A distributional approach to asymptotics of the Spectral Action

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Abstract

The spectral action is the natural and appropriate notion of an action on the space of spectral triples, and it was introduced by Chamseddine and Connes in 1997.
After including some definitions and results concerning the Cesàro theory of distributions and asymptotic analysis, we discuss the asymptotic expansion of the spectral action in the distributional sense for a commutative spectral triple following Estrada, Gracia-Bondía and Várilly.

Resumen

La acción espectral es el concepto natural y apropiado para hablar de una acción en el espacio de triplas espectrales, y fue introducido por primera vez por Chamseddine y Connes en 1997.

Después de incluir definitiones y resultados que conciernen a la teoría de Cesàro para distribuciones y análisis asintótico, discutimos la expansión asintotica de la acción espectral en el sentido distribucional para una tripla espectral conmutativa, siguiendo a Estrada, Gracia-Bondía y Várilly.

Keywords: *spectral action, Cesàro summability, distributions, asymptotic expansion, pseudodifferential operators, spectral theory, noncommutative geometry.*

Palabras clave: *acción espectral, sumabilidad de Cesàro, distribuciones, expansiones asintóticas, operadores pseudidiferenciales, teoría espectral, geometría no conmutativa.*

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Introduction

This thesis is the result of my Master's work in the field of distributions, noncommutative geometry and spectral geometry, that I have been conducting under the supervision of Dr. S. Paycha and Dr. C. Neira-Jiménez. It deals with the distributional approach to asymptotics applied to the spectral action in the setting of the canonical commutative spectral triple. Noncommutative geometry turns out to be so broad and modern that many notions from other mathematical branches meet; this work is an example of these meetings. In a nutshell, noncommutative geometry is about extends the correspondence between commutative algebras and spaces to algebras whose elements do not commute, corresponding to some virtual noncommutative space [Con94; CM08a; GBVF01].

In the past decades it became clear that noncommutative geometry is well suited for physical applications. An example of this is the spectral action principle, defined by Chamseddine and Connes [CC97], which is an universal action functional on spectral triples (see Definition 3.3.1) that applied to a particular type of noncommutative spaces, is the way how one gets the Standard Model coupled to gravity [CCM07]. This model also makes various predictions at unification scale, and there are some phenomenological and cosmological consequences of this purely geometrical approach to unification [Sak11]. In this part, authors like [FFM15; ILV12; MPT11] used heat kernel methods to obtain an asymptotic behavior of the spectral action. The assumption made in these works, that a particular function f from which one builds the spectral action is a Laplace transform, is somewhat restrictive since it does not cover interesting cases like the heat kernel case $f(x) = e^{-x^2}$ (see Remark 3.4.1), from which is possible to recover geometrical information [Gil95, Sections 1.7 and 4.8].

The authors of [EGBV98] have dealt with this problem using an interesting distributional approach to derive an asymptotic expansion of the spectral action. Our contribution with this work is to provide some proofs of various steps leading to their interesting results, not included in their work.

In order to do that, this thesis is divided into three chapters. The first is dedicated to set the stage for a number of mathematical tools from distribution theory and Cesàro summability theory with a distributional interpretation, that will be needed in the rest of this work [Est98; Gru09; EK02; EGBV98]. In Chapter 2, the functional calculus for an unbounded operator in a Hilbert space is rewritten in terms of its spectral density, which is an operator-valued distribution to which a Cesàro behavior can be associated. When the operator is a pseudodifferential operator we reach the heart of the matter, because a formula for the distributional kernel of the spectral density is given and applied to compute the coefficients of its asymptotic expansion on the diagonal, in terms of Wodzicki residues (see Theorem 2.3.11) [EGBV98].

What follows is the actual core of this thesis; the application of the Cesàro behavior of distributions (Ch. 1) and its relation with asymptotic analysis (Ch. 2) to noncommutative geometry and the spectral action (Ch. 3). I cannot emphasize enough that is not meant as —and by far is— an exhausting coverage of noncommutative geometry; in general, we only introduce here (sections 3.1 and 3.2) what will be needed later on (sections 3.3 and 3.4). Due to the nature of this part, sections 3.1 and 3.2 will unfortunately be rather prosaic; it mainly contains definitions and examples. Sections 3.3. and 3.4 cover the Chamseddine-Connes expansion from both approaches: with the Laplace transform assumption [CM08b] and the distributional method [EGBV98].

Contents

Chapter 1

Cesàro behavior of distributions

This chapter provides a short introduction to Cesàro summability ([Har92]) of series and integrals of functions to show that they admit a generalization to distributions (Proposition 1.3.24). The notion of parametric and distributional theory of asymptotic expansions in one variable will be introduced in order to study the behavior of distributions at infinity in the "average" Cesàro sense. The final result, which states equivalences between various notions of asymptotic behavior (Theorem 1.4.10), will be useful in the subsequent chapters. References for this chapter are [EK02; Est98; Gru09; Har92].

1.1 Order notation and asymptotic expansions

Let us recall the **big-$\mathcal{O}$** and **little-o** Landau notation, which provides a precise mathematical formulation of ideas that correspond — roughly — to the 'same order of magnitude' and 'smaller order of magnitude', respectively. We state the definitions for the asymptotic behavior of a real-valued function $f(x)$ as $x \to x_0$, where x is a real parameter. With obvious modifications, similar definitions apply to asymptotic behavior in the limits $x \to x_0^+$, $x \to \infty$, to complex or integer parameters, and other cases. Also, by replacing $|\cdot|$ in Definition1.1.1 with a norm, we can define similar concepts for functions taking values in a normed linear space.

Definition 1.1.1. Let $X = \overline{\mathbb{R}}$ and let x_0 in X. Given two functions $f, g : X \to \mathbb{C}$ we write

$$f(x) = \mathcal{O}(g(x)) \text{ as } x \to x_0, \text{ if } \limsup_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < \infty, \quad (1.1)$$

$$f(x) = o(g(x)) \text{ as } x \to x_0, \text{ if } \lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = 0. \quad (1.2)$$

Remark 1.1.2. Observe that writing $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ is equivalent to say that there are constants C and $r > 0$ such that

$$|f(x)| \leq C|g(x)| \text{ whenever } 0 < |x - x_0| < r.$$

In the same way, we could also say $f = o(g)$ as $x \to x_0$ if for every $\delta > 0$ there is an $r > 0$ such that

$$|f(x)| \leq \delta |g(x)| \text{ whenever } 0 < |x - x_0| < r.$$

Example 1.1.3. A few simple examples are:

1. $\sin(1/x) = \mathcal{O}(1)$ as $x \to 0$;
2. it is not true that $1 = \mathcal{O}(\sin(1/x))$ as $x \to 0$, because $\sin 1/x$ vanishes is every neighborhood of $x = 0$;
3. $x^3 = o(x^2)$ as $x \to 0$, and $x^2 = o(x^3)$ as $x \to \infty$;
4. $x = o(\log x)$ as $x \to 0^+$, and $\log x = o(x)$ as $x \to \infty$.

An asymptotic expansion describes the behavior of a function in terms of an asymptotic sequence of functions, as it is stated in the following definition.

Definition 1.1.4. A sequence of functions $\phi_n : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $n = 0, 1, 2, \ldots$, is an *asymptotic sequence* as $x \to x_0$ if for each $n = 0, 1, 2, \ldots$ we have

$$\phi_{n+1} = o(\phi_n) \qquad \text{as } x \to x_0.$$

If $\{a_n\}$ is a sequence of real numbers, $\{\phi_n\}$ is an asymptotic sequence and $f : \mathbb{R} \setminus x_0 \to \mathbb{R}$ is a function, we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \qquad \text{as } x \to 0 \tag{1.3}$$

if for each $N = 0, 1, 2, \ldots$ we have

$$f(x) - \sum_{n=0}^{N} a_n \phi_n(x) = o(\phi_N) \qquad \text{as } x \to 0.$$

We call Equation (1.3) the *asymptotic expansion* of f with respect to $\{\phi_n\}$ as $x \to 0$.

Example 1.1.5.

1. The functions $\phi_n(x) = x^n$ form an asymptotic sequence as $x \to 0^+$, as do $\phi_n(x) = x^{n/2-5}$. The functions $\phi_n(x) = x^{-n}$ form an asymptotic sequence as $x \to \infty$.
2. The function $\log \sin x$ has an asymptotic expansion as $x \to 0^+$ with respect to the asymptotic sequence $\{\log x, x^2, x^4, \ldots\}$:

$$\log \sin x \sim \log x + \frac{1}{6}x^2 + \cdots \qquad \text{as } x \to 0^+.$$

Remark 1.1.6. If, as is usually the case, the asymptotic sequence of functions ϕ_n do not vanish in a punctured neighborhood of 0, then it follows from Definition 1.1.1 that

$$a_{N+1} = \lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n \phi_n(x)}{\phi_{N+1}(x)}. \tag{1.4}$$

Thus, if a function has an asymptotic expansion with respect to a given sequence of functions, the expansion is unique in terms of the coefficients a_n. Different functions may have the same asymptotic expansion.

Example 1.1.7. For any constant $c \in \mathbb{R}$, we have

$$\frac{1}{1-x} + ce^{-1/x} \sim 1 + x + x^2 + \cdots + x^n + \ldots \qquad \text{as } x \to 0^+,$$

since $e^{-1/x} = o(x^n)$ as $x \to 0^+$ for every $n \in \mathbb{N}$, and by Formula (1.4).

Asymptotic expansions can be added, and — under natural conditions on the asymptotic sequences of functions — multiplied. The term-by-term integration of asymptotic expansions is valid, but differentiation may not be, because small, highly-oscillatory terms can become large when they are differentiated (see [EI18, Remark 2.35]).

1.2 Summability

We will discuss the convergence question, from a general point of view, of the arithmetic and integral means.

1.2.1 Summability of series

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers which defines the formal series $\sum_{n=0}^{\infty} a_n$. It often has no limit, but the limit of averages $\lim_{n\to\infty}(A_0 + \cdots + A_n)/(n+1)$, where A_n is the partial sum $A_n := a_0 + \cdots + a_n$, does. For example, if we consider $a_n = (-1)^n$ then we have $\lim_{n\to\infty}(A_0 + \cdots + A_n)/(n+1) = \frac{1}{2}$, which we shall also denote by $\sum a_n = \frac{1}{2}$ **(C,1)** (see [Har92, Equation (1.2.7)]).

Definition 1.2.1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers with partial sums $A_n = a_0 + \cdots + a_n$. We define recursively for $k \in \mathbb{N}$

$$A_n^0 := A_n \quad \text{and} \quad A_n^{k+1} := A_0^k + A_1^k + \cdots + A_n^k$$

and we construct the means

$$\{H_n^k\}_{n=0}^{\infty} \text{ for } k \in \mathbb{N} \quad \text{where} \quad H_n^0 := A_n \quad \text{and} \quad H_n^{k+1} := \frac{H_0^k + H_1^k + \cdots + H_n^k}{n+1}$$

$$\{C_n^k\}_{n=0}^{\infty} \text{ for } k \in \mathbb{N} \quad \text{where} \quad C_n^k := \frac{A_n^k}{\binom{n+k}{k}} = \frac{n!k!}{(n+k)!} A_n^k.$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be

- **(C)-convergent** towards L of order k if, and only if, $\lim_{n\to\infty} C_n^k = L$.
- **(H)-convergent** towards L of order k if, and only if, $\lim_{n\to\infty} H_n^k = L$.

We also talk of (C,k)-convergence for series that are (C)-convergent of order k.

Remark 1.2.2. Notice that the (H)-convergence is the simplest generalization of the (C,1)-convergence, because the means $\{H_n^k\}$ are obtained through k successive averages. However, for most purposes, this is not the most convenient for computations. The (C)-convergence is obtained by the Cèsaro means $\{C_n^k\}$ which, by definition, are constructed by adding terms of the sequence k times followed by a single division, and hence, it has nicer analytical properties. The good news are that both procedures are equivalent: $\sum a_n$ is (C)-convergent of order k if, and only if, it is (H)-convergent of order k to the same limit (see [Har92, Theorem 49]). Additionally, both methods are *regular*, i.e. the usual convergence implies the (C)-convergence (see [Har92, Theorem 49]), and therefore the (H)-convergence.

Definition 1.2.3. We write E_n^k for the value of A_n^k (as in Definition 1.2.1) when $a_0 = 1$ and $a_n = 0$ for $n = 1, 2, \ldots$.

The following lemmas will be useful for future observations:

Lemma 1.2.4. For every $n, k \in \mathbb{N}$

$$\sum_{j=0}^{n} \frac{(j+k)!}{j!} = \frac{(n+k+1)!}{n!(k+1)} \tag{1.5}$$

Proof. By induction on n:

1. For $n = 0$, $\frac{k!}{0!} = k! = \frac{(k+1)!}{k+1}$ for all $k \in \mathbb{N}$.
2. Assume that Equation (1.5) is true for some $n \in \mathbb{N}$ for all $k \in \mathbb{N}$. We see then

$$\begin{aligned} \sum_{j=0}^{n+1} \frac{(j+k)!}{j!} &= \frac{(n+k+1)!}{n!(k+1)} + \frac{(n+1+k)!}{(n+1)!} \\ &= \frac{(n+k+1)!}{n!} \left(\frac{1}{k+1} + \frac{1}{n+1} \right) \\ &= \frac{(n+k+2)!}{(k+1)(n+1)!} \end{aligned}$$

□

Lemma 1.2.5. For every $n, k \in \mathbb{N}$

$$E_n^k = \binom{n+k}{k} \tag{1.6}$$

Proof. By induction on k:

1. For $k = 0$, $E_n^0 = 1$ for all $n \in \mathbb{N}$.

2. Assume that Equation (1.6) is true for some $k \in \mathbb{N}$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} E_n^{k+1} &= \sum_{j=0}^{n} E_j^k = \sum_{j=0}^{n} \binom{j+k}{j} = \frac{1}{k!}\sum_{j=0}^{n} \frac{(j+k)!}{j!} \\ &= \frac{(n+k+1)!}{(k+1)!n!} = \binom{n+k+1}{k+1}. \end{aligned}$$

where we use Lemma 1.2.4. With a second induction on n and the same schema, we finished the proof. □

We could then set $C_n^k = A_n^k / E_n^k$ in Definition 1.2.1 in accordance with Lemma 1.2.5. Notice now that since $\binom{n+k}{k} = \frac{(n+k)(n+k-1)\cdots(n+1)}{k!}$ then $\binom{n+k}{k} \sim n^k/k!$ as $n \to \infty$ so the (C)-convergence of the series $\sum_{n=0}^{\infty} a_n$ towards L of order k is equivalent to

$$\lim_{n\to\infty} \frac{k! A_n^k}{n^k} = L \tag{1.7}$$

and we will use this characterization in the following sections. Moreover, another advantage of (C)-convergence is that, while the Hölder means are naturally defined for $k \in \mathbb{N}$, the Cesàro means can be defined for $k \in \mathbb{R}$ with $k > -1$: For $|x| < 1$, consider the following identities of formal power series

$$\frac{1}{1-x}\sum_{n=0}^{\infty} A_n^{k-1} x^n = \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} A_n^{k-1} x^n = \sum_{n=0}^{\infty} A_n^k x^n$$

via the Cauchy product of series. Thus, inductively,

$$\begin{aligned} \sum_{n=0}^{\infty} A_n^k x^n &= \frac{1}{(1-x)}\sum_{n=0}^{\infty} A_n^{k-1} x^n \\ &= \frac{1}{(1-x)^2}\sum_{n=0}^{\infty} A_n^{k-2} x^n \\ &\vdots \\ &= \frac{1}{(1-x)^k}\sum_{n=0}^{\infty} A_n x^n \\ &= \frac{1}{(1-x)^{k+1}}\sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

This suggests that for any $k > -1$, we may simply define A_n^k and E_n^k to be the quantities satisfying

$$\begin{gathered} \textstyle\sum_{n=0}^{\infty} E_n^k x^n = \frac{1}{(1-x)^{k+1}}, \text{ and} \\ \textstyle\sum_{n=0}^{\infty} A_n^k x^n = \frac{1}{(1-x)^k}\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x)^{k+1}}\sum_{n=0}^{\infty} a_n x^n. \end{gathered} \tag{1.8}$$

We also have

Proposition 1.2.6. Let a_n and A_n^k be defined as above. Then

$$A_n^k = \sum_{m=0}^{n} \binom{m+k-1}{k-1} A_{n-m} = \sum_{m=0}^{n} \binom{m+k}{k} a_{n-m} = \sum_{m=0}^{n} \binom{n-m+k}{k} a_m.$$

The proof of this follows from Equation (1.8), (see [Har92, Section 5.4]). Note that from the above proposition it is not difficult to see that $E_n^k = \binom{n+k}{k}$ directly from Definition 1.2.3. On the other hand, the last expression of the proposition can be defined for k real by using the Gamma function where the coefficients

$$E_n^k = \binom{n+k}{k} = \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+1)}$$

are nonzero for $k > -1$ and the means $C_n^k = A_n^k / E_n^k$ correctly extend the previous notion of Cesàro means (Definition 1.2.1). Actually, using the meromorphic extension of the Gamma function to the complex plane, we can use the last formula for E_n^k to extend both E_n^k and A_n^k to $\mathbb{C} \setminus \mathbb{Z}_{<0}$. This is not used in the rest of this work.

We want to finish this section with a third averaging procedure equivalent to Cesàro's one: the so-called *Riesz means*. Observe that

$$\begin{aligned} C_n^k &= \binom{n+k}{k}^{-1} \sum_{m=0}^{n} \binom{n-m+k}{k} a_m \\ &= \sum_{m=0}^{n} \frac{(n-m+k)!n!}{(n-m)!(n+k)!} a_m \\ &= \sum_{m=0}^{n} \left(\frac{n+1-m}{n+1}\right) \cdots \left(\frac{n+k-m}{n+k}\right) a_m \\ &= \sum_{m=0}^{n} \left(1 - \frac{m}{n+1}\right) \cdots \left(1 - \frac{m}{n+k}\right) a_m \end{aligned}$$

If we replace all of $n+1,\ n+2,\ \ldots,\ n+k$ by n we obtain a new mean

$$R_n^k := \sum_{m=0}^{n} \left(1 - \frac{m}{n}\right)^k a_m$$

and we could take $\lim_{n\to\infty} R_n^k = L$ as a new definition of the summability of $\sum_{n=0}^{\infty} a_n$ to L. Yet, this definition is quite unsatisfactory since for large k, the R_n^k means do not behave like the Cesàro means C_n^k (see [Har92, Section 5.16]). However, modifying this definition of Riesz means by the introduction of a continuous parameter μ we have:

Definition 1.2.7. The series $\sum_{n=0}^{\infty} a_n$ is said to be **(R)-convergent** towards L of order k if, and only if, $\lim_{\mu\to\infty} \frac{1}{\mu} \sum_{n\le\mu} \left(1 - \frac{n}{\mu}\right)^k a_n = L$.

Remark 1.2.8. $\sum a_n$ is (C)-convergent of order k if, and only if, it is (R)-convergent of order k to the same limit (see [Har92, Section 5.16]).

1.2.2 Summability of integrals

Definition 1.2.9. Let f be locally integrable around 0, i.e. $f_0(x) := \int_0^x f(t)dt$ exists for each $x > 0$.

- We define recursively

$$H_0(x) := f_0(x) = \int_0^x f(t)dt \quad \text{and} \quad H_{k+1}(x) := \frac{1}{x}\int_0^x H_k(t)dt.$$

  The integral $\int_0^\infty f(x)dx$ is said to be **(H)-convergent** of order k towards I if, and only if, $\lim_{x\to\infty} H_k(x) = I$.

- We define recursively

$$C_0(x) := f_0(x) = \int_0^x f(t)dt \quad \text{and} \quad C_{k+1}(x) := \int_0^x C_k(t)dt$$

  and the integral $\int_0^\infty f(x)dx$ is said to be **(C)-convergent** of order k towards I if, and only if, $\lim_{x\to\infty} \frac{k!}{x^k} C_k(x) = I$.

Remark 1.2.10. We can also consider the integral $\int_0^x \left(1 - \frac{t}{x}\right) f(t)dt$ and we use integration by parts taking $u = 1 - t/x$ and $v = f_0(t)$ to compute

$$\int_0^x \left(1-\frac{t}{x}\right) f(t)dt = \left(1-\frac{t}{x}\right) f_0(t)\Bigg|_{t=0}^{t=x} + \frac{1}{x}\int_0^x f_0(t)dt = \frac{1}{x}\int_0^x f_0(t)dt. \tag{1.9}$$

Repeating this argument inductively, we have for $k \in \mathbb{N}$

$$\int_0^x \left(1-\frac{t}{x}\right)^k f(t)dt = \frac{k!C_k(x)}{x^k}. \tag{1.10}$$

So, as expected (see Remark 1.2.2), for integrals the Equation (1.9) implies that (H)-convergence and (C)-convergence are equivalent. Also, Equation (1.10) is analogous to Equation (1.7) and the definition of Riesz summation (Definition 1.2.7), and can be used for $k \in \mathbb{R}$ with $k > -1$ using the Gamma function (see [Har92, Section 5.14]).

1.3 Summability of distributions

This section is based mainly in [Gru09; EGBV98; EK02].

1.3.1 A very short introduction to distributions

Distributions on $\mathbb{R}$ generalize the notion of functions on $\mathbb{R}$ by regarding a function f as an operator T_f acting by integration on "test functions" ϕ like:

$$T_f(\phi) := \int_{-\infty}^{\infty} f(x)\phi(x)dx. \tag{1.11}$$

Clearly, the integral in Equation (1.11) does not always exist. If we want that integral to exist, we can ask ϕ to have compact support (avoiding the divergence) and we can also ask ϕ to be infinitely differentiable in order to generalize the ordinary derivation. In fact, when f is continuously differentiable the functional associated with f' is

$$T_{f'}(\phi) = \int_{-\infty}^{\infty} f'(x)\phi(x)dx$$

and if ϕ satisfies those conditions, integration by parts shows that

$$T_{f'}(\phi) = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx. \tag{1.12}$$

Then, we have now the advantage that the derivative of f does not longer appear and in order to be able to iterate this procedure we should ask for ϕ being infinitely differentiable. We denote the space of these functions (infinitely differentiable with compact support) by $\mathcal{D}(\mathbb{R})$ or simply $\mathcal{D}$. For an open set U of $\mathbb{R}$, we write $\mathcal{D}(U)$ the set of smooth functions with compact support in U. We endow these spaces with the *weak topology* (or topology of point convergence) such that for a sequence $\{\phi_n\}$ in $\mathcal{D}(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$

$$\phi_n \to \phi \ \text{ if, and only if, } \ \phi_n(x) \longrightarrow_{n\to\infty} \phi(x) \ \text{ for all } x \in \mathbb{R}.$$

This topology matches with the *inductive limit topology* on $\mathcal{D}$ [Gru09, Appendix B]. Thus, we have the necessary notions for the following definition

Definition 1.3.1. A **distribution** on $\mathbb{R}$ is a continuous linear functional on $\mathcal{D}$. The vector space of distributions on $\mathcal{D}(\mathbb{R})$ is denoted by $\mathcal{D}'(\mathbb{R})$ or $\mathcal{D}'$. When $T \in \mathcal{D}'(\mathbb{R})$ we denote the value of T in $\phi \in \mathcal{D}$ by $T(\phi)$ or most commonly in this document, $\langle T, \phi \rangle$. [1]

Example 1.3.2. 1. *Delta distribution*: Let a be a real number and let δ_a be the mapping defined on $\mathcal{D}$ by

$$\delta_a(\phi) := \phi(a).$$

It is clear that δ_a is linear. To see the continuity, we follow a convergence argument: Let $\{\phi_n\}$ be a sequence of functions on $\mathcal{D}$ and $\phi \in \mathcal{D}$ such that $\phi_n \to \phi$ in the weak topology. Then

$$\langle \delta_a, \phi_n \rangle = \phi_n(a) \longrightarrow_{n\to\infty} \phi(a) = \langle \delta_a, \phi \rangle.$$

Thus δ_a is a distribution. We will use the notation $\delta(x-a)$ (which is customary in physics texts) for the distribution δ_a motivated by the heuristic calculation

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)dx = \phi(a), \ \text{ for } \phi \in \mathcal{D}(\mathbb{R}). \tag{1.13}$$

[1] This second notation brings to mind the inner product in $L^2(\mathbb{R})$ (the space of square integrable functions on $\mathbb{R}$) expressed by Equation (1.11).

2. It is easy to see that locally integrable functions (functions in the space $L^1_{\text{loc}}(\mathbb{R})$) can be identified as distributions via Equation (1.11). Moreover, if two functions are locally integrable and equal almost everywhere, they define the same distribution on $\mathcal{D}$. The converse of this implication is also true ([Gru09, Lemma 3.2]), so from now on we will make the identification $f \leftrightarrow T_f$ for locally integrable functions. This kind of distributions are said to be **regular** and we shall somewhat say that a distribution is locally integrable based on this identification.

Remark 1.3.3. There are examples of nonregular distributions. In fact, the Dirac delta distribution (Example 1.3.2 (1)) is a simple example of a nonregular distribution. By contradiction, assume that there exists $f \in L^1_{\text{loc}}(\mathbb{R})$ such that the representation $\delta_0 \leftrightarrow f$ holds. The restriction of δ_0 on $\mathbb{R} \setminus \{0\}$ is the zero-distribution, i.e. the distribution that assigns to each $\phi \in \mathcal{D}$ the value 0. Therefore, the function which represents δ_0 is $f \equiv 0$ almost everywhere on $\mathbb{R}$, and thus δ_0 would be the zero-distribution as well. Hence a contradiction.

Definition 1.3.4. Let $T \in \mathcal{D}'(\mathbb{R})$.

