Sobre los problemas de espectro de cofinalidad y $\mathfrak{p} = \mathfrak{t}$

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Yo hago lo imposible, pues lo posible lo hace cualquiera.

Pablo Picasso

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Resumen

La noción de problema de espectro de cofinalidad fue introducida en 2016 por Malliaris y Shelah en [11]. Esta noción permite conectar y dar respuesta a dos antiguos problemas abiertos en dos áreas totalmente distintas: el problema en Teoría de Modelos de determinar la maximalidad de SOP_2 en el orden de Keisler y el problema en Topología Conjuntista de determinar si los cardinales invariantes del continuo $\mathfrak{p}$ y $\mathfrak{t}$ son iguales.

En el presente trabajo hacemos un análisis detallado de la noción de problema de espectro de cofinalidad y su conexión con el problema de $\mathfrak{p} = \mathfrak{t}$. Además, estudiamos algunas aplicaciones topológicas de $\mathfrak{p} = \mathfrak{t}$ y damos respuesta a una pregunta abierta hecha por Todorčević y Veličković en [20] sobre la exisitencia de un conjunto parcialmente ordenado de tamaño $\mathfrak{p}$ sin precalibre $\mathfrak{p}$ como una consecuencia directa de $\mathfrak{p} = \mathfrak{t}$.

Palabras clave: problema de espectro de cofinalidad, número de pseudo-intersección, número de torre, orden de Keisler.

Abstract

The notion of cofinality spectrum problem was introduced by Malliaris and Shelah in [11]. This notion allows to connect and solve two longstanding open problems in quite different areas: the model-theoretic question of determining the maximality of SOP_2-theories in Keisler's order and the set-theoretic Topology problem of determining whether the cardinal invariants of the continuum $\mathfrak{p}$ y $\mathfrak{t}$ are the same.

In the present dissertation we do a detailed analysis of the notion of cofinality spectrum problem and its connection with the problem $\mathfrak{p} = \mathfrak{t}$. Also, we study some topological applications of $\mathfrak{p} = \mathfrak{t}$ and we answer an open question asked by Todorčević y Veličković in [20] about the existence of a poset of size $\mathfrak{p}$ without precaliber $\mathfrak{p}$ as a direct consequence of $\mathfrak{p} = \mathfrak{t}$.

Keywords: cofinality spectrum problem, pseudo-intersection number, tower number, Keisler's order.

Contents

Symbol list

Symbol	**Meaning**
$\mathcal{P}(X)$	the set of all the subsets of X
$\mathcal{P}(\omega)/\mathrm{fin}$	the set of infinite subsets of $\mathbb{N}$
$[\kappa]^{\lambda}$	the set of all the subsets of κ of cardinality λ
$[\kappa]^{<\aleph_0}$	the set of all the finite subsets of κ
ω^{ω}	the set of sequences of natural numbers
$\omega^{<\omega}$	the set of finite sequences of natural numbers
X^{ω}	the set of sequences of elements of a set X
$X^{<\omega}$	the set of finite sequences of elements of a set X
$\dot{a}$	the name of a set $a \in V[G]$
$\check{a}$	the canonical name of a set $a \in V$
$M[G]$	the generic extension of a transitive model M of ZFC
$\mathcal{H}(\aleph_1)$	the class of all sets with countable transitive closure
$\Vdash$	forcing relation
$\vDash$	satisfaction relation
$\equiv$	elementary equivalence
$\mathcal{M} \prec \mathcal{N}$	$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$
MA	Martin's Axiom
$\trianglelefteq$	Keisler's order
$\mathrm{Th}\,(\mathcal{M})$	the set of all sentences true in $\mathcal{M}$
$\mathcal{C}(\mathcal{D})$	the cut spectrum of an ultrafilter
$[(a_i)_{i\in I}]_{\mathcal{D}}$, $a/\mathcal{D}$	the class of equivalence of an element a modulo $\mathcal{D}$

Introduction

In this thesis we study the notion of *cofinality spectrum problem*, introduced in 2016 by Maryanthe Mallaris and Saharon Shelah in [11]. These cofinality spectrum problems allows to connect and solve two problems in different areas: the set-theoretic topology problem of determine whether $\mathfrak{p} = \mathfrak{t}$ and the model-theoretic problem of maximality of SOP_2-theories in Keisler's order. Let us first describe both problems.

A *cardinal invariant of the continuum* is a cardinal which describes a particular property of the real line, either a topological or a combinatorial property, among others. Usually, these cardinals lie between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$, and there are several work done determining which relations between them are provable in ZFC. Also, configurations such as *Cichoń's diagram* or *van Douwen's diagram* show us some known relations between them (see [15, p. 199], among others).

Two of these cardinal invariants are $\mathfrak{p}$ (known as *the pseudo-intersection number*) and $\mathfrak{t}$ (known as *the tower number*), which capture interesting combinatorial properties of the set of infinite subsets of $\mathbb{N}$. By definition, it is straightforward to see that $\mathfrak{p} \leq \mathfrak{t}$; however, the problem to determine whether it was consistent that $\mathfrak{p} < \mathfrak{t}$ remained open for a long time. This problem was solved by Malliaris and Shelah in [11] where they proved that $\mathfrak{p} = \mathfrak{t}$ in ZFC.

The second problem we describe here is the problem to determine a criterion for maximality in Keisler's order on countable complete theories. Keisler's order (proposed by Keisler [9]) is a preorder which uses the relative difficulty of producing saturated regular ultrapowers to compare the complexity of any pair of countable complete theories. Keisler [9] showed that this order has a maximum class and there is a family of ultrafilters that saturates any theory.

The structure of the order on stable theories was first studied by Shelah [16]. Although the structure on unstable theories still remains unknown, Shelah showed that any theory which codifies a linear ordering, or more precisely that satisfies the strict order property, belongs to the maximum class of Keisler's order. Shelah [17] also proved that theories with a property denoted as SOP_3 and which retains many features of linear order, are in the maximum class.

It is surprising that both $\mathfrak{p} = \mathfrak{t}$ and maximality in in Keisler's order are connected by this idea of cofinality spectrum problems. Informally, we could think of a cofinality spectrum problem as a set of orders which capture many of the behaviour of ω as an ordering, and for each one of these orderings we have a set of trees which captures many of the features of the tree of finite functions of natural numbers. In particular, the interesting task is to determine how to translate the realization (or not) of gaps in certain orderings in the search of upper bounds for increasing sequences in certain trees. Therefore, one can think of a cofinality spectrum problem as the precise framework where we can study deep connections between orderings and trees.

The first chapter is dedicated to some preliminaries in set theory and model theory: the first three sections of this chapter are dedicated to review the main concepts about filters, forcing and Martin's Axiom, and we introduce the cardinals $\mathfrak{p}$ and $\mathfrak{t}$. In these three sections, we analyze several results about the forcing of infinite subsets of $\mathbb{N}$ (denoted by $\mathcal{P}(\omega)/\text{fin}$) which will be central in chapter 4. The two remaining sections of this chapter correspond to a brief review of model theory, focusing our attention on the construction of saturated ultraproducts. We dedicate the last section of this chapter to study the main properties of *regular ultrafilters*, a special type of ultrafilters that allow to perform saturation of ultraproducts.

In chapter 2 we study the notion of *peculiar gap*, introduced by Shelah [18]. A peculiar gap is a special kind of gap in ω^ω, and it is possible to ensure the existence of peculiar gaps under the assumption that $\mathfrak{p} < \mathfrak{t}$. Also, we introduce the notion of cofinality spectrum problem emphasizing in three main aspects: in Section 2.2, we study the existence or not of certain special kind of gaps into distinguished orders; in Section 2.3 we analyze the model-theoretic aspect of a cofinality spectrum problem, studying the notion of Or-type and the possibility or not of realizing Or-types in a cofinality spectrum problem, called Or-saturation; and also, we analyze the notion of Gödel codification, building a non-standard arithmetic in any cofinality spectrum problem. Finally, in Section 2.4 we connect these three aspects in the analysis of the main [11, Thm. 8.1], which allows to rule out assymetric gaps. We slightly simplify this proof, specially by omitting the notion of *internal cardinality*.

In chapter 3, we study the *Keisler's order* and a characterization of the maximum class of this order. To do this, we study a special kind of ultrafilters, called *good ultrafilters*. These kind of ultrafilters allows us to transfer the saturation of ultraproducts to any infinite uncountable cardinal. Then, we focus on a special cofinality spectrum problem where we can characterize the maximum class in Keisler's order by good ultrafilters.

In chapter 4 we study the proof of $\mathfrak{p} = \mathfrak{t}$, by analysing a convenient cofinality spectrum problem. Then we present some applications of this result, including an open question asked by Todorčević and Veličković [20] about the existence of a poset of size $\mathfrak{p}$ without precaliber $\mathfrak{p}$. In this dissertation, we give an answer to this open question in Theorem 4.2.7.

1 Preliminaries

This chapter is dedicated to introduce the main basic concepts used along this work. In the first two sections, we give a review about set theory, focusing on *forcing theory*. This technique, developed by Paul Cohen in the early 1960s, is a powerful tool that is mainly used to construct a large number of models of set theory and to prove consistency results. We focus our attention on a particular forcing notion, which will be central in our work: The forcing of infinite subsets of $\mathbb{N}$ with the partial order of *almost inclusion* (usually denoted by $\mathcal{P}(\omega)/\text{fin}$). Also, we define the cardinals $\mathfrak{p}$ and $\mathfrak{t}$. The study of the equality $\mathfrak{p} = \mathfrak{t}$ is the center of this dissertation.

We dedicate section 1.3 to study the *Martin's Axiom*, which is an interesting combinatorial principle with a great number of interesting applications.

In section 1.4, we give a little review of the most basic notions and results in model theory. Many of the results and constructions presented here are classical (e.g. construction of ultraproducts, Łoś's theorem, compactness theorem, Tarski-Vaught test, etc.), and they will be used frequently in this work.

Finally, section 1.5 is dedicated to the construction and analysis of main properties of *regular ultrafilters*, central concept used in chapter 3.

1.1. Basic notions in set theory

This section is dedicated to introduce the basic notions in set theory used along this dissertation. For this section, we follow [3, 7].

1.1.1. Filters and ultrafilters

We begin introducing the concept of *filter*.

Definition 1.1.1 *Let X be a non-empty set and $\mathcal{F} \subseteq \mathcal{P}(X)$. We say that $\mathcal{F}$ is a* ***filter over*** *X if:*

1. $X \in \mathcal{F}$.

2. *If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*

3. *If $A \in \mathcal{F}$ and $A \subseteq C \subseteq X$, then $C \in \mathcal{F}$.*

Example 1.1.2 Let X be a non-empty set.

1. Both $\{X\}$ and $\mathcal{P}(X)$ are filters over X, called the *trivial filter* and the *improper filter*, respectively. A filter $\mathcal{F}$ is called **proper** if it is different from the improper filter.

2. Let $A \subseteq X$. The set $\mathcal{F} := \{Y \subseteq X : A \subseteq Y\}$ is a filter over X, called the *principal filter generated by* A, and denoted by $\langle A \rangle$. This filter is proper if and only if $A \neq \emptyset$. We say that a filter $\mathcal{F}$ is *non-principal* if and only if there is no $A \subseteq X$ such that $\mathcal{F} = \langle A \rangle$.

3. If X is infinite, then the set $\mathrm{Fr}_X := \{A \subseteq X : |X \setminus A| < \aleph_0\}$ is a filter over X, called the *Frèchet filter*. This filter is non-principal.

4. A proper filter $\mathcal{F}$ over X is called **free** if $\bigcap \mathcal{F} = \emptyset$. If X is infinite, we have that $\mathcal{F}$ is free if and only if $\mathcal{F}$ contains the Frèchet filter. Moreover, if $\mathcal{F}$ is a free proper filter, then $\mathcal{F}$ is non-principal.

There is an easy way of building filters from subsets of X: given $G \subseteq \mathcal{P}(X)$, we consider the intersection of all filters over X which contains G, i.e., $\langle G \rangle := \bigcap\{\mathcal{F} : G \subseteq \mathcal{F}$ and $\mathcal{F}$ is a filter$\}$. This is called the *filter generated by* G.

Definition 1.1.3 *Let X be a non-empty set and $F \subseteq \mathcal{P}(X)$. We say that F has the* ***finite intersection property*** *if and only if the intersection of any finite number of elements of F is non-empty.*

Fact 1.1.4 ([3, Prop. 4.1.1]) *Let $G \subseteq \mathcal{P}(X)$ and let $\langle G \rangle$ be the filter generated by G. Then:*

(i) $\langle G \rangle$ is a filter over X.

(ii) $\langle G \rangle$ is the set of all $A \subseteq X$ such that either $A = X$ or, for some $B_1, ..., B_n \in G$, $B_1 \cap \cdots \cap B_n \subseteq A$.

(iii) $\langle G \rangle$ is a proper filter if and only if G has the finite intersection property.

Notice that filters over a set can be ordered by inclusion, so we may wonder for the maximal proper filters.

Definition 1.1.5 *Let X be a non-empty set and $\mathcal{U}$ a filter over X. We say that $\mathcal{U}$ is an* ***ultrafilter over*** *X if and only if $\mathcal{U}$ is a maximal proper filter.*

Ultrafilter can be characterized in many ways, but we use the following one.

Fact 1.1.6 *A proper filter $\mathcal{U}$ over X is an ultrafilter if and only if for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \smallsetminus A \in \mathcal{U}$.*

It is not hard to show that any filter generated by a single point $x \in X$ is a principal ultrafilter over X; moreover, an ultrafilter $\mathcal{U}$ is principal if and only if $\mathcal{U} = \langle\{x\}\rangle$, for some $x \in X$. But there are also non-principal ultrafilters: indeed, any ultrafilter which extends the Frèchet filter is non-principal when X is infinite. The next result allows us to claim that any filter can be extended to an ultrafilter. The following fact presents several properties about ultrafilters.

Fact 1.1.7 ([3, Cor. 4.1.4]) *Let X be a non-empty set and $\mathcal{U}$ be an ultrafilter over X.*

1. *$\mathcal{U}$ is not free if and only if $\mathcal{U}$ is principal.*
2. *(Ultrafilter lemma) Any proper filter $\mathcal{F}$ over X can be extended to an ultrafilter over X.*
3. *If X is infinite, then there is a non-principal ultrafilter $\mathcal{U}$ over X.*

We will focus our attention on non-principal ultrafilters along this dissertation.

1.1.2. Cardinals $\mathfrak{p}$ and $\mathfrak{t}$

Now we define the cardinals $\mathfrak{p}$ and $\mathfrak{t}$. The reader can see [21, 7] for further properties about these cardinals. First, we define a relation between subsets of $\mathbb{N}$, called *almost contention*. From now, the set of all infinite subsets of $\mathbb{N}$ will be denoted by $\mathcal{P}(\omega)/\mathrm{fin}$, and the set of all sequences of natural numbers will be denoted by ω^ω.

Definition 1.1.8 *Let $X, Y \in \mathcal{P}(\mathbb{N})$ and let $f, g \in \omega^\omega$.*

1. *We say that is X is **almost contained** in Y if $|X \smallsetminus Y| < \aleph_0$. We denote this as $X \subseteq^* Y$.*
2. *We say that X is **almost equal** to Y if $X \subseteq^* Y$ and $Y \subseteq^* X$. We denote this as $X =^* Y$.*
3. *We say that g **eventually dominates** f if $|\{n \in \mathbb{N} : g(n) < f(n)\}| < \aleph_0$. We denote this by $f \leq^* g$.*

Note that every finite subset of $\mathbb{N}$ is almost contained in $\emptyset$ (hence, every finite subset of $\mathbb{N}$ is almost equal to $\emptyset$), and every cofinite subset of $\mathbb{N}$ is almost contained in $\mathbb{N}$ (so, every cofinite subset of $\mathbb{N}$ is almost equal to $\mathbb{N}$).

Definition 1.1.9 *Let $\mathcal{F} \subseteq \mathcal{P}(\omega)/\mathrm{fin}$ and $\mathcal{B} \subseteq \omega^\omega$.*

1. *We say that* $P \in \mathcal{P}(\omega)/\mathrm{fin}$ *is a **pseudo-intersection of*** $\mathcal{F}$ *if* $P \subseteq^* F$*, for every* $F \in \mathcal{F}$.

2. *We say that* $\mathcal{F}$ *has the **strong finite intersection property*** (*abbreviated **SFIP***) *if for every* $F_1, F_2, ..., F_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} F_i \in \mathcal{P}(\omega)/\mathrm{fin}$.

3. *We say that* $\mathcal{F}$ *is a **tower** if* $(\mathcal{F}, \supseteq^*)$ *is a well-ordered set with no pseudo-intersection.*

4. *We say that* $\mathcal{B}$ *is **bounded in*** ω^ω *if there is a* $g \in \omega^\omega$ *such that* $f \leq^* g$ *for all* $f \in \mathcal{B}$. *Otherwise, we say that* $\mathcal{B}$ *is **unbounded in*** ω^ω.

Observation 1.1.10 Notice that if a family $\mathcal{F}$ has pseudo-intersection, then $\mathcal{F}$ has SFIP, but not vice versa: any free filter $\mathcal{F} \subseteq \mathcal{P}(\omega)/\mathrm{fin}$ has SFIP, but no non-principal ultrafilter over $\mathbb{N}$ has pseudo-intersection. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and let A be a pseudo-intersection of $\mathcal{U}$, then $A \in \mathcal{U}$. Let $A = \{a_n \,|\, n \in \mathbb{N}\}$ be an enumeration, and consider $B = \{a_{2n} \,|\, n \in \mathbb{N}\}$ and $C = \{a_{2n+1} \,|\, n \in \mathbb{N}\}$. We can see that $A = B \cup C \in \mathcal{U}$, so $B \in \mathcal{U}$ or $C \in \mathcal{U}$, but neither $A \nsubseteq^* B$ nor $A \nsubseteq^* C$ holds, which is absurd.

Thanks to observation 1.1.10, we can give the following definitions.

Definition 1.1.11 1. *The **pseudo-intersection number*** $\mathfrak{p}$ *is the smallest cardinality of a family* $\mathcal{F} \subseteq \mathcal{P}(\omega)/\mathrm{fin}$ *with the SFIP but which does not have a pseudo-intersection; more formally*

$$\mathfrak{p} := \min\{|\mathcal{F}| \,:\, \mathcal{F} \subseteq \mathcal{P}(\omega)/\mathrm{fin} \text{ has the SFIP but no pseudo-intersection}\}. \tag{1-1}$$

2. *The **tower number*** $\mathfrak{t}$ *is the smallest cardinality of a family* $\mathcal{T} \subseteq \mathcal{P}(\omega)/\mathrm{fin}$ *which is a tower, i.e.*

$$\mathfrak{t} := \min\{|\mathcal{T}| \,:\, \mathcal{T} \subseteq \mathcal{P}(\omega)/\mathrm{fin} \text{ is a tower}\}. \tag{1-2}$$

3. *The **unbounding number*** $\mathfrak{b}$ *is the smallest cardinality of an unbounded family* $\mathcal{B} \subseteq \omega^\omega$*, i.e.*

$$\mathfrak{b} := \min\{|\mathcal{B}| \,:\, \mathcal{B} \subseteq \omega^\omega \text{ is an unbounding family}\}. \tag{1-3}$$

We mention some useful facts about these two cardinals.

Proposition 1.1.12 ([21, Thm. 3.1]) 1. $\mathfrak{p}$ *and* $\mathfrak{t}$ *are regular.*

2. $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{c}$.

Proof.

1. Regularity of $\mathfrak{t}$ is immediate from definition; regularity of $\mathfrak{p}$ is due to Szymański (see theorem 3.1(e) [21, p. 116]).

2. For $\aleph_1 \leq \mathfrak{p}$, let $\mathcal{E} = \{X_n \mid X_n \in \mathcal{P}(\omega)/\text{fin}, n \in \omega\}$ with SFIP. Build $K = \{k_n \mid n \in \omega\}$ such that $k_n \in \left(\bigcap_{i<n} X_n\right) \smallsetminus \{k_i \mid i < n\}$. It is clear that $K \subseteq^* X_n$ for all $n \in \omega$, hence K is a pseudo-intersection of $\mathcal{E}$.

   For $\mathfrak{p} \leq \mathfrak{t}$, it is enough to proof that every tower has SFIP. Let $\mathcal{T} \subseteq \mathcal{P}(\omega)/\text{fin}$ be a tower and $T_1, ..., T_n \in \mathcal{T}$. Without loss of generality, we can assume that $T_1 \supseteq^* ... \supseteq^* T_n$, then $T_n \subseteq^* \bigcap_{i=1}^n T_i$, so $\bigcap_{i=1}^n T_n \in \mathcal{P}(\omega)/\text{fin}$, which is what we wanted to show.

   For $\mathfrak{t} \leq \mathfrak{b}$, see theorem 3.1(a) [21, p. 120].

This concludes the proof. ■

We will revisit these cardinals when we study the forcing of infinite subsets of $\mathbb{N}$ in chapter 4.

1.2. Forcing

In this section, we review the forcing theory. Broadly speaking, the forcing method could be described as follows: given a transitive model M (called the *ground model*) of set theory, we extend this model by adjoining a new set G (called a *generic set*) in order to obtain a larger transitive model of set theory $M[G]$ (called a *generic extension*) without increasing the set of ordinals in the model. The generic set is approximated by forcing conditions in the ground model, and a reasonable choice of forcing conditions determines what is true in the generic extension. We recall that a pair $(P, \leq)$ is called a *pre-ordered set* if P is a non-empty set and $\leq$ is a reflexive and transitive relation. We follow [6, 8, 10] for this section.

Definition 1.2.1 *Let $(\mathbb{P}, \leq, 1)$ be a pre-ordered set where $1 \in \mathbb{P}$ is its maximum element, usually abbreviated as $\mathbb{P}$. We call $\mathbb{P}$ a* ***forcing notion****, and the elements of $\mathbb{P}$ are called* ***forcing conditions****.*

If M is a transitive model of ZFC, we have that $\mathbb{P} \in M$ implies $1 \in M$; it is usual to assume that $\leq \in M$.

Definition 1.2.2 *Let $\mathbb{P}$ be a forcing notion and $p, q \in \mathbb{P}$.*

1. *We say that p is* ***stronger than*** *q if $p \leq q$.*

2. *We say that p is* ***compatible with*** *q if there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. We denoted this by $p \not\perp q$. Otherwise, they are* ***incompatible*** *(denoted by $p \perp q$).*

3. *A set $A \subseteq \mathbb{P}$ is called an* ***antichain*** *if its elements are pairwise incompatible. We say that $\mathbb{P}$ has the* ***countable chain condition*** *(abbreviated as* **ccc***) if every antichain of $\mathbb{P}$ is at most countable.*

Example 1.2.3 (Cohen forcing or forcing with finite partial functions) Let I, J be non-empty sets. We define the forcing notion $\mathrm{Fn}(I, J)$ whose conditions are finite partial functions from I to J with finite domain. We establish that a condition f is stronger than g if f extends g (as a function). Notice that a condition f is incompatible with g if and only if $f \cup g$ is not a function. When $I = \omega$ and $J = 2$, the forcing $\mathrm{Fn}(\omega, 2)$ is called the **Cohen forcing**.

Definition 1.2.4 *Let $\mathbb{P}$ be a forcing notion and $G \subseteq \mathbb{P}$.*

1. *We say that G is* ***open*** *if whenever $p \in G$ and $q \leq p$, $q \in G$.*
2. *We say that G is* ***dense in*** *$\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in G$ such that $q \leq p$.*
3. *We say that G is* ***dense below*** *$p \in \mathbb{P}$ if for every $q \leq p$, there is an $r \leq q$ such that $r \in G$.*
4. *We say that G is a* ***filter in*** *$\mathbb{P}$ if*
   (i) $G \neq \emptyset$.
   (ii) If $p \leq q$ and $p \in G$, then $q \in G$.
   (iii) If $p, q \in G$, there is a $r \in G$ such that $r \leq p$ and $r \leq q$.

From definition, it is clear that any dense subset of a forcing is dense below p for all $p \in \mathbb{P}$. Now we define the notion of generic filter over a transitive model of ZFC.

Definition 1.2.5 *Let M be a set, let $\mathbb{P}$ be a forcing notion and let $G \subseteq \mathbb{P}$. We say that G is* ***generic over*** *M (or $\mathbb{P}$-**generic over** M) if*

(i) G is a filter in $\mathbb{P}$.

(ii) If $D \subseteq \mathbb{P}$ is dense and $D \in M$, then $G \cap D \neq \emptyset$.

Notice that the genericity of a set does not depend on the ground model, but depends on the dense subsets of $\mathbb{P}$ which are in M.

In general, a generic set over a transitive model need not exist. However, these generic sets always exist when the ground model M is countable since there would be just countably many dense subsets in M.

Lemma 1.2.6 (Generic filter existence lemma, [10, Lemma IV.2.3]) *If $\mathbb{P}$ is a partially ordered set and $\mathcal{D}$ is a countable collection of dense subsets of $\mathbb{P}$, then there exists a $\mathbb{P}$-generic filter over $\mathcal{D}$. In fact, for every $p \in \mathbb{P}$ there exists a $\mathbb{P}$-generic filter G over $\mathcal{D}$ such that $p \in G$.*

Proof. Let $\mathcal{D} = \{D_n : n \in \omega\}$ be a countable collection of dense subsets of $\mathbb{P}$, with $D_0 = \mathbb{P}$. Let $p_0 = p$, and for each n, choose p_{n+1} such that $p_{n+1} \leq p_n$ and $p_n \in D_n$. Then the set

$$G := \{q \in \mathbb{P} : q \geq p_n \text{ for some } n \in \omega\}$$

is a $\mathbb{P}$-generic filter over $\mathcal{D}$ and $p \in G$. ■

Note that, generally, if M is a countable transitive model of ZFC, then $G \notin M$. So, we can build a new transitive model of ZFC which contains both M and G. This new model will contain all the things we are able to construct from G and elements of M. The elements of this new model will have a *name*, which describes how they were built, and this name will live in M. By recursion, we can define the *value* for a name, and the set of all values for names in M will be called $M[G]$.

Fact 1.2.7 (Generic Model Theorem, [8, Thm. 14.5]) *Let M be a transitive model of* ZFC *and $\mathbb{P}$ be a notion of forcing in M. If $G \subseteq P$ is $\mathbb{P}$-generic over M, then there exists a transitive model $M[G]$ such that:*

1. *$M[G]$ is a model of* ZFC.
2. *$M \subseteq M[G]$ and $G \in M[G]$.*
3. *$ON^{M[G]} = ON^M$, where $ON^M = ON \cap M$ denotes the ordinals of the model M.*
4. *If N is a transitive model of* ZFC *such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.*

Now, if we would want to know if the ground model M is a subclass of $M[G]$, and it is canonically defined, we need to appeal to the canonical names. These canonical names are defined by $\in$-induction as $\check{x} := \{(\check{y}, 1) : y \in x\}$. This tells us that 1 will force that $\check{y} \in \check{x}$ whenever $y \in x$ (in M), so every condition will have to force that as well. Moreover, $\dot{G} := \{(\check{p}, p) : p \in \mathbb{P}\}$ is a name for the filter G in M (even when M does not know about the existence of G).

The next concepts will be crucial in chapter 4, when we analyze the proof of $\mathfrak{p} = \mathfrak{t}$.

Definition 1.2.8 ([8, Def. 15.5]) *Let $\mathbb{P}$ a forcing notion and κ an infinite cardinal. We say that $\mathbb{P}$ is **κ-distributive** if the intersection of κ-many open dense sets is open dense. We say that $\mathbb{P}$ is **$< \kappa$-distributive** if it is λ-distributive for all $\lambda < \kappa$.*

Notice that, in the previous definition, we can work with open dense sets below some condition p when we want to use κ-distributivity of a forcing. Also, forcing notions with κ-distributivity have the convenient property of adding no new sequences of length $\leq \kappa$ in generic extensions.

Theorem 1.2.9 ([8, Thm. 15.6]) *Let κ be an infinite cardinal, let M a transitive model of ZFC, let G a $\mathbb{P}$-generic filter over M and assume that $\mathbb{P}$ is κ-distributive. If $f \in M[G]$ is a function from κ into M, then $f \in M$. In particular, κ has no new subsets in $M[G]$.*

Proof. Let $f : \kappa \to M$, $f \in M[G]$ and let $\dot{f}$ be a name for f. Then there is some $A \in M$ and a condition $p_0 \in G$ such that, in M,

$$p_0 \Vdash_{\mathbb{P}} \text{“}\dot{f} \text{ is a function from } \check{\kappa} \text{ into } \check{A}\text{”}.$$

Working in M, for each $\alpha < \kappa$, the set

$$D_\alpha = \{p \leq p_0 : (\exists x \in A)\, p \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{x}\}.$$

is open dense below p_0. Hence, $D = \bigcap_{\alpha<\kappa} D_\alpha$ is dense below p_0 and, in $M[G]$, there is some $p \in D \cap G$. Back in M, for each $\alpha < \kappa$ there is some x_α such that $p \Vdash f(\check{\alpha}) = \check{x_\alpha}$. Define $g : \kappa \to A$ by $g(\alpha) = x_\alpha$. Notice that, in $M[G]$, $f(\alpha) = x_\alpha = g(\alpha)$ for every $\alpha < \kappa$, therefore $f \in M$. ■

Definition 1.2.10 ([8, Def. 15.7]) *Let $\mathbb{P}$ be a forcing notion and let κ an infinite cardinal. We say that $\mathbb{P}$ is **κ-closed** if for every $\lambda \leq \kappa$, every descending sequence $p_0 \geq p_1 \geq ... \geq p_\alpha \geq ...$, with $\alpha < \lambda$, has a lower bound in $\mathbb{P}$. We say that $\mathbb{P}$ is **$< \kappa$-closed** if it is λ-closed for all $\lambda < \kappa$.*

Usually, verifying closure in a forcing notion is easier than verifying distributivity.

Lemma 1.2.11 ([8, Lemma 15.8]) *Let $\mathbb{P}$ be a forcing notion and let κ an infinite cardinal. If $\mathbb{P}$ is κ-closed, then it is κ-distributive.*

Proof. Let $\{D_\alpha : \alpha < \kappa\}$ be a collection of open dense sets. Clearly, the intersection $D = \bigcap_{\alpha<\kappa} D_\alpha$ is open, so we need only to show that D is dense. For this purpose, let $p \in \mathbb{P}$ be arbitrary. By induction on $\alpha < \kappa$, we construct a descending κ-sequence of conditions $p \leq p_0 \leq p_1 \leq ....$ set p_α as a condition such that $p, p_\zeta \geq p_\alpha$, for all $\zeta < \alpha$ and $p_\alpha \in D_\alpha$. Finally, let q be a condition such that $p_\alpha \geq q$, for all $\alpha < \kappa$. Notice that $q \in D$. ■

Remark 1.2.12 Consider the forcing $\mathbb{P} = \mathcal{P}(\omega)/\text{fin}$, ordered by the relation $\subseteq^*$. From definition of $\mathfrak{t}$, we see that this forcing is $< \mathfrak{t}$-closed, and hence $\mathcal{P}(\omega)/\text{fin}$ is $< \mathfrak{t}$-distributive. Therefore, if M is a countable transitive model of ZFC and G is a generic filter over M, then this generic filter will add no new sequences of length $< \mathfrak{t}^M$ and it will preserve cofinalities and cardinals up to and including $\mathfrak{t}^M$ (we refer the reader interested in these notions to Kunen [10, pp. 263,264]). Moreover, we can ensure that $\mathcal{P}(\mathbb{N})^{M[G]} = \mathcal{P}(\mathbb{N})^M$ and $\mathcal{H}(\aleph_1)^{M[G]} = \mathcal{H}(\aleph_1)^M$.

Lemma 1.2.13 ([6, Prop. 4B]) $\Vdash_{\mathbb{P}} \mathfrak{t} = \check{\mathfrak{t}}$ *and* $\Vdash_{\mathbb{P}} \mathfrak{p} = \check{\mathfrak{p}}$.

