UNIVERSIDAD NACIONAL DE COLOMBIA

El teorema de completación de Atiyah-Segal

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To my father

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Resumen

Esta tesis es sobre el teorema de completacin de Atiyah-Segal. La primera parte estudia los conceptos bsicos de fibrados vectoriales y teora de representacin. Luego, se estudia la K-teora y la K-teora equivariante. Finalmente, se construye y prueba el teorema de Atiyah-Segal, con nfasis en sistemas inversos.

Palabras claves: Atiyah-Segal, teorema, topología algebraica, completación, K-teoría, sistemas inversos.

Abstract

This dissertation is about Atiyah-Segal's theorem of completion. The first part studies the basic concepts of vector bundles and representation theory. Then we study K-theory and equivariant K-theory. Lastly, we construct and prove Atiyah-Segal's theorem with emphasis on inverse systems.

Keywords: Atiyah-Segal, theorem, algebraic topology, completion, K-theory, inverse systems.

Contents

Introduction

K-theory is a theory initially introduced by Alexander Groethendieck in the 50's to study coherent sheaves in algebraic varieties. Sometime later, Michael Atiyah and Friedrich Hirzebruch translated these ideas to algebraic topology so they could be used in the study of vector bundles over compact topological spaces. In this way, K-theory can be defined as the ring $K(X)$ of equivalence classes of vector bundles over a topological space X. In this ring, the sum will be the Whitney sum and the product the tensor product of vector bundles. Using Bott's periodicity, one can define $K^n(X)$ and prove that K-theory corresponds to a generalized cohomology theory that does not satisfy the dimension axiom. We can also use Brown's representability theorem to extend the concept of K-theory to be defined also over spaces that are not necessarily compact.

From this construction, Michael Atiyah invents a theory called the equivariant K-theory, which is the equivariant cohomology version of K-theory. In this theory, we consider equivalence classes of G-vector bundles over a topological space X. Note that there are various relationships between K-theory and equivariant K-theory. One of the most common is when the group G acts freely on X. In this case, we have that $K_G(X) \cong K(X/G)$. Nevertheless, in general, if G does not act freely on X we do not have the previous isomorphism. To solve this problem, instead of considering quotients, we can consider homotopy quotients that are denoted by $X \times_G EG$, where EG is the total space of the universal principal G-bundle. The homotopy quotients are also known as Borel's construction.

On the other hand, given a compact Lie group G, we define the representation ring $R(G)$ as the formal differences of isomorphism classes of linear finite dimensional representations of G over $\mathbb{C}$. The ring structure is given by direct sum and tensor product of representations of the group G. The representation ring also comes equipped with a ring homomorphism called the augmentation map $\varepsilon\colon R(G) \to \mathbb{Z}$. This map sends a formal difference of representations over G to its "virtual dimension". The kernel of this map is called the augmentation ideal and is going to be denoted as

I_G. This ideal gives us a natural filtration of the ring $R(G)$ in the following way

$$R(G) \supseteq I \supseteq I^2 \supseteq \cdots \supseteq I^n \supseteq I^{n+1} \supseteq \cdots.$$

We define the completion of the ring $R(G)$ with respect to the augmentation ideal as the inverse limit $R(G)\hat{}_{I_G} = \varprojlim_n (R(G)/I^n)$. This completion can be done with any $R(G)$-module.

Since the equivariant K-theory of a G-space X is a $R(G)$-module, we can consider its completion with respect to the augmentation ideal $K_G(X)\hat{}_{I_G} = \varprojlim_n (K_G(X)/I_G^n K_G(X))$. Then, Atiyah-Segal's completion theorem tells us that the K-theory of the space $K(X \times_G EG)$ is isomorphic the completion of $K_G(X)$ with respect to the augmentation ideal I_G. This is done through the projection map $\pi \colon X \times EG \to X$. Using this theorem when $X = *$, we can find the K-theory of the classifying space BG when G is a compact Lie group. It gives us that $K(BG)$ is isomorphic to the completion of the ring $R(G)$ with respect to the augmentation ideal.

Atiyah-Segal's completion theorem can be interpreted as a connection between a geometric process and an algebraic process. In this case, the geometrical process can be evidenced in the homotopy quotient of the G-space which makes the action free before passing it to the quotient, and the algebraic process is seen when completing with respect to the augmentation ideal.

This work will focus on studying the components involved in Atiyah-Segals theorem along with its demonstration. Specifically, the study is focused on Atiyah and Segal's article [4]. The proof is done by using the tools of inverse systems rather than pro-objects as done in [4]. This thesis is organized in the following way. In chapter 1, we focus on studying the concepts of bundles and representations. These concepts are key to understanding and defining non-equivariant and equivariant K-theory. In chapter 2 we define K-theory and its equivariant part. We also explore their generalized cohomology theories properties and mention how to extend K theory to non-compact spaces. Lastly, in chapter 3 we see the construction and proof of Atiyah-Segals completion theorem and some examples of the use of this theorem.

1. Bundles and representations

In this chapter we will review some prerequisite knowledge needed for this work, specifically, concepts on vector bundles, Lie groups, principal bundles and representations of Lie groups. We will go over some of the basic definitions and prove some results useful to construct the theory. This will be done in a rather concise manner. The concepts in this chapter will be required to understand Atiyah-Segal's completion theorem, which is the main focus of this work.

1.1. Vector bundles

Definition 1.1. An n-dimensional *complex vector bundle* is a continuous function $p\colon E \to B$ where $p^{-1}(b)$ is a complex vector space for all $b \in B$ such that there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of B together with homeomorphisms $h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ that map $p^{-1}(b)$ to $\{b\} \times \mathbb{C}^n$ by a vector space isomorphism for all $b \in U_\alpha$. The maps h_α are called local trivializations of the vector bundle.

In this definition, the space B is called the base space, E is called the total space and $p^{-1}(b)$ are called the fibers for all $b \in B$. This will often be denoted by E_b when the map is clear. We will also abbreviate the terminology sometimes and call the vector bundle itself E when the function and base space are clear. It is important to note that we can also create the notion of a *real vector bundle* by taking $\mathbb{R}$ instead of $\mathbb{C}$ for the scalar field. In this thesis we will work mostly with complex vector bundles.

Example 1.2.

1. The simplest example of a complex vector bundle can be done by taking $E = B \times \mathbb{C}^n$ and p as the canonical projection of $E \in B$. This is called the *trivial bundle* and is often used.

2. An interesing example of a real vector bundle can be constructed by taking $B = \mathbf{S}^1 = [0,1]/\sim$ where $0 \sim 1$, and $E = ([0,1] \times \mathbb{R})/\sim$ where $(0,x) \sim (1,-x)$ for all $x \in \mathbb{R}$. We consider $p\colon E \to B$ the map induced by the canonical projection $\mathrm{pr}\colon [0,1] \times \mathbb{R} \to [0,1]$. This vector bundle also known as the *Möbius band*.

Definition 1.3. An *isomorphism* between two vector bundles $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ over the same base space B is a homeomorphism $h\colon E_1 \to E_2$ that maps each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism. This will also be denoted by $E_1 \cong E_2$.

Definition 1.4. A *section* of a vector bundle $p\colon E \to B$ is a map $s\colon B \to E$ that assigns to each $b \in B$ a vector $s(b) \in p^{-1}(b)$ such that $p \circ s = \mathrm{id}_B$.

One can note that every vector bundle has a canonical section called the zero section, that assigns to each $b \in B$ the zero vector in the fiber $p^{-1}(b)$. We will briefly see now some results regarding vector bundles that will be important to our discussion.

Lemma 1.5. *A continuous map $h\colon E_1 \to E_2$ between vector bundles over the same base space B, is an isomorphism if it takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism.*

Proof. For h to be a vector bundle isomorphism it needs to be a homeomorphism. Given the hypotheses of the lemma regarding the fibers, it is clear that the function h is both injective and surjective. Therefore we only need to prove that h^{-1} is continuous.

Since B admits a covering for both fibers and the property of being continuous is local, we can restrict our analysis to an open set $U \subseteq B$ where E_1 and E_2 are both trivial. If we denote by $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ our vector bundles, we can construct the following commuting diagram

$$\begin{array}{ccc} p_1^{-1}(U) & \xrightarrow{h} & p_2^{-1}(U) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ U \times \mathbb{C}^n & \xrightarrow{l} & U \times \mathbb{C}^n, \end{array}$$

where $l\colon U \times \mathbb{C}^n \to U \times \mathbb{C}^n$ is defined by $l(x,v) = (x, g_x(v))$ for all $x \in U$ and $v \in \mathbb{R}^n$. We then have that g_x can be thought of as an element of $\mathrm{GL}(n,\mathbb{C})$ that depends continuously on x. Therefore

g_x^{-1} also depends continuously on x since its entries can be expressed algebraically in terms of the entries of g_x. This proves that $l^{-1}(x,v) = (x, g_x^{-1}(v))$ is continuous so h^{-1} is also continuous. □

With this in mind, we can prove the following result.

Proposition 1.6. *An n-dimensional bundle $p\colon E \to B$ is isomorphic to the trivial bundle if and only if there are n sections $s_1, \dots, s_n$ such that the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$.*

Proof. First, note that it is clear that if a vector bundle is isomorphic to the trivial bundle it must have said sections.

Suppose now that we have a vector bundle $p\colon E \to B$ that has n sections $s_1, \dots, s_n$ such that the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$. Let us then define the map $h\colon B \times \mathbb{C}^n \to E$ by $h(b, t_1, \dots, t_n) = \sum_{i=1}^n t_i s_i(b)$ for all $b \in B$ and all $t_1, \dots, t_n \in \mathbb{C}$.

Note that by definition, h is a linear isomorphism in each fiber and it is a continuous function between the trivial bundle and p. By Lemma 1.5, h is a vector bundle isomorphism. □

1.1.1. Pullback and inner products

We will denote by $\mathbf{Vect}_n(B)$ the set of isomorphism classes of n-dimensional vector bundles over B. With the following proposition we will define the pullback of a vector bundle.

Proposition 1.7. *Let $f\colon A \to B$ be a map and $p\colon E \to B$ a vector bundle. Then there exists a vector bundle $p'\colon E' \to A$ together with a map $f'\colon E' \to E$ that maps E'_a isomorphically to $E_{f(a)}$ for all $a \in A$. The map $p'\colon E' \to A$ is unique up to isomorphisms.*

Proof. Let us begin by defining the space

$$E' = \{(a,v) \in A \times E \colon f(a) = p(v)\} \subseteq A \times E.$$

We then have the following diagram

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B, \end{array}$$

where $p'\colon E' \to A$ and $f'\colon E' \to E$ are defined by $p'(a,v) = a$ and $f'(a,v) = v$ for all $(a,v) \in E'$. Note that with these definitions we have that $f \circ p' = p \circ f'\colon E' \to B$. Hence the previous diagram is a commutative diagram. We need to prove that $p'\colon E' \to A$ is a vector bundle. Note that

$$\begin{aligned} E'_a &= \{(a,v) \in E' \colon p'(a,v) = a\} = \{(a,v) \in A \times E \colon p'(a,v) = a \text{ and } f(a) = p(v)\} \\ &= \{a\} \times \{v \in E \colon p(v) = f(a)\} = \{a\} \times E_{f(a)}. \end{aligned}$$

Then E'_a and $E_{f(a)}$ are isomorphic as vector spaces. This proves that $E'_a = p'^{-1}(a)$ has a complex n-dimensional space structure for all $a \in A$.

Let $\{U_\alpha\}_\alpha$ be a covering of B with local trivializations $h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$. Then $f^{-1}(U_\alpha)$ is open in A and $\{f^{-1}(U_\alpha)\}_\alpha$ is an open cover of A. Let $h'_\alpha\colon p'^{-1}(f^{-1}(U_\alpha)) \to f^{-1}(U_\alpha) \times \mathbb{C}^n$ be a function defined by $h'_\alpha(a,v) = (a, \pi_2(h_\alpha(v)))$ where $\pi_2\colon U_\alpha \times \mathbb{C}^n \to \mathbb{C}^n$ is the canonical projection. Then h'_α is a local trivialization making p' a vector bundle.

Lastly, let us prove the uniqueness. Let $p''\colon E'' \to A$ and $f''\colon E'' \to E$ be such that E''_a and $E_{f(a)}$ are isomorphic as vector spaces for all $a \in A$. We have the following commutative diagram

$$\begin{array}{ccc} E'' & \xrightarrow{f''} & E \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \\ {\scriptstyle p'}\uparrow & & {\scriptstyle p}\uparrow \\ E' & \xrightarrow{f'} & E. \end{array}$$

Let us define $\varphi\colon E'' \to E$ by $\varphi(v'') = (p''(v''), f''(v''))$ since $f(p''(v'')) = p(f''(v''))$ the function φ is well defined and continuous. Analyzing what happens to the fibers we note that if $v'' \in E''_a$ then $p(f''(v'')) = f(p''(v'')) = f(a)$, then $\varphi\colon E''_a \to \{a\} \times E_{f(a)} = E'_a$. Hence φ is a vector space isomorphism in each fiber and, by Lemma 1.5, φ is a vector bundle isomorphism. □

Since p' in the previous lemma is unique, we can construct a function $f^*\colon \mathbf{Vect}_n(B) \to \mathbf{Vect}_n(A)$ defined by $f^*([E]) = [E']$. Hence we will use the notations $f^*(p)$ instead of p' and $f^*(E)$ instead of E' for the map and total space of the new bundle. This will be called the *pullback* bundle of E by f.

Definition 1.8. A *Hermitian metric* on a complex vector bundle $p\colon E \to B$ is a continuous function $\langle \cdot, \cdot \rangle\colon E \otimes E \to \mathbb{C}$ that restricted to each fiber is a Hermitian product, meaning, a bilinear form which is $\mathbb{C}$-anti-linear in the first input and $\mathbb{C}$-linear on the second.

Let us remember that a space X is paracompact if it is Hausdorff and every open cover has a partition of unity associated. With this in mind, we have the following lemma.

Lemma 1.9. *If B is paracompact and $p\colon E \to B$ is a vector bundle then there is a Hermitian metric in E.*

Proof. Let us denote $h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ some local trivializations of the vector bundle and let $\{f_\alpha\}_\alpha$ be a partition of unity associated to $\{U_\alpha\}_\alpha$. In each subset $p^{-1}(U_\alpha)$ we can pullback the standard Hermitian metric in $\mathbb{C}^n$ to a Hermitian metric $\langle \cdot, \cdot \rangle_\alpha$ on $p^{-1}(U_\alpha)$. We define the Hermitian metric on the vector bundle by

$$\langle u, v\rangle = \sum_\alpha f_\alpha(p(u))\langle u, v\rangle_\alpha,$$

where $\langle u, v\rangle_\alpha = 0$ when $u, v \notin p^{-1}(U_\alpha)$. It is easy to see that this defines a Hermitian metric on E. ☐

Definition 1.10. Let $p\colon E \to B$ be a vector bundle. A *sub-vector bundle* E_0 is a subspace of E such that the intersection in each fiber of p is a sub-vector space and $p\colon E_0 \to B$ is a vector bundle.

Theorem 1.11. *Let $p\colon E \to B$ be a vector bundle with B paracompact. Then any sub-vector bundle $E_0 \subseteq E$ has an orthogonal complement. In other words, there is a sub-vector bundle $E_0^\perp$ of E such that $E_0^\perp \oplus E_0 \cong E$.*

Proof. By the previous lemma we know that there is a Hermitian metric in E. Let us define

$$E_0^\perp = \{v\colon \langle v, w\rangle = 0 \text{ for all } w \in E_0 \text{ with } p(v) = p(w)\}.$$

Let us see that $E_0^\perp$ is a vector space. Since this is a local property, it is enough to consider the sub-vector bundle $E_0 \subseteq B \times \mathbb{C}^n$. Let $s_1, \ldots, s_m$ be the linearly independent sections of E_0. For any given $b \in B$, there are vectors $v_{m+1}, \ldots, v_n$ such that $s_1(b), \ldots, s_m(b), v_{m+1}, \ldots v_n$ is a basis for $\mathbb{C}^n$. Then, there is an open neighborhood U of $b \in B$ such that the vectors $s_1(x), \ldots, s_m(x), v_{m+1}, \ldots v_n$ are linearly independent for all $x \in U$.

Applying the Gram-Schmidt process we can obtain orthogonal sections $s'_{m+1}, \ldots, s'_n \in U$. These are sections for $E_0^\perp$ and with them we can define the trivializations $\varphi\colon U \times \mathbb{C}^{n-m} \to p^{-1}|_{E_0}(U)$ by $\varphi(b, x_{m+1}, \ldots, x_n) = \sum_{i=m+1}^n x_i s'_i(b)$. ☐

Proposition 1.12. *If $p\colon E \to B$ is a vector bundle and B is compact Hausdorff then there exists $p'\colon E' \to B$ such that $E \oplus E' \cong B \times \mathbb{C}^n$.*

Proof. If $x \in B$ then there is an open set U_x such that $E|_{U_x}$ is trivial. Therefore we have a function $h_x\colon p^{-1}(U_x) \to U_x \times \mathbb{C}^n$. Since $\{x\}$ and $B \setminus U_x$ are closed, by Urysohn's Lemma there is a function $\varphi_x\colon B \to [0,1]$ such that $\varphi_x(B \setminus U_x) = 0$ and $\varphi_x(x) = 1$.

Then the sets $\{\varphi_x^{-1}((0,1])\}_{x\in B}$ cover B. Since B is compact there are U_i for $i = 1,\ldots,m$ such that $\varphi_i^{-1}((0,1]) \subseteq U_i$ cover B. We then define $g_i\colon E \to \mathbb{C}^n$ by

$$g_i(v) = \begin{cases} \varphi_i(p(v))(\pi_i(h_i(v))) & v \in p^{-1}(U_i), \\ 0 & v \notin p^{-1}(\mathrm{Supp}(\varphi_i)), \end{cases}$$

where $h_i\colon U_i \times \mathbb{C}^n \to \mathbb{C}^n$ is the projection to $\mathbb{C}^n$. With this definition, it is clear that $g_i|_{E_b}$ is a linear injection on each fiber over $\varphi_i^{-1}((0,1])$. We can then define $g\colon E \to \mathbb{C}^N$ with $N = nm$ by $g(v) = (g_1(v),\ldots,g_m(v))$ for all $v \in E$. This function is then also a linear injection on each fiber.