(i) We say that T is **null** (is zero, vanishes) on an open set U of $\mathbb{R}$ when

$$\langle T, \phi \rangle = 0 \ \text{ for all } \phi \in \mathcal{D}(U).$$

(ii) The **support** of T is defined as the set

$$\text{supp}\ T = \mathbb{R} \setminus \left(\bigcup \{ U \mid U \text{ open } \subset \mathbb{R}, T \text{ is null on } U \} \right).$$

For example, supp $(\delta_a) = \{a\}$. We will denote by $\mathcal{E}'(\mathbb{R})$ the subspace of distributions of $\mathcal{D}'(\mathbb{R})$ whose support is compact.

Remark 1.3.5. We defined $\mathcal{E}'(\mathbb{R})$ as a subspace of $\mathcal{D}'(\mathbb{R})$, so that every distribution in $\mathcal{E}'(\mathbb{R})$ will act on $\mathcal{D}(\mathbb{R})$. However, every $T \in \mathcal{E}'(\mathbb{R})$ has a well-defined action on the much bigger space $C^\infty(\mathbb{R})$. Thus, it is justified to define $\mathcal{E}'(\mathbb{R})$ as the topological dual (for the weak topology) of the space $\mathcal{E}(\mathbb{R}) := C^\infty(\mathbb{R})$. In fact, let $T \in \mathcal{E}'(\mathbb{R})$, and take $\phi_n \in \mathcal{D}(\mathbb{R})$ properly such that

$$\phi_n(x) = \begin{cases} 1 & x \in (-n, n) \\ 0 & |x| > |n+1| \end{cases}$$

For every $f \in C^\infty$, set $f_n := \phi_n f$ and then $f_n \to f$ as $n \to \infty$. Since T has compact support, there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\langle T, f_n \rangle = \langle T, f_N \rangle.$$

Taking $n \to \infty$, we can then define $\langle T, f \rangle := \langle T, f_N \rangle$. The result does not depend on the ϕ_n because an arbitrary test function $\psi \in \mathcal{D}(\mathbb{R})$ can be written as

$$\psi = \phi_n \psi + (1 - \phi_n)\psi.$$

Finally, we see now the definition of the distributional derivative as it is motivated by Equation (1.12):

Definition 1.3.6. Let $T \in \mathcal{D}'(\mathbb{R})$. The derivative T' of the distribution T is defined by the formula

$$\langle T', \phi \rangle := -\langle T, \phi' \rangle, \ \text{ for all } \phi \in \mathcal{D}(\mathbb{R}). \tag{1.14}$$

Remark 1.3.7. T' is also a distribution, and then it has a derivative T'' given by

$$\langle T'', \phi \rangle = \langle T, \phi'' \rangle.$$

This way, $T^{(n)} \in \mathcal{D}'$ for any $n \in \mathbb{N}$. Then, each distribution T is infinitely differentiable and the nth-derivative of T is defined by

$$\langle T^{(n)}, \phi \rangle := (-1)^n \langle T, \phi^{(n)} \rangle, \tag{1.15}$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

We provide now an interesting example.

Example 1.3.8. Let us denote by H the function on $\mathbb{R}$ defined by

$$H(x) := \begin{cases} 1 & \text{for } x \in (0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

It is called the *Heaviside function*. Since $H \in L^1_{\text{loc}}(\mathbb{R})$, we have that $H \in \mathcal{D}'(\mathbb{R})$. The derivative of H in $\mathcal{D}'(\mathbb{R})$ satisfies:

$$\langle H', \phi \rangle \ = \ -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x) dx = \phi(0) = \langle \delta_0, \phi \rangle \ \text{ for all } \phi \in \mathcal{D}(\mathbb{R}).$$

So we have that

$$H' = \delta_0.$$

in the distributional sense Furthermore, $\text{supp}(H) = \mathbb{R}_+$.

Remark 1.3.9. The product of a distribution and a test function can also be defined. Given $f \in C^\infty(\mathbb{R})$, the multiplication by f defines a continuous operator $M_f : \phi \mapsto f\phi$ on $\mathcal{D}(\mathbb{R})$. Since

$$\int_{-\infty}^{\infty} (f\phi)\psi dx = \int_{-\infty}^{\infty} \phi(f\psi), \text{ where } \phi, \psi \in \mathcal{D}(\mathbb{R}),$$

we define $M_f T = fT$ by

$$\langle fT, \phi \rangle := \langle T, f\phi \rangle, \ T \in \mathcal{D}'(\mathbb{R}), \text{ where } \phi \in \mathcal{D}(\mathbb{R}).$$

Composition with a smooth function can also be handled in this way (seen as a coordinate change). When F is a diffeomorphism from an open subset V of $\mathbb{R}$ onto an open subset U of $\mathbb{R}$, change of variables under the integral gives

$$\int_V \phi \circ F(x) \psi(x) dx = \int_U \phi(x) \psi(F^{-1}(x)) |\det dF^{-1}(x)| dx,$$

for $\phi \in \mathcal{D}(V), \psi \in \mathcal{D}(U)$. Define $T(F) : \mathcal{D}(V) \to \mathcal{D}(U)$ by

$$(T(F)\phi)(y) := \phi(F^{-1}(y)), \text{ where } \phi \in \mathcal{D}(V),\ y \in U.$$

The map $T(F)$ is continuous according to the chain rule and the Leibniz formula and we have that

$$\langle T(F)f, \psi\rangle := \langle f, |\det dF^{-1}(x)|\psi \circ F^{-1}\rangle, \text{ where } \psi \in \mathcal{D}(U),\ f \in \mathcal{D}'(V)$$

defines a continuous linear map $T(F)$ of $\mathcal{D}'(V)$ into $\mathcal{D}'(U)$.

With these definitions in mind, we can move on to the next example.

Example 1.3.10. For $a \in \mathbb{R}^+$, let f be the function $f(x) = x^2 - a^2$. Note that

$$\begin{aligned} H(x^2 - a^2) &= \begin{cases} 1 & x < -a \\ 0 & -a \le x \le a \\ 1 & x > a \end{cases} \\ &= 1 - \{H(x+a) - H(x-a)\} \end{aligned}$$

By differentiation this implies

$$\begin{aligned} 2x\delta(x^2 - a^2) &= -\delta(x+a) + \delta(x-a) \\ \delta(x^2 - a^2) &= -\frac{\delta(x+a)}{-2a} + \frac{\delta(x-a)}{+2a} \\ &= \frac{1}{2|a|}\{\delta(x-a) + \delta(x+a)\} \end{aligned}$$

Let us assume more generally that f is a polynomial such that

$$f(x) = f_0(x - x_1)(x - x_2)\cdots(x - x_n)$$

with f_0 and $x_1 < x_2 < \cdots < x_n$ real constants. Then

$$H(f(x)) = \begin{cases} H(x - x_1) - H(x - x_2) + \cdots - (-1)^n H(x - x_n) \\ 1 - \{H(x - x_1) - H(x - x_2) + \cdots - (-1)^n H(x - x_n)\} \end{cases}$$

depending on $f'(x_1) \gtrless 0$. Differentiating

$$\begin{aligned} f'(x)\delta(f(x)) &= \pm\{\delta(x - x_1) - \delta(x - x_2) + \cdots - (-1)^n \delta(x - x_n)\} \\ \delta(f(x)) &= \sum_{k=1}^{n} \frac{1}{|f'(x_k)|}\delta(x - x_k). \end{aligned} \tag{1.16}$$

Formula (1.16) will be useful in the next chapter.

1.3.2 Primitives of distributions

From now on, we will use lower case letters to denote distributions as well. In order to be able to develop the theory of summability of distributions, it is very important to understand the notion of a primitive of a distribution, i.e.

given $f \in \mathcal{D}'$, find $F \in \mathcal{D}'$ such that $F' = f$. We will see that some results for distributions are the same as those for functions and it will help to simplify the notation. We first consider the case $f = 0$:

Lemma 1.3.11. Let $F \in \mathcal{D}'(\mathbb{R})$. The derivative of F is the zero-distribution if and only if F is constant, i.e. if there exists $C \in \mathbb{R}$ such that $\langle F, \phi \rangle = C \int_{\mathbb{R}} \phi(x)dx$ for all $\phi \in \mathcal{D}(\mathbb{R})$.

Proof. Suppose that $F' = 0$. Consider $I \in \mathcal{D}'(\mathbb{R})$ the distribution

$$\langle I, \phi \rangle := \int_{\mathbb{R}} \phi(x)dx \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Fixing an arbitrary $\phi^* \in \mathcal{D}(\mathbb{R})$ such that $\langle I, \phi^* \rangle = \int_{\mathbb{R}} \phi^* = 1$, we have for every $\phi \in \mathcal{D}(\mathbb{R})$

$$\phi = \psi + \langle I, \phi \rangle \phi^*$$

where $\psi = \phi - \langle I, \phi \rangle \phi^*$. Note that $\int_{\mathbb{R}} \psi = \langle I, \psi \rangle = 0$ and $\psi \in \mathcal{D}(\mathbb{R})$ implies that the function τ^ϕ defined by

$$\tau^\phi(x) := \int_{-\infty}^{x} (\phi(t) - \langle I, \phi \rangle \phi^*(t))dt = \int_{-\infty}^{x} \psi(t)dt \tag{1.17}$$

is in $\mathcal{D}(\mathbb{R})$ and $(\tau^\phi)' = \psi$, so that $\langle F', \tau^\phi \rangle = -\langle F, (\tau^\phi)' \rangle = -\langle F, \psi \rangle$ because $\langle F', \psi \rangle = 0$ by assumption. Then

$$\langle F, \phi \rangle = \langle F, \psi \rangle + \langle I, \phi \rangle \langle F, \phi^* \rangle = -\langle F', \tau^\phi \rangle + \langle I, \phi \rangle \langle F, \phi^* \rangle = \langle I, \phi \rangle \langle F, \phi^* \rangle$$

Setting $C = \langle F, \phi^* \rangle$ yields the constant distribution

$$F : \phi \mapsto C \int_{\mathbb{R}} \phi.$$

The converse is immediate. □

Remark 1.3.12. As in the previous proof, if $\phi \in \mathcal{D}$, then $\phi' \in \mathcal{D}$ and $\int_{\mathbb{R}} \phi' = 0$. Conversely, if $\psi \in \mathcal{D}$ and $\int_{\mathbb{R}} \psi = 0$, then the function

$$\phi(x) = \int_{-\infty}^{x} \psi(t)dt \tag{1.18}$$

lies in $\mathcal{D}$ and yields a primitive of ψ. This motivates us to set

$$\mathcal{D}_0 := \left\{ \psi \in \mathcal{D} \mid \int_{\mathbb{R}} \psi = 0 \right\} = \{\phi' \mid \phi \in \mathcal{D}\}.$$

Theorem 1.3.13. For all $f \in \mathcal{D}'(\mathbb{R})$ there exists a primitive $F \in \mathcal{D}'(\mathbb{R})$ of f, and any primitive of f is of the form $F + C$ where C is a constant.

Proof. Suppose that we have F_1, F_2 primitives of f in $\mathcal{D}'(\mathbb{R})$. Then, $F_1' - F_2' = 0$, and by Lemma 1.3.11 there exists a constant C such that

$$F_1 = F_2 + C.$$

Let us now prove the existence of a primitive of f. If $F \in \mathcal{D}'$ is such that $F' = f$ in the distributional sense, then

$$\langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle \quad \text{for all } \phi \in \mathcal{D}.$$

So we know how F acts on $\mathcal{D}_0$, and what we want to know is how F acts on $\mathcal{D}$. Fixing $\phi \in \mathcal{D}$ we write

$$\psi := \phi - \langle I, \phi \rangle \phi^*$$

using the notation of Lemma 1.3.11. Since $\int_{\mathbb{R}} \psi = 0$ then $\psi \in \mathcal{D}_0$. A primitive of ψ is given by

$$\tau^{\phi}(x) = \int_{-\infty}^{x} (\phi(t) - \langle I, \phi \rangle \phi^*(t)) dt \tag{1.19}$$

and

$$\tau^{\phi'}(x) = \int_{-\infty}^{x} (\phi'(t) - \langle I, \phi' \rangle \phi^*(t)) dt = \phi(x) \quad \text{because } \langle I, \phi' \rangle = 0.$$

On the other hand, $(\tau^{\phi})' = \psi \in \mathcal{D}_0$, so $\tau^{\phi} \in \mathcal{D}$ (see Remark 1.3.12). Let $F : \mathcal{D} \to \mathbb{C}$ be defined by

$$\langle F, \phi \rangle := -\langle f, \tau^{\phi} \rangle.$$

If this is a distribution, then F is a primitive of f since

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle = \langle f, \tau^{\phi'} \rangle = \langle f, \phi \rangle \quad \text{for all } \phi \in \mathcal{D}.$$

To see that F is a distribution it is enough to see that the map $\phi \mapsto \tau^{\phi}$ is linear and continuous. While linearity is immediate by Equation (1.19), the continuity follows from a dominated convergence argument, or alternatively from [Gru09, Lemmas 2.2 and 2.3]. We do not go into details. □

Remark 1.3.14. Note that Equations (1.17), (1.18) and (1.19) suggest to calculate primitives of distributions as primitives of functions. Also, iterating the process of the proof of Theorem 1.3.13, every primitive of order $N \in \mathbb{N}$ of a distribution f will be a distribution of the form "polynomial" distribution[2] of order $N - 1$ plus a primitive of order N of f (i.e. the nth-derivative in the distributional sense gives us f). This leads us to the topic of summability in the next subsection.

1.3.3 Distributional Cesàro summability and basic properties

Let f be an element of $\mathcal{D}'(\mathbb{R})$.

Definition 1.3.15. Let $\beta \in \mathbb{R} \setminus \mathbb{Z}_{<0}$. The distribution f is said to be **big-$\mathcal{O}$ of order x^{β} in the Cesàro sense (for $N \in \mathbb{N}$)** when $x \to \infty$, and we write

[2]That is M_{P_n} like in Remark 1.3.9 with P_n a polynomial of order n.

$f = \mathcal{O}_C(x^\beta)$ if, and only if, there exists $N \in \mathbb{N}$, a primitive f_N of order N of f and P_{N-1} a polynomial of degree at most $N-1$ (see Remark 1.3.14) such that

(i) f_N is regular and corresponds to a locally integrable function in the large[3], that is to say $f_N|_{(A,\infty)}$ is integrable for any large enough A,

(ii) $f_N(x) = P_{N-1}(x) + \mathcal{O}(x^{N+\beta})$ when $x \to \infty$ in the ordinary sense.

Furthermore, we say $f = \mathcal{O}_C(x^{-\infty})$ if $f = \mathcal{O}_C(x^\beta)$ for all $\beta \in \mathbb{R} \setminus \mathbb{Z}_{<0}$. Similarly, one defines $f = o_C(x^\beta)$ and $f = o_C(x^{-\infty})$. When β is a negative integer, primitives of $x^{-1}, x^{-2}, x^{-3}, \ldots$ take place. We do not go into this in depth since it is not within the objectives of this work.

Observe that the Cesàro behavior of a distribution at infinity depends on the behavior for large values of x, so for two distributions f, g which coincide on an interval bounded on the left, then $f = \mathcal{O}_C(x^\beta)$ if, and only if, $g = \mathcal{O}_C(x^\beta)$ and similarly with little-o. So if necessary, we can multiply the distribution by a suitable cut-off function in order to assume that its support is bounded on the left.

We introduce now some basic properties of this Cesàro notion of big-$\mathcal{O}$. Let us start with the following:

Remark 1.3.16. In general, there is no standard way to figure out a primitive of a distribution in $\mathbb{R}$, but if this one has support bounded on the left, there is only one primitive with support bounded on the left [SE13, Theorem 3.1]. Consider then $I(f)$ *the primitive of* f with support bounded on the left when there is one (see for instance, under the conditions of Theorem 1.3.13), and $I_n(f)$ the primitive of f of order n. When f is also regular distribution then, as in Equation (1.18), $I(f)$ will have the form

$$I(f)(x) = \int_{-\infty}^{x} f(t)dt$$

and by Cauchy formula for repeated integration [Fol02, Ex. 4.5.6], $I_n(f)$ will be

$$I_n(f)(x) = \int_{-\infty}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$$

The next lemma will help us for later results. We corrected the typographical mistake in the proof given in [Est98, Theorem 3.4], and we also filled some gaps in the calculations.

Lemma 1.3.17. If $f \in \mathcal{D}'(\mathbb{R})$ has support bounded on the left and $\alpha \in \mathbb{R}$ then for $n \in \mathbb{N}$

$$I_n(x^\alpha f) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha)_j I_j(x^{\alpha-j} f_n) \tag{1.20}$$

where $f_n = I_n(f)$, $(\alpha)_j := \alpha(\alpha-1)\cdots(\alpha-j+1)$ and the product $x^\alpha f$ is determined by Remark 1.3.9.

[3] See Example 1.3.2 2.

Proof. We need to show that the n-th derivative of the right hand side of Equation (1.20) equals to $x^\alpha f(x)$.

$$\begin{aligned}
\frac{d^n}{dx^n} \sum_{j=0}^{n} & (-1)^j \binom{n}{j} (\alpha)_j I_j(x^{\alpha-j} f_n)(x) \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha)_j \frac{d^{n-j}}{dx^{n-j}} (x^{\alpha-j} f_n)(x) \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha)_j \sum_{q=0}^{n-j} \binom{n-j}{q} (\alpha - j)_{n-j-q} x^{\alpha-n+q} f_{n-q}(x) \quad (1.21) \\
&= x^{\alpha-n} \sum_{j=0}^{n} \sum_{q=0}^{n-j} (-1)^j \binom{n}{j} \binom{n-j}{q} (\alpha)_{n-j-q} x^q f_{n-q}(x) \\
&= x^{\alpha-n} \sum_{q=0}^{n} \sum_{j=q}^{n} (-1)^{j-q} \binom{n}{j-q} \binom{n-(j-q)}{q} (\alpha)_{n-q} x^q f_{n-q}(x) \\
&= x^{\alpha-n} \sum_{q=0}^{n} x^q f_{n-q}(x) (\alpha)_{n-q} \sum_{j=q}^{n} (-1)^{j-q} \binom{n}{q} \binom{n-q}{j-q} \quad (1.22) \\
&= x^{\alpha-n} \sum_{q=0}^{n} x^q f_{n-q}(x) (\alpha)_{n-q} \sum_{j=q}^{n} (-1)^{j-q} \binom{n}{j} \binom{j}{q}
\end{aligned}$$

where we use Leibniz rule in Equation (1.21) and the combinatorial property:

$$\binom{n}{k}\binom{n-k}{i} = \binom{n}{i}\binom{n-i}{k}$$

in Equation (1.22). Observe that

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

so

$$\frac{d^q}{dx^q}(1+x)^n = \sum_{j=q}^{n} \binom{n}{j} (j)_q x^{j-q}$$

thus

$$\delta_{q,n} = \binom{n}{q} (1+x)^{n-q} \Big|_{x=-1} = \frac{1}{q!} \frac{d^q}{dx^q} (1+x)^n \Big|_{x=-1} = \sum_{j=q}^{n} \binom{n}{j} \binom{j}{q} (-1)^{j-q} \quad (1.23)$$

where $\delta_{q,n}$ is the Kronecker delta function. Replacing Equation (1.23) in Equation (1.22) we have the result:

$$\frac{d^n}{dx^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha)_j I_j(x^{\alpha-j} f_n) = x^\alpha f.$$

□

Let us list some properties of big-$\mathcal{O}_C$ (and consequently little-o_C), in the notation of Definition 1.3.15. Again we filled some gaps in [Est98, Theorem 3.4].

Proposition 1.3.18. For any $\beta \in \mathbb{R} \setminus \mathbb{Z}_{<0}$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we have

(i) $f = \mathcal{O}_C(x^\beta)$ when $x \to \infty$ (for some $N \in \mathbb{N}$) implies that $f^{(k)} = \mathcal{O}_C(x^{\beta-k})$ when $x \to \infty$ (for $N+k$).

(ii) $f = \mathcal{O}_C(x^\beta)$ when $x \to \infty$ implies that $x^\alpha f = \mathcal{O}_C(x^{\alpha+\beta})$ when $x \to \infty$ for $\alpha + \beta \notin \mathbb{Z}_{<0}$.

Proof. (i) If f_N is a primitive of f of order N, it is also a primitive of $f^{(k)}$ of order $N+k$ so the result is immediate.

(ii) Suppose that $f = \mathcal{O}_C(x^\beta)$ when $x \to \infty$ (for some $N \in \mathbb{N}$) then $I_N(f) = f_N = \mathcal{O}(x^{\beta+N})$. By definition of big-$\mathcal{O}$, there exists $M > 0$ such that

$$|f_N(x)| \leq M|x^{\beta+N}| \quad x \to \infty$$

which is equivalent to the existence of a neighborhood (a, ∞), such that

$$|x^{\alpha-j} f_N(x)| \leq M|x^{\alpha+\beta+N-j}| \quad \text{for all } x \in (a,\infty) \text{ and } j \in \mathbb{Z}. \tag{1.24}$$

By the binomial expansion

$$(x-t)^{j-1} = \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k x^{j-k-1} t^k$$

and Equation (1.24) we have

$$\begin{aligned} |I_j(x^{\alpha-j} f_N(x))| &= \left| \int_{-\infty}^{x} \frac{(x-t)^{j-1}}{(j-1)!} t^{\alpha-j} f_N(t) dt \right| \\ &\leq M \sum_{k=0}^{j-1} \binom{j-1}{k} |x^{j-k-1}| \int_{-\infty}^{x} |t|^{\alpha+\beta+N-j+k} dt \\ &\leq C|x^{\alpha+\beta+N}| \end{aligned}$$

for a suitable constant C. By Lemma 1.3.17, we obtain that $I_N(x^\alpha f) = \mathcal{O}(x^{\alpha+\beta+N})$, and therefore

$$x^\alpha f = \mathcal{O}_C(x^{\alpha+\beta}) \quad \text{as } x \to \infty.$$

□

Remark 1.3.19. The previous proposition is important in the sense that computations with big-$\mathcal{O}$ and little-o notation in the Cesàro sense can be performed exactly as expected for the ordinary big-$\mathcal{O}$ and little-o. For example, it is clear that if a function f is such that $f(x) = \mathcal{O}(1)$ when $x \to \infty$, then there exists some non-zero constant C such that

$$\frac{f(x)}{C} \longrightarrow 1 \quad \text{as } x \to \infty$$

and in that way $x^\alpha f(x) = \mathcal{O}(x^\alpha)$ for all $\alpha \in \mathbb{R}$ as $x \to \infty$. Then, Proposition 1.3.18 (ii) says that this kind of properties are inherited for the estimate in the Cesàro sense. The same similitude is held, for instance, with derivations and Proposition 1.3.18 (i), and with little-o.

We finish this section with the notion of limit in the Cesàro sense:

Definition 1.3.20. For f a distribution we write $C\text{-}\lim_{x\to\infty} f = L$ (for $k \in \mathbb{N}$), if as $x \to \infty$, $f = L + o_C(1)$ (for $k \in \mathbb{N}$).

Remark 1.3.21. Note that if f is an arbitrary continuous function such that $f(x) = L + o(1)$ when $x \to \infty$ in the ordinary sense (see Equation (1.2)), then this is equivalent to say $\lim_{x\to\infty} f(x) = L$. Furthermore, if f is a distribution such that $f = L + o_C(1)$ for $k \in \mathbb{N}$ then there exists a primitive $(f - L)_k$ of $f - L$ of order k and a polynomial P_{k-1} of degree at most $k - 1$ such that $(f - L)_k$ is locally integrable for x large and the relation

$$(f - L)_k(x) = P_{k-1}(x) + o(x^k) \quad \text{as } x \to \infty$$

holds. But this implies that

$$f_k(x) - \frac{Lx^k}{k!} = Q_{k-1}(x) + o(x^k) \quad \text{as } x \to \infty$$

for some polynomial Q_{k-1} of degree at most $k - 1$, and f_k a primitive of f of order k. Note that Q_{k-1} could even not be equals to P_{k-1}, since in the computation of primitives we are adding a constant distribution. So

$$\frac{k!f_k(x)}{x^k} = L + o(1) \quad \text{as } x \to \infty$$

which means

$$C\text{-}\lim_{x\to\infty} f(x) = L \text{ for } k \in \mathbb{N} \text{ if, and only if, } \lim_{x\to\infty} \frac{k!f_k(x)}{x^k} = L \tag{1.25}$$

for f_k a locally integrable primitive of order k of f. Compare Equation (1.25) with Equations (1.7) and (1.10).

1.3.4 Cesàro summability of evaluations

Evaluations of distributions do not *a priori* make sense. For instance, if $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$ then $\langle f, \phi \rangle$ is well defined, but if $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in$

$\mathcal{E}(\mathbb{R})$, $\langle f, \phi \rangle$ is not necessarily well defined (since $\mathcal{D} \subsetneq \mathcal{E}$, see Remark 1.3.5). One solution would be to find a space $X \subseteq \mathcal{E}$ such that $f \in X'$ and $\phi \in X$. Instead, we are going to focus on another approach that we are about to present.

