Dynkin Functions and Its Applications

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Title in English

Dynkin Functions and Its Applications.

Abstract: Dynkin functions were introduced by Ringel as a tool to investigate combinatorial properties of hereditary artin algebras. According to Ringel, a Dynkin function consists of four sequences associated to $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$ and five single values associated to the diagrams $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ and $\mathbb{G}_2$. He also proposes to create an On-line Encyclopedia of Dynkin functions (OEDF) with the same purposes as the famous OEIS. Dynkin functions arise from the context of categorification of integer sequences, which according to Ringel and Fahr it means to consider suitable objects in a category instead of numbers of a given integer sequence. They gave a categorification of Fibonacci numbers by using the Gabriel's universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver. For instance, if Λ denotes a hereditary artin algebra associated to a Dynkin diagram Δ_n then $r(\Delta_n)$ the number of indecomposable modules, $a(\Delta_n)$ the number of antichains in mod Λ, and $t_n(\Delta_n)$ the number of tilting modules are Dynkin functions. In particular, we are focused on the way that some Dynkin functions act on Dynkin diagrams of type $\mathbb{A}_n$.

In this work, we follow the ideas of Ringel regarding Dynkin functions by investigating the number of sections in the Auslander-Reiten quiver of algebras of finite representation type. Dyck paths categories are introduced as a combinatorial model of the category of representations of quivers of Dynkin type $\mathbb{A}_n$ and it is shown an algebraic interpretation of frieze patterns as a direct sum of indecomposable objects of the category of Dyck paths. In particular, it is proved that there is a bijection between some Dyck paths and perfect matchings of some snake graphs. The approach allows us to give formulas for cluster variables in cluster algebras of Dynkin type $\mathbb{A}_n$ in terms of Dyck paths. At last but not least, it is introduced some Brauer configuration algebras such that the dimension of these algebras and its corresponding centers can be obtained via some combinatorial properties of the Catalan triangle.

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Keywords: Auslander-Reiten quiver; categorification; Brauer configuration; Brauer configuration algebra; Catalan triangle; cluster algebras; Dyck paths; Dynkin algebra; Dynkin function; frieze patterns; lattice path; mutation class; perfect matchings; poset; quiver representation; section; triangulations.

Acceptation Note

Thesis Work

“ mention”

Jury

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Agustín Moreno Cañadas

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Dedicated to

My wife, Estefania.
My son, Andrés.
My parents, Gabriel and Claudia.
My grandmother, Elvia.
My brother, Jorge and his family Andrea and Juan.

Acknowledgments

I would like to thank to my wife Estefania, my son Andrés, my parents Claudia and Gabriel, my grandmother Elvia, my brother Jorge and his family Andrea y Juan for being with me throughout these years, for their love and patience.

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Last but not least, I especially express my regards to my partners of the TERENUFIA-UNAL research group for the discussions in the seminar, feedback and, contributions throughout this process.

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Introduction

Dynkin functions were introduced in 2016 by Ringel with the purpose of giving a systematic study of the relationships between integer sequences and invariants of objects in a category mod Λ where Λ is a hereditary artin algebra. A Dynkin function f does not depend on orientation and consists of four sequences $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ and five single values $f(\mathbb{E}_6)$, $f(\mathbb{E}_7)$, $f(\mathbb{E}_8)$, $f(\mathbb{F}_4)$ and $f(\mathbb{G}_2)$ [78]. If Λ is an algebra of Dynkin type $\Delta_n = \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\}$ then the number $\mathbf{r}(\Delta_n)$ of indecomposable modules, the number $\mathbf{a}_n(\Delta_n)$ of exceptional antichains in mod Λ and $t_n(\Delta_n)$ the number of tilting modules are examples of Dynkin functions. Ringel also proposes to create an On-Line Encyclopedia of Dynkin Functions (ODEF) with the same purposes as the famous On-Line Encyclopedia of Integer Sequences (OEIS) which is the main tool dealing with the research of integer sequences.

Dynkin functions are a way to categorify integer sequences. According to Ringel and Fahr a categorification of an integer sequence means to consider instead of numbers in the sequence suitable invariants of objects in a category. Ringel and Fahr gave a categorification of Fibonacci numbers by using the Gabriel's universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver [49, 50]. The categorification of generalized non-crossing partitions (in the sense of Kreweras) of a given finite set has been studied by Hubery, Krausse, Ingalls, Ringel and Thomas amongst others mathematicians [59, 79]. Therefore, researches regarding Dynkin functions not only impact on the theory of representation of algebras if not another fields of the mathematics as combinatorics and number theory, for instance, factorization of numbers associated to invariants of algebras of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$ seems to be very interesting as Ringel quotes in [78].

Although Ringel's ideas regarding categorification of integer sequences are so new, they have inspired different researches of many mathematicians, we recall here works on categorification of different integer sequences obtained by the author, Cañadas, and Giraldo et al who have used Kronecker modules, tiled orders and the theory of representation of posets to categorify some integer sequences [27, 31–35, 41]. In this direction, we use lattice paths connecting points of some suitable posets to investigate the number of sections $S(\Delta_n)$ in the Auslander-Reiten quiver of some algebras as a Dynkin function. Some interesting integer sequences arise from this research, for instance, Fermat numbers (i.e., numbers of the form $2^{2^j} + 1$) is a subsequence of an integer sequence whose some of its elements can be interpreted as the number of some lattice paths via the procedures used in this work. We also give a formula partition for numbers in the sequence A049611 in the OEIS by using sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{A}_n$. Besides, an explicit formula for sections in the Auslander-Reiten quiver of algebras of this type is given, in particular, categorifications of the integer sequences A083329 and A000295 in the OEIS are obtained by interpreting each of its elements as the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{A}_n$ [38].

Another interesting integer sequence with many interpretations in the theory of representation of algebras is the sequence of Catalan numbers, i.e., the sequence whose elements are numbers of the form $C_n = \frac{1}{n+1}\binom{2n}{n}$ [92]. For instance, Gabriel and De la Peña proved that Catalan numbers count the number of discrete subsets contained in the set of representatives of isoclasses of indecomposable finite-dimensional modules over a Dynkin algebra of type $\mathbb{A}_n$ (with $\mathbb{A}_n$ linearly oriented).

In the last few years, researches regarding connections between cluster algebras and different fields of mathematics have been growing. For instance, relationships between cluster algebras, quiver representations, combinatorics and number theory have been reported by Fomin et al., Shiffler et al., K. Baur et al., Assem et al. amongst a great number of mathematicians [4, 9, 19, 22, 52, 54–56].

Perhaps the Catalan combinatorics (which consists of all the enumeration problems whose solutions are Catalan numbers) is the most appropriate environment for the investigation of cluster algebras of Dynkin type $\mathbb{A}_n$. Among all these kinds of problems, for example, it is possible to prove that the Catalan numbers count [92]:

1. The number of plane binary trees with $n+1$ endpoints (or $2n+1$) vertices,
2. The number of ways to parenthesize a string of length $n+1$ subject to a non associative binary operation,
3. The number of paths P in the (x, y)-plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never pass below the x-axis. Such paths are called *Dyck paths*,
4. The number of triangulations of an $(n+3)$ polygon,
5. The number of clusters of a cluster algebra of Dynkin type $\mathbb{A}_n$.

Regarding integer friezes, we point out that Propp in [71] reminds that Conway and Coxeter completely classified the frieze patterns whose entries are positive integers, and showed that these frieze patterns constitute a manifestation of the Catalan numbers. Specifically, that there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labelled vertices. According to Baur and Marsh [9], a connection between cluster algebras and frieze patterns was established by Caldero and Chapoton [18], which showed that frieze patterns can be obtained from cluster algebras of Dynkin type $\mathbb{A}_n$.

Another example of the use of the Catalan combinatorics as a tool to describe the structure of cluster algebras, was given by Schiffler et al. [19, 22, 69], who found out formulas for cluster variables based on its relations with some triangulated surfaces and perfect matchings of snake graphs. They also proved that there is a way of obtaining the number of perfect matchings of a given snake graph by associating a suitable continued fraction defined by the sign function of the graph.

Given a non-negative integer n and a triangulation T of a regular polygon with $(n+3)$ vertices. Caldero, Chapoton and Schiffler [17] gave a realization of the category $\mathcal{C}_C$ of representations of a quiver Q_C associated to a cluster C of a cluster algebra in terms of the diagonals of the $(n+3)$ polygon. They proved that there is a categorical equivalence between the categories C_T and $\operatorname{Mod} Q_T$, where C_T is the category whose objects are positive integral linear combinations of positive roots (i.e., diagonals that does not belong to the triangulation T), whereas $\operatorname{Mod} Q_T$ denotes the category of modules over the quiver Q_T with triangular relations induced by the triangulation T.

Following the ideas of Caldero, Chapoton and Schiffler, in this work, a combinatorial model of the category of representations of Dynkin quivers of type $\mathbb{A}_n$ with relations is developed by using Dyck paths. This approach allows us to realize perfect matchings of snake graphs as objects of suitable Dyck paths categories, and with this machinery a formula for cluster variables based on Dyck paths is obtained.

We show that frieze patterns arise from Dyck paths and they can be written in terms of Dyck path categories. We also introduce a family of Brauer configuration algebras associated to Dyck paths. Combinatorial properties of the Catalan triangle are used to find out formulas for the dimension of this type of algebras and its corresponding centers.

Main results, contributions, papers and conferences

This research regards the categorification of integer sequences and some applications of Dynkin functions in representation theory of algebras and combinatorics.

Contributions

The following are the main contributions:

1. It is given a recurrence formula of the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type via lattice paths connecting minimal and maximal points of suitable posets.
2. It is obtained a categorification of integer sequences arising from sections in the Auslander-Reiten quiver of algebras of Dynkin type in the sense of Ringel and Fahr.
3. Dyck paths categories are introduced and it is proved that there exists an equivalence of categories between the category of Dyck paths and the category of representations of Dynkin quivers of type $\mathbb{A}_n$ with relations.
4. It is given a formula of the cluster variables of cluster algebras associated to quivers of type $\mathbb{A}_n$ by using Dyck paths.
5. It is established a bijective correspondence between Dyck paths and frieze patterns, attaining in this way a new algebraic interpretation of frieze patterns as a direct sum of indecomposable objects of Dyck paths categories.

6. It is defined Dyck-Brauer configuration algebras, and it is given an explicit formulas of the dimension of these Brauer configuration algebras and its corresponding centers in terms of the Catalan triangle.

Papers

Results of this research allowed us to publish the following papers:

1. On the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type [32].
2. Integer sequences arising from Auslander-Reiten quivers of some hereditary artin algebras [38].

Results of this research allowed us to submit the following manuscript:

1. Dyck paths categories and its relationships with cluster algebras.

Conferences

The main results of this research have been presented in the following conferences:

1. Primer encuentro de Álgebra y Topología Universidad Nacional de Colombia. Bogotá-Colombia, 01-2018.
2. UN Encuentro de Matemáticas. Bogotá-Colombia, 06-2018.
3. Third International Colloquium on Representations of Algebras and Its Applications; Alexander Zavadskij. Medellín -Colombia, 06-2018.
4. IV Jornada de Álgebra no Amazonas. Tabatinga-Brasil, 09-2019.
5. Primer Encuentro de Estudiantes de Posgrado en Matemáticas, Medellín -Colombia, 02-2020.
6. Fourth International Colloquium on Representations of Algebras and Its Applications; Alexander Zavadskij, Bogotá-Colombia, 11-2020.

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4. Algebra seminar at the Instituto de Matemáticas, Universidad de Antioquia, Medellín-Colombia, Professor Hernán Giraldo.

This thesis is distributed as follows:

Chapter 1, aims to present a theoretical introduction of representation theory of algebras, sections in the Auslander-Reiten quiver, representation theory of posets, Brauer configuration algebras, Dyck paths, cluster algebras, cluster-tilted algebras, friezes, snake graphs, and the category of diagonals, as well as, definitions and notations to be used throughout the work.

In chapter 2, it is described a family of posets that allows us to find a formula of the number of sections in the Auslander-Quiver quiver of algebras of Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$. These formulas establish a categorification of some integer sequences in the sense of Ringel and Fahr.

In chapter 3, it is introduced the category of Dyck paths as a combinatorial model of the category of representations of a quiver of type $\mathbb{A}_n$ with relations. It is presented a bijective correspondence between a family of words of Dyck paths and the number of perfect matchings of a snake graph. Besides, it is described a formula of cluster variables arising from the Dyck paths of algebras with an underlying graph of type $\mathbb{A}_n$.

In chapter 4, it is defined a basic set called diamond which is used to build frieze patterns, these sets are in bijective correspondence with Dyck paths and triangulations of $(n+3)$ polygons, and it is presented frieze patterns by using indecomposable objects of Dyck paths categories. Dyck-Brauer configuration algebras are introduced and it is given the dimension of these algebras and its corresponding centers.

Finally, appendix A, contains examples of integer sequences arising from the number of sections associated to algebras of type $\mathbb{A}_n$ and $\mathbb{D}_n$. Besides, it is included examples of family of integer sequences and pairs of matrices associated to Dyck-Brauer configuration algebras.

CHAPTER 1

Preliminaries

In this chapter, we present a brief description and important theorems regarding quiver representations in section 1.1. Sections in the infinite translation quiver and Brauer configuration algebras are described in sections 1.2 and 1.3, respectively. Category of representation of ordinary posets and some classical theorems regarding classification of ordinary posets are introduced in section 1.4. In section 1.5 we recall Dyck paths as a Catalan object, whereas some elementary notions of cluster algebras, category of diagonals of an $(n+3)$ polygon, cluster-tilted algebras, and friezes are defined in sections 1.6, 1.7, and 1.8. Finally, some definitions and results regarding snake graphs are given in section 1.9. Throughout the thesis, k denotes an algebraically closed field. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets natural, integer, real, and complex numbers, respectively [2, 3, 17, 19, 39, 43, 45, 54, 60–62, 71, 80, 84, 92, 96, 99].

1.1 Representation Theory of Quivers

In this section, we present some concepts regarding representations of a quiver. We recall theorems that describe algebras of finite and tame representation type [3, 60, 80, 84].

A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called points, or vertices) and Q_1 (whose elements are called arrows), and two maps $s, t : Q_1 \to Q_0$, which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Figure 1.1 shows examples of quivers

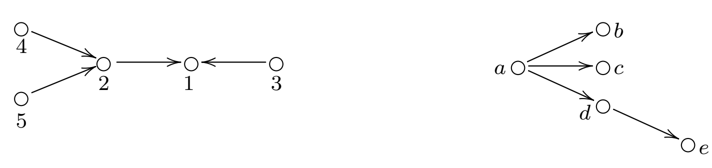


FIGURE 1.1. Examples of quivers.

A *path* of length $l \geq 1$ with source a and target b is a sequence $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ where $\alpha_k \in Q_1$ for all $1 \leq k \leq l$, and we have $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < l$, and finally $t(\alpha_l) = b$. We denote by Q_l the set of all paths in Q of length l. We also agree to associate with each point $a \in Q_0$ a path of length $l = 0$ (denoted by $e_a = (a||a)$).

The *path algebra* kQ of Q is the $k-$algebra whose underlying $k-$vector space has as its basis the set of all paths $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ of length $l \geq 0$ in Q and such that the product of two basis vectors $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ and $(c \mid \beta_1, \ldots, \beta_k \mid d)$ of kQ is equal to zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the composed path $(a \mid \alpha_1, \ldots, \alpha_{l,}\beta_1, \ldots, \beta_k \mid d)$ if $t(\alpha_l) = s(\beta_1)$.

Let Q be a finite and connected quiver. The *two-sided ideal* of the path algebra kQ generated (as an ideal) by the arrows of Q is called the *arrow ideal* of kQ and is denoted by R_Q. In particular, for each $l \geq 1$,

$$R_Q^l = \bigoplus_{m \geq l} kQ_m.$$

R_Q^l is an ideal of kQ. A two-sided ideal I of kQ is said to be *admissible ideal* if there exists an integer $m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

If I is an admissible ideal of kQ, the pair (Q, I) is said to be a *bound quiver*. The quotient algebra kQ/I is said to be the algebra of the bound quiver (Q, I) or, simply, a *bound quiver algebra*.

A *relation* in Q with coefficients in k is a $k-$linear combination of paths of length at least two having the same source and target. Thus, a relation ρ is an element of kQ such that

$$\rho = \sum_{i=1}^{m} \lambda_i w_i,$$

where the λ_i are scalars and the w_i are paths in Q of length at least 2 such that, if $i \neq j$, then the source (resp. the target) of w_i coincides with that of w_j. If $(\rho_j)_{j \in J}$ is a set of relations for a quiver Q such that the ideal they generate $\langle \rho_j \mid j \in J \rangle$ is admissible, we say that the quiver Q is *bounded by the relations* $(\rho_j)_{i \in J}$ or by the relations $\rho_j = 0$ for all $j \in J$.

A *representation* $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of k-vector spaces M_i, one for each vertex $i \in Q_0$, and a collection of k-linear maps $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$, one for each arrow $\alpha \in Q_1$.

Let $M = (M_i, \varphi_\alpha)$, $M' = (M'_i, \varphi'_\alpha)$ be two representations of Q. A *morphism* (or homomorphism) of representations $f : M \to M'$ is a collection $(f_i)_{i \in Q_0}$ of linear maps $f_i : M_i \longrightarrow M'_i$, such that for each arrow $i \xrightarrow{\alpha} j$ in Q_1 the diagram:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j \end{tikzcd} \tag{1.1}$$

commutes, that is $(f_j \circ \varphi_\alpha)(m) = (\varphi'_\alpha \circ f_i)(m)$ for all $m \in M_i$. Let $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ be representations of Q. Then

$$M \oplus M' = \left(M_i \oplus M'_i, \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix} \right)_{i \in Q_0, \alpha \in Q_1},$$

is a representation of Q called the *direct sum* of M and M'.

Rep Q is the category of representations of a quiver Q, rep Q is the full subcategory of Rep Q consisting of the finite dimensional representations. Rep Q and rep Q are abelian k-categories. A representation $M \in$ rep Q is called *indecomposable* if $M \neq 0$ and M cannot be written as a direct sum of two nonzero representations, that is, whenever $M \simeq N \oplus L$ with $N, L \in$ rep Q, then $N = 0$ or $L = 0$. A quiver Q is said to be of *finite representation type* if the number of isoclasses of indecomposable representations of Q is finite. A quiver Q is said to be of *infinite representation type* if Q is not of finite representation type [80].

Theorem 1.1. [3]. *Let $\mathcal{A} = kQ/I$, where Q is a finite connected quiver and I is an admissible ideal of kQ. There exists a $k-$linear equivalence of categories*

$$F: \ Mod\ \mathcal{A} \rightarrow \ Rep\ (Q, I),$$

that restricts to an equivalence of categories $F: \ mod\ \mathcal{A} \rightarrow \ rep\ (Q, I)$.

Gabriel [58] and Nazarova [72] proved the following theorems, respectively.

Theorem 1.2. [3]. *Let Q be a finite, connected, and acyclic quiver; k be an algebraically closed field; and $\mathcal{A} = kQ$ be the path $k-$algebra of Q.*

(a) *The algebra $\mathcal{A}$ is representation-finite if and only if the underlying graph $\overline{Q}$ of Q is one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$.*

(b) *If $\overline{Q}$ is a Dynkin graph, then the mapping $\textbf{dim} : M \rightarrow \textbf{dim}\ M$ induces a bijection between the set of isomorphism classes of indecomposable $\mathcal{A}-$modules and the set $\{x \in \mathbb{N}^n \ ; q_Q(x) = 1\}$ of positive roots of the quadratic form q_Q of Q.*

(c) *The number of the isomorphism classes of indecomposable $\mathcal{A}-$modules equals $\frac{1}{2}n(n+1)$, $n^2 - n$, 36, 63, and 120, if $\overline{Q}$ is the Dynkin graph $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, respectively.*

Theorem 1.3. [84]. *Let Q be a connected quiver without oriented cycles and k be an algebraically closed field. Then kQ is representation-tame if and only if the underlying graph $\overline{Q}$ of Q is one of the extended Dynkin diagrams $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, and $\tilde{\mathbb{E}}_8$.*

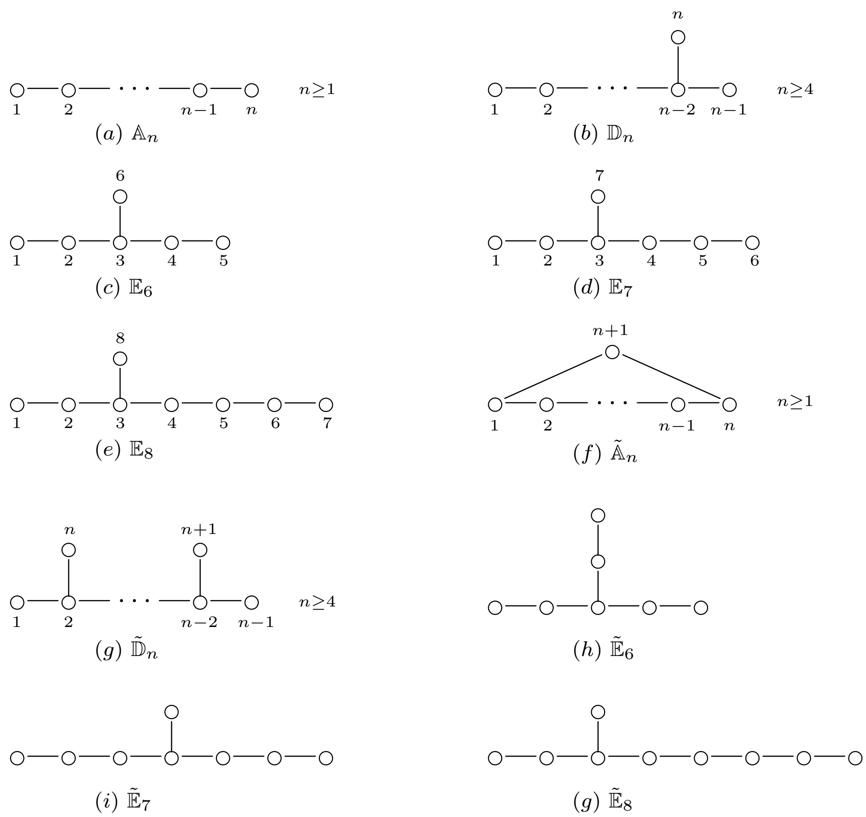


FIGURE 1.2. Dynkin and extended Dynkin diagrams.

1.2 Sections in the Infinite Translation Quiver

In this section, for the sake of clarity we recall the definitions of section, an orbit in an Auslander-Reiten quiver as Assem et al. described in [3].

Let $\Sigma = (\Sigma_0, \Sigma_1)$ be a connected and acyclic quiver. An *infinite translation quiver* $(\mathbb{Z}\Sigma, \tau)$ has the set $(\mathbb{Z}\Sigma)_0 = \mathbb{Z} \times \Sigma_0 = \{(n, x) \mid n \in \mathbb{Z}, x \in \Sigma_0\}$ as its set of vertices, and for each arrow $\alpha : x \to y \in \Sigma_1$ there exist two arrows

$$(n, \alpha) : (n, x) \to (n, y) \quad (n, \alpha') : (n+1, y) \to (n, x) \text{ in } (\mathbb{Z}\Sigma)_1, \tag{1.2}$$

and these are all the arrows in $(\mathbb{Z}\Sigma)_1$. The *translation* τ on $\mathbb{Z}\Sigma$ is given by the formula $\tau(n, x) = (n+1, x)$, and for every $(n, x) \in (\mathbb{Z}\Sigma)_0$ it is defined a bijection between the set of arrows of target (n, x) and the set of arrows of source $(n+1, x)$ by the formulas:

$$\sigma(n, \alpha) = (n, \alpha') \quad \text{and} \quad \sigma(n, \alpha') = (n+1, \alpha), \tag{1.3}$$

Let Σ be a quiver described in Figure 1.3.

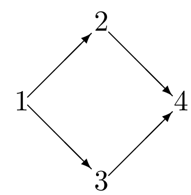


FIGURE 1.3. Quiver Σ.

Then the infinite translation quiver of Σ is given by Figure 1.4.

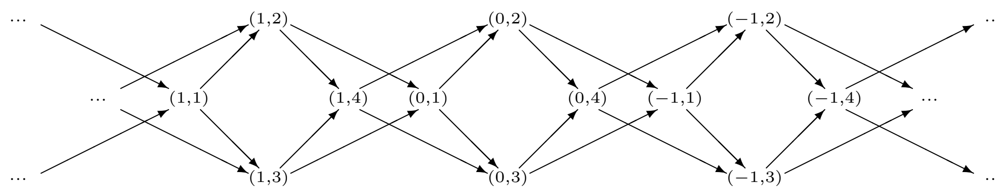


FIGURE 1.4. Infinite translation quiver of Σ.

Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ of Γ is a *section* of Γ if the following conditions are satisfied:

S(1) Σ is acyclic.

S(2) For each $x \in \Gamma_0$, there exists a unique $n \in \mathbb{Z}$ such that $\tau^n x \in \Sigma_0$.

S(3) If $x_0 \to x_1 \to \cdots \to x_t$ is a path in Γ with $x_0, x_t \in \Sigma_0$, then $x_i \in \Sigma_0$ for all i such that $0 \leq i \leq t$.

For a translation quiver (Γ, τ), the τ-*orbit* of a point $x \in \Gamma_0$ is defined to be the set of all points of the form $\tau^n x$ with $n \in \mathbb{Z}$. Thus, any section Σ meets each τ-orbit exactly once.

Arrows in a section of a translation quiver (Γ, τ) satisfy the following conditions:

1. If $x \to y$ is an arrow in Γ and $x \in \Sigma_0$, then $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.
2. If $x \to y$ is an arrow in Γ and $y \in \Sigma_0$, then $x \in \Sigma_0$ or $\tau^{-1} x \in \Sigma_0$.

Sections are useful to characterize representation-finite tilted algebras. Regarding this subject, we recall the Happel and Ringel's criterion which states that a *connected representation-finite algebra B is a tilted algebra if and only if the Auslander-Reiten quiver of B contains a section.*

Henceforth, we let $\mathcal{O}_x$ denote the orbit of a fixed element $x \in \Gamma_0$. In particular, if $\Gamma(\text{Mod } \mathcal{A}) = (\Gamma_0, \Gamma_1)$ is the Auslander-Reiten quiver of an algebra of Dynkin type Δ_n then each element of the τ-orbit of an indecomposable projective module will be denoted τ_i^n, $i \in \mathbb{N}$. We also note that in the case of representation-finite hereditary algebras $\mathcal{A}$ the vertices of the Auslander-Reiten quiver $\Gamma_{\mathcal{A}}$ corresponding to the indecomposable projective modules form in $\Gamma_{\mathcal{A}}$ a section of Dynkin class.

As an example in Figure 1.5 we show an oriented quiver Q of type $\mathbb{A}_3$ and the corresponding Auslander-Reiten quiver of the algebra $\mathcal{A} = kQ$.

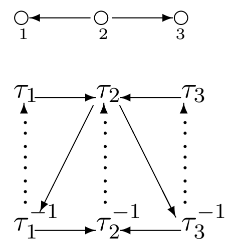


FIGURE 1.5. Quiver Q and the Auslander-Reiten quiver of kQ.

In this case sections are $S_1 = \{\tau_1, \tau_2, \tau_3\}$, $S_2 = \{\tau_1, \tau_2, \tau_3^{-1}\}$, $S_3 = \{\tau_1^{-1}, \tau_2, \tau_3\}$, $S_4 = \{\tau_1^{-1}, \tau_2, \tau_3^{-1}\}$ and $S_5 = \{\tau_1^{-1}, \tau_2^{-1}, \tau_3^{-1}\}$ all of them of type $\mathbb{A}_3$.

1.3 Brauer Configuration Algebras

In 2015 Green and Schroll [61] introduced the concept of Brauer configuration algebra as a generalization of a Brauer graph algebra. In general, these algebras are of wild representation type. They showed that Brauer configuration algebras are finite-dimensional symmetric, multiserial, and others. In this section, we recall definitions of Brauer configuration and its Brauer configuration algebra, we present some properties of these algebras [61, 83].

A *Brauer configuration* is a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$, where:

(B1) Γ_0 is a finite set whose elements are called *vertices*.

(B2) Γ_1 is a finite collection of multisets called *polygons*. In this case, if $V \in \Gamma_1$ then the elements of V are vertices possibly with repetitions, $\mathrm{occ}(\alpha, V)$ denotes the frequency of the vertex α in the polygon V and the *valency* of α denoted $\mathrm{val}(\alpha)$ is defined in such a way that:

$$\mathrm{val}(\alpha) = \sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V). \tag{1.4}$$

(B3) μ is an integer valued function such that $\mu : \Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers, it is called the *multiplicity function*.

(B4) $\mathcal{O}$ denotes an orientation defined on Γ_1 which is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex, including repetitions, we denote S_α such collection of polygons. More specifically, if $S_\alpha = \{V_1^{(\alpha_1)}, V_2^{(\alpha_2)}, \ldots, V_t^{(\alpha_t)}\}$ is the collection of polygons where the vertex α occurs with $\alpha_i = \mathrm{occ}(\alpha, V_i)$ and $V_i^{(\alpha_i)}$ meaning that S_α has α_i copies of V_i then an orientation

$\mathcal{O}$ is obtained by endowing a linear order $<$ to S_α and adding a relation $V_t < V_1$, if $V_1 = \min S_\alpha$ and $V_t = \max S_\alpha$. According to this order the α_i copies of V_i can be ordered as $V_{1,i} < V_{2,i} < \cdots < V_{(\alpha_i-1),i} < V_{\alpha_i,i}$ and S_α can be ordered in the form $V_1^{(\alpha_1)} < V_2^{(\alpha_2)} < \cdots < V_{(t-1)}^{(\alpha_{(t-1)})} < V_{(t)}^{\alpha_t}$.

(B5) Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1.

(B6) Every polygon has at least two vertices.

(B7) Every polygon in Γ_1 has at least one vertex α such that $\mu(\alpha)\mathrm{val}(\alpha) > 1$.

The set $(S_\alpha, <)$ is called the *successor sequence* at the vertex α.

A vertex $\alpha \in \Gamma_0$ is said to be *truncated* if $val(\alpha)\mu(\alpha) = 1$, that is, α is truncated if it occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$. A vertex is *nontruncated* if it is not truncated.

Given a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ we say that the polygon $V \in \Gamma_1$ is a $d-$gon if the number of vertices appearing in V is d. We say that the configuration Γ is *reduced* if and only if every polygon $V \in \Gamma_1$ satisfies one of the following conditions:

(i) $V \cap \mathfrak{F}_\Gamma = \emptyset$,

(ii) if $V \cap \mathfrak{F}_\Gamma \neq \emptyset$, then V is a 2-gon with only one truncated vertex,

where $\mathfrak{F}_\Gamma = \{\alpha \in \Gamma_0 \mid \mu(\alpha)\mathrm{val}(\alpha) = 1\}$.

The Quiver of a Brauer Configuration Algebra

The quiver $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1)$ of a Brauer configuration algebra is defined in such a way that the vertex set $(Q_\Gamma)_0 = \{v_1, v_2, \ldots, v_m\}$ of Q_Γ is in correspondence with the set of polygons $\{V_1, V_2, \ldots, V_m\}$ in Γ_1, noting that there is one vertex in $(Q_\Gamma)_0$ for every polygon in Γ_1.

Arrows in Q_Γ are defined by the successor sequences. That is, there is an arrow $v_i \xrightarrow{s_i} v_{i+1} \in (Q_\Gamma)_1$ provided that $V_i < V_{i+1}$ in $(S_\alpha, <) \cup \{V_t < V_1\}$ for some nontruncated vertex $\alpha \in \Gamma_0$. In other words, for each nontruncated vertex $\alpha \in \Gamma_0$ and each successor V' of V at α, there is an arrow from v to v' in Q_Γ where v and v' are the vertices in Q_Γ associated to the polygons V and V' in Γ_1, respectively.

Ideal of Relations and Definition of a Brauer Configuration Algebra

Fix a polygon $V \in \Gamma_1$ and suppose that $\mathrm{occ}(\alpha, V) = t \geq 1$ then there are t indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. Then the *special* α*-cycles* at v are the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$ where v is the vertex in the quiver of Q_Γ associated to the polygon V. If α occurs only once in V and $\mu(\alpha) = 1$ then there is only one special α-cycle at v.

Let k be a field and Γ a Brauer configuration. The *Brauer configuration algebra associated to* Γ is defined to be the bounded path algebra $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where Q_Γ is the quiver associated to Γ and I_Γ is the *ideal* in kQ_Γ generated by the following set of relations ρ_Γ of type I, II and III.

- *Relations of type* I. For each polygon $V = \{\alpha_1, \dots, \alpha_m\} \in \Gamma_1$ and each pair of nontruncated vertices α_i and α_j in V, ρ_Γ contains all relations of the form $C^{\mu(\alpha_i)} - (C')^{\mu(\alpha_j)}$ or $(C')^{\mu(\alpha_j)} - C^{\mu(\alpha_i)}$ where C is a special α_i-cycle at v and C' is a special α_j-cycle at v.
- *Relations of type* II. The type two relations are all paths of the form $C^{\mu(\alpha)}a$ where C is a special $\alpha-$cycle and a is the first arrow in C.
- *Relations of type* III. These relations are quadratic monomial relations of the form ab in kQ_Γ where ab is not a subpath of any special cycle.

For example, let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ be a Brauer configurations, where $\Gamma_0 = \{1, 2, 3\}$, $\Gamma_1 = \{V_1 = \{1, 1, 3\},\ V_2 = \{1, 2\},\ V_3 = \{2, 3, 3\}\}$, $\mu(1) = \mu(3) = 1$ and $\mu(2) = 2$. The successor sequence of vertex 1 is $V_1 < V_1 < V_2$, the successor sequence of vertex 2 is $V_2 < V_3$, and the successor sequence of vertex 3 is $V_1 < V_3 < V_3$. There are two special 1−cycles at v_1, $a_1a_2a_3$ and $a_2a_3a_1$. There is only one special 3−cycle at v_1, $c_1c_2c_3$. There is one special 1−cycle at v_2, $a_3a_1a_2$. The special 2−cycle at v_2 is b_1b_2. The special 2−cycle at v_3 is b_2b_1. There are two special 3−cycles at v_3, $c_2c_3c_1$ and $c_3c_1c_2$. The ideal I_Γ is generated by following relations in ρ_Γ:

$$
\begin{gathered}
a_1a_2a_3 = a_2a_3a_1 = c_1c_2c_3;\ a_3a_1a_2 = (b_1b_2)^2;\ (b_2b_1)^2 = c_2c_3c_1 = c_3c_1c_2;\\
a_1a_2a_3a_1;\ a_2a_3a_1a_2;\ a_3a_1a_2a_3;\ (b_1b_2)^2b_1;\\
(b_2b_1)^2b_2;\ c_1c_2c_3c_1;\ c_2c_3c_1c_2;\ c_3c_1c_2c_3;\\
a_1c_1;\ c_3a_1;\ a_2b_1;\ c_3a_2;\ a_3c_1;\ b_2a_3;\ b_1c_2;\ b_1c_3;\ c_2b_2;\ c_1b_2.
\end{gathered}
\tag{1.5}
$$

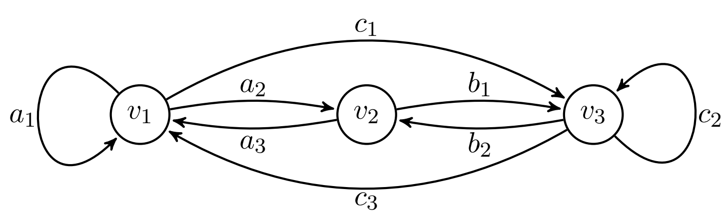


FIGURE 1.6. Quiver Q_Γ associated to the Brauer configuration Γ.

Figures 1.6 and 1.7 show the quiver associated to Γ and the indecomposable projective modules of Λ_Γ.

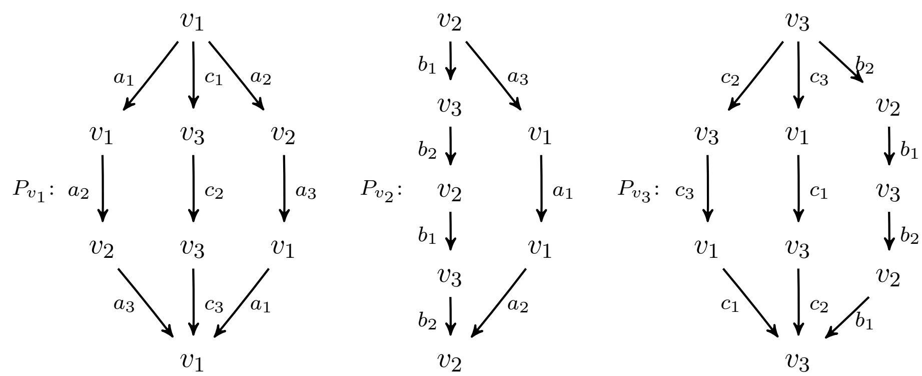


FIGURE 1.7. Indecomposable projective modules of Λ_Γ.

The following results show some properties of Brauer configuration algebras [61].

Theorem 1.4. *Let Λ be a Brauer configuration algebra with Brauer configuration Γ.*

(i) A Brauer configuration algebra is a finite dimensional symmetric algebra.

(ii) Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ is a decomposition of Γ into two disconnected Brauer configurations Γ_1 and Γ_2. Then there is an algebra isomorphism $\Lambda_\Gamma \simeq \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2}$ between the associated Brauer configuration algebra.

(iii) The Brauer configuration algebra associated to a connected Brauer configuration is an indecomposable algebra.

(iv) A Brauer graph algebra is a Brauer configuration algebra.

(v) There is a bijective correspondence between the set of indecomposable projective $\Lambda-$modules and the polygons in Γ.

(vi) If P is an indecomposable projective $\Lambda-$module corresponding to a polygon V in Γ. Then rad (P) is a sum of r indecomposable uniserial modules, where r is the number of (nontruncated) vertices of V and where the intersection of any two of the uniserial modules is a simple $\Lambda-$module.

(vii) A Brauer configuration algebra is a multiserial algebra.

Proposition 1.1. *Let Λ be a Brauer configuration algebra associated to the Brauer configuration Γ and let $C = \{C_1, \dots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that, for $i = 1, \dots, t$, C_i is a special α_i-cycle where α_i is a nontruncated vertex in Γ. Then*

$$dim_k\Lambda = 2|Q_0| + \sum_{C_i \in C} |C_i|(n_i|C_i| - 1),$$

where $|Q_0|$ denotes the number of vertices of Q, $|C_i|$ denotes the number of arrows in the α_i-cycle C_i and $n_i = \mu(\alpha_i)$.

Sierra proved the following result [83].

Theorem 1.5. *Let $\Lambda = kQ_\Gamma/I_\Gamma$ be the Brauer configuration algebra associated to the connected and reduced Brauer configuration Γ. Then*

$$dim_k Z(\Lambda) = 1 + \sum_{\alpha\in\Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#Loops(Q_\Gamma) - |\mathfrak{C}_\Gamma|,$$

where $\mathfrak{C}_\Gamma = \{\gamma \in \Gamma_0 \mid val(\gamma) = 1 \text{ and } \mu(\gamma) > 1\}$.

For the case of Λ_Γ in the previous example, the dimension of Λ_Γ is equal to 24, and the dimension of its center is 7.

1.4 Representation Theory of Ordinary Posets

The theory of representation of posets was introduced and developed by Nazarova, Roiter and their students in Kiev at the 1970s, one of their ideas was to used it as a way of giving a solution of the second Brauer-Thrall conjecture regarding classification of algebras [74, 75, 84]. The main tool to classify posets both ordinary and with additional structures have been the algorithms of differentiation which are functors defined to reduce dimension of the objects of the categories involved in the procedure. The first of these algorithms of differentiation known as the algorithm with respect to a maximal point was introduced by Nazarova and Roiter in 1972, it was used by Kleiner to obtain a criterion to classify posets of finite representation type and by Nazarova in order to classify posets of tame representation type in 1977 [63, 76]. In 1977 as well Zavadskij introduced the algorithm of differentiation with respect to a suitable pair of points which was used by him and Nazarova in 1981 to classify posets of finite growth [77, 84, 97]. We recall that in 1991 Zavadskij introduced an apparatus of differentiation for posets consisting of the algorithms of differentiation DI, DII, DIII, DIV and DV this apparatus was used by him and Bondarenko to classify posets of tame and finite growth with an involution [11, 98] (see in [39]). Particularly in Colombia, Cañadas et al. have studied applications of the theory of representation of posets and its generalizations [24–26, 28–30, 36, 37, 39, 40, 42]. In this section, we introduce some elementary notions of the matrix problems, ordinary posets, and classical theorems regarding classification of ordinary posets [2, 39, 60, 84, 99].

Let **Mat** be a set of finite matrices with coefficients in k which is closed under direct sums and direct summands, where for matrices A, B we set

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Suppose that $\mathcal{G}$ is a set of elementary transformations on rows and columns of matrices in **Mat**. We say that A is $\mathcal{G}-$equivalent to B ($A \sim_\mathcal{G} B$) if B can be obtained from A by applying a sequence of transformations from $\mathcal{G} \cup \mathcal{G}^{-1}$. A class $A/\sim_\mathcal{G}$ represented by A in **Mat** is said to be *decomposable* if there is a matrix B in **Mat** of the form $B = C \oplus D$ such that $A \sim_\mathcal{G} B$. A class $A/\sim_\mathcal{G}$ is *indecomposable* if A is not $\mathcal{G}-$equivalent to zero matrix and $A/\sim_\mathcal{G}$ it is not decomposable. The problem of classifying the indecomposables in the residue class set $\mathbf{Mat}/\sim_\mathcal{G}$ is called a *matrix problem* and we denote it by $(\mathbf{Mat}, \mathcal{G})$. This is equivalent to a reduction of any matrix in **Mat** to a canonical form by applying the transformations in $\mathcal{G} \cup \mathcal{G}^{-1}$. The problem $(\mathbf{Mat}, \mathcal{G})$ is of *finite representation type* if the set of indecomposable $\mathcal{G}-$equivalence classes of matrices in **Mat** is finite [84].

A *poset* is an ordered pair of the form $(\mathcal{P}, \leq)$ of a set $\mathcal{P}$ and a binary relation $\leq$ contained in $\mathcal{P} \times \mathcal{P}$, called the order on $\mathcal{P}$ such that $\leq$ is reflexive, antisymmetric and transitive [46].