Let us define now the function $f\colon E \to B \times \mathbb{C}^N$ by $f(v) = (p(v), g(v))$. Then $f(E)$ can be seen as a sub-vector bundle of $B \times \mathbb{C}^N$. Since f is injective, $f(E) \cong E$. Therefore E is isomorphic to a sub-vector bundle of the trivial bundle $B \times \mathbb{C}^N$. By the previous theorem, we can construct E' with the orthogonal complement and we can consider only the inner product of $\mathbb{R}^N$ in the second component. Hence having the result. □

1.2. *G*-maps and principal bundles

Let us remember that a Lie group G is a group with a smooth manifold structure such that the multiplication and inverse maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$, given by $\mu(g,h) = gh$ and by $\iota(g) = g^{-1}$ for all g and $h \in G$, are smooth.

Definition 1.13. A *smooth left action* of a Lie group G over a smooth manifold M is a smooth map $\sigma\colon G \times M \to M$ such that $\sigma(g, \sigma(h,p)) = \sigma(gh, p)$ for all $g,h \in G$ and all $p \in M$, and $\sigma(e,p) = p$ for all $p \in M$ where $e \in G$ denotes the identify element of G.

With an action σ one can define a map $\sigma_g\colon M \to M$ for $g \in G$ where $\sigma_g(p) = \sigma(g,p)$ for all $p \in M$. It is easy to see that this function satisfies that $(\sigma_g)^{-1} = \sigma_{g^{-1}}$ and $\sigma_g \circ \sigma_h = \sigma_{gh}$. One can also define a right action $\widetilde{\sigma}\colon M \times G \to M$ by $\widetilde{\sigma}(p,g) = \sigma(g^{-1}, p)$ for all $p \in M$ and $g \in G$. We will then say that M is a left G-space if it has a left action defined on it and a right G-space if it has a right action. To simplify notation, we will often make the identification $\sigma(g,p) = gp$ when the action is clear.

Definition 1.14. Let G be group and M a left G-space.

1. An action is *transitive* if $Gp = M$ for all $p \in M$, in other words, an action is transitive if we have that for all p and $q \in M$ there exists $g \in G$ such that $gp = q$.
2. An action is *free* if for every $p \in M$ the isotropy group $\{g \in G\colon\, gp = p\}$ equals $\{e\}$, meaning, if $g \in G$ is such that $gp = p$ for some $p \in M$ then $g = e$.
3. An action is *effective* if the induced homomorphism $\tilde{\sigma}\colon G \to \mathrm{Diff}(M)$ defined by $\tilde{\sigma}(g) = \sigma_g$ for all $g \in G$ is injective.

Definition 1.15. Let M be a smooth manifold and G a Lie group. A *principal G-bundle* is a triplet (P, π, σ) where P is a smooth manifold, $\pi\colon P \to M$ is a surjective map and $\sigma\colon P \times G \to P$ is a right action of G on P such that the following conditions are true.

1. The action σ preserves the fibers of π, meaning, $\pi(pg) = \pi(p)$ for all $p \in P$ and all $g \in G$.
2. For all $m \in M$ there exists an open set $U \subseteq M$ and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times G$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\Phi\quad} & U \times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U, & \end{array}$$

   where $\mathrm{pr}\colon U \times G \to U$ is the canonical projection and $\Phi^{-1}(x, hg) = \Phi^{-1}(x,h)g$ for all $x \in U$ and all $g, h \in G$.

Similarly as for vector bundles, P is called the total space, M is called the base space and Φ is called the local trivialization. Note that the last condition implies that the function Φ has the form $\Phi(p) = (\pi(p), \varphi(p))$ where $\varphi\colon \pi^{-1}(U) \to G$ is a smooth map such that $\varphi(pg) = \varphi(p)g$. We will often refer to the principal G-bundle (P, π, σ) as just $\pi\colon P \to M$ when the action σ is clear.

Also note that the action $\sigma\colon P \times G \to P$ is free. This is because if we have that $pg = p$ then there exist $x \in U$ and $h \in G$ such that $\Phi^{-1}(x,h) = p$. We can then calculate

$$\Phi^{-1}(x,hg) = \Phi^{-1}(x,h)g = pg = p = \Phi^{-1}(x,h).$$

Since Φ is a diffeomorphism this means that $hg = h$ hence $g = e$.

It is also possible to see that $\pi^{-1}(\pi(p)) \cong Gp = \{pg\colon g \in G\}$. This tells us that the action is transitive on the fibers and $\pi^{-1}(x)$ is diffeomorphic to G for all $x \in M$.

Example 1.16.

1. The simplest example of a principal G-bundle can be done with $P = M \times G$ and the canonical projection $\mathrm{pr}\colon M \times G \to M$ with the action $\sigma\colon P \times G \to P$ defined by $\sigma((x,h),g) = (x,hg)$ for all $x \in P$ and $g,h \in G$. Then (P,pr,σ) is a principal G-bundle called the *trivial principal bundle*.
2. Another example can be made for K a Lie subgroup of G. The subgroup K acts on G by right multiplication and we can define $\pi\colon G \to G/K$ the canonical projection, which happens to be a submersion. Then (G,π,μ) is a principal G-bundle.

There is an equivalent notion to trivializations that helps us define bundles in general. Suppose $\pi\colon P \to M$ is a principal G-bundle with trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ for $\alpha \in \Lambda$. Then for all $\alpha,\beta \in \Lambda$ the function

$$\Phi_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G,$$

is a diffeomorphism that maps isomorphically $\{x\} \times G$ to itself for $x \in U_\alpha \cap U_\beta$. One can then see that the function $\Phi_{\alpha\beta}$ must be given by $\Phi_{\alpha\beta}(x,g) = (x, g_{\alpha\beta}(x)g)$ for all $x \in U_\alpha \cap U_\beta$. This map will then be determined by the function $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ and vice-versa. Since the function $\Phi_{\alpha\beta}$ is smooth, so is $g_{\alpha\beta}$. One can easily check that the maps $g_{\alpha\beta}$ satisfy the following properties for all $\alpha,\beta,\gamma \in \Lambda$.

1. $g_{\alpha\alpha}(x) = e$ for all $x \in U_\alpha$.
2. If $U_\alpha \cap U_\beta \neq \emptyset$ then $g_{\alpha\beta}(x) = (g_{\beta\alpha}(x))^{-1}$ for all $x \in U_\alpha \cap U_\beta$.
3. If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then $g_{\gamma\alpha}(X) = g_{\gamma\beta}(g_{\beta\gamma}(x))$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

This properties are called the *co-cycle properties*. We will call the functions $g_{\alpha\beta}$ the *transition functions*.

From a principal G-bundle $\pi\colon P \to M$ we can then form a collection of smooth transition maps $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$. Conversely, for a given open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M and a collection of smooth maps $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ that satisfy the co-cycle properties we can create a principal G-bundle $\pi'\colon P' \to M$ by defining

$$P' = \left(\bigsqcup_{\alpha\in\Lambda} (U_\alpha \times G) \right) / \sim,$$

where $(x,g) \in U_\alpha \times G \sim (x, g_{\alpha\beta}(x)g) \in U_\beta \times G$ for all $x \in U_\alpha \cap U_\beta$, the map $\pi'\colon P' \to M$ will be induced by the natural projection to the first coordinate and the action of G in P' will induced by the multiplication to the second coordinate.

We can see that this indeed defines a principal G-bundle with trivialization functions given by $\Phi_\alpha\colon \pi'^{-1}(U_\alpha) \to U_\alpha \times G$ where $\Phi_\alpha^{-1}(x,g) = [(x,g)]$. We will not go into detail on how these two definitions are equivalent, this can be found in [7]. Nevertheless, we will sometimes use the fact that they can be both used to define the same bundle.

1.2.1. Morphisms and pullback

Definition 1.17.

1. If $\Phi\colon X \to Y$ is a map with X and Y G-spaces such that $\Phi(xg) = \Phi(x)g$ for all $x \in X$ and all $g \in G$ then Φ is called a *G-map.* We also say that Φ is G-equivariant.
2. Let $\pi\colon P \to B$ and $\pi'\colon P' \to B'$ be principal G-bundles. A *principal G-bundle morphism* is a pair $(\tilde{f}, f)\colon \pi' \to \pi$ where $\tilde{f}\colon P' \to P$ is a G-map that maps fibers to fibers and $f\colon B' \to B$ is a continuous function such that the following diagram

$$\begin{array}{ccc} P' & \xrightarrow{\tilde{f}} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

commutes.

3. A principal *G-bundle morphism over B* is a G-map $\varphi\colon \pi' \to \pi$ such that the diagram

$$\begin{array}{ccc} P' & \xrightarrow{\phi} & P \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

commutes.

We will denote $\mathbf{Bund}_G(B)$ the set of isomorphism classes of principal G-bundles over B. There are some interesting properties about principal G-bundles that differ to the notion of maps between vector bundles. The following proposition showcases a big difference between them.

Proposition 1.18. *Every principal G-bundle morphism over B is an isomorphism*

Proof. Let us begin the proof by assuming that both bundles are trivial, meaning $P, P' = B \times G$ and $\pi, \pi' = \mathrm{pr}$. Therefore we have the following commutative diagram

$$\begin{array}{ccc} B \times G & \xrightarrow{\quad f \quad} & B \times G \\ & {\scriptstyle \mathrm{pr}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}} & \\ & B, & \end{array}$$

where f is a principal G-bundle morphism over B. Since by definition we have that f is a G-map we can write $f(b,g) = (b, \tilde{\varphi}(b,g))$ where $\tilde{\varphi} \colon B \times G \to G$ is a continuous function. Therefore we have that $f(b,g) = f(b,e)g = (b, \tilde{\varphi}(b,e))g$.

Define $\varphi \colon B \to G$ by $\varphi(b) = \tilde{\varphi}(b,e)$ for all $b \in B$. This function is continuous and we can write $f(b,g) = (b, \varphi(b)g)$. Now, define the map $h \colon B \times G \to B \times G$ by $h(b,g) = (b, \varphi(b)^{-1}g)$. Then for all $b \in B$ and $g \in G$ we have that

$$h \circ f(b,g) = h(b, \varphi(b)g) = (b, \varphi(b)^{-1}\varphi(b)g) = (b,g).$$

Also,

$$f \circ h(b,g) = f(b, \varphi(b)^{-1}g) = (b, \varphi(b)\varphi(b)^{-1}g) = (b,g).$$

Since h is a continuous G-map, h is a principal G-bundle morphism whose inverse is f.

Suppose now that we have a principal G-bundle morphism f between two arbitrary bundles π and π'. We know that these bundles are locally trivial by definition. Since we also know that the action is free and transitive on the fibers, then the function on the fibers is injective and surjective hence we can write it globally. Therefore we have the result. □

We can also prove the following proposition.

Proposition 1.19. *A principal G-bundle $\pi\colon P\to B$ is trivial if and only if admits a section $s\colon B\to P$ such that $\pi\circ s = id$.*

Proof. First, suppose $\pi\colon P\to B$ is trivial. Then there is a principal G-bundle morphism $\varphi\colon \pi\to \mathrm{pr}$ such that the following diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & B\times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & B & \end{array}$$

commutes. We know that there is a section $f\colon B\to B\times G$ defined by $f(b)=(b,e)$. Then $\varphi^{-1}\circ f$ is a section for π such that $\pi\circ s = \mathrm{id}$.

Suppose now that $s\colon B\to P$ is a section for π, then we can define $\varphi\colon B\times G\to P$ by $\varphi(b,g)=s(b)g$. We then have that

$$\pi\circ\varphi(b,g)=\pi(s(b)g)=\pi(s(b))=b=\mathrm{pr}(b,g),$$

so the following diagram commutes

$$\begin{array}{ccc} B\times G & \xrightarrow{\ \varphi\ } & P \\ & {\scriptstyle\mathrm{pr}}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B. & \end{array}$$

Since φ is a G-map, $\varphi\colon \mathrm{pr}\to\pi$ is a principal G-bundle morphism. Using the previous proposition, φ is an isomorphism. $\square$

Now, let us define the pullback of a principal G-bundle by a function. Similarly as for vector bundles, with two continuous functions $f\colon X\to Z$ and $P\colon Y\to Z$ we can define the set

$$X\times_Z X=\{(x,y)\in X\times Y\colon f(x)=P(y)\},$$

and the functions $\tilde{f}$ and f^*P as the restrictions of the projections such that the following diagram commutes

$$\begin{array}{ccc} X\times_Z X & \xrightarrow{\ \tilde{f}\ } & Y \\ {\scriptstyle f^*P}\downarrow & & \downarrow{\scriptstyle P} \\ X & \xrightarrow{\ f\ } & Z. \end{array}$$

If we have a principal G-bundle $\pi\colon P \to B$ and a continuous function $f\colon B' \to B$ we define a right action of G on the set $B' \times_B P$ by $\overline{(b',p)}g = \overline{(b',pg)}$. It is easy to check that this action is well defined. Denoting $P(f^*\pi) := B' \times_B P$ we have then the following commuting diagram

$$\begin{array}{ccc} P(f^*\pi) & \xrightarrow{\tilde{f}} & P \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B. \end{array}$$

Since π is a principal G-bundle, there is an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ and trivialization maps $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times G \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U_\alpha. & \end{array}$$

Let $U'_\alpha := f^{-1}(U_\alpha) \subseteq B'$. We know that the function $\Phi_\alpha(p) = (\pi(p), \varphi_\alpha(p))$, so we can define local trivializations $\widetilde{\Phi_\alpha}\colon f^*\pi^{-1}(U_\alpha) \to U'_\alpha \times G$ by $\widetilde{\Phi_\alpha}(b',p) = (b', \varphi_\alpha(p))$ for $f^*\pi$. Thus making $f^*\pi\colon P(f^*\pi) \to B$ a principal G-bundle called the *pullback*. Lastly, we will use the following theorem to prove some results in this chapter.

Theorem 1.20 (Homotopy)**.** *Let $\pi\colon P \to B$ be a principal G-bundle and $f_0, f_1\colon B' \to B$ two homotopic functions. Then $f_0^*\pi$ and $f_1^*\pi$ are isomorphic.*

The proof of this result can be seen in [7, Theorem 4.7].

1.3. Milnor's construction

Definition 1.21. An open cover $\{U_i\}_{i\in S}$ of a topological space B is *numerable* given that there exists a locally finite partition of unity $\{U_i\}_{i\in S}$ such that the closure

$$\mathrm{cl}(U_i^{-1}((0,1])) \subseteq U_i,$$

for all $i \in S$. A principal G-bundle π over a base space B is called *numerable* if there is a numerable open cover of B such that π_{U_i} is trivial for all $i \in S$.

Definition 1.22. A principal G-bundle $w = (E_0, p_0, B_0)$ is going to be called *universal* if its numerable and the function $\phi_w \colon [-, B_0] \to \mathbf{Bund}_G$ defined by $\phi_w(X) \colon [X, B_0] \to \mathbf{Bund}_G(X)$ is an isomorphism. This means that $\phi_w(X)$ is a bijection for all spaces X.

Proposition 1.23. *A principal G-bundle $w \colon E_0 \to B_0$ is universal if and only if the following conditions are true.*

1. *For all principal G-bundles π over X there is a map $f \colon X \to B_0$ such that π and f^*w are isomorphic over X.*
2. *If $f, g \colon X \to B_0$ are two maps such that f^*w and g^*w are isomorphic over X, then f and g are homotopic.*

Proof. Note that the first condition says that $\phi_w(X)$ is surjective and the second condition that $\phi_w(X)$ is injective. This proves the theorem. □

Milnor's construction is going to provide us a way to construct a universal G-bundle for any given group G.

Definition 1.24. Let $I = [0,1]$ and X and Y be two topological spaces. We define the *join* $X * Y$ as the quotient space $X * Y = (I \times X \times Y)/\sim$ where $(0, x, y_1) \sim (0, x, y_2)$ and $(1, x_1, y) \sim (1, x_2, y)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Example 1.25. The join of any topological space X with the 0-sphere $\mathbf{S}^0$ gives us the suspension of X, in other words, $X * \mathbf{S}^0 \cong \Sigma X$. In general, we have that the join of spheres gives us a higher dimensional sphere in the following way.

$$\mathbf{S}^m * \mathbf{S}^n \cong \mathbf{S}^{m+n+1}.$$

Let G be a group. We define the space E^nG as the n-th join $G * \cdots * G$. We can then define EG as a direct limit

$$EG = \varinjlim_n E^nG.$$

More information on direct limits can be found in A.2. One can see that this can also be thought of as the infinite join $G * \cdots * G * \cdots$ where we denote an element in EG by

$$\langle x, t \rangle = (t_0x_0, t_1x_1, \ldots, t_kx_k, \ldots),$$

with $x_i \in G$ and only finite nonzero $t_i \in [0,1]$ such that $\sum_i t_i = 1$. We identify in EG the element $\langle x,t\rangle$ with $\langle x',t'\rangle$ if $t_i = t_i'$ for all i and $x_i = x_i'$ when $t_i > 0$.

We can define a topology in EG in the following way. Consider the G-equivariant function $p_{2i}\colon EG \to [0,1]$ defined by $p_{2i}(\langle x,t\rangle) = t_i$ and the G-invariant function $p_{1i}\colon p_{2i}^{-1}((0,1]) \to G$ defined by $p_{1i}(\langle x,t\rangle) = x_i$. The topology of EG will then be the weak topology generated by the structural functions p_{1i} and p_{2i}, meaning, the smallest topology such that those functions are continuous.

To define an action of G on EG, let $\sigma\colon EG \times G \to EG$ be defined for $\langle x,t\rangle \in EG$ and $g \in G$ by $\sigma(\langle x,t\rangle, g) = \langle xg,t\rangle = (t_0x_0g, t_1x_1g, \ldots, t_kx_kg, \ldots)$. Note that this action is continuous by the topology given to EG. To see this, let us remember that if we have the weak topology, g is continuous if and only if the composition with the structural functions is. Using that, note that the following diagram commutes

$$\begin{array}{ccc} EG \times G & \xrightarrow{\sigma} & EG \\ \Big\downarrow{\scriptstyle \text{proy}} & & \Big\downarrow{\scriptstyle p_{2i}} \\ EG & \xrightarrow{p_{2i}} & I, \end{array}$$

where $I = [0,1]$ for all i. Since proy and p_{2i} are continuous then $p_{2i} \circ \sigma$ is continuous. On the other hand, the following diagram also commutes

$$\begin{array}{ccc} p_{2i}^{-1}((0,1]) \times G & \xrightarrow{\sigma|_{p_{2i}^{-1}}} & p_{2i}^{-1}((0,1]) \\ \Big\downarrow{\scriptstyle p_{1i}\times \mathrm{id}} & & \Big\downarrow{\scriptstyle p_{1i}} \\ G \times G & \xrightarrow{\quad m \quad} & G, \end{array}$$

where $m\colon G \times G \to G$ is the Lie group multiplication function. Hence the function is continuous and is an action because it is derived from the group multiplication. One can also see that EG is a contractible space and that G acts freely on it.

