derivatives in (4.19) and using (4.16), we get $2D(xr_2^E(x)\widetilde{v})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\sigma_4^E(x)\right)\widetilde{u}$. In other words,

$$r_2^E(x)\widetilde{v}-x\widetilde{\phi}_2^E(x)\widetilde{u}=-x\widetilde{\phi}_2^E(x)\widetilde{u}+\frac{1}{2}\sigma_4^E(x)\widetilde{u},$$

and as a consequence, $2r_2^E\widetilde{v}=\sigma_4^E(x)\widetilde{u}$.

Remark 76. *Notice that according to Theorem 30 we get* $\widehat{\Psi}(x)=x\left(\left(\sigma_4^E\right)'(x)+2\widetilde{\phi}_2^E(x)\right)$, *and as a consequence, the class of* u *is* $s=2$.

4.2.2 Case $A(x)=2(x^2-\xi_1^2)(x^2-\xi_2^2)$, $\xi_1^2\neq\xi_2^2$

In this case, the following result is obtained in [35].

Theorem 77. *Suppose that* $A(x)=2(x^2-\xi_1^2)(x^2-\xi_2^2)$, $\xi_1^2\neq\xi_2^2$. *Then there exist odd and even polynomials* ψ *and* ϕ, *respectively, with* $\deg\psi\leq 3$ *and* $\deg\phi\leq 4$ *such that*

$$D(\phi v)+\psi v=0. \tag{4.23}$$

As a consequence, v *is a semiclassical linear functional of class at most* 2. *Besides*

$$x\phi(x)u=x(x^2-\xi^2)v, \tag{4.24}$$

holds, where $\xi\in\{\xi_1,\xi_2\}$. *Also,* $(x^2-\xi^2)Dv=-(\phi'(x)+\psi(x))u$.

Multiplying in (4.23) by x, if we define $\psi(x):=x\widetilde{\psi}(x)$, where $\widetilde{\psi}$ is an even polynomial of degree ≤ 2, and using the symmetrization process, after straightforward calculations you get

$$D\left(x\phi^E(x)\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-\phi^E(x)\right)\widetilde{v}, \tag{4.25}$$

$$x\phi^E(x)D(\widetilde{u})=\frac{1}{2}\left(\rho^E(x)+2x\right)\widetilde{v}-\left(2x(\phi^E)'(x)+\frac{1}{2}x\widetilde{\psi}^E(x)+\phi^E(x)\right)\widetilde{u}, \tag{4.26}$$

$$x\rho^E(x)D(\widetilde{v})=-\frac{1}{2}\rho^E(x)\widetilde{v}-\frac{1}{2}x\left(2(\phi^E)'(x)+\widetilde{\psi}^E(x)\right)\widetilde{u}, \tag{4.27}$$

and

$$x\phi^E(x)\widetilde{u}=x(x-\xi^2)\widetilde{v}. \tag{4.28}$$

Notice that $\widetilde{v}$ is semiclassical of class $\widetilde{s}\leq 1$. Next, the class of $\widetilde{v}$ will be analyzed according to the zeros of ϕ^E.

$\widetilde{v}$ semiclassical of class $\widetilde{s}=0$

A1. $\phi^E(x)=x^2$. In this case (4.25) can be written as $D\left(x^3\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x^2\right)\widetilde{v}$. Since $\widetilde{v}$ is classical, we can reduce the degree of the polynomials involved in this relation in one degree, namely $D\left(x^2\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v}+N\delta(x)$. Since

$$0=\left\langle D\left(x^2\widetilde{v}\right),1\right\rangle=-\frac{1}{2}\left\langle\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v},1\right\rangle+N,$$

if $N=0$, equivalently, $\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x\right\rangle=0$, then $D\left(x^2\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v}$. In such a way, it is well known that $\widetilde{v}=\mathcal{B}^{(\alpha)}$.

A2. $\phi^E(x)=x(x-1)$. (4.25) reads $D\left(x(x-1)\widetilde{v}\right)=-\frac{1}{2}\left[\widetilde{\psi}^E(x)+(x-1)\right]\widetilde{v}+N\delta(x)$. Since

$$0=\left\langle D\left(x(x-1)\widetilde{v}\right),1\right\rangle=-\frac{1}{2}\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x-1\right\rangle+N,$$

if $\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x-1\right\rangle=0$, then $D\left(x(x-1)\widetilde{v}\right)=-\frac{1}{2}\left[\widetilde{\psi}^E(x)+(x-1)\right]\widetilde{v}$. This means that $\widetilde{v}=\mathcal{J}^{(\alpha,\beta)}_{(0,1)}$ as well as the integral representation

$$\left\langle\widetilde{v},p(x)\right\rangle=\int_0^1 p(x)(1-x)^\alpha x^\beta dx.$$

A3. $\phi^E(x)=x$. If $N=\left\langle\widetilde{v},\widetilde{\psi}^E(x)+1\right\rangle=0$, then $D\left(x\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+1\right)\widetilde{v}$. As consequence, $\widetilde{v}=\mathcal{L}^{(\alpha)}$.

On one hand, from the symmetrization process and since the class of $\widetilde{v}$ is 0, the class s of v is determined by the polynomial $\Psi(x):=\left(\phi^E\right)'(x)+\widetilde{\psi}^E(x)-\frac{\phi^E(x)}{x}$. Indeed, if $\Psi(0)=0$, then $s=0$. If $\Psi(0)\neq 0$, then $s=1$. In Table 4.1 we describe the conditions leading to $\Psi(0)=0$.

$\widetilde{v}$	$\widetilde{\psi}^E$	$\Psi(x)$	Conditions for $\Psi(0)=0$
$\mathcal{B}^{(\alpha)}$	$-(2\alpha+5)x-4$	$(-2\alpha-4)x-4$	$\Psi(0)\neq 0$ always
$\mathcal{J}^{(\alpha,\beta)}_{(0,1)}$	$(2\alpha+2\beta+3)x-(2\beta+1)$	$(2\alpha+2\beta+4)x-(2\beta+1)$	$\beta=-1/2$
$\mathcal{L}^{(\alpha)}$	$2x-(2\alpha+3)$	$2x-(2\alpha+3)$	$\alpha=-3/2$

Table 4.1: Conditions for v to be classical

Next, we will prove that we can reduce (4.28) in order to obtain

$$\phi^E(x)\widetilde{u}=\rho^E(x)\widetilde{v},$$

where $\rho^E(x):=x-\xi^2$. In general, the Pearson equation is

$$D(\phi^E(x)\widetilde{v})=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+\frac{\phi^E(x)}{x}\right)\widetilde{v}, \tag{4.29}$$

or, equivalently,

$$\phi^E(x)D\widetilde{v}=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+\frac{\phi^E(x)}{x}+2\left(\phi^E\right)'(x)\right)\widetilde{v}, \tag{4.30}$$

under the condition $\left\langle \widetilde{v}, \widetilde{\psi}^E(x) + \frac{\phi^E(x)}{x} \right\rangle = 0$.

The case A1, where $\widetilde{v}$ is the classical Bessel functional, reads as

$$D(x^2\widetilde{v}) = ((\alpha+2)x+2)\,\widetilde{v} = -\frac{1}{2}\left(\widetilde{\psi}^E(x) + x\right)\widetilde{v}.$$

Then $\widetilde{\psi}^E(x) = (-2\alpha-5)\,x - 4$, and the above differential relation can be written as

$$x^2 D(\widetilde{v}) = (\alpha x + 2)\widetilde{v}, \tag{4.31}$$

with the condition $\langle \widetilde{v}, (\alpha+2)x+2\rangle = 0$. Also, in this case, the linear functionals $\widetilde{u}$ and $\widetilde{v}$ are related by $x^3\widetilde{u} = x\rho^E(x)\widetilde{v}$ and, as a consequence,

$$\widetilde{u} = \frac{\rho^E(x)}{x^2}\widetilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta''(x).$$

From (4.27) and (4.31) we get

$$\left(\alpha x + 2 + \frac{1}{2}\rho^E(x)\right)\widetilde{v} - \xi^2 x D(\widetilde{v}) = -\frac{1}{2}x\left(\left(-2\alpha-1\right)x-4\right)\widetilde{u}.$$

The action of the linear functionals of both sides on $p(x) = x$ yields

$$\begin{aligned}\left\langle \widetilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\xi^2 \right\rangle &= \frac{1}{2}\left\langle \widetilde{u}, (2\alpha+1)\,x^3 + 4x^2\right\rangle \\ &= \frac{1}{2}\left\langle \rho^E(x)\widetilde{v}, (2\alpha+1)\,x + 4\right\rangle + 8K_3.\end{aligned}$$

As a consequence,

$$\begin{aligned}8K_3 &= \left\langle \widetilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\xi^2 - \frac{1}{2}\left(2\alpha+1\right)x\rho^E(x) - 2\rho^E(x)\right\rangle \\ &= \left\langle \widetilde{v}, \alpha x^2 + 2x + 2x\xi^2 + \left(-\alpha x - 2\right)\rho^E(x)\right\rangle \\ &= \xi^2\left\langle \widetilde{v}, \left(2+\alpha\right)x + 2\right\rangle.\end{aligned}$$

Thus, $K_3 = 0$. In a similar way, in the case A2 we get

$$\widetilde{u} = \frac{\rho^E(x)}{x(x-1)}\widetilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta(x-1).$$

The action of the linear functional of both hand sides on $p(x) = x - 1$ yields

$$\begin{aligned}&-\frac{1}{2}\left\langle \left(\widetilde{\psi}^E(x) + x - 1 + 4x - 2 - x + \xi^2\right)\widetilde{v}, (x-1)\right\rangle - (1-\xi^2)\left\langle xD\widetilde{v}, x-1\right\rangle \\ = &-\frac{1}{2}\left\langle \widetilde{u}, x(x-1)\left(4x - 2 + \widetilde{\psi}^E(x)\right)\right\rangle,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\left\langle \widetilde{v}, -\frac{1}{2}(x-1)\left(\widetilde{\psi}^E(x) + 4x - 3 + \xi^2\right) - (1-\xi^2)\left(2x-1\right)\right\rangle \\ = &-\frac{1}{2}\left\langle \widetilde{v}, \rho^E(x)\left(4x - 2 + \widetilde{\psi}^E(x)\right)\right\rangle + \frac{1}{2}K_2\left(-2 + \widetilde{\psi}^E(0)\right).\end{aligned}$$

Then

$$\frac{1}{2}K_2\left(-2+\widetilde{\psi}^E(0)\right)=-\frac{1}{2}\left(\xi-1\right)\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x-1\right\rangle.$$

In this case, since $\widetilde{v}=\mathcal{J}^{(\alpha,\beta)}_{(0,1)}$, it is well known that

$$\frac{1}{2}\left(\widetilde{\psi}^E(x)+(x-1)\right)=\left(\alpha+\beta+2\right)x-\left(\beta+1\right).$$

In other words,

$$\widetilde{\psi}^E(x)=\left(2\alpha+2\beta+3\right)x-\left(2\beta+1\right).$$

If $\widetilde{\psi}^E(0)=2$, then $\beta=-3/2$. Notice that this up to for this value $K_2=0$. In the same way, for the case A3, (4.28), becomes

$$x\widetilde{u}=(x-\xi^2)\widetilde{v},$$

when $2+\widetilde{\psi}^E(0)\neq 0$. This means that $\alpha\neq-\frac{1}{2}$.

$\widetilde{v}$ semiclassical of class $\widetilde{s}=1$

From (4.25) the following situations appear.

A. $\deg\left(x\phi^E(x)\right)=3$, $1\leq\deg\left(x\widetilde{\psi}^E(x)-\phi^E(x)\right)\leq 2$.

A1. $\phi^E(x)=x^2$, $\Psi(x)=\frac{1}{2}(x\widetilde{\psi}^E(x)-x^2)$ and $D\left(x^3\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x^2\right)\widetilde{v}$. This corresponds to the case A_{32} in [20], where

$$D\left(x^3\widetilde{v}\right)=x\left(\left(\alpha+2\right)x+2\right)\widetilde{v},$$

with the condition $\Psi'(x)\neq 0$. Since $\Psi'(x)=\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\left(\widetilde{\psi}^E\right)'(x)-2x\right)$, the above condition means $\Psi'(0)=\frac{1}{2}\widetilde{\psi}^E(0)$ and $\widetilde{\psi}^E(0)\neq 0$. In addition, $\widetilde{v}=x^{-1}\mathcal{B}^{(\alpha)}+M\delta(x)$.

A2. $\phi^E(x)=x(x-1)$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)$ and $D\left(x^2(x-1)\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)\widetilde{v}$. It corresponds to the case A_2 in [20], where $\widetilde{v}$ satisfies $D\left(x^2(x-1)\widetilde{v}\right)=-x\left(-\left(\alpha+\beta+3\right)x+\beta+2\right)\widetilde{v}$, and

$$\widetilde{v}=x^{-1}\left(\tau_{1/2}\circ h_{1/2}\right)\mathcal{J}^{(\alpha,\beta+1)}+s\delta(x),\quad s\neq 0,$$

taking into account that for every polynomial p and $\alpha,\beta+1>-1$,

$$\left\langle\mathcal{J}^{(\alpha,\beta+1)},p(x)\right\rangle=\int_{-1}^{1}p(x)(1-x)^{\alpha}(x+1)^{\beta+1}dx.$$

The affine transformation $2t=x+1$ yields

$$\begin{aligned}\left\langle\left(\tau_{1/2}\circ h_{1/2}\right)\mathcal{J}^{(\alpha,\beta+1)},p(x)\right\rangle &= \int_{-1}^{1}p\left(\frac{1}{2}x+\frac{1}{2}\right)(1-x)^{\alpha}(x+1)^{\beta+1}dx,\\ &= \left\langle\mathcal{J}^{(\alpha,\beta+1)}_{[0,1]},p(x)\right\rangle.\end{aligned}$$

As a consequence, $\widetilde{v}=\mathcal{J}^{(\alpha,\beta)}_{[0,1]}+s\delta(x)$, $s\neq 0$.

A3. $\phi^E(x)=(x-1)(x-\zeta)$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)$. It corresponds to the case A_1 in [20], where $\widetilde{v}$ satisfies

$$\begin{aligned}&D\left(x(x-1)(x-\zeta)\widetilde{v}\right)\\ =\;&-\left[-\left(\alpha+\beta+\gamma+3\right)x^2+\left(\left(\alpha+\beta+2\right)\zeta+\alpha+\gamma+2\right)x-\zeta(\alpha+1)\right],\end{aligned}$$

and it has the integral representation

$$\langle \widetilde{v}, p(x) \rangle = \int_0^1 p(x)\,(1-x)^{\alpha}\,x^{\beta}\,|x-\zeta|^{\gamma}\,dx,$$

with the conditions $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > 0$, $\zeta \in (0,1)$.

B. $\deg\left(x\phi^E(x)\right) < 3$, $\deg\left(x\widetilde{\psi}^E(x) - \phi^E(x)\right) = 2$.

B1. $\phi^E(x) = x-1$, $\Psi(x) = \frac{1}{2}\left(x\widetilde{\psi}^E(x) - (x-1)\right)$. It corresponds to the case B_1 in [20], where $\widetilde{v}$ satisfies $D\left(x(x-1)\widetilde{v}\right) = -\left(2\lambda x^2 + (-\alpha-\beta-2\lambda-2)\,x + \beta + 1\right)\widetilde{v}$, and has the integral representation

$$\langle \widetilde{v}, p(x) \rangle = \int_0^1 p(x)\,(1-x)^{\alpha+1}\,x^{\beta+1}e^{-\lambda x}dx,$$

with the conditions $\alpha\beta \neq 0$, $\alpha, \beta > -1$, and $\deg\widetilde{\psi}^E = 1$.

B2. $\phi^E(x) = x$, $\Psi(x) = \frac{1}{2}\left(x\widetilde{\psi}^E(x) - x\right)$. This is the case B_2 in [20], where $\widetilde{v}$ satisfies

$$D\left(x^2\widetilde{v}\right) = -x\,(x-\alpha-2)\,\widetilde{v}.$$

Besides, for $\alpha > -1$

$$\langle \widetilde{v}, p(x) \rangle = \int_0^\infty p(x)x^{\alpha}e^{-x}dx + sp(0),$$

$s \neq 0$.

B3. $\phi^E(x) = 1$, $\Psi(x) = \frac{1}{2}\left(x\widetilde{\psi}^E(x) - 1\right)$. It corresponds to the case B_3 in [20], where $\widetilde{v}$ satisfies

$$D\left(x\widetilde{v}\right) = -\left(2x^2 - \lambda x - 2\mu - 1\right)\widetilde{v},$$

and it has the integral representation

$$\langle \widetilde{v}, p(x) \rangle = \int_0^\infty p(x)x^{2\mu}e^{-x^2-\lambda x}dx,$$

with the conditions $\mu > -1/2$, $\lambda \in \mathbb{R}$ and $\deg\widetilde{\psi}^E = 1$.

Now, we will analyze the reduction of (4.28) in the positive-definite case in order to get integral representations of such linear functionals. Then, we assume that $\widetilde{v}$ has an integral representation in terms of a weight function $\omega_{\widetilde{v}}$ on an interval $[a,b]$ with $a \geq 0$, that is

$$\langle \widetilde{v}, p(x) \rangle = \int_a^b p(x)\omega_{\widetilde{v}}dx.$$

First, we analyze the $A2$ and $B2$ cases. We get the rational relation $x^2\sigma_1(x)\widetilde{u} = x(x-\xi^2)\widetilde{v}$ with $\sigma_1(x) = x-1$ in $A2$ and $\sigma_1(x) = 1$ in $B2$. Besides

$$\langle \widetilde{u}, p(x) \rangle = \int_a^b p(x)\frac{\rho(x)\omega_{\widetilde{v}}}{\phi^E(x)}dx + M_1p(0) + M_2p'(0) + Np(1),$$

where $N = 0$ in B2. By using (4.27) and (4.28) we get

$$\begin{aligned}
\left\langle x\rho^E(x)D\widetilde{v}, p(x) \right\rangle &= -\left\langle \widetilde{v}, \left(x\rho^E p\right)' \right\rangle \\
&= \int_a^b p\rho^E \frac{x\phi^E(x)\omega_{\widetilde{v}}'}{\phi^E(x)}dx \\
&= -\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)}dx - \frac{1}{2}\int_a^b p(x)\omega_{\widetilde{v}}(x)dx.
\end{aligned}$$

Since

$$\begin{aligned} & -\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x)+2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)}dx \\ = & -\frac{1}{2}\left\langle \widetilde{u}, p(x)\left(x\widetilde{\psi}^E(x)+2x\left(\phi^E\right)'\right)\right\rangle \\ & +\frac{1}{2}M_2p(0)\left(\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)\right)+\frac{1}{2}Np(1)\left(\widetilde{\psi}^E(1)+2\left(\phi^E\right)'(1)\right), \end{aligned}$$

you get

$$M_2p(0)\left(\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)\right)+Np(1)\left(\widetilde{\psi}^E(1)+2\left(\phi^E\right)'(1)\right)=0,$$

for every polynomial p. In particular, for $p(x)=x-1$

$$M_2\left(\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)\right)=0.$$

Next we deal with $\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)\neq 0$. When $\phi^E(x)=x(x-1)$, $\widetilde{v}$ is positive definite if $\alpha,(\beta+1)>-1$. Taking into account that in this case $\widetilde{\psi}^E(x)=-\left(2\alpha+2\beta+5\right)x+2\beta+3$, then

$$\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)=2\beta+1,$$

and $M_2=0$ if $\beta\neq -1/2$. In a similar way, we get $\widetilde{\psi}^E(x)=2x-2\alpha-3$ and $\widetilde{v}$ is positive definite if $\alpha>-1$. After straightforward calculations, we obtain

$$\widetilde{\psi}^E(0)+2\left(\phi^E\right)'(0)=-2\alpha-1.$$

Thus $M_2=0$ if $\alpha\neq -1/2$.

In $A3$ and $B1$ we get

$$\langle\widetilde{u},p(x)\rangle=\int_a^b p(x)\frac{\rho(x)\omega_{\widetilde{v}}}{\phi^E(x)}dx+M_1p(0)+M_2p(1)+M_3p(\zeta),$$

where $M_3=0$ in $B1$. An iteration of the above procedure yields

$$\begin{aligned} & -\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x)+2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)}dx \\ = & -\frac{1}{2}\left\langle \widetilde{u}, p(x)\left(x\widetilde{\psi}^E(x)+2x\left(\phi^E\right)'\right)\right\rangle \\ & +\frac{1}{2}M_2p(1)\left(\widetilde{\psi}^E(1)+2\left(\phi^E\right)'(1)\right)+\frac{1}{2}M_3p(\zeta)\left(\zeta\widetilde{\psi}^E(\zeta)+2\zeta\left(\phi^E\right)'(\zeta)\right). \end{aligned}$$

Then

$$\frac{1}{2}M_2p(1)\left(\widetilde{\psi}^E(1)+2\left(\phi^E\right)'(1)\right)+\frac{1}{2}M_3p(\zeta)\left(\zeta\widetilde{\psi}^E(\zeta)+2\zeta\left(\phi^E\right)'(\zeta)\right)=0. \tag{4.32}$$

On one hand, in A3

$$\begin{aligned} & x\widetilde{\psi}^E(x) \\ = & -\left(2\alpha+2\beta+2\gamma+5\right)x^2 \\ & +2\left(\left(\alpha+\beta+2\right)\zeta+\alpha+\gamma+2-\frac{1}{2}(1+\zeta)\right)x-2\zeta(\alpha+1)+\zeta. \end{aligned}$$

Then $\alpha = -\frac{1}{2}$. In this way, the case A3 will not be considered. On the other hand, in the case B1

$$x\widetilde{\psi}^E(x) = 4\lambda x^2 + (-2\alpha - 2\beta - 4\lambda - 3)\, x + (2\beta + 1)\,,$$

and, thus, $\beta = -1/2$ and $\widetilde{\psi}^E(x) = 4\lambda x - 2\alpha - 4\lambda - 2$. Then $\widetilde{\psi}^E(1) = -2\alpha - 2$ and

$$\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1) = -2\alpha.$$

Therefore, $M_2 = 0$ if $\alpha \neq 0$.

In the case B3 we cannot simplify the factor x. However, we get

$$x\widetilde{\psi}^E(x) = 4x^2 - 2\lambda x - 4\mu - 1,$$

and, as a consequence, $\mu = -\frac{1}{4}$. Then $\widetilde{\psi}^E(x) = 4x - 2\lambda$ and $\widetilde{v}$ satisfies $D\left(x\widetilde{v}\right) = -\frac{1}{2}\left(4x^2 - 2\lambda x - 1\right)\widetilde{v}$, as well as

$$\langle \widetilde{v}, p(x)\rangle = \int_0^\infty p(x) x^{-1/2} e^{-x^2 - \lambda x} dx.$$

4.3 Positive-definite symmetric $(1,1)-$coherent pairs (u,v)

According to the functionals $\widetilde{u}$ and $\widetilde{v}$ obtained in the previous section when $A(x) = 2(x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$, or $A(x) = 2\left(x^2 - \xi^2\right)^2$, respectively, the symmetrization process allows us to recover the original symmetric functionals u and v and, as a consequence, we get a classification of symmetric $(1,1)-$coherent pairs. Of course, if we recover one pair (u,v), we must also prove that it is symmetric $(1,1)-$coherent one. For this purpose, we state the next results.

Theorem 78. *([35]). Let u be a symmetric, semiclassical and quasi-definite linear functional of odd class s satisfying*

$$D\left(\phi u\right) + \psi u = 0,$$

where $\deg\phi \leq s+2$, and $\deg\psi \leq s+1$. Notice that, ϕ and ψ are even and odd polynomials, respectively. $\{P_n\}_{n\geq 0}$ will denote the corresponding SMOP, We assume that the linear functional $w = x\phi(x)u$ is quasi-definite, with $\{W_n\}_{n\geq 0}$ as the corresponding SMOP. Then

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{(s+1)/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s+1,$$

with $\eta_{n,n-(s+1)} \neq 0$.

Theorem 79. *Let u be a symmetric, semiclassical and quasi-definite linear functional of even class s satisfying*

$$D\left(\phi u\right) + \psi u = 0,$$

where $\deg\phi \leq s+2$, and $\deg\psi \leq s+1$. Notice that ϕ and ψ are odd and even polynomials, respectively. $\{P_n\}_{n\geq 0}$ will denote the corresponding SMOP. We assume that the linear functional $w = \phi(x)u$ is quasi-definite, with $\{W_n\}_{n\geq 0}$ as the corresponding SMOP. Then

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{s/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s,$$

with $\eta_{n,n-s} \neq 0$.

Proof. It is enough to expand the sequence $\left\{\frac{P'_{n+1}}{n+1}\right\}_{n\geq 0}$ in terms of the basis $\{W_n\}_{n\geq 0}$ and to consider its quasi-orthogonal character described in Theorem 18, B). □

As a consequence of above theorems we get the next result.

Corollary 80. *Let u be as above with class s either 1 or 2. Let v denote a symmetric and quasi-definite linear functional such that there exist even polynomials p and q, with $0 \leq \deg p \leq 4$ and $\deg q = 2$ such that*

$$p(x)u = q(x)v,$$

holds. In addition, let $\{Q_n\}_{n\geq 0}$ be the SMOP associated with v. Then (u,v) is a symmetric $(1,1)-$coherent pair.

