Transitionally Commutative Bundles and Characteristic Classes

Mauricio Cepeda Dávila

Universidad Nacional de Colombia
Faculty of Science
School of Mathematics
Medellín, Colombia
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Mauricio Cepeda Dávila

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Abstract

Let $p : E \to M$ be a principal G-bundle, over a manifold M. Assume that we can find an open cover of M together with trivializations over them in such a way that the corresponding transition functions commute with each other whenever they are simultaneously defined. Such data defines a transitionally commutative (TC) structure on the principal bundle. (We make this precise in Chapter 1.)

In this thesis we developed characteristic classes for TC structures by using an algebraic-geometric method. For this we first obtain generators of the cohomology with real coefficients of the classifying space for TC structures over principal G-bundles, a space known as $B_{\text{com}}G$. We then show that such cohomology is in one to one correspondence with TC characteristic classes. Next, we show how to use Chern-Weil theory to compute the TC characteristic classes for each of these generators. This is done through what we call the k-th associated bundle of the TC structure. After developing this theory, we illustrate in some explicit examples how this can be applied.

Additionally, we show that $B_{\text{com}}GL_n(\mathbb{R})$ can be obtained, up to homotopy equivalence, as a subspace of the Grassmanians.

Keywords: principal bundles, commutative cocycles, characteristic classes, commutative Grassmanians, Chern-Weil theory.

Resumen

Sea $p : E \to M$ un G-fibrado principal sobre una variedad M. Asumamos que se puede encontrar un cubrimiento abierto para M, junto trivializaciones sobre sus abiertos, tal que las funciones de transición correspondientes conmutan entre si en la intersección de sus dominios. Esta información define lo que llamamos una estructura transicionalmente conmutativa (TC) en el fibrado principal. (Detallamos esto en el Capítulo 1.)

En esta tesis desarrollamos las clases características para las estructuras TC al usar un método algebraico-geométrico. Para esto, primero obtenemos un conjunto de generadores de la cohomología con coeficientes reales del espacio clasificante para estructuras TC sobre G-fibrados principales. Este espacio es conocido como $B_{\text{com}}G$. Luego mostramos como existe una correspondencia uno a uno entre dichas clases y las clases características TC. En seguida, mostramos como podemos usar teoría de Chern-Weil para calcular las clases características TC respectivas a cada uno de estos generadores. Para esto definimos los k-ésimos fibrados asociados de la estructura TC. Después de desarrollar esta teoría, mostramos, a través de unos ejemplos explicitos, como puede ser aplicada.

Adicionalmente, mostramos que $B_{\text{com}}GL_n(\mathbb{R})$ puede ser obtenido, salvo equivalencia homotópica, como subespacio de los Grasmannianos.

Titulo en Español: Fibrados transicionalmente conmutativos y clases características.

Palabras Clave: Fibrados principales, cociclos conmutativos, clases características, grasmaniano conmutativo, teoría de Chern Weil.

Contents

Introduction

During the 20th century, algebraic topology saw great advancements in the theory of fiber bundles. These are onto maps $p : E \to M$ such that each fiber is the same up to homeomorphism via local trivializations. Fiber bundles were first introduced by H. Seifert (1907-1966) in his second PhD thesis. In it, he laid the grounds to study Poincare's conjecture as well as fiber bundles, by focusing on the study of the total space E. He worked with 3-dimensional closed manifolds and defined total spaces as sets of curvatures, each having a point over the manifolds. Later on, H. Whitney (1907-1989) switched the focus onto the base space, M, giving rise to invariants for fiber bundles. This started a theory that would lead to important results such as the Riemann-Roch theorem.[1]

A particular kind of fiber bundles are principal G-bundles, where G is a topological group. These are fiber bundles, $p : E \to M$, that can be seen locally as a product of an open set of M and G and have a free G-action on the fibers. For these fiber bundles J.W. Milnor proved that they are classified, up to equivalence, by the homotopy classes of maps from M into a space BG. This space is called the classifying space of G. For simple cases, this allows us to obtain all the possible classes of principal G-bundles. For example, if $G = \mathbb{Z}$, $BG = S^1$ and if $M = S^1$, then all the possible principal bundles are classified by $\left[S^1, S^1\right] \cong \mathbb{Z}$.

A key tool to distinguish between principal G-bundles, up to isomorphism, are the so called characteristic classes. These are natural transformations between principal bundles over a base space M and its cohomology. Therefore, characteristic classes of principal G-bundles are in a one to one correspondence with the cohomology groups of the classifying space BG. This is, for a principal bundle $E \to M$ and a class $c \in H^* (BG, \mathbb{R})$, there is natural transformation given by an assignment $E \mapsto c(E) \in H^* (M, \mathbb{R})$. Even further, when we work on the category of smooth manifolds, one can aim to compute the characteristic classes of principal G-bundles using tools from differential geometry. This is known as Chern-Weil theory (after Shiing-Shen Chern and André Weil). A rather interesting result of this theory is that for compact Lie groups the homotopical and the geometrical methods lead to the same characteristic classes.

Once again, a principal G-bundle over a manifold M can be seen locally as a product of an open set $U \subseteq M$ and G. Such an identification is called a trivialization. The collection of the trivializations of an open cover $\{U_i\}_{i \in I}$ determine the equivalence class of the G-bundle. In an informal way they carry the gluing information of the patches $U \times G$. Formally, this information is carried on the transition maps, or cocycles, obtained from the trivializations. These are functions $\rho : U_i \cap U_j \to G$ satisfying what is known as the cocycle condition. From them one can reconstruct a principal G-bundle up to equivalence. They are also important as they allow us to construct a function $M \to BG$ classifying the bundle up to homotopy. Intrinsically they also carry the information to determine the curvature in the case of smooth bundles.

[1] An excellent recount of the history of fiber bundles can be found in "A History of Manifolds and Fibre Spaces: Tortoises and Hares." by John McCleary.

We are particularly interested in principal G-bundles for which we can find local trivializations for which the corresponding transition functions commute with each other. The motivation to study this came on a seemingly different subject, the sets of commuting n-tuples of a Lie group, $\operatorname{Hom}(\mathbb{Z}^n, G)$. The study of the former, as well as some related concepts, was motivated by their connections with some quantum field theories (see the intro of [**TS**]). Mathematically speaking, one can endow the family $\operatorname{Hom}(\mathbb{Z}^n, G)$, $n \in \mathbb{N}$, with the structure of a simplicial space. By seeing each $\operatorname{Hom}(\mathbb{Z}^n, G)$ as a subspace of G^n, they inherit the simplicial structure of the Bar construction for the classifying space BG. As such, it is natural to study the geometrical realization of the simplicial space $\operatorname{Hom}(\mathbb{Z}^\bullet, G)$, which is called $B_{\text{com}}G$. This space has a natural inclusion into BG.

Recently while studying the cohomology of $B_{\text{com}}G$, Alejandro Adem and Jose Manuel Gómez (see [**AG**]) discovered that the space $B_{\text{com}}G$ serves as a new type of classifying space. Consider a function $f : M \to BG$ classifying a principal G-bundle. They proved that up to homotopy, there is a unique factorization

$$\begin{tikzcd} & B_{\text{com}}G \arrow[d] \\ M \arrow[ur] \arrow[r, "f"] & BG \end{tikzcd}$$

if and only if there is an open cover of M on which the bundle is trivial over each open set and such that on intersections the transition functions commute when they are simultaneously defined (Theorem 2.2 of [**AG**]). The data consisting the bundle, the cover and the trivializations is what we call a TC structure. In [**AG**] Adem and Gómez proved that $B_{\text{com}}G$ serves as a classifying space for TC structures.

In the same work, Adem and Gómez also computed the cohomology of the path connected component of $B_{\text{com}}G$ containing the class of the tuple with only the identity, $(1, 1, \ldots, 1)$. Here, we expand their results by presenting a set of generators when G is either $U(n)$, $SU(n)$ or $\operatorname{Sp}(n)$. For these Lie groups $B_{\text{com}}G$ is path connected. In particular, we consider homomorphisms $\Phi^k : H^*(B_{\text{com}}G, \mathbb{R}) \to H^*(B_{\text{com}}G, \mathbb{R})$ called power maps, and the map $\iota : H^*(BG, \mathbb{R}) \to H^*(B_{\text{com}}G, \mathbb{R})$ induced by the natural inclusion. We show that the subset

$$\left\{\Phi^k \circ \iota(c) \mid c \in H^*(BG, \mathbb{R}), k \in \mathbb{Z} \setminus \{0\}\right\}$$

generates, as an algebra, all of $H^*(B_{\text{com}}G, \mathbb{R})$. We use these results to develop the theory of characteristic classes for TC structures, or TC characteristic classes.

Moreover, the main goal of this thesis is to extend Chern-Weil theory to TC structures when G is either $U(n)$, $SU(n)$ or $\operatorname{Sp}(n)$.To do this we follow the scheme just presented for principal and vector bundles. We first define TC characteristic classes, and then we use $B_{\text{com}}G$ to obtain TC characteristic classes through homotopy theory; we show that they are in a one to one correspondence with $H^*(B_{\text{com}}G, \mathbb{R})$. Then we developed the concepts necessary to use Chern-Weil theory. For a vector bundle $E \to M$ endowed with a TC structure we associate to it a family of bundles $E^k \to M$. These are called k-th associated bundles. Then we consider the TC characteristic class associated to the generator $\Phi^k \circ \iota(c)$. Its class in $H^*(M, \mathbb{R})$ is the same as the class of $c\left(E^k\right)$. This reduces the computation of the TC characteristic class to the computation of $c\left(E^k\right)$, which can be done by using Chern-Weil theory.

Additionally, we found an alternative way to describe the classifying space for TC structures over vector bundles. We called it the commutative Grassmannian. This is achieved by studying the construction of the classifying function of a vector bundle, we were able to obtain the commutative Grassmannian as a subspace of the regular Grassmannian. This approach allowed us to adapt the ideas that make the Grassmannians a classifying space for vector bundles to our new purpose.

This thesis is organized in 5 chapters as follows: In Chapter 1 we lay the ground concepts regarding TC structures. We present a review of the relevant concepts of vector bundles and principal G-bundles that allow us to define TC structures and the equivalence relations between them. In the second chapter we construct the commutative Grassmannian. In Chapter 3 we focus on the cohomology of $B_{\text{com}}G$, for $G = U(n), SU(n)$ and $\text{Sp}(n)$, as previously described. In Chapter 4 we developed the theory of characteristic classes for TC structures over vector bundles. For this we first review characteristic classes for vector bundles as well as the Chern-Weil theory for them. Then we show how to adapt it to TC structures through both the results of Chapter 3 and k-th associated bundles. In the last chapter of this thesis we provide some explicit examples of calculations of TC characteristic classes.

CHAPTER 1

Preliminaries

In this chapter we establish the basic language to talk about what we call Transitionally Commutative (TC) structures. These structures consists of either a vector or principal bundle, with the additional property of having transition maps that commute with each other. Thus, we lay down the definitions and basic results of this theory.

We start this chapter with the basic definition of principal and vector bundles, paying special attention to the transition functions. Then we move towards the definitions of Transitionally Commutative structures and their basic properties. Here we find general results for both of this type of fiber bundles, as well as some particular results for TC structures over vector bundles.

We remark that throughout this chapter M will denote a compact manifold and G denotes a Lie group.

1.1. Basic concepts

In this section we briefly introduce the basic notions and constructions concerning vector and principal bundles. In such spirit we avoid proofs of the results stated here. A good source for vector bundles is the work of Hatcher [**Hatcher II**], while Hussemoller [**Husemoller**] is a good source for the general theory of fiber bundles and principal bundles.

Fiber bundles generalize the idea of a Cartesian product. When F is a topological space, the product $M \times F$ comes naturally endowed with the projection map

$$\begin{aligned} \pi_1 : M \times F &\to M \\ (x, y) &\mapsto x \end{aligned}$$

which has a fixed structure on its fibers $\pi_1^{-1}(x) = \{x\} \times F$. Vector and principal bundles are examples of fiber bundles where the fiber is a vector space and a topological group, respectively. However, they take into consideration the extra structure of these fibers. In a sense we will make precise ahead, vector bundles over M are equivalent to principal $U(n)$-bundles.

Definition 1.1. A **principal G-bundle** is a surjective map $\pi : E \to M$, with a free action of G on E such that there is an open cover $\{U_i\}_{i\in I}$ of M and homeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$. These maps are called **trivializations** and satisfy the following properties:

(1) There is a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[dr, "\pi"'] & U_i \times G \arrow[d, "\pi_1"] \\ & U_i \end{tikzcd}$$

where $\pi_1 : U_i \times \mathbb{C}^n \to U_i$ is the natural projection.

(2) Every map φ_i is a G-equivariant map, where the action on $U_i \times G$ is given by right multiplication on the second component.

The space E is called the **total space** of the bundle, while M is referred to as the **base space**.

Two principal bundles over the same space M, $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$, are isomorphic if there exists a G-equivariant homeomorphism $f : E_1 \to E_2$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr, "p_1"] & E_2 \arrow[d, "p_2"] \\ & M \end{tikzcd}$$

commutes.

Suppose that $\pi : E \to M$ is a principal bundle with a cover $\{U_i\}_{i \in I}$ and trivializations

$$\left\{\varphi_i : \pi^{-1}(U_i) \to U_i \times G\right\}_{i \in I}.$$

By comparing two different trivializations we obtain induced maps $\rho_{ij} : U_i \cap U_j \to G$ called **transition functions** which are characterized by

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times G &\to (U_i \cap U_j) \times G \\ (x, g) &\mapsto (x, \rho_{ij}(x)\, g). \end{aligned}$$

Transition functions could be interpreted as the gluing instructions which determine the bundle completely, as they provide the information to obtain a possibly complex global structure from a locally trivial behavior. Furthermore these functions determine completely a principal bundle as we see next. From the covering $\{U_i\}_{i \in I}$ and functions $\{\rho_{ij} : U_i \cap U_j \to G\}$ we first build the space

$$\bar{E} := \bigsqcup_{i \in I} (U_i \times G) / \sim,$$

where two pairs $(x_i, v) \in U_i \times G$ and $(x_j, w) \in U_j \times G$ are related under $\sim$ if $x_i = x_j \in U_i \cap U_j$, and $\rho_{ij}(x)\, v = w$. This is well defined thanks a property of transition functions called the **cocycle condition**:

$$\rho_{ij} = \rho_{ik}\rho_{kj}.$$

Now we consider the natural map given by the first projection $\bar{\pi} : \bar{E} \to M$. This is a principal bundle where its trivializations are given by the natural maps $U_i \times G \hookrightarrow \bar{\pi}^{-1}(U_i)$ which are naturally homeomorphisms. They induce trivializations having ρ_{ij} as their transition functions. Finally it is not difficult to see that E

and $\bar{E}$ are isomorphic as principal bundles: there is a map $\tilde{f} : \bar{E} \to E$ that makes the following diagram commutes

$$\begin{tikzcd} \bar{E} \arrow[r, "\tilde{f}"] \arrow[d, "\bar{\pi}"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{Id}"] & M \end{tikzcd}$$

and such that $\tilde{f}$ is a G-equivariat homeomorphism. The map $\tilde{f}$ is induced by the natural map

$$\bigsqcup_{i\in I} \varphi_i^{-1} : \bigsqcup_{i\in I} U_i \times G \to E$$

which is constant on the equivalence classes of the relation $\sim$, since the transition functions satisfy

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : U_i \cap U_j \times G &\to U_i \cap U_j \times G \\ (x, g) &\mapsto (x, \rho_{ij}(x)\, g). \end{aligned}$$

The key observation here is that the functions $\{\rho_{ij} : U_i \cap U_j \to G\}$ do not need to come from trivializations to allow us the construction of $\bar{E}$. They only need to satisfy the cocycle condition in order to induce a principal bundle on M. In conclusion a principal bundle can be characterized from either the transition functions or the trivializations of a given cover of the base space.

Strictly speaking vector bundles are different objects than principal bundles, since the fiber is not a group but a vector space. When the fiber is a complex vector bundle of finite dimension, the model of the fiber is $\mathbb{C}^n$.This is, we have an onto map $p : E \to M$, where there is a cover $\{U_i\}$ and trivializations $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$, such that the following diagram is commutative and linear on fibers.

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[dr, "\pi"'] & U_i \times \mathbb{C}^n \arrow[d, "\pi_1"] \\ & U_i, \end{tikzcd}$$

such that they are a linear bijection on each fiber. The linearity implies the compositions $\varphi_j \circ \varphi_i^{-1}$ are of the form

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : U_i \cap U_j \times \mathbb{C}^n &\to U_i \cap U_j \times \mathbb{C}^n \\ (x, g) &\mapsto (x, \rho_{ij}(x)\, g) \end{aligned}$$

where the transition functions take the form $\rho_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$. From this we can construct the frame bundle with structure group $GL_n(\mathbb{C})$: consider the space of n-tuples of linearly independent elements,

$$F_E := \{(e_1, \ldots, e_n) \in E^n \mid p(e_i) = p(e_1) \text{ for all } i \text{ and } \{e_1, \ldots, e_n\} \text{ is L.I.}\} \subseteq E^n.$$

The set F_E is endowed with the subspace topology of E^n. There is a natural surjective mapping $\pi : F_E \to M$ by considering the image under p of the first component. Let's see that this is a principal $GL_n(\mathbb{C})$-bundle. The action of $A \in GL_n(\mathbb{C})$ on $\mathbf{e} = (e_1, \ldots, e_n) \in F_E$ is given by

$$\mathbf{e} \cdot A := \left(\sum_{j=1}^{n} b_{1j} e_j, \ldots, \sum_{j=1}^{n} b_{n,j} e_j \right),$$

where $A^{-1}= [b_{ij}]$. For the trivializations of π, suppose the trivializations of the vector bundle have the form

$$\begin{aligned} \varphi_i : p^{-1}(U_i) &\to U_i \times \mathbb{C}^n \\ e &\mapsto (p(e), h_i(e)), \end{aligned}$$

where the function $h_i : p^{-1}(U_i) \to \mathbb{C}^n$ is a linear bijection over each fiber of p. If $x \in U_i$ and $(e_1, \ldots, e_n) \in \pi^{-1}(x) \subseteq F_E$, then $\{h_i(e_1), \ldots, h_i(e_n)\}$ is a basis of $\mathbb{C}^n$, thus we can consider the change of basis to the standard basis. This allow us to defined a continuous assignment $\pi^{-1}(U_i) \to U_i \times GL_n(\mathbb{C})$, which can be checked to have the properties of a trivialization.

Conversely, if $\pi : E \to M$ is a principal $GL_n(\mathbb{C})$-bundle, one can obtain a vector bundle via the balanced construction. For this consider the right action of $G = GL_n(\mathbb{C})$ on $E \times \mathbb{C}^n$ given for $A \in GL_n(\mathbb{C})$ and $(e, v) \in E \times \mathbb{C}^n$ by

$$(e, v) \cdot A := \left(e \cdot A, A^{-1}v\right).$$

Consider the composition

$$E \times \mathbb{C}^n \xrightarrow{\mathrm{pr}_1} E \xrightarrow{\pi} M,$$

with $\mathrm{pr}_1$the projection on the first component. This composition can be factored through space of orbits of the previous action $E_0 := (E \times \mathbb{C}^n)/G$, giving us a surjective map $p : E_0 \to M$. This map can be proven to be a vector bundle having trivializations over the same cover as the original principal bundle, and so, they have the same transition funcitons. If we now apply the previous construction to this vector bundle we recover the equivalence class of $\pi : E \to M$, and the opposite is also true.

In conclusion having a vector bundle is then equivalent to having a principal $GL_n(\mathbb{C})$-bundle, or just transition funcitons $\rho_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$. Furthermore, when M is compact, one can endow the total space with an euclidean metric in such a way that the trivializations are isometries on the fibers. This in turn implies that the transition functions have values on $U(n)$, that is we can assume $\rho_{ij} : U_i \cap U_j \to U(n)$. One can then repeat the same constructions to see that principal $U(n)$-bundles are equivalent to vector bundles over a compact space or just transition functions on $U(n)$ for a given open cover of M.

1.2. Transitionally commutative vector and principal bundles

Here we develop the basic theory formalizing the study of vector or principal bundles with commutative transition functions.

DEFINITION 1.2. Suppose $\pi : E \to M$ is a vector or principal bundle. We define a **commutative trivializations** on $\pi : E \to M$ as a choice of an open cover $\{U_\alpha\}_{\alpha \in J}$ of M, together with trivializations, $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that the transition functions $\{\rho_{\alpha\beta}\}$ associated to them commute with each other. That is if $x \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\theta$ then

$$\rho_{\alpha\beta}(x)\, \rho_{\gamma\theta}(x) = \rho_{\gamma\theta}(x)\, \rho_{\alpha\beta}(x).$$

REMARK 1.3. By the comments made at the end of the previous section, the previous definition applies as well to vector bundles by taking $U(n)$ or $GL_n(\mathbb{C})$ as the codomain of the transition functions.

DEFINITION 1.4. Suppose $\pi : E \to M$ is a vector or principal bundle. A **Transitionally Commutative (TC)** structure on $\pi : E \to M$ is a choice of a commutative trivialization for it. We denote a TC structure as $\left(\pi : E \to M, \{U_\alpha\}_{\alpha \in I}, \varphi_\alpha\right)$ or more briefly as $\left(E, \{U_\alpha\}_{\alpha \in I}, \varphi_\alpha\right)$ when the map and the base space M are implicit.

REMARK 1.5. Unless stated otherwise, whenever we state a result for TC structures without specifying whether it is on a vector or principal bundle, it means the result applies for both. We do this to avoid unnecessary repetitions of the same argument.

EXAMPLE 1.6. Consider an open cover $\{U_i\}_{i \in I}$ of a space M together with a collection of functions

$$\{\rho_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})\}$$

such that they satisfy the cocycle conditions. If the functions $\{\rho_{ij}\}$ commute with each other then the vector bundle constructed from them has naturally a TC structure.

As we are defining a local condition on bundles, we need to carry this into the notion of equivalent TC structures over the same space M.

DEFINITION 1.7. Let $\left(\pi_1 : E_1 \to M, \{U_\alpha\}_{\alpha \in I}, \varphi_\alpha\right)$ and $\left(\pi_2 : E_2 \to M, \{V_\gamma\}_{\gamma \in J}, \phi_\gamma\right)$ be two TC structures over M. We say that these two structures are equivalent if there is a TC structure on a principal G-bundle or vector bundle $\pi : E \to M \times [0,1]$ accordingly,

$$\left(\pi : E \to M \times [0,1], \{W_i\}_{i \in K}, \zeta_i\right),$$

such that:

- The cover $\{W_i\}_{i \in K}$ intersected to $M \times \{0\}$ is a refinement of $\{U_\alpha\}_{\alpha \in I}$ and its intersecting with $M \times \{1\}$ is a refinement $\{V_\gamma\}_{\gamma \in J}$.
- $\pi^{-1}(M \times \{0\}) = E_1$ and $\pi^{-1}(M \times \{1\}) = E_2$.
- $\pi \mid_{\pi^{-1}(M \times \{0\})} = \pi_1$ and $\pi \mid_{\pi^{-1}(M \times \{1\})} = \pi_2$.
- $\zeta_i \mid_{\pi^{-1}(W_i \cap M \times \{0\})} = \varphi_{\alpha_i} \mid_{\pi_1^{-1}(W_i \cap M \times \{0\})}$ and $\zeta_i \mid_{\pi^{-1}(W_i \cap M \times \{1\})} = \phi_{\gamma_i} \mid_{\pi_2^{-1}(W_i \cap M \times \{1\})}$.

Here we are abusing notation by identifying M with $M \times \{0\}$ and $M \times \{1\}$.

We remark that the above definition defines an equivalence relation. We denote by $\mathrm{Vect}_n^{com}(M)$ to the collection of equivalence classes of TC structures on vector bundles of dimension n over M, while $\mathrm{Bund}_{TC}^{\mathrm{G}}(M)$ refer to those of principal G-bundles.

Notice that this equivalence relationship has stronger requirements than the equivalence relationship for vector and principal bundles. The extra requirements are thought to relate the data of a TC structure with homotopical information of the base space, in a process that will be made clear ahead.

Besides the previous definition, there is an equivalent way to obtain a homotopical notion of TC structure. On the work of Adem and Gómez **[AG]** they show there is a subspace of the classifying space of G, BG, known as the classifying space for commutativity, $B_{\mathrm{com}}G$. Then they showed that for a compact manifold

M, a principal bundle $E \to M$ admits a TC structure if only if its classifying function $f : M \to BG$ admits a lifting, up to homotopy, through $B_{\mathrm{com}}G$,

$$\begin{tikzcd} & B_{\mathrm{com}}G \arrow[d] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BG. \end{tikzcd}$$

It can be shown that a TC structure on a principal G- bundle or vector bundle can be defined in an equivalent way as a lifting up to homotopy for its classifying function.

What follows next is an account of the basic properties of TC structures, some of which are equivalent to those of principal and vector bundles. Let's begin saying that it is immediate from the definition that refinements of covers, with the corresponding restrictions of the trivializations, do not change the equivalence class of the TC structure. Next we will see there are equivalent notions of pullbacks, as well as a criteria to determine when a commutative square

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

gives in fact a pullback structure over N from one on M.

Proposition 1.8. *Let $f : N \to M$ be a continuous map, and $\left(\pi : E \to M, \{U_\alpha\}_{\alpha \in I}, \varphi_\alpha\right)$ a TC structure. Then the pullback $f^*(E) \to N$ has a natural TC structure over the cover $\left\{f^{-1}(U_\alpha)\right\}_{\alpha \in I}$.*

Proof. By definition we have

$$f^*(E) := \{(x, e) \in N \times E \mid f(x) = \pi(e)\}$$

where the principal bundle map is given by the projection $\pi_1(x, y) = x$. Its trivializations come from those of E: if

$$\begin{aligned} \varphi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times G \\ e &\mapsto (\pi(e), \tilde{\varphi}_\alpha(e)) \end{aligned}$$

is a trivialization of E, then by taking

$$\begin{aligned} \tau_\alpha : \pi_1^{-1}\left(f^{-1}(U_\alpha)\right) &\to f^{-1}(U_\alpha) \times G \\ (x, e) &\mapsto (x, \tilde{\varphi}_\alpha(e)) \end{aligned}$$

we obtain a trivialization of $f^*(E)$. By direct computation we see that if $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are the transition functions of φ_α, then the transition functions of τ_α are given by

$$\rho_{\alpha\beta} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to G,$$

which commutes in their common domains. $\square$

Now we continue with a criteria to know when a commutative square gives a pullback of TC structures.