Definition 1.26. Let G be a group. We define the space $BG = EG/G$ which will be also called the classifying space of G. The resulting bundle $\omega_G\colon EG \to BG$ given by the projection map is known as *Milnor's construction*.

We want to see that ω_G is a universal principal G-bundle. For that we will use the following results.

Proposition 1.27. *The bundle ω_G defines a numerable principal G-bundle.*

Proof. To prove that ω_G is a G-bundle, we want to define transition functions. Let us we denote $U_i = \overline{p_{2i}^{-1}((0,1])}$, then U_i is open. Firstly, it is clear that $\cup_{i\geqslant 0} U_i \subseteq BG$. Secondly, note that by our definition of EG, the element $(0x_0, 0x_1, \dots, 0x_k, \dots)$ does not belong to EG since we have the condition that $\sum_i t_i = 1$.

If $\overline{\langle x,t\rangle} \in BG$ then we can choose a class representative $\langle x,t\rangle \in EG$ such that there is a $t_j \in (0,1]$ for some j. Then $\langle x,t\rangle \in p_{2j}^{-1}((0,1])$. This proves that $BG = \cup_{i\geqslant 0} U_i$. Therefore, $\{U_i\}_{i\geqslant 0}$ is an open cover of BG.

Lastly, we define the transition functions $g_{ij}\colon U_i \cap U_j \to G$ by $g_{ij}(\langle x,t\rangle) = p_{1i}(\langle x,t\rangle)p_{2i}(\langle x,t\rangle)^{-1}$ for all $\langle x,t\rangle \in U_i \cap U_j$. To see that they are well defined, suppose that $\langle x,t\rangle = \langle y,s\rangle \in U_i \cap U_j$ then if $t_i = t_j$ and $t_j = s_j$ both nonzero then $x_i = y_i$ and $x_j = y_j$. Therefore we have that

$$g_{ij}(\langle x,t\rangle) = x_i x_j^{-1} = y_i y_j^{-1} = g_{ij}(\langle y,s\rangle).$$

Since the g_{ij} meets the cocycle conditions for all i,j then they are transition functions therefore ω_G is a principal G-bundle.

Let us define a locally finite partition of unity on BG. Let $w_i\colon BG \to [0,1]$ be a map defined for $b \in BG$ by

$$w_i(b) = \max(0, u_i(b) - \sum_{j<i} u_j(b))$$

where $u_i\colon BG \to [0,1]$ are going to be the unique maps with the property that $u_i \circ p = p_{2i}$. Note that $w_i^{-1}(0,1] \subseteq U_i$. For $b \in BG$ let m be the smallest i such that $u_i(b) \neq 0$ and let n be the largest. We then have that $\sum_i u_i(b) = 1$. Then $u_m(b) = w_m(b)$ and BG is covered by the open sets $w_i^{-1}(0,1]$.

Since we have that $u_i(b) = 0$ for all $i > n$, the open covering $\{w_i^{-1}(0,1]\}_i$ is locally finite. We can then consider $v_i = \frac{w_i}{\sum_j w_j}$ and we have that $\{v_i\}_{j\leqslant i}$ is the partition of unity on BG. □

Lemma 1.28. *Let $p\colon X \to B$ be a numerable G-bundle. Then there is a numerable partition of unity $\{u_i\}_{i\in S}$ of B such that $p|_{u_i^{-1}(0,1]}$ is trivial for all $i \in S$.*

Proposition 1.29. *For each numerable principal G-bundle $p\colon E \to B$ there is a map $f\colon B \to BG$ such that p and $f^*\omega_G$ are isomorphic principal G bundles.*

Theorem 1.30. *Let $f_0, f_1\colon X \to BG$ be two maps such that $f_0^*\omega_G$ and $f_1^*\omega_G$ are isomorphic. Then f_0 and f_1 are homotopic.*

The proofs of the results mentioned above can be found in [7, Proposition 12.1], [7, Theorem 12.2] and [7, Theorem 12.4] respectively. Using all of that we have that ω_G is then universal.

Example 1.31. Let $G = \mathbb{Z}/2$. We know that we can make the identification $Z/2 \cong \mathbf{S}^0$. Therefore we have that $E^nG \cong \mathbf{S}^{n-1}$ and $EG \cong \mathbf{S}^\infty$.

Then we have that the classifying space for $\mathbb{Z}/2$ is

$$BG = (EG)/G = (\mathbf{S}^\infty)/\mathbb{Z}/2 \cong \mathbb{R}P^\infty,$$

also known as the infinite real projective space. The universal bundle given by Milnor's construction is

$$\omega_{\mathbb{Z}/2} \colon \mathbf{S}^\infty \to \mathbb{R}P^\infty.$$

1.4. Balanced product and vector bundles

Definition 1.32. Let W be a right G-space and X a left G-space. We define the *balanced product* denoted by $W \times_G X$ as the quotient $W \times X/\sim$ where $(wg,x) \sim (w,gx)$ for all $w \in W$, $x \in X$ and $g \in G$.

Since any left G-space can be thought of as a right G-space we can just consider the action of G on $W \times X$ as the diagonal action $(w,x)g = (wg,xg) = (wg,g^{-1}x)$. Therefore $W \times_G X = (W \times X)/G$ can be thought of as the orbit space of this action.

Example 1.33. These are some things that can be noted from the previous construction.

1. If $X = *$ then the product $W \times_G X = W/G$.
2. If $X = G$ thought of as a left G-space by left multiplication, then the right action of G on itself makes $W \times_G G$ a right G-space and the action induces a G-map $W \times_G G \to W$ that is a homeomorphism.

With the previous construction we can construct a natural bijection between $\mathbf{Bund}_{\mathrm{GL}_n(\mathbb{C})}(B)$ and $\mathbf{Vect}_n(B)$ for any paracompact Hausdorff topological space B. Let us see how this construction can be done.

Let B be a paracompact Hausdorff topological space and $\pi\colon P \to B$ a principal G-bundle with $G = \mathrm{GL}_n(\mathbb{C})$. Let us denote $E = P \times_G \mathbb{C}^n$. Note that the function $\pi \circ \mathrm{pr}\colon P \times \mathbb{C}^n \to B$ defined by

$\pi(\mathrm{pr}(x,z)) = \pi(x)$ for all $x \in P$ and $z \in \mathbb{C}^n$ is continuous and respects the identifications, hence we have a function $p\colon E = P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n \to B$. We can see, from the construction, that for $x \in P$ the fibers $p^{-1}(x) = \pi^{-1}(x) \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n$ have a vector space structure giving us that $p\colon E \to B$ is an n dimensional vector bundle.

Conversely, if we have an n dimensional vector bundle $p\colon E \to B$ we can consider the frame bundle $\mathrm{Fr}(E)$. Note that, if we fix a basis for E, the elements in $\mathrm{Fr}(E)$ can be expressed as linear combinations of the basis. Hence, having an element in $\mathrm{Fr}(E)$ is the same as having a matrix $A \in \mathrm{GL}_n(\mathbb{C})$. The group $G = \mathrm{GL}_n(\mathbb{C})$ then acts on the right of $\mathrm{Fr}(E)$ by matrix multiplication. We can also give $\mathrm{Fr}(E)$ an appropriate topology such that the map $\pi\colon \mathrm{Fr}(E) \to B$ is continuous. With that, we can see that $\pi\colon \mathrm{Fr}(E) \to B$ is a principal G-bundle over B.

We can consider the map $\phi\colon \mathbf{Bund}_{\mathrm{GL}_n(\mathbb{C})}(B) \to \mathbf{Vect}_n(B)$ defined by $\phi([P]) = [P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n]$ for all $[P] \in \mathbf{Bund}_{\mathrm{GL}_n(\mathbb{C})}(B)$. It can be proven that ϕ is well defined and also has a well defined inverse $\psi\colon \mathbf{Vect}_n(B) \to \mathbf{Bund}_{\mathrm{GL}_n(\mathbb{C})}(B)$ given by $\psi([E]) = [\mathrm{Fr}(E)]$ for all $[E] \in \mathbf{Vect}_n(B)$. Therefore ϕ is an isomorphism.

Example 1.34. Let X be a compact G-space where G is a compact Lie group. We have that BG is the classifying space obtained by taking the quotient of EG. Then we can define $X_G = (X \times EG)/G$. This space is also called Borel's construction.

1.5. Representation theory

Definition 1.35. A *representation* of a group G in a complex finite dimensional vector space V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ denotes the set of linear transformations from V to V.

This map gives V a structure of a G-module. The dimension of V is sometimes called the degree of ρ. We often refer to the representation as just V and we denote $\rho(g)(v) = gv$. On the other hand, note that if we have an action $\sigma\colon G \times V \to V$ of a compact Lie group G over a vector space V, then we can construct a representation $\rho\colon G \to \mathrm{GL}(V)$ defined by $\rho(g) = \sigma_g$. This shows that when G is a discrete group, having a representation V is equivalent to having an action of the group G over V.

Definition 1.36. A *map between two G-representations* V and W is a vector space map $\varphi\colon V \to W$

such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \big\downarrow g & & \big\downarrow g \\ V & \xrightarrow{\varphi} & W, \end{array}$$

for all $g \in G$. The map φ is also a G-map and when G is a Lie group, an intertwining map. When this map is bijective then we say that the representations V and W are equivalent or isomorphic.

Definition 1.37. A *sub-representation* of a representation V of G is a G-invariant sub-vector space $W \subseteq V$. A representation that has a proper sub-representation it is called *reducible* and if the only sub-representations are $\{0\}$ and V it is called *irreducible*.

We can create new representations from two given representations V and W such as $V \oplus W$, $V \otimes W$, $\Lambda^k V$, $\mathrm{Sym}^k V$, $V^* = \mathrm{Hom}(V, \mathbb{C})$, $\mathrm{Hom}(V,W)$ and $\overline{V}$. Nevertheless we will not go over how to do these constructions in this thesis. This can be seen in detail in [9, Definition 2.10].

Definition 1.38. Let V be a representation of G, a bilinear form $(\cdot,\cdot) \colon V \times V \to G$ in a Lie group G is called *G-invariant* if $(gv, gv') = (v, v')$ for all $g \in G$ and $v, v' \in V$. The representation is called *unitary* if there is a G-invariant Hermitian metric in V.

We have then the following theorem.

Theorem 1.39. *Every representation for a compact Lie group is unitary*

We will not go over the proof of this result, it can be found in [9, Theorem 2.15]. We will call a representation V *completely irreducible* if it is the direct sum of irreducible sub-modules. We have then the following corollary.

Corollary 1.40. *All the finite dimensional representations over a compact Lie group are completely irreducible.*

Proof. If V is a reducible representation, then there is a proper sub-representation $W \subseteq V$ and we can write $V = W \oplus W^{\perp}$ with respect to the unitary G-invariant inner product that we know exists from Theorem 1.39. Note that $W^{\perp}$ is also G-invariant since $(gw', w) = (w', g^{-1}w) = 0$ for all $w' \in W^{\perp}$ and $w \in W$.

Inductively, we can continue decomposing each sub-representation and since V has finite dimension this process must stop. Therefore, we have the result. □

This implies that any finite dimensional representation V of a compact Lie group G can be written as

$$V \cong \bigoplus_{i=1}^{N} n_i V_i,$$

where the irreducible representations V_i are inequivalent.

Corollary 1.41. *If V is a finite dimensional representation of a compact Lie group G. V is irreducible if and only if* $\dim \mathrm{Hom}_G(V,V) = 1$.

Proof. If V is irreducible, by Schur's Lemma we know that $\dim \mathrm{Hom}_G(V,V) = 1$. On the other hand, suppose that V is reducible, then $V = W \oplus W'$ for some proper sub-representations of V. Then, necessarily $\dim \mathrm{Hom}_G(V,V) \geqslant 2$ since $\pi_1 \colon V \to W$ and $\pi_2 \colon V \to W'$ are two distinct G-maps. Then we have the result. □

Let $\hat{G}$ be the set of equivalence classes of irreducible representations of a compact Lie group G. We will take (π, E_π) as the class representative of the element $[\pi] \in \hat{G}$. We have then the following theorem.

Theorem 1.42 (Canonical decomposition)**.** *Let V be a finite dimensional representation of a compact Lie group G. Then there is an intertwine G-isomorphism*

$$\bigoplus_{[\pi]\in\hat{G}} \mathrm{Hom}_G(E_\pi, V) \otimes E_\pi \to V.$$

The proof of this theorem can be found in [9, Theorem 2.24]. Let us denote by $Rep(G)$ the set of isomorphism classes of all of the finite dimensional representations of the group G. It is easy to see that this is a commutative monoid under the direct sum of representations. Let us define now Groethendieck's group construction for an arbitrary commutative monoid.

Definition 1.43. Let M be a commutative monoid with operation denoted by the sum. We define *Groethendieck's group* for M a the set $K = (M \times M)/\sim$ where $(m_1, m_2) \sim (n_1, n_2)$ if there is a $k \in M$ such that $m_1 + n_2 + k = m_2 + n_1 + k$ for all $m_1, m_2, n_1, n_2 \in M$. This k is not needed when the monoid M has the cancellation property. The sum on K is defined coordinate-wise which makes K an abelian group where the identity is $[(0,0)]$ and the inverse of $[(m_1,m_2)]$ is given by $[(m_2,m_1)]$.

Definition 1.44. Let G be a group. We define the *representation ring of* G, denoted by $R(G)$, as Groethendieck's group for $Rep(G)$.

If we denote $(E_1,F_1) = E_1 - F_2$ then $R(G)$ can be thought of as the set of formal differences of isomorphism classes of finite dimensional representations over G. To see that $R(G)$ has a ring structure, we can define multiplication with the tensor product of representations over the complex numbers.

Note that if we have two compact Lie groups G and H, and a homomorphism of compact Lie groups $i\colon H \to G$, we can think of any G-module M as an H-module by defining $hx = i(h)x$ for all $h \in H$ and $x \in M$. This process then induces a homomorphism of rings $i^*\colon R(G) \to R(H)$ called the induced map.

A particular example can be done when H is a Lie subgroup of a compact Lie group G. Then we have the inclusion map $\iota\colon H \to G$ that is clearly a homomorphism of compact Lie groups. It can be seen that the induced map gives us a relation between the representation rings $R(G) \subseteq R(H)$.

Example 1.45. Let us find the representation ring for some G.

1. If $G = *$ then the representations over G are going to be defined by the dimension of the vector space. Since we are taking now formal differences of those representations, then this gives us Groethendieck's group for $\mathbb{N}$, which is $\mathbb{Z}$.
2. If $G = \mathbb{Z}/2 = \{1.\tau\}$ then one can see that there are two irreducible representations for G. One is given by the trivial representation and the other one is going to be called the sign representation, which maps τ to the sign function. We also have that the sign representation squared is going to correspond to the trivial representation.

   Since all the representations are going to be linear combinations of those two irreducible representations, then we can map the trivial representation to 1 and the sign representation to x obtaining that
   $$R(G) \cong \mathbb{Z}[x]/\langle x^2 - 1\rangle$$
3. If $G = \mathbf{S}^1$, we have that the standard one dimensional representation $\rho\colon G \to GL(\mathbb{C})$ is going to be the multiplication map $\rho(z)$ for $z \in \mathbf{S}^1$ where $\rho(z)(v) = zv$ for all $v \in \mathbb{C}$. We can

also consider the representation $\rho^{-1}\colon G \to GL(\mathbb{C})$ given by the map $\rho(z)$ for $z \in \mathbf{S}^1$ where $\rho^{-1}(z)(v) = z^{-1}v$ for all $v \in \mathbb{C}$. Both of this representations are irreducible.

One can see that the irreducible representations of G are going to be tensor powers of these two representations, giving us that

$$R(G) \cong \mathbb{Z}[x, x^{-1}].$$

2. K-theory and Equivariant K-theory

In this chapter we will study the basic concepts of K-theory and equivariant K-theory that will be important to understand Atiyah-Segal's completion theorem. In this construction, any time a vector bundle is mentioned it will be a complex vector bundle unless otherwise specified. We will also allow the fibers of vector bundles to be vector spaces of different dimensions and will denote by $\mathbf{Vect}(X)$ the set of all the vector bundles over X.

2.1. K-theory

To define the K-theory in general, we want to first define an equivalence relation between vector bundles over a fixed compact Hausdorff base space. Let us denote by $\varepsilon^n : X \times \mathbb{C}^n \to X$ the trivial n-dimensional vector bundle over X.

Definition 2.1. We say that two vector bundles E_1 and E_2 over X are *stably isomorphic*, denoted by $E_1 \sim_s E_2$, if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$ for some $n \in \mathbb{N}$. Similarly, we say that $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^m \cong E_2 \oplus \varepsilon^n$.

One can verify that both $\sim$ and $\sim_s$ define equivalence relations. With the relation $\sim$ we can prove the following result.

Proposition 2.2. *If X is compact Hausdorff, then the set $\widetilde{K}(X) := \mathbf{Vect}(X)/\sim$ of $\sim$-equivalence classes of vector bundles over X together with the direct sum forms an abelian group called the reduced K-theory of X.*

Proof. Let us first verify that the direct sum is well defined. Suppose that $E_1 \sim E_1'$ and $E_2 \sim E_2'$, by definition we have that $E_1 \oplus \varepsilon^{n_1} \cong E_1' \oplus \varepsilon^{n_1'}$ and $E_2 \oplus \varepsilon^{n_2} \cong E_2' \oplus \varepsilon^{n_2'}$ for some $n_1, n_1', n_2, n_2' \in \mathbb{N}$. This implies that $(E_1 \oplus E_2) \oplus \varepsilon^{n_1+n_2} \cong (E_1' \oplus E_2') \oplus \varepsilon^{n_1'+n_2'}$. Hence $E_1 \oplus E_2 \sim E_1' \oplus E_2'$ which proves that the sum is well defined. Furthermore, note that the zero element is the class of ε^0 and the associative and commutative properties are derived from the isomorphism relation.