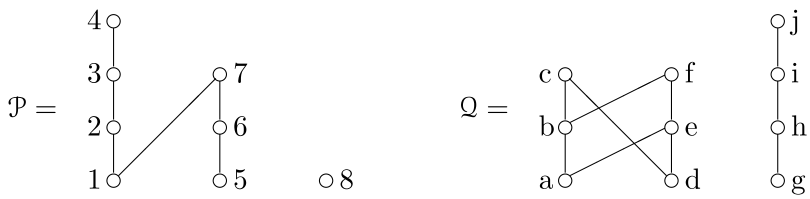


FIGURE 1.8. Hasse diagrams of $\mathcal{P}$ and $\mathcal{Q}$.

A *representation* of $\mathcal{P}$ over a field k is a collection $U = (U_0, U_x \mid x \in \mathcal{P})$, where U_0 is a finite-dimensional $k-$space and U_x is a subspace in U_0 for each $x \in \mathcal{P}$, such that $U_x \subseteq U_y$ if the relation $x \leq y$ holds in $\mathcal{P}$. The *dimension vector* of a representation U is the vector $d = \underline{\dim}\, U = (d_0, d_x \mid x \in \mathcal{P}) \in \mathbb{Z}^{\mathcal{P}}$ where $d_0 = \dim U_0$ and $d_x = \dim U_x/\underline{U}_x$ with rad $U_x = \underline{U}_x = \sum_{y<x} U_y$.

A *morphism* $\varphi : U \to V$ from a representation U to a representation V is any $k-$linear map $\varphi : U_0 \to V_0$ with the condition $\varphi(U_x) \subseteq V_x$ for all $x \in \mathcal{P}$. The *category of representations* of $\mathcal{P}$ over k is denoted by rep $(\mathcal{P}, k) =$ rep $\mathcal{P}$. Two objects U, V are isomorphic in rep $\mathcal{P}$ $(U \simeq V)$ if and only if there exists an isomorphism of $k-$spaces $\varphi : U_0 \to V_0$ such that $\varphi(U_x) = V_x$ for all $x \in \mathcal{P}$. Denote by Ind $\mathcal{P}$ a complete set of pairwise non-isomorphic indecomposable representations of $\mathcal{P}$ over k.

The *direct sum* $U \oplus V$ of two representations $U, V \in \mathcal{P}$ is the representation $U \oplus V = (U_0 \oplus V_0, U_x \oplus V_x \mid x \in \mathcal{P})$. A representation U is said to be *decomposable* if there exist two representations $U' \neq 0$, $U'' \neq 0$ such that $U \simeq U' \oplus U''$. Otherwise, U is an *indecomposable* representation (Krull-Schmidt category). We say that a representation U is trivial if $\dim U_0 = 1$, i.e., $U_0 = k$.

An ordered set C is called a *chain* (or a totally ordered set or a linearly ordered set) if and only if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (i.e., p and q are comparable). On the other hand, an ordered set $\mathcal{P}$ is called an *antichain* if $x \leq y$ in $\mathcal{P}$ only if $x = y$. An antichain consisting exactly of two (resp. three) points is called a *dyad* (resp. *triad*). If some subsets $X_1, \ldots, X_n \subseteq \mathcal{P}$ do not intersect mutually (but may have comparable points), then their union $X_1 \cup \cdots \cup X_n$ is called a *sum* and is denoted by $X_1 + \cdots + X_n$. We denote by $w(\mathcal{P})$ the *width* of a poset $\mathcal{P}$, i.e., the maximal cardinality of its antichains. Accordingly to the known Dilworth's theorem [82], each poset of finite width n is a sum of n chains.

For a point $a \in \mathcal{P}$ and a subset $A \subseteq \mathcal{P}$, we define their *up*- and *down-cones*

$$a^{\triangledown} = \{x \in \mathcal{P} \mid a \leq x\},\ a_{\vartriangle} = \{x \in \mathcal{P} \mid x \leq a\},\ A^{\triangledown} = \bigcup_{a \in A} a^{\triangledown},\ A_{\vartriangle} = \bigcup_{a \in A} a_{\vartriangle}.$$

For any subset $A \subseteq \mathcal{P}$, we define a trivial representation $k(A) = k(A^{\nabla}) = (k; U_x \mid x \in \mathcal{P})$ of $\mathcal{P}$ where

$$U_x = \begin{cases} k, & \text{if } x \in A^{\nabla}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $k(\varnothing) = (k, 0, \ldots, 0)$. We write often $k(X_1, \ldots, X_n)$ instead of $k(X_1 \cup \cdots \cup X_n)$. For example, let $\mathcal{P}_1 = \{a, b, c\}$ be the triad, i.e., three incomparable points, the elements of Ind $(\mathcal{P})$ are $k(\varnothing)$, $k(a)$, $k(b)$, $k(c)$, $k(a, b)$, $k(b, c)$, $k(a, c)$, $k(a, b, c)$ and $U = (k \oplus k, k \oplus 0, 0 \oplus k, (1, 1)k)$ (see [2]).

Attached to each representation U there exists its *matrix representation* $M = M_U$ choosing some basis B_0 in U_0 and for each $x \in \mathcal{P}$, some system B_x of linearly independent generators of U_x modulo the radical subspace rad U_x. Then

$$M = \begin{array}{|c|c|c|}\hline M_{x_1} & \cdots & M_{x_n} \\ \hline \end{array},$$

with entries in k, partitioned horizontally into $n = |\mathcal{P}|$ blocks (strips). The set of all matrix representations of $\mathcal{P}$ is denoted by $\mathbf{Mat}_{\mathcal{P}}$.

If M and M' are matrix representations of a poset $\mathcal{P} = \{x_i \mid 1 \leq i \leq n\}$ given by

$$M = \begin{array}{|c|c|c|}\hline M_{x_1} & \cdots & M_{x_n} \\ \hline \end{array}, \qquad M' = \begin{array}{|c|c|c|}\hline M'_{x_1} & \cdots & M'_{x_n} \\ \hline \end{array}.$$

then its direct sum $M \oplus M'$ is given by the formula

$$M \oplus M' = \begin{array}{|cc|c|cc|}\hline M_{x_1} & 0 & \ldots & M_{x_n} & 0 \\ \hline 0 & M'_{x_1} & \ldots & 0 & M'_{x_n} \\ \hline \end{array}.$$

Two representations M and N of a poset $\mathcal{P}$ are isomorphic if and only if their matrix representations can be turned into each other with help of the following admissible transformations (denoted by $\mathcal{G}_{\mathcal{P}}$):

(i) Elementary transformations of rows of the whole matrix M.

(ii) Elementary transformations of columns within each vertical strip.

(iii) Additions of columns of a strip M_i to columns of a strip M_j if $i \leq j$ in $\mathcal{P}$.

Then we have defined a *matrix problem* $(\mathbf{Mat}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}})$. A poset $\mathcal{P}$ is said to be of *representation-finite* if $(\mathbf{Mat}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}})$ is of finite representation type.

Remark 1.1. *$(\boldsymbol{Mat}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}})$ is a category whose objects are the matrices M in $\boldsymbol{Mat}_{\mathcal{P}}$ and morphisms are pairs of matrices (C, D), where $C \in Gl(|B_0|, k)$ and D is a matrix in $Gl(|B_0| + \cdots + |B_n|, k)$ which is a composition of elementary matrices corresponding to admissible transformations $\mathcal{G}_{\mathcal{P}}$ ($|B_i|$ denote the number of independent generators of U_i, for $0 \leq i \leq n$)* [84].

Figure 1.9 is an example of a matrix representation of the triad [60].

a						b						c					
I	0	0	0	0	0	I	0	0	0	0	0	I	0	0	0	0	0
0	I	0	0	0	0	0	I	0	0	0	0	0	0	0	0	0	0
0	0	I	0	0	0	0	0	0	0	0	0	0	I	0	0	0	0
0	0	0	I	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	I	0	0	0	0	0	0	0	0	0	I	0	0	0
0	0	0	0	0	0	0	0	I	0	0	0	0	0	I	0	0	0
0	0	0	0	0	0	0	0	0	I	0	0	0	0	0	I	0	0
0	0	0	0	0	0	0	0	0	0	I	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	I	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 1.9. Example of a matrix representation of the triad.

Kleiner presented the finite representation type criterion [63].

Theorem 1.6. [2]. *Let $\mathcal{P}$ be a finite poset. Then $\mathcal{P}$ does not contain $\mathcal{K}_1 = (1,1,1,1)$, $\mathcal{K}_2 = (2,2,2)$, $\mathcal{K}_3 = (1,3,3)$, $\mathcal{K}_4 = (N,4)$ or $\mathcal{K}_5 = (1,2,5)$ as a subposet if and only if rep $\mathcal{P}$ has finite representation type.*

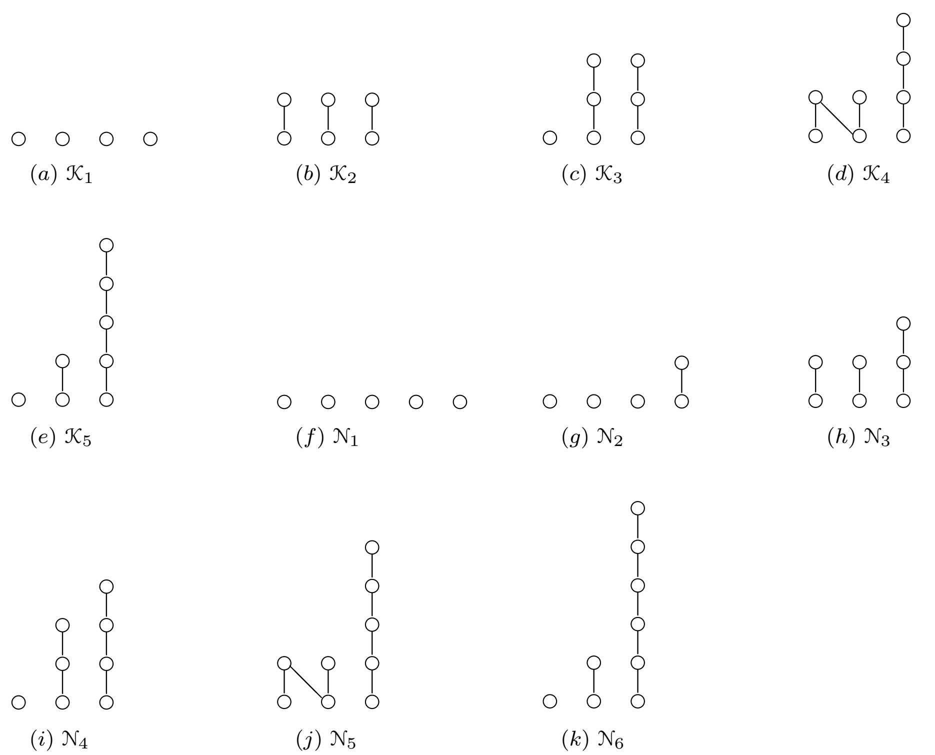


FIGURE 1.10. Kleiner's critical $\mathcal{K}_1 - \mathcal{K}_5$ and Nazarova's critical $\mathcal{N}_1 - \mathcal{N}_6$.

Nazarova extended the result of Kleiner and showed the tame representation type criterion [73].

Theorem 1.7. [2]. *Let $\mathcal{P}$ be a finite poset and k a field. Then rep $\mathcal{P}$ has wild representation type if and only if $\mathcal{P}$ contains $\mathcal{N}_1 = (1,1,1,1,1)$, $\mathcal{N}_2 = (1,1,1,2)$, $\mathcal{N}_3 = (2,2,3)$, $\mathcal{N}_4 = (1,3,4)$, $\mathcal{N}_5 = (N,5)$ or $\mathcal{N}_6 = (1,2,6)$ as a subposet.*

1.5 Dyck Paths

Dyck paths is an important tool in combinatorics which is in relationship with Catalan objects as permutations, binary trees, non-decreasing parking functions, triangulations of a regular polygon, etc [10, 92]. Dyck paths can be defined as lattice paths connecting points in a square lattice that satisfies some conditions in the xy plane. Such Dyck paths are also described by using some Dyck words. In this section, we present the concept of a square lattice, lattice path, Dyck words. A connection between Dyck words and Dyck paths is given as well [6, 12, 48, 92, 96].

A *lattice* $\Lambda = (V, E)$ is a mathematical model of a discrete space. It consists of two sets, a set $V \subset \mathbb{R}^n$ of vertices and a set $E \subset \mathbb{R}^n \times \mathbb{R}^n$ of edges, with no more than two edges between any two vertices. If two vectors are connected via an edge, we call them nearest neighbors.

Let $\Lambda = (V, E)$, an n-step *lattice path* or *lattice walk* or *walk* from $s \in V$ to $x \in V$ is a sequence $w = (w_0, \dots, w_n)$ of elements in V, such that

1. $w_0 = s$, $w_n = x$,
2. $(w_i, w_{i+1}) \in E$.

The length $|w|$ of a lattice path is the number n of steps (edges) in the sequence w.

The *Euclidean lattice* is a lattice where $V = \mathbb{Z}^d$. The edges are mostly defined through a so called step set. On this lattice an alternative definition via the step set can be used. A step set $S \subset \mathbb{Z}^d$ is the fixed and finite set of possible steps. The elements of S are called steps. If the step set S is a subset of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, then we say S is a set of small steps.

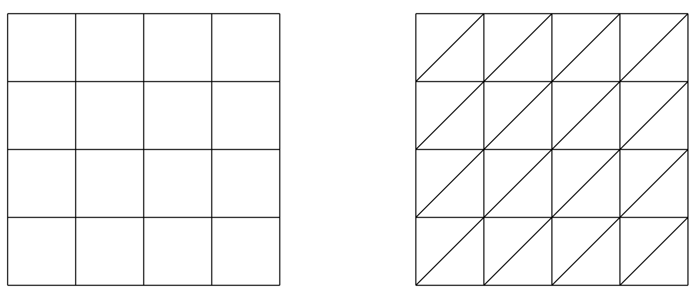


FIGURE 1.11. Square lattice (left) and triangular lattice (right).

For the square lattice and triangular lattice in Figure 1.11, the sets of small steps are $S_1 = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ and $S_2 = \{(1, 0), (0, 1), (-1, 0), (0, -1), (1, 1), (-1, -1)\}$, respectively.

An $n-$step lattice path or lattice walk or walk from $s \in \mathbb{Z}^d$ to $x \in \mathbb{Z}^d$ relative to S is a sequence $w = (w_0, \dots, w_n)$ of elements in $\mathbb{Z}^d$, such that [96]

1. $w_0 = s$, $w_n = x$,
2. $w_{i+1} - w_i \in S$

Let A be the diagonal square lattice where $V_A = \{(x, y) \in \mathbb{Z}^2 \mid x \geq 0, y \geq 0\}$ and $S_A = \{(1, 1), (1, -1)\}$, walks on the diagonal square lattice A are equivalent to walks on the square lattice B with $V_B = \{(x, y) \in \mathbb{Z}^2 \mid x \geq 0, y = x\}$ and $S_B = \{(1, 0), (0, 1)\}$ [12].

A Dyck path is a lattice path in $\mathbb{Z}^2$ with steps $(1, 1)$ and $(1, -1)$, such that the path starts at $(0, 0)$ and ends at $(2n, 0)$ and it does not pass below the $x-$axis. The number of Dyck paths of length 2n is equal to the $n-$th Catalan number ($C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$)[92].

Figure 1.12 shows the set of all lattice paths of length 6 in the square lattice B as described above.

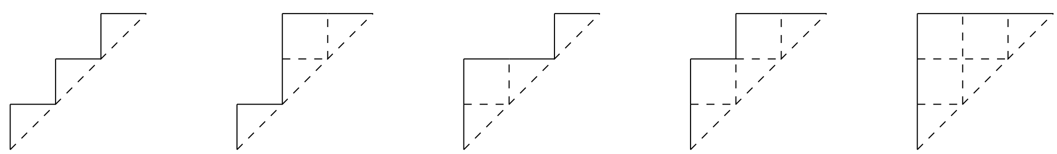


Figure 1.12. Lattice paths from $(0, 0)$ to $(3, 3)$.

Let X be an *alphabet*. We define the *free monoid* generated by X, denoted by X^*, as the set of the finite words written with X's letters. The *product* of $u = u_1 \dots u_p \in X^*$ and $v = v_1 \dots v_q \in X^*$ is defined as the concatenation of these words: $uv = u_1 \dots u_p v_1 \dots v_q$. The word u is called a *left factor* of the word $w = uv$. The *empty word* is denoted by e. The number of occurrences of the letter $a \in X$ in the word w is denoted by $|w|_a$, and the length of w by

$$|w| = \sum_{a \in X} |w|_a, \tag{1.6}$$

The set of *Dyck words* is the set of words $w \in X^* = \{U, D\}^*$ characterized by the following two conditions [6]:

- for any left factor u of w, $|u|_U \geq |u|_D$,
- $|w|_U = |w|_D$.

For example, the set of Dyck words of length 6 is

$$\{UDUDUD, UDUUDD, UUDDUD, UUDUDD, UUUDDD\}. \tag{1.7}$$

The number of Dyck words of length 2n is equal to the $n-$th Catalan number [47].

There is a bijective correspondence between the set of Dyck paths and the set of Dyck words [6].

1.6 Cluster Algebras

In 2002, Fomin and Zelevinsky introduced the term of the cluster algebra [54] as a subalgebra of a field of rational functions generated by the set of cluster variables [52, 55, 56]. The cluster algebras are in connection with different topics as algebraic combinatorics, Lie theory, discrete dynamical systems, tropical geometry, and others. Afterwards, Fomin, Schiffler et al introduced cluster algebras associated to surfaces [19, 52–56, 69].

The definition of a cluster algebra $\mathcal{A}$ starts by introducing its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield, i.e., an abelian multiplicative group endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The group ring $\mathbb{Z}\mathbb{P}$ will be used as a field of scalars (ground ring) for $\mathcal{A}$.

Let J be a finite set of labels, and let Trop $(u_j : j \in J)$ be an abelian group (written multiplicatively) freely generated by the elements u_j. We define the addition $\oplus$ in Trop $(u_j ; i \in J)$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}, \tag{1.8}$$

and call (Trop $(u_j : j \in J), \oplus, \cdot)$ a *tropical semifield.* To illustrate, $u_2 \oplus u_1^2 u_2^{-1} = u_2^{-1}$ in Trop (u_1, u_2). The group ring of Trop $(u_j : j \in J)$ is the ring of *Laurent polynomials* in the variables u_j. If J is empty, we obtain the trivial semifield consisting of a single element 1.

As an *ambient field* for a cluster algebra $\mathcal{A}$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables (here n is the rank of $\mathcal{A}$), with coefficients in $\mathbb{Q}\mathbb{P}$. Note that the definition of $\mathcal{F}$ ignores the auxiliary addition in $\mathbb{P}$.

A *labeled* $Y-$*seed* in $\mathbb{P}$ is a pair $(\mathbf{y}, B)$, where:

- $\mathbf{y}= (y_1, \ldots, y_n)$ is an $n-$tuple of elements of $\mathbb{P}$,
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is skew-symmetrizable.

That is, $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$. A *labeled seed* in $\mathcal{F}$ is a triple $(\mathbf{x}, \mathbf{y}, B)$, where;

- $(\mathbf{y}, B)$ is a labeled $Y-$seed,
- $\mathbf{x}= (x_1, \ldots, x_n)$ is an $n-$tuple of elements of $\mathcal{F}$ forming a free generating set.

That is, $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Q}\mathbb{P}$, and $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_n)$. We refer to $\mathbf{x}$ as the (labeled) *cluster* of a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, to the tuple $\mathbf{y}$ as the *coefficient tuple*, and to the matrix B as the *exchange matrix.*

The (unlabeled) seeds are obtained by identifying labeled seeds that differ from each other by simultaneous permutations of the components in $\mathbf{x}$ and $\mathbf{y}$, and of the rows and columns of B [55].

We use the notation $[x]_+ = \max\,(x, 0)$, $[1, n] = \{1, \ldots, n\}$, and

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed in $\mathcal{F}$, and let $k \in [1, n]$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, B)$ into the labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B)$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases} \tag{1.9}$$

- The coefficient tuple $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is given by

$$y'_j = \begin{cases} y_k^{-1}, & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}}, & \text{if } j \neq k. \end{cases} \tag{1.10}$$

- The cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation

$$x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k}. \tag{1.11}$$

We consider the $n-$*regular tree* $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$, so that the n edges emanating from each vertex receive different labels. We write $t \xrightarrow{k} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation in direction k. The elements of Σ_t are written as follows:

$$\mathbf{x}_t = (x_{1,t}, \ldots, x_{n,t}),\ \mathbf{y}_t = (y_{1,t}, \ldots, y_{n,t}),\ B_t = (b^t_{ij}).$$

A cluster pattern is uniquely determined by each of its seeds, which can be chosen arbitrarily.

For example (case $\mathbb{A}_2$, see [56]), let $n = 2$, then the tree $\mathbb{T}_2$ is an infinite chain. We denote its vertices by $\ldots, t_{-1}, t_0, t_1, \ldots$, and label its edges as follows:

$$\ldots \xrightarrow{2} t_{-1} \xrightarrow{1} t_0 \xrightarrow{2} t_1 \xrightarrow{1} t_2 \xrightarrow{2} \ldots$$

We denote the corresponding seeds by $\Sigma_m = \Sigma_{t_m} = (\mathbf{x}_m, \mathbf{y}_m, B_m)$, for $m \in \mathbb{Z}$. Let the initial seed Σ_0 be

$$\mathbf{x}_0 = (x_1, x_2),\ \mathbf{y}_0 = (y_1, y_2),\ B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.12}$$

We then recursively compute the seeds $\Sigma_1, \ldots, \Sigma_5$ as shown in Table 1.1 with $B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

t	B_t	$\mathbf{y}_t$		$\mathbf{x}_t$	
0	B_0	y_1	y_2	x_1	x_2
1	B_1	$y_1(y_2 \oplus 1)$	$\frac{1}{y_2}$	x_1	$\frac{x_1y_2+1}{x_2(y_2 \oplus 1)}$
2	B_0	$\frac{1}{y_1(y_2 \oplus 1)}$	$\frac{y_1y_2 \oplus y_1 + 1}{y_2}$	$\frac{x_1y_1y_2+y_1+x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$	$\frac{x_1y_2+1}{x_2(y_2 \oplus 1)}$
3	B_1	$\frac{y_1 \oplus 1}{y_1y_2}$	$\frac{y_2}{y_1y_2 \oplus y_1 + 1}$	$\frac{x_1y_1y_2+y_1+x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$	$\frac{y_1+x_2}{x_1(y_1 \oplus 1)}$
4	B_0	$\frac{y_1y_2}{y_1 \oplus 1}$	$\frac{1}{y_1}$	x_2	$\frac{y_1+x_2}{x_1(y_1 \oplus 1)}$
5	B_1	y_2	y_1	x_2	x_1

TABLE 1.1. Seeds for the case $\mathbb{A}_2$.

Cluster Algebra

Given a cluster pattern, we denote by

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} : t \in \mathbb{T}_n, 1 \le i \le n\},$$

the union of clusters of all of the seeds in the pattern. We refer to the elements $x_{i,t} \in \mathcal{X}$ as *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given cluster pattern is the $\mathbb{ZP}-$subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ is any labeled seed in the underlying cluster pattern. A cluster algebra is of *geometric type* if the coefficient semifield $\mathbb{P}$ is a tropical semifield.

We say that a cluster algebra is of finite type if it has finitely many seeds. More specifically, we define the diagram $\Gamma(B)$ associated to an $n \times n$ exchange matrix B to be a weighted directed graph on nodes $v_1, \ldots, v_n$, with v_i directed towards v_j if and only if $b_{ij} > 0$. In that case, we label this edge by $|b_{ij}b_{ji}|$. Then $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is of finite type if and only if $\Gamma(B)$ is mutation-equivalent to an orientation of a *finite type* Dynkin diagram [55]. In this case, we say that B and $\Gamma(B)$ are of finite type. We say that a matrix B (and the corresponding cluster algebra) has finite mutation type if its mutation equivalence class is finite, i.e. only finitely many matrices can be obtained from B by repeated matrix mutations. A classification of all cluster algebras of finite mutation type with skew-symmetric exchange matrices was given by Felikson, Shapiro, and Tumarkin [51] (see [69]).

Cluster Algebras From Quivers

For quivers, cluster algebras are defined as follows:

Fix an integer $n \geq 1$. In this case, a seed (Q, u) consists of a finite quiver Q without loops or 2-cycles with vertex set $\{1, \ldots, n\}$, whereas u is a free-generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$.

Let (Q, u) be a seed and k a vertex of Q. The mutation $\mu_k(Q, u)$ of (Q, u) at k is the seed (Q', u'), where;

(a) Q' is obtained from Q as follows;

  (1) reverse all arrows incident with k,

  (2) for all vertices $i \neq j$ distinct from k, modify the number of arrows between i and j, in such a way that a system of arrows of the form $(i \xrightarrow{r} j, i \xrightarrow{s} k, k \xrightarrow{t} j)$ is transformed into the system $(i \xrightarrow{r+st} j, k \xrightarrow{s} i, j \xrightarrow{t} k)$. And the system $(i \xrightarrow{r} j, j \xrightarrow{t} k, k \xrightarrow{s} i)$ is transformed into the system $(i \xrightarrow{r-st} j, i \xrightarrow{s} k, k \xrightarrow{t} j)$. Where, r, s and t are non-negative integers, an arrow $i \xrightarrow{l} j$, with $l \geq 0$ means that l arrows go from i to j and an arrow $i \xrightarrow{l} j$, with $l \leq 0$ means that $-l$ arrows go from j to i.

(b) u' is obtained form u by replacing the element u_k with

$$u_k = \frac{1}{u_k} \prod_{\text{arrows } i \to k} u_i + \prod_{\text{arrows } k \to j} u_j. \tag{1.13}$$

If there are no arrows from i with target k, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(\mu_k(Q, u)) = (Q, u)$. Thus, if Q is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$, the following interpretations have place:

1. the clusters with respect to Q are the sets u appearing in seeds, (Q, u) obtained from a initial seed (Q, x) by iterated mutation,
2. the cluster variables for Q are the elements of all clusters,
3. the cluster algebra $\mathcal{A}(Q)$ is the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

As example, the cluster variables associated to the quiver $Q = 1 \longrightarrow 2$ are:

$$\Big\{x_1, x_2, \frac{1 + x_2}{x_1}, \frac{1 + x_1 + x_2}{x_1 x_2}, \frac{1 + x_1}{x_2}\Big\}.$$

Cluster Algebra Arising from Unpunctured Surface

Let S be a connected oriented $2-$dimensional Riemann surface with nonempty boundary, and let M be a nonempty finite subset of the boundary of S, such that each boundary component of S contains at least one point of M. The elements of M are called *marked points*. The pairs (S, M) is called a *bordered surface with marked points*. Marked points in the interior of S are called punctures (For technical reasons, we require that (S, M) is not a disk with 1,2 or 3 marked points) [22].

An *arc* γ in (S, M) is a curve in S, considered up to isotopy, such that:

(i) the endpoints of γ are in M,

(ii) γ does not cross itself, except that its endpoints, may coincide,

(iii) except for the endpoints, γ is disjoint from the boundary of S,

(iv) γ does not cut out a monogon or a bigon.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are boundary segments. Note that boundary segments are not arcs. For any two arcs γ, γ' in S, let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs α and α', where α and α' range over all arcs isotopic to γ and γ', respectively. We say that arcs γ and γ' are *compatible* if $e(\gamma, \gamma') = 0$.

A *triangulation* is a maximal collection of pairwise compatible arcs (together with all boundary segments). Triangulations are *connected* to each other by sequences of flips. Each flip replaces a single arc γ in a triangulation T by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in T, forms a new triangulation.

Choose any triangulation T of (S, M), and let $\tau_1, \ldots, \tau_n$ be the n arcs of T. For any triangle Δ in T, we define a matrix $B^{\Delta} = (b^{\Delta}_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ as follows.

- $b^{\Delta}_{ij} = 1$ and $b^{\Delta}_{ji} = -1$ if τ_i and τ_j are sides of Δ with τ_j following τ_i in the clockwise order,
- $b^{\Delta}_{ij} = 0$ otherwise.

Then define the matrix $B_T = (b_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ by $b_{ij} = \sum_{\Delta} b^{\Delta}_{ij}$, where the sum is taken over all triangles in T. Note that B_T is skew-symmetric and each entry b_{ij} is either $0, \pm 1$, or ± 2, since every arc τ is in at most two triangles.

According to Schiffler and Canakci [22], Fomin, Shapiro and Thurston [54] associated a cluster algebra $\mathcal{A}(S, M)$ to any bordered surface with marked points (S, M), and the cluster variables of $\mathcal{A}(S, M)$ are in bijection with the (tagged) arcs of (S, M).

The following theorem regarding relationships between cluster algebras and surface triangulations was obtained Fomin, Shapiro, and Thurston [52, 53].

Theorem 1.8. [69]. *Fix a bordered surface (S, M) and let $\mathcal{A}$ be the cluster algebra associated to the signed adjacency matrix of a tagged triangulation. Then the (unlabeled) seed Σ_T of $\mathcal{A}$ are in bijection with tagged triangulations T of (S, M), and the cluster variables are in bijection with the tagged arcs of (S, M) (so we can denote each by x_γ, where γ is a tagged arc). Moreover, each seed in $\mathcal{A}$ is uniquely determined by its cluster. Furthermore, if a tagged triangulation T' is obtained from another tagged triangulation T by flipping a tagged arc $\gamma \in T$ and obtaining γ', then $\Sigma_{T'}$ is obtained from Σ_T by the seed mutation replacing x_γ by $x_{\gamma'}$.*

1.7 Category of Diagonals and Cluster-tilted Algebras

In 2006 [17], Caldero, Chapoton, and Schiffler introduced the category of diagonals of a polygon with $n+3$ sides associated to a triangulation T, in this case, the diagonals are called *roots* which can be classified as negative or positive, negative roots are those roots belonging to the triangulation T [17, 80].

The combinatorial $\mathbb{C}$-linear additive category C_T is described as follows. The objects are positive integral linear combinations of positive roots, and the space of morphisms from a positive root α to a positive root α' is a quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from α to α'. The subspace which defines the quotient is spanned by the so-called *mesh relations.* For any couple α, α' of positive roots such that α is related to α' by two consecutive pivoting elementary moves with distinct pivots, the mesh relations are given by the identity $P_{v_2'}P_{v_1} = P_{v_1'}P_{v_2}$, where v_1, v_2 (resp. $v_1'v_2'$) are the vertices of α (resp. α') such that $P_{v_1'}P_{v_2} = \alpha'$.

Let T be a triangulation, then one can define a planar tree t_T as follows. Its vertices are the triangles of T and the edges connect adjacent triangles. In the same way, we can define a graph Q_T whose vertices are the inner edges of T and are related to each other by an edge, if they bound the same triangle. An orientation can be defined by using graph Q_T, in such a way that a vertex i connects a vertex j (denoted $i \to j$), if $-\alpha_j$ can be obtained from the diagonal $-\alpha_i$ by rotating anticlockwise about their common vertex.

According to Caldero, Chapoton, and Schiffler [17], one can define a $\mathbb{C}-$linear abelian category Mod Q_T as follows. This is the category of modules over the quiver Q_T with the following relations, called *triangle relations*:

In any triangle, the composition of two successive maps is zero.

These relations are exactly the relations prescribed by [17, Definition 1].

Figure 1.13 shows an example of the tree and the quiver associated to a triangulation.

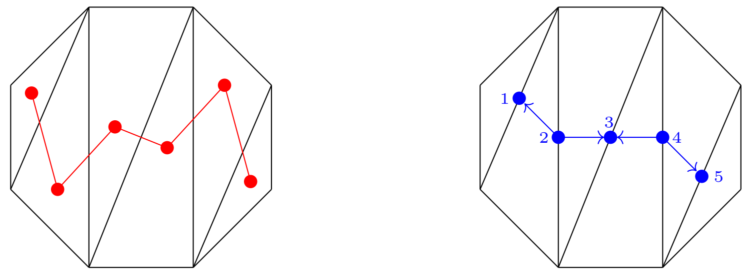


FIGURE 1.13. Tree (left) and quiver (right) associated to a triangulation of the 8-polygon.

The following results regarding the category of diagonals were given by Caldero, Chapoton, and Schiffler in [17].

Theorem 1.9. *There is an equivalence of categories between C_T and Mod Q_T.*

Corollary 1.1. *There exists a bijection φ between Ind Q_T and the diagonals of the polygon not in T. Moreover, for M in Ind Q_T and any vertex i of Q_T, the multiplicity of the simple module S_i in the module M is 1 if $\varphi(M)$ crosses the i^{th} diagonal of T and 0 if not. In particular, for two isoclasses M, M' in Ind Q_T, we have $M = M'$ if and only if $n_i(M) = n_i(M')$ for all i.*

Theorem 1.10. *Let T be a triangulation of the $n+3$ polygon, and let C_T be the corresponding category, then:*

(i) The irreducible morphisms of C_T are direct sums of the generating morphisms given by pivoting elementary moves.

(ii) The mesh relation of C_T are the mesh relations [5] of the Auslander-Reiten quiver of C_T.

(iii) The Auslander-Reiten translate is given on diagonals by r^-.

(iv) The indecomposable projective objects of C_T are diagonals in $r^+(T)$.

(v) The indecomposable injective objects of C_T are diagonals in $r^-(T)$.

with r^+ (resp. r^-) the elementary rotation of the polygon in the positive (resp. negative) direction.

Theorem 1.11. *Let $C = \{u_1, \ldots, u_n\}$ be a cluster of a cluster algebra of type $\mathbb{A}_n$ and let V be the set of all cluster variables of the algebra. Let Q_C be the quiver with relations associated to C and Ind Q_C the set of isoclasses of indecomposable modules. Then there is a bijection*

$$\textit{Ind } Q_C \to V \setminus C, \ \alpha \mapsto w_\alpha,$$

such that

$$w_\alpha = \frac{P(u_1, \ldots, u_n)}{\prod_{i=1}^{n} u_i^{n_i(\alpha)}},$$

where P is a polynomial such that none of the u_i divides P $(i = 1, \ldots, n)$ and $n_i(\alpha)$ is the multiplicity of the simple module α_i in the module α.

The Auslander-Reiten quiver of the quiver shown in Figure 1.13 is given by Figure 1.14.

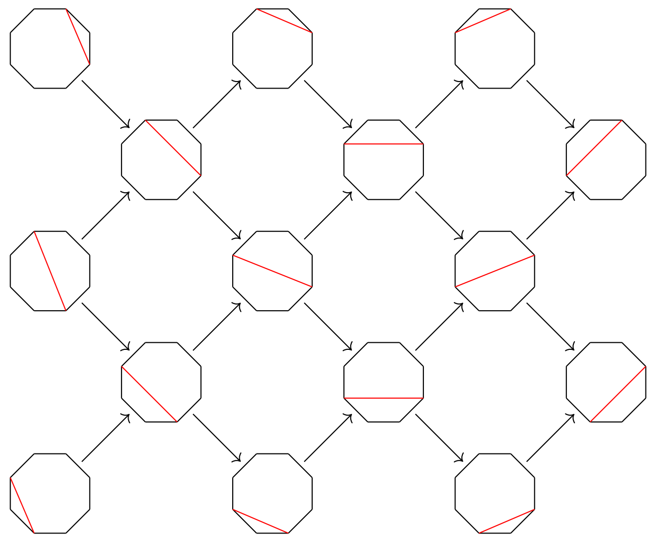


FIGURE 1.14. Auslander-Reiten quiver of Q_T.

Cluster-tilted Algebras of Type $\mathbb{A}_n$

In this section, we recall some results regarding cluster-tilted algebras [95].

The cluster category was introduced independently in [17] for type $\mathbb{A}_n$ and in [13] for the general case. Let $\mathcal{D}^b(\text{mod } H)$ be the bounded derived category of the finitely generated modules over a finite dimensional hereditary algebra H over a field k. In [13] the cluster category was defined as the orbit category $\mathcal{C} = \mathcal{D}^b(\text{mod } H)/\tau^{-1}[1]$, where τ is the Auslander-Reiten translation and [1] the suspension functor. The cluster-tilted algebras are the algebras of the form $\Gamma = \text{End}_{\mathcal{C}}(B)^{op}$, where B is a cluster-tilting object in $\mathcal{C}$ [14].

Let Q be a quiver with no multiple arrows, no loops and no oriented cycles of length two and let Q' be a quiver obtained from Q via mutations. We say that a quiver Q is mutation equivalent to Q', if Q' can be obtained from Q by a finite number of mutations. The mutation class of Q is all quivers mutation equivalent to Q. The mutation class of a Dynkin quiver Q is finite [55].

If Γ is a cluster-tilted algebra, then we say that Γ is of type $\mathbb{A}_n$ if it arises from the cluster category of a path algebra of Dynkin type $\mathbb{A}_n$. Let Q be a quiver of a cluster-tilted algebra Γ, if Q' is obtained from Q by a finite number of mutations, then there is a cluster-tilted algebra Γ' with quiver Q'. Moreover, Γ is of finite representation type if and only if Γ' is of finite representation type. We also have that Γ is of type $\mathbb{A}_n$ if and only if Γ' is of type $\mathbb{A}_n$. We know that a cluster-tilted algebra is up to isomorphism uniquely determined by its quiver [13–17, 95]. It follows from this that to count the number of cluster-tilted algebras of type $\mathbb{A}_n$, it is enough to count the mutation class of any quiver with underlying graph $\mathbb{A}_n$.

We define mutation of a triangulation at a given diagonal, by replacing this diagonal with another one. This can be done in one and only one way. Let Q_T be a quiver corresponding to a triangulation T. Then mutation of Q_T at the vertex i corresponds to mutation of T at the diagonal corresponding to i.

Let $\mathcal{M}_n$ be the mutation class of $\mathbb{A}_n$, i.e. all quivers obtained by repeated mutation from $\mathbb{A}_n$, up to isomorphisms of quivers. Let $\mathcal{T}_n$ be the set of all triangulations of an $n+3$ polygon. We can define a function $\gamma : \mathcal{T}_n \to \mathcal{M}_n$ where we set $\gamma(T) = Q_T$ for any triangulation T in $\mathcal{T}_n$. Note that γ is surjective.

For a triangulation T of an $n+3$ polygon, let us denote by T^i the triangulation obtained from T by rotating T i steps in the clockwise direction. We define an equivalence relation on $\mathcal{T}_n$, where we let $T \sim T^i$ for all i. We define a new function $\overline{\gamma} : (\mathcal{T}_n \backslash \sim) \to \mathcal{M}_n$ induced from $\overline{\gamma}$.

The following results regarding cluster-tilted algebras of type $\mathbb{A}_n$ were obtained by Torkildsen in [95].

Theorem 1.12. *The function $\overline{\gamma} : (\mathcal{T}_n \backslash \sim) \to \mathcal{M}_n$ is bijective for all $n \geq 2$.*

Corollary 1.2. *The number $a(n)$ of non-isomorphic basic cluster-tilted algebras of type $\mathbb{A}_n$ is the number of triangulations of the disk with n diagonals, i.e.*

$$a(n) = C_{n+1}/(n+3) + C_{(n+1)/2}/2 + (2/3)C_{n/3}, \tag{1.14}$$

where C_i is the i-th Catalan number and the second term is omitted if $(n+1)/2$ is not an integer and the third term is omitted if $n/3$ is not an integer.

1.8 Friezes

In this section, we recall the concepts of frieze patterns, a generalization associated to Cartan matrix, vector friezes and its connection with cluster algebras [4, 7, 43–45, 62, 71].

Coxeter introduced frieze patterns in [45] in the early 1970s, inspired by Gauss's pentagramma mirificum. A frieze pattern is a grid of positive integers, with a finite number of infinite rows, where the top and bottom rows are bi-infinite repetition of 0s and the second to top and the second to bottom row are bi-infinite repetitions of 1s, and every four adjacent numbers of the following square

$$\begin{array}{ccc} & b & \\ a & & c \\ & d & \end{array}$$

satisfy the identity $ac - bd = 1$. The sequence of integers in the first non-trivial row, $(m_{ii})_{i\in\mathbb{Z}}$, is called *quiddity sequence*. This sequence completely determines the frieze pattern. Each frieze pattern is also periodic, since it is invariant under glide reflection. The order of the frieze pattern is defined to be the number of rows minus one. It follows that each frieze pattern of order n is $n-$periodic [7, 8]. Conway and Coxeter classified completely the frieze patterns whose entries are positive integers, and show that these frieze patterns constitute a manifestation of the Catalan numbers [43, 44]. Specifically, there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labeled vertices. From every triangulation T of a regular n-gon with vertices cyclically labeled 1 through n, Conway and Coxeter build an n-rowed frieze pattern determined by the numbers $a_1, a_2, \ldots, a_n$ where a_k is the number of triangles in T incident with vertex k. Specifically [71]:

(1) the top row of the array is $\ldots, 0, 0, 0, \ldots$;

(2) the second row (offset from the first) is $\ldots, 1, 1, 1, \ldots$;

(3) the third row is $\ldots, a_1, \ldots, a_n, a_1, \ldots$ (with period n);

(4) each succeeding row (offset from the one before) is determined by the frieze recurrence of the four adjacent numbers given as above.

For instance, given a frieze pattern

$$\begin{array}{ccccccccccccc}
\ldots & & 0 & & 0 & & 0 & & 0 & & 0 & & \ldots \\
& \ldots & & 1 & & 1 & & 1 & & 1 & & \ldots & \\
\ldots & & 2 & & 3 & & 1 & & 2 & & 3 & & \ldots \\
& \ldots & & 5 & & 2 & & 1 & & 5 & & \ldots & \\
\ldots & & 2 & & 3 & & 1 & & 2 & & 3 & & \ldots \\
& \ldots & & 1 & & 1 & & 1 & & 1 & & \ldots & \\
\ldots & & 0 & & 0 & & 0 & & 0 & & 0 & & \ldots
\end{array}$$

this is in relationship with a triangulation of the form

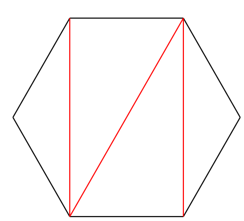


FIGURE 1.15. Example of triangulation associated to a frieze pattern.