PROPOSITION 1.9. *Consider a map $f : N \to M$ and TC structures*

$$\left(\pi_1 : E_1 \to N, \{V_i\}_{i\in I}, \varphi_i\right)$$

and

$$\left(\pi_2 : E_2 \to M, \{U_\alpha\}_{\alpha\in J}, \phi_\alpha\right),$$

such that

- *There is a commutative square*

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

  where $\tilde{f}$ is G-equivariant.
- *The open cover $\left\{f^{-1}(U_\alpha)\right\}_{\alpha\in J}$ is a refinement of $\{V_i\}_{i\in I}$ satisfying that if $f^{-1}(U_\alpha) \subseteq V_{i_\alpha}$, for every $(x,g) \in f^{-1}(U_\alpha) \times G$ we have*

$$\phi_\alpha\left(\tilde{f}\left(\varphi_{i_\alpha}^{-1}(x,g)\right)\right) = (f(x), g).$$

Then $\left(\pi_1 : E_1 \to N, \{V_i\}_{i\in I}, \varphi_i\right)$ and $\left(f^*(E_2) \to N, \left\{f^{-1}(U_\alpha)\right\}_{\alpha\in J}, \phi_\alpha\right)$ are equivalent TC structures.

On this result we have the extra requirement when compared to that equivalent result for regular principal bundles, where the first condition is enough to obtain the same conclusion. Let us look in detailed what this second condition is telling us. If $f^{-1}(U_\alpha) \subseteq V_{i_\alpha}$ the last equality is equivalent to the following commuting diagram

$$\begin{tikzcd} f^{-1}(U_\alpha) \times G \arrow[r, "\varphi_{i_\alpha}^{-1}"] \arrow[dr] & \pi_1^{-1}\left(f^{-1}(U_\alpha)\right) \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"] & \pi_2^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi_2"] & U_\alpha \times G \arrow[dl] \\ & f^{-1}(U_\alpha) \arrow[r, "f"] & U_\alpha. & \end{tikzcd}$$

This tells us that the restriction of the trivializations on V_{i_α} to $f^{-1}(U_\alpha)$ somehow agrees with the trivializations we would give them via the pullback structure. We make this precise in the proof.

PROOF. Since refinements of the covers preserve the equivalence class of the TC structure, the second condition says that we can refine the original TC structure on $\pi_1 : E_1 \to N$ by taking $V_\alpha = f^{-1}(U_\alpha)$.

Now, the pullback of f is given by

$$f^*(E_2) := \{(x,y) \in N \times E_2 \mid f(x) = \pi_2(y)\}.$$

The projection over the first component $\pi : f^*(E_2) \to N$ is the principal bundle, while the rest of the TC structure is given by the cover $\left\{f^{-1}(U_\alpha)\right\}_{\alpha\in J} = \{V_\alpha\}_{\alpha\in J}$ and trivializations

$$\begin{aligned} \phi_\alpha : \pi^{-1}\left(f^{-1}(U_\alpha)\right) &\to f^{-1}(U_\alpha) \times G \\ (x,e) &\mapsto (x, \phi_{\alpha 2}(e)), \end{aligned}$$

where $\phi_{\alpha 2}(e)$ is such that

$$\begin{aligned}\pi_2^{-1}(U_\alpha) &\to U_\alpha \times G\\ e &\mapsto (\pi_2(e), \phi_{\alpha 2}(e)).\end{aligned}$$

We have to show that this TC structure is equivalent to $\left(\pi_1 : E_1 \to N, \left\{f^{-1}(U_\alpha)\right\}_{\alpha\in J}, \varphi_{i_\alpha}|_{f^{-1}(U_\alpha)}\right)$. For this consider the space

$$E := E_1 \times [0,1/2] \sqcup f^*(E_2) \times [1/2,1] / \sim$$

where the relation $\sim$ for $y \in E_1$ is given by

$$(y,1/2) \sim \left(\left(\pi_1(y), \tilde{f}(y)\right), 1/2\right).$$

Then there is a natural map

$$F : E \to N \times [0,1]$$

having a natural TC structure as follows: take the open cover $\{V_\alpha \times [0,1]\}_{\alpha\in J}$ and trivializations

$$\Phi_\alpha : F^{-1}(V_\alpha \times [0,1]) \to V_\alpha \times [0,1] \times G$$

induced by the maps

$$\begin{aligned}\pi_1^{-1}(V_\alpha) \times [0,1/2] &\to V_\alpha \times [0,1/2] \times G\\ (y,t) &\mapsto (\pi_1(y), t, \varphi_{\alpha 2}(y))\end{aligned}$$

and

$$\begin{aligned}\pi^{-1}(V_\alpha) \times [1/2,1] &\to V_\alpha \times [1/2,1] \times G\\ ((x,y),t) &\mapsto (x,t,\phi_{\alpha 2}(y)).\end{aligned}$$

Here by $\varphi_{\alpha 2}$ and $\phi_{\alpha 2}$ we mean the second components of φ_α and ϕ_α, respectively. The conditions $V_\alpha = f^{-1}(U_\alpha)$ and

$$\phi_\alpha\left(\tilde{f}\left(\varphi_\alpha^{-1}(x,g)\right)\right) = (f(x), g)$$

guarantee that the two previous maps indeed induce a well defined map $F^{-1}(V_\alpha \times [0,1]) \to V_\alpha \times [0,1] \times G$. That last condition also tells us that if $\rho_{\alpha\beta}, \tau_{\alpha\beta} : V_\alpha \cap V_\beta \to G$ and are the transition functions related to the trivializations ϕ_α and φ_α, respectively, then it follows that

$$\tau_{\alpha\beta} = f \circ \rho_{\alpha\beta}$$

so it is clear that the map $F : E \to N \times [0,1]$ has a TC structure, and this defines the desired equivalence between TC structures. ☐

COROLLARY 1.10. *Let $\left(\pi : E \to M, \{U_\alpha\}_{\alpha\in I}, \varphi_\alpha\right)$ be a TC structure, and consider maps $f : N \to M$ and $g : W \to N$, then we have*

- *The pullback TC structure over the identity is equivalent to the original TC structure.*
- *The pullback TC structure over $(g\circ f)^*(E)$ with the cover $(g\circ f)^{-1}(U_\alpha)_{\alpha\in I}$ is equivalent to the (double pullback) TC structure on $g^*(f^*(E))$, over the same cover.*

PROPOSITION 1.11. *Let $\left(\pi : E \to M, \{U_i\}_{i\in I}, \varphi_i\right)$ be a TC structure over either a principal or vector bundle, and $f_1, f_2 : N \to M$ be two homotopic functions. Then the TC structures over N obtained via the pullbacks $p_1 : f_1^*(E) \to N$ and $p_2 : f_2^*(E) \to N$ are equivalent.*

PROOF. If we have a homotopy $T : N \times [0,1] \to M$, then we have the pullback structure given by T,

$$\left\{\pi_1 : T^*(E) \to N \times [0,1], \left\{T^{-1}(U_i)\right\}_{i\in I}, \tau_i\right\}.$$

The map $\pi_1 : T^*(E) \to N \times [0,1]$ is the restriction of the projection on the first coordinate,

$$(N \times [0,1]) \times E \to N \times [0,1].$$

And if we consider $\pi_2 : (N \times [0,1]) \times E \to E$ the projection on E, τ_i is the composition

$$\pi_1^{-1}\left(T^{-1}(U_i)\right) \xrightarrow{\pi_2} \pi^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times K,$$

where K is either G or $\mathbb{C}^n$ depending on whether we are considering principal G-bundle or vector bundles, respectively. The TC structure on $T^*(E) \to N \times [0,1]$ defines an equivalence between the TC structures on $p_1 : f_1^*(E) \to N$ and $p_2 : f_2^*(E) \to N$. $\square$

1.2.1. Contraction of transition functions for TC vector bundles: Next we prove another technical result we will need later on. Consider a vector bundle $\pi : E \to M$ with trivializations $\varphi_i : \pi^{-1}(V_i) \to V_i \times \mathbb{C}^n$ where $\varphi_i(e) = (\pi(e), \tilde{\varphi}_i(e))$. If we have a family of functions $\{\kappa_i : V_i \to (0,1]\}$, we can consider new trivializations

$$\begin{aligned} \varphi_i' : \pi^{-1}(V_i) &\to V_i \times \mathbb{C}^n \\ e &\mapsto (\pi(e), \kappa_i(\pi(e))\tilde{\varphi}_i(e)) \end{aligned}$$

which are still trivializations; the function $e \mapsto \kappa_i(\pi(e))\tilde{\varphi}_i(e)$ is still a linear bijection on every fiber of π. Even more, if $\rho_{ij} : V_i \cap V_j \to GL_n(\mathbb{C})$ are the transition functions associated to $\{\varphi_i\}$, then by linearity it is clear that

$$x \mapsto \frac{\kappa_j(x)}{\kappa_i(x)}\rho_{ij}(x)$$

are the transition functions associated to $\{\varphi_i'\}$. So if the original trivilizations gives a TC structure so does the second one. Let's see that these structures are actually equivalent.

PROPOSITION 1.12. *Let $\pi : E \to M$ be a vector bundle with a TC structure over the cover $\{V_i\}_{i\in I}$ and trivializations*

$$\begin{aligned} \varphi_i : \pi^{-1}(V_i) &\to V_i \times \mathbb{C}^n \\ e &\mapsto (\pi(e), \tilde{\varphi}_i(e)). \end{aligned}$$

If we have functions $\{\kappa_i : V_i \to (0,1]\}$, then the TC structure given by the contractions

$$\begin{aligned} \varphi_i' : \pi^{-1}(V_i) &\to V_i \times \mathbb{C}^n \\ e &\mapsto (\pi(e), \kappa_i(\pi(e))\tilde{\varphi}_i(e)) \end{aligned}$$

is equivalent to the original structure.

PROOF. Consider the map

$$\begin{aligned} p : E \times I &\to M \times I \\ (e,t) &\mapsto (\pi(e), t) \end{aligned}$$

with the natural vector bundle structure inherited from π. Now we consider the cover $\{V_i \times [0,1]\}_{i\in I}$ with trivializations

$$\begin{aligned} \tau_i : \pi^{-1}(V_i) \times [0,1] &\to V_i \times [0,1] \times \mathbb{C}^n \\ (e,t) &\mapsto (\pi(e), t, (t + (1-t)\kappa_i \circ \pi(e))\, \tilde{\varphi}_i(e)). \end{aligned}$$

Since $t + (1-t)\kappa_i \circ \pi(e) > 0$ this is indeed a well defined trivialization. Its inverse is given by

$$\tau_i^{-1}(x,t,v) = \left(\frac{1}{(t + (1-t)\kappa_i(x))}\varphi_i^{-1}(x,v), t\right).$$

So if $\rho_{ij} : V_i \cap V_j \to GL_n(\mathbb{C})$ are the transition functions associated to the trivializations φ_i, the transition functions of the trivializations τ_i are given by

$$(x,t) \mapsto \frac{(t + (1-t)\kappa_j(x))}{(t + (1-t)\kappa_i(x))}\rho_{ij}(x).$$

These last functions are commutative on their common domains.

The TC structure on $p : E \times [0,1] \to M \times [0,1]$ defines an equivalence between the structures defined by $\{\varphi_i\}$ and $\{\varphi_i'\}$. $\square$

CHAPTER 2

Commutative Grassmanian

The main goal of this chapter is the construction of a classifying space for TC vector bundles. We call this space a commutative Grassmanian since it is obtained as a subspace of the classical Grassmanian. Thus the content of this chapter is divided in two parts. The first one is the construction itself of the commutative Grassmanian and a vector bundle over it endowed with a TC structure. The second part consists in showing that they classify, up to homotopy, vector bundles endowed with a TC structure.

Throughout this chapter M denotes a compact manifold.

2.1. Commutative structure on Grassmanians

In this section we give the construction of a transitionally commutative version of the Grassmanian. This construction is natural when you consider the proof of the universality of the Grassmanians as a classifying space for vector bundles over compact Hausdorff spaces.

Let us recall the construction of a classifying function for a vector bundle, which gives us the ideas to obtain the Commutative Grassmannian. Consider a vector bundle $p : E \to M$, for M compact and Hausdorff. Now consider the Grassmannian

$$G_{n,3}(\mathbb{C}) := \left\{ l \subseteq \mathbb{C}^{3n} \mid l \text{ is a linear subspace of dimension } n \right\}$$

and the vector bundle

$$E_{n,3}(\mathbb{C}) := \left\{ (l, v) \in G_{n,3}(\mathbb{C}) \times \mathbb{C}^{3n} \mid v \in l \right\}$$

with the projection over the first component $E_{n,k}(\mathbb{C}) \to G_{n,k}(\mathbb{C})$. Then there is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & E_{n,k}(\mathbb{C}) \arrow[d] \\ M \arrow[r, "f"] & G_{n,k}(\mathbb{C}) \end{tikzcd}$$

where g is linear function on the fibers of the fibers of p into the fibers of $E_{n,k}(\mathbb{C}) \to G_{n,k}(\mathbb{C})$. The function $f : M \to G_{n,k}(\mathbb{C})$ is called the **classifying function** of $p : E \to M$, In fact, in order to obtain f it is enough to construct g (See Theorem 1.16 of [**Hatcher II**] for details.). This construction depends on an open cover of M and trivializations over its sets for the vector bundle $p : E \to M$. Since M is compact, we may assume that the cover is finite, and in fact this number determines k. Let us see this in an example in order to motivate the definitions ahead.

Suppose we have a finite open cover $\{U_j\}_{j=1}^{3}$ of M, with trivializations $\varphi_j : p^{-1}(U_j) \to U_j \times \mathbb{C}^n$ for $j = 1, 2, 3$. If we consider the projection over the second components, we get functions $g_j : p^{-1}(U_j) \to \mathbb{C}^n$. Using a partition of unity subordinated to the cover $\{U_j\}_{j=1}^{3}$, $\sigma_j : M \to [0,1]$, we can construct a function $g : E \to (\mathbb{C}^n)^3$ given by

$$g(e) := (\sigma_1(p(e))\, g_1(e), \sigma_2(p(e))\, g_2(e), \sigma_3(p(e))\, g_3(e)).$$

Notice that if $x \in \sigma_1^{-1}(0,1] \cap \sigma_3^{-1}(0,1]$ and $\sigma_2(x) = 0$ then for $e \in p^{-1}(x) \backslash \{0\}$ there are vectors $v_1(e), v_3(e) \in \mathbb{C}^n \setminus \{0\}$, with

$$g(e) = (v_1(e), 0, v_3(e)).$$

If $\rho_{13} : U_1 \cap U_3 \to GL_n(\mathbb{C})$ is the transition function of this cover, then $\rho_{13}(x)\, v_1(e) = v_3(e)$. So if we call $\rho_{13}(x) = A$, and take into account that g_1 is a linear isomorphism on each fiber, then we get

$$g\left(p^{-1}(x)\right) = \left\{(v, 0, Av) \in (\mathbb{C}^n)^3 \mid v \in \mathbb{C}^n\right\}.$$

Under the same reasoning, if $x \in \sigma_1^{-1}(0,1] \cap \sigma_2^{-1}(0,1] \cap \sigma_3^{-1}(0,1]$, then there would be a non singular matrix B such that

$$g\left(p^{-1}(x)\right) = \left\{(v, Bv, Av) \in (\mathbb{C}^n)^3 \mid v \in \mathbb{C}^n\right\}.$$

And if we suppose that the transition functions commute, we have that $AB = BA$. With this ideas in mind, we proceed to the following definitions.

We start with a definition that comes from the properties of transition functions of a TC bundle, where the first condition is no other than the cocycle condition, and the second one is the commutative condition:

Definition 2.1. We say that a family of non singular matrices $\{A_{ij}\}_{i,j=1}^{m} \subseteq GL_n(\mathbb{C})$ is a **commutative cocycle** if they satisfy

(1) $A_{ij} A_{jk} = A_{ik}$.
(2) $A_{ij} A_{kl} = A_{kl} A_{ij}$.

for all i, j, k and l.

Next we consider elements $\mathbf{v} \in (\mathbb{C}^n)^m$ as vectors of the form

$$\mathbf{v} = (v_1, \ldots, v_m)$$

where $v_k \in \mathbb{C}^n$. (So we are not using the identification of $(\mathbb{C}^n)^m$ with $\mathbb{C}^{nm}$ yet.)

Definition 2.2. Let $J = (\alpha_1, \ldots, \alpha_m) \in \{0,1\}^m$ be a set of indices, and $\mathcal{A} = \{A_{kj}\}_{k,j=1}^{m}$ be a commutative cocycle. We say that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\} \subseteq (\mathbb{C}^n)^m$ are $(\mathcal{A}, J)$-**related** if for $\mathbf{v}_i = (v_{i1}, \ldots, v_{im})$ we have that $v_{i\alpha_j} = 0$ if and only if $\alpha_j = 0$ and

$$(\alpha_l \alpha_j)\, A_{lj} v_{il} = v_{ij}$$

for every $1 \le i \le q$ and $1 \le j, l \le m$.

EXAMPLE 2.3. Suppose $\mathcal{A}$ is the family of matrices where all of them are the identity I, and $J = (\alpha_1, \dots, \alpha_m)$ is such that $\alpha_k = 1$ while $\alpha_t = 0$ for $k \neq t$. Take the vectors having all the entries equal to zero except for the k-th one,

$$\mathbf{e}_q^k = (0, \dots, 0, e_q, 0, \dots, 0)$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. $\{\mathbf{e}_1^k, \dots, \mathbf{e}_n^k\}$ is a $(\mathcal{A}, J)$-related set, since the conditions

$$(\alpha_k \alpha_l)\, I e_q = \begin{cases} e_q & l = k, \\ 0 & l \neq k, \end{cases}$$

tell us that each $\mathbf{e}_j^k$ is indeed a $(\mathcal{A}, J)$-related vector.

DEFINITION 2.4. An n-dimensional subspace $l \subseteq (\mathbb{C}^n)^m$ is called a **commutative subspace** if there is a set of binary indices $J \in \{0,1\}^m$ and a commutative cocycle $\mathcal{A}$ such that there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset l$ of $(\mathcal{A}, J)$-related vectors.

DEFINITION 2.5. We denote $G_{n,m}^{com}(\mathbb{C}) \subset G_{n,m}(\mathbb{C})$ as the set of commutative subspaces of $(\mathbb{C}^n)^m$, endowed with the subspace topology. Here by $G_{n,m}(\mathbb{C})$ we mean the Grassmanians, that is the set of subspaces of $(\mathbb{C}^n)^m$ of dimension n.

EXAMPLE 2.6. A trivial example of a commutative subspace are the subspaces

$$l_k = \text{gen}\{\mathbf{e}_1^k, \dots, \mathbf{e}_n^k\}$$

as in the previous example. These spaces will play an important role ahead.

EXAMPLE 2.7. To built a non trivial element of $G_{n,m}^{com}(\mathbb{C})$, consider for example $n = 2$, and $m = 3$. Call $\mathbf{v}_1 = \left((1,0), (0,1), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right)$ and $\mathbf{v}_2 = \left(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0,1)\right)$. If we make

$$T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and take $A_{12} = T(\pi/2)$, $A_{13} = T(\pi/4)$, we can make a commutative cocycle $\mathcal{A} := \{A_{ij}\}_{i,j=1}^2$ using the cocycle conditions for the rest of the matrices. Also take $J = (1,1,1)$. It is clear that $\mathbf{v}_1$ and $\mathbf{v}_2$ are $(\mathcal{A}, J)$-related. They are L.I in $(\mathbb{C}^2)^3$, thus the space $l = \text{gen}\{v_1, v_2\}$ is a commutative subspace of dimension 2.

Just as in the general case, we denote

$$E_{n,m}^{com} := \{(l, \mathbf{v}) \in G_{n,m}^{com} \times (\mathbb{C}^n)^m \mid \mathbf{v} \in l\}$$

and we get a natural map

$$\begin{aligned} \pi_{com} : E_{n,m}^{com} &\to G_{n,m}^{com} \\ (l, \mathbf{v}) &\mapsto l \end{aligned}$$

which is easily seen as the pullback from the inclusion $G_{n,m}^{com} \to G_{n,m}$, thus, it is a vector bundle. We want to show that the bundle π_{com} has a TC structure. But first we need to define the trivializing cover of $G_{n,m}^{\text{com}}$

and state some properties of it before we can prove the existence of its TC structure. To do this we mimic the construction of the general case for Grassmanians. Consider the commutative spaces

$$l_k = \text{gen}\left\{\mathbf{e}_1^k, \ldots, \mathbf{e}_n^k\right\}$$

and take the projections

$$\begin{aligned} \pi_k : (\mathbb{C}^n)^m &\to l_k \\ (v_1, \ldots, v_n) &\mapsto (0, \ldots, 0, v_k, 0, \ldots, 0). \end{aligned}$$

Then let us consider the open set of $G_{n,m}^{com}$ given by

$$U_k := \left\{ l \in G_{n,m}^{com} \mid \pi_k(l) = l_k \right\}.$$

We know this set is open, since it is the intersection of an open set of $G_{n,m}$ with $G_{n,m}^{com}$ (See the proof of Lemma 1.15 of [**Hatcher II**]). We trivialized the bundle over U_k in the following way: if $l \in U_k$ then l is projected isomorphically to

$$l_k := \left\langle \mathbf{e}_1^k, \mathbf{e}_2^k, \ldots, \mathbf{e}_n^k \right\rangle$$

via the projection $\pi_k : (\mathbb{C}^n)^m \to l_k$. Thus, we can use a natural linear isomorphism characterized by $\mu_k : \mathbf{e}_j^k \mapsto e_j \in \mathbb{C}^n$, where again $\{e_1, \ldots, e_n\}$ is the standard basis, to get a linear map

$$\bar{\phi}_k := \mu_k \circ \pi_k : (\mathbb{C}^n)^m \to \mathbb{C}^n$$

such that if $l \in U_k$, $\bar{\phi}_k \mid_l$ is an isomorphism. We get the trivialization

$$\begin{aligned} \phi_k : \pi_{\text{com}}^{-1}(U_k) &\to U_k \times \mathbb{C}^n \\ (l, v) &\mapsto \left(l, \bar{\phi}_k(v)\right). \end{aligned} \tag{2.1.1}$$

Now we need to guarantee these trivializations have indeed commutative transition functions. As a first step towards this we prove the following.

Proposition 2.8. *An n-dimensional commutative space $l \in G_{n,m}^{com}$ belongs to U_j if and only if there is a set of indices*

$$I = (\alpha_1, \ldots, \alpha_m) \in \{0, 1\}^m$$

with $i_j = 1$, a commutative cocycle $\mathcal{A}$ and a basis of $(\mathcal{A}, J)$-related vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset l$. Thus, $l \in U_j \cap U_k$ if only if there is a set of indices J such that $\alpha_j = \alpha_k = 1$, and such that l has a basis of $(\mathcal{A}, J)$-related vectors.

Proof. First assume that there is a sequence $J = (\alpha_1, \ldots, \alpha_m) \in \{0, 1\}^m$ such that $\alpha_j = 1$, and a basis of $(\mathcal{A}, J)$-related vectors of l, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset l$. In particular this gives us that every $\mathbf{v}_t$, $1 \leq t \leq n$, is of the form

$$\mathbf{v}_t = \left(v_1^t, \ldots, v_m^t\right)$$

with $v_j^t \neq 0$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are $(\mathcal{A}, I)$-related and a linearly independent set in $(\mathbb{C}^n)^m$, then $\left\{v_j^1, \ldots, v_j^n\right\}$ is a basis for $\mathbb{C}^n$ as we now see: by the definitions, there are non singular matrices $\{A_{jk}\}_{k=1}^m$ and scalars $\lambda_{jk} \in \{0, 1\}$ such that

$$\lambda_{jk} A_{jk} v_j^t = v_k^t$$

for every $1 \leq k \leq m$ and $1 \leq t \leq n$. Now suppose there are scalars $c_p \in \mathbb{C}$ such that

$$\sum_{p=1}^n c_p v_j^p = 0$$

and so we have

$$\sum_{p=1}^{n} c_p \mathbf{v}_p = \sum_{p=1}^{n} c_p \left(v_1^p, v_2^p, \ldots, v_m^p\right) = \sum_{p=1}^{n} c_p \left(\lambda_{j1} A_{j1} v_j^p, \ldots, \lambda_{jm} A_{jm} v_j^p\right).$$

By linearity we obtain

$$\sum_{p=1}^{n} c_p \mathbf{v}_p = \left(\lambda_{j1} A_{j1} \left(\sum_{p=1}^{n} c_p v_j^p\right), \ldots, \lambda_{jm} A_{jm} \left(\sum_{p=1}^{n} c_p v_j^p\right)\right) = 0.$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent, we conclude that $\alpha_p = 0$ for $1 \leq p \leq n$. So, $\left\{v_j^1, \ldots, v_j^n\right\}$ is linearly independent and since $\pi_j\left(\mathbf{v}_p^I\right) = v_j^p$ it follows that $\pi_j(l) = l_j$.

The opposite implication is obtained via the definitions. Let us assume that $\pi_j(l) = l_j$. By definition of commutative n-dimensional subspaces, there is a basis of $(\mathcal{A}, J)$-related vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset l$, where $J = (\alpha_1, \ldots, \alpha_m) \in \{0,1\}^m$. Also by definition $v_j^t \neq 0$ if only if $\alpha_j = 1$ for all $1 \leq t \leq n$, and since $\pi_j\left(\mathbf{v}_p^I\right) = v_j^p$, we must have $i_j = 1$.

The last statement is a consequence of the first equivalence. $\square$

Proposition 2.9. *Under the previous definitions,* $\left\{\pi_{\text{com}} : E_{n,m}^{com} \to G_{n,m}^{com}, \{U_i\}_{i=1}^m, \phi_i : \pi_{\text{com}}^{-1}(U_i) \to U_i \times \mathbb{C}^n\right\}$ *is a TC structure.*

Proof. So far we already know that $\pi_{\text{com}} : E_{n,m}^{com} \to G_{n,m}^{com}$ is a vector bundle, and that we have an open cover $\{U_i\}_{i=1}^m$ of $G_{n,m}^{com}$ with trivializations given by $\phi_i : \pi_{\text{com}}^{-1}(U_i) \to U_i \times \mathbb{C}^n$. To prove that we indeed have a TC structure we need to find the transition maps associated with the trivializations ϕ_i and then show that they commute with each other.

To find the transition maps, we first need to find the inverse map of

$$\begin{aligned} \phi_j : \pi_{\text{com}}^{-1}(U_j) &\to U_j \times \mathbb{C}^n \\ (l, \mathbf{v}) &\mapsto \left(l, \bar{\phi}_j(\mathbf{v})\right). \end{aligned}$$

constructed in the previous proof. By construction this in turn can be reduced to finding the inverse on a fiber of $\pi_j : \pi_{\text{com}}^{-1}(U_j) \to l_j$. Thus, consider l to be an n-dimensional commutative subspace of $(\mathbb{C}^n)^m$ with a $(\mathcal{A}, J)$-related basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset l$. If $J = (\alpha_1, \ldots, \alpha_m)$ is such that $\alpha_j = 1$, then because of the previous theorem we know that $\pi_j(l) = l_j$. Even further we can obtain the inverse map $l_j \to l$ as follows. Consider an element

$$\mathbf{w} := \sum_{k=1}^{n} c_k \mathbf{e}_k^j = (0, \ldots, 0, v, 0, \ldots, 0) \in l_j.$$

where $v \in \mathbb{C}^n$ is the j-th position.