Lastly, we must prove the existence of inverses. Let $E \in \mathbf{Vect}(X)$ and let us define, for all $i \in \mathbb{N}$ the sets $X_i = \{x \in X : \dim(E_x) = i\}$. Note that these sets are disjoint and open in X, and are such that $X = \bigsqcup_{i\in\mathbb{N}} X_i$. Since X is compact there must be only a finite amount of X_i's. We then construct a bundle that, when added to E, over each X_i is a trivial bundle of a suitable dimension. This is given by Proposition 1.12. Therefore we have that $E \oplus E' \cong \varepsilon^m$ for some m.

□

On the other hand, if we wanted to define a sum operation on $\sim_s$-equivalence classes in the same manner as for $\sim$, only the zero element could have an inverse since $E \oplus E' \sim_s \varepsilon^0$ implies that $E \oplus E' \oplus \varepsilon^n \cong \varepsilon^n$ for some $n \in \mathbb{N}$ which can only happen if both E and E' are 0-dimensional. This tells us that if we want to form an abelian group $K(X)$ with this relation, we have to do a modification.

Definition 2.3. Let X be compact Hausdorff topological space. We define the *K-theory of* X as Groethendieck's group for the set of isomorphism classes of all vector bundles over X under the equivalence relation $\sim_s$.

Note that this definition is possible since the set of isomorphism classes of all vector bundles over X under the equivalence relation $\sim_s$ is an abelian monoid under the direct sum. Also note that we have the cancellation property. It says that $E_1 \oplus E_2 \sim_s E_1 \oplus E_3$ implies $E_1 \sim E_3$ when E_1, E_2, E_3 are vector bundles over a compact base space X. The last assertion is true because we know that we can add to both sides a bundle E_1' such that $E_1 \oplus E_1'$ is trivial. If we denote $(E_1, E_1') = E_1 - E_1'$ and define the direct sum $\overline{(E_1 - E_1')} \oplus \overline{(E_2 - E_2')} = \overline{E_1 \oplus E_2 - E_1' \oplus E_2'}$, then it is easy to see that $K(X)$ is an abelian group.

Example 2.4. Let $X = *$. Note that the only vector bundles that can be defined over X are trivial. We can see that $K(X) = K(*) \cong \mathbb{Z}$ since the function $\Phi\colon K(*) \to \mathbb{Z}$ defined by $\Phi(\overline{\varepsilon^n - \varepsilon^m}) = n - m$ is a group isomorphism. We can also see that $\widetilde{K}(*) = \{0\}$.

A relation between K and $\widetilde{K}$ can be made. First, note that $\overline{E - E'} = \overline{E \oplus E'' - \varepsilon^n}$ where E'' is a vector bundle such that $E' \oplus E'' \cong \varepsilon^n$ given by Proposition 1.12. Therefore, we can write an arbitrary element of $K(X)$ with class representative $E - \varepsilon^n$ for some $E \in \mathbf{Vect}(X)$ and some $n \in \mathbb{Z}$.

We have then the following proposition.

Proposition 2.5. *The homomorphism* $\Phi\colon K(X) \to \widetilde{K}(X)$ *defined as* $\Phi(\overline{E-\varepsilon^n}) = \overline{E}$ *is surjective and* $\ker\Phi = \mathbb{Z}$.

Proof. First we verify that this function is well defined. For that, consider E_1 and E_2 such that $E_1 - \varepsilon^{n_1} \sim_s E_2 - \varepsilon^{n_2}$. This is equivalent to saying that $E_1 \oplus \varepsilon^{n_2+n} \cong E_2 \oplus \varepsilon^{n_1+n}$ which means that $E_1 \sim_s E_2$. Note that this function is a homomorphism because of the way that the sum is defined and it is surjective because $\overline{E} = \Phi(\overline{E-\varepsilon^0})$.

Suppose now that $\overline{E-\varepsilon^n} \in \ker\Phi$, then we have that $\overline{E} = \Phi(\overline{E-\varepsilon^n}) = \overline{\varepsilon^0}$ which is true if and only if $E \oplus \varepsilon^n \cong \varepsilon^0 \oplus \varepsilon^m$. This is equivalent to saying that $E \sim_s \varepsilon^k$ for some k, therefore we have that $\overline{E-\varepsilon^n} = \overline{\varepsilon^k - \varepsilon^n}$. Then the set

$$\ker\Phi = \{\overline{\varepsilon^k - \varepsilon^n}\colon k,n \in \mathbb{Z}\} \cong \{k-n\colon k,n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

□

Corollary 2.6. *For a fixed* $x_0 \in X$ *the following exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } K(X) \xrightarrow{\ \Phi\ } \widetilde{K}(X) \longrightarrow 0,$$

splits. By the splitting lemma we have that $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

Proof. We can define a function $\Psi\colon K(X) \to \mathbb{Z}$ by $\Psi(\overline{E-\varepsilon^n}) = \overline{E|_{x_0} - \varepsilon^n|_{x_0}} \cong \overline{\varepsilon^m - \varepsilon^n}$. It is easy to see that this function is such that $\Psi \circ \iota = \mathrm{id}_{\ker\Phi}$. Then we have the result. □

Note that a straight forward consequence of the previous corollary is that $\widetilde{K}(X) = \ker\Psi$ and that $\widetilde{K}(X)$ is a normal subgroup of $K(X)$. Let us define the tensor product in $K(X)$ by

$$\overline{E_1 - E_1'} \otimes \overline{E_2 - E_2'} := \overline{(E_1 \otimes E_2 \oplus E_1' \otimes E_2)} - \overline{(E_1 \otimes E_2' \oplus E_1' \otimes E_2)}.$$

This product is associative, commutative and distributive. We then have that $K(X)$ is a ring with the direct sum and tensor product. Since Ψ is a homomorphism and $\widetilde{K}(X)$ is an ideal of $K(X)$, it inherits the product hence making $\widetilde{K}(X)$ also a ring with the previously mentioned operations.

We will denote **CRings** as the category of commutative rings and **Top** the category of topological spaces. We can say that $K\colon \textbf{Top} \to \textbf{CRings}$ is a contravariant functor that assigns a compact Hausdorff space X the commutative ring $K(X)$, and a continuous function $f\colon X \to Y$ the pullback map $f^*\colon K(Y) \to K(X)$ defined as $f^*(\overline{E-E'}) = \overline{f^*(E) - f^*(E')}$. Note that the pullback map is a group homomorphism and satisfies the required properties for $K(X)$ to have a functorial structure. In the same way $\widetilde{K}(X)$ also has a functorial structure.

2.1.1. Generalized cohomology properties

In this subsection, we want to see that $\widetilde{K}$ and K define reduced and non-reduced generalized cohomology theories. To do this, we need a notion of suspension and reduced suspension for a given space.

Definition 2.7. Let X be a compact topological space and let $I = [0,1]$.

1. The *cone of* X will be defined by $CX = (X \times I)/(X \times \{0\})$.
2. The *suspension of* X, denoted by SX, is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point.
3. For a fixed $x_0 \in X$ we have the line segment $\{x_0\} \times I$ in SX and we can define the *reduced suspension* denoted by ΣX by collapsing $\{x_0\} \times I$ to a point.

For well behaved spaces, such as CW-complexes or, in our specific setting, topological spaces where the inclusion $\{x_0\} \to X$ is a cofibration, one can see that the suspension and reduced suspension of a space are homotopy equivalent. This property is also known as the point $x_0 \in X$ not being nondegenerate. Hence we can work with either one.

Example 2.8. When $X = \mathbf{S}^1$ we can see that SX is homeomorphic to $\mathbf{S}^2$ and in general we will have that if $X = \mathbf{S}^n$ then $SX \cong \Sigma X \cong \mathbf{S}^{n+1}$.

We will denote by $\mathbf{CW}_*$ the category of pointed CW-complexes and $\mathbf{Ab}$ the category of abelian groups. Remember that for two pointed spaces (X, x_0) and (Y, y_0) we can define the wedge sum $X \vee Y$ as $(X \sqcup Y)/\sim$ where $x_0 \sim y_0$. Then we are set for the following definition.

Definition 2.9. A *reduced cohomology theory* is defined as a sequence of contravariant functors $\{\widetilde{h}^n\colon \mathbf{CW}_* \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ together with natural isomorphisms $\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X)$ for all $X \in \mathbf{CW}_*$ such that the following axioms are true.

1. *Homotopy axiom:* If $f, g\colon (X, x_0) \to (Y, y_0)$ are morphisms and f is homotopic to g preserving the base points, then $f^* = g^*\colon \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$.

2. *Exactness axiom:* If $X \in \mathbf{CW}_*$ and $A \subseteq X$ is a subcomplex, then the sequence

$$A \xhookrightarrow{\iota} X \xrightarrow{\pi} X/A,$$

where ι is the inclusion and π the canonical projection, induces the exact sequence

$$\widetilde{h}^n(X/A) \xrightarrow{\pi^*} \widetilde{h}^n(X) \xrightarrow{\iota^*} \widetilde{h}^n(A).$$

3. *Wedge axiom:* If $X = \bigvee_{\alpha\in\Lambda} V_\alpha$ with inclusions $\iota_\alpha\colon X_\alpha \to X$ for all $\alpha \in \Lambda$, then the ι_α induce a group isomorphism $\widetilde{h}^n(X) \cong \Pi_{\alpha\in\Lambda}\widetilde{h}^n(X_\alpha)$.

Our goal is to see that the reduced K-theory defines a reduced generalized cohomology theory. Let X be a compact Hausdorff space. We define $\widetilde{K}^{-n}(X) := \widetilde{K}(\Sigma^n X) = \widetilde{K}(S^n X)$. By [6, Theorem 2.11] we have that $\widetilde{K}(X) \cong \widetilde{K}(\Sigma^2 X)$. This tells us that the reduced K-theory has period 2. We can then identify $\widetilde{K}^n$ with $\widetilde{K}^{n-2}$. Therefore we have that $\widetilde{K}^{-n}(X) \cong \widetilde{K}^n(X)$ and by definition $\widetilde{K}^{n+1}(\Sigma X) = \widetilde{K}^n(X)$.

Remember that, for a function $f\colon (X, x_0) \to (Y, y_0)$ and $n = 0$ we have an induced map called the pullback $f^*\colon \widetilde{K}^0(Y) \to \widetilde{K}^0(X)$. For $n \neq 0$ we have an induced function $\hat{f}\colon \Sigma X \to \Sigma Y$ that is well defined and in hand induces $\hat{f}_n\colon \Sigma^n X \to \Sigma^n Y$. By doing the pullback to $\hat{f}_n$, we have induced maps $f_n^*\colon \widetilde{K}^{-n}(Y) \to \widetilde{K}^{-n}(X)$. Then the reduced K-theory is a sequence of well defined functors $\{\widetilde{K}^n\colon \mathbf{TopComp}_* \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$, where $\mathbf{TopComp}_*$ denotes the category of pointed topological compact spaces. These functors then map $X \in \mathbf{TopComp}_*$ to the abelian group $\widetilde{K}^n(X)$ and $f\colon (X, x_0) \to (Y, y_0)$ to the pullback f_n^*.

Note that by Theorem 1.20 we can see that the reduced K-theory follows the first axiom. To prove that it follows the second axiom we will need the following lemma.

Lemma 2.10. *If A is a contractible space then the quotient map $\pi\colon X \to X/A$ induces a bijection $\pi^*\colon \mathbf{Vect}(X/A) \to \mathbf{Vect}(X)$.*

The proof of this can be found in [6, Lemma 2.10]. With that, if we denote $Y = X \cup CA$, we can construct the following diagram.

$$\begin{array}{ccccccccccc}
A & \hookrightarrow & X & \longrightarrow & Y & \longrightarrow & Y\cup CX & \longrightarrow & (Y\cup CX)\cup CY & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & Y/CA\cong X/A & & (Y\cup CX)/Y\cong SA & & SX. & &
\end{array}$$

Since each cone is a contractible space, we can use the previous lemma to determine inverses in the following diagram

$$\begin{array}{ccccccccccc}
\widetilde{K}(A) & \longleftarrow & \widetilde{K}(X) & \longleftarrow & \widetilde{K}(Y) & \longleftarrow & \widetilde{K}(Y\cup CX) & \longleftarrow & \widetilde{K}((Y\cup CX)\cup CY) & \longleftarrow & \cdots \\
{\scriptstyle\mathrm{id}}\uparrow & & {\scriptstyle\mathrm{id}}\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\widetilde{K}(A) & \dashleftarrow & \widetilde{K}(X) & \dashleftarrow & \widetilde{K}(X/A) & \dashleftarrow & \widetilde{K}(SA) & \dashleftarrow & \widetilde{K}(SX). & &
\end{array}$$

This tells us that we have a long exact sequence of $\widetilde{K}$-groups

$$\cdots \longrightarrow \widetilde{K}(SX) \longrightarrow \widetilde{K}(SA) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A).$$

This proves that the reduced K-theory follows the second axiom. Lastly, let us prove that it follows the last one. Since we are going to be working with compact topological spaces we can just consider $X = \bigvee_{k=1}^{m} X_k$. For that we will use the following corollary.

Corollary 2.11. *Let $X, Y \in$ **TopComp**$_*$. Then $\widetilde{K}^{-n}(X\vee Y)\cong \widetilde{K}^{-n}(X)\oplus\widetilde{K}^{-n}(Y)$ for all $n\geqslant 0$.*

Proof. Let us define $\iota_1\colon X\to X\vee Y$ and $\iota_2\colon Y\to X\vee Y$ the inclusions and also $r_1\colon X\vee Y\to (X\vee Y)/Y = X$ and $r_2\colon X\vee Y\to Y$ the projections. It is clear that $r_1\circ\iota_1 = \mathrm{id}_X$ and $r_2\circ\iota_2 = \mathrm{id}_Y$. Then we have the following maps

$$\widetilde{K}^{-n}(X)\oplus\widetilde{K}^{-n}(Y) \xrightarrow{r_1^*+r_2^*} \widetilde{K}^{-n}(X\vee Y) \xrightarrow{\iota_1^*\oplus\iota_2^*} \widetilde{K}^{-n}(X)\oplus\widetilde{K}^{-n}(Y),$$

where we define $r_1^*+r_2^*(\overline{E}\oplus\overline{F}) = r_1^*(\overline{E})+r_2^*(\overline{F})$ for $\overline{E}\in\widetilde{K}^{-n}(X)$ and $\overline{F}\in\widetilde{K}^{-n}(Y)$ and define $\iota_1^*\oplus\iota_2^*(\overline{H}) = \iota_1^*(\overline{H})\oplus\iota_2^*(\overline{H})$ for all $\overline{H}\in\widetilde{K}^{-n}(X\vee Y)$.

Note that $(\iota_1^*\oplus\iota_2^*)\circ(r_1^*+r_2^*) = \mathrm{id}$. This tells us that $\iota_1^*\oplus\iota_2^*$ is surjective. We now see $\iota_1^*\oplus\iota_2^*$ is injective. Let $\xi\in\ker(\iota_1^*\oplus\iota_2^*)$. Then $\iota_1^*(\xi)=0$ and $\iota_2^*(\xi)=0$. Since $\xi\in\ker(\iota_2^*)$ we know that there is a $\eta\in\widetilde{K}^{-n}(Y)$ such that $\overline{r_2^*\eta} = \xi$. This tells us that

$$\eta = \iota_2^*\circ r_2^*\eta = \iota_2^*(\xi) = 0.$$

Therefore $\xi = 0$. □

Inductively we have then the isomorphisms

$$\widetilde{K}^{-n}\left(\bigvee_{k=1}^{m} X_k\right) \cong \bigoplus_{k=1}^{m} \widetilde{K}^{-n}(X_k) \cong \prod_{k=1}^{m} \widetilde{K}^{-n}(X_k).$$

This shows us that the reduced K-theory defines a reduced cohomology theory. We now want to do something similar for non-reduced K-theory. Let us denote by **CWPair** the category of topological pairs (X,Y) formed by $X,Y \in \mathbf{CW}$ and Y a sub-complex of X.

Definition 2.12. A *non-reduced generalized cohomology theory*, or simply, a generalized cohomology theory, is a sequence of functors $\{h^n\colon \mathbf{CWPair} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ together with natural transformations $\Delta\colon h^n(A) := h^n(A,\emptyset) \to h^{n+1}(X,A)$ called the connection maps that satisfy the following axioms

1. *Homotopy axiom:* If $f,g\colon (X,A) \to (Y,B)$ are morphisms and f is homotopic to g preserving the base points, then $f^* = g^*\colon h^n(Y,B) \to h^n(X,A)$.
2. *Exactness axiom:* If $(X,A) \in \mathbf{CWPair}$, via the inclusions $\iota\colon A \to X$ and $j\colon (X,\emptyset) \to (X,A)$, we have the exact long sequence
$$\cdots \longrightarrow h^n(X,A) \xrightarrow{j^*} h^n(X) \xrightarrow{\iota^*} h^n(A) \xrightarrow{\Delta} h^{n+1}(X,A) \xrightarrow{j^*} \cdots.$$
3. *Excision axiom:* If X has two subcomplexes A and B such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then the inclusion $\iota\colon (A, A\cup B) \to (X,B)$ induces an isomorphism $\iota^*\colon h^n(X,B) \to h^n(A,A\cap B)$.
4. *Addition axiom:* If $(X,A) = \sqcup_{\alpha\in\Lambda}(X_\alpha,A_\alpha)$ then the inclusions $\iota_\alpha\colon (X_\alpha,A_\alpha) \to (X,A)$ for all $\alpha \in \Lambda$ induce a group isomorphism $h^n(X,A) \cong \Pi_{\alpha\in\Lambda} h^n(X_\alpha,A_\alpha)$.

To prove that the K-theory defines a cohomology theory we will use the following theorem.

Theorem 2.13. *If $\{\widetilde{h}^n\}_{n\in\mathbb{Z}}$ defines a reduced cohomology theory then the collection $\{h^n\}_{n\in\mathbb{Z}}$ defined by $h^n(X,A) = \widetilde{h}^n(X/A)$ is a cohomology theory.*

The proof of this result can be found in [1, Theorem 1.5]. With this in hand, we define

$$K^{-n}(X,Y) = \widetilde{K}^{-n}(X/Y) = \widetilde{K}(\Sigma^n(X/Y)),$$

$$K^{-n}(X) = K^{-n}(X,\emptyset) = \widetilde{K}(\Sigma^n X_+),$$

where X_+ is the disjoint union of X with a point.