In 2010 Assem, Reutenauer and Smith [4] introduced a generalization of friezes associated to Cartan matrix (see [3] 226p.), in the following way, let $C = (C_{i,j})_{n\times n}$ be a Cartan matrix of a connected Quiver Q, then a frieze is a collection of positive integers $a(j, m)$, with $j \in \{1, \ldots, n\}$ and $m \in \mathbb{Z}$, such that

$$a(j, m)a(j, m + 1) = 1 + \left(\prod_{j\to i} a(i, m)^{|C_{i,j}|}\right)\left(\prod_{i\to j} a(i, m + 1)^{|C_{i,j}|}\right). \tag{1.15}$$

For instance, if Q is a Dynkin diagram of type $\mathbb{D}_6$ with any orientation, a frieze associated to $\mathbb{D}_6$ is

$$
\begin{array}{ccccccccccccc}
 & \dots & & 1 & & 3 & & 3 & & 1 & & \dots & \\
\dots & & 2 & & 2 & & 8 & & 2 & & 2 & & \dots \\
 & \dots & & 3 & & 5 & & 5 & & 3 & & \dots & \\
\dots & & 7 & 4 & 7 & 2 & 3 & 2 & 7 & 4 & 7 & & \dots \\
 & \dots & & 4 & & 2 & & 2 & & 4 & & \dots &
\end{array}
$$

Many authors have studied properties of the friezes, and they have found connections with different topics (see examples in [4, 7–9, 57, 62, 66, 67, 71]). In particular, Fontaine and Plamondon [57] obtained the following results.

Theorem 1.13. *The number of friezes of type $\mathbb{D}_n$ is $\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$ where $d(m)$ denotes the number of divisors of m.*

Corollary 1.3. *The number of friezes in type, $\mathbb{B}_n$, $\mathbb{C}_n$, and $\mathbb{G}_2$ is $\sum_{m\leq\sqrt{n+1}} \binom{2n-m^2+1}{n}$, $\binom{2n}{n}$, and 9, respectively.*

Fontaine, Plamondon and Propp (in type $\mathbb{E}_6$) conjectured that the number of friezes in type $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and $\mathbb{F}_4$ is 868, 4400, 26592, and 112, respectively [57, 71].

One way to define friezes is to say that they are ring homomorphisms from a cluster algebra to the ring of integers such that all cluster variables are sent to positive integers [57]. Let Q be a quiver without loops and $2-$cycles and let $\mathcal{A}(Q)$ be the corresponding cluster algebra with trivial coefficients (see [54]).

(i) A *frieze of type Q* is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \to \mathbf{R}$ from the cluster algebra to an integral domain $\mathbf{R}$. The frieze is called *integral* if $\mathbf{R} = \mathbb{Z}$.

(ii) A frieze $\mathcal{F}$ is said to be *unitary* if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that every cluster variable $x \in \mathbf{x}$ is mapped by $\mathcal{F}$ to a unit in R.

(iii) A frieze is said to be *non-zero* if every cluster variable in $\mathcal{A}(Q)$ is mapped by $\mathcal{F}$ to a non-zero element of $\mathbf{R}$.

(iv) An integral frieze is said to be *positive* if every cluster variable in $\mathcal{A}(Q)$ is mapped by $\mathcal{F}$ to a positive integer.

Let $\mathbf{x}= (x_1, \dots , x_n)$ be a cluster of $\mathcal{A}(Q)$.

(i) A vector $(a_1, \dots , a_n) \in \mathbf{R}^n$ is called a *frieze vector* relative to $\mathbf{x}$ if the frieze $\mathcal{F}$ defined by $\mathcal{F}(x_i) = a_i$ has values in $\mathbf{R}$. If the frieze $\mathcal{F}$ is unitary we say that the frieze vector $(a_1, \dots , a_n)$ is *unitary.*

(ii) A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ is called a *positive* frieze vector relative to $\mathbf{x}$ if the frieze $\mathcal{F}$ defined by $\mathcal{F}(x_i) = a_i$ is positive integral.

Gunawan and Schiffler proved the following result [62].

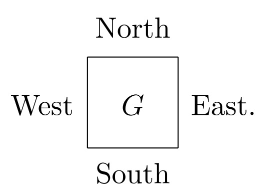
Theorem 1.14. *Let Q be a quiver without loops and $2-$cycles and let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an arbitrary cluster of $\mathcal{A}(Q)$. Then there is a bijection*

$$\begin{array}{rcl} \phi : \{\textit{unordered clusters in } \mathcal{A}(Q)\} & \longrightarrow & \{\textit{positive unitary frieze vectors relative to } \boldsymbol{x}\} \\ \boldsymbol{x}' = \{x'_1, \ldots, x'_n\} & \longmapsto & \phi(\boldsymbol{x}') = (a_1, \ldots, a_n). \end{array}$$

1.9 Snake Graphs

Snake graph is a combinatorial tool that has appeared in cluster algebras. According to Propp, given a triangulation T, we can define a graph whose n vertices correspond to the vertices in T and $n-2$ vertices corresponded to the triangular faces of T [71]. Canakci and Schiffler have studied relationships between snake graphs and continued fractions, introducing a calculus for cluster algebras [19–23] (see other works [68, 81]). In particular, Musiker, Schiffler, and Williams introduced a combinatorial formula for the cluster variables of cluster algebras from surfaces by using snake graphs and its perfect matchings [69]. In this section, we recall the definition of a snake graph, the number of perfect matchings associated to these graphs, and the way that these concepts can be used to find out a formula for the cluster variables of a cluster algebra associated to a surface [19, 21, 22, 69].

A *tile* G is a square of fixed side-length in the plane whose sides are parallel or orthogonal to the fixed basis.



We consider a tile G as a graph with four vertices and four edges in the obvious way. A *snake graph* $\mathcal{G}$ is a connected graph consisting of a finite sequence of tiles $G_1, \ldots, G_d$ with $d \geq 1$, such that for each $i = 1, \ldots, d-1$

(i) G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1}.

(ii) G_i and G_j have no edge in common whenever $|i-j| \geq 2$.

(iii) G_i and G_j are disjoint whenever $|i-j| \geq 3$.

For notation, $\mathcal{G}[i, i+t] = (G_i, \ldots, G_{i+t})$ is a subgraph of $\mathcal{G} = (G_1, \ldots, G_n)$, the $d-1$ edges $e_1, \ldots, e_{d-1}$ which are contained in two tiles are called *interior edges* of $\mathcal{G}$ and the other edges are called *boundary edges.* A *perfect matching* P of a graph G is a subset of the set of edges of G such that each vertex of G is incident to exactly one edge in P. Let $\mathrm{Match}(G)$ denote the set of all perfect matchings of the graph G. The following figure presents some perfect matchings of a snake graph.

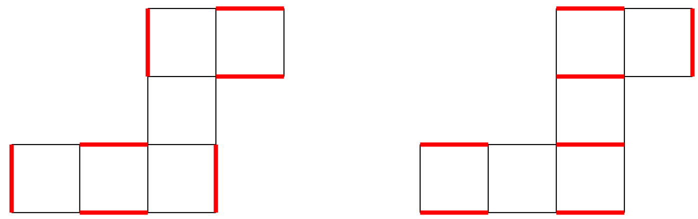


FIGURE 1.16. Perfect matchings of a snake graph.

Snake Graphs and Cluster Algebras

Let T be a triangulation of a surface (S, M) and let γ be an arc in (S, M) which is not in T. Choose an orientation on γ, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. Denote by $s = p_0, p_1, \ldots, p_{d+1} = t$ the ordered points of intersection of γ and T. For $j = 1, 2, \ldots, d$, let τ_{i_j} be the arc of T containing p_j, and let Δ_{j-1} and Δ_j be the two triangles in T on either side of τ_{i_j}. Then, for $j = 1, \ldots, d-1$, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two sides of the triangle Δ_j in T and we define e_j to be the third arc in this triangle.

Let G_j be the quadrilateral in T that contains τ_{i_j} as a diagonal (a tile) whose edges are arcs in T, thus, they are labeled edges. Define a sign function f of the edges $e_1, \ldots, e_d$ by

$$f(e_j) = \begin{cases} +1, & \text{if } e_j \text{ lies on the right of } \gamma \text{ when passing through } \Delta_j, \\ -1, & \text{otherwise.} \end{cases} \tag{1.16}$$

The labeled snake graph $\mathcal{G}_\gamma = (G_1, \ldots, G_d)$ with tiles G_i and sign function f is called the snake graph associated to the arc γ. Each edge e of $\mathcal{G}_\gamma$ is labeled by an arc $\tau(e)$ of the triangulation T. Such an arc defines the weight $x(e)$ of the edge e to be the cluster variable associated to the arc $\tau(e)$. Thus $x(e) = x_{\tau(e)}$.

In [69] Musiker, Schiffler, and Williams showed a combinatorial formula for cluster variables of a cluster algebra of surface type $\mathcal{A}(S, M)$ with principal coefficients $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T)$. In such a case, if γ is an arc, $\mathcal{G}_\gamma$ is its snake graph, and the triangulation T has no self-folded triangles. Then the corresponding cluster variable x_γ is given by the identity

$$x_\gamma = \frac{1}{\mathrm{cross}(\gamma, T)} \sum_{P \in \mathrm{Match}(\mathcal{G}_\gamma)} x(P), \tag{1.17}$$

where the sum runs over all perfect matchings of $\mathcal{G}_\gamma$, the summand $x(P) = \prod_{e \in P} x(e)$ is the weight of the perfect matching P, and $\mathrm{cross}(T, \gamma) = \prod_{j=1}^{d} x_{\tau_{i_j}}$ is the product of all initial cluster variables whose arcs cross γ.

A relationship between cluster variables and continued fractions is described by Schiffler and Canakci in [22], who claimed that, the numerator of a continued fraction is equal to the number of perfect matchings of the corresponding abstract snake graph, and that it can therefore be interpreted as the number of terms in the numerator of the Laurent expansion of an associated cluster variable. Thus, the Laurent polynomials of the cluster variable can be recovered from the continued fraction.

For example, let T be a triangulation, and let γ be a diagonal which is not in T.

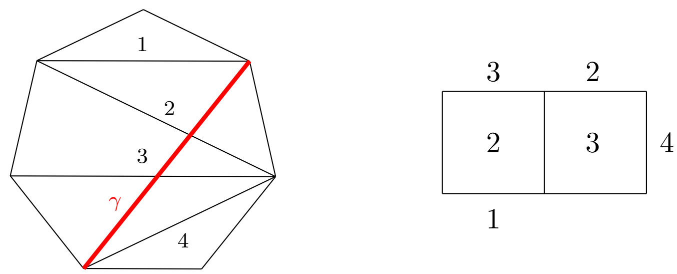


FIGURE 1.17. Triangulation T (left) and snake graph $\mathcal{G}_\gamma$ (right).

We can build the snake graph $\mathcal{G}_\gamma$ associated to γ (see Figure 1.17). The set of all perfect matchings of $\mathcal{G}_\gamma$ are shown in Figure 1.18, and the cluster variable associated to x_γ is given by the identity

$$x_\gamma = \frac{x_4 + x_1x_3x_4 + x_2}{x_2x_3}.$$

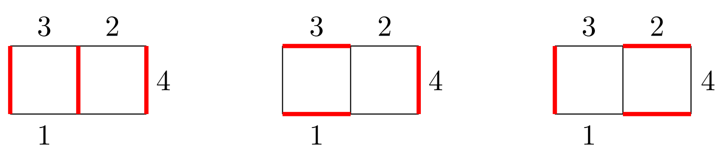


FIGURE 1.18. Perfect matchings of $\mathcal{G}_\gamma$.

CHAPTER 2

Integer Sequences Arising From Auslander-Reiten Quivers

Ringel and Fahr called *categorification* of an integer sequence the process for which numbers in the sequence can been seen as suitable invariants of objects in a category and proposed a categorification of Fibonacci numbers by using the Gabriel's universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver [49, 50]. In this chapter, we study sections in the Auslander-Reiten quiver of algebras of Dynkin type as a tool that provides categorifications of some integer sequences in the Online Encyclopedia of Integer Sequences (OEIS) [85–89]. Posets of type b, d and h and some properties of its lattice paths are introduced in section 2.1. In section 2.2 lattice paths connecting minimal and maximal points in posets of type b, d and h are used to enumerate sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ [32, 38]. As a consequence of this chapter, we conclude that the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type is not a Dynkin function.

2.1 Posets of Type b, d, and h

In this section, we build families of posets (almost all of wild representation type), and we present integer sequences associated to the lattices paths over these posets.

2.1.1 Posets and Lattice Paths

If $\mathcal{P} = (\mathbb{N}^2, \preceq)$ is a poset where $(\mathbb{N}, \leq)$ denotes the set of natural numbers endowed with the usual order and $(x, y) \preceq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Then, a *lattice path* $P \subseteq \mathcal{P}$ is a sequence of points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \subseteq \mathcal{P}$ where $(x_i, y_i) \preceq (x_{i+1}, y_{i+1})$ for any $1 \leq i \leq n-1$, and either $x_{i+1} = x_i+1$ and $y_{i+1} = y_i$ or $y_{i+1} = y_i+1$ and $x_{i+1} = x_i$.

An *order ideal* of a poset $(\mathcal{P}, \leq)$ is a subset I of $\mathcal{P}$ such that if $x \in I$ and $y \leq x$, then $y \in I$ (i.e., *x covers y*). We let $J(\mathcal{P})$ denote the set of all order ideals of $\mathcal{P}$, ordered by inclusion. Note that, m-element antichains in $\mathcal{P}$ correspond to elements of $J(\mathcal{P})$ that cover exactly

m-elements.

Given a finite poset $\mathcal{P}$ with $|\mathcal{P}| = n$ in [93] it is defined an *extension of* $\mathcal{P}$ *to a total order or linear extension of* $\mathcal{P}$ as an order-preserving bijection $\sigma : \mathcal{P} \to \mathbf{n}$. The number of extensions of $\mathcal{P}$ to a total order is denoted $e(\mathcal{P})$. Actually, $e(\mathcal{P})$ is also equal to the number of maximal chains of $J(\mathcal{P})$.

According to Stanley [91, 93] the enumeration of lattice paths is an extensively developed subject, the point in this chapter is that certain lattice path problems are equivalent to determining $e(\mathcal{P})$ for a given poset $\mathcal{P}$, or equivalent to the problem of finding the number of sections in the Auslander-Reiten quiver of some finite-representation algebras. In this fashion, it is possible to establish connections between the theory of partitions, the theory of partially ordered sets and the theory of representation of algebras.

If $\mathcal{M} = \mathbf{2} \times \mathbf{n}$ then it can be shown that the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$, which never rise above the main diagonal $x = y$ of the plane (x, y)-plane equals the number of linear extensions $e(\mathcal{M})$ of the poset $\mathcal{M}$ and that $e(\mathbf{2} \times \mathbf{n}) = \frac{1}{n+1}\binom{2n}{n} = C_n$ [91]. Figure 2.1 shows the number of lattice paths from each point $(x, y) \in \mathcal{M}_3 = \mathbf{2} \times \mathbf{3}$ to the maximal point, note that the number of lattice paths from the minimal to the maximal point is $C_3 = 5$. We will make the same computations for other types of posets in order to enumerate sections in the Auslander-Reiten quiver of some hereditary algebras of finite-representation type.

$\mathcal{M}_3 =$

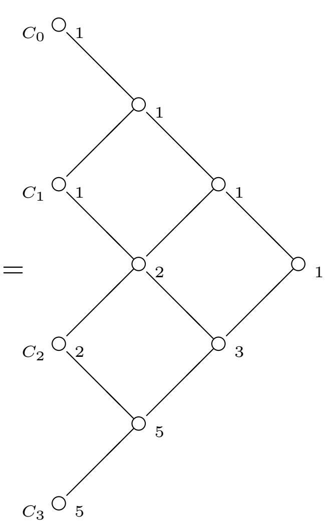


FIGURE 2.1. Number of lattice paths from each C_t to the maximal points C_0 is a Catalan number.

More connections between the theory of partitions and the theory of partially ordered sets via lattice paths have been quoted by Andrews and Stanley in [1, 93]. Firstly by establishing an identity between inversions and $p(m_1, m_2; n)$ the number of partitions of an integer number n into at most m_2 parts no greater than a given integer m_1. And secondly by using $\mathcal{P}$-partitions, i.e., order-preserving maps from a partially ordered set $\mathcal{P}$ to a chain with special rules specifying where equal values may occur. For instance, if $\mathcal{P}$

is a p-element chain, then a $\mathcal{P}$-partition of a positive integer n is equivalent to an ordinary partition of n into at most p parts. Some relationships between $\mathcal{P}$-partitions and the counting of chains in the set of order ideals of $\mathcal{P}$ ordered by inclusion are well described by Stanley in [91, 93]. Actually, he describes in [93] the following relation between the number of some $\mathcal{P}$-partitions of a positive integer n, denoted m_n, and the number $e(\mathcal{P})$ of extensions of $\mathcal{P}$ to a total order. In this case, we have considered that $|\mathcal{P}| = p$:

$$m_n = \frac{e(\mathcal{P})n^{p-1}(1+o(\frac{1}{n}))}{p!(1-p)!} \quad \text{as} \quad n \to \infty.$$

The theory of $\mathcal{P}$-partitions has been used by Petersen in [70] and Stembridge in [94] to investigate peak algebras and descent algebras.

2.1.2 Some Integer Sequences

We will see that the following sequences $\{a_n\}_{n\geq 0}$ and $\{C_m^n\}_{m\geq 0}$ are useful to enumerate the number of sections in the Auslander-Reiten quiver of some algebras of Dynkin type. Sequence $\{a_n\}$ is defined as follows:

$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-2^{x-1}} + a_{n-2^{x-1}+y}, \end{aligned} \tag{2.1}$$

where x stands for the length of the binary expansion of n and y denotes the largest power of 2 associated to a zero occurring in such expansion bearing in mind that $y = 0$ if the binary expansion of n has no 0 's. The following are the first 20 terms of $\{a_n\}$.

$$\{1, 2, 3, 4, 4, 6, 7, 8, 5, 8, 10, 12, 11, 14, 15, 16, 6, 10, 13, 16\}.$$

Note that,

$$a_{2^k+j} = a_{2^{k-1}+j} + a_j, \text{ for each } k \geq 2 \text{ and } 0 \leq j \leq 2^{k-1} - 1. \tag{2.2}$$

In particular,

$$a_{2^{2^{(2^k)}}-1} = 2^{2^k} + 1 \text{ (a Fermat number)}. \tag{2.3}$$

Sequence C_m^n is defined in such a way for $n \geq 3$ fixed it holds that:

$$C_m^n = \begin{cases} 0, & \text{if } m = 0, \\ C_m^{n-1} + C_{m-2^{p_1}-1}^{n-1}, & \text{if } 0 < m < 2^{n-3}, \\ 2C_{m-2^{n-3}}^{n-1} + a_{2^{n-2}-(m-1)}, & \text{if } 2^{n-3} \leq m < 2^{n-2}. \end{cases} \tag{2.4}$$

In this case, for $n > 1$ and $m \geq 0$, p_1 denotes the number of digits in the binary expansion of the number m and $a_{2^{n-2}-(m-1)} \in \{a_n\}$. Besides, for $m \geq 0$, $C_m^2 = 0$, further $C_1^3 = 1$ (see Appendix, Table A.1).

Remark 2.1. *If $\mathbf{n} = \{1, 2, \ldots, n\}$ is an n-point chain then $\mathcal{C}_{(1,n)}$ stands for all admissible subchains $\mathcal{C}$ of $\mathbf{n}$ with min $\mathcal{C} = 1$ and max $\mathcal{C} = n$. For instance, $\{1, 3, 5, 7\}$ and $\{1, 4, 6, 7\}$ are four-point subchains contained in $\mathcal{C}_{(1,7)}$. Note that, the number of admissible chains in $\mathbf{n}$ equals 2^{n-2}. To enumerate admissible subchains is a particular case of another interesting problem in combinatorics which consists of finding the number of chains contained in a poset $(L, \preceq)$ where $\preceq$ is the dominance order defined on the lattice of integer points $(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$. And for fixed nonnegative integers $n_1, n_2, \ldots, n_d$, points $(a_1, a_2, \ldots, a_d) \in L$ are defined in such a way that $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. Stanley proved that in the case $d = 2$ and n_1, n_2 share common value n then the total number of chains in L equals $2^{n+1}d_n$ where d_n denotes the n-th Delannoy number* [92].

2.1.3 Posets of Type $b^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}$

In this section we define the first type of poset we are interested in. Henceforth, we assume that $i_0 = j_0 = 0$, and the set $\{j_1, j_2, \ldots, j_m, i_1, i_2, \ldots, i_k\}$ is an admissible subchain where either $i_1 = 1$ or $j_1 = 1$.

Given the partially ordered set $\mathcal{P} = (\mathbb{Z}^2, \preceq)$ where $(x, y) \preceq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$, it is denoted as $b^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}$ a subposet of $\mathcal{P}$ whose points $i^{\cdot}_r = (x_r, y_r)$, $j^{\cdot}_s = (w_s, z_s)$, $i_{r_{\cdot}} = (j^{\cdot}_r)^t = (z_r, w_r)$ and $j_{s_{\cdot}} = (j^{\cdot}_s)^t = (y_s, x_s)$ satisfy the following conditions:

(1) If $i_1 = 1$ and $k = m = 1$,

$$i^{\cdot}_1 = (x^{\cdot}_1, y^{\cdot}_1) = (n-1, 0) \text{ and } j^{\cdot}_1 = (w^{\cdot}_1, z^{\cdot}_1) = (0, 0).$$

(2) If $i_1 = 1$ and $k > 1$,

$$\begin{aligned}
i^{\cdot}_1 &= (x^{\cdot}_1, y^{\cdot}_1) = \left(\sum_{t=1}^{m} |i_t - j_t|, \sum_{t=1}^{\max\{k,m\}-1} |j_t - i_{t+1}| \right), \\
j^{\cdot}_1 &= (w^{\cdot}_1, z^{\cdot}_1) = (x^{\cdot}_1 - |i_1 - j_1|, y^{\cdot}_1), \\
i^{\cdot}_r &= (x^{\cdot}_r, y^{\cdot}_r) = \left(x^{\cdot}_1 - \sum_{t=1}^{r-1} |i_t - j_t|, y^{\cdot}_1 - \sum_{t=1}^{r-1} |j_t - i_{t+1}| \right), \\
j^{\cdot}_s &= (w^{\cdot}_s, z^{\cdot}_s) = \left(x^{\cdot}_1 - \sum_{t=1}^{s} |i_t - j_t|, y^{\cdot}_1 - \sum_{t=1}^{s-1} |j_t - i_{t+1}| \right),
\end{aligned}$$

for $1 < r \leq k$ and $1 < s \leq m$.

The admissible subchain $\mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq \mathbf{n}$ must satisfy the following constraints for $1 \leq r \leq k$ and $1 \leq s \leq m$:

- If $i_1 = 1$ and $k = m$ then $i_1 < j_1 < \cdots < i_k < j_m = n$.

- If $i_1 = 1$ and $k = m + 1$ then $i_1 < j_1 < \cdots < i_k < j_m < i_k = n$.

- If $j_1 = 1$ and $k = m$ then $j_1 < i_1 < \cdots < j_m < i_k = n$.

- If $j_1 = 1$ and $m = k + 1$ then $j_1 < i_1 < \cdots < j_m < i_k < j_m = n$.

If for s fixed, $1 \leq s \leq m$, it is defined

$$H_{z_s} = \Big\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq y \leq z_s,\, x \geq w_s,\, x + y \leq \max\,\{i_k, j_m\} - 1\Big\}. \tag{2.5}$$

Then

$$b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} = \bigcup_{s=1}^{m} H_{z_s}. \tag{2.6}$$

The following algorithm summarizes the construction of posets of type $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$:

Algorithm 2.1. *(1) If $i_0 = j_0 = 0$, then either $i_1 = 1$ or $j_1 = 1$,*

*(2) Fix $k \in \{m, m+1\}$ and **n**,*

*(3) **For** $1 \leq r \leq k$ and $1 \leq s \leq m$ **do**;*

- ***If** $i_1 = 1$ and either $k = m$ or $k = m+1$ **then** the subchain $\mathcal{C} = \{i_1 < j_1 < \cdots < i_k < j_m = n\}$ or $\mathcal{C} = \{i_1 < j_1 < \ldots < i_k < j_m < i_k = n\}$, respectively,*
- ***Else***
- *If $j_1 = 1$ and either $m = k$ or $m = k + 1$ then the subchain $\mathcal{C} = \{j_1 < i_1 < \cdots < j_m < i_k = n\}$ or $\mathcal{C} = \{j_1 < i_1 < \cdots < j_m < i_k < j_m = n\}$, respectively.*

*(4) **For** $1 \leq s \leq m$ **do**;*

(5) $i^{\bullet}_r$, $j^{\bullet}_s$, $(i^{\bullet}_r)^t$, $(j^{\bullet}_s)^t$, and H_{z_s},

(6) Do $\bigcup\limits_{s=1}^{m} H_{z_s}$.

Remark 2.2. *The main problem regarding posets of type $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ consists of finding the number of lattice paths from $(0, 0)$ to each point $p \in M_b$ where M_b denotes the set of maximal points contained in $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$. Actually, points $p \in M_b$ are solutions of the following linear programming problem:*

Problem 1

***Maximize** $x + y$;*
***Subject to the constraints**;*

$$\begin{aligned} &x \geq w_s,\\ &y \leq z_s,\\ &y \leq n - x - 1,\\ &x \geq 0, \quad y \geq 0. \end{aligned} \tag{2.7}$$

Henceforth, we let $[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}]$ denote such a number and $|P^{(x',y')}_{(x,y)}|$ the number of lattice paths from a point (x, y) to a point (x', y') in a given poset $\mathcal{P}$. $[A]$ will denote the number of lattice paths from the set of minimal points to the set of maximal points of a subset $A \subseteq \mathcal{P}$.

The following example shows the procedure described above to construct the poset b_{025}^{014}.

(1) Firstly we make, $i_0 = j_0 = 0$.

(2) Choose $m = k = 2$ and $n = 5$, thus $i_r, j_s \in \{1, 2\}$.

(3) Select the admissible subchain $i_1 = 1 < j_1 = 2 < i_2 = 4 < j_2 = 5$ among all possible admissible subchains in $\mathcal{C}_{(\mathbf{1},\mathbf{5})}$ satisfying the constraints.

(4) Points $i_r^{\bullet}$, $j_s^{\bullet}$ are given by the following identities:

$$\begin{aligned}
i_1^{\bullet} &= (x_1^{\bullet}, y_1^{\bullet}) = (2, 2),\\
j_1^{\bullet} &= (w_1^{\bullet}, z_1^{\bullet}) = (1, 2),\\
i_2^{\bullet} &= (x_2^{\bullet}, y_2^{\bullet}) = (1, 0),\\
j_2^{\bullet} &= (w_2^{\bullet}, z_2^{\bullet}) = (0, 0).
\end{aligned} \tag{2.8}$$

(5) Subsets H_{z_1} and H_{z_2} are given by the identities:

$$\begin{aligned}
H_{z_1} &= \{(1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (2, 1), (3, 1), (1, 2), (2, 2)\},\\
H_{z_2} &= \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\}.
\end{aligned} \tag{2.9}$$

(6) We conclude finally that:

$$b_{025}^{014} = H_{z_1} \cup H_{z_2} = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (2, 1), (3, 1), (1, 2), (2, 2)\}.$$

Figure 2.2 shows the way points in b_{025}^{014} are connected by lattice paths.

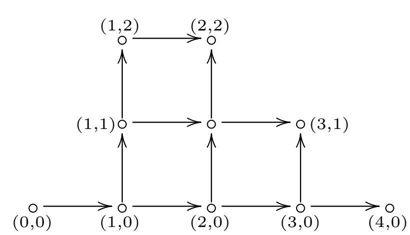


FIGURE 2.2. Lattice paths in poset b_{025}^{014} .

Figure 2.3 shows other examples of lattice paths in posets of type $b_{j_0 j_1 \ldots j_m}^{i_0 i_1 \ldots i_k}$.

Remark 2.3. *Let $\mathcal{P}_b$ be a poset of type $b_{j_0 j_1 \ldots j_m}^{i_0 i_1 \ldots i_k}$ then the derivatives $\frac{\partial \mathcal{P}_b}{\partial x}$ and $\frac{\partial \mathcal{P}_b}{\partial y}$ are defined in such a way that*

$$\begin{aligned}
\frac{\partial \mathcal{P}_b}{\partial x} &= \mathcal{P}_b \backslash \{(n-1, 0)_{\triangle}\},\\
\frac{\partial \mathcal{P}_b}{\partial y} &= \mathcal{P}_b \backslash \{(0, n-1)_{\triangle}\}.
\end{aligned} \tag{2.10}$$

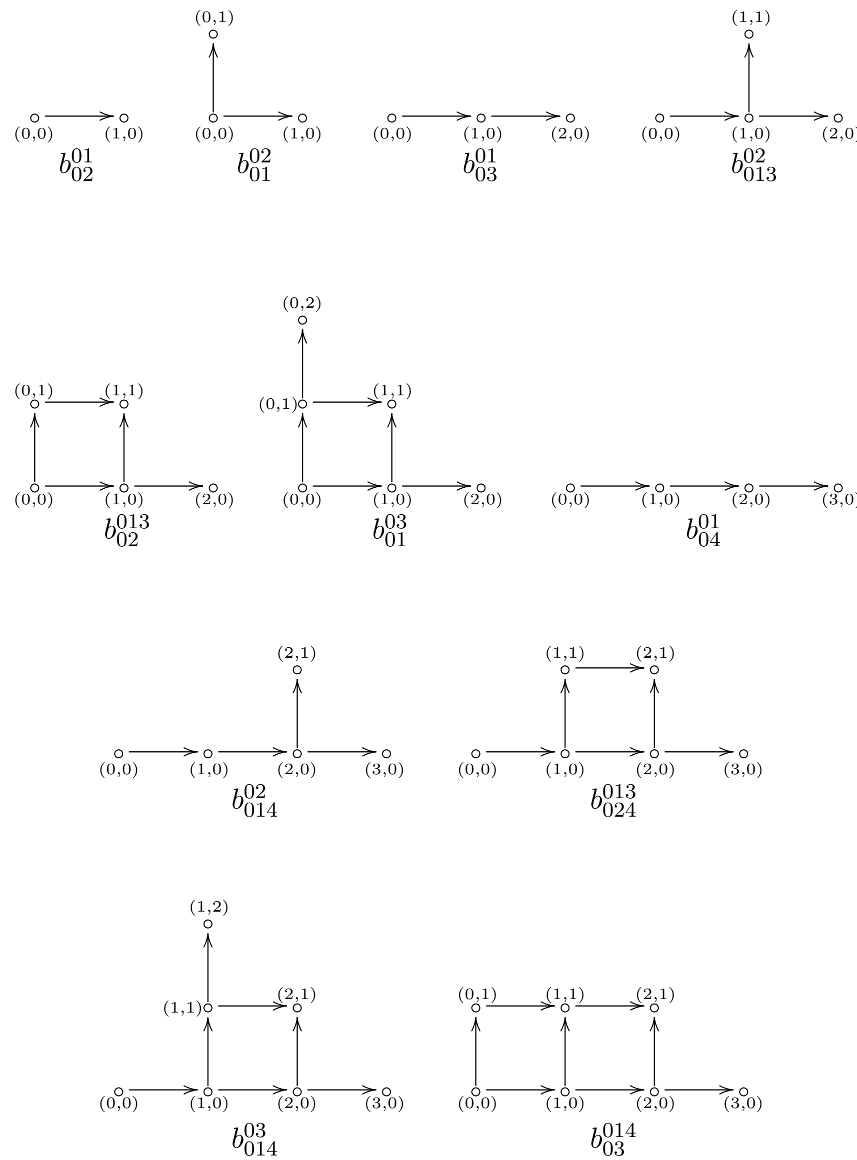


FIGURE 2.3. Examples of oriented lattice paths in posets of type $b_{j_0 j_1 \dots j_m}^{i_0 i_1 \dots i_k}$.

Formulas (2.10) allow to establish the following equalities regarding the number of lattice paths from $(0, 0)$ to each point $p \in M_b$ as follows:

$$[\mathcal{P}_b] = \begin{cases} [\frac{\partial \mathcal{P}_b}{\partial x} + \frac{\partial \mathcal{P}_b}{\partial y}], & \text{if } i_k = n, \\ [\frac{\partial \mathcal{P}_b}{\partial y}], & \text{otherwise.} \end{cases}$$

In posets of type $b_{j_0 j_1 \dots j_m}^{i_0 i_1 \dots i_k}$ these identities have the following interpretations:

Lemma 2.1. *Posets $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ satisfy the following identities:*

$$
\left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \begin{cases}
\left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right], & \text{if } 1 \leqslant i_k < n-1, \\
\left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_{m-1}}\right], & \text{if } i_k = n-1, \\
\left[b^{i_0 i_1 \dots i_{k-1}}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-1} j_m}_{j_0 j_1 \dots j_m - 1}\right], & \text{if } i_k = n,\ j_m = n-1,\ |j_m - i_{k-1}| > 1, \\
\left[b^{i_0 i_1 \dots i_{k-1}}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-2} j_m}_{j_0 j_1 \dots j_{m-1}}\right], & \text{if } i_k = n,\ j_m = n-1,\ |j_m - i_{k-1}| = 1, \\
\left[b^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-1} i_k - 1}_{j_0 j_1 \dots j_m - 1}\right], & \text{if } i_k = n,\ 1 < j_m < n-1,\ |j_m - i_{k-1}| > 1, \\
\left[b^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-2} i_k - 1}_{j_0 j_1 \dots j_{m-1}}\right], & \text{if } i_k = n,\ 1 < j_m < n-1,\ |j_m - i_{k-1}| = 1, \\
\left[b^{i_0 n-1}_{j_0 1}\right] + \left[b^{i_0 n-1}_{j_0 1}\right], & \text{if } i_1 = n,\ j_m = 1.
\end{cases}
$$

The following theorem shows a connection between sequence (2.1) and the number of lattice paths in posets of type $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ from $(0,0)$ to points $p \in M_b$.

Theorem 2.1. *For a given poset $b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ (of type b) associated to an admissible subchain $\{j_1, \dots, j_m, i_1, \dots, i_k\}$ it holds the identity*

$$
\left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = a_r,
$$

where for $n > 1$

$$
r = \begin{cases}
\displaystyle\sum_{t=k-m+2}^{k} 2^{i_t - 1} - \sum_{t=1}^{m-1} 2^{j_t - 1}, & \text{if } 1 < i_k < n, \\
\displaystyle\sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{j_t - 1}, & \text{if } i_k = n, \\
0, & \text{if } i_k = 1.
\end{cases}
$$

Proof. By induction. For $n = 2$ we have two cases. If $i_k = 1$ the associated poset is given by $b^{i_0 2}_{j_0 1} = \{(0,0), (1,0)\}$ and the only lattice path is $(0,0) \longrightarrow (1,0)$. Thus, $\left[b^{i_0 1}_{j_0 2}\right] = 1 = a_0$. On the other hand, if $i_k = 2$ the associated poset is $b^{i_0 2}_{j_0 1} = \{(0,0), (0,1), (1,0)\}$ with two lattice paths. Since $r = 1$ and $a_1 = 1$ it holds that $\left[b^{i_0 2}_{j_0 1}\right] = 2 = a_1$. Suppose now that the case holds for $n \leq p$ with $1 \leq i_k \leq p$.

If $n = p+1$ and $i_k = 1$ then we have that $\left[b^{i_0 1}_{j_0 p+1}\right] = \left[b^{i_0 1}_{j_0 p}\right] = a_1$. And if $1 < i_k < p$ then $\left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots p+1}\right] = \left[b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots p}\right] = a_r$ with $r = \displaystyle\sum_{t=k-m+2}^{k} 2^{i_t - 1} - \sum_{t=1}^{m-1} 2^{j_t - 1}$, if $i_k = p$ we have that $j_{m-1} < p$, thus $\left[b^{i_0 i_1 \dots p}_{j_0 j_1 \dots p+1}\right] = \left[b^{i_0 i_1 \dots p}_{j_0 j_1 \dots j_{m-1}}\right] = a_r$ with $r = \displaystyle\sum_{t=k-m+2}^{k} 2^{i_t - 1} - \sum_{t=1}^{m-1} 2^{j_t - 1}$.

Now if $i_k = p+1$, then the following cases hold:

Case 1. If $j_m = p$, $|j_m - k_{k-1}| > 1$ and $i_{k-1} > 1$, we have that $\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = \left[b^{i_0 i_1 \dots i_{k-1}}_{j_0 j_1 \dots p}\right] + \left[b^{i_0 \dots i_{k-1} p}_{j_0 j_1 \dots p-1}\right] = a_r + a_s$ where $r = \displaystyle\sum_{t=k-m+1}^{k-1} 2^{i_t - 1} - \sum_{t=1}^{m-1} 2^{j_t - 1}$ and $s = \displaystyle\sum_{t=k-m+1}^{k-1} 2^{i_t - 1} -$

$\sum_{t=1}^{m} 2^{j_t-1} + 2^{p-2}$, actually, $r = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1}$ and $s = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1} + 2^{p-2}$. Note that, if $w = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1}$ then the number of digits of w is $x = p$ and the largest power of 2 associated to a zero in the binary expansion of y is 2^{p-2} (see formula (2.1)), thus

$$\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = a_w.$$

Now, if $i_{k-1} = 1$ then $\left[b^{i_0 1(p+1)}_{j_0 p}\right] = \left[b^{i_0 1}_{j_0 p}\right] + \left[b^{i_0 1,p}_{j_0 p-1}\right] = a_r + a_s$ where $r = 0$ and $s = 2^{p-1} - 2^{p-2}$. Since $w = 2^p - 2^{p-1}$ then the number of digits of w is $x = p$ and the largest power of 2 associated to a zero in the binary expansion of y is again 2^{p-2}, therefore,

$$\left[b^{i_0 1(p+1)}_{j_0 p}\right] = a_w.$$

Case 2. If $i_k = p+1$, $j_m = p$ and $|j_m - i_{k-1}| = 1$, then $\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = \left[b^{i_0 i_1 \dots p-1}_{j_0 j_1 \dots p}\right] + \left[b^{i_0 \dots i_{k-2}(p)}_{j_0 j_1 \dots j_{m-1}}\right] = a_r + a_s$ where $r = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1}$ and $s = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1} + 2^{p-2}$, $x = p$ and $y = 2^{p-2}$, therefore

$$\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = a_w.$$

Case 3. If $i_k = p+1$, $1 < j_m < p$ and $|j_m - i_{k-1}| > 1$ then $\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots j_m}\right] = \left[b^{i_0 i_1 \dots p}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-1}(p)}_{j_0 j_1 \dots j_m - 1}\right] = a_r + a_w$ where $r = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1}$, $s = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1} + 2^{j_m-2}$, $x = p$ and $y = 2^{j_m-2}$, thus

$$\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = a_w.$$

Case 4. If $i_k = p+1$, $1 < j_m < p$ and $|j_m - i_{k-1}| = 1$ then $\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots j_m}\right] = \left[b^{i_0 i_1 \dots p}_{j_0 j_1 \dots j_m}\right] + \left[b^{i_0 \dots i_{k-2}(p)}_{j_0 j_1 \dots j_m - 1}\right] = a_r + a_w$ where $r = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1}$, $s = \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1} - 2^{p-1} + 2^{j_m-2}$, in this case, $x = p$ and $y = 2^{j_m-2}$, therefore

$$\left[b^{i_0 i_1 \dots p+1}_{j_0 j_1 \dots p}\right] = a_w.$$

Case 5. If $i_k = p+1$ and $j_1 = 1$, $\left[b^{i_0p+1}_{j_01}\right] = \left[b^{i_0p}_{j_o1}\right] + \left[b^{i_0p}_{j_o1}\right] = a_r + a_r$ where $r = 2^{p-1}-1$, the number of digits of $w = 2^p - 1$ is $x = p$ and w has no zeroes in its binary expansion, that is, $y = 0$. Therefore, $r = w - 2^{x-1}$ and $\left[b^{i_0p+1}_{j_01}\right] = a_w$. □

2.1.4 Posets of Type $d^{i_0i_1\ldots i_k}_{j_0j_1\ldots j_m}$

In this section, we define another type of posets whose lattice paths will allow to enumerate sections in Auslander-Reiten quivers of algebras of Dynkin type $\mathbb{A}_n$.

Points $i^*_r = (x_r, y_r)$, $j^*_s = (x_s, y_s)$, $i_{r*} = (\overline{x_r}, \overline{y_r})$ and $j_{r*} = (\overline{w_s}, \overline{z_s})$ in posets of type $d^{i_0i_1\ldots i_k}_{j_0j_1\ldots j_m} \subset \mathcal{P}$ satisfy the following conditions:

- If $i_1 = 1$

$$\begin{aligned}
i^*_r = (x_r, y_r) &= \left(-\sum_{w=0}^{r-1} l_w, u - \sum_{w=0}^{r-1} d_w\right),\\
j^*_s = (w_s, z_s) &= \left(-\sum_{w=0}^{s-1} l_w, u - \sum_{w=0}^{s} d_w\right),\\
i_{r*} = (\overline{x_r}, \overline{y_r}) &= \left(\sum_{w=0}^{r-1} d_w, -n+1+u+\sum_{w=0}^{r-1} l_w\right),\\
j_{s*} = (\overline{w_s}, \overline{z_r}) &= \left(\sum_{w=0}^{s} d_w, -n+1+u+\sum_{w=0}^{s-1} l_w\right).
\end{aligned}$$

- If $j_1 = 1$

$$\begin{aligned}
i^*_r = (x_r, y_r) &= \left(-\sum_{w=0}^{r} l_w, u - \sum_{w=0}^{r-1} d_w\right),\\
j^*_s = (w_s, z_s) &= \left(-\sum_{w=0}^{s-1} l_w, u - \sum_{w=0}^{s-1} d_w\right),\\
i_{r*} = (\overline{x_r}, \overline{y_r}) &= \left(\sum_{w=0}^{r-1} d_w, -n+1+u+\sum_{w=0}^{r} l_w\right),\\
j_{s*} = (\overline{w_s}, \overline{z_s}) &= \left(\sum_{w=0}^{s-1} d_w, -n+1+u+\sum_{w=0}^{s-1} l_w\right).
\end{aligned}$$

where $1 \le r \le k$ and $1 \le s \le m$.