Proof. We consider the above theorems with $s = 1$ and $s = 2$, respectively. In both cases, we get

$$Q_n(x) = W_n(x) + \beta_n W_{n-2}(x),$$

and

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \lambda_n W_{n-2}(x),$$

where $\beta_n \lambda_n \neq 0$. From the above equations we obtain

$$\frac{P'_{n+1}(x)}{n+1} + \beta_{n-2}\frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})}\frac{P'_{n-1}(x)}{n-1} = Q_n(x) + \lambda_{n-2}\frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})}Q_{n-2}(x),$$

where $\beta_n \neq \lambda_n$ for every n. □

Case $A(x) = 2\left(x^2 - \xi^2\right)^2$

According to Theorem 30, if the class of $\widetilde{u}$ is $\widetilde{s} = 0$, then the class of u is either 0 or 1. The classical cases (Gegenbauer, Hermite) have been analyzed in [34]. We suppose that $s = 1$, i.e.

$$\widehat{\Psi}(0) = \left(\sigma_4^E\right)'(0) + \lim_{x\to 0}\frac{\sigma_4^E(x)}{x} + 2\widetilde{\phi}_2^E(0) \neq 0.$$

i). If $\sigma_4^E(x) = x^2$, assuming that $\widetilde{\phi}_2^E(0) \neq 0$, then $u = \overline{\mathcal{B}}^{(\alpha)}$ satisfies

$$D(x^3 u) = -2\left(\widetilde{\phi}_2(x) + x^2\right)u.$$

ii). If $\sigma_4^E(x) = x(x-1)$, assuming that $\widetilde{\phi}_2^E(0) \neq 1$, then

$$D(x\left(x^2 - 1\right)u) = -\left(2\widetilde{\phi}_2(x) + \frac{1}{2}(x^2 - 1)\right)u.$$

Notice that $u = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$.

iii). If $\sigma_4^E(x) = x$, assuming $\widetilde{\phi}_2^E(0) \neq -1$, then $u = \overline{\mathcal{L}}^{(\alpha)}$ and

$$D(xu) = -\left(2\widetilde{\phi}_2(x) + \frac{1}{2}\right)u,$$

On the other and, if $\widetilde{u}$ is of class $\widetilde{s}=1$, then from the symmetrization theorem we deduce that the class of u is $s=2$. Next we will describe u according to σ_4^E.

i). If $\sigma_4^E(x)=x^2$ and $\widetilde{\phi}_2^E(0)\neq 0$, then $D(x^4u)=-2x\widetilde{\phi}_2(x)u$. Thus $u=x^{-2}\overline{\mathcal{B}}^{(\alpha)}+M\delta(x)$.

ii). If $\sigma_4^E(x)=x(x-1)$, then u satisfies $D(x^2(x^2-1)u)=-2x\widetilde{\phi}_2(x)u$ and it has the integral representation

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)\left(1-x^2\right)^{\alpha}|x|^{2\beta+1}e^{-\frac{\gamma}{x^2}}dx,$$

with the conditions $\alpha\gamma\neq 0$, $\gamma>0$, $\alpha>-1$.

iii). If $\sigma_4^E(x)=(x-1)(x-\zeta)$, with $\zeta\in(0,1)$, then u satisfies $D((x^2-1)(x^2-\zeta)u)=-2x\widetilde{\phi}_2(x)u$. Moreover,

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)\left(1-x^2\right)^{\alpha}|x|^{2\beta+1}\left|x^2-\zeta\right|^{\gamma}dx,$$

with the conditions $\alpha\beta\gamma\neq 0$, $\alpha,\beta,\gamma>-1$.

iv). If $\sigma_4^E(x)=x$, then u satisfies $D(x^2u)=-2x\widetilde{\phi}_2(x)u$ as well as

$$\langle u,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\,|x|^{2\alpha+1}\left(1+x^2\right)^{\beta+1}e^{-x^2+\frac{\beta}{x^2}}dx,$$

with $\beta<0$, $\alpha,\beta>-1$.

v). If $\sigma_4^E(x)=x-1$, then u satisfies $D((x^2-1)u)=-2x\widetilde{\phi}_2(x)u$. Moreover,

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)\left(1-x^2\right)^{\alpha+1}|x|^{2\beta+3}e^{-\lambda x^2}dx,$$

with the conditions $\alpha\beta\neq 0$, $\alpha,\beta>-1$.

vi). If $\sigma_4^E(x)=1$, then u satisfies $Du=-2x\widetilde{\phi}_2(x)u$ and it has the integral representation

$$\langle u,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\,|x|^{4\mu+1}e^{-x^4-\lambda x^2}dx,$$

under the conditions $\mu>-1/2$, $\lambda\in\mathbb{R}$.

Since in the previous cases u and v are related by

$$\sigma_4(x)u=2r_2(x)v,$$

then according to Corollary 80, in each case the pair (u,v) is a symmetric $(1,1)-$coherent pair. Next, the corresponding symmetric $(1,1)-$coherent pairs are described in the positive-definite framework.

Theorem 81. *Let $\{u,v\}$ be a symmetric $(1,1)-$coherent pair satisfying*

$$\sigma_4(x)u=2(x^2-\xi^2)v,$$

such that σ_4 is an even polynomial with $\deg\sigma_4\leq 4$ and u is a semiclassical linear functional of class at most 2. In addition, u and v are positive-definite and $A(x)=(x^2-\xi^2)^2$ in (4.8).

A. u of class $s=1$.

$S_{1,1}$. *If $\sigma_4(x)=x^2(x^2-1)$ and either $\xi^2=0$ or $\xi^2=1$, then*

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)(1-x^2)^{\alpha}\left|x\right|^{2\beta+1}dx,$$

and

$$\begin{aligned}\langle v,p(x)\rangle &= \int_{-1}^{1}p(x)\frac{(1-x^2)^{\alpha+1}\left|x\right|^{2\beta+3}}{(x^2-\xi^2)}dx,\\ &+\frac{M}{2}\left(\delta(x+|\xi|)+\delta(x-|\xi|)\right).\end{aligned}$$

$S_{1,2}$. *If $\sigma_4(x)=x^2$ and $\xi^2=0$ then $u=\overline{\mathcal{L}}^{(\alpha)}$ and $v=\overline{\mathcal{L}}^{(\alpha)}+M\delta(x)$.*

B. u of class $s=2$.

$S_{1,3}$. *If $\sigma_4(x)=x^2(x^2-1)$, $\alpha\gamma\neq 0$, $\gamma>0$ $\alpha>-1$, and either $\xi^2=0$ or $\xi^2=1$, then*

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)\left(1-x^2\right)^{\alpha}\left|x\right|^{2\beta+1}e^{-\frac{\gamma}{x^2}}dx,$$

and

$$\langle v,p(x)\rangle=\int_{-1}^{1}p(x)\frac{\left(1-x^2\right)^{\alpha+1}}{(x^2-\xi^2)}\left|x\right|^{2\beta+3}e^{-\frac{\gamma}{x^2}}dx+\frac{M}{2}\left(p(|\xi|)+p(-|\xi|)\right).$$

$S_{1,4}$. *If $\sigma_4(x)=(x^2-1)(x^2-\zeta)$, with $\zeta\in(0,1)$, $\alpha\beta\gamma\neq 0$, $\alpha,\beta,\gamma>0$, and either $\xi^2=0$ or $\xi^2=1$, then*

$$\langle u,p(x)\rangle=\int_{-1}^{1}p(x)\left(1-x^2\right)^{\alpha}\left|x\right|^{2\beta+1}\left|x^2-\zeta\right|^{\gamma}dx,$$

and

$$\langle v,p(x)\rangle=\int_{-1}^{1}p(x)\frac{\left(1-x^2\right)^{\alpha+1}}{(x^2-\xi^2)}\left|x\right|^{2\beta+1}\left|x^2-\zeta\right|^{\gamma+1}dx+\frac{M}{2}\left(p(|\xi|)+p(-|\xi|)\right).$$

$S_{1,5}$. *If $\sigma_4(x)=x^2$, $\beta\in(-1,0)$, $\alpha>-1$ and $\xi^2=0$, then*

$$\langle u,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\left|x\right|^{2\alpha+1}\left(1+x^2\right)^{\beta+1}e^{-x^2+\frac{\beta}{x^2}}dx,$$

and

$$\langle v,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\left|x\right|^{2\alpha+1}\left(1+x^2\right)^{\beta+1}e^{-x^2+\frac{\beta}{x^2}}dx+Mp(0).$$

$S_{1,6}$. *If $\sigma_4(x)=1$, $\mu>0$, $\lambda\in\mathbb{R}$ and $\xi^2=0$, then*

$$\langle u,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\left|x\right|^{4\mu+1}e^{-x^4-\lambda x^2}dx,$$

and

$$\langle v,p(x)\rangle=\int_{-\infty}^{\infty}p(x)\left|x\right|^{4\mu-1}e^{-x^4-\lambda x^2}dx+Mp(0).$$

$S_{1,7}$. *If* $\sigma_4(x) = x^2 - 1$, $\alpha\beta \neq 0$, $\alpha, \beta > -1$ *and either* $\xi^2 = 0$ *or* $\xi^2 = 1$, *then*

$$\langle u, p(x) \rangle = \int_{-1}^{1} p(x) \left(1 - x^2\right)^{\alpha+1} |x|^{2\beta+3} e^{-\lambda x^2} dx,$$

and

$$\langle v, p(x) \rangle = \int_{-1}^{1} p(x) \frac{\left(1 - x^2\right)^{\alpha+2}}{(x^2 - \xi^2)} |x|^{2\beta+3} e^{-\lambda x^2} dx + \frac{M}{2} \left(p(|\xi|) + p(-|\xi|)\right).$$

4.3.1 Case $A(x) = 2(x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$.

When $\widetilde{v}$ is a semiclassical linear functional of class $\widetilde{s} = 0$, the class of v is either 0 or 1. When the class of v is $s = 0$, we get $\mathcal{J}_{(0,1)}^{(\alpha,-1/2)}$ and $\mathcal{L}^{(-3/2)}$, which are nonpositive-definite linear functionals. Next, we describe the cases when the class of v is $s = 1$, and according to the expression of ϕ^E.

i). If $\phi^E(x) = x^2$, then $v = \overline{\mathcal{B}}^{(\alpha)}$ and it satisfies $D\left(x^3 v\right) = -\left(\widetilde{\psi}(x) + x^2\right) v$. Notice that this is not a positive-definite case.

ii). If $\phi^E(x) = x(x-1)$ and $\beta \neq -1/2$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,\beta)}$, moreover $D\left(x(x^2-1)v\right) = -\left(\widetilde{\psi}(x) + (x^2 - 1)\right) v$.
Notice that

$$\langle v, p(x) \rangle = \int_{-1}^{1} p(x)(1 - x^2)^{\alpha} |x|^{2\beta+1} dx.$$

iii). If $\phi^E(x) = x$, then v satisfies $D(xv) = -\left(\widetilde{\psi}(x) + 1\right) v$ and as a consequence $v = \overline{\mathcal{L}}^{(\alpha)}$. Thus,

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx.$$

If $\widetilde{v}$ is a semiclassical linear functional of class $\widetilde{s} = 1$, notice that, according to Theorem 30, v must be semiclassical of class $s = 2$. Next we describe the possible choices for v.

iv). If $\phi^E(x) = x^2$, then v satisfies $D\left(x^4 v\right) = -x\widetilde{\psi}(x)v$, i.e. $v = x^{-2}\overline{\mathcal{B}}^{(\alpha)} + M\delta(x)$. Notice that this is not a positive definite case.

v). If $\phi^E(x) = x(x-1)$, then $v = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)} + s\delta(x)$, $s \neq 0$, and v satisfies

$$D\left(x^2(x^2-1)v\right) = -x\widetilde{\psi}(x)v = -x\left(-\left(2\alpha + 2\beta + 5\right)x + 2\beta + 3\right)v,$$

with $\beta \neq -1/2$.

vi). If $\phi^E(x) = x - 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, then v satisfies

$$D\left((x^2-1)v\right) = -2x\left(2\lambda x^2 - \alpha - 2\lambda - 1\right)v,$$

i.e.

$$\langle v, p(x) \rangle = \int_{-1}^{1} p(x) \left(1 - x^2\right)^{\alpha+1} x^2 e^{-\lambda x^2} dx.$$

vii). If $\phi^E(x) = x$, $\alpha > -1$, $\alpha \neq -1/2$ and $s \neq 0$, then

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx + sp(0),$$

and $D\left(x^2 v\right) = -x\left(2x^2 - 2\alpha - 3\right)v$.

viii). If $\phi^E(x) = 1$, then v satisfies $Dv = -x(4x^2 - 2\lambda)v$ and

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x) e^{-x^4-\lambda x^2} dx.$$

Moreover,

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\left(x^2 - \zeta^2\right) e^{-x^4-\lambda x^2} dx + Mp(0).$$

In Cases from i) to v) and vii) we will assume that $\xi^2 = 0$. From (4.26), we get

$$x\phi^E(x)D(\widetilde{u}) = \frac{3}{2}x\widetilde{v} - \left(2x(\phi^E)'(x) + \frac{1}{2}x\widetilde{\psi}^E(x) + \phi^E(x)\right)\widetilde{u}.$$

Taking into account $\phi^E(x)\widetilde{u} = x\widetilde{v}$, then

$$D(x\phi^E(x)\widetilde{u}) = \left(\frac{5}{2}\phi^E(x) - (x\phi^E(x))' - \frac{1}{2}x\widetilde{\psi}^E(x)\right)\widetilde{u}.$$

As a consequence, $\widetilde{u}$ is semiclassical of class at most 1. According to Theorem 30 and Corollary 80, since $\phi^E(0) = 0$, then the class of u must be at most 2 and the pairs (u,v) are symmetric $(1,1)-$coherent. For Cases vi) and viii) we get $x^2u = x^2(x^2 - \xi^2)v$, and $x^2u = x^2v$, respectively. Then it is enough to apply the arguments of the above lemma but by using the fact that v is of class $s \leq 2$.

For the positive-definite case, the previous analysis is summarized next

Theorem 82. *Let $\{u,v\}$ be a symmetric $(1,1)-$coherent pair satisfying*

$$x\phi(x)u = x(x^2 - \xi^2)v$$

such that ϕ is an even polynomial with $\deg\phi(x) \leq 4$ and v is semiclassical of class at most 2. In addition, let assume that u and v are positive-definite as well as in (4.8) $A(x) = (x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$.

A. v classical.

$S_{2,1}$. If $\phi(x) = x^2(x^2-1)$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,-1/2)} = \mathcal{G}^{(\lambda)}$, $\lambda > -1$, $\lambda \neq 0$, i.e. the classical Gegenbauer functional. Thus

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\lambda-1/2}dx + M_1p(0) + \frac{M_2}{2}\left(p(1) + p(-1)\right).$$

B. v of class 1.

$S_{2,2}$. If $\phi(x) = x^2(x^2-1)$, $\beta \neq -1/2$, then

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha}\,|x|^{2\beta+1}\,dx,$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha-1}\,|x|^{2\beta+1}\,dx + M_1 p(0) + \frac{M_2}{2}\,(p(1)+p(-1))\,.$$

$S_{2,3}$. *If* $\phi(x) = x^2$ *then*

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\,|x|^{2\alpha+1}\,e^{-x^2}dx,$$

and

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\,|x|^{2\alpha+1}\,e^{-x^2}dx + Mp(0).$$

C. v of class 2.

$S_{2,4}$. *If* $\phi(x) = x^2(x^2-1)$, $\beta \neq -1/2$, *then*

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha}\,|x|^{2\beta+1}\,dx + Mp(0),$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha-1}\,|x|^{2\beta+1}\,dx + Mp(0) + \frac{N}{2}\,(p(1)+p(-1))\,.$$

$S_{2,5}$. *If* $\phi(x) = x^2$, $\alpha > -1$, $\alpha \neq -1/2$ *and* $M \neq 0$, *then*

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\,|x|^{2\alpha+1}\,e^{-x^2}dx + Mp(0),$$

and

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\,|x|^{2\alpha+1}\,e^{-x^2}dx + Mp(0).$$

$S_{2,6}$. *If* $\phi(x) = x^2-1$, $\xi^2 = 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, *then*

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha+1} x^2 e^{-\lambda x^2}dx,$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha} e^{-\lambda x^2}dx + Mp(0).$$

$S_{2,7}$. *If* $\phi(x) = 1$, *then*

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)e^{-x^4-\lambda x^2}dx,$$

as well as

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x^2-\xi^2|e^{-x^4-\lambda x^2}dx + Mp(0).$$

4.4 Inverse problem associated with a non-coherence relation

Let $\{u, v\}$ be a symmetric $(1,1)-$coherent pair satisfying (4.2). By using the inverse process to the symmetrization, we can obtain

$$\frac{A'_{n+2}(x)}{n+2} + b_{2n+1}\frac{A'_{n+1}(x)}{n+1} = \widetilde{B}_{n+1}(x) + b_{2n+1}\widetilde{B}_n(x), \quad n \geq 0,$$

and

$$\frac{2x\widetilde{A}'_{n+1}(x) + \widetilde{A}_{n+1}(x)}{2n+3} + b_{2n}\frac{2x\widetilde{A}'_n(x) + \widetilde{A}_n(x)}{2n+1} = B_{n+1}(x) + a_{2n}B_n(x), \tag{4.33}$$

$n \geq 0$, where

$$\begin{aligned} P_{2n}(x) &= A_n(x^2), \quad P_{2n+1}(x) = x\widetilde{A}_n(x^2), \\ R_{2n}(x) &= B_n(x^2), \quad R_{2n+1}(x) = x\widetilde{B}_n(x^2). \end{aligned}$$

Notice that the first one is related with the concept of $(1,1)-$coherent pair. However, the second one is not a coherence relation and as far as we know, this kind of algebraic relation has not been still studied. Then, in order to give an equivalent expression of (4.33), it is well known that, (see [31]),

$$x\widetilde{A}_n(x) = A_{n+1}(x) - \frac{A_{n+1}(0)}{A_n(0)}A_n(x), \quad n \geq 0.$$

Taking derivatives

$$x\widetilde{A}'_n(x) + \widetilde{A}_n(x) = A'_{n+1}(x) - \frac{A_{n+1}(0)}{A_n(0)}A'_n(x), \tag{4.34}$$

and replacing in (4.33) we obtain

$$\begin{aligned} & \frac{2A'_{n+2}(x) - 2\frac{A_{n+2}(0)}{A_{n+1}(0)}A'_{n+1}(x) - \widetilde{A}_{n+1}(x)}{2n+3} \\ & +b_{2n}\frac{2A'_{n+1}(x) - 2\frac{A_{n+1}(0)}{A_n(0)}A'_n(x) - \widetilde{A}_n(x)}{2n+1} \\ = \; & B_{n+1}(x) + a_{2n}B_n(x), \end{aligned}$$

or, equivalently,

$$\begin{aligned} & A'_{n+2}(x) + \left(\frac{b_{2n}\,(2n+3)}{(2n+1)} - \frac{A_{n+2}(0)}{A_{n+1}(0)}\right)A'_{n+1}(x) \\ & -\frac{b_{2n}\,(2n+3)}{(2n+1)}\frac{A_{n+1}(0)}{A_n(0)}A'_n(x) - \frac{1}{2}\widetilde{A}_{n+1}(x) - \frac{b_{2n}\,(2n+3)}{2\,(2n+1)}\widetilde{A}_n(x) \\ = \; & \frac{2n+3}{2}\left(B_{n+1}(x) + a_{2n}B_n(x)\right). \end{aligned}$$

The above relation leads us to define, in a general approach, the non-coherence relation

$$\begin{aligned} & P^{[i]}_{n+1}(x) + a^{[1]}_n P^{[i]}_n(x) + a^{[2]}_n P^{[i]}_{n-1}(x) \\ & +b_n\left(Q_{n+1}(x) + c_nQ_n(x)\right) \\ = \; & (1+b_n)\,R_{n+1}(x) + d_nR_n(x), \end{aligned} \tag{4.35}$$

where the sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w, respectively, with $P_k^{[i]}(x) := \dfrac{P_{k+i}^{(i)}(x)}{(k+1)_i}$, $i = 0, 1$, and $a_n^{[i]} b_n c_n d_n \neq 0$, $n \geq 0$. Besides, the functionals u and v are related through the rational relation

$$\rho(x)u = v, \tag{4.36}$$

where ρ is a monic polynomial of degree m.

4.4.1 Case $i = 0$

In this section we consider (4.35) with $i = 0$, and therefore we study the algebraic relation

$$\begin{aligned} & P_{n+1}(x) + a_n^{[1]} P_n(x) + a_n^{[2]} P_{n-1}(x) + b_n \left(Q_{n+1}(x) + c_n Q_n(x)\right) \\ = \;& \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \end{aligned} \tag{4.37}$$

where $(1 + b_n) \neq 0$. Here the sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w respectively, as well as the linear functionals u and v are related by the rational relation

$$\rho(x)u = v,$$

where $\deg \rho \geq 0$. If we define the sequence $\{T_n\}_{n\geq 0}$ as

$$T_{n+1}(x) := \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \tag{4.38}$$

then let $\{\mathbf{t}_n\}_{n\geq 0}$, $\{\mathbf{p}_n\}_{n\geq 0}$, $\{\mathbf{q}_n\}_{n\geq 0}$, and $\{\mathbf{r}_n\}_{n\geq 0}$ be the corresponding dual bases associated with the sequences $\{T_n(x)\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, respectively. First, from (4.36), if $\mathbf{q}_n = \sum \langle \mathbf{q}_n, p_k(x)\rangle \mathbf{p}_k$, then

$$\langle \mathbf{q}_n, P_k(x)\rangle = \left\langle \frac{Q_n(x)}{\|Q_n\|_v^2} v, P_k(x) \right\rangle = \frac{1}{\|Q_n\|_v^2} \langle v, Q_n(x) P_k(x)\rangle,$$

and $\langle \mathbf{q}_n, p_k(x)\rangle = 0$ for $k < n$. Here $\|Q_n\|_v^2 := \langle v, Q_n^2(x)\rangle$. Moreover, if $k \geq n$,

$$\langle \mathbf{q}_n, P_k(x)\rangle = \frac{1}{\|Q_n\|_v^2} \langle u, P_k(x) Q_n(x) \rho(x)\rangle.$$

If $k > n + m$, by orthogonality of u we get $\langle \mathbf{q}_n, P_k(x)\rangle = 0$. In this way, we get

$$\mathbf{q}_n = \sum_{k=n}^{n+m} \eta_k \mathbf{p}_k, \tag{4.39}$$

where we have defined $\eta_{n,k} := \langle \mathbf{q}_n, P_k(x)\rangle$. In particular, for $n \geq 0$,

$$\eta_{n,n+m} = \langle \mathbf{q}_n, P_{n+m}(x)\rangle = \frac{1}{\|Q_n\|_v^2} \langle u, P_{n+m}(x) Q_n(x) \rho(x)\rangle = \frac{\|P_{n+m}\|_u^2}{\|Q_n\|_v^2} > 0.$$

On the other hand, expanding $\mathbf{q}_n$ in terms of the basis $\{\mathbf{t}_k\}_{k\geq 0}$, it is clear that $\langle \mathbf{q}_n, T_k(x)\rangle = 0$ for $n > k$. Now, if $k \geq n$, by using (4.39) we get

$$\begin{aligned}
&\langle \mathbf{q}_n, T_k(x)\rangle \\
= &\left\langle \mathbf{q}_n, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x)\right\rangle \\
&+b_{k-1}\langle \mathbf{q}_n, Q_k(x) + c_{k-1}Q_{k-1}(x)\rangle \\
= &\left\langle \sum_{j=n}^{n+m} \eta_{n,j}\mathbf{p}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x)\right\rangle \\
&+b_{k-1}\langle \mathbf{q}_n, Q_k(x) + c_{k-1}Q_{k-1}(x)\rangle .
\end{aligned}$$

If $k > n+m+2$, then

$$\left\langle \sum_{j=n}^{n+m} \eta_{n,j}\mathbf{p}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x)\right\rangle = 0.$$

Besides, $\langle \mathbf{q}_n, Q_k(x) + c_{k-1}Q_{k-1}(x)\rangle = 0$ if $k \neq n, n+1$. In this way

$$\mathbf{q}_n = \sum_{k=n}^{n+m+2} \xi_{n,k}\mathbf{t}_k. \tag{4.40}$$

Here $\xi_{n,k} := \left\langle \sum_{j=n}^{n+m} \eta_{n,j}\mathbf{p}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x)\right\rangle$ if $n+2 \leq k \leq n+m+2$. On the other hand $\xi_{n,n+1} = \eta_{n,n}a_n^{[1]} + \eta_{n,n+1} + b_n c_n$ and $\xi_{n,n} = \eta_{n,n} + b_{n-1}$. Notice that $\xi_{n,k}$ depends on m. In particular, for $n \geq 0$

$$\begin{aligned}
&\xi_{n,n+m+2} \\
= &\left\langle \sum_{j=n}^{n+m} \eta_{n,j}\mathbf{p}_j, P_{n+m+2}(x) + a_{n+m+1}^{[1]} P_{n+m+1}(x) + a_{n+m+1}^{[2]} P_{n+m}(x)\right\rangle \\
= &\ \eta_{n,n+m}a_{n+m+1}^{[2]} \neq 0.
\end{aligned}$$

If we assume that $\mathbf{r}_n = \sum \langle \mathbf{r}_n, T_k(x)\rangle \mathbf{t}_k$, then

$$\langle \mathbf{r}_n, T_k(x)\rangle = \langle \mathbf{r}_n, (1+b_{k-1}) R_k(x) + d_{k-1}R_{k-1}(x)\rangle .$$

If $k \neq n\,, n+1$, then $\langle \mathbf{r}_n, T_k(x)\rangle = 0$ and as a consequence

$$\mathbf{r}_n = (1+b_{n-1})\,\mathbf{t}_n + d_n\mathbf{t}_{n+1}, \quad n \geq 0. \tag{4.41}$$

If we consider the above relation for $n = 0, .., m+1$, and (4.40) for $n = 0$, we obtain a system of $m+3$ linear equations with the $m+3$ unknowns $\{\mathbf{t}_k\}_{k=0}^{m+2}$, namely

$$D_m \begin{pmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{m+2} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{m+1} \\ \mathbf{q}_0 \end{pmatrix},$$

where the entries of $D_m := \left(h_{i,j}^{[m]}\right)_{i,j=1}^{m+3}$ are

$$h_{i,j}^{[m]} = \begin{cases} 0, & if \;\; m+3 > i > j \text{ and } i+1 < j, \\ 1+b_{i-2}, & if \;\; i = j \neq m+3, \\ d_{j-2}, & if \;\; i+1 = j, \\ \xi_{0,j-1}, & if \;\; i = m+3, \end{cases} ,$$

where $b_{-1} := 0$. If we suppose that $|D_m| \neq 0$, then we can find a matrix $(\mu_{i,j})_{i,j=0}^{m+2}$ such that if $k = 0, ..., m+2$, each $\mathbf{t}_{n+k}$ can be written as

$$\mathbf{t}_k = \mu_{k,0}\mathbf{q}_0 + \mu_{k,1}\mathbf{r}_0 + \mu_{k,2}\mathbf{r}_1 + \cdots + \mu_{k,m+1}\mathbf{r}_m + \mu_{k,m+2}\mathbf{r}_{m+1}. \tag{4.42}$$

Now, multiplying (4.40), (with $n = 1$), and (4.41), (with $n = m+2$), by d_{m+2} and $\xi_{m+3} := \xi_{1,m+3}$, respectively, and then subtracting, we obtain

$$d_{m+2}\mathbf{q}_1 - \delta_{m+3}\mathbf{r}_{m+2} = \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mathbf{t}_k,$$

where $\widetilde{\xi}_{m,k} := d_{m+2}\xi_{1,k}$, for $1 \leq k \leq m+1$, and $\widetilde{\xi}_{m,m+2} := d_{m+2}\xi_{1,m+2} - \xi_{m+3}\left(1 + b_{m+1}\right)$. Now, replacing (4.42) in above relation, we get

$$d_{m+2}\mathbf{q}_1 - \xi_{m+3}\mathbf{r}_{m+2} = \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k} \left(\mu_{k,0}\mathbf{q}_0 + \sum_{p=0}^{m+1} \mu_{k,p+1}\mathbf{r}_p \right),$$

or, equivalently,

$$d_{m+2}\mathbf{q}_1 - \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,0}\mathbf{q}_0 = \xi_{m+3}\mathbf{r}_{m+2} + \sum_{p=0}^{m+1} \left(\sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,p+1} \right) \mathbf{r}_p.$$

From (1.12) and (4.39) we get the next

Theorem 83. *Let u, v and w be quasi-definite linear functionals and let $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be, respectively, the corresponding SMOP. Assume that the functionals u and v are related by*

$$\rho(x)u = v,$$

where ρ is a monic polynomial with $\deg \rho = m \geq 0$. Also, suppose that the relation (4.37) holds. If the matrix $D_m = \left(h_{i,j}^{[m]}\right)_{i,j=1}^{m+3}$, with entries

$$h_{i,j}^{[m]} = \begin{cases} 0, & if \;\; m+3 > i > j \text{ and } i+1 < j, \\ 1+b_{i-2}, & if \;\; i = j \neq m+3, \\ d_{j-2}, & if \;\; i+1 = j, \\ \xi_{0,j-1}, & if \;\; i = m+3. \end{cases} ,$$

is non-singular, then there exist polynomials ϕ_{m+2}, φ_1 with $\deg(\phi_{m+2}) = m+2$, $\deg(\varphi_1) = 1$, such that

$$\phi_{m+2}(x)w = \varphi_1(x)v = \varphi_1(x)\rho(x)u.$$

Here

$$\varphi_1(x)=\frac{d_{m+2}}{\left\|Q_1(x)\right\|_v}Q_1(x)-\sum_{k=1}^{m+2}\widetilde{\xi}_{m,k}\mu_{k,0},$$

$$\varphi_{m+2}(x)=\frac{\xi_{m+3}}{\left\|R_{m+2}(x)\right\|_w}R_{m+2}(x)+\sum_{p=0}^{m+1}\frac{\left(\sum_{k=1}^{m+2}\widetilde{\xi}_{m,k}\mu_{k,p+1}\right)}{\left\|R_p(x)\right\|_w}R_p(x).$$

For instance, when $\rho(x)=1$, i. e., $u=v$, (4.37) becomes

$$P_{n+1}(x)+a_n^{[1]}P_n(x)+a_n^{[2]}P_{n-1}(x)=R_{n+1}(x)+d_nR_n(x).$$

Of course, this is a $(2,1)-$coherence relation. According to the above theorem, since

$$D_0=\begin{bmatrix}1 & d_0 & 0\\ 0 & 1+b_0 & d_1\\ \xi_{0,0} & \xi_{0,1} & \xi_{0,2}\end{bmatrix},$$

where $\xi_{0,0}:=1$, $\xi_{0,1}:=a_0^{[1]}$, and $\xi_{0,2}:=a_1^{[2]}$, if $|D_0|=d_0d_1-a_0^{[1]}d_1+a_1^{[2]}(1+b_0)\neq 0$, then the pair (u,v) satisfies

$$\phi_2(x)w=\varphi_1(x)u.$$

This is the same result obtained in [106] for the particular case $M=1$ and $N=2$. Notice this case has been deeply studied in [11].