By the previous proof we know that if

$$\mathbf{v}_t = \left(v_1^t, \ldots, v_m^t\right)$$

then $\left\{v_j^1, \ldots, v_j^n\right\}$ is a basis of $\mathbb{C}^n$ and so there are scalars $\beta_i \in \mathbb{C}$ such that

$$\sum_{k=1}^{n} \beta_k v_j^k = v.$$

So, if we consider the vector

$$\mathbf{z} = (z_1, \ldots, z_m) := \sum_{k=1}^{n} \beta_k \mathbf{v}_k,$$

we get that

$$z_j = \sum_{k=1}^{n} \beta_k v_j^k = v.$$

So it follows that

$$\pi_j \left(\sum_{k=1}^{n} \beta_k \mathbf{v}_k \right) = \mathbf{w}.$$

This means that the inverse function of the projection $\pi_j : l \to l_j$ can be described by

$$\left(0, \ldots, 0, \underbrace{\sum_{k=1}^{n} \beta_k v_k^j}_{j\text{th}-\text{position}}, 0, \ldots, 0 \right) \in l_j \mapsto \sum_{k=1}^{n} \beta_k \mathbf{v}_k.$$

Now take $1 \leq q \leq n$. If we call $\lambda_{jq} = \alpha_j \alpha_q$, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are $(\mathcal{A}, J)$-related it follows that $v_q^k = \lambda A_{jq} v_j^k$, and so

$$\begin{aligned} \pi_q \left(\sum_{k=1}^{n} \beta_k \mathbf{v}_k \right) &= \sum_{k=1}^{n} \beta_k v_q^k = \sum_{k=1}^{n} \beta_k \lambda_{jq} A_{jq} v_j^k \\ &= \lambda_{jq} A_{jq} \left(\sum_{k=1}^{n} \beta_k v_j^k \right) = \lambda_{jq} A_{jq} v. \end{aligned}$$

We conclude that if $l \in U_j \cap U_s$, then

$$\begin{aligned} \phi_s \circ \phi_j^{-1} : (U_j \cap U_s) \times \mathbb{C}^n &\to (U_j \cap U_s) \times \mathbb{C}^n \\ (l, v) &\mapsto (l, A_{sj} v) . \end{aligned}$$

This means that if $\rho_{jk} : U_j \cap U_k \to GL_n(\mathbb{C})$ are the transition function associated to the trivializatons $\{\phi_i\}_{i \in \mathbb{N}}$, we get

$$\rho_{jk}(l) = A_{jk}.$$

In conclusion $\pi_{com} : E_{n,m}^{com} \to G_{n,m}^{com}$, $\{U_j\}_{j \in \mathbb{N}}$ and $\phi_j : \pi_{com}^{-1}(U_j) \to U_j \times \mathbb{C}^n$ has transition functions that are commutative, since by construction the matrices $\{A_{ij}\}$ commute with each other. □

At this point it should be clear that we are following the steps of the construction of the Grassmanians for vector bundles of dimension n. We have achieved the construction of a commutative Grassmanian $G_{n,m}^{com}$ for arbitrary $n, m \in \mathbb{N}$, where m allows to "classify" vector bundles with covers with m elements. In order to get rid of this last parameter we consider the inclusions $G_{n,m}^{com} \subset G_{n,m+1}^{com}$ coming themselves from the natural inclusions $(\mathbb{C}^n)^m \subset (\mathbb{C}^n)^{m+1}$, where $(\mathbb{C}^n)^m$ is taken as the subspace of $(\mathbb{C}^n)^{m+1}$ the $(m+1)$- vector coordinate equal to zero. Then we use the colimit to define

$$G^{com}_{n,\infty} := \mathrm{Colim}_{m\to\infty} G^{com}_{n,m},$$

which is just the set of n-dimensional commutative subspaces of $(\mathbb{C}^n)^\infty$.

Then again we get the vector bundle

$$\pi_{com} : E^{com}_{n,\infty} \to G^{com}_{n,\infty}$$

where

$$E^{com}_{n,\infty} := \left\{(l, v) \in G^{com}_{n,\infty} \times (\mathbb{C}^n)^\infty \mid v \in l\right\}$$

and π_{com} is the projection over the first component. Just as before, we can consider the vectors $\mathbf{e}^j_k \in (\mathbb{C}^n)^\infty$ as the sequence having zero in all its entries, except in the j-th entry where it has the vector e_k of the canonical basis. Then l_j is the n-dimensional subspace generated by $\left\{\mathbf{e}^j_1, \ldots, \mathbf{e}^j_n\right\}$. Just as in the finite case, we can consider the open cover $\{U_j\}_{j=1}^\infty$ where U_j is the set of commutative n-dimensional subspaces whose projection over l_j is one to one. The proof that the previous data gives rise to a TC structure uses the same arguments as in the finite case, since in the colimits we are consider vectors with only a finite amount of non-zero entries. Thus, we obtain the main result of this section, and one of the goals of this chapter:

THEOREM 2.10. *The projection over the first component $\pi_{com} : E^{com}_{n,m} \to G^{com}_{n,m}$ with $1 \le m \le \infty$, together with the open cover $\{U_j\}_{j=1}^m$ of $G^{com}_{n,m}$ and trivializations inherited from the general Grassmanians define a transitional commutative structure.*

To close this section let us show that our definitions are not empty. First, not every n-dimensional subspace $l \subseteq (\mathbb{C}^n)^m$ is $(\mathcal{A}, J)$-related. There are clearly many examples of non commutative subspaces. For example consider in $\left(\mathbb{C}^2\right)^3$ the 2-dimensional subspace

$$l := \mathrm{gen}\left\{(e_1, 0, e_2), (e_1, e_2, e_2)\right\}.$$

By Proposition 2.8, every basis of a commutative space must be $(\mathcal{A}, J)$-related for a certain J, and every non zero entry of J must have elements giving rise to a basis of $\mathbb{C}^2$. This is not the case for $\{(e_1, 0, e_2), (e_1, e_2, e_2)\}$, because of the second entry. Thus this l is not a 2-dimensional commutative subspace.

Secondly, unless we have a TC structure, there might be a classifying function $f : M \to G_{n,m}(\mathbb{C})$ such that there is $x \in M$ with $f(x) \notin G^{\mathrm{com}}_{n,m}(\mathbb{C})$. However every $l = f(x)$ will satisfy the condition of having a basis of $(\mathcal{A}, J)$-related vectors, where the family $\mathcal{A}$ satisfies the cocycle condition but not necessarily the commutative condition. We illustrate this in the following example.

EXAMPLE 2.11. (Non commutative subspace)

Here we want to see that the commutativity for linear subspaces is not a void condition. Not every set of trivializations lead to a commutative subspace. Consider the space $(\mathbb{C}^n)^3$. For $\theta \ne m\pi$ with $m \in \mathbb{N}$, and take

$$A_{12} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and

$$A_{13} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where the rest of the family $\mathcal{A} = \{A_{jk}\}^3_{j,k=1}$ is determined by the cocycle conditions. This family is non commutative since

$$A_{12}A_{13} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \neq \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} = A_{13}A_{12}.$$

Now take $\{e_1, e_2\} \subset \mathbb{C}^2$ the standard basis and consider the vectors

$$\mathbf{v}_1 = (e_1, A_{12}e_1, e_2) \text{ and } \mathbf{v}_2 = (e_2, A_{12}e_2, e_1)$$

which are $(\mathcal{A}, (1,1,1))$-related. The space

$$l := \text{gen}\{\mathbf{v}_1, \mathbf{v}_2\}$$

is non commutative but it can be in the image of a classifying function of a vector bundle.

2.2. Universality of $E^{com}_{n,\infty} \to G^{com}_{n,\infty}$

In this section we complete the goal of the chapter by showing that the commutative Grassmanian classifies TC structures on vector bundles up to homotopy. That is we prove the following theorem:

THEOREM 2.12. *(Universality of $G^{com}_{n,\infty}$)*

Let M be a Hausdorff and compact space. The assignment

$$\begin{aligned} \Psi : \left[M, G^{com}_{n,\infty}\right] &\to \text{Vect}^{com}_n(M) \\ [f] &\mapsto \left[f^*\left(E^{com}_{n,\infty}\right)\right] \end{aligned}$$

is a well defined bijective map, where $\left[M, G^{com}_{n,\infty}\right]$ is the set of homotopy classes of functions from M to $G^{com}_{n,\infty}$.

In order to prove this theorem, there are three facts needed to be checked. We need to check first that this map is well defined, that is, we need to show that the choice of the representative of a homotopy class $[f]$ does not change the class of the pullback. This is true thanks to Proposition 1.11 on page 11. The other two facts are surjectivity and injectivity of Ψ, which we do in two separate sub sections.

2.2.1. Surjectivity of Ψ. It is worth mentioning that for this proof, it is enough to require M to be paracompact and Hausdorff.

Given a TC structure $(\pi : E \to M, \{V_j\}, \varphi_j)$, we have to find a function $f : M \to G^{\text{com}}_{n,\infty}$ such that the TC structure on $f^*\left(E^{com}_{n,\infty}\right) \to M$ is equivalent to the previous TC structure on M. Such function is obtained as

$$f(x) = g\left(\pi^{-1}(x)\right) \in G^{\infty}_n,$$

where $g : E \to \mathbb{C}^{\infty}$ is a function constructed with the trivializations φ_j. For this purpose we may assume that $\{V_j\}$ is a locally finite and countable open cover of M. Let us consider a partition of the unity $\{\sigma_j\}$ subordinated to the cover $\{V_j\}$. In particular we have $\sigma_j^{-1}((0,1]) \subseteq V_j$, and since the support of a function is an open set, we have a refinement of the open cover $\{V_i\}_{i\in\mathbb{N}}$.

Call $\tilde{g}_j : \pi^{-1}(V_j) \to \mathbb{C}^n$ the composition of φ_j with the projection on the second component, $V_j \times \mathbb{C}^n \to \mathbb{C}^n$. We can now take $g_i : \pi^{-1}(V_i) \to (\mathbb{C}^n)^\infty$ as the function whose i-th (vector) entry is $\tilde{g}_i$, while the rest entries are zero. We can define $g : E \to (\mathbb{C}^n)^\infty$ by

$$g(e) := \sum_{j=1}^{\infty} \sigma_j(\pi(e))\, g_j(e),$$

where we are using the convention that g_j is zero outside $\pi^{-1}(V_j)$. This if of course well defined since the closure of the support of each σ_j is contained in V_j. Also note that g is a continuous function, as it has continuous component functions. Also for a given point $x \in \sigma_j^{-1}((0,1]) \subseteq V_j$ the image of $\pi^{-1}(x)$ under g_j is exactly

$$l_j := \left\langle \mathbf{e}^j_1, \mathbf{e}^j_2, \ldots, \mathbf{e}^j_n \right\rangle$$

where $\mathbf{e}^j_k$ have all its entries equal to zero except the j-th one, which is e_k, with $\{e_k\}_{k\in\mathbb{N}}$ the standard basis of $\mathbb{C}^\infty$. This is true because g_j is a 1-1 linear function on the fibers of π. For this reason we have that

$$f^{-1}(U_j) \subseteq V_j$$

for every $j \in \mathbb{N}$.

Let's see that for every $x \in M$, we have that $f(x)$ is in $G^{com}_{n,\infty}$. Let $\{j_1, \ldots, j_p\} \subset \mathbb{N}$ be the set of indices such that $x \in \operatorname{supp}(\sigma_{j_k})$. Also, let $\rho_{rs} : U_{j_r} \cap U_{j_s} \to GL_n(\mathbb{C})$ be the transition functions. Define

$$A_{rs} := \rho_{rs}(x)$$

for which is clear that for $v \in \pi^{-1}(x)$

$$\tilde{g}_{j_s}(v) = A_{rs}\tilde{g}_{j_r}(v).$$

The matrices $\{A_{rs}\}$ satisfy the relations of the definition of commutative cocycle. So, $g(v)$ is in fact a $(\mathcal{A}, J)$-related vector. Since the matrices $\{A_{rs}\}$ do not depend on the vectors, but only the fiber they are on, we conclude that $f(x) := g\left(\pi^{-1}(x)\right)$ is in fact a commutative n-dimensional subspace of $(\mathbb{C}^n)^q$, thus $f(x) \in G^{com}_{n,\infty}$ for all $x \in M$. Notice that the continuity of f follows from the fact that it is the classifying function of $\pi : E \to M$, so when we restrict its codomain, it remains continuous since $G^{\mathrm{com}}_{n,\infty}$ has the subspace topology with respect to the regular Grassmanians.

Now we have to check that indeed the pull back structure obtained from the pullpack of $f : M \to G^{com}_n(\mathbb{C})$ is equivalent to the original structure as TC structures. For this we are going to apply Proposition 1.9 to see that the TC structure given by f is equivalent to the original TC structure over M, up to a contraction on the trivializations, which we also know it does not change the TC structure class by Proposition 1.12.

We need to construct a function $\tilde{f} : E \to E^{\mathrm{com}}_n$ which is an isomorphism on the fibers, such that we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E^{\mathrm{com}}_n \arrow[d, "\pi_{\mathrm{com}}"] \\ M \arrow[r, "f"] & G^{\mathrm{com}}_n. \end{tikzcd} \tag{2.2.1}$$

This function is obtained in a natural way, since we just need to define

$$\begin{aligned} \tilde{f} :&E \to E^{\text{com}}_n \\ &e \mapsto (f \circ \pi (e), g(e)) \end{aligned}$$

which is well defined by construction. It is clear that it satisfies the above conditions. Next, we need to see that the second condition of Proposition 1.9 is satisfied, where the first condition of this item is already being achieved, since we saw that $f^{-1}(U_j) \subseteq V_j$. Let us recall first how we get trivializations for the TC structure $\pi_{\text{com}} : E^{\text{com}}_{n,\infty} \to G^{\text{com}}_{n,\infty}$. Again, if $l \in U_j$ then l is projected isomorphically to

$$l_j := \left\langle \mathbf{e}^j_1, \mathbf{e}^j_2, \dots, \mathbf{e}^j_n \right\rangle$$

via the projection $\pi_j : \mathbb{C}^\infty \to l_j$. So we can use natural linear bijection $\tilde{\phi}_j : e^j_k \mapsto e_k \in \mathbb{C}^n$, where again $\{e_1, \dots, e_n\}$ is the standard basis, to get a linear map

$$\bar{\phi}_j := \tilde{\phi}_j \circ \pi_j : \mathbb{C}^\infty \to \mathbb{C}^n$$

such that if $l \in U_j$, $\bar{\phi}_j \mid_l$ is an isomorphism. We get the trivializations

$$\begin{aligned} \phi_j : \pi^{-1}_{\text{com}} (U_j) &\to U_j \times \mathbb{C}^n \\ (l, v) &\mapsto \left(l, \bar{\phi}_i (v)\right). \end{aligned}$$

Applying these constructions we have that if $e \in \pi^{-1}\left(f^{-1}(U_j)\right) \subseteq \pi^{-1}(V_j)$ then

$$\phi_i \circ \tilde{f}(e) = (f(\pi(e)), \sigma_i(\pi(e)) \tilde{g}_i(e)).$$

Now recall the trivialization over V_j, $\varphi_j : \pi^{-1}(V_j) \to V_j \times \mathbb{C}^n$ have the form

$$\varphi_j(e) = (\pi(e), \tilde{g}_j(e)),$$

so we get that if $(x, v) \in f^{-1}(U_i) \times \mathbb{C}^n$ then

$$\phi_j \circ \tilde{f}\left(\varphi^{-1}_j (x, v)\right) = (f(x), \sigma_j(x) v).$$

Linearity over the fibers imply that

$$\phi_j \circ \tilde{f}\left(\varphi^{-1}_j \left(x, \sigma_j(x)^{-1} v\right)\right) = (f(x), v) \tag{2.2.2}$$

and so if we replace the trivializations

$$\begin{aligned} \varphi_j : \pi^{-1}(V_j) &\to V_j \times \mathbb{C}^n \\ e &\mapsto (\pi(e), \tilde{g}_j(e)) \end{aligned}$$

by

$$\begin{aligned} \varphi'_j : \pi^{-1}(V_j) &\to V_j \times \mathbb{C}^n \\ e &\mapsto (\pi(e), \sigma_j(\pi(e)) \tilde{g}_j(e)) \end{aligned}$$

we get that

$$\phi_j \circ \tilde{f}\left(\varphi'^{-1}_j (x, v)\right) = (f(x), v).$$

We then apply Proposition 1.9 to see that the TC structure $\left(\pi : E \to M, \{V_j\}_{j\in\mathbb{N}}, \varphi'_j\right)$ is equivalent to the TC pullback structure given by the map $f : M \to G^{com}_{n,\infty}$. Finally we apply Proposition 1.12 to get that said pullback is equivalent to the original TC structure $\left(\pi : E \to M, \{V_j\}_{j\in\mathbb{N}}, \varphi_j\right)$.

2.2.2. Injectivity of Ψ. Suppose we have two functions $f_1, f_2 : M \to G^{com}_{n,\infty}$, such that their pullbacks TC structures

$$\left(p_1 : f_1^*\left(G^{com}_{n,\infty}\right) \to M, \left\{f_1^{-1}\left(U_j\right)\right\}_{j\in\mathbb{N}}, \varphi_{1j}\right)$$

and

$$\left(p_2 : f_2^*\left(G^{com}_{n,\infty}\right) \to M, \left\{f_2^{-1}\left(U_j\right)\right\}_{j\in\mathbb{N}}, \varphi_{2j}\right)$$

are equivalent. We have to show that f_1 and f_2 are homotopic in order to conclude that Ψ is injective. Recall that the definition of equivalence of TC structures implies that there is a TC structure

$$\left\{p : E \to M \times [0,1], \{W_k\}, \phi : p^{-1}\left(W_k\right) \to W_k \times \mathbb{C}^n\right\}$$

such that

(1) $f_1^*\left(G^{com}_{n,\infty}\right) = p^{-1}\left(M \times \{0\}\right)$ and $f_2^*\left(G^{com}_{n,\infty}\right) = p^{-1}\left(M \times \{1\}\right)$.
(2) $p_1 = p\mid_{M\times\{0\}}$ and $p_2 = p\mid_{M\times\{1\}}$.
(3) For each $k \in \mathbb{N}$ there are $j_k, s_k \in \mathbb{N}$ such that $p^{-1}\left(W_k \cap M \times \{0\}\right) \subseteq f_1^{-1}\left(U_{j_k}\right)$, $p^{-1}\left(W_k \cap M \times \{1\}\right) \subseteq f_2^{-1}\left(U_{s_k}\right)$,
$$\phi\mid_{p^{-1}(W_k\cap M\times\{0\})} = \varphi_{1j_k}\mid_{p_1^{-1}(W_k\cap M\times\{0\})}$$
and $\phi\mid_{p^{-1}(W_k\cap M\times\{1\})} = \varphi_{2s_k}\mid_{p_2^{-1}(W_k\cap M\times\{0\})}$.

Now, using the trivializations over $\{W_k\}$ we can build up a function (as we did to prove surjectivity) $G : E \to (\mathbb{C}^n)^\infty$ that is an injective linear map on the fibers of p, thus it induces a function $F : M \times [0,1] \to G^{com}_{n,\infty}$. This construction is done in such a way such that the functions $g_1 = G\mid_{p^{-1}(M\times\{0\})}$ and $g_2 = G\mid_{p^{-1}(M\times\{1\})}$ could also be obtained using the trivializations $\varphi_{1,j_k}\mid_{p_1^{-1}(W_k\cap M\times\{0\})}$ and $\varphi_{2,j_k}\mid_{p_2^{-1}(W_k\cap M\times\{0\})}$, respectively. This means that for $\tilde{f}_1 = F\mid_{M\times\{0\}}$ and $\tilde{f}_2 = F\mid_{M\times\{1\}}$, the assignments $\tilde{f}_1(x) = g_1\left(p^{-1}(x,0)\right)$ and $\tilde{f}_2(x) = g_2\left(p^{-1}(x,1)\right)$ also hold. Even further we have that $\tilde{f}_1$ and $\tilde{f}_2$ are homotopic through F. So if we prove f_i is homotopic to $\tilde{f}_i$ $(i = 1, 2)$, it will follow that f_1 is homotopic to f_2, since homotopy is an equivalence relation. Thus the proof reduces to proving that given a function $f : M \to G^{com}_{n,\infty}$, if we have a refinement $\{V_k\}_{k\in\mathbb{N}}$ of $\left\{f^{-1}\left(U_j\right)\right\}_{j\in\mathbb{N}}$, and we build a function $\tilde{f}$ with the trivializations of $f^*\left(E^{com}_{n,\infty}\right)$ restricted to the sets of $\{V_k\}_{k\in\mathbb{N}}$, then $\tilde{f}$ is homotopic to f.

Before dealing with the construction of $\tilde{f}$, let us examine both the covers and the trivialization coming from the pullback structure of f. On one hand recall that by construction we have a filtration in $G^{com}_{n,\infty}$ given by

$$G^{com}_{n,1} \subseteq G^{com}_{n,2} \subseteq \cdots \subseteq G^{com}_{n,m} \subseteq G^{com}_{n,m+1} \cdots$$

such that $G^{com}_{n,\infty}$ has the colimit topology with respect to this sequence. As such, any compact subset of $G^{com}_{n,\infty}$ is contained in one of the members of this filtration. In particular, since M is compact and f is continuous, there is some $N \in \mathbb{N}$ such that $f(M) \subseteq G^{com}_{n,N} \subseteq G^{com}_{n,\infty}$. This means that for every $x \in M$ if $\mathbf{v} \in f(x)$, this vector is of the form

$$\mathbf{v} = (v_1, \ldots, v_N, 0, 0, \ldots)$$

where $v_i \in \mathbb{C}^n$.

On the other hand we have a natural function

$$\begin{aligned} g : f^*\left(E^{com}_{n,\infty}\right) &\to (\mathbb{C}^n)^\infty \\ (x, (l, \mathbf{v})) &\mapsto \mathbf{v} \in f(x) \end{aligned}$$

which satisfies the property $g\left(\pi_2^{-1}(x)\right) = f(x)$. In conclusion there are N functions $g_i, \dots, g_N : f^*\left(E^{com}_{n,\infty}\right) \to \mathbb{C}^n$ such that

$$g(e) = (g_1(e), g_2(e), \dots, g_N(e), 0, 0, \dots).$$

This also implies that $\left\{f^{-1}(U_j)\right\}_{j=1}^{N}$ is a cover of M, and that $f^{-1}(U_j) = \emptyset$ for $j > N$. Now consider the trivializations $\varphi_j : \pi_1^{-1}\left(f^{-1}(U_j)\right) \to f^{-1}(U_j) \times \mathbb{C}^n$ coming with the pullback structure. By construction the second component of φ_j is determined by the projection $\mathrm{pr}_j : (\mathbb{C}^n)^\infty \to l_j$. This means that if $(x,(l,\mathbf{v})) \in \pi_1^{-1}\left(f^{-1}(U_j)\right)$, then we have the assignment

$$\mathbf{v} = (v_1, \dots, v_N, 0, 0, \dots) \in l \mapsto \mathrm{pr}_j\,|_l\,(\mathbf{v}) = (0, \dots 0, v_j, 0, 0, \dots) \in l_j \cong \mathbb{C}^n.$$

This agrees with the definition of g_j, that is $\mathrm{pr}_j\,|_l\,(\mathbf{v}) = g_j(x,(l,\mathbf{v}))$.

Now suppose we have a refinement $\{V_k\}_{k\in J}$ of $\left\{f^{-1}(U_j)\right\}_{j\in\mathbb{N}} = \left\{f^{-1}(U_j)\right\}_{j=1}^{N}$. Since M is compact we can assume that such refinement is finite, even more we may take it to be of the form

$$\mathcal{V} = \{V_{11}, V_{12}, \dots, V_{1m_1}, \dots, V_{N1}, V_{N2}, \dots, V_{Nm_N}\}$$

where $V_{j,k} \subseteq f^{-1}(U_j)$ for all the indices. Now we proceed with the construction of $\tilde{f}$ with the cover $\mathcal{V}$. The trivializations on the elements of $\mathcal{V}$ are given by the restrictions $\varphi_j\,|_{V_{jk}} : \pi_1^{-1}(V_{jk}) \to V_{jk} \times \mathbb{C}^n$. Once again, we obtain a new function $\tilde{g} : f^*\left(E^{com}_{n,\infty}\right) \to (\mathbb{C}^n)^\infty$ by considering the second components of the trivializations $\varphi_j\,|_{V_{jk}}$, which are given by $g_j\,|_{V_{jk}}$. We also need a partition of the unity $\{\sigma_{jk} : M \to [0,1]\}$ such that $\sigma_{jk}^{-1}(0,1] \subseteq V_{jk}$. For simplicity let us call $\tau_{jk} := \sigma_{jk} \circ \pi_1$, then we have

$$\tilde{g} = (\tau_{11}\cdot g_1, \dots, \tau_{1m_1}\cdot g_1, \tau_{21}\cdot g_2, \dots, \tau_{2m_2}\cdot g_2, \dots, \tau_{N1}\cdot g_N, \dots, \tau_{Nm_N}\cdot g_N).$$

Notice that both g and $\tilde{g}$ map every $\pi_1^{-1}(x)$ into (different) commutative n-dimensional subspaces, for $x \in M$. In particular we can now take $\tilde{f} : M \to G^{com}_{n,\infty}$ by defining

$$\tilde{f}(x) = \tilde{g}\left(\pi_1^{-1}(x)\right).$$

In order to construct a homotopy between f and $\tilde{f}$ we see first that g and $\tilde{g}$ are homotopic via

$$G : f^*\left(E^{com}_{n,\infty}\right) \times [0,1] \to (\mathbb{C}^n)^\infty,$$

satisfying that for every $(x,t) \in M \times [0,1]$, $G\left(\pi_1^{-1}(x), t\right) \subseteq (\mathbb{C}^n)^\infty$ is a n-dimensional commutative subspace, with $G(e,0) = g(x)$ and $G(x,1) = \tilde{g}(x)$. This way G induces a homotopy $F : M \times [0,1] \to G^{com}_{n,\infty}$ given by

$$F(x,t) := G\left(\pi_1^{-1}(x), t\right),$$

such that $F(x,0) = f(x)$ and $F(x,1) = \tilde{f}(x)$.