Definition 1.3.22. [EGBV98, Definition 2.3] Let $f \in \mathcal{D}'(\mathbb{R})$ with support bounded on the left and $\phi \in \mathcal{E}(\mathbb{R})$. We say that the evaluation $\langle f, \phi \rangle$ takes the value L in the Cesàro sense, and write $\langle f, \phi \rangle_C = L$ if, and only if, there is a primitive $I(f\phi)$ of the distribution $f\phi$ satisfying

$$C\text{-}\lim_{x\to\infty} I(f\phi)(x) = L.$$

In case f is a regular distribution, the last equation can be seen as

$$C\text{-}\lim_{x\to\infty} \int_{-\infty}^{x} f(t)\phi(t)dt = L.$$

Also, when f has its support bounded on the right, a similar definition applies by taking $x \leftrightarrow -x$ for f and the resulting distribution has support bounded on the left. If f is a distribution that admits a decomposition $f = f_1 + f_2$ where f_1 has bounded support on the left and f_2 has bounded support on the right, we say that

$$\langle f, \phi \rangle_C := L \quad \text{if} \quad \langle f_i, \phi \rangle_C = L_i \quad (i = 1, 2) \quad \text{and} \quad L = L_1 + L_2, \text{ for } \phi \in \mathcal{E}.$$

This definition is independent of the decomposition. Indeed, if $f = \hat{f}_1 + \hat{f}_2$ is another decomposition then $f_1 - \hat{f}_1 = \hat{f}_2 - f_2$ has compact support. Since $\phi \in \mathcal{E}(\mathbb{R})$

$$\langle f_1 - \hat{f}_1, \phi \rangle_C = \langle \hat{f}_2 - f_2, \phi \rangle_C$$

then

$$\langle f_1 + f_2, \phi \rangle_C = \langle \hat{f}_1 + \hat{f}_2, \phi \rangle_C$$

which implies that

$$\langle \hat{f}_1, \phi \rangle_C + \langle \hat{f}_2, \phi \rangle_C = \langle f_1, \phi \rangle_C + \langle f_2, \phi \rangle_C = L_1 + L_2 = L.$$

Remark 1.3.23. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. The series

$$\sum_{n=1}^{\infty} a_n \delta(x - n)$$

makes sense as a distribution in $\mathcal{D}'$. Note that if $f(x) = \sum_{n=1}^{\infty} a_n \delta(x - n)$, then $\langle f, \phi \rangle$ should be equal to the sum of the series $\sum_{n=1}^{\infty} a_n \phi(n)$:

$$\langle f(x), g(x) \rangle = \left\langle \sum_{n=1}^{\infty} a_n \delta(x - n), \phi(x) \right\rangle = \sum_{n=1}^{\infty} a_n \langle \delta(x - n), \phi(x) \rangle = \sum_{n=1}^{\infty} a_n \phi(n)$$

where the second equality is by the continuity of f used on the partial sums, and the last one by definition of the distribution δ. Notice that the series on the right hand side converges since ϕ has bounded support. Analogously, if f is locally integrable and supported in (a, ∞) then $\langle f, \phi \rangle$, when defined, should be equal to $\int_a^\infty f(x)\phi(x)dx$.

In the same vein as the previous remark, we want to see that for suitable choices of f, the notion of evaluation in the Cesàro sense comprises both notions of summability of series and integrals of Section 1.2. The proof of the following Proposition is an original work of ours.

Proposition 1.3.24. (i) $\langle \sum_{n=1}^\infty a_n \delta(x-n), \phi(x) \rangle_C = L$ if, and only if, $\sum_{n=1}^\infty a_n \phi(n)$ is (C)-convergent towards L.

(ii) If f is locally integrable and supported in (a, ∞) then $\langle f, \phi \rangle_C = L$ if, and only if, $\int_a^\infty f(x)\phi(x)dx$ is (C)-convergent towards L.

Remark 1.3.25. Before starting the proof of the last proposition, let us mention how we found that Radon measures fit in here. The theory of Radon measures provides a notion of measure on a topological space that is compatible with the topology in some sense. For instance, restricting to locally compact Hausdorff spaces, one can relate Radon measures with positive linear functionals on the space of smooth functions with compact support. It is precisely in this correspondence that this observation is addressed: if $\mathcal{M}(\mathbb{R})$ is the vector space of Radon measures on $\mathbb{R}$ and $\mathcal{D}(\mathbb{R})$ is the space of test functions, the previous property results in the existence of the isometric isomorphism

$$\begin{aligned} \mathcal{M}(\mathbb{R}) &\longrightarrow \quad \mathcal{D}'(\mathbb{R}) \\ \mu &\longmapsto I_\mu : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R} \\ &\qquad\qquad \varphi \longmapsto I_\mu(\varphi) := \int_{\operatorname{supp}\varphi} \varphi d\mu = \int_{\mathbb{R}} \varphi d\mu \end{aligned}$$

which allows us to identify Radon measures μ on $\mathbb{R}$ with continuous linear functionals I_μ on $\mathcal{D}(\mathbb{R})$. This gives rise to an injection of $L_{1,\text{loc}}(\mathbb{R})$ in $\mathcal{M}(\mathbb{R})$ where a function locally integrable f defines the Radon measure μ_f by the formula

$$\mu_f(K) = \int_K f\, dx \quad \text{for} \quad K \subseteq \mathbb{R} \text{ compact.}$$

Also, the Riesz Representation Theorem [FF99, Theorem 7.2] states

$$\int f\varphi\, dx = \int \varphi\, d\mu_f \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R})$$

so the distributions $I_f : \varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)dx$ and I_{μ_f} coincide. As we usually write f instead of I_f (compare with Example 1.3.2(ii)), then we also write

$$I_f(\varphi) = \langle I_f, \varphi \rangle = \langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx.$$

Let us consider, as an example, the point measure μ_n that has the value 1 on the set $\{n\}$ and the value 0 on compact sets disjoint from n. Then μ_n is a

Radon measure so

$$\langle \delta(x-n), \varphi \rangle = \int \varphi d\mu_n$$

is well defined.

Proof of Proposition 1.3.24:

Without loss of generality, let us assume that f is supported on $[0, \infty)$ since the divergence of the improper integral only depends on the open right ray. Let μ be a Radon measure with support on $[0, \infty)$[4]. Note that for some $k \in \mathbb{N}$

$$\langle \mu(x), 1 \rangle_C = C\text{-} \lim_{x \to \infty} \int_0^x d\mu(t) = \lim_{x \to \infty} \int_0^x \left(1 - \frac{t}{x}\right)^k d\mu(t)$$

where the last equality holds by Equations (1.10) and (1.25). As $\int_0^\infty d\mu(x)$ is (C)-convergent of order k towards L if $\lim_{x \to \infty} \int_0^x \left(1 - \frac{t}{x}\right)^k d\mu(t) = L$, we have seen that $\int_0^\infty d\mu(x)$ is (C)-convergent of order k towards L if and only if $\langle \mu(x), 1 \rangle$ has the value L for some k in the Cesàro sense.

If we consider the Radon measure μ_f in Remark 1.3.25, we get (ii) by direct use of the previous reasoning. Also with Definition 1.2.7, we have that the series $\sum_{n=0}^\infty a_n = L$ is (R)-convergent towards L of order k if, and only if,

$$\left\langle \sum_{n=0}^\infty a_n \delta(x-n), 1 \right\rangle_C = L$$

for some $k \in \mathbb{N}$, which means that the (R)-convergence of $\sum a_n$ is equivalent to the Cesàro evaluation with the Radon measure $\mu = \sum_{n=0}^\infty a_n \mu_n$ where μ_n is the point measure. Then

$$\left\langle \sum_{n=0}^\infty a_n \delta(x-n), \varphi \right\rangle_C = L \text{ if, and only if } \sum_{n=0}^\infty a_n \varphi(n) = L \text{ (in (R)-convergence)}$$

both for the same $k \in \mathbb{N}$. As (R)-convergence and (C)-convergence are equivalent (Remark 1.2.8), we conclude (i). □

1.4 The space of distributions $\mathcal{K}'(\mathbb{R})$ and the moment asymptotic expansion

In this section we discuss what a moment asymptotic expansion is. Then we define the spaces of "symbols" $\mathcal{K}$ and we show (Theorem 1.4.10) that distributions which satisfy the moment asymptotic expansion are precisely those belonging to $\mathcal{K}'$. This in turn relates to the Cesàro behavior of distributions we have been studying until this point.

[4] Support of the corresponding distribution in $\mathcal{D}'$.

1.4.1 The theory of moments

We start with the following definition

Definition 1.4.1. The space $\mathcal{S}(\mathbb{R})$ of *test functions of rapid decay* contains the functions ϕ having the following properties:

1. ϕ is infinitely differentiable; i.e. $\phi \in C^{\infty}(\mathbb{R})$.

2. ϕ, as well as its derivatives of all orders, vanish at infinity faster than the reciprocal of any polynomial; i.e.

$$|x^k \phi^{(n)}(x)| < C_{k,n,\phi}$$

where $k, n \in \mathbb{N}$ and $C_{k,n,\phi}$ is a constant depending on k, n, ϕ.

Remark 1.4.2. It is evident that $\mathcal{D} \subsetneq \mathcal{S}$, because all test functions in $\mathcal{D}$ vanish identically outside a finite interval, whereas those in $\mathcal{S}$ merely decrease rapidly at infinity. For instance, the function $e^{-|x|^2}$ belongs to $\mathcal{S}$ but not to $\mathcal{D}$. More generally, if p is any polynomial, then $p(x)e^{-|x|^2}$ belongs to $\mathcal{S}$. So that, the space of *tempered distributions* $\mathcal{S}'$, satisfies $\mathcal{S}' \subset \mathcal{D}'$.

Example 1.4.3. As we saw in Remark 1.3.23, the series

$$\sum_{n=0}^{\infty} \delta^{(n)}(x-n)$$

where $\delta^{(n)}$ is the nth derivative of the Dirac Delta distribution, converges to a distribution $\mathcal{D}'$, but, it neither converges in $\mathcal{S}'$ nor defines a tempered distribution.

Now, let us briefly recall the notion of moments and why it is important in this context. Let f be a continuous function in $\mathbb{R}$ whose 'moments' μ_n

$$\mu_n := \langle f(x), x^n \rangle = \int_{-\infty}^{\infty} f(x) x^n dx \qquad n = 0, 1, 2, 3 \ldots \tag{1.26}$$

are well-defined. Let us consider $\phi \in \mathcal{S}(\mathbb{R})$ a test function which is analytic i.e. that is defined by a convergent Taylor expansion:

$$\phi(x) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) x^n}{n!}$$

and, let us assume, that this series converges absolutely in all $\mathbb{R}$. It follows then from Equation (1.26) that

$$\langle f, \phi \rangle = \left\langle f, \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) x^n}{n!} \right\rangle = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) \mu_n}{n!} \tag{1.27}$$

where the last equality comes from the fact that f is continuous.

On the other hand, both properties hold for the Dirac Delta distribution:[5]

$$\langle \delta(x-a), \phi(x)\rangle = \phi(a) \quad \text{and} \quad \langle \delta^{(n)}(x), \phi(x)\rangle = -\langle \delta^{(n-1)}(x), \phi'(x)\rangle$$

So in general, $\phi^{(n)}(0) = (-1)^n \langle \delta^{(n)}(x), \phi(x)\rangle$, and Equation (1.27) can be written as

$$\langle f, \phi\rangle = \left\langle \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}}{n!}, \phi \right\rangle.$$

Thus,

$$f = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}}{n!}. \tag{1.28}$$

Definition 1.4.4. Let f be a distribution on $\mathbb{R}$. For $\lambda > 0$, we define the parametrised distribution

$$\begin{aligned} f(\lambda \cdot) : \mathcal{D}(\mathbb{R}) &\to \mathbb{R} \\ \phi &\mapsto f(\lambda\cdot)(\phi) := \langle f, \phi_\lambda\rangle \end{aligned}$$

where $\phi_\lambda(x) := \lambda^{-1}\phi\left(\frac{x}{\lambda}\right)$.

Remark 1.4.5. Note that this is a distribution precisely because f is a distribution and ϕ_λ is also a compactly supported function as long as ϕ is. This definition is motivated for the substitution rule in integrals. Indeed, if f is a locally integrable function, taking $t = \lambda x$ we have

$$\langle f(\lambda\cdot), \phi\rangle = \int_{\mathbb{R}} f(\lambda x)\phi(x)dx = \lambda^{-1}\int_{\mathbb{R}} f(t)\phi\left(\frac{t}{\lambda}\right) dt = \langle f, \phi_\lambda\rangle.$$

Example 1.4.6. From

$$\delta(\lambda x) = \frac{1}{\lambda}\delta(x) \quad \text{for } \lambda > 0 \text{ and for every } n \in \mathbb{N},$$

it follows that

$$\delta^{(n)}(\lambda x) = \frac{1}{\lambda^{n+1}}\delta^{(n)}(x) \qquad \text{for } \lambda > 0.$$

Combining Example 1.4.6 and Equation (1.28) we get the asymptotic formula

$$f(\lambda\cdot) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}}{n!\lambda^{n+1}}, \quad \lambda \to \infty \tag{1.29}$$

and we call it *the moment asymptotic expansion* of f.

1.4.2 The space $\mathcal{K}'$

The space $\mathcal{K}$ of Grossman-Loupias-Stein symbols was first introduced in [GLS68] for quantum mechanics considerations. Ricardo Estrada ([Est98; EK02]) later

[5] See Example 1.3.2(i) and Equation (1.14).

proposed a relationship between the dual space $\mathcal{K}'$, the Cesàro behavior at infinity of distributions and the parametric or distributional behavior at infinity (Theorem 1.4.10). We complete his proof with Lemma 1.4.8 inspired by [EKL90, Theorem 1] and [EKL90, Lemma 3]. Let us start with the following definition:

Definition 1.4.7. Let $\gamma \in \mathbb{N}$. The space of **Grossman-Loupias-Stein symbols of order** γ in one variable ([GLS68]) is defined by

$$\mathcal{K}_\gamma := \left\{ \phi \in C^\infty(\mathbb{R}) : \text{ for each } k \in \mathbb{N}, |\phi^{(k)}(x)| = \mathcal{O}(|x|^{\gamma-k}) \text{ as } |x| \to \infty \right\}.$$

On this space we define the seminorms:

$$\|\phi\|_{k,\gamma} := \max \left\{ \sup_{|x|\leq 1} \{\phi^{(k)}(x)\}, \sup_{|x|\geq 1} \{|x|^{k-\gamma}\phi^{(k)}(x)\} \right\} \quad n \in \mathbb{N}, \gamma \in \mathbb{N}.$$

The family $\{\mathcal{K}_\gamma\}_{\gamma\in\mathbb{N}}$ is a family of *Fréchet spaces* i.e. every $\mathcal{K}_\gamma$ is a locally convex vector space defined by the *countable* number of seminorms $\|\cdot\|_{k,\gamma}$ ($k \in \mathbb{N}$) that is sequentially complete, in the sense that a sequence that is Cauchy with respect to all $\|\cdot\|_{k,\gamma}$ converges. In particular, we define

$$\mathcal{K} := \bigcup_{\gamma=1}^{\infty} \mathcal{K}_\gamma$$

and the topology of $\mathcal{K}$ is the inductive limit topology of the $\mathcal{K}_\gamma$ as $\gamma \to \infty$ (see for references [Gru09, Appendix B]). Particularly, $\mathcal{D} \subset \mathcal{K}$.

Polynomials lie in the space $\mathcal{K}$. Also, classical symbols with constant coefficients [Gru09, Definition 7.2] of order γ lie in $\mathcal{K}_\gamma$. We observe that a distribution $f \in \mathcal{K}'$ has well defined moments $\mu_k(f)$ given by

$$\mu_k := \langle f, x^k \rangle \quad \text{for each } \; k \in \mathbb{N}. \tag{1.30}$$

so, in some sense, the existence of these moments of all orders is an indication of the fact that f "decays rapidly at infinity".

The proof of the following Lemma is an original work of ours.

Lemma 1.4.8. If f is a distribution such that $f \in \mathcal{K}'$ then, there exist constants $\mu_0, \mu_1, \mu_2, \ldots$ such that

$$f(\lambda\cdot) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}}{n!\lambda^{n+1}}, \quad \lambda \to \infty$$

in the weak sense, meaning that for any $\phi \in \mathcal{K}(\mathbb{R})$, we have as $\lambda \to \infty$

$$\langle f(\lambda\cdot), \phi(x) \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right)$$

for every $N \in \mathbb{N}$.

Proof. Let us denote

$$M_t := \{\phi \in \mathcal{K}(\mathbb{R}) : \phi^{(n)}(0) = 0,\ n \le t\}$$

for $t \in \mathbb{N}$ and let $\phi \in \mathcal{K}_q(\mathbb{R})$ for some q. If there exists $t \le q$ such that $\phi \in M_t$ then

$$\left\|\phi\left(\frac{x}{\lambda}\right)\right\|_{q,0} = \mathcal{O}\left(\frac{1}{\lambda^t}\right).$$

In fact, by definition of $\mathcal{K}_q$ and M_t, there exists a constant K such that

$$\begin{array}{rcll} |\phi(x)| & \le & K|x|^t & |x| \le 1 \\ |\phi(x)| & \le & K|x|^q & |x| \ge 1. \end{array}$$

Hence, if $\lambda^{-1} \le 1$

$$\left\|\phi\left(\frac{x}{\lambda}\right)\right\|_{q,0} \le \frac{K}{\lambda^t}.$$

Notice that if $n \le t$, $(n \in \mathbb{N})$, and $\phi \in M_t$ then $\phi^{(n)} \in M_{t-n}$ and thus

$$\left\|\phi\left(\frac{x}{\lambda}\right)\right\|_{q,n} = \left\|\frac{1}{\lambda^n}\phi^{(n)}\left(\frac{x}{\lambda}\right)\right\|_{q,0} = \frac{1}{\lambda^n}\mathcal{O}\left(\frac{1}{\lambda^{t-n}}\right) = \mathcal{O}\left(\frac{1}{\lambda^t}\right)$$

while if $n > t$, $\phi^{(n)} \in M_0$ and thus

$$\left\|\phi\left(\frac{x}{\lambda}\right)\right\|_{q,n} = \frac{1}{\lambda^n}\left\|\phi^{(n)}\left(\frac{x}{\lambda}\right)\right\|_{q,0} = \frac{1}{\lambda^n}\mathcal{O}(1) = \mathcal{O}\left(\frac{1}{\lambda^n}\right).$$

So, if $\phi \in \mathcal{K}_q$ let $P_N(x) := \sum_{n=0}^{N} \frac{\phi^{(n)}(0)}{n!}x^n$ be the Taylor polynomial of ϕ of order $N < q$. Then, if $f \in \mathcal{K}'(\mathbb{R})$

$$\begin{array}{rcl} \langle f(\lambda\cdot), \phi\rangle & = & \langle f(\lambda\cdot), P_N\rangle + \langle f(\lambda\cdot), \phi - P_N\rangle \\ & = & \displaystyle\sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + R_N(\lambda) \end{array}$$

where $R_N(\lambda)$ is given by $R_N(\lambda) = \langle f(\lambda\cdot), \phi - P_N\rangle$ and μ_n are the well-defined moments in Equation (1.26). We observe that $\phi_N := \phi - P_N \in M_{N+1}$. By the same argument as before, for every seminorm $\|\phi_N(\frac{x}{\lambda})\|_{q,j}$ with $j \in \mathbb{N}$ there is some constant K such that

$$\left\|\phi_N\left(\frac{x}{\lambda}\right)\right\|_{q,j} \le \frac{K}{\lambda^{N+1}}.$$

Since f is a continuous linear functional, then for a suitable choice of constant C

$$|R_N| = |\langle f(\lambda\cdot), \phi_N(x)\rangle| = \frac{1}{\lambda}\left|\left\langle f, \phi_N\left(\frac{x}{\lambda}\right)\right\rangle\right| \le \frac{C}{\lambda}\sum_{j=0}^{N+1}\left\|\phi_N\left(\frac{x}{\lambda}\right)\right\|_{q,j} = \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right).$$

The topology of $\mathcal{K}$ being the inductive topology of the $\mathcal{K}_q$ as $q \to \infty$ implies that for $\phi \in \mathcal{K}$ and $\lambda \to \infty$

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n! \lambda^{n+1}} + \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right)$$

for every $N \in \mathbb{N}$. □

For the following Lemma, we complete some steps that were not given in the original result.

Lemma 1.4.9. [EK02, Lemma 6.7.1] A distribution f with support bounded on the left such that $f = o_C(x^{-\infty})$ as $x \to \infty$, lies in $f \in \mathcal{K}'$.

Proof. Since $\mathcal{K}'(\mathbb{R})$ is the projective limit of the spaces $\mathcal{K}'_\gamma(\mathbb{R})$ as $\gamma \to \infty$, we want to show that if $f = o_C(x^{-\infty})$ as $x \to \infty$ then $f \in \mathcal{K}'_\gamma(\mathbb{R})$ for each γ. Since $f = o_C(x^{-\infty})$ as $x \to \infty$ means that $f = o_C(x^{-\beta})$ for all β, then, in particular, choosing $\gamma \in \mathbb{N}$, we have that $f = \mathcal{O}_C(x^{-\gamma-2})$ as $x \to \infty$ remains true (since little-o implies big-$\mathcal{O}$). In this way, there exists $N \in \mathbb{N}$ and f_N a primitive of f of order N with bounded support on the left such that f_N is locally integrable for x large and $f_N(x) = P_{N-1}(x) + \mathcal{O}(x^{-\gamma-2+N})$ as $x \to \infty$ where P_{N-1} is a polynomial of order at most $N-1$.

Note that if $\langle f, x^j \rangle = 0$ for all j with $0 \leq j \leq N-1$ then $P_{N-1}(x) = 0$. In fact,

$$0 = \langle f, x^j \rangle = \langle f_N^{(N)}, x^j \rangle = (-1)^j (j)! \langle f_N^{(N-j)}(x), 1 \rangle$$

so, the coefficient of $P_{N-1}(x)$ corresponding to the power x^j equals 0.

Let $g \in C_c(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} f(x) x^n dx = \int_{supp(g)} g(x) x^n dx \quad \text{for all } 0 \leq n \leq N-1$$

i.e. $\mu_n = \langle f(x), x^n \rangle = \langle g(x), x^n \rangle$. Let g_N be the primitive of g of order N with support bounded on the left, so, by the previous reasoning,

$$f_N(x) - g_N(x) = \mathcal{O}(x^{-\gamma-2+N}) \quad \text{as } x \to \infty. \tag{1.31}$$

If $\phi \in \mathcal{K}_{\gamma-N}$ then in particular $|\phi(x)| = \mathcal{O}(x^{\gamma-N})$ as $x \to \infty$, so, in view of Equation (1.31), there exist $a, C, K \in \mathbb{R}_{\geq 1}$ such that

$$|(f_N - g_N)(x)| < C|x|^{-\gamma-2+N} \quad \text{and} \quad |\phi(x)| < K|x|^{\gamma-N} \quad \text{for } x \in [a, \infty).$$

Therefore,

$$\int_{|x| \geq a} |(f_N - g_N)(x)||\phi(x)| dx < CK \int_{|x| \geq a} |x|^{-2} dx < \infty$$

and, since $(f_N - g_N)$ and ϕ are continuous, $\int_{|x|\leq a} |(f_N - g_N)(x)||\phi(x)|dx < \infty$. Then

$$\int_{-\infty}^{\infty} (f_N - g_N)(x)\phi(x)dx$$

converges, and thus $f_N - g_N \in \mathcal{K}'_{\gamma-N}$. On the other hand, by definition, if $\psi \in \mathcal{K}_{\gamma}(\mathbb{R})$ then $\psi^{(N)} \in \mathcal{K}_{\gamma-N}(\mathbb{R})$. So, the equality

$$\langle (f_N - g_N)^{(N)}, \psi \rangle = (-1)^N \langle (f_N - g_N), \psi^{(N)} \rangle$$

implies that $(f_N - g_N)^{(N)} \in \mathcal{K}'_{\gamma}(\mathbb{R})$, and finally $f = (f_N - g_N)^{(N)} + g \in \mathcal{K}'_{\gamma}(\mathbb{R})$. □

Now that we have closed the gaps, thanks mainly to the Lemma 1.4.8. The following, which is the most important result of this section, has its full proof.

Theorem 1.4.10. Let f be a distribution on $\mathbb{R}$. The following are equivalent:

(i) $f \in \mathcal{K}'(\mathbb{R})$.

(ii) f satisfies

$$f = o_C(|x|^{-\infty}) \quad \text{as } |x| \to \infty. \tag{1.32}$$

(iii) There exist constants $\mu_0, \mu_1, \mu_2, \ldots$ such that

$$f(\lambda\cdot) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!\lambda^{n+1}}, \quad \lambda \to \infty \tag{1.33}$$

in the weak sense, it means that for any $\phi \in \mathcal{K}(\mathbb{R})$, we have as $\lambda \to \infty$

$$\langle f(\lambda\cdot), \phi(x) \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right)$$

for every $N \in \mathbb{N}$.

Proof. Directions $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$ follow from Lemma 1.4.8 and Lemma 1.4.9 respectively. The rest of the proof consists of showing $(iii) \Rightarrow (ii)$.

Note that if f is a distribution such that for any $\phi \in \mathcal{K}(\mathbb{R})$ as $\lambda \to \infty$ it satisfies

$$\langle f(\lambda\cdot), \phi \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right) \tag{1.34}$$

for every $N \in \mathbb{N}$, this is equivalent to say that for any $\phi \in \mathcal{K}(\mathbb{R})$ as $\lambda \to \infty$ it satisfies

$$\langle f(\lambda\cdot), \phi \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + o\left(\frac{1}{\lambda^{N+1}}\right) \tag{1.35}$$

for every $N \in \mathbb{N}$. Therefore it is enough to show that for all $f \in \mathcal{D}'$

$$f(\lambda\cdot) = \mathcal{O}(\lambda^{\alpha}) \quad \text{as } \lambda \to \infty \text{ distributionally for some real } \alpha \tag{1.36}$$

implies that $f = \mathcal{O}_C(|x|^\alpha)$ as $|x| \to \infty$. Indeed, Equation (1.34) for all N implies that Equation (1.36) holds for all negative integer powers α, and using this claim we can say $f = \mathcal{O}_C(|x|^{-\infty})$ as $|x| \to \infty$. This last statement is equivalent to $f = o_C(|x|^{-\infty})$ as a consequence of Equation (1.35) and the properties given in Proposition 1.3.18. Thus, suppose that $f \in \mathcal{D}'$ has support bounded on the left and satisfies Equation (1.36) i.e.

$$\langle f(\lambda\cdot), \phi \rangle = \mathcal{O}(\lambda^\alpha) \quad \text{as } \lambda \to \infty \ \text{ for all } \phi \in \mathcal{D}.$$

Taking a derivative

$$\langle f'(\lambda\cdot), \phi(x) \rangle = -\langle f(\lambda\cdot), \phi'(x) \rangle = \mathcal{O}(\lambda^\alpha)$$

because $\phi' \in \mathcal{D}$. In the same vein, there exists $N \in \mathbb{N}$ such that the primitives of order N of $f(\lambda x)$ are bounded by $M\lambda^\alpha$ for $|x| \leq 1$, some suitable $M > 0$ and λ large enough. Notice that if F is a primitive of order N of f (see Remark 1.3.16), then $\lambda^{-N} F(\lambda\cdot)$ is a primitive of order N of $f(\lambda\cdot)$. Then,

$$|F(\lambda x)| \leq M\lambda^{\alpha+N}, \ \text{ distributionally with } \ |x| \leq 1, \ \lambda \text{ large enough.}$$

Taking $x = 1$ and replacing λ by x we obtain

$$|F(x)| \leq Mx^{\alpha+N}, \ \text{ distributionally with } \ x \ \text{ large enough.}$$

So that

$$F(x) = \mathcal{O}(x^{\alpha+N}) \quad \text{as } x \to \infty$$

and thus

$$f = \mathcal{O}_C(x^\alpha) \quad \text{as } x \to \infty.$$

The corresponding result also holds if f has support bounded on the right. A general case follows by using a decomposition $f = f_1 + f_2$, where f_1 has support bounded on the left and f_2 has support bounded on the right. So finally,

$$f = \mathcal{O}_C(|x|^\alpha), \quad \text{as } |x| \to \infty.$$

□

Finally, the following interesting result shows that the existence of the moments in the Cesàro sense of a distribution implies that the distribution belongs to $\mathcal{K}'$. The proof use some of the propositions and remarks studied up to this point.