Proof. To prove $\Vdash_{\mathbb{P}} \mathfrak{t} = \check{\mathfrak{t}}$, let $(a_\xi)_{\xi<\mathfrak{t}}$ be a $\subseteq^*$-decreasing family in $\mathcal{P}(\omega)/\text{fin}$ with no lower bound in $\mathcal{P}(\omega)/\text{fin}$. Then

$\Vdash_{\mathbb{P}} (\check{a}_\xi)_{\xi<\check{\mathfrak{t}}}$ is a $\subseteq^*$ -decreasing family in $\mathcal{P}(\omega)/\text{fin}$.

Since $\Vdash_{\mathbb{P}} \mathcal{P}(\mathbb{N}) = \mathcal{P}\check{(\mathbb{N})}$, $\Vdash_{\mathbb{P}} \{\check{a}_\xi : \xi < \check{\mathfrak{t}}\}$ has no lower bound in $\mathcal{P}(\omega)/\text{fin}$, so $\Vdash_{\mathbb{P}} \mathfrak{t} \leq \check{\mathfrak{t}}$. Suppose that $\kappa < \mathfrak{t}$ and let $p \in \mathbb{P}$ and let $(\dot{a}_\xi)_{\xi<\kappa}$ be a family of $\mathbb{P}$-names such that

$p \Vdash (\dot{a}_\xi)_{\xi<\kappa}$ is a $\subseteq^*$ -decreasing family in $\mathcal{P}(\omega)/\text{fin}$.

Since $\mathbb{P}$ is $< \mathfrak{t}$-closed, there are a $q \leq p$ and a family $(a_\xi)_{\xi<\kappa}$ in $\mathcal{P}(\mathbb{N})$ such that $q \Vdash_{\mathbb{P}} \dot{a}_\xi = \check{a}_\xi$ for every $\xi < \kappa$. Now, it is clear that

$q \Vdash (\check{a}_\xi)_{\xi<\check{\kappa}}$ is a $\subseteq^*$ -decreasing family in $\mathcal{P}(\omega)/\text{fin}$,

then $(a_\xi)_{\xi<\kappa}$ is a $\subseteq^*$-decreasing family in $\mathcal{P}(\omega)/\text{fin}$, and as $\kappa < \mathfrak{t}$, there is a lower bound a of $\{a_\xi : \xi < \kappa\}$ in $\mathcal{P}(\omega)/\text{fin}$, and

$\Vdash_{\mathbb{P}} \check{a}$ is a lower bound of $\{\check{a}_\xi : \xi < \check{\kappa}\}$ in $\mathcal{P}(\omega)/\text{fin}$

thus

$q \Vdash_{\mathbb{P}} \check{a}$ is a lower bound of $\{\dot{a}_\xi : \xi < \kappa\}$ in $\mathcal{P}(\omega)/\text{fin}$

Since $(\dot{a}_\xi)_{\xi<\kappa}$ is arbitrary, then $\Vdash_{\mathbb{P}} \check{\kappa} < \mathfrak{t}$, and we can conclude that $\Vdash_{\mathbb{P}} \check{\mathfrak{t}} \leq \mathfrak{t}$, hence $\Vdash_{\mathbb{P}} \check{\mathfrak{t}} = \mathfrak{t}$. For proving $\Vdash_{\mathbb{P}} \check{\mathfrak{p}} = \mathfrak{p}$, we do an analogous argument using the fact that $\mathfrak{p} \leq \mathfrak{t}$, and then $\mathbb{P}$ is $\mathfrak{p}$-closed. ■

To finish this section, we give another useful result about the generic filters of the forcing $\mathcal{P}(\omega)/\text{fin}$: these generic filters are non-principal Ramsey ultrafilters over $\mathbb{N}$.

Proposition 1.2.14 ([6, Prop. 4C]) *Let $\dot{G}$ be the name $\{(\check{A}, A) : A \in \mathcal{P}(\omega)/\text{fin}\}$. Then $\Vdash_{\mathbb{P}}$ $_{\mathbb{P}}\dot{G}$ is a non-principal ultrafilter over $\mathbb{N}$.*

Proof. Showing $\Vdash_{\mathbb{P}}$ "$\dot{G}$ is a filter" does not require too much work; now, let $A \in \mathcal{P}(\omega)/\text{fin}$ and $\dot{C}$ a name such that $A \Vdash_{\mathbb{P}} \dot{C} \in \mathcal{P}(\omega)/\text{fin}$, then there are a $C \in \mathcal{P}(\omega)/\text{fin}$ and an infinite $A' \subseteq^* A$ such that $A' \Vdash_{\mathbb{P}} \dot{C} = \check{C}$ (since $\mathcal{P}(\omega)/\text{fin}$ adds no new subsets of $\mathbb{N}$); now, if $A' \cap C$ is infinite, then $A' \cap C \Vdash_{\mathbb{P}} \dot{C} \in \dot{G}$; otherwise, $A' \setminus C$ is infinite and $A' \setminus C \Vdash_{\mathbb{P}} \mathbb{N} \setminus \dot{C} \in \dot{G}$. Hence, $\Vdash_{\mathbb{P}}$ "$\dot{G}$ is a ultrafilter". Finally, for $n \in \mathbb{N}$, we have that $\mathbb{N} \subseteq^* \mathbb{N} \setminus \{n\} \Vdash_{\mathbb{P}} \mathbb{N} \setminus \{n\} \in \dot{G}$, therefore $\Vdash_{\mathbb{P}}$ "$\dot{G}$ is non-principal". ■

1.3. Martin's Axiom

In this section, we give a short review of Martin's Axiom (abbreviated MA). This combinatorial principle is useful when one wants to introduce several combinatorial complexities, usually found in forcing. We will mention some basic results which relate MA and the cardinal $\mathfrak{p}$, defined en section 1.1. We follow [2, 10] for this section.

Definition 1.3.1 *For any infinite cardinal κ, $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is the following statement: For any family $\mathcal{D}$ of dense sets in $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$, for $D \in \mathcal{D}$.*

Following the convention used in [10], $\mathsf{MA}(\kappa)$ is the statement saying that $\mathsf{MA}_{\mathbb{P}}(\kappa)$ holds for any ccc forcing $\mathbb{P}$; and MA is the statement that $\mathsf{MA}(\kappa)$ holds for all $\kappa < \mathfrak{c}$.

Also, from the condition $|\mathcal{D}| \leq \kappa$ in definition 1.3.1, we can ensure that $\lambda \leq \kappa$ implies that $\mathsf{MA}_{\mathbb{P}}(\kappa) \longrightarrow \mathsf{MA}_{\mathbb{P}}(\lambda)$ and $\mathsf{MA}(\kappa) \longrightarrow \mathsf{MA}(\lambda)$. Moreover, we can conclude that

Fact 1.3.2 ([10, Lemma III.3.13]) $\mathsf{MA}(\kappa) \longrightarrow \kappa < \mathfrak{c}$

In light of the previous fact, we wonder for the first cardinal κ in which $\mathsf{MA}(\kappa)$ fails, and how we can relate this cardinal with those studied in section 1.2.

Definition 1.3.3 ([10, Def. III.3.16]) *$\mathfrak{m}$ is the least cardinal κ such that $\neg\,\mathsf{MA}(\kappa)$.*

Lemma 1.3.4 ([10, Lemma III.3.22]) $\aleph_1 \leq \mathfrak{m} \leq \mathfrak{p}$

Proof. It is clear that $\aleph_1 \leq \mathfrak{m}$ by Lemma 1.2.6.
Let κ be such that $\mathsf{MA}(\kappa)$ holds (i.e. $\kappa < \mathfrak{m}$) and fix $\mathcal{F} \subseteq \mathcal{P}(\omega)/\text{fin}$ such that $\mathcal{F}$ has the SFIP and $|\mathcal{F}| = \kappa$. We want to find a $K \in \mathcal{P}(\omega)/\text{fin}$ such that K is a pseudo-intersection of $\mathcal{F}$.
Define the forcing notion $\mathbb{P}$ whose conditions are set of pairs $p = (s_p, \mathcal{W}_p)$ such that $s_p \in [\mathbb{N}]^{<\aleph_0}$ and $\mathcal{W}_p \in [\mathcal{F}]^{<\aleph_0}$, and ordered as follows: we will say that $q \leq p$ if

(i) s_p is an initial segment of s_p.

(ii) $\mathcal{W}_q \supseteq \mathcal{W}_p$.

(iii) For all $Z \in \mathcal{W}_p$, $(s_q \setminus s_p) \subseteq Z$.

Broadly speaking, the s_p are finite approximations of K, and the $\mathcal{W}_p$ are witnessing that $K \subseteq^* Z$, for all $Z \in \mathcal{W}_p$ (this forcing is known as *Mathias-Prikry forcing*, as it is usually denoted by $\mathbb{M}_{\mathcal{F}}$). We leave to the reader the remaining details of this proof. ■

Martin's Axiom also has a topological statement, which gives us a bound on the number of closed nowhere dense sets that can cover a compact Hausdorff space. Recall that a topological space X has the ccc if the forcing notion $\Omega(X)$ of non-empty open sets of X, ordered by $\subseteq$, has the ccc property.

Theorem 1.3.5 ([10, Lemma III.3.18]) *Let X be a ccc compact Hausdorff space. Assume $\mathsf{MA}(\kappa)$ and let H_α, for $\alpha < \kappa$, be closed nowhere dense sets. Then $\bigcup_{\alpha<\kappa} H_\alpha \neq X$.*

Proof. Let G be a filter in $\Omega(X)$. Since G is a filter, then $p, q \in G$ implies $p \cap q \in G$, so G has the finite intersections property, hence $F_G := \bigcap \{\overline{p} : p \in G\} \neq \emptyset$ because X is compact. Let $D_\alpha := \{q \in \Omega(X) : \overline{q} \cap H_\alpha = \emptyset\}$. Notice that each D_α is dense: given $p \in \Omega(X)$, then $p \smallsetminus H_\alpha$ is a non-empty open set of X, and as X is compact Hausdorff, there is an $r \in \Omega(X)$ such that $r \subseteq \overline{r} \subseteq p \smallsetminus H_\alpha$. Thus $r \in D_\alpha$ and $r \leq p$.
Since $\mathsf{MA}(\kappa)$ holds, we can choose a filter G such that $G \cap D_\alpha \neq \emptyset$, for all $\alpha < \kappa$, and this implies that $F_G \cap H_\alpha = \emptyset$, for all $\alpha < \kappa$. ■

Now we want to consider a strengthening of ccc property and the restriction of MA to it.

Definition 1.3.6 ([10, Def. III.3.23]) *Let $\mathbb{P}$ be a forcing notion and $C \subseteq \mathbb{P}$. We say that C is **centred** if and only if for all $n \in \omega$ and $p_1, ..., p_n \in \mathbb{P}$, there is a $q \in \mathbb{P}$ such that $q \leq p_i$ for all $i \leq n$. We say that $\mathbb{P}$ is **σ-centred** if and only if $\mathbb{P} = \bigcup_{m\in\omega} C_m$, where each C_m is centred.*

The notion of “centred” is the poset analogous of the finite intersections property for families of sets. In topological terms, a compact Hausdorff space X is separable (meaning there is a countable dense subset of X) if and only if the forcing $\Omega(X)$ is σ-centred.

Studying MA restricted to σ-centred forcing gives us a characterization of the cardinal $\mathfrak{p}$. This result is due to Bell [2], and it is called *Bell's theorem.*

Theorem 1.3.7 (Bell, [2, Thm. 1.2],[10, Lemma III.3.61]) *Let $\mathfrak{m}_\sigma$ be the least cardinal κ such that $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is false for some σ-centred forcing $\mathbb{P}$. Then $\mathfrak{m}_\sigma = \mathfrak{p}$.*

Proof. Notice that Mathias-Prikry forcing is σ-centred, so we have that $\mathfrak{m}_\sigma \leq \mathfrak{p}$. The reader can find the rest of the proof in [2, pp. 151-152] or [10, pp. 187-188]. ■

Observation 1.3.8 According to theorems 1.3.5 and 1.3.7, we can see that there is no separable compact Hausdorff space X which can be covered by fewer than $\mathfrak{p}$-many closed nowhere dense sets.

1.4. Model theory

In this section, we give the basic concepts in model theory used in this work. We assume the reader has some basic knowledge about Mathematical Logic, first order languages, $\mathcal{L}$-structures and $\mathcal{L}$-theories. Most of the definitions and results in this section can be found in [3, 12, 19]. Unless stated otherwise, all languages considered along this dissertation are countable first-order languages.

1.4.1. Some basic results

Recall that if $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq \mathcal{M}$, then $\mathcal{L}_A$ is the language obtained by adding to $\mathcal{L}$ constant symbols for each $a \in A$. It is possible to consider $\mathcal{M}$ as an $\mathcal{L}_A$-structure by interpreting the new symbols in the obvious way, i.e. as elements of A. Also, we consider $\mathrm{Th}_A(\mathcal{M})$ the set of all true $\mathcal{L}_A$-sentences in $\mathcal{M}$.

Definition 1.4.1 ([12, Def. 4.1.1]) *Let p be the set of $\mathcal{L}_A$-formulas with free variables $v_1, ..., v_n$. We say that p is an n-**type** if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that p is a **complete** n-**type** if $\varphi \in p$ or $\neg\varphi \in p$ for any $\mathcal{L}_A$-formula $\varphi(v_1, ..., v_n)$. By $S_n^{\mathcal{M}}(A)$ we denote the set of all complete n-types.*

The following result, called the *Tarski-Vaught test*, allows us to build small elementary substructures.

Fact 1.4.2 (Tarski-Vaught test, [12, Prop. 2.3.5]) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ if and only if, for any formula $\varphi(x, \overline{y})$ and $\overline{a} \in M$, where $\overline{y}$ represents a finite tuple of variables, if there is a $b \in N$ such that $\mathcal{N} \vDash \varphi(b, \overline{a})$, then there is a $c \in M$ such that $\mathcal{N} \vDash \varphi(c, \overline{a})$.*

Also, we work here with the notion of complete theory.

Definition 1.4.3 *Let T be an $\mathcal{L}$-theory. We say that T is **complete** if $\mathcal{M} \equiv \mathcal{N}$, for all $\mathcal{M}, \mathcal{N} \vDash T$.*

Definition 1.4.4 *Let T be a $\mathcal{L}$-theory with models of size κ, for κ an infinite cardinal. We say that T is κ-**categorical** if any two models of T of size κ are isomorphic.*

The next result gives us a criterion for finding complete theories. We recall that an $\mathcal{L}$-theory T is called *satisfiable* if T has a model.

Fact 1.4.5 (Vaught test, [12, Thm. 2.2.6]) *Let T be a satisfiable theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete.*

Along this dissertation, we work with countable completes theories over countable languages.

1.4.2. Ultraproducts

Now we give a brief review of the construction of ultraproducts of models $\mathcal{M}_i$ modulo an ultrafilter $\mathcal{F}$.

Suppose I is a non-empty set, $\mathcal{F}$ is a filter over I and let $(\mathcal{M}_i)_{i\in I}$ be a family of $\mathcal{L}$-structures, each with universe M_i. For $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} M_i$, define the relation $\sim_{\mathcal{F}}$ as follows:

$$f \sim_{\mathcal{F}} g \quad \Longleftrightarrow \quad \{i \in I \,|\, a_i = b_i\} \in \mathcal{F} \tag{1-4}$$

The relation $\sim_{\mathcal{F}}$ is an equivalence relation, and thus the set $\prod_{i\in I} M_i/{\sim_{\mathcal{F}}}$ is called the **reduced product of the family** $(\mathcal{M}_i)_{i\in I}$, and it is denoted by $\prod_{i\in I} M_i/\mathcal{F}$. When $\mathcal{M}_i = \mathcal{M}$, for all $i \in I$, this product is called the **reduced power of** $\mathcal{M}$.

Now, when $\mathcal{F}$ is an ultrafilter, in $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/\mathcal{F}$ we define the interpretations of symbols in $\mathcal{M}$:

(i) If c is a symbol constant, then $c^{\mathcal{M}} := [(c^{\mathcal{M}_i})_{i\in I}]$.

(ii) If F is an n-ary function symbol, then

$$F^{\mathcal{M}}\left([(a_i^1)_{i\in I}], ..., [(a_i^n)_{i\in I}]\right) := [(F^{\mathcal{M}_i}(a_i^1, ..., a_i^n))_{i\in I}]$$

(iii) If R is a k-ary relation symbol, then

$$R^{\mathcal{M}} := \left\{ ([(a_i^1)_{i\in I}], ..., [(a_i^k)_{i\in I}]) \in \left(\prod_{i\in I} M_i/\mathcal{F}\right)^k \,:\, \{i \in I \,:\, (a_i^1, ..., a_i^k) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \right\}$$

Notice that both $F^{\mathcal{M}}$ and $R^{\mathcal{M}}$ are well defined, hence the reduced product $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/\mathcal{F}$ is called the **ultraproduct of** $\mathcal{M}_i$ **modulo** $\mathcal{F}$, and it is often denoted by $\prod_{\mathcal{F}} \mathcal{M}_i$. When $\mathcal{M}_i = \mathcal{M}$ for every $i \in I$, the ultraproduct is called the **ultrapower of** $\mathcal{M}$ **modulo** $\mathcal{F}$, and it is denoted by $\mathcal{M}^I/\mathcal{F}$.

The next theorem is the fundamental result about ultraproducts, and gives us a way to determine when a formula is valid in an ultraproduct.

Theorem 1.4.6 (Łoś's theorem) *Let $\mathcal{F}$ be an ultrafilter over I, let $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$ be the ultraproduct of $(\mathcal{M}_i)_{i\in I}$ and $\varphi(x_1, ..., x_n)$ be an $\mathcal{L}$-formula. Then*

$$\mathcal{M} \vDash \varphi([g_1]_{\sim_{\mathcal{F}}}, ..., [g_n]_{\sim_{\mathcal{F}}}) \quad \Longleftrightarrow \quad \{i \in I \,:\, \mathcal{M}_i \vDash \varphi(g_1(i), ..., g_n(i))\} \in \mathcal{F}$$

The following is an useful corollary of Łoś's theorem, and it will be used several times in this work.

Fact 1.4.7 ([8, Cor. 12.5]) *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{U}$ be an ultrafilter on I. The canonical embedding $j : \mathcal{M} \to \mathcal{M}^I/\mathcal{U}$, defined by $j(a) := [(a)_{i\in I}]$, is an elementary embedding.*

With Łoś's theorem, it is possible to prove the Compactness theorem.

Fact 1.4.8 (Compactness theorem, [12, Thm. 2.1.4]) *An $\mathcal{L}$-theory T is satisfiable if and only if every finite subset of T is satisfiable.*

Another important notion in model theory is called *saturation*, and it will be crucial in chapter 3, when we study saturation of ultraproducts.

Definition 1.4.9 ([12, Def. 4.3.1]) *Let κ be an infinite cardinal. We say that $\mathcal{M}$ is **κ-saturated** if, for all $A \subseteq M$ such that $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in M. We say that M is **saturated** if it is $|M|$-saturated.*

Saturation allows to realize types over small subsets of the universe of a given model. It is usual to relate saturation with another notion called *homogeneity* (the reader who wants to read about homogeneity can find some detailed information in [12, 19]).

1.4.3. Quantifier elimination

Now we study some theories which have an important property, called *quantifier elimination*, such as dense linear orders (shortly, DLO) and discrete linear orders.

Definition 1.4.10 ([12, Def. 3.1.1]) *Let T be a $\mathcal{L}$-theory. We say that T has **quantifier elimination** if for every formula $\varphi(x_1, ..., x_n)$ there is a quantifier-free formula $\psi(x_1, ..., x_n)$ such that $T \vDash \varphi \leftrightarrow \psi$*

The next theorem gives us a criterion to determine whether a theory has quantifier elimination or not. We recall that a formula φ is called *simple existential* if $\varphi := \exists x\, \psi$, where ψ is a quntifier-free formula. If ψ is a conjunction of atomic formulas or negation of atomic formulas, then we say that φ is *primitive existential.*

Fact 1.4.11 ([19, Lemma 3.2.4]) *Suppose that T is an $\mathcal{L}$-theory. Then T has quantifier elimination if and only if every simple existential formula $\varphi(\overline{v})$ is equivalent to a quantifier-free $\mathcal{L}$-formula.*

Some important theories have quantifier elimination, such as algebraically closed fields (shortly, ACF) and dense linear orders (shortly, DLO). The reader can view [12, 19] for more interesting and fascinating examples.

Now we define the theory of **discrete linear orders with minimum element and without maximum element**, which appears in chapters 2 and 3. Later, we show that this theory has quantifier elimination.

Definition 1.4.12 *Let $\mathcal{L} = \{s, <, 0\}$ be a language with a unary function symbol s, a binary relation symbol $<$ and a constant symbol 0. The $\mathcal{L}$-theory T of **discrete linear orders with minimum element and without maximum element** is defined by the following axioms:*

1. $\forall x \, \neg(x < x)$.
2. $\forall x \, \forall y \, \forall z \, (x < y \wedge y < z \rightarrow x < z)$.
3. $\forall x \, \forall y \, (x < y \vee y < x \vee x = y)$.
4. $\forall x \, \neg(x < 0)$.
5. $\forall x \forall y \, (s(x) = s(y) \rightarrow x = y)$.
6. $\forall y \, (y \neq 0 \rightarrow \exists x \, (y = s(x)))$.
7. $\forall x \, (\neg \exists y \, (x < y \wedge y < s(x)))$.
8. $\forall x \, (x < s(x))$.

Theorem 1.4.13 *The theory T described in definition 1.4.12 has quantifier elimination.*

Proof. See [4, Thm 32A, pp. 195-196]. ■

1.5. Regular ultrafilters

This section is dedicated to study a special kind of ultrafilters which allows us to characterize saturation of ultraproducts. In this section, we follow [3, 9, 16].

Definition 1.5.1 *Let I be a non-empty set, $\mathcal{D}$ a filter over I and λ an infinite cardinal.*

1. *A family $\mathcal{X} = \{X_i : i < \lambda\}$ of subsets of I is called **regular** if for every $u \subseteq \lambda$, we have*

$$\bigcap_{i \in u} X_i \neq \emptyset \quad \text{if and only if} \quad |u| < \aleph_0$$

*Also, we say that the family $\mathcal{X}$ **regularizes** $\mathcal{D}$ (or $\mathcal{X}$ is a **regularizing family**) if it is regular and $X_i \in \mathcal{D}$, for all $i < \lambda$.*

2. *The filter $\mathcal{D}$ is called λ-**regular** if $\mathcal{D}$ has a regularizing family $\mathcal{X}$ of size λ. We say that $\mathcal{D}$ is **regular** if it is $|I|$-regular.*

3. *We say that $\mathcal{D}$ is λ-**complete** if for every $X_i \in \mathcal{D}$, with $i < \alpha < \lambda$, we have that $\bigcap_{i<\alpha} X_i \in \mathcal{D}$.*

Observation 1.5.2 Note that a filter $\mathcal{D}$ over a set I is λ-regular if there is a subset $X \subseteq \mathcal{D}$ such that $|X| = \lambda$ and each $i \in I$ belongs only to finitely members of X.

Let us give some several useful facts about regular (ultra)filters.

Fact 1.5.3 ([16, Lemma 1.3, Chapter VI]) *Let $\mathcal{D}$ be a filter over an infinite set I.*

1. *If $\mathcal{D}$ is λ-regular and $\mu \leq \lambda$, then $\mathcal{D}$ is μ-regular.*

2. *$\mathcal{D}$ is not $|I|^+$-regular.*

3. *For every cardinal $\lambda \geq \aleph_0$, there is a non-principal λ-regular ultrafilter $\mathcal{D}$ over a set I of size λ.*

Proof.

1. Immediate from the definitions.

2. If $\{X_i : i < |I|^+\} \subseteq \mathcal{D}$ is regular, then choose $j_i \in X_i$. For some $k \in I$, we have that $|\{i < |I|^+ : j_i = k\}| = |I|^+$, however we have that $\bigcap\{X_i : j_i = k\} \supseteq \{k\} \neq \emptyset$, which is a contradiction.

3. Without loss of generality, let $I = [\lambda]^{<\aleph_0}$. For each $\alpha \in \lambda$, consider $X_\alpha := \{u \in [\lambda]^{<\aleph_0} : \alpha \in u\} \subseteq [\lambda]^{<\aleph_0}$. We can see that the family $\mathcal{X} = \{X_\alpha : \alpha \in \lambda\}$ has the finite intersection property: if $\alpha_1, ..., \alpha_n \in \lambda$, then $\{\alpha_1, ..., \alpha_n\} \in X_{\alpha_1} \cap \cdots \cap X_{\alpha_n}$. Thus $\mathcal{X}$ can be extended to an ultrafilter $\mathcal{D}$ over I. Moreover, this ultrafilter is λ-regular: if $L \subseteq \lambda$ is infinite and $\bigcap_{\alpha \in L} X_\alpha \neq \emptyset$, then there exists an $x \in [\lambda]^{<\aleph_0}$ such that $x \in X_\alpha$ for all $\alpha \in L$. Therefore, we have that $\alpha \in x$, for all $\alpha \in L$, which is absurd since x is finite. For proving that $\mathcal{D}$ is non-principal, just notice that $\bigcap_{\alpha \in L} X_\alpha = \emptyset$.

Here concludes the proof. ■

We will focus our attention on $\aleph_0$-regular ultrafilters (also called **ω-regular ultrafilters**) and $\aleph_1$-incomplete ultrafilters (also called **countably incomplete ultrafilters**). These kind of ultrafilters can be characterized in the following way.

Proposition 1.5.4 *Let $\mathcal{D}$ be an ultrafilter over I. The following statements are equivalent:*

(i) $\mathcal{D}$ is ω-regular.

(ii) $\mathcal{D}$ is countable incomplete.

(iii) There is a decreasing countable chain

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$$

of elements $I_n \in \mathcal{D}$ such that $\bigcap_{n\in\omega} I_n = \emptyset$.

Proof. (i) $\implies$ (ii): Suppose $\mathcal{D}$ is ω-regular, then there is a $\mathcal{X} \subseteq \mathcal{D}$ such that $|\mathcal{X}| = \aleph_0$ and each $i \in I$ belongs to only many finitely $X \in \mathcal{X}$. Then $\bigcap \mathcal{X} = \emptyset \notin \mathcal{D}$, so $\mathcal{D}$ is countably incomplete.
(ii) $\implies$ (iii): Let $\mathcal{D}$ be a countably incomplete ultrafilter over I, so there is a countable family $\{X_n : n \in \omega\} \subseteq \mathcal{D}$ such that $\bigcap_{n\in\omega} X_n \notin \mathcal{D}$. Consider

$$\begin{cases} I_0 = X_0 \setminus \bigcap_{n\in\omega} X_n \\ I_{n+1} = I_n \cap X_{n+1}. \end{cases}$$

It is easy to see that $I_n \in \mathcal{D}$, since $(\bigcap_{n\in\omega} X_n)^c \in \mathcal{D}$ (because $\mathcal{D}$ is an ultrafilter). Moreover, it is clear that $I_{n+1} \subseteq I_n$, for all $n \in \omega$. Now, we have that

$$\bigcap_{n\in\omega} I_n = \bigcap_{n\in\omega} X_n \setminus \bigcap_{n\in\omega} X_n = \emptyset.$$

So $\{I_n : n \in \omega\} \subseteq \mathcal{D}$ is a decreasing chain with empty intersection.
(iii) $\implies$ (i): Let $\{I_n : n \in \omega\} \subseteq \mathcal{D}$ be a decreasing chain with empty intersection. If $L \subseteq \mathbb{N}$ is infinite, then $\bigcap_{n\in L} I_n = \emptyset$, hence $\mathcal{D}$ is ω-regular. ■

Countably incomplete ultrafilters are very important in model theory, because these allow us to build $\aleph_1$-saturated ultraproducts.

Theorem 1.5.5 ([3, Thm. 6.1.1]) *Let $\mathcal{L}$ be a countable language, and let $\mathcal{D}$ be a countably incomplete ultrafilter over a set I. Then for every family $(\mathcal{M}_i)_{i\in I}$ of $\mathcal{L}$-structures, the ultraproduct $\prod_{\mathcal{D}} \mathcal{M}_i$ is $\aleph_1$-saturated.*

Proof. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_i$, let $\Sigma(x)$ be a set of formulas (with one free variable) of $\mathcal{L}$. It is enough to prove that if $\Sigma(x)$ is finitely satisfiable in $\mathcal{M}$, then $\Sigma(x)$ is satisfiable in $\mathcal{M}$.
Suppose that each finite subset of $\Sigma(x)$ is realized in $\mathcal{M}$. Since $\mathcal{L}$ is countable, then we know that $\Sigma(x)$ is countable. Therefore, let $\Sigma(x) = \{\sigma_n(x) : n \in \omega\}$ be an enumeration of $\Sigma(x)$.

Since $\mathcal{D}$ is countably incomplete, there is a decreasing chain $I = I_0 \supseteq I_1 \supseteq \ldots$ such that $\bigcap_{n\in\omega} I_n = \emptyset$ by proposition 1.5.4. Now, let $X_0 = I$ and for all $n \in \omega$, define

$$X_n = I_n \cap \{i \in I \,:\, \mathcal{M}_i \vDash \exists x\,(\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))\}$$

Then, by Łoś's theorem, we have that $\{i \in I \,:\, \mathcal{M}_i \vDash \exists x\,(\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))\} \in \mathcal{D}$, thus $X_n \in \mathcal{D}$ for all $n \in \omega$. Moreover, it is clear that $\bigcap_{n\in\omega} X_n = \emptyset$ and $X_n \supseteq X_{n+1}$. Hence, for each $i \in I$, we can find a largest $n(i) < \omega$ such that $i \in X_{n(i)}$.
Now, choose $f \in \mathcal{M}$ as follows: If $n(i) = 0$, let $f(i)$ be some arbitrary element in $\mathcal{M}_i$. If $n(i) > 0$, choose $f(i) \in \mathcal{M}_i$ such that

$$\mathcal{M}_i \vDash \sigma_1(f(i)) \wedge \ldots \wedge \sigma_{n(i)}(f(i))$$

Notice that for any $i \in X_n$, we have that $n \leq n(i)$ and therefore $\mathcal{M}_i \vDash \sigma_n(f(i))$. Thus, by Łoś's theorem, we have that $\mathcal{M} \vDash \sigma_n([f])$ for all $n > 0$, and hence $[f]$ satisfies $\Sigma(x)$ in $\mathcal{M}$. ■

We have seen that countably incomplete ultrafilters give us certain amount of saturation of ultraproducts. Regular ultrafilters will allow us to preserve saturation of ultrapowers which have elementary equivalent ultraroots (recall that an elementary $\mathcal{L}$-substructure $\mathcal{N} \prec \mathcal{M}$ of a model of a first-order theory is an *ultraroot* of $\mathcal{M}$ if $\mathcal{M}$ is isomorphic to an ultrapower of $\mathcal{N}$).