In these cases;

$$u = \begin{cases} \displaystyle\sum_{l=0}^{t-1} |i_{k-l} - j_{m-l}|, & \text{if } 1 \le i_k < n \text{ and } t = \min\{k, m\},\\ \displaystyle n - 1 - \sum_{l=0}^{t-1} |i_{k-l} - j_{m-l}|, & \text{if } i_k = n. \end{cases}$$

Numbers d_w and l_w are given by the following relations:

(1) If $i_1 = 1$ and

- $k = m = 1$ then $d_1 = |j_1 - i_1|$ and $l_0 = 0$.
- $k = m > 1$ then $d_r = |j_r - i_r|$ and $l_s = |i_{s+1} - j_s|$ for $1 \leq r \leq k$ and $1 \leq s \leq k-1$.
- $1 < k = m+1$ then $d_r = |j_r - i_r|$ and $l_s = |i_{s+1} - j_s|$ for $1 \leq r \leq k-1$ and $1 \leq s \leq k-1$.

If $j_i = 1$ and

- $k = m = 1$ then $d_0 = 0$ and $l_1 = |j_1 - i_1|$.
- $k = m > 1$ then $d_s = |j_{s+1} - i_s|$ and $l_r = |j_r - i_r|$ for $1 \leq r \leq k-1$ and $1 \leq s \leq k$.
- $1 < k = m+1$ then $d_s = |j_{s+1} - i_s|$ and $l_r = |j_r - i_r|$ for $1 \leq r \leq m-1$ and $1 \leq s \leq m-1$.

$l_0 = 0$ and $d_0 = 0$.

(2) $d^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m} = A_1 \cup A_2$ where

$$A_1 = \bigcup_{r=1}^{k} H_{y_r} \text{ and } A_2 = \bigcup_{s=1}^{m-1} H_{\overline{z_s}},$$

with

$$H_{y_r} = \Big\{(x,y) \in \mathbb{Z}^2 \mid 0 \leq y \leq y_r,\, x \geq x_r,\, x+y \leq u\Big\} \text{ and}$$

$$H_{\overline{z_s}} = \Big\{(x,y) \in \mathbb{Z}^2 \mid \overline{z_s} \leq y \leq 0,\, x \leq \overline{w_s},\, x+y \geq -n+1+u\Big\}.$$

The following algorithm summarizes the construction of posets of type $d^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}$.

Algorithm 2.2. *(1) Fix* $i_0 = 0 = j_0$*,*

(2) Define $k \in \{m, m+1\}$*,* $\boldsymbol{n}$ *and either* $i_1 = 1$ *or* $j_1 = 1$*,*

(3) ***If*** $i_1 = 1$ *and* $k = m$ *or* $k = m+1$ ***then*** *the subchain* $\mathcal{C} = \{i_1 < j_1 < i_2 < j_2 < \cdots < j_{k-1} < i_k < j_k\}$ *or* $\mathcal{C} = \{i_1 < j_1 < i_2 < j_2 < \cdots < j_{k-2} < i_{k-1} < j_{k-1} < i_k\}$*,*

(4) ***Else*** $j_1 = 1$ *and* $m = k$ *or* $m = k+1$ *then the subchain* $\mathcal{C} = \{j_1 < i_1 < j_2 < i_2 < \cdots < i_{k-1} < j_k < i_k\}$ *or* $\mathcal{C} = \{j_1 < i_1 < j_2 < i_2 < \cdots < i_{k-1} < j_k < i_k < j_{k+1}\}$*,*

(5) ***If*** $i_1 = 1$ ***then*** *for* $1 \leq r \leq k$ *and* $1 \leq s \leq k-1$ *or* $1 \leq r \leq k-1$ *and* $1 \leq s \leq k-1$ *do* d_r *and* l_s*,*

(6) ***Else*** $j_1 = 1$ *and for* $1 \leq r \leq k-1$ *and* $1 \leq s \leq k$ *or* $1 \leq r \leq k-1$ *and* $1 \leq s \leq k-1$ ***do****;* d_s *and* l_r*,*

(7) Compute u*,*

(8) ***For*** $1 \leq r \leq k$ *and* $1 \leq s \leq m$ ***do*** i^*_r*,* j^*_s*,* i_{r^*}*,* j_{s^*}*,*

(9) ***For*** $1 \leq r \leq k$ *and* $1 \leq s \leq m-1$ ***do*** H_{y_r}*,* $H_{\overline{z_s}}$*,*

(10) Compute A_1, A_2 and $A_1 \cup A_2$.

As an example, we build the poset d^{013}_{025} by using the Algorithm 2.2.

(1) $i_0 = j_0 = 0$.

(2) $k = m = 2$, $n = 5$, $i_1 = 1$.

(3) It is constructed the admissible subchain $1 < 2 < 3 < 5$ with $i_1 = 1$, $j_1 = 2$, $i_2 = 3$ and $j_2 = 5$,

(4)
$$d_1 = |j_1 - i_1| = 1 \ , \ d_2 = |j_2 - i_2| = 2 \text{ and } l_1 = |i_2 - j_1| = 1,$$

(5) $3 = i_2 < 5$ and $t = \min\{k, m\} = 2$
$$u = |i_2 - j_2| + |i_1 - j_1| = 3,$$

(6) $1 \leq r, s \leq 2$

$$i_1^* = (x_1, y_1) = (-l_0, 3 - d_0) = (0, 3),$$

$$i_2^* = (x_2, y_2) = (-l_1, 3 - d_1) = (-1, 2),$$

$$i_{1*} = (\overline{x_1}, \overline{y_1}) = (d_0, -1 + l_0) = (0, -1),$$

$$i_{2*} = (\overline{x_2}, \overline{y_2}) = (d_1, -1 + l_1) = (1, 0),$$

$$j_1^* = (w_1, z_1) = (l_0, 3 - d_1) = (0, 2),$$

$$j_2^* = (w_2, z_2) = (-l_1, 3 - d_1 - d_2) = (-1, 0),$$

$$j_{1*} = (\overline{w_1}, \overline{z_1}) = (d_1, -1 + l_0) = (1, -1),$$

$$j_{2*} = (\overline{w_2}, \overline{z_2}) = (d_1 + d_2, -1 + l_1) = (3, 0).$$

(7)

$$H_{y_1} = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (0,2), (1,2), (0,3)\},$$

$$H_{y_2} = \{(-1,0), (0,0), (1,0), (2,0), (3,0), (-1,1), (0,1), (1,1), (2,1), (-1,2), (0,2), (1,2)\},$$

$$H_{\overline{z_1}} = \{(1,-1), (0,-1), (1,0), (0,0), (-1,0)\}.$$

(8) $A_1 = H_{y_1} \cup H_{y_2}$, $\quad A_2 = H_{\overline{z_1}}$, $\quad d^{013}_{025} = A_1 \cup A_2$.

The following is the linear programming problem associated to posets of type d.

Problem 2

Maximize (Minimize) $x+y$;
Subject to the constraints;

$$\begin{aligned} x_r \leq x \leq \overline{w_s},\\ z_s \leq y \leq y_r,\\ -n+1 \leq x+y-u \leq 0. \end{aligned} \tag{2.11}$$

As for posets of type b, if M^d denotes the set of minimal points in a poset of the form $d^{i_0i_1\ldots i_k}_{j_0j_1\ldots j_m}$ then the main problem for this kind of posets (of type d) consists of finding the number $[\alpha^{i_0i_1\ldots i_k}_{j_0j_1\ldots j_m}]$ of lattice paths from points $p \in M^d$ to points $q \in M_d$ satisfying the following conditions:

$$\begin{aligned} &x_i \leq x \leq w_{i-1},\ y = y_i \text{ for any } 2 \leq i \leq k,\ \text{if } i_1 = 1,\\ &x_i \leq x \leq w_i,\ y = y_i \text{ for any } 1 \leq i \leq k,\ \text{if } j_1 = 1. \end{aligned} \tag{2.12}$$

Figure 2.4 shows lattice paths linking points in d^{013}_{025}.

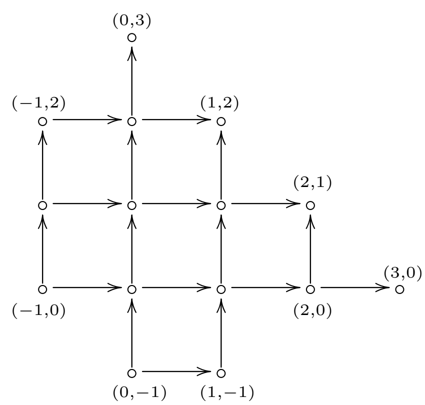
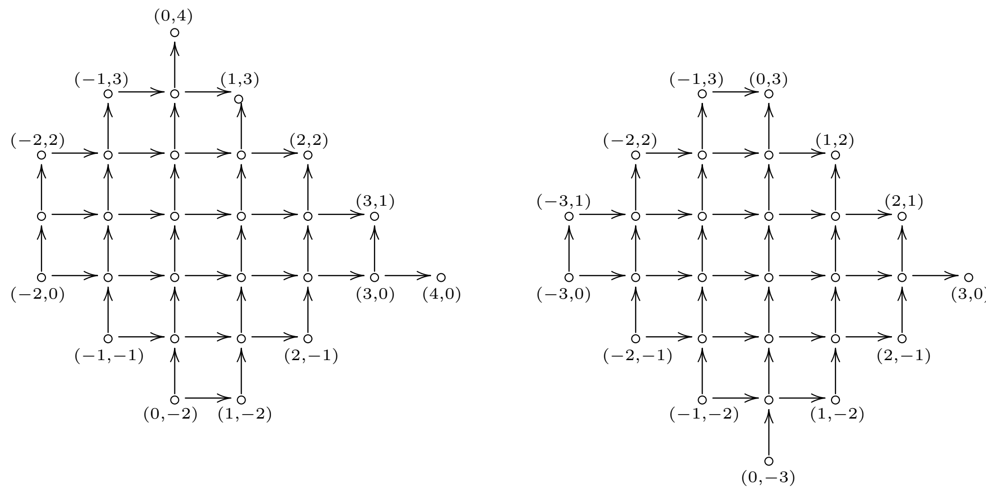


FIGURE 2.4. Lattice paths in poset of type d^{013}_{025}.

Figure 2.5 below shows other examples of lattice paths in posets of type d^{0135}_{0247} and d^{0246}_{01357}.



d^{0135}_{0247} $\qquad$ d^{0246}_{01357}

FIGURE 2.5. Examples of oriented lattice paths in posets of type $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$.

Lemma 2.2. *Numbers $[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}]$ with $j_m = n$ satisfy the following identities:*

$$\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \begin{cases} 0, & \text{if } i_k = 1, \\ 1, & \text{if } i_k = 2, \\ \left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right] + \left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right], & \text{if } 2 < i_k \leq n-2, i_k - j_{m-1} > 1, \\ \left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right] + \left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right], & \text{if } 2 < i_k \leq n-2, i_k - j_{m-1} = 1, \\ 2\left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right] + \left[b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k}\right], & \text{if } 2 < i_k = n-1, j_{m-1} = n-2, \\ 2\left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right] + \left[b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k}\right], & \text{if } 2 < i_k = n-1, j_{m-1} < n-2. \end{cases}$$

Proof. We have the following cases:

Case 1. If $i_k = 1$ then $\left[\alpha^{i_0 1}_{j_0 n}\right] = 0$.

Case 2. If $i_k = 2$ then $\left[\alpha^{i_0 2}_{j_0 1 n}\right] = \left|P^{(0,n-2)}_{(-1,0)}\right| = 1$.

Case 3. If $2 < i_k \leq n-2$, $i_k - j_{m-1} > 1$, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3$ with

$$A_1 = \bigcup_{r=1}^{k} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid 1 \leq y \leq y_r,\, x \geq x_r,\, x+y \leq u\},$$

$$A_2 = \bigcup_{s=1}^{m-1} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid \overline{z_s} + 1 \leq y \leq 1,\, x \leq \overline{w_s},\, x+y \geq -n+2+u\},$$

$$B_1 = \bigcup_{r=1}^{k-1} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid 0 \leq y \leq y_r,\, x \geq x_r,\, x+y \leq u\},$$

$B_2 = \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid 0 \leq y \leq y_k,\, x \geq x_k + 1,\, x + y \leq u\}$ and

$$B_3 = \bigcup_{s=1}^{m-1} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid \overline{z_s} + 1 \leq y \leq 0,\, x \leq \overline{w_s},\, x + y \geq -n + 2 + u\},$$

then the maps $g_1 : d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1} \longrightarrow A$ and $g_2 : d^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1} \longrightarrow B$ such that:

$$\begin{aligned} g_1(x,y) &= (x, y+1), \\ g_2(x,y) &= (x,y), \end{aligned} \tag{2.13}$$

are isomorphisms.

If $C = \{(x,y) \in b^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m},\, |x+y = -n+1+u\}$ then the union $C \cup A$ is a poset with relations of type $(x,y) \preceq (x, y+1)$ for $(x,y) \in C$ and $(x, y+1) \in A$. Since $\left|P^{(e,f)}_{(x,y-1)}\right| = \left|P^{(e,f)}_{(x,y)}\right|$ where all of paths $P^{(e,f)}_{(x,y)}$ contain at least one point satisfying conditions (2.12) and points (x,y), (e,f) are chosen in such a way that, $x + y - 1 = -n + 1 + u$ and $e + f = u$ then $\left[C \cup A\right] = \left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right]$.

Now, we define the poset $C \cup B$ with relations of the form $(x,y) \preceq (x+1, y)$ with $(x,y) \in C$ and $(x+1, y) \in B$. Since $\left|P^{(e,f)}_{(x-1,y)}\right| = \left|P^{(e,f)}_{(x,y)}\right|$ then $\left[C \cup B\right] = \left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right]$. Thus,

$$\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \left[C \cup A\right] + \left[C \cup B\right] = [\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\Big] + \left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right].$$

Case 4. If $2 < i_k \leq n - 2$, $i_k - j_{m-1} = 1$, $(A, C$ described as before) and $D = B_1 U C_1$ where

$$C_1 = \bigcup_{s=1}^{m-2} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid \overline{z_s} + 1 \leq y \leq 0,\, x \leq \overline{w_s},\, x + y \geq -n + 2 + u\},$$

then the maps $g_3 : d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1} \longrightarrow A$ and $g_4 : d^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1} \longrightarrow D$ defined as g_1 and g_2, respectively are isomorphisms, sets $C \cup A$ and $C \cup D$ are posets with the same relations as in Case 3 for points $(x,y) \in C \cup A$. Whereas, relations between points $(x,y) \in C \cup D$ are of the form $(x,y) \preceq (x+1, y)$ for $(x,y) \in C$ and $(x+1, y) \in D$. Thus $\left[C \cup A\right] = \left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right]$ and $\left[C \cup D\right] = \left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right]$. Therefore, $\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \left[C \cup A\right] + \left[C \cup D\right] = \left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}\right] + \left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right]$.

Case 5. If $2 < i_k = n - 1$, $i_{m-1} = n - 2$, D is described as before and

$$E = \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid y \geq y_k,\, x_k \leq x\},$$

then maps g_4 and

$$\begin{array}{rccc} g_5 : & b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k} & \longrightarrow & E \\ & (x,y) & \longmapsto & \left(x - \sum_{t=1}^{k} l_t, y + 1\right) \end{array}$$

where for each edge $((x,y),(w,z))$, it holds that

$$g_5((x,y),(w,z)) = \left(\left(x - \sum_{t=1}^{k} l_t, y+1\right), \left(w - \sum_{t=1}^{k} l_t, z+1\right)\right), \tag{2.14}$$

are isomorphisms.

Now, the set $C \cup D$ is a poset with lattice paths induced by $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ and $\left|P^{(e,f)}_{(x,y-1)}\right| + \left|P^{(e,f)}_{(x-1,y)}\right| = 2\left|P^{(e,f)}_{(x,y)}\right|$ where paths $P^{(e,f)}_{(x,y)}$ satisfy conditions as described in Case 3. Therefore $\left[C \cup D\right] = 2\left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right]$.

The set $\{(-n+1+u,0)\} \cup E$ is a poset whose lattice paths are induced by $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ and $\left|P^{(e,f)}_{(-n+1+u,0)}\right| = \left|P^{(e,f)}_{(-n+1+u,1)}\right|$ then $\left[\{(-n+1+u,0)\} \cup E\right] = \left[b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k}\right]$. Thus,

$$\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \left[C \cup D\right] + \left[\{(-n+1+u,0)\} \cup E\right] = 2\left[\alpha^{i_0 i_1 \dots i_{k-1}}_{j_0 \dots j_{m-2} j_m - 1}\right] + \left[b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k}\right].$$

Case 6. If $2 < i_k = n-1$, $i_{m-1} < n-2$, B, E, C, g_2, g_5 and $\{(-n+1+u,0)\} \cup E$ are described as before then the set $C \cup B$ is a poset with lattice paths induced by $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$. Then $\left[C \cup B\right] = 2\left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right]$. Thus,

$$\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = \left[C \cup D\right] + \left[\{(-n+1+u,0)\} \cup E\right] = 2\left[\alpha^{i_0 i_1 \dots i_k - 1}_{j_0 j_1 \dots j_m - 1}\right] + \left[b^{j_0 j_1 \dots j_{m-1}}_{i_0 i_1 \dots i_k}\right].$$

□

If $r = 0$ and $r = 1$ are associated to numbers $\left[\alpha^{i_0 1}_{j_0 n}\right]$ and $\left[\alpha^{i_0 2}_{j_0 1 n}\right]$ respectively. Then the following result shows a relationship between numbers $\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right]$ and elements in the integer sequence C^n_m (see formula (2.4)).

Theorem 2.2. *Let $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ be a poset of type d with $2 < i_k < n$ and $j_m = n$ then*

$$\left[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\right] = C^{j_m}_r,$$

where $r = \sum_{t=k-m+2}^{k} 2^{i_t - 1} - \sum_{t=1}^{m-1} 2^{j_t - 1}.$

Proof. (Induction). If $n = 4$ and $2 \le i_k < 4$ then $j_{m-1} = 1$, and $\left[\alpha^{i_0 3}_{j_0 14}\right] = 2\left[\alpha^{i_0 2}_{j_0 13}\right] + \left[b^{j_0 1}_{i_0 3}\right] = 2C^3_1 + a_0 = C^4_3$ with $r = 2^3 - 1$. If $j_{m-1} = 2$ $\left[\alpha^{i_0 13}_{j_0 24}\right] = 2\left[\alpha^{i_0 1}_{j_0 3}\right] + \left[b^{j_0 2}_{i_0 13}\right] = 2C^3_0 + a_1 = C^4_2$ with $r = 2^2 - 2$.

Suppose that the hypothesis holds for $n \le p$ with $2 \le i_k < p$. Then if $n = p+1$ the following cases have place:

Case 1. If $2 < i_k \leq p-1$ and $|i_k - j_{m-1}| > 1$ we have that $\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = \left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots p}\right] + \left[\alpha^{i_0 i_1 \ldots i_k - 1}_{j_0 j_1 \ldots p}\right] = C^p_w + C^p_s$ where $w = \sum\limits_{t=k-m+2}^{k} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1}$ and $s = \sum\limits_{t=k-m+2}^{k-1} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1} + 2^{i_k - 2}$. Since, it is easy to see that the number of digits in the binary expansion of $w = s + 2^{i_k - 2} < 2^{p-2}$ is $x = i_k - 1$ then we conclude that,

$$\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = C^{p+1}_w.$$

Case 2. If $2 < i_k \leq p-1$ and $|i_k - j_{m-1}| = 1$ then we have that $\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = \left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots p}\right] + \left[\alpha^{i_0 i_1 \ldots i_{k-1}}_{j_0 \ldots j_{m-2} p}\right] = C^p_w + C^p_s$, in this case, $w = \sum\limits_{t=k-m+2}^{k} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1}$ and $s = \sum\limits_{t=k-m+2}^{k-1} 2^{i_t - 1} - \sum\limits_{t=1}^{m-2} 2^{j_t - 1} + 2^{i_k - 2}$ then $s = w - 2^{i_k - 2}$ and the number of digits in the binary expansion of w is $x = i_k - 1$, thus,

$$\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = C^{p+1}_w.$$

Case 3. If $2 < i_k = p$ and $i_{m-1} = p-1$, it follows that $\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = 2\left[\alpha^{i_0 i_1 \ldots i_{k-1}}_{j_0 \ldots j_{m-2} p}\right] + \left[b^{j_0 j_1 \ldots p-1}_{i_0 i_1 \ldots p}\right] = 2C^p_w + a_s$, in this case, if $k = m$ then $w = \sum\limits_{t=k-m+2}^{k-1} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1} = r - 2^{p-2}$ and $s = 2^{p-1} - (r+1)$ with $r = \sum\limits_{t=1}^{k} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1}$. On the other hand, if $k = m+1$ then $w = r - 2^{p-2}$ and $s = 2^{p-1} - (r+1)$ with $r = \sum\limits_{t=1}^{k} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1}$, therefore,

$$\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = C^{p+1}_r.$$

Case 4. If $2 < i_k = p$ and $i_{m-1} < p-1$, it holds that $\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = 2\left[\alpha^{i_0 i_1 \ldots i_k - 1}_{j_0 j_1 \ldots p}\right] + \left[b^{j_0 j_1 \ldots j_{m-1}}_{i_0 i_1 \ldots p}\right] = 2C^p_w + a_s$ where $w = \sum\limits_{t=k-m+2}^{k-1} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1} + 2^{p-2} = r - 2^{p-2}$ and $s = \sum\limits_{t=k-m+1}^{m-1} 2^{j_t - 1} - \sum\limits_{t=1}^{k-1} 2^{i_t - 1} = 2^{p-1} - (r+1)$, with $r = \sum\limits_{t=k-m+2}^{k} 2^{i_t - 1} - \sum\limits_{t=1}^{m-1} 2^{j_t - 1}$. In this case, we also have that $\left[\alpha^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}\right] = C^{p+1}_r$. $\square$

2.1.5 Posets of Type $h^{i_0 i_1 \ldots i_k}_{j_0 j_1 \ldots j_m}$

We let $h^{j_0 j_1 \ldots j_m}_{i_0 i_1 \ldots i_k}$ denote a poset which can be defined by following the steps:

1. Follow steps 1-5 of Algorithm 2.1.
2. Define
$$G=\bigcup_{s=1}^{m}\{\{(x,y)\in\mathbb{Z}^2 \mid a\le y\le z_s,\ x=w_s\}\cup\{(x,y)\in\mathbb{Z}^2 \mid y=z_s,\ w_s\le x\le b\}\},$$
where
$$a=\begin{cases} z_{s+1}, & \text{if } s<m,\\ 0, & \text{if } s=m,\end{cases}\qquad b=\begin{cases} w_{s-1}, & \text{if } s>1,\\ x_1, & \text{if } s=1.\end{cases}$$
3. $h^{j_0j_1\dots j_m}_{i_0i_1\dots i_k}=\{(x,y)\in\mathbb{Z}^2 \mid x=w-i,\ y=z+i,\ (w,z)\in G\}$ for $0\le i\le \max\{i_k,j_m\}$.

The following are examples of lattice paths in posets of type $h^{j_0j_1\dots j_m}_{i_0i_1\dots i_k}$:

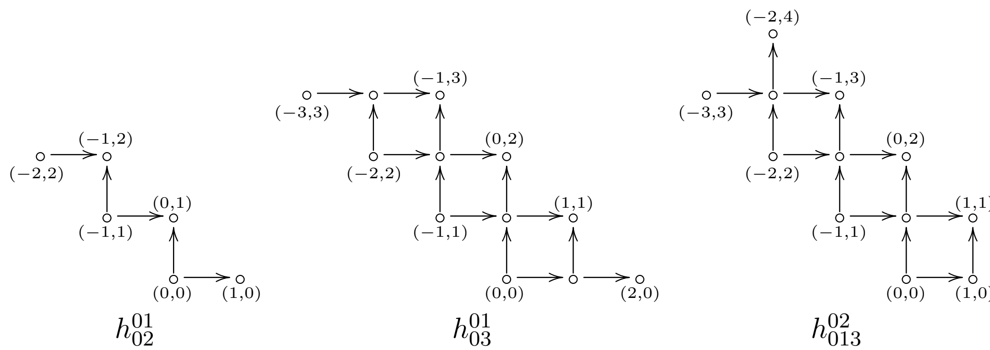


FIGURE 2.6. Examples of oriented lattice paths of posets of type $h^{j_0j_1\dots j_m}_{i_0i_1\dots i_k}$.

Now, we intend to enumerate the number of lattice paths $\left|P^{(e,f)}_{(x,y)}\right|$ connecting points (x,y) and (e,f) with $y=-x$ and $e+f=\max\{i_k,j_m\}$, $(x,y),(e,f)\in h^{j_0j_1\dots j_m}_{i_0i_1\dots i_k}$, and conclude that
$$\left[h^{j_0j_1\dots j_m}_{i_0i_1\dots i_k}\right]=\sum\left|P^{(e,f)}_{(x,y)}\right|.$$

Numbers b_{R_t} and b_{L_r} are defined as follows for cases $i_1=1$, $j_1=1$:

- If $j_1=1$ then
$$b_{R_t}=\begin{cases}\left[b^{j_0\dots j_tj_m}_{i_0\dots i_{t-1}(j_t+p_1)}\right], & \text{for } 1\le t\le k,\ 1\le p_1\le |R_t|,\\ \left[b^{j_0j_m}_{i_01}\right], & \text{for } t=0,\end{cases}$$
$$b_{L_r}=\begin{cases}\left[b^{i_0\dots i_{r-1}j_m}_{j_0\dots j_{r-1}(i_{r-1}+p_2)}\right], & \text{for } 1\le r\le m,\ 1\le p_1\le |L_m| \text{ y } p_2\ne |L_m|\\ \left[b^{i_01_1\dots i_k}_{j_0j_1\dots j_m}\right], & \text{for } p_2=|L_m| \text{ and } r=m,\end{cases}$$
with $|R_t|=|i_t-j_t|$, $|R_0|=1$, $|L_r|=|j_r-i_{r-1}|$ and $|L_1|=1$ for $1\le t\le k$ and $1<r\le m$.
- If $i_1=1$

$$b_{R_t} = \left[b^{j_0\ldots j_{t-1}j_m}_{i_0\ldots i_{t-1}(j_{t-1}+p_1)}\right] \text{ para } 1 \leq t \leq k,\ 1 \leq p_1 \leq |R_t|,$$

$$b_{L_r} = \begin{cases} \left[b^{i_0\ldots i_r j_m}_{j_0\ldots j_{r-1}(i_r+p_2)}\right], & \text{for } 1 \leq r \leq m,\ 1 \leq p_1 \leq |L_m| \text{ and } p_2 \neq |L_m|, \\ \left[b^{i_0 1_1\ldots i_k}_{j_0 j_1\ldots j_m}\right], & \text{for } p_2 = |L_m| \text{ and } r = m, \\ \left[b^{j_0 j_m}_{i_0 1}\right], & \text{for } r = 0, \end{cases}$$

with $|R_t| = |i_t - j_{t-1}|$, $|R_1| = 1$, $|L_r| = |j_r - i_r|$ and $|L_0| = 1$ for $1 \leq t \leq k$ and $1 \leq r \leq m$.

Theorem 2.3.

$$\left[h^{j_0 j_1\ldots j_m}_{i_0 i_1\ldots i_k}\right] = \sum_{t,p_1} b_{R_t} + \sum_{r,p_1,p_2} b_{L_r}.$$

Proof. Let $h^{j_0 j_1\ldots j_m}_{i_0 i_1\ldots i_k}$ be a poset with $j_m = n$ and $m \geq k$, and sets b_u defined in such a way that:

$$b_u = \{(x,y) \in h^{j_0 j_1\ldots j_m}_{i_0 i_1\ldots i_k} \mid x \geq x_m - u,\ y \geq y_m + u,\ x + y \leq n - 1\},$$

with $0 \leq u \leq n$. If $j_1 = 1$, it is possible to build the sets $R_t = \{y \in \mathbb{Z} | z_t \leq y < y_t\}$, $R_0 = \{y_1\}$, $L_r = \{x \in \mathbb{Z} | x_r \leq x \leq w_{r-1}\}$ and $L_1 = \{x_1\}$ for $1 \leq t \leq k$ y $1 < r \leq m$ where $|R_t| = |z_t - y_t| = |i_t - j_t|$, $|R_0| = 1$, $|L_r| = |x_r - w_{r-1}| = |j_r - i_{r-1}|$ and $|L_1| = 1$. If $\sum_{a=0}^{k-t} |R_{k+1-a}| \leq u < \sum_{a=0}^{k+1-t} |R_{k+1-a}|$ ($|R_{k+1}| = 0$), then there exists an isomorphism

$$\begin{array}{rccl} f_u: & b^{j_0\ldots j_t j_m}_{i_0\ldots i_{t-1}(j_t+p_1)} & \longrightarrow & b_u \\ & (x,y) & \longmapsto & (y-u, x+u) \end{array}$$

with

$$f_u\big((x,y),(z,w)\big) = \big(((y-u, x+u),(w-u, z+u))\big) \text{ for any edge } ((x,y),(z,w)). \quad (2.15)$$

For $1 \leq p_1 \leq |R_t|$, where $u = \sum_{a=0}^{k+1} |R_{k+1-a}|$ then there exists an isomorphism $f_u : b^{j_0 j_m}_{i_0 1} \longrightarrow$ b_u defined as in (2.15). Similarly, if $\sum_{a=0}^{k\mp1} |R_{k+1-a}| + \sum_{a=0}^{r-1} |L_a| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + \sum_{a=0}^{r} |L_a|$ ($|L_0| = 0$) or $u = |L_m|$, it is possible to define an isomorphism defined as in (2.15) with

$$g_u : b^{i_0\ldots i_{r-1} j_m}_{j_0\ldots j_{r-1}(i_{r-1}+p_2)} \longrightarrow b_u, \text{ for } 1 \leq p_2 \leq |L_r|.$$

The same can be done for a homomorphism

$$g_n : b^{i_0 1_1\ldots i_k}_{j_0 j_1\ldots j_m} \longrightarrow b_n, \text{ if } r = m \text{ and } |L_m| = p_2.$$

Thus,

$$\left[h^{j_0 j_1 \ldots j_m}_{i_0 i_1 \ldots i_k}\right] = \sum_{t,p_1} b_{R_t} + \sum_{r,p_1,p_2} b_{L_r}. \tag{2.16}$$

If $i_1 = 1$ then we can construct sets $R_t = \{y \in \mathbb{Z} | z_t \leq y < y_{t-1}\}$, $R_1 = \{z_1\}$, $L_r = \{x \in \mathbb{Z} | x_r \leq x < w_r\}$ and $L_0 = \{w_1\}$ for $1 < t \leq k$ and $1 \leq r \leq m$, where $|R_t| = |z_t - y_{t-1}| = |i_t - j_{t-1}|$, $|R_1|1$, $|L_r| = |x_r - w_r| = |j_r - i_r|$ and $|L_0| = 1$.

We conclude that the theorem holds provided that isomorphisms of the following types:

$$\begin{aligned}
f_u &: b^{j_0 \ldots j_{t-1} j_m}_{i_0 \ldots i_{t-1}(j_{t-1}+p_1)} \longrightarrow b_u, \\
g_u &: b^{j_0 j_m}_{i_0 1} \longrightarrow b_u, \\
h_u &: b^{i_0 \ldots i_r j_m}_{j_0 \ldots j_{r-1}(i_r+p_2)} \longrightarrow b_u, \\
i_n &: b^{i_0 1_1 \ldots i_k}_{j_0 j_1 \ldots j_m} \longrightarrow b_n
\end{aligned} \tag{2.17}$$

can be defined respectively according to the following cases for u, p_1 and p_2:

$$\sum_{a=0}^{k-t} |R_{k+1-a}| \leq u < \sum_{a=0}^{k+1-t} |R_{k+1-a}| \text{ and } 1 \leq p_1 \leq |R_t|, \tag{2.18}$$

$$\sum_{a=0}^{k\mp 1} |R_{k+1-a}| + |L_0| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + |L_0| + |L_1|, \tag{2.19}$$

$$\sum_{a=0}^{k\mp 1} |R_{k+1-a}| + \sum_{a=0}^{r-1} |L_a| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + \sum_{a=0}^{r} |L_a|, | \text{ and } 1 \leq p_2 \leq |L_r|, p_2 \neq |L_m|, \tag{2.20}$$

$$r = m \text{ and } p_2 = |L_m|. \tag{2.21}$$

We are done. $\square$

Remark 2.4. *On sets $\{b_{R_t}\}$ (resp. $\{b_{L_m}\}$) it is defined a partial order such that $b_{R_t} < b_{R_s}$ (resp. $b_{L_t} < b_{L_s}$) if and only if $i_{k_s} < i_{k_t}$ (resp. $i_{k_t} < i_{k_s}$) with $<$ the relation induced by the usual order of natural numbers, thus elements in the set $\{\{b_{R_t}\}, \{b_{L_m}\}\}$ can be written as a vector*

$$\overline{h^{j_0 j_1 \ldots j_m}_{i_0 i_1 \ldots i_k}} = (v_0, \ldots, v_n),$$

where if $b_{R_t} < b_{R_s}$ then $v_u = b_{R_t}$ and $v_{u+1} = b_{R_s}$ for $0 \leq u < |\{b_{R_t}\}| - 1$, and if $b_{L_t} < b_{L_s}$ then $v_r = b_{L_t}$ and $v_{r+1} = b_{L_s}$ for $|\{b_{R_t}\}| \leq r < |\{b_{R_t}\}| + |\{b_{L_m}\}|$.

As an example, we have the following identities:

$$
\begin{aligned}
\overline{h_{01}^{02}} &= \left([b_{01}^{02}], [b_{01}^{02}], [b_{02}^{01}]\right),\\
\overline{h_{01}^{03}} &= \left([b_{01}^{03}], [b_{01}^{03}], [b_{02}^{013}], [b_{03}^{01}]\right),\\
\overline{h_{02}^{013}} &= \left([b_{02}^{013}], [b_{01}^{03}], [b_{01}^{03}], [b_{013}^{02}]\right),\\
\overline{h_{01}^{04}} &= \left([b_{01}^{04}], [b_{01}^{04}], [b_{02}^{014}], [b_{03}^{014}], [b_{04}^{01}]\right),\\
\overline{h_{02}^{014}} &= \left([b_{02}^{014}], [b_{01}^{04}], [b_{01}^{04}], [b_{013}^{024}], [b_{014}^{02}]\right),\\
\overline{h_{013}^{024}} &= \left([b_{013}^{024}], [b_{01}^{04}], [b_{01}^{04}], [b_{02}^{014}], [b_{024}^{013}]\right),\\
\overline{h_{03}^{014}} &= \left([b_{03}^{014}], [b_{02}^{014}], [b_{01}^{04}], [b_{01}^{04}], [b_{014}^{03}]\right),\\
\overline{h_{01}^{05}} &= \left([b_{01}^{05}], [b_{01}^{05}], [b_{02}^{015}], [b_{03}^{015}], [b_{04}^{015}], [b_{05}^{01}]\right),\\
\overline{h_{02}^{015}} &= \left([b_{02}^{015}], [b_{01}^{05}], [b_{01}^{05}], [b_{013}^{025}], [b_{014}^{025}], [b_{015}^{02}]\right),\\
\overline{h_{013}^{025}} &= \left([b_{013}^{025}], [b_{01}^{05}], [b_{01}^{05}], [b_{02}^{015}], [b_{024}^{0135}], [b_{025}^{013}]\right),\\
\overline{h_{03}^{015}} &= \left([b_{03}^{015}], [b_{02}^{015}], [b_{01}^{05}], [b_{01}^{05}], [b_{014}^{035}], [b_{015}^{03}]\right),\\
\overline{h_{014}^{035}} &= \left([b_{014}^{035}], [b_{01}^{05}], [b_{01}^{05}], [b_{02}^{015}], [b_{03}^{015}], [b_{035}^{014}]\right),\\
\overline{h_{024}^{0135}} &= \left([b_{024}^{0135}], [b_{02}^{015}], [b_{01}^{05}], [b_{01}^{05}], [b_{013}^{025}], [b_{0135}^{024}]\right),\\
\overline{h_{014}^{025}} &= \left([b_{014}^{025}], [b_{013}^{025}], [b_{01}^{05}], [b_{01}^{05}], [b_{02}^{015}], [b_{025}^{014}]\right),\\
\overline{h_{04}^{015}} &= \left([b_{04}^{015}], [b_{03}^{015}], [b_{02}^{015}], [b_{01}^{05}], [b_{01}^{05}], [b_{015}^{04}]\right).
\end{aligned}
$$

2.2 Sections in the Auslander-Reiten Quiver of Algebras of Dynkin Type

In this section we use quivers of type b, d and h in order to enumerate the number of sections in the Auslander-Reiten quiver of algebras of type $\mathbb{A}_n$ and $\mathbb{D}_n$. We present the same description for the case of $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$.

2.2.1 Sections in the Auslander-Reiten Quivers of Algebras of Type $\mathbb{A}_n$

Let $\mathcal{A} = k\mathbb{A}_n$ be a path algebra induced by an oriented Dynkin diagram of type $\mathbb{A}_n$ with k sinks and m sources, $\Gamma(\text{mod}\,\mathcal{A})$ be the corresponding Auslander-Reiten quiver and $S_{(\mathbb{A}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$ the number of sections in $\Gamma(\text{mod}\,\mathcal{A})$ of these kind of algebras, where i_t represents the location of a sink for $1 \leq t \leq k$ and j_s represents the location of a source for $1 \leq s \leq m$, points i_t and j_s follow conditions defined in steps 1-3 of Algorithm 2.1. If we have an algebra $\mathcal{B}$ with only one point then the number of sections in the Auslander-Reiten quiver will be denoted $S_{(\mathbb{A}_1)^{1}_{-}}$. For vertices in Dynkin diagrams of type $\mathbb{A}_n$, we assume the numbering described in Figure 1.2 (Section 1.1).

If $\mathcal{A}$ is an algebra as described above then $\Gamma(\text{mod}\,\mathcal{A})$ is isomorphic to the quiver $\overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$ obtained from $d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ by orienting each edge $((x, y), (x', y'))$ as $(x, y) \to (x', y')$. Such an isomorphism can be defined by associating to each $\tau-$orbit of a given vertex $x_t \in \Gamma_0$ points $(x, y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$ such that $x + y = u + 1 - t$ for $1 \leq t \leq n$. If $\mathcal{A}$ is an algebra of type $\mathbb{A}_n$ with sinks located at points $\{i_1 \dots, i_k\}$ and sources located at points $\{j_1, \dots, j_m\}$ and $\mathcal{B}$ is an algebra of type $\mathbb{A}_n$ with sinks at $\{j_1, \dots, j_m\}$ and sources at $\{i_1, \dots, i_k\}$ then there exists an isomorphism:

$$\begin{array}{rccc} \varphi: & d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} & \longrightarrow & d^{j_0 j_1 \dots j_m}_{i_0 i_1 \dots i_k} \\ & (x, y) & \longmapsto & (-x, -y) \end{array}$$

where $\varphi\big((x,y),(z,w)\big) = \big((-x,-y),(-z,-w)\big)$. Henceforth, a quiver of the form $\overrightarrow{d^{j_0 j_1 \dots j_m}_{i_0 i_1 \dots i_k}}$ is said to be the *conjugate quiver* of $\overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$, and

$$S_{(\mathbb{A}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}} = \sum_{p,q} |P^q_p|,$$

where $p = (a,b), q = (c,d) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}$.

Note that,

$$S_{(\mathbb{A}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}} = S_{(\mathbb{A}_n)^{j_0 j_1 \dots j_m}_{i_0 i_1 \dots i_k}}.$$

The arguments described above allow us to give the following result regarding the number of sections in algebras of Dynkin type $\mathbb{A}_n$ with $i_k < n$.

Theorem 2.4. *Let $\mathcal{A}$ be an algebra of type $\mathbb{A}_n$ with sinks and sources located at points of the sets $\{i_1, \dots, i_k < n\}$ and $\{j_1, \dots, j_m\}$, respectively. Then*

$$S_{(\mathbb{A}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}} = \begin{cases} 2S_{(\mathbb{A}_{n-1})^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots (j_m)-1}} + \big[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\big], & \text{if } i_k < n-1, \\ 2S_{(\mathbb{A}_{n-1})^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_{m-1}}} + \big[\alpha^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}\big], & \text{if } i_k = n-1. \end{cases}$$

Proof. Suppose that the algebra $\mathcal{A}$ satisfies hypothesis of the theorem then since $\Gamma(\text{mod}\,\mathcal{A})$ is isomorphic to the quiver $\overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$, we choose the subquiver B whose vertices B_0 can be written in the form $B_0 = B_1 \cup B_2$ where

$$B_1 = \bigcup_{r=1}^{k} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid 0 \leq y \leq y_r - 1,\, x \geq x_r,\, x + y \leq u - 1\},$$

$$B_2 = \bigcup_{s=1}^{m-1} \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} \mid \overline{z_s} \leq y \leq 0,\, x \leq \overline{w_s},\, x + y \geq -n + u\}.$$

If $|y_k - 1 - z_m| > 1$ $(i_k < n-1)$ then $i'_t = (x, y-1) = j'_s = (x, y-1)$ and $j'_m = j^*_m$ for $1 \leq t \leq k$ and $1 < s < m$, therefore $B_0 = d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}$, and $B = \overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}}$.

If $|y_k - 1 - z_m| = 1$ $(i_k = n-1)$ it holds that $i'_t = (x, y-1) = j'_s$ and $i'_k = j^*_m$ for $1 \leq t < k$ and $1 < s < m$, thus as before $B_0 = d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_{m-1}}$ and $B = \overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m - 1}}$.

If $C = \{(x,y) \in d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m} | x + y = u\}$ and it is defined the quiver $C \cup B$ with arrows induced by $\overrightarrow{d^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$ then since $\big|P^{(x,y)}_{(c,d)}\big| = 2$ we have

$$\Big[C \cup B\Big] = \sum_{a,b,x,y} \big|P^{(x,y)}_{(a,b)}\big| = 2 \sum_{a,b,c,d} \big|P^{(c,d)}_{(a,b)}\big| = 2\Big[B\Big],$$

where $a+b=u-n$ and $c+d=u-1$.