One can see that this definition for the K-theory coincides with the definition that we had. Then, by the previous theorem we have that the K-theory defines a cohomology theory. One can also create a reduced cohomology theory from a generalized cohomology theory. This can be seen by the following theorem.

Theorem 2.14. *From a non-reduced cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$ we can define a reduced cohomology theory for (X,x_0) the functor $\widetilde{h}^n(X) = h^n(X,x_0)$.*

The proof of this result can be found in [1, Theorem 1.4]. We can also define the graded K-groups as the direct sums $K^*(X) = K^0(X) \oplus K^1(X)$ regarded as $\mathbb{Z}/2$-graded groups. Under the tensor product this then makes $K^*(X)$ a $\mathbb{Z}/2$-graded commutative ring.

Lastly, note that the previous construction for K-theory only lets us work with compact spaces. We will now briefly mention the homotopic representation for K-theory. For that, first let us remember that if we have two topological spaces pointed A and B with base points a and b respectively, then we can consider $\langle A,B\rangle$, which denotes the set of homotopy classes of maps from A to B that preserve the base points. We can also consider $[A,B]$, which is the set of homotopy classes of maps that don't necessarily preserve a base point. Then we can see the following result.

Theorem 2.15 (Brown representability theorem)**.** *Every reduced cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$ on the category of pointed CW-complexes and base point-preserving maps has the form $h^n(X) = \langle X,K_n\rangle$ for some Ω-spectrum $\{K_n\}_{n\in\mathbb{Z}}$.*

The proof of this theorem can be found in [5, Theorem 4E.1]. We can use this result and prove that for reduced K-theory we have that

$$\widetilde{K}^0(X) \cong \langle X, BU\rangle,$$

where BU is the classifying space of $U = \varinjlim_{n\in\mathbb{N}} U(n)$ where $U(n)$ is the group of n-dimensional unitary matrices. In the same manner, since we have that $K(X) \cong \widetilde{K}(X)\times\mathbb{Z}$, then we can see that for a connected space X there is the following isomorphism

$$K^0(X) \cong [X,BU]\times\mathbb{Z} \cong [X,\mathbb{Z}\times BU].$$

We then know that we can construct K^1 with the suspension which lets us extend this result to K^*. This result is important because this gives us a way to define the K-theory for a spaces that are not necessarily compact.

2.2. Equivariant K-theory

We can create a very similar theory to K-theory for spaces equipped with an action of a topological group.

Definition 2.16. Let G be a topological group and X a left G-space. A *G-vector bundle over X* consists of a G-space E and a G-map $p\colon E \to X$ such that $p\colon E \to X$ is a complex vector bundle and for all $g \in G$ and $x \in X$ we have that the map $g\colon E_x \to E_{gx}$ is a vector space isomomorphism.

If G acts trivially on X then a G-vector bundle can be thought of as a family of representations E_x of G that are parametrized by the points $x \in X$ and vary continuously with x.

Definition 2.17. A *section* of a G-vector bundle $p\colon E \to X$ is a map $s\colon X \to E$ such that $p \circ s = \mathrm{id}$. We can form a vector space denoted ΓE of all the sections of the bundle $p\colon E \to X$.

If a section is additionally a G-map we call it equivariant. It can be seen that we can form a vector subspace $\Gamma^G E \subseteq \Gamma E$ of all the equivariant sections of a given bundle. We also have that if E and F are two G-vector bundles over X then we can also form new G-vector bundles such as $E \oplus F$, $E \otimes F$ and $\mathrm{Hom}(E,F) = E^* \otimes F$.

Example 2.18. If $p\colon E \to X$ is a vector bundle, then the tensor product of k copies of E, $E \otimes \cdots \otimes E$, is naturally a S_k-vector bundle over X where S_k is the symmetric group that permutes the factors of the product and S_k acts trivially on X.

Definition 2.19. A *G-vector bundle homomorphism* is a G-map $f\colon E \to F$ that induces a vector space homomorphism $f_x\colon E_x \to F_x$ for all $x \in X$.

These homomorphisms form a vector space isomorphic to $\Gamma^G \mathrm{Hom}(E,F)$, this can be seen with detail in [8]. Note that if $\varphi\colon Y \to X$ is a G-space G-map and E is a vector bundle over X, then we can form a G-vector bundle $\varphi^* E$ over Y such that $(\varphi^* E)_y = E_{\varphi(y)}$. This G-vector bundle is also known as the pullback. Even more, if Y is an H-space, X is a G-space, $\alpha\colon H \to G$ is an homomorphism and $\varphi\colon Y \to X$ is such that $\varphi(hy) = \alpha(h)y$, then $\varphi^* E$ is an H-vector bundle over Y. Particularly, if $Y \subseteq X$ and ι is the inclusion, we will denote $\iota^* E$ as $E|_Y$.

Let E be a G-vector bundle over a compact G-space X. We will give ΓE the compact open topology. Then G acts continuously over ΓE in the following way. For a section $s\colon X \to E$ in ΓE we can define $gs\colon X \to E$ by $(gs)(x) = gs(g^{-1}x)$ for all $g \in G$ and $x \in X$ and this action is clearly continuous. It can also be seen that ΓE is Hausdorff, locally convex and complete.

Proposition 2.20. *Let E be a G-vector bundle in a compact G-space X and A a closed G-subspace of X. Then an equivariant section of $E|_A$ can be extended to an equivariant section of E.*

Proof. Let $p\colon E \to X$ be a G-vector bundle and $\bar{s}\colon A \to E|_A$ an equivariant section of $E|_A$. Since the map $\iota^*(p)\colon E|_A \to A$ is locally trivial, we know that there is an open set $U_x \subseteq A$ containing x, such that for each $x \in A$ the map $\psi_x\colon U_x \times \mathbb{C}^k \to (\iota^*(p))^{-1}(U_x)$ is an isomorphism. Then we have that

$$\bar{s}|_{U_x}\colon U_x \to (\iota^*(p))^{-1}(U_x) \cong U_x \times \mathbb{C}^k.$$

By Tietze's extension theorem, we have that for all $x \in X$ there is an open set U_x containing x and $t_x \in \Gamma(E|_{U_x})$ such that t_x and $\bar{s}$ coincide in $U \cap A$ and this cover is finite since X is compact. Let us denote the open cover $\{U_{x_i}\}_{i=1}^m$ and let $\{\phi_{x_i}\}_{i=1}^m$ be a partition of unity associated to the open cover. If we define

$$s_x(x) = \begin{cases} \phi_x(x)t_x(x) & x \in U_x \\ 0 & x \notin U_x, \end{cases}$$

then $s = \sum_{i=1}^m s_{x_i}$ is an equivariant section over E. □

Proposition 2.21. *Let E be a G-vector bundle over a compact G-space X and $A \subseteq X$ a closed G-subspace. If F is a principal G-bundle over X and $f\colon E_A \to F_A$ is an isomorphism, then there is a G-neighborhood U of A in X and an isomorphism $f\colon E_U \to F_U$ that extends to f.*

Proof. Note that we can see f as an equivariant section of $\mathrm{Hom}(E,F)|_A$. By the previous proposition we can extend f to an equivariant section $\hat{f}$ of $\mathrm{Hom}(E,F)$.

Let U be the subset of X consisting of all the points $x \in X$ such that $\hat{f}(x)$ is an isomorphism. Then we can see that U is a G-neighborhood of A in X and clearly $\hat{f}(U)$ gives is the wanted isomorphism. □

Proposition 2.22. *If $\varphi_0, \varphi_1\colon Y \to X$ are G-homotopic G-maps, Y is compact and E is a G-vector bundle over X then $\varphi_0^*E \cong \varphi_1^*E$.*

The proof of this proposition can be found in [3, Proposition 1.3]. With the previous information we can then define the equivariant K-theory of a compact space X.

Definition 2.23. The *equivariant K-theory of* X, denoted $K_G(X)$, is the Groethendieck group of the set of isomorphism classes of G-vector bundles over X. The tensor product of G-vector bundles induces a commutative ring structure on $K_G(X)$.

Let $\varphi\colon Y \to X$ be a G-map of compact G-spaces. We know that for every G-vector bundle E we can construct a G-vector bundle φ^*E over Y. In this manner we have a induced ring homomorphism $\varphi^*\colon K_G(X) \to K_G(Y)$ that assigns E the bundle φ^*E. Therefore we can think of K_G as a contravariant functor of compact G-spaces to commutative rings.

The homomorphisms $\alpha\colon H \to G$ induce a "restriction" homomorphism $K_G(X) \to K_H(X)$ and if $\varphi\colon Y \to X$ is a map from a H-space to a G-space compatible with α, meaning that $\varphi(hy) = \alpha(h)y$ for all $n \in H$ and $a \in Y$, then we get the induced map $\varphi^*\colon K_G(X) \to K_H(Y)$.

Example 2.24. Here are some interesting examples of equivariant K-theory.

1. If $G = 1$ then we have that $K_G(X) = K(X)$.
2. Let $X = *$. Then one can see that $K_G(X) \cong R(G)$. In general, we can think of $K_G(X)$ as an algebra over $R(G)$ since there always exists a natural map $X \to *$.
3. Let us consider the co-sets $G/H = \{gH\colon g \in G\}$. We can determine all the G-vector bundles over G/H when H is a closed subgroup of G. There is an equivalence between the G-vector bundles over G/H and the H-modules. Then $K_G(G/H) \cong R(H)$.
4. Let X be a compact H-space where H is a closed subgroup of G. We can then form a new G-space $G \times_H X$ where G acts by $g(g',x) = (gg',x)$ for $g, g' \in G$ and $x \in X$. Then one can see that $K_G(G \times_H X) \cong K_H(X)$.

Proposition 2.25. *If a Lie group G acts freely over X then* $\mathrm{pr}^*\colon K(X/G) \to K_G(X)$ *induced by* $\mathrm{pr}\colon X \to X/G$ *is an isomorphism. More generally, if N is a normal subgroup of G that acts freely over X such that G/N is a Lie group. Then the induced map* $\mathrm{pr}^*\colon K_{G/N}(X/N) \to K_G(X)$ *is an isomorphism.*

Proof. First, note that the map $\mathrm{pr}\colon X \to X/G$ induces a map $\mathrm{pr}^*\colon K(X/G) \to K_G(X)$. This is given from the fact that if we have a vector bundle $p\colon E \to X/G$ then the pullback $\mathrm{pr}^*(E) \subseteq X \times E$ can be thought of as a G-bundle where G acts only on the first component of $X \times E$.

Let us see that it is an isomorphism. Suppose $\overline{E}, \overline{E'} \in K(X/G)$ are such that $\overline{\mathrm{pr}^*E} = \overline{\mathrm{pr}^*E'} \in K_G(X)$.

Then we have a G-vector bundle isomorphism between them. In particular, since they are vector bundles, then $\overline{E} = \overline{E'}$.

Now, let E be a G-vector bundle over X. We can construct the following diagram

$$\begin{array}{ccccc} E & \xrightarrow{p} & X & \xrightarrow{\mathrm{pr}} & X/G \\ \big\downarrow{\scriptstyle \pi} & & & \nearrow_{p'} & \\ E/G, & & & & \end{array}$$

where $p' \colon E/G \to X/G$ is defined by $p'(\overline{e}) = \overline{p(e)}$. We can see that this function is well defined and the diagram is commutative. Even more, this defines a vector bundle because E/G is locally trivial as long as X is a compact Lie group. This concludes that pr^* is an isomorphism.

Suppose now that N is a normal subgroup of G such that G/N is a Lie group. We can consider the projection $\mathrm{pr}\colon X \to X/N$ that induces a map $\mathrm{pr}^* \colon K_{G/N}(X/N) \to K_G(X)$. We want to construct an inverse map to prove that it is an isomorphism.

Let E be a G-vector bundle over X. We can construct a G/N-vector bundle from E/N like this

$$\begin{array}{ccccc} E & \longrightarrow & X & \longrightarrow & X/N \\ \big\downarrow & & & \nearrow_{p'} & \\ E/N. & & & & \end{array}$$

This diagram is also commutative and gives a locally trivial G-vector bundle because it is also G-equivariant. Even more, it is a G/N-vector bundle because G/N acts over E/G by $(gN)e = ge$ and over X/N by $(gN)x = gx$. And if $g \notin N$ then

$$p'((gN)e) = p'(ge) = gp'(e).$$

It is easy to see that this is an inverse, then $\mathrm{pr}^* \colon K_{G/N}(X/N) \to K_G(X)$ is an isomorphism. □

Proposition 2.26. *If a Lie group G acts trivially over a G-space X then the natural map*

$$\mu \colon R(G) \otimes K(X) \to K_G(X)$$

is a ring isomorphism.

Proof. We know by Theorem 1.42 that if W is a representation of G then we have an isomorphism

$$\bigoplus_{[\pi]\in\hat{G}} \mathrm{Hom}_G(E_\pi, E)\otimes E_\pi \to W.$$

The map μ is given by $\mu(\overline{v}\otimes\overline{E}) = \overline{v\times X}\otimes\overline{E}\in K_G(X)$ for $\overline{v}\in R(G)$ and $\overline{E}\in K(X)$. This is because we have the two homomorphisms $K(X)\to K_G(X)$ where we think that G acts trivially and $R(G)\to K_G(X)$. We want to construct an inverse of this map.

Let $p\colon E\to X$ be a G-vector bundle. Since X is a trivial G-space, we can think of the fibers of E a representations of G. Let V_ρ be an irreducible representation of G. We can construct the G-vector bundle $V_\rho\times X$ over X. Since $\mathrm{Hom}_G(V_\rho\times X, E)$ is a G-vector bundle whose fibers are $\mathrm{Hom}_G(V_\rho, E_x)$ and they are vector spaces with dimension equal to the number of times the representation V_ρ is in E_x, we obtain that $V_\rho\times X\otimes\mathrm{Hom}_G(V_\rho, E)$ is a G-vector bundle. By Theorem 1.42 we have the isomorphism

$$\alpha\colon E\to\bigoplus_{\rho\in\hat{G}}\mathrm{Hom}_G(V_\rho\times X, E)\otimes(V_\rho\times X).$$

We also have that $\bigoplus_{\rho\in\hat{G}}(V_\rho\times X)\cong R(G)$ and $\bigoplus_{\rho\in\hat{G}}\mathrm{Hom}_G(V_\rho\times X, E)\cong K(X)$. Then, for all $e\in E$ we have that $\alpha(e)\in R(G)\otimes K(X)$ and this gives us an inverse for μ. □

Example 2.27. We know that if E is a vector bundle over X we can construct a S_k-vector bundle $E^{\otimes k} = E\otimes\cdots\otimes E\to X$ thinking of X as a trivial S_k-space and S_k acts over $E^{\otimes K}$ permuting the coordinates. Then the map $E\to E^{\otimes k}$ induces a natural transformation $K(X)\to K_{S_k}(X)$.

Let us recall the following result.

Theorem 2.28 (Peter-Weyl)**.** *Let Γ be a topological locally convex Hausdorff complete vector space. If G acts continuously over Γ and Γ_α, the image of Γ under the cannonical injection*

$$\bigoplus_{[\mu\in\hat{G}]}(M\otimes\mathrm{Hom}_G(\mu,\Gamma)\to\Gamma,$$

is the union of the invariant finite dimensional subspaces of Γ. Then Γ_α is dense in Γ.

This result can be found in [8, Theorem 2.5]. With the previous theorem one can see the following proposition.

Proposition 2.29. *Let E be a G-vector bundle over X. Then there is a G-module M and a G-vector bundle $E^\perp$ such that $E\oplus E^\perp\cong M\times X$.*

Proof. By using Peter-Weyl's Theorem 2.28, it is enough to find the G-module M. This is because we can embbed E in M and with that, we can choose a Hermitian inner product in M such that $E^{\perp}$ is the orthogonal complement.

Let $\Gamma = \Gamma E$ be the Banach section space of E. For each $x \in X$ we can choose a finite number of sections of E, namely σ_x, such that $\{s(x)\}_{s\in\sigma_x}$ generates E_x.

Since we know that Γ_α is dense in Γ and the evaluating map $\Gamma \to E_x$, defined by mapping s to $s(x)$, is continuous, we can suppose that $\sigma_x \subseteq \Gamma_\alpha$. The set $\{s(y)\}_{s\in\sigma_x}$ then generates E_y for all $y \in U_x$ some neighborhood of x.

Suppose $U_{x_1}, \dots, U_{x_n}$ covers X. We define $\sigma = \bigcup_{i=1}^{n} \sigma_{x_i}$ and M the finite dimensional G-subspace of Γ generated by σ. Then the evaluating map $\alpha\colon X \times M \to E$ defined by $\alpha(x,s) = s(x)$ is sujective and we choose $E^{\perp} = \ker\alpha$. □

Definition 2.30. Two G-vector bundles E and E' are said to be *equivalent* if there exist two G-modules M and M' such that

$$E \oplus (M \times X) \cong E' \oplus (M' \times X).$$

The previous proposition implies that with this equivalence relation, we can form an abelian group unter the direct sum. We will call this group the reduced equivariant K-theory of X denoted by $\widetilde{K}_G(X)$ and can be seen as a quotient group of $K_G(X)$.

2.2.1. Generalized equivariant cohomology properties

Just as for cohomology theories, we have certain axioms that define a generalized equivariant cohomology theory. We say that two functions f and g are G-homotopic if the homotopy between them is a G-map.

A generalized equivariant cohomology theory, very similarly to what we had in the non-equivariant case, can be seen as a collection of functors $\{h^n\colon G-\mathbf{CWPair} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ together with natural transformations $\Delta\colon h^n_G(A) := h^n_G(A,\emptyset) \to h^{n+1}_G(X,A)$ called the connection maps that satisfy the following axioms.

1. *Homotopy axiom:* If $f,g\colon (X,A) \to (Y,B)$ are morphisms and f is G-homotopic to g, then $f^* = g^*\colon h^n_G(Y,B) \to h^n_G(X,A)$.