We built G as a composition of several homotopies. Let's see first that we can move the components of g to the right leaving zeros behind the previous positions. We do this via linear homotopies moving one component at the time

$$\begin{aligned} H_1(-,t) &= (1-t)(g_1, g_2, \dots, g_N, 0, \dots) + t(g_1, g_2, \dots, g_{N-1}, 0, g_N, 0, \dots) \\ &= (g_1, g_2, \dots, g_{N-1}, tg_N, (1-t)g_N, 0, \dots). \end{aligned}$$

This homotopy has the desires properties. If $e \in f^*\left(E^{com}_{n,\infty}\right)$ then the vector $g(e)$ is $(\mathcal{A}, J)$-related for an appropriate family of commutative cocycles $\mathcal{A} = \{A_{ij}\}$ and set of indices $J = (i_1, \dots, i_N, 0, \dots)$. Then for every $t \in (0,1)$, $H_1(e,t)$ is $(\mathcal{A}', I')$-related with $I' = (i_1, \dots, i_N, i_N, 0, \dots)$ and

$$\mathcal{A}' = \{A_{ij}\}_{i,j=1}^{N-1} \cup \{B_{iN}\}_{i=1}^{N-1} \cup \{B_{iN+1}\}_{i=1}^{N-1} \cup \{B_{NN+1}\}$$

where $B_{iN} = tA_{iN}$, $B_{iN+1} = (1-t)\,A_{iN}$ and $B_{NN+1} = \frac{1-t}{t} I_n$, with I_n the identity matrix. The rest of the matrices are given by the cocycle conditions. It is immediate that $\mathcal{A}'$ is a commutative cocycle. The cases $t \in \{0,1\}$ are handle in an easier manner. Using induction we see that g is homotopic through commutative n-subspaces to a function h_1 with

$$h_1 = \left(\underbrace{0,0,\ldots,0}_{M-\text{times}}, g_1, g_2, \ldots, , g_N, 0, 0, \ldots\right)$$

where $M := \sum_{k=1}^{j} m_k$.

Now we use yet another linear homotopy

$$\begin{aligned} H_2(-,t) = {} & (t\tau_{11} \cdot g_1, \ldots, t\tau_{1m_1} \cdot g_1, \ldots, t\tau_{N1} \cdot g_N, \ldots, t\tau_{Nm_N} \cdot g_N, 0, 0, \ldots) \\ & + (1-t)\left(\underbrace{0,0,\ldots,0}_{M}, g_1, g_2, \ldots, , g_N, 0, 0, \ldots\right) \end{aligned}$$

which again go through commutative n-subspaces. This is clearly a homotopy from $\tilde{g}$ to h_1, so we get that g and $\tilde{g}$ are homotopic through commutative n-subspaces.

CHAPTER 3

Classifying space for TC structures

On this chapter we describe the construction of the classifying space for TC structures over principal bundles, $B_{\mathrm{com}}G$, as well as its cohomology with real coefficients. We pay special attention to the construction of power maps on the classifying space and their effect on the cohomology ring, as well as its relation with the regular classifying space. This work is based on that of Adem and Gómez [**AG**].

In this chapter G will denote a connected Lie group.

3.1. Bar construction for TC structures

We start this section with the definition of classifying space for commutativity.

Definition 3.1. We say a space $B_{com}G$ is a classifying space for commutativity, if there is a TC structure $\left(p : E_{com}G \to B_{com}G, \{U_i\}_{i\in\mathbb{N}}, \varphi_i : p^{-1}(U_i) \to U_i \times G\right)$ such that for every paracompact and Hausdorff space X there is a natural bijection

$$\begin{aligned} \Psi_X : [X, B_{com}G] &\to \mathrm{Bund}_G^{com}(X) \\ [f] &\mapsto [f^*(E_{com}G)] \end{aligned}$$

where $[X, B_{com}G]$ denotes the set of homotopy classes of functions $f : X \to B_{com}G$. Under this conditions, if the TC structure over $f^*(E_{com}G)$ is equivalent to a given TC structure, we will call f the classifying function of the TC structure.

By naturality we mean that if $g : Y \to X$ is a continuous function, then there is a commutative diagram

$$\begin{tikzcd} {[X, B_{com}G]} \arrow[r, "\Psi_X"] \arrow[d, "g^*"] & \mathrm{Bund}_G^{com}(X) \arrow[d, "g^*"] \\ {[Y, B_{com}G]} \arrow[r, "\Psi_Y"] & \mathrm{Bund}_G^{com}(Y) \end{tikzcd}$$

where the left vertical map is the postcomposition $f \circ g : Y \to B_{com}G$ and the right vertical map is simply the pullback of TC structures.

Let us define a simplicial space whose n-th level is given by $\mathrm{Hom}(\mathbb{Z}^n, G)$, which is the subspace of G^n consisting of all commuting n-tuples. This is $(g_1, \ldots, g_n)$ such that $g_ig_j = g_jg_i$ for every $1 \le i, j. \le n$. Its face maps $\delta_i : \mathrm{Hom}(\mathbb{Z}^n, G) \to \mathrm{Hom}(\mathbb{Z}^{n-1}, G)$ are

$$\delta_i(g_1, \dots, g_n) := \begin{cases} (g_2, \dots, g_n) & i = 0, \\ (g_1, \dots, , g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & 1 \le i \le n-1, \\ (g_1, \dots, g_{n-1}) & i = n, \end{cases}$$

and the degeneracies $s_i : \mathrm{Hom}(\mathbb{Z}^n, G) \to \mathrm{Hom}(\mathbb{Z}^{n+1}, G)$ are given by

$$s_i(g_1, \dots, g_n) = (g_1, \dots, , g_i, 1, g_{i+1}, \dots, g_n).$$

It is routine to see they satisfy the simplicial identities. It is also important to mention that for this construction the fat realization is homotopically equivalent to the geometrical realization. Thus, we work with the former. The following theorem is proved in [**AG**].

THEOREM 3.2. *The fat realization of the previous simplicial space. i.e.*

$$B_{com}G := \|\mathrm{Hom}(\mathbb{Z}^\bullet, G)\|$$

has a TC structure on it making it a classifying space for TC structures over finite CW-spaces.

In fact the above theorem holds in general for any CW complex or in general for any paracompact and Hausdorff space. Also it is worth pointing out that in case $G = GL_n(\mathbb{C})$, $B_{\mathrm{com}}G$ is homotopically equivalent to the commutative grassmannian, $G^{com}_{n,\infty}$. This is true since both of them are classifying spaces, there are functions $f : B_{\mathrm{com}}G \to G^{com}_{n,\infty}$ and $g : G^{com}_{n,\infty} \to B_{\mathrm{com}}G$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity on $G^{com}_{n,\infty}$ and the identity on $B_{\mathrm{com}}G$, respectively.

In what follows we need to define a subspace of $B_{\mathrm{com}}G$, obtained by considering the connected component of $\mathrm{Hom}(\mathbb{Z}^m, G)$ containing the element $(1, 1, \dots, 1)$. We denote this by $\mathrm{Hom}(\mathbb{Z}^m, G)_1$. It is clear that we can restrict the face and degeneracy maps to obtain a simplicial space $\mathrm{Hom}(\mathbb{Z}^\bullet, G)_1$.

DEFINITION 3.3. The fat realization of the simplicial space $\mathrm{Hom}(\mathbb{Z}^\bullet, G)_1$ is denoted by $B_{\mathrm{com}}G_1$.

This distinction makes it clear that $\mathrm{Hom}(\mathbb{Z}^m, G)$ is in general not path connected. However Adem and Cohen showed in Corollary 2.4 of [**AC**] that $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected when G is either $U(n)$, $SU(n)$ or $\mathrm{Sp}(n)$. It is also worth mentioning that when $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected, then every m-th tuple is contained in a maximal torus. (See Lemma 4.2 of [**Baird**].)

3.1.1. The classifying function: For our purposes besides Theorem 3.2 itself, we need to examine the construction of the classifying function of a TC structure $(\pi : E \to X, \{U_i\}_{i=1}^m, \varphi_i)$ on a finite CW-complex X. Assume $\mathcal{U} := \{U_i\}_{i=1}^m$ is a good cover and we have the transition functions $\rho_{ij} : U_i \cap U_j \to G$. Then we consider the construction of the nerve of the cover:

$$\mathcal{N}(\mathcal{U})_l = \bigsqcup (U_{i_0} \cap U_{i_1} \cdots \cap U_{i_l}).$$

Then we have a simplicial function $f_l : \mathcal{N}(\mathcal{U})_l \to \mathrm{Hom}(\mathbb{Z}^l, U(n))$ given by

$$f_l(x) := (\rho_{i_0 i_1}(x), \rho_{i_2 i_3}(x), \dots, \rho_{i_{l-1} i_l}(x)).$$

This induces a function $f : \mathcal{N}(\mathcal{U}) \to B_{\mathrm{com}}G$, where $\mathcal{N}(\mathcal{U}) := \|\mathcal{N}(\mathcal{U})_\bullet\|$ is the nerve of the cover. Since $\mathcal{U}$ is a good cover, X and $\mathcal{N}(\mathcal{U})$ are homotopy equivalent (See [**Hatcher**], Corollary 4G.3). Thus, there is a bijection

$$[\mathcal{N}(\mathcal{U}), B_{\mathrm{com}}G] \cong [X, B_{\mathrm{com}}G],$$

which allows us to construct the classifying function from the nerve instead of X. As we mentioned before, Adem and Gómez also showed that if $F : X \to BG$ is the classifying function for the principal bundle $\pi : E \to X$, this is if we ignore the TC structure as whole, the following diagram commutes up to homotopy

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F"'] & B_{com}G \arrow[d] \\ & BG, \end{tikzcd}$$

where the vertical map is the natural inclusion.

3.2. Power maps and cohomology of $B_{com}G_1$

In this section we explain both the construction of the power maps as well as the reasoning behind the computation of $H^*(B_{com}G_1, \mathbb{R})$ made in [**AG**]. We tackle them together since our main objective is to show the effect of the power maps on cohomology. Thus we track such effect in every step of the computation. To make notation simpler, we assume $B_{com}G_1 = B_{com}G$, which is true when G is either $U(n)$, $SU(n)$ or $\mathrm{Sp}(n)$, as mentioned before. We also fix a maximal torus $T \subseteq G$ with Weyl group W and we write $H^*(Y)$ to refer to the cohomology of Y with real coefficients.

First let us introduced the construction of **power maps**. Consider once again the space of commutative m-tuples of G, $\mathrm{Hom}(\mathbb{Z}^m, G)$. For each $k \in \mathbb{Z}$ we define maps

$$\begin{aligned} \Phi_m^k : \mathrm{Hom}(\mathbb{Z}^m, G) &\to \mathrm{Hom}(\mathbb{Z}^m, G) \\ (g_1, \ldots, g_m) &\mapsto \left(g_1^k, \ldots, g_m^k\right). \end{aligned}$$

This map is well defined since the power of commuting elements is still commutative. Commutativity is needed in order for them to induce simplicial maps. By this we mean maps commuting with the face and degeneracy maps. More precisely we need the equality

$$(g_i g_{i+1})^k = g_i^k g_{i+1}^k$$

to hold. Thus, only for commuting tuples we guarantee the existence of the k-th power map $\Phi^k : B_{com}G \to B_{com}G$. In the general Bar construction for G, the power maps do not necessarily induced simplicial maps.

On Section 7 of [**AG**] they proved that for a maximal torus T of G with Weyl group W we have

$$H^*(B_{com}G) \cong \left(H^*(BT) \otimes H^*(BT)\right)^W / J,$$

where J is the ideal generated by the set

$$\{f(x) \otimes 1 \in H^*(BT) \otimes H^*(BT) \mid f \text{ is of positive degree polynomial and } n \cdot f(x) = f(x) \text{ for all } n \in W\}.$$

In order to reach the description of the induced power maps Φ^k on cohomology, we need to consider some auxiliary maps that are used in [**AG**] to compute the cohomology with real coefficients of $B_{\mathrm{com}}G$. In this process we will see what their relationship with the power maps. First, since all the tuples of T^m have commuting elements, we can consider the power maps for the torus $\psi^k : H^*(BT) \to H^*(BT)$. This is the map induced in the m-level the by

$$\begin{aligned} \Phi_m^k : \mathrm{Hom}(\mathbb{Z}^m, T) &\to \mathrm{Hom}(\mathbb{Z}^m, T) \\ (g_1, \ldots, g_m) &\mapsto \left(g_1^k, \ldots, g_m^k\right). \end{aligned}$$

Also consider

$$\begin{aligned}\varphi_m : G \times T^m &\to \mathrm{Hom}\left(\mathbb{Z}^m, G\right)\\ (g, t_1, \ldots, t_n) &\mapsto \left(g t_1 g^{-1}, \ldots, g t_m g^{-1}\right).\end{aligned}$$

Because $\mathrm{Hom}\left(\mathbb{Z}^m, G\right)$ is path connected, an m-tuple $(g_1, \ldots, g_m)$ has commuting elements if and only if there is a maximal tori containing all g_i (See Lemma 4.2 of [**Baird**]). Since every maximal tori is conjugated to T, the previous map is surjective. We also have an action of the normalizer of T in G, $N_G(T)$, on $G \times T^m$, where for $\eta \in N_G(T)$ we have

$$\eta \cdot (g, t_1, \ldots, t_m) = \left(g\eta^{-1}, \eta t_1 \eta^{-1}, \ldots, \eta t_m \eta^{-1}\right).$$

On the other hand, consider the Flag variety G/T. It is easy to verify that the maps φ_m factor through the product $G/T \times T^m$ giving us a commutative diagram

$$\begin{tikzcd} G \times T^m \arrow[r, "\varphi_m"] \arrow[d] & \mathrm{Hom}\left(\mathbb{Z}^m, G\right), \\ G/T \times T^m \arrow[ur] & \end{tikzcd}$$

such that the diagonal map is also surjective. We call it φ_m as well. These family of maps give rise to a simplicial map

$$\varphi_\bullet : G/T_\bullet \times T^\bullet \to \mathrm{Hom}\left(\mathbb{Z}^\bullet, G\right).$$

Here $G/T_\bullet$ is the trivial simplicial space with G/T on every level, and $T^\bullet$ is the simplicial space obtained by the Bar construction for the classifying space applied to T.

Furthermore using representatives of the Weyl group $[\eta] \in W = N_G(T)/T$, we have a well define action on $G/T \times T^m$ given by

$$[\eta] \cdot ([g], t_1, \ldots, t_m) = \left(\left[g\eta^{-1}\right], \eta t_1 \eta^{-1}, \ldots, \eta t_m \eta^{-1}\right).$$

It is easy to see that this action makes φ_m W-invariant. Also we can construct a simplicial space, $G/T \times_W T^\bullet$, having the space of orbits $G/T \times_W T^m$ on the m-th level. Where the simplicial structure is inherit form $G/T_\bullet \times T^\bullet$, giving us a simplicial map $\pi_\bullet : G/T_\bullet \times T^\bullet \to G/T \times_W T^\bullet$ where on each level we have the natural quotient map. Then we have a commuting diagram

$$\begin{tikzcd} G/T_\bullet \times T^\bullet \arrow[r, "\varphi_\bullet"] \arrow[d, "\pi_\bullet"] & \mathrm{Hom}\left(\mathbb{Z}^\bullet, G\right), \\ G/T \times_W T^\bullet \arrow[ur, "\bar{\varphi}_\bullet"] & \end{tikzcd}$$

where $\bar{\varphi}_m : G/T \times_W T^m \to \mathrm{Hom}\left(\mathbb{Z}^m, G\right)$ is the induced map. Finally, we have maps

$$\begin{aligned}P_m^k : G/T \times T^m &\to G/T \times T^m\\ ([g], t_1, \ldots, t_m) &\mapsto \left([g], t_1^k, \ldots, t_m^k\right).\end{aligned}$$

By direct computation it can be seen that these maps are compatible with the simplicial structure. They are also W-equivariant, that is

$$[\eta] \cdot P_m^k (g, t_1, \ldots, t_m) = P_m^k ([\eta] \cdot (g, t_1, \ldots, t_m)),$$

This is true since, $\left(\eta t \eta^{-1}\right)^k = \eta t^k \eta^{-1}$ for $t \in T$.

PROPOSITION 3.4. *If G is a compact connected Lie group such that $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected for every non negative integer m. Then for the cohomology with real coefficients we have a commutative diagram*

$$\begin{tikzcd} H^*(\mathrm{Hom}(\mathbb{Z}^m, G)) \arrow[r, "\varphi_m^*"] \arrow[d, "(\Phi_m^k)^*"'] & H^*(G/T \times T^m)^W \arrow[d, "(P_m^k)^*"] \\ H^*(\mathrm{Hom}(\mathbb{Z}^m, G)) \arrow[r, "\varphi_m^*"] & H^*(G/T \times T^m)^W . \end{tikzcd} \tag{3.2.1}$$

where the horizontal maps are isomorphisms.

PROOF. Under this setting, Theorem 3.3 of [**Baird**] is applied to conclude that we have the following natural isomorphisms

$$H^*(\mathrm{Hom}(\mathbb{Z}^m, G)) \overset{(\bar{\varphi}_m)^*}{\cong} H^*(G/T \times_W T^m) \overset{\pi^*}{\cong} H^*(G/T \times T^m)^W . \tag{3.2.2}$$

Now let us see how the power maps are related to this constructions so far. We have maps

$$\begin{aligned} P_m^k &: G/T \times T^m \to G/T \times T^m \\ &([g], t_1, \ldots, t_m) \mapsto ([g], t_1^k, \ldots, t_m^k) . \end{aligned}$$

By direct computation it can be seen that these maps are compatible with the simplicial structure. They are also W-equivariant, that is

$$[\eta] \cdot P_m^k(g, t_1, \ldots, t_m) = P_m^k([\eta] \cdot (g, t_1, \ldots, t_m)) ,$$

This is true since, $(\eta t \eta^{-1})^k = \eta t^k \eta^{-1}$ for $t \in T$. Thus, they induced a well define map $\bar{P}_m^k : G/T \times_W T^m \to G/T \times_W T^m$, and we get the following commuting diagram

$$\begin{tikzcd} H^*(G/T \times_W T^m) \arrow[r, "\pi^*"] \arrow[d, "(\bar{P}_m^k)^*"] & H^*(G/T \times T^m) \arrow[d, "P^k"] \\ H^*(G/T \times_W T^m) \arrow[r, "\pi^*"] & H^*(G/T \times T^m) . \end{tikzcd}$$

We know that the homomorphism $\pi^* : H^*(G/T \times_W T^m) \to H^*(G/T \times T^m)$ actually has image equal to $H^*(G/T \times T^m)^W$, since $H^*(G/T \times_{\mathcal{W}} T^m) \overset{\pi^*}{\cong} H^*(G/T \times T^m)^W$. Thus, we actually have the diagram

$$\begin{tikzcd} H^*(G/T \times_W T^m) \arrow[r, "\pi^*"] \arrow[d, "(\bar{P}_m^k)^*"] & H^*(G/T \times T^m)^W \arrow[d, "P^k"] \\ H^*(G/T \times_W T^m) \arrow[r, "\pi^*"] & H^*(G/T \times T_m)^W . \end{tikzcd}$$

where the horizontal maps are isomorphism. This implies that $(P_m^k)^*$ preserves W-invariance:

$$(P_m^k)^* \left(H^*(G/T \times T^m)^W \right) \subseteq H^*(G/T \times T^m)^W .$$

Also, by direct computation form the definitions and the fact that $\left(gtg^{-1}\right)^k = gt^kg^{-1}$, it follows that $\varphi_m \circ P_m^k = \Phi_m^k \circ \varphi_m$ holds. And since $\left(P_m^k\right)^*$ preserves W-invariance, we obtain the following commuting diagram

$$\begin{tikzcd} H^*\left(\mathrm{Hom}\left(\mathbb{Z}^m, G\right)\right) \arrow[r, "\varphi_m^*"] \arrow[d, "\left(\Phi_m^k\right)^*"'] & H^*\left(G/T \times T^m\right)^W \arrow[d, "\left(P_m^k\right)^*"] \\ H^*\left(\mathrm{Hom}\left(\mathbb{Z}^m, G\right)\right) \arrow[r, "\varphi_m^*"] & H^*\left(G/T \times T^m\right)^W . \end{tikzcd}$$

Here the horizontal maps are isomorphism as they can be factored by the isomorphisms

$$\bar{\varphi}_m^* : H^*\left(\mathrm{Hom}\left(\mathbb{Z}^m, G\right)\right) \to H^*\left(G/T \times_W T^m\right)$$

and

$$\pi^* : H^*\left(G/T \times_W T^m\right) \to H^*\left(G/T \times T^m\right)^W .$$

□

PROPOSITION 3.5. *Let $X_\bullet$ and $Y_\bullet$ be two simplicial spaces with a simplicial map $f : X_\bullet \to Y_\bullet$. Suppose also that there is a finite group K with an action on every level X_q compatible with the simplicial structure, such that there is an isomorphism $H^p\left(C^*\left(X_q\right)\right)^K \cong H^p\left(C^*\left(Y_q\right)\right)$ induce on every level by the maps of f. Then there is natural isomorphism*

$$\|f\|^* : H^*\left(\|Y\|\right) \to H^*\left(\|X\|\right)^K ,$$

where $\|X\|$ and $\|Y\|$ are the fat realizations.

PROOF. For this consider the bimodule

$$C^{p,q}\left(X_\bullet\right) := C^p\left(X_q\right),$$

where $C^p\left(X_q\right)$ are the p-cochains of the space n-th level of the simplicial space $X_\bullet$. Then call $C^*\left(X_\bullet\right)$ the total complex of this bimodule. Then by Theorem 5.15 of [**Dupont**], there is a natural isomorphism

$$H^*\left(\|X\|\right) \cong H^*\left(C^*\left(X_\bullet\right)\right).$$

Naturality means that if $f_\bullet : X_\bullet \to Y_\bullet$ is a simplicial map, then there is a commutative diagram

$$\begin{tikzcd} H^*\left(\|Y\|\right) \arrow[r, "\|f\|^*"] \arrow[d] & H^*\left(\|X\|\right) \arrow[d] \\ H^*\left(C^*\left(Y_\bullet\right)\right) \arrow[r, "\left(f_\bullet\right)^*"] & H^*\left(C^*\left(X_\bullet\right)\right), \end{tikzcd}$$

where $\left(f_\bullet\right)^*$ is the map induce on the total cohomology by $f_\bullet$ and $\|f\|^*$is the map induce on cohomology by the realization map $\|f\|$.

Since there are isomorphims $H^p\left(C^*\left(X_q\right)\right)^K \cong H^p\left(C^*\left(Y_q\right)\right)$ induced by the level maps of $f_\bullet$, then Theorem 1.19 of [**Dupont**], implies that

$$\left(f_\bullet\right)^* : H^*\left(C^*\left(Y_\bullet\right)\right) \to H^*\left(C^*\left(X_\bullet\right)\right)^K$$

is an isomorphims. Naturality then implies that we have a commutative diagram

$$\begin{tikzcd} H^*(\|Y\|) \arrow[r, "\|f\|^*"] \arrow[d] & H^*(\|X\|)^K \arrow[d] \\ H^*(C^*(Y_\bullet)) \arrow[r, "(f_\bullet)^*"] & H^*(C^*(X_\bullet))^K , \end{tikzcd}$$

where the vertical maps and the bottom map are isomorphism. This implies that $\|f\|^* : H^*(\|Y\|) \to H^*(\|X\|)^K$ is a natural isomorphism. ☐

PROPOSITION 3.6. *If G is a compact connected Lie group such that $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected for every non negative integer m. Then for the cohomology with real coefficients we have a commutative diagram*

(3.2.3)
$$\begin{tikzcd} H^*(B_{\mathrm{com}}G) \arrow[r, "\pi^*"] \arrow[d, "\Phi^k"'] & H^*(\|(G/T)_\bullet \times BT_\bullet\|)^W \arrow[d, "P^k"] \\ H^*(B_{\mathrm{com}}G) \arrow[r, "\pi^*"] & H^*(\|(G/T)_\bullet \times BT_\bullet\|)^W , \end{tikzcd}$$

where the horizontal maps are isomorphisms. Here we are abusing notation by using the same names for the power map and its induce map on cohomology.

PROOF. Because of Proposition 3.4 the conditions of Proposition 3.5 can be applied to conclude that $\pi^* : H^*(B_{\mathrm{com}}G) \to H^*(\|(G/T)_\bullet \times BT_\bullet\|)^W$ is an isomorphism. Then Diagram 3.2.1 implies that Diagram 3.2.3 commutes. ☐

THEOREM 3.7. *If G is a compact connected Lie group such that $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected for every non negative integer m. Then for the cohomology with real coefficients we have a commutative diagram*

$$\begin{tikzcd} H^*(B_{com}G) \arrow[r] \arrow[d, "\Phi^k"] & (H^*(BT) \otimes H^*(BT))^W /J \arrow[d, "\overline{\mathrm{Id}\otimes\psi^k}"] \\ H^*(B_{com}G) \arrow[r] & (H^*(BT) \otimes H^*(BT))^W /J, \end{tikzcd}$$

where the horizontal maps are the same isomorphism given above, and Φ^k are the power maps on cohomology.

REMARK 3.8. Adem and Gomez proved in [**AG**] that $H^*(B_{com}G)$ is isomorphic to $(H^*(BT) \otimes H^*(BT))^W /J$. Here we go through their proof to show that the previous commutative diagram also holds. This will allow us to compute the effect of the power maps.

PROOF. Here we continue the results of Proposition 3.6. The realization of the simplicial product are naturally isomorphic to the product of the realizations of each of the simplicial spaces involved (see Theorem 14.3 of [**May**]). This is, we have a natural isomorphism

$$\|(G/T)_\bullet \times BT_\bullet\| \cong \|(G/T)_\bullet\| \times BT.$$

Now recall that we have a map

$$\mathrm{Id} \times \psi^k : \|(G/T)_\bullet\| \times BT \to \|(G/T)_\bullet\| \times BT$$

where $\psi^k : BT \to BT$ is the natural power map on BT. From this we can use the naturality of the Kunneth formulas to obtain the following commutative diagram

$$\begin{tikzcd} H^*\left(\|G/T_\bullet \times BT_\bullet\|\right)^W \arrow[r, "\pi^*"] \arrow[d, "P^k"] & \left(H^*(G/T) \otimes H^*(BT)\right)^W \arrow[d, "\mathrm{Id}\otimes\psi^k"] \\ H^*\left(\|G/T_\bullet \times BT_\bullet\|\right)^W \arrow[r, "\pi^*"] & \left(H^*(G/T) \otimes H^*(BT)\right)^W . \end{tikzcd}$$

Combining this diagram Diagram 3.2.3 we obtain the following commutative diagram

$$\begin{tikzcd} H^*(B_{com}G) \arrow[r] \arrow[d, "\Phi^k"] & \left(H^*(G/T) \otimes H^*(BT)\right)^W \arrow[d, "\mathrm{Id}\otimes\psi^k"] \\ H^*(B_{com}G) \arrow[r] & \left(H^*(G/T) \otimes H^*(BT)\right)^W , \end{tikzcd} \tag{3.2.4}$$

where the horizontal maps are still isomorphisms.

