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Abstract

The notion of a Whitney dual for a graded partially ordered set (poset) P with a minimum element $\hat{0}$ has been introduced recently by González D'León and Hallam with some interesting connections to other areas of algebra and combinatorics. We say that two posets are Whitney duals to each other if (the absolute value of) their Whitney numbers of the first and second kind are interchanged between the two posets. Some families of familiar posets such as the poset Π_n of partitions of the set $\{1, 2, 3..., n\}$ have Whitney duals. This has been proved by defining a suitable edge labeling λ on the edges of the Hasse diagram of Π_n satisfying certain conditions. Such an edge labeling is called a Whitney labeling and González D'León - Hallam proved that every graded poset that admits a Whitney labeling has a Whitney dual.

We study the Whitney duality property for two families of operadic posets, finding Whitney labelings and constructing combinatorial descriptions of their Whitney duals. One is known as the family of posets of weighted partitions Π_n^k, studied by González D'León and Wachs related to the operad $\mathcal{C}om^k$ of commutative algebras with k totally commutative products, and the other is the family of posets of pointed partitions $\Pi_n^\bullet$, studied by Chapoton and Vallette associated to the operad $\mathcal{P}erm$ of $\mathcal{P}erm$-algebras. We prove that a labeling, previously defined by González D'León, for Π_n^k is a Whitney labeling and prove that its associated Whitney dual is a poset of colored Lyndon forests. We also find a Whitney labeling for $\Pi_n^\bullet$ and then use this labeling to show that its associated Whitney dual is a poset of pointed Lyndon forests. For the case $k = 2$, it turns out that the families Π_n^2 and $\Pi_n^\bullet$ have the same Whitney numbers of the first and second kind. Our results imply that there are multiple non-isomorphic Whitney duals for these two families in this case.

Resumen

Título: Duales de Whitney de posets operádicos

González D'León y Hallam introdujeron recientemente la noción de duales de Whitney para un conjunto parcialmente ordenado (poset) graduado P con un elemento mínimo $\hat{0}$ con algunas conexiones interesantes a otras áreas del álgebra y la combinatoria. Decimos que dos posets son duales de Whitney entre sí, si (el valor absoluto de) sus números de Whitney del primer y segundo tipo se intercambian entre los dos posets. Algunas familias de posets familiares como el poset Π_n de particiones del conjunto $\{1, 2, 3..., n\}$ tienen duales de Whitney. Esto se ha demostrado definiendo un etiquetamiento adecuado λ en las aristas del diagrama de Hasse de Π_n que satisface ciertas condiciones. A tal etiquetamiento de aristas se le llama etiquetamiento de Whitney y González D'León - Hallam demostraron que todo poset graduado que admite un etiquetamiento de Whitney tiene un dual de Whitney.
Estudiamos la propiedad de dualidad de Whitney para dos familias de posets operadicos, por medio de etiquetamientos de Whitney y de la construcción de descripciones combinatorias de sus duales de Whitney. Una de las familias es la familia de posets de particiones con pesos Π_n^k, estudiadas por González D'León y Wachs, relacionadas con el operad $\mathcal{C}om^k$ de álgebras conmutativas con k productos totalmente conmutativos, y la otra es la familia de posets de particiones punteadas $\Pi_n^\bullet$, estudiadas por Chapoton y Vallette asociadas al operad $\mathcal{P}erm$ de $\mathcal{P}erm$-álgebras. Demostramos que un etiquetamiento, previamente definido por González D'León, para Π_n^k es un etiquetamiento de Whitney y demostramos que su dual de Whitney asociado es un poset de bosques de Lyndon coloreados. También encontramos un etiquetamiento de Whitney para $\Pi_n^\bullet$ y luego usamos este etiquetamiento para mostrar que su dual de Whitney asociado es un poset de bosques de Lyndon punteados. Para el caso $k = 2$, resulta que las familias Π_n^2 y $\Pi_n^\bullet$ tienen los mismos números de Whitney del primer y segundo tipo. Nuestros resultados implican que hay múltiples duales de Whitney no isomórfos entre sí para estas dos familias en este caso.

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1 Introduction

A *partially ordered set* or *poset* is a pair $(P, \leq)$ where P is a set and $\leq$ is a relation that is reflexive, antisymmetric and transitive. In general, we will abuse notation and say that P is a poset. We denote $x < y$ whenever $x, y \in P$ are such that $x \leq y$ but $x \neq y$. If $x < y$ and there is no $z \in P$ such that $x < z < y$, we say that y *covers* x and we denote it by $x \lessdot y$. A subset C of P is said to be a *chain* if for every pair of elements $x, y \in C$ either $x \leq y$ or $y \leq x$ (that is, x and y are *comparable*). We say that an element $x \in P$ is *minimal* if there is no $z \in P$ such that $z < x$. If there is a unique minimal element we call it the *minimum* of P and we denote it by $\hat{0}$. A *maximal chain* of P is a chain C in P such that for every $z \in P \setminus C$ the subset $C \cup \{z\}$ is not a chain. A poset whose maximal chains are all of the same cardinality is said to be *graded*. It is known that in a graded poset we can define a function $\rho : P \to \mathbb{N}$ such that $\rho(x) = 0$ when x is a minimal element and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y$. We call ρ the *rank function* of P. In what follows we will assume that every poset P is finite, graded with rank function ρ and has a $\hat{0}$. For undefined concepts and notation about posets the reader could consult [16].

1.1 Whitney duals of a graded poset

The notion of a Whitney dual for a graded poset P with a minimum element $\hat{0}$ has been introduced recently by González D'León and Hallam [7] with some interesting connections to other areas of algebra and combinatorics.
The *Möbius function* is an important invariant of a poset P, defined recursively for $x \leq y$ in P as

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, y) & \text{if } x \neq y. \end{cases}$$

Two other important invariants of P are the *k-th Whitney numbers* of the *first* and *second kind*, denoted respectively as $w_k(P)$ and $W_k(P)$, and that are defined for any $k \in \mathbb{N}$ as

$$w_k(P) = \sum_{\rho(x)=k} \mu(\hat{0}, x)$$
$$W_k(P) = \sum_{\rho(x)=k} 1.$$

Whitney numbers play a meaningful role in many areas of mathematics. It has been proved that Whitney numbers are equal to the coefficients of the chromatic polynomial of a finite

graph [18] and hence they are useful in the counting of the number of acyclic orientations of a graph [15]. Whitney number also are useful in counting the number of regions in the complement of a hyperplane arrangement [19].
González D'León and Wachs noticed in [6] that for a pair of posets, namely the poset of weighted partitions Π_n^2 introduced by Dotsenko and Khoroshkin in [5], and the poset of spanning increasing forest $\mathcal{SF}_n$ studied by Reiner in [13] and Sagan in [14], the Whitney numbers of the first and second kind appeared swapped between the two posets. This motivated the following definition in [7].

Definition 1.1.1 ([7]). Let P and Q be two graded posets with a $\hat{0}$. We say that P and Q are *Whitney duals* if for all $k \geq 0$

$$|w_k(P)| = W_k(Q) \text{ and } W_k(P) = |w_k(Q)|.$$

Some families of familiar posets, such as the poset Π_n of partitions of the set $[n] := \{1, 2, 3..., n\}$, the poset NC_n of noncrossing partitions of $[n]$ and the poset of weighted partitions Π_n^2 have been shown in [7] to have Whitney duals. That these posets have Whitney duals was proved by defining a suitable edge labeling λ on the edges of the Hasse diagram of Π_n satisfying certain conditions (*EW-labelings*). There is also a more general version in [7] that instead of edge labelings, considers labelings of pairs (c, e) where c is a maximal chain and e is an edge in c (*CW-labelings*). Any of these two types of labelings are in general called *Whitney labelings*. The following is the main theorem in [7] that allows to conclude Whitney duality for the posets mentioned above.

Theorem 1.1.2 ([7]). *A finite graded poset P with a $\hat{0}$ that admits a Whitney labeling λ has a Whitney dual. Moreover, a Whitney dual $Q_\lambda(P)$ can be constructed explicitly depending on λ.*

1.2 A longstanding conjecture

Two geometric properties that the can be satisfied by a sequence of numbers are the following. A sequence $(a_0, a_1, \ldots, a_n)$ is said to be *unimodal* if for some value of c we have that

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{n-1} \geq a_n.$$

A sequence $(a_0, a_1, \ldots, a_n)$ is said to be *log-concave* if for all i we have that

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

In sequences of positive numbers, log-concavity is a stronger property that implies unimodality.
A longstanding conjecture of Rota, Heron and Welsh (and originally of Read and Hoggar in the case of chromatic polynomials of graphs) asserted that for any geometric lattice the

absolute values of the Whitney numbers of the first kind form a log-concave sequence. This conjecture was recently proved by Adiprasito, Huh and Katz [1] using a combinatorial version of Hodge Theory. Since it was proved in [7] that geometric lattices are Whitney labelable, in recent work of González D'León, Hallam and Samper [9] this and other conjetures have been generalized to the realm of Whitney labelable and Whitney dualizable posets.

Conjecture 1.2.1 ([9])**.** *The Whitney numbers of the first and second kind of a graded poset with a* $\hat{0}$ *and a Whitney labeling* λ *are log-concave.*

In order to verify this and other conjectures regarding the theory of Whitney labelings, finding good examples to the theory becomes extremely relevant.

1.3 Partition posets associated to operads

An *operad* is an algebraic object that encodes types of algebras. In [12] Mendez and Yang, and later independently Vallette in [17] defined a family of decorated partition posets associated to a *set operad*. In particular, in [17] the author describes a technique to prove Koszulity of the operad by showning that the operadic partition poset is Cohen-Macaulay. For the context in operad theory and partition posets associated to an operad the reader could consult [10].

In [17] Vallette defined and later in [2] Chapoton and Vallette studied the family of pointed partition posets $\Pi_n^\bullet$ associated to the $\mathcal{P}erm$ operad that models $\mathcal{P}erm$-algebras. In [4] González D'León defined and studied the family of weighted partition posets Π_n^k associated to the $\mathcal{C}om^k$ operad of algebras with k totally commutative operations. In the case $k = 1$ the poset Π_n^1 is in fact the poset of partitions Π_n, and in the case $k = 2$ this poset was already defined in [5] by Dotsenko and Khoroshkin and studied in [6] by González D'León and Wachs.

1.4 Results

In this work we study the Whitney duality property for the families Π_n^k and $\Pi_n^\bullet$ associated to the operads $\mathcal{C}om^k$ and $\mathcal{P}erm$. We find and describe Whitney labelings for these posets and find combinatorial descriptions of their Whitney duals. In the case $k = 2$, it turns out that the families Π_n^2 and $\Pi_n^\bullet$ have the same Whitney numbers of the first and second kind (what we call *Whitney twins*). Our results imply that there are multiple non-isomorphic Whitney duals for these two families of posets.

In [4] the author defined an edge labeling λ_E for the poset Π_n^k. We prove the following theorem.

Theorem 1.4.1. *For every* $n \geq 1$ *and* $k \geq 1$ *the labeling* λ_E *is an EW-labeling. Hence,* Π_n^k *has a Whitney dual.*

Using Theorems 1.4.1 and 1.1.2 we follow the construction of $Q_\lambda(P)$ in [7] and prove the following theorem.

Theorem 1.4.2. *For $n \geq 1$ and $k \geq 1$ the poset $Q_{\lambda_E}(\Pi_n^k)$ is isomorphic to the poset $\mathcal{FL}yn_{n,k}$ of colored Lyndon forests defined in Section 3.1.2.*

For the particular case $k = 2$ is has been already concluded in [8] that for $n \geq 3$ $Q_{\lambda_E}(\Pi_n^2) \ncong \mathcal{SF}_n$, so $\mathcal{SF}_n$ and $\mathcal{FL}yn_{n,2}$ are two non-isomorphic Whitney duals for Π_n^2 for $n \geq 3$.

Theorem 1.4.3. *For any $n \geq 1$ There is an EW-labeling λ_P of $\Pi_n^\bullet$. Hence, $\Pi_n^\bullet$ has a Whitney dual.*

Using Theorems 1.4.3 and 1.1.2 we also prove the following theorem.

Theorem 1.4.4. *For $n \geq 1$ the poset $Q_{\lambda_P}(\Pi_n^\bullet)$ is isomorphic to the poset $\mathcal{FL}yn_{n,\bullet}$ of pointed Lyndon forests defined in Section 3.2.2.*

We also prove the following two theorems about the relation of the new poset $\mathcal{FL}yn_{n,\bullet}$ to the two known Whitney duals of Π_n^2, namely, $\mathcal{SF}_n$ and $\mathcal{FL}yn_{n,2}$.

Theorem 1.4.5. *For $n = 3$ we have that the posets $\mathcal{FL}yn_{3,\bullet}$ and $\mathcal{FL}yn_{3,2}$ are isomorphic. For $n \geq 4$ the posets $\mathcal{FL}yn_{n,\bullet}$ and $\mathcal{FL}yn_{n,2}$ are not isomorphic.*

Theorem 1.4.6. *For $n \geq 3$ the posets $\mathcal{FL}yn_{n,\bullet}$ and $\mathcal{SF}_n$ are not isomorphic.*

Our results imply that the two posets Π_n^2 and $\Pi_n^\bullet$ for $n \geq 4$ have three families of non-isomorphic Whitney duals, $\mathcal{SF}_n$, $\mathcal{FL}yn_{n,\bullet}$ and $\mathcal{FL}yn_{n,2}$, illustrating the nonuniqueness of Whitney duality. It is still an open question to determine whether uniqueness occurs in a more restrictive setting. For example if the pair (P, λ) of graded poset P together with its Whitney labeling λ have a unique dual pair (Q, λ^*), where Q is a Whitney dual to P and λ^* is another labeling with characteristics yet to be determined.

1.5 Organization of this thesis

This thesis is organized as follows:
In Chapter 2 we give the preliminaries about posets, Whitney duality and operadic partition posets that will be used in the rest of this work. In particular, we describe the theory of Whitney labelings and Whitney duality defined by González D'León and Hallam in [7] and give a few examples.
In Chapter 3 we study the property of Whitney duality for the families $\Pi_n^\bullet$ and Π_n^k proving the main theorems outlined in the previous section.
In Chapter 4 we provide some open questions and directions that we would like to undertake in the near future.

2 Preliminaries

2.1 Partially ordered sets

A *partially ordered set* or *poset* is a pair $(P, \leq)$ where P is a set and $\leq$ is a relation that satisfies:

- For every $x \in P$ we have $x \leq x$ (reflexivity).
- For every $x, y \in P$, $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry).
- For every $x, y, z \in P$, $x \leq y$ and $y \leq z$ implies $x \leq z$(transitivity).

In general, we will abuse notation and say that P is a poset. We denote $x < y$ whenever $x, y \in P$ are such that $x \leq y$ but $x \neq y$. If $x < y$ and there is no $z \in P$ such that $x < z < y$, we say that y *covers* x and we denote it by $x \lessdot y$. A *subposet* Q of P is a poset whose element set is a subset of P and for every $x, y \in Q$ we have that $x \leq y$ in Q if and only if $x \leq y$ in P.
Given $x, y \in P$, the (closed) *interval* $[x, y]$ between x and y is the subposet of P with element set given by

$$[x, y] := \{z \in P : x \leq z \leq y\}.$$

Similarly we define an *open interval* as $(x, y) := \{z \in P : x < z < y\}$. A subposet C of P is said to be a *chain* if for every pair of elements $x, y \in C$ either $x \leq y$ or $y \leq x$ (that is, x and y are *comparable*). A *maximal chain* of P is a chain in P such that for every $z \in P \setminus C$ the subposet $C \cup \{z\}$ is not a chain. A maximal chain in an interval $[x, y]$ of P is said to be *saturated.*
We say that an element $x \in P$ is *minimal* if there is no $z \in P$ such that $z < x$. We denote $\mathcal{M}in(P)$ the set of minimal elements of P. If $\mathcal{M}in(P)$ has a unique element we call it the *minimum* of P and we denote it by $\hat{0}$. In a similar manner we call $x \in P$ a *maximal* element if there is no $z \in P$ such that $x < z$. We denote $\mathcal{M}ax(P)$ the set of maximal elements of P and if $\mathcal{M}ax(P)$ has a unique element we call it the *maximum* of P and we denote it by $\hat{1}$.
We say that the poset P is *finite* if the underlying set of elements of P is finite. We say that P is *graded* if all its maximal chains have the same length. It is not hard to show that when P is graded there exists a function $\rho : P \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, satisfying that $\rho(x) = 0$ whenever $x \in \mathcal{M}in(P)$ and if $x \lessdot y$ then $\rho(y) = \rho(x) + 1$. The *rank* $\rho(P)$ of P is defined as $\rho(x)$ for any $x \in \mathcal{M}ax(P)$. All the posets which we will be working

with are finite, graded (with rank function denoted ρ) and have a minimum element $\hat{0}$. So throughout this paper P will denote such a poset.

Example 2.1.1. Let Π_n be the set of all partitions of the set $[n] = \{1, 2, ...n\}$. Given a pair of elements $\pi, \pi' \in \Pi_n$ we say $\pi \leq \pi'$ whenever every block in π is contained in a block of π'. This order relation is known as *refinement* and we call the poset Π_n the *partition lattice*. It is not hard to show that the cover relation $\pi \lessdot \pi'$ will hold whenever exactly two different blocks of π have been merged to form a block of π' while all the other blocks remain the same. The *Hasse diagram* of P is a directed graph whose vertices are the elements of P and there is a directed edge going upward for each cover relation. In Figure **2-1** we can see an example of the Hasse diagram for the poset Π_3.

2.2 The Möbius function and the Whitney numbers of the first and second kind

For $x \leq y$ in P we define the *Möbius function* recursively as follows:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, y) & \text{if } x \neq y. \end{cases} \tag{2-1}$$

The *kth Whitney numbers* of the *first* and *second* kind, denoted respectively as $w_k(P)$ and $W_k(P)$, are defined for any $k \in \mathbb{N}$ as

$$w_k(P) = \sum_{\rho(x)=k} \mu(\hat{0}, x)$$
$$W_k(P) = \sum_{\rho(x)=k} 1.$$

In Figure **2-1** the Möbius numbers $\mu(\hat{0}, x)$ appear to the right of each element $x \in \Pi_3$, whereas the Whitney numbers of the first and second kind are displayed in the table.