Now, we assume $\rho(x)=x$ and u with orthogonality interval $[a,b]$, $a\geq 0$. We also suppose that the TTRR, (three term recurrence relation), associated with $\{P_n\}_{n\geq 0}$ is

$$xP_n(x)=P_{n+1}(x)+\alpha_nP_n(x)+\beta_nP_{n-1}(x),\quad n\geq 0,$$

and that $\{Q_n\}_{n\geq 0}$ satisfies $xQ_n(x)=P_{n+1}(x)-\sigma_nP_n(x)$, $n\geq 0$, where $\sigma_n=\frac{P_{n+1}(0)}{P_n(0)}$. If the TTRR associated to $\{R_n(x)\}_{n\geq 0}$ is given by $xR_n(x)=R_{n+1}(x)+\lambda_nR_n(x)+\gamma_nR_{n-1}(x)$, $n\geq 1$, from (4.37) we get the algebraic relation

$$\begin{aligned}
&(1+b_n)\,P_{n+2}(x)+\left[\alpha_{n+1}+a_n^{[1]}+b_n\left(c_n-\sigma_{n+1}\right)\right]P_{n+1}(x)\\
&+\left[\beta_{n+1}+a_n^{[1]}\alpha_n+a_n^{[2]}-b_nc_n\sigma_n\right]P_n(x)\\
&+\left[\beta_na_n^{[1]}+a_n^{[2]}\alpha_{n-1}\right]P_{n-1}(x)+\beta_{n-1}a_n^{[2]}P_{n-2}(x)\\
=\;&(1+b_n)\,R_{n+2}(x)+\left[\lambda_{n+1}\left(1+b_n\right)+d_n\right]R_{n+1}(x)+\left[\gamma_{n+1}\left(1+b_n\right)+\lambda_nd_n\right]R_n(x)\\
&+\gamma_nd_nR_{n-1}(x).
\end{aligned}$$

Assuming $b_n\neq -1$ for every n, as a consequence $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ satisfy a $(4,3)-$coherence relation, namely

$$P_{n+2}(x)+\sum_{k=1}^{4}A_{n,k}P_{n+2-k}(x)=R_{n+2}(x)+\sum_{k=1}^{3}B_{n,k}R_{n+2-k}(x),$$

with $A_{n,4}B_{n,3}\neq 0$. Besides, according to [106], u and w must be related through

$$\widetilde{\phi}(x)u=\widetilde{\psi}(x)v,$$

for certain polynomials $\widetilde{\phi}$ and $\widetilde{\psi}$ such that $\deg(\widetilde{\psi}) = 4$ and $\deg(\widetilde{\phi}) = 3$. Here

$$D_1 = \begin{bmatrix} 1 & d_0 & 0 & 0 \\ 0 & 1+b_0 & d_1 & 0 \\ 0 & 0 & 1+b_1 & d_2 \\ 1 & a_0^{[1]} + \|P_1\|_u^2 & a_1^{[2]} + \|P_1\|_u^2 a_1^{[1]} & \|P_1\|_u^2 a_2^{[2]} \end{bmatrix}.$$

From the above theorem, if

$$\begin{aligned} & |D_1| \\ = & \left(d_0 - a_0^{[1]} - \|P_1\|_u^2\right) d_1 d_2 + \left(\left(a_1^{[2]} + \|P_1\|_u^2 a_1^{[1]}\right) d_2 - \|P_1\|_u^2 a_2^{[2]} (1+b_1)\right)(1+b_0) \\ \neq & \ 0, \end{aligned}$$

then the pair $\{u, w\}$ satisfies

$$\phi_3(x) w = x\varphi_1(x) u.$$

Thus, under the conditions of such a particular framework, it is possible to improve the result obtained in [106].

4.4.2 Case $i = 1$

In this section we consider the relation (4.35) with $i = 1$. Then

$$\begin{aligned} & \frac{P'_{n+2}(x)}{n+2} + a_n^{[1]} \frac{P'_{n+1}(x)}{n+1} a_n^{[2]} \frac{P'_n(x)}{n} + b_n \left(Q_{n+1}(x) + c_n Q_n(x)\right) \\ = & \ (1+b_n) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \end{aligned}$$

holds. We assume that $(1 + b_n) \neq 0$. Here $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are orthogonal with respect to u, v and w, respectively. Besides, let ρ be a polynomial such that $\deg \rho = m \geq 1$ and $\rho(x) u = v$. Additionally, in the sequel we assume that u is semiclassical of class at most s. Under these conditions (see Theorem 18) there exists a polynomial σ, with degree $t \leq s + 2$, such that

$$\sigma(x) \frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} \beta_{n,k} P_k(x), \tag{4.43}$$

where $\beta_{n,k} = \dfrac{\left\langle u, \sigma(x) \frac{P'_{n+1}(x)}{n+1} P_k(x) \right\rangle}{\|P_k\|_u^2}$, and $\beta_{n,n-s} \neq 0$. As in the previous section, let us define $T_{n+1}(x) := (1 + b_n) R_{n+1}(x) + d_n R_n(x)$, and $\{\mathbf{t}_n\}_{n\geq 0}$ be the dual basis associated with the sequence $\{T_n\}_{n\geq 0}$. Now, we consider the expansion of $\sigma \mathbf{q}_n$, in terms of the basis $\{\mathbf{t}_n\}_{n\geq 0}$, where

$$\begin{aligned} & \langle \sigma \mathbf{q}_n, T_k \rangle \\ = & \left\langle \sigma \mathbf{q}_n, \frac{P'_{k+1}(x)}{k+1} + a_{k-1}^{[1]} \frac{P'_k(x)}{k} + a_{k-1}^{[2]} \frac{P'_{k-1}(x)}{k-1} \right\rangle \\ & + b_{k-1} \langle \sigma \mathbf{q}_n, Q_k(x) + c_{k-1} Q_{k-1}(x) \rangle . \end{aligned}$$

As a consequence of (4.43) we get

$$\begin{aligned}
&\sigma(x)\frac{P'_{k+1}(x)}{k+1}+a^{[1]}_{k-1}\sigma(x)\frac{P'_k(x)}{k}+a^{[2]}_{k-1}\sigma(x)\frac{P'_{k-1}(x)}{k-1}\\
=\;&\left(\sum_{j=k-s}^{k+t}\beta_{k,j}P_j(x)\right)+a^{[1]}_{k-1}\left(\sum_{j=k-s-1}^{k+t-1}\beta_{k-1,j}P_j(x)\right)\\
&+a^{[2]}_{k-1}\left(\sum_{j=k-s-2}^{k+t-2}\beta_{k-2,j}P_j(x)\right)\\
=\;&\sum_{j=k-s-2}^{k+t}\beta^*_{k,j}P_j(x),
\end{aligned}$$

where

$$\beta^*_{k,j}:=\begin{cases}
\beta_{k,j}, & \text{if } j=k+t,\\
\beta_{k,j}+a^{[1]}_{k-1}\beta_{k-1,j}, & \text{if } j=k+t-1,\\
\beta_{k,j}+a^{[1]}_{k-1}\beta_{k-1,j}+a^{[2]}_{k-1}\beta_{k-2,j}, & \text{if } k+t-2\le j\le k-s,\\
a^{[1]}_{k-1}\beta_{k-1,j}+a^{[2]}_{k-1}\beta_{k-2,j}, & \text{if } j=k-s-1,\\
a^{[2]}_{k-1}\beta_{k-2,j}, & \text{if } j=k-s-2.
\end{cases}$$

Then

$$\begin{aligned}
&\left\langle\sigma\mathbf{q}_n,\frac{P'_{k+1}(x)}{k+1}+a^{[1]}_{k-1}\frac{P'_k(x)}{k}+a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1}\right\rangle\\
=\;&\frac{1}{\|Q_n\|_v^2}\left\langle v,\sum_{j=k-2-s}^{k+t}\beta^*_{k,j}Q_n(x)P_j(x)\right\rangle\\
=\;&\frac{1}{\|Q_n\|_v^2}\sum_{j=k-2-s}^{k+t}\beta^*_{k,j}\left\langle v,Q_n(x)P_j(x)\right\rangle,
\end{aligned}$$

as well as

$$\begin{aligned}
&\left\langle\sigma\mathbf{q}_n,\frac{P'_{k+1}(x)}{k+1}+a^{[1]}_{k-1}\frac{P'_k(x)}{k}+a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1}\right\rangle\\
=\;&\left\langle\frac{Q_n(x)}{\|Q_n\|_v^2}\rho u,\sum_{j=k-2-s}^{k+t}\beta^*_{k,j}P_j(x)\right\rangle\\
=\;&\frac{1}{\|Q_n\|_v^2}\sum_{j=k-2-s}^{k+t}\beta^*_{k,j}\left\langle u,P_j(x)\rho(x)Q_n(x)\right\rangle.
\end{aligned}$$

Thus, we have proved the following

Lemma 84. *For $k<n-t$ and $k>n+m+s+2$*

$$\left\langle\sigma\mathbf{q}_n,\frac{P'_{k+1}(x)}{k+1}+a^{[1]}_{k-1}\frac{P'_k(x)}{k}+a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1}\right\rangle=0.$$

On the other hand,

$$\begin{aligned}&\langle\sigma\mathbf{q}_n, Q_k(x)+c_{k-1}Q_{k-1}(x)\rangle\\ =\ &\frac{1}{\|Q_n\|_v^2}\langle v, Q_n(x)\sigma(x)\left(Q_k(x)+c_{k-1}Q_{k-1}(x)\right)\rangle\end{aligned}$$

and, as a consequence, the following result holds.

Lemma 85. *For $k<n-t$ and $k>n+t+1$*

$$\langle\sigma\mathbf{q}_n, Q_k(x)+c_{k-1}Q_{k-1}(x)\rangle=0.$$

Since $t\leq m+s+1$, from the above lemmas we deduce

Proposition 86.

$$\sigma\mathbf{q}_n=\sum_{k=n-t}^{n+m+s+2}\mu_{n,k}\mathbf{t}_k,\qquad n\geq t,\tag{4.44}$$

where $\mu_{n,k}=\langle\mathbf{q}_n,\sigma T_k\rangle$.

Since $\beta^*_{t+m+s+2,m+t}=a^{[2]}_{t+m+s+1}\beta_{t+m+s,m+t}\neq 0$, we point out that in the particular case $n=t$ we get

$$\begin{aligned}\mu_{t,t+m+s+2}\ &=\ \left\langle\sigma\mathbf{q}_t,\frac{P'_{t+m+s+3}(x)}{t+m+s+3}+a^{[1]}_{t+m+s+1}\frac{P'_{t+m+s+2}(x)}{t+m+s+2}+a^{[2]}_{t+m+s+1}\frac{P'_{t+m+s+1}(x)}{t+m+s+1}\right\rangle\\ &\quad +b_{t+m+s+1}\langle\sigma\mathbf{q}_t, Q_{t+m+s+2}(x)+c_{t+m+s+1}Q_{t+m+s+1}(x)\rangle\\ &=\ \frac{1}{\|Q_t\|_v^2}\sum_{j=m+t}^{2t+m+s+2}\beta^*_{t+m+s+2,j}\langle u, P_j(x)\rho(x)Q_t(x)\rangle\\ &=\ \frac{1}{\|Q_t\|_v^2}\beta^*_{t+m+s+2,m+t}\langle u, P_{m+t}(x)\rho(x)Q_t(x)\rangle\\ &\neq\ 0.\end{aligned}$$

We consider (4.41) with $n=0,1,....,t+s+m+1$, and (4.44) with $n=t$. Thus we get a system of linear equations for the $t+m+s+3$ unknowns $\{\mathbf{t}_k\}_{k=0}^{t+m+s+2}$. Indeed,

$$D_{m,s,t}\begin{pmatrix}\mathbf{t}_0\\ \mathbf{t}_1\\ \vdots\\ \mathbf{t}_{t+m+s+2}\end{pmatrix}=\begin{pmatrix}\mathbf{r}_0\\ \mathbf{r}_1\\ \vdots\\ \mathbf{r}_{t+m+s+1}\\ \sigma\mathbf{q}_t\end{pmatrix},$$

where the entries of $D_{m,s,t}=\left(h^{[m,s,t]}_{i,j}\right)_{i,j=1}^{m+s+t+3}$ are

$$h^{[m,s,t]}_{i,j}=\begin{cases}0, & if\ \ m+s+t+3>i>j \text{ and } i+1<j,\\ 1+b_{j-2} & if\ \ i=j\neq m+s+t+3,\\ d_{i-1}, & if\ \ i+1=j,\\ \mu_{t,j-1} & if\ \ i=m+s+t+3,\end{cases}$$

with $b_{-1} := 0$. Assuming that $|D_{m,s,t}| \neq 0$, the system of linear equations has solution and, as a consequence, there exists a matrix $(\zeta_{k,i})_{k,i=0}^{t+m+s+2}$ such that each $\mathbf{t}_k$, $k = 0,..., m+s+t+2$, can be written in terms of the functionals $\{\mathbf{r}_k\}_{k=0}^{m+s+t+1}$ and the functional $\sigma\mathbf{q}_t$ as follows

$$\mathbf{t}_k = \zeta_{k,0}\sigma\mathbf{q}_t + \zeta_{k,1}\mathbf{r}_0 + \zeta_{k,2}\mathbf{r}_1 + \cdots + \zeta_{k,t+m+s+2}\mathbf{r}_{t+s+m+1} = \zeta_{k,0}\sigma\mathbf{q}_t + \sum_{j=1}^{t+m+s+2} \zeta_{k,j}\mathbf{r}_{j-1}.$$

If we consider (4.44) with $n = t+1$, (4.41) with $n = t+m+s+2$, multiplying them by $d_{t+m+s+2}$ and $\mu_{t+1,t+m+s+3}$, respectively, and subtracting the corresponding expressions we get

$$d_{t+m+s+2}\sigma\mathbf{q}_{t+1} - \mu_{t+1,t+m+s+3}\mathbf{r}_{t+m+s+2} = d_{t+m+s+2} \sum_{k=1}^{t+m+s+2} \widetilde{\mu}_{t+1,k}\mathbf{t}_k,$$

where $\widetilde{\mu}_{t+1,k} := \mu_{t+1,k}$ for $1 \leq k \leq t+m+s+1$ and $\widetilde{\mu}_{t+1,t+m+s+2} := d_{t+m+s+2}\mu_{t+1,t+m+s+2} - (1+b_{t+m+s+2})\,\mu_{t+1,t+m+s+3}$. Finally, replacing $\mathbf{t}_k$, with $k = 0, ..., t+m+s+2$, in the above relation, we get

$$\begin{aligned} & d_{t+m+s+2}\sigma(x)\mathbf{q}_{t+1} - \left(\sum_{k=1}^{t+m+s+2} d_{t+m+s+2}\widetilde{\mu}_{t+1,k}\zeta_{k,0}\sigma(x) \right) \mathbf{q}_t \\ = \ & \mu_{t+1,t+m+s+3}\mathbf{r}_{t+m+s+2} + \sum_{j=1}^{t+m+s+2} \left(\sum_{k=1}^{t+m+s+2} d_{t+m+s+2}\widetilde{\mu}_{t+1,k}\zeta_{k,j} \right) \mathbf{r}_{j-1}. \end{aligned}$$

Thus,

Theorem 87. *Let be u a semiclassical linear functional of class s. Assume that the relation (4.35) holds and there exists a monic polynomial ρ, with $\deg \rho = m \geq 1$ such that $\rho(x)u = v$, as well as the matrix $\left(h_{i,j}^{[m,s,t]}\right)_{i,j=1}^{m+s+t+3}$ with entries*

$$h_{i,j}^{[m,s,t]} = \begin{cases} 0, & if \quad m+s+t+3 > i > j \ and \ i+1 < j, \\ 1+b_{j-2} & if \ \ i = j \neq m+s+t+3, \\ d_{i-1}, & if \ \ i+1 = j, \\ \mu_{t,j-1} & if \ \ i = m+s+t+3, \end{cases} \tag{4.45}$$

is nonsingular, then there exist polynomials ϕ_{t+1}, $\psi_{t+m+s+2}$, with $\deg(\phi_{t+1}) = t+1$ and $\deg(\psi_{t+m+s+2}) = t+m+s+2$, such that

$$\phi_{t+1}(x)\sigma(x)v = \phi_{t+1}(x)\sigma(x)\rho(x)u = \psi_{t+m+s+2}(x)w, \tag{4.46}$$

where

$$\phi_{t+1}(x) = \frac{d_{t+m+s+2}}{\|Q_{t+1}\|_v^2} \left(Q_{t+1}(x) - \frac{\|Q_{t+1}\|_v^2}{\|Q_t\|_v^2} \left(\sum_{k=1}^{t+m+s+2} \widetilde{\mu}_{t+1,k}\zeta_{k,0} \right) Q_t(x) \right),$$

and

$$\begin{aligned} & \psi_{t+m+s+2}(x) \\ = \ & \frac{\mu_{t+1,t+m+s+3}}{\|R_{t+m+s+2}\|_w^2} R_{t+m+s+2}(x) + d_{t+m+s+2} \sum_{j,k=1}^{t+m+s+2} \frac{\widetilde{\mu}_{t+1,k}\zeta_{k,j}}{\|R_{j-1}\|_w^2} R_{j-1}(x), \end{aligned}$$

where σ is the polynomial satisfying (4.43).

Remark 88. *The above result is based on the choice $n = t$ in (4.44) and $n = 0, 1, ..., t+s+m+1$, in (4.41). In a general way, we also could consider $n = t+i$ in (4.44) and $n = i, 1, ..., t+i+s+m+1$ in (4.41) for $i \in \mathbb{Z}^+ \cup \{0\}$. Thus (4.46) becomes*

$$\phi_{t+i+1}(x)\sigma(x)\rho(x)u = \psi_{t+i+m+s+2}(x)w. \tag{4.47}$$

Remark 89. *When $m = 0$, (4.44) can be written as*

$$\sigma \mathbf{q}_n = \sum_{k=n-t}^{n+\Phi_{t,s}} \mu_{n,k}\mathbf{t}_k, \qquad n \geq t, \tag{4.48}$$

where $\Phi_{t,s} = \max\{t+1, s+2\}$. As above, we can consider (4.41) with $n = 0, 1, ..., \Phi_{t,s}$ and (4.48) with $n = t$. In this way, we get the $t + \Phi_{t,s} + 1$ unknowns $\{\mathbf{t}_k\}_{k=0}^{t+\Phi_{t,s}}$, namely, we have the system

$$D_{s,t}\left(\mathbf{t}_0, \mathbf{t}_1, ..., \mathbf{t}_{t+\Phi_{t,s}}\right)^T = \left(\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{t+\Phi_{t,s}-1}, \sigma\mathbf{q}_t\right)^T,$$

where $D_{s,t}$ has the same structure as in (4.45) with $\Phi_{t,s}$ instead of $m+s+2$. In this way, we get

$$\phi_{t+1}(x)\sigma(x)u = \psi_{t+\Phi_{t,s}}(x)w. \tag{4.49}$$

Now, we want to study the relation between the formal Stieltjes series associated with the functionals u and w. Let $\mathbf{p}_k^{[1]}$ be the $k-th$ term of the dual basis associated with the sequence $\left\{\frac{P'_{n+1}}{n+1}\right\}_{n\geq 0}$. Then we expand the linear functional $\sigma\mathbf{p}_n$ in terms of such a basis. From (1.16) we get

$$\left\langle \sigma\mathbf{p}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = \left\langle \mathbf{p}_n, \sum_{j=k-s}^{k+t} \beta_{k,j}P_j(x) \right\rangle, \ \ \beta_{k,k-s} \neq 0.$$

If either $k+t < n$ or $k-s > n$, then $\left\langle \sigma\mathbf{p}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = 0$,. Therefore, from $\lambda_{n,k} := \left\langle \sigma\mathbf{p}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = \left\langle \mathbf{p}_n, \sum_{j=k-s}^{k+t} \beta_{k,j}P_j(x) \right\rangle$, we get

$$\sigma\mathbf{p}_n = \sum_{k=n-t}^{n+s} \lambda_{n,k}\mathbf{p}_k^{[1]}. \tag{4.50}$$

In particular, $\lambda_{t,t+s} = \left\langle \mathbf{p}_t, \sum_{j=t}^{2t+s} \beta_{t+s,j}P_j(x) \right\rangle = \beta_{t+s,t} \neq 0$. For $n = t$ and taking derivatives, in the distributional sense, from (1.11) the above relation becomes

$$D\left(\sigma\mathbf{p}_t\right) = \xi_{t+s+1}(x)u, \tag{4.51}$$

where $\xi_{t+s+1}(x) := \sum_{k=0}^{t+s} \widetilde{\lambda}_{t,k}P_{k+1}(x)$ and $\widetilde{\lambda}_{t,k} = -\frac{(k+1)}{\|P_{k+1}\|_u^2}\lambda_{t,k}$. In particular, notice that $\widetilde{\lambda}_{t,t+s} = -\frac{(t+s+1)}{\|P_{t+s+1}\|_u^2}\lambda_{t,t+s} \neq 0$ and, therefore, $\deg(\xi_{t+s+1}) = t+s+1$. Multiplying both sides of (4.51) by $\phi_{t+1}\rho$, from (4.46) we get

$$\phi_{t+1}(x)\rho(x)D\left(\frac{1}{\|P_t\|_u^2}P_t(x)\sigma(x)u\right) = \phi_{t+1}(x)\rho(x)\xi_{t+s+1}(x)u.$$

As a consequence, from (4.46) we obtain

Theorem 90. *The linear functionals w and u are related through the differential relation*

$$D(P_t(x)\psi_{t+m+s+2}(x)w) = \omega_{2t+m+s+2}(x)u, \tag{4.52}$$

where

$$\omega_{2t+m+s+2}(x) = \|P_t\|_u^2 \, \phi_{t+1}(x)\rho(x)\xi_{t+s+1}(x) + (\phi_{t+1}(x)\rho(x))' \, P_t(x)\sigma(x),$$

and $\deg(\omega_{2t+m+s+2}) = 2t + m + s + 2$.

From (4.46) and (4.52) we can deduce an upper bound for the class of w. Indeed, after straightforward calculations we get

Corollary 91. *The linear functional w satisfies the differential relation*

$$D(\widetilde{\phi}w) = \widetilde{\psi}w,$$

where $\widetilde{\phi}(x) = \phi_{t+1}(x)\sigma(x)\rho(x)P_t(x)\psi_{t+m+s+2}(x)$ *and*

$$\widetilde{\psi}(x) = \psi_{t+m+s+2}(x) \left((\phi_{t+1}(x)\sigma(x)\rho(x))' \, P_t(x) + \omega_{2t+m+s+2}(x) \right).$$

Thus, w is semiclassical of class at most $2m + 2s + 3t + 3$.