In the proof of Proposition 7.1 of [**AG**] they show that there is an isomorphism

$$H^*(G/T) \cong H^*(BT)/J_0,$$

where J_0 is the ideal generated by the elements of positive degree in the image of the map induced by the inclusion $H^*(BG) \to H^*(BT)$. Now using the natural projection we have a commutative diagram

$$\begin{tikzcd} H^*(BT) \otimes H^*(BT) \arrow[r, "\pi"] \arrow[d, "\mathrm{Id}\otimes\Phi^k"] & H^*(BT)/J_0 \otimes H^*(BT) \arrow[d, "\mathrm{Id}\otimes\psi^k"] \\ H^*(BT) \otimes H^*(BT) \arrow[r, "\pi"] & H^*(BT)/J_0 \otimes H^*(BT) \end{tikzcd}$$

where the horizontal maps are of course surjective. This in turn gives us a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & H^*(BT) \otimes H^*(BT) \arrow[r, "\pi\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\Phi^k"] & H^*(BT)/J_0 \otimes H^*(BT) \arrow[r] \arrow[d, "\mathrm{Id}\otimes\psi^k"] & 0 \\ 0 \arrow[r] & I \arrow[r] & H^*(BT) \otimes H^*(BT) \arrow[r, "\pi\otimes\mathrm{Id}"] & H^*(BT)/J_0 \otimes H^*(BT) \arrow[r] & 0 \end{tikzcd}$$

where I is the kernel of the map $\pi \otimes \mathrm{Id} : H^*(BT) \otimes H^*(BT) \to H^*(BT)/J_0 \otimes H^*(BT)$ and the first vertical map is the restriction of the middle vertical map. Furthermore the exactness of the rows is preserved for W-invariance and thus, if we take $J = I^W$ we get

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "\mathrm{Id}\otimes\Phi^k|"] & \left(H^*(BT) \otimes H^*(BT)\right)^W \arrow[r, "\pi\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\Phi^k"] & \left(H^*(BT)/J_0 \otimes H^*(BT)\right)^W \arrow[r] \arrow[d, "\mathrm{Id}\otimes\psi^k"] & 0 \\ 0 \arrow[r] & J \arrow[r] & \left(H^*(BT) \otimes H^*(BT)\right)^W \arrow[r, "\pi\otimes\mathrm{Id}"] & \left(H^*(BT)/J_0 \otimes H^*(BT)\right)^W \arrow[r] & 0 \end{tikzcd}$$

which we can combine with Diagram 3.2.4 to get

$$\begin{tikzcd} H^*(B_{com}G) \arrow[r] \arrow[d, "\Phi^k"] & (H^*(BT) \otimes H^*(BT))^W / J \arrow[d, "\overline{\mathrm{Id} \otimes \psi^k}"] \\ H^*(B_{com}G) \arrow[r] & (H^*(BT) \otimes H^*(BT))^W / J \end{tikzcd}$$

where by $\overline{\mathrm{Id} \otimes \psi^k}$ we mean the induced map for the quotients, giving us the desired conclusion. □

3.2.1. Power maps on the torus: The last result is important since it tell us that in order to obtain the effect of power maps on cohomology of $B_{\mathrm{com}}G^1$, we need to understand their effect when the Lie group is a torus, $T = \left(S^1\right)^n$. We now explore this.

THEOREM 3.9. *Consider the k-th power map*

$$\psi^k : T \to T$$
$$(t_1, \ldots, t_n) \mapsto \left(t_1^k, \ldots, t_n^k\right).$$

Then by identifying $H^*(BT) \cong \mathbb{R}[x_1, \ldots, x_n]$, the induced k-th power map is characterized by

$$\mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$$
$$x_i \mapsto kx_i.$$

PROOF. On a circle the k-th power of its elements induces the multiplication by k on the fundamental group: if

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

then the k-th power map is given by

$$\eta : S^1 \to S^1$$
$$z \mapsto z^k$$

which is know to be a map of degree k. This means that if identify $\pi_1\left(S^1\right) \cong \mathbb{Z}$ then the k-th power maps induces multiplication by k on the fundamental group.

Consider the projections

$$p_i : \left(S^1\right)^n \to S^1$$
$$(z_1, \ldots, z_n) \mapsto z_i.$$

It is well known that the map $q : \pi_1\left(\left(S^1\right)^n\right) \to \pi_1\left(S^1\right)^n$ given by

$$q([\alpha]) := ([p_1 \circ \alpha], \ldots, [p_n \circ \alpha])$$

[1] When $\mathrm{Hom}(\mathbb{Z}^m, G)$ is path connected for every m.

is an isomoprihsm, since S^1 is path connected. Since the power map $\psi^k : T \to T$ considers the k-th power component wise, it follows that we have a commutative diagram

$$\begin{tikzcd} \pi_1(T) \arrow[r, "q"] \arrow[d, "\psi_*^k"] & \pi_1\left(S^1\right)^n \arrow[d, "\prod \eta_*"] \\ \pi_1(T) \arrow[r, "q"] & \pi_1\left(S^1\right)^n . \end{tikzcd}$$

Since the horizontal maps are isomorphisms, this implies that

$$\begin{aligned} \psi_*^k : \pi_1(T) &\to \pi_1(T) \\ \alpha &\mapsto k\alpha \end{aligned}$$

where we see $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $k\alpha = (k\alpha_1, \ldots, k\alpha_n)$.

Now let us consider the fiber sequence of the classifying space of the torus

$$T \to ET \to BT.$$

This induces a exact sequence on homotopy

$$\cdots \pi_m(ET) \to \pi_m(BT) \to \pi_{m-1}(T) \to \cdots \pi_1(ET) \to \pi_1(BT) \to$$
$$\to \pi_0(T) \to \pi_0(ET) \to \pi_0(BT) \to 0$$

but since ET in null homotopic, we get an isomorphism $\pi_m(BT) \to \pi_{m-1}(T)$. In particular we get

$$\pi_m(BT) = \begin{cases} \mathbb{Z}^n & m = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since the exact sequence is natural, we get that the power map on BT induces the multiplication by k on the second homotopy group. Furthermore since BT is simply connected, by Hurewicz´s theorem we get that $H_2(BT, \mathbb{Z}) \cong \pi_2(BT)$, and once again because of naturality the effect on the second homology is multiplication by k.

We now apply the universal coefficients theorem to get that

$$H^2(BT) \cong \operatorname{Hom}(H_2(BT, \mathbb{Z}), \mathbb{R}) \cong \mathbb{R}^n.$$

Naturality allow us to conclude that the effect of the k-th power map is once again multiplication by k. Finally it is known that the real cohomology of BT is the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ where $x_i \in H^2(BT)$ for $1 \leq i \leq n$ (see [**Dupont**], Proposition 8.11). Since we know that the effect of the k power map is multiplication by k on the x_i, this determines the effect on the whole cohomology ring. □

As corollary of Theorem 3.7 and Theorem 3.9 we obtain the following:

THEOREM 3.10. *By identifying the real cohomology ring of an n-Torus with $\mathbb{R}[x_1, \ldots, x_n]$, we get that if G is a Lie group such that* $\operatorname{Hom}(\mathbb{Z}^m, G) = \operatorname{Hom}(\mathbb{Z}^m, G)_1$ *for every m, then*

$$H^*(B_{com}G) \cong (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^W / J.$$

Where J is the ideal generated by the invariant polynomials of positive degree on the x_i under the action of the Weyl group, W. Further, the power maps $\Phi^k : H^*(B_{com}G, F) \to H^*(B_{com}G, F)$ are induced by the homomorphism characterized by sending $x_i \mapsto x_i$ and $y_i \mapsto ky_i$ for every $1 \leq i \leq n$.

Next we are going to examine particular cases of Lie groups satisfying the stated property. From this we see that possible differences between the different cases depend entirely on the Weyl group and its action on the cohomology of BT.

3.3. Generators of $H^*(B_{com}G,\mathbb{R})$ for $G=U(n), \mathrm{Sp}(n)$ and $SU(n)$

If $B_{\mathrm{com}}G_1 = B_{\mathrm{com}}G$ we know that

$$H^*(B_{com}G,\mathbb{R}) \cong \left(\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n]\right)^W/J.$$

While in general it is known that for a compact and connected Lie group G

$$H^*(BG,\mathbb{R}) \cong \mathbb{R}[z_1,\dots,z_n]^W,$$

where W is its Weil group. It action is induced by adjunction. That is, if $\mathfrak{g}$ is the Lie algebra of G, $\mathbb{R}[z_1,\dots,z_n]$ can be identified as the polynomial algebra of $\mathfrak{g}$. An element $[n]\in W\cong N_G(T)/T$ has a well defined action given by adjunction, $\mathrm{ad}(n):\mathfrak{g}\to\mathfrak{g}$. This in turn induces an action of W on $\mathbb{R}[z_1,\dots,z_n]$.

There is a natural inclusion $B_{\mathrm{com}}G\hookrightarrow BG$, inducing a map

$$\iota: H^*(BG,\mathbb{R})\to H^*(B_{com}G,\mathbb{R}).$$

In terms of the previous identifications, ι is induced by the homomorphism ([**Gritschacher**], Corollary A.2.)

$$\begin{aligned}\mathbb{R}[z_1,\dots,z_n]&\to\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n]\\ z_i&\mapsto x_i+y_i.\end{aligned}$$

Additionally we saw in the previous section that the power maps, $\Phi^k: H^*(B_{com}G,\mathbb{R})\to H^*(B_{com}G,\mathbb{R})$ are induced by the map characterized by sending $x_i\mapsto x_i$ and $y_i\mapsto ky_i$ for every $1\le i\le n$.

Definition 3.11. We call the subalgebra generated by $\left\{\Phi^k(\mathrm{Im}\iota)\mid k\in\mathbb{Z}\setminus\{0\}\right\}\subset H^*(B_{com}G,\mathbb{R})$ by

$$\mathcal{S} := \left\langle \Phi^k(\mathrm{Im}\iota)\mid k\in\mathbb{Z}\setminus\{0\}\right\rangle.$$

On this section we use the previous maps to see that if $G=U(n),\mathrm{Sp}(n)$ and $SU(n)$ then $\mathcal{S}$ is all of $H^*(B_{\mathrm{com}}G,\mathbb{R})$. We do this by dealing with the explicit descriptions of their actions and the specific Weyl groups on each case.

Before dealing with each individual case, it is worth proving

Lemma 3.12. *The subalgebra $\mathcal{S}$ is closed under the power maps.*

Proof. This is true since Φ^k is a $\mathbb{R}$-homomorphism of algebras, and also $\Phi^k\circ\Phi^l=\Phi^{kl}$. This last statement comes from

$$\Phi^k\circ\Phi^l(x_i+y_i)=\Phi^k(x_i+ly_i)=x_i+kly_i=\Phi^{kl}(x_i+y_i).$$

This implies that for $q_j\in\mathbb{R}[z_1,\dots,z_n]$, $\alpha_j\in\mathbb{R}$

$$\Phi^k\left(\sum_{l=1}^{s}\alpha_j\Phi^{k_j}\circ\iota(q_j)\right)=\sum_{l=1}^{s}\alpha_j\Phi^{kk_j}\circ\iota(q_j)\in\mathcal{S}.$$

$\square$

3.3.1. Generators of $H^*(B_{com}U(n),\mathbb{R})$: For this case recall that the Weyl group of $U(n)$ is isomorphic to the symmetric group S_n. By the previous section we know that

$$H^*(B_{com}U(n),\mathbb{R}) = (\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^{S_n}/J$$

where S_n acts diagonally on the tensor product, permuting the variables of each factor. J is the ideal generated by the symmetric polynomials of positive degree on the x_i. It is also known that

$$H^*(BU(n),\mathbb{R}) = (\mathbb{R}[x_1,\ldots,x_n])^{S_n},$$

where the action is once again by permuting variables. $H^*(BU(n),\mathbb{R})$ is generated by the power polynomials

$$p_m := z_1^m + z_2^m + \cdots + z_n^m,$$

which are clearly invariant under the action of S_n. These polynomials have their counterparts on two variables polynomials in the form of

$$P_{a,b}(n) := x_1^a y_1^b + x_2^a y_2^b + \cdots + x_n^a y_n^b,$$

where $1 \le a+b \le n$. These generate the algebra $(\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^{S_n}$ (See **[Vaccarino]**, Theorem 1). Thus to prove that $\mathcal{S}$ is all of $H^*(B_{\mathrm{com}}U(n),\mathbb{R})$ it is enough to see that the multisymmetric polynomials (modulo J) are in fact in $\mathcal{S}$. To see it, we first need a couple of lemmas.

LEMMA 3.13. *For every $n\in\mathbb{N}$ and $1\le a+b\le n$ with $a,b\ge 0$ we have $\Phi^k(P_{a,b}(n)) = k^b P_{a,b}(n)$.*

PROOF. Since Φ^k is a homomorphism of algebras, we have

$$\Phi^k(P_{a,b}(n)) = \Phi^k\left(x_1^a y_1^b + x_2^a y_2^b + \cdots + x_n^a y_n^b\right) = \sum_{i=1}^n \Phi^k\left(x_i^a y_i^b\right).$$

But we have

$$\Phi^k\left(x_i^a y_i^b\right) = \Phi^k(x_i)^a\,\Phi^k(y_i)^b = k^b x_i^a y_i^b.$$

Where the last equality is true since we already saw that $\Phi^k(x_i) = x_i$ and $\Phi^k(x_i) = k y_i$ for every $1\le i\le n$. $\square$

To prove the goal of this subsection, we illustrate explicitly the cases $n=2$ and $n=3$.

- Suppose first that $n=2$.
  We want to show that the following multisymmetric polynomials are indeed in $\mathcal{S}$
  - $P_{0,1}(2) = y_1 + y_2$,
  - $P_{1,1}(2) = x_1y_1 + x_2y_2$ and
  - $P_{0,2}(2) = y_1^2 + y_2^2$.

  We ignore $P_{1,0}(2) = x_1 + x_2$ since this is zero modulo J. For this first observe that

  $$\iota(z_1+z_2) = (x_1+y_1)+(x_2+y_2) = (x_1+x_2)+(y_1+y_2) = P_{1,0}(2)+P_{0,1}(2)$$

  clearly belongs to $\mathcal{S}$. Since $P_{1,0}(2) = 0 \bmod J$ we are done. For $P_{1,1}(2)$ and $P_{0,2}(2)$ notice that the total degree (the sum of the power of each term) is 2, thus we have to consider $\iota(p_2)$:

  $$\begin{aligned}\iota\left(z_1^2+z_2^2\right) &= (x_1+y_1)^2 + (x_2+y_2)^2\\ &= \left(x_1^2+x_2^2\right) + 2(x_1y_1+x_2y_2) + \left(y_1^2+y_2^2\right).\end{aligned}$$

  This can be rewritten as

  $$\iota\left(z_1^2+z_2^2\right) = P_{1,0}(2) + 2P_{1,1}(2) + P_{0,2}(2).$$

Then we consider

$$\Phi^{-1}\left(\iota\left(z_1^2+z_2^2\right)\right) = \left(x_1^2+x_2^2\right) - 2\left(x_1y_1+x_2y_2\right) + \left(y_1^2+y_2^2\right)$$

giving us that

$$\iota\left(z_1^2+z_2^2\right) + \Phi^{-1}\left(\iota\left(z_1^2+z_2^2\right)\right) = 2\left(y_1^2+y_2^2\right) \bmod J$$

meaning that $P_{0,2}(2) \in \mathcal{S}$, since $\mathcal{S}$ is a subalgebra closed under power maps. Finally modulo J we get

$$P_{1,1}(2) = \frac{\iota\left(z_1^2+z_2^2\right) - P_{0,2}(2)}{2} \in \mathcal{S}$$

which finishes the proof for $n = 2$.

- Suppose now that $n = 3$.

  The arguments used in the case $n = 2$ can be used to obtained the first two of the next equalities, where once again they are taken to be modulo J:
  (1) $P_{0,1}(3) = \iota(z_1+z_2+z_3)$.
  (2) $P_{0,2}(3) = \frac{1}{2}\left(\iota\left(z_1^2+z_2^2+z_3^2\right) + \Phi^{-1}\left(\iota\left(z_1^2+z_2^2+z_3^2\right)\right)\right)$.
  (3) $P_{1,1}(3) = \frac{1}{2}\left(\iota\left(z_1^2+z_2^2+z_3^2\right) - P_{0,2}\right)$.
  We are left to obtain $P_{a,b}(3)$ such that $a+b=3$. For this we can reorder to see that

$$\begin{aligned}\iota\left(z_1^3+z_2^3+z_3^3\right) &= (x_1+y_1)^3 + (x_2+y_2)^3 + (x_3+y_3)^3 \\ &= \left(x_1^3+x_2^3+x_3^3\right) + 3\left(x_1^2y_1+x_2^2y_2+x_3^2y_3\right) \\ &\quad + 3\left(x_1y_1^2+x_2y_2^2+x_3y_3^2\right) + \left(y_1^3+y_2^3+y_3^3\right)\end{aligned}$$

  which amounts to

$$\iota\left(z_1^3+z_2^3+z_3^3\right) = 3P_{2,1} + 3P_{1,2} + P_{0,3} \bmod J.$$

  We use the power maps to get that

$$\Phi^{-1}\left(\iota\left(z_1^3+z_2^3+z_3^3\right)\right) = -3P_{2,1} + 3P_{1,2} - P_{0,3} \bmod J.$$

  By adding the last two equalities we get

$$P_{1,2} \bmod J = \frac{1}{6}\left(\Phi^{-1}\left(\iota\left(z_1^3+z_2^3+z_3^3\right)\right) + \iota\left(z_1^3+z_2^3+z_3^3\right)\right) \in \mathcal{S}.$$

  Thus we have $\iota\left(z_1^3+z_2^3+z_3^3\right) - 3P_{1,2} \bmod J \in \mathcal{S}$, and by closure under power maps we obtain

$$8P_{0,3} \bmod J = \Phi^2\left(\iota\left(z_1^3+z_2^3+z_3^3\right) - 3P_{1,2}\right) - 6\left(\iota\left(z_1^3+z_2^3+z_3^3\right) - 3P_{1,2}\right) \in \mathcal{S}$$

  from we conclude that $P_{0,3} \bmod J \in \mathcal{S}$. We finally have

$$P_{2,1} = \frac{1}{3}\left(\iota\left(z_1^3+z_2^3+z_3^3\right) - 3P_{1,2} - P_{0,3}\right) \bmod J$$

  which finishes the case $n = 3$.

In the previous two examples we see that for non negative numbers a and b, we proved that $P_{a,b}(n)$ belongs to $\mathcal{S}$ using induction on the value $a+b$. This was done in such a way that the induction process did not depend on n. These arguments can be generalized more methodically to obtain.

THEOREM 3.14. *The algebra $H^*(B_{com}U(n);\mathbb{R})$ is equal to the subalgebra*

$$\mathcal{S} := \left\langle \Phi^k(\mathrm{Im}\iota) \mid k \in \mathbb{Z}\setminus\{0\}\right\rangle.$$

PROOF. For this proof we will be working modulo J. Also, for an arbitrary n consider a fixed $m \in \{1,2,\ldots,n\}$. Now take

$$p_m := z_1^m + z_2^m + \cdots + z_n^m.$$

An easy reordering gives us

$$\begin{aligned}\iota(p_m) &= \left(\sum_{i=1}^{n}(x_i+y_i)^m\right) = \sum_{i=1}^{n}\sum_{j=0}^{m}\binom{m}{j}x_i^{m-j}y_i^j \\ &= \sum_{j=0}^{m}\binom{m}{j}P_{m-j,j}(n) = \sum_{j=1}^{m}\binom{m}{j}P_{m-j,j}(n),\end{aligned} \tag{3.3.1}$$

where the last equality holds because we are working modulo J. From this point we will use the power maps Φ^k to obtain the various $P_{m-j},_j(n)$. First we use the following recursion to get first $P_{0,m}(n)$ from 3.3.1: Let $A_0 := \iota(p_m)$,

$$A_1 := \Phi^2(A_0) - 2A_0 = \sum_{j=2}^{m}\left(2^j-2\right)\binom{m}{j}P_{m-j,j}(n)$$

and

$$A_2 := \Phi^3(A_1) - 3^2A_1 = \sum_{j=3}^{m}\left(2^j-2\right)\left(3^j-3^2\right)\binom{m}{j}P_{m-j,j}(n).$$

In general for $1 \le k \le m-1$ we define

$$A_k := \Phi^{k+1}(A_{k-1}) - (k+1)^k A_{k-1}.$$

Notice that every A_k has non zero coefficients only for $P_{m-j,j}(n)$ for $k+1 \le j \le m$. Since $A_0 \in \mathcal{S}$ by definition and every A_k is defined in terms of the power maps and A_{k-1}, induction implies that $A_k \in \mathcal{S}$ for every $1 \le k \le m-1$. Some easy calculations allow us to obtain that

$$P_{0,m}(n) = \left(\prod_{k=2}^{m}\left(k^m - k^{k-1}\right)\right)^{-1} A_{m-1} \in \mathcal{S}.$$

And thus we obtain that

$$\iota(p_m) - P_{0,m}(n) = \sum_{j=1}^{m-1}\binom{m}{j}P_{m-j,j}(n) \in \mathcal{S}.$$

Then we can apply a new recursion to conclude that $P_{1,m-1}(n) \in \mathcal{S}$. By continuing with this backwards recursion we obtain that $P_{a,b}(n) \in \mathcal{S}$ for all positive a,b such that $a+b=m$. Since we picked $m \in \{1,2,\ldots,n\}$ arbitrarily, this finishes the proof. ☐

3.3.2. Generators of $H^*(B_{com}SU(n);\mathbb{R})$: To obtain that

$$H^*(B_{com}SU(n),\mathbb{R}) = \left\langle \Phi^k(\mathrm{Im}\iota) \mid k \in \mathbb{Z}\setminus\{0\}\right\rangle,$$

we use a different presentation of $H^*(BT,\mathbb{R})$. A maximal torus of $SU(n)$ is the set of diagonal matrices with entries in $S^1 \subseteq \mathbb{C}$, such that their product equals one. Under such presentation it is routine to show that

$$H^*(BT,\mathbb{R}) \cong (\mathbb{R}[z_1,\dots,z_n]/\langle z_1+\cdots+z_n\rangle)$$

where the Weyl group is then S_n acting by permutation. This implies that

$$H^*(BSU(n),\mathbb{R}) \cong (\mathbb{R}[z_1,\dots,z_n]/\langle z_1+\cdots+z_n\rangle)^{S_n},$$

but since $p_1 := z_1+\cdots+z_n$ is already invariant, the previous ring is isomorphic to

$$H^*(BSU(n),\mathbb{R}) \cong \mathbb{R}[z_1,\dots,z_n]^{S_n}/\langle z_1+\cdots+z_n\rangle.$$

Since $\mathbb{R}[z_1,\dots,z_n]^{S_n}$ is itself a polynomial algebra on $p_i = z_1^i+\cdots+z_n^i$, (see **[Humphrey]**, Chapter 3.5: Chevalley's Theorem), we finally get that

$$H^*(BSU(n),\mathbb{R}) \cong \mathbb{R}[p_1,\dots,p_n]/\langle p_1\rangle \cong \mathbb{R}[p_2,\dots,p_n].$$

We will use this to conclude the following

THEOREM 3.15. *The real cohomology of $B_{\mathrm{com}}SU(n)$ can be given by*

$$H^*(B_{com}SU(n),\mathbb{R}) \cong (\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n])^{S_n}/\tilde{J},$$

where $\tilde{J}$ is the ideal generated by $x_1^i+\cdots+x_n^i$, $1\le i\le n$ and $y_1^1+\cdots+y_n^1$.

PROOF. We saw in Theorem 3.7 that

$$H^*(B_{com}SU(n),\mathbb{R}) \cong (H^*(BT)\otimes H^*(BT))^{S_n}/J,$$

where J is the ideal generated by the S_n-invariants on the first component. The previous reasoning then gives us

$$H^*(B_{com}SU(n),\mathbb{R}) \cong (\mathbb{R}[x_1,\dots,x_n]/\langle x_1+\cdots+x_n\rangle\otimes\mathbb{R}[y_1,\dots,y_n]/\langle y_1+\cdots+y_n\rangle)^{S_n}/J.$$

Notice that this is well defined since the S_n-invariance of $x_1+\cdots+x_2$ and $y_1+\cdots+y_2$ allow us to have a well define action of S_n on

$$R := \mathbb{R}[x_1,\dots,x_n]/\langle x_1+\cdots+x_n\rangle\otimes\mathbb{R}[y_1,\dots,y_n]/\langle y_1+\cdots+y_n\rangle.$$

Consider first the map

$$p : \mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n]\to R,$$

which is induced by the projection

$$\mathbb{R}[x_1,\dots,x_n]\times\mathbb{R}[y_1,\dots,y_n]\to\mathbb{R}[x_1,\dots,x_n]/\langle x_1+\cdots+x_n\rangle\times\mathbb{R}[y_1,\dots,y_n]/\langle y_1+\cdots+y_n\rangle.$$

The map p is naturally S_n-equivariant, thus it induces a map

$$\tilde{p} : (\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n])^{S_n}\to R^{S_n}.$$

Also, since p is surjective, and the action is diagonal, we have that $\tilde{p}$ is also onto. We can further consider the composition with the quotient by J to obtain a surjective map

$$q : (\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n])^{S_n}\to(R)^{S_n}/J.$$

It is easy to see that the kernel of this map is what we called $\tilde{J}$, so the result follows. $\square$

Even further, since the map

$$\begin{aligned}\mathbb{R}[z_1,\ldots,z_n] &\to \mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n]\\ z_i &\mapsto x_i+y_i.\end{aligned}$$

induces the map $\iota : H^*(BSU(n),\mathbb{R}) \to H^*(B_{com}SU(n),\mathbb{R})$, we still have the same characterization under the identifications given above. That is, ι can be seen as the map

$$\iota : (\mathbb{R}[z_1,\ldots,z_n]/\langle z_1+\cdots+z_n\rangle)^{S_n} \to (\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^{S_n}/\tilde{J}$$

induce by $z_i \mapsto x_i+y_i$. The k-th power maps on $(\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^{S_n}/\tilde{J}$ is also still induce by the assignment $x_i\mapsto x_i$ and $y_i \mapsto ky_i$. Thus, with slight changes we can still apply the arguments given in the proof of Theorem 3.14, to obtain the main result.

THEOREM 3.16. *The algebra $H^*(B_{com}SU(n);\mathbb{R})$ is equal to the subalgebra*

$$\mathcal{S} := \left\langle \Phi^k(\mathrm{Im}\iota) \mid k\in\mathbb{Z}\setminus\{0\}\right\rangle,$$

where Φ^k is the k-th power map.

3.3.3. Generators of $H^*(B_{com}\mathrm{Sp}(n);\mathbb{R})$: In this section $\mathbb{Z}_2$ will denote the multiplicative group $\{-1,1\}$.