Example 2.2.1. Let T be a tree whose vertices are labeled with distinct non-negative integers. We consider T as a rooted tree considering the smallest vertex of T as a distinguished vertex or *root*. We say that T is increasing if when going along any path starting from the root, we follow an increasing sequence of vertex labels. We call an increasing spanning forest a set of increasing trees whose vertex labels form a partition for some $[n]$ $(n \geq 1)$. Here we will denote by ISF_n the set of all increasing spanning forests on $[n]$. We define a cover relation in ISF_n as follows:
Let $F, F' \in ISF_n$, we say that $F \lessdot F'$ if we can obtain F' by connecting exactly two trees in F when joining their roots through an edge, the root of the new tree will then be the smallest of the two roots which were joined. See Figure **2-2** for the example of ISF_3.

k	$w_k(\Pi_3)$	$W_k(\Pi_3)$
0	1	1
1	-3	3
2	2	1

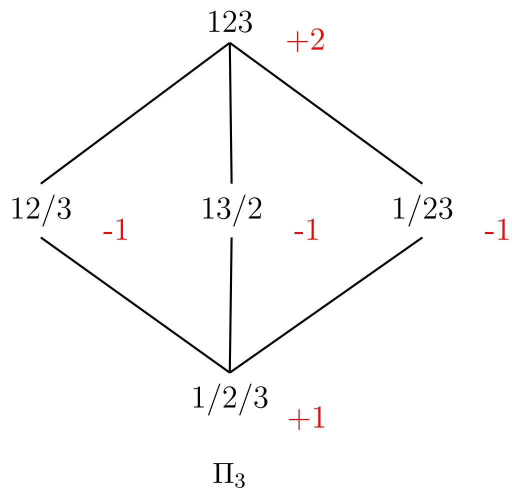


Figure **2-1**: Π_3 and its Whitney numbers.

k	$w_k(ISF_3)$	$W_k(ISF_3)$
0	1	1
1	-3	3
2	1	2

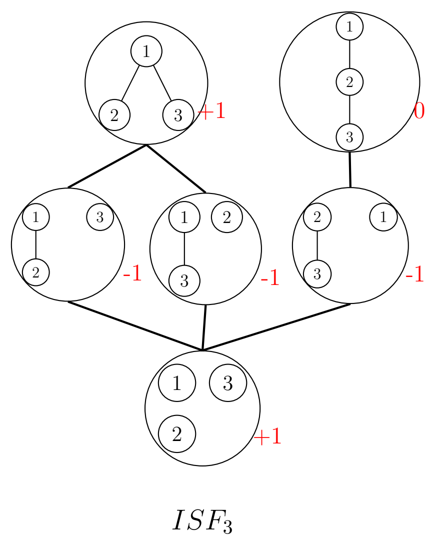
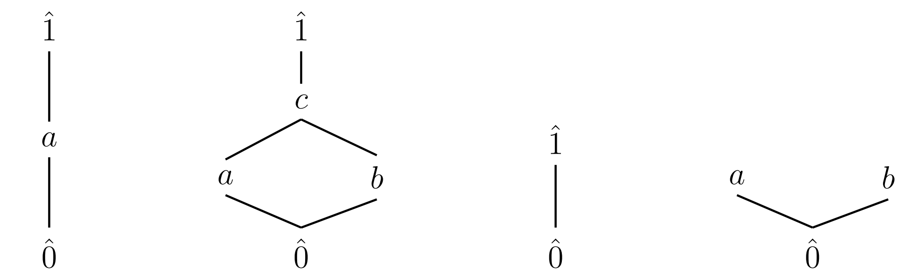


Figure **2-2**: ISF_3 and its Whitney numbers.



No Whitney dual **(a)** No Whitney dual **(b)** Whitney self-dual **(c)** Whitney self-dual **(d)**

Figure **2-3**: Examples of posets with or without Whitney duals.

2.3 Whitney duals and Whitney labelings

When comparing the two tables from Figure **2-1** and Figure **2-2**, we see a relation between the Whitney numbers of the first and second kind for Π_3 and ISF_3. We see that for each k ranging from 0 to 2, $|w_k(\Pi_3)| = W_k(ISF_3)$ and $W_k(\Pi_3) = |w_k(ISF_3)|$. In fact, as it is illustrated by the authors in [7], we have that for any $n \geq 1$

$$|w_k(\Pi_n)| = W_k(ISF_n) \text{ and } W_k(\Pi_n) = |w_k(ISF_n)|.$$

This example motivates the following definition.

Definition 2.3.1 ([7] Definition 1.3)**.** Let P and Q be two graded posets with a $\hat{0}$. We say that P and Q are Whitney duals if for all $k \geq 0$

$$|w_k(P)| = W_k(Q) \text{ and } W_k(P) = |w_k(Q)|.$$

Remark 2.3.2. After Definition 2.3.1 we say that Π_n and ISF_n are Whitney duals for any $n \geq 1$.

In general, to determine whether for a poset P there exists another poset Q satisfying Definition 2.3.1 is not a trivial fact. Not all graded posets have a Whitney dual. For example, the posets (a) and (b) in Figure **2-3** cannot have Whitney duals since $\mu(\hat{0}, \hat{1}) = 0$. A poset P which is its own Whitney dual is said to be *Whitney self-dual*. The posets (c) and (d) in Figure **2-3** are Whitney self-dual. In particular, an *Eulerian poset* is a graded poset P such that for every $x \leq y$ in P it happens that $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$. All Eulerian posets are Whitney self-dual. The poset $\mathbb{B}_n$ of subsets of $[n]$ ordered by set inclusion (known as the *boolean algebra*) is one example of such Eulerian poset and hence is Whitney self-dual.
The authors in [7] found sufficient conditions that we can impose in the poset P that guarantees the existence of a Whitney dual Q. Their technique involves a suitable type of edge labeling on the Hasse diagram of P, satisfying certain conditions and then using such labeling and a technique with quotient posets to construct a suitable Q. For concepts not defined

here and notation the reader may refer to [7] since we will be following closely the results contained there.
We call a map $\lambda : \mathcal{E}(P) \to \Lambda$ an *edge labeling* of P, where $\mathcal{E}(P)$ is the set of edges (cover relations of P) of the Hasse diagram of P and Λ is a poset. We will call Λ the *poset of labels*. Given an edge labeling λ of P and a saturated chain

$$c : (x = x_0 \lessdot x_1 \lessdot x_2 \lessdot .... \lessdot x_n = y)$$

in an interval $[x, y]$ of P, we define the *word of labels* of c as

$$\lambda(c) := \lambda(x_0 \lessdot x_1)\lambda(x_1 \lessdot x_2)...\lambda(x_{n-1} \lessdot x_n).$$

We say that c is an *increasing* chain if for $i \in \{1, 2, ...n-2\}$, $\lambda(x_i \lessdot x_{i+1}) < \lambda(x_{i+1} \lessdot x_{i+2})$. Similarly, we say that c is an *ascent-free* chain if for $i \in \{1, 2, ...n-2\}$, $\lambda(x_i \lessdot x_{i+1}) \not< \lambda(x_{i+1} \lessdot x_{i+2})$.

Remark 2.3.3. Note that in any interval $[x, y]$ there will be increasing and ascent-free saturated chains as well as other chains that are neither of those, i.e. chains that in some parts are increasing and in other parts are not.

Definition 2.3.4. An edge labeling of P is said to be an *ER-labeling* if for every $x, y \in P$ such that $x \leq y$, there exists a unique saturated chain in $[x, y]$ that is increasing.

Definition 2.3.5. Let λ be an ER-labeling. We say that λ satisfies the *rank two switching property* if for every pair $x, y \in P$ such that $\rho(y) - \rho(x) = 2$, if ab is the unique increasing word of labels in $[x, y]$, then there exists a unique chain in $[x, y]$ with word of labels ba.

Definition 2.3.6. Let $\lambda : \mathcal{E}(P) \to \Lambda$ be an *ER-labeling* of P. We say that λ is an *EW-labeling* or a *Whitney labeling* if it satisfies:

- The rank two switching property.
- In each interval each ascent-free maximal chain has a unique word of labels.

Remark 2.3.7. It is worth highlighting that in [7] the authors defined a more general family of Whitney labelings based on what is known in the literature as *chain-edge labelings* or *C-labelings*. These labelings are more technically involved and they go beyond the scope of this project since EW-labeling will be enough for the applications that we have in this work.

Example 2.3.8. In [7] the following labeling for Π_n was shown to be EW. Let $\pi, \pi' \in \Pi_n$ such that $\pi \lessdot \pi'$, and define

$$\begin{aligned} \lambda \colon \mathcal{E}(\Pi_n) &\longrightarrow [n] \times [n] \\ (\pi \lessdot \pi') &\longmapsto \lambda(\pi \lessdot \pi') := (a, b), \end{aligned}$$

where $a < b$ are the minimum elements of the blocks that were merged in π to get to π' and the poset $[n] \times [n]$ has the lexicographic order. See Figure **2-4**

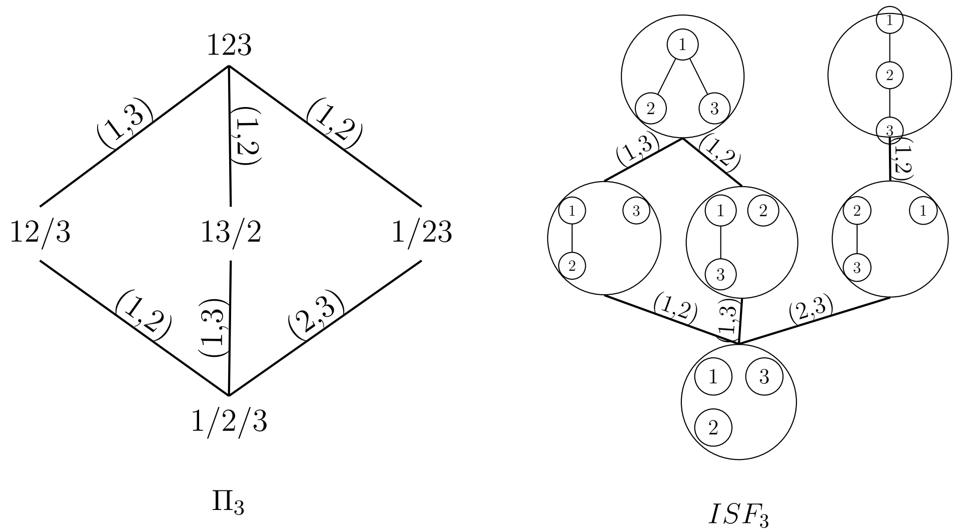


Figure **2-4**: Example of edge labelings on Π_3 and ISF_3.

The following is the main theorem in [7].

Theorem 2.3.9 (c.f. Theorem 1.6 in [7])**.** *Let P be a poset with an EW-labeling. Then P has a Whitney dual. Moreover, using λ we can explicitly construct a Whitney dual $Q_\lambda(P)$.*

Remark 2.3.10. After Example 2.3.8 and Theorem 2.3.9 we conclude that Π_n has a Whitney dual.

2.4 The construction of the Whitney dual $Q_\lambda(P)$

In this section we will describe, for a poset P with an EW-labeling λ, the construction of the Whitney dual $Q_\lambda(P)$.

Definition 2.4.1. An edge labeling of P is said to be an *ER^*-labeling* if for every $x, y \in P$ such that $x \leq y$, there exists a unique saturated chain in $[x, y]$ that is ascent-free.

Example 2.4.2. In [7] the following labeling of ISF_n was shown to be an ER*-labeling. Let $F, F' \in ISF_n$ such that $F \lessdot F'$ and define

$$\begin{aligned} \lambda^*\colon \mathcal{E}(ISF_n) &\longrightarrow [n] \times [n] \\ (F \lessdot F') &\longmapsto \lambda^*(F \lessdot F') := (a, b), \end{aligned}$$

where $a < b$ are the roots of the trees which were merged in F in order to get F' and $[n] \times [n]$ has the lexicographic order. See Figure **2-4**.

The following Theorem of Stanley relates ER and ER*-labelings with the computation of the Möbius values in a poset P.

Theorem 2.4.3 (c.f. Theorem 3.14.2 in [16]). *Let P be a graded poset with an ER-labeling (ER*-labeling). Then*

$$\mu(x,y) = (-1)^{\rho(y)-\rho(x)}|\{c|c \text{ is an ascent-free (increasing) maximal chain in } [x,y]\}|.$$

Theorem 2.4.3 shows that there is a dual role between the increasing maximal chains and the ascent-free maximal chains according to ER and ER*-labelings. In an ER-labeling the increasing maximal chains from $\hat{0}$ identify the elements of the poset P (that in turn correspond to the Whitney numbers of the second kind) while the ascent-free maximal chains correspond to the Möbius values (that in turn correspond to the Whitney numbers of the first kind). On the other hand, in an ER*-labeling the ascent-free maximal chains from $\hat{0}$ identify the elements of the poset P while the increasing maximal chains correspond to the Möbius values. Following this observation, if one has two posets P and Q, with an ER-labeling λ and an ER*-labeling λ^* respectively, and such that there is a label preserving bijection between the sets of saturated chains from $\hat{0}$ of P and Q, then Theorem 2.4.3 implies that the poset P and Q are Whitney duals. This is precisely the main idea behind the proof of Theorem 2.3.9 in [7].
To construct $Q_\lambda(P)$ from P and an EW-labeling λ, the authors in [7], initially consider the poset $C(P)$ whose elements are all saturated chains in P with minimum $\hat{0}$ ordered by inclusion. Then they consider an equivalence relation $\sim_\lambda$ on $C(P)$ defined as follows. We say that that two chains c_1 and c_2 in $C(P)$ are related by a *quadratic exchange* if they have the same maximal element and they differ in one interval of rank two in which c_1 has an increasing step ab and c_2 has the opposite subword ba predicted by the rank two switching property. The relation $\sim_\lambda$ is then defined by saying that $c_1 \sim_\lambda c_2$ if c_1 and c_2 are related by a quadratic exchange and then the relation is extended to all saturated chains in $C(P)$ by transitivity. The resulting quotient poset $Q_\lambda(P) := C(P)/\sim$ is then shown to have an ER*-labeling λ^* induced by λ and furthermore, that there is an bijection between the saturated chains from $\hat{0}$ between P and $Q_\lambda(P)$ that preserves the labels. An application of Theorem 2.4.3 then implies that P and $Q_\lambda(P)$ are Whitney duals to each other. This construction is illustrated in Figure **2-5** for the poset $P = \Pi_n$. Note in this example that $ISF_3 \simeq Q_\lambda(\Pi_3)$.

2.5 Operadic partition posets

An *operad* is a mathematical object that models types of algebras. In an operad, we do not focus on the elements of the algebra but instead on the relations between the operations applied to these elements. The idea of an operad was introduced in the 70's in work of P. May in [11] and others, however, the idea of an operad was already present in different forms in the existing literature. In particular, operads have appeared in different contexts. There are set operads, topological operads, algebraic operads, etc., depending on the category where they are defined. For a further background on the theory of operads and, in particular, algebraic operads the reader can visit [10].

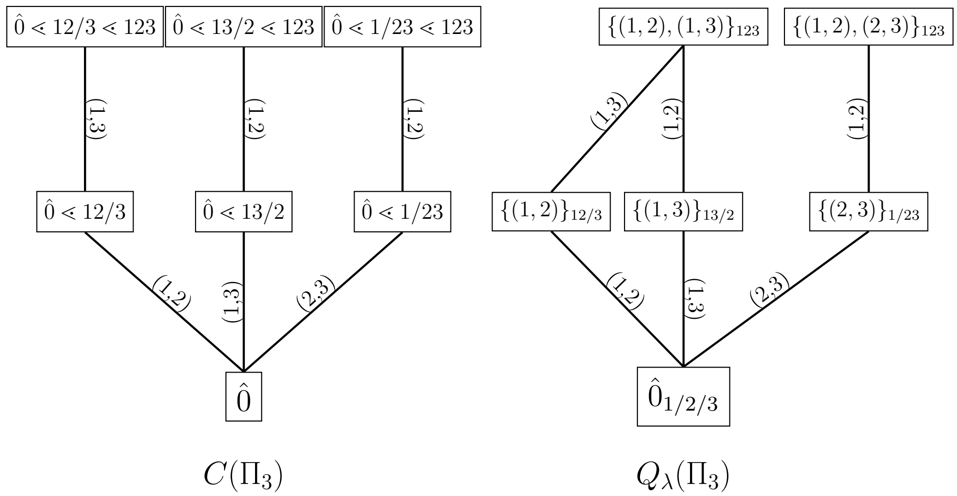


Figure **2-5**: Example of the construction of $Q_\lambda(\Pi_3)$.

Formally, a *set operad* is a monoidal functor $\mathcal{P} : \mathbb{B} \to \mathbb{S}et$ from the category $\mathbb{B}$ of finite sets and bijections to the category $\mathbb{S}et$ of finite sets and functions, together with an associative composition μ and a unit map η.
Méndez and Yang in [12] introduced the concept of a partition poset associated to a set operad (monoidal species in their language). This type of poset was later discovered and used independently by Vallette [17] in order to give a criterion to determine whether a quadratic algebraic operad obtained as the linearization of a set operad is *Koszul*.

Definition 2.5.1. Given an operad $\mathcal{P}$ we define a $\mathcal{P}$*-partition* of $[n]$ as a collection $\alpha = \{p_1, p_2, \ldots, p_k\}$ where $p_i \in \mathcal{P}[A_i]$ for all $i \in [k]$ and $\widetilde{\alpha} = \{A_1, A_2, \ldots, A_k\} \in \Pi_n$ is a partition of $[n]$. We say that $\widetilde{\alpha}$ is the *underlying (regular) partition* of α. We denote $\Pi_n^{\mathcal{P}}$ the set of $\mathcal{P}$*-partitions* of $[n]$. We can define an order relation in $\Pi_n^{\mathcal{P}}$ as follows: given two $\mathcal{P}$-partitions α and β in $\Pi_n^{\mathcal{P}}$ we say that $\alpha \leq \beta$ whenever:

- $\alpha = \{p_1, p_2, \ldots, p_k\}$ where $p_i \in \mathcal{P}[A_i]$ for all i and $\widetilde{\alpha} = \{A_1, A_2, \ldots, A_k\} \in \Pi_n$, and $\beta = \{q_1, q_2, \ldots, q_l\}$ where $p_i \in \mathcal{P}[B_i]$ for all i and $\widetilde{\beta} = \{B_1, B_2, \ldots, B_l\} \in \Pi_n$.
- $\widetilde{\alpha} \leq \widetilde{\beta}$ in Π_n.
- If $B_h = A_{j_1} \cup \cdots \cup A_{j_s}$ then $q_h = \mu(e; p_{j_1}, \ldots, p_{j_s})$ where $e \in \mathcal{P}[\{A_{j_1}, \ldots, A_{j_s}\}]$.
- if $B_h = A_i$ then $q_h = p_i$.

Example 2.5.2. The operad $\mathcal{C}om$, that models the behaviour of commutative algebras, is such that for any $n \geq 1$ there is a unique structure $\star_{[n]}$ in $\mathcal{C}om[[n]]$. Hence, since there is a unique way to merge any set of blocks of a $\mathcal{C}om$-partition, the poset $\Pi_n^{\mathcal{C}om} \cong \Pi_n$.

Example 2.5.3. In [4] González D'León studied the poset Π_n^k of weighted partition defined in Section 3.1. Dotsenko and Koroshkin in [5] showed that $\Pi_n^2 \cong \Pi_n^{\mathcal{C}om^2}$ where $\mathcal{C}om^2$ is the operad of commutative algebras with two totally commutative products. With the same arguments given in [5] it follows that $\Pi_n^k \cong \Pi_n^{\mathcal{C}om^k}$ where $\mathcal{C}om^k$ is the operad of commutative algebras with k totally commutative products.

Example 2.5.4. The operad $\mathcal{P}erm$ was studied by Vallette in [17] where he showed that the poset $\Pi_n^{\mathcal{P}erm} \cong \Pi_n^{\bullet}$, where $\Pi_n^{\bullet}$ is the poset of pointed partitions defined in Section 3.2.