Remark 92. *As above, when $m = 0$ we get*

$$D(P_t(x)\psi_{t+\Phi_{t,s}}(x)w) = \omega_{2t+s+2}(x)u,$$

with

$$\omega_{2t+s+2}(x) = \|P_t\|_u^2 \, \phi_{t+1}(x)\xi_{t+s+1}(x) + \phi'_{t+1}(x)P_t(x)\sigma(x).$$

Remark 93. *Notice that we can write the differential relation obtained in the above theorem as follows*

$$D(P_{t+i}(x)\psi_{t+i+m+s+2}(x)w) = \omega_{2t+2i+m+s+2}(x)u, \qquad i \in \mathbb{Z}^+ \cup \{0\}. \tag{4.53}$$

Let $S_u(z) = -\sum_{n\geq 0} \frac{u_n}{z^{n+1}}$ and $S_w(z) = -\sum_{n\geq 0} \frac{w_n}{z^{n+1}}$ be the formal Stieltjes series associated with u and w respectively. Here $\{u_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ denote the corresponding moment sequences of such a functionals, respectively. Taking into account the particular cases $i = 0$ and $i = 1$ in (4.53) and, as a consequence of Theorem 7 we get

Corollary 94. *Under the same conditions of the above theorem, the Stieltjes series $S_u(z)$ and $S_w(z)$ satisfy*

$$\omega_{2t+2n+m+s+2}(z)S_u(z) - (\psi_{t+n+m+s+2}(z)P_{t+n}(z)S_w(z))' = A_n^{[m,t,s]}(z),$$

where

$$A_n^{[m,t,s]}(z) = (w\theta_0 \left(P_{t+n}\psi_{t+n+m+s+2}\right))'(z) - \left(u\theta_0\omega_{2t+2n+m+s+2}\right)(z).$$

On the other hand, $S_w(z)$ satisfies the following first order non-homogeneous linear differential equation

$$B_1(z)S'_w(z) + B_0(z)S_w(z) = C(z),$$

where, for $j = 0, 1$,

$$\begin{aligned} &B_j(z) \\ = \; &\binom{1}{j} \left(\omega_{2t+m+s+2}(z)P_{t+1}(x)\psi^{(1-j)}_{t+m+s+3}(z) - \omega_{2t+m+s+4}(z)P_t(x)\psi^{(1-j)}_{t+m+s+2}(z) \right), \end{aligned}$$

and

$$C(z) = \omega_{2t+m+s+4}(z)A_0^{[m,t,s]}(z) - \omega_{2t+m+s+2}(z)A_1^{[m,t,s]}(z).$$

In order to illustrate the results of this section, we consider $\rho(x)=x$, $s=0$, and $u=\mathcal{L}^{(\alpha)}$, the classical Laguerre functional with $\alpha>-1$. Notice that $\sigma(x)=x$. Thus,

$$L_{n+1}^{\alpha+1}(x)+\frac{a_n^{[1]}+b_nc_n}{1+b_n}L_n^{\alpha+1}(x)+\frac{a_n^{[2]}}{1+b_n}L_{n-1}^{\alpha+1}(x)=R_{n+1}(x)+\frac{d_n}{1+b_n}R_n(x),\ n\geq 1. \tag{4.54}$$

Also, the SMOP $\left\{L_{n+1}^{\alpha+1}\right\}_{n\geq 0}$ satisfies the TTRR

$$xL_n^{\alpha+1}(x)=L_{n+1}^{\alpha+1}(x)+(2n+\alpha+2)\,L_n^{\alpha+1}(x)+n(n+\alpha+1)L_{n-1}^{\alpha+1}(x),\ n\geq 1, \tag{4.55}$$

and $\|L_n^{\alpha}\|^2=n!\Gamma(n+\alpha+1)$. We also assume that

$$xR_n(x)=R_{n+1}(x)+\lambda_nR_n(x)+\gamma_nR_{n-1}(x),\quad n\geq 0, \tag{4.56}$$

with $R_{-1}(x)=0$, is the TTRR that $\{R_n\}_{n\geq 0}$ satisfies. From (4.54), (4.55) and (4.56) we get

$$\begin{aligned}
&-\frac{a_{n-1}^{[1]}+b_{n-1}c_{n-1}}{1+b_{n-1}}xL_{n-1}^{\alpha+1}(x)-\frac{a_{n-1}^{[2]}}{1+b_{n-1}}xL_{n-2}^{\alpha+1}(x)\\
&+\left(\frac{a_n^{[1]}+b_nc_n}{1+b_n}-(2n+\alpha+2)\right)L_n^{\alpha+1}(x)+\left(\frac{a_n^{[2]}}{1+b_n}-n(n+\alpha+1)\right)L_{n-1}^{\alpha+1}(x)\\
=&\left(\frac{d_n}{1+b_n}-\lambda_n\right)R_n(x)-\frac{d_{n-1}}{1+b_{n-1}}xR_{n-1}(x)-\gamma_nR_{n-1}(x).
\end{aligned}$$

The comparison of the coefficients of x^n and x^{n-1} in both hand sides yields explicit expressions for the recurrence coefficients of $\{R_n\}_{n\geq 0}$. Indeed, for $n\geq 0$, and after straightforward calculations, we get

$$\lambda_n=\theta_{n-1}-\theta_n+(2n+\alpha+2) \tag{4.57}$$

and

$$\begin{aligned}
&\gamma_n\\
=&\ \Lambda_{n-1}-\Lambda_n-n(n+\alpha+1)\,(2n+\alpha+1)\\
&+\left(\lambda_n-\frac{d_n}{1+b_n}\right)\Omega_n+\frac{d_{n-1}}{1+b_{n-1}}\Omega_{n-1},
\end{aligned} \tag{4.58}$$

where

$$\theta_n=\frac{a_n^{[1]}+b_nc_n-d_n}{1+b_n},\qquad \Lambda_n=\frac{a_n^{[2]}-\left(a_n^{[1]}+b_nc_n\right)n\,(n+\alpha+1)}{1+b_n}, \tag{4.59}$$

and

$$\Omega_n=n\,(n+\alpha+1)-\theta_{n-1}. \tag{4.60}$$

If

$$a_n^{[1]}=\frac{1}{n+1},\ a_n^{[2]}=\frac{n+2}{n+1},\ b_n=\frac{3}{2n+1},\ c_n=d_n=\frac{n+1}{n+2},\quad n\geq 0,$$

with $a_{-1}^{[1]}=a_{-1}^{[2]}=b_{-1}=c_{-1}=d_{-1}:=0$, then the recurrence coefficients $\{\lambda_n,\gamma_n\}_{n\geq 0}$ are completely determined by (4.57), (4.58), (4.59) and (4.60). For instance, if $\alpha=0$ we get

$$\lambda_0=\frac{3}{2},\ \lambda_1=\frac{17}{4},\ \lambda_2=\frac{299}{48},\ \lambda_3=\frac{9799}{1200},\ \lambda_4=\frac{6073}{600},\ \lambda_5=\frac{35551}{2940},$$

and

$$\gamma_1 = \frac{19}{24}, \ \gamma_2 = \frac{13559}{384}, \ \gamma_3 = \frac{5518523}{57600}, \ \gamma_4 = \frac{4003607}{20000}, \ \gamma_5 = \frac{254335129}{705600}.$$

In this way, from (4.56) the elements the sequence $\{R_n\}_{n\geq 0}$ can be deduced. As an example, from the above data we get

$$R_0(x) = 1, \quad R_1 = x - \frac{3}{2}$$

$$R_2(x) = x^2 - \frac{23}{4}x + \frac{67}{12}, \quad R_3(x) = x^3 - \frac{575}{48}x^2 + \frac{2339}{384}x + \frac{41\,899}{2304},$$

$$R_4(x) = x^4 - \frac{4029}{200}x^3 + \frac{2431}{300}x^2 + \frac{239\,311\,997}{460\,800}x - \frac{125\,968\,831}{184\,320},$$

and

$$R_5(x) = x^5 - \frac{454}{15}x^4 + \frac{11\,351}{960}x^3 + \frac{32\,662\,826\,593}{11\,520\,000}x^2 - \frac{1979\,414\,728\,109}{276\,480\,000}x + \frac{1812\,077\,996\,999}{552\,960\,000}.$$

On the other hand, according to Theorem 12 u and w are related through

$$\phi_2(x)x^2 u = \psi_4(x)w,$$

and from Theorem 15 satisfy

$$D(L_1^{\alpha}(x)\psi_4(x)w) = \omega_5(x)u.$$

As a consequence, for a fixed value α, we can explicitly obtain the polynomials ϕ_2, ψ_4 and ω_5. On one hand, according to definitions of ϕ_2, ψ_4 we get

$$\begin{aligned} &\left\|L_n^{\alpha+1}\right\|^2 \mu_{n,k} \\ = &\left\langle \mathcal{L}^{(\alpha+1)}, xL_n^{\alpha+1}(x)\left(\left(1+b_{k-1}\right)L_k^{\alpha+1}(x) + \left(a_{k-1}^{[1]} + b_{k-1}c_{k-1}\right)L_{k-1}^{\alpha+1}(x) + a_{k-1}^{[2]}L_{k-2}^{\alpha+1}(x)\right)\right\rangle, \end{aligned}$$

On the other hand, let consider

$$\left(h_{i,j}^{[1,0,1]}\right)_{i,j=1}^{5} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 4 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 2 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{8}{5} & \frac{4}{5} \\ 1 & \frac{37}{2} & \frac{109}{6} & \frac{301}{30} & \frac{29}{4} \end{bmatrix}.$$

This matrix is non-singular and, as a consequence, the numbers $(\zeta_{k,i})_{k,i=0}^{4}$ are given by

$$\begin{bmatrix} \frac{4859}{4874} & -\frac{2707}{19\,496} & \frac{134}{7311} & -\frac{2175}{77\,984} & \frac{15}{4874} \\ \frac{15}{2437} & \frac{2707}{9748} & -\frac{268}{7311} & \frac{2175}{38\,992} & -\frac{15}{2437} \\ -\frac{90}{2437} & -\frac{405}{2437} & \frac{536}{2437} & -\frac{6525}{19\,496} & \frac{90}{2437} \\ \frac{240}{2437} & \frac{1080}{2437} & \frac{1820}{2437} & \frac{2175}{2437} & -\frac{240}{2437} \\ -\frac{480}{2437} & -\frac{2160}{2437} & -\frac{3640}{2437} & -\frac{5215}{9748} & \frac{480}{2437} \end{bmatrix}.$$

Finally, and as it is well known, (see [31]), $\|R_n\|_w^2 := \prod_{k=0}^{n} \gamma_k$, with $\gamma_0 := 1$. Besides, the polynomials $R_i(x)$, $i = 0,1,2,3,4$, will be needed. On the other hand, according to definitions of ω_5, through straightforward calculations you can deduce the values $\beta_{n,k} = \dfrac{\left\langle \mathcal{L}^{(\alpha+1)}, L_n^{\alpha+1}(x)L_k^{\alpha}(x)\right\rangle}{\left\|L_k^{\alpha}\right\|^2}$ and $\|L_n^{\alpha}\|^2 \lambda_{n,k} = \left\langle \mathcal{L}^{(\alpha+1)}, L_n^{\alpha}(x)L_k^{\alpha+1}(x)\right\rangle.$

Chapter 5

Asymptotics for Sobolev polynomials

The aim of this Chapter is to study asymptotic properties of Sobolev polynomials $\{S_n^\lambda\}_{n\geq 0}$ orthogonal with respect to the inner product

$$\langle p(x), q(x)\rangle_S := \int_{\mathbb{R}} p(x)q(x)e^{-x^2}dx + \lambda \int_{\mathbb{R}} p'(x)q'(x)\frac{x^2+a}{x^2+b}e^{-x^2}dx, \quad \lambda > 0,$$

where the pair $\{\mu_0, \mu_1\}$, $d\mu_0 = e^{-x^2}dx$, $d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx$, is a symmetric $(1,1)-$coherent pair found in [34] when μ_0 is classical. In order to do this we deduce a connection formula between the Hermite and Sobolev polynomials, namely, we get

$$S_{n+3}^\lambda(x) + \eta_n(\lambda)S_{n+1}^\lambda(x) = H_{n+3}(x) + \frac{n+3}{n+1}b_n H_{n+1}(x), \qquad n \geq 0.$$

In first section we analyze, in a general way, the recurrence properties of the Sobolev coefficients $\{\eta_n(\lambda)\}_{n\geq 0}$ as well as the Sobolev norms $\left\{\left\|S_n^\lambda\right\|_S^2\right\}_{n\geq 0}$. In section 5.3 we study the limit behavior of $\eta_n(\lambda)$ and in section 5.4 the relative outer asymptotic is deeply studied .

5.1 Sobolev Polynomials and Sobolev Coefficients

Let $\{u, v\}$ be a symmetric $(1,1)-$coherent pair with $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as their respective SMOP and satisfying (4.2). We assume that u and v are positive-definite with μ_0 and μ_1 as the respective positive Borel measures and let $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ be the respective moment sequences. Then we consider the Sobolev inner product

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0. \tag{5.1}$$

Besides let $\{S_n^\lambda\}_{n\geq 0}$ be the Sobolev orthogonal polynomials associated with (5.1). The above product also will be written as

$$\begin{aligned} \langle p, q\rangle_s &= \langle p, q\rangle_{\mu_0} + \lambda \langle p', q'\rangle_{\mu_1} \\ &= \langle u, pq\rangle + \lambda \langle v, p'q'\rangle . \end{aligned}$$

For $n \geq 1$, we consider the expansion $S_n^\lambda(x) = x^n + \sum_{j=0}^{n-1} c_{n,j}^\lambda x^j$, and let $\Delta_{S,n} = \det\left[\mu_{i,j}\right]_{i,j=0}^n$ be the principal submatrix associated with the moments $\mu_{i,j} := \left\langle x^i, x^j\right\rangle_S$. According to (5.1) if

$i+j=0,1,$ then $\mu_{i,j}=u_{i+j}$, and if $i+j\geq 2$ we have

$$\begin{aligned}\mu_{i,j} &= \left\langle x^i, x^j\right\rangle_S = \int_{\mathbb{R}} x^{i+j} d\mu_0(x) + ij\lambda \int_{\mathbb{R}} x^{i+j-2} d\mu_1(x) \\ &= u_{i+j} + ij\lambda v_{i+j-2},\end{aligned}$$

and if $i+j$ is odd or $ij=0$ then $\mu_{i,j}=u_{i+j}$. It is well known that

$$S_n^\lambda(x) = \frac{1}{\Delta_{S,n-1}} \begin{vmatrix} 1 & 0 & u_2 & \cdots & u_n \\ 0 & \mu_{1,1} & 0 & \cdots & \mu_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & \mu_{n-1,1} & \mu_{n-1,2} & \cdots & 0 \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

and moreover $c_{n,j}^\lambda = \dfrac{(-1)^{n+2+j}\Delta_{S,n-1}^j}{\Delta_{S,n-1}}$ where $\Delta_{S,n-1}^j$ is obtained deleting the $j-th$ column and the $(n+1)-th$ row of the matrix $[\mu_{i,j}]_{i,j=0}^n$.

Example 95. *For $n=1$ we get $S_1^\lambda(x)=x$. Moreover,*

$$S_2^\lambda(x) = x^2 + \frac{\begin{vmatrix} 0 & u_2 \\ \mu_{1,1} & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & \mu_{1,1} \end{vmatrix}} = x^2 - u_2, \tag{5.2}$$

so $c_{2,0}^\lambda = -u_2$.

Using properties of the determinants and cumbersome calculations yields

$$\begin{aligned}\Delta_{S,n-1} &= \begin{vmatrix} 1 & 0 & \cdots & u_{n-1} \\ 0 & \mu_{1,1} & \cdots & \mu_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & \cdots & u_{n-1} \\ 0 & u_2+\lambda v_0 & \cdots & u_n+(n-1)\lambda v_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_n+(n-1)\lambda v_{n-2} & \cdots & u_{2n-2}+(n-1)^2\lambda v_{2n-4} \end{vmatrix} \\ &= \left((n-1)!\right)^2 \Delta_{n-2}^v \lambda^{n-1} + \cdots + \Delta_{n-1}^u.\end{aligned}$$

So every coefficient of S_n^λ can be seen as a rational function in λ where the degree of the numerator is at most as large as the denominator. Then it makes sense to define the sequence $\{W_n\}_{n\geq 0}$ in the next way:

$$W_n(x) := \lim_{\lambda\to\infty} S_n^\lambda(x),$$

where, as consequence of the symmetry, if n is even, (resp. odd) W_n is a even function, (resp. odd).

On the other hand

$$\left\langle S_{n+1}^{\lambda}, q\right\rangle_S = \int_{\mathbb{R}} S_{n+1}^{\lambda}(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} \left(S_{n+1}^{\lambda}\right)'(x)q'(x)d\mu_1(x) = 0,$$

if $\deg(q) \leq n$. And when $\lambda \to \infty$ we obtain,

$$\int_{\mathbb{R}} W_{n+1}'(x)q'(x)d\mu_1(x) = 0,$$

i.e.

$$W_{n+1}'(x) = (n+1)R_n(x). \tag{5.3}$$

Moreover, for $n \geq 0$

$$\left\langle S_{n+1}^{\lambda}, 1\right\rangle_S = \int_{\mathbb{R}} S_{n+1}^{\lambda}(x)d\mu_0(x) = 0.$$

Then, when $\lambda \to \infty$ we get

$$\int_{\mathbb{R}} W_{n+1}(x)d\mu_0(x) = 0. \tag{5.4}$$

From (4.2) and by using (5.3) we have

$$\frac{W_{n+3}(x)}{n+3} + b_n\frac{W_{n+1}(x)}{n+1} = \frac{P_{n+3}(x)}{n+3} + a_n\frac{P_{n+1}(x)}{n+1} + k_n, \qquad n \geq 0.$$

Integrating respect to the measure μ_0, and using (5.4), we get $k_n = 0$, as well as

$$W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x) = P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x), \qquad n \geq 0. \tag{5.5}$$

where

$$\widetilde{a}_n = a_n\frac{n+3}{n+1}, \qquad \widetilde{b}_n = b_n\frac{n+3}{n+1}, \quad n \geq 0.$$

Now we consider the expansion of W_n by using the basis $\left\{S_n^{\lambda}\right\}_{n\geq 0}$

$$W_n(x) = S_n^{\lambda}(x) + \sum_{j=0}^{n-1}\sigma_{n,j}S_j^{\lambda}(x). \tag{5.6}$$

Notice that

$$\sigma_{n,j} = \frac{\left\langle W_n, S_j^{\lambda}\right\rangle_S}{\left\|S_j^{\lambda}\right\|_S^2} = \frac{\int_{\mathbb{R}} W_n(x)S_j^{\lambda}(x)d\mu_0(x) + \lambda\int_{\mathbb{R}} W_n'(x)\left(S_j^{\lambda}\right)'(x)d\mu_1(x)}{\left\|S_j^{\lambda}\right\|_S^2},$$

and $\left\|S_j^{\lambda}\right\|_S^2 := \left\langle S_j^{\lambda}, S_j^{\lambda}\right\rangle_S$. In the same way $W_{n+3}(x) = S_{n+3}^{\lambda}(x) + \sum_{j=0}^{n+2}\sigma_{n+3,j}S_j^{\lambda}(x)$, and multiplying by $\widetilde{b}_n$ in (5.6) we get:

$$\widetilde{b}_n W_{n+1}(x) = \widetilde{b}_n S_{n+1}^{\lambda}(x) + \sum_{j=0}^{n}\widetilde{b}_n\sigma_{n+1,j}S_j^{\lambda}(x).$$

As a consequence,

$$\begin{aligned} W_{n+3}(x)+\widetilde{b}_n W_{n+1}(x) &= S_{n+3}^{\lambda}(x)+\sigma_{n+3,n+2}S_{n+2}^{\lambda}(x)+\left(\sigma_{n+3,n+1}+\widetilde{b}_n\right)S_{n+1}^{\lambda}(x) \\ &\quad +\sum_{j=0}^{n}\left(\sigma_{n+3,j}+\widetilde{b}_n\sigma_{n+1,j}\right)S_j^{\lambda}(x). \end{aligned}$$

Taking into account the polynomials W_{n+3} and W_{n+1} are either even or odd functions, necessarily $\sigma_{n+3,n+2}=0$. Then

$$W_{n+3}(x)+\widetilde{b}_n W_{n+1}(x)=S_{n+3}^{\lambda}(x)+\sum_{j=0}^{n+1}\eta_{n,j}(\lambda)S_j^{\lambda}(x),$$

where every coefficient $\eta_{n,j}(\lambda)$, $j\le n$, can be written as:

$$\begin{aligned} &\eta_{n,j}(\lambda) \\ =\ &\sigma_{n+3,j}+\widetilde{b}_n\sigma_{n+1,j} \\ =\ &\frac{\int_{\mathbb{R}}\left(W_{n+3}(x)+\widetilde{b}_nW_{n+1}(x)\right)S_j^{\lambda}(x)d\mu_0(x)+\lambda\int_{\mathbb{R}}\left(W_{n+3}'(x)+\widetilde{b}_nW_{n+1}'(x)\right)\left(S_j^{\lambda}\right)'(x)d\mu_1(x)}{\left\|S_j^{\lambda}\right\|_S^2}, \end{aligned}$$

and $\sigma_{n+1,n+1}:=1$.

By using (5.5) we obtain

$$\int_{\mathbb{R}}\left(W_{n+3}(x)+\widetilde{b}_nW_{n+1}(x)\right)S_j^{\lambda}(x)d\mu_0(x)=\int_{\mathbb{R}}\left(P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x)\right)S_j^{\lambda}(x)d\mu_0(x)=0,$$

for $j=0,...,n$, and the relation (5.3) allows us to deduce

$$\int_{\mathbb{R}}\left(W_{n+3}'(x)+\widetilde{b}_nW_{n+1}'(x)\right)\left(S_j^{\lambda}\right)'(x)d\mu_1(x)=0,$$

for $j=0,...,n$, and, as a consequence,

$$W_{n+3}(x)+\widetilde{b}_nW_{n+1}(x)=S_{n+3}^{\lambda}(x)+\eta_{n,n+1}(\lambda)S_{n+1}^{\lambda}(x). \tag{5.7}$$

Equivalently,

$$S_{n+3}^{\lambda}(x)+\eta_{n,n+1}(\lambda)S_{n+1}^{\lambda}(x)=P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x),\quad n\ge 0. \tag{5.8}$$

Taking derivatives

$$\frac{\left(S_{n+3}^{\lambda}\right)'(x)}{n+3}+\eta_{n,n+1}(\lambda)\frac{\left(S_{n+1}^{\lambda}\right)'(x)}{n+3}=P_{n+2}(x)+a_nP_n(x). \tag{5.9}$$

Notice that

$$\begin{aligned} \eta_{n,n+1}(\lambda) &= \sigma_{n+3,n+1}^{\lambda}+\widetilde{b}_n \\ &= \frac{\int_{\mathbb{R}}W_{n+3}(x)S_{n+1}^{\lambda}(x)d\mu_0(x)+\lambda\int_{\mathbb{R}}W_{n+3}'(x)\left(S_{n+1}^{\lambda}\right)'(x)d\mu_1(x)}{\left\|S_{n+1}^{\lambda}\right\|_S^2}+\widetilde{b}_n, \end{aligned}$$

and, again, using (5.3) we obtain

$$\int_{\mathbb{R}} W'_{n+3}(x)\left(S_{n+1}^{\lambda}\right)'(x)d\mu_1(x)=0,$$

thus

$$\eta_n(\lambda):=\eta_{n,n+1}(\lambda)=\frac{\int_{\mathbb{R}} W_{n+3}(x)S_{n+1}^{\lambda}(x)d\mu_0(x)}{\left\|S_{n+1}^{\lambda}\right\|_S^2}+\widetilde{b}_n. \tag{5.10}$$

We summarize the above results in the next

Theorem 96. *Let $\{\mu_0,\mu_1\}$ be a symmetric $(1,1)-$coherent pairs of measures with $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as their respective SMOP satisfying (4.2) and let $\left\{S_n^{\lambda}\right\}_{n\geq 0}$ be the Sobolev polynomials orthogonal with respect to (5.1). Then*

$$S_{n+3}^{\lambda}(x)+\eta_n(\lambda)S_{n+1}^{\lambda}(x)=P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x), \tag{5.11}$$

holds with

$$\eta_n(\lambda)=\frac{\int_{\mathbb{R}} W_{n+3}(x)S_{n+1}^{\lambda}(x)d\mu_0(x)}{\left\|S_{n+1}^{\lambda}\right\|_S^2}+\widetilde{b}_n.$$

The coefficients $\eta_n(\lambda)$ will be called Sobolev Coefficients.

Lemma 97. *For $n\geq 0$*

$$\eta_n(\lambda)=\frac{\widetilde{b}_n(n+1)^2\left\langle v,R_n^2\right\rangle\lambda+\widetilde{a}_n\left\langle u,P_{n+1}^2\right\rangle}{\left\|S_{n+1}^{\lambda}\right\|_S^2} \tag{5.12}$$

Proof. From (4.2), multiplying by R_n and using the measure μ_1 we get

$$\begin{aligned}
b_n\left\langle v,R_n^2\right\rangle &= \left\langle\frac{P'_{n+3}(x)}{n+3},R_n(x)\right\rangle_{\mu_1}+a_n\left\langle\frac{P'_{n+1}(x)}{n+1},R_n(x)\right\rangle_{\mu_1}\\
&= \left\langle\frac{W'_{n+3}(x)}{n+3}+b_n\frac{W'_{n+1}(x)}{n+1}-a_n\frac{P'_{n+1}(x)}{n+1},R_n(x)\right\rangle_{\mu_1}+a_n\left\langle\frac{P'_{n+1}(x)}{n+1},R_n(x)\right\rangle_{\mu_1}\\
&= \left\langle\frac{W'_{n+3}(x)}{n+3}+b_n\frac{W'_{n+1}(x)}{n+1},R_n(x)\right\rangle_{\mu_1}\\
&= \frac{1}{\lambda}\left(\left\langle\frac{W_{n+3}(x)}{n+3}+b_n\frac{W_{n+1}(x)}{n+1},\frac{W_{n+1}(x)}{n+1}\right\rangle_S-\left\langle\frac{W_{n+3}(x)}{n+3}+b_n\frac{W_{n+1}(x)}{n+1},\frac{W_{n+1}(x)}{n+1}\right\rangle_{\mu_0}\right)\\
&= \frac{1}{\lambda}\left(\left\langle\frac{S_{n+3}^{\lambda}(x)}{n+3}+\eta_n(\lambda)\frac{S_{n+1}^{\lambda}(x)}{n+3},\frac{W_{n+1}(x)}{n+1}\right\rangle_S-\left\langle\frac{P_{n+3}(x)}{n+3}+a_n\frac{P_{n+1}(x)}{n+1},\frac{W_{n+1}(x)}{n+1}\right\rangle_{\mu_0}\right)\\
&= \frac{1}{\lambda}\left(\frac{\eta_n(\lambda)}{(n+1)(n+3)}\left\|S_{n+1}\right\|_S^2-\frac{a_n}{(n+1)^2}\left\langle u,P_{n+1}^2\right\rangle\right).
\end{aligned}$$

Then

$$\lambda b_n(n+1)(n+3)\left\langle v,R_n^2\right\rangle+\frac{a_n}{(n+1)}(n+3)\left\langle u,P_{n+1}^2\right\rangle=\eta_n(\lambda)\left\|S_{n+1}\right\|_S^2,$$

and the result follows. □

On the other hand, we use (4.2), (5.5), (5.8) and the notation $\langle u, p(x)q(x)\rangle := \langle p(x), q(x)\rangle_{\mu_0}$, (i.e. we express u in terms of the associated bilinear form) in order to obtain

$$\begin{aligned}
&\left\langle W_{n+3}(x), S_{n+1}^{\lambda}(x)\right\rangle_{\mu_0} \\
= &\left\langle P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x) - \widetilde{b}_n W_{n+1}(x), P_{n+1}(x) + \widetilde{a}_{n-2} P_{n-1}(x) - \eta_{n-2}(\lambda) S_{n-1}^{\lambda}(x)\right\rangle_{\mu_0} \\
= &\ \widetilde{a}_n \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left\langle W_{n+1}(x), P_{n-1}(x)\right\rangle_{\mu_0} \\
&+\widetilde{b}_n \eta_{n-2}(\lambda) \left\langle W_{n+1}(x), S_{n-1}^{\lambda}(x)\right\rangle_{\mu_0},
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle W_{n+1}(x), P_{n-1}(x)\right\rangle_{\mu_0} \\
= &\left\langle P_{n+1}(x) + \widetilde{a}_{n-2} P_{n-1}(x) - \widetilde{b}_{n-2} W_{n-1}(x), R_{n-1}(x)\right\rangle_{\mu_0} \\
= &\left(\widetilde{b}_{n-2} - \widetilde{a}_{n-2}\right) \left\langle u, P_{n-1}^2\right\rangle.
\end{aligned}$$

Then, from to above relations, for $n \geq 2$ we get

$$\begin{aligned}
\left\langle W_{n+3}(x), S_{n+1}^{\lambda}(x)\right\rangle_{\mu_0} = &\left(\widetilde{a}_n - \widetilde{b}_n\right) \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right) \left\langle u, P_{n-1}^2\right\rangle \\
&+\widetilde{b}_n \eta_{n-2}(\lambda) \left\langle W_{n+1}(x), S_{n-1}^{\lambda}(x)\right\rangle_{\mu_0}.
\end{aligned}$$