Theorem 1.5.6 ([9, Thm. 2.1]) *Let $\mathcal{D}$ be a regular ultrafilter over a set I, with $|I| = \lambda$. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{N}^I/\mathcal{D}$ is λ^+-saturated, then $\mathcal{M}^I/\mathcal{D}$ is λ^+-saturated.*

Proof. Let $\Sigma(x)$ be a collection of formulas finitely satisfiable in $\mathcal{M}^I/\mathcal{D}$ such that $|\Sigma(x)| \leq \lambda$. Since $\mathcal{D}$ is a regular ultrafilter, there is some regularizing family $\mathcal{X} = \{X_i \,:\, i \in I\}$ for $\mathcal{D}$, and since $|\Sigma(x)| \leq |\mathcal{X}| = \lambda$, let j be an injection from $\Sigma(x)$ into $\mathcal{X}$. We define the following sets

$$\begin{aligned} \Sigma(i) &:= \{\sigma \in \Sigma(x) \,:\, i \in j(\sigma)\} \\ X(i) &:= j(\Sigma(i)) = \{j(\sigma) \,:\, \sigma \in \Sigma(i)\} \end{aligned}$$

Notice that $\Sigma(i)$ is finite: otherwise, it would be possible to find an infinite collection of elements of $\mathcal{X}$ with non-empty intersection, contradicting the regularity of $\mathcal{D}$. Moreover, we have that $|X(i)| = |\Sigma(i)| < \aleph_0$ since j is an injection from $\Sigma(x)$ to $\mathcal{X}$.
For each $i \in I$, let

$$\psi_i(\overline{a}_\sigma(i) \,:\, \sigma \in \Sigma(i)) := \bigwedge_{\substack{w \subseteq \Sigma(i) \\ w \neq \emptyset}} \varphi_w(\overline{a}_\sigma(i) \,:\, \sigma \in w),$$

where $\varphi_w(\overline{a}_\sigma(i) : \sigma \in w)^{e_w} := \exists x \bigwedge_{\sigma\in w} \sigma(x, \overline{a}_\sigma(i))$, with

$$e_w := \begin{cases} 0 & \text{if} \quad \mathcal{M} \vDash \varphi_w(\overline{a}_\sigma(i) : \sigma \in w) \\ 1 & \text{if} \quad \mathcal{M} \vDash \neg\, \varphi_w(\overline{a}_\sigma(i) : \sigma \in w) \end{cases}$$

with $\varphi_w^0 = \varphi$ and $\varphi_w^1 = \neg\, \varphi$.
Since $\mathcal{M} \vDash \psi_i(\overline{a}_\sigma(i) : \sigma \in \Sigma(i))$ and $\mathcal{M} \equiv \mathcal{N}$, we can find $\langle \overline{b}_\sigma(i) : \sigma \in \Sigma(i)\rangle$ such that $\mathcal{N} \vDash \psi_i(\overline{b}_\sigma(i) : \sigma \in \Sigma(i))$. For $\sigma \subseteq \Sigma(i)$, choose $\overline{b}_\sigma(i) \in N$ arbitrary, and consider $\overline{b}_\sigma := [(b_\sigma(i))_{i\in I}]$.
Notice that $\{\sigma(x, \overline{b}_\sigma) : \sigma \in \Sigma(x)\}$ is finitely satisfiable in $\mathcal{N}^I/\mathcal{D}$: let $w \subseteq \Sigma(x)$ be non-empty and finite. Since $\Sigma(x)$ is finitely satisfiable in $\mathcal{M}^I/\mathcal{D}$, then $\mathcal{M}^I/\mathcal{D} \vDash \varphi_w(\overline{a}_\sigma : \sigma \in w)$, so

$$K_w := \left\{ i \in \bigcap_{\sigma\in w} j(\sigma) : \mathcal{M} \vDash \varphi_w(\overline{a}_\sigma(i) : \sigma \in w) \right\} \in \mathcal{D}.$$

Now fix $i \in K_w$, then $w \subseteq \Sigma(i)$ and $\mathcal{M} \vDash \varphi_w(\overline{a}_\sigma(i) : \sigma \in w)$, therefore $\mathcal{N} \vDash \varphi_w(\overline{b}_\sigma(i) : \sigma \in w)$, i.e. $\mathcal{N} \vDash \exists x \bigwedge_{\sigma\in w} \sigma(x, \overline{b}_\sigma(i))$, hence

$$K_w \subseteq \left\{ i \in I : \mathcal{N} \vDash \exists x \bigwedge_{\sigma\in w} \varphi_w(x, \overline{b}_\sigma(i) : \sigma \in w) \right\} \in \mathcal{D},$$

then by Łoś's theorem we have that $\mathcal{N}^I/\mathcal{D} \vDash \exists x \bigwedge_{\sigma\in w} \sigma(x, \overline{b}_\sigma)$.
Since $\mathcal{N}^I/\mathcal{D}$ is λ^+-saturated, there is some $b^* \in \mathcal{N}^I/\mathcal{D}$ such that $\mathcal{N}^I/\mathcal{D} \vDash \sigma(b^*, \overline{b}_\sigma)$ for all $\sigma \in \Sigma(x)$. Fix $i \in I$ and let

$$W_i := \{\sigma \in \Sigma(i) : \mathcal{N} \vDash \sigma(b^*(i), \overline{b}_\sigma(i))\}$$

Recall that $\mathcal{M} \vDash \varphi_{W_i}(\overline{a}_\sigma(i) : \sigma \in W_i)$ if and only if $\mathcal{N} \vDash \varphi_{W_i}(\overline{b}_\sigma(i) : \sigma \in W_i)$, but this is true with $e_w = 0$ for all $\sigma \in W_i$, so choose some $a^*(i) \in \mathcal{M}$ such that $\mathcal{M} \vDash \sigma(a^*(i), \overline{a}_\sigma(i))$ for all $\sigma \in W_i$.
Finally, let us see that $\mathcal{M} \vDash \sigma(a^*, \overline{a}_\sigma)$ for all $\sigma \in \Sigma(x)$. Since $\mathcal{N}^I/\mathcal{D} \vDash \sigma(b^*, \overline{b}_\sigma)$, then

$$J_\sigma := \{i \in j(\sigma) : \mathcal{N} \vDash \sigma(b^*(i), \overline{b}_\sigma(i))\} \in \mathcal{D}.$$

Fix some $i \in J_\sigma$, so $\sigma \in W_i$ and $\mathcal{M} \vDash \sigma(a^*(i), \overline{a}_\sigma(i))$. Therefore,

$$J_\sigma \subseteq \{i \in I : \mathcal{M} \vDash \sigma(a^*(i), \overline{a}_\sigma(i))\} \in \mathcal{D},$$

hence by Łoś's theorem, we conclude that $\mathcal{M} \vDash \sigma(a^*, \overline{a}_\sigma)$ for all $\sigma \in \Sigma(x)$. ∎

In chapter 3, we discuss how to extend $\aleph_1$-saturation of ultraproducts via a new type of ultrafilters.

2 Cofinality spectrum problems

In this chapter, we present the notion of *cofinality spectrum problem*, the central concept of this dissertation.

Before introducing the notion of cofinality spectrum problem, we focus on a special type of gaps, called *peculiar gaps*. This type of gaps will be very important when we study the proof of $\mathfrak{p} = \mathfrak{t}$. section 2.1 is completely dedicated to introduce peculiar gaps.

Section 2.2 is dedicated exclusively to study main definition and results about cofinality spectrum problems.

In section 2.3 we study two important aspects of cofinality spectrum problems: a model-theoretical aspect (related to a special kind of types and conditions about realizations of these types, called *local saturation*); and a recursive aspect (related to the possibility of carrying out codifications of trees in a convenient arithmetic).

Finally, in section 2.4 we study the characterization of $\mathcal{C}^{\mathrm{ct}}(\mathbf{s}) = \emptyset$, using the tools developed in the previous sections.

The results and definitions we give in sections 2.2, 2.3 and 2.4 are compiled in Mallaris-Shelah [11, §2-§9].

2.1. Peculiar gaps

In this section, we discuss the main properties of gaps in ω^ω. In this section we follow [5, 11, 13, 18]. First, we recall the definition of gap in a poset.

Definition 2.1.1 ([13, Def. 1.3]) *Let $P = (P, \leq)$ be a preorder and β, γ ordinals. A (β, γ)-**gap in P** is a pair of sequences $\langle p_\xi : \xi < \beta \rangle$ and $\langle q_\eta : \eta < \gamma \rangle$ in P such that*

(i) $\langle p_\xi : \xi < \beta \rangle$ *is $\leq$-increasing.*

(ii) $\langle q_\eta : \eta < \gamma \rangle$ *is $\leq$-decreasing.*

(iii) $p_\xi \leq q_\eta$ *for all $\xi < \beta$ and $\eta < \gamma$.*

(iv) there is no $r \in P$ such that $p_\xi \leq r \leq q_\eta$ for all $\xi < \beta$ and $\eta < \gamma$.

*If the pair only satisfies (i)-(iii), we say that it is a (β, γ)-**pregap**.*

It is clear that if $(\langle p_\xi : \xi < \beta\rangle, \langle q_\eta : \eta < \gamma\rangle)$ represents a (β, γ)-gap in P, then by reversing the order in P we will obtain a (γ, β)-gap in $(P, \geq)$.

We focus our attention in a very special (and useful) type of gaps in $(\omega^\omega, \leq^*)$.

Definition 2.1.2 (Shelah, [18, Def. 1.10]) *Let κ_1, κ_2 be infinite regular cardinals. A (κ_1, κ_2)-**peculiar gap** in ω^ω is a pair of sequences $(\langle f_\alpha : \alpha < \kappa_1\rangle, \langle g_\beta : \beta < \kappa_2\rangle)$ of functions in ω^ω such that:*

(i) $(\langle f_\alpha : \alpha < \kappa_1\rangle, \langle g_\beta : \beta < \kappa_2\rangle)$ is a (κ_1, κ_2)-gap in $(\omega^\omega, \leq^*)$.

(ii) If $h \in \omega^\omega$ is such that $f_\alpha \leq^* h$, for all $\alpha < \kappa_1$, then $g_\beta \leq^* h$ for some $\beta < \kappa_2$.

(iii) If $h \in \omega^\omega$ is such that $h \leq^* g_\beta$, for all $\beta < \kappa_2$, then $h \leq^* f_\alpha$ for some $\alpha < \kappa_1$.

Now we state some very useful results, which give us convenient bounds if we want to build peculiar gaps.

Proposition 2.1.3 ([5, Thm. 2.2]) *Let κ, λ be infinite cardinals. If both $\kappa, \lambda < \mathfrak{p}$, then there is no (κ, λ)-peculiar gap.*

Proof. Suppose that $(\langle f_\alpha : \alpha < \kappa\rangle, \langle g_\beta : \beta < \lambda\rangle)$ represents a (κ, λ)-peculiar gap. Let $\mathbb{P}$ be the forcing notion whose conditions are ordered pairs (s, F) such that $s \in \omega^{<\omega}$ and $F \in [\lambda]^{<\aleph_0}$, and ordered as follows: $(s, F) \leq (s', F')$ if $s' \subseteq s$, $F' \subseteq F$ and for all $n \in \operatorname{dom}(s) \smallsetminus \operatorname{dom}(s')$ and for all $\alpha \in F'$, $s(n) < g_\alpha(n)$. It is clear that $\mathbb{P}$ is a σ-centred forcing notion, and for $\alpha < \kappa$ and $n < \omega$, consider the sets

$$D_{\alpha,n} := \{(s, F) \in \mathbb{P} : \exists i \geq n\, (s(i) \geq f_\alpha(i))\}$$

By Bell's theorem, we can choose a filter G such that it intersects $D_{\alpha,n}$ for all $\alpha < \kappa$ and $n < \omega$. Hence, for some filter G we have a function $h := \bigcup\{s \in \omega^\omega : (\exists F \in [\lambda]^{<\aleph_0})((s, F) \in G)\}$ such that $h \leq^* g_\beta$ for all $\beta < \lambda$, but $h \not\leq^* f_\alpha$ for all $\alpha < \kappa$. ■

Proposition 2.1.4 ([18, Prop. 1.11]) *If $\kappa < \mathfrak{b}$, then there is no $(\kappa, \aleph_0)$-peculiar gap.*

Proof. Assume that $(\langle f_\alpha : \alpha < \kappa\rangle, \langle g_n : n < \omega\rangle)$ represents a $(\kappa.\aleph_0)$-peculiar gap. Given $h \in \omega^\omega$ increasing such that $h(0) = 0$, let us consider the function g_h given by

$$g_h := \bigcup_{n<\omega} g_n \restriction_{[h(n),h(n+1))} .$$

Let us prove that, for a convenient h, we can find a g_h that fills the gap.
Define $f \in \omega^\omega$ recursively by $f(0) = 0$ and $f(n+1) := \min\{k < \omega : f(n) < k \wedge (\forall k' \geq k)\, g_{n+1}(k') \leq g_n(k')\}$; notice that whenever $f \leq^* h$, $g_h \leq^* g_n$ for all $n < \omega$.
Now for each $\alpha < \kappa$, recursively define $h_\alpha \in \omega^\omega$ by $h_\alpha(0) = 0$ and $h_\alpha(n) := \min\{k < \omega : h_\alpha(n-1) < k \wedge (\forall k' \geq k)\, f_\alpha(k') \leq g_n(k')\}$ for $n > 0$; notice that whenever $h_\alpha \leq^* h$, $f_\alpha \leq^* g_h$ for all $\alpha < \kappa$.
Since the family $\{h_\alpha : \alpha < \kappa\} \cup \{f\}$ has $< \mathfrak{b}$ elements, it is possible to find an $h \in \omega^\omega$ almost dominating all of them. Hence, g_h will fill the gap, a contradiction. ∎

Following [18], it is possible to ensure the existence of some peculiar gaps in ω^ω under the assumption of $\mathfrak{p} < \mathfrak{t}$.

Theorem 2.1.5 ([18, Thm. 1.12]) *Assume $\mathfrak{p} < \mathfrak{t}$. Then for some regular cardinal κ, there exists a $(\kappa, \mathfrak{p})$-peculiar gap in ω^ω with $\aleph_1 \leq \kappa < \mathfrak{p}$.*

In chapter 4, we will revisit this result when we analyze the proof of $\mathfrak{p} = \mathfrak{t}$.

2.2. A motivating example and main definitions

This section is dedicated to a complete study of the notion of cofinality spectrum problem, introduced by Malliaris and Shelah in [11].

First, we give some definitions about ultrafilters.

Definition 2.2.1 ([11, Def. 10.14]) *Let $\mathcal{D}$ be an ultrafilter over a non-empty set I. We define the **cut spectrum of** $\mathcal{D}$ as follows:*

$$\mathcal{C}(\mathcal{D}) := \{(\kappa_1, \kappa_2) \in |I|^+ \times |I|^+ : \kappa_1, \kappa_2 \text{ are regular and } (\omega, <)^I/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2)\text{-gap}\}.$$

Remark 2.2.2 Suppose that κ_1 is an infinite regular cardinal and κ_2 is finite. By Łoś theorem, we know that every $0 \neq a \in \omega^I/\mathcal{D}$ has an immediate predecessor; therefore, we can affirm that $(\kappa_1, \kappa_2) \notin \mathcal{C}(\mathcal{D})$. With a similar argument, we can conclude that $(\kappa_2, \kappa_1) \notin \mathcal{C}(\mathcal{D})$.

Definition 2.2.3 ([11, Def. 10.13]) *Let $\mathcal{D}$ be an ultrafilter over I and κ a regular cardinal.*

(i) *We say that $\mathcal{D}$ has κ-**treetops** if for any κ-saturated model $\mathcal{M}$ which interprets a tree $(\mathcal{T}_{\mathcal{M}}, \trianglelefteq)$, $\mathcal{N} = \mathcal{M}^I/\mathcal{D}$, $\gamma = \mathrm{cf}\,(\gamma) < \kappa$ and for any $\trianglelefteq$-increasing sequence $\langle a_i : i < \gamma \rangle$ in $(\mathcal{T}_{\mathcal{N}}, \trianglelefteq_{\mathcal{N}})$, there is an $a \in \mathcal{T}_{\mathcal{N}}$ such that $a_i \trianglelefteq a$, for all $i < \gamma$.*

(ii) *We say that $\mathcal{D}$ has $< \kappa$-**treetops** if $\mathcal{D}$ has θ-treetops for any infinite cardinal $\theta = \theta < \kappa$.*

The following lemma gives us a first approach on how λ^+-treetops in regular ultrafilters have some essential information about the existence (or not) of some special kind of gaps.

Lemma 2.2.4 ([11, Lemma 2.2]) *Suppose $\mathcal{D}$ is a regular ultrafilter on λ with λ^+-treetops, and $\kappa < \lambda^+$ is regular. Then $\mathcal{C}(\mathcal{D})$ has no (κ, κ)-gaps, i.e. $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$.*

Proof. We use the same notation as in [11]. Let $M = (\mathbb{N}, <)$ and $M_1 = M^\lambda/\mathcal{D}$. We suppose that in M_1 there is a (κ, κ)-gap, represented by $(\overline{a}, \overline{b}) = (\langle a_\alpha : \alpha < \kappa \rangle, \langle b_\alpha : \alpha < \kappa \rangle)$, such that $M_1 \vDash a_\beta < a_\alpha < b_\alpha < b_\beta$, for all $\beta < \alpha < \kappa$ and there is no $c \in M_1$ satisfying $a_\alpha < c < b_\alpha$, for all $\alpha < \kappa$.
As our main assumption refers to trees and not to orders, we have to build a convenient tree which models the gap, and then, by the λ^+-treetops condition, we will be able to find an element in M_1 that fills the gap, and thus we will have a contradiction. In this order of ideas, it should be natural to consider the tree $(\mathcal{T}, \trianglelefteq)$ of finite sequences of pairs of natural numbers, ordered in its natural way, i.e. $f \trianglelefteq g \in \mathcal{T}$ if and only if $f = g \restriction \mathrm{dom}(f)$.
To be able to talk about this tree, the symbol $<$ is not powerful enough to capture all the information lying in the tree. So, we have to add in an adequate way some relation and function symbols, with their interpretations in the new language. For this reason, we expand the model M to an adequate model M^+ in which we could find these new symbols. Besides, thanks to the fact that ultrapowers commute with reduced products, we can transfer naturally these new symbols into the ultrapower M_1^+ of M^+ (obviously, with their interpretations in this ultrapower)[1]. In this ultrapower, we can build an increasing sequence (via transfinite induction), and the treetops condition will give us an upper bound of this sequence that will fill the gap, arriving to the desired contradiction.
So, let us perform the first stage of the proof: find an expansion of M. Notice that if we want to talk about $\mathcal{T}$ in M, we could just talk about ordered sequences, but these are not the only kind of sequences we can have in M. Also, if we want to talk about $\mathcal{T}$, implicitly we need to refer to the following: for $x \in \mathcal{T}$, we have

(a) a length function, $\lg(x)$.

(b) a function that gives us the maximum of $\mathrm{dom}\,(x)$, $\max(\mathrm{dom}(x)) := \lg(x) - 1$.

(c) for each $n < \max(\mathrm{dom}(x))$, an evaluation function, $x(n)$.

[1]In the case of lemma 2.2.4, we are not adding new symbols: what we do is consider a richer symbol ($\in$), which allows us to talk about more things than with $<$.

(d) for each $n < \max(\mathrm{dom}(x))$, projections functions, $x(n,0)$ and $x(n,1)$.

Also, by a brief computation using cardinal arithmetic, we can see that $\mathcal{T}$ is a countable set. After this digression, now it should be natural to consider the expansion $M^+ = (\mathcal{H}(\omega_1), \in)$, where $\mathcal{H}(\omega_1)$ is the class of sets whose transitive closures are at most countable. It is important to notice that, in M^+, $\mathbb{N}$ and $\mathcal{T}$ are definable sets, $\trianglelefteq$ is a definable relation, and the items (a)-(d) are also definable in M^+.
Having found an expansion for M, we ask for an expansion on M_1. For this purpose, we can use the fact that ultrapowers commute with reducts to find a convenient expansion on M_1. In this case, this expansion would be $M_1^+ = (\mathcal{H}(\omega_1), \in)^\lambda/\mathcal{D}$. Moreover, thanks to Łoś's theorem, we have that $M \preceq M_1$ and $M^+ \preceq M_1^+$. Also, it's important to notice the following: in the process of finding an expansion for M_1, we are not adding new symbols, but these symbols acquire a new interpretation in the ultrapower (in the case of lemma 2.2.4, we didn't add new symbols).
We have performed the first stage of the proof. Now we move to $M_1^+ = (M^+)^\lambda/\mathcal{D}$ and consider the version $\mathcal{T}^{M_1^+}$ of $\mathcal{T}$ in M_1^+. In $\mathcal{T}^{M_1^+}$, we consider the subtree $\mathcal{T}_*$ defined by the formula

$$\varphi(x) : x \in \mathcal{T}^{M_1^+} \wedge (\forall n < m < (\lg(x)))(x(n,0) < x(m,0) < x(m,1) < x(n,1)). \tag{2-1}$$

The subtree $\mathcal{T}_*$ is infinite, but more important is that this subtree represents the supposed gap. Also, note that if $M_1^+ \vDash \varphi(c)$, then $M_1^+ \vDash \varphi(c \restriction_n)$, for all $n \leq \max(\mathrm{dom}(c))$.
By transfinite induction, we will construct a convenient branch $(c_\alpha)_{\alpha<\kappa}$ of $\mathcal{T}_*$ and $n_\alpha \in \mathbb{N}^{M_1^+}$ such that

(i) for all $\beta < \alpha < \kappa$, $M_1^+ \vDash c_\beta \trianglelefteq c_\alpha$.

(ii) for all $\alpha < \kappa$, $n_\alpha = \max(\mathrm{dom}(c_\alpha))$.

(iii) for all $\alpha < \kappa$, $c_\alpha(n_\alpha, 0) = a_\alpha$ and $c_\alpha(n_\alpha, 1) = b_\alpha$.

Let us perform this induction.

1. (Base case) Let $c_0 = \langle (a_0, b_0) \rangle$ and $n_0 = 0$.

2. (Inductive step for $\alpha = \beta + 1$) Assume that c_β and n_β are defined. We could just concatenate (a_α, b_α) to the tail of the sequence. So, $c_\alpha := c_\beta{}^\frown\langle (a_\beta, b_\beta) \rangle$ and $n_\alpha := n_\beta + 1$.

3. (Inductive step for α limit) By the treetops hypothesis, we can find a $c_* \in \mathcal{T}_*$ such that $c_\beta \trianglelefteq c_*$, for all $\beta < \alpha$. Let $n_* = \max(\mathrm{dom}(c_*))$, then for every $\beta < \alpha$, we have that

$$a_\beta = c_\beta(n_\beta, 0) = c_*(n_\beta, 0) < c_*(n_*, 0) < c_*(n_*, 1) < c_*(n_\beta, 1) = c_\beta(n_\beta, 1) = b_\beta. \tag{2-2}$$

However, we may have the case that $a_\alpha < c_*(n_*, 0) < c_*(n_*, 1) < b_\alpha$, in which case c_* is not the best upper bound we wish to find. To solve this potential issue, observe that the set

$$\{n \leq n_* \,:\, c_*(n,0) < a_\alpha \wedge b_\alpha < c_*(n, 1)\} \subseteq M_1^+ \tag{2-3}$$

contains n_β for all $\beta < \alpha$, and it also is definable in M_1^+ and bounded above by n_*. So, thanks to Łoś's theorem, we can find the maximum of this set, let us called it m_*. In this way, if we consider $c_* \restriction_{m_*}$, we note that $c_\beta \trianglelefteq c_* \restriction_{m_*}$ for all $\beta < \alpha$, and thus we can concatenate (a_α, b_α) to the tail of the sequence. Thus, let $c_\alpha = c \restriction_{m_*}{}^\frown\langle(a_\alpha, b_\alpha)\rangle$ and $n_\alpha = m_*$. Also, notice that $c_\alpha \in \mathcal{T}_*$: the way we buily this sequence guarantees this fact.

Finally, by the treetops hypothesis, we can find a $d \in \mathcal{T}_*$ such that $c_\alpha \trianglelefteq d$ for all $\alpha < \kappa$, and let $N = \max(\mathrm{dom}(d))$. Then, by the definition of φ, we have that

$$a_\alpha = c_\alpha(n_\alpha, 0) = d(n_\alpha, 0) < d(N, 0) < d(N, 1) < d(n_\alpha, 1) = c_\alpha(n_\alpha, 1) = b_\alpha \tag{2-4}$$

Then, the elements $d(N, 0)$ and $d(N, 1)$ are elements in M_1^+ (and in fact, in M_1) that realizes the gap, which is absurd. ■

We would like to do some important comments about the previous proof.

- ✓ First, as we worked with ultraproducts the appearance of Łoś's theorem should not be surprising. However, Łoś's theorem allowed us to find suitable expansions of the models M and M_1, and also preserve the validity of some statements. In particular, we could observe that all the non-empty, definable and bounded subsets of $\mathbb{N}^{M_1^+}$ have first and last element.
- ✓ The fact that ultrapowers commute with reducts allows us to find a natural expansion of M_1 given an expansion for M, while not adding new relation or function symbols to the new expansion.

With these observations in mind, we will give the main definitions on which we are going to work on this thesis. These definitions should be seen according to lemma 2.2.4.

Definition 2.2.5 (Enough set theory for trees -ESTT-) *Let M_1 be a model and Δ a non-empty set of formulas in the language of M_1. We say that (M_1, Δ) has **enough set theory for trees** when the following conditions are true.*

1. *Δ consists of first-order formulas $\varphi(\overline{x}, \overline{y}; \overline{z})$, with $\lg(\overline{x}) = \lg(\overline{y})$.*
2. *For each $\varphi \in \Delta$ and each parameter $\overline{c} \in {}^{\lg(\overline{z})}M_1$, $\varphi(\overline{x}, \overline{y}; \overline{c})$ defines a discrete linear order on $\{\overline{a} : M_1 \models \varphi(\overline{a}, \overline{a}; \overline{c})\}$ with first and last element.*

3. *The family of all linear orders defined in this way will be denoted by* $\mathrm{Or}(M_1, \Delta)$. *Specifically, each* $\mathbf{a} \in \mathrm{Or}(M_1, \Delta)$ *is a tuple* $(X_{\mathbf{a}}, \leq_{\mathbf{a}}, \varphi_{\mathbf{a}}, \bar{c}_{\mathbf{a}}, d_{\mathbf{a}})$, *where:*
   (a) $X_{\mathbf{a}}$ *denotes the underlying set* $\{\bar{a} : M_1 \vDash \varphi_{\mathbf{a}}(\bar{a}, \bar{a}; \bar{c}_{\mathbf{a}})\}$.
   (b) $\bar{x} \leq_{\mathbf{a}} \bar{y}$ *abbreviates the formula* $\varphi_{\mathbf{a}}(\bar{x}, \bar{y}; \bar{c}_{\mathbf{a}})$ *and* $\varphi_{\mathbf{a}} \in \Delta$.
   (c) $d_{\mathbf{a}} \in X_{\mathbf{a}}$ *is a bound for the length of elements in the associated tree; it is often, but not always,* $\max X_{\mathbf{a}}$. *If* $d_{\mathbf{a}}$ *is finite, we call* $\mathbf{a}$ *trivial.*

4. *For each* $\mathbf{a} \in \mathrm{Or}(M_1, \Delta)$, $(X_{\mathbf{a}}, \leq_{\mathbf{a}})$ *is pseudofinite, meaning that any bounded, nonempty,* M_1*-definable subset has* $\leq_{\mathbf{a}}$*-greatest and* $\leq_{\mathbf{a}}$*-least element.*

5. *For each pair* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathrm{Or}(M_1, \Delta)$, *there is a* $\mathbf{c} \in \mathrm{Or}(M_1, \Delta)$ *such that:*
   (a) *There exists an* M_1*-definable bijection* $\mathrm{Pr} : X_{\mathbf{a}} \times X_{\mathbf{b}} \to X_{\mathbf{c}}$ *such that the coordinate projections are* M_1*-definable.*
   (b) *If* $d_{\mathbf{a}}$ *is not finite in* $X_{\mathbf{a}}$ *and* $d_{\mathbf{b}}$ *is not finite in* $X_{\mathbf{b}}$, *then also* $d_{\mathbf{c}}$ *is not finite in* $X_{\mathbf{c}}$.

6. *For some nontrivial* $\mathbf{a} \in \mathrm{Or}(M_1, \Delta)$, *there is a* $\mathbf{c} \in \mathrm{Or}(M_1, \Delta)$ *such that* $X_{\mathbf{c}} = \mathrm{Pr}(X_{\mathbf{a}} \times X_{\mathbf{a}})$ *and the ordering* $\mathbf{c}$ *satisfies*

   $$M_1 \vDash (\forall x \in X_{\mathbf{a}})(\exists y \in X_{\mathbf{c}})(\forall x_1, x_2 \in X_{\mathbf{a}})(\max\{x_1, x_2\} \leq_{\mathbf{a}} x \Leftrightarrow \mathrm{Pr}(x_1, x_2) \leq_{\mathbf{c}} y)$$

7. *To the family of distinguished orders, we associate a family of trees as follows. For each formula* $\varphi(\bar{x}, \bar{y}; \bar{z})$ *in* Δ *there are formulas* ψ_0, ψ_1, ψ_2 *of the language of* M_1 *such that for any* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ *with* $\varphi_{\mathbf{a}} = \varphi$*:*
   (a) $\psi_0(\bar{x}; \bar{c}_{\mathbf{a}})$ *defines a set, denoted* $\mathcal{T}_{\mathbf{a}}$, *of partial functions from* $X_{\mathbf{a}}$ *to* $X_{\mathbf{a}}$.
   (b) $\psi_1(\bar{x}, \bar{y}; \bar{c})$ *defines a function* $\lg_{\mathbf{a}} : \mathcal{T}_{\mathbf{a}} \to X_{\mathbf{a}}$ *satisfying:*
      (i) *For all* $b \in \mathcal{T}_{\mathbf{a}}$, $\lg_{\mathbf{a}}(b) \leq_{\mathbf{a}} d_{\mathbf{a}}$.
      (ii) *For all* $b \in \mathcal{T}_{\mathbf{a}}$, $\lg_{\mathbf{a}}(b) = \max(\mathrm{dom}(b)) + 1$.
   (c) $\psi_2(\bar{x}, \bar{y}; \bar{c})$ *defines a function from* $\{(b, a) : b \in \mathcal{T}_{\mathbf{a}}, a \in X_{\mathbf{a}}, a <_{\mathbf{a}} \lg_{\mathbf{a}}(b)\}$ *into* $X_{\mathbf{a}}$ *whose value is called* $\mathrm{val}_{\mathbf{a}}(b, c)$, *and abbreviated* $b(a)$, *such that*
      (i) *If* $c \in \mathcal{T}_{\mathbf{a}}$, $\lg_{\mathbf{a}}(c) <_{\mathbf{a}} d_{\mathbf{a}}$ *and* $a \in X_{\mathbf{a}}$, *then* $c^\frown\langle a\rangle$ *exists, i.e. there is a* $c_0 \in \mathcal{T}_{\mathbf{a}}$ *such that* $\lg_{\mathbf{a}}(c_0) = \lg_{\mathbf{a}}(c) + 1$, $c_0(\lg_{\mathbf{a}}(c)) = a$, *and* $(\forall a <_{\mathbf{a}} \lg_{\mathbf{a}}(c))(c(a) = c_0(a))$.
      (ii) *if* $b_1 \neq b_2 \in \mathcal{T}_{\mathbf{a}}$, $\lg_{\mathbf{a}}(b_1) = \lg_{\mathbf{a}}(b_2)$ *then for some* $n <_{\mathbf{a}} \lg_{\mathbf{a}}(b_1)$, $b_1(n) \neq b_2(n)$.
   (d) $\psi_3(\bar{x}, \bar{y}; \bar{c})$ *defines the partial order* $\trianglelefteq_{\mathbf{a}}$ *on* $\mathcal{T}_{\mathbf{a}}$ *given by initial segment, that is, such that that* $b_1 \trianglelefteq_{\mathbf{a}} b_2$ *implies:*
      (i) $\lg_{\mathbf{a}}(b_1) \leq_{\mathbf{a}} \lg_{\mathbf{a}}(b_2)$.
      (ii) $(\forall a <_{\mathbf{a}} \lg_{\mathbf{a}}(b_1))(b_2(a) = b_1(a))$.

The family of all $\mathcal{T}_{\mathbf{a}}$ defined this way will be denoted $\mathrm{Tr}(M_1, \Delta)$. *We refer to elements of this family as trees.*

Now we are ready to give the central definition of this work.

Definition 2.2.6 (Cofinality spectrum problems) *We say that*

$\mathbf{s} := (M^{\mathbf{s}}, M_1^{\mathbf{s}}, M^{+,\mathbf{s}}, M_1^{+,\mathbf{s}}, T^{\mathbf{s}}, \Delta^{\mathbf{s}})$

is a ***cofinality spectrum problem*** *when*

1. $M^{\mathbf{s}} \preceq M_1^{\mathbf{s}}$.