3 Whitney duals of some operadic posets

3.1 The poset of weighted partitions

In this section we will take a look at a family of posets Π_n^k of set partitions with weights given by integral vectors of length k. These posets were defined by González D'León in [4]. For the case $k = 1$ we obtain back the poset of partitions Π_n and the case $k = 2$ is isomorphic to the poset of weighted partitions introduced originally by Dotsenko and Khoroshkin in [5]. There Π_n^2 is denoted Π_n^w but in this work we will always stick to our convention above.
The authors in [6] noticed that the Whitney numbers of the first and second kind of Π_n^2 were (up to sign) switched with respect to the Whitney numbers of a poset of rooted spanning forests $\mathcal{SF}_n$ on $[n]$ studied initially by Reiner [13] and then by Sagan [14]. Hence, we know that both, Π_n^2 and $\mathcal{SF}_n$ are Whitney duals. Even though this was demonstrated through direct comparison of the two pairs of sequences of Whitney numbers, González D'León and Hallam [8] provided a different proof where they show that there is a CW-labeling λ_C for Π_n^2, where $Q_{\lambda_C}(\Pi_n^2)$ was proved to be isomorphic to $\mathcal{SF}_n$. In their work they also considered an EW-labeling λ_E for Π_n^2, originally defined in [6] by González D'León and Wachs, and they noticed that the Whitney dual $Q_{\lambda_E}(\Pi_n^2)$ was **not** isomorphic to $\mathcal{SF}_n$.
A sequence $\mu = (\mu(1), \mu(2), \cdots, \mu(k))$ of non-negative integers with $|\mu| := \sum_{i=1}^{i=k} \mu(i) = n$ is called a *weak composition* of n of length k. If all μ_i are positive then we say $(\mu(1), \mu(2), .., \mu(k))$ is a *composition* of n. Given n, we denote the set of all weak compositions of n of length k as $wcomp_{n,k}$.
Given weak compositions μ and ν of length k, we say $\mu \leq \nu$ iff $\mu(i) \leq \nu(i)$ for $i = 1, \ldots, k$. Addition and subtraction between two weak compositions ν and μ of equal length k is defined component-wise.

Definition 3.1.1. A *weighted partition* of $[n]$ is a collection $\{B_1^{\mu_1}, B_2^{\mu_2}, ..., B_l^{\mu_l}\}$ where the collection $\{B_1, B_2, ..., B_l\}$ is a partition of $[n]$ and $\mu_i \in wcomp_{|B_i|-1,k}$. Let Π_n^k be the poset whose elements are weighted partitions of $[n]$ with weights given by compositions of length k and with cover order relation $\{A_1^{\nu_1}, A_2^{\nu_2}, ..., A_l^{\nu_l}\} \lessdot \{B_1^{\mu_1}, B_2^{\mu_2}, ..., B_m^{\mu_m}\}$ whenever

- $\{A_1, A_2, ..., A_l\} \lessdot \{B_1, B_2, ..., B_m\} \in \Pi_n$.
- if $B_h = A_i \cup A_j$ then $\mu_h - (\nu_i + \nu_j) = e_r$ where $e_r = (0_1, ..., 0_{r-1}, 1_r, 0_{r+1}, ..., 0_k)$.
- if $B_h = A_i$ then $\mu_h = \nu_i$.

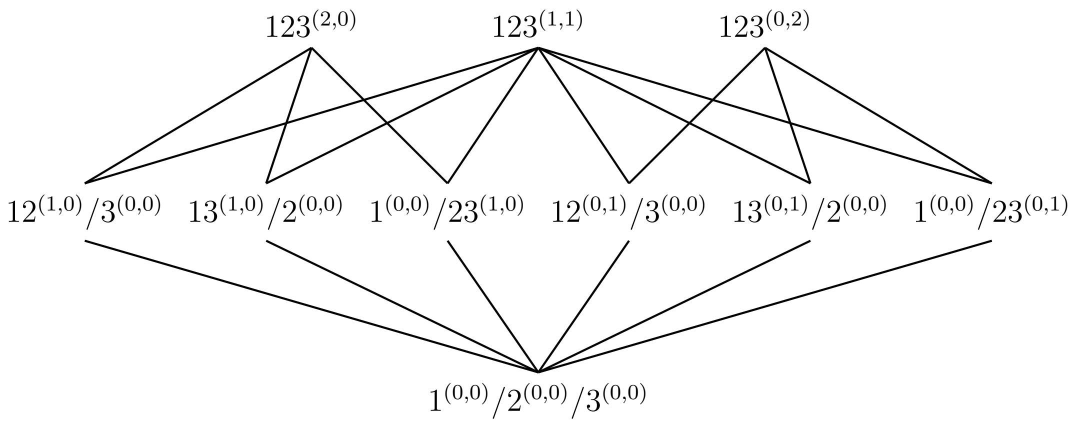


Figure **3-1**: Π_3^2.

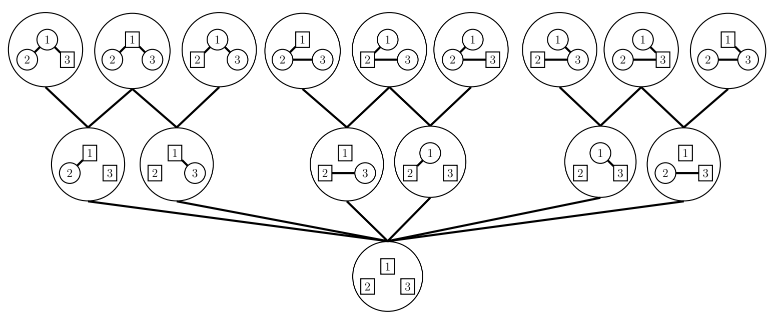


Figure **3-2**: SF_3 (the roots are represented by squares).

In the definition above we will say that $\{B_1^{\mu_1}, B_2^{\mu_2}, ..., B_m^{\mu_m}\}$ is obtained by *r-merging* the blocks $A_i^{\nu_i}$ and $A_j^{\nu_j}$ of $\{A_1^{\nu_1}, A_2^{\nu_2}, ..., A_l^{\nu_l}\}$.
Recall that a *spanning forest* of the complete graph on vertex set $[n]$ is a graph on $[n]$ that is free of loops and cycles. Each of the connected components of a spanning forest is called a *spanning tree* and we say that these are *rooted* if every tree has a unique marked vertex called the *root*. Let $\mathcal{SF}_n$ denote the poset of rooted spanning forests on $[n]$ where the cover relation $F_1 \lessdot F_2$ is defined whenever F_2 can be attained by joining two roots $\{x, y\}$ in F_1 through an edge, where one of the two roots will remain as a root for the resulting tree in F_2.

Theorem 3.1.2 (González D'león-Wachs [6], González D'león-Hallam [8])**.** *The posets Π_n^2 and $\mathcal{SF}_n$ are Whitney duals. Moreover, there exists a CW-labeling λ_C such that $Q_{\lambda_C}(\Pi_n^2)$ is isomorphic to $\mathcal{SF}_n$.*

In [8] González D'león-Hallam also showed that a labeling λ_E previously given by González D'león-Wachs in [6] is an EW-labeling. The poset $Q_{\lambda_E}(\Pi_3^2)$ was also computed in [8] and it is illustrated in Figure **3-3**. It is immediately evident that the posets $\mathcal{SF}_3$ in Figure **3-2** and

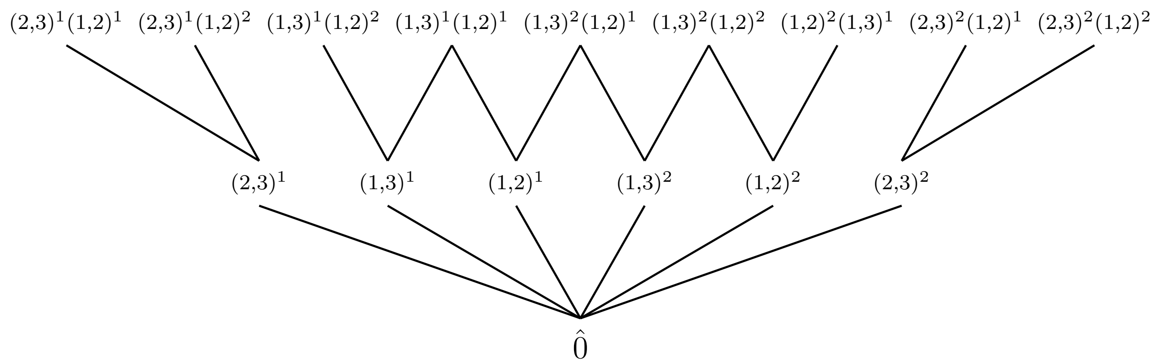


Figure **3-3**: $Q_{\lambda_E}(\Pi_3^2)$.

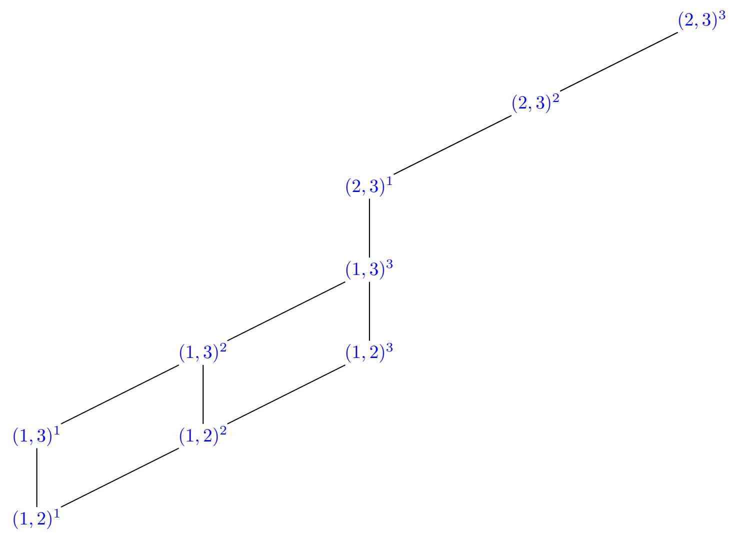


Figure **3-4**: Poset of labels Λ_3^3.

$Q_{\lambda_E}(\Pi_3^2)$ in Figure **3-3** are not isomorphic. Hence, the following theorem.

Theorem 3.1.3 (c.f. Theorem 5.22 in [8])**.** *There exists a poset P and two Whitney labelings λ_1 and λ_2 of P such that the posets $Q_{\lambda_1}(P)$ and $Q_{\lambda_2}(P)$ are not isomorphic.*

3.1.1 A Whitney labeling for the poset of weighted partitions

Our goal now is to prove that a more general ER-labeling $\lambda_E : \mathcal{E}(\Pi_n^k) \to \Lambda_n^k$ (that in fact is also an EL-labeling) given by González D'león in [4] is an EW-labeling. The new labeling generalizes the labeling λ that was given for Π_n^2 in [6] and proven to be an EW-labeling in [7]. We will also provide a combinatorial charaterization of the Whitney dual $Q_{\lambda_E}(\Pi_n^k)$.
Let $\alpha, \alpha' \in \Pi_n^k$ such that $\alpha \lessdot \alpha'$, and α' was obtained by merging two blocks A^x and B^y in α in order to get a new block $(A \cup B)^{x+y+e_r}$ with $r \in [k]$, while all the other blocks remain equal and $min(A) < min(B)$.

We define the following labeling:

$$\begin{aligned}\lambda_E \colon \mathcal{E}(\Pi_n^k) &\longrightarrow \Lambda_n^k\\ (\alpha \lessdot \alpha') &\longmapsto \lambda_E(\alpha \lessdot \alpha') = (min(A), min(B))^r,\end{aligned}$$

where Λ_n^k is the poset of labels defined as follows. We consider first posets of the form Γ_a for $a \in [n-1]$ with set of elements $\Gamma_a = \{(a,b)^u : a < b \leq n, u \in [k]\}$ and partial order $(a,b)^u \leq (a,c)^v$ if $b \leq c$ and $u \leq v$. We then define Λ_n^k to be the ordinal sum $\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_{n-1}$. See Figure **3-4** for the example of Λ_3^3.
González D'León proved in [4] that λ_E is an ER-labeling (in fact an EL-labeling).

Theorem 3.1.4. *(cf. Theorem 3.6 in [4]) The labeling λ_E is an ER-labeling.*

We now prove that λ_E is an EW-labeling.

Theorem 3.1.5. *The labeling $\lambda_E : \mathcal{E}(\Pi_n^k) \to \Lambda_n^k$ considered above is an EW-labeling.*

Proof. From the previous theorem we already know that λ_E is an ER-labeling. Furthermore note that when given any set of labels which corresponds to a saturated chain from $\hat{0}$, we can easily restore the chain itself, i.e. through the sequence of labels we uniquely determine the chain. We need to now prove that the rank two switching property holds. We will be contemplating all possible cases for an arbitrary rank two interval $[\alpha, \alpha'] \in \Pi_n^k$. When referring to the blocks A, B, C, D in each of the cases contemplated below we will assume that $a = min(A) < b = min(B) < c = min(C) < d = min(D)$.

First Case: We merge two pairs of distinct blocks $\{A^\eta, B^\beta\}$ and $\{C^\theta, D^\gamma\}$ in α in order to obtain α'. The open interval (α, α') is of the form $\{K_1, K_2\}$, where $K_1 = AB^{\eta+\beta+e_{r_1}}/C^\theta/D^\gamma$ and $K_2 = A^\eta/B^\beta/CD^{\theta+\gamma+e_{r_2}}$ with $r_1, r_2 \in [k]$. The two chains which appear in our interval $[\alpha, \alpha']$ will have as set of labels; $(a,b)^{r_1}(c,d)^{r_2}$ and $(c,d)^{r_2}(a,b)^{r_1}$ which is what we wanted.

Second Case: If we merge three distinct blocks $\{A^\eta, B^\beta, C^\theta\}$ of α to obtain α', so that when merging two of the blocks, for $r \in [k]$ the vector e_r which we add to the total weight remains fixed. The open interval (α, α') is of the form $\{K_1, K_2, K_3\}$ where $K_1 = AB^{\eta+\beta+e_r}/C^\theta$, $K_2 = A^\eta/BC^{\beta+\theta+e_r}$ and $K_3 = AC^{\eta+\theta+e_r}/B^\beta$. We once again see that the rank two switching property holds because of the chains $c_1 :\ \alpha \lessdot K_1 \lessdot \alpha'$ and $c_2 :\ \alpha \lessdot K_3 \lessdot \alpha'$ with word of labels $\lambda_E(c_1) = (a,b)^r(a,c)^r$ and $\lambda_E(c_2) = (a,c)^r(a,b)^r$ respectively.

Third Case: If we merge three distinct blocks $\{A^\eta, B^\beta, C^\theta\}$ of α to obtain α', so that when merging two of the blocks we either add e_{r_1} or e_{r_2} to the total weight and $r_1 < r_2$. The open interval (α, α') is of the form $\{K_1, K_2, K_3, K_4, K_5, K_6\}$ where $K_1 = AB^{\eta+\beta+e_{r_1}}/C^\theta$, $K_2 = AC^{\eta+\theta+e_{r_1}}/B^\beta$, $K_3 = A^\eta/BC^{\beta+\theta+e_{r_1}}$, $K_4 = AB^{\eta+\beta+e_{r_2}}/C^\theta$, $K_3 = AC^{\eta+\theta+e_{r_2}}/B^\beta$, $K_6 = A^\eta/BC^{\beta+\theta+e_{r_2}}$. We see that $c_1 :\ \alpha \lessdot K_1 \lessdot \alpha'$ is an increasing chain with word of

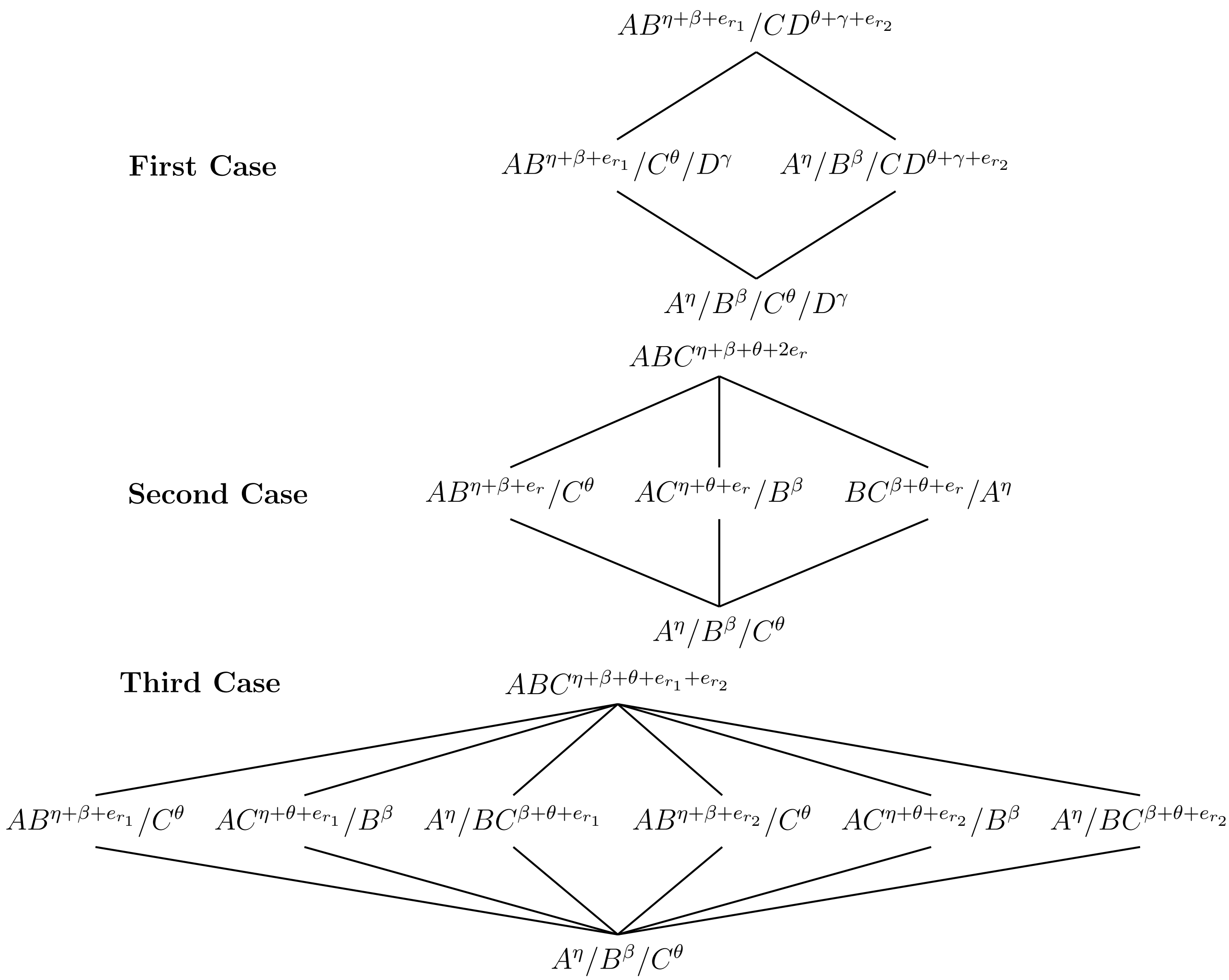


Figure **3-5**: Rank two intervals in Π_n^k.

labels $\lambda_E(c_1) = (a,b)^{r_1}(a,c)^{r_2}$ and $c_2 :\ \alpha \lessdot K_5 \lessdot \alpha'$ is a decreasing chain with word of labels $\lambda_E(c_2) = (a,c)^{r_2}(a,b)^{r_1}$ and no other pair of chains over $[\alpha, \alpha']$ satisfy this property.