2. *Exactness axiom:* If we have a pair $(X,A) \in G-\mathbf{CWPair}$, via the inclusions $\iota\colon A \to X$ and $j\colon (X,\emptyset) \to (X,A)$, we have the long exact sequence
$$\cdots \longrightarrow h_G^n(X,A) \xrightarrow{j^*} h_G^n(X) \xrightarrow{\iota^*} h_G^n(A) \xrightarrow{\Delta} h_G^{n+1}(X,A) \xrightarrow{j^*} \cdots.$$
3. *Addition axiom:* If $X = \vee_{\alpha\in\Lambda} X_\alpha$ then the inclusions $\iota_\alpha\colon X_\alpha \to X$ for all $\alpha \in \Lambda$ induce a group isomorphism $h_G^n(X) \cong \Pi_{\alpha\in\Lambda} h_G^n(X_\alpha)$.

We then define $K_G^n(X) = K_G(\Sigma^n X_+)$ and with this it can be seen that the equivariant K-theory of a space is a generalized equivariant cohomology theory.

In general, if H is a generalized cohomology theory, for example singular cohomology with coefficients in $\mathbb{R}$, one can define a generalized G-equivariant cohomology theory from it by defining $H_G^n(X) = H^n(X_G)$. Nevertheless, in our specific case of K-theory, that construction will not correspond to the equivariant K-theory of a space. We will see more on that in the next chapter.

Lastly, just as it happened in the case of K-theory, we can represent homotopically the equivariant K-theory in the following way
$$K_G^0(X) \cong [X, \mathbb{Z} \times B_G U]_G.$$
In this case, now we have that $B_G U$ is the G-equivariant version of BU and $[-,-]_G$ will correspond to G-equivariant homotopy classes of maps. Hence giving us a way to define equivariant K-theory for non-compact spaces.

3. Atiyah-Segal's completion theorem

We previously mentioned that an equivariant cohomology theory could be defined from a cohomology theory using Borel's construction. On the other hand, we have equivariant K-theory which is constructed out of equivariant bundles. Those two constructions do not coincide in the case of K-theory. Nevertheless, Atiyah and Segal provided a way to relate these two constructions. They stated that the completion with respect to the augmentation ideal of the equivariant K-theory of a topological G-space X and the equivariant K-theory of the space $X \times EG$ are isomorphic. This theorem then gives us an alternative way to find the K-theory of a given topological G-space. This will be particularly useful in the case where $X = *$ and shall be discussed later in this chapter.

3.1. The construction

We know that if X is a compact G-space we have Borel's construction X_G which is not necessarily compact. In this chapter, however, we will assume that X_G is compact. Note that we can think of X_G as a space fibered over BG with fiber X.

Let $p\colon F \to X$ be a G-vector bundle over X. Then we can construct a vector bundle over X_G with total space $F_G = (F \times EG)/G$. This tells us that the map $F \to (F \times EG)/G$, has the induced homomorphism $\alpha\colon K_G(X) \to K(X_G)$ defined by $\alpha([F]) = [(F \times EG)/G]$ for all $[F] \in K_G(X)$. Let us verify that this induced homomorphism is well defined. Suppose $[E] = [F] \in K_G(X)$, then there is a bijective G-map $\phi : F \to E$ such that $\phi_x\colon F_x \to E_x$ is a linear isomorphism for all $x \in X$. We can then define the function $\psi\colon F_G \to E_G$ by $\psi([(f,e)] = [(\phi(f),e)]$ for all $[(f,e)] \in F_G$. This function is continuous and bijective by definition. It is also well defined since for $[(f,e)] = [(f',e')]$ there is a $g \in G$ such that

$$\psi([(f',e')]) = [(\phi(f'),e')] = [(\phi(f)g, g^{-1}e)] = [(\phi(f),e)] = \psi([(f,e)]).$$

Lastly, it can be seen that ψ_x is linear by noting that $\psi_x = \mathrm{pr}(\phi_x \times \mathrm{id})$.

Now, we would like to give $K_G(X)$ and $K(X_G)$ appropriate topologies such that the induced homomorphism α is continuous, $K(X_G)$ is complete and α induces a homeomorphism from the completion $K_G(X)\hat{}_{I_G}$ to $K(X_G)$. We know that G acts freely on $X \times EG$, hence, by Proposition 2.25, we have that

$$K_G(X \times EG) \cong K((X \times EG)/G) = K(X_G).$$

Then α becomes the ring homomorphism $\alpha\colon K_G(X) \to K_G(X \times EG)$ induced by the projection $X \times EG \to X$.

Let us denote $BG^n = EG^n/G$ and let $x_0 \in X$ be a fixed point. We know that the inclusion $\iota\colon x_0 \to X$ induces a map in K-theory $\iota^*\colon K(X) \to K(x_0) \cong \mathbb{Z}$. The map ι^* is called the *augmentation map* in K-theory. We also know, by Corollary 2.6, that $\tilde{K}(X) = \ker \iota^*$ so if we set $X = BG^n$ we have that $\tilde{K}(BG^n) = \ker(\iota^*\colon K(BG^n) \to \mathbb{Z})$.

On the other hand, we can consider the map $EG^n \to *$. With this map, we obtain the equivariant K-theory induced map

$$\alpha_n\colon K_G(*) \cong R(G) \to K_G(EG^n).$$

Since G acts freely on EG^n we once more have, by Proposition 2.25, that

$$K_G(EG^n) \cong K(EG^n/G) = K(BG^n).$$

Let us consider the composition $\iota^* \circ \alpha_n\colon R(G) \to \mathbb{Z}$. From the definition of the composition $\iota^* \circ \alpha_n$, it can be verified that this map corresponds to the augmentation map $\varepsilon\colon R(G) \to \mathbb{Z}$ for the representation ring defined by $\varepsilon([V]-[W]) = \dim V - \dim W$ for all $[V],[W] \in R(G)$. Let us denote $I_G = \ker(\varepsilon\colon R(G) \to \mathbb{Z})$. Then we know that α_n factors through $R(G)/I_G^n$.

Note that $BG^n = \bigcup_{i=1}^n U_i$ where each U_i is the set where the i-th join coordinate does not vanish. This shows then that the product of n elements in $\widetilde{K}(BG^n)$ is zero because we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{K}(BG^n) \otimes \cdots \otimes \tilde{K}(BG^n) & \longrightarrow & \tilde{K}(BG^n) \\
\uparrow & & \uparrow \\
K(BG^n, U_1) \otimes \cdots \otimes K(BG^n, U_n) & \longrightarrow & K(BG^n, \cup_{i=1}^n U_i).
\end{array}$$

This tells us that $I_G^n \subseteq \ker \alpha_n$ so the map α_n factors through $R(G)/I_G^n$. We denote the factored map $\tilde{\alpha}_n\colon R(G)/I_G^n \to K_G(EG^n)$. Since we also have that for any G-space X, $K_G(X)$ is a $R(G)$ module, we can consider the set $I_G^n K_G(X)$.

On the other hand, we have the projection map $\mathrm{pr}\colon X \times EG^n \to *$. This map is such that the following diagram commutes

$$\begin{array}{ccccc} & & X \times EG^n & & \\ & \swarrow & \downarrow{\scriptstyle \mathrm{pr}} & \searrow & \\ X & & & & EG^n \\ & \searrow & \downarrow & \swarrow & \\ & & *. & & \end{array}$$

We can then consider the induced map $\mathrm{pr}^*\colon K_G(*) \to K_G(X \times EG^n)$. Note that this map is such that the following diagram commutes

$$\begin{array}{ccccc} & & K_G(*) & & \\ & \swarrow & \downarrow{\scriptstyle \mathrm{pr}^*} & \searrow & \\ K_G(X) & & & & K_G(EG^n) \\ & \searrow & \downarrow & \swarrow & \\ & & K_G(X \times EG^n). & & \end{array}$$

Since pr^* is natural, it factors, which gives us the map

$$\beta_n\colon \frac{K_G(X)}{I_G^n K_G(X)} \longrightarrow K_G(X \times EG^n).$$

If $[F] \in K_G(X)$ then we will denote by $\overline{[F]}$ an element of $\frac{K_G(X)}{I_G^n K_G(X)}$. The main result of the theorem we will prove in the next section will then be that the homomorphisms β_n induce an inverse limit isomorphism $\beta\colon K_G(X)\hat{}_{I_G} \to \varprojlim K_G(X \times EG^n)$.

Before we move on to the statement of the theorem and its proof, we have two special cases worth mentioning.

Firstly, suppose G acts freely over X. Then X_G is fibered over X/G. In this case, the theorem is reduced to proving that

$$K_G(X) \cong K(X/G).$$

Secondly, if G acts trivially over X then

$$X_G = (X \times EG)/G = X \times EG/G = X \times BG.$$

Since we also have that

$$K_G(X) \cong K(X) \otimes R(G)$$

then the theorem is a composition of the Küneth formula $K(X \times BG) \cong K(X) \otimes K(BG)$ and the isomorphism $K(BG) \cong R(G)\hat{}_{I_G}$.

3.2. The theorem

The main result of this thesis is then the following theorem.

Theorem 3.1 (Atiyah-Segal's completion theorem). *Let X be a compact G-space such that $K_G(X)$ is finite as a module over $R(G)$. Then the homomorphisms*

$$\alpha_n \colon \frac{K_G(X)}{I_G^n K_G(X)} \to K_G(X \times EG^n),$$

induce an inverse limit isomorphism.

To prove this theorem, we will first prove a result that will help us for the case when the group is $T = \mathbf{S}^1$, then with T^m and lastly with the unitary group $U(m)$. Finally, we will finish the proof with a compact Lie group G in general, by embedding it in a sufficiently big unitary group and that will finish the proof.

Suppose E is a G-vector bundle over a compact G-space X. Let us denote by φ the metric of E. This means that we have a metric on each fiber E_x that varies continuously with $x \in X$. We can consider the unit disk bundle $D(E)$, which is the set of points $e \in E$ such that $\varphi(e,e) \leqslant 1$ in each fiber. We can also consider the unit sphere bundle $S(E)$, which is the set of points $e \in E$ such that $\varphi(e,e) = 1$ in each fiber. Then we have the following definition.

Definition 3.2. Let E be a G-vector bundle over a compact G-space X. Then the *Thom space* $T(E)$ is the quotient space $D(E)/S(E)$.

Note that, since X is a compact space, this definition can also be regarded as the one-point compactification of E. Using this space, let us recall the following technical result.

Theorem 3.3. *Thom's homomorphism*

$$\varphi \colon K_G^*(X) \to K_G^*(T(E))$$

is an isomorphism for all G-vector bundles E in a compact G-space X.

The proof of this result can be found in [8, Proposition 3.3]. Now, let us see the following lemma.

Lemma 3.4. *Let G be a compact Lie group and X a compact G-space such that $K_G(X)$ is finite as a module over $R(G)$. Denote $T = \mathbf{S}^1$ and let $\theta\colon G \to T$ be a homomorphism so that G acts over ET. Then the homomorphisms*

$$\delta_n\colon \frac{K_G(X)}{I_T^n K_G(X)} \to K_G(X \times ET^n),$$

induced by the projections $X \times E_T^n \to X$, induce an inverse limit isomorphism.

Proof. Consider the projection $X \times ET^n \to X$ which induces the map $\mathrm{pr}^*\colon K_G(X) \to K_G(X \times ET^n)$.

Note that through the map $\theta\colon G \to T$ we can see $K_G(X)$ as a $R(T)$-module. This is because if we have a representation $\rho\colon T \to \mathrm{GL}(V)$ of T, we can construct a representation of G by doing $\rho \circ \theta$. This shows that we can consider $I_T^n K_G(X)$. Even more, $I_T^n K_G(X) \subseteq \ker(\mathrm{pr}^*)$ and the map $\delta_n\colon \frac{K_G(X)}{I_T^n K_G(X)} \to K_G(X \times ET^n)$ is well defined since the following diagram commutes.

$$\begin{array}{ccccc} & & X \times ET^n & & \\ & \overset{\mathrm{pr}}{\swarrow} & & \overset{\mathrm{pr}}{\searrow} & \\ X & & & & ET^n \\ & \searrow & & \swarrow & \\ & & *. & & \end{array}$$

Let us form the inverse system $A_n = \left\{ \frac{K_G(X)}{I_T^n K_G(X)} \right\}_{n \in \mathbb{N}}$ with structural maps $a_n\colon \frac{K_G(X)}{I_T^n K_G(X)} \to \frac{K_G(X)}{I_T^{n-1} K_G(X)}$ defined by $a_n(\overline{[E]}) = \overline{[E]}$. Let us also form the inverse system $B_n = \{K_G(X \times ET^n)\}_{n \in \mathbb{N}}$ with structural maps $b_n\colon K_G(X \times ET^n) \to K_G(X \times ET^{n-1})$. These maps are induced by the inclusion maps $\widetilde{b_n}\colon X \times ET^{n-1} \to X \times ET^n$ defined by $\widetilde{b_n}((x,(y_0t_0,\dots,y_{n-2}t_{n-2}))) = (x,(y_0t_0,\dots,y_{n-2}t_{n-2},0))$ for all $(x,(y_0t_0,\dots,y_{n-2}t_{n-2})) \in X \times ET^{n-1}$.

We will now see that $\delta_n\colon A_n \to B_n$ satisfies the hypotheses of Lemma A.5. This means that we will see that the following diagram commutes

$$\begin{array}{ccc} A_n & \xrightarrow{\delta_n} & B_n \\ \downarrow a_n & & \downarrow b_n \\ A_{n-1} & \xrightarrow{\delta_{n-1}} & B_{n-1}. \end{array}$$

Let $\overline{[F]} \in A_n = \frac{K_G(X)}{I_T^n K_G(X)}$. We know that $\delta_n(\overline{[F]}) = [F \times ET^n]$. Then $b_n(\delta_n(\overline{[F]})) = [\widetilde{b_n}^*(F \times ET^n)]$, where $\widetilde{b_n}^*(F \times ET^n)$ is the pullback determined by the following commutative diagram

$$\begin{array}{ccc} \widetilde{b_n}^*(F \times ET^n) & \xrightarrow{\mathrm{pr}_2} & F \times ET^n \\ \downarrow \mathrm{pr}_1 & & \downarrow \\ X \times ET^{n-1} & \xrightarrow{\widetilde{b_n}} & X \times ET^n. \end{array}$$

On the other hand, note that $\delta_{n-1}(a_n(\overline{[F]})) = \delta_{n-1}(\overline{[F]}) = [F \times ET^{n-1}]$. It can be seen, from the definition of the pullback, that $[F \times ET^{n-1}] = [\widetilde{b_n}^*(F \times ET^n)]$. Hence we have that $\delta_n \colon A_n \to B_n$ is an inverse system homomorphism.

We now have to construct the map $g_n \colon B_{k(n)+n} \to A_n$ as in the hypotheses for Lemma A.5. For this, note that $ET^n = T * \cdots * T$ can be identified with $\mathbf{S}^{2n-1}$, which is the unit sphere in $\mathbb{C}^n$. Since $\mathbf{D}^{2n}$ is contractible we have that $X \times \mathbf{D}^{2n}$ is homotopic to X, hence $K_G(X \times \mathbf{D}^{2n}) \cong K_G(X)$. On the other hand, we can consider Thom's associated space for $E = X \times \mathbb{C}^n$ given by

$$T(E) = (X \times \mathbf{D}^n)/(X \times \mathbf{S}^{n-1}) \cong (X \times \mathbb{C}^n)^+ \cong (X \times \mathbf{S}^n)/(X \times \{\infty\}) = \Sigma^n(X_+).$$

By Thom's isomorphism from Theorem 3.3 we have that $K_G(X \times \mathbf{D}^{2n}, X \times \mathbf{S}^{2n-1}) \cong K_G(X)$. Then, if we consider the following long exact sequence

$$\cdots \longrightarrow K_G(X \times \mathbf{D}^{2n}, X \times \mathbf{S}^{2n-1}) \longrightarrow K_G(X \times \mathbf{D}^{2n}) \longrightarrow K_G(X \times \mathbf{S}^{2n-1}) \longrightarrow \cdots,$$

the map $K_G(X \times \mathbf{D}^{2n}, X \times \mathbf{S}^{2n-1}) \to K_G(X \times \mathbf{D}^{2n})$ can be identified with the map $K_G(X) \to K_G(X)$ given by multiplication by $\xi^n = (1-\rho)^n$ where ρ is the standard one dimensional representation of T. Since ξ generates I_T^n then ξ^n generates I_T^n and we can construct the following short exact sequence

$$0 \longrightarrow K_G(X)/(\xi^n K_G(X)) \longrightarrow K_G(X \times \mathbf{S}^{2n-1}) \longrightarrow K_n \longrightarrow 0,$$

where we denote $K_n = \{x \in K_G(X)\colon \xi^n x = 0\}$. Since $K_G(X)$ is a finitely generated ring and $R(G)$ is Noetherian, there is a k such that $K_k = K_{k+1} = K_{k+2} = \cdots$. Therefore $\xi^k K_{n+k} = 0$ for all $n \in \mathbb{N}$ and we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_G(X)/(\xi^{n+k}K_G(X)) & \longrightarrow & K_G(X \times \mathbf{S}^{2n+2k-1}) & \longrightarrow & \xi^{n+k}K_G(X) & \longrightarrow & 0 \\
 & & \downarrow & \swarrow g_n & \downarrow & & \downarrow \times\xi^k & & \\
0 & \longrightarrow & K_G(X)/(\xi^{n}K_G(X)) & \longrightarrow & K_G(X \times \mathbf{S}^{2n-1}) & \longrightarrow & \xi^{n}K_G(X) & \longrightarrow & 0,
\end{array}$$

that is also exact in the rows. Then by Lemma A.5 we have the result. □

Now, for T^m we will do some inductive reasoning. Let G be a compact Lie group and X a compact G-space such that $K_G(X)^*$ is finite over $R(G)$.

Lemma 3.5. *Let $\theta\colon G \to T^m$ be a homomorphism so that G acts over ET^m. Then the homomorphisms*

$$\gamma_n\colon \frac{K_G(X)}{I^n_{T^m}K_G(X)} \to K_G\left(X \times (ET^m)^n\right),$$

induce an inverse limit isomorphism.

Proof. We will prove this result using induction. First, note that when $m = 1$ we have the result by Lemma 3.4.