The Weyl group, W, of the simplectic group $\mathrm{Sp}(n)$ is isomorphic to the semidirect product $\mathbb{Z}_2^n \rtimes S_n$, where $\sigma\in S_n$ acts on $(a_1,\ldots,a_n)\in\mathbb{Z}_2^n$ by

$$\sigma\cdot(a_1,\ldots,a_n) = \left(a_{\sigma(1)},\ldots,a_{\sigma(n)}\right).$$

Under these identifications, if $f\in\mathbb{R}[x_1,\ldots,x_n]\cong H^*(T)$ and $g=((a_1,\ldots,a_n),\sigma)\in\mathbb{Z}_2^n\rtimes S_n$ we have

$$g\cdot f(x_1,\ldots,x_n) = f\left(a_1x_{\sigma(1)},\ldots,a_nx_{\sigma(n)}\right).$$

Recall that

$$H^*(B_{com}\mathrm{Sp}(n);\mathbb{R}) \cong (\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^W/J$$

where W acts diagonally: for $n\in W$ and $p(x)\otimes q(y)\in\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n]$ we have

$$n\cdot(p(x)\otimes q(y)) := (n\cdot p(x))\otimes(n\cdot q(y)).$$

J is the ideal generated by the symmetric polynomials on the variables x_i^2. For brevity, let us call $R := (\mathbb{R}[x_1,\ldots,x_n]\otimes\mathbb{R}[y_1,\ldots,y_n])^W$ the **signed multisymmetric polynomials**.

Once again we want to see that $\mathcal{S} := \left\langle\Phi^k(\mathrm{Im}\iota)\mid k\in\mathbb{Z}\setminus\{0\}\right\rangle$ is equal to all of $H^*(B_{com}\mathrm{Sp}(n);\mathbb{R})$. For this let us see first that the set

$$\{P_{a,b}(n) \mid a,b\geq 0 \text{ and } a+b\in 2\mathbb{Z}\}$$

generates all of the signed multisymmetric polynomials as an algebra. This will allow us to use the same arguments used in the case of $U(n)$ to obtain that $\mathcal{S} = H^*(B_{com}\mathrm{Sp}(n);\mathbb{R})$. We need the following lemmas, where the first has a straightforward proof.

LEMMA 3.17. *Let $\mu : \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n] \to R$ be the operator defined as*

$$\mu(f) = \frac{1}{|W|} \sum_{g \in W} g \cdot f.$$

*This is a well defined $\mathbb{R}$-linear map, where $|W|$ is the cardinality of the Weyl group. We call this operator the **symmetrization operator**.*

LEMMA 3.18. *If $f \in R$ and $h \in \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$, then $\mu(f) = f$ and $\mu(fh) = f \cdot \mu(h)$.*

PROOF. Since f is invariant, we have that $g \cdot f = f$ for all $g \in W$, thus

$$\mu(f) = \frac{1}{|W|} \sum_{g \in W} g \cdot f = \frac{|W|}{|W|} f = f.$$

Also, since by definition $g \cdot (fh) = (g \cdot f)(g \cdot h)$ for every $g \in W$ and $f, h \in \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$. In particular if f is invariant it follows that

$$\mu(fh) = \frac{1}{|W|} \sum_{g \in W} g \cdot (fh) = \frac{f}{|W|} \sum_{g \in W} g \cdot h = f \cdot \mu(h).$$

□

In order to prove our objective we need to analyze the summands (or monomials) of a signed multisymmetric polynomials first. For this consider sets of indices $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbb{N}^n$ (including zero as a natural number) and let us denote

$$x^I y^J := x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n}.$$

DEFINITION 3.19. We say a pair of multi indices $(I, J) \in \mathbb{N}^n \times \mathbb{N}^n$ is **odd** if there if $1 \leq k \leq n$ such that $i_k + j_k$ is odd. Such a pair is **even** if it is not odd.

LEMMA 3.20. *If a pair of multi indices (I, J) is odd, then $\mu\left(x^I y^J\right) = 0$.*

PROOF. Let $(I, J) = ((i_1, \ldots, i_n), (j_1, \ldots, j_n))$ and let's assume $i_k + j_k$ is odd. Let

$$h_k := \left(\left(1, \ldots, 1, \underbrace{-1}_{k-\text{position}}, 1, \ldots, 1\right), e\right) \in W,$$

where e is the identity permutation. Denote by $H \subseteq W$ the subgroup generated by h_k and the partition by right cosets $\{Hg_1, \ldots, Hg_m\}$ of W. Since h_k has order 2

$$W = \{g_1, \ldots, g_m\} \cup \{h_k g_1, \ldots, h_k g_m\}$$

and thus

$$\mu\left(x^I y^J\right) = \frac{1}{|W|} \sum_{l=1}^{m} \left(g_l x^I y^J + h_k \left(g_l x^I y^J\right)\right).$$

Notice that in general if $g = ((a_1, \ldots, a_n), \sigma)$, then since $i_k + j_k$ is odd we get

$$\begin{aligned}
h_k\left(g \cdot x^I y^J\right) &= h_k\left(a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n}\right) \\
&= (-1)^{i_k+j_k} a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n} \\
&= -a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n} \\
&= -g \cdot x^I y^J.
\end{aligned}$$

This implies that

$$\mu\left(x^I y^J\right) = \frac{1}{|W|} \sum_{l=1}^{m} \left(g_l x^I y^J - g_l x^I y^J\right) = 0.$$

□

THEOREM 3.21. *If a polynomial is signed multisymmetric then its monomials have all even multi indices.*

PROOF. An element $f \in \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$ can be uniquely written as

$$f = c_0 + \sum_{k=1}^{m} c_k x^{I_k} y^{J_k}.$$

Where $c_0 \in \mathbb{R}$ and for $k > 0$, $c_k \in \mathbb{R} \setminus \{0\}$, I_k and J_k are multi indices of n variables, not all of them zero. If f is signed multisymmetric,

$$f = \mu(f) = c_0 + \sum_{k=1}^{m} c_k \mu\left(x^{I_k} y^{J_k}\right).$$

These two last expressions for f imply that

$$\sum_{k=1}^{m} c_k x^{I_k} y^{J_k} = \sum_{k=1}^{m} c_k \mu\left(x^{I_k} y^{J_k}\right). \tag{3.3.2}$$

But by the previous lemma, we know that if (I_t, J_t) is odd for a given t, then $\mu\left(x^{I_t} y^{J_t}\right) = 0$. Since $\mu\left(x^{I_k} y^{J_k}\right)$ is itself a sum of monomials, the expression

$$\sum_{k=1}^{m} c_k \mu\left(x^{I_k} y^{J_k}\right)$$

must have only monomials with an even set of multi indices. Since all the coefficients in

$$\sum_{k=1}^{m} c_k x^{I_k} y^{J_k}$$

are non zero, the last equality and the uniqueness of the expression for non zero coefficients of a polynomial, allow us to conclude that (I_k, J_k) is even for every $1 \leq k \leq m$. □

In particular this proof allows us to obtain

COROLLARY 3.22. *Every signed multisymmetric polynomial can be written in the form*

$$f = c_0 + \sum_{k=1}^{m} c_k \mu\left(x^{I_k} y^{J_k}\right),$$

where (I_k, J_k) *is even for every* $1 \leq k \leq m$.

If a multisymmetric polynomial has monomials with even multi indices, such polynomial is signed symmetric, meaning that is invariant under the action of elements of the form $((a_1, \ldots, a_n), e) \in W$. In particular we can now conclude:

THEOREM 3.23. *A multisymmetric polynomial is signed symmetric if only if all its monomials have even multi indices.*

This result grant us the frame work to obtain generators for the algebra $H^*(B_{com}\mathrm{Sp}(n); \mathbb{R})$. Recall that multisymmetric are generated by the power polynomials

$$P_{a,b} := \sum_{i=1}^{n} x_i^a y_i^b.$$

On the other hand, due to the last result we know $P_{a,b}$ is signed multi symmetric if and only if $a+b$ is even. Let's see that they in fact generate all of the signed multisymmetric polynomials.

THEOREM 3.24. $(\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{\mathbb{Z}_2^n \rtimes S_n}$ *is generated as an algebra by the set*

$$\mathcal{G} := \left\{ P_{a,b} := \sum_{i=1}^{n} x_i^a y_i^b \mid 0 \leq a, b \text{ and } a+b \in 2\mathbb{Z} \right\}.$$

PROOF. By Corollary 3.22 is enough to show that for even multi indices (I, J), $\mu\left(x^I y^J\right) \in \mathrm{gen}\mathcal{G}$. To see this, note that any permutation of the set of indices have the same symmetrization. This is, for $k_1, \ldots, k_p \in \{1, \ldots, n\}$ all mutually different, $p \leq n$, we have

$$\mu\left(x_{k_1}^{i_1} \cdots x_{k_p}^{i_p} y_{k_1}^{j_1} \cdots y_{k_p}^{j_p}\right) = \mu\left(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}\right).$$

So it is enough to show that

$$\mu\left(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}\right) \in \mathrm{gen}\mathcal{G},$$

where of course $i_k + j_k$ is even for every $1 \leq k \leq p$. We do it using induction on p. The cases $p = 1$ is immediate, since in this case $\mu\left(x^I y^J\right)$ is a scalar multiple of even power polynomials of the form $P_{a,0}, P_{0,b}$ or $P_{a,b}$.

Next, assume we know $\mu\left(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}\right) \in \mathrm{gen}\mathcal{G}$ for $1 \leq p \leq k$. By reordering we have

$$\begin{aligned}\mu\left(x_1^{i_1} y_1^{j_1}\right) \mu\left(x_2^{i_2} \cdots x_{k+1}^{i_{k+1}} y_2^{j_2} \cdots y_{k+1}^{j_{k+1}}\right) &= c\mu\left(x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} y_1^{j_1} \cdots y_{k+1}^{j_{k+1}}\right) \\ &\quad + \sum_{r=2}^{k+1} c_r \mu\left(x_2^{i_2} \cdots x_r^{i_r + i_1} \cdots x_{k+1}^{i_{k+1}} y_2^{j_2} \cdots y_r^{j_r + j_1} \cdots y_{k+1}^{j_{k+1}}\right)\end{aligned}$$

where $c_2,\dots,c_{k+1}$ are integers, and c is a non zero integer. This implies that

$$\begin{aligned}\mu\left(x_1^{i_1}\cdots x_{k+1}^{i_{k+1}}y_1^{j_1}\cdots y_{k+1}^{j_{k+1}}\right)&=\frac{1}{c}\mu\left(x_1^{i_1}y_1^{j_1}\right)\mu\left(x_2^{i_2}\cdots x_k^{i_k}y_2^{j_2}\cdots y_k^{j_k}\right)\\&\quad-\frac{1}{c}\sum_{r=2}^{k+1}c_r\mu\left(x_2^{i_2}\cdots x_r^{i_r+i_1}\cdots x_k^{i_k}y_2^{j_2}\cdots y_r^{j_r+j_1}\cdots y_k^{j_k}\right).\end{aligned}$$

By the induction hypothesis all of the terms in the right are in $\mathrm{gen}\mathcal{G}$, which implies that

$$\mu\left(x_1^{i_1}\cdots x_{k+1}^{i_{k+1}}y_1^{j_1}\cdots y_{k+1}^{j_{k+1}}\right)$$

belongs to $\mathrm{gen}\mathcal{G}$. □

With the last result at hand we can imitate the reasoning in the proof of Theorem 3.14 to obtain the main result of this part.

THEOREM 3.25. *The algebra $H^*(B_{com}\mathrm{Sp}(n);\mathbb{R})$ is equal to the subalgebra*

$$\mathcal{S}:=\left\langle\Phi^k(\mathrm{Im}\iota)\mid k\in\mathbb{Z}\setminus\{0\}\right\rangle.$$

Where Φ^k are the power maps and $\iota: H^*(B\mathrm{Sp}(n);\mathbb{R})\to H^*(B_{com}\mathrm{Sp}(n);\mathbb{R})$ is the map induced by the homomorphism

$$\begin{aligned}\mathbb{R}[z_1,\dots,z_n]&\to\mathbb{R}[x_1,\dots,x_n]\otimes\mathbb{R}[y_1,\dots,y_n]\\z_i&\mapsto x_i+y_i.\end{aligned}$$

PROOF. Take once again

$$p^m=z_1^m+z_2^m+\cdots+z_n^m\in\mathbb{R}[z_1,\dots,z_n]$$

for m even. We also work modulo J, the ideal generated by the x_i^2. Recall that

$$\iota(p^m)=\sum_{j=1}^m\binom{m}{j}P_{m-j,j}(n).$$

Since $(m-j)+j=m$, all of the power polynomials $P_{m-j,j}(n)$ are even. Now we use recursion to get first $P_{0,m}(n)$ from the last equality: for this we name $A_0:=\iota(p_m)$, then we take

$$A_1:=\Phi^2(A_0)-2A_0=\sum_{j=2}^m\left(2^j-2\right)\binom{m}{j}P_{m-j,j}(n)$$

and

$$A_2:=\Phi^3(A_1)-3^2A_1=\sum_{j=3}^m\left(2^j-2\right)\left(3^j-3^2\right)\binom{m}{j}P_{m-j,j}(n).$$

In general for $1\le k\le m-1$ we define

$$A_k:=\Phi^{k+1}(A_{k-1})-(k+1)^kA_{k-1}.$$

Notice that every A_k has non zero coefficients only for $P_{m-j,j}(n)$ for $k+1 \leq j \leq m$. Since $A_0 \in \mathcal{S}$ by definition and every A_k is defined in terms of the power maps and A_{k-1}, induction implies that $A_k \in \mathcal{S}$ for every $1 \leq k \leq m-1$. In particular we have

$$P_{0,m}(n) = \left(\prod_{k=2}^{m}\left(k^m - k^{k-1}\right)\right)^{-1} A_{m-1} \in \mathcal{S}.$$

We now can apply a similar procedure to

$$\iota(p_m) - P_{0,m}(n) = \sum_{j=1}^{m-1}\binom{m}{j} P_{m-j,j}(n) \in \mathcal{S}$$

to conclude that if $m = 2k$, and $P_{a,b}$ is such that $a+b=m$ then $P_{a,b} \in \mathcal{S}$. □

CHAPTER 4

Chern-Weil theory for TC structures

In this chapter we achive the main objective of this thesis, the development of characteristic classes for TC structures on principal bundles. Our central goal is to obtain characteristic classes for TC structures using Chern-Weil theory. Specifically, we will develop this for TC structures over vector bundles whose structural group is either $U(n)$ or $SU(n)$.

The structure of this chapter is as follows: we start with a review of the theory of characteristic classes for vector bundles, including both the homotopy and geometrical points of view, and see how they relate to each other. Then we define the characteristic classes for TC structrues, develop first the homotopic construction and then the geometrical one.

In this chapter we write $H^*(-)$ to denote the cohomology with real coefficients.

4.1. Characteristic Classes for Vector Bundles

In this section we review the basic theory of characteristic classes for vector bundles. Consider a manifold M. Let $\mathrm{Vect}_n(M)$ be the set of equivalence classes of complex vector bundles of dimension n over M. This defines a contravariant functor from the category of manifolds to the category of sets. Also, $H^*(-)$ is a functor from the category of manifods to the category of algebras.

It is also important to mention that developing characteristic classes for vector bundles with structural group $GL_n(\mathbb{C})$ is the same as with structural group $U(n)$. This is true since $U(n)$ is homotopy equivalent to $GL_n(\mathbb{C})$. Even more this theory can also be applied to vector bundles whose structure group is $SU(n)$. Thus, for brevity, in this section G will denote $U(n)$ or $SU(n)$ unless otherwise stated.

Definition 4.1. A **characteristic class** for vector bundles is a natural transformation $\eta : \mathrm{Vect}_n(-) \to H^*(-)$.

Characteristic classes are equivalent to the cohomology classes of $H^*(BG)$. This is done in a constructive way, so we recall its proof.

Theorem 4.2. *Let $EG \to BG$ be the classifying vector bundle of dimension n. A natural transformation $\eta : \mathrm{Vect}_n(-) \to H^*(-)$ is determined uniquely by $\eta([EG]) \in H^*(BG)$, and every class of $H^*(BG)$ determines a natural transformation.*

PROOF. Let $c := \eta([EG]) \in H^*(BG)$. Recall there is a natural bijection

$$\begin{aligned} \varphi : [M, BG] &\to \mathrm{Vect}_n(M) \\ [f] &\mapsto [f^*(EG)], \end{aligned}$$

between the set of homotopy classes of functions from M to BG, $[M, BG]$, and the set of equivalence classes of vector bundles. Here $f^*(EG)$ refers to the pullback vector bundle.

Now take $[E] \in \mathrm{Vect}_n(M)$ and $\varphi^{-1}([E]) = [f]$. Since η is a natural transformation we obtain that

$$f^*(\eta([EG])) = \eta([f^*(EG)]).$$

But since $[E] = [f^*(EG)]$, and by definition $f^*(c) = f^*(\eta([EG]))$ it follows that

$$f^*(c) = \eta([E]) \in H^*(M).$$

Since the class $f^*(c) \in H^*(M)$ depends only on the homotopy class of f and the class c, we obtain in turn that $f^*(c)$ is uniquely determined by $[E]$ and c.

On the other hand it also follows that given a class $\tilde{c} \in H^*(BG)$, the assignment

$$\begin{aligned} \mathrm{Vect}_n(M) &\to H^*(M) \\ [E] &\mapsto [f^*(\tilde{c})], \end{aligned}$$

where $\varphi^{-1}([E]) = [f]$, defines a natural transformation. □

From this proof we see that given a vector bundle $E \to M$ and a natural transformation η, to determined $\eta([E])$ is equivalent to find the homotopy class classifying E, $\varphi^{-1}([E]) = [f]$. This method is thus known as the homotopic method to determine characteristic classes.

4.1.1. Chern-Weil theory: When we work on the smooth category, there is a geometrical way to obtain characteristic classes. Here we give a small review of this construction for vector bundles (see Chapter 5 of [**Morita**] for details). For this, let $p : E \to M$ be a smooth vector bundle over a manifold. Also let $\Gamma(E)$ be the set of smooth sections of p, and $\mathfrak{X}(M)$ the set of vector fields over M.

DEFINITION 4.3. Let $p : E \to M$ be a smooth vector bundle over a manifold. A **connection** for E is smooth map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying the following conditions: for $X, Y \in \mathfrak{X}(M)$, $s, s' \in \Gamma(E)$ and $f \in C^\infty(M)$ we have

- $\nabla_{fX+Y}(s) = f\nabla_X(s) + \nabla_Y(s)$.
- $\nabla_X(s + s') = \nabla_X(s) + \nabla_X(s')$.
- $\nabla_X(fs) = f\nabla_X(s) + X(f)s$.

The existence of a connection can always be guaranteed. In particular consider a trivial bundle $M \times \mathbb{C}^n \to M$ and the sections $s_i : M \to M \times \mathbb{C}^n$ given by $s_i(x) = (x, e_i)$, where $\{e_1, \ldots, e_n\}$ is the standard basis. Then we have a trivial connection given by $\nabla_X s_i = 0$ for $1 \le i \le n$ and every $X \in \mathfrak{X}(M)$. For any other section s, consider that they can be written as

$$s = \Sigma_{i=1}^n f_i s_i$$

where $f_i \in C^\infty(M)$. With the second condition of the definition we obtain that

$$\nabla_X s = \nabla_X(\Sigma_{i=1}^n f_i s_i) = \Sigma_{i=1}^n \nabla_X(f_i s_i),$$

and then by the third condition we have

$$\nabla_X (f_i s_i) = \nabla_X s_i + X(f_i) s_i = X(f_i) s_i.$$

Thus, we obtain that

$$\nabla_X s = \Sigma_{i=1}^n X(f_i) s_i. \tag{4.1.1}$$

Now consider an arbitrary vector bundle $p : E \to M$ with an open cover $\{U_i\}$ of M and trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^n$. We can obtain linearly independent sections over U_i, $s_i(x) := \varphi_i^{-1}(x, e_i)$, $1 \leq i \leq n$. Following the previous construction, we can define a connection over U_i, ∇^i by the previous equation. Then, by taking a partition of the unity subordinated to $\{U_i\}$, $\{g_i\}$, we define a connection over all M by

$$\nabla_X s := g_i \nabla_X^i s \mid_{U_i} .$$

This can be easily checked to be a well defined connection.

Let us go back to the trivial connection over the trivial bundle, $M \times \mathbb{C}^n \to M$. Notice that for every section s it follows that for the Lie bracket of two vector fields X and Y, $[X, Y]$, we have

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = 0.$$

This equality does not hold for every connection of an arbitrary vector bundle.

Definition 4.4. Consider a smooth vector bundle $E \to M$ with a connection ∇. The **curvature** of the connection is the smooth map $R : \mathfrak{X}(M)^2 \times \Gamma(E) \to \Gamma(E)$ given by

$$R_{(X,Y)} s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for $(X, Y) \in \mathfrak{X}(M)^2$ and $s \in \Gamma(E)$.

Now consider an open cover $\{U_i\}_{i \in \mathbb{N}}$ with trivializations and transition functions $\rho_{ij} : U_i \cap U_j \to G$ associated to these trivializations. The curvature R can be expressed locally in every U_i as a matrix Ω^i, where every entry is an a differential 2-form over M. The relation between this local forms on the intersection $U_i \cap U_j$ is given by the formula

$$\Omega^j = \rho_{ij}^{-1} \Omega^i \rho_{ij}.$$

Now consider the Lie algebra $\mathfrak{g}$ of G. Let p be an invariant polynomial under matrix conjugation of the polynomial algebra of $\mathfrak{g}$. Since $p\left(\rho_{ij}^{-1} \Omega^i \rho_{ij}\right) = p\left(\Omega^i\right)$, $p(\Omega)$ is a well defined two form of M. In fact, Chern-Weil theory guarantees that this gives us a well define characteristic class. See Propositions 5.27, 5.28 and 5.29 of [**Morita**].

Proposition 4.5. *The form $p(\Omega)$ is a closed form of M. This endows us with a well defined natural transformation*

$$\begin{aligned} \mathrm{Vect}_n(M) &\to H^*(M) \\ [E] &\mapsto [p(\Omega)] . \end{aligned}$$

Also, the class $[p(\Omega)]$ is independent of the choice of connection and its curvature.

This construction is related to the homotopical construction via the Chern-Weil homomorphism. That is

THEOREM 4.6. *Let G be either $U(n)$ or $SU(n)$. Also let $I(\mathfrak{g})$ be the algebra of invariant polynomials under conjugation of the polynomial algebra of the Lie algebra of G, $\mathfrak{g}$. Then there exists an isomorphism,*

$$\Psi : I(\mathfrak{g}) \to H^*(BG)$$

called the Chern-Weil isomomophism. Even further let Ω be the curvature of a vector bundle $E \to M$ and $c := \Psi(p)$ be the class given by an invariant polynomial $p \in I(\mathfrak{g})$. Then if η_c is the characteristic class defined by c, then we have the equality

$$\eta_c(E) = [p(\Omega)].$$

Finally let's consider the polynomials σ_i on the entries of a $n \times n$ matrix X characterized by the equality

$$\det(I - tX) = 1 + t\sigma_1(X) + t^2\sigma_2(X) + \cdots + t^n\sigma_n(X).$$

It is easy to see that the polynomial σ_i are invariant under matrix conjugation. Even further, σ_i, $1 \le i \le n$ are generators (as an algebra) of the whole algebra of invariant polynomials over matrices both with complex or real coefficients. (See Theorem 5.26 of [**Morita**]) In fact it is well known that the class of the closed forms given by $\left(\frac{i}{2\pi}\right)^i \sigma_i(\Omega)$ have integer values.

DEFINITION 4.7. The characteristic class given by the polynomial $c_i = \left(\frac{i}{2\pi}\right)^i \sigma_i$ is called the i-th Chern class.

Chern classes are important since they generate the rest of the classes as an algebra. This means that the rest of the classes can be obtained as linear combinations of products of Chern classes.

4.2. Characteristic classes for TC structures

In this section we first define characteristic classes for TC structures. Then we adapt the homotopy construction to show how $H^*(B_{\mathrm{com}}G, \mathbb{R})$ is in one to one correspondence with them. We only consider here vector bundles with structure group $SU(n)$ or $U(n)$.

In this section G will denote $U(n)$ or $SU(n)$.

DEFINITION 4.8. A **characteristic class for a TC structure** or **TC characteristic class** is natural transformation between the functors $\mathrm{Top} \to \mathrm{Bund}_G^{com}(-)$ and $\mathrm{Top} \to H^*(-, \mathbb{R})$.

While this definition does not require further restrictions, we are interested in working only with manifolds. The classifying space for commutativity gives us a natural way to construct characteristic classes:

PROPOSITION 4.9. *There is a one to one correspondence between classes $p \in H^*(B_{com}G, \mathbb{R})$ and characteristic classes η_p for TC structures on principal G-bundles over a manifold.*

PROOF. For brevity let us call $\mathcal{U}$ the TC structure of $B_{com}G$ making it a classifying space for TC structures themselves. Also, suppose we have a natural transformation $\eta : \mathrm{Bund}_G^{com}(-) \to H^*(-)$, and take $c := \eta(\mathcal{U}) \in H^*(B_{com}G, \mathbb{R})$.

If M is a manifold there is a natural bijection

$$[M, B_{com}G] \to \mathrm{Bund}_G^{com}(M)$$
$$[f] \mapsto [f^*(\mathcal{U})].$$

This means that for an equivalence class $\xi \in \mathrm{Bund}_G^{com}(M)$ there is a unique classifying function $f : M \to B_{com}G$, up to homotopy, that represents that given structure. That is, the pullback $f^*(\mathcal{U})$ is in the class ξ. This implies, by naturality, that $\eta(\xi) = f^*(\eta(\mathcal{U})) = f^*(c)$. Here we are abusing notation by also calling

$$f^* : H^*(B_{com}G, \mathbb{R}) \to H^*(M, \mathbb{R})$$

the induced map on cohomology. In conclusion c determines completely the characteristic class η.

On the other hand, it is clear that every element of $c \in H^*(B_{com}G, \mathbb{R})$ gives rise to a characteristic class for TC structures. This is, the assignment $\xi \in \mathrm{Bund}_G^{com}(M) \to f^*(c) \in H^*(M)$ is natural. where f is the classifying function of ξ. □

Now let's see how we can use the results of Chapter 3 to obtain TC characteristic classes through the ordinary characteristic classes. Each of the elements of the algebra $H^*(B_{\mathrm{com}}G, \mathbb{R})$ represents a TC characteristic class, thanks to the previous theorem. However, we saw that these algebras can be generated by a smaller generating set (as an algebra). Recall that we have the k-th power maps $\Phi^k : H^*(B_{\mathrm{com}}G) \to H^*(B_{\mathrm{com}}G)$, and a natural inclusion $\iota : H^*(BG) \to H^*(B_{\mathrm{com}}G)$. So if we consider

$$\mathcal{S} := \left\langle \Phi^k \circ \iota(c) \mid c \in H^*(BG), k \in \mathbb{Z} \setminus \{0\} \right\rangle,$$

the algebra of classes generated by the images of $\Phi^k \circ \iota$, $k \in \mathbb{Z}$, we already proved in Chapter 3 that

THEOREM 4.10. *For G equal to $U(n)$ or $SU(n)$, then $H^*(B_{\mathrm{com}}G) = \mathcal{S}$.*

This means that given a class in $H^*(B_{\mathrm{com}}G)$, it can be written as a sum of finite products of elements of the form $\Phi^k \circ \iota(c)$, $c \in H^*(BG), k \in \mathbb{Z} \setminus \{0\}$.