If we denote $I_k(\lambda) := \left\langle W_k(x), S_{k-2}^{\lambda}(x)\right\rangle_{\mu_0}$, the above relation can be written as

$$I_{n+3}(\lambda) = \left(\widetilde{a}_n - \widetilde{b}_n\right) \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right) \left\langle u, P_{n-1}^2\right\rangle + \widetilde{b}_n \eta_{n-2}(\lambda) I_{n+1}(\lambda). \quad (5.13)$$

Moreover, from (5.12) we get

$$I_{n+3}(\lambda) = \widetilde{b}_n (n+1)^2 \lambda \left\langle v, R_n^2\right\rangle + \widetilde{a}_n \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \left\| S_{n+1}^{\lambda}\right\|_S^2. \quad (5.14)$$

Taking into account the expressions of $I_{n+1}(\lambda)$ and $\eta_{n-2}(\lambda)$, by using (5.13) we obtain

$$\begin{aligned}
&I_{n+3}(\lambda) \\
= &\left(\widetilde{a}_n - \widetilde{b}_n\right) \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right) \left\langle u, P_{n-1}^2\right\rangle \\
&+\widetilde{b}_n \eta_{n-2}(\lambda) \left(\eta_{n-2}(\lambda) - \widetilde{b}_{n-2}\right) \left\| S_{n-1}^{\lambda}\right\|_S^2
\end{aligned}$$

and from (5.14) we get

$$\begin{aligned}
&\left(\widetilde{a}_n - \widetilde{b}_n\right) \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right) \left\langle u, P_{n-1}^2\right\rangle + \widetilde{b}_n \eta_{n-2}(\lambda) \left(\eta_{n-2}(\lambda) - \widetilde{b}_{n-2}\right) \left\| S_{n-1}^{\lambda}\right\|_S^2 \\
= &\ \widetilde{b}_n (n+1)^2 \lambda \left\langle v, R_n^2\right\rangle + \widetilde{a}_n \left\langle u, P_{n+1}^2\right\rangle - \widetilde{b}_n \left\| S_{n+1}^{\lambda}\right\|_S^2.
\end{aligned}$$

We have proved the next

Lemma 98. *For* $n \geq 2$

$$\begin{aligned} & \left\| S_{n+1}^{\lambda} \right\|_S^2 \\ = \; & (n+1)^2 \lambda \left\langle v, R_n^2 \right\rangle + \left\langle u, P_{n+1}^2 \right\rangle + \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2} \right) \left\langle u, P_{n-1}^2 \right\rangle \\ & - \eta_{n-2}(\lambda) \left(\eta_{n-2}(\lambda) - \widetilde{b}_{n-2} \right) \left\| S_{n-1}^{\lambda} \right\|_S^2 . \end{aligned} \tag{5.15}$$

The above formula is useful in order to compute the norms $\left\| S_n^{\lambda} \right\|_S^2$ if the Sobolev coefficients are known. We are going to describe the initial conditions which are needed. When $n = 0$ we get

$$\eta_0(\lambda) = \frac{\widetilde{b}_0 \lambda + \widetilde{a}_0 \left\langle u, P_1^2 \right\rangle}{\left\| S_1^{\lambda} \right\|_S^2}.$$

But $\left\langle u, P_1^2 \right\rangle = \left\langle u, x^2 \right\rangle$. Moreover, we know that $S_1^{\lambda}(x) = x$. Then,

$$\begin{aligned} \left\| S_1^{\lambda} \right\|_S^2 &= \left\langle S_1^{\lambda}, S_1^{\lambda} \right\rangle_S \\ &= \int_{\mathbb{R}} x^2 d\mu_0(x) + \lambda \int_{\mathbb{R}} d\mu_1(x) \\ &= \left\langle u, P_1^2 \right\rangle + \lambda, \end{aligned}$$

and, as consequence,

$$\left\| S_1^{\lambda} \right\|_S^2 = \lambda + \left\langle u, P_1^2 \right\rangle, \qquad \eta_0(\lambda) = \frac{\widetilde{b}_0 \lambda + \widetilde{a}_0 \left\langle u, P_1^2 \right\rangle}{\lambda + \left\langle u, P_1^2 \right\rangle}. \tag{5.16}$$

On the other hand, for $n = 1$,

$$\eta_1(\lambda) = \frac{4\widetilde{b}_1 \left\langle v, R_1^2 \right\rangle \lambda + \widetilde{a}_1 \left\langle u, P_2^2 \right\rangle}{\left\| S_2^{\lambda} \right\|_S^2},$$

and we know that $P_2(x) = x^2 - \lambda_2$, (here $\{\lambda_n\}_{n \geq 0}$ are the recurrence coefficients of the sequence $\{P_n\}_{n \geq 0}$), but since $u_0 = \lambda_1 = 1$, we get $\left\langle u, P_1^2 \right\rangle = \left\langle u, x^2 \right\rangle = \lambda_1 \lambda_2 = \lambda_2$. Therefore, $P_2(x) = x^2 - \left\langle u, P_1^2 \right\rangle$ and from (5.2) we get

$$\begin{aligned} \left\| S_2^{\lambda} \right\|_S^2 &= \left\langle S_2^{\lambda}, S_2^{\lambda} \right\rangle_S \\ &= \int_{\mathbb{R}} \left[x^4 - 2x^2 u_2 + u_2^2 \right] d\mu_0(x) + 4\lambda \int_{\mathbb{R}} x^2 d\mu_1(x) \\ &= \int_{\mathbb{R}} \left[\left(x^2 - \left\langle u, P_1^2 \right\rangle \right)^2 + 2x^2 \left(\left\langle u, P_1^2 \right\rangle - u_2 \right) + u_2^2 - \left\langle u, P_1^2 \right\rangle^2 \right] d\mu_0(x) + 4\lambda \left\langle v, R_1^2 \right\rangle \\ &= \left\langle u, P_2^2 \right\rangle + \left\langle u, P_1^2 \right\rangle^2 - 2u_2 \left\langle u, P_1^2 \right\rangle + u_2^2 + 4\lambda \left\langle v, R_1^2 \right\rangle \\ &= 4 \left\langle v, R_1^2 \right\rangle \lambda + \left\langle u, P_2^2 \right\rangle . \end{aligned}$$

As a consequence

$$\left\| S_2^{\lambda} \right\|_S^2 = 4 \left\langle v, R_2^2 \right\rangle \lambda + \left\langle u, P_2^2 \right\rangle + \left(\left\langle u, P_1^2 \right\rangle - u_2 \right)^2, \quad \eta_1(\lambda) = \frac{4\widetilde{b}_1 \left\langle v, R_1^2 \right\rangle \lambda + \widetilde{a}_1 \left\langle u, P_2^2 \right\rangle}{4 \left\langle v, R_1^2 \right\rangle \lambda + \left\langle u, P_2^2 \right\rangle}. \tag{5.17}$$

In this way, using (5.16) and for $n=2$ in (5.15) we obtain $\left\|S_3^\lambda\right\|_S^2$, and then by using of (5.12) we find $\eta_2(\lambda)$. In the same way successively for $n=4,6,8,10,....$ in (5.15) we obtain $\left\|S_{2k+1}^\lambda\right\|_S^2$ and $\eta_{2k}(\lambda)$, for every $k\in\mathbb{N}$. Similarly, using (5.17) we can obtain recurrently $\left\|S_{2k+2}^\lambda\right\|_S^2$ and $\eta_{2k+1}(\lambda)$, for every $k\in\mathbb{N}$. In the next section we study a recurrence formula for the coefficients $\eta_n(\lambda)$.

5.1.1 The Sobolev Coefficients

We define $T_{n+1}(x)=W_{n+1}(x)+\widetilde{b}_{n-2}W_{n-1}(x)$. Through straightforward calculations it is not difficult to prove that

$$\eta_n(\lambda)=\frac{\langle T_{n+3}(x),T_{n+1}(x)\rangle_S}{\langle T_{n+1}(x),T_{n+1}(x)\rangle_S-\eta_{n-2}(\lambda)\langle T_{n-1}(x),T_{n+1}(x)\rangle_S}. \tag{5.18}$$

This expression is well defined since the denominator is non zero. Indeed

$$\begin{aligned}
&\langle T_{n+1}(x),T_{n+1}(x)\rangle_S-\eta_{n-2}(\lambda)\langle T_{n-1}(x),T_{n+1}(x)\rangle_S\\
=&\left\langle T_{n+1}(x)-\eta_{n-2}(\lambda)T_{n-1}(x),S_{n+1}^\lambda(x)+\eta_{n-2}(\lambda)S_{n-1}^\lambda(x)\right\rangle_S\\
=&\left\langle S_{n+1}^\lambda(x),S_{n+1}^\lambda(x)\right\rangle_S+\eta_{n-2}^2(\lambda)\left\langle S_{n-1}^\lambda(x),S_{n-1}^\lambda(x)\right\rangle_S-\eta_{n-2}^2(\lambda)\left\langle T_{n-1}(x),S_{n-1}^\lambda(x)\right\rangle_S\\
=&\left\langle S_{n+1}^\lambda(x),S_{n+1}^\lambda(x)\right\rangle_S\neq 0.
\end{aligned}$$

We will express each term in (5.18) in a more simple form.

$$\begin{aligned}
&\langle T_{n+3}(x),T_{n+3}(x)\rangle_S\\
=&\langle P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x),P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x)\rangle_{\mu_0}\\
&+\lambda\left\langle W_{n+3}'(x)+\widetilde{b}_nW_{n+1}'(x),W_{n+3}'(x)+\widetilde{b}_nW_{n+1}'(x)\right\rangle_{\mu_1}\\
=&\left\langle u,P_{n+3}^2\right\rangle+\widetilde{a}_n^2\left\langle u,P_{n+1}^2\right\rangle\\
&+\lambda\left\langle (n+3)R_{n+2}(x)+\widetilde{b}_n(n+1)R_n(x),(n+3)R_{n+2}(x)+\widetilde{b}_n(n+1)R_n(x)\right\rangle_{\mu_1}\\
=&\left\langle u,P_{n+3}^2\right\rangle+\widetilde{a}_n^2\left\langle u,P_{n+1}^2\right\rangle+\lambda\left((n+3)^2\left\langle v,R_{n+2}^2\right\rangle+\widetilde{b}_n^2(n+1)^2\left\langle v,R_n^2\right\rangle\right)\\
=&\,p_{n+3}+\widetilde{a}_n^2p_{n+1}+\lambda\left((n+3)^2r_{n+2}+\widetilde{b}_n^2(n+1)^2r_n\right).
\end{aligned}$$

Here we have used the notation $r_n=\left\langle v,R_n^2\right\rangle$ and $p_n=\left\langle u,P_n^2\right\rangle$. Also

$$\begin{aligned}
&\langle T_{n+1}(x),T_{n+3}(x)\rangle_S\\
=&\langle T_{n+1}(x),\widetilde{a}_nP_{n+1}(x)\rangle_{\mu_0}\\
&+\lambda\left\langle (n+1)R_n(x)+\widetilde{b}_{n-2}(n-1)R_{n-2}(x),(n+3)R_{n+2}(x)+\widetilde{b}_n(n+1)R_n(x)\right\rangle_{\mu_1}\\
=&\,\widetilde{a}_np_{n+1}+\lambda\widetilde{b}_n(n+1)^2r_n,
\end{aligned}$$

and replacing in (5.18) we get for $n\geq 1$,

$$\begin{aligned}
&\eta_n(\lambda)\\
=&\frac{\widetilde{a}_np_{n+1}+\lambda\widetilde{b}_n(n+1)^2r_n}{p_{n+1}+\lambda(n+1)^2r_n+\widetilde{a}_{n-2}^2p_{n-1}+\lambda\widetilde{b}_{n-2}^2(n-1)^2r_{n-2}-\eta_{n-2}(\lambda)\left[\widetilde{a}_{n-2}p_{n-1}+\lambda\widetilde{b}_{n-2}(n-1)^2r_{n-2}\right]},
\end{aligned}$$

where $a_{-n} = b_{-n} = 0$ for $n \in \mathbb{N}$. Furthermore

$$
\begin{aligned}
&\eta_n(\lambda) \\
= &\ \frac{\dfrac{\widetilde{b}_n(n+1)^2 r_n \left[\dfrac{\widetilde{a}_n p_{n+1}}{\widetilde{b}_n(n+1)^2 r_n}+\lambda\right]}{\eta_{n-2}(\lambda)\widetilde{b}_{n-2}(n-1)^2 r_{n-2}\left[\dfrac{\widetilde{a}_{n-2}p_{n-1}}{\widetilde{b}_{n-2}(n-1)^2 r_{n-2}}+\lambda\right]}}{\dfrac{\left[(n+1)^2 r_n+\widetilde{a}_{n-2}^2 p_{n-1}+\widetilde{b}_{n-2}^2(n-1)^2 r_{n-2}\right]\left[\lambda+\frac{p_{n+1}}{\left[(n+1)^2 r_n+\widetilde{a}_{n-2}^2 p_{n-1}+\widetilde{b}_{n-2}^2(n-1)^2 r_{n-2}\right]}\right]}{\eta_{n-2}(\lambda)\widetilde{b}_{n-2}(n-1)^2 r_{n-2}\left[\dfrac{\widetilde{a}_{n-2}p_{n-1}}{\widetilde{b}_{n-2}(n-1)^2 r_{n-2}}+\lambda\right]}-1} \\
= &\ \frac{A_n\lambda + B_n}{C_n\lambda + D_n - \eta_{n-2}(\lambda)\left[A_{n-2}\lambda + B_{n-2}\right]},
\end{aligned}
$$

where we have defined for $n \geq 1$

$$
A_n = \widetilde{b}_n(n+1)^2 r_n, \qquad B_n = \widetilde{a}_n p_{n+1},
$$

$$
C_n = (n+1)^2 r_n + \widetilde{a}_{n-2}^2 p_{n-1}, \qquad D_n = p_{n+1},
$$

$C_1 = 4r_1$. With this notation we can prove the next

Theorem 99. *There exist sequences of polynomials* $\{Q_n(\lambda)\}_{n\geq 0}$ *and* $\left\{\widetilde{Q}_n(\lambda)\right\}_{n\geq 0}$, *with* $\deg(Q_n) = \deg(\widetilde{Q}_n) = n$ *for every* n, *such that satisfy the three term recurrence relations*

$$
Q_{n+1}(\lambda) = (C_{2n}\lambda + D_{2n})\, Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2\, Q_{n-1}(\lambda), \tag{5.19}
$$

and

$$
\widetilde{Q}_{n+1}(\lambda) = (C_{2n+1}\lambda + D_{2n+1})\, \widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2\, \widetilde{Q}_{n-1}(\lambda), \tag{5.20}
$$

respectively and with the initial conditions $Q_0(\lambda) = \widetilde{Q}_0(\lambda) = 1$, $Q_1(\lambda) = \lambda + \left\langle u, P_1^2 \right\rangle$, *and* $\widetilde{Q}_1(\lambda) = 4\left\langle v, R_1^2\right\rangle\lambda + \left\langle u, P_2^2\right\rangle$. *Furthermore, the Sobolev coefficients are rational functions in terms of such polynomials, namely*

$$
\eta_{2n}(\lambda) = (A_{2n}\lambda + B_{2n})\, \frac{Q_n(\lambda)}{Q_{n+1}(\lambda)}, \tag{5.21}
$$

and

$$
\eta_{2n+1}(\lambda) = (A_{2n+1}\lambda + B_{2n+1})\, \frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)}. \tag{5.22}
$$

Proof. The initial conditions are obtained according to the definition of $\eta_0(\lambda)$ and $\eta_1(\lambda)$. Suppose that $\eta_{2n-2}(\lambda) = (A_{2n-2}\lambda + B_{2n} - 2)\, \dfrac{Q_{n-1}(\lambda)}{Q_n(\lambda)}$, then

$$
\begin{aligned}
\eta_{2n}(\lambda) &= \frac{A_{2n}\lambda + B_{2n}}{C_{2n}\lambda + D_{2n} - (A_{2n-2}\lambda + B_{2n-2})^2\, \dfrac{Q_{n-1}(\lambda)}{Q_n(\lambda)}} \\
&= \frac{(A_{2n}\lambda + B_{2n})\, Q_n(\lambda)}{(C_{2n}\lambda + D_{2n})\, Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2\, Q_{n-1}(\lambda)},
\end{aligned}
$$

then (5.19) holds being the denominator $Q_{n+1}(\lambda)$. In an analogous way $\eta_{2n-1}(\lambda) = (A_{2n-1}\lambda + B_{2n-1})\,\dfrac{\widetilde{Q}_{n-1}(\lambda)}{\widetilde{Q}_n(\lambda)}$ and we get

$$\eta_{2n+1}(\lambda) = \frac{(A_{2n+1}\lambda + B_{2n+1})\,\widetilde{Q}_n(\lambda)}{(C_{2n+1}\lambda + D_{2n+1})\,\widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2\,\widetilde{Q}_{n-1}(\lambda)},$$

and if the denominator is defined as $\widetilde{Q}_{n+1}(\lambda)$ we get (5.20). □

Remark 100. *The relations 5.19 and 5.20 are well known in the literature as R_{II} type recurrence relations and they were studied for the first time in [56]. On the matter, the references [22], [57] and [113] are highly recommended.*

Remark 101. *Notice that $B_n = \widetilde{b}_n r_{n+1} \neq 0$, and if $\widetilde{a}_n = 0$ then $A_n = 0$, for every n, as a consequence (4.2) turns into*

$$P_{n+2}(x) = \frac{R'_{n+3}(x)}{n+3} + b_n \frac{R'_{n+1}(x)}{n+1},$$

and, according to Favard's theorem the recurrence relations (5.19) and (5.20) mean that $\{Q_n(\lambda)\}_{n\geq 0}$ and $\left\{\widetilde{Q}_n(\lambda)\right\}_{n\geq 0}$ are orthogonal in the standard sense.

5.2 Asymptotics for Hermite $(1,1)-$ coherent pairs

In this section we study asymptotic properties of Sobolev polynomials $\{S_n^{\lambda}\}_{n\geq 0}$ orthogonal with respect to the inner product

$$\langle p(x), q(x)\rangle_S := \int_{\mathbb{R}} p(x)q(x)e^{-x^2}dx + \lambda \int_{\mathbb{R}} p'(x)q'(x)\frac{x^2+a}{x^2+b}e^{-x^2}dx, \ \lambda > 0,$$

where the pair

$$d\mu_0 = e^{-x^2}dx, \qquad d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx. \qquad a, b \in \mathbb{R}^+, \quad a \neq b, \tag{5.23}$$

is a symmetric $(1,1)-$coherent pair presented in [34]. As a consequence there exist sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, such that the algebraic relation

$$H_n(x) + b_{n-2}H_{n-2}(x) = Q_n(x) + a_{n-2}Q_{n-2}(x), \qquad n \geq 2, \tag{5.24}$$

is satisfied, where $\{H_n\}_{n\geq 0}$ represents the classical sequence of monic Hermite polynomials and $\{Q_n\}_{n\geq 0}$ is the MOPS associated with $d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx$. In [101], some outer relative asymptotic results of the sequence $\{Q_n\}_{n\geq 0}$ with respect to the classical Hermite polynomials are obtained. With respect to the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ in [34] it is proved the next

Proposition 102. *One of the following situations holds*

1. *If $a_0 = b_0$ then $H_n(x) = Q_n(x)$ and $a_n = b_n$ for every $n \in \mathbb{N}$.*
2. *If $a_0 \neq b_0$ and $a_1 \neq b_1$ then $H_n(x) \neq Q_n(x)$ for $n \geq 2$.*

3. *If $a_0 \neq b_0$ and $a_1 = b_1$ then $H_{2n}(x) \neq Q_{2n}(x)$ for $n \geq 1$, $H_3(x) = Q_3(x)$ and if there exists $N > 1$ such that $H_{2N+1} \neq Q_{2N+1}$ then $H_{2n+1} \neq Q_{2n+1}$ for $n \geq N$.*

It is well known that the monic Hermite polynomials satisfy the recurrence relation

$$H_{n+1}(x) = xH_n(x) - \frac{n}{2}H_{n-1}(x), \quad n \geq 2, \tag{5.25}$$

with $H_0(x) = 1$, $H_1(x) = x$, as well as

$$\langle u, H_n^2 \rangle = \frac{\sqrt{\pi}n!}{2^n}.$$

The zeros of H_n are real, simple and symmetrically located around the origin, i.e. if $H_n(x_0) = 0$ then $H_n(-x_0) = 0$. Let $\{x_{n,k}\}_{n=1}^{[n/2]}$ be the positive zeros of H_n on increasing order. Also it is well known that the zeros of H_n and H_{n-1} interlace. Moreover, taking into account that J_α has a countably infinite number of real and positive zeros if $\alpha > -1$, as a consequence of Mehler-Heine formulas and the Hurwitz's theorem if $n \to \infty$ and $k \geq 1$ then

$$2\sqrt{n}x_{2n,k} \to j_{-1/2,k}, \qquad 2\sqrt{n}x_{2n+1,k} \to j_{1/2,k}, \tag{5.26}$$

i.e. $x_{n,k} \sim \dfrac{c_k}{\sqrt{n}}$, where $c_k > 0$ and $\{j_{\alpha,k}\}_{n\geq 0}$ are the zeros of J_α, $\alpha > -1$.

We will assume the recurrence relation for the sequence of monic orthogonal polynomials $\{Q_n\}_{n\geq 0}$ is

$$Q_{n+1}(x) = xQ_n(x) - \widetilde{\gamma}_n Q_{n-1}(x), \quad n \geq 0, \quad Q_0(x) = 1, \quad Q_{-1}(x) = 0 \ . \tag{5.27}$$

Explicit relations between recurrence coefficients and the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ can be seen in [34].

On the other hand, according to (5.11) there exists an algebraic relation between $\{H_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$. Indeed, there exists a sequence $\{\eta_n(\lambda)\}_{n\geq 0}$, (the Sobolev coefficients), such that

$$S_{n+3}^\lambda(x) + \eta_n(\lambda)S_{n+1}^\lambda(x) = H_{n+3}(x) + \frac{n+3}{n+1}b_n H_{n+1}(x), \qquad n \geq 0. \tag{5.28}$$

Moreover, from (5.12) we get

$$\frac{(n+1)}{(n+3)}\eta_n(\lambda) = \frac{\lambda\,(n+1)^2\,a_n\,\langle v, Q_n^2\rangle + b_n\,\langle u, H_{n+1}^2\rangle}{\langle S_{n+1}^\lambda, S_{n+1}^\lambda\rangle_S}, \quad n \geq 0. \tag{5.29}$$

As an important tool that we will need for the purpose of this paper, we define the positive definite symmetric linear functional $\widetilde{v}$, associated with the measure $d\mu_1^* = \dfrac{e^{-x^2}}{x^2+b}dx$ and let $\{\widetilde{Q}_n\}_{n\geq 0}$ be the corresponding SMOP. Taking into account the algebraic connection of this sequence and Hermite polynomials, the next result is proved in [9].

Lemma 103. *There exists a sequence of real numbers $\{\sigma_n\}_{n\geq 0}$ such that*

$$\widetilde{Q}_n(x) = H_n(x) + \sigma_n H_{n-2}(x), n \geq 2, \tag{5.30}$$

with

$$\sigma_n = \frac{\left\langle \widetilde{v}, \widetilde{Q}_n^2 \right\rangle}{\left\langle u, H_{n-2}^2 \right\rangle}. \tag{5.31}$$

Moreover

$$\lim_{n\to\infty} \sqrt{\left[\frac{n}{2}\right]} \left(\frac{2\sigma_n}{n} - 1\right) = -\sqrt{b}, \tag{5.32}$$

and

$$\lim_{n\to\infty} \frac{\sigma_n}{n} = \frac{1}{2}. \tag{5.33}$$

Concerning relative asymptotics the next result is proved in [102].

Theorem 104. *It holds*

$$\lim_{n\to\infty} \sqrt{\left[\frac{n}{2}\right]} \frac{\widetilde{Q}_n(x)}{H_n(x)} = \begin{cases} -ix + \sqrt{b}, & x \in \mathbb{C}_+ \\ ix + \sqrt{b}, & x \in \mathbb{C}_-, \end{cases} . \tag{5.34}$$

uniformly on compact subsets of half planes $\mathbb{C}_+$ *and* $\mathbb{C}_-$*, respectively.*

5.3 Asymptotics of Sobolev coefficients

In order to obtain an algebraic relation between the sequences $\{Q_n\}_{n\geq 0}$ and $\{\widetilde{Q}_n\}_{n\geq 0}$ we have

$$(x^2 + a)Q_n(x) = \widetilde{Q}_{n+2}(x) + c_n\widetilde{Q}_n(x), n \geq 0, \tag{5.35}$$

where $c_n = \frac{\left\langle \widetilde{v}, (x^2+a)Q_n\widetilde{Q}_n \right\rangle}{\left\langle \widetilde{v}, \widetilde{Q}_n^2 \right\rangle} = \frac{\left\langle v, Q_n^2 \right\rangle}{\left\langle \widetilde{v}, \widetilde{Q}_n^2 \right\rangle}$. If $x = i\sqrt{a}$, then we get $c_n = -\frac{\widetilde{Q}_{n+2}(i\sqrt{a})}{\widetilde{Q}_n(i\sqrt{a})}$ $= -\frac{n}{2} \frac{\sqrt{\left[\frac{n+2}{2}\right]} \frac{\widetilde{Q}_{n+2}(i\sqrt{a})}{H_{n+2}(i\sqrt{a})} \frac{2H_{n+2}(i\sqrt{a})}{nH_n(i\sqrt{a})}}{\sqrt{\left[\frac{n+2}{2}\right]} \frac{\widetilde{Q}_n(i\sqrt{a})}{H_n(i\sqrt{a})}}$, and from (1.31) and (5.34) we get the next

Lemma 105.

$$\lim_{n\to\infty} \frac{c_n}{n} = \frac{1}{2}. \tag{5.36}$$

With respect to the functional v and the recurrence coefficients $\{\widetilde{\gamma}_n\}_{n\geq 0}$, we get the following

Lemma 106. *The recurrence coefficients* $\widetilde{\gamma}_n$ *satisfy*

$$\lim_{n\to\infty} \frac{\widetilde{\gamma}_n}{n} = \frac{1}{2}. \tag{5.37}$$

Proof. From $\widetilde{\gamma}_n = \dfrac{\left\langle v, Q_n^2\right\rangle}{\left\langle v, Q_{n-1}^2\right\rangle} = \dfrac{\langle \widetilde{v}, Q_n(x^2+a)Q_n\rangle}{\langle \widetilde{v}, Q_{n-1}(x^2+a)Q_{n-1}\rangle}$ and by using (5.35), we get

$$\begin{aligned}\widetilde{\gamma}_n &= \frac{\left\langle \widetilde{v}, Q_n\left(\widetilde{Q}_{n+2}(x)+c_n\widetilde{Q}_n(x)\right)\right\rangle}{\left\langle \widetilde{v}, Q_{n-1}\left(\widetilde{Q}_{n+1}(x)+c_{n-1}\widetilde{Q}_{n-1}(x)\right)\right\rangle} \\ &= \frac{c_n\left\langle \widetilde{v}, \widetilde{Q}_n^2\right\rangle}{c_{n-1}\left\langle \widetilde{v}, \widetilde{Q}_{n-1}^2\right\rangle} \\ &= \frac{c_n\left\langle \widetilde{v}, \widetilde{Q}_n^2\right\rangle}{c_{n-1}\left\langle u, H_{n-2}^2\right\rangle}\frac{\left\langle u, H_{n-3}^2\right\rangle}{\left\langle \widetilde{v}, \widetilde{Q}_{n-1}^2\right\rangle}\frac{\left\langle u, H_{n-2}^2\right\rangle}{\left\langle u, H_{n-3}^2\right\rangle} \\ &= \frac{c_n}{c_{n-1}}\sigma_n\sigma_{n-1}^{-1}\frac{(n-2)}{2}.\end{aligned}$$

Thus, from (5.33) and (5.36) the result follows. □

As a consequence of above results, if we define $\delta_n = \frac{\langle v,Q_n^2\rangle}{\langle u,H_n^2\rangle} = \frac{\langle v,Q_n\widetilde{Q}_n\rangle}{\langle u,H_n^2\rangle}$ and taking into account (5.35), $\delta_n = c_n\frac{\langle \widetilde{v},\widetilde{Q}_n^2(x)\rangle}{\langle u,H_n^2\rangle} = c_n\frac{\langle \widetilde{v},\widetilde{Q}_n^2(x)\rangle}{\langle u,H_{n-2}^2\rangle}\frac{\langle u,H_{n-2}^2\rangle}{\langle u,H_n^2\rangle} = 4\frac{c_n}{n}\frac{\sigma_n}{n}\frac{n}{(n-1)}$. Thus, from (5.33) and (5.36) we get the next

Lemma 107. *Let* $\delta_n = \dfrac{\langle v, Q_n^2\rangle}{\langle u, H_n^2\rangle}$, $n \geq 0$. *Then, the sequence* $\{\delta_n\}_{n\geq 0}$ *converges to* 1.