Theorem 1.4.11. [EK02, Theorem 6.7.3] A distribution f in $\mathcal{D}'$ lies in $\mathcal{K}'(\mathbb{R})$ if, and only if, for all $n \in \mathbb{N}$ the moments $\mu_n = \langle f, x^n \rangle_C$ exist.

Proof. Since Equation (1.30) always holds for $f \in \mathcal{K}'$, we restrict to the converse. Let us assume without loss of generality that f has support bounded on the left. By definition, if $\langle f(x), x^n \rangle_C = \mu_n$ exists for all $n \in \mathbb{N}$ then

$$C\text{-}\lim_{x\to\infty} \int_{-\infty}^{x} f(t) t^n dt = \mu_n. \tag{1.37}$$

With the notation of Remark 1.3.16, we can rewrite Equation (1.37) as

$$I(x^n f(x)) = \mu_n + o_C(1) \quad \text{as } x \to \infty. \tag{1.38}$$

By Remark 1.3.19, we see that Equation (1.38) particularly implies that

$$I(x^n f(x)) = \mu_n + \mathcal{O}_C(x^{1/2}) \quad \text{as } x \to \infty.$$

So, by Proposition 1.3.18 (i)

$$x^n f(x) = \mathcal{O}_C(x^{-1/2}) \quad \text{as } x \to \infty$$

and by Proposition 1.3.18 (ii)

$$f(x) = \mathcal{O}_C(x^{-n-1/2}) \quad \text{as } x \to \infty.$$

Since n is arbitrary, $f(x) = \mathcal{O}_C(x^{-\infty})$. Theorem 1.4.10 implies that $f \in \mathcal{K}'(\mathbb{R})$. □

Remark 1.4.12. Let $f \in \mathcal{K}'$. The moment asymptotic expansion (1.33) allows us to obtain the small-t behavior of functions $\Phi(t)$ written as

$$\Phi(t) = \langle f, g(t\cdot) \rangle$$

with $g \in \mathcal{K}$, which gives

$$\Phi(t) = \sum_{j=0}^{\infty} \frac{\mu_j g^{(j)}(0) t^j}{j!} \qquad \text{as } t \to \infty$$

with the well-defined moments $\mu_j(f)$ given by

$$\mu_j := \langle f, x^j \rangle \quad \text{for each } \ j \in \mathbb{N}.$$

We will be interested in the cases where $f(\lambda \cdot)$ is the kernel of the spectral density of a pseudodifferential operator P, but those are topics of the following chapter.

Chapter 2

Spectral densities

2.1 A warm up with spectral measures

First, let us revisit diagonalization of a normal operator in finite dimension, i.e. a normal matrix. Suppose that T is a linear operator acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ of finite dimension n. Assume that T is normal so there exists an orthonormal basis $e_1, e_2, \ldots, e_n$ for $\mathcal{H}$ consisting of eigenvectors of T,

$$Te_k = \lambda_k e_k, \quad \text{for each } k = 1, \ldots, n,$$

where $\mathrm{Sp}(T) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is the spectrum of T. The eigenspaces W_k of T corresponding to the eigenvalues λ_k satisfy $\mathcal{H} = W_1 \oplus \cdots \oplus W_n$, and the orthogonal projections $P_1, \ldots, P_n$ associated to this decomposition satisfy

$$I = \sum_{k=1}^{n} P_k \quad \text{and} \quad T = \sum_{k=1}^{n} \lambda_k P_k \tag{2.1}$$

where the right part of the last equality is called the *spectral decomposition* of the operator T. This decomposition depends on the basis $e_1, e_2, \ldots, e_n$, and it is useful to reformulate this decomposition in a basis-free way, which can be done as follows. For every $\lambda \in \mathbb{C}$, let W_λ be the eigenspace

$$W_\lambda := \{\xi \in \mathcal{H} : T\xi = \lambda\xi\}.$$

The subspaces $\{W_\lambda : \lambda \in \mathbb{C}\}$ sum to $\mathcal{H}$ (in the sense of direct sums), they are mutually orthogonal, each of them is invariant under T, and W_λ is nonzero if, and only if, $\lambda \in \mathrm{Sp}(T)$ where $\mathrm{Sp}(T)$ denote the spectrum of T. Following the ideas of [RS80, Chapter VII], these remarks can be converted into a statement about only T in the next way: let P_λ be the projection of $\mathcal{H}$ onto W_λ, then, the family $\{P_\lambda\}$ of spectral projections forms a system of mutually orthogonal projections in $\mathcal{B}(\mathcal{H})$, again they sum to I, $P_\lambda \neq 0$ if, and only if, $\lambda \in \mathrm{Sp}(T)$, and we also have

$$T = \sum_{\lambda \in \mathrm{Sp}(T)} \lambda P_\lambda \tag{2.2}$$

(analogous to Equation (2.1)). As expected, functions of T can be expressed like

$$f(T) = \sum_{\lambda \in \mathrm{Sp}(T)} f(\lambda) P_\lambda.$$

Thus, Formula (2.2) expresses the operator T as a "spectral sum" where the right side represents the integral of the complex-valued function $f(z) = z$, $z \in \mathrm{Sp}(T)$, against the projection-valued measure $P(A) = \sum_{\lambda \in A} P_\lambda$ with $A \subseteq \mathbb{C}$.

Formula (2.2) can be generalized to normal operators acting on an infinite-dimensional Hilbert space $\mathcal{H}$. For this porpuse, let us consider $\mathcal{B}$ as the algebra of Borel sets in $\mathbb{C}$ and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$.

Definition 2.1.1. A function $P : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ taking values in projections $P(A)$ ($A \in \mathcal{B}$), such that

(i) $P(\emptyset) = 0$,

(ii) $P(\mathbb{C}) = I$

(iii) for every sequence $A_1, A_2, \ldots$ of mutually disjoint sets, we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \text{s-}\lim_{N\to\infty} \sum_{n=1}^{N} P(A_n)$$

is called a *spectral measure* on $\mathbb{C}$. The last limit is interpreted as the limit in the strong topology of the sequence of partial sums $\sum_{n=1}^{N} P(A_n)$.

The next step is to give meaning of the spectral integral $\int f dP$ where $P : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a spectral measure and $f : \mathbb{C} \to \mathbb{C}$ is a bounded Borel function (i.e. $f^{-1}(A) \subset \mathbb{C}$ is a bounded Borel set for any open set A). This can be done by interpreting $\int f dP$ as a weak integral as follows:

Theorem 2.1.2. [RS80, Theorem VII.7] For every spectral measure P defined on $\mathbb{C}$ and taking values in the set of projections of $\mathcal{B}(\mathcal{H})$, and for every bounded Borel function $f : \mathbb{C} \to \mathbb{C}$ there exists a unique operator $\int f dP$ such that

$$\left\langle \left(\int f dP \right) \xi, \eta \right\rangle = \int_{\mathbb{C}} f(\lambda) \langle P(d\lambda)\xi, \eta \rangle, \qquad \xi, \eta \in \mathcal{H} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$.

In order to understand how this operator behaves, let us give an overview of the proof: we start with a couple of vectors ξ, η in $\mathcal{H}$ and define a complex-valued measure $\mu(\xi, \eta)$ by $\mu(\xi, \eta)(A) := \langle P(A)\xi, \eta \rangle$ on $\mathbb{C}$. This one is a countably additive complex-valued measure on $\mathcal{B}$ and satisfy the estimate

$$\|\mu(\xi, \eta)\| \leq \|\xi\| \|\eta\|, \qquad \xi, \eta \in \mathcal{H},$$

where $\| \cdot \|$ is the norm associated to the inner product in $\mathcal{H}$. Also, the application from $\mathcal{H} \times \mathcal{H}$ into the space of measures on $\mathbb{C}$ given by

$$(\xi, \eta) \mapsto \mu(\xi, \eta)$$

is linear in ξ and antilinear in η, which implies that the mapping $[\cdot,\cdot]$ on $\mathcal{H}\times\mathcal{H}$

$$[\xi,\eta] := \int_{\mathbb{C}} f d\mu(\xi,\eta)$$

is a bounded sesquilinear form with a straightforward estimate

$$|[\xi,\eta]| \leq \sup_{\lambda\in\mathbb{C}} |f(\lambda)| \|\xi\| \|\eta\| = \|f\|_\infty \|\xi\| \|\eta\|.$$

By the Riesz lemma, there is a unique operator $\pi(f)$ in $\mathcal{B}(\mathcal{H})$ such that

$$\langle \pi(f)\xi,\eta\rangle \qquad \text{for } \xi,\eta \text{ in } \mathcal{H}, \text{ and } \qquad \|\pi(f)\| \leq \|f\|_\infty.$$

This defines the operator $\pi(f)$ as a weak integral, and yields an interpretation of $\int f dP$.

As it is shown in [RS80, Chapter VIII], spectral measures on the real line generalise beyond bounded operators. In fact, if a spectral measure P has compact support in the sense that there is a compact subset $K \subseteq \mathbb{R}$ with $P(\mathbb{R}\setminus K) = 0$, one can associate P with a self-adjoint operator as follows: since P is not null only in K, $f(\lambda) = \lambda$ is bounded almost everywhere with respect to P, so that

$$T = \int_{-\infty}^{\infty} \lambda dP(\lambda) = \int_K \lambda dP(\lambda)$$

defines a self-adjoint operator on $\mathcal{H}$. For a bounded Borel function f

$$f(T) = \int_{-\infty}^{\infty} f(\lambda) dP(\lambda)$$

where

$$\langle f(T)\xi,\eta\rangle = \int_{\infty}^{\infty} f(\lambda)\langle dP(\lambda)\xi,\eta\rangle, \qquad \xi,\eta\in\mathcal{H}.$$

Thus, spectral integrals yield another way of looking at the functional calculus for Borel functions. Now, suppose f is an unbounded complex-valued Borel function and let

$$D_f := \left\{\xi\in\mathcal{H} : \int_{-\infty}^{\infty} |f(\lambda)|^2\langle dP(\lambda)\xi,\xi\rangle < \infty\right\}$$

Then, D_f is dense $\mathcal{H}$ and an operator $f(T)$ is defined on D_f by

$$\langle f(T)\xi,\xi\rangle := \int_{-\infty}^{\infty} f(\lambda)\langle dP(\lambda)\xi,\xi\rangle$$

and we write symbolically

$$f(T) = \int f(\lambda) dP_\lambda.$$

If f is real-valued, then $f(T)$ is self-adjoint on D_f. Summarizing all this section:

Theorem 2.1.3. [RS80, Spectral Theorem VIII.6] There exists a one to one correspondence between self-adjoint operators T and spectral measures $\{P_\lambda\}$ on $\mathcal{H}$, given by

$$T = \int_{-\infty}^{\infty} \lambda dP_\lambda$$

and if f is a real-valued Borel function on $\mathbb{R}$, then

$$f(T) = \int_{-\infty}^{\infty} f(\lambda) dP_\lambda$$

defined on a dense domain D_f is self-adjoint, i.e. $f(T)$ is essentially self-adjoint on $\mathcal{H}$.

2.2 Spectral densities

Let $\mathcal{H}$ be a Hilbert space and $T : \mathrm{Dom}(T) \to \mathcal{H}$ a (possibly unbounded) self-adjoint operator. This operator defines a "smooth structure" on a subset of $\mathcal{H}$:

Definition 2.2.1. The subspace $\mathcal{H}^\infty \subset \mathcal{H}$ of *smooth vectors* in $\mathcal{H}$ (relative to T) is defined as

$$\mathcal{H}^\infty = \bigcap_{m \in \mathbb{N}} \mathcal{H}^m \quad \text{where for every } m \in \mathbb{N} \ \ \mathcal{H}^m := \mathrm{Dom}(T^m).$$

It can be shown that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$ in the sense of the norm inherited by the inner product of $\mathcal{H}$ (see [Kna86, Theorem 3.15]). By the Spectral Theorem (2.1.3), the operator T admits a spectral decomposition $\{P_\lambda\}_{\lambda=-\infty}^{\infty}$ such that

$$T = \int_{-\infty}^{\infty} \lambda dP_\lambda(T) \tag{2.4}$$

where we use the symbol $dP_\lambda(T)$ to mean integration with respect to the spectral measures associated to T. Also, if g is a real-valued Borel function on $\mathbb{R}$, then

$$g(T) = \int_{-\infty}^{\infty} g(\lambda) dP_\lambda(T) \tag{2.5}$$

is essentially self-adjoint on $\mathcal{H}$. Spectral integrals (2.4) and (2.5) make sense in the functional calculus for Borel functions.

If T moreover has a property of compact resolvent[1], from Equation (2.4) we may write

$$T = \sum_k \lambda_k |\xi_k\rangle\langle\xi_k| \tag{2.6}$$

[1] i.e. the operator $(T - \lambda I)^{-1}$ is a compact operator, then the spectrum of T is a discrete subset of $\mathbb{C}$

where $(\xi_k)_k$ is the orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of T with eigenvalues λ_k (repeated in the sum (2.6) in case of multiplicity), and $|\xi_k\rangle\langle\xi_k|$ denotes the orthogonal projection onto $\mathrm{Span}(\xi_k)$.

Remark 2.2.2. Although $\mathcal{H}^\infty$ is not a Hilbert space, in slightly different form both of the following aspects are fulfilled by $\mathcal{H}^\infty$:

1. $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$, and each $T^m : \mathcal{H}^\infty \to \mathcal{H}$, $m \in \mathbb{N}$, is an unbounded operator.

2. $\mathcal{H}^\infty$ is a Fréchet space, and each $T^m : \mathcal{H}^\infty \to \mathcal{H}$ is a continuous linear map.

Furthermore, T^m maps $\mathcal{H}^n$ into $\mathcal{H}^{n-m}$ for $n \geq m$, so that $\mathcal{H}^\infty$ is stable under T^m for all m. Notice that we could introduce also pseudodifferential operators of order n (defined in Appendix A) by those operators $P : \mathcal{H}^\infty \to \mathcal{H}^\infty$ such that they extend to bounded operators $\mathcal{H}^m \to \mathcal{H}^{m-n}$ for $m \geq n$.

Definition 2.2.3. The spectral decomposition (2.4), determines the *spectral density*:

$$d_T(\lambda) := \frac{dP_\lambda(T)}{d\lambda}$$

understood as a distribution from $\mathcal{D}'$ valued in $\mathfrak{B}(\mathrm{Dom}(T), \mathcal{H})$ (bounded operators acting from $\mathrm{Dom}(T)$ into $\mathcal{H}$), such that

$$\langle d_T(\lambda), \phi(\lambda)\rangle_\lambda := \int_{-\infty}^{\infty} \phi(\lambda) dP_\lambda(T). \tag{2.7}$$

Example 2.2.4. 1. The identity operator is

$$I = \langle d_T(\lambda), 1\rangle = \int_{-\infty}^{\infty} dP_\lambda(T).$$

2. Following (2.4), the operators T reads

$$T = \langle d_T(\lambda), \lambda\rangle = \int_{-\infty}^{\infty} \lambda dP_\lambda(T)$$

i.e.

$$(T\xi|\eta) := (\langle d_T(\lambda), \lambda\rangle_\lambda \xi|\eta) = \int_{-\infty}^{\infty} \lambda d(P_\lambda \xi|\eta)$$

for all $\xi \in \mathrm{Dom}(T)$ and $\eta \in \mathcal{H}$ (here we write the inner product as $(\cdot|\cdot)$ to avoid confusion with the evaluation of a distribution, $\langle\cdot,\cdot\rangle$).

3. As any function $\phi \in \mathcal{D}$ is a Borel function, one gets $\phi(T) = \langle d_T, \phi\rangle_\lambda$. Following Equation (1.13), we use the notation

$$d_T(\lambda) := \delta(\lambda - T),$$

so that, if the operator T has compact resolvent, from Equation (2.6) we have

$$d_T(\lambda) = \sum_k |\xi_k\rangle\langle\xi_k|\delta(\lambda - \lambda_k). \tag{2.8}$$

Remark 2.2.5. In accordance with Chapter 1, if $d_T(\lambda)$ is viewed as a distribution from $\mathcal{D}'$ valued in $\mathfrak{B}(\mathcal{H}^\infty, \mathcal{H})$, then we have

$$\langle d_T(\lambda), \lambda^n \rangle = T^n \quad \text{for all } n$$

which means, by Theorem 1.4.11, that actually $\delta(\lambda - T) \in \mathcal{K}'(\mathbb{R}, \mathfrak{B}(\mathcal{H}^\infty, \mathcal{H}))$. This implies, by Theorem 1.4.10, that

$$\delta(\Lambda\lambda - T) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Lambda^{n+1}} T^n \delta^{(n)}(\lambda), \quad \Lambda \to \infty \tag{2.9}$$

and moreover, $\delta(\lambda - T)$ vanishes to infinite order at $\pm\infty$ in the Cesàro sense, namely

$$\delta(\lambda - T) = o_C(|\lambda|^{-\infty}) \quad \text{as } |\lambda| \to \infty. \tag{2.10}$$

Note that $d_T(\lambda)$ is not a density in the usual sense (as a Radon-Nikodym derivative, see [Yos12, Theorem III.8]) because it does not have a punctual value for all λ. On the contrary, it is important to emphasize that $d_T(\lambda)$ should be understood as an operator-valued distribution, i.e. an element of the space $\mathcal{D}'(\mathbb{R}, \mathcal{B}(\mathrm{Dom}(T), \mathcal{H}))$ acting on $\mathcal{D}(\mathbb{R})$ as

$$\phi(T) = \langle d_T(\lambda), \phi(\lambda) \rangle_\lambda.$$

2.3 Spectral densities of elliptic pseudodifferential operators

For the purposes of this thesis, we are mostly interested in the case of an elliptic pseudodifferential operator T with smooth coefficients on a smooth Riemannian manifold $\mathcal{M}$. For further information about pseudodifferential operators, please review Appendix A. The corresponding Hilbert space is $\mathcal{H} = L^2(\mathcal{M})$ and $Dom(T)$ is the domain corresponding to suitable boundary conditions. The operator T will be usually positive, but for this section it will only be assumed to be self-adjoint.

2.3.1 Motivating examples and results

Since the space $\mathcal{D}(\mathcal{M})$ of smooth functions with compact support on $\mathcal{M}$ is a subspace of $\mathcal{H}^\infty$ [Nee10, Remark 4.11], by the Schwartz kernel theorem, operators P acting on $\mathcal{D}(\mathcal{M})$ can be represented as integrals of distributional kernels $K_P(x,y)$ in $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$ by

$$(P\phi)(x) = \int K_P(x,y)\phi(y)dy = \langle K_P(x,y), \phi(y) \rangle_y. \tag{2.11}$$

Example 2.3.1. The distributional kernel corresponding to the identity operator I is $\delta(x-y)$, and the one for T is $T\delta(x-y)$ because

$$\langle \delta(x-y), \phi(y)\rangle_y = I\phi(x) \quad \text{and}$$
$$\langle T\delta(x-y), \phi(y)\rangle = \langle \delta(x-y), T\phi(y)\rangle = T\phi(x).$$

Similarly, $d_T(\lambda)$ has an associated kernel $d_T(x,y;\lambda) \in \mathcal{K}'(\mathbb{R}, \mathcal{D}'(\mathcal{M} \times \mathcal{M}))$, such that

$$\langle\langle d_T(x,y;\lambda), \phi(y)\rangle_y, \psi(\lambda)\rangle_\lambda = \langle d_T(\lambda), \psi(\lambda)\rangle_\lambda \phi(x) = \psi(T)\phi(x)$$

with $\phi \in \mathcal{D}$ and $\psi \in \mathcal{K}$. Since T is elliptic, it follows that $d_T(x,y;\lambda)$ is smooth in (x,y). The expansions (2.9) and (2.10) hold in $\mathcal{H}^\infty$, and thus in $\mathcal{D}(\mathcal{M})$, so observing that

$$\langle d_T(x,y;\lambda), \lambda^n\rangle_\lambda = T^n\delta(x-y)$$

we have

$$d_T(x,y;\Lambda\lambda) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Lambda^{n+1}} (T^n\delta)(x-y)\delta^{(n)}(\lambda), \quad \Lambda \to \infty \tag{2.12}$$

and

$$d_T(x,y;\lambda) = o_C(|\lambda|^{-\infty}) \quad \text{as } |\lambda| \to \infty. \tag{2.13}$$

Proposition 2.3.2. Expansion (2.12) and Relation (2.13) away from the diagonal of $\mathcal{M} \times \mathcal{M}$, are valid in the sense of uniform convergence of all derivatives on compact sets and hence, pointwise outside of the diagonal of $\mathcal{M} \times \mathcal{M}$.

Proof. The expansion (2.12) can not hold pointwise in both variables x and y in all $\mathcal{M} \times \mathcal{M}$, since we can not set $x = y$ in the distribution $\delta(x-y)$. Let U, V be open sets in $\mathcal{M}$ with $U \cap V = \emptyset$. Due to the fact that T is elliptic, $d_T(x,y;\lambda)$ is smooth in $(x,y) \in U \times V$, so if $f \in \mathcal{D}'(\mathcal{M})$ and $\phi \in \mathcal{D}(\mathbb{R})$, then $\phi(T)$ is a smoothing pseudodifferential operator, and $\phi(T)f$ is smooth in $\mathcal{M}$ for all $f \in \mathcal{D}'(\mathcal{M})$. Thus, if $f \in \mathcal{D}'(\mathcal{M})$ with $\operatorname{supp} f \subseteq U$ and $g \in \mathcal{D}'(\mathcal{M})$ with $\operatorname{supp} g \subseteq V$ then

$$\langle d_T(x,y;\lambda), f(x)g(y)\phi(y)\rangle = \langle \phi(T)f(x), g(x)\rangle$$

is well-defined, and so $d_T(x,y;\lambda)$ belongs to $\mathcal{D}'(\mathbb{R}, \mathcal{E}(U \times V))$. Moreover,

$$\langle d_T(x,y;\lambda), f(x)g(y)\lambda^n\rangle = \langle T^n f(x), g(x)\rangle = 0,$$

so actually $d_T(x,y;\lambda)$ belongs to $\mathcal{K}'(\mathbb{R}, \mathcal{E}(U \times V))$, and all its moments vanish. In consequence,

$$d_T(x,y;\Lambda\lambda) = o(\Lambda^{-\infty}) \quad \text{as} \quad \Lambda \to \infty$$

in $\mathcal{K}'(\mathbb{R}, \mathcal{E}(U \times V))$. Similarly, Relation (2.13) also holds in $\mathcal{E}(U \times V)$. As convergence in $\mathcal{E}(U \times V)$ implies pointwise convergence on $U \times V$, we have the second part of what we want to prove. Also convergence in $\mathcal{E}(U \times V)$

implies uniform convergence of all derivatives on compact sets, thus (2.12)-(2.13) hold uniformly on compacts subsets of $U \times V$. Hence, the expansion can be differentiated as many times as necessary with respect to x or y. $\square$

On the other hand, similar to the previous proof, we will also show the following result on the behavior of the density kernel.

Proposition 2.3.3. Let T_1 and T_2 be pseudodifferential operators acting on $\mathcal{M}$ with spectral density kernels $d_1(x, y; \lambda)$, $d_2(x, y; \lambda)$ respectively. Let U be an open set on $\mathcal{M}$ and assume that $T_1 - T_2$ is a smoothing operator in U. Then,

$$d_1(x, y; \lambda) = d_2(x, y; \lambda) + o_c(\lambda^{-\infty}) \quad \text{as} \quad \lambda \to \infty,$$

in the topology of the space $\mathcal{E}(U \times U)$ and, in particular, pointwise on $(x, y) \in U \times U$.