2. $\Delta^{\mathbf{s}}$ *is a set of formulas in the language of* $M^{\mathbf{s}}$*, i.e., we are interested in studying the orders of* $\mathcal{L}(M^{\mathbf{s}}) = \mathcal{L}(M_1^{\mathbf{s}})$ *in the presence of the additional structure of* $\mathcal{L}(M^{+,\mathbf{s}}) = \mathcal{L}(M_1^{+,\mathbf{s}})$.

3. $\sigma^{M^{\mathbf{s}}} \in T^{\mathbf{s}}$ *for all* $\sigma \in \mathrm{Th}\,(M^{\mathbf{s}})$*, where* $\sigma^{M^{\mathbf{s}}}$ *denotes the* ***relativization*** *of* σ *to* $M^{\mathbf{s}}$*, see [8, definition 12.6]. (Notice that the language of* $T^{\mathbf{s}}$ *is* $\mathcal{L}(M^{+,\mathbf{s}})$ *and* $M^{\mathbf{s}}$ *is definable in* $\mathcal{L}(M^{+,\mathbf{s}})$*)*

4. $M^{+,\mathbf{s}}, M_1^{+,\mathbf{s}}$ *expand* $M^{\mathbf{s}}, M_1^{\mathbf{s}}$ *respectively so that* $M^{+,\mathbf{s}} \preceq M_1^{+,\mathbf{s}} \models T^{\mathbf{s}}$ *and* $(M_1^{+,\mathbf{s}}, \Delta)$ *has enough set theory for trees.*

When the context is clear, we often omit the upper index $\mathbf{s}$.

In this work, when we consider a cofinality spectrum problem $\mathbf{s}$, the model M given in definition 2.2.6 will be called the *ground model* of $\mathbf{s}$, and the model M_1^+ will be called the *main expansion* of $\mathbf{s}$. This convention is just used to simplify some aspects about the behaviour of any cofinality spectrum problem.

Remark 2.2.7 In [11, p. 249], there is mentioned the following example of a cofinality spectrum problem in relation to lemma 2.2.4: "Consider $M = (\mathbb{N}, <)$. Then there are a set of $\mathcal{L}$-formulas $\Delta \supseteq \{x < y\}$, an expanded language $\mathcal{L}^+$, and an $\mathcal{L}^+$-theory $T \supseteq \mathrm{Th}\,(M)$ such that $(M, M_1, M^+, M_1^+, T, \Delta)$ is a cofinality spectrum problem. For instance, we may take $T = \mathrm{Th}\,(\mathcal{H}(\omega_1), \in)$ and identify $\mathbb{N}$ with ω". This is not true, as we have that $\mathrm{Th}\,(\mathcal{H}(\omega_1), \in) \not\supseteq \mathrm{Th}\,(M)$: indeed, the sentence σ : "$\{1\}$ is not a set" is true in $\mathrm{Th}\,(M)$, but is not in $\mathrm{Th}\,(\mathcal{H}(\omega_1), \in)$. Actually, it is just necessary that the whole theorems of $\mathrm{Th}\,(M)$ relativized in M hold in M^+.

As cofinality spectrum problems are conformed by models and sets of formulas, it is natural to wonder how to compare any pair of cofinality spectrum problems.

Definition 2.2.8 *Let* $\mathbf{s}_1 = (M^{\mathbf{s}_1}, M_1^{\mathbf{s}_1}, M^{+,\mathbf{s}_1}, M_1^{+,\mathbf{s}_1}, T^{\mathbf{s}_1}, \Delta^{\mathbf{s}_1})$ *and* $\mathbf{s}_2 = (M^{\mathbf{s}_2}, M_1^{\mathbf{s}_2}, M^{+,\mathbf{s}_2}, M_1^{+,\mathbf{s}_2}, T^{\mathbf{s}_2}, \Delta^{\mathbf{s}_2})$ *be cofinality spectrum problems. We say that* $\mathbf{s}_2$ *is* ***more complex***[2] ***than*** $\mathbf{s}_1$ *(we denote this by* $\mathbf{s}_1 \leq \mathbf{s}_2$*) if*

(i) $M^{\mathbf{s}_1} = M^{\mathbf{s}_2}$, $M_1^{\mathbf{s}_1} = M_1^{\mathbf{s}_2}$, *i.e. the ground models are the same.*

(ii) $\mathcal{L}(M^{+,\mathbf{s}_1}) \subseteq \mathcal{L}(M^{+,\mathbf{s}_2})$, *i.e. in the problem* $\mathbf{s}_2$ *we might require new symbols, and also* $T^{\mathbf{s}_1} \subseteq T^{\mathbf{s}_2}$.

(iii) $(M^{+,\mathbf{s}_2}) \restriction_{\mathcal{L}(M^{+,\mathbf{s}_1})} \cong M^{+,\mathbf{s}_1}$.

(iv) $(M_1^{+,\mathbf{s}_2}) \restriction_{\mathcal{L}(M_1^{+,\mathbf{s}_1})} \cong M_1^{+,\mathbf{s}_1}$, *i.e. the main model of* $\mathbf{s}_1$ *is the main model of* $\mathbf{s}_2$ *restricted to an adequate vocabulary.*

(v) $\Delta^{\mathbf{s}_1} \subseteq \Delta^{\mathbf{s}_2}$.

Definition 2.2.9 *Let* $\mathbf{s}$ *be a cofinality spectrum problem. We define the following:*

1. $\mathrm{Or}(\mathbf{s}) = \mathrm{Or}(M_1^{\mathbf{s}}, \Delta^{\mathbf{s}})$, *but* $X_{\mathbf{a}}$ *and* $\mathcal{T}_{\mathbf{a}}$ *are interpreted in* $M_1^{+,\mathbf{s}}$ *when* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$.
2. $\mathcal{C}^{\mathrm{ct}}(\mathbf{s}) = \{(\kappa_1, \kappa_2) :$ *for some* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$, $(X_{\mathbf{a}}, \leq_{\mathbf{a}})$ *has a* (κ_1, κ_2)*-gap*$\}$ *(this set will be called the* ***cut spectrum of*** $\mathbf{s}$*).*
3. $\mathrm{Tr}(\mathbf{s}) = \{\mathcal{T}_{\mathbf{a}} : \mathbf{a} \in \mathrm{Or}(\mathbf{s})\} = \mathrm{Tr}(M_1^s, \Delta^s)$, *and moreover,* $\mathcal{T}_{\mathbf{a}}$ *is interpreted in* $M_1^{+,\mathbf{s}}$ *when* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$.
4. $\mathcal{C}^{\mathrm{ttp}}(\mathbf{s}) = \{\kappa : \kappa \geq \aleph_0,\ \mathbf{a} \in \mathrm{Or}(\mathbf{s}),$ *and there is in the tree* $\mathcal{T}_{\mathbf{a}}$ *a strictly increasing sequence of cofinality* κ *with no upper bound*$\}$ *(this set will be called the* ***treetops of*** $\mathbf{s}$*).*
5. *Let* $\mathfrak{t}_{\mathbf{s}} = \min \mathcal{C}^{\mathrm{ttp}}(\mathbf{s})$ *and* $\mathfrak{p}_{\mathbf{s}} = \min \{\kappa : (\kappa_1, \kappa_2) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$ *and* $\kappa = \kappa_1 + \kappa_2\}$.
6. *For an infinite cardinal* λ, *write*

$$\mathcal{C}(\mathbf{s}, \lambda) = \{(\kappa_1, \kappa_2) : \kappa_1 + \kappa_2 < \lambda, (\kappa_1, \kappa_2) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})\} \tag{2-5}$$

Remark 2.2.10 Let us give some pertinent observations about definition 2.2.9.

1. Given an ultrafilter $\mathcal{D}$ on ω, we may define the following:
   (1) Let $\mathrm{Po}_{<\omega}(\mathcal{D})$ be the class of preorders of the form $(\mathbf{P}, \leq) := \prod_{n\in\omega} (P_n, \leq_n)/\mathcal{D}$, where $(P_n, \leq_n)$ is a finite non-empty partial order, for all $n \in \omega$.
   (2) Let $\mathrm{Lo}_{<\omega}(\mathcal{D})$ be the class of linear orders of the form $(\mathbf{L}, \leq) := \prod_{n\in\omega} (L_n, \leq_n)/\mathcal{D}$, where $(L_n, \leq_n)$ is a finite non-empty linear order, for all $n \in \omega$.

[2] This name actually is not mentioned in [11]. This is just a name which we thought it could be precise for naming this relation.

(3) Define $\mathfrak{t}_{\mathcal{D}}$ as the least ordinal γ such that there is an unbounded $\leq$-increasing sequence of length γ in some $(\mathbf{P}, \leq) \in \mathrm{Po}_{<\omega}(\mathcal{D})$.

(4) Define $\mathfrak{p}_{\mathcal{D}}$ as the least ordinal γ such that there is some $\beta < \gamma$ such that in some $(\mathbf{L}, \leq) \in \mathrm{Lo}_{<\omega}(\mathcal{D})$ it is possible to find a (β, γ)-gap.

There are a lot of results known about $\mathfrak{p}_{\mathcal{D}}$ and $\mathfrak{t}_{\mathcal{D}}$: e.g, both $\mathfrak{p}_{\mathcal{D}}$ and $\mathfrak{t}_{\mathcal{D}}$ are uncountable regular cardinal; or if $\mathcal{D}$ is regular, then $\mathfrak{t}_{\mathcal{D}} \leq \mathfrak{p}_{\mathcal{D}}$ and there are no (κ, λ)-gaps with $\kappa, \lambda < \mathfrak{t}_{\mathcal{D}}$ (see [6, 13, 14] for details). So, roughly speaking, cofinality spectrum problems are general frames in which we can analyze the relations between performing certain gaps in linear orders and finding bounds for increasing sequences in some distinguished trees. Also, it is natural to think in $\mathfrak{p}_{\mathbf{s}}$ and $\mathfrak{t}_{\mathbf{s}}$ as natural generalizations of the cardinals $\mathfrak{p}_{\mathcal{D}}$ and $\mathfrak{t}_{\mathcal{D}}$, respectively.

2. Another characterization of $\mathfrak{t}_{\mathbf{s}}$ is given as follows: if κ is a regular cardinal such that $\kappa = \mathfrak{t}_{\mathbf{s}}$, then there is a definable linear order which has a (κ, κ)-gap (see [11, Lemma 6.2]). This useful characterization will be use in chapter 3.

Following remark 2.2.10, the cardinals $\mathfrak{p}_{\mathbf{s}}$ and $\mathfrak{t}_{\mathbf{s}}$ are regular. The purpose of this work is the study of the cut spectrum below $\mathfrak{t}_{\mathbf{s}}$ of a cofinality spectrum.

Definition 2.2.11 *Let* $\mathbf{s}$ *be a cofinality spectrum problem and* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$.

1. *Write* $0_{\mathbf{a}}$ *for the* $\leq_{\mathbf{a}}$*-least element of* $X_{\mathbf{a}}$.

2. *For any natural number* k *and any* $a \in X_{\mathbf{a}}$, *let* $S^k_{\mathbf{a}}(a)$ *denote the k-th successor of* a *in the discrete linear order* $\leq_{\mathbf{a}}$, *if defined, and likewise let* $S^{-k}_{\mathbf{a}}(a)$ *denote the k-th predecessor of* a, *if defined. When the context is clear, we will generally write* $S^k(a)$ *and* $S^{-k}(a)$.

3. *Say that* $c \in \mathcal{T}_{\mathbf{a}}$ *is* ***below the ceiling*** *if* $S^k(\lg(c)) <_{\mathbf{a}} d_{\mathbf{a}}$ *for all* $k < \omega$, *i.e. if these successors exist and the statement is true.*

As this point, we should do some pertinent remarks.

✓ The orders we work within a cofinality spectrum problem can be seen as a generalization of non-standard natural numbers: sets with a definable order relation, a distinguished element (the so-called $d_{\mathbf{a}}$, non reachable via finite successors of $0_{\mathbf{a}}$) and in which their non-empty definable bounded sets have first and last element.

✓ Following lemma 2.2.4, we can see that the study of certain gaps in a cofinality spectrum problem could be analyzed considering conditions in certain distinguished trees.

✓ Conditions 5. and 6. in definition 2.2.5 could be seen as artificial conditions inside a cofinality spectrum problem, but these conditions will allow us later to define an arithmetic in a cofinality spectrum problem, which will be very useful, and will let us consider more sophisticated orders.

The next two results are shown having in mind lemma 2.2.4. When we realized the construction of the sequence $(c_\alpha)_{\alpha<\kappa}$ in $\mathcal{T}_*$, we consider a suitable upper bound $c_* \in \mathcal{T}_*$ which will allow us deduce a contradiction. We are giving the tools that allow us find these convenient upper bounds to certain sequences in a cofinality spectrum problem.

Fact 2.2.12 ([11, Claim 2.14]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem,* $M_1^+ = M_1^{+,s}$. *Let* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$, *so* $\mathcal{T}_\mathbf{a} \in \mathrm{Tr}(\mathbf{s})$. *Let* φ *be a formula, possibly with parameters in* M_1^+, *and let* $(\mathcal{T}, \trianglelefteq_\mathbf{a})$ *be the subtree of* $(\mathcal{T}_\mathbf{a}, \trianglelefteq_\mathbf{a})$ *defined by* φ *in* M_1^+. *Let* $\langle c_\alpha : \alpha < \kappa \rangle$ *be a* $\trianglelefteq_\mathbf{a}$*-increasing sequence of elements of* $\mathcal{T}$, *with* $\kappa = cf(\kappa) < \mathfrak{t}_\mathbf{s}$. *Then there is a* $c_* \in \mathcal{T}$ *such that for all* $\alpha < \kappa$, $c_\alpha \trianglelefteq_\mathbf{a} c_*$.

Proof. By definition of $\mathfrak{t}_\mathbf{s}$, we may find an element $c \in \mathcal{T}_\mathbf{a}$ (not necessarily in $\mathcal{T}$) such that $c_\alpha \trianglelefteq_\mathbf{a} c$ for all $\alpha < \kappa$. Now the set $\{\lg(c') : c' \trianglelefteq_\mathbf{a} c \text{ and } c' \in \mathcal{T}\}$ is a nonempty definable subset of $X_\mathbf{a}$, hence it contains a last member a_*, and consider $c_* := c \restriction_{a_*}$: it is clear that $c_* \trianglelefteq_\mathbf{a} c$, $\lg(c_*) = a_*$ and $c_* \in \mathcal{T}$ is an upper bound of $\{c_\alpha : \alpha < \kappa\}$. ■

Fact 2.2.13 ([11, Lemma 2.15]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem,* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$, $\kappa < \min\{\mathfrak{p}_\mathbf{s}, \mathfrak{t}_\mathbf{s}\}$. *Let* $\mathcal{T} \subseteq \mathcal{T}_\mathbf{a}$ *a definable subtree and* $\bar{c} = \langle c_\alpha : \alpha < \kappa \rangle$ *be a strictly* $\trianglelefteq_\mathbf{a}$*-increasing sequence of elements of* $\mathcal{T}$. *Then there is a* $c_{**} \in \mathcal{T}$ *such that for all* $\alpha < \kappa$, $c_\alpha \trianglelefteq_\mathbf{a} c_{**}$ *and* c_{**} *is below the ceiling.*

Proof. Let $c_* \in \mathcal{T}$ be such that $c_\alpha \trianglelefteq_\mathbf{a} c_*$, just as given by fact 2.2.12. Since $\bar{c}$ is strictly increasing, for each $\alpha < \kappa$ the element c_α is below the ceiling. If c_* is also below the ceiling, we are done. Otherwise, notice that $(\{\lg(c_\beta) : \beta < \alpha\}, \{S^{-k}(\lg(c_*)) : k < \omega\})$ represents a pre-gap in $X_\mathbf{a}$, which cannot be a gap, since $\aleph_0 \leq \kappa < \mathfrak{p}_\mathbf{s}$. Therefore, we may choose some $a \in X_\mathbf{a}$ that realizes this pre-gap, and consider $c_{**} := c_* \restriction_a$. ■

2.2.1. The lower cofinality function lcf in a cofinality spectrum problem

In lemma 2.2.4, we have seen there are some kind of gaps we cannot realize in a specific cofinality spectrum problem (the so-called *symmetric gaps*, i.e. (κ, κ)-gaps). However, a question emerges immediately: given $\mathbf{s}$ a cofinality spectrum problem and κ a regular cardinal, can we find an adequate cardinal λ such that $(\kappa, \lambda) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$? Under which conditions can we possibly find this λ?

Lemma 2.2.14 ([11, Lemma 3.1]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. If* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ *is nontrivial, then for each infinite regular* $\kappa \leq \mathfrak{p}_{\mathbf{s}}$*:*

(1) There is a strictly decreasing κ*-indexed sequence* $\overline{a} = \langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* $X_{\mathbf{a}}$ *such that*

$$(\{S^k(0_{\mathbf{a}}) : k < \omega\}, \{a_\alpha : \alpha < \kappa\})$$

represents a pre-gap (and possibly a gap) in $X_{\mathbf{a}}$*.*

(2) There is a strictly increasing κ*-indexed sequence* $\overline{a} = \langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* $X_{\mathbf{a}}$ *such that*

$$(\{a_\alpha : \alpha < \kappa\}, \{S^{-k}(d_{\mathbf{a}}) : k < \omega\})$$

represents a pre-gap (and possibly a gap) in $X_{\mathbf{a}}$*.*

(3) There is at least one infinite regular θ *such that* $(\kappa, \theta) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$*, witnessed by a* (κ, θ)*-gap in* $X_{\mathbf{a}}$*.*

(4) There is at least one infinite regular θ' *such that* $(\theta', \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$*, witnessed by a* (θ', κ)*-gap in* $X_{\mathbf{a}}$*.*

Proof. We will prove (2) and (4) following the ideas on the proofs of (1) and (3) done in [11]

(2) By induction on $\alpha < \kappa$ we choose elements $a_\alpha \in X_{\mathbf{a}}$ such that:

(i) for each $\alpha < \kappa$ and each $k < \omega$, $a_\alpha <_{\mathbf{a}} S^{-k}(d_{\mathbf{a}})$.

(ii) $\beta < \alpha$ implies $a_\beta <_{\mathbf{a}} a_\alpha$.

For $\alpha = 0$, let $a_0 = 0_{\mathbf{a}}$. As $\mathbf{a}$ is nontrivial, condition (i) is satisfied. For $\alpha = \beta + 1$, consider $a_\alpha = S^1(a_\beta)$. As any non-empty definable subset of $X_{\mathbf{a}}$ has a least element, the successor of any element not equal to $d_{\mathbf{a}}$ is well defined. Since (i) holds for β, it will still hold for $\beta + 1$. Now, for α limit, we know that

$$(\{a_\beta : \beta < \alpha\}, \{S^{-k}(d_{\mathbf{a}}) : k < \omega\})$$

is a pre-gap, which can't be a gap: otherwise, $(\mathrm{cf}\,(\alpha), \aleph_0) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$ and $\alpha < \kappa \leq \mathfrak{p}_{\mathbf{s}}$, contradicting the definition of $\mathfrak{p}_{\mathbf{s}}$. So, we are able to choose a_α that fills this pre-gap, finishing the proof.

(4) Let $\overline{a}$ be the κ-indexed strictly decreasing $\leq_{\mathbf{a}}$-monotonic sequence of elements of $X_{\mathbf{a}}$ given by (1). By construction, $B = \{b \in X_{\mathbf{a}} : \alpha < \kappa \text{ implies } b <_{\mathbf{a}} a_\alpha\} \neq \emptyset$ and it does

not have maximum. Let $\theta' = \mathrm{cf}\,(B)$. By the way we consider the sequence $\overline{a}$, we can conclude that

$$(\{b_\gamma : \gamma < \theta'\}, \{a_\alpha : \alpha < \kappa\})$$

represents a gap, for any cofinal increasing sequence $\langle b_\gamma : \gamma < \theta' \rangle \subseteq B$. Hence, θ' is an infinite regular cardinal such that $(\theta', \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

This concludes the proof. ■

Note that lemma 2.2.14 just provides to us the existence of some θ, θ' such that $(\kappa, \theta), (\theta', \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$. The following theorem will let us deduce that, in fact, $\theta = \theta'$ whenever $\kappa < \mathfrak{t}_{\mathbf{s}}$.

Theorem 2.2.15 ([11, Thm. 3.1]) *Let $\mathbf{s}$ be a cofinality spectrum problem. Then for each regular $\kappa \leq \mathfrak{p}_{\mathbf{s}}$, $\kappa < \mathfrak{t}_{\mathbf{s}}$*

(1) there is one and only one λ regular such that $(\kappa, \lambda) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

(2) $(\kappa, \lambda) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$ if and only if $(\lambda, \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

Proof. We are following the same notation used in [11]. Let κ be as in the hypothesis of the theorem. Lemma 2.2.14 allows us to ensure that there are some infinite regular cardinals θ_1, θ_2 such that $(\kappa, \theta_1), (\theta_2, \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$. Therefore, we will prove that given $\mathbf{a}, \mathbf{b} \in \mathrm{Or}(\mathbf{s})$, if

- ✓ $(\langle a^1_\alpha : \alpha < \kappa \rangle, \langle b^1_\gamma : \gamma < \theta_1 \rangle)$ represents a (κ, θ_1)-gap in $(X_{\mathbf{a}}, <_{\mathbf{a}})$.
- ✓ $(\langle b^2_\gamma : \gamma < \theta_2 \rangle, \langle a^2_\alpha : \alpha < \kappa \rangle)$ represents a (θ_2, κ)-gap in $(X_{\mathbf{b}}; <_{\mathbf{b}})$,

then $\theta_1 = \theta_2$.
The main problem in this proof is that we have to model two gaps (in two different orders) instead of one. However, we can consider the order $\mathbf{c} \in \mathrm{Or}(\mathbf{s})$ given by $X_{\mathbf{c}} = X_{\mathbf{a}} \times X_{\mathbf{b}}$. In this way, the associated tree $\mathcal{T}_{\mathbf{c}}$ will model both gaps, and the treetops condition (implicit in proof, as we know that $\kappa < \mathfrak{t}_{\mathbf{s}}$) will witness the realization or omission of theses gaps.
So, first we are giving a simple description of $\mathcal{T}_{\mathbf{c}}$: given $x \in \mathcal{T}_{\mathbf{c}}$, for each $n \leq \max(\mathrm{dom}(x))$, $x(n) = (x(n,1), x(n,2))$, where $x(n,1) \in X_{\mathbf{a}}$ and $x(n,2) \in X_{\mathbf{b}}$.
Our next step is to describe both gaps in a suitable subtree. For this, consider $\mathcal{T}_0 \subseteq \mathcal{T}_{\mathbf{c}}$ defined by

$$n < m < \max(\mathrm{dom}(x)) \quad \text{implies} \quad (x(n,1) <_{\mathbf{a}} x(m,1)) \wedge (x(m,2) <_{\mathbf{b}} x(n,2)) \tag{2-6}$$

Now, by transfinite induction, we choose $\langle c_\alpha : \alpha < \kappa \rangle \subseteq \mathcal{T}_0$ and $\langle n_\alpha : \alpha < \kappa \rangle \subseteq X_{\mathbf{c}}$ as follows:

(i) $\beta < \alpha$ implies $M_1^+ \vDash c_\beta \trianglelefteq_{\mathbf{c}} c_\alpha$.

(ii) $n_\alpha = \lg(c_\alpha) - 1$, so $\max(\mathrm{dom}(c_\alpha))$ is well defined.

(iii) n_α is below the ceiling (see definition 2.2.11 (3)).

(iv) $c_\alpha(n_\alpha, 1) = a^1_\alpha$ and $c_\alpha(n_\alpha, 2) = a^2_\alpha$.

Let's perform the induction:

- ✓ (Case $\alpha = 0$) Let $c_0 = \langle a^1_0, a^2_0 \rangle$ and $n_0 = 0_{\mathbf{c}}$.
- ✓ (Case $\alpha = \beta + 1$) Note that c_β is below the ceiling, and thanks to conditions (ii)-(iv), we can concatenate, so define $c_\alpha = c_\beta{}^\frown \langle a^1_\beta, a^2_\beta \rangle$ and $n_\alpha = n_\beta + 1$.
- ✓ (Case $\alpha < \kappa$ limit) Since $\mathrm{cf}(\alpha) < \min\{\mathfrak{p}_{\mathbf{s}}, \mathfrak{t}_{\mathbf{s}}\}$, by fact 2.2.13 we can find $c \in \mathcal{T}_0$ such that $M_1^+ \vDash c_\beta \trianglelefteq_{\mathbf{c}} c$, for any $\beta < \alpha$ with $\mathrm{dom}\, c$ below the ceiling. Let $n = \max(\mathrm{dom}(c))$. Note that we might have the case that $a^1_\alpha <_{\mathbf{a}} c(n,1)$ and $c(n,2) <_{\mathbf{b}} a^2_\alpha$. However, the set

  $$\{n \,:\, n < \lg(c), M_1^+ \vDash (c(n,1) <_{\mathbf{a}} a^1_\alpha) \wedge (a^2_\alpha <_{\mathbf{b}} c(n,2))\} \tag{2-7}$$

  is a non-empty, definable and bounded set, so it has a maximal element n_*. We can concatenate since n and all its initial segments are below the ceiling. In that way, we're able to define $c_\alpha = (c\restriction_{n_*})^\frown \langle a^1_\alpha, a^2_\alpha \rangle$ and $n_\alpha = n_*$.

Using fact 2.2.12, there is a $c \in \mathcal{T}_0$ such that $c_\alpha \trianglelefteq_{\mathbf{c}} c$ for any $\alpha < \kappa$. Let $n_{**} = \lg(c) - 1 \in X_{\mathbf{c}}$, and define

$$n_{\gamma,1} = \max\{n \leq_{\mathbf{c}} n_{**} \,:\, c(n,1) <_{\mathbf{a}} b^1_\gamma\}$$
$$n_{\gamma,2} = \max\{n \leq_{\mathbf{c}} n_{**} \,:\, b^2_\gamma <_{\mathbf{b}} c(n,2)\}$$

Note that $\alpha < \kappa$ implies $n_\alpha <_{\mathbf{c}} n_{\gamma,1}$ and $n_\alpha <_{\mathbf{c}} n_{\gamma,2}$. By the choice of sequences witnessing the original gaps, we can conclude that for $l = 1, 2$

$$(\langle n_\alpha \,:\, \alpha < \kappa \rangle, \langle n_{\gamma,l}; \,:\, \gamma < \theta_l \rangle) \tag{2-8}$$

represents a gap in $X_{\mathbf{c}}$: otherwise, if there were an m realizing this pre-gap, we would have that

$$a^1_\alpha = c(n_\alpha, 1) <_{\mathbf{a}} c(m,1) \quad \text{and} \quad c(m,1) <_{\mathbf{b}} c(n_\gamma, 1) < b^1_\gamma \tag{2-9}$$

so $c(m,1)$ would realize the first of the original gaps, which is absurd. The same argument applies for $l = 2$. This previous fact and the regularity of θ_1, θ_2 (given by hypothesis) guarantees us that $\theta_1 = \theta_2$: indeed, consider the map $f : \theta_1 \to \theta_2$ such that $f(\zeta) := \min\{\gamma < \theta_2 \,:\, n_{\gamma,2} < n_{\zeta,1}\}$. Notice that f is well defined, since for every $n_{\zeta,1}$ there is a $n_{\gamma,2}$ such that

$n_{\gamma,2} < n_{\zeta,1}$. Moreover, f is cofinal in θ_2, since if $\zeta < \theta_2$, then $n^1_\gamma < n^2_\zeta$ for some $\gamma < \theta_1$, because $(\langle n_\alpha : \alpha < \kappa\rangle, \langle n_{\gamma,1} : \gamma < \theta_1\rangle)$ witnesses a (κ, θ_1)-gap. Therefore, we may conclude that $f(\gamma) > \zeta$. Hence $\theta_2 = \operatorname{cof}(\theta_2) \leq \theta_1$. An analog argument allows to conclude that $\theta_1 \leq \theta_2$. ■

Thanks to the theorem 2.2.15, we can give the following definition.

Definition 2.2.16 (The lower cofinality $\operatorname{lcf}(\kappa, \mathbf{s})$**)** *Let* $\mathbf{s}$ *be a cofinality spectrum problem. Given a regular cardinal* $\kappa \leq \mathfrak{p}_\mathbf{s}$, $\kappa < \mathfrak{t}_\mathbf{s}$, *we define* $\operatorname{lcf}(\kappa, \mathbf{s})$ *to be the unique* θ *such that* $(\kappa, \theta) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

According to lemma 2.2.14 and theorem 2.2.15, the following corollaries are immediate.

Corollary 2.2.17 *Let* $\mathbf{s}$ *be a cofinality spectrum problem and* κ *a regular cardinal,* $\kappa \leq \mathfrak{p}_\mathbf{s}$, $\kappa < \mathfrak{t}_\mathbf{s}$. *Then the following are equivalent:*

(1) $\operatorname{lcf}(\kappa, \mathbf{s}) = \theta$.

(2) $(\kappa, \theta) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

(3) $(\theta, \kappa) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$.

Corollary 2.2.18 *Let* $\mathbf{s}_1, \mathbf{s}_2$ *be cofinality spectrum problems and suppose that* $M^{\mathbf{s}_1} = M^{\mathbf{s}_2}$, $M_1^{+,\mathbf{s}_1} = M_1^{+,\mathbf{s}_2}$ *up to the language* $\mathcal{L}(M_1^{+,\mathbf{s}_1}) \cap \mathcal{L}(M_1^{+,\mathbf{s}_2})$. *If there is a non-trivial order* $\mathbf{a} \in \operatorname{Or}(\mathbf{s}_1) \cap \operatorname{Or}(\mathbf{s}_2)$ *(i.e.* $\Delta^{\mathbf{s}_1} \cap \Delta^{\mathbf{s}_2} \neq \emptyset$*) then, for all regular* κ *with* $\kappa \leq \min\{\mathfrak{p}_{\mathbf{s}_1}, \mathfrak{p}_{\mathbf{s}_2}\}$, $\kappa < \min\{\mathfrak{t}_{\mathbf{s}_1}, \mathfrak{t}_{\mathbf{s}_2}\}$, $\operatorname{lcf}(\kappa, \mathbf{s}_1) = \operatorname{lcf}(\kappa, \mathbf{s}_2)$. *Moreover, the same conclusion holds if* $\mathbf{s}_1 \leq \mathbf{s}_2$.

Proof. If $\operatorname{lcf}(\kappa, \mathbf{s}_1) = \theta$, then, by theorem 2.2.15, there is a (κ, θ)-gap in $X_\mathbf{a}$ (in $\mathbf{s}_1$). As $M^{\mathbf{s}_1} = M^{\mathbf{s}_2}$ and $M_1^{+,\mathbf{s}_1} = M_1^{+,\mathbf{s}_2}$, then this gap in $X_\mathbf{a}$ is also detected in the problem $\mathbf{s}_2$, thus $\operatorname{lcf}(\kappa, \mathbf{s}_2) = \theta$. ■

Remark 2.2.19 Corollary 2.2.18 is really useful: it provides us a kind of "invariance" of the function lcf under the relation $\leq$. Also, this result tells us that certain gaps in $X_\mathbf{a}$ in a cofinality spectrum problem $\mathbf{s}$ will remain the same in any other cofinality spectrum problem $\mathbf{s}'$ that contains $\mathbf{a}$ and with the same ground model and main expansion of $\mathbf{s}$.

2.3. Local saturation and Gödel codes

In this section we study the logic aspects in a cofinality spectrum problem. First, we study a notion of Model-theoretic saturation, called *local saturation*. Then, we focus our efforts in building an adequate arithmetic which will allow us carry out a Gödel codification.