4.3. Chern-Weil theory for TC structures

Here we show how to expand Chern-Weil theory to TC characteristic classes. Even further, we will see that this can be done without introducing new geometrical concepts, thanks to the commutative property of the transition functions. We only consider TC structures over vector bundles whose structure group G is $U(n)$ or $SU(n)$.

4.3.1. The k-th associated bundles: Let $\left(p : E \to M, \{U_\alpha\}_{\alpha\in J}, \varphi_\alpha\right)$ be a TC structure with structure group G and transition functions $\{\rho_{\alpha\beta}\}$. By definition if $x \in U_{\alpha\beta} \cap U_{\gamma\eta}$ then

$$\rho_{\alpha\beta}(x)\,\rho_{\gamma\eta}(x) = \rho_{\gamma\eta}(x)\,\rho_{\alpha\beta}(x).$$

These transition functions satisfy the cocycle condition as well, that is,

$$\rho_{\alpha\gamma}(x) = \rho_{\alpha\beta}(x)\,\rho_{\beta\gamma}(x).$$

In particular these two properties imply that for $k \in \mathbb{Z}$ we have

$$\rho_{\alpha\gamma}(x)^k = \left(\rho_{\alpha\beta}(x)\,\rho_{\beta\gamma}(x)\right)^k = \rho_{\alpha\beta}(x)^k\,\rho_{\beta\gamma}(x)^k.$$

This tell us that the collection of functions $\rho^k_{\alpha\beta} : U_{\alpha\beta} \to G$ defined as

$$\rho^k_{\alpha\beta}(x) := \rho_{\alpha\beta}(x)^k$$

also satisfy the cocycle condition. Thus we can construct a new principal bundle $p(k) : E^k \to M$ with trivializations over the same open cover $\{U_\alpha\}_{\alpha\in J}$ (See Chapter 1). We call it the **k-th associated bundle** of E.

THEOREM 4.11. *(Classifying functions for k- th associated bundles)*

If $f : M \to B_{com}G$ is a bundle with a TC structure, and $f^k : M \to B_{com}G$ is the classifying function of the k-th associated bundle, then the following map diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f^k"'] & B_{com}G \arrow[d, "\Phi^k"] \\ & B_{com}G. \end{tikzcd} \tag{4.3.1}$$

Where $\Phi^k : B_{com}G \to B_{com}G$ are the power maps.

PROOF. As it was explained before, to obtain the classifying functions for the k-th associated bundle $p(k) : E^k \to M$ we need to consider a simplicial map $f^k_l : \mathcal{N}(\mathcal{U})_l \to \operatorname{Hom}\left(\mathbb{Z}^l, G\right)$. The components of this function are given by the transition functions: if $x \in U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_{l+1}}$, we take

$$f^k_l(x) = \left(\rho^k_{i_0 i_1}(x), \ldots, \rho^k_{i_{l-1} i_l}(x)\right) = \left(\rho_{i_0 i_1}(x)^k, \ldots, \rho_{i_{l-1} i_l}(x)^k\right).$$

This can be rewritten using the power functions as

$$f^k_l = \Phi^k_l \circ f_l.$$

The desired result is obtained after passing to the geometrical realization. ∎

4.3.2. Chern-Weil theory for TC structures: We finally are able to reach our main goal of developing a theory that uses Chern-Weil theory to obtain characteristic classes. For this, we develop an algebraic and geometrical method that uses the fact the every TC characteristic class is uniquely represented by an element of $H^*(B_{\text{com}}G, \mathbb{R})$.

For the rest of this chapter let $\varepsilon \in \text{Vect}_n^{\text{com}}(M)$ be an equivalence class with an underlying smooth vector bundle $E \to M$ and structure group $U(n)$ or $SU(n)$. For an element $p \in H^*(B_{\text{com}}G, \mathbb{R})$ we denote by $p(\varepsilon) \in H^*(M)$ the value of the TC characteristic class on the TC equivalence class ε. Also, recall that via the Chern-Weil isomorphism, if $\mathfrak{g}$ is the Lie algebra of G, then $H^*(BG) \cong I(\mathfrak{g})$. Here $I(\mathfrak{g})$ is the subalgebra of invariant polynomials under conjugation of the polynomial algebra of $\mathfrak{g}$. Under this identification, every characteristic class for vector bundles -having G as its structure group- can be identify with a polynomial $c \in I(\mathfrak{g})$.

Now recall that for a smooth vector bundle $F \to M$ with curvature Ω, the value on F of the characteristic class associated to c is equal to $c(\Omega) \in H^*(M)$. Under these terms, we are now able to compute the TC characteristic classes associated to the set of generators of $H^*(B_{\text{com}}G)$, $\left\{\Phi^k \circ \iota(c) \mid 1 \le i \le n, k \in \mathbb{Z} \setminus \{0\}\right\}$. Here, we take ι to be a map from $I(\mathfrak{g})$ to $H^*(B_{\text{com}}G)$.

Theorem 4.12. *Consider $\varepsilon \in \text{Vect}_n^{\text{com}}(M)$ an equivalence class with an underlying smooth vector bundle $E \to M$, and structure group $U(n)$ or $SU(n)$. Also let Ω_k be the curvature of E^k, the k-th associated bundle of E. Then for $c \in I(\mathfrak{g})$ and $p = \Phi^k \circ \iota(c) \in H^*(B_{\text{com}}G)$, the TC characteristic class $p(\varepsilon)$ has same class in $H^*(M)$ as the characteristic class for vector bundles $c\left(E^k\right)$. This implies that*

$$p(\varepsilon) = c(\Omega_k) \in H^*(M).$$

Proof. This is straight forward. First, by Theorem 4.11 we know that if f and f_k the the classifying functions for TC structures over $E \to M$ and $E^k \to M$, respectively, then there is the following commuting diagram

$$\begin{tikzcd} H^*(B_{\text{com}}G) \arrow[r, "f^*"] & H^*(M) \\ H^*(B_{\text{com}}G). \arrow[u, "\Phi^k"] \arrow[ur, "f_k^*"'] & \end{tikzcd}$$

This means that for $c \in H^*(BG)$ we have the identity $f^*\left(\Phi^k \circ \iota(c)\right) = f_k^*(\iota(c))$ in $H^*(M)$.

In turn, since the composition $f_k^* \circ \iota$ is a classifying function for the vector bundle $E^k \to M$, we can apply the Chern-Weil isomorphism. That is, we can consider the curvature Ω_k of E_k to obtain that

$$f_k^*(\iota(c)) = c(\Omega_k).$$

The conclusion of the theorem then follows by transitivity. $\square$

Theorem 4.13. *(Chern-Weil theory for TC structures)*

Consider $\varepsilon \in \text{Vect}_n^{\text{com}}(M)$ an equivalence class with an underlying smooth vector bundle $E \to M$, and structure group $U(n)$ or $SU(n)$. Also let Ω_k be the curvature of E^k, the k-th associated bundle of E. Then every TC characteristic class can be obtained as a linear combinations of products of the form

$$s_1(\Omega_{k_1}) \cdot s_2(\Omega_{k_2}) \cdots s_m(\Omega_{k_m}) \in H^*(M),$$

where $s_i \in H^*(BG)$ and $k_i \in \mathbb{Z}$. Each $s_i(\Omega_{k_1})$ is the characteristic class of the vector bundle $E^k \to M$ computed using its curvature.

PROOF. Recall that if we set $\mathcal{S}$ as the subalgebra of $H^*(B_{\text{com}}G)$ generated by

$$\left\{\Phi^k \circ \iota(s) \mid 1 \le i \le n, k \in \mathbb{Z} \setminus \{0\}, s \in H^*(BG)\right\}$$

then we have $\mathcal{S} = H^*(B_{\text{com}}G)$. Thus, every element of $H^*(B_{\text{com}}G)$ can be written as a linear combination of products of the form

$$\Phi^{k_1}(\iota(s_1)) \cdot \Phi^{k_m}(\iota(s_2)) \cdots \Phi^{k_m}(\iota(s_m)).$$

Then we can apply the previous theorem to obtain $\Phi^{k_i}(\iota(s_i)) = s_i(\Omega_{k_i})$. □

As suggested by the name of the theorem, we are now able to compute TC characteristic classes by using Chern-Weil theory. This is done in a three steps process for a class in $s \in H^*(B_{\text{com}}G)$ and a TC structure ξ over a vector bundle $E \to M$: first we need to decompose s in terms of the generators in $\left\{\Phi^k \circ \iota(c) \mid 1 \le i \le n, k \in \mathbb{Z} \setminus \{0\}\right\}$. Secondly, for each of the generators $\Phi^k \circ \iota(c)$ in the decomposition of s we use the curvature of the k-th associated bundle, Ω_k, to compute the characteristic class associated to it, $c(\Omega_k) \in H^*(M)$ (this class is equal to the TC class given by $\left(\Phi^k \circ \iota(c)\right)(\xi)$). Finally we replace the values of each $\left(\Phi^k \circ \iota(c)\right)(\xi)$ to obtain $s(\xi) \in H^*(M)$.

Recall from Chapter 3 that when G is equal to $U(n)$, then

$$H^*(B_{\text{com}}G, \mathbb{R}) \cong \left(\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]\right)^{S_n} / J$$

where S_n acts by permutation on their indexes and J is the ideal generated by the invariant polynomials of positive degree on the x_i. When G is $SU(n)$ is the same description for $H^*(B_{\text{com}}G, \mathbb{R})$ except J is generated by the invariant polynomials of positive degree on x_i and the polynomial $y_1 + \cdots y_n$.

We also have the identifications

$$H^*(BU(n), \mathbb{R}) \cong \mathbb{R}[z_1, \ldots, z_n]^{S_n}$$

and

$$H^*(BSU(n), \mathbb{R}) \cong \mathbb{R}[z_1, \ldots, z_n]^{S_n} / \langle z_1 + \cdots + z_n \rangle.$$

Then we have that the polynomials

$$p_i = z_1^i + \cdots + z_n^i \in \mathbb{R}[z_1, \ldots, z_n]$$

generated all of $H^*(BG, \mathbb{R})$, when G is $U(n)$ or $SU(n)$. Even further for $a, b \in \mathbb{N} \cup \{0\}$ such that $0 < a + b$ then

$$P_{a,b}(n) := \sum_{i=1}^{n} x^a y^b \bmod J.$$

generated all of $H^*(B_{\text{com}}G, \mathbb{R})$ as an algebra. We also saw in the proof of Theorem 3.14 there every $P_{a,b}(n)$ can be obtain, via a recursive procedure, as a linear combination of elements of the form $\Phi^k(\iota(p_i))$. With that recursive procedure and the previous theorem, we can compute the TC characteristic classes corresponding to each $P_{a,b}(n)$.

Recall that another set of generators for $H^*(BG, \mathbb{R})$, when G is $U(n)$ or $SU(n)$ is given by the polynomials σ_i, characterized by the equation

$$\det(I - tX) = 1 + t\sigma_1(X) + t^2\sigma_2(X) + \cdots + t^n\sigma_n(X).$$

These generators are more commonly used instead of the p_i, as σ_i are used in the definition of Chern classes.

EXAMPLE 4.14. In Chapter 3 we saw that for $G = U(3)$ we have the equalities

(1) $y_1 + y_2 + y_3 = \iota(z_1 + z_2 + z_3)$.
(2) $y_1^2 + y_2^2 + y_3^2 = \frac{1}{2}\left(\iota\left(z_1^2 + z_2^2 + z_3^2\right) + \Phi^{-1}\left(\iota\left(z_1^2 + z_2^2 + z_3^2\right)\right)\right)$.
(3) $x_1y_1 + x_2y_2 + x_3y_3 = \frac{1}{4}\left(\iota\left(z_1^2 + z_2^2 + z_3^2\right) - \Phi^{-1}\left(\iota\left(z_1^2 + z_2^2 + z_3^2\right)\right)\right)$.

Now consider a TC strcutrure ξ with underlying vector bundle $E \to M$, with curvature Ω and Ω_k is the curvature of the k-th associated bundle. Now since we have that $p_1 = \sigma_1$ and that

$$p_2 = \sigma_1^2 - 2\sigma_2$$

we obtain that

(1) $(y_1 + y_2 + y_3)(\xi) = \sigma_1(\Omega)$.
(2) $\left(y_1^2 + y_2^2 + y_3^2\right)(\xi) = \frac{1}{2}\left(\sigma_1(\Omega)^2 + \sigma_1(\Omega_{-1})^2\right) - (\sigma_2(\Omega) + \sigma_2(\Omega_{-1}))$.
(3) $(x_1y_1 + x_2y_2 + x_3y_3)(\xi) = \frac{1}{4}\left(\sigma_1(\Omega)^2 - \sigma_1(\Omega_{-1})^2\right) + \frac{1}{2}(\sigma_2(\Omega_{-1}) - \sigma_2(\Omega))$.

For $G = U(n)$ we know that $H^*(BG)$ is a polynomial algebra generated by the Chern classes c_i, $1 \le i \le n$. Thus it follows that $\mathcal{S}$ is generated by the set $\left\{\Phi^k \circ \iota(c_i) \mid 1 \le i \le n, k \in \mathbb{Z} \setminus \{0\}\right\}$.

DEFINITION 4.15. We call the classes of the form $c_i^k := \Phi^k \circ \iota(c) \in H^*(B_{\mathrm{com}}U(n))$ the **TC Chern classes**. Also, for a TC structure ε with underlying vector bundle $E \to M$ we call

$$c_i^k(\varepsilon) := f^*\left(c_i^k\right) \in H^*(M)$$

the TC (i,k)-Chern class. Here $f : M \to B_{\mathrm{com}}U(n)$ is the classifying function of the TC structure.

From the previous theorem we have the immediate following corollary:

COROLLARY 4.16. *Let $E \to M$ by the underlying bundle of a TC structure structure ε, and let Ω_k be the curvature of the k-th associated bundle. Then $c_i^k(\epsilon) = c_i(\Omega_k)$.*

It is immediate from our results that $\left\{c_i^k \mid k \in \mathbb{Z},\ i \in \mathbb{N}\right\}$ generates all of $B_{\mathrm{com}}U(n)$ as an algebra. That is, every class in $H^*(B_{\mathrm{com}}U(n))$ can be written in the form

$$s = \sum_{j=1}^{m} \alpha_j C_j$$

where $\alpha_j \in \mathbb{R}$ and

$$C_j = \prod_{t=1}^{m_j} c_{i_{j,t}}^{k_{j,t}},$$

where $k_{j,t} \in \mathbb{Z}$ and $i_{j,t} \in \mathbb{N}$. Then it follows that if ξ is a TC structure with underlying vector bundle $E \to M$, with curvature its Ω and Ω_k the curvature of the k-th associated bundle, then

$$s(\xi) = \sum_{j=1}^{m} \alpha_j \left(\prod_{t=1}^{m_j} c_{i_{j,t}}\left(\Omega_{k_{j,t}}\right)\right) \in H^*(M).$$

At this point it is worth mentioning that $H^*(B_{\text{com}}G)$ is in general not a polynomial algebra. For example when $G = U(n)$, the TC Chern classes are not algebraically independent. However, the relationships governing them are rather complicated. As such, their values in a given TC structure can vary significantly. We see an example of this in the next chapter.

REMARK 4.17. The concepts developed in this chapter can also be applied to vector bundles on the quaternions. In this case, the structural group is the simplectic group $\mathrm{Sp}(n)$. The main ideas we needed to developed TC characteristic classes also hold for this group. As we saw in Chapter 3 we also have power maps on cohomology, and $H^*(B_{\text{com}}\mathrm{Sp}(n), \mathbb{R})$ is also generated by as an algebra by $\left\{\Phi^k \circ \iota(c_i) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\}\right\}$. Where again $\iota : H^*(B\mathrm{Sp}(n), \mathbb{R}) \to H^*(B_{\text{com}}\mathrm{Sp}(n), \mathbb{R})$ is induced by the natural inclusion $B_{\text{com}}\mathrm{Sp}(n) \to B\mathrm{Sp}(n)$. Also, since $\mathrm{Sp}(n)$ is a compact group, the Chern-Weil homomorphism is in fact an isomorphism. Thus, most of the ideas we used through out this chapter can be used.

CHAPTER 5

Examples

In this final chapter we exhibit explicit calculations of examples using Chern-Weil theory to compute TC characteristic classes. In particular we show there is a TC structure ξ such that $c_i(\xi) = 0$ for every $i \in \mathbb{N}$ while $c_2^{-1}(\xi) \neq 0$. This shows that a TC Chern class c_i^n does not necessarily determines another TC Chern class c_i^m, if $m \neq n$. This confirms that the underlying vector bundle of a TC structure does not determine completely the TC structure.

In this chapter we present two examples, both of which are TC structures over spheres with a two sets open cover with trivializations. As such, we start with the most general calculations to continue considering the more specific conditions our examples need. The first example is the computation of the TC Chern classes for the tautological linear bundle over the sphere. This follows the first general calculations. The second example is the one mentioned previously, consisting of a non trivial TC structure over a trivial vector bundle.

This treatment is based on the concepts presented as in [**Morita**], Chapter 5. A small review is presented in the first section of Chapter 4.

5.1. Connection for a vector bundle with a two sets cover with trivializations

Let $\pi : E \to M$ denote a smooth vector bundle over $\mathbb{C}$ of dimension n, with M a closed manifold. Assume we can find an open cover $\{U_1, U_2\}$ of M together with trivializations

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\to U_i \times \mathbb{C}^n \\ e &\mapsto (\pi(e), h_i(e)) . \end{aligned}$$

Suppose these trivializations have structure group a Lie group of matrices G. This is, we have a function $\rho : U_1 \cap U_2 \to G \subseteq GL_n(\mathbb{C})$ characterized by

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{C}^n &\to U_1 \cap U_2 \times \mathbb{C}^n \\ (x, v) &\mapsto (x, \rho(x) v) . \end{aligned}$$

These trivializations induce smooth sections

$$\begin{aligned} s_{ij} : U_i &\to \pi^{-1}(U_i) \\ x &\mapsto \varphi_j^{-1}(x, e_j) , \end{aligned}$$

where e_j is the j-th vector of the standard basis of $\mathbb{C}^n$, and $i = 1, 2$. This setting implies that for $x \in U_i$the set $\{s_{i1}(x), s_{i2}(x), \ldots, s_{in}(x)\} \subset \pi^{-1}(x)$ is a basis. Under these conditions, for a point $x \in U_1 \cap U_2$ it follows that

$$s_{1k}(x) = \sum_{l=1}^{n} \rho_{lk}(x) s_{2l}(x) \tag{5.1.1}$$

where we take $\rho = [\rho_{lk}]_{k,l=1}^{n}$.

Now let $\{f_1, f_2\}$ be a partition of the unity subordinated to $\{U_1, U_2\}$, as well as the trivial connections over each U_i, ∇^i (See Section 1 of Chapter 4 for details). We can now define the connection

$$\nabla_X s := f_1 \nabla_X^1 s + f_2 \nabla_X^2 s.$$

This means that for a vector field X and a section s, we consider their restriction to U_i in order to evaluate ∇_X^i. That is, we need first to consider the decomposition of s in terms of the basis $\{s_{i1}, \ldots, s_{in}\}$, which means that there are smooth functions $\alpha_i^j : U_i \to \mathbb{C}$ such that for $x \in U_i$

$$s\mid_{U_i}(x) = \sum_{j=1}^{n} \alpha_j^i(x) s_{ij}(x).$$

Then, applying the product rule and the definition, we have

$$\nabla_X s := \sum_{i,j} f_i X\left(\alpha_j^i\right) s_{ij}.$$

Recall that with n-linearly independent sections $\{s_1, \ldots, s_n\}$ we have the local expressions for both the connection and the curvature, $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(E)$. There exists 1-forms ω_{ij} and two forms Ω_{ij} such that we can write

$$\nabla_X s_i = \sum_i \omega_{ij}(X) s_j$$

and

$$R(X, Y)(s_i) = \sum_j \Omega_{ij}(X, Y) s_j$$

which gives rise to the local connection and curvature matrices

$$\omega := [\omega_{ij}] \text{ and } \Omega := [\Omega_{ij}].$$

These local forms are related to the transition function in the following way. From Equality 5.1.1 we get that

$$\nabla_X s_{1k} = \sum_{l=1}^{n} f_2 X(\rho_{lk}(x)) s_{2l}(x).$$

From differential geometry we know that for a function $f : M \to \mathbb{R}$, $X(f) = df(X)$ holds, where d is the external derivation. Thus we get the expression

$$\nabla_X s_{1k} = \sum_{l=1}^{n} f_2 d(\rho_{lk}(x))(X) s_{2l}(x),$$

which allow us to write

$$\nabla s_{1k} = \sum_{l=1}^{n} f_2 d(\rho_{lk}(x)) s_{2l}(x).$$

By the properties of cocycles we also know that

$$s_{2l}(x) = \sum_{t=1}^{n} \rho_{tl}^{-1}(x)\, s_{1t}(x),$$

where by ρ_{tl}^{-1}we mean the components of the matrix ρ^{-1}. Thus, we can write

$$\nabla s_{1k} = \sum_{t=1}^{n} \left(f_2 \left(\sum_{l=1}^{n} \rho_{tl}^{-1}(x)\, d\left(\rho_{lk}(x)\right) \right) \right) s_{1t}.$$

By comparing this expression with the local form, we conclude that

$$\omega^1 = f_2 \rho^{-1} d\rho. \tag{5.1.2}$$

Our next step is to obtain the local form of the curvature. For this we use the structural equation (see [**Morita**] Theorem 5.21.)

$$\Omega^i = d\omega^i + \omega^i \wedge \omega^i.$$

Consider the equality $\rho^{-1}\rho = I$. An application of the product rule allow us to write:

$$0 = dI = d\left(\rho^{-1}\right)\rho + \rho^{-1} d\rho.$$

This in turn implies that

$$d\left(\rho^{-1}\right)\rho = -\rho^{-1} d\rho \Rightarrow d\left(\rho^{-1}\right) = -\rho^{-1}\left(d\rho\right)\rho^{-1}.$$

Since $dd = 0$, we obtain $d\left(\rho^{-1} d\rho\right) = d\left(\rho^{-1}\right) \wedge d\left(\rho\right)$, which allow us to conclude that

$$d\omega^1 = \left(\left(df_2\right)\rho^{-1} d\rho - f_2 \rho^{-1} d\rho \wedge \rho^{-1} d\rho \right).$$

On the other hand

$$\omega^1 \wedge \omega^1 = \left(f_2 \rho^{-1} d\rho\right) \wedge \left(f_2 \rho^{-1} d\rho\right) = f_2^2 \rho^{-1} d\rho \wedge \rho^{-1} d\rho,$$

which finally gives us

$$\Omega^1 = \left(\left(df_2\right)\rho^{-1} d\rho - f_2 \rho^{-1} d\rho \wedge \rho^{-1} d\rho \right) + f_2^2 \rho^{-1} d\rho \wedge \rho^{-1} d\rho. \tag{5.1.3}$$

Observe that in a point $x \notin U_1 \cap U_2$, Ω^1 is zero since the closure of the support of f_2 is contained in U_2. Similarly, an analogue formula can be deduce for the local form of the curvature in U_2, and deduced that it is also zero outside $U_1 \cap U_2$. Thus, we can conclude that

PROPOSITION 5.1. *Let $\pi : E \to M$ be a smooth vector bundle with $\{U_1, U_2\}$ an open cover of M, both having trivializations of E, φ_1 and φ_2, respectively. Let $\{f_1, f_2\}$ be a partition of unity associated to $\{U_1, U_2\}$, respectively. If ρ is the transition function associated to $\varphi_2 \circ \varphi_1^{-1}$, then the curvature Ω_k of the k-th associated bundle is given by*

$$\left(\Omega_k\right)_x = \begin{cases} \left(\Omega_k^1\right)_x & x \in U_1 \cap U_2. \\ 0 & x \notin U_1 \cap U_2. \end{cases}$$

Where

$$\Omega_k^1 = \left(df_2\right)\rho^{-k} d\left(\rho^k\right) + \left(f_2^2 - f_2\right)\rho^{-k} d\left(\rho^k\right) \wedge \rho^{-k} d\left(\rho^k\right) \tag{5.1.4}$$

is the local expression on U_1.

PROOF. Since the k-th associated vector bundle has the same cover associated to its TC structure, with transition functions equal to ρ^k, the previous discussion provides a proof of the theorem. $\square$

It is worth mentioning that it is possible to deduce a similar formula to 5.1.3 for a arbitrary number of sets in an open cover, but we will not need this.

5.2. Calculations over the spheres

5.2.1. TC Chern classes of the tautological vector bundle over the sphere. As a simple illustration and fist example, we compute the TC Chern Classes for the Tautological vector bundle over $\mathbb{C}P^1 \cong S^2$. Here we can consider the vector bundle as a TC structure since the structure group, $\mathbb{C}^* \simeq S^1$, is abelian. This implies that $B_{\text{com}}S^1 = BS^1$. So, by the homotopy classification of TC structures and vector bundles we obtain that every vector bundle with this structure group is a TC structure. In this case the TC equivalence class is independent of the open cover and its trivializations.