Proof. Let $f, g \in \mathcal{E}'(U)$. As $T_1 - T_2$ over U is a smoothing operator, if $\phi \in \mathcal{D}(\mathbb{R})$ then $\phi(T_1) - \phi(T_2)$ is a smoothing operator. Therefore, $\langle d_1(x, y; \lambda) - d_2(x, y; \lambda), f(x)g(y)\rangle$ is a well-defined element of $\mathcal{D}'(\mathbb{R})$ given by

$$\langle\langle d_1(x, y; \lambda) - d_2(x, y; \lambda), f(x)g(y)\rangle, \phi(\lambda)\rangle = \langle(\phi(T_1) - \phi(T_2))f, g\rangle.$$

If supp $f \subset U$ and supp $g \subset U$ then all the moments

$$\langle\langle d_1(x, y; \lambda) - d_2(x, y; \lambda), f(x)g(y)\rangle, \lambda^n\rangle = \langle(T_1^n - T_2^n)f, g\rangle$$

exist because $T_1 - T_2$ is smoothing in U. Consequently,

$$\langle d_1(x, y; \lambda) - d_2(x, y; \lambda), f(x)g(y)\rangle \in \mathcal{K}'(\mathbb{R})$$

and by Theorem (1.4.10),

$$\langle d_1(x, y; \lambda) - d_2(x, y; \lambda), f(x)g(y)\rangle = o_c(\lambda^{-\infty}) \quad \text{as} \quad \lambda \to \infty.$$

$\square$

Let us exemplify the described behavior with a very illustrative example:

Example 2.3.4. [EKL90, Example 160] Let H be the operator $Hy = -y''$ considered on the domain $\mathcal{X} := \{y \in C^2[0, \pi] : y(0) = y(\pi) = 0\}$ in $L^2[0, \pi]$. The eigenvalues are $\lambda_n = n^2$, $n = 1, 2, 3, \dots$ with normalised eigenfunctions $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Therefore, taking the distributional derivative as in Equation (2.8), the spectral density is determined by

$$d_H(x, y; \lambda) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \sin ny \delta(\lambda - n^2),$$

where $0 < x < \pi$, $0 < y < \pi$. Then, by Equation (2.9)

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \sin ny \delta(\Lambda\lambda - n^2) \sim \sum_{j=0}^{\infty} \frac{\delta^{(2j)}(x - y)\delta^{(j)}(\lambda)}{j!\Lambda^{j+1}} \qquad \text{as } \Lambda \to \infty$$

in $\mathcal{D}'(\mathbb{R}, \mathcal{D}'((0,\pi)\times(0,\pi)))$. If $x \neq y$, then

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \sin nx \sin ny \delta(\Lambda\lambda - n^2) = o(\Lambda^{-\infty})$$

and, on the other hand

$$d_H(x,x;\lambda) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin^2 nx \delta(\lambda - n^2).$$

In Appendix C we show

$$\begin{aligned} \sum_{n=1}^{\infty} \cos 2nx &= -\frac{1}{2} \qquad (C) \\ \sum_{n=1}^{\infty} n^{2k}\cos 2nx &= 0 \qquad (C), \qquad k = 1,2,3,\ldots \end{aligned}$$

Thus, if $0 < x < \pi$

$$\begin{aligned} d_H(x,x;\Lambda\lambda) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(1-\cos 2nx)\delta(\Lambda\lambda - n^2) \\ &= \frac{1}{\pi}\delta(\Lambda\lambda - n^2) + \frac{1}{2\pi\Lambda}\delta(\lambda) + o(\Lambda^{-\infty}) \qquad \Lambda \to \infty \end{aligned}$$

since $\sum_{n=1}^{\infty}\cos(2nx)\delta(\lambda - n^2)$ belongs to $\mathcal{K}'$ if $0 < x < \pi$ with moments $\mu_0 = -1/2$ y $\mu_k = 0$ for $k \geq 1$.

From the Ramanujan's formula[2]

$$\sum_{n=1}^{\infty}\phi(\epsilon n^2) = \frac{1}{2\pi\sqrt{\epsilon}}\int_0^{\infty} x^{-1/2}\phi(x)dx - \frac{1}{2}\phi(0) + o(\epsilon^{\infty}) \text{ as } \epsilon \to 0^+.$$

we infer

$$d_H(x,x;\lambda) = \frac{1}{2\pi\sqrt{\lambda}} + o_C(\lambda^{-\infty}) \qquad \lambda \to \infty,$$

which means, $d_H(x,x;\lambda) \sim (1/2\pi)\lambda^{-1/2}$, as $\lambda \to \infty$ in the Cesàro sense.

2.3.2 The Cesáro asymptotic behavior of the spectral density: constant coefficient case

When T is an elliptic operator with constant coefficients defined in all $\mathbb{R}^n$, it admits a selfadjoint extension which we will also denote as T. Let $p = \sigma(T)$ be the total symbol of T. Then the spectral projection of T is given by

$$P(x,y,;\lambda) = \frac{1}{(2\pi)^n}\int_{p(\xi)<\lambda} e^{i(x-y)\cdot\xi}d\xi$$

[2][EGBV98, Lemma 2.11] applied to $g(x) := \phi(x^2)$ for $\phi \in \mathcal{S}$.

so that the spectral density can be written as

$$d_T(x, y; \lambda) = \frac{1}{(2\pi)^n} \left\langle e^{i(x-y)\cdot\xi}, \delta(p(\xi) - \lambda) \right\rangle. \tag{2.14}$$

The definition of $\delta(p(\xi) - \lambda)$ is similar to that of Formula (1.16).

In order to obtain the Cesáro asymptotic behavior of $d_T(x, y; \lambda)$ we shall start with the parametric behavior of $d_T(x, y; \Lambda\lambda)$ as $\Lambda \to \infty$. Setting $t = 1/\Lambda$ leads to the function

$$\Psi(t) := \langle d_T(x, y; \lambda), \phi(t\lambda) \rangle_\lambda$$

where $\phi(\lambda)$ is a function in $\mathcal{K}(\mathbb{R})$ (see Remark 1.4.12). Because of Relation (2.14)

$$\Psi(t) = \frac{1}{(2\pi)^n} \left\langle e^{i(x-y)\cdot\xi}, \phi(tp(\xi)) \right\rangle_\xi,$$

so when $x \neq y$ are fixed, $e^{i(x-y)\cdot\xi}$ is in $\mathcal{K}'$ as a function of ξ. This also holds distributionally in (x, y). The expansion of $\Psi(t)$ is therefore a consequence of the next lemma.

Lemma 2.3.5. Let $f \in \mathcal{K}'(\mathbb{R}^n)$, so that it satisfies the multidimensional moment asymptotic expansion

$$f(\lambda \mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{R}^n} \frac{(-1)^{|\mathbf{k}|} \mu_\mathbf{k} D^\mathbf{k} \delta(\mathbf{x})}{\mathbf{k}! \lambda^{|\mathbf{k}|+n}} \quad \text{as} \quad \lambda \to \infty$$

where $\mu_\mathbf{k}$ are the moments of the generalized function f,

$$\mu_\mathbf{k} = \langle f(\mathbf{x}), \mathbf{x}^\mathbf{k} \rangle,$$

meaning that if $\phi \in \mathcal{K}(\mathbb{R}^n)$, then

$$\langle f(\lambda \cdot), \phi \rangle = \sum_{|\mathbf{k}|=0}^{N} \frac{\mu_\mathbf{k} D^\mathbf{k} \phi(\mathbf{0})}{\mathbf{k}! \lambda^{|\mathbf{k}|+n}} + \mathcal{O}\left(\frac{1}{\lambda^{N+n+1}} \right),$$

as $\lambda \to \infty$. If p is an elliptic polynomial and $\phi \in \mathcal{K}$, then

$$\langle f(x), \phi(tp(x)) \rangle \sim \sum_{m=0}^{\infty} \frac{\langle f(x), p(x)^m \rangle \phi^{(m)}(0)}{m!} t^m \quad \text{as} \quad t \to 0. \tag{2.15}$$

Proof. We want to show that the Taylor expansion in t

$$\phi(tp(x)) = \sum_{m=0}^{N} \frac{\phi^{(m)}(0) p(x)^m t^m}{m!} + \mathcal{O}(t^{N+1})$$

not only holds pointwise but actually converges in the topology of $\mathcal{K}(\mathbb{R}^n)$. The remainder in this expansion is

$$R_N(x,t) = \frac{\phi^{(N+1)}t_0 p(x) p(x)^{N+1} t^{N+1}}{(N+1)!},$$

for some $t_0 \in (0,t)$. Since $\phi \in \mathcal{K}$, there exists $q \in \mathbb{R}$ such that $\phi^{(j)}(x) = \mathcal{O}(|x|^{q-j})$ as $x \to \infty$. As p has degree m, it follows that

$$|R_N(x,t)| \leq \frac{M \max\{1, |x|^{mq}\} t^{N+1}}{(N+1)!}$$

for some constant M and the convergence of the Taylor expansion in the topology of the space $\mathcal{K}$ follows. □

Remark 2.3.6. Observe that applying Equation (2.15) with $f(x) = e^{i(x-y)\cdot\xi}$ for $x \neq y$, we obtain

$$\Psi(t) \sim \frac{1}{(2\pi)^n} \sum_{k=0}^{\infty} \frac{\langle e^{i(x-y)\cdot\xi}, p(\xi)^k \rangle \phi^{(k)}(0) t^k}{k!},$$

or even more

$$\Psi(t) \sim \sum_{k=0}^{\infty} \frac{T^k \delta(x-y) \phi^{(k)}(0) t^k}{k!}.$$

Consequently,

$$d_T(x,y;\Lambda\lambda) \sim \sum_{k=0}^{\infty} \frac{(-1)^k T^k \delta(x-y) \delta^{(k)}(\lambda)}{k! \Lambda^{k+1}}, \quad \text{as} \quad \Lambda \to \infty$$

which goes in accordance with the general result (2.12).

Remark 2.3.7. Observe also that by Proposition 2.3.3 if T_1 is any operator with the same symbol, considered in some open set of $\mathcal{M}$ with some boundary conditions, then its spectral density $d_1(x,y;\lambda)$ satisfies

$$d_1(x,y;\lambda) = \frac{1}{(2\pi)^n} \langle e^{i(x-y)\cdot\xi}, \delta(p(\xi) - \lambda) \rangle + o_C(\lambda^{-\infty})$$

As in the Example 2.3.4 we illustrate the previous reasoning with the following example.

Example 2.3.8. Let M be a region of $\mathbb{R}^n$ and H any selfadjoint extension of the negative Laplacian $-\Delta$ with suitable boundary conditions on M. The symbol of this operator is $|\xi|^2$. Let $d_H(x,y;\lambda)$ be its spectral density. Then

$$d_H(x,y;\lambda) = \frac{1}{(2\pi)^n} \langle \delta(|\xi|^2 - \lambda), e^{i(x-y)\cdot\xi} \rangle + o_C(\lambda^{-\infty})$$

As $|\xi|^2$ has only one zero in $|\xi| \in \mathbb{R}$, we use the formula

$$\delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

where f has the only zero x_0. Passing to polar coordinates with $\xi = rw$ in which $r = |\xi|$, $w = (w_1, \ldots, w_n)$ with $|w| = 1$, $d\xi = r^{n-1}drdw$ and dw representing the measure on the sphere in $\mathbb{R}^n$, we have

$$\begin{aligned}
\frac{1}{(2\pi)^n}\langle \delta(|\xi|^2-\lambda), e^{i(x-y)\cdot\xi}\rangle &= \frac{1}{(2\pi)^n}\int_{|w|=1}\int_0^\infty \langle \delta(r^2-\lambda), e^{i(x-y)\cdot w}\rangle r^{n-1}drd\sigma(w) \\
&= \frac{\lambda^{n/2-1}}{2(2\pi)^n}\int_{|w|=1} e^{i\lambda^{1/2}(x-y)\cdot w}d\sigma(w) \\
&= \frac{\lambda^{n/2-1}}{2(2\pi)^n}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int_{-1}^1 e^{i\lambda^{1/2}u|x-y|}(1-u^2)^{\frac{n-3}{2}}du \\
&= \frac{\lambda^{n/4-1/2}J_{n/2-1}(\lambda^{1/2}|x-y|)}{2^{n/2+1}\pi^{n/2}|x-y|^{n/2-1}},
\end{aligned}$$

with $J_p(x)$ the Bessel function of order p. Thus

$$d_H(x,y;\lambda) = \frac{\lambda^{n/4-1/2}J_{n/2-1}(\lambda^{1/2}|x-y|)}{2^{n/2+1}\pi^{n/2}|x-y|^{n/2-1}} + o_C(\lambda^{-\infty}), \qquad \lambda\to\infty,$$

uniformly on compact sets of $M \times M$.

2.3.3 The Cesáro asymptotic behavior of the spectral density: general case

In this subsection we pass from spectral densities of pseudodifferential operators on the flat space $\mathbb{R}^n$ to the ones defined on a closed Riemannian manifold. This is posible since the Cesàro behavior of the spectral densities is a local property. Let us start with the following observation:

Let P be an elliptic pseudodifferential operator of positive integer order d with total symbol $p(x,\xi) = \sigma(P)$ on a Riemannian manifold $\mathcal{M}$ of dimension n acting on scalar functions. The Schwartz kernel (2.11) of P is by definition:

$$K_P(x,y) := \frac{1}{(2\pi)^n}\langle e^{i(x-y)\cdot\xi}, p(x,\xi)\rangle_\xi.$$

So, in particular on the diagonal

$$K_P(x,x) := \frac{1}{(2\pi)^n}\langle 1, p(x,\xi)\rangle_\xi.$$

Additionally, for a pseudodifferential operator Q with symbol q, the following composition rule holds

$$p(x,\xi)\sharp q(x,\xi) - p(x,\xi)q(x,\xi) \sim \sum_{|\alpha|>0} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p(x,\xi)\partial_x^\alpha q(x,\xi) \tag{2.16}$$

with $p\sharp q$ meaning the symbol of the composite operator PQ. From this, in the constant coefficient case we have that

$$\sigma(P^n) = \sigma(P)^n = p^n, \quad \text{for } n \in \mathbb{N}.$$

Remark 2.3.9. When the resolvent operator is well-defined (the construction of a parameter dependent parametrix for $(P-\lambda)$ is treated in [See67]), we have that

$$\sigma((P-\lambda))^{-1}) = (\sigma(P)-\lambda))^{-1}.$$

Using the representation through the resolvent, if ϕ is a Borel-function

$$\phi(P) = \int_{\mathrm{Sp}(P)} \phi(\lambda)(P-\lambda)^{-1}\,d\lambda$$

where $\mathrm{Sp}(P)$ is the spectrum of P. So that if P has constant coefficients, then for the symbol of $\phi(P)$ we have

$$\sigma[\phi(P)] = \int_{\mathrm{Sp}(P)} \phi(\lambda)\sigma[(P-\lambda)^{-1}]\,d\lambda = \int_{\mathrm{Sp}(P)} \phi(\lambda)(\sigma(P)-\lambda)^{-1}\,d\lambda = \phi(\sigma(P)).$$

We set

$$\sigma(\delta(\lambda - P)) := \delta(\lambda - \sigma(P)) \tag{2.17}$$

so that in particular, if $\phi(x) = x^n$ for $n \in \mathbb{N}$, one has

$$\sigma(P^n) = \sigma(P)^n = \int \lambda^n \delta(\lambda - \sigma(P))d\lambda. \tag{2.18}$$

In the same spirit as Remark 2.3.9, we define the symbol of a spectral density in the general case of nonconstant coefficients, as the distributional symbol that satisfies the relations

$$\langle \sigma[\delta(\lambda - P)], \lambda^k\rangle_\lambda = \sigma(P^k) \quad \text{for } k \in \mathbb{N}. \tag{2.19}$$

Ansatz 2.3.10. Since we want an asymptotic expansion for the symbol of a spectral density, as we already have for the spectral density itself and its kernel, in the light of Equations (2.10), (2.12), (2.16) and (2.17) we make the Ansatz for the symbol:

$$\sigma[\delta(\lambda - P)] \sim_{\lambda\to\infty} \delta(\lambda - \sigma(P)) + \sum_{n=1}^{\infty} (-1)^n c_n \delta^{(n)}(\lambda - \sigma(P)) \ \text{(C)} \tag{2.20}$$

where the coefficients c_n depend on P but not on λ.

To justify this expansion and the expression of the coefficients c_n, we revisite (2.20) via the computation of nth-moments with nth-powers of λ, for each $n \in \mathbb{N}$, i.e. $\int \lambda^n \sigma[\delta(\lambda - H)]d\lambda$ for $n = 1, 2, 3, \ldots$. We first recall the formula for the successive derivatives of the δ-function. For $\phi \in \mathcal{D}$

$$\begin{aligned}
\int \phi(\lambda)\delta'(\lambda - t)d\lambda &= -\int \phi'(\lambda)\delta(\lambda - t)d\lambda = -\phi'(t) \\
\int \phi(\lambda)\delta''(\lambda - t)d\lambda &= -\int \phi'(\lambda)\delta'(\lambda - t)d\lambda \\
&= (-1)^2 \int \phi''(\lambda)(\lambda - t)d\lambda = (-1)^2\phi''(x) \\
&\vdots \\
\int \phi(\lambda)\delta^{(n)}(\lambda - t)d\lambda &= (-1)^n \phi^{(n)}(x)
\end{aligned}$$

Using this formula of successive derivatives of δ, we infer that any power of λ will pair nontrivially with a finite number of terms in Equation (2.20). Thus, we get

$$\begin{aligned}
\sigma(P) &= \langle \sigma[\delta(\lambda - P)], \lambda \rangle = \int \lambda\delta(\lambda - \sigma(P))d\lambda - c_1 \int \lambda\delta'(\lambda - \sigma(P))d\lambda + 0 \\
&= \sigma(P) - c_1(-1) \cdot 1 = \sigma(P) + c_1
\end{aligned}$$

from which $c_1 = 0$. Similarly,

$$\sigma(P^2) = (\sigma(P))^2 - c_1(-1)2\sigma(P) + c_2(-1)^2 \cdot 2$$

which gives us $c_2 = \frac{\sigma(P^2) - \sigma(P)^2}{2}$. Successively

$$c_3 = \frac{1}{6}(\sigma(P^3) - 3\sigma(P^2)\sigma(P) + 2\sigma(P)^3)$$

and so on. It is clear how to solve iteratively to derive c_n.

We have thus arrived at one of the two most important results of this chapter, which has lots of similarities with Example (2.3.8):

Theorem 2.3.11. ([EGBV98, Equation (4.5)]) Let P be a positive elliptic pseudodifferential operator of order $k > 0$ on a d-dimensional compact manifold M. Then, the Schwartz kernel $d_P(x, x; \lambda)$ of the operator $\delta(\lambda - P)$ enjoys the following Cesàro expansion on the diagonal:

$$d_P(x, x; \lambda) \sim_{\lambda \to \infty} \sum_{n=0}^{\infty} a_n(x)\lambda^{(d-k-n)/k} \ \text{(C)}, \tag{2.21}$$

where $\int_M a_k(x)|dx| = \frac{1}{k}\text{WRes}P^{-d/k}$. (the Wodzicki residue WRes and the local residue wres_x are defined in Appendix A).

Proof. From the Ansatz (2.3.10) we have

$$d_P(x, x; \lambda) \sim (2\pi)^{-n}\langle 1, \delta(\lambda - p(x, \xi)) + c_2(x, \xi)\delta''(\lambda - p(x, \xi)) - \cdots \rangle_\xi \ \text{(C)}$$

In polar coordinates, $\xi = |\xi| w$ with $|w| = 1$, this gives us

$$(2\pi)^{-n} \int_{|w|=1} dw \langle |\xi|^{n-1}, \delta(\lambda - p(x, |\xi| w)) + c_2(x, |\xi| w)\delta''(\lambda - p(x, |\xi| w)) - \cdots \rangle_{|\xi|}$$

In Formula (1.16), where $|\xi|$ is taken positive, we shall assume that the equation $p(x, |\xi| w) = \lambda$ has a unique positive solution $|\xi|(x, w; \lambda)$, so we need to compute

$$(2\pi)^{-n} \int_{S^{n-1}} dw \frac{|\xi|^{n-1}(x, w; \lambda) + \frac{\partial^2}{\partial \lambda^2} \left(c_2(x, |\xi|(x, w; \lambda) w) |\xi|^{n-1}(x, w; \lambda) \right) - \ldots}{p'(x, |\xi|(x, w; \lambda) w)}$$

with p' meaning the derivative with respect to the $|\xi|$ variable. From the asymptotic expansion of the total symbol

$$p(x, |\xi| w) \sim p_d(x, w) |\xi|^d + p_{d-1}(x, w) |\xi|^{d-1} + p_{d-2}(x, w) |\xi|^{d-2} + \cdots$$

We see that $p(x, |\xi| w) = \lambda$ can be solved by series reversion. As a first step, let us assume a short while that P has order $d = 1$, then

$$|\xi|(x, w; \lambda) \sim \frac{1}{p_1(x, w)} \lambda - \frac{p_0(x, w)}{p_1(x, w)} - p_{-1}(x, w) \lambda^{-1} + \cdots$$

Integrating over $|w| = 1$ gives us an expansion of the type

$$d_P(x, x; \lambda) \frac{1}{(2\pi)^n} (a_0(x) \lambda^{n-1} + a_1(x) \lambda^{n-2} + a_2(x) \lambda^{n-3} + \cdots) \quad (C)$$

where $a_k(x)|d^n x| = \mathrm{wres}_x P^{k-n}$. While the calculation of $a_1(x)$, $a_2(x)$, and so on, is intrincate and goes beyond our purposes, we will give an idea of why $a_0(x) = \mathrm{wres}_x P^{-n}$. Since $c_2 = \frac{\sigma(P^2) - \sigma(P)^2}{2}$ do not contribute in the $a_0(x)$ coefficient, nor the subsequent terms, it depends then only on the principal symbol. Therefore, in the expansion of

$$\int_{|w|=1} \frac{|\xi|^{n-1}(x, w; \lambda)}{p_1(x, w)}$$

the coefficient of λ^{n-1} will be

$$\int_{|w|=1} p_1(x, w)^{-n} dw$$

which actually means that

$$a_0(x)|dx| = \mathrm{wres}_x P^{-n}$$

Finally, if P is of order $d > 1$ then $A := P^{1/d}$ is an operator of order 1. Setting $\mu := \lambda^{1/d}$,

$$\delta(\lambda - P) = \delta(\mu^d - A^d) = \frac{\delta(\mu - A)}{d\mu^{d-1}} = \frac{\delta(\lambda^{1/d} - P^{1/d})}{d\lambda^{(d-1)/d}},$$

and then

$$d_P(x, x; \lambda) \sim \frac{1}{(2\pi)^n}(a_0(x)\lambda^{(n-d)/d} + a_1(x)\lambda^{(n-d-1)/d} + \cdots) \ \ (C)$$

with the leading term in the expansion given by

$$a_0(x)|dx| = \frac{1}{d}\mathrm{wres}_x P^{-n/d}$$

which ends the proof. □

2.4 Hadamard finite part and an 'assumption'

In this section, we introduce the Hadamard regularization technique and discuss how it is used in order to get the small-t behavior of Green functions associated to tempered distributions f with support bounded on the left. These are of the form

$$\Psi(t) = \langle f(\lambda), \phi(t\lambda)\rangle_\lambda,$$

where, in this section, ϕ belongs (or can be extended) to the space $\mathcal{S}(\mathbb{R})$. Assuming a (C) asymptotic expansion of a distribution[3], we will see that $\Psi(t)$ has an ordinary asymptotic expansion as $t \to 0^+$. This will be used strongly in Section 3.4.

Our interest in Hadamard finite part is to give a meaning to divergent integrals of the type

$$F(\phi) := \int_0^\infty \frac{\phi(x)}{x^k}dx \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}), \text{ and } k \in \mathbb{N}.$$

Let us consider the "cut-off" integral

$$F_\epsilon(\phi) := \int_\epsilon^\infty \frac{\phi(x)}{x^k}dx \quad \text{for } \epsilon > 0.$$

To extract a finite part when $\epsilon \to 0$, we take the following proposition into account:

Proposition 2.4.1. [Pay12, Proposition 1.21] F_ϵ can be expressed in the form

$$F_\epsilon(\phi) = a_0(\epsilon) + \sum_{j=1}^{k-1} \frac{a_j}{\epsilon^j} + b\ln(\epsilon)$$

[3] See Definition 2.4.5.

where the a_j's with $j = 1, 2 \ldots, k-1$, and b are constants and the function $a_0(\epsilon)$ has a finite limit when $\epsilon \to 0$.

Proof. Let $a \in (\epsilon, \infty)$ arbitrary but fixed. Dividing the interval of integration of $F_\epsilon(\varphi)$ and adding and subtracting appropiated terms we have

$$\begin{aligned}
\int_\epsilon^\infty \frac{\phi(x)}{x^k}dx &= \int_a^\infty \frac{\phi(x)}{x^k}dx + \int_\epsilon^a \frac{1}{x^k}\left(\phi(x) - \sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)}{j!}x^j dx\right)dx + \int_\epsilon^a \sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)}{j!}x^{j-k} \\
&= \int_a^\infty \frac{\phi(x)}{x^k}dx + \int_\epsilon^a \frac{1}{x^k}\left(\phi(x) - \sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)}{j!}x^j\right)dx \\
&\quad + \sum_{j=0}^{k-2}\frac{\phi^{(j)}(0)}{j!}\int_\epsilon^a x^{j-k}dx + \int_\epsilon^a \frac{\phi^{(k-1)}(0)}{(k-1)!}x^{-1}dx \\
&= \int_a^\infty \frac{\phi(x)}{x^k}dx + \int_\epsilon^a \frac{1}{x^k}\left(\phi(x) - \sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)}{j!}x^j\right)dx \\
&\quad - \sum_{j=0}^{k-2}\frac{\phi^{(j)}(0)}{j!(k-j-1)}\left(\frac{1}{a^{k-j-1}} - \frac{1}{\epsilon^{k-j-1}}\right) + \frac{\phi^{(k-1)}(0)}{(k-1)!}(\ln(a) - \ln(\epsilon)).
\end{aligned}$$

Choosing appropiately $a_0(\epsilon)$ and a_j's one has the result. □

Definition 2.4.2. The *finite part of $F(\phi)$ in the sense of Hadamard* is defined as

$$\mathrm{fp}F(\phi) := \lim_{\epsilon\to 0} a_0(\epsilon).$$

Remark 2.4.3. Note that the result in Proposition 2.4.1 does not depend on the value of a. Henceforth, we will take $a = 1$. Then

$$\mathrm{F.p.}F(\phi) = \int_1^\infty \phi(x)x^{-k}dx + \int_0^1 \left(\phi(x) - \sum_{j=0}^{k-1}\frac{\phi^{(j)}(0)}{j!}x^j\right)x^{-k}dx - \sum_{j=0}^{k-2}\frac{\phi^{(j)}(0)}{j!(k-j-1)}. \tag{2.22}$$

Then, considering $H(x)$ the Heaviside function as defined in Example 1.3.8, it is clear that if we denote $\mathrm{Pf}[x^{-k}H(x)]$ as

$$\begin{aligned}
\mathrm{Pf}[x^{-k}H(x)] : \mathcal{D}(\mathbb{R}) &\to \mathbb{R} \\
\phi(x) &\mapsto \langle \mathrm{Pf}[x^{-k}H(x)], \phi(x)\rangle := \mathrm{F.p.}F(\phi) = \mathrm{F.p.}\int_0^\infty \phi(x)x^{-k}dx
\end{aligned}$$

it defines a distribution.