2.3.1. Or-**types**

In this section, we discuss the property of the cofinality spectrum problems of having some amount of local saturation. As indicated in [11], the "local" sense here means partial types satisfied in a distinguished order.

Definition 2.3.1 *Let* $\mathbf{s}$ *be a cofinality spectrum problem, let* λ *be a regular cardinal and* $p = p(x_0, ..., x_{n-1})$ *be a consistent partial type with parameters in* M_1^+*.*

(1) We say that p *is an* Or-***type over*** M_1^+ *if* p *is a consistent partial type in* M_1^+ *and for some* $\mathbf{a}_0, ..., \mathbf{a}_{n-1} \in \mathrm{Or}(\mathbf{s})$*, we have that*

$$p \vdash \bigwedge_{i<n} x_i \in X_{\mathbf{a}_i} \tag{2-10}$$

(2) An Or*-type* p *over* M_1^+ *is* ***realized in*** M_1^+ *if there are* $\overline{a} = (a_0, ..., a_{n-1}) \in |M_1^+|$ *such that* $M_1^+ \vDash \varphi(\overline{a})$ *for all* $\varphi(\overline{x}) \in p$*.*

(3) We say that M_1^+ *is* λ-Or-***saturated*** *if every* Or*-type* p *over* M_1^+ *with parameters in some* $A \subseteq M_1^+$ *of size* $< \lambda$ *is realized in* M_1^+*.*

(4) Finally, we say that $\mathbf{s}$ *is* λ-Or-***saturated*** *if* M_1^+ *is.*

In the previous definition, we may assume that $p = p(x)$, where $p \vdash x \in X_{\mathbf{a}}$, for some $a \in \mathrm{Or}(\mathbf{s})$ (by the closure under -finite- Cartesian products).

Theorem 2.3.2 ([11, Thm. 4.1]) *If* $\mathbf{s}$ *is a cofinality spectrum problem and* $\kappa < \min\{\mathfrak{p}_{\mathbf{s}}, \mathfrak{t}_{\mathbf{s}}\}$*, then* $\mathbf{s}$ *is* κ^+-Or*-saturated.*

Proof. This proof will be done by induction over $\kappa < \min\{\mathfrak{p}_{\mathbf{s}}, \mathfrak{t}_{\mathbf{s}}\}$. Let us first give a little description of the proof.
As in lemma 2.2.4 and theorem 2.2.15, given a convenient order $\mathbf{a} \in \mathrm{Or}$ and its associated tree $\mathcal{T}_{\mathbf{a}} \in \mathrm{Tr}(\mathbf{s})$, we have to find a definable subtree which models, in this case, the realization or omission of Or-types.
So, suppose either $\kappa = \aleph_0$ or that the theorem holds for any $\mu < \kappa$. Let $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ and let $p = \{\varphi_i(x; \overline{a}_i) : i < \kappa\}$ be finitely satisfiable in $X_{\mathbf{a}}$. Doing a new induction (called "internal") on $\alpha \leq \kappa$, we choose $c_\alpha \in \mathcal{T}_{\mathbf{a}}$ and $n_\alpha \in X_{\mathbf{a}}$ as follows:

(1) $\beta < \alpha$ implies $c_\beta \trianglelefteq_{\mathbf{a}} c_\alpha$.

(2) $n_\alpha = \lg(c_\alpha) - 1$.

(3) c_α is below the ceiling.

(4) if $i < \beta \leq \alpha$ and $n_\beta \leq_{\mathbf{a}} n \leq_{\mathbf{a}} n_\alpha$, then $M_1^+ \vDash \varphi_i(c_\alpha(n), \overline{a}_i)$.

Let's perform the induction.

✓ (Case $\alpha = 0$) Since p is finitely satisfiable, we can find some $d \in X_{\mathbf{a}}$ such that $M_1^+ \vDash \varphi_0(d, \overline{a}_0)$. Let $c_0 := \langle d \rangle$ and $n_0 := 0$.

✓ (Case $\alpha = \beta + 1$) We analyze two cases.

- If $\kappa = \aleph_0$ and $\beta < \omega$, then $\{\varphi_i(x, \overline{a}_i) : i \leq \beta\}$ is finite, and by the finitely satisfability of p we have that there is some $d \in X_{\mathbf{a}}$ such that $M_1^+ \vDash \varphi_i(d, \overline{a}_i)$ for all $i \leq \beta$. By hypothesis (3) we can do concatenations, so define $c_\alpha := c_\beta{}^\frown\langle d \rangle$ and $n_\alpha := n_\beta + 1$.
- Otherwise, by external inductive hypothesis, we can find a realization $d \in X_{\mathbf{a}}$ of $\{\varphi_i(x, \overline{a}_i) : i \leq \beta\}$ (since $|\alpha| < \kappa$), and by the the internal inductive hypothesis (3), we can do concatenations, so define $c_\alpha := c_\beta{}^\frown\langle d \rangle$ and $n_\alpha := n_\beta + 1$.

✓ (Case $\alpha \leq \kappa$ limit) By fact 2.2.13, we can find $c_* \in \mathcal{T}_{\mathbf{a}}$ such that $\beta < \alpha$ implies $c_\beta \trianglelefteq_{\mathbf{a}} c_*$, c_* is below the ceiling and let $n_* := \lg(c_*) - 1$. As before, we need to refine this value to satisfy condition (4). We will do this as follows: for each $i < \alpha$, define

$$n(i)_{\mathbf{a}} := \max\{n \leq_{\mathbf{a}} n_* : M_1^+ \vDash \varphi_i(c_*(m), \overline{a}_i) \text{ for all } m \text{ such that } n_i <_{\mathbf{a}} m \leq_{\mathbf{a}} n\} \tag{2-11}$$

We can see this is a nonempty bounded subset of $X_{\mathbf{a}}$, so $n(i)$ exists for each $i < \alpha$, and the internal inductive hypothesis (4) guarantees us that $n(i)_{\mathbf{a}} > n_\beta$ for each $i, \beta < \alpha$. Thus,

$$(\{n_\beta : \beta < \alpha\}, \{n(i)_{\mathbf{a}} : i < \alpha\}) \tag{2-12}$$

represents a pre-gap in $X_{\mathbf{a}}$. Let γ be a co-initial subsequence of $\{n(i)_{\mathbf{a}} : i < \alpha\}$. Note that $|\gamma| \leq \kappa < \mathfrak{p}_{\mathbf{s}}$, and besides $(\mathrm{cf}(\alpha), \gamma) \notin \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$: otherwise, we would contradict the definition of $\mathfrak{p}_{\mathbf{s}}$. Thus, let $n_{**} \in X_{\mathbf{a}}$ realizing this pre-gap, and define $c_\alpha = c \restriction_{n_{**}}$ and $n_\alpha = \lg(c_\alpha) - 1$. This completes the induction. Also, note that $M_1^+ \vDash \varphi_i(c_\alpha(n), \overline{a}_i)$ for each $i < \alpha$ and $n_i \leq n \leq n_\alpha$.

As the limit case also included $\alpha = \kappa$, we claim that $c_\kappa(n_\kappa)$ is the realization of p we desired. ■

2.3.2. Arithmetic in a cofinality spectrum problem

Until now, the study of the cofinality spectrum problems has been centred in analyzing branches of certain trees of distinguished orders. These branches capture some properties we want to know about these trees, i.e. realization or omission of certain gaps. Nevertheless, complexity of these trees could be getting more difficult when we need to capture more information about certain gaps in orders. So, how can we reduce the complexity of theses trees? That is the main task of the Gödel codifications.

The following proposition allows us to refine orders in a cofinality spectrum problems without modifying the essence of gaps.

Proposition 2.3.3 ([11, Cor. 3.9]) *Let* $\mathbf{s} = (M^{\mathbf{s}}, M_1^{\mathbf{s}}, M^{+,\mathbf{s}}, M_1^{+,\mathbf{s}}, T^{\mathbf{s}}, \Delta^{\mathbf{s}})$ *be a cofinality spectrum problem. Then we may assume* $\mathrm{Or}(\mathbf{s})$ *is closed under definable subsets of* $X_{\mathbf{a}}$*, i.e. whenever* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$*,* $\psi(x)$ *a formula in the language of* M_1 *such that* $\psi(x) \vdash x \in X_{\mathbf{a}}$*, there is* $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$ *with* $\leq_{\mathbf{b}} = \leq_{\mathbf{a}}$ *and*

$$X_{\mathbf{b}} = \{a \in X_{\mathbf{a}} \, : \, M_1 \vDash \psi(a)\} \tag{2-13}$$

For definiteness, we specify that $d_{\mathbf{b}} = \min\{d_{\mathbf{a}}, \max X_{\mathbf{b}}\}$.

Proof. Let $\mathbf{s}$ be a cofinality spectrum problem. We consider the set of formulas

$$\Delta' := \{\chi(x, y, \overline{z}) = \varphi(x, y, \overline{z}_1) \wedge \psi(x, \overline{z}_2) \, : \, \varphi(x, y, \overline{z}_1) \in \Delta^{\mathbf{s}}, \psi(x, \overline{z}_2) \in \mathcal{L}(M), \, \overline{z} := \overline{z}_1{}^\frown \overline{z}_2\}$$

It is clear that $\Delta^{\mathbf{s}} \subseteq \Delta'$, hence applying Corollary 2.2.18, we claim that the cofinality spectrum problem $\mathbf{s}' := (M^{\mathbf{s}}, M_1^{\mathbf{s}}, M^{+,\mathbf{s}}, M_1^{+,\mathbf{s}}, T^{\mathbf{s}}, \Delta')$ is more complex than $\mathbf{s}$. Now, since $\Delta^{\mathbf{s}} \subseteq \Delta'$, we have that $\mathfrak{p}_{\mathbf{s}'} \leq \mathfrak{p}_{\mathbf{s}'}$ and $\mathfrak{t}_{\mathbf{s}'} \leq \mathfrak{t}_{\mathbf{s}'}$. Now suppose that $X_{\mathbf{b}} = \{a \in X_{\mathbf{a}} \, : \, M_1 \vDash \psi(a)\}$. If $(\overline{a}, \overline{b})$ is a gap in $X_{\mathbf{b}}$, then it is also a gap in $X_{\mathbf{a}}$. If not, and $c \in X_{\mathbf{a}}$ fills the gap, then $c' := \min\{a \in X_{\mathbf{a}} \, : \, M_1^+ \vDash \psi(a) \wedge c \leq a\} \in X_{\mathbf{b}}$ and fills the gap. Likewise, if $\overline{\eta}$ is an increasing unbounded sequence in $\mathcal{T}_{\mathbf{b}}$, then it is also in $\mathcal{T}_{\mathbf{a}}$. ■

Now let us describe how Gödel codifications of trees works. Let $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$ be a non-trivial order. We define the formulas associated to the sum, the multiplication and the exponentiation (in the language of M_1^+) on $X_{\mathbf{b}}$. It is important to remark that $\varphi_+^{\mathbf{b}}$, $\varphi_\times^{\mathbf{b}}$ and $\varphi_{\exp}^{\mathbf{b}}$ are defined along a branch of $\mathcal{T}_{\mathbf{b}}$. The variables range over elements of $X_{\mathbf{b}}$ unless otherwise indicated.

(1) Define the sum $\varphi_+^{\mathbf{b}}(x, y, z)$ as follows

$$\begin{aligned}(\exists \eta \in T_{\mathbf{b}})[&\lg(\eta) = y + 1 \wedge \eta(0) = x \wedge \eta(y) = z \\ &\wedge (\forall i \in X_{\mathbf{b}})(i < y \to \eta(S(i)) = S(\eta(i))].\end{aligned}$$

(2) Define the product $\varphi_\times^{\mathbf{b}}(x, y, z)$ as follows

$$\begin{aligned}(\exists \eta \in T_{\mathbf{b}})[&\lg(\eta) = y + 1 \wedge \eta(0) = x \wedge \eta(y) = z \\ &\wedge (\forall i \in X_{\mathbf{b}})(i < y \to \varphi_+^{\mathbf{b}}(\eta(i), x, \eta(S(i)))].\end{aligned}$$

(3) Define the exponentiation $\varphi^{\mathbf{b}}_{\exp}(x,y,z)$ as follows

$$(\exists \eta \in T_{\mathbf{b}})[\lg(\eta) = y+1 \,\wedge\, \eta(0) = x \,\wedge\, \eta(y) = z \\ \wedge (\forall i \in X_{\mathbf{b}})(i < y) \,\rightarrow\, \varphi^{\mathbf{b}}_{\times}(\eta(i), x, \eta(S(i)))].$$

With help of the sum and the product, we can define the formula "x divides y":

$\varphi^{\mathbf{b}}_{\mathrm{div}}(x,y) : x < y \,\wedge\, \exists z \in X_{\mathbf{b}}\, \varphi^{\mathbf{b}}_{\times}(x,z,y)$,

also, we can define "x is prime" as follows:

$\varphi^{\mathbf{b}}_{\mathrm{prime}}(x) : x > 1_{\mathbf{b}} \,\wedge\, \forall y\, (\varphi^{\mathbf{b}}_{\mathrm{div}}(x,y) \iff (y = 1_{\mathbf{b}} \,\wedge\, y = x))$.

Having the notion of prime number in $X_{\mathbf{b}}$, we can define the formula "y is the first prime above x":

$\varphi^{\mathbf{b}}_{\mathrm{fp}}(x,y) : x < y \,\wedge\, \varphi^{\mathbf{b}}_{\mathrm{prime}}(y) \,\wedge\, \neg\exists p\, (x < p < y \,\wedge\, \varphi^{\mathbf{b}}_{\mathrm{prime}}(p))$,

and the formula "x is the nth prime":

$\psi^{\mathbf{b}}(x,n) : \exists \eta \in \mathcal{T}_{\mathbf{b}}\, [\lg(\eta) = n+1 \,\wedge\, \eta(0) = 2 \,\wedge\, \left(\forall k < n\, \varphi^{\mathbf{b}}_{\mathrm{fp}}(\eta(k), \eta(S(k))) \,\wedge\, \eta(n) = x]\right)$,

Why do we have to define all these formulas? Informally speaking, we want to imitate two well-known processes in $\mathbb{N}$: prime decomposition and Fundamental theorem of Arithmetic. Moreover, since induction is valid in $X_{\mathbf{b}}$, then all the notations we are about to introduce make sense as long as objects remain below $d_{\mathbf{b}}$. Also, the choice of $d_{\mathbf{b}}$ will give us a bound for the length of the branches of $\mathcal{T}_{\mathbf{b}}$.

Let us describe first how we can perform prime decomposition of elements of $X_{\mathbf{b}}$: define the formula $\chi_1(x,n,m)$ saying that "x is divisible by the nth prime exactly m times":

$$\chi_1(x,n,m) : \exists \eta \in \mathcal{T}_{\mathbf{b}}\, \exists p\, [\psi^{\mathbf{b}}(p,n) \,\wedge\, \lg(\eta) = m+1 \,\wedge\, \eta(0) = x \\ \wedge \neg\varphi^{\mathbf{b}}_{\mathrm{div}}(p, \eta(m)) \,\wedge\, \forall k < m\, \varphi^{\mathbf{b}}_{\times}(\eta(S(k)), p, \eta(k)].$$

Having prime decomposition, it is natural to think whether this decomposition is unique. To ensure this, define the formula staying that there is an $\eta \neq \emptyset$ such that $\eta \in \mathcal{T}_{\mathbf{b}}$, $x > 2$, and for all $i < \lg(\eta)$, x is divisible by the ith prime exactly $\eta(i) + 1$ times:

$$\exists\, !\eta^{\mathbf{b}}_{\mathrm{pr}} \in \mathcal{T}_{\mathbf{b}}\, [\eta^{\mathbf{b}}_{\mathrm{pr}}(0) = 2 \,\wedge\, (\forall k < \max\mathrm{dom}\,(\eta^{\mathbf{b}}_{\mathrm{pr}}))\, \varphi^{\mathbf{b}}_{\mathrm{fp}}(\eta^{\mathbf{b}}_{\mathrm{pr}}(k), \eta^{\mathbf{b}}_{\mathrm{pr}}(k+1)) \\ \wedge (\forall p \le d_{\mathbf{b}})\, (\varphi^{\mathbf{b}}_{\mathrm{prime}}(p) \Rightarrow \exists k < \lg(\eta^{\mathbf{b}}_{\mathrm{pr}})\, (p = \eta^{\mathbf{b}}_{\mathrm{pr}}(k)))].$$

(Last formula allows us to enumerate all the prime below $d_{\mathbf{b}}$ in order). Now, let $\chi_2(x,\eta)$ be the formula saying that $x \in X_{\mathbf{b}}$ is a Gödel code for η:

$$\chi_2(x, \eta) : \forall k < \lg(\eta^{\mathbf{b}}_{\mathrm{pr}}) \, \big(\chi_1(x, k, \eta(k)) + 1\big) \wedge \lg(\eta) = \lg(\eta^{\mathbf{b}}_{\mathrm{pr}}\big) .$$

Now, we notice that each element of $X_{\mathbf{b}}$ is the code of some $\eta \in \mathcal{T}_{\mathbf{b}}$, but not all the elements of in $\mathcal{T}_{\mathbf{b}}$ will have a Gödel code. So, consider the set

$$X_{\mathbf{a}} := \{y \in X_{\mathbf{b}} : (\exists x \leq d_{\mathbf{b}})(\exists \eta \in \mathcal{T}_{\mathbf{b}})(\exists k < \lg(\eta)) \, \chi_2(x, \eta) \wedge y = \eta(k)\}. \tag{2-14}$$

Notice that $d_{\mathbf{a}} := \max X_{\mathbf{a}} \leq d_{\mathbf{b}}$ and $X_{\mathbf{a}}$ is an initial segment of $X_{\mathbf{b}}$, i.e. $X_{\mathbf{a}} := \{x \in X_{\mathbf{b}} : x \leq d_{\mathbf{a}}\}$. By proposition 2.3.3, then $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ and $\mathbf{b}$ contains all the Gödel codes of elements of $\mathcal{T}_{\mathbf{a}}$. What we wanted to do here was to describe the mechanism of Gödel codification.

Lemma 2.3.4 ([11, Lemma 5.3]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem and* $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$*. Then there is* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ *such that* $X_{\mathbf{a}}$ *is an initial segment of* $X_{\mathbf{b}}$ *and all Gödel codes for elements of* $\mathcal{T}_{\mathbf{a}}$ *belong to* $X_{\mathbf{b}}$*. In particular, we may identify* $\mathcal{T}_{\mathbf{a}} = \{\eta \in \mathcal{T}_{\mathbf{b}} : \exists x \leq d_{\mathbf{b}} : \chi_2(x, \eta)\}$*.*

Lemma 2.3.4 is fundamental in our context: it provides us the tools to work with more complex trees viewing them as definable subsets of some suitable order without changing the cofinality spectrum problem we are working on. Besides, along with Corollaries 2.2.17 and 2.2.18, the study of certain gaps in some order of a cofinality spectrum problem will remain invariant when we carry out the codification. From now on, we can consider richer trees that model more conditions about certain orders.

Definition 2.3.5 (Covers, [11, Def. 5.4]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem and* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$*.*

(1) Say that $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$ *is a* ***cover*** *for* $\mathbf{a}$ *if all Gödel codes for elements of* $\mathcal{T}_{\mathbf{a}}$ *belong to* $X_{\mathbf{b}}$*. The usual case is when* $X_{\mathbf{a}}$ *is an initial segment of* $X_{\mathbf{b}}$*, so is itself an element of* $\mathrm{Or}(\mathbf{s})$ *by proposition 2.3.3.*

(2) We define k***-coverable*** *by induction on* $k < \omega$*.*

  (a) Say that $\mathbf{a}$ *is* 0*-coverable if* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ *is nontrivial.*

  (b) Say that $\mathbf{a}$ *is* $(k+1)$*-coverable if there exists* $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$ *such that* $\mathbf{b}$ *is a cover for* $\mathbf{a}$ *and* $\mathbf{b}$ *is itself* k*-coverable.*

(3) Say that $\mathbf{a}$ *is* ***coverable*** *if it is* 1*-coverable; this will be our main case.*

(4) Say that $\mathbf{a}$ *is* ***coverable as a pair*** *by* $\mathbf{d} \in \mathrm{Or}(\mathbf{s})$ *when*

  (a) there is a $\mathbf{c} \in \mathrm{Or}(\mathbf{s})$ *such that* $X_{\mathbf{c}} = X_{\mathbf{a}} \times X_{\mathbf{a}}$ *and definition 2.2.5 (6) holds of* $\mathbf{a}, \mathbf{c}$*.*

(b) $\mathbf{c}$ *is coverable by* $\mathbf{d}$.

By lemma 2.3.4, we have that there is some k-coverable $\mathbf{a}$ for any k (likewise for coverable as a pair). So, we have the necessary tools to reach our main goal.

2.4. Characterization of $\mathcal{C}(\mathrm{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$

After analyzing some useful properties of cofinality spectrum problems (such as local saturation and Gödel codifications), we have new elements to continue with our main goal: characterize the cut spectrum $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$.

Remark 2.4.1 In lemma 2.2.4 we study how to rule out symmetric gaps in regular ultrafilters. Actually, it is natural to think in bringing this result to the context of cofinality spectrum problem and indeed, it is possible to rule out (κ, κ)-gaps in any cofinality spectrum problem, for $\kappa \leq \mathfrak{p}_{\mathbf{s}}$, $\kappa < \mathfrak{t}_{\mathbf{s}}$ (see [11, Lemma 6.1]). The proof is similar to lemma 2.2.4, so we leave the details to the reader.

Fact 2.4.2 ([11, Claim 8.2]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem,* $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$*,* $\kappa \leq \mathfrak{p}_{\mathbf{s}}$*,* $(\kappa, \lambda) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$ *and let* $f : X_{\mathbf{a}} \to X_{\mathbf{a}}$ *be multiplication by* 2*. Then we may choose sequences* $\langle d_\epsilon : \epsilon < \kappa \rangle$*,* $\langle e_\alpha : \alpha < \lambda \rangle$ *of elements of* $X_{\mathbf{a}}$ *such that* $(\langle d_\epsilon : \epsilon < \kappa \rangle, \langle e_\alpha : \alpha < \lambda \rangle)$ *represents a* (κ, λ)*-gap, and moreover, for all* $\alpha < \lambda$ *we have that* $f(e_{\alpha+1}) <_{\mathbf{a}} e_\alpha$.

Proof. Suppose that there is a (κ, λ)-gap in $X_{\mathbf{a}}$, for some $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$. Since $(\kappa, \lambda) \in \mathcal{C}(s, \mathfrak{t}_{\mathbf{s}})$, by definition we have that $\kappa + \lambda < \mathfrak{t}_{\mathbf{s}}$. Consider the definable subtree $\mathcal{T}_4$ of $\mathcal{T}_{\mathbf{a}}$ whose elements x are such that:

(i) If $n_1 <_{\mathbf{a}} n_2 <_{\mathbf{a}} \lg(x)$, then $f(x(n_1)) <_{\mathbf{a}} x(n_2)$.

(ii) If $n_1 <_{\mathbf{a}} n_2 <_{\mathbf{a}} \lg(x)$, then $x(n_1) <_{\mathbf{a}} x(n_2)$.

Notice that this tree is nonempty and contains arbitrarily long finite branches. Now, by induction on $\epsilon < \kappa$ we construct a sequence $c_\epsilon \in \mathcal{T}_4$, $n_\epsilon := \lg(c_\epsilon) - 1 \in X_{\mathbf{a}}$, and $d_\epsilon \in X_{\mathbf{a}}$ as follows:

(a) $c_\beta \trianglelefteq c_\epsilon$ for all $\beta < \epsilon$.

(b) $c_\epsilon(n_\epsilon) = d_\epsilon$.

(c) c_ϵ is below the ceiling.

(d) for each $n <_{\mathbf{a}} n_\epsilon$ and $m < \omega$, there are $x_0, ..., x_m \in X_{\mathbf{a}}$ such that $c_\epsilon(n) = x_0$, $f(x_k) < x_{k+1}$ for all $k < m$, and $c_\epsilon(n) < x_1 < ... < x_n$.

We leave the details of the induction to the reader (see [11, pp. 268-269]).
Since $\kappa < \mathfrak{t}_{\mathbf{s}}$, by fact 2.2.12 we can find an upper bound $c \in \mathcal{T}_4$ of the sequence $\langle c_\epsilon : \epsilon < \kappa \rangle$. By theorem 2.2.15 and the assumption that $(\kappa, \lambda) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$, there is a decreasing sequence $\langle m_\alpha : \alpha < \lambda \rangle$ of elements of $\{n \in X_{\mathbf{a}} : n <_{\mathbf{a}} \lg(c)\}$ such that $(\langle n_\epsilon : \epsilon < \kappa \rangle, \langle m_\alpha : \alpha < \lambda \rangle)$ represents a (κ, λ)-gap and $m_0 \leq \lg(c) - 1$. Define the sequence $\langle e_\alpha : \alpha < \lambda \rangle$ in $X_{\mathbf{a}}$ by $e_\alpha := c(m_\alpha)$. It is clear that $d_\epsilon <_{\mathbf{a}} e_\alpha$ for all $\epsilon < \kappa$ and $\alpha < \lambda$, and by hypothesis (d) of the construction of $\mathcal{T}_4$, we have that $f(e_{\alpha+1}) <_{\mathbf{a}} e_\alpha$ for all $\alpha < \lambda$. Hence, the sequences $\langle d_\epsilon : \epsilon < \kappa \rangle$, $\langle e_\alpha : \alpha < \lambda \rangle$ satisfy the conditions of the fact. ■

The idea of the previous fact is to determine how far apart we can choose the elements of the right side of a gap, and in this way, having sufficient space in the interval $(e_{\alpha+1}, e_\alpha)$.

Fact 2.4.3 ([11, Fact 8.4]) *For every κ, there is some symmetric function $g : \kappa^+ \times \kappa^+ \to \kappa$ such that for every $W \in [\kappa^+]^{\kappa^+}$ we have that* $\sup(\operatorname{ran}(g \restriction_{W \times W})) = \kappa$.

Proof. Since for all $\alpha < \kappa^+$ there is an injection from α into κ, then let g be such that for all $\beta < \gamma < \alpha$, $g(\beta, \alpha) \neq g(\gamma, \alpha)$. Now let $W \in [\kappa^+]^{\kappa^+}$. Then we choose $\alpha \in W$ such that $|\alpha \cap W| = \kappa$. Hence for all distinct $\gamma, \beta \in \alpha \cap W$, we have that $g(\gamma, \alpha) \neq g(\beta, \alpha)$. Therefore, $\sup(\operatorname{ran}(g \restriction_{W \times W})) = \kappa$. ■

Now we are ready to prove the main result in [11] which allows us to rule out asymmetric gaps below $\mathfrak{t}_{\mathbf{s}}$.

Theorem 2.4.4 ([11, Thm. 8.1]) *Let $\mathbf{s}$ be a cofinality spectrum problem. Suppose that κ, λ are regular and $\kappa < \lambda \leq \mathfrak{p}_{\mathbf{s}}$ and $\lambda < \mathfrak{t}_{\mathbf{s}}$. Then $(\kappa, \lambda) \notin \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$.*

Proof. Let $\mathbf{a} \in \operatorname{Or}(\mathbf{s})$ coverable as a pair by some $\mathbf{a}'$ and suppose that $(\kappa, \lambda) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$. Without loss of generality, in $X_{\mathbf{a}}$, this gap is represented by

$$(\langle d_\epsilon : \epsilon < \kappa \rangle, \langle e_\alpha : \alpha < \lambda \rangle),$$

and the sequences $\langle d_\epsilon : \epsilon < \kappa \rangle$, $\langle e_\alpha : \alpha < \lambda \rangle$ can be chosen as in fact 2.4.2 (we often abbreviate this cut as $(\overline{d}, \overline{e})$). To complete our preliminaries, let $g : \kappa^+ \times \kappa^+ \to \kappa$ be a symmetric function as in fact 2.4.3.
Now we define a convenient tree that models this gap: let $\mathbf{b} \in \operatorname{Or}(\mathbf{s})$ be such that $X_{\mathbf{b}} = X_{\mathbf{a}} \times X_{\mathbf{a}} \times X_{\mathbf{a}} \times X_{\mathbf{a}'} \times X_{\mathbf{a}'} \times X_{\mathbf{a}'}$, and consider the subtree $\mathcal{T}_6 \subseteq \mathcal{T}_{\mathbf{b}}$ given as follows: $x \in \mathcal{T}_{\mathbf{b}}$ if and only if:

(1) $n_1 <_{\mathbf{b}} n_2 <_{\mathbf{b}} \lg(x)$ implies

$$x(n_2, 0) \leq_{\mathbf{a}} x(n_2, 1) <_{\mathbf{a}} x(n_2, 2) <_{\mathbf{a}} x(n_1, 0)$$

(these first three coordinates witness the progression of the leftward towards the gap,).

(2) $x(n,3)$ is a non-empty subset of $X_{\mathbf{a}}$ (i.e. $x(n,3) \in \mathcal{T}_{\mathbf{a}}$ such that $x(n,3)(k) < 2_{\mathbf{a}}$ for all $k < \lg(x(n,3)))$ such that $|x(n,3)| + |x(n,3)| \leq d_{\mathbf{a}}$, where $|x(n,3)| \in X_{\mathbf{a}}$ is below the ceiling and satifies

$$\exists \eta\, [\lg(\eta) = \lg(x(n{,}3)) + 1 \wedge \eta(0) = 0 \wedge \eta(\lg(\eta) - 1) = |x(n,3)| \\ \wedge (\forall k < \lg(\eta) - 1)\, (\eta(k+1) = \eta(k) + x(n,3)(k))].$$

(3) $x(n,4)$ is a symmetric 2-place function with domain $x(n,3) \times x(n,3)$ and range $\subseteq X_{\mathbf{a}}$, which we call a distance estimate function (this function provides a lower bound for condition (4)).

(4) $x(n,5)$ is an injective function from $x(n,3)$ into the interval $(x(n,1), x(n,2))_{\mathbf{a}}$ such that:

$a \neq b \in x(n,3)$ implies $x(n,4)(a,b) \leq |x(n,5)(a) - x(n,5)(b)|$

where $|\cdot|$ represents the usual absolute value.

(5) If $n_1 <_{\mathbf{b}} n_2 <_{\mathbf{b}} \lg(x)$ and $a, b \in X_{\mathbf{a}}$ are such that $(\forall m)((n_1 \leq_{\mathbf{b}} m \leq_{\mathbf{b}} n_2) \to \{a,b\} \subseteq x(m,3))$ then $x(n_1,4)(a,b) = x(n_2,4)(a,b)$ (the distance estimate of two elements in the domain of $x(n,5)$ is constant if these elements remain continously in the sequence of these domains).

Now we will choose $c_\alpha \in \mathcal{T}_6$ and $n_\alpha = \max(\mathrm{dom}(c_\alpha))$ by induction on $\alpha < \lambda$. When $\alpha < \kappa^+$ is a successor, then we will also choose a constant y_α. They satisfy the following:

✓ For all $\alpha < \lambda$:

(1) $\beta < \alpha$ implies $c_\beta \trianglelefteq c_\alpha$.

(2) $\beta < \alpha$ implies

$$e_{\alpha+1} \leq_{\mathbf{a}} c_\alpha(n_\alpha, 0) \leq_{\mathbf{a}} c_\alpha(n_\alpha, 1) <_{\mathbf{a}} c_\alpha(n_\alpha, 2) <_{\mathbf{a}} e_{\beta+1}$$

and if $\alpha = \beta + 1$, then in addition $c(n_\alpha, 0) = e_{\alpha+1}$.

(3) For all $\gamma < \min\{\alpha, \kappa^+\}$:

(a) $(\forall m)[n_{\gamma+1} \leq_{\mathbf{b}} m \leq_{\mathbf{b}} n_\alpha \to y_{\gamma+1} \in c_\alpha(m,3)]$ (i.e. all constants of small index are in the domain of $c_\alpha(n,5)$, and they will remain in the domains of $c_\beta(n,5)$ for all $\beta > \alpha$).