Hence we have that λ_E has the rank two switching property. □

3.1.2 A poset of colored Lyndon forests

Thus far we have seen a combinatorial description of a Whitney dual for Π_n and for Π_n^2 where for the former we considered the poset of increasing spanning forests $\mathcal{ISF}_n$ and for the latter the poset of spanning forests $\mathcal{SF}_n$. The authors in [7] computed the poset $Q_{\lambda_E}(\Pi_n^2)$ for $n = 3$. This poset is shown in Figure **3-3**. We will give a combinatorial characterization

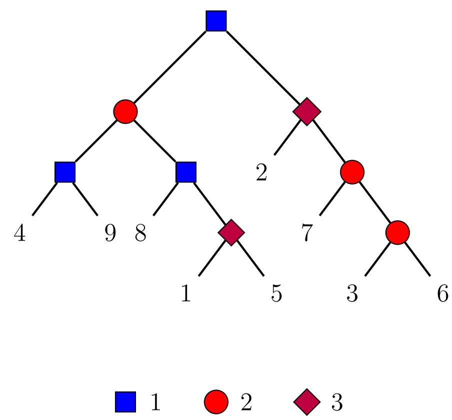


Figure **3-6**: Colored Binary Tree $T \in \mathcal{BT}_{(3,3,2)}$.

of $Q_{\lambda_E}(\Pi_n^2)$ for any n. In fact we will provide a general combinatorial description of $Q_{\lambda_E}(\Pi_n^k)$ for any pair of positive integers n and k.

A *tree* is an undirected graph in which any two vertices are connected by exactly one path. We say a tree is rooted if there is a distinguished node (the *root*). If in order to travel through the unique path from a node b to the root we need to pass through node a, we say that a is an *ancestor* of b. If in particular, $\{a, b\}$ is an edge we say that a is the *parent* of b (or equivalently, b is a *child* of a). Every node in a rooted tree T which has at least one child is considered an *internal node*, if it has no child we say it is a *leaf*. A *planar tree* is a rooted tree in which the set of children of each internal node comes equipped with a total order. A *binary tree* is a rooted planar tree in which every internal node has two children, a *left child* and a *right child*. All the trees we consider from now on are both rooted and planar, so we will be referring to them (informally) as "trees" when it is clear from the context. We say a binary tree is a *colored binary tree* if there is a function **color** that assigns to each internal node x a number $\mathbf{color}(x) \in \mathbb{N}$ (a *color*). We will depict trees graphically from top to bottom where the root sits on top, see Figure **3-6** for an example.

If we take an element $\mu \in wcomp_{n-1,k}$, we denote $\mathcal{BT}_\mu$ the set of colored binary trees with n leaves and $\mu(j)$ internal nodes with color j for each j. We also denote

$$\mathcal{BT}_{n,k} := \bigcup_{\mu \in wcomp_{n-1,k}} \mathcal{BT}_\mu.$$

We are going to be interested in colored binary trees whose set of leaf labels is a subset of $[n]$ for some $n \geq 1$, and if we call A such subset and $\mu \in wcomp_{|A|-1,k}$, then we denote the respective sets of colored binary trees on A as $\mathcal{BT}_{A,\mu}$ and $\mathcal{BT}_{A,k}$.

A *linear extension* of a tree T is a listing $v_1, v_2, \cdots, v_{n-1}$ of the internal nodes of T such that each node precedes its parent.

Let T be a colored binary tree and x a node of T. We define the valency $v(x)$ of x to be the smallest leaf label of the subtree rooted at x. Note that, by this definition, if y is an

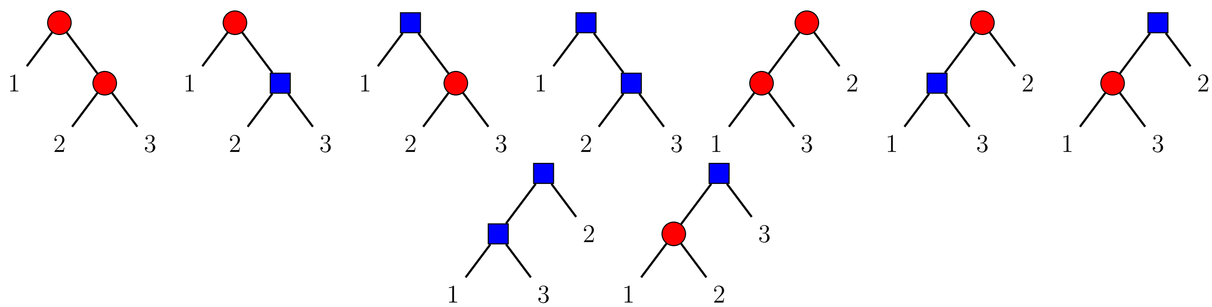


Figure **3-7**: $\mathcal{L}yn_{3,2}$.

ancestor of x we have that $v(x) \geq v(y)$. Hence, since the set $[n]$ is totally ordered, there is a unique linear extension $v_1, v_2, \cdots, v_{n-1}$ of T such that

$$v(v_1) \geq v(v_2) \geq \cdots \geq v(v_{n-1}). \tag{3-1}$$

We will call this linear extension the *reverse-minimal* linear extension of the internal nodes of T. Figure **3-8** depicts the reverse-minimal linear extension of the tree in Figure **3-7**.
Let x be an internal node for a labeled binary tree T. We denote as $L(x)$ the left child of x and as $R(x)$ the right child of x. We say that T is *normalized* if for every internal node x we have

$$v(x) = v(L(x)).$$

Whenever T is normalized we say that an internal node x is a *Lyndon node* if it satisfies

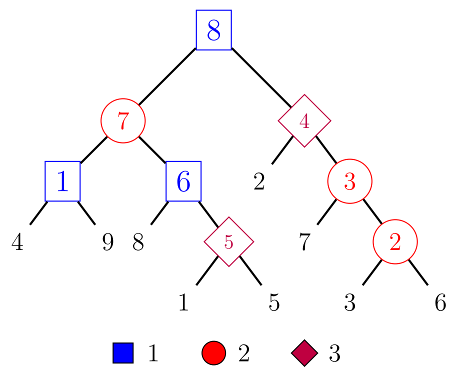
$$v(R(L(x))) > v(R(x)). \tag{3-2}$$

Let T be a normalized colored binary tree, we say that T is a *colored Lyndon tree* if for each internal node x of T, if x is not a Lyndon node then

$$\mathit{color}(L(x)) > \mathit{color}(x). \tag{3-3}$$

Given $\mu \in wcomp_{n-1}$, we denote as $\mathcal{L}yn_\mu$ the set of colored Lyndon trees in $\mathcal{BT}_\mu$, and $\mathcal{L}yn_n = \cup_{\mu \in wcomp_{n-1}} \mathcal{L}yn_\mu$. Moreover, it will be convenient to generalize this concept to proper subsets $A \subsetneq [n]$, denoting by $\mathcal{L}yn_{A,\nu}$ the set of colored Lyndon trees in $\mathcal{BT}_{A,\nu}$ with $\nu \in wcomp_{|A|-1}$.
Given that we will be also concerned with the *length* (or number of entries) $\ell(\mu)$ of the composition μ (which will represent the number of colors), we denote $\mathcal{L}yn_{n,k} = \cup_{\mu \in wcomp_{n-1,k}} \mathcal{L}yn_\mu$.
See the example given in Figure **3-7** for all the elements in $\mathcal{L}yn_{3,2}$.
Given a poset P, we denote as $M(P)$ the set of maximal chains of P.

Definition 3.1.6. For $T \in \mathcal{BT}_\mu$ and $t \in [n-1]$, let $T_t =: L_t \wedge R_t$ be the subtree of T rooted at the t-th node listed in the reverse-minimal linear extension, where L_t and R_t are respectively the left and right subtrees at the t-th node. The chain $c(T) \in M([\hat{0}, [n]^\mu])$ is the



$$123456789^{(3,3,2)}$$
$$1458^{(2,1,1)}/2367^{(0,2,1)}$$
$$158^{(1,0,1)}/2367^{(0,2,1)}/49^{(1,0,0)}$$
$$15^{(0,0,1)}/2367^{(0,2,1)}/49^{(1,0,0)}/8^{(0,0,0)}$$
$$1^{(0,0,0)}/2367^{(0,2,1)}/49^{(1,0,0)}/5^{(0,0,0)}/8^{(0,0,0)}$$
$$1^{(0,0,0)}/2^{(0,0,0)}/367^{(0,2,0)}/49^{(1,0,0)}/5^{(0,0,0)}/8^{(0,0,0)}$$
$$1^{(0,0,0)}/2^{(0,0,0)}/36^{(0,1,0)}/49^{(1,0,0)}/5^{(0,0,0)}/7^{(0,0,0)}/8^{(0,0,0)}$$
$$1^{(0,0,0)}/2^{(0,0,0)}/3^{(0,0,0)}/49^{(1,0,0)}/5^{(0,0,0)}/6^{(0,0,0)}/7^{(0,0,0)}/8^{(0,0,0)}$$
$$1^{(0,0,0)}/2^{(0,0,0)}/3^{(0,0,0)}/4^{(0,0,0)}/5^{(0,0,0)}/6^{(0,0,0)}/7^{(0,0,0)}/8^{(0,0,0)}/9^{(0,0,0)}$$

Figure **3-8**: $c(T) \in M[\hat{0}, [9]^{(3,3,2)}]$ for $T \in \mathcal{BT}_{(3,3,2)}$ (internal nodes ordered with the reverse-minimal linear extension).

one whose rank t weighted partition is obtained from the rank $t-1$ weighted partition by j_t-merging the blocks $A(L_t)^{\mu(L_t)}$ and $A(R_t)^{\mu(R_t)}$, where j_t is the color of the t-th node and $A(T)$ is the set of leaf-labels of the colored tree T and $\mu(T)$ is the weak composition such that $\mu(T)(i)$ is the number of internal nodes of T of color i. See Figure **3-8** for an example.

The following theorem gives a relation between the set of *colored Lyndon trees* $\mathcal{L}yn_\mu$ and the set of ascent-free maximal chains in Π_n^k.

Theorem 3.1.7. *(cf. Theorem 3.11 in [4]) The set* $\{c(T) | T \in \mathcal{L}yn_\mu\}$ *is the set of ascent-free maximal chains of the EW-labeling of* $[\hat{0}, [n]^\mu]$ *given in Theorem 3.1.5.*

In order to model maximal chains in an interval $[\hat{0}, \alpha]$, for any $\alpha \in \Pi_n^k$, we will need to move from trees to the concept of a forest and generalize to forests some of the definitions above. A *colored binary forest* is a set of colored binary trees whose sets of leaf labels form a partition for some $[n]$. We denote $\mathcal{BF}_\mu$ the set of colored binary forests with colors determined by μ. If for $F \in \mathcal{BF}_\mu$ all trees are colored Lyndon trees we say that F is a *colored Lyndon forest* and denote $\mathcal{FL}yn_\mu$ the set of all such forests.
Note that for any $F \in \mathcal{BF}_\mu$ there exists also a unique *reverse-minimal linear extension* of the internal nodes of F with the same condition as for colored trees given by equation (3-1). We denote $\mathcal{BF}_{n,k}$ the set of colored binary forests whose set of leaf labels form a partition

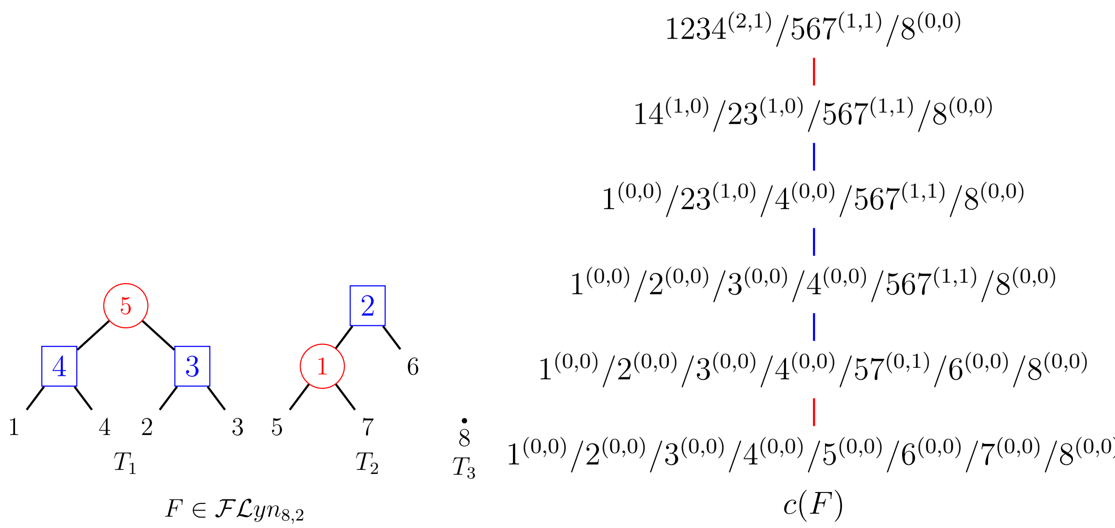


Figure **3-9**: Example of $c(F)$ for $F \in \mathcal{FL}yn_{8,2}$.

for some $[n]$ and whose colors are determined by $[k]$. We also denote

$$\mathcal{FL}yn_{n,k} := \{F \in \mathcal{BF}_{n,k} \mid F \text{ is a colored Lyndon forest}\}.$$

Definition 3.1.8. Given $F \in \mathcal{BF}_{n,k}$ we define $\pi(F)$ to be the weighted partition in Π_n^k given by

$$\pi(F) = \{A(T)^{\mu(T)} \mid T \in F\}.$$

Example 3.1.9. For the colored forest $F \in \mathcal{BF}_{8,2}$ of Figure **3-9** we have that $\pi(F) = 1234^{(2,1)}/567^{(1,1)}/8^{(0,0)}$.

Definition 3.1.10. For $F \in \mathcal{FL}yn_\mu$ and $t \in [n-1]$, let $T_t = L_t \wedge R_t$ be the subtree of F rooted at the t-th node listed in the reverse-minimal linear extension. The maximal chain $c(F) \in M([\hat{0}, \pi(F)])$ is the one whose rank t weighted partition is obtained from the rank $t-1$ weighted partition by j_t-merging the blocks $\pi(L_t)$ and $\pi(R_t)$, where j_t is the color of the t-th node. See Figure **3-9** for an example.

Proposition 3.1.11. *For $F \in \mathcal{FL}yn_\mu$ we have that the chain $c(F)$ is ascent-free according to the EW-labeling of Π_n^k given in Theorem 3.1.5.*

The same proof of Theorem 3.1.7 in [4] proves the following more general theorem.

Theorem 3.1.12. *For a given weighted partition $\alpha \in \Pi_n^k$, the set*

$$\{c(F) | F \in \mathcal{FL}yn_{n,k} \text{ and } \pi(F) = \alpha\}$$

is the set of ascent-free maximal chains in the interval $[\hat{0}, \alpha]$ according to the EW-labeling of Π_n^k given in Theorem 3.1.5.

Definition 3.1.13. Let $F \in \mathcal{FL}yn_{n,k}$ such that $\pi(F)$ has two or more blocks (i.e. $\pi(F)$ is not a maximal element in Π_n^k) and consider two colored Lyndon trees $T_1, T_2 \in F$ and $j \in [k]$, such that $\min(\pi(T_1)) < \min(\pi(T_2))$. We define the *j-merge* $T = T_1 \wedge_j T_2$ of the trees T_1 and T_2 using the following recursive process:

(1) If when we merge the roots of T_1 and T_2 respectively using a new root x of color $\mathbf{color}(x) = j$ such that $L_x = T_1$ and $R_x = T_2$, the resulting tree T is a colored Lyndon tree then $T_1 \wedge_j T_2 := T$

(2) Otherwise, if the resulting tree T is not a colored Lyndon tree, which can only happen because the root x of T is not a Lyndon node and $\mathbf{color}(L(x)) \leq \mathbf{color}(x)$ (condition (3-3) is not satisfied), we define the j-merge of T_1 and T_2 as $T_1 \wedge_j T_2 := (L_r \wedge_j T_2) \wedge_{\mathbf{color}(r)} R_r$ where r is the root of T_1 and, L_r and R_r are, respectively, the left and right subtrees of r in T_1.

Remark 3.1.14. Note that the tree obtained in step (2) of Definition 3.1.13 is also a colored Lyndon tree since by the recursivity in the definition we have that either

- the right subtree of $(L_r \wedge_j T_2)$ is T_2 and $\min(T_2) > \min(R_r)$, or
- the right subtree of $(L_r \wedge_j T_2)$ is $R_{L(r)}$ and the color of the root of $(L_r \wedge_j T_2)$ is $\mathbf{color}(L(r))$, and in this case the pairs $(\mathbf{color}(r), R_r)$ and $(\mathbf{color}(L(r)), R_{L(r)})$ already satisfied the colored Lyndon condition in T_1.

An example of a j-merge of two colored Lyndon trees is illustrated in Figure **3-10**.

Definition 3.1.15 (Poset of colored Lyndon forests)**.** From now on we will denote by $\mathcal{FL}yn_{n,k}$ the poset of colored Lyndon forests with order relation given by $F \lessdot F'$ whenever F' is obtained from F when exactly two trees of F are j-merged for some $j \in [k]$ to obtain one tree of F' while every other tree in F is also in F'.

Figure **3-11** shows the example of $\mathcal{FL}yn_{3,2}$.
Let Λ be a poset of labels and let $\alpha = \alpha_1\alpha_2\cdots\alpha_{n-1}\alpha_n$ be a word of labels where $\alpha_i \in \Lambda$ for all $i \in [n]$. If we assume that whenever $\alpha_i < \alpha_{i+1}$ we can exchange the order of the two letters in α,

$$\alpha = \alpha_1\cdots\alpha_{i-1}\alpha_i\alpha_{i+1}\alpha_{i+2}\cdots\alpha_n \to \alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\alpha_i\alpha_{i+2}\cdots\alpha_n$$

then it is easy to see that when performing all possible exchanges there will be a unique ascent free word α' associated to α in this demeanor. We define $sort(\alpha) = \alpha'$ this end result. Note that if α is already an ascent free word then $sort(\alpha) = \alpha$.

Definition 3.1.16. (cf. Definition 4.3 in [7]) Let P be a poset with an EW-labeling λ. Let $R_\lambda(P)$ be the poset whose elements are the ascent-free saturated chains from $\hat{0}$ and such that $c \lessdot c'$ whenever $\max(c) \lessdot \max(c')$ and $\lambda(c') = sort(\lambda(c)\lambda(c', \max(c) \lessdot \max(c')))$.