Assumption 2.4.4. Let $f \in \mathcal{S}'_+(\mathbb{R})$ (that is, tempered distributions supported on $(0, \infty)$) and *assume* the following Cesàro asymptotic expansion:

$$f(\lambda) \sim_{\lambda\to\infty} \sum_{n=1}^\infty c_n\lambda^{\alpha_n} + \sum_{j=1}^\infty b_j\lambda^{-j} \quad (\mathrm{C})$$

where $\alpha_n \in \mathbb{R} \setminus \mathbb{Z}_{<0}$ constitute a decreasing sequence, and, $\{c_n\}$, $\{b_j\}$ sequences of real numbers.

Let us first look at what this assumption means. In order to give a rigorous presentation of the ideas contained in [EGBV98], we formulate the following definition

Definition 2.4.5. Let $f \in \mathcal{D}'(\mathbb{R})$. We say that f is asymptotic to $\sum_{n=0}^{\infty} a_n x^n$ as $x \to \infty$ in the Cesàro sense and write

$$f \sim_{x\to\infty} \sum_{n=0}^{\infty} a_n x^n \quad (C), \tag{2.23}$$

if $\{a_n\}$ is a sequence of real numbers, and we have $f - \sum_{n=0}^{N} a_n x^n = o_C(x^N)$ for any $N \in \mathbb{N}$ as $x \to \infty$.

Remark 2.4.6. Translating Assumption 2.4.4 into the language of Definition 2.4.5, we would then have that for every $N = 0, 1, 2, 3 \ldots$

$$f(\lambda) - \sum_{n=0}^{N} c_n \lambda^{\alpha_n} - \sum_{j=0}^{[\![|\alpha_N|]\!]} b_j \lambda^{-j} = o_C(\lambda^{-|\alpha_N|}) \quad \text{as } \lambda \to \infty \tag{2.24}$$

where $[\![m]\!]$ means the floor function of m. Since a distribution is said to be $o_C(x^{-\infty})$ if it is $o_C(x^{\beta})$ for all $\beta \in \mathbb{R} \setminus \mathbb{Z}_{<0}$, in particular, Equation (2.24) implies that the formal distribution $f(\lambda) - \sum_{n=0}^{\infty} c_n \lambda^{\alpha_n} - \sum_{j=1}^{\infty} b_j \lambda^{-j}$ is such that

$$f(\lambda) - \sum_{n=0}^{\infty} c_n \lambda^{\alpha_n} - \sum_{j=1}^{\infty} b_j \lambda^{-j} = o_C(\lambda^{-\infty}) \text{ as } \lambda \to \infty. \tag{2.25}$$

Now we take a detour in [EKL90]. If $f \in \mathcal{S}'(\mathbb{R})$ has bounded support on the left and

$$f(\lambda) = \sum_{n=1}^{M} b_n \lambda^{\beta_n} + O(\lambda^{\beta}) \qquad \text{as } x \to \infty \text{ for some } M \in \mathbb{N},$$

with $-(k+1) > \beta > -(k+2)$ for some $k \in \mathbb{N}$ and $\beta_1 > \beta_2 > \cdots > \beta_M > \beta$, then the generalized moments μ_j, with $0 \le j \le k$ are defined by

$$\mu_j(f) := \left\langle \left(f - \sum_{i=1}^{M} b_i g_i(\lambda) \right), \lambda^j \right\rangle, \tag{2.26}$$

where $g_j(\lambda) = \lambda_+^{\beta_j}$ if $\beta_j \notin \mathbb{Z}_{<0}$ and $g_j(\lambda) = \mathrm{Pf}(\lambda^j H(\lambda))$ if $\beta_j \in \mathbb{Z}_{<0}$. Since Equation (2.25) is valid under the assumption, by Theorem 1.4.10 we have $f(\lambda) - \sum_{n=0}^{\infty} c_n \lambda^{\alpha_n} - \sum_{j=1}^{\infty} b_j \lambda^{-j} \in \mathcal{K}'(\mathbb{R})$, and then f has the parametric development in Λ:

$$f(\Lambda\cdot) \sim_{\Lambda\to\infty} \sum_{n=1}^{\infty} c_n(\lambda_+\Lambda)^{\alpha_n} + \sum_{j=1}^{\infty} b_j \lambda^{-j} \mathrm{Pf}[(\lambda\Lambda)^{-j} H(\lambda)] + \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k \delta^{(k)}(\lambda)}{k!\Lambda^{k+1}}, \tag{2.27}$$

where $\mu_k = \langle f - \sum_{n=1}^{\infty} c_n \lambda_+^{\alpha_n} - \sum_{j=1}^{\infty} b_j \mathrm{Pf}[\lambda^{-j} H(\lambda)], \lambda^k \rangle$ are the generalised moments, like in Equation (2.26), interpreted as

$$\mu_k = \langle f - \sum_{n=1}^{M} c_n \lambda_+^{\alpha_n} - \sum_{j=1}^{[\![|\alpha_M|]\!]} b_j \mathrm{Pf}[\lambda^{-j} H(\lambda)], \lambda^k \rangle$$

for M large enough.

The previous remark lead us to the following theorem, which is used in the main result of this thesis.

Theorem 2.4.7. Let $f \in \mathcal{S}'$ supported in $[0, \infty)$. Then, under the Assumption 2.4.4, f has the following parametric asymptotic expansion in Λ

$$f(\Lambda \cdot) \sim_{\Lambda \to \infty} \sum_{n=1}^{\infty} c_n (\lambda_+ \Lambda)^{\alpha_n} + \sum_{j=1}^{\infty} b_j \lambda^{-j} \mathrm{Pf}[(\lambda\Lambda)^{-j} H(\lambda)] + \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(\lambda)}{n! \Lambda^{n+1}}, \tag{2.28}$$

where $\mu_n = \langle f - \sum_{n=1}^{\infty} c_n x_+^{\alpha_n} - \sum_{j=1}^{\infty} b_j \mathrm{Pf}[x^{-j} H(x)], x^n \rangle$ are the "generalised moments". So that,

$$\begin{aligned} \langle f, \phi(t\lambda) \rangle_\lambda \quad &\sim_{t \downarrow 0} \quad \sum_{k=1}^{\infty} c_k t^{-\alpha_k - 1} \mathrm{F.p.} \int_0^{\infty} \lambda^{\alpha_k} \phi(\lambda) d\lambda \\ &+ \quad \sum_{j=1}^{\infty} b_j t^j \left[\mathrm{F.p.} \int_0^{\infty} \frac{\phi(\lambda)}{\lambda^j} d\lambda - \frac{\phi^{(j-1)}(0)}{(j-1)!} \log t \right] + \sum_{n=0}^{\infty} \frac{\mu_n \phi^{(n)}(0)}{n!} t^n. \end{aligned} \tag{2.29}$$

for every $\phi \in \mathcal{S}$.

Proof. We argue the first part of the proof in Remark 2.4.6. For the second part, we can see that taking the finite part implies letting a logarithmic term proportional to $\phi^{(j-1)}(0)$. This leads a loss of homogeneity in $\mathrm{Pf}[\lambda^{-j} H(\lambda)]$ failing to be homogeneous of degree $-j$ by the logarithmic term; indeed

$$\mathrm{Pf}[(\Lambda\lambda)^{-j} H(\Lambda\lambda)] = \Lambda^{-j} \mathrm{Pf}[\lambda^{-j} H(\lambda)] + \frac{(-1)^j \delta^{(j-1)}(\lambda) \log \Lambda}{(j-1)! \Lambda^j}$$

The result follows comparing term-by-term between Equation (2.27) and Equation (2.29), by replacing $t \leftrightarrow 1/\Lambda$. □

Formula 2.29 can be used to obtain an asymptotic expansion of the spectral action in the commutative case (for the canonical spectral triple), as we will see in Section 3.4.

Remark 2.4.8. Under the conditions of Assumption 2.4.4, suppose that $f(\lambda)$ has a Cesàro expansion in falling powers of λ and suppose also that the Green kernel $\Psi(t) = \langle f, e^{-t\lambda} \rangle_\lambda$ has an asymptotic expansion as $t \to 0^+$ without the $\log t$ terms. Then, from Equation (2.29) it follow that $b_j = 0$ for all j, i.e. there are no powers which belongs to $\mathbb{Z}_{<0}$ in the exponents of the Cesàro behavior of f. So that, the μ_n are the usual moments of f. Hence, Equation (2.29) is simplified to

$$\Psi(t) = \langle f, e^{-t\lambda} \rangle_\lambda \sim \sum_{k=1}^{\infty} c_k \Gamma(\alpha_k + 1) t^{-\alpha_k - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n t^n}{m!}$$

This case will be also useful in Section 3.4.

Chapter 3

The spectral action and the distributional approach

In this chapter we give an introduction to some of the main concepts of noncommutative geometry, and in the last section we show how the results studied in Chapters 1 and 2 can be applied to this context. We start with basic prerequisites in C^*-algebras to understand the motivating example, the canonical spectral triple, which describes a Riemannian spin manifold. Subsequently, we give the definition of the spectral action and calculate the heat expansion for the canonical spectral triple. Our aim is to show a distributional approach of the asymptotics of the spectral action based on the Cesàro theory developed in Chapters 1 and 2.

3.1 The beginning of a story: from C^*-algebras to Spectral Triples

We give a brief review of the theory C^*-algebras.

Definition 3.1.1. An associative algebra $\mathcal{A}$ is a

(i) **-algebra* if it is equipped with an involution, i.e. an antilinear map $(\cdot)^* : \mathcal{A} \to \mathcal{A}$ satisfying

$$\text{for all } a, b \in \mathcal{A} : \quad a^{**} = a \;\text{ and }\; (ab)^* = b^* a^*.$$

(ii) *Banach algebra* if it is a $\mathbb{C}$-algebra equipped with a norm $\|\cdot\|$ for which it is complete and satisfies

$$\text{for all } a, b \in \mathcal{A} : \quad \|ab\| \leq \|a\| \|b\|,$$

(iii) $\mathcal{A}$ is a *C^*-algebra* if it is a Banach *-algebra that satisfies

$$\text{for all } a \in \mathcal{A} : \quad \|a^* a\| \leq \|a\|^2.$$

Example 3.1.2. The algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices with the norm

$$\|A\| := \max \{ \|Ax\| : x \in \mathbb{C}^n, \; \|x\| = 1 \}$$

and equipped with the involution given by the conjugate transpose $A^* = \overline{A^t}$, is a C^*-algebra.

Example 3.1.3. Let M be a compact Hausdorff topological space and $C(M)$ the algebra of continuous complex valued functions on M equipped with pointwise multiplication

$$(\phi\psi)(x) := \phi(x)\psi(x)$$

as the algebra multiplication, complex conjugation

$$\overline{\phi}(x) := \overline{\phi(x)}$$

as the involution, and the supremum norm

$$\|\phi\| := \sup_{x\in M} |\phi(x)|.$$

Then $C(M)$ is a C^*-algebra.

Example 3.1.4. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then, for each bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ there exists a unique map $T^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle T(u), v\rangle = \langle u, T^*(v)\rangle \text{ for all } u, v \in \mathcal{H}$$

which is also a bounded linear operator. So, the map $T \to T^*$ is an involution in the algebra $\mathfrak{B}(\mathcal{H})$ of linear bounded operators on $\mathcal{H}$, and $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra. Every closed subalgebra of $\mathfrak{B}(\mathcal{H})$ is also a C^*-algebra.

Note that this last example is a generalization of Examples 3.1.2 and 3.1.3 taking $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{H} = L^2(M)$ respectively. In fact, all C^*-algebras are of the form of Example 3.1.4:

Theorem 3.1.5. (*Gelfand-Naimark-Segal*). Every C^*-algebra is $*$-isometric-isomorphic to a closed subalgebra of $\mathfrak{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

Definition 3.1.6. Let $\mathcal{A}$ be a unital commutative Banach algebra.

(i) The *Gelfand spectrum* of $\mathcal{A}$ is the set $\hat{\mathcal{A}}$ of all non-zero homomorphisms (of algebras) $\mathcal{A} \to \mathbb{C}$. This set can be regarded as a compact Hausdorff topological space.[1]

(ii) For each $x \in \mathcal{A}$, we define

$$\begin{aligned} \hat{x} : \hat{\mathcal{A}} &\to \mathbb{C} \\ \phi &\mapsto \hat{x}(\phi) := \phi(x). \end{aligned}$$

[1] To do that, we need first to consider the topological dual space $\mathcal{A}^*$ of continuous linear functionals $\psi : \mathcal{A} \to \mathbb{C}$. On $\mathcal{A}^*$ we shall consider next the so-called weak* topology, which is the topology of pointwise convergence on elements of $\mathcal{A}$. In the light of the Example 3.1.3, if $\mathcal{A} = C(M)$ then $\mathcal{A}^*$ is the space of complex measures on M with its standard topology. The Banach Alaoglu Theorem states that the unit ball $B_1(\mathcal{A}^*)$ of $\mathcal{A}^*$ is compact in the weak* topology. Then, the **Gelfand Topology** (the topology of $\hat{\mathcal{A}}$) is the relative topology determined by the inclusion $\hat{\mathcal{A}} \hookrightarrow B_1(\mathcal{A}^*)$. See [GBVF01, Section 1.2] for references.

The map

$$\begin{aligned} \Gamma : \mathcal{A} &\to C(\hat{\mathcal{A}}) \\ x &\mapsto \Gamma(x) := \hat{x} \end{aligned}$$

is the *Gelfand transformation* and is a homomorphism of Banach algebras.

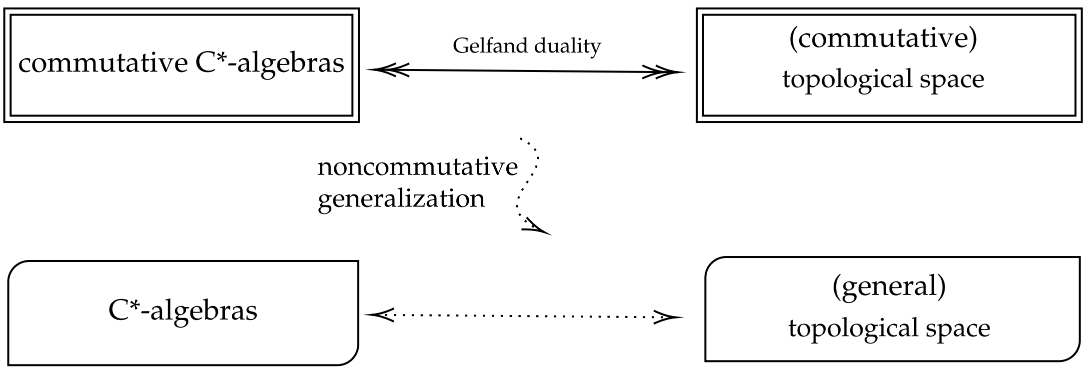
This raises the question: when is the Gelfand transformation an isomorphism? Well, that is exactly the result we are interested in:

Theorem 3.1.7. (*Gelfand-Naimark, unital case*) Every C^*-algebra is $*$-isometric-isomorphic to $C(M)$, for some compact Hausdorff topological space M. In fact, the Gelfand transformation

$$\Gamma : \mathcal{A} \to C(\hat{\mathcal{A}})$$

is a $*$-isometric-isomorphism of C^*-algebras.

Conversely, every compact Hausdorff topological space M is homeomorphic to $\widehat{C(M)}$. So, we have now a duality between unital commutative C^*-algebras and compact Hausdorff spaces, the so-called *Gelfand duality*. Theorem 3.1.7 states that studying commutative C^*-algebras, in some sense, is the same as studying "commutative" topological spaces. In the language of categories we have a categorical equivalence.



The first key idea of *noncommutative topology* is to study C^*-algebras as generalizations of topological spaces, or in other words, noncommutative analogues of such spaces. Also the main point of *Noncommutative geometry* is to extend the same idea from topology to geometry. Although this is not possible for any Riemannian manifold, one of the contributions of Alain Connes ([Con94]) is that by enriching the structure to a Riemannian *spin manifold*, this step can be achieved. The main tools for spin geometry are described in Appendix B.

In the above correspondence, a Cartesian product of topological spaces corresponds to the tensor product of the corresponding commutative C^*-algebras and viceversa. This idea can be extended to noncommutative spaces associated to noncommutative algebras.

The second concept of noncommutative geometry is that of vector bundles over differentiable manifolds. An algebraic description of these vector bundles arises naturally considering continuous sections $\Gamma(E)$ of a vector bundle $E \to M$. The sections $\Gamma(E)$ form a right module over the algebra of functions $C(M)$, but these are not enough to reconstruct vector bundles over M. The key observation is that $\Gamma(E)$ is a *finitely generated projective* module over $C(M)$, and even more:

Theorem 3.1.8. (*Serre-Swan*) The functor Γ from vector bundles over a compact space M to finitely generated projective modules over $C(M)$ is an equivalence of categories.

The third and last concept is to rewrite the notion of distance between two points in algebraic terms. The usual geodesic distance between the points x and y in a Riemannian manifold is given by

$$d(x,y) = \inf \int_\gamma ds$$

where $ds^2 = g_{ij}dx^i dx^j$ in local coordinates with Riemannian metric g and the infimum is taken over all paths γ from x to y. In the case of the real line with usual metric we have then $d(x,y) = |x-y|$, however the distance on the real line can also be expressed as a supremum taken over continuous derivable functions on $\mathbb{R}$

$$d(x,y) = \sup_{f \in C^1(\mathbb{R})} \{|f(x) - f(y)| : |f'(p)| \leq 1 \text{ for all } p \in \mathbb{R}\}. \tag{3.1}$$

The fact that one defines a distance as a supremum whereas the usual definition uses an infimum is due to the fact hat the primary objects are "observables" f rather than points x and y. The advantage of this method is that it does not use the concept of path and thus can be generalized to algebraic terms, as explained further.

First, for $f \in C^1(\mathbb{R})$ we associate a state $\omega_x(f) := f(x)$ to each $x \in \mathbb{R}$, so that a point is considered as an evaluator for observables. This is called a state in accordance with the mathematical definition of positive linear functional on an algebra, which in the above case goes from $C^1(\mathbb{R})$ to $\mathbb{R}$. We consider each function f as an operator acting by pointwise multiplication on the space of square integrable functions $L^2(\mathbb{R})$; that is to say $(f\psi)(p) = f(p)\psi(p)$ for any function $\psi \in L^2(\mathbb{R})$. Condition $|f'(p)| \leq 1$ corresponds to $\left\| \left[\frac{d}{dp}, f\right] \right\| \leq 1$ where $[,]$ denotes the commutator of operators and $\| \cdot \|$ the operator norm. Therefore, the supremum expression can be rewritten in the operator framework as

$$d(x,y) = \sup_{f \in C^1(\mathbb{R})} \left\{ |\omega_x(f) - \omega_y(f)| : \left\| \left[\frac{d}{dp}, f\right] \right\| \leq 1 \right\}.$$

Several elements introduced in Appendix B now come into play: let us consider the (canonical) Dirac operator $\not{D}$ on a Riemannian spin manifold M, let $\mathcal{H} = L^2(M,S)$ be the Hilbert space of square integrable spinors, with the algebra $\mathcal{A} = C^\infty(M)$ acting on $\mathcal{H}$ as multiplication operators by the pointwise product $(a\phi)(x) := a(x)\phi(x)$. In this context, it is shown (see [CM08a, Prop. 1.119] that the expression

$$d(x,y) = \sup_{a\in\mathcal{A}} \{|a(x) - a(y)| : \| [\not{D}, a] \| \le 1\}$$

corresponds to the geodesic distance. The metric g_{ij}, which is not explicitly present in this expression, is recovered by the distance. In other terms, $\not{D}$ encodes the metric of the Riemannian structure.

Consequently, a generalised notion of geometry is obtained from the Dirac operator instead of the metric, which motivates the key notions of noncommutative geometry: that of a **spectral triple**.

3.2 Spectral Triples

Definition 3.2.1. A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ consists of a Hilbert space $\mathcal{H}$, an involutive unital algebra $\mathcal{A}$ and an operator $\mathcal{D}$ on $\mathcal{H}$ such that

- $\mathcal{A}$ has a faithful representation as bounded operators on $\mathcal{H}$,
- $\mathcal{D} : \mathcal{H} \to \mathcal{H}$ is self-adjoint and has compact resolvent, and,
- for all $a \in \mathcal{A}$, the commutator $[\mathcal{D}, a]$ is a bounded operator on $\mathcal{H}$.

Note that we allow the algebra to be noncommutative!

Example 3.2.2. (Canonical Spectral Triple) In the case of a compact Riemannian spin manifold M without boundary with vector bundle $E \to M$ on M,

$$(C^\infty(M), L^2(M,E), \not{D})$$

is a particular example of a spectral triple (see Appendix B for details). Most of the requirements are met: $C^\infty(M)$ is involutive by Example 3.1.3, which indeed has a representation as bounded operators on $\mathcal{H}$ by Remark B.1.3. The operator $\not{D}$ is self-adjoint and $[\not{D}, a]$ is bounded by Remark B.2.2.

Example 3.2.3. The triple

$$(M_n(\mathbb{C}), M_n(\mathbb{C}), 0),$$

where $M_n(\mathbb{C})$ acts on itself by matrix multiplication, and the inner product on $M_n(\mathbb{C})$ is given by the Hilbert–Schmidt inner product:

$$\langle A, B\rangle = \mathrm{Tr}(A^* B),$$

is a spectral triple.

One can confront these two examples with Examples 3.1.3 and 3.1.2 respectively (with the difference that the algebra $M_n(\mathbb{C})$ is represented on $\mathbb{C}^n$ in Example 3.1.2 and on $M_n(\mathbb{C})$ here).

Spectral triples can be further enriched:

Definition 3.2.4. The tuple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ is an even spectral triple if there exists an operator Γ on $\mathcal{H}$ with the properties

$$\Gamma^* = \Gamma, \ \ \Gamma^2 = 1, \ \ \Gamma\mathcal{D} + \mathcal{D}\Gamma = 0, \ \ \Gamma a - a\Gamma = 0$$

for every $a \in \mathcal{A}$. If such an operator does not exist, then the triple is said to be odd.

The properties of Γ imply that $\mathcal{H}$ splits into the eigenspaces $E_\pm$ of Γ corresponding to the eigenvalues ± 1.

Example 3.2.5. If M is an n-dimensional (n is even) Riemannian spin manifold with chirality element γ (B.3), then

$$\left(C^\infty(\mathcal{M}), L^2(M, E), \not{D}, \gamma\right)$$

is an even spectral triple.

Remark 3.2.6. Spectral triples can be endowed with more structure. In [Con13], A. Connes formulated eight axioms that a certain spectral triple must meet in order to be called a noncommutative (spin) geometry. We will not cover all of them here, since some will not be needed in the present work. Instead we refer to [GBVF01, Ch. 10] for details. However, we should talk about what a *real* structure is, because it is needed in Section 3.3 in order to mention the Einstein-Yang-Mills Lagrangian.

According to the previous observation, it is important to highlight the following result. The proof is very involved, which establishes a duality between commutative spectral triples and spin manifolds.

Theorem 3.2.7. (**Connes' reconstruction theorem**) A commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfying the eight axioms mentioned in Remark 3.2.6 is isomorphic to the canonical commutative spectral triple $(C^\infty(M), L^2(M, E), \not{D})$ from Example 3.2.2 and viceversa.[2]

Consequently, we have a notion of noncommutative geometry: a noncommutative spin manifold is the object described by a noncommutative spectral triple.

Now, as it is announced earlier, we state the following definition:

Definition 3.2.8. A real structure of KO-dimension $n \in \mathbb{Z}_8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \varepsilon; \ \ J\mathcal{D} = \varepsilon'\mathcal{D}J; \ \ J\Gamma = \varepsilon'' J\Gamma \text{ (even case)}$$

[2]With spectral triple isomorphism I mean the *-algebra isomorphism $\mathcal{A} \simeq C^\infty(M)$ with the differential structure constructed on the spectrum of $\mathcal{A}$ whenever the spectral triple satisfies the eight called "axioms".

where the numbers ε, ε', ε'' are functions of n (mod 8) given by

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \quad \text{for all } a, b \in \mathcal{A}$$

where

$$b^0 = Jb^* J^{-1} \quad \text{for all } b \in \mathcal{A},$$

and the operator $\mathcal{D}$ satisfies the one-order condition

$$[[\mathcal{D}, a], b^0] = 0 \quad \text{for all } a, b \in \mathcal{A}.$$

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ endowed with a real structure J is called a real spectral triple. In particular, the canonical triple defines a real even spectral triple of KO-dimension m mod 8 with m =dim M.