(b) for all $\zeta < \gamma$ and for all m such that $n_{\gamma+1} \leq_{\mathbf{b}} m \leq_{\mathbf{b}} n_\alpha$,

$$c_\alpha(m,4)(y_{\zeta+1}, y_{\gamma+1}) = d_{g(\zeta+1,\gamma+1)}.$$

(the distance estimate function gives a lower bound to the distance of the $c_\alpha(m,5)$ images of $y_{\zeta+1}$ and $y_{\gamma+1}$, and this distance is not larger than $d_{g(\zeta+1,\gamma+1)}$. Meanwhile $y_{\zeta+1}$ and $y_{\gamma+1}$ stay in the domain of $c_\alpha(m,5)$ for $n_{\gamma+1} \leq m \leq n_\alpha$, so the distance estimate will remain the same by (5)).

✓ When $\alpha = \beta + 1 < \kappa^+$, then in addition :

(4) $y_{\beta+1} \in X_{\mathbf{a}} \setminus \{y_{\gamma+1} : \gamma < \beta\}$.

(5) $y_{\beta+1} \in c_\alpha(n_\alpha, 3)$.

(6) $y_{\beta+1} \notin c_\beta(n_\beta, 3)$.

(7) $|c_\alpha(n_\alpha, 3)| + |c_\alpha(n_\alpha, 3)| \leq d_{\mathbf{a}}$.

(8) for all $\gamma + 1 < \beta + 1$ and all n such that $n_{\gamma+1} \leq_{\mathbf{b}} n \leq_{\mathbf{b}} n_{\beta+1}$,

$$x(n,4)(y_{\gamma+1}, y_{\beta+1}) = d_{g(\gamma+1,\beta+1)}$$

Let us perform the induction.

✓ (Case $\alpha = 0$): Let $c_0(n_0,0), c_0(n_0,1), c_0(n_0,2) \in X_{\mathbf{a}}$ such that

$$e_1 \leq_{\mathbf{a}} c_0(n_0,0) \leq_{\mathbf{a}} c_0(n_0,1) <_{\mathbf{a}} c_0(n_0,2) <_{\mathbf{a}} e_0.$$

and $n_0 = 0$. Also, define $c_0(n_0,3) := \{0_{\mathbf{a}}\}$, $c_0(n_0,4)(0_{\mathbf{a}}, 0_{\mathbf{a}}) := 0_{\mathbf{a}} \in X_{\mathbf{a}}$ and $c_0(n_0,5)(0_{\mathbf{a}}) := 0$.

✓ (Case $\alpha = \beta + 1$, when in addition $\alpha < \kappa^+$): If $\alpha = \beta + 1 < \kappa^+$, we first define $y_\alpha = y_{\beta+1}$. By inductive hypothesis, we have that $M_1^+ \vDash c_\beta \in \mathcal{T}_6$, hence by (2) in the definition of $\mathcal{T}_6$, then we claim that $X_{\mathbf{a}} \setminus c_\beta(n_\beta, 3) \neq \emptyset$. Choose $y_{\beta+1} \in X_{\mathbf{a}} \setminus c_\beta(n_\beta, 3)$. Notice that since the size of $c_\beta(n_\beta, 3) \cup \{y_{\beta+1}\}$ is no larger than its complement in $X_{\mathbf{a}}$ and below the ceiling, then we can choose $c_\alpha(n_\alpha, 3)$ in such a way that it remains small enough.

It is clear that conditions (4) and (6) of the inductive hypothesis hold, therefore by condition (5) in our definition of $\mathcal{T}_6$, we will be able to freely choose the value of $c_\alpha(n_\alpha, 4)$ on any pair which includes $y_{\beta+1}$ (*). This remark will be useful later. This step continues below.

✓ (Case $\alpha = \beta + 1$ for arbitrary $\alpha < \lambda$): This is the key part of the induction. Suppose that we have already chosen $y_{\beta+1}$ for all $\beta < \min\{\alpha, \kappa^+\}$ and continue the induction for $\alpha = \beta + 1 < \lambda$.

At this step, we are interested in defining $c_\alpha(n_\alpha, l)$ for $l < 6$. The nontrivial cases, as expected, are $l = 3, 4, 5$. This will be done by showing it may be expressed as a consistent partial Or-type, and then applying local saturation (see theorem 2.3.2). Recall that we suppose that $\lambda < \mathfrak{t}_\mathbf{s}$.

Let $p := p(x_0, x_1, x_2, x_3, x_4, x_5)$ be the partial type stating the following.

(P1) $x_0, x_1, x_2 \in X_\mathbf{a}$ and $x_3, x_4, x_5 \in X_{\mathbf{a}'}$.

(P2) $x_0 = e_{\alpha+1} \leq_\mathbf{a} x_1 <_\mathbf{a} x_2 <_\mathbf{a} e_\alpha$.

(P3) $x_3 = \mathrm{dom}\,(x_4) \subseteq X_\mathbf{a}$ is below the ceiling and $|x_3| + |x_3| \leq d_\mathbf{a}$.

(P4) x_4 is a definable symmetric 2-place function from x_3 to $X_\mathbf{a}$.

(P5) x_5 is a definable injection from x_3 into the interval $(e_{\alpha+1}, e_\alpha)_\mathbf{a}$ such that $a \neq b \in x_3$ implies that $x_4(a, b) < |x_5(a) - x_5(b)|$.

(P6) $x_1 < \min\,(\mathrm{ran}\,(x_5))$, $x_2 > \max\,(\mathrm{ran}\,(x_5))$.

(P7) if $a, b \in c_\beta(n_\beta, 3) \cap x_3$ then $x_4(a, b) = c_\beta(n_\beta, 4)(a, b)$.

For $\gamma < \min\{\alpha, \kappa^+\}$ we also consider

$(\mathrm{P8})_\gamma$ $y_{\gamma+1} \in x_3$.

For $\zeta < \gamma < \min\{\alpha, \kappa^+\}$ we consider

$(\mathrm{P9})_{\zeta,\gamma}$ $x_4(y_{\zeta+1}, y_{\gamma+1}) = d_{g(\zeta+1,\gamma+1)}$. (Notice that the remark (*) of case $\alpha = \beta + 1 < \kappa^*$ allows us to choose x_4 in this way when the pair includes $y_{\beta+1}$, and by (4) of the inductive hypothesis for all other pairs of y's.)

Notice that p depends on the parameters $\{e_{\alpha+1}, e_\alpha, c_\beta\} \cup \{y_{\gamma+1} : \gamma < \min\{\alpha, \kappa^+\}\} \cup \{d_{g(\zeta+1,\gamma+1)} : \zeta, \gamma < \min\{\alpha, \kappa^+\}\}$, and also we have that $|\alpha| < \mathfrak{p}_\mathbf{s}$, since $\lambda \leq \mathfrak{p}_\mathbf{s}$. Therefore, to show that p is an Or-type it is enough to prove that p is finitely satisfiable in $X_\mathbf{b}$, recalling that since we chose a partial injection x_5, the domain of this partial injection x_3 and a distance estimate function x_4, then the conditions (P8) and (P9) are restrictions that forces us to find certain elements in the domain x_3, with an estimated distance previously fixed and, when necessary, certain new distances set.

We will see that p is finitely satisfiability by using a compactness argument. Let $\Gamma \subseteq \alpha$ be a non-empty finite subset and let $p_0 \subseteq p$ be finite and such that p_0 implies (P1)-(P7), p_0 implies $(\mathrm{P8})_\gamma$ for each $\gamma \in \Gamma$ and p_0 implies $(\mathrm{P9})_{\gamma,\zeta}$ for each $\gamma, \zeta \in \Gamma$. Let us prove that p_0 is satisfiable. Define b_3, b_4, b_5, b_1, b_2 as follows.

(i) Let $b_3 := \{y_{\gamma+1} : \gamma \in \Gamma\}$.

(ii) Let b_4 be the symmetric 2-place function on b_3 defined by $b_4(y_{\zeta+1}, y_{\zeta+1}) := d_{g(\zeta+1,\gamma+1)}$.

(iii) Let $d = \max\{d_{g(\zeta+1,\gamma+1)} : \zeta \neq \gamma \in \Gamma\}$. It is clear that $d < e_\alpha$ for all $\alpha < \lambda$, because of the way we chose the gap at the beginning of the proof.

(iv) Let $\{\gamma_0, ..., \gamma_n\}$ be an increasing enumeration of Γ without repetitions, and define b_5 by $b_5(y_{\gamma_i+1}) := e_{\alpha+1} +_{\mathbf{a}} 1 +_{\mathbf{a}} i \cdot d$ for $i = 0, 1, ..., n$. Because of the way we chose the sequence $\bar{e}$, we have that $\max(\operatorname{ran}(b5)) <_{\mathbf{a}} e_\alpha$.

(v) Let b_0, b_1, b_2 be defined from b_3, b_4, b_5 by conditions (P1), (P2) and (P6) of the definition of p.

We want to show that $(b_0, ..., b_5) \vDash p_0$. It is clear that condition (P1) holds. Conditions (P2) and (P6) follow from the fact that by (iv), we claim that $\max(\operatorname{ran}(b_5)) <_{\mathbf{a}} e_\alpha$. Condition (P3) holds since b_3 is finite and $X_{\mathbf{a}}$ is not. Conditions (P4), (P5), and (P6) follow immediately because of the way we chose the elements b_i, and for $\zeta < \gamma \leq \beta$ conditions $(\mathrm{P8})_\gamma$ and $(\mathrm{P9})_{\zeta,\gamma}$ also follow immediately. So, it remains to analyze condition (P7), knowing that $c_\beta(n_\beta, 3) \cap b_3 \subseteq b_3$. By inductive hypothesis (1) and (5), we have that $c_\beta(n_\beta, 3) \cap b_3 = \{y_{\gamma+1} : \gamma \in \Gamma,\, \gamma \neq \beta\}$. Therefore, if $\zeta < \gamma < \beta$, condition (P7) for $a = y_{\zeta+1}, b = y_{\gamma+1}$ holds by condition (P9), and if $\zeta < \gamma$ then condition (p.7) for $a = y_{\zeta+1}, b = y_{\beta+1}$ is immediate since $y_{\beta+1} \notin c_\beta(n_\beta, x_3)$.

Since p_0 is realized, then we have that p is an Or-type in the sense of definition 2.3.1 over a set of size $< \mathfrak{p}_{\mathbf{s}}$, and by theorem 2.3.2 we can find a realization $\langle b_i^* : i < 6\rangle$ of p. By inductive hypothesis, we can concatenate, therefore we consider $c_\alpha := c_\beta{}^\frown\langle b_0^*, b_1^*, b_2^*, b_3^*, b_4^*, b_5^*\rangle$ and $n_\alpha := n_\beta + 1$. With this, the successor step is done.

✓ (Case α limit): Since $\operatorname{cf}(\alpha) < \lambda \leq \mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$, then by fact 2.2.13 there is a $c \in \mathcal{T}_6$ such that $c_\beta \trianglelefteq c$ for all $\beta < \alpha$ and c is below the ceiling. As in proof of lemma 2.2.4, we want to refine this bound c by choosing a convenient initial segment c_α in such way that conditions (1), (2) and (3a) of the inductive hypothesis will be satisfied. Conditions (3b-3c) will follow directly from definition of $\mathcal{T}_6$). First, let

$$n_* := \max\{n : n \leq_{\mathbf{b}} \lg(c),\, M_1^+ \vDash e_\alpha <_{\mathbf{a}} c(n, 0))\}.$$

Notice that $\lg(c_\beta) <_{\mathbf{b}} n_*$ for all $\beta < \alpha$. Now, for each $\beta < \max\{\alpha, \kappa^+\}$, let

$$n(\beta) := \max\{n \leq_{\mathbf{b}} n_* : y_{\beta+1} \in c(n, 3)\}.$$

For each $\gamma < \beta < \alpha$, by the inductive hypothesis (3) for β, we may ensure that $y_{\gamma+1} \in c_\beta(n_\beta, 3)$. Therefore, $(\{n_\beta : \beta < \alpha \cap \kappa^+\}, \{n(\beta) : \beta < \alpha \cap \kappa^+\})$ represents a (κ_1, κ_2)-pre-gap in $X_{\mathbf{b}}$, for some regular $\kappa_1, \kappa_2 \in |\alpha \cap \kappa^+|$. It cannot represent a gap, since $\kappa_1 + \kappa_2 \leq |\alpha| < \lambda \leq \mathfrak{p}_{\mathbf{s}}$, contradicting the definition of $\mathfrak{p}_{\mathbf{s}}$. Hence, we can choose $n_{**} \leq_{\mathbf{b}} n_*$ such that $n_\gamma <_{\mathbf{b}} n_{**} <_{\mathbf{b}} n(\beta)$ for all $\gamma, \beta < \alpha \cap \kappa^+$. Consider $n_\alpha := n_{**}$ and $c_\alpha := c \restriction_{n_\alpha+1}$. This completes the limit case, and hence we have finished the inductive construction of the sequences $\langle c_\alpha : \alpha < \lambda\rangle$ and $\langle n_\alpha : \alpha < \lambda\rangle$.

Having built the sequence $\langle c_\alpha : \alpha < \lambda\rangle$ and as $\lambda < \mathfrak{t}_{\mathbf{s}}$, we can find $c_\lambda \in \mathcal{T}_6$ such that $c_\alpha \trianglelefteq c_\lambda$ for all $\alpha < \lambda$, with $n_\lambda = \max(\operatorname{dom}(c_\lambda))$. Note that c_λ is a function from $X_{\mathbf{b}}$ into $X_{\mathbf{b}}$, so

$\langle n_\alpha : \alpha < \lambda \rangle$ is a strictly increasing sequence in $X_{\mathbf{b}}$ below n_λ, so by theorem 2.2.15 we can find a strictly decreasing sequence $\langle m_\epsilon : \epsilon < \kappa \rangle$ such that $(\langle n_\alpha : \alpha < \lambda \rangle, \langle m_\epsilon : \epsilon < \kappa \rangle)$ represents a (λ, κ)-gap in $X_{\mathbf{b}}$ (again, sometimes we will refer this gap as $(\overline{n}, \overline{m})$. Since $c_\lambda(n, 2)$ is strictly decreasing in $X_{\mathbf{a}}$ as n is too large, then we may ensure that $c_\lambda(m_\epsilon, 2) \leq_{\mathbf{a}} d_\gamma$ for some γ: since $c_\alpha(m_\epsilon, 2) \leq_{\mathbf{a}} e_\alpha$ for all $\alpha < \lambda$ and $(\overline{d}, \overline{e})$ represents a (κ, λ)-gap in $X_{\mathbf{a}}$, then $c_\lambda(m_\epsilon, 2) <_{\mathbf{a}} d_\gamma$ for some γ. Hence, we can choose a map $\zeta : \kappa \to \kappa$ such that $c_\lambda(m_\epsilon, 2) <_{\mathbf{a}} d_{\zeta(\epsilon)}$.

What about the constants $y_{\beta+1}$? Well, for $\beta < \kappa^+$, consider the set

$$X_\beta := \{n : n \leq_{\mathbf{b}} n_\lambda, (\forall n')(n_{\beta+1} \leq_{\mathbf{b}} n' \leq_{\mathbf{b}} n \to y_{\beta+1} \in c_\lambda(n', 3))\}.$$

It is clear that X_β is a subset of $X_{\mathbf{b}}$ that includes $[n_{\alpha_1}, n_{\alpha_2}]_{\mathbf{b}}$, for all $\beta < \alpha_1 < \alpha_2 < \lambda^+$. Therefore, each X_β has a maximal element above all the n_α's, and since $(\overline{n}, \overline{m})$ is a (λ, κ)-cut, then for some $\epsilon(\beta) < \kappa$ we have that $[n_{\beta+1}, m_{\epsilon(\beta)}]_{\mathbf{b}} \subseteq X_{\mathbf{b}}$. Since there are κ^+-many β's, then we may assure the existence of some $W \subseteq \kappa^+$ of size κ^+ and $\epsilon^* < \kappa$ such that $\epsilon(\beta) = \epsilon^*$ for all $\beta \in W$.

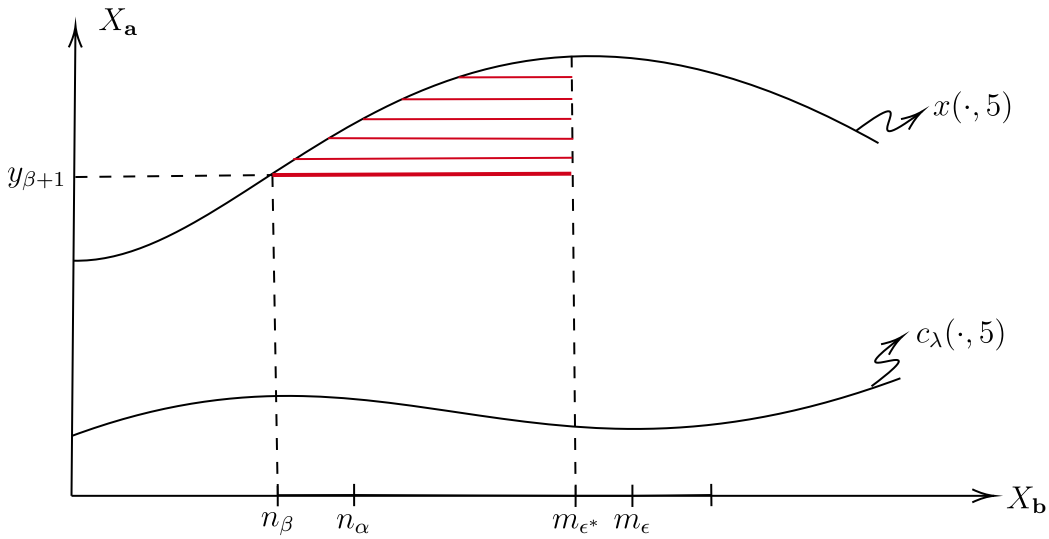

Figure 2-1: Sketch which describes how to obtain the desired bound m_{ϵ^*}.

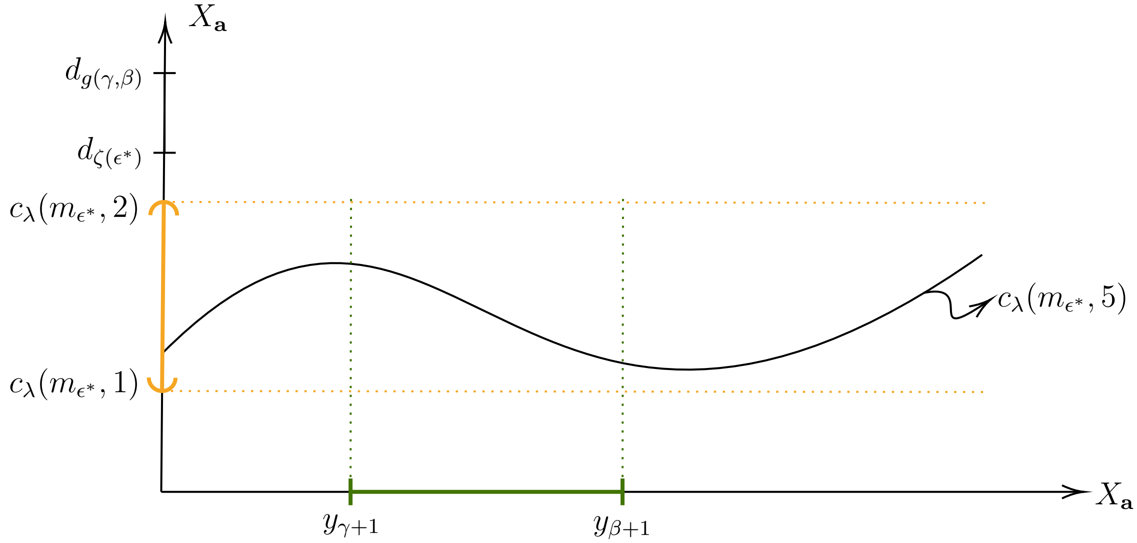

Figure 2-2: Behaviour of $c_\lambda(n,5)$ in m_{ϵ^*}.

By construction, for every $\beta \neq \gamma \in W$, we have that $F(y_\gamma, y_\beta) = d_{g(\gamma,\beta)}$, and also, by the choice of g, we can find $\gamma, \beta \in W$ such that $\zeta(\epsilon^*) < g(\gamma,\beta)$, so

$$\begin{aligned} c_\lambda(m_{\epsilon^*},2) \ &<_{\mathbf{a}} \ d_{\zeta(\epsilon^*)} <_{\mathbf{a}} d_{g(\gamma,\beta)} = c_\lambda(m_{\epsilon^*},4)(y_{\gamma+1}, y_{\beta+1}) \\ &\leq_{\mathbf{a}} \ |c_\lambda(m_{\epsilon^*},5)(y_{\gamma+1}) - c_\lambda(m_{\epsilon^*},5)(y_{\beta+1})| <_{\mathbf{a}} c_\lambda(m_{\epsilon^*},2), \end{aligned}$$

a contradiction. ■

Theorem 2.4.5 ([11, Central theorem 9.1]) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. Then* $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$. *In particular,* $\mathfrak{t}_{\mathbf{s}} \leq \mathfrak{p}_{\mathbf{s}}$.

Proof. If $\mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$, without loss of generality suppose that $\kappa \leq \lambda$ are such that $\kappa + \lambda = \mathfrak{p}_{\mathbf{s}}$ and $(\kappa, \lambda) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}})$. We have seen that neither the case $\kappa = \lambda$ (lemma 2.2.4 and remark 2.4.1) nor $\kappa < \lambda$ (theorem 2.4.4) can occur. So this case cannot occur. So, we have that $\mathfrak{t}_{\mathbf{s}} \leq \mathfrak{p}_{\mathbf{s}}$. By definition of $\mathfrak{p}_{\mathbf{s}}$, then $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$. ■

3 Keisler's order

Keisler's order was first introduced by Keisler in 1967 (see [9]). This order uses the notion of saturations of ultrapowers to compare the complexity of any two countable complete first-order theories, giving a set theoretic characterization of the maximal theories in terms of the combinatorial properties of the ultrafilters which saturate them. Several facts are known about this order: for example, Keisler's order restricted to stable theories is linear (see [16], chapter VI), but its complete structure is still unknown.

Keisler [9] showed that there is a maximum class for this order. Later, Shelah [16] proved that any theory of linear order, or more precisely with the *strict order property* (abbreviated as SOP), belongs to the maximum class; this was weakened to the *strong order property* SOP_3, (a weak version of linear orders). Malliaris and Shelah [11] showed that theories with SOP_2 belong to the maximum class of Keisler's order. In this chapter, define a convenient cofinality spectrum problem related to a regular ultrafilter $\mathcal{D}$. In this cofinality spectrum problem, it is possible to characterize $\mathcal{C}(\mathbf{s}, \mathbf{t}_\mathbf{s})$ by combinatorial properties of $\mathcal{D}$, such as goodness $\mathcal{D}$ or existence of treetops. The aim of this chapter is to analyze how these combinatorial properties of $\mathcal{D}$ are related to $\mathcal{C}(\mathbf{s}, \mathbf{t}_\mathbf{s})$.

Unless stated otherwise, we will work with countable languages and first-order complete theories.

3.1. Definition of Keisler's order

In Section 1.5, we have studied several properties about countably incomplete ultrafilters and regular ultrafilters. For regular ultrafilters, we can wonder about saturation of an ultrapower beyond $\aleph_1$.

Definition 3.1.1 *Let $\mathcal{D}$ be a regular ultrafilter over I, T an $\mathcal{L}$-theory and $\mathcal{M} \vDash T$. We say that $\mathcal{D}$* ***saturates*** *$\mathcal{M}$ if $\mathcal{M}^I/\mathcal{D}$ is $|I|^+$-saturated.*

The main property of regular ultrafilters, described in Theorem 1.5.6, allows us to preserve saturation of ultrapowers when we have elementary equivalent ultraroots. Therefore, if $\mathcal{M}$ is a model of a complete theory T (see definition 1.4.3) and $\mathcal{D}$ is a regular ultrafilters that saturates $\mathcal{M}$, then $\mathcal{D}$ saturates any model of T, so we can talk about *saturation of complete theories.*

Definition 3.1.2 ([11, Def. 10.5]) *Let T be a countable complete first order theory. If $\mathcal{D}$ is a regular ultrafilter over a set I, we say that $\mathcal{D}$ **saturates** T if $\mathcal{D}$ saturates $\mathcal{M}$, for every $\mathcal{M} \models T$.*

The previous definitions gives us an effective method to compare any pair of first order theories via regular ultrafilters.

Definition 3.1.3 ([11, Def. 10.6]) *Let T_1, T_2 be countable complete first order theories. We say that $T_1 \trianglelefteq T_2$ if for any cardinal λ and any regular ultrafilter $\mathcal{D}$ over λ, if $\mathcal{D}$ saturates T_2 then $\mathcal{D}$ saturates T_1.*

The relation $\trianglelefteq$ given in definition 3.1.3 is known as **Keisler's order**.

3.2. Good ultrafilters

This section is dedicated to the study of good ultrafilters. This ultrafilters allow to transfer $\aleph_1$-saturation of ultraproducts to uncountable cardinal above $\aleph_1$. In this section, given I a non-empty set and κ an infinite cardinal, we will say that a function $f : [\kappa]^{<\aleph_0} \to \mathcal{P}(I)$ is *monotone* if for every $u, v \in [\kappa]^{<\aleph_0}$ such that $u \subseteq v$, then $f(u) \supseteq f(v)$ (i.e. in this context, "monotone" means antimonotone).

Definition 3.2.1 ([11, Def. 10.8]) *Let $\mathcal{D}$ be a filter over a non-empty set I. We say that $\mathcal{D}$ is λ-**good** if for every $\kappa < \lambda$, every monotone function $f : [\kappa]^{<\aleph_0} \to \mathcal{D}$ has a multiplicative refinement, i.e., there is $g : [\kappa]^{<\aleph_0} \to \mathcal{D}$ such that:*

(1) If $u \in [\kappa]^{<\aleph_0}$, then $g(u) \subseteq f(u)$.

(2) If $u, v \in [\kappa]^{<\aleph_0}$, then $g(u) \cap g(v) = g(u \cup v)$.

*We say that $\mathcal{D}$ is **good** if it is $|I|^+$-good.*

The following theorem guarantees the existence of good ultrafilters for any non-empty set.

Fact 3.2.2 ([3, Thm. 6.1.4]) *Let I be a set of cardinality λ. Then there is a λ^+-good ω-regular ultrafilter $\mathcal{D}$ over I.*

Proof. Without loss of generality, consider $I = \lambda$. Let $\{f_\xi : \xi < 2^\lambda\}$ be an enumeration of all monotone function from $[\lambda]^{<\aleph_0}$ to $\mathcal{P}(\lambda)$ Define by transfinite induction two sequences, $(\Pi_\xi)_{\xi<2^\lambda}$, $(F_\xi)_{\xi<2\lambda}$, as follows:

(i) If $\eta < \xi < 2^\lambda$, $F_\xi \supseteq F_\eta$ and $\Pi_\xi \subseteq \Pi_\eta$.

(ii) $|\Pi_\xi| = 2^\lambda$

(iii) $|\Pi_\xi \smallsetminus \Pi_{\xi+1}| < \aleph_0$

(iv) For δ limit, $\Pi_\delta = \bigcap_{\eta<\delta} \Pi_\eta$ and $F_\delta = \bigcup_{\eta<\delta} F_\eta$.

(v) (Π_ξ, F_ξ) is consistent for $\eta < 2^\lambda$ (see [3, lemma 6.1.7]).

Then, by construction, it is possible to see that $F := \bigcup_{\xi<2^\lambda} F_\xi$ is the λ^+-good countably incomplete ultrafilter seeked. We leave the remaining details of this proof to the reader (see [3, Thm. 6.1.4]). ■

Example 3.2.3 Let $\mathcal{D}$ be an ultrafilter over $\mathbb{N}$ and let $f : [\mathbb{N}]^{<\aleph_0} \to \mathcal{D}$ be monotone. Given $A \in [\mathbb{N}]^{<\aleph_0}$, define $n(A) := \min\{n \in \mathbb{N} : A \subseteq n\} \in [\mathbb{N}]^{<\aleph_0}$, with $n = \{0, 1, ..., n-1\}$. Now, consider $g : [\mathbb{N}]^{<\aleph_0} \to \mathcal{D}$ defined by $g(A) = f(n(A))$. Since $A \subseteq n(A)$ and f is monotone, then we have that $g(A) = f(n(A)) \subseteq f(A)$ for $A \in [\mathbb{N}]^{<\aleph_0}$, and thus $g \leq f$. To prove that g is multiplicative, let $A, B \in [\mathbb{N}]^{<\aleph_0}$. Then $n(A \cup B) = n(A) \cup n(B) = \max\{n(A), n(B)\}$, hence

$$g(A \cup B) = f(n(A \cup B)) = f(\max\{n(A), n(B)\}) = f(n(A)) \cap f(n(B)) = g(A) \cap g(B)$$

Therefore, $\mathcal{D}$ is a $\aleph_1$-good ultrafilter over $\mathbb{N}$.

Proposition 3.2.4 ([16, Claim 2.4, chapter VI]) *Let $\mathcal{D}$ be an ultrafilter over I and let λ be an infinite regular cardinal. If $\mathcal{D}$ is λ^+-good and countably incomplete, then $\mathcal{D}$ is λ-regular.*

Proof. Since $\mathcal{D}$ is countably incomplete, by proposition 1.5.4, there is a decreasing countable chain $\{I_n \in \mathcal{D} : n < \omega\rangle$ such that $\bigcap_{n<\omega} I_n = \emptyset$. Define $f : [\lambda]^{<\aleph_0} \to \mathcal{D}$ by $f(w) := I_{|w|}$, and since $\mathcal{D}$ is λ^+-good, let $g : [\lambda]^{<\aleph_0}$ be the multiplicative refinement of f. It is clear that $g(\{\alpha\}) \in \mathcal{D}$, for all $\alpha < \lambda$.
Suppose that $\mathcal{D}$ is not λ-regular, then there is an infinite $w \subseteq \lambda$ such that $t \in \bigcap_{\alpha\in w} g(\{\alpha\})$, for some $t \in I$. For each $n < \omega$, choose $w(n) \subseteq w$ such that $|w(n)| = n$, then $t \in \bigcap_{\alpha\in w(n)} g(\{\alpha\}) = g(w(n)) \in I_n$, which contradicts the fact that $\bigcap_{n<\omega} I_n = \emptyset$. Hence, $\bigcap_{\alpha\in w} g(\{\alpha\}) = \emptyset$, and we may conclude that $\{g(\{\alpha\}) : \alpha < \lambda\}$ is a λ-regularizing family for $\mathcal{D}$. ■

The importance of good ultrafilters is that they give us essential data about theories with saturated ultraproducts.

Theorem 3.2.5 ([3, Thm. 6.1.8]) *Let λ be an infinite and let $\mathcal{D}$ be a λ-good ω-regular ultrafilter over I. Suppose that $\{\mathcal{M}_i : i \in I\}$ is a family of $\mathcal{L}$-structures, with $|\mathcal{L}| < \lambda$. Then the $\mathcal{L}$-structure $\prod_{\mathcal{D}} \mathcal{M}_i$ is λ-saturated.*

Proof. Without loss of generality, let $p = p(x) = \{\varphi_\mu(x) : \mu < \kappa\}$ be a type in $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_i$ without parameters: $p(x)$ has parameters $(a_\mu)_{\mu<\kappa}$), then consider the language $\mathcal{L}' = \mathcal{L} \cup \{c_i : i < \mu\}$, where c_i is a constant symbol. We should prove that if $p(x)$ is finitely satisfiable in $(\mathcal{M}_{a_m})_{m<\omega}$, where $(\mathcal{M}_{a_m})_{m<\omega}$ is the expansion of $\mathcal{M}$ by interpretating each element a_m as a constant symbol $c_i \in \mathcal{L}'$, then $p(x)$ is satisfiable in $(\mathcal{M}_{a_m})_{m<\omega}$. Notice that if $a_m = [(a_m(i)_{i\in I}] \in \mathcal{M}$, then

$$\left(\prod_{\mathcal{D}} (\mathcal{M}_i)_{a_m}\right)_{m<\omega} = \prod_{\mathcal{D}} ((\mathcal{M}_i)_{a_m})_{m<\omega}$$

Since $\mathcal{L}$ and $\mathcal{L}'$ are of size les than λ, then it is enough to prove that if $p(x)$ is finitely satisfiable (without parameters) in $\mathcal{M}$, then $p(x)$ is satisfiable in $\mathcal{M}$.