Suppose now that the result is true for T^k with $k < m$. Let us write $T^m = T \times H$ where $H = T^{m-1}$. If we consider the projections $\mathrm{pr}_1\colon T \times H \to H$ and $\mathrm{pr}_2\colon T \times H \to H$, then with the compositions $\mathrm{pr}_1 \circ \rho$ and $\mathrm{pr}_2 \circ \rho$ we can see G acting over both ET and EH. Note that $ET \times EH$ is also a contractible space where T^m acts freely, so we can consider $ET \times EH$ for the total space ET^m. We also have that $ET \times EH$ has a natural filtration of the form $\{ET^n \times EH^l\}_{n,l \in \mathbb{N}}$. Hence we can consider the map that induces the same inverse limit as γ_n as follows

$$\gamma_{n,l}\colon \frac{K_G(X)}{I^n_{T^m}K_G(X)} \to K_G(X \times ET^n \times EH^l).$$

On the other hand, note that both $R(T)$ and $R(H)$ can be included in $R(T^m)$ in the following way. If $\rho\colon T \to \mathrm{GL}(V)$ is a representation of T, then we have that $\rho \circ \mathrm{pr}_1\colon T \times H \to \mathrm{GL}(V)$ is a representation of $T \times H$. Similarly for a representation of H using the map pr_2. Then we have that $I_T, I_H \subseteq R(T^m)$.

Now, let a and b be the ideals generated by I_T and I_H in $R(T^m)$ respectively. Let $[V_1]-[V_2] \in I_{T^m}$ with maps $\rho_1 \colon T \times H \to \mathrm{GL}(V_1)$ and $\rho_2 \colon T \times H \to \mathrm{GL}(V_2)$ for the class representatives. Then we necessarily have that $\dim V_1 = \dim V_2$. Let $\iota_1 \colon T \to T \times H$ and $\iota_2 \colon H \to T \times H$ be the inclusions in T^m of T and H respectively. Then $\rho_1 = \rho_1 \circ \iota_1 + \rho_1 \circ \iota_2$ and $\rho_2 = \rho_2 \circ \iota_1 + \rho_2 \circ \iota_2$ from the definition of a homomorphism. Also, note that $\overline{\rho_1 \circ \iota_1} + \overline{\rho_2 \circ \iota_1} \in a$ and $\overline{\rho_1 \circ \iota_2} + \overline{\rho_2 \circ \iota_2} \in b$, therefore we have that $[V_1]-[V_2] \in a+b$ thus $I_{T^m} \subseteq a+b$. On the other hand, since $a, b \subseteq I_{T^m}$ we have that $a+b \subseteq I_{T^m}$. This then concludes that $I_{T^m} = a+b$.

Note that I_{T^m} can be filtered through $\{I^n_{T^m}\}_{n\in\mathbb{N}}$ and $a+b$ through $\{a^n+b^n\}_{n\in\mathbb{N}}$. This gives both I_{T^m} and $a+b$ an inverse system structure where the structural maps are defined by the canonical projections. On one hand, we have that $a^n+b^n \subseteq I^n_{T^m}$, so we can define the following quotient map

$$f_n \colon \frac{K_G(X)}{(a^n+b^n)K_G(X)} \to \frac{K_G(X)}{I^n_{T^m}K_G(X)}.$$

This is then an inverse system morphism and is well defined. On the other hand, we have that $(a+b)^{2n-1} \subseteq a^n+b^n$. Then, we can also define the following quotient map

$$g_n \colon \frac{K_G(X)}{I^{2n-1}_{T^m}K_G(X)} \to \frac{K_G(X)}{(a^n+b^n)K_G(X)}.$$

This map is well defined and is also an inverse system morphism. Since the following diagram commutes

$$\begin{array}{ccc}
\frac{K_G(X)}{(a^{2n-1}+b^{2n-1})K_G(X)} & \xrightarrow{f_{2n-1}} & \frac{K_G(X)}{I^{2n-1}_{T^m}K_G(X)} \\
\downarrow & \swarrow_{g_n} & \downarrow \\
\frac{K_G(X)}{(a^n+b^n)K_G(X)} & \xrightarrow{f_n} & \frac{K_G(X)}{I^n_{T^m}K_G(X)},
\end{array}$$

we have that both filtrations induce isomorphic inverse limits. This means that we can use the filtration $\{a^n+b^n\}_{n\in\mathbb{N}}$ in $\gamma_{n,l}$. Even more, we can use Lemma A.7 to prove that the filtrations given by $\{K_G(X)/(a^n+b^n)K_G(X)\}_{n\in\mathbb{N}}$ and $\{K_G(X)/(a^n+a^l)K_G(X)\}_{n,l\in\mathbb{N}}$ induce isomorphic inverse limits, therefore we will use the latter.

We then have the map $\delta_{n,l} \colon \frac{K_G(X)}{(a^n+b^l)K_G(X)} \to K_G(X \times ET^n \times EH^l)$ which induces the same inverse limit map as the map γ_n. Since we know that $ET^n \cong \mathbf{S}^{2n-1}$ we can construct the short exact sequence

$$0 \longrightarrow \frac{K_G(X)}{\xi^n K_G(X)} \longrightarrow K_G(X \times \mathbf{S}^{2n-1}) \longrightarrow K_n \longrightarrow 0$$

from the proof of Lemma 3.4, where $K_n = \{x \in K_G(X) \colon \xi^n x = 0\}$ with $\xi^n = (1-\rho)^n$ and ρ the standard one dimensional representation. Since we have that $K_G(X)$ is Noetherian, then both $\frac{K_G(X)}{\xi^n K_G(X)}$ and K_n are finitely generated. Even more, since the previous sequence is exact, we have that $K_G(X \times \mathbf{S}^{2n-1}) = K_G(X \times ET^n)$ is finitely generated over $R(G)$.

We also know that for any R-module A we have an isomorphism $\omega_A \colon A/IA \to A \otimes_R R/I$, where I is an ideal of R. Since the map $\delta_{n,l}$ is induced by the projection $X \times ET^n \times ET^l \to X$, and this projection can be factored as $X \times ET^n \times EH^l \to X \times ET^n \to X$ we then have that the map $\delta_{n,l}$ can be factored in the following way

$$\begin{array}{ccccc}
\frac{K_G(X)}{(a^n+b^l)K_G(X)} & & \xrightarrow{\quad\delta_{n,l}\quad} & & K_G(X \times ET^n \times EH^l) \\
\downarrow{\scriptstyle \omega_{K_G(X)}} & & & & \uparrow{\scriptstyle \gamma_l} \\
\frac{K_G(X)}{a^n K_G(X)} \otimes_{R(T^m)} R(T^m)/b^l & & & & \frac{K_G(X \times ET^n)}{b^l K_G(X \times ET^n)} \\
& \searrow{\scriptstyle \delta_n \otimes \mathrm{id}} & & \nearrow{\scriptstyle \omega^{-1}_{K_G(X \times ET^n)}} & \\
& & K_G(X \times ET^n) \otimes_{R(T^m)} R(T^m)/b^l, & &
\end{array}$$

where δ_n is given by Lemma 3.4 and γ_l by the induction step. Therefore, since the previous diagram is commutative, we have that $\delta_{n,l}$ induces an isomorphism in the inverse limits.

□

By using holomorphic induction we obtain the following general result.

Theorem 3.6. *Let us consider $j \colon T \to U$ the inclusion of the maximal torus T in the unitary group $U = U(m)$. For every compact U-space X let $j^* \colon K_U(X) \to K_T(X)$ be the induced map by j. Then there is a functorial homomorphism of $K_U(X)$-modules*

$$j_* \colon K_T(X) \to K_U(X)$$

that is a left inverse of J^*.

The proof of this theorem can be found in [2, Proposition 4.9]. Note that if we replace X with ΣX we can extend this proposition to K^*.

We know that $K_G(X)$ is a natural direct sumand of $K_T(X)$. We can then construct the following diagram

$$\begin{array}{ccc} \frac{K_G(X)}{I_G^n K_G(X)} & \xrightarrow{\alpha_n} & K_G(X \times EG^n) \\ j_* \big\uparrow \big\downarrow j^* & & j_* \big\uparrow \big\downarrow j^* \\ \frac{K_T(X)}{I_G^n K_T(X)} & \xrightarrow{\eta_n} & K_T(X \times EG^n). \end{array}$$

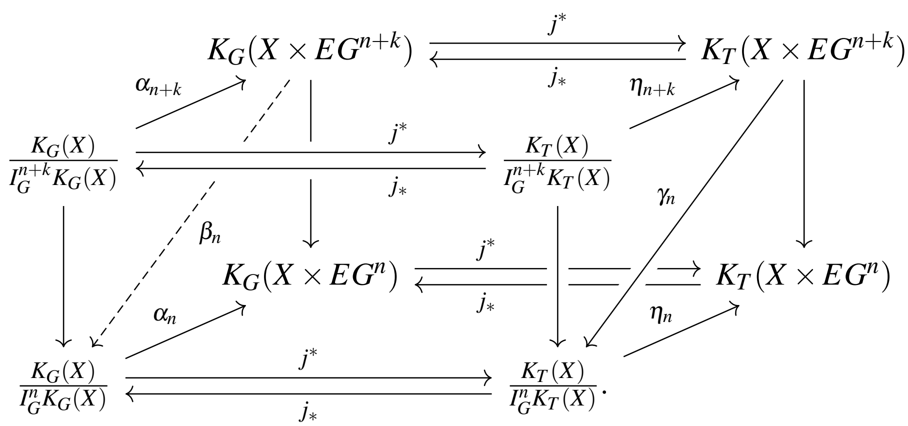
We can also construct the following diagram

$$\begin{array}{ccc} \frac{K_T(X)}{I_G^n K_T(X)} & \xrightarrow{\eta_n} & K_T(X \times EG^n) \\ \big\downarrow \lambda_n & & \big\downarrow \rho_n \\ \frac{K_T(X)}{I_T^n K_T(X)} & \xrightarrow{a_n} & K_T(X \times ET^n). \end{array}$$

By the inductive reasoning, λ_n defines an isomorphism since the I_G and I_T-adic topologies coincide in any $R(T)$-module. We also have that EG^n and ET^n can both be seen as T-spaces and they define the same inverse limit, hence ρ_n defines an isomorphism. Hence, since a_n defines an isomorphism by Lemma 3.5, the commutative diagram gives us that η_n also defines an isomorphism. This tells us that there is a map γ_n such that the following diagram commutes

$$\begin{array}{ccc} \frac{K_T(X)}{I_G^{n+k} K_T(X)} & \xrightarrow{\eta_{n+k}} & K_T(X \times EG^{n+k}) \\ \big\downarrow & \swarrow_{\gamma_n} & \big\downarrow \\ \frac{K_T(X)}{I_G^n K_T(X)} & \xrightarrow{\eta_n} & K_T(X \times EG^n), \end{array}$$

for some $k \in \mathbb{Z}$. Note that we can form the following diagram



We denote the map $\beta_n = j_* \circ \gamma_n \circ j^* \colon K_G(X \times EG^{n+k}) \to \frac{K_G(X)}{I_G^n K_G(X)}$. Given that $j^* \circ j_* = \mathrm{id}$ then it is easy to see that α_n and β_n satifiy the conditions of Lemma A.5, therefore we have that α_n induces an inverse limit isomorphism.

With this in hand, now we can finally see the proof of the theorem.

Proof. Theorem 3.1. Let G be a compact Lie group and X a compact G-space. We know that we can embed G in a sufficiently large unitary group U. Then $U \times_G X$ is a compact U-space. We then have that

$$K_U(U \times_G X) \cong K_G(X),$$

as $R(U)$ modules if and only if $K_G(X)$ is finite over $R(G)$ since $R(G)$ is finite over $R(U)$.

Then we have from the previous Lemma that the following function defines isomorphic inverse limits

$$\begin{array}{ccc} \frac{K_U(U \times_G X)}{I_U^n K_U(U \times_G X)} & \longrightarrow & K_U(U \times_G X \times EU^n) \\ \downarrow & & \downarrow \\ \frac{K_G(X)}{I_G^n K_G(X)} & \xrightarrow{\alpha_n} & K_G(X \times EG^n). \end{array}$$

And then the result follows. □

Lastly, we want to see what is the relation between $\varprojlim_n K_G(X \times EG^n)$ and $K_G(X \times EG)$. For that, we have the following proposition.

Proposition 3.7. *If the space X is the limit of an expanding sequence of compact subspaces X_n then there is a natural sequence*

$$0 \longrightarrow \varprojlim{}^1 K_G^{q-1}(X_n) \longrightarrow K_G^q(X) \longrightarrow \varprojlim K_G^q(X_n) \longrightarrow 0.$$

The proof of this result can be found in [4, Proposition 4.1].

Note that in our case we have $X_n = X \times EG^n$, which is an expanding sequence of compact subspaces, and $X = X \times EG$ is its direct limit. Then by the previous proposition we have that there is a natural sequence

$$0 \longrightarrow \varprojlim{}^1 K_G^{q-1}(X \times EG^n) \longrightarrow K_G^q(X \times EG) \longrightarrow \varprojlim K_G^q(X \times EG^n) \longrightarrow 0.$$

Since the structure maps for the inverse system in Theorem 3.1

$$a_n \colon \frac{K_G(X)}{I_G^n K_G(X)} \longrightarrow \frac{K_G(X)}{I_G^{n-1} K_G(X)},$$

are surjective, by Lemma A.6 we have that the system $K_G(X \times EG^n)$ satisfies the Mittag-Leffler condition. This then gives us that $\varprojlim{}^1 K^*(X \times EG^n) = 0$. From the natural sequence we then get the isomorphism

$$K_G(X \times EG) \cong \varprojlim K_G(X \times EG^n).$$

With this result we then have that, when G acts freely on X, $K_G(X)$ is complete and Hausdorff.

3.3. Examples

Now we will dedicate the remaining of this chapter to explore some applications of the previous theorem when the base space is $X = *$. We have seen that for a compact Lie group G we have the identification $K_G(*) \cong R(G)$. Also, note that $K_G(* \times EG) \cong K(EG/G) \cong K(BG)$. Therefore, Atiyah-Segal's completion theorem states that

$$K^*(BG) \cong R(G)\hat{}_{I_G}.$$

Example 3.8. Let us then see what happens for specific compact Lie groups G.

1. Let $G = *$. Then we have that $K_G^*(X) \cong R(G) \cong \mathbb{Z}$. We also have that the augmentation ideal $I = \{0\}$ since the augmentation map is an isomorphism. Note that when we take EG^n, in

this case we get a contractible space. It is easy to see that the limit EG is also a contractible space. Then we have that $K_G^*(X \times EG) = K^*(X \times EG) \cong K^*(X)$. Therefore the theorem in this case tells us that

$$K^*(X) \cong \mathbb{Z},$$

which is a result that we already had.

2. Let $G = \mathbb{Z}/2$. We saw before that $BG = \mathbb{R}P^\infty$. We also know that $R(G) \cong \mathbb{Z}[x]/\langle x^2 - 1\rangle$. Then the theorem establishes that

$$K^*(\mathbb{R}P^\infty) \cong \big(\mathbb{Z}[x]/\langle x^2-1\rangle\big)\hat{}_{I_G}.$$

Note that in $R(G)$, both irreducible representations are one dimensional, hence both x and 1 are mapped by the augmentation map to 1. It is then easy to see that $I_G = \langle x-1\rangle$.

To find the completion with respect to the augmentation ideal of $R(G)$, let us do a change of variables with $t = x-1$. Then we have that $R(G) \cong \frac{\mathbb{Z}[t]}{\langle t^2+2t\rangle}$. Let us denote $R = \frac{\mathbb{Z}[t]}{\langle t^2+2t\rangle}$. Note that R is a Noetherian ring. By example A.13 from the appendix, we have that

$$R\hat{}_{I_G} \cong \frac{R[\![y]\!]}{\langle y-t\rangle} \cong \frac{\mathbb{Z}[\![t]\!]}{\langle t^2+2t\rangle}.$$

Hence,

$$K^*(\mathbb{R}P^\infty) \cong \frac{\mathbb{Z}[\![t]\!]}{\langle t^2+2t\rangle}.$$

3. Let $G = \mathbf{S}^1$. Then we know that $R(G) \cong \mathbb{Z}[x, x^{-1}]$. In this case, we also have that the augmentation ideal is $I = \langle x-1\rangle$. By making the same change of variable as in the previous example we have that $R(G) \cong \mathbb{Z}[t,(t+1)^{-1}]$ and the augmentation ideal is generated by t. Then, we have that the completion is going to be given by

$$\mathbb{Z}[t,(t+1)^{-1}][\![y]\!]/\langle y-t\rangle.$$

To see this completion, let us consider the map $f\colon \mathbb{Z}[\![t]\!] \to \mathbb{Z}[t,(t+1)^{-1}][\![y]\!]/\langle y-t\rangle$ defined by sending t to $\bar{y}$. It is clear that this function is well defined. Now, let us define the function $\widetilde{g}\colon \mathbb{Z}[t,(t+1)^{-1}][\![y]\!] \to \mathbb{Z}[\![t]\!]$ by sending t to t, $(t+1)^{-1}$ to $(t+1)^{-1}$ and y to t. Since $(t+1)^{-1}$ is a power series this function is well defined. Also note that this map, by definition, factors in the following way

$$\begin{array}{ccc} \mathbb{Z}[t,(t+1)^{-1}][\![y]\!] & \xrightarrow{\widetilde{g}} & \mathbb{Z}[\![t]\!] \\ \downarrow & \overset{g}{\nearrow} & \\ \frac{\mathbb{Z}[t,(t+1)^{-1}][\![y]\!]}{\langle y-t\rangle}. & & \end{array}$$

Hence we have the well defined map $g\colon \mathbb{Z}[t,(t+1)^{-1}][\![y]\!]/\langle y-t\rangle \to \mathbb{Z}[\![t]\!]$. It is easy to verify that $f\circ g = \mathrm{id}_{\mathbb{Z}[\![x]\!]}$ and $g\circ f = \mathrm{id}_{\mathbb{Z}[t,(t+1)^{-1}][\![y]\!]/\langle y-t\rangle}$. Hence we have that

$$R(G)\hat{}_{I_G} \cong \mathbb{Z}[\![t]\!].$$

On the other hand, we have seen that $EG^n \cong \mathbf{S}^{2n-1}$. Then, we have that $EG \cong \mathbf{S}^\infty$ thus giving us $BG = EG/G \cong \mathbf{S}^\infty/\mathbf{S}^1 \cong \mathbb{C}P^\infty$. Therefore, Atiyah-Segal's theorem tells us that

$$K^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\![t]\!].$$

A. Appendix

In this appendix, we will see some basic concepts on direct limits, inverse limits and completion. These concepts were mentioned and used in the previous chapters. More specifically, the concepts of inverse limits were used in the proof of Atiyah-Segal's theorem as an alternate way than the one proposed in [4] using pro-objects.