If we consider the relation

$$\widetilde{Q}_n(x) = Q_n(x) + \xi_n Q_{n-2}, \qquad n \geq 2, \tag{5.38}$$

where $\xi_n = \dfrac{\left\langle \widetilde{v}, \widetilde{Q}_n^2\right\rangle}{\left\langle v, Q_{n-2}^2\right\rangle} = \dfrac{\left\langle \widetilde{v}, \widetilde{Q}_n^2\right\rangle}{\left\langle u, H_{n-2}^2\right\rangle}\dfrac{\left\langle u, H_{n-2}^2\right\rangle}{\left\langle v, Q_{n-2}^2\right\rangle} = \sigma_n\delta_n^{-1}$, then as a straightforward consequence we get

$$\lim_{n\to\infty}\frac{\xi_n}{n} = \frac{1}{2}. \tag{5.39}$$

Notice that from (5.24), (5.30) and (5.38), for $n \geq 0$

$$\begin{aligned}\widetilde{Q}_{n+2}(x) &= H_{n+2}(x) + \sigma_{n+2}H_n(x) \\ &= Q_{n+2}(x) + a_nQ_n(x) - b_nH_n(x) + \sigma_{n+2}H_n(x),\end{aligned}$$

and then, $Q_{n+2}(x) + \xi_{n+2}Q_n = Q_{n+2}(x) + a_nQ_n(x) - b_nH_n(x) + \sigma_{n+2}H_n(x)$, or, equivalently,

$$\left(\xi_{n+2} - a_n\right)Q_n(x) = \left(\sigma_{n+2} - b_n\right)H_n(x).$$

Thus, $\xi_{n+2} - a_n = \sigma_{n+2} - b_n = 0$, and, as a consequence we get the next

Theorem 108. *The sequences* $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$ *can be defined as* $a_n = \xi_{n+2} = \dfrac{\left\langle \widetilde{v}, \widetilde{Q}_{n+2}^2\right\rangle}{\langle v, Q_n^2\rangle}$, *and* $b_n = \sigma_{n+2} = \dfrac{2^n\left\langle \widetilde{v}, \widetilde{Q}_{n+2}^2\right\rangle}{\sqrt{\pi}n!}$. *Then*

$$\lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{b_n}{n} = \frac{1}{2}. \tag{5.40}$$

On the other hand, for every n from the extremal properties of orthogonal polynomials with respect to the positive definite linear functional $\widetilde{v}$ we get $\left\langle u, H_n^2 \right\rangle = \left\langle \widetilde{v}, (x^2+b)H_n^2 \right\rangle = \left\langle \widetilde{v}, (xH_n)^2 \right\rangle + b\left\langle \widetilde{v}, H_n^2 \right\rangle > \left\langle \widetilde{v}, (xH_n)^2 \right\rangle \geq \left\langle \widetilde{v}, \widetilde{Q}_{n+1}^2 \right\rangle$. Then $b_n = \frac{\left\langle \widetilde{v}, \widetilde{Q}_{n+2}^2 \right\rangle}{\left\langle u, H_n^2 \right\rangle} < \frac{\left\langle u, H_{n+1}^2 \right\rangle}{\left\langle u, H_n^2 \right\rangle} = \frac{n+1}{2}$, and, as a consequence, for every $n \geq 0$

$$\frac{b_n}{n+1} < \frac{1}{2}. \tag{5.41}$$

A similar argument yields

$$\frac{a_n}{n+1} = \frac{\left\langle \widetilde{v}, \widetilde{Q}_{n+2}^2 \right\rangle}{(n+1)\left\langle v, Q_n^2 \right\rangle} < 1. \tag{5.42}$$

On the other hand, from the above section, and in the Hermite case we get

$$\eta_n(\lambda) = \frac{\Lambda_n(\lambda)}{B_n + \lambda A_n - \eta_{n-2}(\lambda)\Lambda_{n-2}(\lambda)}, \quad n \geq 2, \tag{5.43}$$

where,

$$A_n = (n+1)^2 \left\langle v, Q_n^2 + a_{n-2}^2 Q_{n-2}^2 \right\rangle > 0, \qquad n \geq 2, \tag{5.44}$$

with $A_0 = A_1 = 1$, as well as

$$B_n = \left\langle u, H_{n+1}^2 + \widetilde{b}_{n-2}^2 H_{n-1}^2 \right\rangle > 0, \quad n \geq 1, \tag{5.45}$$

and

$$\Lambda_n(\lambda) = (n+3)(n+1)a_n \left\langle v, Q_n^2 \right\rangle \lambda + \widetilde{b}_n \left\langle u, H_{n+1}^2 \right\rangle, \qquad n \geq 0, \tag{5.46}$$

i.e. Λ_n is a polynomial of degree one in the variable λ. From (5.29), $\frac{1}{3}\eta_0(\lambda) = \frac{\lambda a_0 \left\langle v, 1 \right\rangle + b_0 \left\langle u, H_1^2 \right\rangle}{\left\langle x, x \right\rangle_S} = \frac{\lambda a_0 \widetilde{\gamma}_1 + b_0 \frac{\sqrt{\pi}}{2}}{\frac{\sqrt{\pi}}{2} + \lambda \widetilde{\gamma}_1}$, and, $\frac{1}{2}\eta_1(\lambda) = \frac{4\lambda a_1 \left\langle v, Q_1^2 \right\rangle + b_1 \left\langle u, H_2^2 \right\rangle}{\left\langle S_2^\lambda, S_2^\lambda \right\rangle_S} = \frac{4\lambda a_1 \widetilde{\gamma}_1^2 + b_1 \frac{\sqrt{\pi}}{2}}{\frac{\sqrt{\pi}}{2} + 4\lambda \left\langle v, Q_1^2 \right\rangle}$. As a consequence, we also get the next result.

Theorem 109. *There exist sequences of polynomials* $\{\Omega_n(\lambda)\}_{n\geq 0}$ *and* $\{\widetilde{\Omega}_n(\lambda)\}_{n\geq 0}$ *such that*

$$\eta_{2n}(\lambda) = \frac{\Lambda_{2n}(\lambda)}{A_{2n}} \frac{\Omega_n(\lambda)}{\Omega_{n+1}(\lambda)}, \qquad n \geq 1,$$

and

$$\eta_{2n+1}(\lambda) = \frac{\Lambda_{2n+1}(\lambda)}{A_{2n+1}} \frac{\widetilde{\Omega}_n(\lambda)}{\widetilde{\Omega}_{n+1}(\lambda)}, \qquad n \geq 1,$$

with $\eta_0(\lambda) = \frac{3a_0 \widetilde{\gamma}_1 \lambda + b_0 \frac{3\sqrt{\pi}}{2}}{\widetilde{\gamma}_1 \lambda + \frac{\sqrt{\pi}}{2}}$ *and* $\eta_1(\lambda) = \frac{8a_1 \widetilde{\gamma}_1^2 \lambda + b_1 \sqrt{\pi}}{4\left\langle v, Q_1^2 \right\rangle \lambda + \frac{\sqrt{\pi}}{2}}$. *On the other hand, the above two sequences are defined recursively by*

$$\Omega_{n+1}(\lambda) = \left(\lambda + \frac{B_{2n}}{A_{2n}}\right)\Omega_n(\lambda) - \frac{\Lambda_{2n-2}^2(\lambda)}{A_{2n}A_{2n-2}}\Omega_{n-1}(\lambda), \qquad n \geq 1, \tag{5.47}$$

and

$$\widetilde{\Omega}_{n+1}(\lambda) = \left(\lambda + \frac{B_{2n+1}}{A_{2n+1}}\right)\widetilde{\Omega}_n(\lambda) - \frac{\Lambda_{2n-1}^2(\lambda)}{A_{2n+1}A_{2n-1}}\widetilde{\Omega}_{n-1}(\lambda), \qquad n \geq 1, \tag{5.48}$$

with $A_0 = A_1 = 1$, $\Omega_0(\lambda) = \widetilde{\Omega}_0(\lambda) = 1$, $\Omega_1(\lambda) = \widetilde{\gamma}_1 \lambda + \frac{\sqrt{\pi}}{2}$ *and* $\widetilde{\Omega}_1(\lambda) = 4\left\langle v, Q_1^2 \right\rangle \lambda + \frac{\sqrt{\pi}}{2}$.

We will study the asymptotic behavior of the functions $p_n(\lambda) = \lambda + \dfrac{B_n}{A_n}$ and $q_n(\lambda) = \dfrac{\Lambda_n(\lambda)}{\sqrt{A_n A_{n+2}}}$.
First, $\frac{\langle u,H_{n+1}^2\rangle}{\langle v,Q_n^2\rangle} = \frac{n+1}{2}\frac{\langle u,H_n^2\rangle}{\langle v,Q_n^2\rangle} = \frac{n+1}{2}\frac{\langle u,H_n^2\rangle}{\langle \widetilde{v},\widetilde{Q}_{n+2}^2\rangle}\frac{\langle \widetilde{v},\widetilde{Q}_{n+2}^2\rangle}{\langle v,Q_n^2\rangle} = \frac{n+1}{2}\frac{a_n}{b_n}$. Then

$$\begin{aligned}
\Lambda_n(\lambda) &= \left\langle v, Q_n^2\right\rangle \left((n+3)(n+1)a_n\lambda + \frac{n+3}{n+1} b_n \frac{\left\langle u, H_{n+1}^2\right\rangle}{\left\langle v, Q_n^2\right\rangle}\right) \\
&= \left\langle v, Q_n^2\right\rangle \left((n+3)(n+1)a_n\lambda + \frac{n+3}{2} a_n\right) \\
&\sim \left\langle v, Q_n^2\right\rangle \frac{n^3}{2}\lambda,
\end{aligned}$$

and, $A_n = (n+1)^2 \left\langle v, Q_n^2\right\rangle \left(1 + \frac{a_{n-2}^2}{\widetilde{\gamma}_n\widetilde{\gamma}_{n-1}}\right) \sim 2\left\langle v, Q_n^2\right\rangle n^2$. As a consequence,

$$\begin{aligned}
&\frac{\Lambda_n^2(\lambda)}{A_{n+2}A_n} \\
\sim\ & \frac{\left(\left\langle v, Q_n^2\right\rangle \frac{n^3}{2}\right)^2 \lambda^2}{2\left\langle v, Q_{n+2}^2\right\rangle (n+2)^2 2\left\langle v, Q_n^2\right\rangle n^2} \\
=\ & \frac{\frac{n^6}{4}}{4\frac{\langle v,Q_{n+2}^2\rangle}{\langle v,Q_{n+1}^2\rangle}\frac{\langle v,Q_{n+1}^2\rangle}{\langle v,Q_n^2\rangle}(n+2)^2 n^2}\lambda^2 \\
\sim\ & \frac{\frac{n^2}{4}}{4\widetilde{\gamma}_n\widetilde{\gamma}_{n+1}}\lambda^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
B_n &= \left\langle u, H_{n+1}^2\right\rangle \left(1 + \frac{(n+1)^2}{(n-1)^2} b_{n-2}^2 \frac{\left\langle u, H_{n-1}^2\right\rangle}{\left\langle u, H_{n+1}^2\right\rangle}\right) \\
&= \left\langle u, H_{n+1}^2\right\rangle \left(1 + \frac{(n+1)}{(n-1)^2 n} 4b_{n-2}^2\right) \\
&\sim 2\left\langle u, H_{n+1}^2\right\rangle.
\end{aligned}$$

Taking into account $A_n \sim 2\left\langle v, Q_n^2\right\rangle n^2$ we get

$$\begin{aligned}
\frac{B_n}{A_n} &\sim \frac{\left\langle u, H_{n+1}^2\right\rangle}{2\left\langle v, Q_n^2\right\rangle n^2} \\
&= \frac{1}{2n^2}\widetilde{\gamma}_n \frac{\left\langle \widetilde{v}, \widetilde{Q}_{n+3}^2\right\rangle}{\left\langle v, Q_{n+1}^2\right\rangle} \frac{\left\langle u, H_{n+1}^2\right\rangle}{\left\langle \widetilde{v}, \widetilde{Q}_{n+3}^2\right\rangle} \\
&= \frac{1}{2n^2}\widetilde{\gamma}_n \frac{a_{n+1}}{b_{n+1}} \\
&\sim \frac{1}{2n},
\end{aligned}$$

and, finally, $\dfrac{\Lambda_n(\lambda)}{A_n} \sim \dfrac{\left\langle v, Q_n^2 \right\rangle \frac{n^3}{2}\lambda}{2\left\langle v, Q_n^2 \right\rangle n^2} = \dfrac{n\lambda}{4}$. We summarize the above results in the following

Proposition 110. *For $\lambda > 0$ and according to (5.44), (5.45) and (5.46) we get $A_n \sim 2\left\langle v, Q_n^2 \right\rangle n^2$, $B_n \sim 2\left\langle u, H_{n+1}^2 \right\rangle$ and $\Lambda_n(\lambda) \sim \left\langle v, Q_n^2 \right\rangle \frac{n^3}{2}\lambda$. As a consequence,*

$$\lim_{n\to\infty} \frac{\Lambda_n^2(\lambda)}{A_{n+2}A_n} = \frac{1}{4}\lambda^2, \tag{5.49}$$

$$\lim_{n\to\infty} \frac{B_n}{A_n} = 0, \tag{5.50}$$

and

$$\lim_{n\to\infty} \frac{\Lambda_n(\lambda)}{nA_n} = \frac{\lambda}{4}. \tag{5.51}$$

For our purpose it is very important to study the asymptotic behavior of the ratios of polynomials associated with the recurrence relations (5.47) and (5.48). Indeed, the next theorem describes the asymptotic behavior of the ratio of the solutions of general three term recurrence relations whose coefficients are analytic functions in a prescribed region.

Theorem 111. *([18]). Consider the sequence of functions $\{w_n\}_{n\geq 0}$ satisfying the recurrence relation*

$$w_{n+1}(z) = p_n(z)w_n(z) - q_n^2(z)w_{n-1}(z), \tag{5.52}$$

where $p_n(z) \to p(z)$ and $q_n(z) \to q(z)$ locally uniformly on a domain G, besides $p_n(z) \neq 0$ for $z \in G$. We define, $I(z) := p(z)$, and $\Delta(z) := q^2(z)$. From

$$\Gamma = \left\{ z \in G \middle| \left| I(z) + \sqrt{I^2(z) - 4\Delta(z)} \right| = \left| I(z) - \sqrt{I^2(z) - 4\Delta(z)} \right| \right\},$$

$$P(z) = \frac{I(z) + \sqrt{I^2(z) - 4\Delta(z)}}{2}$$

and $E = \{z \in G \mid P(z) = 0\}$, the ratio $\dfrac{w_{n+1}}{w_n}$ converges locally uniformly on each compact subset of $G\backslash(\Gamma \cup E)$ to the zero of greatest absolute value of the equation $x^2 - p(z)x + \Delta(z) = 0$.

From (5.49) and (5.50) we get $p(z) = z$ and $q(z) = \dfrac{z}{2}$. However, we can not use directly the above theorem taking into account $I(z) = z$, $I^2(z) - 4\Delta(z) = 0$ for every $z \in \mathbb{C}$. To solve this problem, for $\xi > 0$ let $\{\Psi_n(x;\xi)\}_{n\geq 0}$ be the sequence of monic polynomials defined by

$$\Psi_{n+1}(\lambda;\xi) = \left(\lambda + \frac{B_{2n}}{A_{2n}}\right)\Psi_n(\lambda;\xi) - \frac{\widetilde{\Lambda}_{2n-2}^{\xi}(\lambda)}{A_{2n}A_{2n-2}}\Psi_{n-1}(\lambda;\xi), \qquad n \geq 1, \tag{5.53}$$

with $A_0 = A_1 = 1$, $\Psi_0(\lambda) = 1$, $\Psi_1(\lambda) = \widetilde{\gamma}_1\lambda + \frac{\sqrt{\pi}}{2}$. Moreover

$$\widetilde{\Lambda}_{2n-2}^{\xi}(\lambda) = \Lambda_{2n-2}^2(\lambda) - \frac{\xi}{4}(2n-2)^4 \left\langle v, Q_{2n-2}^2 \right\rangle^2.$$

In such a way, for n fixed we get

$$\lim_{\xi\to 0} \Psi_n(\lambda;\xi) = \Omega_n(\lambda).$$

Since $\Lambda^2_{2n-2}(\lambda) \sim \dfrac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6$ then for $\lambda \neq 0, \pm\sqrt{\xi}$ we get that

$$\begin{aligned}
&\frac{\Lambda^2_{2n-1}(\lambda) - \frac{\xi}{4}(2n-2)^6 \left\langle v, Q^2_{2n-2} \right\rangle^2}{\frac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6 - \frac{\xi}{4}(2n-2)^6 \left\langle v, Q^2_{2n-2} \right\rangle^2} \\
= &\frac{\dfrac{\Lambda^2_{2n-1}(\lambda)}{\frac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6} - \dfrac{\frac{\xi}{4}(2n-2)^6 \left\langle v, Q^2_{2n-2} \right\rangle^2}{\frac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6}}{1 - \dfrac{\frac{\xi}{4}(2n-2)^6 \left\langle v, Q^2_{2n-2} \right\rangle^2}{\frac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6}} \\
= &\frac{\dfrac{\Lambda^2_{2n-1}(\lambda)}{\frac{\lambda^2}{4} \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6} - \dfrac{\xi}{\lambda^2}}{1 - \dfrac{\xi}{\lambda^2}}.
\end{aligned}$$

The above expression tends to 1 when $n \to \infty$ and, as a consequence, $\widetilde{\Lambda}^{\xi}_{2n-2}(\lambda) \sim \left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6 \dfrac{\left(\lambda^2 - \xi\right)}{4}$. Thus,

$$\begin{aligned}
&\frac{\widetilde{\Lambda}^{\xi}_{2n-2}(\lambda)}{A_{2n}A_{2n-2}} \\
\sim &\frac{\left\langle v, Q^2_{2n-2} \right\rangle^2 (2n-2)^6 \frac{\left(\lambda^2-\xi\right)}{4}}{4 \left\langle v, Q^2_{2n} \right\rangle \left\langle v, Q^2_{2n-2} \right\rangle (2n)^2 (2n-2)^2} \\
= &\frac{\left\langle v, Q^2_{2n-2} \right\rangle}{4 \left\langle v, Q^2_{2n} \right\rangle} \frac{(2n-2)^4}{(2n)^2} \frac{\left(\lambda^2 - \xi\right)}{4} \\
= &\frac{1}{4 \frac{\widetilde{\gamma}_{2n}\widetilde{\gamma}_{2n-1}}{(2n)(2n-1)}} \frac{(2n-2)^4}{(2n)^3(2n-1)} \frac{\left(\lambda^2 - \xi\right)}{4}
\end{aligned}$$

and we get

$$\lim_{n\to\infty} \frac{\widetilde{\Lambda}^{\xi}_{2n-2}(\lambda)}{A_{2n}A_{2n-2}} = \frac{\lambda^2}{4} - \frac{\xi}{4}. \tag{5.54}$$

With this result in mind, we will use the above theorem in the next step. Taking into account (5.53), we define $q^{\xi}_n(z) = \sqrt{\dfrac{\widetilde{\Lambda}^{\zeta}_{2n-2}(z)}{A_{2n}A_{2n-2}}}$ and $q^{\xi}(z) = \sqrt{\frac{z^2}{4} - \frac{\xi}{4}}$. In this way, $I(z) = p(z) = z$, $\Delta(z) = \left(q^{\xi}\right)^2(z) = \frac{1}{4}z^2 - \frac{\xi}{4}$, and

$$I(z) \pm \sqrt{I^2(z) - 4\Delta(z)} = z \pm \sqrt{z^2 - 4\left(\frac{1}{4}z^2 - \frac{\xi}{4}\right)} = z \pm \sqrt{\xi}.$$

Then, $P(z) = \dfrac{z \pm \sqrt{\xi}}{2}$. We consider the region $G = \mathbb{C} \setminus \left\{0, \pm\sqrt{\xi}\right\}$. If we define $\Gamma = \{iw \mid w \in \mathbb{R}\}$, it is clear that,

$$\left| I(z) + \sqrt{I^2(z) - 4\Delta(z)} \right| = \left| I(z) - \sqrt{I^2(z) - 4\Delta(z)} \right|,$$

if and only if $z \in \Gamma$. Also, $P(z) = \dfrac{I(z) + \sqrt{I^2(z) - 4\Delta(z)}}{2} = \frac{z+\sqrt{\xi}}{2}$, and then $E = \left\{-\sqrt{\xi}\right\}$. Finally, $x^2 - zx + \Delta(z) = x^2 - zx + \left(\frac{1}{4}z^2 - \frac{\xi}{4}\right)$, i.e. their zeros are $x = \dfrac{z \pm \sqrt{\xi}}{2}$. As a consequence of above theorem we can state

Theorem 112. *Given the sequences* $\{\Psi_n(z;\xi)\}_{n\geq 0}$, *the ratio* $\dfrac{\Psi_{n+1}(z;\xi)}{\Psi_n(z;\xi)}$ *converges to* $h^{\xi}(\lambda) = \dfrac{\lambda + \sqrt{\xi}}{2}$ *uniformly on compact subsets of* $\mathbb{C} \setminus \left(\left\{\pm\sqrt{\xi}\right\} \cup \{iw \mid w \in \mathbb{R}\}\right)$.

We can do an analogous work with a convenient sequence of polynomials $\{\widetilde{\Psi}_n(\lambda;\eta)\}_{n\geq 0}$, $\eta > 0$, such that $\lim_{\eta \to 0} \widetilde{\Psi}_n(\lambda;\eta) = \widetilde{\Omega}_n(\lambda)$ and the convergence of $\dfrac{\widetilde{\Psi}_{n+1}(z;\xi)}{\widetilde{\Psi}_n(z;\xi)}$ to $h^{\eta}(\lambda) = \dfrac{\lambda + \sqrt{\eta}}{2}$ follows. As a consequence,

Theorem 113. *For* $\lambda > 0$ *we get*

$$\lim_{n\to\infty} \frac{\Omega_{n+1}(\lambda)}{\Omega_n(\lambda)} = \lim_{\xi \to 0} \left(\lim_{n\to\infty} \frac{\Psi_{n+1}(\lambda;\xi)}{\Psi_n(\lambda;\xi)} \right) = \lim_{n\to\infty} \frac{\widetilde{\Omega}_{n+1}(\lambda)}{\widetilde{\Omega}_n(\lambda)} = \lim_{\eta\to 0} \left(\lim_{n\to\infty} \frac{\widetilde{\Psi}_{n+1}(\lambda;\eta)}{\widetilde{\Psi}_n(\lambda;\eta)} \right) = \frac{\lambda}{2}.$$

As above, we can also work with the sequence $\{\widetilde{\Omega}_n(\lambda)\}_{n\geq 0}$. Then for $\lambda > 0$ we find, $\lim_{n\to\infty} \dfrac{\widetilde{\Omega}_{n+1}(\lambda)}{\widetilde{\Omega}_n(\lambda)} = \dfrac{\lambda}{2}$. According to the above results and from (5.51) we get

Theorem 114 (Sobolev coefficients)**.** *For* $\lambda \in \mathbb{R}^+$

$$\lim_{n\to\infty} \frac{\eta_n(\lambda)}{n} = \frac{1}{2}. \tag{5.55}$$

5.4 Asymptotics of Sobolev Polynomials

5.4.1 Relative Outer Asymptotics

From (5.28) and its successive application it is not difficult to prove the next lemma that gives us a connection between the Sobolev and Hermite orthogonal polynomials.

Lemma 115. *For* $n \in \mathbb{N}$,

$$S_{n+2}^{\lambda}(x) = T_{n+2}(x) + \sum_{k=1}^{\left[\frac{n+2}{2}\right]} (-1)^k \rho_{n,k}(\lambda) T_{n+2-2k}(x), \tag{5.56}$$

where, $\rho_{n,k}(\lambda) := \prod_{j=1}^{k} \eta_{n+1-2j}(\lambda)$, $k \geq 1$, and

$$T_{n+2}(x) := H_{n+2}(x) + (n+2)\frac{b_{n-1}}{n} H_n(x) = S_{n+2}^{\lambda}(x) + \eta_{n-1}(\lambda) S_n^{\lambda}(x). \tag{5.57}$$

On the other hand, for every $n \in \mathbb{N}$

$$\begin{aligned}\left\langle S_n^\lambda, S_n^\lambda \right\rangle_S &= \int_{\mathbb{R}} \left(S_n^\lambda(x)\right)^2 e^{-x^2} dx + \lambda \int_{\mathbb{R}} \left(\left(S_n^\lambda(x)\right)'\right)^2 \frac{x^2+a}{x^2+b} e^{-x^2} dx \\ &\geq \left\langle u, H_n^2 \right\rangle + \lambda n^2 \left\langle v, Q_{n-1}^2 \right\rangle.\end{aligned}$$

By using (5.29), (5.41) and (5.42) we get

$$\begin{aligned}&\frac{|\eta_n(\lambda)|}{n+3} \\ \leq\ & \frac{1}{(n+1)} \frac{\lambda (n+1)^2 |a_n| \left\langle v, Q_n^2 \right\rangle + |b_n| \left\langle u, H_{n+1}^2 \right\rangle}{\left\langle S_{n+1}^\lambda, S_{n+1}^\lambda \right\rangle_S} \\ \leq\ & \frac{\frac{|a_n|}{n+1} \lambda (n+1)^2 \left\langle v, Q_n^2 \right\rangle + \frac{|b_n|}{n+1} \left\langle u, H_{n+1}^2 \right\rangle}{\left\langle u, H_{n+1}^2 \right\rangle + \lambda (n+1)^2 \left\langle v, Q_n^2 \right\rangle} \\ <\ & 1.\end{aligned}$$

Summarizing, we get the next

Lemma 116. *For every n*

$$\frac{|\eta_n(\lambda)|}{n+3} < 1. \tag{5.58}$$

Thus

Theorem 117. *The Sobolev polynomials $\{S_n^\lambda(x)\}_{n\geq 0}$ satisfy*

$$\lim_{n\to\infty} \frac{S_n^\lambda(x)}{H_n(x)} = 0,$$

uniformly on compact sets of $\mathbb{C}\backslash\mathbb{R}$.