Since $\mathcal{D}$ is countably incomplete, there is a decreasing chain $\{I_n : n \in \omega\} \subseteq \mathcal{D}$ such that $\bigcap_{n\in\omega} I_n = \emptyset$. Define $f : [p]^{<\aleph_0} \to \mathcal{D}$ as follows:

$$f(u) := \begin{cases} I & \text{if } u = \emptyset \\ I_{|u|} \cap \left\{ i \in I : \mathcal{M} \vDash \exists x \bigwedge_{\varphi\in u} \varphi(x) \right\} & \text{if } u \neq \emptyset \end{cases} \tag{3-1}$$

It is clear that f is monotone: if $u \subseteq v$, then $I_{|u|} \supseteq I_{|v|}$, and

$$\left\{ i \in I : \mathcal{M} \vDash \exists x \bigwedge_{\varphi\in u} \varphi(x) \right\} \supseteq \left\{ i \in I : \mathcal{M} \vDash \exists x \bigwedge_{\varphi\in v} \varphi(x) \right\}$$

Since $\mathcal{D}$ is λ-good, there is a multiplicative refinement $g : [p]^{<\aleph_0} \to \mathcal{D}$ of f. For each $i \in I$, consider

$$\sigma(i) := \{\varphi(x) \in p(x) : i \in g(\{\varphi(x)\})\}$$

First of all, let us prove that $\sigma(i)$ is finite, for every $i \in I$: otherwise, if $|\sigma(i)| \geq n$, then choose n distinct elements $\varphi_1, ..., \varphi_n \in \sigma(i)$. Thus, we will have that $i \in g(\{\varphi_j\})$ for $j \in \{1, ...n\}$, and by multiplicativity if $\mathcal{D}$, we may assure that

$$\begin{aligned} i \in g(\{\varphi_1\}) \cap ... \cap g(\{\varphi_n\}) &= g(\{\varphi_1 \cup ... \cup \varphi_n\}) \\ &\subseteq f(\{\varphi_1 \cup ... \cup \varphi_n\}) \subseteq I_n \end{aligned}$$

If $\sigma(i)$ were not finite, then $i \in I_n$ for infinitely many n, a contradiction.

Thus $\sigma(i)$ is finite and $i \in g(\{\varphi\})$ for finitely many $\varphi \in p$. Moreover, for $i \in I$, we have that

$$i \in \bigcap \{g(\{\varphi(x)\}) : \varphi(x) \in \sigma(i)\} = g(\sigma(i)) \subseteq f(\sigma(i))$$

Now we will build a suitable $h_{\mathcal{D}}$ which satisfies $p(x)$ in $\prod_{\mathcal{D}} \mathcal{M}_i$. If $\sigma(i) = \emptyset$, choose any

$h(i) \in M_i$; otherwise, since (3-1) holds, for $i \in I$, choose $h(i) \in M_i$ such that

$$\mathcal{M}_i \vDash \bigwedge_{\varphi \in \sigma(i)} \varphi(h(i))$$

We will see that $[(h(i))_{i \in I}]$ is a realization of $p(x)$. Let $\varphi(x) \in p(x)$. Since $i \in g(\{\varphi(x)\})$ implies $\varphi(x) \in \sigma(i)$, we have that

$$\{i \in I : \mathcal{M}_i \vDash \varphi(h(i))\} \supseteq g(\{\varphi(x)\} \in \mathcal{D}$$

By Łoś theorem, we can conclude that $[(h(i))_{i \in I}]$ realizes $p(x)$. ■

In Theorem 1.5.5 we showed that countably incomplete ultrafilters produce $\aleph_1$-saturated ultraproducts. Good ultrafilters allow us to transfer the saturation of ultraproducts to any uncountable cardinal λ.

The next result is a useful characterization of good ultrafilters via saturation of ultrapowers.

Theorem 3.2.6 ([16, Thm. 2.2, chapter VI]) *Let $\mathcal{D}$ be an ultrafilter over I and λ be a cardinal. The following statements are equivalent:*

(i) $\mathcal{D}$ is λ-good.

(ii) For any family of λ-saturated models $(\mathcal{M}_t)_{t \in I}$, the ultraproduct $\prod_{\mathcal{D}} \mathcal{M}_t$ is λ-saturated.

(iii) For every $\mu < \lambda$ and every elementary λ-saturated extension $\mathcal{M}$ of $\mathcal{M}_\mu := ([\mu]^{<\aleph_0}, \subseteq, P)$, with $P(w) \Leftrightarrow w \neq \emptyset$, the ultrapower $\mathcal{M}^I/\mathcal{D}$ is λ-saturated.

Proof. (i) $\Longrightarrow$ (ii): Let $\mu < \lambda$, let $\mathcal{N} = \prod_{\mathcal{D}} \mathcal{M}_t$, let $A \subseteq \mathcal{N}$ such that $|A| < \lambda$ and let $p(x) := \{\varphi_\alpha(\overline{x}, \overline{a}_\alpha) : \alpha < \mu\}$ be a type over N with parameters $\overline{a}_\alpha := ([a_1], ..., [a_n]) \in A^{<\omega}$, for $\alpha < \mu$. For any $\varphi(\overline{x}, \overline{a}_\alpha) \in p$ and any $w \in [\mu]^{<\aleph_0}$, we define the following map $f : [\mu]^{<\aleph_0} \to \mathcal{D}$.

$$f(w) := \left\{ t \in I : \mathcal{M}_t \vDash \exists \overline{x} \left(\bigwedge_{\alpha \in w} \varphi_\alpha(\overline{x}; \overline{a}_\alpha(t)) \right) \right\},$$

with $\overline{a}_\alpha(t) := (a_1(t), ..., a_n(t))$. Notice that $f(w)$ is well defined by Łoś's Theorem. Since p is finitely satisfiable in $\mathcal{N}$, then there is some $\overline{c}_w \in N$ such that $\overline{c}_w \vDash \{\varphi_\alpha(\overline{x}; \overline{a}_\alpha) : \alpha \in w\}$. Hence, we may conclude by Łoś's Theorem that $X^\alpha_{\overline{c}_w} := \{t \in I : \mathcal{M}_t \vDash \varphi_\alpha(\overline{c}_w(t); \overline{a}_\alpha(t))\} \in \mathcal{D}$, and therefore we have that $\bigcap_{\alpha \in w} X^\alpha_{\overline{c}_w} \subseteq f(w)$, so $f(w) \in \mathcal{D}$.

Since $\mathcal{D}$ is λ-good, there is a multiplicative function $g : [\mu]^{<\aleph_0} \to \mathcal{D}$ which refines f. For each $t \in I$, consider the set $w(t) := \{\alpha < \mu : t \in g(\{\alpha\})\}$. Notice that for every finite subset

$u \subseteq w(t)$, $\alpha \in u$ implies that $t \in g(\{\alpha\}) \subseteq f(\{\alpha\})$. So,

$$\mathcal{M}_t \vDash \exists \overline{x} \left(\bigwedge_{\alpha \in u} \varphi_\alpha(\overline{x}; \overline{a}_\alpha(t)) \right)$$

and since this holds for every finite $u \subseteq w(t)$, then $p_t = \{\varphi_\alpha(\overline{x}; \overline{a}_\alpha(t)) : \alpha \in w(t)\}$ is finitely satisfiable in $\mathcal{M}_t$. Since $\mathcal{M}_t$ is λ-saturated, there is $\overline{c}(t) \in M_t$ such that $\overline{c}(t) \vDash p_t$. Consider $\overline{c} := [(c(t))_{t \in I}]$, and let us prove that $\overline{c} \vDash p$: for any $\alpha < \mu < \lambda$, we have

$$\{t \in I : \mathcal{M}_t \vDash \varphi_\alpha(\overline{c}(t); \overline{a}_\alpha(t))\} \supseteq g(\{\alpha\}) \in \mathcal{D}$$

thus $\{t \in I : \mathcal{M}_t \vDash \varphi_\alpha(\overline{c}(t); \overline{a}_\alpha(t))\} \in \mathcal{D}$, and by Łoś's Theorem, we claim that $\mathcal{N} \vDash \varphi_\alpha(\overline{c}, \overline{a}_\alpha)$.

(ii) $\Longrightarrow$ (iii): Let $\mathcal{M}$ be a λ-saturated extension of $\mathcal{M}_\mu$. Then, by hypothesis, if we consider $\mathcal{M}_t := \mathcal{M}$, for all $t \in I$, we have that $\prod_{\mathcal{D}} \mathcal{M} = \mathcal{M}^I/\mathcal{D}$ is λ-saturated.

(iii) $\Longrightarrow$ (i): Let $\mu < \lambda$, let $f : [\mu]^{<\aleph_0} \to \mathcal{D}$ be a monotone function and let $\mathcal{M}$ be a λ-saturated extension of $\mathcal{M}_\mu$ such that $\mathcal{M}^I/\mathcal{D}$ is λ-saturated. For $t \in I$ and $\alpha < \mu$, define $a_\alpha(t) \in M$ as follows: for every $w \in [\mu]^{<\aleph_0}$,

$$\mathcal{M} \vDash \exists x \left[\bigwedge_{a \in w} (x \subseteq a_\alpha(t)) \wedge P(x) \right] \quad \text{if and only if} \quad t \in f(w) \tag{3-2}$$

(This formula says that x is a non-empty common subset of some finitely many $a_\alpha(t)$, which can be chosen from $\mathcal{M}_\mu$. Since $\mathcal{M}$ is λ-saturated, then it is possible to choose all $a_\alpha(t)$, for all $\alpha < \mu$.)

Hence, let $a_\alpha := [(a_\alpha(t))_{t \in I}] \in \mathcal{M}^I/\mathcal{D}$ and $p = \{x \subseteq a_\alpha : \alpha < \mu\} \cup \{P(x)\}$ in $\mathcal{M}^I/\mathcal{D}$. We want to prove that p is finitely satisfiable in $\mathcal{M}^I/\mathcal{D}$. Let $q \in [p]^{<\aleph_0}$, then for some $u \in [\mu]^{<\aleph_0}$ we have that $q \subseteq \{x \subseteq a_\alpha : \alpha \in u\} \cup \{P(x)\}$, and since (3-2) holds, we can claim that

$$\left\{ t \in I : \mathcal{M} \vDash \exists x \left[\bigwedge_{\alpha \in u} x \subseteq a_\alpha(t)) \wedge P(x) \right] \right\} = f(u) \in \mathcal{D}$$

Therefore, by Łoś's Theorem, $\mathcal{M}^I/\mathcal{D} \vDash (\exists x)[\bigwedge_{\alpha \in u} x \subseteq a_\alpha \wedge P(x)]$, and thus, p is finitely satisfiable in $\mathcal{M}^I/\mathcal{D}$. Now, by hypothesis there is a $c \in \mathcal{M}^I/\mathcal{D}$ which realizes p (because $\mathcal{M}^I/\mathcal{D}$ is λ-saturated).

Now, let us define $g : [\mu]^{<\aleph_0} \to \mathcal{D}$ as follows:

$$g(w) := \left\{ t \in I : \mathcal{M} \vDash \left[\bigwedge_{\alpha \in w} c(t) \subseteq a_\alpha(t) \wedge P(c(t)) \right] \right\}$$

We may claim that g is a multiplicative refinement of f: given $u, v \in [\mu]^{<\aleph_0}$, we have that

$$\begin{aligned}
g(u \cup v) &= \left\{ t \in I \, : \mathcal{M} \vDash \left[\bigwedge_{\alpha \in u \cup v} c(t) \subseteq a_\alpha(t) \wedge P(c(t)) \right] \right\} \\
&= \left\{ t \in I \, : \mathcal{M} \vDash \left[\bigwedge_{\alpha \in u} c(t) \subseteq a_\alpha(t) \wedge P(c(t)) \right] \right\} \\
&\quad \cap \left\{ t \in I \, : \mathcal{M} \vDash \left[\bigwedge_{\alpha \in v} c(t) \subseteq a_\alpha(t) \wedge P(c(t)) \right] \right\} \\
&= g(u) \cap g(v)
\end{aligned}$$

Besides, for all $u \in [\mu]^{<\aleph_0}$, we have that

$$\begin{aligned}
g(u) &= \left\{ t \in I \, : \mathcal{M} \vDash \left[\bigwedge_{\alpha \in u} c(t) \subseteq a_\alpha(t) \wedge P(c(t)) \right] \right\} \\
&\subseteq \left\{ t \in I \, : \mathcal{M} \vDash \exists x \left[\bigwedge_{\alpha \in u} x \subseteq a_\alpha(t) \wedge P(x) \right] \right\} = f(u)
\end{aligned}$$

So $g(u) \subseteq f(u)$. Hence, we have shown that $\mathcal{D}$ is λ-good ultrafilter over I. ■

Fact 3.2.7 Consider DLO the theory of dense linear orders without first or last element. As in Theorem 3.2.6, there is a characterization of good ultrafilters via saturated models of DLO (this characterization is [16, Thm. 2.6, chapter VI]). If $\mathcal{D}$ is an ultrafilter over I and $\lambda > \aleph_0$, then the following are equivalent:

(i) $\mathcal{D}$ is λ-good.

(ii) For every λ-saturated model $\mathcal{M}$ of DLO, with $\mathcal{M} = (M, \leq)$, $\mathcal{M}^I/\mathcal{D}$ is λ-saturated.

(iii) For every λ-saturated model $\mathcal{M}$ of DLO, and for every set $A \subseteq M^I/\mathcal{D}$ linearly ordered by $\leq$, then every 1-type p over A in $\mathcal{M}^I/\mathcal{D}$, with $|p| < \lambda$, is realized in $\mathcal{M}^I/\mathcal{D}$.

The proof of this fact is similar to the proof of Theorem 3.2.6 so we leave the details to the reader (see [16, pp. 337-341]).

The following result is a characterization of the maximal theories of Keisler's order in terms of saturation of ultrapowers by good ultrafilters.

Theorem 3.2.8 ([9, Thm. 3.2]) *A theory T is maximal in Keisler's order if and only if for any $\lambda > \aleph_0$, any model $\mathcal{M} \vDash T$ and any regular ultrafilter $\mathcal{D}$ over λ, the following holds*

($\star$) *$M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if $\mathcal{D}$ is λ^+-good*

Proof. ($\Longleftarrow$) Let T' be a theory and $\mathcal{N} \models T'$. Also, suppose that $\mathcal{M} \models T$. If $\mathcal{M}^\lambda/\mathcal{D}$ is λ^+-saturated, then by htpothesis we may conclude that $\mathcal{D}$ is λ^+-good. By theorem 3.2.5, we conclude that $\mathcal{N}^\lambda/\mathcal{D}$ is λ^+-saturated, and thus $T' \trianglelefteq T$. Therefore, T is maximal in the pre-order $\trianglelefteq$.

($\Longrightarrow$) Notice that by theorem 3.2.5, if $\mathcal{D}$ is a λ^+-good regular ultrafilter over λ, then $\mathcal{M}^I/\mathcal{D}$ is λ^+-saturated. Hence, let T be maximal, $\mathcal{D}$ be a ω-regular ultrafilter over λ and $\mathcal{M} \models T$ such that $\mathcal{M}^\lambda/\mathcal{D}$ is λ^+-saturated. By maximality of T, we have that $\mathrm{Th}\left([\mu]^{<\aleph_0}, \subseteq, P)\right) \trianglelefteq T$, for any $\mu < \lambda^+$, and then $([\mu]^{<\aleph_0})^\lambda/\mathcal{D}$ is λ^+-saturated. According to Theorem 3.2.6 (iii), we can conclude that $\mathcal{N}^\lambda/\mathcal{D}$ is λ^+-saturated, for every elementary extension $\mathcal{N}$ of $([\mu]^{<\aleph_0}, \subseteq, P)$. Therefore, by Theorem 3.2.6 (i), we may assure that the ultrafilter $\mathcal{D}$ is λ^+-good. ■

Remark 3.2.9 Let $\mathcal{D}$ be a regular ultrafilter over I, $\lambda > \aleph_0$ and $\mathcal{M} \models \mathsf{DLO}$. By fact 3.2.7, we have that $\mathcal{M}^\lambda/\mathcal{D}$ is λ-saturated if and only if $\mathcal{D}$ is λ-good. Now, by Theorem 3.2.8, we conclude that DLO is maximal in Keisler's order. In particular, since $(\mathbb{Q}, \leq) \models \mathsf{DLO}$, then we may claim that $\mathrm{Th}\,(\mathbb{Q}, \leq)$ is maximal in Keisler's order.

3.3. Cofinality spectrum problem associated to a regular ultrafilter

Until now, we characterized maximality in Keisler's order via good ultrafilters: indeed, maximal theories are those that are saturated by good ultrafiter, and we showed that DLO is maximal in $\trianglelefteq$. Now, we connect characterization of Keisler's order given by Theorem 3.2.8 with the notion of cofinality spectrum problem, studied in chapter 2: indeed, given a regular ultrafilter $\mathcal{D}$ over a non-empty set I, we may define a cofinality spectrum problem $\mathbf{s}$ associated to $\mathcal{D}$, and in this cofinality spectrum problem we may characterize good ultrafilters via $\mathcal{C}(\mathbf{s}, \mathfrak{t}_\mathbf{s})$.

First, we study an useful characterization of good ultrafilters via its cut spectrum (see definition 2.2.1). Essentially, good ultrafilters are those whose allow us to saturate $\omega^\lambda/\mathcal{D}$ with absence of certain gaps.

Theorem 3.3.1 ([11, Fact 1.3]) *Let $\mathcal{D}$ a regular ultrafilter over λ. Then $\mathcal{C}(\mathcal{D}) = \emptyset$ if and only if $\mathcal{D}$ is λ^+-good.*

Proof. ($\Longleftarrow$) If $\mathcal{D}$ is λ^+-good, then $\omega^\lambda/\mathcal{D}$ is λ^+-saturated by theorem 3.2.5. Therefore, by remark 2.2.10 we may conclude that $\mathcal{C}(\mathcal{D}) = \emptyset$.
($\Longrightarrow$) Suppose that $\mathcal{C}(\mathcal{D}) = \emptyset$. We need to show that $\omega^\lambda/\mathcal{D}$ is λ^+-saturated. To prove this, we expand the language $\mathcal{L} = \{<\}$ by adding a constant symbol 0 (which will be interpreted as the

minimum element of ω) and an unary function symbol s (which will be the interpretation of the successor function). Now, we consider the theory T of discrete linear orders with minimum element and without maximum (see definition ??). Let p be a type in $\omega^\lambda/\mathcal{D}$ in the expanded language $\{0, s, <\}$ with parameters in some set $A \subseteq \omega^\lambda/\mathcal{D}$ of cardinality $< \lambda^+$. Since T has quantifier elimination (see Theorem 1.4.13), we may suppose that $p(x)$ only contains atomic formulas with parameters in A.
Fix $A_1 := \{a \in A : (x > a) \in p\}$ and $A_2 := \{a \in A : (x < a) \in p\}$, and we have to consider the following cases:

(i) There are $a \in A_1$ and $n \in \omega$ such that $\{s^n(a) = x\} \in p$. Then we can see that p is realized by $s^n(a)$. (if $a \in A_2$, then p is realized by $s^{-n}(a)$).

(ii) There are $a \in A_2$ and $n \in \omega$ such that $\{s^n(x) = a\} \in p$. Then we can see that p is realized by $s^{-n}(a)$.

(iii) If both A_1 and A_2 are infinite: suppose that $|A_1| = \lambda_1$, $|A_2| = \lambda$ and $\lambda_1 + \lambda_2 \leq \lambda$. We consider two cases here:

   (a) Both λ_1, λ_2 are regular. Since $\mathcal{C}(\mathcal{D}) = \emptyset$, then it is possible to find a realization b of p.

   (b) If λ_1 is not regular, consider a cofinal sequence $\langle a^1_\alpha : \alpha < \operatorname{cof}(\lambda_1)\rangle$ in A_1. It is clear that $\operatorname{cof}(\lambda_1)$ is regular and $\operatorname{cof}(\lambda_1) < \lambda_1$, then it is possible to find a realization b of p, since $\mathcal{C}(\mathcal{D}) = \emptyset$.

   (c) If λ_1 is not regular, we conclude as in (b).

(iv) If A_1 is finite and A_2 is infinite: by cases (i) and (ii), we may assume that there is $a \in A_1 \cup A_2$ and $n \in \omega$ such that neither $(s^n(a) = x) \in p(x)$ or $s^n(x) = a) \in p(x)$. Thus, let $a_1 \in A_1$ be its maximum. Since $\mathcal{C}(\mathcal{D}) = \emptyset$, then $(\langle s^n(a_1) : n \in \omega\rangle, \langle a_2 : a_2 \in A_2\rangle$ does not represent a gap, hence there is a $b \in \omega^I/\mathcal{D}$ such that $s^n(a_1) < b < a_2$ for all $n \in \omega$ and $a_2 \in A_2$. Then b realizes p.

(v) If A_1 is infinite and A_2 is finite, we conclude as in (iv).

In any case, we can find a realization of p in $\omega^\lambda/\mathcal{D}$. Therefore, we conclude that $\omega^\lambda/\mathcal{D}$ is λ^+-saturated. ■

The following lemma gives us a clue of the form of the orders in the cofinality spectrum problem we want to define later in this section.

Lemma 3.3.2 ([11, Claim 10.17]) *Let $\mathcal{D}$ be a regular ultrafilter on I, with $|I| = \lambda$. For any $n < \omega$, let $<_n$ denote the usual order on ω restricted to the set $\{0, 1, ..., n-1\}$. Then there is a sequence $\overline{n} = \overline{n}(\mathcal{D}) = \langle n_t : t \in I\rangle \in \omega^I$ such that for all regular cardinals $\kappa_1, \kappa_2 \leq \lambda$, the following are equivalent:*

(i) $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathcal{D})$.

(ii) The linear order $\prod_{\mathcal{D}}(n_t, <_{n_t})$ *has a* (κ_1, κ_2)*-gaps.*

Proof. Note that it is enough to prove (i) $\Longrightarrow$ (ii).
Let $\mathcal{X} = \{X_i : i < \lambda\}$ be a λ-regularizing family for $\mathcal{D}$, and fix $n_t := |\{i < \lambda : t \in X\}| + 1$, for some $X \in \mathcal{X}$. Since $\mathcal{D}$ is λ-regular, then $n_t \in \omega$, for all $t \in I$. Now suppose that $(\langle a_\alpha : \alpha < \kappa_1\rangle, \langle b_\beta : \beta < \kappa_2\rangle)$ represent a (κ_1, κ_2)-gap in $\omega^\lambda/\mathcal{D}$. Since $\kappa_1 + \kappa_2 \leq \lambda$, we have an injective function

$$d : (\kappa_1 \times \{0\}) \cup (\kappa_2 \times \{1\}) \to \mathcal{D}$$

such that for all $t \in I$, we have that $|\{x \in \mathrm{dom}\,(d) : t \in d(x)\}| < n_t$, since $\mathcal{D}$ is regular. Now consider

$$Y_t := \{a_\alpha(t) : t \in d((\alpha, 0))\} \cup \{b_\beta(t) : t \in d((\beta, 1))\}$$

we can notice that $|Y_t| < n_t$. If we denote $<_{Y_t}$ the restriction of the linear order on ω to Y_t, then we are able to choose an injective map $h_t : (Y_t, <_{Y_t}) \to (n_t, <_{n_t})$ such that its image is an interval of n_t and h_t preserves the order. Consider

$$h := \prod_{t \in I} h_t : \prod_{t \in I} (Y_t, <_{Y_t}) \to \prod_{t \in I} (n_t, <_{n_t}),$$

and let us show that $(\langle h(a_\alpha) : \alpha < \kappa_1\rangle, \langle h(b_\beta) : \beta < \kappa_2\rangle)$ represent a (κ_1, κ_2)-cut in $\prod_{\mathcal{D}}(n_t, <_{n_t})$. If $\gamma < \alpha < \kappa_1$, note that

$$\{t \in I : h_t(a_\gamma(t)) < h_t(a_\alpha(t))\} \supseteq d((\alpha, 0)) \cap d((\gamma, 0)) \cap \{t \in I : a_\gamma(t) < a_\alpha(t)\} \in \mathcal{D}.$$

Therefore, the sequence $(h(a_\alpha)_{\alpha<\kappa_1})$ is increasing in $\prod_{\mathcal{D}}(n_t, <_{n_t})$. A similar argument will show that the sequence $(h(b_\beta)_{\beta<\kappa_2})$ is decreasing in $\prod_{\mathcal{D}}(n_t, <_{n_t})$. Since h preserves the order, the we can conclude that $(\langle h(a_\alpha) : \alpha < \kappa_1\rangle, \langle h(b_\beta) : \beta < \kappa_2\rangle)$ represent a (κ_1, κ_2)-gap in $\prod_{\mathcal{D}}(n_t, <_{n_t})$; otherwise, it would be possible to find $d \in \omega^I/\mathcal{D}$ such that $a_\alpha < d < b_\beta$, for all $\alpha < \kappa_1$ and $\beta < \kappa_2$, contradicting the fact that $(\langle a_\alpha : \alpha < \kappa_1\rangle, \langle b_\beta : \beta < \kappa_2\rangle)$ represent a (κ_1, κ_2)-gap in $\omega^\lambda/\mathcal{D}$. ■

Definition 3.3.3 ([11, Def. 10.18]) *Let* $\mathcal{D}$ *be a regular ultrafilter on* I*,* M *a model extending* $(\omega, <)$*. If* $\langle n_t : t \in I\rangle \in \omega^I$ *is a sequence satisfying lemma 3.3.2 for* $\mathcal{D}$ *and* $(X, <_X) \subseteq M^I/\mathcal{D}$ *is given by*

$$(X, <_X) = \prod_{t \in I} (n_t, <_{n_t})/\mathcal{D}$$

then we say $(X, <_X)$ ***captures pseudofinite gaps****.*

Now we are ready to define the cofinality spectrum problem in which we will work on to determine the sufficient conditions of a theory to be in the maximum class in Keisler's order. Definition 3.3.3 gives us a main property of the orders of this cofinality spectrum problem.

Fact 3.3.4 *Let* $\mathcal{D}$ *be a regular ultrafilter on* I*, with* $|I| = \lambda$*. Let* M *expand* $(\omega, <)$ *and let* $M_1 = M^I/\mathcal{D}$*. Then there exist expansions* M^+, M_1^+ *of* M, M_1 *respectively such that* $M_1^+ = (M^+)^I/\mathcal{D}$ *and a set of formulas* $\Delta \supseteq \{x < y < z\}$ *of the language of* M *such that*

(1) $\mathbf{s} = (M, M_1, M^+, M_1^+, (\mathrm{Th}\,(M^+))^M, \Delta)$ *is a cofinality spectrum problem.*

(2) some nontrivial $\mathbf{a} \in \mathrm{Or}\,(\mathbf{s})$ *captures pseudofinite gaps in the sense of definition 3.3.3.*

Proof. Since ultrapowers commute with reducts (see [11, Thm. A, p. 274]), for (1) we can choose any expansion M^+ of M which codifies sufficient set theory for trees in the sense of definition 2.2.5, e.g. an expansion to a model of $(\mathcal{H}(\chi), \in)$, for some sufficiently large χ such that $M \in \mathcal{H}(\chi)$. Also, we can consider $M_1^+ = (M^+)^I/D$.
For (2), let $\langle n_t : t \in I\rangle$ be given by lemma 3.3.2. By the way this sequence is built, the linear order $\prod_{t\in I}(n_t, <_{n_t})/\mathcal{D}$ is Δ-definable in M_1 and captures pseudofinite gaps (see proof of lemma 3.3.2). It will correspond to some nontrivial $\mathbf{a} \in \mathrm{Or}\,(\mathbf{s})$, and for this order we can choose $d_{\mathbf{a}}$ to not be a natural number. ■

Observation 3.3.5 In [11, p. 277], the theory of the cofinality spectrum problem defined in fact 3.3.4 requires that $T = \mathrm{Th}\,(M^+)$. But by remark 2.2.7, considering the whole theory of the expansion of M leads us to a contradiction. Because of that, we take $T := (\mathrm{Th}\,(M^+))^M$.

The following results characterizes the κ-treetops (see definition 2.2.3) in the cofinality spectrum problem $\mathbf{s}$ defined in fact 3.3.4.

Lemma 3.3.6 ([11, Claim 10.22]) *Let* $\mathcal{D}$ *be a regular ultrafilter over* I*, with* $|I| = \lambda$*,* $\mathcal{M}$ *expanding* $(\omega, <)$ *and* $\mathcal{M}_1 = \mathcal{M}^I/\mathcal{D}$*. Let* $\mathbf{s}$ *be the cofinality spectrum problem given by fact 3.3.4. Given* $\kappa = \mathrm{cf}\,(\kappa) \leq \lambda$*, the following are equivalent:*

(i) $\mathcal{D}$ *has* κ^+*-treetops.*

(ii) $\kappa^+ \leq \mathfrak{t}_{\mathbf{s}}$*.*

Proof. For definition of $\mathfrak{t}_{\mathbf{s}}$, it is immediate that (i) $\Longrightarrow$ (ii).
To prove (ii) $\Longrightarrow$ (i), we prove the contrapositive. Let $(\mathcal{T}, \trianglelefteq_{\mathcal{T}})$ be a given tree (not necessarily in $\mathrm{Tr}\,(\mathbf{s})$ definable in M^+, and let $\bar{c} = \langle c_\alpha : \alpha < \kappa \rangle$ be an increasing sequence in $\mathcal{T}$ with no upper bound. Since $\mathcal{D}$ is regular, then for each $\kappa \leq \lambda$ there is a function $d : \kappa \to \mathcal{D}$ whose image is a regularizing family. So, by Łoś's Theorem, we may assume that there is a sequence of finite trees $(\mathcal{T}_t, \trianglelefteq^t_{\mathcal{T}})$ for $t \in I$ such that $\prod_{t\in I} (\mathcal{T}_t, \trianglelefteq^t_{\mathcal{T}})/\mathcal{D}$ is a subtree of $(\mathcal{T}, \trianglelefteq_{\mathcal{T}})^{M_1^+} := (\mathcal{T}, \trianglelefteq_{\mathcal{T}})^I/\mathcal{D}$ which includes the sequence $\bar{c}$. Thus, let $\mathbf{a} \in \mathrm{Or}\,(\mathbf{s})$ be given by fact 3.3.4.
As in the proof of lemma 3.3.2, we may choose at every (or almost every) $t \in I$ a function $f_t : (\mathcal{T}_t \trianglelefteq^t_{\mathcal{T}}) \to (\mathcal{T}_{\mathbf{a}}^{M^+}, \trianglelefteq_{\mathbf{a}}^{M^+})$ with the following property $(\star)$: "f_t is injective and respects the partial ordering, i.e. for $x, y \in \mathrm{dom}\,(f_t)$ we have that $x \trianglelefteq^t_{\mathcal{T}} y$ if and only if $f_t(x) \trianglelefteq_{\mathbf{a}}^{M^+} f_t(y)$".
Now let $f := [(f_t)_{t\in I}]$ and suppose by contradiction that $\langle b_\alpha := f(c_\alpha) : \alpha < \kappa \rangle$ has an upper bound in $\mathcal{T}_{\mathbf{a}}$, call it b_*. Consider the map

$$\begin{array}{llll} d_0 : & \kappa & \longrightarrow & \mathcal{D} \\ & \alpha & \longmapsto & d(\alpha) \cap \{t \in I : b_\alpha(t) \trianglelefteq b_*(t)\} \cap \{t \in I : f_t \text{ satisfies } (\star)\} \end{array}$$

Notice that for each $t \in I$, the set $B_t := \{b_\alpha(t) : \alpha < \kappa \,\text{and}\, t \in d_0(\alpha)\}$ is finite and linearly ordered by $\trianglelefteq$, because of the way we chose b_*. For each $t \in I$, let b_t be the maximal element of B_t under this linear ordering. Then by Łoś's Theorem and the choice of the functions f_t, we have that the element $c_* := [(f_t^{-1}(b_t))_{t\in I}]$ is well defined. By Łoś's Theorem, we claim that $c_* \in \mathcal{T}^{M_1^+}$, and c_* is an upper bound for the sequence $\bar{c}$ in $\mathcal{T}^{M_1^+}$, which is a contradiction. Hence, we showed that $\langle b_\alpha : \alpha < \kappa \rangle$ is an increasing sequence in $\mathcal{T}_{\mathbf{a}}$ with no upper bound. Here concludes the proof. ■

The κ-treetops property allows us to prove the goodness of a given ultrafilter.