Since $\sum\limits_{a,b,h,p}\left|P^{(h,p)}_{(a,b)}\right|$ contains at least one point of $(C\cup B)^c$ for $a+b=u-n+1$, $h+p=u$ and $(a,b),(h,p)\in a^{i_0i_1\dots i_k}_{j_0j_1\dots j_m}$ then $S_{(\mathbb{A}_n)^{i_0i_1\dots i_k}_{j_0j_1\dots j_m}}=2\Big[B\Big]+\Big[\alpha^{i_0i_1\dots i_k}_{j_0j_1\dots j_m}\Big]$. $\square$

Remark 2.5. *Numbers $S_{(\mathbb{A}_n)^{i_0i_1\dots i_k}_{j_0j_1\dots j_m}}$ with $i_k<n$ are shown in Table A.2 (see Appendix), where rows represents values $j_m=n$ and columns are given by the identities:*

$$r=\begin{cases}\sum\limits_{t=k-m+2}^{k}2^{i_t-1}-\sum\limits_{t=1}^{m-1}2^{j_t-1}, & \text{if } i_k>1,\\ 0, & \text{if } i_k=1.\end{cases}\tag{2.22}$$

For instance, if $\mathcal{A}$ is an algebra of type $\mathbb{A}_7$ with sinks and sources at points of the sets $\{1,4,7\}$ and $\{3,5\}$ respectively then it holds that $S_{(\mathbb{A}_7)^{0147}_{035}}=S_{(\mathbb{A}_7)^{035}_{0147}}$,

$$r=2^4+2^2-2^3-1=11,$$

and

$$S_{(\mathbb{A}_7)^{0147}_{035}}=S_{(\mathbb{A}_7)^{035}_{0147}}=2S_{(\mathbb{A}_7)^{035}_{0146}}+\alpha^{035}_{0147}=2(64)+17=145.$$

We recall that sections in algebras of type $\mathbb{A}_n$ with sinks and sources at sets $\{1,3\}$ and $\{2,n\}$, respectively categorifies the sequence $A176448=\{5,12,26,54,110,\dots\}$ in the OEIS [88].

The following corollaries dealing with oriented Dynkin diagram of type $\mathbb{A}_n$ with only sink were reported by the author, Cañadas and Giraldo in [32]. Henceforth, we assume the notation $S_{(\mathbb{A}_n)^m_-}$ for $S_{(\mathbb{A}_n)^{0m}_{01n}}$.

Corollary 2.1. $S_{(\mathbb{A}_n)^m_-}=2\left(S_{(\mathbb{A}_{n-1})^m_-}\right)+\sum\limits_{i=0}^{m-2}\binom{n-2}{i}$ *for $n\geq 3$, $1\leq m<n$ with* $S_{(\mathbb{A}_n)^1_-}=S_{(\mathbb{A}_n)^n_-}=2^{n-1}$.

Corollary 2.2.

$$S_{(\mathbb{A}_n)^m_-}=S_{(\mathbb{A}_{n-1})^{m-1}_-}+S_{(\mathbb{A}_{n-1})^m_-}+S_{(\mathbb{A}_{n-2})^1_-},\tag{2.23}$$

for $n\geq 3$, $1\leq m<n$.

Proof. We proceed by induction on n taking into account that $S_{(\mathbb{A}_n)^1_-}=S_{(\mathbb{A}_n)^n_-}=2^{n-1}$. If $n=3$ and $m=2$ we have that

$$S_{(\mathbb{A}_3)^2_-}=2(S_{(\mathbb{A}_2)^2_-})+1=S_{(\mathbb{A}_2)^2_-}+S_{(\mathbb{A}_2)^1_-}+S_{(\mathbb{A}_1)^1_-}.$$

Suppose that the assertion is true for $3\leq k\leq n$ and $2\leq m\leq n-1$. Thus

$$\begin{aligned}
S_{(\mathbb{A}_k)^m_-} &= 2(S_{(\mathbb{A}_k)^m_-}) + \sum_{i=0}^{m-2}\binom{k-1}{i}\\
&= 2(S_{(\mathbb{A}_k)^m_-}) + \sum_{i=0}^{m-2}\binom{k-2}{i} + \sum_{i=0}^{m-3}\binom{k-2}{i}.
\end{aligned}$$

Therefore

$$\begin{aligned}
S_{(\mathbb{A}_{k+1})^m_-} &= 2(S_{(\mathbb{A}_{k-1})^{m-1}_-} + S_{(\mathbb{A}_{k-1})^m_-} + 2^{k-3}) + \sum_{i=0}^{m-2}\binom{k-2}{i} + \sum_{i=0}^{m-3}\binom{k-2}{i}\\
&= S_{(\mathbb{A}_k)^{m-1}_-} + S_{(\mathbb{A}_k)^m_k} + S_{(\mathbb{A}_{k-1})^1_-}. \quad \square
\end{aligned}$$

Identity (2.23) allows to give a partition-formula for numbers in the sequence A049611 or A084851 [87].

Corollary 2.3. $c_n = \sum_{h=1}^{n} S_{(\mathbb{A}_n)^h_-} = 2^{n-4}(n^2+5n+2)$, $n \geq 4$.

Proof. Rows in the next table show the number of sections in the Auslander-Reiten quiver of an algebra $\mathcal{A}$ associated to a Dynkin graph of type $\mathbb{A}_n$ with an unique sink allocated at the hth position (from the left to the right), $1 \leq h \leq n$ and $1 \leq n \leq 7$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 2 & & 2 & & & & & \\
 & & & & 4 & & 5 & & 4 & & & & \\
 & & & 8 & & 11 & & 11 & & 8 & & & \\
 & & 16 & & 23 & & 26 & & 23 & & 16 & & \\
 & 32 & & 47 & & 57 & & 57 & & 47 & & 32 & \\
64 & & 95 & & 120 & & 130 & & 120 & & 95 & & 64
\end{array}$$

For instance, according to the Corollary 2.2 we see that,

$$\begin{aligned}
95 &= 16+32+47,\\
120 &= 16+47+57,\\
130 &= 57+57+16,
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
688 &= 2(64)+2(95)+2(120)+130\\
&= 2(2^6)+2(2^5)+9(2^4)+16(2^3)+28(2^2)+48(2^1)+16(1).
\end{aligned} \tag{2.25}$$

Actually, it is easy to see that in the case $n = k \geq 4$, it holds that

$$\textstyle\sum_{h=1}^{k} S_{(\mathbb{A}_k)^h_-} = 2(2^k) + 2^{k-3} + \sum_{j=4}^{k}((k+2)-(k-j))2^{k-j}.2^{j-3} = 2(2^k) + 2^{k-3} + \sum_{i=6}^{n+2} i2^{k-3}.$$

Thus

$\sum_{h=1}^{k} S_{(\mathbb{A}_k)^h_-} = [\frac{(k+2)(k+3)}{2} - 14]2^{k-3} + 2^k + 2^{k-1} = [\frac{(k+2)(k+3)}{2} - 14]2^{k-3} + 3(2^{k-1}) = 2^{k-4}[k^2+5k-22+24] = 2^{k-4}[k^2+5k+2]$. Since $k \geq 4$ is arbitrary we are done. □

Corollary 2.4. $S_{(\mathbb{A}_n)^2_-} = 3(2^{n-2}) - 1$.

Proof. $S_{(\mathbb{A}_3)^2_-} = 5 = 2+2+1$ and $S_{(\mathbb{A}_4)^2_-} = S_{(\mathbb{A}_3)^2_-} + (2^2) + (2^1) = 5 + (2^2) + (2^1) = (2^2) + (2^0) + (2^2) + (2^1) = (2^3) + (2^1) + (2^0)$. Thus for any $k \geq 3$ it holds that

$$S_{(\mathbb{A}_k)^2_-} = S_{(\mathbb{A}_{k-1})^2_-} + (2^{k-1}) + (2^{k-2}). \tag{2.26}$$

Therefore

$$\begin{aligned} S_{(\mathbb{A}_k)^2_-} &= 2(2^{k-2}) + 2(2^{k-4}) + 2(2^{k-5}) + \cdots + 2(2^2) + 3(2^1) = (2^{k-1}) + \sum_{j=0}^{k-3} 2^j \\ &= 2^{k-1} + 2^{k-2} - 1 = 3(2^{k-2}) - 1. \quad \square \end{aligned} \tag{2.27}$$

Corollaries 2.2 and 2.4 allow us to establish the following result.

Corollary 2.5. $S_{(\mathbb{A}_k)^h_-} = (h+1)2^{k-2} - \sum_{j=1}^{\lfloor \frac{h}{2} \rfloor} j\binom{k+1}{h-2j}$.

Proof. (Induction on h) Firstly, we recall the following identities:

$$\begin{aligned} h\sum_{j=h+1}^{k-1} 2^{j-2} &= h(2^{k-2}) - h2^{h-2}, \\ \sum_{j=h-2}^{k-3} 2^j &= (2^{k-2}) - 2^{h-2}, \\ \sum_{j=h+1}^{k-1} \binom{j+1}{i} &= \binom{k+1}{i+1} - \binom{h+1}{i+1}. \end{aligned} \tag{2.28}$$

Corollaries 2.2 and 2.4 induce the following identity where $S_{(\mathbb{A}_{h+1})^{h+1}_-} = 2^h$:

$$S_{(\mathbb{A}_k)^{h+1}_-} = \sum_{j=h+1}^{k-1} S_{(\mathbb{A}_k)^h_-} + \sum_{j=h-2}^{k-3} 2^j + S_{(\mathbb{A}_{h+1})^{h+1}_-}. \tag{2.29}$$

Now if we assume that the theorem is true for any fixed k, $k \geq 1$ and $1 \leq s \leq h$ then the theorem holds for $s = h+1$ if identities (2.28) are applied to the summands in (2.29). □

Remark 2.6. *We note that*

1. *For $n \geq 2$, the sequence $a_n = S_{(\mathbb{A}_n)^2_-}$ appears in the OEIS as A083329* [85].

2. For $n \geq 3$, the sequence $b_n = S_{(\mathbb{A}_n)^3_-}$ appears in the OEIS as A000295 [86].

2.2.2 Sections in the Auslander-Reiten Quiver of Algebras of Type $\mathbb{D}_n$

Let $\mathcal{A}$ be an algebra with underlying diagram of type $\mathbb{D}_n$, with k sinks and m sources. We assume the numbering for Dynkin diagrams of type $\mathbb{D}_n$ described in Figure 1.2 (section 1.1).

Let $\mathcal{B}$ be an algebra of type $\mathbb{D}_n$ $(n \geq 4)$ whose sinks and sources are located at points of the sets $\{j_1, \dots, j_m\}$ and $\{i_1, \dots, i_k\}$ respectively. Then, if there exists an irreducible morphism in $\Gamma(\text{mod } \mathcal{A})$ of the form $\tau_a^{-s} \longrightarrow \tau_b^{-r}$ then there exists an irreducible morphism in $\Gamma(\text{mod } B)$ of the form $\tau_b^{-s} \longrightarrow \tau_a^{-r}$ for some $s, r \in \mathbb{Z}$, $\mathcal{B}$ denotes the conjugate quiver of $\mathcal{A}$ and the following identity has place:

$$S_{(\mathbb{D}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}} = S_{(\mathbb{D}_n)^{j_0 j_1 \dots j_m}_{i_0 i_1 \dots i_k}}.$$

$\overrightarrow{h^{j_0 j_1 \dots j_w}_{i_0 i_1 \dots i_r}}$ with $r \leq k$ and $w \leq n$ is a subquiver of $\Gamma(\text{mod } \mathcal{A})$, where each $\tau-$orbit of a point $x_t \in \Gamma_0$ has associated points $(x, y) \in h^{i_0 i_1 \dots i_r}_{j_0 j_1 \dots j_w}$ with $x + y = n - 1 - t$ for $1 \leq t \leq n - 2$. According to these arguments it suffices to consider the subquiver $\mathbb{A}'_{n-2}$ with vertices $1 \dots n-2$ a sink at the vertex $n-2$. Thus, we can enumerate sections in $\Gamma(\mathcal{A})$ via the following three cases described in Theorem 2.5:

Theorem 2.5. *Let $\mathcal{A}$ be an algebra of type $\mathbb{D}_n$ with sink and sources located at the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_m\}$. If the subquiver $\mathbb{A}'_{n-2}$ has a sink at the vertex $n-2$ then:*

$$S_{(\mathbb{D}_n)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}} = \begin{cases} 4\displaystyle\sum_{s=0}^{n-3} v_s + v_{n-2}, & \textit{if } n, n-1 \textit{ are sources,} \\ 4\displaystyle\sum_{s=1}^{n-3} v_s + 2(v_{n-2} + v_0), & \textit{if } n \textit{ is a source and n-1 is a sink, (viceversa),} \\ 4\displaystyle\sum_{s=1}^{n-2} v_s + v_0, & \textit{if } n, n-1 \textit{ are sinks,} \end{cases}$$

where $v_p \in \overline{h^{i_0 i_1 \dots i_r}_{j_0 j_1 \dots j_w}}$ with $r \leq k$ and $w \leq n$ (see Theorem 2.3 and Remark 2.4).

Proof. If the algebra $\mathcal{A}$ satisfies the hypothesis and the subquiver $\mathbb{A}'_{n-2}$ has a sink at the vertex $n-2$. Then, we can take a subquiver B of $\Gamma(\text{mod } \mathcal{A})$ which is isomorphic to $\overrightarrow{h^{i_0 i_1 \dots i_r}_{j_0 j_1 \dots j_w}}$ with $r \leq k$ and $w \leq m$, we note that each orbit $\tau_{n-2}^{-s} \in \Gamma(\text{mod } \mathcal{A})$ has associated the point $(-n-2+s, n-2-s) \in h^{i_0 i_1 \dots i_r}_{j_0 j_1 \dots j_w}$ with $0 \leq s \leq n-2$, thus $\sum \left| P^{(a,b)}_{(-n-2+s, n-2-s)} \right| = v_{n-2-s}$ for $a + b = n - 3$. Now, we have three cases:

- If n and $n-1$ are sources for each τ_{n-2}^{-s} with $1 \leq s \leq n-2$ then there exist the irreducible morphisms $\tau_{n-1}^{-(s-1)} \longrightarrow \tau_{n-2}^{-s}$, $\tau_n^{-(s-1)} \longrightarrow \tau_{n-2}^{-s}$, $\tau_{n-2}^{-s} \longrightarrow \tau_{n-1}^{-s}$, and

$\tau_{n-2}^{-s} \longrightarrow \tau_n^{-s}$ and for τ_{n-2}^{0} there exist two irreducible morphisms of $\tau_{n-2}^{0} \longrightarrow \tau_{n-1}^{0}$ and $\tau_{n-2}^{0} \longrightarrow \tau_n^{0}$, then each $\left|P_{(-n-2+s,n-2-s)}^{(a,b)}\right|$ is multiplied by the 4 combinations of the irreducible morphisms of the vertices n and $n-1$ if $1 \leq s \leq n-2$ and by 1 if $s=0$, thus

$$S_{(\mathbb{D}_n)_{j_0 j_1 \cdots j_m}^{i_0 i_1 \cdots i_k}} = 4\sum_{s=0}^{n-3} v_s + v_{n-2}.$$

- If n is a source and $n-1$ is a sink (or viceversa), for each τ_{n-2}^{-s} with $1 \leq s \leq n-3$ then we can associate irreducible morphisms $\tau_{n-1}^{-s} \longrightarrow \tau_{n-2}^{-s}$, $\tau_n^{-(s-1)} \longrightarrow \tau_{n-2}^{-s}$, $\tau_{n-2}^{-s} \longrightarrow \tau_{n-1}^{-(s+1)}$ and $\tau_{n-2}^{-s} \longrightarrow \tau_n^{-s}$. Whereas, associated to the translation τ_{n-2}^{0} there are associated irreducible morphisms $\tau_{n-1}^{0} \longrightarrow \tau_{n-2}^{0}$, $\tau_{n-2}^{0} \longrightarrow \tau_n^{0}$ and $\tau_{n-2}^{0} \longrightarrow \tau_{n-1}^{-1}$. Finally, translation $\tau_{n-2}^{-(n-2)}$ has associated irreducible morphisms $\tau_n^{-(n-3)} \longrightarrow \tau_{n-2}^{-(n-2)}$, $\tau_{n-1}^{-(n-2)} \longrightarrow \tau_{n-2}^{-(n-2)}$ and $\tau_{n-2}^{-(n-2)} \longrightarrow \tau_{n-1}^{-(n-1)}$, then each number $\left|P_{(-n-2+s,n-2-s)}^{(a,b)}\right|$ can be multiplied by the 4 combinations induced by the irreducible morphisms of vertices n and $n-1$ if $1 \leq s \leq n-3$ and for the 2 combinations of the irreducible morphisms of vertices n and $n-1$ if $s=0$ or $s=n-2$, thus

$$S_{(\mathbb{D}_n)_{j_0 j_1 \cdots j_m}^{i_0 i_1 \cdots i_k}} = 4\sum_{s=1}^{n-3} v_s + 2(v_{n-2} + v_0).$$

- If n and $n-1$ are sinks, for each τ_{n-2}^{-s} with $0 \leq s \leq n-3$ then there are associated irreducible morphisms $\tau_{n-1}^{-s} \longrightarrow \tau_{n-2}^{-s}$, $\tau_n^{-s} \longrightarrow \tau_{n-2}^{-s}$, $\tau_{n-2}^{-s} \longrightarrow \tau_{n-1}^{-(s+1)}$ and $\tau_{n-2}^{-s} \longrightarrow \tau_n^{-(s+1)}$, as well as for $\tau_{n-2}^{-(n-2)}$ there are associated irreducible morphisms $\tau_n^{-(n-2)} \longrightarrow \tau_{n-2}^{-(n-2)}$ and $\tau_{n-1}^{-(n-2)} \longrightarrow \tau_{n-2}^{-(n-2)}$ then each number $\left|P_{(-n-2+s,n-2-s)}^{(a,b)}\right|$ is multiplied for the 4 combinations of the irreducible morphisms of the vertices n and $n-1$ if $0 \leq s \leq n-3$ and by 1 if $s=n-2$, thus

$$S_{(\mathbb{D}_n)_{j_0 j_1 \cdots j_m}^{i_0 i_1 \cdots i_k}} = 4\sum_{s=1}^{n-2} v_s + v_0.$$

□

For example, let Q_1 be a quiver of type $\mathbb{D}_n$ whose sinks and sources are located at points of the sets $\{1,4,6,7\}$ and $\{3,5\}$ respectively, since the vertex 5 is not a sink, we take the conjugate quiver of Q_1, Formula A.1 and Table A.3 (see Appendix) establishes that $r=4$ and

$$S_{(\mathbb{D}_7)_{035}^{01467}} = 284.$$

In the same way, the author, Cañadas and Giraldo showed the next recurrence formula for algebras of Dynkin type $\mathbb{D}_n$ and $\mathbb{D}_n$ with just only one sink (by notation $S_{(\mathbb{D}_n)_{-}^{m}} = S_{(\mathbb{D}_n)_{01n-1n}^{0m}}$) [32].

Corollary 2.6. $S_{(\mathbb{D}_n)_{-}^{m}} = S_{(\mathbb{D}_{n-1})_{-}^{m-1}} + S_{(\mathbb{D}_{n-1})_{-}^{m}} + 3\left(2^{n-3}\right)$ *for* $n \geq 5$, $1 \leq m < n-2$ *with* $S_{(\mathbb{D}_n)_{-}^{1}} = 2^{n-3}(2n-1)$ *and* $S_{(\mathbb{D}_n)_{-}^{n-2}} = 2^{n-2}(n+1) - 3$.

Corollary 2.6 allows to build the following triangular table where the rows give the number of sections in the Auslander-Reiten quiver of an algebra $\mathcal{A}$ associated to a Dynkin graph $\mathbb{D}_n$ with a unique sink allocated at the hth position, $1 \leq h \leq n-2$:

14	17					
36	43	45				
88	103	112	109			
208	239	263	269	253		
480	543	598	628	618	573	
1088	1215	133	1418	1438	1383	1277

Remark 2.7. *Sequence $c_n = S_{(\mathbb{D}_n)^{1}_{-}}$ for $n \geq 4$ appears in the OEIS as A052951* [89].

2.2.3 Sections in the Auslander-Reiten Quiver of Algebras of Type $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$

In order to give the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$. Let $\mathcal{A}$ be a path algebra induced by an oriented Dynkin diagram of type $\mathbb{E}_l$ $(l = 6, 7, 8)$ with k sinks and m sources. Let $\Gamma(\operatorname{mod} \mathcal{A})$ be the Auslander-Reiten quiver of $\mathcal{A}$ and $S_{(\mathbb{E}_l)^{i_0 i_1 \dots i_k}_{j_0 j_1 \dots j_m}}$ is the number of section in $\Gamma(\operatorname{mod} \mathcal{A})$. We assume the numbering described in Figure 1.2 (Section 1.1).

Let $\mathbb{A}'_{i-1}$ be the subgraph of the vertices $\{1, \dots, l-1\}$ of $\mathbb{E}_l$ and suppose that $i_k < i-1$, we define the vectors $S_{\mathbb{E}_l \uparrow} = (v_0, \dots, v_{2^{l-3}-1})$, and $S_{\mathbb{E}_l \downarrow} = (w_0, \dots, w_{2^{l-3}-1})$ in the same way:

$$
\begin{aligned}
S_{\mathbb{E}_6\uparrow} &= (124, 134, 136, 132, 146, 150, 146, 134), \\
S_{\mathbb{E}_6\downarrow} &= (124, 139, 146, 147, 136, 145, 146, 139), \\
S_{\mathbb{E}_7\uparrow} &= (408, 430, 436, 434, 460, 472, 468, 450, 454, 470, 470, \\
&\quad 456, 478, 478, 466, 438), \\
S_{\mathbb{E}_7\downarrow} &= (412, 443, 458, 465, 448, 468, 472, 462, 452, 473, 478, \\
&\quad 477, 452, 461, 458, 439), \\
S_{\mathbb{E}_8\uparrow} &= (1520, 1566, 1580, 1584, 1632, 1662, 1660, 1636, 1628, \\
&\quad 1664, 1668, 1650, 1692, 1698, 1680, 1632, 1614, 1654, \\
&\quad 1662, 1650, 1698, 1712, 1698, 1694, 1676, 1698, 1692, \\
&\quad 1656, 1690, 1678, 1650, 1590), \\
S_{\mathbb{E}_8\downarrow} &= (1532, 1595, 1626, 1647, 1620, 1663, 1674, 1663, 1624 \\
&\quad 1676, 1696, 1694, 1652, 1673, 1670, 1637, 1616, 1673, \\
&\quad 1698, 1703, 1664, 1694, 1696, 1748, 1653, 1693, 1702, \\
&\quad 1681, 1632, 1637, 1626, 1583),
\end{aligned}
\tag{2.30}
$$

therefore

$$S_{(\mathbb{E}_l)^{i_1\cdots i_k}_{j_1\cdots j_m}} = \begin{cases} v_r, & \text{if } \ l \ \text{ is sink,} \\ w_r, & \text{if } \ l \ \text{ is source,} \end{cases} \tag{2.31}$$

with r given by formulas 2.22.

For example, if we take an algebra $\mathcal{A} = kQ$ associated to Figure 2.7, then $S_{(\mathbb{E}_6)^{135}_{246}} = S_{(\mathbb{E}_6)^{246}_{135}} = v_5 = 150$.

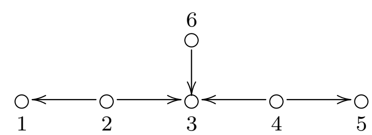


FIGURE 2.7. Quiver of type $\mathbb{E}_6$.

CHAPTER 3

Dyck Paths Categories And Its Relationships With Cluster Algebras

In this chapter, we introduce Dyck paths categories as a combinatorial model of the category of representations of Dynkin quivers of type $\mathbb{A}_n$. These categories help us to find an alternative formula of cluster variables of cluster algebras based on Dyck paths. In Section 3.1, we define Dyck paths categories and some of its main categorical properties are given in Section 3.2. In section 3.3, relationships between objects of the categories of Dyck paths, perfect matchings, and cluster algebras are given.

3.1 Dyck Paths Category

In this section, we introduce the category of Dyck paths of length $2n$.

3.1.1 Elementary Shifts

Let $\mathfrak{D}_{2n}$ be the set of all Dyck paths of length $2n$ and let $UWD = Uw_1 \dots w_{n-1}D$ be a Dyck path in $\mathfrak{D}_{2n}$ with $w_i \in A = \{UD, DU, UU, DD\}$ for $1 \leq i \leq n-1$.

The *support* of UWD (denoted by $\operatorname{Supp} UWD \subseteq \{1, 2, \dots, n-1\} = \mathbf{n\text{-}1}$) is a set of indices such that

$$\operatorname{Supp} UWD = \{q \in \mathbf{n\text{-}1} \mid w_q = UD \text{ or } w_q = UU,\ 1 \leq q \leq n-1\}.$$

A map $f : A \longrightarrow A$ such that for any $w \in A$, it holds that $f(w) = f(ab) = w^{-1} = ba$, $a, b \in \{U, D\}$ is said to be a *shift*.

For i fixed, $1 \leq i \leq n-1$, a *unitary shift* is a map $f_i : \mathfrak{D}_{2n} \longrightarrow \mathfrak{D}_{2n}$ such that

$$f_i(Uw_1 \dots w_{i-1}w_iw_{i+1} \dots w_{n-1}D) = Uw_1 \dots w_{i-1}f(w_i)w_{i+1} \dots w_{n-1}D.$$

We will denote a unitary shift by a vector of maps from $\mathfrak{D}_{2n}$ to itself of the form $(1_1, \dots, 1_{i-1}, f_i, 1_{i+1}, \dots, 1_{n-1})$, where 1_k is the identity map associated to the i-th coordinate.

An *elementary shift* is a unitary shift or composition of different unitary shifts. A *shift path* of length m $UWD \longrightarrow UW_1D \longrightarrow \cdots \longrightarrow UW_mD \longrightarrow UVD$ from UWD to UVD is a composition of elementary shifts. The set of all Dyck paths in a shift path between UWD and UVD will be denoted by J. For notation, we introduce the *identity shift* as the elementary shift $(1_1, \dots, 1_{n-1})$.

Irreversibility condition. Consider a relation $R \subset \mathfrak{D}_{2n} \times \mathfrak{D}_{2n}$ consisting of a set of pairs of Dyck paths of the form (UWD, UVD), where UVD is obtained from UWD by applying an elementary shift. Then, R is said to be an *irreversible relation*, if for any $(UWD, UVD) \in R$, it holds that $(UVD, UWD) \notin R$.

Henceforth, if $(UWD, UVD) \in R$ then we will write $UVD = R(UWD)$.

Shift Relation. Suppose that $UWD, UW'D, UW''D, UVD \in \mathfrak{D}_{2n}$. And that there are elementary shifts $F : UWD \to UW'D$, $F' : UWD \to UW''D$, $G : UW'D \to UVD$, $G' : UW''D \to UVD$ in an irreversible relation R. Then if the compositions $G \circ F$ and $G' \circ F'$ are shift paths (of length 2) transforming the Dyck path UWD into the Dyck path UVD (see the diagram below),

$$\begin{tikzcd}
& UW'D \arrow[dr, "G"] & \\
UWD \arrow[ur, "F"] \arrow[dr, "F'"'] & & UVD, \\
& UW''D \arrow[ur, "G'"'] &
\end{tikzcd}$$

with $W' \neq W''$. Then $G \circ F$ is said to be related with $G' \circ F'$ (denoted $G \circ F \sim_R G' \circ F'$) whenever $G' = F$ and $G = F'$.

Category of Dyck paths of length $2n$. As for the case of diagonals [17], we can also define a k-linear additive category $(\mathfrak{D}_{2n}, R)$ based on Dyck paths, in this case, *objects* are k-linear combinations of Dyck paths in $\mathfrak{D}_{2n}$ with *space of morphisms* from a Dyck path UWD to a Dyck path UVD over an irreversible relation R being the set

$$\mathrm{Hom}_{(\mathfrak{D}_{2n}, R)}(UWD, UVD) = \langle \{ g \mid g \text{ is a shift path over } R \} \rangle / \langle \sim_R \rangle.$$

The *vector space* $\mathrm{Hom}_{(\mathfrak{D}_{2n}, R)}(UWD, UVD) \neq 0$ if and only if there are shift paths from UWD to UVD and

$$\bigcap_{i \in J} \mathrm{Supp}\ UW^iD \neq \varnothing, \tag{3.1}$$

for each shift path, with UWD and UVD in $\mathfrak{D}_{2n}$.

Figure 3.1 shows the elementary shifts over $(\mathfrak{D}_6, R)$ associated to an irreversible relation R defined over the set of all Dyck paths of length 6. And such that,

$$R(UWD) = \begin{cases} f_1(UWD), & \text{if } w_1 = UD, \\ f_2(UWD), & \text{if } w_2 = DU. \end{cases} \tag{3.2}$$

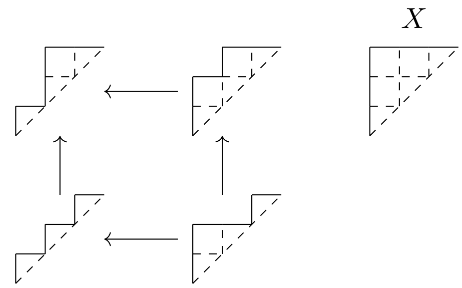


FIGURE 3.1. Elementary shifts in $(\mathfrak{D}_6, R)$. Notice that, there is no elementary shift transforming the Dyck path X into the others in the diagram.

3.1.2 Relations of Type $R^{i_1 \dots i_k}_{j_1 \dots j_m}$

Fix an admissible subchain $\mathcal{C} = \{j_1, \dots, j_m, i_1, \dots, i_k\} \subseteq \mathbf{n\text{-}1}$ (see algorithm 2.1, item (3)), and a Dyck path $UWD \in \mathfrak{D}_{2n}$.

Let $\sigma : \{i_1, j_1\} \to \{0, 1\}$ be a map such that $\sigma(i_1) = 1$ and $\sigma(j_1) = 0$. For $a \in \{i_1, j_1\}$, we assume $i_r, i_{r+\sigma(a)} \in \{i_1, \dots, i_k\}$ and $j_{r+1-\sigma(a)}, j_r \in \{j_1, \dots, j_m\}$. The following indices are defined by using intervals $[i_r, j_{r+1-\sigma(a)}]$(resp. $[j_r, i_{r+\sigma(a)}]$), where for a fixed admissible chain $\mathcal{C}$, an interval $I = [x, y]$ is a subset of $\mathbf{n\text{-}1}$, for which $\min I = x \in \mathcal{C}$ and $\max I = y \in \mathcal{C}$.

- $t = \min\{\ s \mid i_r \leq s \leq j_{r+1-\sigma(a)}, w_s = UD\ \}$ $\big(t = \max\{\ s \mid j_r \leq s \leq i_{r+\sigma(a)}, w_s = UD\ \}\big)$,
- $p = \min\{\ s \mid t < s \leq j_{r+1-\sigma(a)}, w_s = DU\ \}$ $\big(p = \max\{\ s \mid j_s \leq s < t, w_s = DU\ \}\big)$.

We introduce the following elementary shifts:

ES1. If $w_s = UD$ for all $s \in [i_r, j_{r+1-\sigma(a)}]$ ($s \in [j_r, i_{r+\sigma(a)}]$),

$$[j_{r-\sigma(a)}, i_r][i_r, j_{r+1-\sigma(a)}][j_{r+1-\sigma(a)}, i_{r+1}],$$

$$(\text{resp. } [i_{r+\sigma(a)-1}, j_r][j_r, i_{r+\sigma(a)}][i_{r+\sigma(a)}, j_{r+1}]),$$

then

$$g(UWD) = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UWD),$$

if there exists $s \in \mathbb{Z}^+$ such that $j_{r-\sigma(a)} \leq s \leq i_r$, $|s - j_r| \geq 1$, $w_x = UD$ if $s \leq x \leq i_r$ over $[j_{r-\sigma(a)}, i_r]$ and

$$w_y = \begin{cases} UD, & \text{if } y = j_{r+1-\sigma(a)}, \\ DU, & \text{otherwise,} \end{cases} \tag{3.3}$$

over $[j_{r+1-\sigma(a)}, i_{r+1}]$ for $j_{r+1-\sigma(a)} \neq n-1$ or the first condition over $[j_{r-\sigma(a)}, i_r]$ for $j_{r+1-\sigma(a)} = n-1$.

$$\Big(g(UWD) = f_{i_{r+\sigma(a)}} \circ \cdots \circ f_{j_r}(UWD),$$

if there exists $s \in \mathbb{Z}^+$ such that $i_{r+\sigma(a)} \leq s \leq j_{r+1}$, $|s - i_{r+\sigma(a)}| \geq 1$, $w_x = UD$ if $i_{r+\sigma(a)} \leq x \leq s$ over $[i_{r+\sigma(a)}, j_{r+1}]$ and

$$w_y = \begin{cases} UD, & \text{if } y = j_r, \\ DU, & \text{otherwise,} \end{cases} \tag{3.4}$$

over $[i_{r+\sigma(a)-1}, j_r]$ for $j_r \neq 1$ or the first condition over $[i_{r+\sigma(a)}, j_{r+1}]$ for $j_r = 1\Big)$, with $i_r \neq 1$ ($i_{r+\sigma(a)} \neq n-1$).

ES2. If $t = 1$ or $n-1$ then $g(UWD) = f_t(UWD)$. .

ES3. If $i_r < t < j_{r+1-\sigma(a)}$ ($j_r < t < i_{r+\sigma(a)}$) then $g(UWD) = f_t(UWD)$.

ES4. If $p = j_{r+1-\sigma(a)}(j_r)$ then

$$g(UWD) = \begin{cases} f_{i_{r+1}} \circ \cdots \circ f_{j_{r+1-\sigma(a)}}(UWD) & \text{if } j_{r+1-\sigma(a)} \neq n-1, \\ f_{j_{r+1-\sigma(a)}}(UWD) & \text{if } j_{r+1-\sigma(a)} = n-1. \end{cases}$$

$$\left(g(UWD) = \begin{cases} f_{i_{r+\sigma(a)-1}} \circ \cdots \circ f_{j_r}(UWD) & \text{if } j_r \neq 1, \\ f_{j_r}(UWD) & \text{if } j_r = 1. \end{cases} \right)$$

ES5. If $t < p < j_{r+1-\sigma(a)}$ ($j_r < p < t$) then $g(UWD) = f_p(UWD)$.

For a given subchain $\mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq$ **n-1**, two Dyck paths D and D' of length $2n$ are said to be *related by a relation of type* $R^{i_1 \cdots i_k}_{j_1 \cdots j_m}$ if there is an elementary shift ESi, $1 \leq i \leq 5$ which transforms either D into D' or D' into D.

Henceforth, the notation $\underbrace{w_r \ldots w_s}_{XY}$ means that all the elements w_i covered by the brace have the same steps XY.

Proposition 3.1. *The relation* $R^{i_1 \cdots i_k}_{j_1 \cdots j_m}$ *is irreversible.*

Proof. Suppose that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from a Dyck path UWD to a Dyck path UVD and that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from UVD to a UWD, then we have five cases:

(i) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES1 over $[i_r, j_{r+1-\sigma(a)}]$. Elementary shifts ES2, ES3 and ES5 allow to conclude that from UVD to a UWD, $f_t = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}$ or $f_p = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}$ and this is a contradiction. If ES1 is an elementary shift from UVD to UWD, then two cases arise: If $j_{r+1-\sigma(a)} \neq n-1$, thus UVD equals

$$Uv_1 \dots v_{j_{\sigma(a)}} \dots \underbrace{v_{j_s} \dots v_{i_r-1}}_{UD} \underbrace{v_{i_r} \dots v_{j_{r+1-\sigma(a)}}}_{UD} \underbrace{(v_{j_{r+1-\sigma(a)}}+1) \dots v_{i_{r+1}}}_{DU} v_{r_{i+1}+1} \dots v_{n-1}D,$$

it turns out that $f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)$ has the form

$$Uw_1 \dots w_{j_{\sigma(a)}} \dots \underbrace{w_{j_s} \dots (w_{i_r}-1)}_{UD} \underbrace{w_{i_r} \dots w_{j_{r+1-\sigma(a)}}}_{DU} \underbrace{(w_{j_{r+1-\sigma(a)}}+1) \dots w_{i_{r+1}}}_{DU} w_{r_{i+1}+1} \dots v_{n-1}D,$$

which is a contradiction. If $j_{r+1-\sigma(a)} = n-1$, UVD is equal to

$$Uv_1 \dots v_{j_{\sigma(a)}} \dots \underbrace{v_{j_s} \dots v_{i_r-1}}_{UD} \underbrace{v_r \dots v_{j_{r+1-\sigma(a)}}}_{UD} D,$$

and $f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)$ has the shape

$$Uw_1 \dots w_{j_{\sigma(a)}} \dots \underbrace{w_{j_s} \dots w_{i_r-1}}_{UD} \underbrace{w_r \dots w_{j_{r+1-\sigma(a)}}}_{DU} D,$$

again a contradiction. We also get a contradiction if an elementary shift is done by using ES4 from UVD to a UWD, indeed, in these cases it holds that, if $j_{r+1-\sigma(a)} = n-1$, there are t and p such that $p = j_{r+1-\sigma(a)} < t \leq i_{r+1}$ and UVD is equal to

$$Uv_1 \dots v_{i_r-1} \underbrace{v_{i_r} \dots v_{j_{r+1-\sigma(a)}}}_{DU} \underbrace{v_{j_{r+1-\sigma(a)+1}} \dots v_t}_{UD} v_{t+1} \dots v_{n-1}D,$$

and $f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)$ is

$$Uw_1 \dots w_{i_r-1} \underbrace{w_{i_r} \dots w_{j_{r+1-\sigma(a)}}}_{UD} \underbrace{w_{j_{r+1-\sigma(a)+1}} \dots w_t}_{UD} w_{t+1} \dots w_{n-1}D.$$

If $v_{j_{r+1-\sigma(a)}} = n-1$ $f_{r+1-\sigma(a)} = f_{r+1-\sigma(a)} \circ \cdots \circ f_{i_r}$ but this is a contradiction.

(ii) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES2 over $[i_1, j_1]$ then we cannot use elementary shifts defined in cases ES1, ES4, ES5 or ES3, provided that, $i_1 \neq 1$, $t \neq p$ or $1 < t < j_1$. Therefore, ES2 guarantees the existence of a walk from UVD to UWD such that;

$$U \underbrace{v_1}_{UD} \dots v_{j_1} \dots v_{n-1}D,$$

and $f_1(UWD)$ has the form

$$U \underbrace{w_1}_{DU} \dots w_{j_1} \dots w_{n-1}D,$$

which is a contradiction (if $t = n-1$, the proof is dual).

(iii) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES3 over $[i_r, j_{r+1-\sigma(a)}]$, provided that, $i_r < t < p < j_{r+1-\sigma(a)}$, we conclude that it is not possible to use ES1, ES2, ES4 nor ES5. In the case of ES3 from UVD to a UWD, UVD equals

$$Uv_1 \dots \underbrace{v_{i_r} \dots v_{t-1}}_{DU} \underbrace{v_t}_{UD} \dots v_{j_{r+1-\sigma(a)}} \dots v_{n-1}D,$$

and $f_t(UVD)$ has the shape

$$Uw_1 \dots \underbrace{w_{i_r} \dots w_t}_{DU} w_{t+1} \dots w_{j_{r+1-\sigma(a)}} \dots w_{n-1}D,$$

but this is a contradiction.

(iv) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES4 over $[i_r, j_{r+1-\sigma(a)}]$, provided that $t < p$, we do not use ES2, ES3 nor ES5. If $j+1-\sigma(a) = n-1$, we cannot use ES1. If $j+1-\sigma(a) \neq n-1$ we can use ES1 from UVD to a UWD (Note that, it is not necessary with $v_m = UD$ for all $s \in [j_{r+1-\sigma(a)}+1, i_{r+1}]$) UVD is equal to

$$Uv_1 \dots v_{i_r-1} \underbrace{v_{i_r} \dots v_t \dots v_{p-1}}_{DU} \underbrace{v_p v_{j_{r+1-\sigma(a)}+1} \dots v_{i_{r+1}}}_{UD} v_{i_{r+1}+1} \dots v_{n-1}D,$$

it turns out that $g(UVD)$ has the form

$$Uw_1 \dots w_{i_r-1} \underbrace{w_{i_r} \dots w_t \dots w_{p-1} w_p w_{j_{r+1-\sigma(a)}+1} \dots w_{i_{r+1}}}_{DU} w_{i_{r+1}+1} \dots w_{n-1}D,$$

which is a contradiction. Using ES5 from UVD to UWD, if $j_{r+1-\sigma(a)} \neq n-1$, UVD is equal to

$$Uv_1 \dots v_{i_r} \dots \underbrace{v_t \dots v_{p-1}}_{UD} \underbrace{v_p}_{DU} v_{j_{r+1-\sigma(a)}+1} \dots v_{i_{r+1}} v_{i_{r+1}+1} \dots v_{n-1}D,$$

and UWD has the shape

$$Uw_1 \dots w_{i_r} \dots \underbrace{w_t \dots w_p}_{UD} \underbrace{w_{j_{r+1-\sigma(a)}+1} \dots w_{i_{r+1}}}_{f(ab)} w_{i_{r+1}+1} \dots w_{n-1}D,$$

again a contradiction. If $j_{r+1-\sigma(a)} = n-1$, UVD is equal to

$$Uv_1 \dots v_{i_r} \dots v_{t-1} \underbrace{v_t \dots v_{p-1}}_{UD} \underbrace{v_p}_{DU} D,$$

it turns out that UWD has the shape

$$Uw_1 \dots w_{i_r} \dots w_{t-1} \underbrace{w_t \dots v_p}_{UD} D,$$

this is a contradiction.

(v) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES5 over $[i_r, j_{r+1-\sigma(a)}]$. Then we cannot use ES1, ES2, ES3 nor ES4, because $f_p \neq f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}$ and $t < p$. Using ES5 from UVD to a UWD, we observe that UVD is equal to

$$Uv_1 \dots v_{i_r} \dots v_{t-1} \underbrace{v_t \dots v_{p-1}}_{UD} \underbrace{v_p}_{DU} \dots v_{j_{r+1-\sigma(a)}} \dots v_{n-1}D,$$

and $f_p(UWD)$ has the form

$$Uw_1 \dots w_{i_r} \dots w_{t-1} \underbrace{w_t \dots w_p}_{UD} v_{p+1} \dots v_{j_{r+1-\sigma(a)}} \dots v_{n-1}D,$$

again this is a contradiction.