A.1. Direct and inverse limits

Definition A.1. Let I be an indexing set and $(A_i)_{i\in I}$ a group family. Suppose there exists a homomorphism family $f_{ij}\colon A_i \to A_j$ for all $i \leqslant j$ called the bonding maps, such that

1. For all i we have that the map $f_{ii} = \mathrm{id}_{A_i}$.
2. For all $i \leqslant j \leqslant k$ the maps $f_{ik} = f_{ij} \circ f_{jk}$.

Then the pair $(\{A_i\}_{i\in I}, \{f_{ij}\}_{i\leqslant j\in I})$ is called a *direct system* of groups and morphisms over I. We also call f_{ij} the transition maps of the system.

Definition A.2. Let $(\{A_i\}_{i\in I}, \{f_{ij}\}_{i\leqslant j\in I})$ be a direct system of groups over I. The *direct limit*, also known as the colimit, is defined by

$$\varinjlim_{i\in I} A_i = \bigsqcup_i A_i / \sim,$$

where $x_i \sim x_j$ for $x_i \in A_i$ and $x_j \in A_j$ if and only if there is some $k \in I$ with $i \leqslant K$ and $j \leqslant k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$.

The direct limit can then be also thought as two elements in the disjoint union are equivalent if and only if they eventually become the same in the direct system.

Definition A.3. Let I be an indexing set and $(A_i)_{i\in I}$ a group family. Suppose there exists a homomorphism family $f_{ij}\colon A_j \to A_i$ for all $i \leqslant j$ called the bonding maps, such that

1. For all i the map $f_{ii} = \mathrm{id}_{A_i}$.
2. For all $i \leqslant j \leqslant k$ the maps $f_{ik} = f_{ij} \circ f_{jk}$.

Then the pair $(\{A_i\}_{i\in I}, \{f_{ij}\}_{i\leqslant j\in I})$ is called an *inverse system* of groups and morphisms over I. We also call f_{ij} the structure maps of the system.

Definition A.4. Let $(\{A_i\}_{i\in I}, \{f_{ij}\}_{i\leqslant j\in I})$ be an inverse system. The *inverse limit*, also known as limit, is a particular subgroup of the direct product of all the A_i's in the following way.

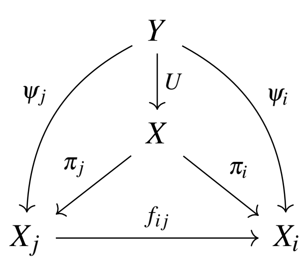
$$A = \varprojlim_{i\in I} A_i = \left\{ \overline{a_i} \in \prod_{i\in I} A_i \colon a_i = f_{ij}(a_j) \quad \text{for all} \quad i \leqslant j \in I \right\}.$$

This set comes equipped with natural projections $\pi_j\colon A \to A_j$ that satisfy a universal property.

In general, if we have an inverse system (X_i, f_{ij}) of objects and morphisms in a category $\mathscr{C}$ where $i, j \in I$, with I a well ordered indexing set. This means that the maps $f_{ij}\colon X_j \to X_i$ have the properties $f_{ii} = \mathrm{id}_{X_i}$ and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leqslant j \leqslant k$. The inverse limit of this system is an object X in $\mathscr{C}$ together with morphisms $\pi_i\colon X \to X_i$ called the projections such that for all $i \leqslant j$ we have

$$\pi_i = f_{ij} \circ \pi_j.$$

The pair (X, π_i) is universal. This means, if there is another pair (Y, ψ_i) such that $\psi_i\colon Y \to X_i$ satisfies that $\psi_i = f_{ij} \circ \psi_j$ for all $i \leqslant j$, then there is a unique morphism $U\colon Y \to X$ such that the following diagram commutes



Lemma A.5. *Suppose that $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are inverse systems. Suppose that $f_n\colon A_n \to B_n$ is a group homomorphism such that the following diagram commutes for all $n \geqslant 1$.*

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle b_n} \\ A_{n-1} & \xrightarrow[f_{n-1}]{} & B_{n-1}. \end{array}$$

Moreover, suppose that we have a map $g_n\colon B_{k(n)+n} \to A_n$ *such that for every* $n \in \mathbb{N}$ *there is a* $l > \max\{k(n)+n, k(n-1)+n-1\}$ *such that the following diagram commutes*

$$\begin{array}{cccc} & B_{k(n)+n} & \xrightarrow{g_n} & A_n \\ \nearrow & & & \\ B_l & & & \downarrow{\scriptstyle a_n} \\ \searrow & & & \\ & B_{k(n-1)+n-1} & \xrightarrow{g_{n-1}} & A_{n-1}. \end{array}$$

If the following diagram commutes for all $n \geqslant 0$

$$\begin{array}{ccc} A_{k(n)+n} & \xrightarrow{f_{k(n)+n}} & B_{k(n)+n} \\ {\scriptstyle a_*}\downarrow & {\scriptstyle g_n}\swarrow & \downarrow{\scriptstyle b_*} \\ A_n & \xrightarrow[f_n]{} & B_n, \end{array}$$

where a_* *and* b_* *denote the appropriate composition of the structure maps. Then the induced map* $f\colon \varprojlim A_n \to \varprojlim B_n$ *is an isomorphism.*

Proof. Note that the map induced by f_n in the inverse limit $f\colon \varprojlim A_n \to \varprojlim B_n$ is given by $f(\overline{(x_n)}) = \overline{(f_n(x_n))}$ for $\overline{x_n} \in \varprojlim A_n$ and the inverse limit map $g\colon \varprojlim B_n \to \varprojlim A_n$ induced by g_n is given by $g(\overline{(y_n)}) = \overline{(g_n(y_{k(n)+n}))}$ for $\overline{y_n} \in \varprojlim B_n$. Let us verify that they are inverses of each other. On one hand, for $\overline{y_n} \in \varprojlim B_n$ we have that

$$f \circ g(\overline{y_n}) = f(\overline{(g_n(y_{k(n)+n}))}) = \overline{(f_n \circ g_n(y_{k(n)+n}))} = \overline{(b_*(y_{k(n)+n}))} = \overline{(y_n)}.$$

On the other hand, for $\overline{(x_n)} \in \varprojlim A_n$ we have that

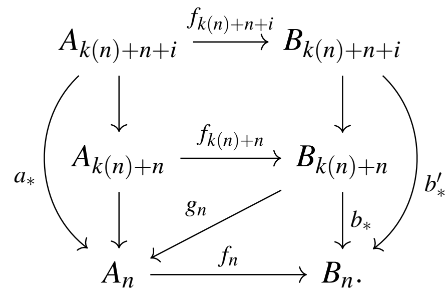
$$g \circ f(\overline{x_n}) = g(\overline{f_n(x_n)}) = \overline{(g_n \circ f_n(x_n))} = \overline{(a_*(x_{k(n)+n})} = \overline{(x_n)}.$$

Then f is an isomorphism. $\square$

Lemma A.6. *If $\{A_n\}_n$ is an inverse system such that the structure maps $a_n \colon A_n \to A_{n-1}$ are surjective and both $\{A_n\}$ and $\{B_n\}$ satisfy the hypotheses of Lemma A.5, then $\{B_n\}$ satisfies Mittag-Leffler's condition.*

Proof. We want to see that $\{B_n\}_n$ satisfies the Mittag-Leffler's condition. This means that for all n there is a m such that $b(B_m) = b(B_k) \subseteq B_n$ for all $k \geqslant m$.

If $k \geqslant m$ we know that $b(B_k) \subseteq b(B_m)$. If we make $m = k(n) + n$ then we have the following commutative diagram for all $i > 0$.



We know that $\operatorname{Im}(B_{k(n)+n+i}) \subseteq \operatorname{Im}(B_{k(n)+n})$, now let us see that $\operatorname{Im}(B_{k(n)+n}) \subseteq \operatorname{Im}(B_{k(n)+n+i})$. Let $x \in \operatorname{Im}(B_{k(n)+n}) \subseteq B_n$, this means there exists a $y \in B_{k(n)+n}$ such that $b_*(y) = x$ where we denote $b_* = b_{n+1} \circ b_{n+2} \circ \cdots \circ b_{k(n)+n}$. Let $b = g_n(y) \in A_n$. Since the structural maps are surjective we know that there is a $c \in A_{k(n)+n+i}$ such that $a_*(c) = b$ where $a_* = a_{n+1} \circ a_{n+2} \circ \cdots \circ a_{k(n)+n+i}$. Hence we have that

$$f_n(a_*(c)) = f_n(b) = f_n(g_n(y)) = b_*(y) = x$$

On the other hand, let $z = f_{k(n)+n+i}(c) \in B_{k(n)+n+i}$. Note that

$$b'_*(z) = b'_*(f_{k(n)+n+i}(c)) = f_n(a_*(c)) = x$$

Hence $x \in \operatorname{Im}(B_{k(n)+n+i})$. This proves the result. □

Lemma A.7. *Let $\{A_{nm}\}_{n,m\in\mathbb{N}}$ be an inverse system with double index. Then the following inverse limits are isomorphic*

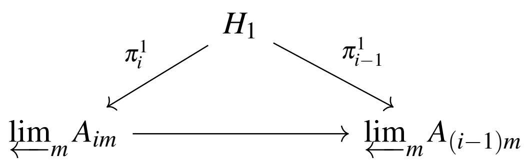
$$L = \varprojlim_{n,m} A_{nm}$$

$$D = \varprojlim_{n} A_{nn}$$

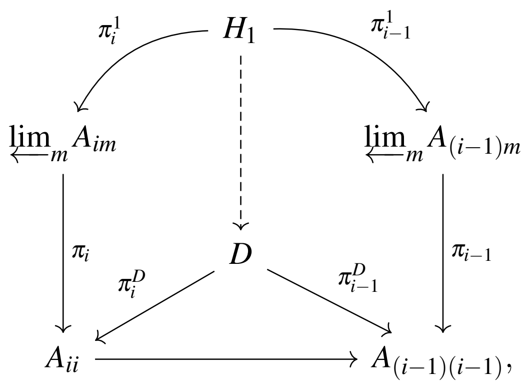
$$H_1 = \varprojlim_{n} \varprojlim_{m} A_{nm}$$

$$H_2 = \varprojlim_m \varprojlim_n A_{nm}$$

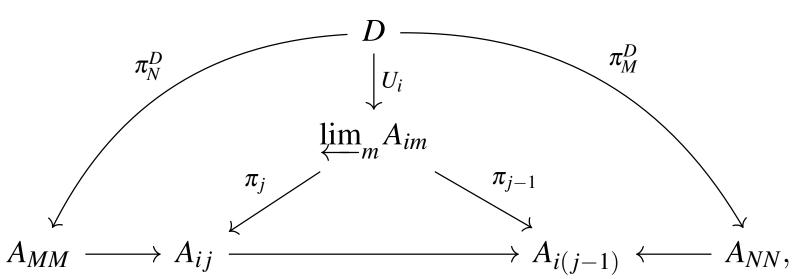
Proof. We will denote by $\pi^L, \pi^D, \pi^1, \pi^2$ and π the natural projections of L, D, H_1, H_2 and $\varprojlim_n A_n$ respectively, that satisfy the universal property. Let us start by proving that $D \cong H_1$. For that, remember that we have maps for H_1 in the following way



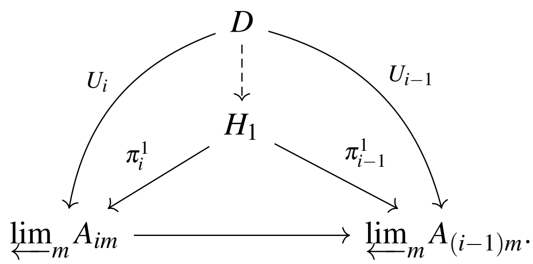
Hence we can make the following commuting diagram



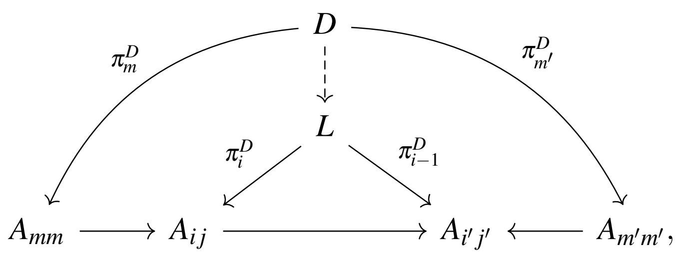
which gives us the unique morphism $H_1 \to D$ by the universal property. On the other hand, we can also construct the following diagram



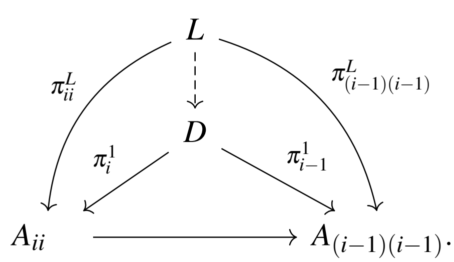
where $M = \max\{i, j\}$ and $N = \max\{i, j-1\}$. Therefore we can construct the following diagram



Therefore $D \cong H_1$. In the same manner one can prove that $D \cong H_2$. Lastly, let us prove that $D \cong L$. For that, note that we can construct the following commutative diagrams



where $m = \max i, j$ and $M' = \max i', j'$, and



Therefore $L \cong D$ and we have the result. □

Corollary A.8. *Let $f\colon \{A_{nm}\}_{n,m\in\mathbb{N}} \to \{B_{nm}\}_{n,m\in\mathbb{N}}$ be an inverse system morphism such that for all fixed n the function $f_n\colon \{A_{nm}\}_{m\in\mathbb{N}} \to \{B_{nm}\}_{m\in\mathbb{N}}$ induces isomorphic inverse limits, then f induces isomorphic inverse limits.*

A.2. Completion

Now, let us explore the concept of completion. Let G be an abelian topological group. We first want to see that the topology of the group G is uniquely determined by the neighborhoods of $0 \in G$. For that we need the following lemma.

Lemma A.9. *Let H be the intersection of all the neighborhoods of $0 \in G$. Then*

1. *H is a subgroup.*
2. *H is the closure of $\{0\}$.*
3. *G/H is Hausdorff.*
4. *G is Hausdorff if and only if $H = 0$.*

Proof.

1. This follows from the continuity of the group operations.
2. Let U be a neighborhood of $0 \in G$. If $a \in H$ then $0 \in a - U = \{a - x \colon x \in U\}$. In other words, $\{0\} \cap (a - U) \neq \emptyset$. Therefore $V \cap \{0\} \neq \emptyset$ for all neighborhoods V of a. Thus $a \in \overline{\{0\}}$.
3. By (2) we know that H is closed. Hence $a + H = \{a + h \colon h \in H\} = \{\overline{a}\}$ is also closed. Then G/H is Hausdorff.
4. By (3), if $H = 0$ then G is Hausdorff. On the other hand, if G is Hausdorff then by definition $H = \{0\}$.

□

Definition A.10. A *Cauchy sequence in G* is a sequence (x_n) in G such that for every neighborhood U of 0 there is an integer s_U such that $x_n - x_m \in U$ for all $n, m \geqslant s_U$. Two Cauchy sequences (x_n) and (y_n) are called equivalent if $x_n - y_n$ tends to 0 in G.

We will denote by $\tilde{G}$ the set of equivalence classes of Cauchy sequences in G. Note that $\tilde{G}$ is an abelian group defining $(x_n) + (y_n) = (x_n + y_n)$. We can also define $\phi \colon G \to \tilde{G}$ by $\phi(x) = (x)$ and we have an abelian group homomorphism. The $\ker \phi = \cap U$ for all U neighborhood of 0.

Let H be another topological group and $f \colon G \to H$ an continuous homomorphism. Then f induces a homomorphism $\tilde{f} \colon \tilde{G} \to \tilde{H}$ that assigns each term of a sequence its image under f. Since f is continuous, this is well defined. Even more, if $f\colon G \to H$ and $g\colon H \to K$ are continuous then $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

Suppose now that the fundamental system of neighborhoods of $0 \in G$ consists of subgroups. Then we have a subgroup sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots .$$

If $g \in G_n$ then $g + G_n$ is a neighborhood of g because $g + G_n \subseteq G_n$ and then G_n is open. Then $h + G_n$ is open and the set

$$\bigcup_{h \notin G_n} h + G_n,$$

is also open. Then the compement of G_n is closed.

Let (x_m) be a Cauchy sequence on G. The projection of (x_m) into G/G_n is eventually constant, let's say ξ_n. If we pass from $n+1$ to n it is clear that ξ_{n+1} tends to ξ_n under

$$\theta_{n+1}\colon G/G_{n+1} \to G/G_n.$$

Then (x_m) defines a coherent sequence (ξ_n), meaning, $\theta_{n+1}(\xi_{n+1}) = \xi_n$ for all n and this does not depend on the class representatives.

Then, $\tilde{G}$ can be defined as the set of coherent sequences (ξ_n) with the obvious group structure. This is also called the inverse limit of the system hence $\tilde{G} \cong \varprojlim G/G_n$.

Definition A.11. If M is an R-module and I is an ideal from R then the *I-adic completion* of M is defined by

$$\hat{M_I} = \varprojlim_n \frac{M}{I^n M}.$$

This is also known as the completion by the ideal I.

Example A.12. Let us see the completion of $\mathbb{Z}[x]$ by the ideal $\langle x \rangle$. For that, we are going to define the following map $p\colon Z[\![x]\!] \to \varprojlim_n \mathbb{Z}[x]/\langle x^n \rangle$ by

$$p(\sum a_i x^i) = \overline{\left(\sum_{i=1}^{n} a_i x^i \right)},$$

for $\sum a_i x^i \in \mathbb{Z}[\![x]\!]$. It can be readily seen that this map is injective, we can also easily verify that it is surjective by recursively choosing appropriate representatives which are partial sums of a power series. Then we have that

$$\varprojlim_n \frac{\mathbb{Z}[x]}{\langle x^n \rangle} = \mathbb{Z}[\![x]\!].$$

Example A.13. More generally, given a Noetherian ring R and an ideal $I = \langle f_1, \dots, f_n \rangle$, the completion of R with respect to I can be seen as the quotient $R[\![x_1, \dots, x_n]\!]/\langle x_1 - f_1, \dots, x_n - f_n \rangle$.

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