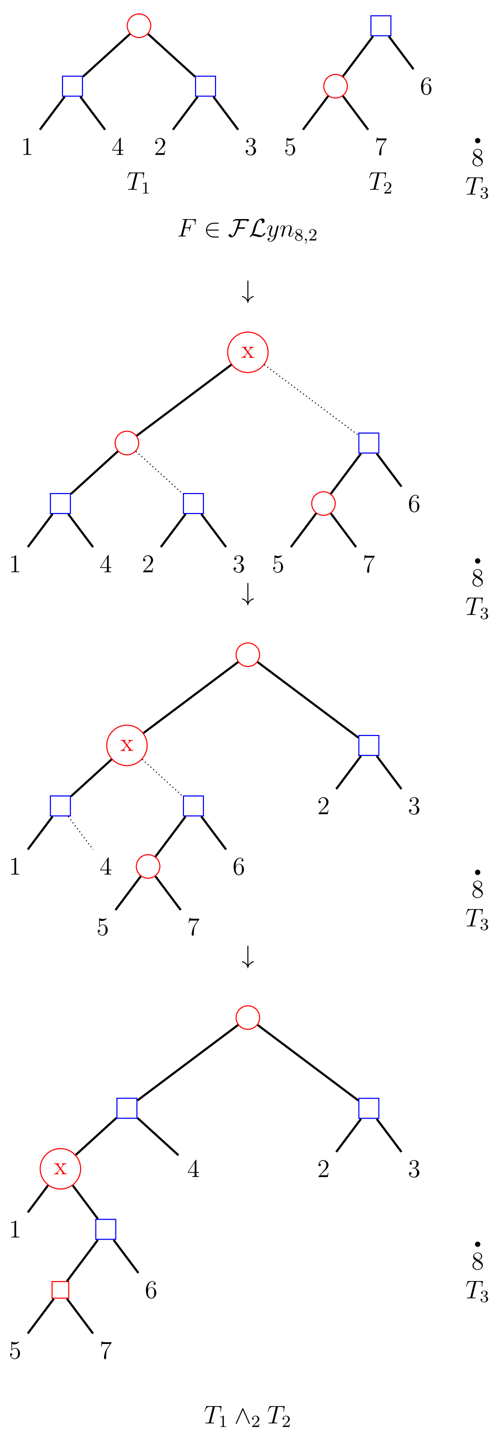


Figure **3-10**: Example of 2-merging two colored Lyndon trees.

Theorem 3.1.17 (cf. Theorem 4.5 in [7]). *If λ is an EW-labeling of P, then $R_\lambda(P) \simeq Q_\lambda(P)$*

Proposition 3.1.18. *Let $F \in \mathcal{FL}yn_{n,k}$ such that $\pi(F)$ is not a maximal element in Π_n^k. Let $T_1, T_2 \in F$ be such that $min(T_1) = t_1 < t_2 = min(T_2)$ and $F' \in \mathcal{FL}yn_{n,k}$ be obtained from F by j-merging T_1 and T_2 for some $j \in [k]$, that is, $F \lessdot F'$ in $\mathcal{FL}yn_{n,k}$. Then we have that*

$$\lambda_E(c(F')) = sort(\lambda_E(c(F))(t_1,t_2)^j). \tag{3-4}$$

Proof. To simplify the proof we will assume that F is a colored Lyndon forest that only consists of the two trees T_1 and T_2. Indeed, recall that in $c(F)$, at step t we j_t-merge the blocks $A(L_t)^{\mu(L_t)}$ and $A(R_t)^{\mu(R_t)}$, and hence we get that the t-th label of $\lambda_E(c(F))$ is $(\min(A(L_t), \min(A(R_t))^{j_t}$. When we j-merge trees T_1 and T_2 in F to obtain F', we include a new label $(t_1,t_2)^j$ in the word of labels $\lambda_E(c(F'))$. Note that if F is composed of s trees $T_1, T_2, \dots T_s$, since the sets $A(T_i)$ are all disjoint, the labels of $\lambda_E(c(T_1))$, $\lambda_E(c(T_2))$ and the label $(t_1,t_2)^j$ are all comparable in Λ_n^k with the labels in the words $\lambda_E(c(T_i))$ for all $i \neq 1,2$. Since $\lambda_E(c(F))$ and $\lambda_E(c(F'))$ are ascent-free, the relative order in them among the labels in $\lambda_E(c(T_1))$, $\lambda_E(c(T_2))$ and $(t_1,t_2)^j$, and the labels in the words $\lambda_E(c(T_i))$ for all $i \neq 1,2$ is uniquely determined.
Now if $F' = T_1 \wedge_j T_2 \in \mathcal{FL}yn_{n,k}$ we have that the set of labels in $\lambda_E(c(F'))$ is equal to the set of labels of $\lambda_E(c(F))$ together with the label $(t_1,t_2)^j$. Denote

$$\lambda_E(c(F)) = (a_1,b_1)^{j_1}(a_2,b_2)^{j_2} \cdots (a_k,b_k)^{j_k}$$

and note that $a_k = t_1$ since the k-th node in the linear extension is the root of T_1. We have one of the following two cases according to Definition 3.1.13:

(1) When joining the roots of T_1 and T_2 through a node x of color j and the resulting tree is already a colored Lyndon tree. In this case, we have that either $t_2 = min(T_2) < min(R_k) = b_k$ or $t_2 = min(T_2) > min(R_k) = b_k$ and $j < j_k$, Both cases imply that $(a_k,b_k)^{j_k} \not\leq (t_1,t_2)^j$ and it follows that

$$\begin{aligned}
\lambda_E(c(T_1 \wedge_j T_2)) &= \lambda_E(c(F))(t_1,t_2)^j \\
&= (a_1,b_1)^{j_1}(a_2,b_2)^{j_2} \cdots (a_{k-1},b_{k-1})^{k-1}(a_k,b_k)^{j_k}(t_1,t_2)^j \\
&= sort((a_1,b_1)^{j_1}(a_2,b_2)^{j_2} \cdots (a_{k-1},b_{k-1})^{k-1}(a_k,b_k)^{j_k}(t_1,t_2)^j) \\
&= sort(\lambda_E(c(F))(t_1,t_2)^j).
\end{aligned}$$

(2) In the case where $T_1 \wedge_j T_2 := (L_k \wedge_j T_2) \wedge_{j_k} R_k$ with $t_2 = min(T_2) > min(R_k) = b_k$ and $j \geq j_k$, we have that $(a_k,b_k)^{j_k} < (t_1,t_2)^j$ and then, using an induction on the size

of the tree T_1, we have then that

$$\begin{aligned}
\lambda_E(c(T_1 \wedge_j T_2)) &= \lambda_E(c(L_k \wedge_j T_2))(min(L_k), min(R_k))^{j_k} \\
&= \lambda_E(c(L_k \wedge_j T_2))(a_k, b_k)^{j_k} \\
&= sort((a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(t_1, t_2)^j)(a_k, b_k)^{j_k} \\
&= sort((a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(a_k, b_k)^{j_k}(t_1, t_2)^j) \\
&= sort(\lambda_E(c(F))(t_1, t_2)^j).
\end{aligned}$$

In both cases we conclude that $\lambda_E(c(T_1 \wedge_j T_2)) = sort(\lambda_E(c(F))(t_1, t_2)^j)$ which completes the proof. □

Proposition 3.1.18 and Theorem 3.1.17 imply then the following theorem.

Theorem 3.1.19. *For $n \geq 1$ and $k \geq 1$ we have that $Q_{\lambda_E}(\Pi_n^k) \cong \mathcal{FL}yn_{n,k}$.*

Remark 3.1.20. When $k = 1$ we have that $\Pi_n^1 \cong \Pi_n$ and the labeling λ_E coincides with the labeling λ defined on Π_n in Example 2.3.8. Hence, Theorem 3.1.19 implies that $ISF_n \cong \mathcal{FL}yn_{n,1}$.

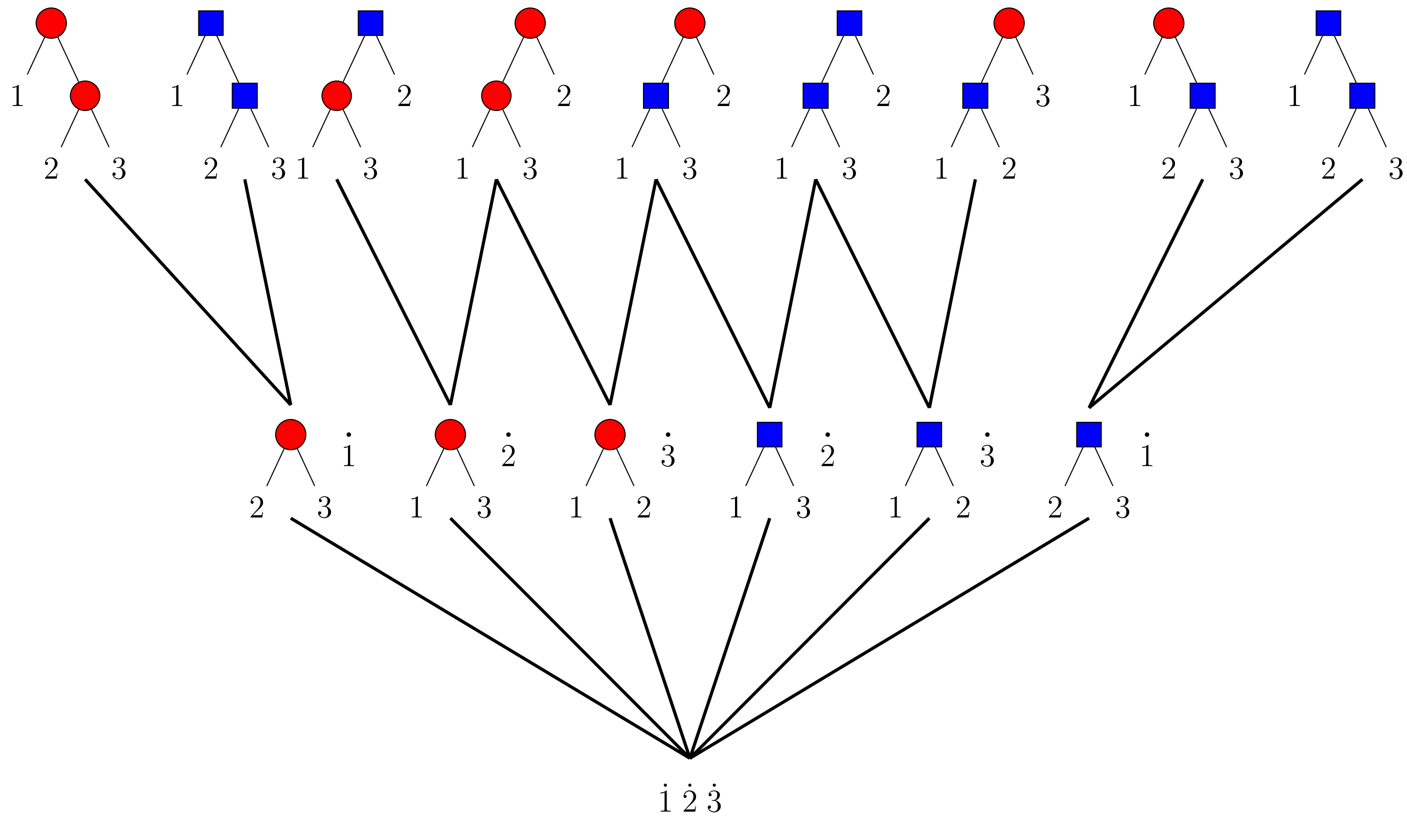


Figure **3-11**: $\mathcal{FL}yn_{3,2}$.

3.2 The poset of pointed partitions

In this section we study the poset of pointed partitions $\Pi_n^\bullet$ previously introduced and studied by Chapoton and Vallette in [2]. In [6] González D'León and Wachs showed that this poset has the same Whitney numbers of the first and second kind than Π_n^2. We define a new EW-labeling for $\Pi_n^\bullet$ and construct the associated Whitney dual applying the quotient poset technique we have already discussed in the previous sections.

3.2.1 The poset of pointed partitions

Definition 3.2.1. A *pointed set* is a pair (A, p) where A is a nonempty set and $p \in A$. We will sometimes use the notation A^p to denote the pointed set (A, p). A *pointed partition* of $[n]$ is a collection $\{B_1^{p_1}, B_2^{p_2}, ..., B_m^{p_m}\}$ where $\{B_1, B_2, ..., B_m\}$ is a partition of $[n]$ and $B_i^{p_i}$ are pointed sets. The poset of pointed partitions $\Pi_n^\bullet$ will be the set of all pointed partitions of $[n]$ and with cover order relation $\{A_1^{q_1}, A_2^{q_2}, ..., A_l^{q_l}\} \lessdot \{B_1^{p_1}, B_2^{p_2}, ..., B_m^{p_m}\}$ whenever

- $\{A_1, A_2, ..., A_l\} \lessdot \{B_1, B_2, ..., B_m\} \in \Pi_n$.
- if $B_h = A_i \cup A_j$ then $p_h \in \{q_i, q_j\}$.
- if $B_h = A_i$ then $p_h = q_i$.

In the definition above, calling $\alpha = \{A_1^{q_1}, A_2^{q_2}, ..., A_l^{q_l}\}$ and $\alpha' = \{B_1^{p_1}, B_2^{p_2}, ..., B_m^{p_m}\}$, we will say that α' was obtained by *j-merging* the blocks $A_i^{q_i}$ and $A_j^{q_j}$ of α, with $\min A_i < \min A_j$ and where $j = 1$ whenever $q_h = p_i$ and $j = -1$ whenever $q_h = p_j$.
In order to facilitate the notation, in the following we also denote $\{e_1, \ldots, e_{i-1}, \tilde{e}_i, e_{i+1}, \ldots, e_l\}$ or simply $e_1 \cdots e_{i-1}\tilde{e}_i e_{i+1} \cdots e_l$, a pointed set of the form $\{e_1, \ldots, e_l\}^{e_i}$, i.e., the pointed element will be denoted using a tilde. We will also be denoting the set $\{1, 2, ...n, \tilde{1}, \tilde{2}, ...\tilde{n}\}$ by $[n, \tilde{n}]$. The example of $\Pi_3^\bullet$ is illustrated in Figure **3-12**.
Let us now define an edge labeling λ_P on $\Pi_n^\bullet$ which we shall prove it is an EW-labeling. Let $\alpha, \alpha' \in \Pi_n^\bullet$ such that $\alpha \lessdot \alpha'$, and assume that the two pointed blocks in α which were j-merged to get α' are A^q and B^p, with $a = min(A) < b = min(B)$. We then define

$$\lambda_P \colon \mathcal{E}(\Pi_n^\bullet) \longrightarrow \Lambda_n^\bullet$$

by

$$\lambda_P(\alpha \lessdot \alpha') = \begin{cases} (\tilde{a}, b) & j = 1, \text{ i.e., if block A preserves the pointed element} \\ (a, \tilde{b}) & j = -1, \text{ i.e., if block B preserves the pointed element.} \end{cases} \tag{3-5}$$

Where $\Lambda_n^\bullet$ is a poset of labels on the set $\{(a, \tilde{b}) \mid 1 \le a < b \le n\} \cup \{(\tilde{a}, b) \mid 1 \le a < b \le n\} \subseteq [n, \tilde{n}] \times [n, \tilde{n}]$. To define the order relation let A_a be the antichain (a poset where every pair of elements are not comparable) $A_a = \{(a, \tilde{b}) \mid a < b \le n\}$ and let C_a be the chain on the set

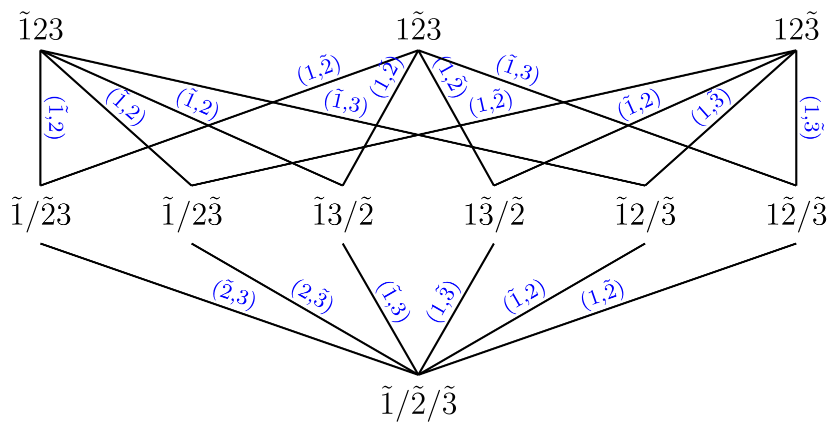


Figure **3-12**: $\Pi_3^\bullet$ with its edge labeling λ_P.

$\{(\tilde{a}, b) \mid a < b \leq n\}$ where $(\tilde{a}, b) < (\tilde{a}, c)$ whenever $b < c$. Then we define $\Lambda_n^\bullet$ as the ordinal sum

$$\Lambda_n^\bullet = A_1 \oplus C_1 \oplus A_2 \oplus C_2 \oplus \cdots \oplus A_{n-1} \oplus C_{n-1}.$$

In Figure **3-12** we illustrate along each edge of the Hasse diagram the labeling λ_P for each cover relation in $\Pi_3^\bullet$.
Note that a label $\lambda_P(\alpha \lessdot \alpha')$ completely determines which two blocks of α merge to form a block of α' and what is the resulting pointed element. This means that starting in any element $\alpha \in \Pi_n^\bullet$ a sequence of valid labels completely determines a chain starting at α.