Taking into account that if $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from $ES1$, $ES2$, $ES3$, $ES4$ and $ES5$ over $[i_r, j_{r+\sigma(a)}]$ then same arguments as described above applied dually allow to conclude the proposition. We are done. □

3.1.3 $\mathbb{A}_{n-1}$-Dyck Paths Categories

For $n \geq 2$ fixed, an $\mathbb{A}_{n-1}$-Dyck paths category is a category of Dyck paths $(\mathfrak{D}_{2n}, R)$ where R is a relation of type $R^{i_1 \dots i_k}_{j_1 \dots j_m}$ as described before. As an example we let $(\mathfrak{D}_8, R^1_3)$ denote the $\mathbb{A}_3$-Dyck paths category with the admissible subchain $1 < 3$. Figure 3.2 shows all the elementary shifts of $(\mathfrak{D}_8, R^1_3)$.

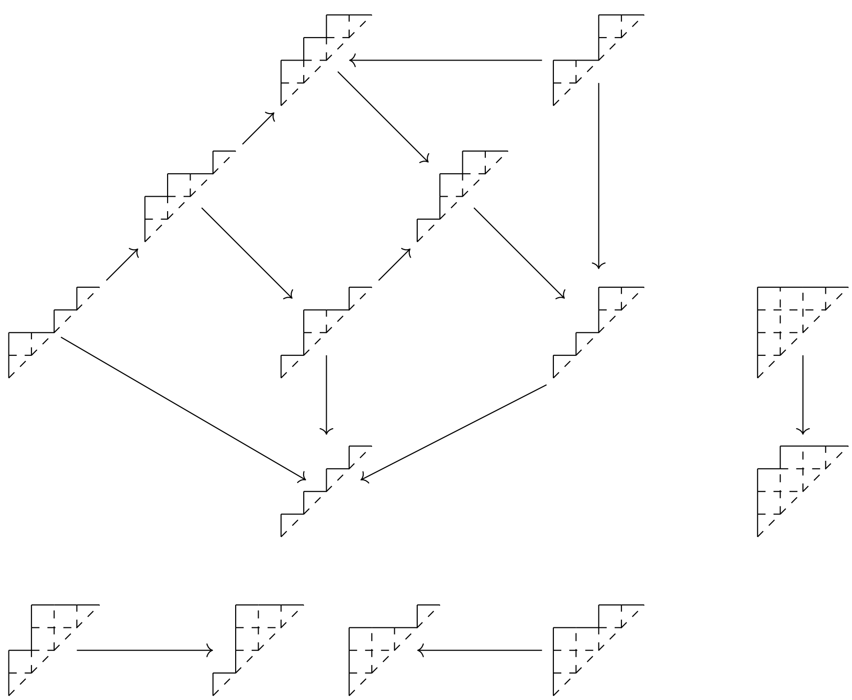


FIGURE 3.2. Elementary shifts in an $\mathbb{A}_3$-Dyck paths category.

We let S denote the set of all Dyck paths with exactly $n-1$ peaks. The following propositions and lemmas describe some properties of the set S in the category $(\mathfrak{D}_{2n}, R^{i_1 \dots i_k}_{j_1 \dots j_m})$.

Proposition 3.2. *Let UWD be a Dyck path of length $2n$, then $UWD \in S$ if and only if there is a unique sequence $w_l w_{l+1} \dots w_{r'-1} w_{r'}$ such that*

$$w_i = \begin{cases} UD, & if\, l \leq i \leq r', \\ DU, & otherwise. \end{cases} \tag{3.5}$$

Proof. Firstly, let δ be a map $\delta : \{), (\} \to \{U, D\}$ where left bracket is associated to the letter U and right bracket is associated to the letter D, suppose $UWD \in S$, then there exist bracket-subchains such that UWD can be written in the following form

$$(\underbrace{)(}_{1} \underbrace{)(}_{2} \cdots \underbrace{)(}_{l-2} \underbrace{)(}_{l-1} \; (\underbrace{()}_{l} \cdots \underbrace{()}_{r'} \;) \underbrace{)(}_{r'+1} \cdots \underbrace{)(}_{n-2} \underbrace{)(}_{n-1} \;),$$

therefore $w_i = UD$ if $l \le i \le r'$ and $w_i = DU$. On the other hand, suppose UWD has a unique subsequence $w_l w_{l+1} \dots w_{r'-1} w_{r'}$ that satisfies (3.5), then if we apply δ^{-1} to UWD, the sequence

$$\underbrace{()}_{1} \underbrace{()}_{2} \cdots \underbrace{()}_{l-1} \; (\underbrace{()}_{l} \cdots \underbrace{()}_{r'} \;) \underbrace{()}_{r'+1} \cdots \underbrace{()}_{n-2} \underbrace{()}_{n-1},$$

is obtained, therefore $UWD \in S$. We are done. $\square$

Lemma 3.1. *Let UWD be a Dyck path in S, and integers r', l defined as in* Proposition 3.2 *with $|r' - l| > 0$, then there exists an elementary shift from UWD to another Dyck path with exactly $n-1$ peaks.*

Proof Let UWD be a Dyck path in S, let l and r' be positive integers such that $w_m = UD$ for $l \le m \le r'$. Let $l \in [i_r, j_{r+1-\sigma(a)}]$, we have the following cases:

(1) If $l = i_r = 1$, then

$$g(UWD) = U \underbrace{f(w_1)}_{DU} \underbrace{w_2 \dots w_{r'}}_{UD} w_{r'+1} \dots w_{n-1} D \in S.$$

(2) If $l = i_r \ne 1$, then there is a $p = l - 1$ over $[j_{r-\sigma(a)}, i_r]$ such that

$$g(UWD) = Uw_1 \dots \underbrace{f(w_p) w_l \dots w_m}_{UD} \dots w_{n-1} D \in S.$$

(3) If $i_r < l < j_{r+1-\sigma(a)}$, then

$$g(UWD) = Uw_1 \dots \underbrace{f(w_l)}_{DU} \underbrace{w_{l+1} \dots w_{r'}}_{UD} w_{r'+1} \dots w_{n-1} D \in S.$$

(4) If $l = j_{r+1-\sigma(a)}$ and $|l - r'| > 0$, then $r' \in [i_{r_1}, j_{r_1+1-\sigma(a)}]$ with $|r_1 - r| > 0$ and the following cases hold:

(4.1) If $i_{r_1} \le r' < j_{r_1+1-\sigma(a)}$, there is a $p = r' + 1$ such that, if $p \ne j_{r_1+1-\sigma(a)}$ then

$$g(UWD) = uw_1 \dots \underbrace{w_l \dots w_{r'} f(w_p)}_{UD} \dots w_{n-1} D \in S,$$

if $p = j_{r_1+1-\sigma(a)} = n - 1$, then

$$g(UWD) = Uw_1 \dots \underbrace{w_l \dots w_{r'} f(w_p)}_{UD} D \in S,$$

or if $p = j_{r_1+1-\sigma(a)} \neq n-1$ then

$$g(UWD) = Uw_1 \ldots \underbrace{w_l \ldots w_{r'} f(w_p) \ldots f(w_{i_{r_1+1}})}_{UD} \ldots w_{n-1}D \in S.$$

(4.2) If $r' = j_{r_1+1-\sigma(a)}$

$$g(UWD) = Uw_1 \ldots \underbrace{w_l \ldots w_{i_{r_1}-1}}_{UD} \underbrace{f(w_{i_{r_1}}) \ldots f(w_{r'})}_{DU} \ldots D \in S.$$

(4.3) Now, if $|r_1 - r| > 1$ or $r_1 = r+1$ and $r' > i_{r+1} + 2$ then

$$g(UWD) = Uw_1 \ldots \underbrace{f(w_l) \ldots f(w_{i_{r+1}})}_{DU} \underbrace{w_{i_{r+1}+1} \ldots w_{r'}}_{UD} \ldots D \in S.$$

For $r' \in [j_{r_1+1-\sigma(a)}, i_{r_1+1}]$ with $|r_1 - r| \geq 0$ we have that:

(4.4) If $s = t = i_{r_1+1} = n-1$, then

$$g(UWD) = Uw_1 \ldots \underbrace{w_l \ldots w_{r'-1}}_{UD} \underbrace{f(w_{r'})}_{DU} D \in S.$$

On the other hand, if $s = t = i_{r_1+1} \neq n-1$, then there is a $p \in [i_{r_1+1}, j_{r_1+2-\sigma(a)}]$ satisfying first condition of (4.1). Thus, if $j_{r_1+1-\sigma(a)} < s < i_{r_1+1}$, it holds that

$$g(UWD) = Uw_1 \ldots \underbrace{w_l \ldots w_{r'-1}}_{UD} \underbrace{f(w_{r'})}_{DU} \ldots w_{n-1}D \in S.$$

(4.5) If $s = j_{r_1+1-\sigma(a)}$ then $|r_1 - r| > 0$ (If $|r_1 - r| = 0$, $|l - f| = 0$ which is a contradiction)

$$g(UWD) = Uw_1 \ldots \underbrace{w_l \ldots w_{i_{r_1}-1}}_{UD} \underbrace{f(w_{i_{r_1}}) \ldots f(w_{r'})}_{DU} w_{r'+1} \ldots w_{n-1}D \in S.$$

(4.6) Now, suppose that in UWD $|r_1 - r| > 0$, then it satisfies the first condition in (4.3).

In case that $l \in [j_r, i_{r+\sigma(a)}]$, we have the following cases:

(5) If $j_r < l \leq i_r + \sigma(a)$, then there exists $p = l+1$ such that, if $p \neq j_r$ then

$$g(UWD) = Uw_1 \ldots w_{j_r} \ldots \underbrace{f(w_p) w_l \ldots w_{r'}}_{UD} \ldots w_{n-1}D \in S.$$

Note that, if $p = j_r = 1$ then

$$g(UWD) = U \underbrace{f(w_p) w_l \ldots w_{r'}}_{UD} \ldots w_{n-1}D \in S,$$

or if $p = j_r \neq 1$, then

$$g(UWD) = Uw_1 \dots \underbrace{f(w_{i_r-1+\sigma(a)}) \dots f(w_p)w_l \dots w_{r'}}_{UD} \dots w_{n-1}D \in S.$$

(6) If $l = j_r$ and $|l - r'| > 0$, then $r' \in [j_{r_1}, i_{r_1+\sigma(a)}]$ with $|r_1 - r| \geq 0$, then the following cases hold:

(6.1) If $j_{r_1} + 2 \leq r' \leq i_{r_1+\sigma(a)}$, then there exists p satisfying (4.4).

(6.2) If $j_{r_1} \leq r' < j_{r_1} + 2$, then $|r_1 - r| > 0$ and if $r = j_{r_1+1}$ satisfies (6.1), or if $r = j_{r_1}$ then UWD satisfies (4.5).

(6.3) Now, if $|r_1 - r| > 0$ then

$$g(UWD) = U \dots \underbrace{f(w_l) \dots f(w_{i_r+\sigma(a)})}_{DU} \underbrace{w_{i_r+\sigma(a)+1} \dots w_s}_{UD} \dots D \in S,$$

or $r' \in [i_{r_1+\sigma(a)}, j_{r_1+1}]$ with $|r_1 - 1| \geq 0$ satisfies conditions (4.1), (4.2) and (4.3) for $i_{r_1+\sigma(a)} \leq r' \leq j_{r_1+1}$.

Same arguments are used for the cases $r' \in [i_r, j_{r+1-\sigma(a)}]([j_r, i_{r+\sigma(a)}])$ to conclude the lemma. We are done. $\square$

Lemma 3.2. *Suppose that UWD is a Dyck path in S and that integers l and r' as defined in* Proposition 3.2 *are such that $l = r'$, then the following statements hold:*

(a) If $l \notin \{j_s\}$ then there is an elementary shift to a Dyck path with exactly $n-1$ peaks.

(b) If $l \in \{j_s\}$ then there is an elementary shift from a Dyck path with exactly $n-1$ peaks to UWD.

Proof. Let UWD be a Dyck path in S, and positive integers l and r' with $l = r'$.

(a) Suppose $l \notin \{j_s\}$ and $l \in [i_r, j_{r+1-\sigma(a)}]$. If $i_r \leq l < j_{r+1-\sigma(a)}$, then UWD satisfies (4.1) and (4.2) of Lemma 3.1. In particular, if $l = i_r \neq 1$ there is a $p' = l - 1$ in $[j_{r-\sigma(a)}, i_r]$ that satisfies the first condition of (5) of Lemma 3.1. The case $l \in [j_r, i_{r+\sigma(a)}]$ is dual.

(b) Suppose $l = j_{r+1-\sigma(a)}$, we have the following cases:

(i) If $|i_r - j_{r+1-\sigma(a)}| = 1$ (or $|i_{r+1} - j_{r+1-\sigma(a)}| = 1$) and $i_r = 1$ (or $i_{r+1} = n-1$), then there is a UVD which is equal to

$$U \underbrace{w_1}_{UD} w_l \dots D \in S \text{ (or } U \dots w_l \underbrace{w_{n-1}}_{UD} D \in S),$$

and

$$Uf(w_1)w_l \dots D = UWD \text{ (or } U \dots w_l f(w_{n-1})D = UWD).$$

(ii) If $|i_r - j_{r+1-\sigma(a)}| = 1$ (or $|i_{r+1} - j_{r+1-\sigma(a)}| = 1$) and $i_r \neq 1$ (or $i_{r+1} \neq n-1$) then there is a $l' = j_{r-\sigma(a)}$ and $r'' = j_{r+1-\sigma(a)}$ (or $l' = j_{r+1-\sigma(a)}$ and $r'' = j_{r+2-\sigma(a)}$) such that UVD is equal to

$$U \dots \underbrace{w_{l'} \dots w_{r''-1} w_l}_{UD} \dots D \text{ (or } U \dots \underbrace{w_l w_{l'+1} \dots w_{r''}}_{UD} \dots D) \in S,$$

and

$$U \dots f(w_{l'}) \dots f(w_{r''-1}) w_l \dots D \text{ (or } U \dots w_l f(w_{l'+1}) \dots f(w_{r''}) \dots D) = UWD.$$

(iii) If $|i_r - j_{r+1-\sigma(a)}| > 1$ (or $|i_{r+1} - j_{r+1-\sigma(a)}| > 1$) then there is a UVD which is equal to

$$U \dots \underbrace{w_{l-1}}_{UD} w_l \dots D \text{ (or } U \dots w_l \underbrace{w_{l+1}}_{UD} \dots D \in S),$$

and

$$U \dots f(w_{l-1}) w_l \dots D = UWD \text{ (or } U \dots w_l f(w_{l+1}) \dots D = UWD).$$

Similar arguments dually applied can be used to obtain the lemma in the case $l = j_r$. We are done. □

Remark 3.1. *Note that, in general there is an elementary leftshift and an elementary rightshift over S, and these elementary shifts are disjoint, i.e. if $f_{p_1} \circ \dots \circ f_{p_q}$ and $f_{p'_1} \circ \dots \circ f_{p'_{q'}}$ are elementary left and right shifts, respectively. Then*

$$\{p_1, \dots, p_q\} \cap \{p'_1, \dots, p'_{q'}\} = \varnothing,$$

these elementary shifts are unique according to Lemma 3.1 and Lemma 3.2. If $F^p = f_{p_1} \circ \dots \circ f_{p_q}$ is an elementary leftshift (rightshift) we write F^p_l (F^p_r).

Proposition 3.3. *Let $\mathcal{C} = \{i_1, \dots i_k, j_1, \dots j_m\}$ be an admissible subchain, then all Dyck paths of S constitute a connected quiver Q whose set of vertices is in correspondence with the set of all Dyck paths in S and there is an arrow from $UWD \in S$ to $UVD \in S$ if there is an elementary shift transforming UWD into UVD.*

Proof. It suffices to prove that Q is connected, to do that, consider Dyck paths UWD and UVD of S. Thus, if there is a shift path between UWD and UVD then they are connected. Otherwise, Lemmas 3.1 and 3.2 allow to define a Dyck path $UW^{(1)}D$ and a shift path $F^{(1)} = F^{(1)}_{p_1} \circ \dots \circ F^{(1)}_1$ with $F^{(1)}_m = f^{(1)}_{m_1} \circ \dots \circ f^{(1)}_{m_{q_1}}$ such that

$$UWD \xrightarrow{F^{(1)}_1} \dots \xrightarrow{F^{(1)}_{p_1}} UW^{(1)}D,$$

and if there is a shift path from UVD to a $UW^{(1)}D$ then they are connected. If there is not a shift path from UVD to $UW^{(1)}D$, then there is a Dyck path $UW^{(2)}D$ and a shift path $F^{(2)} = F^{(2)}_{p_2} \circ \dots \circ F^{(2)}_1$ with $F^{(2)}_m = f^{(1)}_{m_1} \circ \dots \circ f^{(2)}_{m_{q_2}}$ such that

$$UW^{(2)}D \xrightarrow{F^{(2)}_1} \dots \xrightarrow{F^{(2)}_{p_2}} UW^{(1)}D \xleftarrow{F^{(1)}_{p_1}} \dots \xleftarrow{F^{(1)}_1} UWD,$$

again, if there is a shift path from $UW^{(2)}D$ to a UVD then they are connected. Since S is finite, the procedure ends in such a way that UWD and UVD are connected and with this argument we are done. □

Henceforth, we let $\mathfrak{C}_{2\mathfrak{n}}$ denote the full subcategory of $(\mathfrak{D}_{2n}, R^{i_1\dots i_k}_{j_1\dots j_m})$ whose objects are $k-$linear combinations of Dyck paths of S. Lemma 3.3 and Proposition 3.4 give some properties of the Hom-spaces of this category.

Lemma 3.3. *Let UWD, $UW'D$, $UW''D$ and UVD be Dyck paths in $\mathfrak{C}_{2\mathfrak{n}}$ and let $F = F^1_l \circ F^2_r$ (resp. $F^1_r \circ F^2_l$) be a shift path $UWD \xrightarrow{F^2_r} UW'D \xrightarrow{F^1_l} UVD$ (resp. $UWD \xrightarrow{F^2_l} UW'D \xrightarrow{F^1_r} UVD$), if there is another shift path $G = G^1 \circ G^2$ such that $UWD \xrightarrow{G^2} UW''D \xrightarrow{G^1} UW''D$ with $UW'D \neq UW''D$ then $G^2 = F^1_l$ and $G^1 = F^2_r$ (resp. $G^2 = F^1_r$ and $G^1 = F^2_l$).*

Proof. Let $F = F^1_l \circ F^2_r$ be a shift path such that

$$U \dots w_{l_1} \dots w_{r_1} \dots D \xrightarrow{F^2_r} U \dots w'_{l_2} \dots w'_{r_2} \dots D \xrightarrow{F^1_l} U \dots v_{l_3} \dots v_{r_3} \dots D,$$

with $l_1 = l_2$ and $r_2 = r_3$ and suppose that there is another shift path $G = G^1 \circ G^2$ such that

$$U \dots w_{l_1} \dots w_{r_1} \dots D \xrightarrow{G^2} U \dots w''_{l_4} \dots w''_{r_4} \dots D \xrightarrow{G^1} U \dots v_{l_3} \dots v_{r_3} \dots D,$$

with $UW'D \neq UW''D$. Given the elementary rightshift F^2_r, then since $G \neq F^2_r$, it holds that UWD satisfies the conditions of $UW'D$ in order to apply the same elementary leftshift F^2_l, i.e., $F^2_l = G^2$ and $l_3 = l_4$. Since $r_1 = r_4$, UWD and $UW''D$ satisfy the conditions to apply the same elementary rightshift, i.e., $F^1_r = G^1$. Case $F^1_r \circ F^2_l$ is obtained via a dual argument. □

Proposition 3.4. *If* $\mathrm{Hom}_{\mathfrak{C}_{2\mathfrak{n}}}(UWD, UVD) \neq 0$ *then* $\dim_k \mathrm{Hom}_{\mathfrak{C}_{2\mathfrak{n}}}(UWD, UVD) = 1$.

Proof. Suppose that $\mathrm{Hom}_{\mathfrak{C}_{2\mathfrak{n}}}(UWD, UVD) \neq 0$, then there is a shift path F of the form

$$UWD \xrightarrow{F^0_{x_0}} \dots \xrightarrow{F^{i-1}_{x_{i-1}}} UW^{i-1}D \xrightarrow{F^{i-1}_{x_{i-1}}} UW^iD \xrightarrow{F^i_{x_i}} UW^{i+1}D \xrightarrow{F^{i+1}_{x_{i+1}}} \dots \xrightarrow{F^m_{x_m}} UVD,$$

with $x_i \in \{l, r\}$ and for some $m \in \mathbb{Z}^+$. Now, for each pair $F^i_{x_i} \circ F^{i-1}_{x_{i-1}}$ with $x_{i-1} = l$ and $x_i = r$ ($x_{i-1} = r$ and $x_i = l$) that satisfies conditions described in Lemma 3.3 there is another shift path F' of the form

$$UWD \xrightarrow{F^0_{x_0}} \dots \xrightarrow{F^{i-1}_{x_{i-1}}} UW^{i-1}D \xrightarrow{F^i_{x_i}} UW^{i'}D \xrightarrow{F^{i-1}_{x_{i-1}}} UW^{i+1}D \xrightarrow{F^{i+1}_{x_{i+1}}} \dots \xrightarrow{F^m_{x_m}} UVD,$$

transforming UWD and UVD. Thus $F \sim_{R^{i_1\dots i_k}_{j_1\dots j_m}} F'$. □

3.2 A Categorical Equivalence

In this section, we establish an equivalence between the full category $\mathfrak{C}_{2\mathfrak{n}}$ and the category of representations of a quiver of Dynkin type $\mathbb{A}_n$.

3.2.1 The Θ Functor

Given an admissible subchain $\mathcal{C} = \{j_1, \dots, j_m, i_1, \dots, i_k\}$, $\mathfrak{C}_{2n}$ the full subcategory of $(\mathfrak{D}_{2n}, R^{i_1 \dots i_k}_{j_1 \dots j_m})$ and Q a quiver of type $\mathbb{A}_{n-1}$ with $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_m\}$ being the sets of sinks and sources, respectively. Then the k-linear additive functor $\Theta : \mathfrak{C}_{2n} \longrightarrow \operatorname{rep} Q$ is defined in such a way that, for an object $UWD \in \mathfrak{C}_{2n}$, it holds that,

$$\Theta(UWD) = (\Theta(w_i), \varphi_{\Theta(w_i, w_{i+1})}),$$

where

$$\Theta(w_i) = \begin{cases} k, & \text{if } w_i = UD, \\ 0, & \text{if } w_i = DU. \end{cases} \tag{3.6}$$

If $i, i+1 \in [i_r, j_{r+1-\sigma(a)}]$ $\big([j_r, i_{r+\sigma(a)}]\big)$ then $s(\Theta(w_i, w_{i+1})) = i+1$, is the starting point of the corresponding arrow, whereas $t(\Theta(w_i, w_{i+1})) = i$ is the ending vertex of the corresponding arrow $\big(s(\Theta(w_i, w_{i+1})) = i,\ t(\Theta(w_i, w_{i+1})) = i+1\big)$ and,

$$\varphi_{\Theta(w_i, w_{i+1})} : \Theta(w_{s(\Theta(w_i, w_{i+1}))}) \longrightarrow \Theta(w_{t(\Theta(w_i, w_{i+1}))}),$$

$$\varphi_{\Theta(w_i, w_{i+1})} = \begin{cases} 1_k, & \text{if } w_i = UD = w_{i+1}, \\ 0, & \text{if } w_i = DU \text{ or } w_{i+1} = DU. \end{cases} \tag{3.7}$$

Functor Θ acts on morphisms as follows;

Let

$$f_{q_2} \circ \cdots \circ f_{q_1} = (1_1, \dots, 1_{q_1 - 1}, f_{q_1}, \dots, f_{q_2}, 1_{q_2+1}, \dots 1_{n-1}),$$

be an elementary shift between UWD and UVD, then:

$$\begin{gathered} \Theta((1_1, \dots, 1_{q_1-1}, f_{q_1}, \dots, f_{q_2}, 1_{q_2+1}, \dots 1_{n-1})), \\ (\Theta(1_1), \dots, \Theta(1_{q_1-1}), \Theta(f_{q_1}), \dots, \Theta(f_{q_2}), \Theta(1_{q_2+1}), \dots, \Theta(1_{n-1})), \end{gathered}$$

where $\Theta(f_m) = 0$ and,

$$\Theta(1_{m_1}) = \begin{cases} 1_k, & \text{if } w_{m_1} = UD = v_{m_1}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

for $1 \leq m_1 \leq q_1 - 1$, $q_1 \leq m \leq q_2$ and $q_2 + 1 \leq m_1 \leq n-1$.

Remark 3.2. *Note that, it is easy to see that Θ is an additive covariant functor.*

Lemma 3.4. *Let UWD and UVD be Dyck paths of $\mathfrak{C}_{2n}$. If $\operatorname{Hom}_{\mathfrak{C}_{2n}}(UWD, UVD) \neq 0$ then $\operatorname{Hom}_{\operatorname{rep} Q}(\Theta(UWD), \Theta(UVD)) \neq 0$.*

Proof. Suppose $\operatorname{Hom}_{\mathfrak{C}_{2n}}(UWD, UVD) \neq 0$, and let F be a shift path $UW^0D \xrightarrow{F^0} UW^1D \xrightarrow{F^1} \dots \xrightarrow{F^{m-2}} UW^{m-1}D \xrightarrow{F^{m-1}} UW^mD$ from $UWD = UW^0D$ to $UVD =$

UW^mD for some $m \in \mathbb{Z}^+$, then there exist q_1 and q_2 such that

$$\{q_1, q_1+1, \ldots, q_2-1, q_2\} = \bigcap_{i \in J} \operatorname{Supp} UW^iD,$$

applying Θ we obtain the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, dash] & \Theta(w^0_{q_1-1}) \arrow[r, dash, "a^0_{q_1-1}"] \arrow[d, "c^0_{q_1-1}"'] & k \arrow[r, dash, "1"] \arrow[d, "1"] & \cdots \arrow[r, dash, "1"] & k \arrow[r, dash, "a^0_{q_2}"] \arrow[d, "1"'] & \Theta(w^0_{q_2+1}) \arrow[r, dash] \arrow[d, "d^0_{q_2+1}"] & \cdots \\
\cdots \arrow[r, dash] & \Theta(w^1_{q_1-1}) \arrow[r, dash, "a^1_{q_1-1}"] \arrow[d, "c^1_{q_1-1}"'] & k \arrow[r, dash, "1"] \arrow[d, "1"] & \cdots \arrow[r, dash, "1"] & k \arrow[r, dash, "a^1_{q_2}"] \arrow[d, "1"'] & \Theta(w^1_{q_2+1}) \arrow[r, dash] \arrow[d, "d^1_{q_2+1}"] & \cdots \\
 & \vdots \arrow[d, "c^{m-2}_{q_1-1}"'] & \vdots \arrow[d, "1"] & & \vdots \arrow[d, "1"'] & \vdots \arrow[d, "d^{m-2}_{q_2+1}"] & \\
\cdots \arrow[r, dash] & \Theta(w^{m-1}_{q_1-1}) \arrow[r, dash, "a^{m-1}_{q_1-1}"] \arrow[d, "c^{m-1}_{q_1-1}"'] & k \arrow[r, dash, "1"] \arrow[d, "1"] & \cdots \arrow[r, dash, "1"] & k \arrow[r, dash, "a^{m-1}_{q_2}"] \arrow[d, "1"'] & \Theta(w^{m-1}_{q_2+1}) \arrow[r, dash] \arrow[d, "d^{m-1}_{q_2+1}"] & \cdots \\
\cdots \arrow[r, dash] & \Theta(w^m_{q_1-1}) \arrow[r, dash, "a^{m+1}_{q_1-1}"] & k \arrow[r, dash, "1"] & \cdots \arrow[r, dash, "1"] & k \arrow[r, dash, "a^m_{q_2}"] & \Theta(w^m_{q_2+1}) \arrow[r, dash] & \cdots
\end{tikzcd}$$

Diagram 1.

where $c^i_{q_1} - 1$, $a^i_{q_1-1}, a^i_{q_2}, d^i_{q_2+1} \in \{0, k\}$, squares in the diagram are commutative between q_1 and q_2 (independently of the chosen orientation). For the sub-shift path $F^{(x,y)}$ to F with $0 \leq x \leq y \leq m-1$ there exist positive integers $q_1^{(x,y)}$ and $q_2^{(x,y)}$ such that

$$S^{(x,y)} = \{q_1^{(x,y)}, q_1^{(x,y)}+1, \ldots, q_2^{(x,y)}-1, q_2^{(x,y)}\} = \bigcap_{i \in J^{(x,y)}} \operatorname{Supp} UW^iD,$$

and for the diagrams

$$\begin{tikzcd}
\Theta(w^x_{q_1^{(x,y)}-1}) \arrow[r, dash, "a^x_{q_1^{(x,y)}-1}"] \arrow[d, "0"'] & k \arrow[d, "1"] \\
\Theta(w^y_{q_1^{(x,y)}-1}) \arrow[r, dash, "a^y_{q_1^{(x,y)}-1}"] & k
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
k \arrow[r, dash, "a^x_{q_2^{(x,y)}}"] \arrow[d, "1"'] & \Theta(w^x_{q_2^{(x,y)}+1}) \arrow[d, "0"] \\
k \arrow[r, dash, "a^y_{q_2^{(x,y)}}"] & \Theta(w^y_{q_2^{(x,y)}+1})
\end{tikzcd}$$

Diagram 2. Diagram 3.

we have the following cases:

(1) If $q_1^{(x,y)} \in [i_r, j_{r+1-\sigma(a)}]$ ($i_r < q_1^{(x,y)} \leq j_{r+1-\sigma(a)}$) four cases must be considered.

- (1.1) If $\Theta(w^x_{q_1^{(x,y)}-1}) = k$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = k$, $q_1^{(x,y)}$ belong to $S^{(x,y)}$, which is a contradiction.
- (1.2) If $\Theta(w^x_{q_1^{(x,y)}-1}) = k$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = 0$, then the Diagram 2 commutes.

(1.3) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = k$, then there is an elementary shift $f_{q_1^{(x,y)}-1}$ on the interval and this is again a contradiction.

(1.4) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = 0$, then the Diagram 2 commutes.

(2) If $q_1^{(x,y)} \in [j_{r+1-\sigma(a)}, i_{r+1}]$ $(j_{r+1-\sigma(a)} < q_1^{(x,y)} \leq i_{r+1})$, the conditions (1.1)-(1.4) are satisfied on the interval.

(2.1) If $\Theta(w^x_{q_1^{(x,y)}-1}) = k$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = 0$, then they satisfy condition (1.3).

(2.2) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = k$, then they satisfy condition (1.2).

(3) Case $q_2^{(x,y)} \in [i_r, j_{r+1-\sigma(a)}]$ is similar to case (2) for the Diagram 3.

(4) Case $q_2^{(x,y)} \in [j_{r+1-\sigma(a)}, i_{r+1}]$ is similar to case (1) for the Diagram 3.

therefore the Diagram 1 commutes. Since the cases over $[j_r, i_{r+\sigma(a)}]$ can be showed by using dual arguments. We are done. $\square$

Lemma 3.5. *Functor Θ is faithful and full.*

Proof. Let ϕ be the map

$$\phi : \mathrm{Hom}_{\mathfrak{C}_{2n}}(UWD, UVD) \rightarrow \mathrm{Hom}_{\text{ rep } Q}(\Theta(UWD), \Theta(UVD)),$$

such that $\phi(\lambda F) = \lambda\Theta(F)$ with $F = (1_1, \ldots, 1_{q_1-1}, f_{q_1}, \ldots f_{q_2}, 1_{q_2+1}, \ldots, 1_{n-1})$, for some $1 \leq q_1, q_2 \leq n-1$ and $\lambda \in k$. Note, ϕ is well defined and Lemma 3.4 allows us to observe that the image of a non-zero morphism in $\mathfrak{C}_{2n}$ is a non-zero morphism in rep Q. Thus, ϕ is surjective and injective. $\square$

Theorem 3.1. *Functor Θ is a categorical equivalence between the categories $\mathfrak{C}_{2n}$ and* rep Q*.*

Proof. Lemma 3.5 implies that functor Θ is faithful and full. Now, let $(M_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ be an indecomposable representation in rep Q of the form

$$0 \text{ --- } \cdots \text{ --- } \overbrace{k}^{q_1} \xrightarrow{\;1\;} k \xrightarrow{\;1\;} \cdots \xrightarrow{\;1\;} k \xrightarrow{\;1\;} \overbrace{k}^{q_2} \text{ --- } \cdots \text{ --- } 0$$

with $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_m\}$ the sets of sinks and sources respectively. Let $\varphi_1 : \{0, k\} \rightarrow \{DU, UD\}$ be a map such that $\varphi_1(k) = UD$ and $\varphi_1(0) = DU$. Define the Dyck path UWD such that

$$UWD = U \underbrace{w_1 \ldots w_{q_1-1}}_{DU} \underbrace{w_{q_1} \ldots w_{q_2}}_{UD} \underbrace{w_{q_2+1} \ldots w_{n-1}}_{DU} D.$$

Proposition 3.2 allows us to observe that UWD has $n-1$ peaks over $\{j_1, \ldots, j_m, i_1, \ldots, i_k\}$ and $\Theta(UWD) = (M_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$. Thus, Θ is essentially surjective. $\square$

Corollary 3.1. *There exists a bijection φ between the set of representatives of indecomposable representations of* rep Q *and the set of Dyck paths of length $2n$ with exactly $n-1$ peaks.*

Proof. The Narayana number with exactly $n-1$ peaks over all Dyck paths of length $2n$ is the triangular number $T_{n-1} = \frac{(n-1)(n)}{2}$, which is equal to the number of indecomposable representations of rep Q, then we define $\varphi : S \to \text{Ind}\,(\text{rep}\,Q)$ such that $\varphi(UWD) = \Theta(UWD)$. □

Corollary 3.2. *The category $\mathfrak{C}_{2n}$ is an abelian category.*

3.2.2 Properties of the Category $\mathfrak{C}_{2n}$

In this section, we introduce some properties of $\mathfrak{C}_{2n}$ regarding simple, projective and injective indecomposable objects, we also construct the Auslander-Reiten quiver for algebras of Dynkin type $\mathbb{A}_{n-1}$. Some conditions for morphisms between objects of the category are introduced as well.

Theorem 3.2. *Let $\mathcal{C} = \{j_1, \dots, j_m, i_1, \dots, i_k\}$ be an admissible subchain, and let $\mathfrak{C}_{2n}$ be the corresponding category, then*

(i) Indecomposable simple objects of $\mathfrak{C}_{2n}$ are objects of the form

$$S(x) = US(w_1^x) \dots S(w_n^x)D,$$

where

$$S(w_y^x) = \begin{cases} UD, & \text{if } x = y, \\ DU, & \text{otherwise.} \end{cases} \tag{3.9}$$

(ii) Indecomposable projective objects of $\mathfrak{C}_{2n}$ have the form $P(x) = UP(w_1^x) \dots P(w_n^x)D$ where

$$P(w_x^y) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}]\ ([j_r, i_{r+\sigma(a)})\ \text{and } y \leq x\ (x \leq y), \\ DU, & \text{otherwise.} \end{cases} \tag{3.10}$$

(iii) Indecomposable injective objects of $\mathfrak{C}_{2n}$ have the form $I(i) = UI(w_1^x) \dots I(w_n^x)D$ where

$$I(w_x^y) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}]\ ([j_r, i_{r+\sigma(a)}])\ \text{and } x \leq y\ (y \leq x), \\ DU, & \text{otherwise.} \end{cases} \tag{3.11}$$

Proof. (i) Let $S(x) = (S(x)_y, \varphi_\alpha)$ be an indecomposable simple object of rep Q such that $S(x)_y = k$ if $x = y$ and $S(x)_y = 0$ if $x \neq y$. Functor Θ allows us to observe that, there is a $UWD \in \mathfrak{C}_{2n}$ satisfying the required conditions.

(ii) Let $P(x) = (P(x)_y, \varphi_\alpha)$ be an indecomposable projective object of rep Q, if $P(x)_y = k$ then there is a path from x to y, as well as, a source $j_{r+1-\sigma(a)}$ (j_r) and a sink i_r $(i_{r+\sigma(a)})$ such that $i_r \leq y \leq x \leq j_{r+1-\sigma(a)}$ $(j_r \leq x \leq y \leq i_{r+\sigma(a)})$, and $P(x)_y = 0$. Thus, there is not a path between x and y, then functor Θ determines an object UWD of $\mathfrak{C}_{2n}$ with

$i_1, \dots i_k, j_1, \dots j_m$ being an admissible subchain satisfying the required conditions. Case (iii) follows by dually applying the arguments used in the case (ii). □

Corollary 3.3. *The indecomposable simple objects of $\mathfrak{C}_{2n}$ have exactly a subsequence $UUDD$.*

Proof. Let $S(x)$ be an indecomposable simple object of $\mathfrak{C}_{2n}$, then the identity

$$S(x) = U \dots S(w^x_{x-1})S(w^x_x)S(w^x_{x+1}) \dots D = U \dots DU \dots \underbrace{DU}_{x-1}\underbrace{UD}_{x}\underbrace{DU}_{x+1} \dots DU \dots D$$

has place as a consequence of Theorem 3.2. □

Remark 3.3. *The Auslander-Reiten translate can be obtained by using the Coxeter transformation and the dimension vector associated to the support of a Dyck path in $\mathfrak{C}_{2n}$.*

Figure 3.3 describes the Auslander-Reiten quiver of rep Q of the quiver Q given by Figure 1.13.

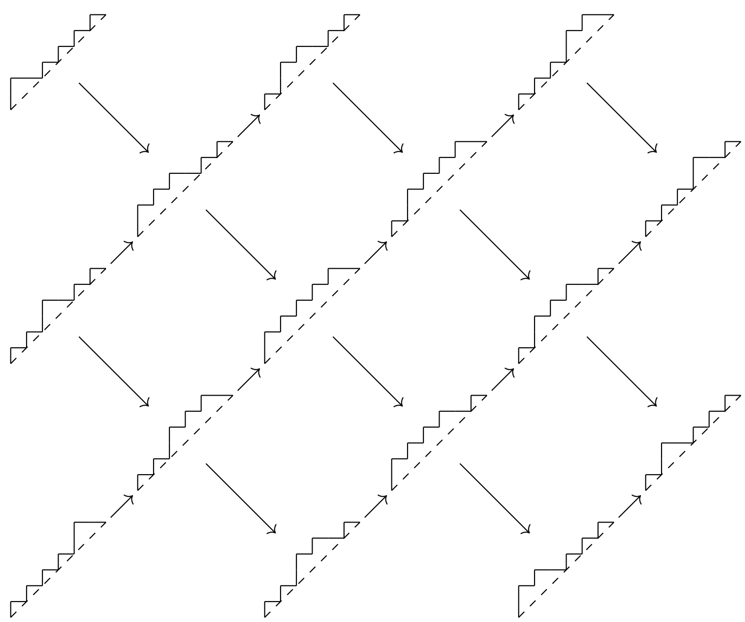


FIGURE 3.3. Quiver Q and the Auslander-Reiten quiver of rep Q.

Morphisms in $\mathfrak{C}_{2n}$ also have the following properties.

Let UWD be a Dyck path of $\mathfrak{C}_{2n}$, then

- $p_{UWD} = t$ and $b_{UWD} = \max \{s \mid i_r \leq s \leq j_{r+1-\sigma(a)},\, w_s = UD\}$ over $[i_r, j_{r+1-\sigma(a)}]$,
- $p^{UWD} = \min \{s \mid j_r \leq s \leq i_{r+\sigma(a)},\, w_s = UD\}$ and $b^{UWD} = t$ over $[j_r, i_{r+\sigma(a)}]$.

Theorem 3.3. *The vector space* $\mathrm{Hom}_{\mathfrak{C}_{2n}}(UWD, UVD) \neq 0$ *if and only if* *(i)* $Supp(UWD) \cap Supp(UVD) \neq \varnothing$,

(ii) $p_{UWD} \leq p_{UVD}$ *and* $b_{UWD} \leq b_{UVD}$ *over* $[i_r, j_{r+1-\sigma(a)}]$,

(iii) $p^{UWD} \geq p^{UVD}$ *and* $b^{UWD} \geq b^{UVD}$ *over* $[j_r, i_{r+\sigma(a)}]$,

for all $[i_r, j_{r+1-\sigma(a)}]$, $[j_r, i_{r+\sigma(a)}]$ *such that* $i_r \leq q \leq j_{r+1-\sigma(a)}$ *and* $j_r \leq q \leq i_{r+\sigma(a)}$ *with* $q \in Supp(UWD) \cap Supp(UVD)$.

Proof. The result follows as a consequence of the definition of the functor Θ and the construction of Lemma 3.1. $\square$

3.2.3 A Relationship with Some Nakayama Algebras

In [65] Marczinzik, Rubey and Stump presented a connection between the Auslander-Reiten quiver of Nakayama algebras and Dyck paths. In such a work for a Nakayama algebra $\mathcal{A}$, they associated the vector space dimension of the indecomposable projective modules $e_i\mathcal{A}$ to a Dyck path, this vector is called the Kupisch series. If we take a Nakayama algebra $\mathcal{A} = kQ/I$, with $I = \langle x_3x_4, x_1x_2x_3 \rangle$,

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, "x_1"] & \underset{2}{\circ} \arrow[r, "x_2"] & \underset{3}{\circ} \arrow[r, "x_3"] & \underset{4}{\circ} \arrow[r, "x_4"] & \underset{5}{\circ} \end{tikzcd}$$

FIGURE 3.4. Quiver Q of type $\mathbb{A}_5$.

then the Kupisch series of kQ/I is $[3, 3, 2, 2, 1]$, and the Auslander-Reiten quiver of kQ/I has the shape described in Figure 3.5.

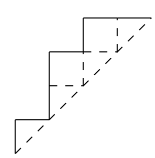


FIGURE 3.5. Dyck path associated to kQ/I.

Let $\mathfrak{C}_{2(n+1)}$ be the category with the admissible subchain $1 < n$, $j_1 = 1$ and $i_1 = n$, and let D_i be the sets

$$\begin{aligned} D_1 &= \{X \in Ob(\mathfrak{C}_{2(n+1)}) \mid w_1 = UD\}, \\ D_i &= \{X \in Ob(\mathfrak{C}_{2(n+1)}) \mid w_m = DU,\ 1 \leq m \leq i-1\}, \end{aligned} \tag{3.12}$$

for $1 < i \leq n$. Then, we take the subset $D_{i,j} \subseteq D_i$,

$$D_{i,j_i} = \{Y \in D_i \mid i \leq r_Y \leq m(i, j_i) + i - 1\}, \tag{3.13}$$

such that the vector $v = (n - (m(i, j_i) + i - 1))_{i=1}^{n}$ constitutes an integer partition with n parts. Now, let $\mathfrak{N}_v$ be the subcategory of $\mathfrak{C}_{2(n+1)}$ whose objects are $k-$linear combinations of the Dyck paths in the following set

$$\mathcal{L} = \bigcup_{i=1}^{n} D_{i,j_i}, \tag{3.14}$$

and morphisms defined by the category $\mathfrak{C}_{2n(n+1)}$.