Remark 3.2.9. Let us mention a few more facts about Gelfand duality. Heuristically speaking, a Cartesian product of topological spaces corresponds to the tensor product of the corresponding commutative C*-algebras and viceversa. It is therefore natural to consider **tensor product of spectral triples**, for instance, with Examples 3.2.2 and 3.2.3: let $\mathcal{A}$ be the algebra of $n \times n$ matrices with entries smooth functions on the manifold M (i.e. $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$), $\mathcal{H}$ the vector space of $n \times n$ matrices with entries square integrable spinors (i.e $\mathcal{H} = L^2(M, E) \otimes M_n(\mathbb{C})$), and $\mathcal{D}$ the Dirac operator $\not{D}$ acting on each matrix entry. Thus, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is the tensor product of the canonical spectral triple and a **finite-dimensional spectral triple** $(M_n(\mathbb{C}), M_n(\mathbb{C}), 0)$ (i.e. the Hilbert space is finite dimensional and hence the algebra is also finite dimensional).

A spectral triple which is the product of the canonical spectral triple and a finite-dimensional spectral triple, so whose noncommutative structure is finite dimensional is called an **almost commutative spectral triple**. They are interpreted to describe a (commutative) geometry, together with a finite noncommutative structure at each point.

3.3 The spectral action and the heat kernel approach of its asymptotics

The notion of action plays an essential role in physics, for instance, the Einstein-Hilbert action in gravity or the Yang-Mills-Higgs action in particle physics. The spectral action offers a generalisation in the context of spectral triples, introduced by Chamseddine and Connes in [CC97] and defined as follows:

Definition 3.3.1. The **spectral action** of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined by

$$\mathcal{S}(\mathcal{D}, f, \Lambda) := \mathrm{Tr}\left(f(\mathcal{D}/\Lambda)\right) \tag{3.2}$$

where $\Lambda \in \mathbb{R}^+$ plays the role of a cut-off parameter and f is positive even function assuming $f(\mathcal{D}/\Lambda)$ is a trace-class operator.

Example 3.3.2. Taking $f(x) = e^{-x}$, one has the celebrated heat kernel trace for an elliptic pseudodifferential operator. We will discuss this example further in the following.

3.3.1 The heat kernel expansion of the spectral action

In this subsection we use the heat kernel expansion to expand the spectral action in Λ. The details on this subject can be found in [Gil95, Sections 1.7 and 4.8].

Let M be an n-dimensional manifold without boundary and let E be a vector bundle over M. Assume that $P : \Gamma(M, E) \to \Gamma(M, E)$ is a generalised Laplacian on the bundle E i.e. P is a second order elliptic differential operator on the bundle E that locally can be written as

$$P = -(g^{\mu\nu} I \partial_\mu \partial_\nu + A^\mu \partial_\mu + B) = -\left(\sum_{i,j} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_k A_k \frac{\partial}{\partial x_k} + B\right) \tag{3.3}$$

where I is the identity matrix, A^μ (i.e. A^k) and B are endomorphisms of the bundle E[3] and g represents the metric. For differential operators as P in (3.3), we have the following expansion in $t \to 0^+$, known as the heat expansion:

$$\mathrm{Tr}\left(e^{-tP}\right) \sim_{t\to 0^+} \sum_{k\geq 0} t^{\frac{k-n}{2}} a_k(P), \tag{3.4}$$

where n is the dimension of the manifold, Tr represents the usual trace on the Hilbert space $L^2(M, E)$ and the coefficients of the expansion are given by

$$a_k(P) := \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M a_k(x, P) \sqrt{g} d^n x$$

with $\sqrt{g} d^n x$ given by the volume form, which clearly depends on the metric g. The coefficients $a_k(x, H)$ are called the **Seeley-DeWitt coefficients** and give geometric information of the manifold M (see [Gil95, Sections 1.7 and 4.8]).

In [Gil95, Lemma 4.8.1] is stated that for an operator of the same form as P in Equation (3.3), there is a unique connection ∇ on E and a unique endomorphism F of the bundle E such that

$$P = \nabla^* \nabla - F \tag{3.5}$$

where $\nabla^* \nabla$ is the connection Laplacian of the connection ∇. We have

[3] See [Gil95, p. 314].

Theorem 3.3.3. ([Gil95, Theorem 4.8.16]) For P of the form (3.5), the Seeley-DeWitt coefficients are

$$\begin{aligned} a_k(x,P) &= 0 \text{ for } k \text{ odd}, \\ a_0(x,P) &= \frac{1}{(4\pi)^{n/2}} \mathrm{Tr}_{E_x}(\mathrm{Id}), \\ a_2(x,P) &= \frac{1}{(4\pi)^{n/2}} \mathrm{Tr}_{E_x}\left(-\frac{R}{6}\mathrm{Id} + F\right), \\ a_4(x,P) &= \frac{1}{360(4\pi)^{n/2}} \mathrm{Tr}_{E_x}(-12R^{\mu}_{;\mu} + 5R^2 - 2R^{\mu\nu}_{\mu\nu} \\ &+ 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RF + 180F^2 + 60F^{\mu}_{;\mu} + 30\Omega_{\mu\nu}\Omega^{\mu\nu}), \\ &\vdots \end{aligned}$$

where $\Omega_{\mu\nu}$ is the curvature of the differential operator ∇.

We continue now with the most important result of this section:

Theorem 3.3.4. (**Heat expansion of the spectral action**) If $\mathcal{D}^2$ is of the form (3.3), and f admits a representation as a Laplace transform of another function, then the spectral action (3.2) can be expanded as

$$\begin{aligned} \mathrm{Tr}\, f\left(\frac{\mathcal{D}}{\Lambda}\right) \sim_{\Lambda\to\infty} & \sum_{\substack{k=0 \\ k \text{ even}}}^{n-2} \Lambda^{n-k} a_k(D^2) \frac{2f_{n-k}}{\Gamma(\frac{n-k}{2})} + a_n(D^2) f(0) \\ + & \sum_{\substack{k>n \\ k \text{ even}}}^{\infty} (-1)^{\frac{k-n}{2}} \frac{(\frac{k-n}{2})!}{(k-n)!} f^{(k-n)}(0) \Lambda^{n-k} a_k(\mathcal{D}^2) \end{aligned} \tag{3.6}$$

where $f_j = \int_0^\infty f(v) v^{j-1} dv$ are the *moments* of the function f for $j > 0$, and a_k as in Equation (3.4).

Proof. This proof is based on [CM08b, Theorem 1.145] with some reformulation. Since the function f is even, there exists a function g such that

$$f(u) = g(u^2). \tag{3.7}$$

Writing this function as a Laplace transform of a function h

$$g(u) = \int_0^\infty e^{-su} h(s) ds \tag{3.8}$$

and therefore, the functional calculus allows us to formally write

$$g(t\mathcal{D}^2) = \int_0^\infty e^{-st\mathcal{D}^2} h(s) ds$$

with the parameter t being considered to be a small expansion parameter (that is $t \to 0^+$). The heat-kernel expansion of the operator $\mathcal{D}^2/\Lambda^2$ (where we take $t = s\Lambda^{-2}$ to be the expansion variable) is

$$\mathrm{Tr}\, e^{-s\mathcal{D}^2/\Lambda^2} \sim_{\Lambda\to\infty} \sum_{k=0}^{\infty} s^{\frac{k-n}{2}} \Lambda^{n-k} a_k(\mathcal{D}^2). \tag{3.9}$$

Formally interchanging the trace, the integral and the infinite sum, we obtain the spectral action expanded as

$$\begin{aligned} \mathrm{Tr}\, f\left(\frac{\mathcal{D}}{\Lambda}\right) &= \int_0^\infty \mathrm{Tr}\, e^{-s\mathcal{D}^2/\Lambda^2} h(s)ds \\ &\sim \sum_{\substack{k\geq 0 \\ k \text{ even}}} \Lambda^{n-k} a_k(\mathcal{D}^2) \int_0^\infty s^{\frac{k-n}{2}} h(s)ds \quad \text{as } \Lambda \to \infty \end{aligned}$$

since $a_k(\mathcal{D}^2) = 0$ for odd k. We must then calculate the integral in three different cases:

(i) For $k = n$ we use

$$\int_0^\infty h(s)ds = \lim_{v\to 0} \int_0^\infty e^{-sv} h(s)ds = f(0).$$

(ii) For even $k < n$, we use the fact that for $\alpha < 0$ one has

$$s^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-sv} v^{-\alpha-1} dv \tag{3.10}$$

via the *Mellin transform*. Taking $\alpha = \frac{k-n}{2}$, multiplying both sides of (3.10) by $h(s)$ and integrating over s then gives

$$\begin{aligned} \int_0^\infty s^{\frac{k-n}{2}} h(s)ds &= \frac{1}{\Gamma(\frac{n-k}{2})} \int_0^\infty v^{\frac{n-k}{2}-1} \left(\int_0^\infty s^{-sv} h(s)ds \right) dv \\ &= \frac{1}{\Gamma(\frac{n-k}{2})} \int_0^\infty v^{\frac{n-k}{2}-1} g(v)dv \\ &= \frac{2}{\Gamma(\frac{n-k}{2})} \int_0^\infty u^{n-k-1} f(u)du \\ &= \frac{2}{\Gamma(\frac{n-k}{2})} f_{n-k} \end{aligned}$$

using (3.7) and (3.8), substituting $v = u^2$ and setting $f_j := \int_0^\infty f(u)u^{j-1}du$.

(iii) Finally, for even $k > n$

$$\begin{aligned} \int_0^\infty s^{\frac{k-n}{2}} h(s)ds &= (-1)^{\frac{k-n}{2}} \int_0^\infty (-s)^{\frac{k-n}{2}} h(s)ds \\ &= (-1)^{\frac{k-n}{2}} g^{(\frac{k-n}{2})}(0) \\ &= (-1)^{\frac{k-n}{2}} \frac{(\frac{k-n}{2})!}{(k-n)!} f^{(k-n)}(0) \end{aligned}$$

using that for even α

$$f^{(\alpha)}(0) = \frac{\alpha!}{(\frac{\alpha}{2})!} g^{(\alpha/2)(0)}.$$

So finally we have (3.6). □

Remark 3.3.5. The Einstein-Yang-Mills system illustrates an application of the spectral action. This is the simplest possible noncommutative modification of an ordinary 4-dimensional manifold, namely the almost noncommutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ obtained by the tensor product of spectral triples in Remark (3.2.9). Instead of the operator $\mathcal{D}$, we consider the *perturbed Dirac operator* $\mathcal{D}_A = \mathcal{D} + A$ with self-adjoint

$$A \in \left\{ \sum_j a_j[\mathcal{D}, b_j] : a_j, b_j \in \mathcal{A} \right\}$$

(where the sums are finite). Chamseddine and Connes ([CC97]) were the first ones to construct this triple and to calculate the spectral action, and showed that one gets, among other terms, the Yang-Mills Lagrangian for an $SU(n)$-gauge theory (see [CM08a, Section 11.4]). Thus, all the information needed to derive the Lagrangian ($\mathcal{L}(g, A)$), actually comes from the spectral triple. The manifold M plays the role of space-time, so taking $n = 4$ in the asymptotic expansion and ignoring the $\mathcal{O}(\Lambda^{-2})$ terms we have:

$$\begin{aligned} \operatorname{Tr} f\left(\frac{\mathcal{D}}{\Lambda}\right) &= 2f_4\Lambda^4 a_0(\mathcal{D}^2) + 2f_2\Lambda^2 a_2(\mathcal{D}^2) \\ &+ a_4(D^2)f(0) + \mathcal{O}(\Lambda^{-2}) \\ &= \frac{1}{4\pi^2}\int_M \mathcal{L}(g, A)\sqrt{g}d^4x + \mathcal{O}(\Lambda^{-2}) + \text{t.t.} \end{aligned}$$

where $\mathcal{L}$ is the Lagrangian and "t.t." represents a topological term: proportional to the Euler characteristic of M. Note that the second term $2f_2\Lambda^2 a_2(\mathcal{D}^2)$ recovers the Einstein-Hilbert action since it is proportional to $a_2(\mathcal{D}^2)$ in Theorem 3.3.3.

3.4 Cesàro asymptotics of the spectral action

The asymptotic behavior (3.6) was proved for a smooth function f which is a Laplace transform of another function h in Equation (3.8). In this section, we use the Cesàro tools developed in Chapters 1 and 2. In Section 2.4, we discussed the small-t behavior of Green functions associated to a tempered distribution $f(\lambda)$. We use here similar functions associated to an elliptic pseudodiferencial operator P as announced in Remark 1.4.12. These are integral kernels of operator-valued functions of P, of the form

$$\Psi(t, x, y) = \langle d_P(x, y; \lambda), \phi(t\lambda) \rangle_\lambda,$$

where ϕ belongs to the space $\mathcal{S}(\mathbb{R})$. From Theorem 2.3.11, we know that a Cesàro asymptotic expansion holds for the spectral density kernel $d_P(x,x;\lambda)$ similar to the one of Assumption 2.4.4. Then $\Psi(t,x,x)$ has an ordinary asymptotic expansion as $t \to 0^+$, and integrating this kernel with respect to x, yields an asymptotic expansion for the spectral action. We consider in this case not the assumption of f being a Laplace transform but ϕ being a rapid decay test function.

Let us start with the following remark:

Remark 3.4.1. According to [Zem66, Theorem 9], a function ϕ is a Laplace transform of a distribution in $\mathcal{S}'_+$ (that is, tempered distributions supported on $(0,\infty)$) if, and only if ϕ is analytic in the right half complex plane and there exists a polynomial p with $|\phi(s)| \leq p(|s|)$ for any s in the right half complex plane. It would advantageous to use the Laplace transform to deduce the behavior of $\mathrm{Tr}e^{-tP^2}$ from that of $\mathrm{Tr}e^{-tP}$ as we did in the previous section for the spectral action. However, the function $\phi(x) = e^{-x^2}$ does not meet the required bound because $|\phi(x+iy)| = e^{-x^2+y^2}$. Therefore, the inverse Laplace transform does not exist even as a distribution.

Hence, the Laplace transform assumption is too restrictive for our means since it does not cover, for instance, the heat kernel case where $\phi(x) = e^{-x^2}$. The important point here is that one can generalise asymptotics for the spectral action in the Cesàro sense for $\phi \in \mathcal{S}$. By Remark 1.4.2, $\phi(x) = e^{-x^2} \in \mathcal{S}$.

We have arrived at the theorem that gives its name to this work.

Theorem 3.4.2. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be the canonical spectral triple with $\dim M = 4$. For the generalised Laplacian $P = \mathcal{D}^2$ we get that

$$d_P(x,x;\lambda) \sim_{\lambda\to\infty} c_0\lambda + c_1(x) \ \ (\mathrm{C})$$

for some coefficients $c_0, c_1 \in C^\infty(\mathbb{R})$. Given $\phi \in \mathcal{S}(\mathbb{R})$ we get then

$$\mathrm{Tr}\ \phi(\mathcal{D}^2/\Lambda^2) \sim_{\Lambda\to\infty} c_0\Lambda^4 \int_0^\infty d\lambda\ \lambda\phi(\lambda) \ + \ c_2\Lambda^2 \int_0^\infty d\lambda\phi(\lambda) \ + \ \sum_{n=0}^{\infty} (-1)^n \phi^{(n)}(0) c_{2n+4}\Lambda^{-2n}.$$

Proof. Because of the form of the heat kernel expansion in Equation (3.4), we have that for a generalized Laplacian it is fullfilled precisely the same case that the one mentioned in Remark 2.4.8. If k is odd, integer multiples of $1/2$ will appear as powers of λ in the spectral density, so logarithmic terms in the heat kernel expansion are discarded. By Theorem 2.3.11,

$$d_{\mathcal{D}^2}(x,x;\lambda) \sim_{\lambda\to\infty} \sum_{n=0}^{\infty} a_{2n}(x)\lambda^{1-n} \ \ (\mathrm{C}). \tag{3.11}$$

Given $\phi \in \mathcal{S}$, note that, in the Cesàro sense

$$
\begin{aligned}
&\int_M \int_0^\infty d_{\mathcal{D}^2}(x,x;\lambda)\phi(\lambda\Lambda^{-2})d\lambda dx \\
&= \frac{1}{2(2\pi)^4}\int_M \int_0^\infty \left(\sum_{n=0}^{\infty} a_{2n}(x)\lambda^{1-n}\right)\phi(\lambda\Lambda^{-2})d\lambda dx \\
&= \frac{1}{2(2\pi)^4}\int_M \int_0^\infty (a_0(x)\lambda\phi(\lambda\Lambda^{-2}) + a_2(x)\phi(\lambda\Lambda^{-2}) + \cdots)d\lambda dx \\
&= \frac{\Lambda^2}{2(2\pi)^4}\int_M \int_0^\infty (a_0(x)\nu\Lambda^2\phi(\nu) + a_2(x)\phi(\nu) + \cdots)d\nu dx \\
&= \frac{\Lambda^2}{2(2\pi)^4}\int_M \int_0^\infty a_0(x)\nu\Lambda^2\phi(\nu)d\nu dx \\
&\qquad + \frac{\Lambda^2}{2(2\pi)^4}\int_M \int_0^\infty a_2(x)\phi(\nu)d\nu dx \cdots \\
&= \frac{\Lambda^4}{2(2\pi)^4}\int_M a_0(x)dx \int_0^\infty \lambda\phi(\lambda)d\lambda \\
&\qquad + \frac{\Lambda^2}{2(2\pi)^4}\int_M a_2(x)dx \int_0^\infty \phi(\lambda)d\lambda \\
&\qquad\quad + \sum_{k=0}^{\infty}\frac{\Lambda^{-2k}}{2(2\pi)^4}\int_M a_{2k+4}(x)dx \int_0^\infty \lambda^{-k}\phi(\lambda)d\lambda \cdots
\end{aligned}
$$

where in the third equality, we made the substitution $\lambda = \Lambda^2\nu$. Since D^{2d} is a differential operator, its local Wodzicki residue (defined in Appendix A) vanishes for $d \in \mathbb{N}$, so that the Cesàro development (3.11) ends up at the term with λ^0. Then,

$$
\begin{aligned}
\int_M \int_0^\infty d_{\mathcal{D}^2}(x,x;\lambda)\phi(\lambda\Lambda^{-2})d\lambda dx \quad &= \quad \frac{\Lambda^4}{2(2\pi)^4}\int_M a_0(x)dx \int_0^\infty \lambda\phi(\lambda)d\lambda \\
&\qquad + \frac{\Lambda^2}{2(2\pi)^4}\int_M a_2(x)dx \int_0^\infty \phi(\lambda)d\lambda.
\end{aligned}
$$

We plug this expression in Formula (2.29) of Theorem 2.4.7 taking $t = \Lambda^{-2}$ to obtain for the spectral action ($S(P,\phi,\lambda)$):

$$
\begin{aligned}
S(P,\phi,\lambda) \quad &= \quad \mathrm{Tr}\,\phi\left(\frac{\mathcal{D}^2}{\Lambda^2}\right) = \int_M \langle d_{\mathcal{D}^2}(x,x;\lambda), \phi(\lambda\Lambda^{-2})\rangle_\lambda dx \\
&\sim_{\Lambda\to\infty} \quad c_0\Lambda^4\int_0^\infty \lambda\phi(\lambda)d\lambda + c_2\Lambda^2\int_0^\infty \phi(\lambda)d\lambda \\
&\qquad + \sum_{m=0}^{\infty}\frac{\mu_m\phi^{(m)}(0)}{m!}\Lambda^{-2m} \\
&= \quad c_0\Lambda^4\int_0^\infty \lambda\phi(\lambda)d\lambda + c_2\Lambda^2\int_0^\infty \phi(\lambda)d\lambda \\
&\qquad + \sum_{m=0}^{\infty}(-1)^m c_{2m+4}\phi^{(m)}(0)\Lambda^{-2m}
\end{aligned}
$$

by choosing c_{2m+4} properly. $\square$

Chapter 4

Conclusions and open questions

The aim of this work was to provide rigorous proofs for some of the formal results included in [EGBV98], particularly in the presentation of the Cesàro theory of distributions and its application to the distributional asymptotic expansion of the spectral action.

To achieve this, we have introduced in Chapter 1 some useful and basic preliminaries in the theory of distributions, innovating with the inclusion of primitives of distributions, the Cesàro theory of distributions, the space $\mathcal{K}$ of Grossman-Loupias-Stein symbols and its relationship to the above. Our merit in that process is the original proofs we have developed and many of the calculations omitted in the literature.

In Chapter 2, we have brought together results of some papers which work on asymptotic analysis for spectral densities, in order to give a sturdy support for applications we are looking for in noncommutative geometry.

Finally, in Chapter 3, we briefly introduce concepts of noncommutative geometry to define what the spectral action is. Part of our work was the interpretation of the proof of the asymptotic expansion of the spectral action with the heat-kernel expansion, given by Connes and Marcolli in the language of noncommutative integration [CM08b], into the standard language of the analysis on manifolds. We give also some calculations omitted in the original article of [EGBV98].

Here there are some of the questions that remained open after the elaboration of this work:

1. Although Theorem 1.4.10 is already proved, its formal use in Chapter 2 is not rigourously justified (see equations (2.9) and (2.10)), since it was previously proved for real-valued distributions and not with operator-valued distributions. About the conditions that should be assumed in order to have a rigorous proof of this fact, it is still unknown to us.

2. After conversations with professors Ricardo Estrada, Bruno Iochum and Joseph Várilly about the formulation of Ansatz 2.3.10, we have concluded that there is no proof of such expansion. Professor Estrada suggests that it is possible to impose some conditions on a differential

operator, so that its spectral density has the form of Expansion (2.20). We still do not see them.

3. While the heat-kernel asymptotic expansion of the spectral action contains coefficients with a useful meaning in physics, in the coefficients obtained in the distributional expansion the physical meaning is still unknown to us.

4. We saw that the heat-kernel asymptotic expansion of the spectral action was well suited not only for the canonical spectral triple, but also for the almost commutative spectral triples (see Remark 3.2.9), which are the noncommutative example closest to the commutative case. One would like to be able to apply the distributional expansion of the Spectral Action (see 3.4.2) to the noncommutative case. To this end, one would need to reformulate some of the results of Chapter 2 appropriately, and to know in which cases those are valid. This is a very challenging problem to address.

Appendix A

Pseudodifferential Operators and Wodzicki's Residue

In this appendix we want to give a brief presentation of the basic tools in classical pseudodifferential operators in order to define the Wodzicki residue by using local terms of the expansion of its symbol (integral of local expressions). We follow [Gru09] and [Fed+96].

A.1 Classical elliptic pseudodifferential operators

Let U be an open subset of $\mathbb{R}^n$. Given $a \in \mathbb{C}$ we denote by $S^a(U)$ the set of complex valued smooth functions

$$\begin{array}{rcl} \sigma : U \times \mathbb{R}^n & \to & \mathbb{C} \\ (x, \xi) & \mapsto & \sigma(x, \xi) \end{array}$$

satisfying the following growth condition: given any compact subset K of U and any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ in $\mathbb{N}^n$, there exists a constant $C_{K,\alpha,\beta}$ such that

$$\|D_x^\alpha D_\xi^\beta \sigma(x, \xi)\| \leq C_{K,\alpha,\beta}(1 + \|\xi\|)^{\mathrm{Re}(a) - |\beta|}, \tag{A.1}$$

for all $x \in K$, $\xi \in \mathbb{R}^n$ where $\mathrm{Re}(a)$ is the real part of a, $\| \cdot \|$ the norm in $\mathbb{R}^n$ and $|\beta| = \beta_1 + \cdots + \beta_n$. An element belonging to $S^a(U)$ is said to be a **symbol of order** a on U.

The space $S^a(U)$ is a Fréchet space with the seminorms defined as the least constants entering in (A.1) for each choice of α, β and K, and clearly

$$S^a(U) \subseteq S^{a'}(U) \quad \text{for all } a' \geq a.$$

A **smoothing symbol** is an element of the set

$$S^{-\infty}(U) = \bigcap_{m \in \mathbb{N}} S^{-m}(U)$$

and the relation

$$\sigma \sim \overline{\sigma} \quad \text{if, and only if,} \quad \sigma - \overline{\sigma} \in S^{-\infty}(U)$$

defines an equivalence relation on $S^a(U)$.

The **leading** part of the symbol $\sigma \in S^a(U)$ is defined by

$$\sigma_a(x,\xi) := \lim_{t\to\infty} \frac{\sigma(x,t\xi)}{t^a}$$

whenever it exists. A symbol of order a is a **classical symbol** if there exist $\sigma_{a-j} \in S^{a-j}(U)$, $j \in \mathbb{N}$, such that

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{a-j}(x,\xi)$$

meaning that, for each N, $\sigma(x,\xi) - \sum_{j=0}^{N} \sigma_{a-j}(x,\xi) \in S^{a-N-1}$, and

$$\sigma_{a-j}(x,t\xi) = t^{a-j}\sigma_{a-j}(x,\xi) \quad \text{for} \quad |\xi|, |t| \geq 1.$$

To each symbol $\sigma \in S^a(U)$ one can associate a **pseudodifferential operator** P of order a on U, i.e. a map

$$\begin{aligned} P : C_o^\infty(U) &\to C^\infty(U) \\ u &\mapsto Pu \quad \text{with} \quad Pu(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{u}(\xi)d\xi, \end{aligned} \tag{A.2}$$

where $\cdot$ denote the inner product in $\mathbb{R}^n$ and $\hat{u}$ is the Fourier transform of the complex valued smooth function u with compact support in U. Using the definition of the Fourier transform, Pu can be written as

$$Pu(x) = \int_{\mathbb{R}^n \times U} e^{i(x-y)\cdot\xi}\sigma(x,\xi)u(y)dyd\xi \tag{A.3}$$

We can induce different types of pseudodifferential operators from the various classes of symbols arising in Formula (A.2). For instance, a **classical pseudodifferential operator** is a pseudodifferential operator whose symbol is a classical symbol and a **smoothing pseudodifferential operator** is a pseudodifferential operator given by a smoothing symbol. A smoothing operator can be represented by a smooth kernel, i.e. P is smoothing if and only if there exists $K \in C^\infty(U \times U)$ such that for any $u \in C_o^\infty(U)$:

$$Pu(x) = \int_U K(x,y)u(y)dy \quad \text{for all } x \in U.$$

Also, a pseudodifferential operator P of order a is said to be **elliptic** if its leading symbol σ_a is such that $\sigma_a(x,\xi) \neq 0$ for all $(x,\xi) \in U \times \mathbb{R}^n \setminus \{0\}$. Note that the ellipticity is not affected by adding a smoothing symbol to the symbol of an elliptic operator.