Proposition 3.2.2. *The labeling λ_P of equation* (3-5) *is injective on saturated chains from $\hat{0}$.*

Lemma 3.2.3. *Let $\alpha = \{B_1^{p_1}, \ldots, B_l^{p_l}\} \in \Pi_n^\bullet$ with $\min B_1 < \cdots < \min B_l$ and $U(\alpha) := \{\beta \in \Pi_n^\bullet \mid \beta \geq \alpha\}$ the* (principal) upper filter *generated by α. There is an isomorphism*

$$\Phi : U(\alpha) \to \Pi^\bullet_{\{\min B_1, \cdots, \min B_l\}}$$

defined on pointed sets: If $A = B_{j_1} \cup \cdots \cup B_{j_r}$ with $j_1 < \cdots < j_r$ and $q = p_{j_s}$ for some $s \in [r]$ we have that $\Phi(A^q) := \{\min B_{j_1} \cup \cdots \cup \min B_{j_r}\}^{\min B_{j_s}}$, and then for any $\beta \in U(\alpha)$ defined as $\Phi(\beta) := \{\Phi(A^q) | A^q \in \beta\}$. Furthermore, the isomorphism Φ preserves the labeling λ_P defined in equation (3-5)*, i.e., for any $\beta \lessdot \beta'$ in $U(\alpha)$ we have that*

$$\lambda_P(\Phi(\beta) \lessdot \Phi(\beta')) = \lambda_P(\beta \lessdot \beta').$$

Proof. We will show first that the function Φ preserves the j-merging of two blocks $j \in \{-1, 1\}$. Let $A_1 = B_{j_1} \cup \cdots \cup B_{j_r}$ with $j_1 < j_2 < \cdots < j_r$ and $q_1 = p_{j_s}$ for some $s \in [r]$ and let $A_2 = B_{k_1} \cup \cdots \cup B_{k_t}$ with $k_1 < k_2 < \cdots < k_t$ and $q_2 = p_{k_u}$ for some $u \in [t]$. Without loss

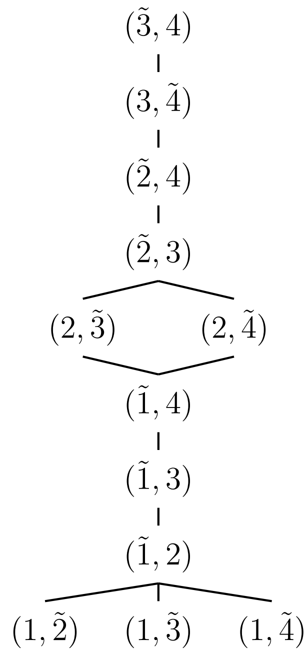


Figure **3-13**: Poset of labels $\Lambda_4^\bullet$.

of generality we assume $j_1 < k_1$ so $\min A_1 < \min A_2$. We denote $A_1^{q_1} \cup_j A_2^{q_2} = (A_1 \cup A_2)^q$ the j-merging of the pointed blocks $A_1^{q_1}$ and $A_2^{q_2}$ where $q = q_1$ if $j = 1$ and $q = q_2$ if $j = -1$.

$$\begin{aligned}\Phi(A_1^{q_1} \cup_j A_2^{q_2}) &= \Phi(\{B_{j_1} \cup \cdots \cup B_{j_r} \cup B_{k_1} \cup \cdots \cup B_{k_t}\}^q) \\ &= \{\min B_{j_1} \cup \cdots \cup \min B_{j_r} \cup \min B_{k_1} \cup \cdots \cup \min B_{k_t}\}^{\tilde{q}} \\ &= \{\min B_{j_1} \cup \cdots \cup \min B_{j_r}\}^{\min B_{j_s}} \cup_j \{\min B_{k_1} \cup \cdots \cup \min B_{k_t}\}^{\min B_{k_u}} \\ &= \Phi(A_1^{q_1}) \cup_j \Phi(A_2^{q_2}),\end{aligned}$$

where $\tilde{q} = \min B_{j_s}$ if $j = 1$ and $\tilde{q} = \min B_{k_u}$ if $j = -1$. Since the blocks of α are in bijection with the blocks of $\min B_1 | \cdots | \min B_l$ and all elements of $U(\alpha)$ are obtained uniquely by a sequence of j-merges of blocks of α and the elements of $\Pi^\bullet_{\{\min B_1, \ldots, \min B_l\}}$ are obtained uniquely by a sequence of j-merges of the blocks of $\min B_1 | \cdots | \min B_l$, we conclude that Φ is a bijection in which Φ and Φ^{-1} preserve cover relations and hence, a poset isomorphism. Now, to see that the labeling according to λ_P of equation (3-5)is preserverd, note that in a cover relation where we j-merge the blocks $A_1^{q_1}$ and $A_2^{q_2}$ the label is

$$(\min A_1, \min A_2, j) = (\min B_{j_1}, \min B_{k_1}, j),$$

that is the same obtained by j-merging the blocks $\Phi(A_1^{q_1})$ and $\Phi(A_2^{q_2})$. □

Proposition 3.2.4. *The labeling λ_P of equation* (3-5) *is an ER-labeling of* $\Pi_n^\bullet$.

Proof. Let $\alpha, \alpha' \in \Pi_n^\bullet$ such that $\alpha \leq \alpha'$. We want to show that there is a unique increasing saturated chain on $[\alpha, \alpha']$.
Assume first that $\alpha = \hat{0}$ and $\alpha' = [n]^p$, so $[\alpha, \alpha'] = [\hat{0}, [n]^p]$ is a maximal interval. We will construct an increasing saturated chain in $[\hat{0}, [n]^p]$ and show that such chain is the only increasing saturated chain in $[\hat{0}, [n]^p]$. The named chain $c_{[n]^p}$, is the one determined (see Proposition 3.2.2)by the sequence of labels

$$\lambda_P(c_{[n]^p}) = (1, \tilde{p})(\tilde{1}, 2)...(\tilde{1}, p-1)(\tilde{1}, p+1)...(\tilde{1}, n), \tag{3-6}$$

where in the case $p = 1$ we just consider $(1, \tilde{1})$ and $(\tilde{1}, 1)$ as empty labels, and our sequence of labels would then be

$$\lambda_P(c_{[n]^1}) = (\tilde{1}, 2)(\tilde{1}, 3)...(\tilde{1}, n-1)(\tilde{1}, n). \tag{3-7}$$

Such chain $c_{[n]^p}$ will merge the blocks $\{\tilde{1}\}$ and $\{\tilde{p}\}$ choosing p as pointed element in the first step and then keep merging the other pointed blocks to this resulting block one by one keeping p pointed, hence is a valid saturated chain in the interval $[\hat{0}, [n]^p]$. The chain is also increasing since $(1, \tilde{p})$ is smaller than any element in $C_1 \subseteq \Lambda_n^\bullet$ and the remaining values are increasing in C_1.
We now show that chain $c_{[n]^p}$ is indeed the only increasing chain in $[\hat{0}, [n]^p]$. Note that if c' is any other chain in $[\hat{0}, [n]^p]$ it must have as final label either $(1, \tilde{a})$ or $(\tilde{1}, a)$ for some other value a since in the last step the block with minimal label 1 always get involved. In order for $\lambda_P(c')$ to be increasing then all labels have to be of this form, otherwise because of the structure of $\Lambda_n^\bullet$ there would be a descent at some point. Hence c' has to be constructed by step by step merging blocks to the block that contains the element 1. Hence, the labels in the second component will be some permutation of the elements $2, 3, \dots, n$. Since p has to be pointed we will have at some point the label $(1, \tilde{p})$ appearing. Note that if there is another label of the form $(1, \tilde{a})$ then these two labels are not comparable and c' could not be increasing. Hence, all other labels are of the form $(\tilde{1}, a)$ and the only way to order them increasingly is as in equation (3-6). Since by Proposition 3.2.2 λ_P is injective, we have that $c' = c_{[n]^p}$, as we wanted to show.
Now, we consider an interval of the form $[\hat{0}, \alpha]$ where $\alpha \in \Pi_n^\bullet$ and $|\alpha| \geq 2$. Let $\alpha = \{B_1^{p_1}, \dots, B_l^{p_l}\}$ where $\min B_1 < \cdots < \min B_l$. We will consider now for each $i = 1, \dots, l$ the word of labels for the uniquely increasing chain $c_{B_i^{p_i}}$ in $[\hat{0}, B_i^{p_i}]$, that is $\lambda_P(c_{B_i^{p_i}})$ (possibly empty if $|B_i| = 1$), and let c_α be the chain in $[\hat{0}, \alpha]$ with word of labels

$$\lambda_P(c_\alpha) = \lambda_P(c_{B_1^{p_1}})\lambda_P(c_{B_2^{p_2}}) \cdots \lambda_P(c_{B_l^{p_l}}).$$

Note that this chain is increasing because of the increasing order on $\min B_i$ and is unique because of Proposition 3.2.2. In fact, c_α is the chain in $[\hat{0}, \alpha]$ that first merges the elements with labels in B_1 as in $c_{B_1^{p_1}}$, then merges the elements with labels in B_2 as in $c_{B_2^{p_2}}$, and so on. In order in to see that $\lambda_P(c_\alpha)$ is the unique increasing chain in $[\hat{0}, \alpha]$, let for any other increasing chain c' in this interval and for every $i = 1, \ldots, l$, be

$$w_i = \lambda_P(c')_{j_1} \lambda_P(c')_{j_2} \cdots \lambda_P(c')_{j_{|B_i|}}$$

the subword of $\lambda_P(c')$ whose labels belong to the steps in c' where blocks with elements in B_i were merged. Since w_i is a subword of an increasing word it must also be increasing. But by the discussion in the paragraph above we concluded that there is a unique form to do the merges to get an increasing word and such word is $\lambda_P(c_{B_i^{p_i}})$. There is also a unique way to have an increasing word $\lambda_P(c')$ with subwords $\lambda_P(c_{B_1^{p_1}}), \lambda_P(c_{B_2^{p_2}}), \ldots, \lambda_P(c_{B_l^{p_l}})$, so we have that $c' = c_\alpha$.
Finally for an interval of the form $[\alpha, \alpha']$ in $\Pi_n^\bullet$ we have, by Lemma 3.2.3 that $[\alpha, \alpha']$ is isomorphic to an interval $[\hat{0}, \beta']$ in some poset $\Pi^\bullet_{\{\min B_1, \ldots, \min B_l\}}$ where this isomorphism preserves the labels of the maximal chains. Hence by the discussion in the paragraph before we have that there is a unique increasing chain in the interval $[\hat{0}, \beta']$ and hence in $[\alpha, \alpha']$, completing the proof. □

Example 3.2.5. To see an example of the unique increasing maximal chain in the proof of Proposition 3.2.4, let $n = 5$ and consider the interval $[\hat{0}, [5]^4] \in \Pi_5^\bullet$. Since 4 is the pointed element on the top element of the increasing maximal chain the word of labels associated to $c_{[5]^4}$ is

$$\lambda_P(c_{[5]^4}) = (1, \tilde{4})(\tilde{1}, 2)(\tilde{1}, 3)(\tilde{1}, 5),$$

which corresponds to the chain

$$c_{[5]^4} = (\hat{0} \lessdot 1\tilde{4}/2/3/5 \lessdot 12\tilde{4}/3/5 \lessdot 123\tilde{4}/5 \lessdot 123\tilde{4}5).$$

Proposition 3.2.6. *The labeling λ_P of equation* (3-5) *satisfies the rank two switching property.*

Proof. Let $[\alpha, \alpha']$ be an interval of rank 2 on $\Pi_n^\bullet$. In order to simplify the proof we will conveniently refer to $(\tilde{a}, b)$ as $(a, b)^1$ and $(a, \tilde{b})$ as $(a, b)^{-1}$ and whenever we use the blocks A, B, C, D we assume $a = min(A) < b = min(B) < c = min(C) < d = min(D)$. We have the following possible two types of intervals of rank two.

Type I: We merge two pairs of distinct blocks $\{A^{p_A}, B^{p_B}\}$ and $\{C^{p_C}, D^{p_D}\}$ in α in order to obtain α'. The open interval (α, α') is of the form $\{K_1, K_2\}$, where $K_1 = AB^{p_1}/C^{p_C}/D^{p_D}$ for $p_1 \in \{p_A, p_B\}$ and $K_2 = A^{p_A}/B^{p_B}/CD^{p_2}$ for $p_2 \in \{p_C, p_D\}$. The two chains that appear in our interval $[\alpha, \alpha']$ will have as their respective increasing and decreasing word of labels $\lambda_P(\alpha \lessdot K_1 \lessdot \alpha') = (a, b)^{j_1}(a, c)^{j_2}$ and $\lambda_P(\alpha \lessdot K_2 \lessdot \alpha') = (a, c)^{j_2}(a, b)^{j_1}$ for $j_1, j_2 \in \{-1, 1\}$.

Type II: We merge three distinct blocks $\{A^{p_A}, B^{p_B}, C^{p_C}\}$ of α to obtain α'. The open interval (α, α') is of the form $\{K_1, K_2, K_3, K_4\}$ where $K_2 = AB^{p_1}/C^{p_C}$ for $p_1 \in \{p_A, p_B\}$ and

$K_3 = AC^{p_2}/B^{p_B}$ for $p_2 \in \{p_A, p_C\}$. K_1 and K_4 will vary upon the selection of the pointed element as can be appreciated in Figure **3-14**. The reader can easily verify that the increasing and decreasing words of labels are $\lambda_P(\alpha \lessdot K_1 \lessdot \alpha') = (a,b)^{j_1}(a,c)^{j_2}$ and $\lambda_P(\alpha \lessdot K_2 \lessdot \alpha') = (a,c)^{j_2}(a,b)^{j_1}$ where we note that if block A preserves the pointed element, then $j_1 = 1$ and $j_2 = 1$, if block B preserves the pointed element, then $j_1 = -1$ and $j_2 = 1$ and lastly if block C preserves the pointed element then $j_1 = 1$ and $j_2 = -1$.

Hence we have that λ_P has the rank two switching property. □

Because of Propositions 3.2.2, 3.2.4 and 3.2.6 we have that the labeling λ_P on $\Pi_n^\bullet$ satisfies the requirement of an EW-labeling which proves our main theorem of this section.

Theorem 3.2.7. *The labeling λ_P is an EW-labeling of $\Pi_n^\bullet$.*

3.2.2 Poset of pointed Lyndon trees

We are interested in giving a combinatorial description of the Whitney dual of $\Pi_n^\bullet$ associated to the Whitney labeling of Theorem 3.2.7. In this case, as it was true in the section on the poset of weighted partitions, the indexing set will be a special subset of $\mathcal{BT}_{n,2}$. We will abuse notation in this section, for convenience on the naming conventions, and from now on we will consider $\mathcal{BT}_{n,2}$ and $\mathcal{BF}_{n,2}$ as the sets defined in the previous section with the exception that instead of referring as *colored* nodes with color 1 and 2 we will say that these nodes are *pointed* or *not pointed* and a new function called **pointed** will take the place of the function **color** of the previous section.

Let T be a normalized binary tree and let **pointed** be a function from the set of internal nodes of T to the set $\{-1, 1\}$, where if for an internal vertex $v \in T$ if $\textbf{pointed}(v) = -1$ we will say that v is *pointed* and if $\textbf{pointed}(v) = 1$ we will say that v is *not pointed*. We say that T is a *pointed Lyndon tree* if for each internal node x of T, $\textbf{pointed}(L(x)) \geq \textbf{pointed}(x)$, and $\textbf{pointed}(L(x)) = \textbf{pointed}(x) = 1$ can only occur when

$$v(R(L(x))) > v(R(x)), \tag{3-8}$$

that is, if both x and $L(x)$ are not pointed then x has to be a Lyndon node.

Remark 3.2.8. Note that the set of pointed Lyndon trees where all internal nodes are not pointed is precisely the set of classical Lyndon trees.

We denote as $\mathcal{L}yn_{n,\bullet}$ the set of pointed Lyndon trees. In Figure **3-16** we illustrate the pointed Lyndon trees in $\mathcal{L}yn_{3,\bullet}$.

Let T be a pointed Lyndon tree. For a node x of T we define its *pointed valency* $\tilde{v}(x)$ as its label if x is a leaf or if x is internal

$$\tilde{v}(x) = \begin{cases} \tilde{v}(L(x)) & \text{if } \textbf{pointed}(x) = 1 \\ \tilde{v}(R(x)) & \text{if } \textbf{pointed}(x) = -1. \end{cases}$$

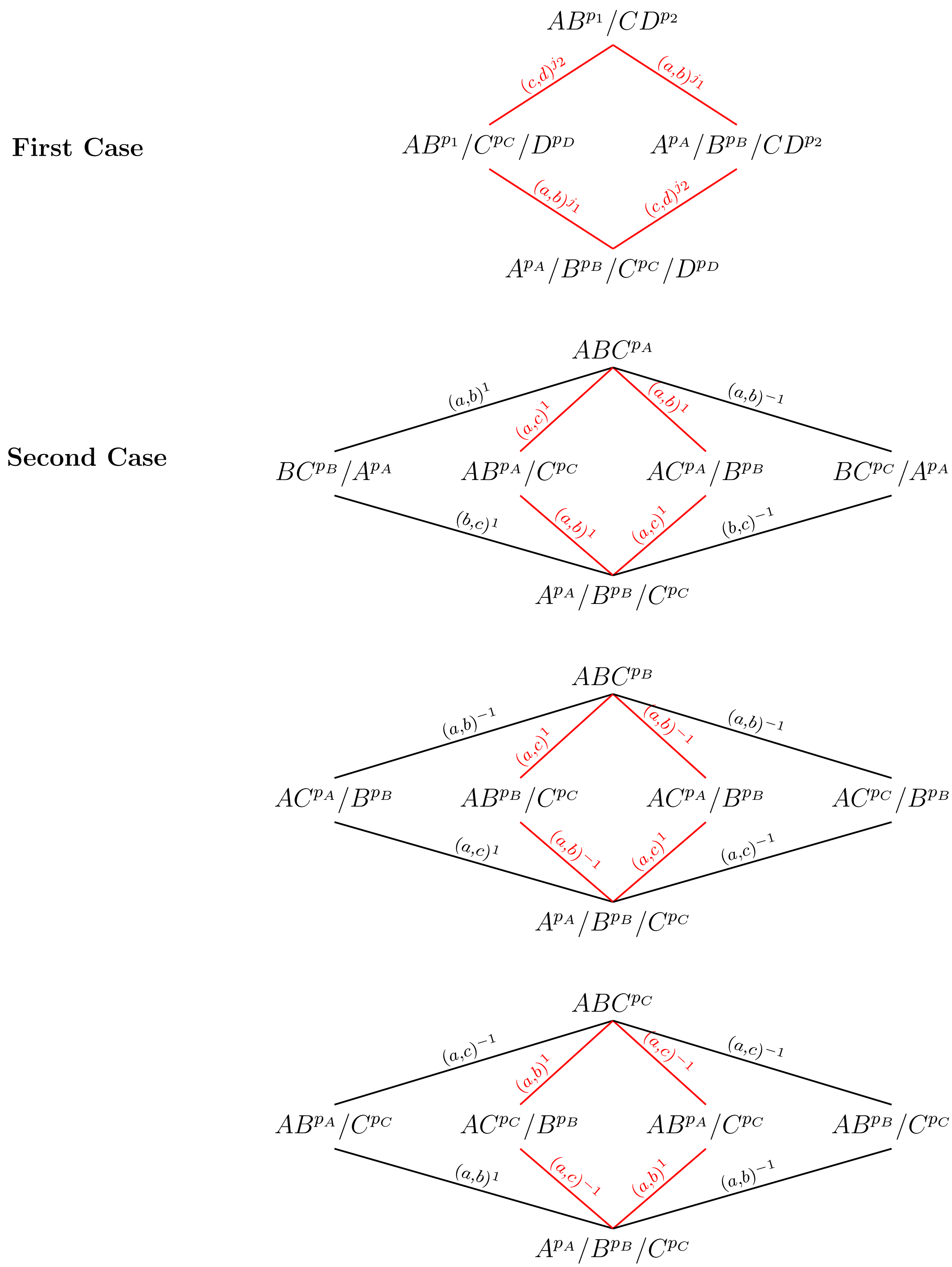


Figure **3-14**: Rank two intervals in $\Pi_3^\bullet$.