We assume the following numbering and orientation for a quiver Q associated to a Nakayama algebra

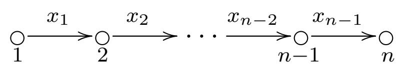


FIGURE 3.6. Quiver Q of type $\mathbb{A}_n$.

The functor Θ' between the category $\mathfrak{N}_v$ and the category of representations of (Q, I) where kQ/I is a Nakayama algebra with Kupisch series $[m(1, j_1), \ldots, m(n, j_n)]$ is defined in such a way that, $\Theta'(UWD) = \Theta(UWD)$ and $\Theta'(F) = \Theta(F)$ for $UWD \in \mathcal{L}$ and F being an elementary shift in $\mathfrak{N}_v$.

Corollary 3.4. *The functor Θ' is an equivalence of categories.*

Proof. It is a direct consequence of Theorem 3.1. □

As an example, Figure 3.7 shows the Auslander-Reiten quiver of the Nakayama algebra $\mathcal{A} = kQ/I$ associated to the quiver Q shown in Figure 3.4 with $I = \langle x_3x_4, x_1x_2x_3 \rangle$.

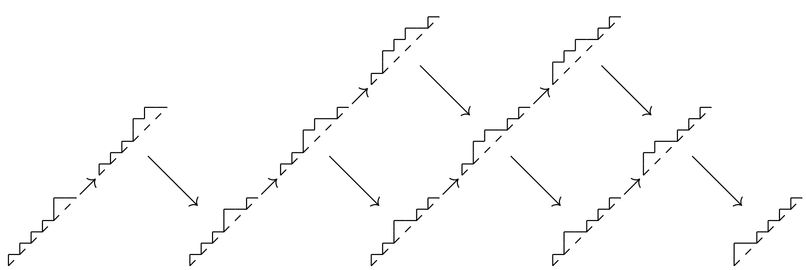


FIGURE 3.7. Auslander-Reiten quiver of mod kQ/I.

3.3 Cluster Variables Associated to Dyck Paths

In this section, we construct an alphabet associated to Dyck paths. And it is given a formula for cluster variables of cluster algebras associated to Dynkin diagrams of type $\mathbb{A}_n$.

3.3.1 An Alphabet for Dyck Paths

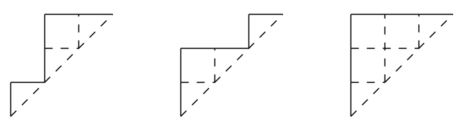
For $n > 2$, let $U_1^i = u_1 \dots u_{2n}$ and $U_2^i = u'_1 \dots u'_{2n}$ be Dyck paths in $\mathfrak{D}_{2n}$ with the following form:

$$u_j = \begin{cases} U, & \text{if } 1 \le j \le i+1 \text{ or } j = 2(i+1)+k \le 2n, \\ D, & \text{if } i+2 \le j \le 2(i+1) \text{ or } j = 2(i+1+k) \le 2n, \end{cases} \tag{3.15}$$

and

$$u'_j = \begin{cases} U, & \text{if } 2i < j \le i+n \text{ or } j = 1+2k \le 2i, \\ D, & \text{if } i+n < j \le 2n \text{ or } j = 2k \le 2n, \end{cases} \tag{3.16}$$

for $k > 0$ and $i \le n-2$. The *alphabet* H_n is the union of the set $\{U_r^j \mid r = 1, 2 \text{ and } 1 \le i \le n-2\}$ and the Dyck path with exactly one peak in $\mathfrak{D}_{2n}$ (denoted E_n). Figure 3.8 shows the alphabet H_3.



(a) U_1^1 (b) U_2^1 (c) E_3

FIGURE 3.8. Alphabet H_3.

Let $\mathcal{C} = \{i_1, \dots, i_k, j_1, \dots, j_m\}$ be an admissible subchain of **n-1**. We fix two different relations of concatenation ϑ_1 and ϑ_2 over H_n such that

$$\vartheta_1(V_i) = \begin{cases} E_n, & \text{if } V_i = E_n \text{ or } V_i = U_1^i, \\ U_2^{i+1}, & \text{if } V_i = E_n \text{ or } V_i = U_1^i, \\ U_1^{i+1}, & \text{if } V_i = U_2^i, \end{cases} \tag{3.17}$$

and

$$\vartheta_2(V_i) = \begin{cases} E_n, & \text{if } V_i = U_2^i, \\ U_1^{i+1}, & \text{if } V_i = E_n \text{ or } V_i = U_1^i, \\ U_2^{i+1}, & \text{if } V_i = U_2^i. \end{cases} \tag{3.18}$$

Then, we take the set of words $V = V_1 \dots V_{n-2}$ in H_n^* such that

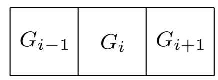
$$V_i = \begin{cases} \vartheta_1(V_{i-1}), & \text{if } i \notin \mathcal{C}, \\ \vartheta_2(V_{i-1}), & \text{if } i \in \mathcal{C} - \{1, n-1\}, \end{cases} \tag{3.19}$$

for $1 < i \le n-2$, $n \ge 4$. This set is denoted by $\mathbb{X}_{\mathcal{C}}$, in particular case $\mathbb{X}_{\{1,2\}} = H_3$.

3.3.2 Dyck Words and Perfect Matchings

Let $\mathcal{G} = (G_1, \ldots, G_{n-1})$ be a snake graph, then we can associate to $\mathcal{G}$ an admissible subchain $\mathcal{C}$ of **n-1** in the following way:

If G_{i-1}, G_i and G_{i+1} denote tiles of the following snake graph



then, $i \in \mathcal{C}$ for $1 < i < n-1$. For example, for the snake graph $\mathcal{G}$ shown in Figure 3.9

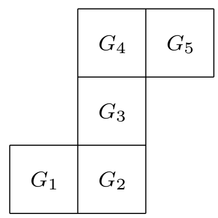


FIGURE 3.9. Snake graph $\mathcal{G}$.

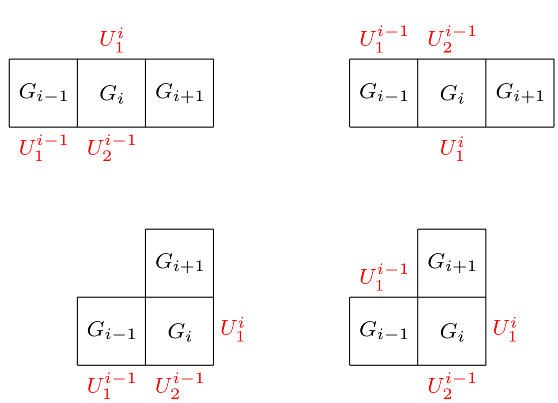
it holds that the corresponding admissible subchain is given by the identity $\{1, 3, 5\} = \{i_1, j_1, i_2\} = \{j_1, i_1, j_2\}$. By notation, $\mathcal{G}$ can be written as $\mathcal{G}_{\mathcal{C}}$.

The following result establishes a relationship between the alphabet $\mathbb{X}_{\mathcal{C}}$ and perfect matchings of snake graphs.

Lemma 3.6. *Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of **n-1**. Then, there is a bijective correspondence between the set $\mathbb{X}_{\mathcal{C}}$ and the perfect matchings of $\mathcal{G}_{\mathcal{C}}$.*

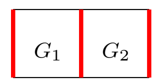
Proof. Let $\mathcal{C}$ be an admissible subchain of **n-1**, $\mathbb{X}_{\mathcal{C}}$ be a set of words, and $\mathcal{G}_{\mathcal{C}}$ be a snake graph associated to $\mathcal{C}$. Assume a numbering over the edges of $\mathcal{G}_{\mathcal{C}}$ in the following way:

For boundary edges of G_i, we have the following four possibilities

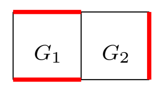


with $1 < i < n-1$ (labeling is given by recurrence). The other edges are labeled with the letter E_n. Now, a perfect matching P of $\mathcal{G}_{\mathcal{C}}$ can be written as a vector $v = (v_1, \ldots, v_n)$, where each v_i corresponds to an edge of $\mathcal{G}_{\mathcal{C}}$ (this vector is unique up to permutation). Define a map $f : \mathbb{X}_{\mathcal{C}} \to \mathrm{Match}(\mathcal{G}_{\mathcal{C}})$ such that $f(V_1 \ldots V_{n-2}) = (E_n, V_1, \ldots, V_{n-2}, E_n)$. Firstly, we will prove that f is well defined by induction over n. To start note that for $n = 3$, we have the following three cases:

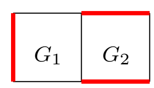
(I) If $V_1 = E_3$, it turns out that $F(V_1) = (E_3, E_3, E_3)$, which is given by



(II) If $V_1 = U_1^1$, it holds that $f(U_1^1) = (E_3, U_1^1, E_3)$, which is equal to



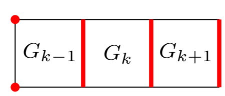
(III) If $V_1 = U_2^1$, then $f(U_2^1) = (E_3, U_2^1, E_3)$, which is of the form



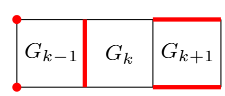
Suppose that the result holds for $n = k$. Let $n = k+1$, by hypothesis $(E_{k+1}, V_1, \ldots V_k)$ are disjoint sets containing all the previous tiles in $\mathcal{G}_{\mathcal{C}}$, then there are two possibilities for k.

(I) for $k \in \mathcal{C} - \{1, k+1\}$, we have the following conditions:

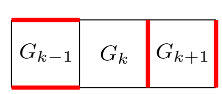
(1.1) If $V_{k-1} = E_{k+1}$, then $f(V_1 \ldots E_{k+1} E_{k+1}) = (E_{k+1}, V_1, \ldots, E_{k+1}, E_{k+1}, E_{k+1})$ and $f(V_1 \ldots E_{k+1} U_2^k) = (E_{k+1}, V_1, \ldots, E_{k+1}, U_2^k, E_{k+1})$, which are given by



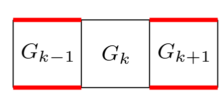
and



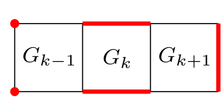
(1.2) If $V_{k-1} = U_1^{k-1}$, then $f(V_1 \ldots U_1^{k-1} E_{k+1}) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, E_{k+1}, E_{k+1})$ and $f(V_1 \ldots U_1^{k-1} U_2^k) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, U_2^k, E_{k+1})$, which are equal to



and

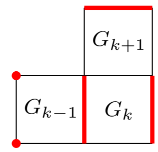


(1.3) If $V_{k-1} = U_2^{k-1}$, then $f(V_1 \ldots U_2^{k-1} U_1^k) = (E_{k+1}, V_1, \ldots, U_2^{k-1}, U_1^k, E_{k+1})$ which is of the form

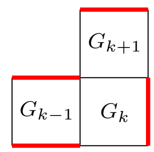


(II) for $k \notin \mathcal{C}$, there are the following cases:

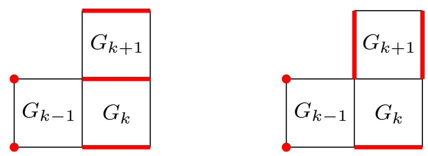
(2.1) If $V_{k-1} = E_{k+1}$, then $f(V_1 \dots E_{k+1}U_1^k) = (E_{k+1}, V_1, \dots, E_{k+1}, U_1^k, E_{k+1})$, which is given by



(2.2) If $V_{k-1} = U_1^{k-1}$, then $f(V_1 \dots U_1^{k-1}U_1^k) = (E_{k+1}, V_1, \dots, U_1^{k-1}, U_1^k, E_{k+1})$, which is equal to



(2.3) If $V_{k-1} = U_2^{k-1}$, then $f(V_1 \dots U_2^{k-1}E_{k+1}) = (E_{k+1}, V_1, \dots, U_2^{k-1}, E_{k+1}, E_{k+1})$ and $f(V_1 \dots U_2^{k-1}U_2^k) = (E_{k+1}, V_1, \dots, U_2^{k-1}, U_2^k, E_{k+1})$, which are of the form



Dual arguments prove the result for the other labelings. We also note that by definition map f is injective and surjective. $\square$

Remark 3.4. *Each perfect matching of $\mathcal{G}_{\mathcal{C}}$ is in correspondence with just only one object of the $\mathbb{A}_{n-1}-$Dyck paths category associated to the admissible subchain $\mathcal{C} = \{i_1, \dots, i_k, j_1, \dots, j_m\}$.*

For each Dyck path $Y = y_1 \dots y_{2n}$ with $n-1$ peaks, we construct a family of words $Y \cap \mathbb{X}_{\mathcal{C}} \in H_n^*$ such that:

$$Y \cap \mathbb{X}_{\mathcal{C}} = \{Y \cap V^z \mid V^z \in \mathbb{X}_{\mathcal{C}}\}, \tag{3.20}$$

where

$$Y \cap V^z = \begin{cases} V^z, & \text{if there exists } j \text{ such that } y_j = v_j^z \text{ for } 1 < j < 2n, \\ E_n, & \text{otherwise,} \end{cases} \tag{3.21}$$

with $V^z = v_1^z \dots v_{2n}^z$ in $\mathbb{X}_{\mathcal{C}}$. For the set $Y \cap \mathbb{X}_{\mathcal{C}}$, it can be defined a relation $\backsim$ such that

$$Y \cap V^{z_1} \backsim Y \cap V^{z_2} \text{ if and only if } Y \cap V^{z_1} \text{ and } Y \cap V^{z_2} \text{ are the same word.} \tag{3.22}$$

In this case, $\backsim$ is an equivalence relation and $(Y \cap \mathbb{X}_{\mathcal{C}})/\backsim$ is denoted by $[Y \cap \mathbb{X}_{\mathcal{C}}]$. Also, we remind that a Dyck path Y can be written as the word $UWD = Uw_1, \dots w_{n-1}D$, where $y_1 = U$, $y_{2n} = D$ and, $w_i = y_{2i}y_{2i+1}$.

Lemma 3.7. *Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of* ***n-1*** *and let Y a Dyck path of length $2n$ with exactly $n-1$ peaks. Then, there is a bijective correspondence between the set $[Y \cap \mathbb{X}_{\mathcal{C}}]$ and the set of perfect matchings of the snake graph belonging to $\mathcal{G}_{\mathcal{C}}$ and induced by the words $w_t = UD$ in Y.*

Proof. Let $\mathcal{C}$ be an admissible subchain of **n-1** and $Y = UWD$ be a Dyck path in S, then by Proposition 3.2 there are $l, r \in \mathbf{Z}_{>0}$ with $1 \leq l \leq r \leq n-1$ such that $w_t = UD$ for $l \leq t \leq r$ and $w_t = DU$ otherwise. Now, let $\mathcal{G}_{\mathcal{C}^{l,r}} = \mathcal{G}[l, d]$ be a snake graph belonging to $\mathcal{G}_{\mathcal{C}}$ induced by Y. Define a map $g : [Y \cap \mathbb{X}_{\mathcal{C}}] \to \mathrm{Match}(\mathcal{G}_{\mathcal{C}^{l,r}})$ such that:

(I) If $1 < l \leq r < n-1$, then $g([Y \cap V^i]) = g(E_n \ldots E_n V^i_{l-1} \ldots V^i_r E_n \ldots E_n) = (V^i_{l-1}, \ldots, V^i_r)$.

(II) If $l = 1$ and $1 = l \leq r < n-1$, then $g([Y \cap V^i]) = g(V^i_1 \ldots V^i_r E_n \ldots E_n) = (E_n, V^i_l, \ldots, V^i_r)$.

(III) If $r = n-1$ and $1 < l \leq r = n-1$, then $g([Y \cap V^i]) = g(E_n \ldots E_n V^i_{l-1} \ldots V^i_{n-2}) = (V^i_{l-1}, \ldots, V^i_{n-2}, E_n)$.

(IV) If $l = 1$ and $r = n-1$, then $g = f$.

Since in the four cases g is a restriction of f. It follows that g is a bijection as a consequence of Lemma 3.6. □

3.3.3 Cluster Variables Formula Based on Dyck Paths Categories

In this section, Dyck paths categories are used to give a formula for cluster variables of cluster algebras of Dynkin type $\mathbb{A}_n$, to do that, we use the category of Dyck paths associated to an admissible subchain. We also present a connection between cluster variables of algebras of type $\mathbb{A}_{n-1}$ and Dyck paths with $n-1$ peaks.

Let $\mathcal{C} = \{i_1, \ldots i_k, j_1, \ldots j_m\}$ be an admissible subchain of **n-1** and let $Y = UWD$ be a Dyck path in S, then we define the monomials

$$\eta_Y = \prod_{UD = w_i \in Y} x_i, \tag{3.23}$$

and

$$X_V = \prod_{m \in M_V} x_m, \tag{3.24}$$

with M_V being the set of indices m such that

$$m = \begin{cases} i+1, & \text{if } U^i_1 \in V, \\ i, & \text{if } U^i_2 \in V, \\ 0, & \text{if } E_n \in V, \end{cases} \tag{3.25}$$

$V \in [Y \cap \mathbb{X}_{\mathcal{C}}]$. For this case $x_0 = 1$.

The following theorem gives the cluster variable associated to a Dyck path in the set S and its connection with cluster algebras of type $\mathbb{A}_{n-1}$.

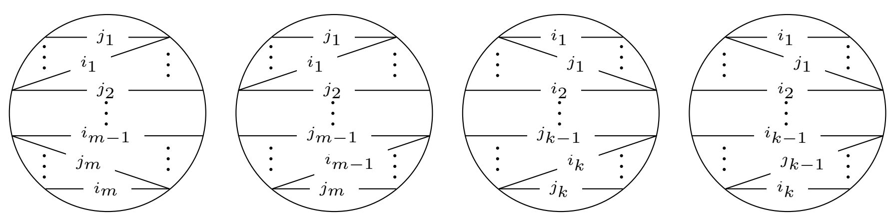
Theorem 3.4. *Let $\mathcal{C} = \{i_1, \dots, i_k, j_1, \dots, j_m\}$ be an admissible subchain of **n-1**, $Y = UWD$ a Dyck path with $n-1$ peaks and M the set of all cluster variables of a cluster algebra of type $\mathbb{A}_{n-1}$ with $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_m\}$ the sets of sinks and sources, respectively. Then:*

(i) The cluster variable associated to Y in the category $\mathfrak{C}_{2n}$ is given by

$$X_Y = (\eta_Y)^{-1}\left(\sum_{V \in [Y \cap \mathbb{X}_{\mathcal{C}}]} X_V\right). \tag{3.26}$$

(ii) There exists a bijective correspondence between Dyck paths with $n-1$ peaks and the set $M \setminus \boldsymbol{x}_0$ with $\boldsymbol{x}_0$ the initial seed.

Proof. Let $\mathcal{C} = \{i_1, \dots, i_k, j_1, \dots, j_m\}$ be an admissible subchain of **n-1**, and let $T_{\mathcal{C}}$ be the triangulation of the polygon with $n+2$ vertices given by $\mathcal{C}$.



Let $\alpha_{l,r}$ be a diagonal that is not in $T_{\mathcal{C}}$ that cuts the diagonals $\alpha_l, \dots \alpha_r \in T_{\mathcal{C}}$. We define a functor $\chi : \mathcal{C}_{T_{\mathcal{C}}} \to \mathfrak{C}_{2n}$ such that $\chi(\alpha_{l,r}) = UW_{l,r}D$, where

$$w_j = \begin{cases} UD, & \text{if } l \leq j \leq r, \\ DU, & \text{otherwise,} \end{cases} \tag{3.27}$$

and for any pivoting elementary move $E : \alpha_{r,l} \to \alpha'_{r',l'}$, $\chi(E)$ is the elementary shift $F = f_{t_1} \circ \cdots \circ f_{t_k}$ from $UW_{l,r}D$ to $UW_{l',r'}D$. Theorems 1.9 and 3.1 allow us to establish the following sequence of equivalences:

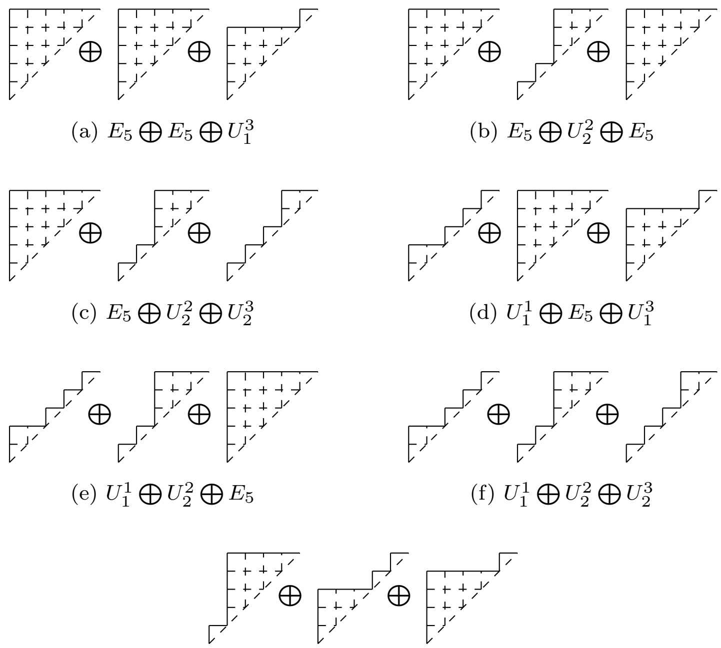
$$\mathcal{C}_{T_{\mathcal{C}}} \backsimeq \text{Mod } Q_{T_{\mathcal{C}}} \backsimeq \mathfrak{C}_{2n}, \tag{3.28}$$

therefore χ is a categorical equivalence. Thus,

(i) Functor χ and Lemma 3.7, allow to establish that $x_\gamma = X_Y$.

(ii) The map $\psi : S \to M \setminus \mathbf{x}_0$ such that $\psi(Y) = X_Y$ is a bijection as a consequence of Theorem 1.8 and the definition of functor χ. We are done. $\square$

For instance, let $\mathcal{C} = \{j_1 = 1, i_1 = 2, j_2 = 4\}$ be an admissible subchain of $\mathbf{4}$, the set $\mathbb{X}_{\mathcal{C}}$ is in correspondence with the objects of $\mathfrak{C}_{10}$ shown in Figure 3.10.



(a) $E_5 \bigoplus E_5 \bigoplus U_1^3$ (b) $E_5 \bigoplus U_2^2 \bigoplus E_5$

(c) $E_5 \bigoplus U_2^2 \bigoplus U_2^3$ (d) $U_1^1 \bigoplus E_5 \bigoplus U_1^3$

(e) $U_1^1 \bigoplus U_2^2 \bigoplus E_5$ (f) $U_1^1 \bigoplus U_2^2 \bigoplus U_2^3$

(g) $U_2^1 \bigoplus U_1^2 \bigoplus U_1^3$

FIGURE 3.10. Objects in $\mathfrak{C}_{10}$.

Then, for $Y = UDUUDUDDUD$, we define the set $Y \cap \mathbb{X}_{\mathcal{C}}$ such that

$$[Y \cap \mathbb{X}_{\mathcal{C}}] = \{E_5E_5U_1^3, E_5U_2^2E_5, U_2^1U_1^2U_1^3\}. \tag{3.29}$$

Thus, identities (3.23), (3.24) and (3.25) define the polynomials

$$\eta_Y = x_2x_3,\ X_{E_5E_5U_1^3} = x_0x_0x_4,\ X_{E_5U_2^2E_5} = x_0x_2x_0,\ X_{U_2^1U_1^2U_1^3} = x_3x_1x_4, \tag{3.30}$$

therefore, the cluster variable associated to the Dyck path Y is given by the expression

$$X_Y = \frac{x_4 + x_2 + x_3x_1x_4}{x_2x_3}. \tag{3.31}$$

CHAPTER 4

Some Applications Of Catalan Numbers

In this chapter, we describe the way that Dyck paths are used in different kind of algebraic structures. In section 4.1, we prove that frieze patterns arise from Dyck paths, to do that, diamonds of $\mathbb{A}_n$ are introduced, in particular, we prove that some new diamonds are in bijective correspondence with Dyck paths, triangulations of an $(n+3)$ polygon, and a family of frieze vectors. This approach allows us to write frieze patterns as a direct sum of indecomposable objects of the category of Dyck paths and it is also given a categorification of the Catalan triangle in the sense of Ringel and Fahr [90]. In section 4.2, we define Brauer configuration algebras whose indecomposable projective modules are in bijective correspondence with Dyck paths, some combinatorial properties of the Catalan triangle allow us to establish formulas for the dimension of these algebras and its corresponding centers.

4.1 Frieze Patterns Arising from Dyck Paths

In this section, we introduce a basic set called diamond which is used to build frieze patterns associated to triangulations of a polygon.

4.1.1 Diamonds of $\mathbb{A}_n$

Let $\mathbf{R}$ be an integral domain, a *diamond* $A = (a_{i,j})$ of $\mathbb{A}_n$ is an array

$$\begin{array}{ccccccc}
 & a_{2,0} & & & & & \\
a_{1,1} & & a_{2,1} & & & & \\
 & a_{1,2} & & \ddots & & & \\
 & & & & a_{2,n-1} & & \\
 & & \ddots & & & & \\
 & & & a_{1,n} & & a_{2,n} & \\
 & & & & a_{1,n+1} & &
\end{array}$$

that satisfies the following conditions:

(D1) $a_{2,0} = a_{1,n+1} = 1$,

(D2) $a_{1,j}a_{2,j} - a_{2,j-1}a_{1,j+1} = 1$ for $1 \leq j \leq n$,

with $a_{i,j} \in \mathbf{R}$ and 1 the identity element of $\mathbf{R}$.

If $\mathbf{R} = \mathbb{Z}$, A is called integral diamond, if it also satisfies

(D3) $a_{1,1} = a$ (or $a_{1,1} = a + m_a$), $a_{2,1} = a + m_a$ (or $a_{2,1} = a$) and $a_{1,2} = a^2 + am_a - 1$, with $1 \leq a \leq \lfloor \frac{n+2}{2} \rfloor$, $1 \leq m_1 \leq n$ and $0 \leq m_a \leq n + 2(1-a)$ if $a > 1$,

A is called positive integral diamond.

Two diamonds A and B of $\mathbb{A}_n$ are a *couple* if and only if $a_{2,j} = b_{1,j}$ for $1 \leq j \leq n$ (denoted by $A \models B$). A set $\{A^t\}_{t\geq 0}$ is an $\mathbb{A}_n-$*sequence of couples* of $\mathbb{A}_n$ if and only if $A^r \models A^{r+1}$ for $r \geq 0$. An $\mathbb{A}_n-$sequence of couples $\{A^t\}_{t\geq 0}$ is a $p-$*cycle* if there is a $p \in \mathbb{N}$ such that $A^t = A^{t+p}$.

For example, let $\mathbf{R} = \mathbb{Z}$, the sets $\{A^t\}_{t\geq 0}$ and $\{B^t\}_{t\geq 0}$ are $\mathbb{A}_1-$sequences of couples which are $2-$cycles with $A^{2k} = B^{2k+1} = A$, $A^{2k+1} = B^{2k} = B$ and $k \geq 0$.

$$A = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & 1 & \end{array} \qquad\qquad B = \begin{array}{ccc} & 1 & \\ 2 & & 1 \\ & 1 & \end{array}$$

In general, it can be written an $\mathbb{A}_n-$sequence of couples $\{A^t\}_{t\geq 0}$ as an $\mathbb{A}_n-$array $C_{A^t} = (c_{i,j})$ such that $c_{t+1,j} = a^t_{1,j}$ and $c_{t+1,0} = c_{t+1,n+1} = 1$, for $t \geq 0$. For the previous example,

$$C_{A^t} = \begin{array}{ccccccccc} & 1 & & 1 & & 1 & & \dots & \\ 1 & & 2 & & 1 & & 2 & & \dots \\ & 1 & & 1 & & 1 & & \dots & \end{array} \qquad C_{B^t} = \begin{array}{ccccccccc} & 1 & & 1 & & 1 & & \dots & \\ 2 & & 1 & & 2 & & 1 & & \dots \\ & 1 & & 1 & & 1 & & \dots & \end{array}$$

C_{A^t} and C_{B^t} are $\mathbb{A}_1-$arrays associated to $\{A^t\}_{t\geq 0}$ and $\{B^t\}_{t\geq 0}$, respectively.

If the $\mathbb{A}_n-$sequence of couples is finite of length m, it can be associated an infinity $\mathbb{A}_n-$array, $C^m_{A^t} = (c^m_{i,j})$ such that

$$c^m_{(t+1)+km,j} = a^t_{1,j},\ c^m_{(t+1)+km,0} = c^m_{(t+1)+km,n+1} = 1, \tag{4.1}$$

for $k \in \mathbb{Z}$. For any $\mathbb{A}_n-$sequence of couples $\{A^t\}_{t\geq 0}$, we can take an $\mathbb{A}_n-$*subsequence* $\{B^z\}_{z\geq 0}$ for $B^z = A^{x+z}$ and some $x \geq t$. In particular, if $\{A^t\}_{t\geq 0}$ is a $p-$cycle , we take the subsequence $\{B^{s_0}\}_{0\leq s_0\leq p-1}$ such that $B^{s_0} = A^t$. This subsequence is called *minimal* $p-$*cycle* of $\{A^t\}_{t\geq 0}$.

Henceforth, we present main results regarding diamonds of $\mathbb{A}_n$.

Proposition 4.1. *Let $\{A^t\}_{t\geq 0}$ be a $p-$cycle and let $B = \{B^{s_0}\}_{0\leq s_0\leq p-1}$ be its minimal $p-$cycle . Then, the C_B^p is a frieze pattern of order $n+3$. In particular, p divides $n+3$.*

Proof. Let $C_B^p = (c_{ij}^p)$ be the infinity $\mathbb{A}_n-$array associated to $\{B^{s_0}\}_{0\leq s_0\leq p-1}$, identity (4.1) implies that

$$c^p_{(s_0+1)+kp,j} = a^{s_0}_{1,j}, c^p_{(s_0+1)+kp,0} = c^p_{(s_0+1)+kp,n+1} = 1,$$

for $k \in \mathbb{Z}$, given that $\{A^t\}_{t\geq 0}$ is a $p-$cycle, then, C_B^p is a frieze pattern. □

Proposition 4.2. *Let $\{A^t\}_{t\geq 0}$ be a $p-$cycle of length $2p$, then the subsequences $\{B^{s_i}\}_{0\leq s_i\leq p-1}$ generate the same frieze pattern of order $n+3$, for $0 \leq i \leq p-1$, and $B^{s_i} = A^{i+s_i}$.*

Proof. Let $\{A^t\}_{t\geq 0}$ be a $p-$cycle of length $2p$, let $C_A^p = (c_{ij}^p)$ and $C_B^p = (c_{ij}^{p'})$ be the infinity arrays of the subsequences $A = \{B^{s_i}\}_{0\leq s_i\leq p-1}$ and $B = \{B_{s_{i'}}\}_{0\leq s_{i'}\leq p-1}$ of $\{A^t\}_{t\geq 0}$ for $0 \leq i < i' \leq p-1$. Applying the translation $s_{i'} = s_i - |i'-i|$, $c^p_{s_{i'}+1+kp,j} = a^{s_{i'}+i'}_{1j} = a^{s_i-|i'-i|+i'}_{1j} = a^{s_i+i}_{ij} = c^p_{s_i+1+kp,j}$. We are done. □

Lemma 4.1. *Let $\{A^t\}_{t\geq 0}$ be a sequence of couples, then $\{A^t\}_{t\geq 0}$ is generated by A^0. In particular, A^0 generates a $p-$cycle for some $p > 0$.*

Proof. Let $\{A^t\}_{t\geq 0}$ be a sequence of couples, then

$$a^x_{2,j} = \frac{1 + (a^x_{2,j-1})(a^{x-1}_{2,j+1})}{a^{x-1}_{2,j}}, \tag{4.2}$$

for $1 \leq j \leq n$, and $x \geq t$, then $a^x_{2,j}$ can be written by using the set $\{a^0_{2,j}\}_{1\leq j\leq n}$ for $x > 0$. In particular, the set $\{a^0_{2,1}, \dots, a^0_{2,n}\}$ is a seed of the cluster algebra associated to the quiver shown in Figure 3.6. Since the cluster variables are finite in the case $\mathbb{A}_n$, then there is $p = n+3$ (in some cases, it is not minimal) such that $A^0 = A^{n+3}$. □

Theorem 4.1. *Let A be a diamond of $\mathbb{A}_n$, then A generates a frieze pattern.*

Proof. It is a direct consequence of Lemma 4.1, and Proposition 4.1. □

For instance, the diamonds A and B generate the following frieze pattern.

$$\begin{array}{ccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & & 1 & & 2 & & 1 & & 2 & & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & \dots
\end{array}$$

4.1.2 Seed Vectors

In this section, we give an algorithm to build a family of positive integral frieze vectors associated to the quiver shown in Figure 3.6. These vectors help to find a connection between the positive integral diamonds of $\mathbb{A}_n$, triangulations, and Dyck paths.

Let A be a diamond of $\mathbb{A}_n$, we can write the first column as a vector $v_A = (a_1, \ldots, a_n)$ where $a_j = a_{1,j}$.

Proposition 4.3. *If $v = (a_1, \ldots, a_n)$ is a vector associated to a positive integral diamond of $\mathbb{A}_n$ with $a_n = 1$, then the vector $v' = (a_1, \ldots, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_{n-1})$ generates a positive integral diamond of $\mathbb{A}_n$, for $1 \leq i < n$.*

Proof. Let $v_A = (a_1, \ldots, a_n)$ be a vector associated to a positive integral diamond $A = (a_{j,m})$ of $\mathbb{A}_n$, we take the vector $v_{A+i} = (a_1, \ldots, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_{n-1})$ and the array $A + i$ of the following form:

$$b_{1,m} = \begin{cases} a_{1,m}, & \text{if } m \leq i, \\ a_{1,i} + a_{1,i+1}, & \text{if } m = i+1, \\ a_{1,m-1}, & \text{if } m > i+1, \end{cases} \tag{4.3}$$

and

$$b_{2,m} = \begin{cases} a_{2,m}, & \text{if } m \leq i-1, \\ a_{2,i-1} + a_{2,i}, & \text{if } m = i, \\ a_{2,m-1}, & \text{if } m \geq i+1, \end{cases} \tag{4.4}$$

then $b_{1,m}b_{2,m} - b_{2,m-1}b_{2,m+1} = 1$, for $1 \leq m \leq n$ and $1 \leq i < n$. Therefore $A + i$ is a positive integral diamond of $\mathbb{A}_n$. □

Proposition 4.4. *The vector $v_{n,z} = (a_1, \ldots, a_n)$ with*

$$a_i = \begin{cases} z + 1 - i, & \text{if } i < z, \\ 1, & \text{if } i \geq z, \end{cases} \tag{4.5}$$

is in bijective correspondence with a positive integral diamond of $\mathbb{A}_n$, for $z \in \{1, \ldots, n+1\}$.

Proof. Let $v_{n,z}$ be a vector and let z be a natural number between 1 and $n+1$, we define a positive integral diamond A with $a_{1,i} = a_i$ and $a_{2,i} = b_i$ where

$$b_i = \begin{cases} 1, & \text{if } i < z, \\ i + 2 - z, & \text{if } i \geq z, \end{cases} \tag{4.6}$$

then $a_{1,i}a_{2,i} - a_{2,i-1}a_{2,i+1} = 1$ for $1 \leq i \leq n$. □

Remark 4.1. *$v_{n,z}$ is called a seed vector. The vector $v^{n,z} = (b_1, \ldots, b_n)$ defines a positive integral diamond B of $\mathbb{A}_n$ such that $b_{2,i}$ satisfies the following identity*

$$b_{2,i} = \begin{cases} i - 1, & \text{if } i < z - 1, \\ (b_{1,i} + 1)z - 1, & \text{if } z - 2 < i < n, \\ z, & \text{if } i = n, \end{cases} \tag{4.7}$$

and $b_i = b_{1,i}$ is defined as in 4.6.

Proposition 4.5. *The positive integral diamonds A and B generated by $v_{n,z}$ and $v^{n,z}$ respectively are a couple.*

Proof. It is a direct consequence of Proposition 4.4 and Lemma 4.1. □

The number of ways of applying recursively Proposition 4.3 to a vector $w_A = (a_1, \ldots, a_{z-1}, 1, \ldots, 1) \in \mathbb{N}^n$ is given by the next identity (denoted by $f_{n,z}$),

$$f_{n,z} = \begin{cases} \sum_{i=z-1}^{n} f_{n-1,i}, & \text{if } z > 1, \\ \sum_{i=1}^{n} f_{n-1,i}, & \text{if } z = 1, \end{cases} \tag{4.8}$$

where it is included the trivial move $w_{A+0} = w_A$, for $n > 1$, and any $z \in \{1, \ldots, n+1\}$. In fact, we represent these numbers by the following triangle

$$\begin{array}{lllll} f_{1,2} & f_{1,1} & & & \\ f_{2,3} & f_{2,2} & f_{2,1} & & \\ f_{3,4} & f_{3,3} & f_{3,2} & f_{3,1} & \\ f_{4,5} & f_{4,4} & f_{4,3} & f_{4,2} & f_{4,1} \\ \vdots & & & & \ddots \end{array} \tag{4.9}$$

for any vector as before. Since the first possibilities are $v_{1,1} = (1)$ and $v_{1,2} = (2)$, then $f_{1,1} = 1$ and $f_{1,2} = 1$. The previous triangle appears in the OEIS as A009766 (Catalan triangle [90]). In particular, we generate all positive integral diamonds of $\mathbb{A}_n$ via the seed vectors $v_{n,z}$. For example, for $n = 3$, all vectors that generate positive integral diamonds of $\mathbb{A}_3$ are:

$$\begin{array}{llll} (1,1,1) & (2,1,1) & (3,2,1) & (4,3,2) \\ (1,1,2) & (2,1,2) & (3,2,3) & \\ (1,2,1) & (2,3,1) & (3,5,2) & \\ (1,2,3) & (2,3,4) & & \\ (1,3,2) & (2,5,3) & & \end{array}$$

Let $G = UD \ldots UD \ldots$ be a Dyck path of length $2n$ and let m_i be the number of U's before of i-th D in G, then, G can be written as a vector $v_G = (v_1, \ldots, v_{n-1})$ where $v_i = m_i - i + 1$. If G is the Dyck path shown in Figure 4.1 then G has associated the vector $v_G = (5, 4, 3, 3, 5, 4, 3, 2)$.

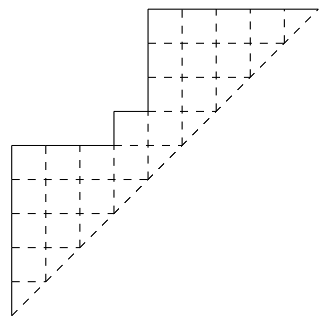


FIGURE 4.1. Dyck path of length 18.

If we take the case $n = 3$, all the vectors are

$$\begin{array}{cccc} (1,1,1) & (2,1,1) & (3,2,1) & (4,3,2) \\ (1,1,2) & (2,1,2) & (3,2,2) & \\ (1,2,1) & (2,2,1) & (3,3,2) & \\ (1,2,2) & (2,2,2) & & \\ (1,3,2) & (2,3,2) & & \end{array}$$

Note that, the number of generating vectors is given by the Catalan numbers.

In what follows, it is defined a map between the vectors associated to positive integral diamonds of $\mathbb{A}_n$ and Dyck paths by using a relation over the coordinates of a vector $u = (a_1, \ldots, a_m)$. The map T_i is defined in such a way that, $T_i : \mathbb{N}^m \to \mathbb{N}$ and:

- If $a_i - a_k > 0$ for some $k \in \{1, \ldots, i\}$, we take max $\{k\}$ and we write $r_1 = a_i - a_k$. Again, we take max $\{k\}$ such that $r_1 - a_k > 0$ and we write $r_2 = r_1 - a_k$, this process ends when there is no a k such that $r_t - a_k > 0$, then, $T_i(u) = r_t + t$. for some t.
- If $a_i - a_k \leq 0$ for all $k \in \{1, \ldots, i\}$, then $T_i(u) = a_i$.

For instance, we take a vector $u = (14, 52, 4, 23, 9, 2)$, then $T_1(u) = 14$, $T_2(u) = 13$, $T_3(u) = 4$, $T_4(u) = 8$, $T_5(u) = 3$, and $T_6(u) = 2$.