Remark A.1.1. [Gru09, Section 8.2] The notion of a pseudodifferential operator can be locally transferred to smooth manifolds in the following way: let M be a closed oriented Riemannian manifold of dimension n, then $P : C^\infty(M) \to C^\infty(M)$ is a pseudodifferential operator of order d on M, if for any local chart (U,ϕ) of M such that $\phi : U \to V$ is a diffeomorphism of U with an open set V in $\mathbb{R}^n$, the operator $P|_U$ defined by the diagram

$$\begin{array}{ccc} C_o^\infty(U) & \xrightarrow{P} & C^\infty(U) \\ \uparrow \phi^* & & \phi^* \uparrow \\ C_o^\infty(V) & \xrightarrow{P|_U} & C^\infty(V) \end{array}$$

is a pseudo-differential operator of order d on V.

Following [Fed+96], let us denote by $\Psi^d(M)$ the space of all classical pseudo-differential operators of order d on M. Thus, a smoothing pseudodifferential operator on M is an operator in

$$\Psi^{-\infty}(M) := \bigcap_{m\in\mathbb{N}} \Psi^{-m}(M),$$

and there is a short exact sequence

$$0 \longrightarrow \Psi^{-\infty}(M) \longrightarrow \Psi^m(M) \longrightarrow S^m(M) \longrightarrow 0\,.$$

There is a notion of the product of two pseudodifferential operators (see, for instance, [Gru09] Theorem 7.13) and the space of all classical pseudodifferential operators on M with integer order,

$$\Psi^\infty(M) := \bigcup_{m\in\mathbb{Z}} \Psi^m(M)$$

is an associative algebra. Let U be an arbitrary local coordinate chart of M. Let P, Q be pseudodifferential operators on U, with total symbols $\sigma(P), \sigma(Q)$, respectively. Then we have

$$\sigma(PQ) - \sigma(P)\sigma(Q) \sim \sum_{|\alpha|>0} \frac{1}{\alpha!} D_\xi^\alpha \sigma(P)\partial_x^\alpha \sigma(Q) \tag{A.4}$$

according to the definition of the product of symbols in open sets of $\mathbb{R}^n$ (see [Gru09, Theorem 7.13]). From this, it could be inferred that the product of two elliptic pseudodifferential operators is an elliptic pseudodifferential operator.

A.2 Wodzicki residue

Again let M be a closed Riemannian manifold of dimension n and U be an arbitrary local coordinate chart. When we consider the form n-differential form

$$\mathrm{d}^n x = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

it induces an orientation of U and also induce an orientation for $\mathbb{R}^n$ given by

$$d\xi = d\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_n.$$

The main result of this appendix given without proof (see [Fed+96] Theorem 1.4) is the following:

Theorem A.2.1. (**Wodzicki**) For $P \in \Psi^\infty(M)$, the local density $\mathrm{wres}_x P$ defined by the local formula

$$\mathrm{wres}_x P := \left(\int_{|\xi|=1} \sigma_{-n}(P)(x,\xi) d\xi \right) \mathrm{d}^n x \quad \text{at } x \in M$$

where $\sigma_{-n}(P)(x,\xi)$ denotes the homogeneous part of degree $-n$ in the asymptotic expansion of the total symbol of P over $\{(x,\xi) : |\xi| = 1\}$, is independent of the local representation of the symbol of P, so that

$$\mathrm{WRes} P = \int_M \mathrm{wres}_x P \tag{A.5}$$

is well-defined, and is called the **Wodzicki residue** of the operator P.

Remark A.2.2. Among the many remarkable properties of the Wodzicki residue, relevant in this work are its **traciality** on $\Psi^\infty(M)$, i.e. $\mathrm{WRes}([P,Q]) = 0$ for any $P, Q \in \Psi^\infty(M)$[1], **locality**, i.e. it can be described as an integral of local expressions involving the symbol of the operator, and **trivial in finite dimensions**, i.e. if $P \in \Psi^\infty(M)$ is of finite rank or its order is less than $-n$, then $\mathrm{WRes} P = 0$.

[1] $[P,Q]$ denotes the commutator operator $PQ - QP$.

Appendix B

Clifford Algebras and the Dirac operator

B.1 Clifford Algebras

This appendix is a brief introduction to spin geometry. The field of this topic is vast and intricate. We will merely touch upon some of the most important concepts and results without proofs. For proofs and more background see [BGV03; LM89; GBVF01].

Definition B.1.1. Let V be a finite-dimensional $\mathbb{R}$-vector space and g be a quadratic form on V. The real Clifford algebra $\mathrm{Cl}(V,g)$ is the algebra generated by elements of V under multiplication $\cdot$ (Clifford multiplication), with the condition

$$v \cdot w + w \cdot v = 2g(v,w) \quad \text{for all} \;\; v,w \in V.$$

We can extend the definition of a Clifford algebra to a vector space over the complex numbers by *complexification*: for a real vector space V we take

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \simeq V \oplus iV.$$

Then we can extend a given bilinear symmetric map $g : V \times V \to \mathbb{R}$ to $g^{\mathbb{C}} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ by

$$g^{\mathbb{C}}(v_1 + iv_2, v_3 + iv_4) := g(v_1, v_3) - g(v_2, v_4) + i(g(v_2, v_3) + g(v_1, v_4)).$$

So that, using $g^{\mathbb{C}}$, we can in a similar fashion as for a real vector space, construct a Clifford algebra $\mathrm{Cl}(V^{\mathbb{C}}, g^{\mathbb{C}})$. In this case, we write $\mathbb{C}\mathrm{l}(V)$ for short.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for V (such that $e_i e_j = -e_j e_i$ and $e_i^2 = 1$ for all i,j). Then the products of the form $e_{i_1} \cdots e_{i_k}$ with $k \in \mathbb{N}$, $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, form a basis of $\mathbb{C}\mathrm{l}(V,g)$. It means that any element u of the Clifford algebra $\mathbb{C}\mathrm{l}(V,g)$ can be written as $u = \sum_I u_I e_I$ where $e_I := e_{i_1} \cdots e_{i_k}$ for all strictly ordered sets $I = \{i_1 < \cdots < i_k\} \subseteq \{1,\ldots,n\}$ and $u_I \in \mathbb{C}$ for each I. With this notation, we define an inner product $\langle .,. \rangle$:

$\mathbb{C}\mathrm{l}(V) \times \mathbb{C}\mathrm{l}(V) \to \mathbb{C}$ by

$$\langle u, v \rangle := \sum_I \overline{u_I} v_I. \tag{B.1}$$

In that same notation, we can make $\mathbb{C}\mathrm{l}(V)$ involutive, by defining for any $u = \sum_I u_I e_I \in \mathbb{C}\mathrm{l}(V)$:

$$u^* := \sum_I \overline{u_I} e_I^!, \tag{B.2}$$

where $e_I^!$ represents the total reversal of the order of the elements appearing in e_I.

For each Clifford algebra $\mathbb{C}\mathrm{l}(V)$ we can define a *Chirality element* γ that is given in terms of an orthonormal basis $\{e_1, ..., e_n\}$ of V by

$$\gamma := (-i)^m e_1 e_2 \cdots e_n \tag{B.3}$$

where $n = 2m$ or $n = 2m + 1$ according to whether n is even or odd. This definition guarantees that $\gamma^* = \gamma$ and $\gamma^2 = 1$.

Let us present an example of the preceding concepts. Over the course of this appendix (M, g) will denote a compact Riemannian manifold with metric g.

Example B.1.2. Consider T_pM the tangent space of M at a point p. For a chart $\{(U, x)\}$ and $p \in U$ we define $\frac{\partial}{\partial x_i}\big|_p$ by

$$\frac{\partial}{\partial x^i}\Big|_p(f) := D_i(f \circ x^{-1})\big|_{x(p)}$$

for each $f \in C^\infty(M)$, where D_i represents the i-th derivative of a map from $\mathbb{R}^n$ to $\mathbb{R}$. Since

$$\left\{ \frac{\partial}{\partial x^i}\Big|_p : i = 1, \ldots, n \right\} \quad \text{is a basis for } T_pM$$

whose dual vectors $\{\mathrm{d}x^i\big|_p : i = 1, \ldots, n\}$ defined by

$$\mathrm{d}x^i\big|_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \frac{\partial}{\partial x^j}\Big|_p x^i = D_j(x^i \circ x^{-1})\big|_{x(p)}$$

i.e. $\mathrm{d}x^i\big|_p \left(\frac{\partial}{\partial x^j}\big|_p \right) = \delta_{i,j}$ with $\delta_{i,j}$ being the Kronecker delta function, form a basis of T_p^*M. We can therefore construct a Clifford algebra $\mathbb{C}\mathrm{l}(T_p^*M)$ generated by the complexifications of $\mathrm{d}x^\nu|_p$ modulo the Clifford algebra relation featuring the complexification of the symmetric bilinear form using the complexification of $g^{\mu\nu}(p) := g(\mathrm{d}x^\mu|_p, \mathrm{d}x^\nu|_p)$.

Thus we can associate a complex Clifford algebra to each point of the manifold. We write $\Gamma(M, \mathbb{C}\mathrm{l}(T^*M))$ (or just $\Gamma(\mathbb{C}\mathrm{l}(T^*M))$) for the space of "Clifford algebra-valued functions", that is for an element $\alpha \in \Gamma(\mathbb{C}\mathrm{l}(T^*M))$ we have $\alpha(x) \in \Gamma(\mathbb{C}\mathrm{l}(T_x^*M))$ for each $x \in M$. Note that, as the Clifford algebra itself, $\Gamma(\mathbb{C}l(T^*M))$ is an algebra by "pointwise multiplication": for $\alpha, \beta \in \Gamma(\mathbb{C}l(T^*M))$

$$(\alpha\beta)(x) := \alpha(x)\beta(x) \in \mathbb{C}l(T_x^*M)$$

i.e. $\alpha\beta \in \Gamma(\mathbb{C}l(T^*M))$ as well. In a similar way we can make $\Gamma(\mathbb{C}l(T^*M))$ involutive ($\alpha^*(x) := \alpha(x)^*$, where $\alpha(x)^*$ is given by Equation (B.2)) and it can be endowed with a Hermitian pairing

$$\begin{array}{rcl}\langle\cdot,\cdot\rangle : \Gamma(\mathbb{C}l(T^*M)) \times \Gamma(\mathbb{C}l(T^*M)) & \to & C^\infty(M) \\ (\alpha,\beta)(x) & \mapsto & \langle\alpha(x),\beta(x)\rangle\end{array}$$

where $\langle\cdot,\cdot\rangle$ is the inner product (B.1) on the Clifford algebra. Last, we introduce a chirality element γ on $\Gamma(\mathbb{C}l(T^*M))$ by

$$\gamma(x) := \gamma$$

where with right hand side represents the chirality element γ on each Clifford algebra (cf. (B.3)).

We introduce now the space of "spinor-valued functions" as follows: consider the space of sections $\Gamma(M,E)$ of a hermitian vector bundle $E \to M$ which is a left $C(M)$-module i.e.,

$$(f\psi)(x) := f(x)\psi(x) \in E_x \quad \text{for all } f \in C(M),\ \psi \in \Gamma(M,E)$$

where E_x is the fiber of x in E. From the compactness of M we infer that elements of $C(M)$ act as multiplication of bounded operators on $\Gamma(M,E)$. We will write $\Gamma(E)$ for $\Gamma(M,E)$. We assume E_x is a representation of the so called spin^c - group of $\mathbb{C}l(T_x^*M)$ that consists of elements that are a product of an even number of unitary elements of $\mathbb{C}l(T_x^*M)$. If the dimension of M is even, E_x splits in two irreducible representations of equal dimension (eigenspaces of the chirality element γ with eigenvalues ± 1) that are denoted by E_x^+ and E_x^- respectively.

This $C(M)$-module of spinor-valued functions can be equipped with a Hermitian pairing

$$\begin{array}{rcl}(\cdot,\cdot) : \Gamma(E) \times \Gamma(E) & \to & C(M) \\ (\alpha,\beta) & \mapsto & (\alpha,\beta)(x) := \langle\alpha,\beta\rangle_{E_x}\end{array}$$

where $\langle\cdot,\cdot\rangle_{E_x} : E_x \times E_x \to \mathbb{C}$ means the complex inner product on the spinor representation (i.e. on the fiber E_x).

If there exists a $C(M)$-linear algebra homomorphism

$$c : \Gamma(\mathbb{C}l(T^*M)) \to \mathrm{End}_{C(M)}(\Gamma(E))$$

(the **spin homomorphism**), the elements $\gamma^\mu := c(\mathrm{d}x^\mu)$ act on the spinor-valued functions $\Gamma(E)$ by $(\gamma^\mu\psi)(x) := \gamma^\mu(x)\psi(x)$ where this latter term is just matrix multiplication.

Thus $\Gamma(E)$ is not only a left $C(M)$-module, but (via the homomorphism c) a left $\Gamma(\mathbb{C}l(T^*M))$-module as well. These two module structures are compatible, for elements of the two different algebras commute with each other.

As from here on we will work in the smooth category, yet we write $C(M)$, $\Gamma(E)$, $\Gamma(\mathbb{C}l(T^*M))$ instead of $C^\infty(M)$, $\Gamma^\infty(E)$, $\Gamma^\infty(\mathbb{C}l(T^*M))$ respectively, for the sake of simplicity.

Remark B.1.3. On the space of smooth spinor-valued functions $\Gamma(E)$ we can define an inner product

$$\begin{aligned} \langle\cdot,\cdot\rangle : \Gamma(E)\times\Gamma(E) &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto \langle\alpha,\beta\rangle := \int_M (\alpha,\beta)(x)\sqrt{g}\mathrm{d}^n x = \int_M \langle\alpha,\beta\rangle_{E_x}\sqrt{g}\mathrm{d}^n x. \end{aligned}$$

With the completion of $\Gamma(E)$ with respect to this inner product, we get the Hilbert space $L^2(M,E)$ which is the **space of square integrable spinors** on $E\to M$.

B.2 Dirac Operator

There is a certain type of manifolds that allows the construction of spinor-valued functions, these are called **spin manifolds**. We shall not go into what the requirements are for such a spin manifold to exist. We do mention that an n-dimensional compact Riemannian manifold is indeed a spin manifold as long as it is orientable, the *second Stiefel-Whitney class* of $E\to M$ vanishes (See e.g. [LM89], Theorem 1.7) and it is posible to define a bijective antilinear map $C:\Gamma(E)\to\Gamma(E)$ satisfying certain conditions ([GBVF01], §9.2). Recall that every Riemannian manifold (M,g) admits a unique *Levi-Civita connection* ∇^g. If M is also spin, the Levi–Civita connection induces a unique Hermitian connection ∇^E (**spin connection**) on *spinors* of the spinor bundle (i.e. sections of $\Gamma(E)$) that commutes with C and satisfies the following Leibniz rule:

$$\nabla^E(c(\alpha)\psi) = c(\nabla^g\alpha)\psi + c(\alpha)\nabla^E\psi$$

where $\alpha\in\Gamma(\mathbb{C}l(M))$ and $\psi\in\Gamma(E)$.

Using the spin-homomorphism c we can construct another map $\hat{c}:\Gamma(\mathbb{C}l(T^*M))\otimes\Gamma(E)\to\Gamma(E)$ defined as

$$\hat{c}(\alpha\otimes\psi) := c(\alpha)\psi \quad \text{for all } \alpha\in\Gamma(\mathbb{C}l(T^*M)), \psi\in\Gamma(E).$$

Combining $\hat{c}$ and ∇^E we get an operator on $\Gamma(E)$:

Definition B.2.1. The Dirac operator on $\Gamma(E)$ is given by

$$D\!\!\!/ := i\hat{c}\circ\nabla^E, \tag{B.4}$$

where we have tacitly used the embedding of $\Gamma(T^*M)$ in $\Gamma(\mathbb{C}l(M))$.

It is not hard to find a local expression of the Dirac operator, once you know the one for the spin connection. However, we avoid them for being out of our interests.

Remark B.2.2. Here are some of many remarkable properties of the Dirac operator, relevant for this work:

1. $\not{D}$ is a selfadjoint operator on $L^2(M, E)$ ([GBVF01], §9.4).
2. The commutator $[\not{D}, a]$ for any $a \in C^\infty(M)$ is a bounded operator on $L^2(M, E)$.
3. The square of the Dirac operator is

$$\not{D}^2 = \Delta^E + \frac{1}{4}R$$

where Δ^E is the spinor Laplacian and R is the scalar curvature.

Appendix C

Some Cesàro summations

This appendix contains some of Cesàro's classical summation exercises, which are intended to be both examples for a good understanding of the subject, and a complement to a couple of examples from Chapter 2. In general, the definition of what a convergent series is, narrows the vision of possible alternative ways of looking at convergence. For instance, it is well-known that

$$1 + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1. \tag{C.1}$$

It is possible to interpret Equation (C.1) in a more general sense by removing the restriction of the convergence interval [Har92, Page 2]. In fact, if s is the sum of the infinite series interpreted in this formal sense, it implies that

$$s = 1 + x + x^2 + \cdots = 1 + x(1 + x + \cdots) = 1 + xs \ \text{ implies } \ s = \frac{1}{1-x}.$$

Now, let us write $x = e^{i\theta}$ with $0 < \theta < 2\pi$. Then, using some trigonometrical identities

$$\begin{aligned} \frac{1}{1-e^{i\theta}} &= \frac{1}{e^{\frac{i\theta}{2}}e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}e^{\frac{i\theta}{2}}} = \frac{1}{e^{\frac{i\theta}{2}}\left(e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}\right)} = \frac{1}{e^{\frac{i\theta}{2}}\left(-2i\sin\left(\frac{\theta}{2}\right)\right)} \\ &= \frac{ie^{\frac{-i\theta}{2}}}{2\sin\left(\frac{\theta}{2}\right)} = \frac{i\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right) \end{aligned}$$

so that

$$1 + e^{i\theta} + e^{2i\theta} + \cdots = \frac{1}{1-e^{i\theta}} = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right). \tag{C.2}$$

After expanding $e^{i\theta}$, $e^{2i\theta}$, ... etc., from Equation (C.2), we can compare real and imaginary parts and we get

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots = 0. \tag{C.3}$$

$$\sin\theta + \sin 2\theta + \cdots = \frac{1}{2}\cot\left(\frac{\theta}{2}\right)$$

If we differentiate Equation (C.3) repeatedly an even number of times with respect to θ, we get the series

$$\sum_{n=1}^{\infty} n^{2k} \cos n\theta = 0, \qquad k = 1, 2, \ldots \tag{C.4}$$

Up to this point, all the calculations we have made (in particular those to get Equations (C.3) and (C.4)) are of a formal nature, since we have not yet talked about conditions for convergence. What we will observe now is that these calculations are valid, at least, in the Cesàro sense, as defined in Chapter 1.

Using the notation of Definition 1.2.1, let $A_0 := \frac{1}{2}$, $A_n := \frac{1}{2} + \sum_{j=1}^{n} \cos jx$ for $n \geq 1$. The scheme of the proof is as follows: by writing $A_n = \frac{1}{2}\sum_{j=-n}^{n} e^{ijx}$ and adding up the geometric series, we will show that $C_n^1 = \frac{A_n^1}{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} A_j \to 0$ as $n \to \infty$ for all $x \not\equiv 0 \bmod 2\pi$, and so

$$\sum_{j=1}^{\infty} \cos jx = -\frac{1}{2} \quad \textbf{(C,1)}. \tag{C.5}$$

Let $w = e^{ix}$, then

$$\begin{aligned}
\sum_{j=-n}^{n} e^{ijx} &= \sum_{j=0}^{n} e^{ijx} + \sum_{j=-n}^{-1} e^{ijx} = \sum_{j=0}^{n} w^j + \sum_{j=-n}^{-1} w^j = \frac{1-w^{n+1}}{1-w} + \frac{w^{-n}-1}{1-w} \\
&= \frac{w^{-n} - w^{n+1}}{1-w} = \frac{w^{-n}(1-w^{2n+1})}{1-w} = \frac{w^{-n}(w^{\frac{2n+1}{2}} w^{\frac{-(2n+1)}{2}} - w^{\frac{2n+1}{2}} w^{\frac{2n+1}{2}})}{w^{\frac{1}{2}} w^{\frac{-1}{2}} - w^{\frac{1}{2}} w^{\frac{1}{2}}} \\
&= \frac{w^{-n} w^{\frac{2n+1}{2}} (w^{\frac{-(2n+1)}{2}} - w^{\frac{2n+1}{2}})}{w^{\frac{1}{2}}(w^{\frac{-1}{2}} - w^{\frac{1}{2}})} = \frac{-2i \sin\left((n+\frac{1}{2})x\right)}{-2i\sin\left(\frac{x}{2}\right)} = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}.
\end{aligned}$$

Since

$$\frac{e^{ijx} + e^{-ijx}}{2} = \frac{2\cos jx}{2} = \cos jx$$

then

$$A_n = \frac{1}{2}\sum_{j=-n}^{n} e^{ijx} = \frac{\sin\left((n+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{1}{2} + \sum_{j=1}^{n} \cos jx.$$

Again, in the notation of Definition 1.2.1,

$$C_n^1 = \frac{A_n^1}{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} A_j = \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left((j+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)}. \tag{C.6}$$

We have to show that the Cesàro mean (C.6) tends to 0 as $n \to \infty$.

$$\begin{aligned}
C_n^1 &= \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin\left((j+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)} \times \frac{2\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin jx\cos\left(\frac{x}{2}\right)2\sin\left(\frac{x}{2}\right)+\cos jx\sin\left(\frac{x}{2}\right)2\sin\left(\frac{x}{2}\right)}{4\sin^2\left(\frac{x}{2}\right)} \\
&= \frac{1}{4(n+1)\sin^2\left(\frac{x}{2}\right)}\sum_{j=0}^{n}[\underbrace{\sin jx\sin x}_{\cos jx\cos x-\cos(j+1)x} + \underbrace{2\sin^2\left(\frac{x}{2}\right)}_{1-\cos x}\cos jx] \\
&= \frac{1}{4(n+1)\sin^2\left(\frac{x}{2}\right)}\sum_{j=0}^{n}[\cos jx\cos x-\cos(j+1)x+(1-\cos x)\cos jx] \\
&= \frac{1}{4(n+1)\sin^2\left(\frac{x}{2}\right)}\sum_{j=0}^{n}[\cos jx-\cos(j+1)x] \\
&= \frac{1}{4(n+1)\sin^2\left(\frac{x}{2}\right)}[1-\cos x+\cos x-\cdots+\cos nx-\cos(n+1)x] \\
&= \frac{1-\cos(n+1)x}{4(n+1)\sin^2\left(\frac{x}{2}\right)} \\
&= \frac{2\sin^2\left(\frac{n+1}{2}\right)x}{4(n+1)\sin^2\left(\frac{x}{2}\right)} \\
&= \frac{2}{n+1}\left(\frac{\sin\left(\frac{n+1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\right)^2.
\end{aligned}$$

Moreover, for all $x \not\equiv 0 \bmod 2\pi$ we have

$$\frac{2}{n+1}\left(\frac{\sin\left(\frac{n+1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\right)^2 \le \frac{1}{2(n+1)}\frac{1}{\sin^2\left(\frac{x}{2}\right)} \le \frac{1}{2(n+1)}\frac{\pi^2}{l(x)^2} \longrightarrow 0 \quad \text{(C.7)}$$

as $n \to \infty$ for fixed x, where $l(x)$ is a function such as $|\sin x|$ is greater than the absolute value of the vertical displacement of the wave (see Figure C.1). For example, for $x \in (0, \frac{\pi}{2}]$, $\sin x \ge \frac{2}{x}$, and for $x \in (\frac{\pi}{2}, \pi]$, $|\sin x| \ge \frac{2}{\pi}|x-\pi|$. More generally, for $\frac{k\pi}{2} < x < \frac{(k+1)\pi}{2}$,

$$|\sin x| \ge \frac{2}{\pi}|x-(k+1)\pi| \qquad \text{for } k \in \mathbb{N}.$$

Then, a estimation of the form $\frac{1}{2(n+1)}\frac{1}{\sin^2\left(\frac{x}{2}\right)} \le \frac{1}{2(n+1)}\frac{\pi^2}{l(x)^2}$ where $l(x) = \frac{x}{2}-(k+1)\pi$ exists for evey $k \in \mathbb{N}$, and for fixed x this goes to zero as $n \to \infty$.

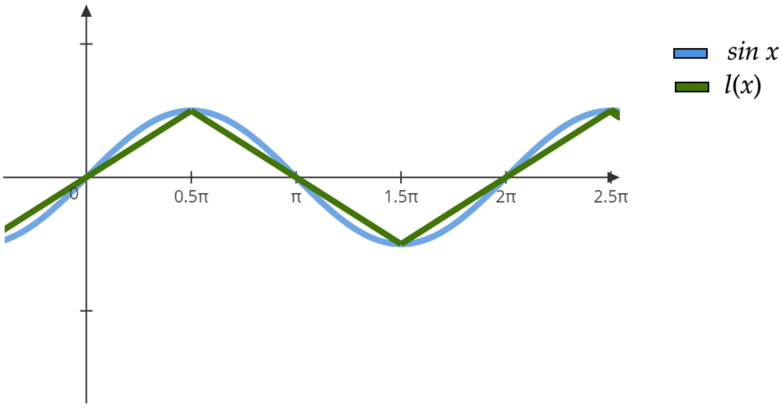


FIGURE C.1: Graph of $l(x)$.

Finally, (C.7) shows that

$$\sum_{n=1}^{\infty} \cos nx = -\frac{1}{2} \qquad \textbf{(C,1)}.$$

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