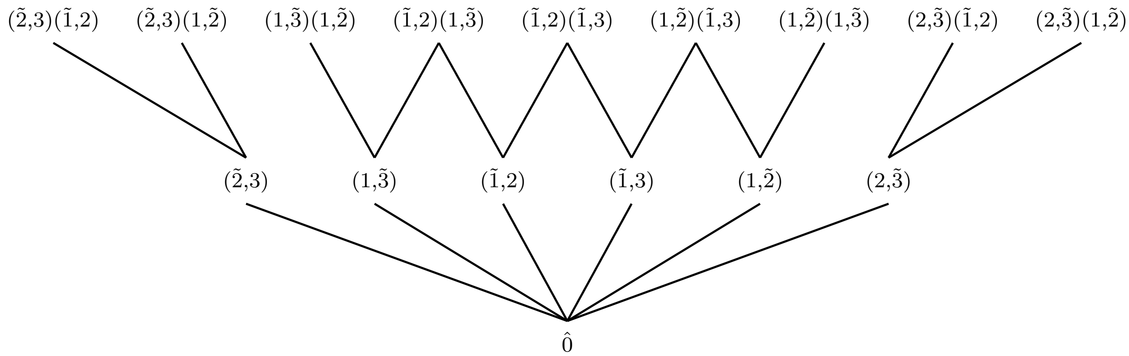


Figure **3-15**: $Q_{\lambda_P}(\Pi_3^\bullet)$.

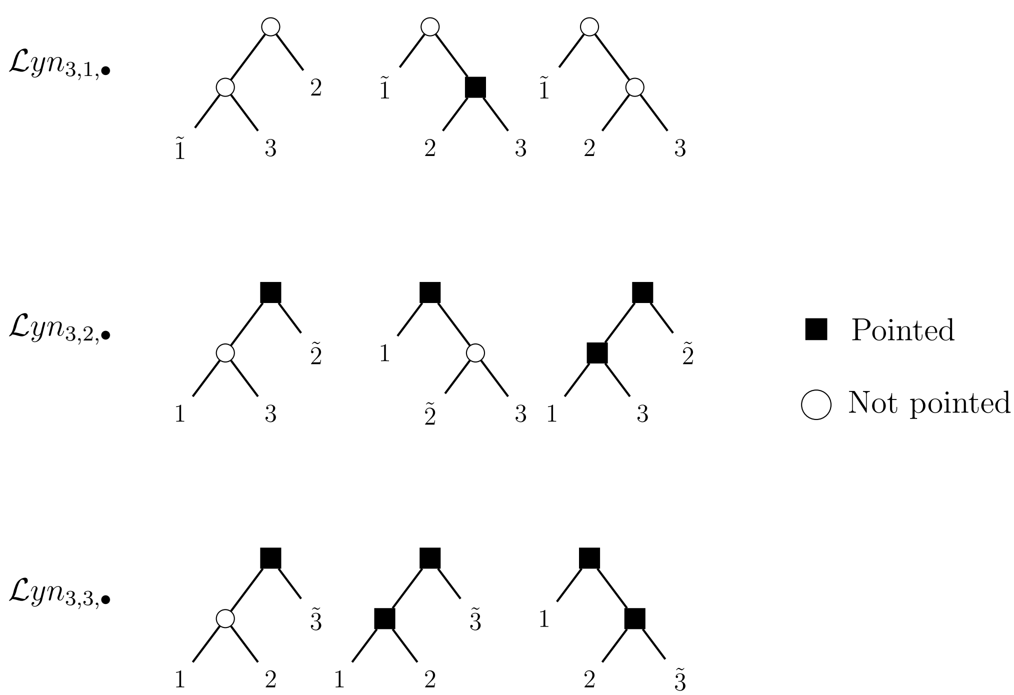


Figure **3-16**: $\mathcal{L}yn_{3,\bullet}$.

Remark 3.2.9. Note that for trees without pointed internal vertices $\tilde{v} = v$.

The *pointed valency* $\tilde{v}(T)$ of T is defined as the pointed valency of the root of T. We denote

$$\mathcal{L}yn_{n,p,\bullet} := \{T \in \mathcal{L}yn_{n,\bullet} \mid \tilde{v}(T) = p\}.$$

In Figure **3-16** the pointed valency of each tree has been indicated by a "tilde" over the leaf with the same label.
If for $F \in \mathcal{BF}_{n,2}$ all trees are pointed Lyndon trees we say that F is a *pointed Lyndon forest* and denote $\mathcal{FL}yn_{n,\bullet}$ the set of all such forests.

Definition 3.2.10. Given $F \in \mathcal{BF}_{n,2}$ we define $\pi(F)$ to be the pointed partition in $\Pi_n^\bullet$ given by

$$\pi(F) = \{A(T)^{\tilde{v}(T)} \mid T \in F\}.$$

Example 3.2.11. For the pointed Lyndon forest $F \in \mathcal{BF}_{8,2}$ of Figure **3-17** we have that $\pi(F) = 1\tilde{2}34/5\tilde{6}7/\tilde{8}$.

Definition 3.2.12. For $F \in \mathcal{BF}_{n,2}$ and $t \in [n-1]$ and a linear extension τ of the internal nodes, let $T_t = L_t \wedge R_t$ be the subtree of F rooted at the t-th node listed according to τ. The chain $c(F,\tau) \in M([\hat{0}, \pi(F)])$, is the one whose rank t pointed partition is obtained from the rank $t-1$ pointed partition by j_t-merging the blocks $A(L_t)^{\tilde{v}(L_t)}$ and $A(R_t)^{\tilde{v}(R_t)}$, where $j_t = \mathbf{pointed}(t)$ and $A(T)$ is the set of leaf-labels of the pointed tree T and $\tilde{v}(T)$ is the pointed valency of T. In particular, if τ_{RM} is the reverse-minimal linear extension of F we denote $c(F) := c(F, \tau_{RM})$. See Figure **3-17** for an example.

The following theorem gives a relation between the set of *pointed Lyndon forests* $\mathcal{FL}yn_{n,\bullet}$ and the set of ascent-free maximal chains in the EW-labeling of of $\Pi_n^\bullet$ given in Theorem 3.2.7. In order to simplify the writing of the labels, we will use the notation $(a, \tilde{b}) = (a, b)^{-1}$ and $(\tilde{a}, b) = (a, b)^1$.

Theorem 3.2.13. *For a given pointed partition $\alpha \in \Pi_n^\bullet$, the set*

$$\{c(F) | F \in \mathcal{FL}yn_{n,\bullet} \text{ and } \pi(F) = \alpha\}$$

is the set of ascent-free maximal chains in the interval $[\hat{0}, \alpha]$ according to the EW-labeling of $\Pi_n^\bullet$ given in Theorem 3.2.7.

Proof. We will first prove that if $F \in \mathcal{FL}yn_{n,\bullet}$ then $c = c(F)$ is ascent-free. We will denote by x_k the $k-$th internal node listed in the reverse-minimal linear extension, then by definition

$$v(x_1) \geq v(x_2) \geq \cdots \geq v(x_{n-r}), \tag{3-9}$$

where r is the number of trees in F. Since each tree of F is normalized, $v(L(x_k)) = v(x_k)$.

Let $\lambda_P(c)_k$ denote the $k-$th letter in the word of labels of $\lambda_P(c)$, i.e.,

$$\lambda_P(c)_k = (v(L(x_k)), v(R(x_k)))^{j_k} = (v(x_k), v(R(x_k)))^{j_k}$$

If we assume that there exists an ascent at $\lambda(c)_k\lambda(c)_{k+1}$, then

$$(v(x_k), v(R(x_k)))^{j_k} < (v(x_{k+1}), v(R(x_{k+1})))^{j_{k+1}} \tag{3-10}$$

We know that $v(x_k) = v(x_{k+1})$ because of (3-9). This equality of valencies imply that $x_k = L(x_{k+1})$ and therefore, expression (3-10) will occur if $j_k = -1 < 1 = j_{k+1}$ or if $j_k = j_{k+1} = 1$ and $v(R(x_k)) < v(R(x_{k+1}))$. If the former case occurs, then

$$-1 = \mathbf{pointed}(x_k) = \mathbf{pointed}(L(x_{k+1})) < \mathbf{pointed}(x_{k+1}) = 1$$

which contradicts that F is a pointed Lyndon forest. If the latter case occurs, then

$$\mathbf{pointed}(L(x_{k+1})) = \mathbf{pointed}(x_{k+1}) = 1 \text{ and } v(R(L(x_{k+1}))) < v(R(x_{k+1})),$$

which violates condition (3-8), therefore F cannot be a pointed Lyndon forests which is also a contradiction.
Now lets assume that c is an ascent-free maximal chain in $[\hat{0}, \alpha]$ for some $\alpha \in \Pi_n^\bullet$. It is not difficult to see that any such chain is of the form $c = c(F, \tau)$ for some $F \in \mathcal{BF}_{n,2}$ and some linear extension τ of the internal nodes of F. We can also assume that F is normalized (since the normalization condition is just a way to consider any leaf-labeled binary tree as a planar tree) and given that c is ascent-free, condition (3-9) must also hold. Suppose that F is not a pointed Lyndon forest. So there must be a tree T of F that is not a pointed Lyndon tree. The following cases can happen then:

Case I: There is an internal node x_k such that $-1 = \mathbf{pointed}(L(x_k)) < \mathbf{pointed}(x_k) = 1$. Given that T is normalized and τ satisfies (3-9) we get that $x_{k-1} = L(x_k)$. We then have that $v(x_{k-1}) = v(L(x_{k-1})) = v(L(x_k))$ and therefore

$$(v(L(x_{k-1})), v(R(x_{k-1})))^{-1} < (v(L(x_k)), v(R(x_k)))^1,$$

i.e. there would be an ascent in $\lambda_P(c)$ at k.

Case II: There is an internal node x_k such that $\mathbf{pointed}(L(x_k)) = \mathbf{pointed}(x_k) = 1$ and $v(R(L(x_k))) < v(R(x_k))$. As in Case I we have that $x_{k-1} = L(x_k)$. Then $v(R(x_{k-1})) < v(R(x_k))$ so we see that at k there is an ascent in $\lambda_P(c)$ given by

$$(v(L(x_{k-1})), v(R(x_{k-1})))^1 (v(L(x_k)), v(R(x_k)))^1.$$

In any of the cases above we have that $\lambda_P(c)$ would not be an ascent-free chain, that is a contradiction. Hence, F has to be a pointed Lyndon forest. □

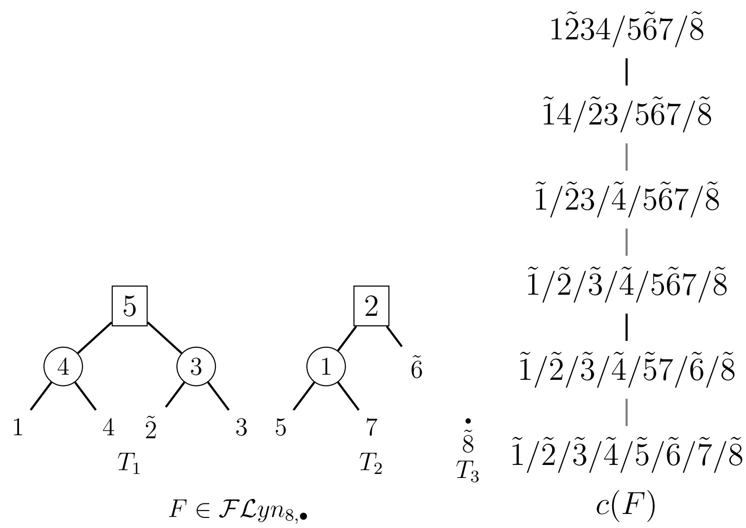


Figure **3-17**: $c(F)$ for $F \in \mathcal{FL}yn_{8,\bullet}$.

Remark 3.2.14. If in Theorem 3.2.13 we consider intervals of the form $[\hat{0}, [n]^p]$ in $\Pi_n^\bullet$, that is $\alpha = [n]^p$, the set of forests that index the ascent-free maximal chains are actually pointed Lyndon trees in $\mathcal{L}yn_{n,p}^\bullet$.

Definition 3.2.15. Let $F \in \mathcal{FL}yn_{n,\bullet}$ such that $\pi(F)$ has two or more blocks (i.e. $\pi(F)$ is not a maximal element in $\Pi_n^\bullet$) and consider two pointed Lyndon trees $T_1, T_2 \in F$, such that $\min(\pi(T_1)) < \min(\pi(T_2))$, and $j \in \{-1, 1\}$. We define the *j-merge* $T = T_1 \wedge_j T_2$ of the trees T_1 and T_2 using the following recursive process:

(1) If when we merge the roots of T_1 and T_2 respectively using a new root x with $\textbf{pointed}(x) = j$ such that $L_x = T_1$ and $R_x = T_2$, the resulting tree T is a pointed Lyndon tree then $T_1 \wedge_j T_2 := T$.

(2) Otherwise, if the resulting tree T is not a pointed Lyndon tree, which can only happen because $-1 = \textbf{pointed}(L(x)) < \textbf{pointed}(x) = 1$ or $1 = \textbf{pointed}(L(x)) = \textbf{pointed}(x)$ and $v(R(L(x)) < v(R(x))$, we define the j-merge of T_1 and T_2 as $T_1 \wedge_j T_2 := (L_r \wedge_j T_2) \wedge_{\textbf{pointed}(r)} R_r$ where r is the root of T_1 and, L_r and R_r are, respectively, the left and right subtrees of r in T_1.

An example of a $j-$merge of two pointed Lyndon trees is illustrated in Figure **3-18**.

Definition 3.2.16 (Poset of pointed Lyndon forests)**.** From now on we will denote $\mathcal{FL}yn_{n,\bullet}$ the poset of pointed Lyndon forests with order relation given for $F, F' \in \mathcal{FL}yn_{n,\bullet}$ we say that $F \lessdot F'$ whenever F' is obtained from F when exactly two trees of F are j-merged for some $j \in \{-1, 1\}$ to obtain one tree of F' while every other tree in F is also in F'.

We illustrate the example of $\mathcal{FL}yn_{3,\bullet}$ in Figure **3-19** .

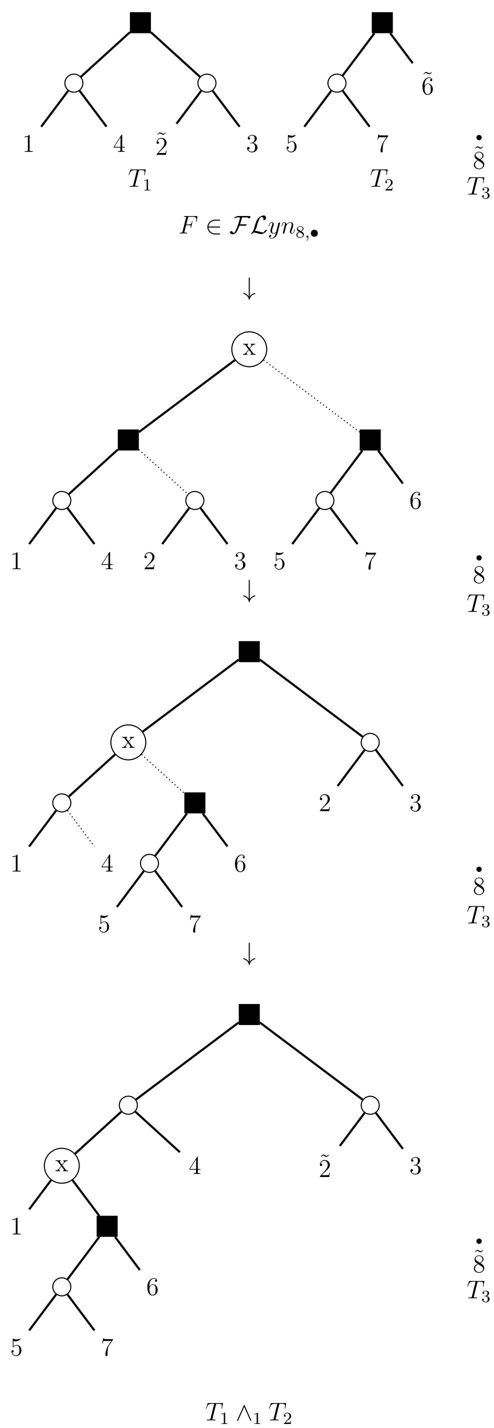


Figure **3-18**: $F' \in \mathcal{FL}yn_{8,\bullet}$ obtained by a $1-$merge of T_1 and T_2, both pointed Lyndon trees in F.

Proposition 3.2.17. *Let $F \in \mathcal{FL}yn_{n,\bullet}$ such that $\pi(F)$ is not a maximal element in $\Pi_n^\bullet$. Let $T_1, T_2 \in F$ be such that $min(T_1) = t_1 < t_2 = min(T_2)$ and $F' \in \mathcal{FL}yn_{n,\bullet}$ be obtained from F by j-merging T_1 and T_2 for some $j \in \{-1, 1\}$, that is, $F \lessdot F'$ in $\mathcal{FL}yn_{n,\bullet}$. Then we have that*

$$\lambda_P(c(F')) = sort(\lambda_P(c(F))(t_1, t_2)^j). \tag{3-11}$$

Proof. To simplify the proof we will assume that F is a pointed Lyndon forest that only consists of the two trees T_1 and T_2. Indeed, a similar reasoning as the one used in Proposition 3.1.18 can easily be given to justify this fact.
Now if $F' = T_1 \wedge_j T_2 \in \mathcal{FL}yn_{n,\bullet}$ we have that the set of labels in $\lambda_P(c(F'))$ is equal to the set of labels of $\lambda_P(c(F))$ together with the label $(t_1, t_2)^j$. Denote

$$\lambda_P(c(F)) = (a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_k, b_k)^{j_k}$$

and note that $a_k = t_1$ since the k-th node in the linear extension is the root of T_1. We have one of the following two cases according to Definition 3.2.15:

(1) When joining the roots of T_1 and T_2 through a node x such that **pointed**$(x) = j$ the resulting tree is already a pointed Lyndon tree. This happens if either $j = -1$, or $j_k = j = 1$ and $b_k = min(R_k) > min(T_2) = t_2$. Both cases imply that $(a_k, b_k)^{j_k} \not\leq (t_1, t_2)^j$ and it follows that

$$\begin{aligned}\lambda_P(c(T_1 \wedge_j T_2)) &= \lambda_P(c(F))(t_1, t_2)^j \\ &= (a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(a_k, b_k)^{j_k}(t_1, t_2)^j \\ &= sort((a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(a_k, b_k)^{j_k}(t_1, t_2)^j) \\ &= sort(\lambda_P(c(F))(t_1, t_2)^j).\end{aligned}$$

(2) In the case where $T_1 \wedge_j T_2 := (L_k \wedge_j T_2) \wedge_{j_k} R_k$ we have that either $j_k = -1 < 1 = j$ or $j_k = j = 1$ and $t_2 = min(T_2) > min(R_k) = b_k$, thus $(a_k, b_k)^{j_k} < (t_1, t_2)^j$ and then, using an induction on the size of the tree T_1, we have then that

$$\begin{aligned}\lambda_P(c(T_1 \wedge_j T_2)) &= \lambda_P(c(L_k \wedge_j T_2))(min(L_k), min(R_k))^{j_k} \\ &= \lambda_P(c(L_k \wedge_j T_2))(a_k, b_k)^{j_k} \\ &= sort((a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(t_1, t_2)^j)(a_k, b_k)^{j_k} \\ &= sort((a_1, b_1)^{j_1}(a_2, b_2)^{j_2} \cdots (a_{k-1}, b_{k-1})^{k-1}(a_k, b_k)^{j_k}(t_1, t_2)^j) \\ &= sort(\lambda_P(c(F))(t_1, t_2)^j).\end{aligned}$$

In both cases we conclude that $\lambda_P(c(T_1 \wedge_j T_2)) = sort(\lambda_P(c(F))(t_1, t_2)^j)$ which proves the theorem. □

Proposition 3.2.17 and Theorem 3.1.17 imply then the following theorem.