Proof. If $\left[\frac{n+2}{2}\right] \geq k \geq 1$, then from (5.57)

$$T_{n-2(k-1)}(x) = H_{n-2(k-1)}(x) + \frac{(n-2(k-1))}{n-2k} b_{n-2k-1} H_{n-2k}(x).$$

If we define $\theta_{n,k} := (n-1)(n-3)\cdots(n-2k+1)$ and $\{f_{n,k}(x)\}_{n\geq 0}$ as $f_{n,k}(x) := (-1)^k \rho_{n,k}(\lambda) T_{n+2-2k}(x)$, then

$$\begin{aligned}&\rho_{n,k}(\lambda) \frac{T_{n-2(k-1)}(x)}{H_{n+2}(x)} \\ =\ & \rho_{n,k}(\lambda) \frac{H_{n-2(k-1)}(x)}{H_{n+2}(x)} + \rho_{n,k}(\lambda) \frac{(n-2(k-1))}{n-2k} b_{n-2k-1} \frac{H_{n-2k}(x)}{H_{n+2}(x)} \\ =\ & \frac{\rho_{n,k}(\lambda)}{\theta_{n,k}} \times \\ & \left(2^k \theta_{n,k} \prod_{i=1}^{k} \frac{H_{n-2(i-1)}(x)}{2H_{n-2(i-2)}(x)} + \frac{(n-2(k-1))}{n-2k} \frac{2^{k+1} b_{n-2k-1}}{n-2k-1} \theta_{n,k+1} \prod_{i=0}^{k} \frac{H_{n-2i}(x)}{2H_{n-2(i-1)}(x)}\right).\end{aligned}$$

When $n \to \infty$ and using (1.31), (5.40) and (5.55), we get $f_{n,k}(x) \to 0$ uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. Moreover, since $\{(n-2j+4)\frac{T_{n-2(k-1)}(x)}{H_{n+2}(x)}\}_{n\geq 0}$ is uniformly bounded on every compact subset of $\mathbb{C}\backslash\mathbb{R}$ and from (5.58)

$$\left|\rho_{n,k}(\lambda)\frac{T_{n+2-2k}(x)}{H_{n+2}(x)}\right| = \prod_{j=1}^{k}\frac{|\eta_{n+1-2j}(\lambda)|}{(n-2j+4)}\prod_{j=1}^{k}\left|(n-2j+4)\frac{T_{n+2-2k}(x)}{H_{n+2}(x)}\right| \leq M,$$

where M does not depend on n (but it depends on the compact subset) and $\lim_{n\to\infty}\frac{T_{n+2}(x)}{H_{n+2}(x)} = 0$, uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$, then $\lim_{n\to\infty}\frac{S_n^{\lambda}(x)}{H_n(x)} = \lim_{n\to\infty}\dfrac{T_{n+2}(x)}{H_{n+2}(x)} + \sum_{k=1}^{\infty}\lim_{n\to\infty} f_{n,k}(x)$, and the result holds. □

In general, the above result does not provide a precise information about of relative outer asymptotic for Sobolev orthogonal polynomials. However, we will improve this result. First, the sequence $\{T_n(x)\}_{n\geq 0}$, defined in (5.57), satisfies

Lemma 118. *(See [29], [102])*

$$\lim_{n\to\infty}\sqrt{\left[\frac{n}{2}\right]}\frac{T_n(x)}{H_n(x)} = \begin{cases} -ix+\sqrt{b}, & x\in\mathbb{C}_+ \\ ix+\sqrt{b}, & x\in\mathbb{C}_-, \end{cases} \tag{5.59}$$

uniformly on compact subsets of the half planes $\mathbb{C}_+$ *and* $\mathbb{C}_-$*, respectively, and*

$$\lim_{n\to\infty}\frac{nT_n(x)}{T_{n+2}(x)} = -2, \tag{5.60}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

Now, from (5.57) we get $\dfrac{S_{n+2}^{\lambda}(x)}{T_{n+2}(x)} = 1 - \dfrac{\eta_{n-1}(\lambda)S_n^{\lambda}(x)}{T_{n+2}(x)}$. Next we study the asymptotic behavior of the sequence $\left\{\dfrac{\eta_{n-1}(\lambda)S_n^{\lambda}}{T_{n+2}}\right\}_{n\geq 1}$. From (5.56) we get

$$\begin{aligned}\frac{\eta_{n-1}(\lambda)S_n^{\lambda}(x)}{T_{n+2}(x)} &= \frac{\eta_{n-1}(\lambda)T_n(x)}{T_{n+2}(x)} + \sum_{k=1}^{\left[\frac{n}{2}\right]}(-1)^k\rho_{n-2,k}(\lambda)\eta_{n-1}(\lambda)\frac{T_{n-2k}(x)}{T_{n+2}(x)} \\ &= \frac{\eta_{n-1}(\lambda)}{n}\frac{nT_n(x)}{T_{n+2}(x)} + \sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{n+1}\frac{\rho_{n,k}(\lambda)}{\theta_{n,k}}\theta_{n+1,k}\prod_{j=1}^{k+1}\frac{T_{n-2-2(k-j)}(x)}{T_{n-2(k-j)}(x)},\end{aligned}$$

then as above, we can show that the sequence $\left\{\frac{(-1)^k}{n+1}\frac{\rho_{n,k}(\lambda)}{\theta_{n,k}}\theta_{n+1,k}\prod_{j=1}^{k+1}\frac{T_{n-2-2(k-j)}}{T_{n-2(k-j)}}\right\}_{n\geq 1}$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. Moreover, from (5.60), it converges to 0. Then,

Lemma 119.

$$\lim_{n\to\infty}\frac{\eta_{n-1}(\lambda)S_n^{\lambda}(x)}{T_{n+2}(x)} = -1, \tag{5.61}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

As a consequence, we can state

Proposition 120.

$$\lim_{n\to\infty}\frac{S_{n+2}^{\lambda}(x)}{2T_{n+2}(x)}=1,$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

Since $\dfrac{\sqrt{\left[\frac{n+2}{2}\right]}S_{n+2}^{\lambda}(x)}{H_{n+2}(x)}=\dfrac{S_{n+2}^{\lambda}(x)}{T_{n+2}(x)}\dfrac{\sqrt{\left[\frac{n+2}{2}\right]}T_{n+2}(x)}{H_{n+2}(x)}$, from the above lemmas we get the next theorem that constitutes the main result of this section

Theorem 121.

$$\lim_{n\to\infty}\frac{\sqrt{\left[\frac{n}{2}\right]}S_{n}^{\lambda}(x)}{2H_{n}(x)}=\begin{cases}-ix+\sqrt{b}, & x\in\mathbb{C}_{+}\\ ix+\sqrt{b}, & x\in\mathbb{C}_{-},\end{cases}\tag{5.62}$$

uniformly on compact subsets of half planes $\mathbb{C}_{+}$ *and* $\mathbb{C}_{-}$

5.4.2 Scaled relative Asymptotics

In this section we obtain asymptotic properties when we scale the sequence $\{S_n^{\lambda}(x)\}_{n\geq 0}$ based on the corresponding results of scaled Hermite polynomials described in (1.38).

From (5.57), $\dfrac{T_{n+2-2k}(x)}{H_{n+2}(x)}=\dfrac{H_{n+2-2k}(x)}{H_{n+2}(x)}+(n+2-2k)\,\dfrac{b_{n-1-2k}}{n-2k}\dfrac{H_{n-2k}(x)}{H_{n+2}(x)}$. By using (1.38) and (5.40) we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sqrt{n}\right)^{2k}\frac{T_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}\\
=\;&\lim_{n\to\infty}\left(\sqrt{n}\right)^{2k}\frac{H_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}+\lim_{n\to\infty}\frac{(n+2-2k)}{n}\lim_{n\to\infty}\frac{b_{n-1-2k}}{n-2k}\lim_{n\to\infty}\left(\sqrt{n}\right)^{2k+2}\frac{H_{n-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}\\
=\;&\left(\frac{\sqrt{2}}{\varphi\left(x/\sqrt{2}\right)}\right)^{2k}+\frac{1}{2}\left(\frac{\sqrt{2}}{\varphi\left(x/\sqrt{2}\right)}\right)^{2k+2}.
\end{aligned}$$

Thus,

Lemma 122. *For* $j\in\mathbb{Z}_{+}$

$$\lim_{n\to\infty}\left(\sqrt{n+j}\right)^{2k}\frac{T_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}=\left(\frac{2}{\varphi^{2}\left(x/\sqrt{2}\right)}\right)^{k}\left(\frac{\varphi^{2}\left(x/\sqrt{2}\right)+1}{\varphi^{2}\left(x/\sqrt{2}\right)}\right),\tag{5.63}$$

holds locally uniformly on compact subsets of $\mathbb{C}\backslash\left[-\sqrt{2},\sqrt{2}\right]$.

Now, from (5.56), if $1\leq k\leq\left[\frac{n+2}{2}\right]$, then we define for $n\in\mathbb{N}$,

$$g_{n+2,k}(x):=(-1)^{k}\left(\prod_{j=1}^{k}\eta_{n+1-2j}(\lambda)\right)\frac{T_{n+2-2k}(x)}{H_{n+2}(x)},$$

with $k \geq 1$. If K is a compact subset of $\mathbb{C}\setminus\left[-\sqrt{2},\sqrt{2}\right]$, then the sequence $\{g_{n+2,k}(\sqrt{n}x)\}_{n\geq 0}$ is uniformly bounded on K. Namely, if $j \geq 2$, then from (5.58) we get

$$\begin{aligned}
&\left|g_{n+2,k}(\sqrt{n}x)\right| \\
&= \prod_{j=1}^{k}|\eta_{n+1-2j}(\lambda)|\,\frac{1}{(n+i)^k}\left|\left(\sqrt{n+j}\right)^{2k}\frac{T_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}\right| \\
&= \prod_{j=1}^{k}\frac{|\eta_{n+1-2j}(\lambda)|}{(n+1-2j+3)}\left(\prod_{j=1}^{k}\frac{(n+1-2j+3)}{(n+j)}\right)\left|\left(\sqrt{n+j}\right)^{2k}\frac{T_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}\right| \\
&\leq M,
\end{aligned}$$

where the constant M only depends on K. Besides, since, $\displaystyle\prod_{j=1}^{k}\frac{|\eta_{n+1-2j}(\lambda)|}{(n+1-2j+3)}\times$ $\displaystyle\prod_{j=1}^{k}\frac{(n+1-2j+3)}{(n+j)} < 1$, we get

$$\begin{aligned}
g_k(x) :&= \lim_{n\to\infty} g_{n+2,k}(\sqrt{n}x) \\
&= \lim_{n\to\infty}(-1)^k\prod_{j=1}^{k}\frac{\eta_{n+1-2j}(\lambda)}{n+1-2j+3}\lim_{n\to\infty}\prod_{j=1}^{k}\frac{(n+1-2j+3)}{(n+j)} \\
&\quad\times\lim_{n\to\infty}\left(\sqrt{n+j}\right)^{2k}\frac{T_{n+2-2k}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)} \\
&= \left(-\frac{1}{\varphi^2\left(x/\sqrt{2}\right)}\right)^k\left(\frac{\varphi^2\left(x/\sqrt{2}\right)+1}{\varphi^2\left(x/\sqrt{2}\right)}\right),
\end{aligned}$$

locally uniformly on compact subsets of $\mathbb{C}\setminus\left[-\sqrt{2},\sqrt{2}\right]$, and

$$\begin{aligned}
\lim_{n\to\infty}\frac{S^{\lambda}_{n+2}(x)}{H_{n+2}(x)} &= \lim_{n\to\infty}\frac{T_{n+2}(x)}{H_{n+2}(x)} + \lim_{n\to\infty}\sum_{k=1}^{\left[\frac{n+2}{2}\right]} g_{n+2,k}(\sqrt{n}x) \\
&= \lim_{n\to\infty}\frac{T_{n+2}(x)}{H_{n+2}(x)} + \sum_{k=1}^{\infty} g_k(x) \\
&= \left(\frac{\varphi^2\left(x/\sqrt{2}\right)+1}{\varphi^2\left(x/\sqrt{2}\right)}\right)\sum_{k=0}^{\infty}\left(-\frac{1}{2\varphi^2\left(x/\sqrt{2}\right)}\right)^k.
\end{aligned}$$

Since, $\left|\frac{1}{2\varphi^2\left(x/\sqrt{2}\right)}\right| < 1$ we get,

Theorem 123.

$$\lim_{n\to\infty}\frac{S^{\lambda}_{n+2}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)} = 1,$$

locally uniformly on compact subsets of $\mathbb{C}\setminus\left[-\sqrt{2},\sqrt{2}\right]$.

5.4.3 Mehler-Heine type formulas

In this section we obtain Mehler-Heine type formulas for scaled polynomials. Taking into account, $\frac{1}{n+2}T'_{n+2}(x) = H_{n+1}(x) + b_{n-1}H_{n-1}(x)$, for $n \in \mathbb{N}$, we get $\widetilde{Q}_n(x) = \frac{T'_{n+1}(x)}{n+1}$. By using a result in [102],

Proposition 124. *It holds*

$$\lim_{n\to\infty} \frac{(-1)^n}{(n-1)!}\widetilde{Q}_{2n}\left(\frac{x}{2\sqrt{n+j}}\right) = \sqrt{\frac{b}{\pi}}\cos(x), \tag{5.64}$$

and

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n+j}}{n!}\widetilde{Q}_{2n+1}\left(\frac{x}{2\sqrt{n+j}}\right) = \sqrt{\frac{b}{\pi}}\sin(x), \tag{5.65}$$

both locally uniformly on compact subsets of $\mathbb{C}$.

Then, in the even case we get

$$\begin{aligned}
&\frac{(-1)^{n-1}\sqrt{n+j}}{(n-1)!}\frac{\left(S_{2n}^{\lambda}\right)'\left(\frac{x}{2\sqrt{n+j}}\right)}{2n} \\
= \; &\frac{(-1)^{n-1}\sqrt{n+j}}{(n-1)!}\widetilde{Q}_{2n-1}\left(\frac{x}{2\sqrt{n+j}}\right) \\
&+\sum_{k=1}^{n-1}\left(\prod_{j=1}^{k}\frac{\eta_{2n-1-2j}(\lambda)}{2n-2j+2}\right)2^k\frac{(-1)^{n-k-1}\sqrt{n+j}}{(n-k-1)!}\widetilde{Q}_{2(n-k-1)+1}\left(\frac{x}{2\sqrt{n+j}}\right).
\end{aligned}$$

Notice that the sequence

$$\left\{\left(\prod_{j=1}^{k}\frac{\eta_{2n-1-2j}(\lambda)}{2n-2j+2}\right)2^k\frac{(-1)^{n-k-1}\sqrt{n+j}}{(n-k-1)!}\widetilde{Q}_{2(n-k-1)+1}\left(\frac{x}{2\sqrt{n+j}}\right)\right\}_{n\geq 0}$$

is uniformly bounded on compact subsets of $\mathbb{C}$. Moreover,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\prod_{j=1}^{k}\frac{\eta_{2n-1-2j}(\lambda)}{2n-2j+2}\right)2^k\frac{(-1)^{n-k-1}\sqrt{n+j}}{(n-k-1)!}\widetilde{Q}_{2(n-k-1)+1}\left(\frac{x}{2\sqrt{n+j}}\right) \\
= \; &\sqrt{\frac{b}{\pi}}\sin(x),
\end{aligned}$$

and, as a consequence of (5.65),

$$\lim_{n\to\infty}\frac{(-1)^{n-1}\sqrt{n+j}}{(n-1)!}\frac{\left(S_{2n}^{\lambda}\right)'\left(\frac{x}{2\sqrt{n+j}}\right)}{2n} = \sqrt{\frac{b}{\pi}}\sin(x). \tag{5.66}$$

The odd case follows in a similar way taking into account that

$$\begin{aligned}
&\frac{(-1)^n}{(n-1)!}\frac{\left(S_{2n+1}^{\lambda}\right)'\left(\frac{x}{2\sqrt{n+j}}\right)}{2n+1}\\
=\;&\frac{(-1)^n}{(n-1)!}\widetilde{Q}_{2n}\left(\frac{x}{2\sqrt{n+j}}\right)\\
&+\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\frac{\eta_{2n-2j}(\lambda)}{2n-2j+3}\right)2^k\frac{\prod_{j=1}^{k}(n-j+3/2)\,(n-k-1)!\,(n-k+1/2)}{(n+1/2)(n-1)!}\\
&\times\frac{(-1)^{n-k}}{(n-k-1)!}\widetilde{Q}_{2n-2k}\left(\frac{x}{2\sqrt{n+j}}\right).
\end{aligned}$$

As a consequence of (5.64) we get

$$\lim_{n\to\infty}\frac{(-1)^n}{(n-1)!}\frac{\left(S_{2n+1}^{\lambda}\right)'\left(\frac{x}{2\sqrt{n+j}}\right)}{2n+1}=\sqrt{\frac{b}{\pi}}\cos(x). \tag{5.67}$$

Since in (5.66) and (5.67) the convergence is uniform, integrating on every compact subset we get the next

Theorem 125. *It holds*

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n+j}}{n!}S_{2n+1}^{\lambda}\left(\frac{x}{2\sqrt{n+j}}\right)=\sqrt{\frac{b}{\pi}}\sin(x), \tag{5.68}$$

and

$$\lim_{n\to\infty}\frac{(-1)^{n-1}}{(n-1)!}S_{2n}^{\lambda}\left(\frac{x}{2\sqrt{n+j}}\right)=-\sqrt{\frac{b}{\pi}}\cos(x), \tag{5.69}$$

both locally uniformly on compact subsets of $\mathbb{C}$.

According to Hurwitz's theorem, from (7.12) and (7.13) we obtain information about the behavior of positive zeros of Sobolev orthogonal polynomials when $n\to\infty$. Indeed, if $\{s_{n,k}^{\lambda}\}_{k=1}^{n^*}$, $n^*\le n$, are the positive zeros of the polynomial $S_n^{\lambda}(x)$, in an increasing order, then

Corollary 126. *Let* $\{s_{n,k}^{\lambda}\}_{k=1}^{n^*}$ *be the positive zeros of* $S_n^{\lambda}(x)$. *Then for* $1\le j\le n^*$

$$\lim_{n\to\infty}2\sqrt{n}s_{2n+1,j}^{\lambda}=j\pi,\qquad \lim_{n\to\infty}2\sqrt{n}s_{2n,j}^{\lambda}=(2j-1)\frac{\pi}{2}.$$

Notice that in [28] the behavior of zeros of Sobolev orthogonal polynomials as well as their asymptotic properties for companion measures of the normal distribution are studied.

Chapter 6

Applications

In this chapter we present an application of the techniques and results used throughout of above chapters. We exhibit an algorithm to compute Fourier coefficients in expansions of functions that belong to the Sobolev space $W_2^1[\mathbb{R}, \mu_0, \mu_1]$ by using Sobolev polynomials. We follow the ideas presented in [55].

6.1 Algorithm for Sobolev-Fourier coefficients

In this section we will describe an algorithm to compute the Fourier coefficients in expansions of Sobolev polynomials orthogonal with respect to

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0.$$

For $f \in W_2^1[\mathbb{R}, \mu_0, \mu_1] = \left\{f | f \in L^2(\mu_0), f' \in L^2(\mu_1)\right\}$ we can expand f in terms of Sobolev orthogonal polynomials $\left\{S_n^\lambda\right\}_{n\geq 0}$, namely

$$f(x) \sim \sum_{n=0}^{\infty} \frac{\left\langle f, S_n^\lambda\right\rangle_S}{\|S_n^\lambda\|_S^2} S_n^\lambda(x).$$

To avoid making the text more cumbersome we define $p_n := \left\langle u, P_n^2\right\rangle$, $r_n := \left\langle v, R_n^2\right\rangle$, $s_n^\lambda := \left\|S_n^\lambda\right\|_S^2$, $f_n^\lambda := \left\langle f, S_n^\lambda\right\rangle_S$ and $F_n^\lambda := f_n^\lambda / s_n$. F_n^λ is said to be the $n-th$ Sobolev-Fourier coefficient. To prepare the tools necessary to implement the algorithms, we deduce the following result.

Lemma 127.

$$f_{n+2}^\lambda + \eta_{n-1}(\lambda) f_n^\lambda = w_n(f), \quad n \geq 0, \tag{6.1}$$

holds, where

$$w_n(f) = \left\langle f, P_{n+2}(x) + a_{n-1}\frac{n+2}{n} P_n(x)\right\rangle_{\mu_0} + \lambda \left\langle f', P'_{n+2}(x) + a_{n-1}\frac{n+2}{n} P'_n(x)\right\rangle_{\mu_1}, \tag{6.2}$$

with the initial conditions $\eta_{-1}(\lambda) = 0$, $f_0^\lambda = \langle f, 1\rangle_S = \langle f, 1\rangle_{\mu_0}$. $f_1^\lambda = \langle f, x\rangle_{\mu_0} + \lambda \langle f', 1\rangle_{\mu_1}$ *and*

$$w_0(f) := \langle f, P_2(x)\rangle_{\mu_0} + \lambda \left\langle f', P'_2(x)\right\rangle_{\mu_1}.$$

Proof. By using (5.8) and (4.2) we get

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x)$$

$$S^{\lambda}_{n+3}(x) + \eta_n(\lambda) S^{\lambda}_{n+1}(x) = P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x)$$

$$\begin{aligned}
\langle f, S_{n+2} \rangle_S &= -\eta_{n-1} \langle f, S_n \rangle_S + \left\langle f, P_{n+2}(x) + a_{n-1} \frac{n+2}{n} P_n(x) \right\rangle_S \\
&= -\eta_{n-1} \langle f, S_n \rangle_S + \left\langle f, P_{n+2}(x) + a_{n-1} \frac{n+2}{n} P_n(x) \right\rangle_{\mu_0} \\
&\quad + \lambda \left\langle f', P'_{n+2}(x) + a_{n-1} \frac{n+2}{n} P'_n(x) \right\rangle_{\mu_1} \\
&= -\eta_{n-1} \langle f, S_n \rangle_S + \left\langle f, P_{n+2}(x) + a_{n-1} \frac{n+2}{n} P_n(x) \right\rangle_{\mu_0} \\
&\quad + \lambda (n+2) \left\langle f', R_{n+1}(x) + b_{n-1} R_{n-1}(x) \right\rangle_{\mu_1},
\end{aligned}$$

and the result follows. □

Now we will summarize the necessary results that, together with (6.1), constitute the structure of the algorithm.

- For $n \geq 1$
$$A_n = \widetilde{b}_n (n+1)^2 r_n, \qquad B_n = \widetilde{a}_n p_{n+1}, \qquad D_n = p_{n+1}, \tag{6.3}$$
  and for $n \geq 2$
$$C_n = (n+1)^2 r_n + \widetilde{a}^2_{n-2} p_{n-1}, \tag{6.4}$$
  with $a_{-n} = b_{-n} = 0$ for $n \in \mathbb{N}$, $C_1 = 4r_1$, furthermore, for $n \geq 0$, $\widetilde{a}_n = \dfrac{n+3}{n+1} a_n$ and $\widetilde{b}_n = \dfrac{n+3}{n+1} b_n$.

- With the initial conditions $\widetilde{Q}_{-1}(\lambda) = 0$, $Q_0(\lambda) = \widetilde{Q}_0(\lambda) = 1$, $Q_1(\lambda) = \lambda + p_1$, $\eta_0(\lambda) = \dfrac{\widetilde{b}_0 \lambda + \widetilde{a}_0 p_1}{\lambda + p_1}$, and $\eta_1(\lambda) = \dfrac{4\widetilde{b}_1 r_1 \lambda + \widetilde{a}_1 p_2}{4 r_1 \lambda + p_2}$,
$$\eta_{2n}(\lambda) = (A_{2n}\lambda + B_{2n}) \frac{Q_n(\lambda)}{Q_{n+1}(\lambda)}, \; n \geq 1, \tag{6.5}$$
  and
$$\eta_{2n+1}(\lambda) = (A_{2n+1}\lambda + B_{2n+1}) \frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)}, \quad n \geq 1, \tag{6.6}$$
  with
$$Q_{n+1}(\lambda) = (C_{2n}\lambda + D_{2n}) Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2 Q_{n-1}(\lambda), \tag{6.7}$$
  and
$$\widetilde{Q}_{n+1}(\lambda) = (C_{2n+1}\lambda + D_{2n+1}) \widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2 \widetilde{Q}_{n-1}(\lambda). \tag{6.8}$$

- With initial conditions $s_1^\lambda = \lambda + p_1$

$$\begin{aligned} s_{n+1}^\lambda &= (n+1)^2\lambda r_n + p_{n+1} + \widetilde{a}_{n-2}\left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right)p_{n-1} \\ &\quad -\eta_{n-2}(\lambda)\left(\eta_{n-2}(\lambda) - \widetilde{b}_{n-2}\right)s_{n-1}^\lambda. \end{aligned} \tag{6.9}$$

In order to describe the algorithms, it is assumed that the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are known as well as the sequences of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$.

Algorithm 128 (Even order)**.** *For n even, the Fourier–Sobolev coefficients* $F_n^\lambda = \dfrac{f_n^\lambda}{s_n^\lambda} = \dfrac{\langle f, S_n\rangle_S}{\|S_n^\lambda\|_S^2}$ *can be computed using the following algorithm.*

Starting data. Initial conditions λ, f_0^λ, η_{-1}, s_0^λ, $\widetilde{Q}_{-1}$, $\widetilde{Q}_0, w_0(f), A_{-1}, B_{-1}, C_1$ and D_1.

Step 1. Using the starting data to compute f_2^λ with the relation (6.1) and $n = 0$, s_2^λ through (6.9) with $n = 1$ and finally F_2^λ.

Step 2. Using the starting data and the information in step 1 compute: $\widetilde{Q}_1$ taking $n = 0$ in (6.8) , A_1, B_1 with 6.3 and $n = 0$, $\eta_1(\lambda)$ taking $n = 0$ in (6.6), w_2 with $n = 2$ in 6.2, f_4^λ through (6.1) with $n = 2$, and finally s_4^λ taking $n = 3$ in (6.9). Then to compute F_4^λ.

Step k. For $k \geq 3$, using the starting data and the information in steps 1 to $k-1$ we can compute A_{2k-3}, B_{2k-3}, C_{2k-1} and D_{2k-1} with $n = k-1$ in 6.3 and 6.4, $\widetilde{Q}_k$ taking $n = k-1$ in (6.8), A_{2k-1}, B_{2k-1} with 6.3 and $n = k-1$, $\eta_{2k-1}(\lambda)$ taking $n = k-1$ in (6.6), w_{2k} with $n = 2k$ in 6.2, f_{2k+2}^λ through (6.1) with $n = 2k$, and finally s_{2k+2}^λ taking $n = 2k+1$ in (6.9). Then, compute F_{2k+2}^λ.