Theorem 3.3.7 ([11, Thm. 10.1]) *Let $\mathcal{D}$ be a regular ultrafilter over I, where $|I| = \lambda \leq \aleph_0$. If $\mathcal{D}$ has λ^+-treetops, then $\mathcal{D}$ is λ^+-good.*

Proof. Consider the cofinality spectrum problem given by fact 3.3.4, associated to $\mathcal{D}$. By Theorem 2.4.5, $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$, and by lemma 3.3.6 and λ^+-treetops hypothesis, we have that $\mathcal{C}(\mathbf{s}, |I|^+) = \emptyset$. Therefore, $\mathcal{C}(\mathcal{D}) = \emptyset$, and by Theorem 3.3.1, we can conclude that $\mathcal{D}$ is λ^+-good. ■

Lemma 3.3.8 ([11, Lemma 10.24]) *Let $\mathcal{D}$ be a regular ultrafilter over I, with $|I| = \lambda$. Then the following are equivalent:*

(i) $\kappa = \mathrm{cf}\,(\kappa) \leq \lambda$ *implies* $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$.

(ii) $\mathcal{D}$ *has* λ^+*-treetops.*

Proof. (ii) $\Longrightarrow$ (i): If $\mathcal{D}$ has λ^+-treetops, then by Theorem 3.3.7 $\mathcal{D}$ is λ^+-good, and by Theorem 3.3.1, we have that $\mathcal{C}(\mathcal{D}) = \emptyset$.

(i) $\Longrightarrow$ (ii): Consider the cofinality spectrum problem given by fact 3.3.4. Since $\mathcal{D}$ has no λ^+-treetops, then there is a tree $T_{\mathbf{a}} \in \mathrm{Tr}\,(\mathbf{s})$ which contains a branch of length κ with no upper bound. By remark 2.2.10 (ii), we can take $\kappa = \mathfrak{t}_{\mathbf{s}}$ and hence we can find a definable (in $\mathcal{M}_1^+$) linear order which has a (κ, κ)-gap. Therefore, $(\kappa, \kappa) \in \mathcal{C}(\mathcal{D})$. ■

Now we can put together all the information about treetops and good ultrafilters in the following theorem.

Theorem 3.3.9 ([11, Main Theorem 10.25]) *Let $\mathcal{D}$ be a regular ultrafilter over I, with $|I| = \lambda$. Then the following are equivalent:*

(i) $\mathcal{D}$ is λ^+-good.

(ii) $\mathcal{D}$ has λ^+-treetops.

(iii) For every $\kappa \leq \lambda$, $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$.

(iv) $\mathcal{C}(\mathcal{D}) = \emptyset$.

Proof. (ii) $\Longleftrightarrow$ (iii): Lemma 3.3.8.
(iv) $\Longrightarrow$ (iii): Immediate: if $\mathcal{C}(\mathcal{D}) = \emptyset$, then in particular $\mathcal{C}(\mathcal{D})$ does not contain symmetric gaps, for all $\kappa \leq \lambda$.
(ii) $\Longrightarrow$ (i): Theorem 3.3.7.
(i) $\Longleftrightarrow$ (iv): Theorem 3.3.1. ■

Remark 3.3.10 Although we give a characterization of maximality in Keisler's order in the cofinality spectrum problem $\mathbf{s}$ defined in fact 3.3.4 (by characterizing the good ultrafilters in $\mathbf{s}$), this characterization does not give us too much information about maximality of SOP_2-theories (see [11, Def. 11.1, p. 280]). Actually, the model-theoretical techniques used by showing that SOP-theories are quite far from the interests of this thesis. References such as [11, 16, 17] have a complete analysis of the model-theoretical tools developped for proving maximality of SOP-theories and SOP_2-theories in Keisler's order.

4 $\mathfrak{p} = \mathfrak{t}$ and some applications in Topology

In this chapter, we study the proof given by Malliaris-Shelah [11] of $\mathfrak{p} = \mathfrak{t}$ by using the tool of cofinality spectrum problems (studied in chapter 2).

In section 4.1, we review the proof of $\mathfrak{p} = \mathfrak{t}$. For this purpose, we define a convenient cofinality spectrum problem and we use many of the properties of the forcing $([\mathbb{N}]^{\aleph_0}, \subseteq^*)$.

In section 4.2, we focus on some applications about $\mathfrak{p}$ and $\mathfrak{t}$. In particular, we give a positive answer to an open question asked by Todorčević and Veličković [20] about the existence of forcings of size $\mathfrak{p}$ without precaliber $\mathfrak{p}$.

4.1. $\mathfrak{p} = \mathfrak{t}$

We focus on the problem $\mathfrak{p} = \mathfrak{t}$. We already defined these cardinal invariants in chapter 1 (see Definition 1.1.11) and we showed that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{c}$ (see proposition 1.1.12).

Definition 4.1.1 ([11, Def. 14.3]) *Let V a countable transitive model of* ZFC*. In this section, we fix the following conventions:*

1. *Let $M = (\mathcal{H}(\aleph_1)^V, \in)$.*
2. *Let $\mathbb{P} = ((\mathcal{P}(\omega)/\mathrm{fin}) \cap V, \subseteq^*) \in V$ (see section 1.2).*
3. *Let G be a generic subset of $\mathbb{P}$ over V (this set G exists since V is countable), and let $\dot{G} := \{(\check{A}, A) : A \in (\mathcal{P}(\omega)/\mathrm{fin}) \cap V\}$ be its canonical $\mathbb{P}$-name.*
4. *For $f \in V$, we denote by $\check{f}$ the $\mathbb{P}$-name for f.*

Having in mind the previous conventions and remark 1.2.12 (i), we define a ***generic ultrapower*** *in the extension $V[G]$ as follows:*

5. *By the generic ultrapower M^ω/G in $V[G]$ we will mean the model $\mathcal{N} \in V[G]$ with universe $\{f/G : f \in (M^\omega)^V\}$ such that*
   - ✓ $\mathcal{N} \vDash$ "$f_1/G = f_2/G$" *if and only if* $\{n : f_1(n) = f_2(n)\} \in G$.

✓ *$\mathcal{N} \models$ "$f_1/G \in f_2/G$" if and only if $\{n : f_1(n) \in f_2(n)\} \in G$.*

6. *In V, we consider the $\mathbb{P}$-name $\widetilde{\mathcal{N}}$ of the generic ultrapower $M^\omega/\check{G}$.*

Observation 4.1.2 Let us make a few comments about Definition 4.1.1.

1. Since $\mathbb{P}$ is a $< \aleph_1$-closed, by remark 1.2.12, we have that $(M^\omega)^{V[G]} = (M^\omega)^V$. This is important, as we will be moving between V and $V[G]$ when necessary.

2. Since $\mathbb{P}$ is $< \mathfrak{t}$-complete and $\mathfrak{p} \leq \mathfrak{t}$, by lemma 1.2.13, moving from V to $V[G]$ will not affect the analysis of $\mathfrak{p} < \mathfrak{t}$. Concretely, by lemma 1.2.13 we have that $\mathfrak{p}^{V[G]} = \mathfrak{p}^V$ and $\mathfrak{t}^{V[G]} = \mathfrak{t}^V$.

3. Notice that $\Vdash_{\mathbb{P}}$ "$\dot{G}$ is a non-principal ultrafilter" (see proposition 1.2.14).

Now we are able to build the cofinality spectrum problem in $V[G]$, where we will work from now. Following the conventions fixed above, we consider $M = M^+ = (\mathcal{H}(\aleph_1), \in)$ and $M_1 = M_1^+ = \mathcal{N} = M^\omega/G$. Also, by Łoś theorem, let $\mathbf{j}$ be an elementary embedding from M into $\mathcal{N}$.

Definition 4.1.3 ([11, Def. 14.4]) *Working in $V[G]$, let $M, \mathcal{N}$ be as in Definition 4.1.1. Let Δ_{psf} be the set of all first-order formulas $\varphi(x, y, \bar{z})$ in the vocabulary of M (i.e., $\{\in, =\}$) such that if $\bar{c} \in M^{\ell(\bar{z})}$ then $\varphi(x, y, \bar{c})$ is a linear order on the set $A^M_{\varphi,\bar{c}} := \{a : M \models \varphi(a, a, \bar{c})\}$, denoted by $\leq_{\varphi,\bar{c}}$. Moreover, we demand that, in M, $A^M_{\varphi,\bar{c}}$ is finite.*

Fact 4.1.4 ([11, Obs. 14.5]) *Let $M, \mathcal{N}$ be as in definition 4.1.1 and $\varphi \in \Delta_{\text{psf}}$. Then, by Łoś's theorem.*

(a) for each $\bar{c} \in \mathcal{N}^{\ell(\bar{z})}$, $\varphi(x, y, \bar{c})$ is a discrete linear order on the set $\{a \in \mathcal{N} : \mathcal{N} \models \varphi(a, a, \bar{c})\}$.

(b) each non-empty $\mathcal{N}$-definable subset of $A^{\mathcal{N}}_{\varphi,\bar{c}}$ has a first and last element.

(c) in $\mathcal{N}$, we can identify $(A^{\mathcal{N}}_{\varphi,\bar{c}}, \leq_{\varphi,\bar{c}})$ with the ultraproduct

$$\langle (A^M_{\varphi,\bar{c}_n}, \leq_{\varphi,\bar{c}_n}), : n \in \omega \rangle / G$$

where each $A^M_{\varphi,\bar{c}_n}$ is finite and linearly ordered by $\leq_{\varphi,\bar{c}_n}$.

Definition 4.1.3 and fact 4.1.4 give us the structure of the orders of the cofinality spectrum problem we are working on: essentially, an order in this cofinality spectrum problem can be seen as an ultraproduct (modulo G) of finite linear orderings. The following theorem gives us the structure of the trees of this cofinality spectrum problem.

Theorem 4.1.5 ([11, Claim 14.6]) *Working in $V[G]$, $\mathbf{s} := (M, \mathcal{N}, \mathrm{Th}\,(M), \Delta_{\text{psf}})$ is a cofinality spectrum problem.*

Proof. It is clear that conditions (1)-(3) of definition 2.2.6 hold, so we prove that Δ_{psf} has ESTT, in sense of definition 2.2.5. Notice that definition 4.1.3 and fact 4.1.4 give us the conditions (1)-(4) in definition 2.2.5; the only remaining data we have to specify is that each $d_{\mathbf{a}}$ is the maximum element of $(X_{\mathbf{a}}, \leq_{\mathbf{a}})$.
Let us see that $\mathbf{s}$ is closed under finite Cartesian products: let $\mathbf{a}, \mathbf{b} \in \text{Or}(\mathbf{s})$, with $X_{\mathbf{a}} = \{a : \mathcal{N} \models \varphi_1(a, a, \bar{c}_1)\}$ and $X_{\mathbf{b}} = \{a : \mathcal{N} \models \varphi_2(a, a, \bar{c}_2)\}$, where $\varphi_1, \varphi_2 \in \Delta_{\text{psf}}$. When $\mathbf{a} = \mathbf{b}$, let $\theta(x_1 y_1, x_2 y_2, \bar{c}_1 {}^\frown \bar{c}_2)$ be the formula which says that $x_1, x_2 \in X_{\mathbf{a}}$, $y_1, y_2 \in X_{\mathbf{b}}$ and

$$\begin{aligned} &(\max\{x_1, y_1\} < \max\{x_2, y_2\}) \quad \vee \\ &[(\max\{x_1, y_1\} = \max\{x_2, y_2\}) \quad \wedge \quad (x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2))] \end{aligned}$$

i.e first we order $(x_1, x_2), (y_1, y_2)$ by maximum (i.e $\max\{x_1, x_2\} \leq \max\{y_1, y_2\}$), then lexicographically; if $\mathbf{a} \neq \mathbf{b}$, then θ is just declared as the lexicographic order. In any case, we can see that θ describes a linear order in $X_{\mathbf{a}} \times X_{\mathbf{b}}$, thus $\theta(x_1 y_1, x_2 y_2, z_1 z_2) \in \Delta_{\text{psf}}$. This allows us to conclude that (5)-(6) in definition 2.2.5 hold.
Finally, let $\mathbf{a} \in \text{Or}(\mathbf{s})$ given and let $\varphi_{\mathbf{a}} = \varphi(x, y, z)$. Let $\psi_{\mathbf{a}}(\eta, \bar{c})$ be the formula which says that η is a function of domain $\{x \in X_{\mathbf{a}} : x \leq_{\mathbf{a}} x_0\}$, for some $x_0 \in X_{\mathbf{a}}$ with $x_0 <_{\mathbf{a}} d_{\mathbf{a}}$, such that $\eta(x) \in X_{\mathbf{a}}$ for all $x \leq_{\mathbf{a}} x_0$ in $X_{\mathbf{a}}$. Now, for each $\bar{c} \in M^{\lg(\bar{z})}$ we have that $\mathcal{T}^M_{\varphi, \bar{c}} := \{\eta : M \models \psi(\eta, \bar{c})\}$ is the set of finite sequences of members of $A^M_{\varphi, \bar{c}} = \{a : M \models \varphi(a, a, \bar{c})\}$ of length $\leq \max A^M_{\varphi, \bar{c}}$, and let

$$\trianglelefteq := \{(\eta, \nu) : \eta, \nu \in \mathcal{T}^M_{\varphi, \bar{c}} \text{ and } \eta \text{ is an initial segment of } \nu\}.$$

We can define the functions lg (the length of a sequence in $\mathcal{T}^M_{\varphi, \bar{c}}$) and val (the evaluation function) as usual. Hence, by Łoś's theorem, we can extend these definitions to $\mathcal{N}$, and they will have the same behaviour as in M. This gives us the structure of the trees in $\mathbf{s}$. ■

Now we are ready to analyze the proof of $\mathfrak{p} = \mathfrak{t}$. Recall that we have $\mathfrak{p} \leq \mathfrak{t}$ (this is immediate from proposition 1.1.12). So, we will assume that $V \models \mathfrak{p} < \mathfrak{t}$, and this will lead us to a contradiction. From the rest of this section, $\mathbf{s}$ will denote the cofinality spectrum problem described in theorem 4.1.5.

Theorem 4.1.6 ([11, Claim 14.7]) *In $V[G]$, $\mathfrak{t} \leq \mathfrak{t}_{\mathbf{s}}$, i.e., if $\mathbf{a} \in \text{Or}(\mathbf{s})$, then any decreasing sequence of cofinality $\kappa < \mathfrak{t}$ in $(\mathcal{T}_{\mathbf{a}}, \trianglelefteq_{\mathbf{a}})^{\mathcal{N}}$ has an upper bound.*

Proof. Working in V, let $\theta = \text{cf}(\theta) < \mathfrak{t}$ be given and let $B \in \mathbb{P}$ ($B \in G$) be such that

$$B \Vdash_{\mathbb{Q}} \text{“}\langle \dot{f}_\alpha / \dot{G} : \alpha < \theta \rangle \text{ is an increasing sequence in } (\omega^{<\omega}, \trianglelefteq)^{\tilde{\mathcal{N}}}\text{”}$$

Since the forcing $\mathbb{P}$ adds no new sequences of length $< \mathfrak{t}$, without loss of generality we can claim that there is some $\langle f_\alpha : \alpha < \theta \rangle$ such that $B \Vdash_{\mathbb{P}}$ “$\dot{f}_\alpha = \check{f}_\alpha$” for all $\alpha < \theta$ and moreover, we assume that $B \subseteq^* \{n \in \mathbb{N} : f_\alpha(n) \trianglelefteq f_\beta(n)\}$ for all $\alpha < \beta < \theta$.

Why can we work in the tree $(\omega^{<\omega}, \trianglelefteq)$? Recalling definition 4.1.3 and fact 4.1.4, each order $\mathbf{a} = (X_{\mathbf{a}}, <_{\mathbf{a}})$ in $\mathbf{s}$ is an ultraproduct of some finite linear orders $\langle (X_n, <_n) : n < \omega \rangle$ in M, and each $\mathcal{T}_{\mathbf{a}} = (\mathcal{T}_{\mathbf{a}}, \trianglelefteq_{\mathbf{a}})$ is the ultraproduct of the corresponding trees of finite sequences $\langle (\mathcal{T}_n, \trianglelefteq_n) : n < \omega \rangle$ of each X_n in $\mathcal{N}$. Thus, by Łoś's theorem, we may claim that $X_{\mathbf{a}}$ is a linear order in $\mathcal{N}$ and $\mathcal{T}_{\mathbf{a}}$ is the corresponding tree of finite sequences of $X_{\mathbf{a}}$.
Now, it is possible to find an isomorphism between each $\mathcal{T}_n$ and a definable downward closed subset of $(\omega^{<\omega}, \trianglelefteq)^M$. So, all these isomorphisms induce an isomorphism of $\mathcal{T}_{\mathbf{a}}$ onto a definable downward closed subset of $(\omega^{<\omega}, \trianglelefteq)^{\mathcal{N}}$.
Since $\mathfrak{t} \leq \mathfrak{b}$ by proposition 1.1.12, we can find an increasing function $g \in \omega^\omega$ such that for each $\alpha < \theta$ there is an $n_\alpha \in \omega$ such that if $n \geq n_\alpha$, then

$$g(n) > \lg(f_\alpha(n)) + \sum \{f_\alpha(n)(i) : i < \lg(f_\alpha(n))\}$$

Now, consider $\overline{s} = \langle s_n : n < \omega \rangle$, where s_n denotes the tree $g(n)^{\leq g(n)}$. First, note that

- ✓ each s_n is a finite non-empty subset of $\omega^{<\omega}$.
- ✓ if $\alpha < \theta$, for all but finitely many n, we have that $f_\alpha(n) \in s_n$, since $f_\alpha(n)$ and its length are dominated by $g(n)$.

Now we will build a convenient tower: for each $\alpha < \theta$, define $\langle Y_\alpha : \alpha < \theta \rangle$ as follows

$$Y_\alpha := \bigcup \left\{ \{n\} \times \left(s_n \cap (\omega^{<\omega})^{[f_\alpha(n)]}\right) : n \in B \right\}$$

where $(\omega^{<\omega})^{[\nu]} := \{\eta \in \omega^{<\omega} : \nu \trianglelefteq \eta\}$ is the cone above ν (The set Y_α is a disjoint union of cones in s_n above $f_\alpha(n)$). Also, define $Y_* := \bigcup \{\{n\} \times s_n : n \in B\}$. It is clear that $Y_\alpha \subseteq Y_*$ for all $\alpha < \theta$, and Y_α is a countably infinite subset of $B \times \omega^{<\omega}$. Moreover, $Y_\beta \subseteq^* Y_\alpha$ if $\alpha < \beta$: since for $\alpha < \beta$, $f_\alpha(n) \trianglelefteq f_\beta(n)$ for all but finitely many $n \in B$. Therefore, we have that $(\omega^{<\omega})^{[f_\beta(n)]} \subseteq (\omega^{<\omega})^{[f_\alpha(n)]}$ for all but finitely many $n \in B$.
Since $\theta < \mathfrak{t}$, we can find a pseudo-intersection Z of $\langle Y_\alpha : \alpha < \theta \rangle$ such that $Z \subseteq Y_*$, and since each s_n is finite, $B_1 = \{n \in B : Z \cap (\{n\} \times s_n) \neq \emptyset\}$ must be infinite. For $n \in B_1$, choose any element ν_n such that $(n, \nu_n) \in Z \cap (\{n\} \times s_n)$; otherwise, choose $\nu_n = \langle 0 \rangle$ for $n \in \mathbb{N} \setminus B$. Since we choose $\nu_n \in Z$, then we can notice that $f_\alpha(n) \trianglelefteq \nu_n(n)$ for all but finitely many $n \in B_1$, and hence, we have shown that

$$B_1 \Vdash_{\mathbb{P}} \text{“}\langle \nu_n : n \in \omega \rangle / \dot{G} \text{ is an upper bound for } \langle f_\alpha / \dot{G} : \alpha < \theta \rangle \text{ in } (\omega^{<\omega}, \trianglelefteq)^{\mathcal{N}}\text{”}.$$

This completes the proof. ■

Corollary 4.1.7 ([11, Conclusion 14.9]) *Working in $V[G]$, let* $\mathbf{s}$ *be the cofinality spectrum problem defined in definition 4.1.5. Then* $\mathcal{C}(\mathbf{s}, \mathfrak{t}) = \emptyset$.

Proof. Immediate from theorems 2.4.4 and 4.1.6. ■

Now we connect the possibility of performing peculiar cuts with the context of cofinality spectrum problems. In section 2.1, we ruled out some peculiar cuts. Shelah [18] showed that, assuming $\mathfrak{p} < \mathfrak{t}$, then it is possible to find a regular cardinal κ such that there is a $(\kappa, \mathfrak{p})$-peculiar gap in ω^ω, with $\aleph_1 \leq \kappa < \mathfrak{p}$ (see theorem 2.1.5). The following result claims that, assuming $\mathfrak{p} < \mathfrak{t}$, it is possible to find a distinguished order in $\mathbf{s}$ where we can detect a $(\kappa, \mathfrak{p})$-gap.

Theorem 4.1.8 ([11, Claim 14.13]) *In V, suppose $\mathfrak{p} < \mathfrak{t}$. Then for some regular κ with $\aleph_1 \leq \kappa < \mathfrak{p}$, we have that $V[G] \vDash (\kappa, \mathfrak{p}) \in \mathcal{C}(\mathbf{s}, \mathfrak{t})$.*

Proof. By theorem 2.1.5, if we assume $\mathfrak{p} < \mathfrak{t}$, then there is a $(\kappa, \mathfrak{p})$-peculiar gap in ω^ω, with $\aleph_1 \leq \kappa < \mathfrak{p}$. Now, we will show that this gap can be found in some $X_{\mathbf{a}}$ in $\mathcal{N}$. So, let $(\langle g_\alpha : \alpha < \kappa \rangle, \langle f_\beta : \beta < \mathfrak{p} \rangle)$ be a $(\kappa, \mathfrak{p})$-peculiar gap, with $\aleph_1 \leq \kappa < \mathfrak{p}$. Since $f_\beta \leq^* f_0$, for all $\beta < \mathfrak{p}$, let us consider

$$I = \prod_{n<\omega} [0, f_0(n)]/G$$

Notice that I is an ultraproduct of some finite linear orders, then by construction of $\mathbf{s}$, we can identify $I = X_{\mathbf{a}}$ for some $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$. Then the peculiar gap forms a pre-gap (possibly, a gap) in I. Suppose that this pre-gap is not a gap, i.e., there are an infinite $B \in G$ and $h \in \omega^\omega$ such that, in V,

$$B \Vdash_{\mathbb{P}} \text{“} g_\alpha/\dot{G} \leq h/\dot{G} \leq f_\beta/\dot{G} \text{ for all } \alpha < \kappa, \beta < \mathfrak{p} \text{”}.$$

Since $\mathbb{P}$ is $< \mathfrak{t}$-closed, there is some $B' \in G$ such that $B' \subseteq^* B$ and $B' \subseteq^* \{n : g_\alpha(n) \leq h(n)\}$ for all $\alpha < \kappa$ and $B' \subseteq^* \{n : h(n) < f_\beta(n)\}$ for all $\beta < \mathfrak{p}$, but this contradicts the definition of peculiar cut: if we consider the function h_* defined by $h_*(n) = h(n)$ for $n \in B'$; and $h_*(n) = f_0(n)$ for $n \notin B'$, we can notice that $h_* \geq^* g_\alpha$ for each $\alpha < \kappa$, but it is not the case that $h_* \geq^* f_\beta$ for some $\beta < \mathfrak{p}$, because B' is infinite. ■

We can now state the main result of this chapter.

Theorem 4.1.9 ([11, Thm. 14.1]) $\mathfrak{p} = \mathfrak{t}$.

Proof. It is immediate that $\mathfrak{p} \leq \mathfrak{t}$. Now suppose that $\mathfrak{p} < \mathfrak{t}$. Then it is possible to find a countable transitive model V of (a large finite fragment of) ZFC such that $V \vDash \mathfrak{p} < \mathfrak{t}$. Now, in $V[G]$, let $\mathbf{s}$ be the cofinality spectrum problem from definition 4.1.5. By Corollary 4.1.7, which does not assume $\mathfrak{p} < \mathfrak{t}$, we have that $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$. But by theorem 4.1.8, which does assume $\mathfrak{p} < \mathfrak{t}$, we have that $\mathcal{C}(\mathbf{s}, \mathfrak{t}_\mathbf{s}) \neq \emptyset$, a contradiction. ■

4.2. Some applications of $\mathfrak{p} = \mathfrak{t}$

Now we explore some interesting consequences of theorem 4.1.9. Two of the applications presented here are some topological results. Also, we give an affirmative answer to a question about σ-linked posets, asked by Todorčević and Veličković in [20].

In theorem 1.3.7 we presented *Bell's theorem* (see [2]), which claims that $\mathfrak{p}$ is the smallest cardinal such that $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is false for some σ-centred forcing $\mathbb{P}$. In topological language, this means that there is no separable compact Hausdorff space X which can be covered by fewer than $\mathfrak{p}$-many closed nowhere dense sets (see observation 1.3.8). According to theorems 1.3.7 and 4.1.9, we can give the following characterization of $\mathfrak{t}$.

Theorem 4.2.1 $\mathfrak{m}_\sigma = \mathfrak{t}$.

Proof. Immediate from theorems 1.3.7 and 4.1.9. ■

Now, for the next application of $\mathfrak{p} = \mathfrak{t}$ let us give some definitions, which can be found in [21]. Unless otherwise stated, all topological spaces consider from now are Hausdorff spaces.

Definition 4.2.2 *Let X be a topological space, $A \subseteq X$ and $x \in X$.*

1. *We say that A **converges to** x if each neighbourhood of x contains all but finitely many points of A. We denote this by $A \to x$.*

2. *We say that X is **countably compact** if each countable infinite subset of X has an accumulation point.*

3. *We say that X is **sequentially compact** if each countable infinite set has an infinite subset which converges to some point of X.*

It is clear from the definition that sequentially compact spaces are countably compact. The next result gives us a partial reciprocal: we recall that *a local base at $x \in X$* is a collection $\mathcal{B}_x$ of open neighbourhoods of x such that for all open set U with $x \in U$ there is a $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$; the minimal cardinality of a local base at x is called the **character of X at x**, and is denoted by $\chi(x, X)$.

Theorem 4.2.3 ([1, Thm. 2.5]) *Let X be a countably compact topological with $\chi(x, X) < \mathfrak{p}$ for all $x \in X$. Then X is sequentially compact.*

Proof. Suppose that S is an countable infinite subset in X, and let $x \in X$ be a proper accumulation point of S. Let $\mathcal{B} = \{U_\alpha : \alpha < \lambda\}$ be a local base of open sets at x, where $\lambda < \mathfrak{p}$. Then the family $\mathcal{B} \cap S := \{U_\alpha \cap S : \alpha < \mathfrak{p}\}$ is a family of subsets of S with the strong finite intersection property. By the definition of $\mathfrak{p}$, there is an infinite subset $T \subseteq S$ such that $T \subseteq^* U_\alpha$ for each $\alpha < \mathfrak{p}$. Hence, we have that $T \to x$. ■

Corollary 4.2.4 *Let X be a countably compact topological with $\chi(x, X) < \mathfrak{t}$ for all $x \in X$. Then X is sequentially compact.*

Proof. Immediate from theorems 4.1.9 and 4.2.3. ■

Now we study an interesting result about σ-linked posets. We recall that given a poset $\mathcal{P}$ and $L \subseteq \mathcal{P}$, we say that L is **linked** if and only if $p_1 \not\perp p_2$, for all $p_1, p_2 \in L$. Besides, we say that $\mathcal{P}$ is **σ-linked** if $\mathcal{P}$ is a countable union of linked subsets.

Theorem 4.2.5 ([20, Thm. 1.3]) *There is a σ-linked poset $\mathcal{P}$ of size $\mathfrak{t}$ without centred subsets of size $\mathfrak{t}$.*

Proof. Let $\{a_\xi : \xi < \mathfrak{t}\}$ be a tower. For $x, y \subseteq \mathbb{N}$ distinct, define $\Delta(x, y) := \min(x \Delta y)$ (the least point in the symmetric difference of x and y). Define the poset $\mathcal{P}$ as follows: $F \in \mathcal{P}$ if and only if $F \in [\mathfrak{t}]^{<\aleph_0}$ and $|a_F \cap k| \geq |\Delta_F \cap k|$, for all $k < \omega$, where $\Delta_F := \{\Delta(a_\xi, a_\eta) : \xi, \eta \in F, \xi \neq \eta\}$ and $a_F := \bigcap\{a_\xi : \xi \in F\}$. The order is reverse inclusion. Then $\mathcal{P}$ is σ-linked and has no centred subsets of size $\mathfrak{t}$ (we leave the details to the reader). ■

According to theorems 4.1.9 and 4.2.5, we give a proof of the following result.

Theorem 4.2.6 ([20, Thm. 1.5]) *There is a poset $\mathcal{P}$ of size $\mathfrak{p}$ which is σ-linked but not σ-centred.*

Proof. Consider the poset $\mathcal{P}$ described in theorem 4.2.5. By theorem 4.1.9, then $\mathcal{P}$ has size $\mathfrak{t}$, and by theorem 4.2.5 we may assure that $\mathcal{P}$ has no centred subsets of size $\mathfrak{t}$. By theorem 4.1.9, $\mathcal{P}$ has size $\mathfrak{p}$ and it has no centred subsets of size $\mathfrak{p}$, and thus $\mathcal{P}$ is not σ-centred. ■

In [20, Question 1.6], it remained the open question to determine the existence of a σ-linked poset without precaliber $\mathfrak{p}$ (we recall that an infinite cardinal κ is a **precaliber** for a poset $\mathcal{P}$ if and only if whenever $p_\alpha \in \mathcal{P}$, for $\alpha < \kappa$, there is a $B \in [\kappa]^\kappa$ such that $\{p_\alpha : \alpha \in B\}$ is centred). We give a positive answer to this question.

Theorem 4.2.7 *There is a σ-linked poset $\mathcal{P}$ without precaliber $\mathfrak{p}$.*

Proof. By Theorems 4.1.9 and 4.2.5, there is a σ-linked poset $\mathcal{P}$ of size $\mathfrak{p}$ without centred subsets of size $\mathfrak{p}$. ■

Observation 4.2.8 The topological applications studied in this dissertation can be considered as trivial: given a topological space with a combinatorial property in terms of $\mathfrak{p}$ (e.g. character less than $\mathfrak{p}$), then we just change $\mathfrak{p}$ for $\mathfrak{t}$ and we obtain the same combinatorial property but in terms of $\mathfrak{t}$ and viceversa. Until now, we have not found non-trivial topological consequences of $\mathfrak{p} = \mathfrak{t}$ yet. Following [21, 11] and other references, the problem $\mathfrak{p} = \mathfrak{t}$ was one of the most important problems on cardinal invariants of the continuum. However, we are still on the survey of interesting applications of $\mathfrak{p} = \mathfrak{t}$ in General Topology.

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