Theorem 3.2.18. *For $n \geq 1$ we have that $Q_{\lambda_P}(\Pi_n^\bullet) \cong \mathcal{FL}yn_{n,\bullet}$.*

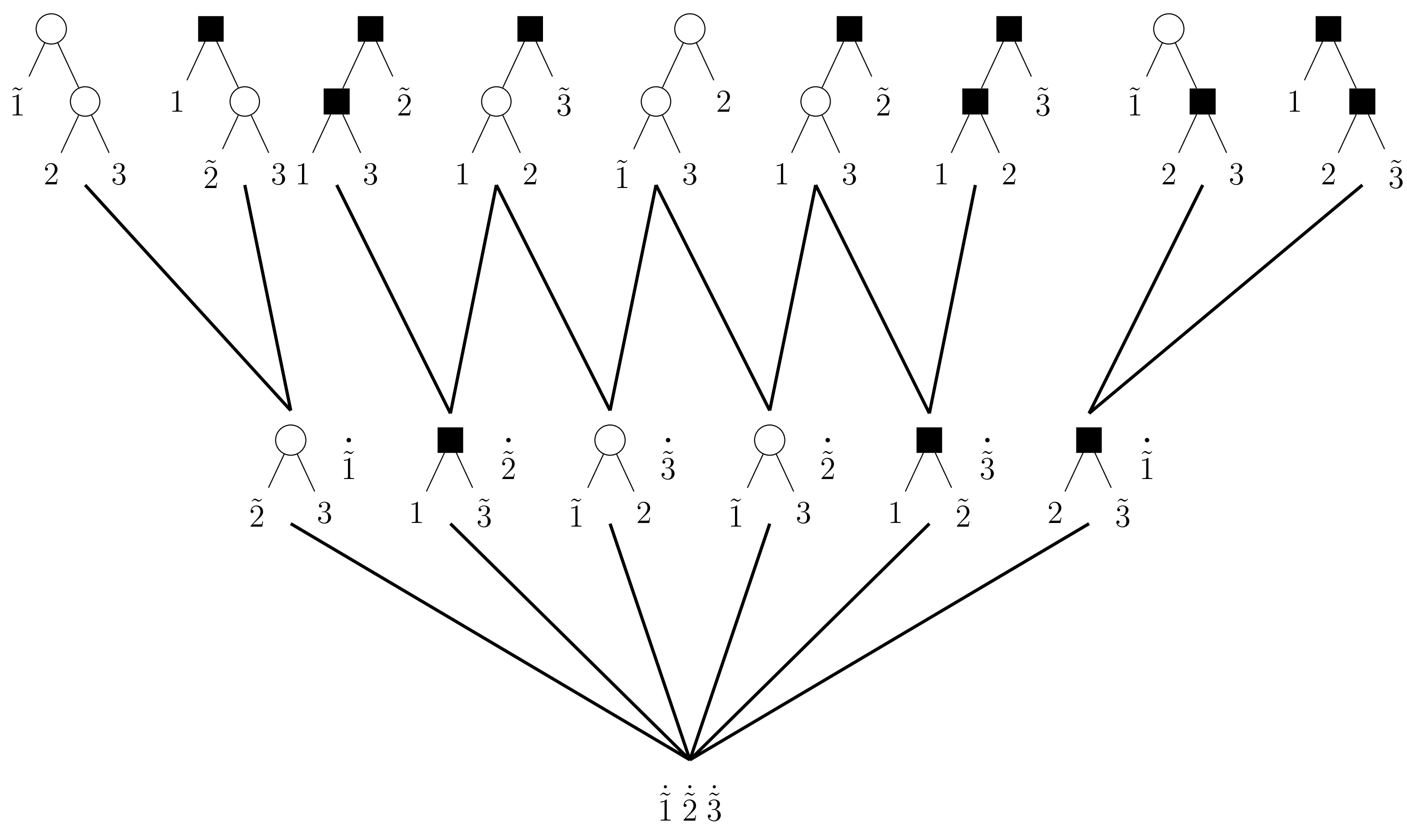


Figure **3-19**: $\mathcal{F}\mathcal{L}yn_{3,\bullet}$.

3.3 Whitney twins: a discussion

The authors in [6] proved that the Whitney numbers of the first and second kind for $\Pi_n^\bullet$ are identical to the ones for Π_n^2. This motivates the following definition.

Definition 3.3.1. Two graded posets P and Q with a $\hat{0}$ are said to be *Whitney twins* if their Whitney numbers of the first and second kind are the same, i.e., they satisfy

$$w_k(P) = w_k(Q) \text{ and } W_k(P) = W_k(Q).$$

Theorem 3.3.2 ([6])**.** *For all $n \geq 1$, the posets $\Pi_n^\bullet$ and Π_n^2 are Whitney twins.*

In [8] González D'León and Hallam show that $\mathcal{SF}_n$ and $\mathcal{L}yn_{n,2}$ are two different (non-isomorphic for $n \geq 3$) Whitney duals for Π_n^2. Hence, because of Definitions 2.3.1 and 3.3.1 we conclude that $\mathcal{SF}_n$ and $\mathcal{FL}yn_{n,2}$ are also Whitney duals for $\Pi_n^\bullet$. With the new EW-labeling of $\Pi_n^\bullet$ of Theorem 3.2.7 found in this section we face the relevant question on how this new Whitney dual $\mathcal{FL}yn_{n,\bullet}$ compares to $\mathcal{SF}_n$ and $\mathcal{FL}yn_{n,2}$.
Note that the Whitney dual $Q_{\lambda_P}(\Pi_3^\bullet)$ in Figure **3-15** is isomorphic to the Whitney dual $Q_{\lambda_E}(\Pi_3^2)$ in Figure **3-3**. However the corresponding posets $\Pi_3^\bullet$ and Π_3^2 are not isomorphic, and their corresponding EW-labelings λ_P and λ_E even have two non-isomorphic posets of labels. We summarize this conclusion as a theorem about Whitney labelings.

Theorem 3.3.3. *There exists two non-isomorphic posets $\Pi_3^\bullet$ and Π_3^2 each with a Whitney labeling λ_P and λ_E respectively, with non-isomorphic posets of labels and such that their Whitney duals $Q_{\lambda_P}(\Pi_3^\bullet)$ and $Q_{\lambda_E}(\Pi_3^2)$ are isomorphic.*

Theorem 3.3.4. *For $n \geq 4$, $\mathcal{FL}yn_{n,2}$ and $\mathcal{FL}yn_{n,\bullet}$ are not isomoprhic.*

Proof. Note first that there are natural copies of $\mathcal{FL}yn_{4,\bullet}$ in $\mathcal{FL}yn_{n,\bullet}$ and of $\mathcal{FL}yn_{4,2}$ in $\mathcal{FL}yn_{n,2}$ for $n \geq 5$. Indeed, we just need to restrict to the subposets that involve merges using only four particular elements in $[n]$.
We will prove that no rank three interval from the bottom $\hat{0}$ in $\mathcal{FL}yn_{n,\bullet}$ for $n \geq 4$ is isomorphic to the rank three maximal interval $I = [\hat{0}, T]$ in $\mathcal{FL}yn_{4,2}$ shown in Figure **3-20**. It is not difficult to show that for $F = \{T_1, T_2, \cdots, T_k\} \in \mathcal{FL}yn_{n,\bullet}$ we have the isomorphism $[\hat{0}, F] \cong [\hat{0}, T_1] \times \cdots \times [\hat{0}, T_k]$, where $[\hat{0}, T_i]$ is a maximal interval in $\mathcal{FL}yn_{A(T_i),\bullet}$ for all $i \in [k]$. Let I' be a bottom interval in $\mathcal{FL}yn_{n,\bullet}$ isomorphic to I. Note that the interval shown in Figure **3-20** is not a product of two non trivial posets, hence if, for some $n \geq 4$, there is an isomorphism $f : \mathcal{FL}yn_{n,2} \to \mathcal{FL}yn_{n,\bullet}$, the image $f(I')$ cannot be a product and hence must be isomorphic to a maximal interval in $\mathcal{FL}yn_{4,\bullet}$. We will show then that there is no maximal interval in $\mathcal{FL}yn_{4,\bullet}$ isomorphic to I.
Suppose that in $\mathcal{FL}yn_{4,\bullet}$ there is a maximal interval $[\hat{0}, T']$ isomorphic to I. Note that since $\mathcal{FL}yn_{4,\bullet} \cong Q_{\lambda_P}(\Pi_4^\bullet)$, any maximal chain in an interval of the form $[\hat{0}, T']$ has the same set of labels according to the induced labeling λ_P^* in $Q_{\lambda_P}(\Pi_n^\bullet)$ (see the construction of $Q_\lambda(P)$

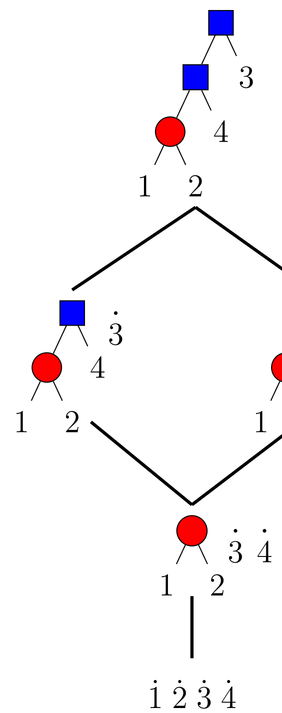


Figure **3-20**: An interval of rank 3 in $\mathcal{FL}yn_{4,2}$.

in [7]). It is not difficult to see that in $Q_{\lambda_P}(\Pi_n^\bullet)$ the labeling λ_P^* is given as follows: if $F \lessdot F'$, that is, F' was obtained by j-merging two pointed Lyndon trees T_1 and T_2 of F with $\min A(T_1) < \min A(T_2)$, then we have that

$$\lambda_P^*(F \lessdot F') = (\min A(T_1), \min A(T_2)))^j. \tag{3-12}$$

Since $[\hat{0}, T']$ has rank three, let us assume that the labels are $a^{j_a}, b^{j_b}, c^{j_c}$ with $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ as is depicted in Figure **3-21**. Because the labels b and c are in the top, in fact we have that $b_1 = c_1 = 1$, and it is not difficult to show that this will also imply that $a_1 = 1$, otherwise, one of the labels b or c would involve elements of $[4]$ that are completely disjoint to the ones in a which will imply that there is another element of rank 1 in the interval different to A. We conclude that the labels are of the form $a = (1, a_2)$, $b = (1, b_2)$ and $c = (1, c_2)$.
The maximal chains $F_A \lessdot F_B \lessdot T'$ and $F_A \lessdot F_C \lessdot T'$ have words of labels $(1, b_2)^{j_b}(1, c_2)^{j_c}$ and $(1, c_2)^{j_c}(1, b_2)^{j_b}$ according to the labeling λ_P^* induced on $Q_{\lambda_P}(\Pi_n^\bullet)$. Since this labeling λ_P^* is an ER*-labeling, we have that in the interval $[F_A, T']$ can only be exactly one ascent-free chain. This implies that the two labels $(1, b_2)^{j_b}$ and $(1, c_2)^{j_c}$ must be comparable in $\Lambda_n^\bullet$, otherwise there would be two ascent-free chains in $[F_A, T']$. This in turn, implies that the two values j_b and j_c cannot be -1 at the same time (for the labels to be comparable in $\Lambda_n^\bullet$). Assume then that $(1, b_2)^{j_b} < (1, c_2)^{j_c}$ and so $j_c = 1$.

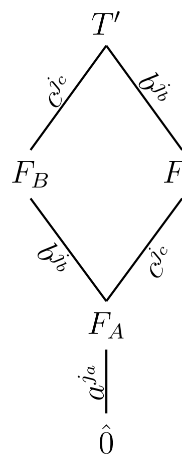


Figure **3-21**: Rank 3 interval used in the proof of Theorem 3.3.4.

Since in each interval from $\hat{0}$ of $Q_{\lambda_P}(\Pi_n^\bullet)$ there must be a unique ascent-free chain according to the labeling λ_P^* (see the construction of $Q_\lambda(P)$ in [7]) then we must have that $(1,a_2)^{j_a} \not\leq (1,c_2)^1$. This implies that $j_a = 1$ and $a_2 > c_2$. Note that chains of labels $(1,c_2)^1(1,a_2)^1$ and $(1,a_2)^1(1,c_2)^1$ both take us to the same element F_C in $\mathcal{FL}yn_{4,\bullet}$ which is a contradiction since our supposition involves only one saturated chain from $\hat{0}$ to F_C. □

Theorem 3.3.5. *For $n \geq 3$, $\mathcal{FL}yn_{n,\bullet}$ and $\mathcal{SF}_n$ are not isomoprhic.*

Proof. Note first that $\mathcal{SF}_n$ is an *uniform* graded poset according to the definition in [4], that is, if $F \in \mathcal{SF}_n$ is an element of rank $\rho(F) = i$ then the filter $U(F)$ in $\mathcal{SF}_n$ is isomorphic to $\mathcal{SF}_{n-i}$. Indeed, the rules of merging in the filter $U(F)$ are only dependent on the roots of F and any $F \in \mathcal{SF}_n$ of rank $\rho(F) = i$ has $n - i$ roots.
When $n = 3$, the posets $\mathcal{SF}_3$ and $\mathcal{FL}yn_{3,\bullet}$ are clearly non-isomorphic as can be appreciated from Figures **3-2** and **3-16**, so let us assume that $n \geq 4$. Consider the pointed Lyndon forest F of Figure **3-22**. It is not hard to see, since the root of the nontrivial tree in F is a Lyndon node with minimal element of the right subtree 4 that is larger than 2 and 3, that the filter $U(F)$ in $\mathcal{FL}yn_{n,\bullet}$ is isomorphic to $\mathcal{FL}yn_{3,\bullet}$.
Now, if there is an isomorphism $f : \mathcal{FL}yn_{n,\bullet} \to \mathcal{SF}_n$, this induces an isomorphism $U(F) \cong U(f(F)) \cong \mathcal{SF}_3$ since the element $f(F)$ has rank $n-3$, but this is a contradiction. □

Remark 3.3.6. Theorems 3.3.4 and 3.3.5, together with the fact that $\mathcal{SF}_n$ and $\mathcal{FL}yn_{n,2}$ are not isomorphic for $n \geq 3$, imply that the Whitney twins Π_n^2 and $\Pi_n^\bullet$ have at least three non-isomorphic Whitney duals for $n \geq 4$, $\mathcal{SF}_n$, $\mathcal{FL}yn_{n,2}$ and $\mathcal{FL}yn_{n,\bullet}$, which are at the same time Whitney twins to each other.

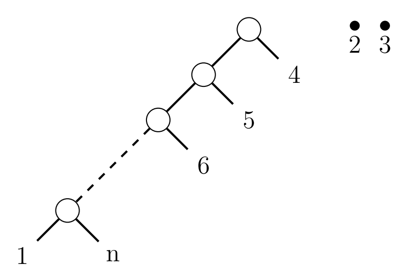


Figure **3-22**: Pointed Lyndon forest in the proof of Theorem 3.3.5.

4 Open questions and future work

In the previous sections some interesting observations have awaken our curiosity and we will leave some open questions that are yet to be answered.

4.1 On the necessity of Whitney labelings

González D'León and Hallam in [7] developed a method to construct Whitney duals of posets that have an associated Whitney labeling. The first question that naturally arises is whether Whitney labelings can be omitted to provide a more general characterization of Whitney duals. In particular, there are examples of Whitney dualizable posets that do not have a Whitney dual. The poset given in Figure **4-1** is one of these examples. This poset is Whitney self-dual, but if it had a Whitney labeling, in particular the interval $[a, f]$ of rank 2 would not satisfy the rank two switching property. There exists however, a poset with the same Whitney numbers as the poset of Figure **4-1** with an EW-labeling. Indeed, the poset P in Figure **4-2** is at the same time a Whitney twin and a Whitney dual of the poset in Figure **4-1** and the labeling depicted in Figure **4-2** is a Whitney labeling. In the figure we have that $\lambda(a,b) = \lambda(a,d) = \lambda(c,e) = \lambda(c,f) = 1$ and $\lambda(a,c) = \lambda(b,e) = \lambda(d,f) = 2$ and our poset of labels is such that $1 < 2$.

Question 4.1.1. *Is there a way to characterize Whitney duals without the need of a Whitney labeling?*

Question 4.1.2. *Is there a Whitney twin replacement that is Whitney labelable for every Whitney dualizable poset?*

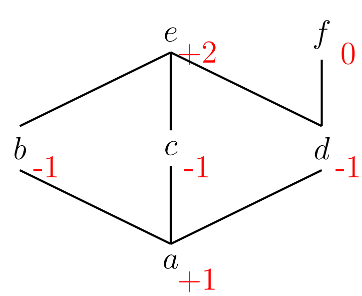


Figure **4-1**: A Whitney dualizable poset without Whitney labeling.

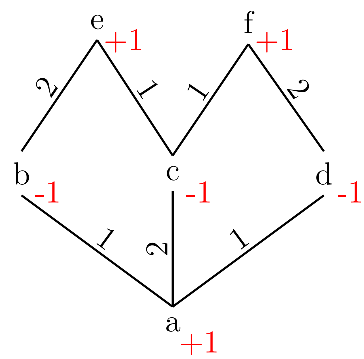


Figure **4-2**: Example of a Whitney labelable poset that is a Whitney twin of the poset of Figure **4-1**.

4.2 On the uniqueness of Whitney duals and other examples

Given that in chapter 3 we noticed that $Q_{\lambda_E}(\Pi_n^2)$ and $Q_{\lambda_P}(\Pi_n^\bullet)$ are isomorphic for $n = 3$ but not for $n \geq 4$. We would like to know if there are labelings for these two families of posets that in fact yield isomorphic Whitney duals.

Question 4.2.1. *Could there be two Whitney labelings λ_1 and λ_2 for Π_n^2 and $\Pi_n^\bullet$ respectively such that $Q_{\lambda_E}(\Pi_n^2) \simeq Q_{\lambda_P}(\Pi_n^\bullet)$ for all n?*

We have observed some other intriguing posets that are a variant of the poset of pointed partitions $\Pi_n^\bullet$ such as the poset of semi-pointed partitions $\widehat{\Pi}_{n,p}$ studied by Bérénice Delcroix-Oger in [3] and the poset of multi-pointed partitions Π_n^B studied by Chapoton and Vallette in [2]. Since both happen to be a small variant of $\Pi_n^\bullet$ the following question arises.

Question 4.2.2. *Are there Whitney labelings defined on the poset of semi-pointed partitions $\widehat{\Pi}_{n,p}$ and on the poset of multi-pointed partitions Π_n^B?*

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