Algorithm 129 (Odd order)**.** *For n even, the Fourier–Sobolev coefficients* $F_n^\lambda = \dfrac{f_n^\lambda}{s_n^\lambda} = \dfrac{\langle f, S_n\rangle_S}{\|S_n^\lambda\|_S^2}$ *can be computed using the following algorithm.*

Starting data. Initial conditions λ, f_1^λ, η_0, s_1^λ, Q_0, Q_1, $w_1(f)$, A_0, B_0, C_2 and D_2.

Step 1. Using the starting data compute f_3^λ through (6.1) with $n = 1$, s_3^λ through (6.9) with $n = 2$ and then to compute F_3^λ.

Step 2. Using the starting data and the information in step 1, compute Q_2 taking $n = 1$ in (6.7), A_2, B_2 with 6.4 and $n = 1$, $\eta_2(\lambda)$ taking $n = 1$ in (6.5), w_3 with $n = 3$ in 6.2, f_5^λ through (6.1) with $n = 3$, and finally s_5^λ taking $n = 4$ in (6.9). Then compute F_5^λ.

Step k. For $k \geq 3$, using the starting data and the information in steps 1 to $k-1$ we can compute A_{2k-2}, B_{2k-2}, C_{2k} and D_{2k} with $n = k$ in 6.3 and 6.4, Q_{k+1} taking $n = k$ in (6.8) , A_{2k}, B_{2k} with 6.4 and $n = k$, $\eta_{2k}(\lambda)$ taking $n = k$ in (6.6), w_{2k+1} with $n = 2k+1$ in 6.2, f_{2k+3}^λ through (6.1) with $n = 2k+1$, and finally s_{2k+3}^λ taking $n = 2k+2$ in (6.9). Then, compute F_{2k+3}^λ.

6.1.1 Numerical examples

Next, with the help of MATHEMATICA 10, we carry out some numerical experiments where the algorithms described above are implemented.

Example 130. *(Gegenbauer Polynomials). In [34] the Symmetric* $(1,1)-$*coherent pairs, when* u *is the classical Gegenbauer functional, are exhibited. In particular the pair*

$$d\mu_0 = (1-x^2)^{\eta-1/2}dx, \quad d\mu_1 = \frac{x^2+a}{x^2+b}(1-x^2)^{\eta-1/2}dx,$$

$a,b\in\mathbb{R}^+$, $a\neq b$, $\eta>-1/2$, $x\in[-1,1]$ *is found. As it is usual, let* $\left\{C_n^{(\eta)}\right\}_{n\geq 0}$ *be the monic sequence of Gegenbauer polynomials, orthogonal with respect to the inner product*

$$\langle p,q\rangle_\eta = \int_{-1}^{1} p(x)q(x)(1-x^2)^{\eta-1/2}dx.$$

Also, the Gegenbauer polynomials satisfy the TTRR

$$C_{n+1}^{(\eta)}(x) = xC_n^{(\eta)}(x) - \frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)}C_{n-1}^{(\eta)}(x), \quad n\geq 1,$$

and $C_0^{(\eta)}(x)=1$ *and* $C_1^{(\eta)}(x)=x$*. The corresponding norm is defined as*

$$\left\|C_n^{(\eta)}\right\|_\eta^2 = \frac{4^{n+\eta}\left(\Gamma(n+\eta+1/2)\right)^2\Gamma\left(n+2\eta\right)}{2\left(n+\eta\right)\left(\Gamma(2n+2\eta)\right)^2}n!.$$

According to results of the above sections, if $d\mu_0=(1-x^2)^{\eta-1/2}dx$*, then we have the symmetric* $(1,1)-$*coherent relation*

$$C_n^{(\eta+1)}(x) + b_{n-2}C_{n-2}^{(\eta+1)}(x) = Q_n(x) + a_{n-2}Q_{n-2}(x), \qquad n\geq 2.$$

Moreover, from (5.8) we get

$$S_{n+3}^{\lambda}(x) + \eta_n(\lambda)S_{n+1}^{\lambda}(x) = C_{n+3}^{(\eta)}(x) + \frac{n+3}{n+1}b_nC_{n+1}^{(\eta)}(x), \qquad n\geq 0.$$

Explicit relations between recurrence coefficients and the sequences $\{a_n\}_{n\geq0}$ *and* $\{b_n\}_{n\geq0}$ *can be seen in [34]. To be more precise, we get*

$$b_0 - \frac{1}{2(\eta+1)} = a_0 - \widetilde{\gamma}_1,$$

$$\frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)} + b_{n-2} - b_{n-1} = \widetilde{\gamma}_n + a_{n-2} - a_{n-1}, \quad n\geq 2,$$

$$b_{n-2}\frac{(n-2)(n+2\eta-3)}{4(n+\eta-3)(n+\eta-2)} = b_{n-3}\left(\frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)} + b_{n-2} - b_{n-1}\right), \quad n\geq 5,$$

and

$$a_{n-2}\widetilde{\gamma}_{n-2} = a_{n-3}\left(\widetilde{\gamma}_n + a_{n-2} - a_{n-1}\right), \quad n\geq 5.$$

In addition, the sequence $\{b_n\}_{n\geq0}$ *satisfies the quadratic difference equation*

$$\begin{aligned} & b_{n+1} \\ = \; & \frac{1}{4(n+\eta+1)}\left(\frac{(n+1)(n+2\eta)}{4(n+\eta)} + \frac{(n+2)(n+2\eta+1)}{(n+\eta+2)}\right) \\ & + \left(\frac{b_3-2}{1-\dfrac{3}{2b_2}}\right) - \frac{n(n+1)(n+2\eta)(n+2\eta-1)}{16(n+\eta)^2((n+\eta)^2-1)b_{n-1}}, \end{aligned}$$

for $n\geq 3$.

With an initial value for b_0 we can compute the sequences of parameters $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ as long as the recurrence coefficients are known. A priori we do not know the recurrence coefficients $\{\widetilde{\gamma}_n\}_{n\geq 0}$, however it is possible to compute them with the desired precision through an efficient algorithm. For instance, the algorithms 1 and 4 in [47] meet this specific case, where there is a rational perturbation.

We will use the function $f(x) = e^{-100(x-0.2)^2}$. It can be seen that $f \in W_2^1[\mathbb{R}, \mu_0, \mu_1]$. On one hand, in order to see the graphic behavior of some partial sums, we choose $\eta = 5$, $\lambda = 0.001$, $a = 1$, $b = 2$. In Table 6.1 we get the first 16 Fourier-Gegenbauer-Sobolev coefficients.

n	a_n	b_n	$\eta_n(\lambda)$	s_n^λ	f_n^λ	F_n^λ
0	1.003	1	3	0.71	0.1391	0.1963
1	1.5062	1.5	3	0.051	0.0263	0.5195
2	2.0084	2	3.3	0.0059	-0.004	-0.7336
3	2.21	2.2	3.073	0.00086	-0.0036	-4.221
4	2.2084	2.1985	2.94	0.00014	-0.0001	-0.691
5	2.2199	2.21	2.03	0.000026	0.0005	20.22
6	2.228	2.2177	1.57	5.09×10^{-6}	0.0001	22.253
7	2.234	2.224	0.051	1.44×10^{-6}	-0.00007	-48.261
8	2.2387	2.2288	0.038	3.77×10^{-7}	-0.00003	-81.427
9	2.2427	2.2328	0.018	2.457×10^{-7}	-0.00005	-20.938
10	2.246	2.2361	0.019	7.243×10^{-7}	-0.00004	-48.874
11	2.2487	2.2388	0.02	3.395×10^{-7}	0.00002	62.233
12	2.251	2.2411	0.021	7.26×10^{-8}	0.00002	268.769
13	2.253	2.2431	0.0211	1.59×10^{-8}	-5.21×10^{-7}	-32.765
14	2.2547	2.2447	0.022	3.521×10^{-9}	-3.71×10^{-6}	-1053.25
15	2.256	2.2462	0.0221	7.92×10^{-10}	-5.47×10^{-7}	-690.56

Table 6.1: Fourier-Gegenbauer-Sobolev coefficients with $\eta = 5$, $\lambda = 0.001$, $a = 1$, $b = 2$.

Furthermore, in Figure 6.1 we show the partial sums for $n = 4, 7, 11, 15$ and 17.

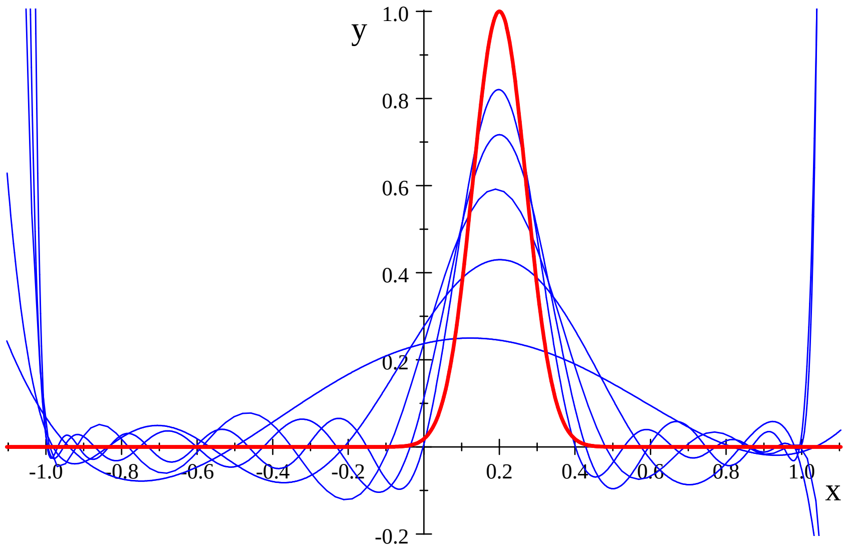

Figure 6.1: Partial sums for $n = 4$, 7, 11, 15 and 17, moreover $\eta = 1$, $\lambda = 0.5$, $a = 1$, $b = 2$. f in red.

On the other hand, in order to analyze the variation of the partial sums with respect to the parameter η. In Figure 6.2 we set $\lambda = 0.7$, $a = 2$, $b = 1$ and $n = 16$. In particular we show the partial sums for $\eta = 0.5, 1, 2$ and 2.5.

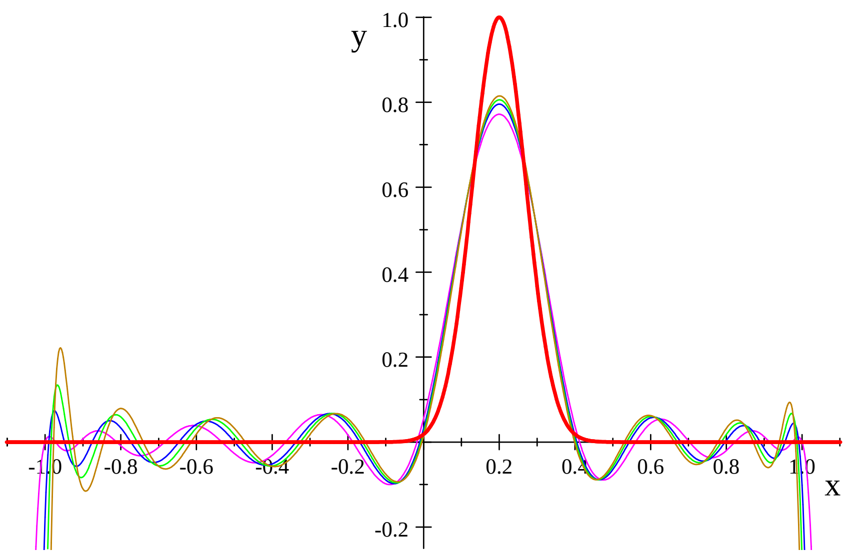

Figure 6.2: $16-th$ partial sums for $\eta = 0.5$ (magenta), 1.5 (blue), 2 (green) and 2.5 (siena), when $\lambda = 0.7$, $a = 2$, $b = 1$, f in red.

Finally, setting $\eta = 1$, $n = 16$, $a = 1$, $b = 3$, in Figure 6.4 we exhibit the partial sums for $\lambda = 0.1$, 0.8, 1.8 and 10.

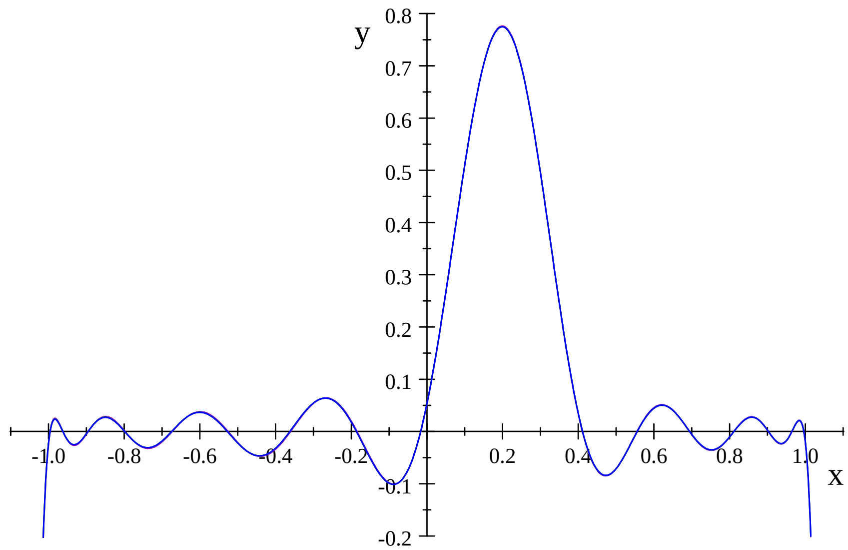

Figure 6.3: $16-th$ partial sums for $\lambda = 0.1$ (purple), 0.8 (cyan), 1.8 (green) and 10 (blue), when $\eta = 1$, $a = 1$, $b = 3$

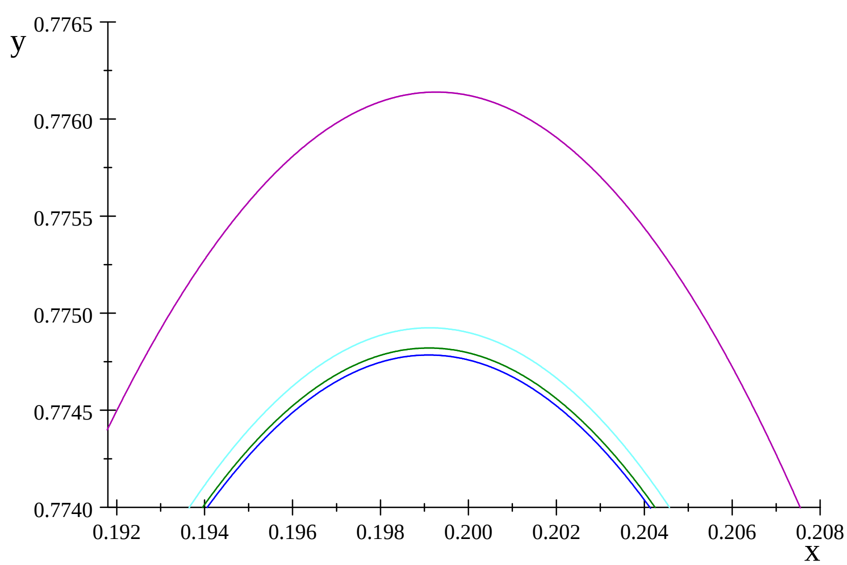

Figure 6.4: $16-th$ partial sums, (Zoom), for $\lambda = 0.1$ (purple), 0.8 (cyan), 1.8 (green) and 10 (blue), when $\eta = 1$, $a = 1$, $b = 3$

Chapter 7

Conclusions and Open Problems

Let u and v denote two symmetric quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ will be their respective SMOP. Suppose that there exist sequences of non-zero real numbers $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, with $a_n b_n \neq 0$, such that

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0, \tag{7.1}$$

holds. Then the pair $\{u, v\}$ is said to be a *Symmetric* $(1,1)-$*Coherent Pair.* This concept is the main topic of this dissertation. In this chapter we summarize the main results and present some open problems as a result of this work.

7.1 Conclusions

- It is possible to give a partial classification of symmetric $(1,1)-$coherent pairs in the positive definite sense. Indeed, in [35] is proved that there exist polynomials A, B and C with $\deg(A) = 4$, $\deg(B) \leq 5$, and $\deg(C) \leq 6$ such

  $$xC(x)u = xA(x)v, . \tag{7.2}$$

  Notice that depending on the nature of the zeros of A it is possible to refine (7.2). Namely, on one hand, if $A(x) = 2(x^2 - \xi_1^2)(x - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$ in [35] is proved that v is a semiclassical linear functional of class at most 2. In this way, through the symmetrization process described in Section 1.4, we can transform the relation 7.2 into

  $$x\phi^E(x)\widetilde{u} = x(x - \xi^2)\widetilde{v}, \tag{7.3}$$

  where u and v are the simmetrizated of $\widetilde{u}$ and $\widetilde{v}$, respectively. Then, according to the class of $\widetilde{v}$, we obtain pairs $\{\widetilde{u}, \widetilde{v}\}$ such that (7.3) is satisfied. By using Theorem 30, it is possible to obtain the corresponding pairs $\{u, v\}$, and through Collorary 80 we identify which ones are $(1,1)$-coherent. In such a framework, in Theorem 81 we present the classification of positive-definite symmetric $(1,1)-$coherent pairs. On the other hand, if $A(x) = 2\left(x^2 - \xi^2\right)^2$, again, by using symmetrization process we show that u is a semiclassical linear functional of class at most 4 and, as a consequence, the class of $\widetilde{u}$ is at most 2. Then, only when class of $\widetilde{u}$ is either 1 or 2, and by using the same techniques as in the above case, in Theorem 82, we obtain the classification of symmetric $(1,1)-$coherent pairs in the positive-definite framework.

- The symmetrization process described in Section 1.4 plays an important role in the search of symmetric $(1,1)-$coherent pairs. In this way it is possible to obtain the interesting general non-coherence relation

$$\begin{aligned} & P_{n+1}^{[i]}(x)+a_n^{[1]}P_n^{[i]}(x)+a_n^{[2]}P_{n-1}^{[i]}(x)+b_n\left(Q_{n+1}(x)+c_nQ_n(x)\right) \\ = \; & \left(1+b_n\right)R_{n+1}(x)+d_nR_n(x), \end{aligned} \tag{7.4}$$

where the sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w, respectively, with $P_k^{[i]}(x) := \dfrac{P_{k+i}^{(i)}(x)}{(k+1)_i}$, $i=0,1$, and $a_n^{[i]}b_nc_nd_n \neq 0$, $n \geq 0$. Besides, the functionals u and v are related through the rational relation

$$\rho(x)u=v, \tag{7.5}$$

where ρ is a monic polynomial of degree m. In Section 4.4, the inverse problem associated to 7.4 is studied for the cases $i=0,1$. Thus, for the case $i=0$ we prove that there exist polynomials ϕ_{m+2}, φ_1 with $\deg(\phi_{m+2})=m+2$, $\deg(\varphi_1)=1$, such that

$$\phi_{m+2}(x)w=\varphi_1(x)v=\varphi_1(x)\rho(x)u.$$

Anagously for the case $i=1$ there exist polynomials ϕ_{t+1}, $\psi_{t+m+s+2}$ with $\deg(\phi_{t+1})=t+1$ and $\deg(\psi_{t+m+s+2})=t+m+s+2$ such that

$$\phi_{t+1}(x)\sigma(x)v=\phi_{t+1}(x)\sigma(x)\rho(x)u=\psi_{t+m+s+2}(x)w. \tag{7.6}$$

So, a relation between the formal Stieltjes series associated with the functionals u and w is described in the Corollary 94. Of course, the rational relation between the functionals u and v, the standard Christoffel formula or the assumption of their semiclassical character, turn 7.4 into a coherence relation in the terms discussed in [106], however in such a case, notice that the number of terms is not optimal in order to determine the degrees of the polynomials involved in the relation between v and w.

- A particular case of a symmetric $(1,1)-$coherent pair $\{\mu_0,\mu_1\}$ is given by $\{\mu_0,\mu_1\}$, $d\mu_0 = e^{-x^2}dx$, $d\mu_1=\frac{x^2+a}{x^2+b}e^{-x^2}dx$. This pair is deduced in [34]. If we consider the inner product

$$\langle p,q\rangle_S=\int_{\mathbb{R}}p(x)q(x)d\mu_0(x)+\lambda\int_{\mathbb{R}}p'(x)q'(x)d\mu_1(x), \quad \lambda>0, \tag{7.7}$$

then we can obtain asymptotic results for the corresponding SMOP $\{S_n^\lambda\}_{n\geq 0}$. First, if $\{Q_n\}_{n\geq 0}$ is the SMOP corresponding to $d\mu_1$, and the $s-$coherence is given by the relation

$$H_n(x)+b_{n-2}H_{n-2}(x)=Q_n(x)+a_{n-2}Q_{n-2}(x), \qquad n\geq 2, \tag{7.8}$$

then, it is possible to connect the Hermite and Sobolev polynomials through

$$S_{n+3}^\lambda(x)+\eta_n(\lambda)S_{n+1}^\lambda(x)=H_{n+3}(x)+\frac{n+3}{n+1}b_nH_{n+1}(x), \qquad n\geq 0. \tag{7.9}$$

In Section 5.3 we deduce the limit behaviour of *Sobolev Coefficients* $\{\eta_n(\lambda)\}_{n\geq 0}$, namely

$$\lim_{n\to\infty}\frac{\eta_n(\lambda)}{n}=\frac{1}{2}. \tag{7.10}$$

Through this result and the well known asymptotic behavior of classical Hermite polynomials, in Section 5.4 we obtain the outer relative asymptotics

$$\lim_{n\to\infty}\frac{\sqrt{\left[\frac{n}{2}\right]}S_n^{\lambda}(x)}{2H_n(x)}=\begin{cases} -ix+\sqrt{b}, & x\in\mathbb{C}_+\\ ix+\sqrt{b}, & x\in\mathbb{C}_-,\end{cases} \tag{7.11}$$

uniformly on compact subsets of half planes $\mathbb{C}_+$ and $\mathbb{C}_-$, as well as

$$\lim_{n\to\infty}\frac{S_{n+2}^{\lambda}(\sqrt{n}x)}{H_{n+2}(\sqrt{n}x)}=1,$$

locally uniformly on compact subsets of $\mathbb{C}\backslash\left[-\sqrt{2},\sqrt{2}\right]$ and, finally, the Mehler-Heine type formulas

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n+j}}{n!}S_{2n+1}^{\lambda}\left(\frac{x}{2\sqrt{n+j}}\right)=\sqrt{\frac{b}{\pi}}\sin(x), \tag{7.12}$$

and

$$\lim_{n\to\infty}\frac{(-1)^{n-1}}{(n-1)!}S_{2n}^{\lambda}\left(\frac{x}{2\sqrt{n+j}}\right)=-\sqrt{\frac{b}{\pi}}\cos(x), \tag{7.13}$$

both locally uniformly on compact subsets of $\mathbb{C}$.

- In [55] the concepts of coherent pair and symmetric coherent pair are introduced in the Sobolev orthogonal polynomials framework. Indeed, the concept of symmetric $(1,1)-$ coherent pair is both inspired by and represents a generalization of the symmetric coherent pairs. In particular, it is implemented an algorithm that allows to evaluate efficiently expansions of functions on the *Sobolev Space*

$$W^{1,2}\left[E,\mu_0,\mu_1\right]=\left\{f:E\to\mathbb{R}|f\in L^2(E;\mu_0),\ f'\in L^2(E;\mu_1)\right\},$$

in terms of the Sobolev polynomials orthogonal with respect to (2.5), without the knowledge of the Sobolev polynomials explicitly. We show that it is possible to extend the algorithm to the symmetric (1,1)-coherent framework. Indeed, in Section 6.1 we accomplish its implementation in Algorithm 128 and Algorithm 129. In addition, we present some numerical examples by using Mathematica Software.

7.2 Open Problems

- An inverse problem arises in a natural way from (7.1). Given a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, orthogonal with respect to a certain quasi-definite linear functional, we define the sequence of polynomials $\{R_n\}_{n\geq 0}$ through (7.1), with $R_0(x)=1$ and $R_1(x)=x$. Find necessary and sufficient conditions in order to the sequence $\{R_n\}_{n\geq 0}$ is a SMOP with respect to a quasi-definite symmetric linear functional v. Notice that the problem can be also raised assuming that $\{R_n\}_{n\geq 0}$ is orthogonal with respect to v.

- In Chapter 5, asymptotics for Sobolev polynomials are studied when the particular symmetric $(1,1)-$coherent pair $\{\mu_0, \mu_1\}$, $d\mu_0 = e^{-x^2}dx$, $d\mu_1 = \frac{x^2+a}{x^2+b}e^{-x^2}dx$, is considered. Find asymptotics for other cases, when u is not classical, is an interesting problem. Other symmetric $(1,1)-$coherent pairs are obtained in Section 4.3 when u and v are positive definite.

- Taking into account the generalization of a coherent pair given in [59] through the so called $(M,N)-$ coherent pair of order (m,n), to extend the concept of symmetric $(1,1)-$coherent pair. In that sense, for instance, consider, with the natural assumptions, the algebraic relation

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-2i}^{[m]}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-2i}^{[k]}(x),$$

where $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are orthogonal with respect to symmetric quasi-definite linear functionals u and v, and to solve the associated inverse problem.

- In Chapter 4 we deal with the inverse problem associated with the non coherence relation

$$\begin{aligned} & P_{n+1}^{[i]}(x) + a_n^{[1]} P_n^{[i]}(x) + a_n^{[2]} P_{n-1}^{[i]}(x) + b_n \left(Q_{n+1}(x) + c_n Q_n(x)\right) \\ = \; & \left(1+b_n\right) R_{n+1}(x) + d_n R_n(x), \end{aligned} \tag{7.14}$$

where the sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w, respectively, and $a_n^{[i]} b_n c_n d_n \neq 0$, $n \geq 0$. Besides, the functionals u and v are related through the rational relation $\rho(x)u = v$, where ρ is a monic polynomial of degree m. It will be interesting to consider this kind of non-coherent relation from a more general perspective, for instance, consider the relation

$$\begin{aligned} & P_n(x) + \sum_{i=1}^{\nu} a_{n,i} P_{n-i}(x) + b_n \left(Q_n(x) + \sum_{i=1}^{\nu-1} c_{n,i} Q_{n-i}(x) \right) \\ = \; & (1+b_n) R_n(x) + \sum_{i=1}^{\nu} c_{n,i} R_{n-i}(x), \end{aligned}$$

under similar conditions associated with (7.14) Of course, with the assumptions of the rational relation between functionals u and v, besides their semiclassical character, through the corresponding structure relations it is possible to arrive to coherence traditional. It seems a hard and interesting challenge to consider the inverse problem associated to the non-coherent relation 7.14 without such assumptions.

- Study properties of the zeros of the Sobolev polynomials $\left\{S_n^{\lambda}\right\}_{n\geq 0}$. In particular, their location with respect to the zeros of the polynomials orthogonal with respect to μ_0 and the distance between consecutive zeros, as well as the asymptotic behavior of the respective zero counting measure.

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