Consider the set

$$\tau := \left\{ \left([z_1, z_2], (w_1, w_2) \right) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid (w_1, w_2) \in [z_1, z_2] \text{ or } (w_1, w_2) = 0 \right\}$$

and the map $\pi : \tau \to \mathbb{C}P^1$ given by the projection on the first component. We have an open covering given by $U_1 = \left\{ [z, 1] \in \mathbb{C}P^1 \mid z \in \mathbb{C} \right\}$ and $U_2 = \left\{ [1, z] \in \mathbb{C}P^1 \mid z \in \mathbb{C} \right\}$. It is routine to check that the functions

$$\begin{aligned} \varphi_1 : \tau^{-1}(U_1) &\to U_1 \times \mathbb{C} \\ \left([z, 1], (w_1, w_2) \right) &\mapsto \left([z, 1], w_2 \right) \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : \tau^{-1}(U_1) &\to U_1 \times \mathbb{C} \\ \left([1, z], (w_1, w_2) \right) &\mapsto \left([1, z], w_1 \right) \end{aligned}$$

are well defined trivializations. Also it is easy to check that $U_1 \cap U_2 = \left\{ [z, 1] \in \mathbb{C}P^1 \mid z \neq 0 \right\}$, where every class can be uniquely be represented by a pair $[z, 1]$ with $z \neq 0$. Under these conditions we get that

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{C} &\to U_1 \cap U_2 \times \mathbb{C} \\ \left([z, 1], \lambda \right) &\mapsto \left([z, 1], z\lambda \right), \end{aligned}$$

which implies that the transition function is given by

$$\begin{aligned} \rho : U_1 \cap U_2 &\to \mathbb{C}^* \\ [z, 1] &\mapsto z. \end{aligned}$$

Since we are dealing with one dimensional matrices, the expression $\rho^{-k} d\left(\rho^k\right) \wedge \rho^{-k} d\left(\rho^k\right)$ has commuting terms. This means that

$$\rho^{-k} d\left(\rho^k\right) \wedge \rho^{-k} d\left(\rho^k\right) = \left(\rho^{-k}\right)^2 d\left(\rho^k\right) \wedge d\left(\rho^k\right) = 0.$$

The formula for the curvature of the k-th associated bundle gives us $\Omega_k = (df_2)\, \rho^{-k} d\left(\rho^k\right)$, where $\{f_1, f_2\}$ is a partition of the unity subordinated to $\{U_1, U_2\}$, respectively. Now, since we can parameterize $U_1 \cap U_2 \cong \mathbb{C}^*$ via polar coordinates, $(r, \theta) \mapsto re^{\theta i}$, we get $\rho^k(r, \theta) = r^k e^{k\theta i}$. Thus

$$d\left(\rho^k\right) = kr^{k-1} e^{k\theta i} dr + ikr^k e^{k\theta i} d\theta$$

and so

$$\rho^{-k} d\left(\rho^k\right) = k\left(\frac{1}{r} dr + i d\theta\right).$$

Informally if we think of $\mathbb{C}P^1$ as the sphere, each $U_1$and U_2 are the whole sphere minus one point. These two points are antipodal, so we can think of them as the north and south poles. Then, it is easy to see that the partition of the unity can be made to depend only on latitude lines. This means that in the parameterization of $U_1 \cap U_2$ we are considering, f_2 depends only on r, its value must be 0 in a neighborhood around $r = 0$ and its constant and equal to 1 from certain given value $r_o > 0$. This give us that

$$d(f_2) = \frac{\partial f_2}{\partial r} dr + 0 d\theta$$

and

$$\int_0^\infty \frac{\partial f_2}{\partial r} dr = \lim_{x\to\infty} f_2(x) - f_2(0) = 1.$$

Then we conclude that

$$\Omega_k = (df_2)\rho^{-k} d\left(\rho^k\right) = ki\frac{\partial f_2}{\partial r} dr \wedge d\theta,$$

and so the first Chern class is given by

$$\frac{i}{2\pi}\int \Omega_k = \frac{i}{2\pi}\int_0^\infty \int_0^{2\pi} ki\frac{\partial f_2}{\partial r} d\theta dr = -k.$$

In particular the first Chern class of the tautological line bundle is equal to -1, as it is well known. In conclusion we have that

$$c_1^k(\tau) = -k.$$

5.2.2. Second Chern class for clutching functions with values on $SU(2)$. Suppose that we have a vector bundle $p : E \to M$ in such a way that we can find an open cover $\{U_1, U_2\}$ of M together with a transition function $\rho : U_1 \cap U_2 \to SU(2)$. First, we are going to compute the determinant of the curvature form in terms of the components of the matrices in $SU(2)$,

$$SU(2) := \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \mid |z|^2 + |w|^2 = 1 \right\}.$$

So let us take

$$\rho = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix},$$

for which we want to compute the curvature

$$\Omega = (df_2)\rho^{-1}d(\rho) + \left(f_2^2 - f_2\right)\rho^{-1}d(\rho)\wedge\rho^{-1}d(\rho).$$

Since $z\bar{z} + w\bar{w} = 1$, we get by differentiating that

$$0 = (\bar{z}dz + \bar{w}dw) + (zd\bar{z} + wd\bar{w}) \Rightarrow zd\bar{z} + wd\bar{w} = -(\bar{z}dz + \bar{w}dw)$$

and so we have

$$\tau := \rho^{-1}d\rho = \begin{bmatrix} \bar{z}dz + \bar{w}dw & \bar{w}d\bar{z} - \bar{z}d\bar{w} \\ -wdz + zdw & -(\bar{z}dz + \bar{w}dw) \end{bmatrix}.$$

Now take $\theta := \rho^{-1}d\rho \wedge \rho^{-1}d\rho$. Using that $\tau_{22} = -\tau_{11}$ and $|z|^2 + |w|^2 = 1$ we get that

$$\theta = \begin{bmatrix} (\bar{w}d\bar{z} - \bar{z}d\bar{w}) \wedge (-wdz + zdw) & 2d\bar{z}\wedge d\bar{w} \\ -2dz\wedge dw & -(\bar{w}d\bar{z} - \bar{z}d\bar{w}) \wedge (-wdz + zdw) \end{bmatrix}.$$

which is the same as expressing it as

$$\theta = \begin{bmatrix} \tau_{12}\wedge\tau_{21} & \theta_{12} \\ \theta_{21} & -\tau_{12}\wedge\tau_{21} \end{bmatrix}.$$

Now, by making $f := f_2$ and $g := (f-1)f$ we may express the curvature as

$$\Omega = df\tau + g\theta = \begin{bmatrix} df\tau_{11} + g\tau_{12}\wedge\tau_{21} & df\tau_{12} + g\theta_{12} \\ df\tau_{21} + g\theta_{21} & -(df\tau_{11} + g\tau_{12}\wedge\tau_{21}) \end{bmatrix},$$

and its determinant is then given by

$$\det(\Omega) = -(df\tau_{11} + g\tau_{12}\wedge\tau_{21})\wedge(df\tau_{11} + g\tau_{12}\wedge\tau_{21}) - (df\tau_{21} + g\theta_{21})\wedge(df\tau_{12} + g\theta_{12}).$$

In order to reduce this expression, we recall that the wedge product of a one form with itself is zero. Also, one forms commute with two forms, so we get:

$$\det(\Omega) = -g^2\theta_{12}\wedge\theta_{21} - gdf\wedge(\tau_{12}\wedge\theta_{21} + \tau_{21}\wedge\theta_{12} + 2\tau_{11}\wedge\tau_{12}\wedge\tau_{21}).$$

By recalling that $\tau_{11}\wedge\tau_{12} = d\bar{z}\wedge d\bar{w}$ we get:

$$\tau_{11}\wedge\tau_{12}\wedge\tau_{21} = -(wdzd\bar{z}d\bar{w} + zd\bar{z}dwd\bar{w}),$$
$$\tau_{12}\wedge\theta_{21} = 2(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw)$$

and

$$\tau_{21}\wedge\theta_{12} = -2(zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}).$$

Now take

$$A := \tau_{12}\wedge\theta_{21} + \tau_{21}\wedge\theta_{12} + 2\tau_{11}\wedge\tau_{12}\wedge\tau_{21}$$

then

$$A = 2[(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw) - (zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}) - (wdzd\bar{z}d\bar{w} + zd\bar{z}dwd\bar{w})]$$

which gives us

$$A = 2(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw - 2(zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}))$$

which we can now replace to have

$$\det(\Omega) = 4(f_2-1)^2 f_2^2 dzd\bar{z}dwd\bar{w} - (f_2-1)f_2 df_2\wedge A. \tag{5.2.1}$$

Now we are going to use this formula to find the second Chern class in terms of a smooth Clutching function $\varphi : S^3 \to SU(2)$ (see [**Hatcher II**], Chapter 1). Consider the sets

$$S^4 = \left\{\mathbf{x} = (x_1,\ldots,x_5)\in\mathbb{R}^5 \mid \|\mathbf{x}\| = 1\right\},$$
$$D_+ = \left\{(x_1,\ldots,x_5)\in S^4 \mid x_5 \geq 0\right\},$$
$$D_- = \left\{(x_1,\ldots,x_5)\in S^4 \mid x_5 \leq 0\right\}$$

and the open set

$$V = \left\{(x_1,\ldots,x_5)\in S^4 \mid -1/3 < x_5 < 1/3\right\}.$$

Also let $U_1 := D_+\cup V$, $U_2 := D_-\cup V$ and identify S^3 with the equator $\left\{(x_1,\ldots,x_5)\in S^4 \mid x_5 = 0\right\}$.

Using "bump" functions we can obtain a partition of the unity $f_1, f_2 : S^4 \to [0,1]$ such that they depend only on the "height" x_5 and $f_i\mid_{U_i\setminus V}\equiv 1$. Also the clutching function $\varphi : S^3 \to SU(2)$ can be composed with a smooth "perpendicular" retraction of V to S^3, to obtain a transition function $\rho : V \to SU(2)$ independent of x_5.

Under this conditions is clear that

- $df_2 = \frac{\partial f_2}{\partial r}dr$, and

- If
$$\rho = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$$
any four form depending on z,$\bar{z}$, w and $\bar{w}$ is zero, since these functions depend only on three variables.

We are in position to apply the previous results to obtain that

$$\det(\Omega) = 4(f_2 - 1)^2 f_2^2 dz d\bar{z} dw d\bar{w} - (f_2 - 1) f_2 df_2 \wedge A.$$

Where $A = 2(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw - 2(zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}))$. However, by construction we have that $dzd\bar{z}dwd\bar{w} = 0$ and so

$$\int_{S^4} \det(\Omega) = \left(\int_{-1}^{1} \left((1 - f_2) f_2 \frac{\partial f_2}{\partial r} \right) dr \right) \int_{S^3} A.$$

First, notice that by construction it follows that

$$\int_{-1}^{1} \left((1 - f_2) f_2 \frac{\partial f_2}{\partial r} \right) dr = -\frac{1}{6}.$$

Finally since the second Chern class in this case is the determinant of the curvature times $\left(\frac{i}{2\pi}\right)^2$, we get

PROPOSITION 5.2. *The second Chern class associated to a clutching function $\varphi : S^3 \to SU(2)$ is given by*

$$c_2 = \frac{1}{24\pi^2} \int_{S^3} A.$$

Here A is a 3-form given by

$$2(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw - 2(zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}))$$

and the functions $z, w : S^3 \to SU(2)$ are determined by the clutching function, $\varphi = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$.

5.3. A non trivial TC structure over a trivial vector bundle

It is already known that there are trivial vector bundles with non trivial TC structures over them. In this section we are going to use such a structure to show that:

THEOREM 5.3. *There exists a TC structure $\xi = \left\{E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2)\right\}$ such that $E \to S^4$ is a trivial bundle, and such that $c_2^{-1}(\xi) = -1$, implying that the TC structure is non trivial.*

This in particular highlights how the TC characteristic classes depends on the TC structure and not on the equivalence class of their underlying bundle.

Now, to prove this theorem we are based on the construction made by D. Ramras and B. Villareal ([**RV**], Chapter 3). In what follows, we first define the vector bundle by defining an open cover on S^4 and transition functions on them. This defines a TC structure

$$\xi = \left\{E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2)\right\}.$$

Then by considering the (-1)-powers of these transition functions we also obtain the (-1)-th associated bundle, E^{-1}.

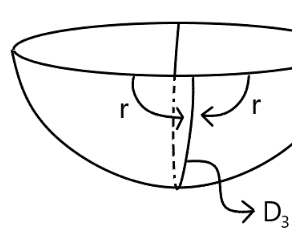

FIGURE 5.3.1. Retraction $r : D_- \to D_3$

Next we are going to use Lemma 3.1 of **[RV]** to show that both vector bundles obtained can be described, up to isomorphism, by a given clutching functions. Then on one hand by showing that the clutching function associated to E is trivial, we conclude that E is trivial. On the other hand, we use the clutching function associated to E^{-1} together with the formulas of the previous sections, to conclude that $c_2^{-1}(\xi) = -1$.

5.3.1. Description of the TC structure: We outline how their initial construction can be made in the smooth category, which allows us to reduce the problem of computing the Chern class by using Clutching functions.

We are constructing a TC structure on a vector bundle defined over S^4 in terms of a triple open cover $\{U_1, U_2, U_3\}$ and transition functions between them. These transition functions themselves will be described in terms of two functions

$$\rho_1, \rho_2 : D_3 \to SU(2),$$

where D_3 is the 3-dimensional closed disk of radius 1.

For this, take

$$S^4 = \left\{\mathbf{x} = (x_1, \ldots, x_5) \in \mathbb{R}^5 \mid \|\mathbf{x}\| = 1\right\}$$

and for $1/5 > \epsilon > 0$ consider the triple open cover

$$U_1 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 > -\epsilon\right\},$$

$$U_2 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 < 0, x_4 > -\epsilon\right\}$$

and

$$U_3 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 < 0, x_4 < \epsilon\right\}.$$

Also call $D_- = \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0\right\}$ and identify the closed 3-dimensional disk with

$$D_3 = \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0, x_4 = 0\right\}.$$

There is a natural retraction $r : D_- \to D_3$ leaving D_3 fixed (See Figure 5.3.1). This is a smooth function almost everywhere.

Take $V = D_3 \cap U_1$. Then we get that

$$V = \{(x_1, \ldots, x_5) \in D_3 \mid x_5 > -1/3\}.$$

Now suppose that the functions $\rho_1, \rho_2 : D_3 \to SU(2)$ are smooth functions such that:

- They are independent of the radius in D_3 in V.
- They are commutative in the closure of V.

We define the transition functions $\rho_{ij} : U_i \cap U_j \to SU(2)$ by

- $\rho_{12} := \rho_1 \circ r$.
- $\rho_{23} := \rho_2 \circ r$.
- $\rho_{13} := (\rho_1 \circ r)(\rho_2 \circ r)$.

Since $r(U_1 \cap U_2 \cap U_3) \subseteq V$ by construction, the previous cocycles commute with each other in their common domain $U_1 \cap U_2 \cap U_3$. This transition functions allow us to construct a smooth vector bundle $E \to S^4$, and so we have constructed a TC structure

$$\xi = \left\{E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2)\right\}.$$

5.3.2. Associated clutching functions: Before dealing with the result we need, it is important to highlight the following. Suppose $E_1 \to M$ and $E_2 \to M$ are smooth vector bundles with classifying functions $f_i : M \to BSU(n)$, $i = 1, 2$. If there is a (non necessarily continous) homotopy between f_1 and f_2, and there is class $c \in H^*(BSU(n))$, it follows that $f_1^*(c) = f_2^*(c) \in H^*(M)$. Now consider the curvatures Ω_1 and Ω_2 for E_1 and E_2, respectively. By the Chern-Weil isomorphism, we get that $c(\Omega_1) = f_1^*(c)$ and $c(\Omega_2) = f_2^*(c)$, and thus $c(\Omega_1) = c(\Omega_2)$. In particular if there is a continuous (but not smooth) isomorphism of vector bundles between E_1 and E_2, their classifying functions will be homotopic and their characteristic classes will coincide.

Now consider the closed sets

$$C_1 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 \geq 0\right\},$$

$$C_2 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0, x_4 \geq 0\right\}$$

and

$$C_2 := \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0, x_4 \leq 0\right\}.$$

It is clear that there is a retraction $r_i : U_i \to C_i$ leaving $C_i$fixed, for $i = 1, 2, 3$. Notice that by applying on $U_2 \cap U_3$ r_2 first and then r_3, we obtain a retraction $r_{23} : U_2 \cap U_3 \to C_2 \cap C_3$ leaving $C_2 \cap C_3$ fixed. For $U_1 \cap U_2$ we apply first r_2 and then r_3, we obtain a retraction $r_{12} : U_1 \cap U_2 \to C_1 \cap C_2$ leaving $C_1 \cap C_2$ fixed, and similarly we obtain $r_{13} : U_1 \cap U_3 \to C_1 \cap C_3$ leaving $C_1 \cap C_3$ fixed. Via this restrictions of ρ_{ij} we obtain transition functions for the closed cover $\{C_1, C_2, C_3\}$:

$$\tilde{\rho}_{ij} : C_i \cap C_j \to SU(2).$$

This new transition functions are clearly homotopic to ρ_{ij} via the retractions r_{ij}. Thus, they characterized vector bundles over S^4 whose classifying functions are homotopic.

Consider the identification $S^3 \cong \left\{(x_1, \ldots, x_5) \in S^4 \mid x_5 = 0\right\}$. This setting allow us to apply Lemma 3.1 of [**RV**]. There they show that the bundle induced by these three cocycles is isomorphic to the vector bundle with clutching function $\varphi : S^3 \to SU(2)$ defined for $\mathbf{x} = (x_1, \ldots, x_5)$ by

$$\varphi(\mathbf{x}) := \begin{cases} \rho_1(r(\mathbf{x}))\rho_2(r(\mathbf{x})) & x_4 \geq 0. \\ \rho_1(r(\mathbf{x}))\rho_2(r(\mathbf{x})) & x_4 \leq 0. \end{cases}$$

The function φ can clearly be extended continuously to the whole disk D_-, since we defined r on D_-. This implies that φ is null homotopic, and thus, the vector bundle given by these cocycles is trivial.

Now lets consider the same construction but using the cocycles given by $\sigma_{ij} = \rho_{ij}^{-1}$. They give rise to the (-1)-th associated bundle by definition. Once again allow us to use Lemma 3.1 of [**RV**]. We conclude that this bundle can be obtain, up to isomorphims, by the clutching function given by

$$\phi(y) := \begin{cases} \rho_1^{-1}(r(\mathbf{x}))\,\rho_2^{-1}(r(\mathbf{x})) & x_4 \geq 0. \\ \rho_2^{-1}(r(\mathbf{x}))\,\rho_1^{-1}(r(\mathbf{x})) & x_4 \leq 0. \end{cases}$$

In this case this function cannot be extended continuously to D_- if ρ_1 and ρ_2 do not commute everywhere in D_3. So ϕ is not necessarily null homotopic.

5.3.3. Existence of a non trivial TC structure: From the previous part, we need to show that it is possible to obtain a non null homotopic clutching function ϕ. For this it is enought to display two functions $\rho_1, \rho_2 : D_3 \to SU(2)$ such that they commute in $\partial D_3 \cong S^3$, giving us a non zero Chern class for the bundle with clutching function $\phi : S^3 \to SU(2)$.

We can describe ϕ in terms of the northern and southern hemispheres of S^3, D_+ and D_-, respectively. Each of them can be identify with the 3-dimensional disc D_3. Then we get that

$$\phi(y) := \begin{cases} \rho_1^{-1}\rho_2^{-1} & \text{in } D_+, \\ \rho_2^{-1}\rho_1^{-1} & \text{in } D_-. \end{cases}$$

For brevity allow us to write the matrices of $SU(2)$ as

$$(a,b) := \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}.$$

PROPOSITION 5.4. *Consider D_3 under spherical coordinates and take*

$$\rho_1(\alpha,\beta,r) := \begin{cases} \left(\sin\left(\frac{\pi}{2}r\right)e^{i\alpha}, \cos\left(\frac{\pi}{2}r\right)\right), & 0 \leq \beta \leq \pi/2. \\ \left(\sin(r\beta)\,e^{i\alpha}, \cos(r\beta)\right) & \pi/2 \leq \beta \leq \pi. \end{cases}$$

$$\rho_2(\alpha,\beta,r) := \begin{cases} \left(-\cos(\pi r)\,e^{2i\beta}, \sin(\pi r)\right), & 0 \leq \beta \leq \pi/2. \\ (\cos(\pi r), \sin(\pi r)) & \pi/2 \leq \beta \leq \pi. \end{cases}$$

then the second Chern class of ϕ is $c_2(\phi) = -1$.

PROOF. Recalled from the previous section that if we make $\phi = (z,w)$, the second Chern class of ϕ is then given by

$$c_2 = \frac{1}{24\pi^2}\int_{S^3} A.$$

Where A is a three form given by

$$2\left(\bar{z}dzdwd\bar{w} + \bar{w}dzd\bar{z}dw - 2\left(zd\bar{z}dwd\bar{w} + wdzd\bar{z}d\bar{w}\right)\right).$$

We can split this integral as

$$\int_{S^3} A = \int_{D_-} A + \int_{D_+} A.$$

Now, call $\rho_1^{-1}\rho_2^{-1} = (z_1, w_1)$ and $\rho_2^{-1}\rho_1^{-1} = (z_2, w_2)$. Because of orientations, we get

$$\int_{S^3} A = \int_{D_3} A_2 - \int_{D_3} A_1 = \int_{D_3} (A_2 - A_1)$$

where

$$A_1 = 2\left(\bar{z}_1 dz_1 dw_1 d\bar{w}_1 + \bar{w}_1 dz_1 d\bar{z}_1 dw_1 - 2\left(z_1 d\bar{z}_1 dw_1 d\bar{w}_1 + w_1 dz_1 d\bar{z}_1 d\bar{w}_1\right)\right)$$
$$A_2 = 2\left(\bar{z}_2 dz_2 dw_2 d\bar{w}_2 + \bar{w}_2 dz_2 d\bar{z}_2 dw_2 - 2\left(z_2 d\bar{z}_2 dw_2 d\bar{w}_2 + w_2 dz_2 d\bar{z}_2 d\bar{w}_2\right)\right).$$

From this we have for $0 \leq \beta \leq \pi/2$ that

$$\begin{aligned}(z_2, w_2) = \rho_2^{-1}\rho_1^{-1} = \Big(&-\sin\left(\frac{\pi}{2}r\right)\cos(\pi r)\, e^{-(\alpha+2\beta)i} - \sin(\pi r)\cos\left(\frac{\pi}{2}r\right),\\ &\cos(\pi r)\cos\left(\frac{\pi}{2}r\right) e^{2i\beta} - \sin(\pi r)\sin\left(\frac{\pi}{2}r\right) e^{-i\alpha}\Big)\end{aligned}$$

$$\begin{aligned}(z_1, w_1) = \rho_1^{-1}\rho_2^{-1} = \Big(&-\sin\left(\frac{\pi}{2}r\right)\cos(\pi r)\, e^{-(\alpha+2\beta)i} - \sin(\pi r)\cos\left(\frac{\pi}{2}r\right),\\ &\cos(\pi r)\cos\left(\frac{\pi}{2}r\right) e^{-2i\beta} - \sin(\pi r)\sin\left(\frac{\pi}{2}r\right) e^{i\alpha}\Big),\end{aligned}$$

while for $\pi/2 \leq \beta \leq \pi$ we have

$$\begin{aligned}(z_2, w_2) = \rho_2^{-1}\rho_1^{-1} = \big(&\sin(r\beta)\cos(\pi r)\, e^{-\alpha i} - \sin(\pi r)\cos(r\beta),\\ &-\sin(\pi r)\sin(r\beta)\, e^{-i\alpha} - \cos(\pi r)\cos(r\beta)\big)\end{aligned}$$

$$\begin{aligned}(z_1, w_1) = \rho_1^{-1}\rho_2^{-1} = \big(&\sin(r\beta)\cos(\pi r)\, e^{-\alpha i} - \sin(\pi r)\cos(r\beta),\\ &-\sin(\pi r)\sin(r\beta)\, e^{i\alpha} - \cos(\pi r)\cos(r\beta)\big).\end{aligned}$$

Observe that in both cases we have that $z_1 = z_2$ y $w_1 = \bar{w}_2$. Then we have to integrate the form

$$A_2 - A_1 = 4\underbrace{\left(2z_1 d\bar{z}_1 - \bar{z}_1 dz_1\right) dw_1 d\bar{w}_1}_{B_1} + 6\underbrace{\left(w_1 d\bar{w}_1 - \bar{w}_1 dw_1\right) dz_1 d\bar{z}_1}_{B_2}.$$

Now consider the decomposition $z_1 = z = x + yi$ and $w_1 = w = u + vi$, where x, y, u and v are real functions. Then it follows that

$$\left(2z_1 d\bar{z}_1 - \bar{z}_1 dz_1\right) = (xdx + ydy) + 3\,(ydx - xdy)\, i,$$
$$dw_1 d\bar{w}_1 = -2i du \wedge dv,$$
$$\left(w_1 d\bar{w}_1 - \bar{w}_1 dw_1\right) = 2\,(vdu - udv)\, i$$

and $dzd\bar{z} = -2idx \wedge dy$. This gives us

$$B_1 = 6\,(ydx - xdy)\, du \wedge dv - 2i\,(xdx + ydy)\, du \wedge dv$$

and

$$B_2 = 4\,(vdu - udv)\, dx \wedge dy.$$

Since we only need to compute the real part of the first form, we consider the form $6\,(ydx - xdy)\, du \wedge dv$ instead of all of B_1. Then by definition we get

$$ydx - xdy = \left(y\frac{\partial x}{\partial \alpha} - x\frac{\partial y}{\partial \alpha}\right) d\alpha + \left(y\frac{\partial x}{\partial \beta} - x\frac{\partial x}{\partial \beta}\right) d\beta + \left(y\frac{\partial x}{\partial r} - x\frac{\partial y}{\partial r}\right) dr,$$

$$vdu - udv = \left(v\frac{\partial u}{\partial \alpha} - u\frac{\partial v}{\partial \alpha}\right) d\alpha + \left(v\frac{\partial u}{\partial \beta} - u\frac{\partial x}{\partial \beta}\right) d\beta + \left(v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r}\right) dr,$$

$$du \wedge dv = \left(\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \beta}\right) d\alpha d\beta + \left(\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial r}\right) d\alpha dr + \left(\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \beta}\frac{\partial u}{\partial r}\right) d\beta dr$$

and

$$dx \wedge dy = \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha}\frac{\partial x}{\partial \beta}\right) d\alpha d\beta + \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial r} - \frac{\partial y}{\partial \alpha}\frac{\partial x}{\partial r}\right) d\alpha dr + \left(\frac{\partial x}{\partial \beta}\frac{\partial y}{\partial r} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial r}\right) d\beta dr.$$

Now call $J_1 := (ydx - xdy)\, du \wedge dv$ and $J_2 := (ydx - xdy)\, du \wedge dv$. Then we have

$$\begin{aligned} J_1 = \Bigg[& \left(y\frac{\partial x}{\partial \alpha} - x\frac{\partial y}{\partial \alpha}\right)\left(\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \beta}\frac{\partial u}{\partial r}\right) - \left(y\frac{\partial x}{\partial \beta} - x\frac{\partial x}{\partial \beta}\right)\left(\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial r}\right) \\ & + \left(y\frac{\partial x}{\partial r} - x\frac{\partial y}{\partial r}\right)\left(\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \beta}\right)\Bigg] d\alpha \wedge d\beta \wedge dr, \\ J_2 = \Bigg[& \left(v\frac{\partial u}{\partial \alpha} - u\frac{\partial v}{\partial \alpha}\right)\left(\frac{\partial x}{\partial \beta}\frac{\partial y}{\partial r} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial r}\right) - \left(v\frac{\partial u}{\partial \beta} - u\frac{\partial x}{\partial \beta}\right)\left(\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial r}\right) \\ & + \left(v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r}\right)\left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha}\frac{\partial x}{\partial \beta}\right)\Bigg] d\alpha \wedge d\beta \wedge dr. \end{aligned}$$

Then by replacing we get $A_2 - A_1 = 24\,(J_1 + J_2)$, and even further

$$c_2(\phi) = \frac{1}{\pi^2}\int (J_1 + J_2)\,.$$

Where by using computational software we obtain that $\int (J_1 + J_2) = -\pi^2$, giving us $c_2(\phi)$ =-1.

We conclude that $c_2^{-1}(\xi) = -1$ for our TC structure, implying that the TC structure is non trivial. $\square$

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