Proposition 4.6. *Let $v_{n,z}$ be a seed vector, then $(T_1(v_{n,z}), \ldots, T_n(v_{n,z}))$ describes a Dyck path of length $2(n + 1)$.*

Proof. For any $z \in \{1, \ldots, n+1\}$, $T_i(v_{n,z}) = a_i$ with a_i given by identity (4.5), then there is a word $G_{v_{n,z}} = w_1 \ldots w_{2(n+1)} \in \{U, D\}^*$ such that

$$G_{v_{n,z}} = \underbrace{U \ldots U}_{z-1}\underbrace{D \ldots D}_{z-1} UDUD \ldots UDUD, \tag{4.10}$$

for any left factor u_s in $G_{v_{n,z}}$ of length $s \in \{1, \ldots, 2(n + 1)\}$, $0 \leq |u_s|_U - |u_s|_D \leq z - 1$, therefore $G_{v_{n,z}} \in \mathfrak{D}_{2(n+1)}$. $\square$

Proposition 4.7. *Let $v_A = (a_1, \ldots, a_n)$ be a vector associated to a positive integral diamond A of $\mathbb{A}_n$ with $a_n = 1$, such that $(T_1(v_A), \ldots, T_n(v_A))$ describes a Dyck path in $\mathfrak{D}_{2(n+1)}$. Then $(T_1(v_{A+i}), \ldots, T_n(v_{A+i}))$ describes a Dyck path in $\mathfrak{D}_{2(n+1)}$.*

Proof. Let $v_A = (a_1, \ldots, a_n)$ be a vector associated to a positive integral diamond A with $a_n = 1$, then there exists a Dyck path $G_{v_A} \in \mathfrak{D}_{2(n+1)}$ such that any left factor u_s of length s satisfies $|u_s|_U \geq |u_s|_D$ for $1 \leq s \leq 2(n+1)$. Let v_{A+i} be a vector associated to the positive integral diamond $A + i$ with

$$T_m(v_{A+i}) = \begin{cases} T_m(v_A), & \text{if } 1 \leq m \leq i, \\ T_m(v_A) + 1, & \text{if } m = i + 1, \\ T_{m-1}(v_A), & \text{if } m > i + 1, \end{cases} \tag{4.11}$$

then there is a word $G_{A+i} = w'_1, \ldots, w'_{2(n+1)}$ in $\{U, D\}^*$, we take the index m_1 of the i-th D in G_{A+i}, any left factor u'_s in G_{A+i} satisfies the identities

$$|u'_s|_U = \begin{cases} |u_s|_U, & \text{if } 1 \leq s \leq m_1, \\ |u_{m_1}|_U + 1, & \text{if } s = m_1 + 1, \\ |u_{s-2}|_U + 1, & \text{if } s \geq m_1 + 2, \end{cases} \tag{4.12}$$

and

$$|u'_s|_D = \begin{cases} |u_s|_D, & \text{if } 1 \leq s \leq m_1, \\ |u_{m_1}|_D, & \text{if } s = m_1 + 1, \\ |u_{s-2}|_U + 1, & \text{if } s \geq m_1 + 2, \end{cases} \tag{4.13}$$

then, we have the following possibilities:

- If $1 \leq s \leq m_1$, $|u'_s|_U = |u_s|_U \geq |u_s|_D = |u'_s|_D$.
- If $s = m_1 + 1$, $|u'_{m_1+1}|_U = |u_{m_1}|_U + 1 > |u_{m_1}|_D = |u'_{m_1+1}|_D$.
- If $m_1 + 2 \leq s \leq 2(n+1)$, $|u'_s|_U = |u_{s-2}|_U + 1 \geq |u_{s-2}|_D + 1 = |u'_s|_D$.

Therefore, $G_{A+i} \in \mathfrak{D}_{2(n+1)}$. □

Lemma 4.2. *There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of $\mathbb{A}_n$ and the set of all Dyck paths of length $2(n+1)$.*

Proof. Let $\mathbb{D}_{\mathbb{A}_n}$ be the set of all vectors associated to positive integral diamonds of $\mathbb{A}_n$ and let $\mathfrak{D}_{2(n+1)}$ be the set of all Dyck paths of length $2(n+1)$, then, we define a map $f : \mathbb{D}_{\mathbb{A}_n} \to \mathfrak{D}_{2(n+1)}$ with $f(u_A) = \big(T_1(u_A), \ldots, T_n(u_A)\big)$, Propositions 4.6 and 4.7 allow us to establish that f is well defined. We should prove that the map f is one to one. Suppose that u_A different from v_B, we take the minimum k such that $u_k \neq v_k$. If $k = 1$ then $T_1(u_A) \neq T_1(v_B)$. If $k > 1$, $u_k = m(u_{k-1}) + a$ and $v_k = m'(u_{k-1}) + a$ with $m \neq m'$ is a consequence of Proposition 4.3 then $r_{t_{u_k}} \neq r_{t_{v_k}}$, therefore $T_k(u_A) \neq T_k(v_B)$. □

Figure 4.2 shows a positive integral diamond of $\mathbb{A}_4$ and its corresponding Dyck path.

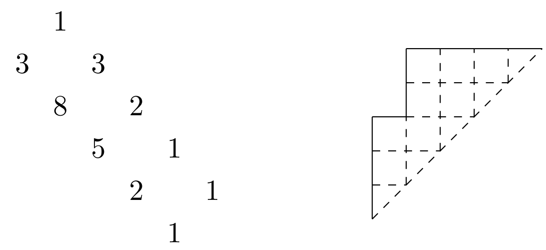


FIGURE 4.2. Diamond (left) and its corresponding Dyck path (right).

An alternative way of writing a Dyck path $G \in \mathfrak{D}_{2(n+1)}$ can be defined by using a vector $\lambda_G = (\lambda_1, \ldots, \lambda_n)$ where λ_i is the number of D's before of $(n+2-i)$-th U in G (see [64,92]), for example, Dyck path of Figure 4.2 has associated the following vector $\lambda_G = (1, 1, 0, 0)$. In the case $n = 3$, all vectors are

$$\begin{array}{cccc}
(3,2,1) & (3,2,0) & (3,0,0) & (0,0,0) \\
(2,2,1) & (2,2,0) & (2,0,0) & \\
(3,1,1) & (3,1,0) & (1,0,0) & \\
(2,1,1) & (2,1,0) & & \\
(1,1,1) & (1,1,0) & &
\end{array}$$

Let λ be a vector associated to a Dyck path of length $2(n+1)$, a triangulation of an $n+3$ polygon can be defined through the use of λ as follows:

- Fix a labeling in the vertices of polygon $K_0^{n+3} = (v_0^{n+3}, \ldots, v_{n+2}^{n+3})$ with $v_i^{n+3} = i$, for $0 \leq i \leq n+2$.
- For λ_i, we draw a diagonal $l_i^{\lambda_i}$ between λ_i and $\lambda_i + 2$. After that, we label the last polygon with $n+3-i$ vertices $K_i^{n+3-i} = (v_0^{n+3-i}, \ldots, v_{n+2-i}^{n+3-i})$, and

$$v_j^{n+3-i} = \begin{cases} v_j^{n+3-(i-1)}, & \text{if } j \leq \lambda_i, \\ v_{j+1}^{n+3-(i-1)} - 1, & \text{if } j > \lambda_i, \end{cases}$$

for $i = 1, \ldots, n$.

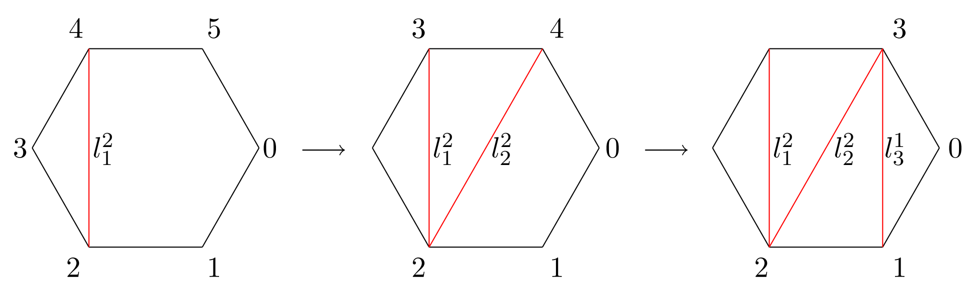


FIGURE 4.3. Triangulation of an hexagon.

The previous algorithm describes that if $l_i^{\lambda_i}$ is a diagonal then it does not cross the diagonals $l_1^{\lambda_1}, \ldots, l_{i-1}^{\lambda_{i-1}}$ for $1 \leq i \leq n$. For instance, let $\lambda_G = (2, 2, 1)$ be the vector associated to $G = UDUDUUDD$, then the triangulation of λ_G is shown in Figure 4.3.

If we fix a labeling K over all vertices of a polygon with $n+3$ vertices, a triangulation T is written as a sequence $T = (l_1^{v_1}, \ldots, l_n^{v_n})$, where v_i belongs to the set of vertices.

Lemma 4.3. *There is a bijective correspondence between the set of all triangulations of a polygon with $n+3$ vertices and the set of all Dyck paths of length $2(n+1)$.*

Proof. Let $\mathcal{T}_n$ be the set of all triangulations of a polygon with $n+3$ vertices, then, we can define a map $g : \mathfrak{D}_{2(n+1)} \to \mathcal{T}_n$ with $g(\lambda) = T_\lambda$. We should prove that g is one to one. Fix a labeling K and suppose $g(\lambda_G) = g(\sigma_{G'})$, then $(l_1^{\lambda_1}, \ldots, l_n^{\lambda_n}) = (l_1^{\sigma_1}, \ldots, l_n^{\sigma_n})$, provided that $l_j^{\lambda_j} = l_j^{\sigma_j}$, there are diagonals $\lambda_j \to (\lambda_j + 2)$ and $\sigma_j \to (\sigma_j + 2)$, therefore $\lambda_j = \sigma_j$ for $j = 1, \ldots, n$. $\square$

The next theorem presents the main result regarding the positive integral diamonds of $\mathbb{A}_n$ and the triangulations of an $n+3$ polygon.

Theorem 4.2. *There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of $\mathbb{A}_n$ and triangulations of a polygon with $n+3$ vertices.*

Proof. We fix a labeling K in a polygon with $n+3$ vertices, the map $F : \mathbb{D}_{\mathbb{A}_n} \to \mathcal{T}_n$ defined by $F(v_A) = (g \circ f)(v_A)$ is a bijection (Lemmas 4.2 and 4.3). $\square$

Figure 4.4 presents the bijective correspondence between a positive integral diamond of $\mathbb{A}_4$, a Dyck path of length 10, and a triangulation of a polygon with 7 vertices.

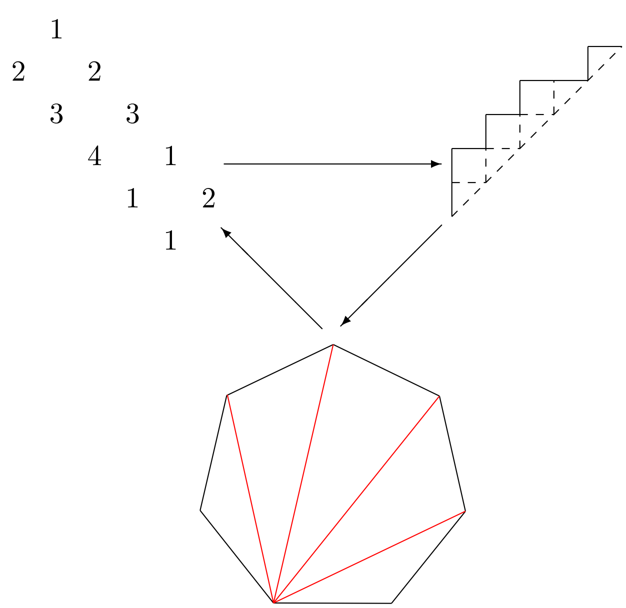


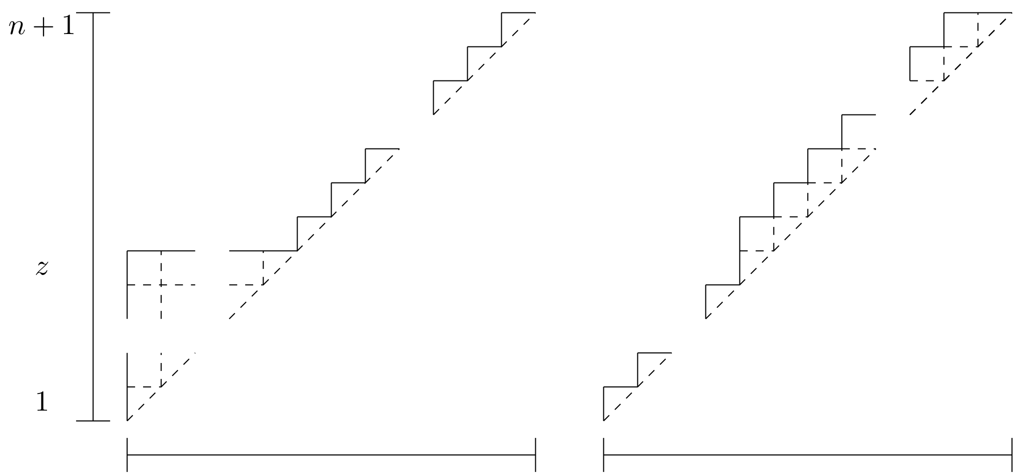
FIGURE 4.4. Connection via the map F.

4.1.3 Frieze Patterns and Dyck Paths

In this section, we describe an algebraic interpretation of frieze patterns as a direct sum of Dyck Paths.

Lemma 4.4. *The vectors $v_{n,z}$ and $v^{n,z}$ generate the same triangulation except for one anti-clockwise rotation.*

Proof. Let $v_{n,z}$ and $v^{n,z}$ be frieze vectors, fixed a labeling K_1 in an $n+3$ polygon, applying map F



$(a)\ f(v_{n,z})$ $(b)\ f(v^{n,z})$

$F(v_{n,z}) = (\underbrace{n}_{1}, \dots, \underbrace{z}_{n-z}, \underbrace{0}_{n-z+1}, \dots, \underbrace{0}_{n})$ and $F(v^{n,z}) = (\underbrace{n-1}_{1}, \dots, \underbrace{z-1}_{n-z}, \underbrace{z-1}_{n-z+1}, \dots \underbrace{1}_{n})$, if we change K_1 by K_2 in the following way:

- the vertex $k \in K_1$ is $k-1 \in K_2$ for $1 \leq k \leq n+2$,
- the vertex $0 \in K_1$ is $n+2 \in K_2$,

the diagonals from 0 to r_1 in K_1 are diagonals from $r_1 - 1$ to $n+2$ in K_2, and the diagonals from r_2 in K_1 are diagonals from $r_2 - 1$ in K_2, for $0 \leq r_1 \leq z \leq r_2 \leq n$. Therefore $F(v_{n,z}) \in K_1$ is equal to $F(v^{n,z}) \in K_2$. $\square$

Note that, there exists a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n-z-1 & n-z & n-z+1 & n-z+2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-z-1 & n-z & n & n-1 & \dots & n-z+2 & n-z+1 \end{pmatrix},$$

in S_n that describes a bijection between the coordinates of the vector $F(v_{n,z}) = (u_1, \dots, u_n)$ and the vector $F(v^{n,z}) = (u'_1, \dots, u'_n)$ such that $\sigma(F(v_{n,z})) = (u_{\sigma(1)}, \dots, u_{\sigma(n)}) = (u'_1, \dots, u'_n) = F(v^{n,z})$ in K_2. In general, if v and w generate the same triangulation except for one anti-clockwise rotation, then there exists a permutation $\sigma' \in S_n$ such that $\sigma'(F(v)) = F(w)$ in K_2.

Lemma 4.5. *Let A and B be positive integral diamonds of $\mathbb{A}_n$, A and B are a couple, and $v_A = (a_1, \ldots, a_z, \ldots, a_n)$ with $a_t = 1$ for $z \leq t \leq n$. If v_A and v_B generate the same triangulation except for one anti-clockwise rotation. Then $v_{A+i} = (a_1, \ldots, a_{i-1}, a_{i-1} + a_i, a_{i+1}, \ldots, a_{n-1})$ and $v_{B+i-1} = (b_1, \ldots, b_{i-2}, b_{i-2}+b_{i-1}, b_{i-1}, \ldots, b_{n-1})$ generate the same triangulation except for one anti-clockwise rotation for $z-1 \leq i \leq n$, $i \geq 2$.*

Proof. Let v_A and v_B are vectors, since v_A and v_B generate the same triangulation except for one anti-clockwise rotation, then, there exists a permutation $\sigma \in S_n$ such that $\sigma(F(v_A)) = F(v_B)$ in K_2. The following options arise from the map f, such that:

(1) If $i > z \geq 1$, $f(v_A) = (\ldots, \underbrace{1}_{i-1}, \underbrace{1}_{i}, \ldots)$, and $f(v_B) = (\ldots, \underbrace{d}_{i-2}, \underbrace{2}_{i-1}, \underbrace{2}_{i}, \ldots)$ (see Figure 4.5)

(1.1) If $d = 1$, $F(v_A) = (\ldots, \underbrace{i}_{n-i}, \underbrace{i-1}_{n+1-i}, \ldots)$, $F(v_B) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-2}_{n+1-i}, \ldots)$, and σ satisfies the expression,

$$\sigma(r) = \begin{cases} r, & \text{if } r \leq n+1-i, \\ m, & \text{otherwise,} \end{cases} \tag{4.14}$$

for some $m > n+1-i$. Applying F to v_{A+i} and v_{B+i-1},

$$F(v_{A+i}) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-1}_{n+1-i}, \ldots), \text{ and } F(v_{B+i-1}) = (\ldots, \underbrace{i-2}_{n-i}, \underbrace{i-2}_{n+1-i}, \ldots),$$

then there exits $\sigma' \in S_n$ such that $\sigma' = \sigma$ and $\sigma'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.6).

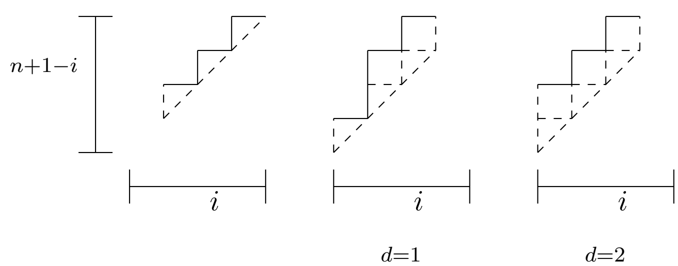


FIGURE 4.5. Dyck paths associated to v_A and v_B for $i > z$.

(1.2) If $d = 2$, this case is equal to the previous case.

(1.3) If $d = 3$, A and B do not generate the same triangulation.

Note that, if $z = 1$, this case satisfies the condition (1.1) and (1.2) without d.

(2) If $i = z \geq 2$, $f(v_A) = (\ldots, \underbrace{2}_{i-1}, \underbrace{1}_{i}, \ldots)$ and $f(v_B) = (\ldots, \underbrace{b}_{i-2}, \underbrace{a}_{i-1}, \underbrace{2}_{i}, \ldots)$ (see Figure 4.7).

(2.1) If $a = 1$ and $b = 1$, $F(v_A) = (\ldots, \underbrace{i}_{n-i}, \ldots)$, $F(v_B) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-1}_{n+1-i}, \ldots)$ and σ_1 is equal to

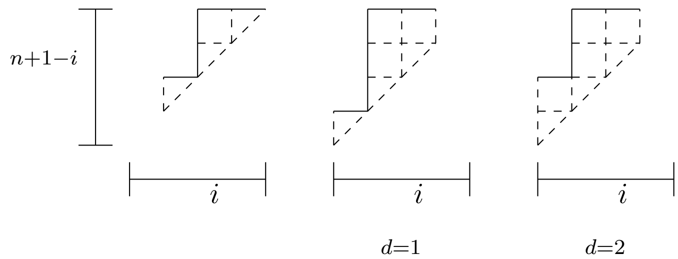


FIGURE 4.6. Dyck paths associated to v_{A+i} and v_{B+i-1} for $i > z$.

$$\sigma_1(r) = \begin{cases} r, & \text{if } r \leq n-i, \\ n+1-i, & \text{if } r = n, \\ m, & \text{otherwise,} \end{cases} \tag{4.15}$$

for some $m > n+1-i$. Applying F, we take

$$F(v_{A+i}) = (\dots, \underbrace{i-1}_{n-i}, \dots) \text{ and } F(v_{B+i-1}) = (\dots, \underbrace{i}_{n-i}, \underbrace{i-2}_{n+1-i}, \dots),$$

then there exits $\sigma_1' \in S_n$ that satisfies

$$\sigma_1'(r) = \begin{cases} n-i, & \text{if } r = n, \\ n+1-i, & \text{if } r = n-i, \\ \sigma_1(r), & \text{otherwise,} \end{cases} \tag{4.16}$$

therefore $\sigma_1'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.8).

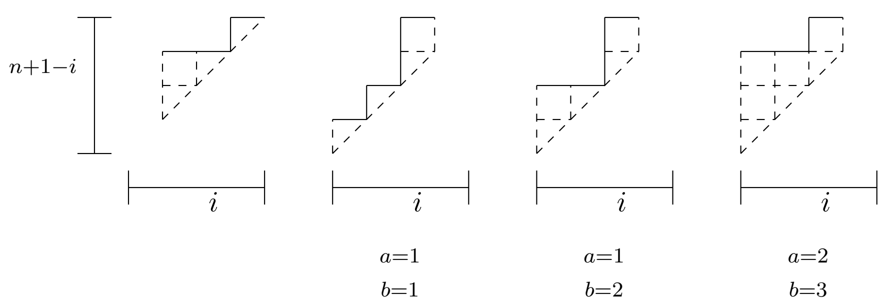


FIGURE 4.7. Dyck paths associated to v_A and v_B for $i = z$.

(2.2) If $a = 1$ and $b = 2$, this case satisfies the conditions of (2.1).

(2.3) If $a = 2$ and $b = 1$ or $b = 2$, these cases are contradictions.

(2.4) If $a = 2$ and $b = 3$, $F(v_B) = (\dots, \underbrace{i-1}_{n-i}, \dots)$ and $\sigma_2 = \sigma$. Applying F to v_{B+i-1}, it holds that $F(v_{B+i-1}) = (\dots, \underbrace{i-2}_{n-i}, \dots)$ then there exits $\sigma_2' \in S_n$ such that $\sigma_2' = \sigma$ and $\sigma_2'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.8).

(2.5) If $a = 3$ and $b = 1, 2, 3$, these cases are equal to case (2.3).

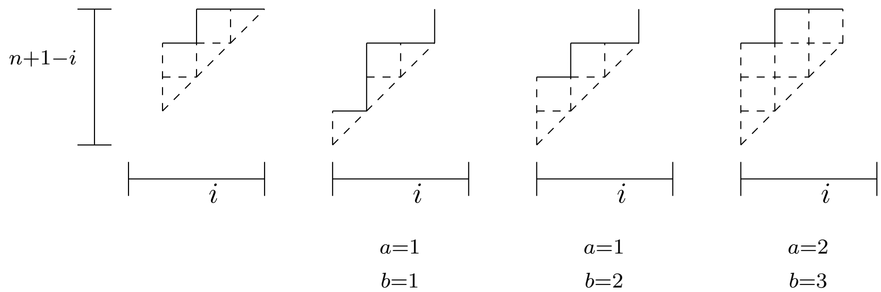


FIGURE 4.8. Dyck paths associated to v_{A+i} and v_{B+i-1} for $i = z$.

Note that, if $z = 2$, this case satisfies the same conditions for $a = 1, 2$ without b.

(3) If $i = z - 1 \geq 3$, $f(v_A) = (\dots, \underbrace{2}_{i}, \underbrace{1}_{i+1}, \dots)$ and $f(v_B) = (\dots, \underbrace{b}_{i-1}, \underbrace{a}_{i}, \underbrace{2}_{i+1}, \dots)$ (see Figure 4.9).

(3.1) If $a = 1$ and $b = 1$, $F(v_A) = (\dots, \underbrace{i+1}_{n-i-1}, \dots)$, $F(v_B) = (\dots, \underbrace{i}_{n-i-1}, \underbrace{i}_{n-i}, \underbrace{i-1}_{n+1-i}, \dots)$, and

$$\sigma_3(r) = \begin{cases} r, & \text{if } r \leq n-i-1, \\ n-i, & \text{if } r = n, \\ n+1-i, & \text{if } r = n-1, \\ m, & \text{otherwise}, \end{cases} \tag{4.17}$$

for some $m > n + 1 - i$. Provided that

$$F(v_{A+i}) = (\dots, \underbrace{i-1}_{n-i-1}, \dots) \text{ and } F(v_{B+i-1}) = (\dots, \underbrace{i+1}_{n-i-1}, \underbrace{i}_{n-i}, \underbrace{i-2}_{n+1-i}, \dots),$$

then, there exits $\sigma_3' \in S_n$ such that

$$\sigma_3'(r) = \begin{cases} n-i-1, & \text{if } r = n, \\ n-i, & \text{if } r = n-1, \\ n+1-i, & \text{if } r = n-i-1, \\ \sigma_3(r), & \text{otherwise}, \end{cases} \tag{4.18}$$

then $\sigma_3'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.10).

(3.2) If $a = 1$ and $b = 2$. Applying F to v_B, it holds that $F(v_B) = (\dots, \underbrace{i}_{n-i-1}, \underbrace{i}_{n-i}, \dots)$, σ_4 is described by

$$\sigma_4(r) = \begin{cases} r, & \text{if } r \leq n-i-1, \\ n-i, & \text{if } r = n, \\ m, & \text{otherwise}, \end{cases} \tag{4.19}$$

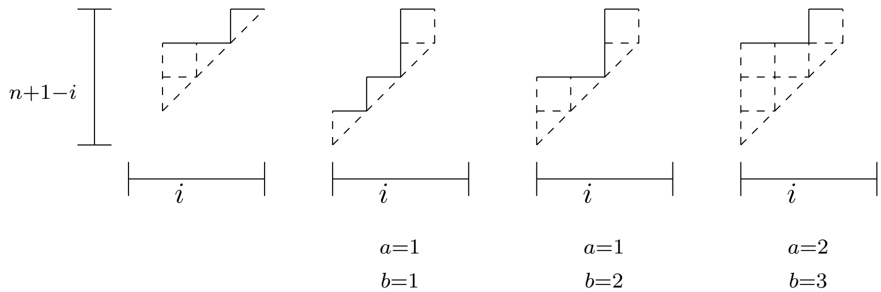


FIGURE 4.9. Dyck paths associated to v_A and v_B for $i-1=z$.

for some $m > n-i$. Applying F to v_{B+i-1}, $F(v_{B+i-1}) = (\dots, \underbrace{i+1}_{n-i-1}, \underbrace{i-2}_{n-i}, \dots)$, then there exist σ_4' with

$$\sigma_4'(r) = \begin{cases} n-i-1, & \text{if } r \leq n, \\ n-i, & \text{if } r = n-i-1, \\ \sigma_4(r), & \text{otherwise,} \end{cases} \tag{4.20}$$

therefore $\sigma_4'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.10).

(3.3) If $a=2$ and $b=3$. $F(v_B) = (\dots, \underbrace{i}_{n-i-1}, \dots)$, since σ_5 is

$$\sigma_5(r) = \begin{cases} r, & \text{if } r \leq n-i-1, \\ m, & \text{otherwise,} \end{cases} \tag{4.21}$$

for some $m > n-i$. Applying F to v_{B+i-1}, $F(v_{B+i-1}) = (\dots, \underbrace{i-2}_{n-i-1}, \dots)$, and there exits $\sigma_5' = \sigma_5$ such that $\sigma_5'(F(v_{A+i})) = F(v_{B+i-1})$ in K_2 (see Figure 4.10).

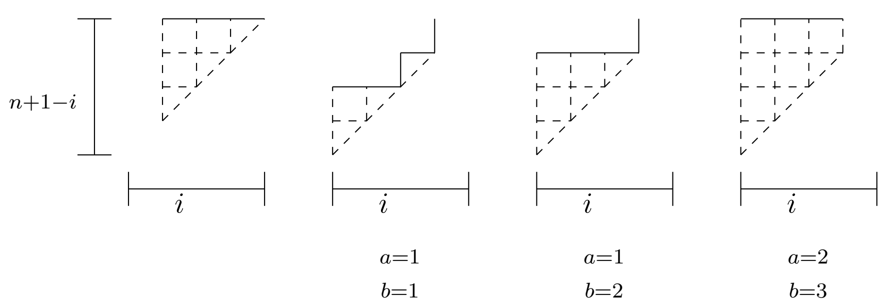


FIGURE 4.10. Dyck paths associated to v_{A+i} and v_{B+i-1} for $i-1=z$.

Same arguments are used for the remaining cases (see item (2) of this proof). □

Proposition 4.8. *Two positive integral diamonds of $\mathbb{A}_n$ are in the same minimal $p-$cycle if their triangulations are in the same mutation class.*

Proof. It is a direct consequence of Theorem 1.12, Proposition 4.3, Lemmas 4.4 and 4.5 □

The following result shows a way to build frieze patterns.

Theorem 4.3. *Let A^0 be a positive integral diamond of $\mathbb{A}_n$ and let $\{A^t\}_{0\leq t\leq p-1}$ be the minimal $p-$cycle generated by A^0. Then:*

(i) A^0 and $F(v_{A^0})$ generate the same frieze pattern.

(ii) $\{A^t\}_{0\leq t\leq p-1}$ is in surjective correspondence with a direct sum of p indecomposable objects of a Dyck paths category.

Proof. Let $\mathbb{D}_{\mathbb{A}_n}$ be the set of all vectors associated to positive integral diamonds of $\mathbb{A}_n$, let A^0 be a positive integral diamond of $\mathbb{A}_n$, and let $\{A^t\}_{0\leq t\leq p-1}$ be the minimal $p-$cycle generated by A^0.

(i) Let K be a labeling of an $(n+3)$ polygon, Theorem 4.2 implies that

$$\begin{aligned} F(v_{A^0}) &= g((a^0_{11}, T_2(v_{A^0}), \dots, T_n(v_{A^0}))) \\ &= g(\lambda_{(a^0_{11},T_2(v_{A^0}),\dots,T_n(v_{A^0}))}) \\ &= g((\lambda_1, \dots, \lambda_{n+1-a^0_{11}}, \underbrace{0,\dots,0}_{a^0_{11}})) \\ &= (l_1^{v_1}, \dots, l_{n+1-a^0_{11}}^{v_{n+1-a^0_{11}}}, l^0_{n-a^0_{11}}, \dots, l^0_n), \end{aligned} \tag{4.22}$$

then, there are $a^0_{11}-1$ diagonals from the vertex 0 to other vertices, i.e., there are a^0_{11} triangles incident with vertex 0. Proposition 4.8 allows us to establish that a^i_{11} is the number of triangles incident with the vertex i, for $1\leq i\leq n+3$, $i=pm$ and $1\leq m\leq p \mid (n+3)$. Therefore A^0 and $F(v_{A^0})$ generate the same frieze pattern.

(ii) Let $(\mathfrak{D}_{2(n+1)}, R)$ be any Dyck paths category, we take objects of $(\mathfrak{D}_{2(n+1)}, R)$ defined by the following identity

$$\overline{Ob}\ (\mathfrak{D}_{2n}, R) = \left\{ \bigoplus_{G_i\in\mathfrak{D}_{2n}} G_i \;\middle|\; g(\lambda_{G_i}) \text{ and } g(\lambda_{G_j}) \text{ are in the same mutation class} \right\}, \tag{4.23}$$

we define the map $\varphi : \mathbb{D}_{\mathbb{A}_n} \to \overline{Ob}\ (\mathfrak{D}_{2n}, R)$, such that

$$\varphi(v_{A^0}) = f(v_{A^0}) \oplus \cdots \oplus f(v_{A^{p-1}}),$$

with $\{A^t\}_{0\leq t\leq p-1}$, φ is surjective as a consequence of Theorem 4.2 and Proposition 4.8. □

For example, let D be an object of any Dyck paths category $(\mathfrak{D}_{2(n+1)}, R)$ shown in Figure 4.11,

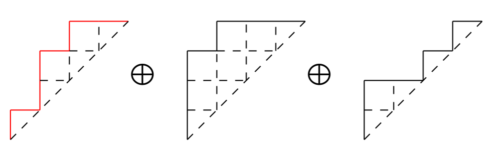


FIGURE 4.11. Examples of objects in a Dyck paths category.

then, D has associated the following frieze pattern

$$\begin{array}{ccccccccccccc}
 & \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
\dots & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & \dots \\
 & \dots & & 2 & & 5 & & 1 & & 2 & & 5 & & \dots \\
\dots & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & \dots \\
 & \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots
\end{array}$$

4.2 Dyck-Brauer Configuration Algebras

In this section, we present a Brauer configuration and its Brauer configuration algebra whose indecomposable projective modules are in bijective correspondence with Dyck paths.

4.2.1 Brauer Configuration and its Brauer Configuration Algebra Associated to Dyck Paths

For n fixed, let $M_n = \{\alpha_{n_1}^{n_2}\}_{0\leq n_1\leq n-1,\ n_1<n_2\leq n}$ and $N_n = \{\beta_{n_1}^{n_2}\}_{0\leq n_1\leq n-1,\ n_1<n_2\leq n}$ be the sets of letters, we define an alphabet Γ_0^n such that

$$\Gamma_0^n = \{\delta \mid \delta \in M_n \text{ or } \delta \in N_n\}. \tag{4.24}$$

For Γ_0^n, we define a concatenation c in the following way:

$$c(\delta) = \begin{cases} \alpha_i^{j+1}, & \text{if } \delta = \alpha_i^j, \\ \beta_i^j, & \text{if } \delta = \alpha_i^j, \\ \alpha_{i+1}^{j+1}, & \text{if } \delta = \beta_i^j, \\ \beta_{i+1}^j, & \text{if } \delta = \beta_i^j, \end{cases} \tag{4.25}$$

for some t,i and j. We take the set of the words $V = \delta_1 \dots \delta_{2n}$ where $\delta_1 = \alpha_0^1$ and $\delta_j = c(\delta_{j-1})$ (this set is denoted by Γ_1^n). We will say that $V <_{\mathcal{O}} V'$ if and only if there exist $r \in \mathbb{Z}_{>0}$ such that

$$\begin{array}{l} N_V^{r_g} = N_{V'}^{r_g} \text{ if } 0 < r_g < r, \\ N_V^{r_g} > N_{V'}^{r_g} \text{ if } r_g = r, \end{array} \tag{4.26}$$

where $N_V^{r_g}$ is the number of alpha words before of the r_g-beta word in V (($\Gamma_1, <_{\mathcal{O}}$) is a linear order. For notation, the words V in Γ_1 are labeling respect to $<_{\mathcal{O}}$, i.e., $V_1 = \alpha_0^1 \dots \alpha_0^n \beta_0^n \dots \beta_{n-1}^n$, $V_2 = \alpha_0^1 \dots \alpha_0^{n-1} \beta_0^{n-1} \alpha_1^n \beta_2^n \dots \beta_{n-1}^n$, etc.

Let $\Gamma_n = (\Gamma_0^n, \Gamma_1^n, \mu, \mathcal{O})$ be a Brauer configuration, where Γ_0^n, Γ_1^n as before, $\mathcal{O}$ is induced by $(\Gamma_1, <_{\mathcal{O}})$, and the multiplicity function μ is defined as follows:

$$\begin{aligned} \mu: \ \Gamma_0 &\longrightarrow \ \mathbb{N} \\ \delta &\longmapsto \ \mu(\delta) = \begin{cases} 1, & \text{if } val(\delta) > 1, \\ 2, & \text{if } val(\delta) = 1, \end{cases} \end{aligned} \tag{4.27}$$

for $\delta \in \Gamma_0$. Γ_n is called a Dyck-Brauer configuration. The occurrence of $\delta = \alpha_{n_1}^{n_2}$ (resp. $\delta = \beta_{n_1}^{n_2}$) in a polygon V is given by the row $n(n_1) + n_2$ in the recurrence matrix $A_n = (a_{u,w}^n)$ (resp. $B_n = (b_{u,w}^n)$) of $T_n \times f_{n,2}$ where $A_1 = (1)$ (resp. $B_1 = (1)$) and A_n (resp. B_n) is described in Table A.5 (see identity A.2).

We define paths $a_{j_1}^i \dots a_{j_k}^i$ (resp. $b_{j'_1}^{i'} \dots b_{j'_{k'}}^{i'}$) where $\{j_1, \dots, j_k\}$ (resp. $\{j'_1, \dots, j'_{k'}\}$) are indices of the matrix A_n (resp. B_n) such that $a_{i,j_r} = 1$ (resp. $b_{i',j'_{r'}} = 1$) with $j_r < j_{r+1}$ (resp. $j'_{r'} < j'_{r'+1}$) for $1 \leq j_r$(resp. $j'_{r'}$) $\leq f_{n,2}$ and $1 \leq i \leq T_n$. If $val(\delta) = 1$, the path is equal to $a_j^i a_j^i$ (resp. $b_{j'}^{i'} b_{j'}^{i'}$). Paths $a_{j_1}^i \dots a_{j_k}^i$ and $b_{j'_1}^{i'} \dots b_{j'_{k'}}^{i'}$ induce special δ_i-cycles ν_i at v_t in such a way that:

$$\nu_i = \begin{cases} a_t^i \dots a_{j_k}^i a_{j_1}^i \dots a_{t-1}^i, & \text{if } a_t^i = 1 \text{ and } \delta \in M_n, \\ b_t^i \dots b_{j'_{k'}}^i b_{j'_1}^i \dots b_{t-1}^i, & \text{if } b_t^i = 1 \text{ and } \delta \in N_n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.28}$$

In the same way, the relations in ρ_{Γ_n} are described by the following cases:

- Relations of type I.

$$\begin{gathered} a_{j_{s_1}}^{i_1} \dots a_{j_{k_1}}^{i_1} a_{j_1}^{i_1} \dots a_{j_{s_1-1}}^{i_1} = \dots = a_{j_{s_t}}^{i_t} \dots a_{j_{k_t}}^{i_t} a_{j_1}^{i_t} \dots a_{j_{s_t-1}}^{i_t}, \\ a_{j_{s_t}}^{i_t} \dots a_{j_{k_t}}^{i_t} a_{j_1}^{i_t} \dots a_{j_{s_t-1}}^{i_t} = b_{j'_{x_1}}^{i'_1} \dots b_{j'_{k'_1}}^{i'_1} b_{j'_1}^{i'_1} \dots b_{j'_{x_1-1}}^{i'_1}, \\ b_{j'_{x_1}}^{i'_1} \dots b_{j'_{k'_1}}^{i'_1} b_{j'_1}^{i'_1} \dots b_{j'_{x_1-1}}^{i'_1} = \dots = b_{j'_{x_h}}^{i'_h} \dots b_{j'_{k'_h}}^{i'_h} b_{j'_1}^{i_h} \dots b_{j'_{x_h-1}}^{i'_h}, \end{gathered} \tag{4.29}$$

  if $j_{s_1} = \dots = j_{s_t} = j'_{x_1} = \dots = j'_{x_h}$ for $1 \leq j_{s_1} \leq f_{n,2}$, $\{i_1, \dots, i_t, i'_1, \dots, i'_h\} \in \{1, \dots, T_n\}$ and $t, h \in \mathbb{Z}_{>0}$.

- Relations of type II.

$$\begin{gathered} a_{j_{s_1}}^{i_r} \dots a_{j_{k_1}}^{i_r} a_{j_1}^{i_r} \dots a_{j_{s_1-1}}^{i_r} a_{j_{s_1}}^{i_r}, \\ b_{j'_{x_1}}^{i'_p} \dots b_{j'_{k'_1}}^{i'_p} b_{j'_1}^{i'_p} \dots b_{j'_{x_1-1}}^{i'_p} b_{j'_{x_1}}^{i'_p}, \end{gathered} \tag{4.30}$$

  for some $i_r \in \{i_1, \dots, i_t\}$ ($i'_p \in \{i'_1, \dots, i_h\}$)

- Relations of type III.

$$a_{j_s}^{i_r} a_{j_x}^{i_p}, \ a_{j_s}^{i_r} b_{j'_x}^{i'_p}, \ b_{j'_x}^{i'_p} a_{j_s}^{i_r}, \ b_{j'_s}^{i'_r} b_{j'_s}^{i'_p}, \tag{4.31}$$

  for all possible combinations.

I_{Γ_n} is generated by ρ_{Γ_n} and *Dyck-Brauer configuration algebra* Λ_{Γ_n} is defined by $kQ_{\Gamma_n}/I_{\Gamma_n}$. For instance, let $\Gamma_2 = (\Gamma_0^2, \Gamma_1^2, \mu, \mathcal{O})$ be the Dyck-Brauer configuration where $\Gamma_0 =$

$\{\alpha_0^1, \alpha_0^2, \alpha_1^2, \beta_0^1, \beta_0^2, \beta_1^2\}$ and $\Gamma_1 = \{V_1 = \{\alpha_0^1, \alpha_0^2, \beta_0^2, \beta_1^2\},\ V_2 = \{\alpha_0^1, \beta_0^1, \alpha_1^2, \beta_1^2\}\}$ with $V_1 <_{\mathcal{O}} V_2$. Since the matrices A_2 and B_2 are

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then $\mu_{\alpha_0^1} = \mu(\beta_1^2) = 1$ and $\mu(\delta) = 2$ for $\delta \in \{\alpha_0^2, \alpha_1^2, \beta_0^1, \beta_0^2\}$. $V_1 <_{\mathcal{O}} V_2$ is the successor sequence of vertex α_0^1, $V_2 <_{\mathcal{O}} V_1$ is the successor sequence of vertex β_1^2, V_1 (resp. V_2) is the successor sequence of vertices α_0^2 and β_0^2 (resp. α_1^2 and β_0^1). Figure 4.12 shows the quiver Q_{Γ_2}.

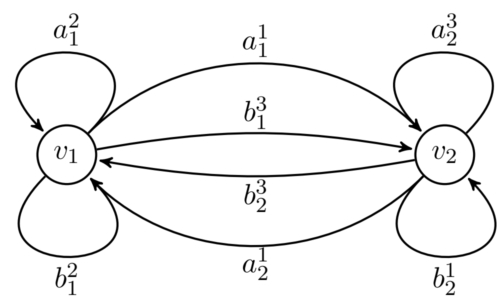


FIGURE 4.12. Quiver of Dyck-Brauer configuration algebra Λ_{Γ_2}.

Identities (4.29), (4.30), and (4.31) induce the following relations,

$$\begin{gathered}
a_1^1a_2^1 = (a_1^2)^2 = (b_1^2)^2 = b_1^3b_2^3,\ a_2^1a_1^1 = (a_2^3)^2 = (b_2^1)^2 = b_2^3b_1^3, \\
a_1^1a_2^1a_1^1,\ (a_1^2)^3,\ (b_1^2)^3,\ b_1^3b_2^3b_1^3,\ a_2^1a_1^1a_2^1,\ (a_2^3)^3,\ (b_2^1)^3,\ b_2^3b_1^3b_2^3, \\
a_1^1a_2^3,\ a_2^3a_2^1,\ a_2^1a_1^2,\ a_1^2a_1^1,\ b_1^3b_2^1,\ b_2^1b_2^3,\ b_2^3b_1^2,\ b_1^2b_1^3, \\
a_1^1b_2^1,\ a_1^1b_2^3,\ a_2^3b_2^1,\ a_2^3b_2^3,\ a_2^1b_1^2,\ a_2^1b_1^3,\ b_1^2b_1^2,\ a_1^2b_2^3, \\
b_1^3a_2^3,\ b_1^3a_2^1,\ b_2^1a_2^3,\ b_1^2a_2^1,\ b_2^3a_1^2,\ b_2^3a_1^1,\ b_1^2a_1^2,\ b_1^2a_1^1.
\end{gathered} \tag{4.32}$$

Dyck-Brauer configuration algebra Λ_{Γ_2} is defined in such a way that $\Lambda_{\Gamma_2} = kQ_{\Gamma_2}/I_{\Gamma_2}$ with $I_{\Gamma_2} = \langle \rho_{\Gamma_2} \rangle$. Figure 4.13 shows the indecomposable projective Λ_{Γ_2}-modules.

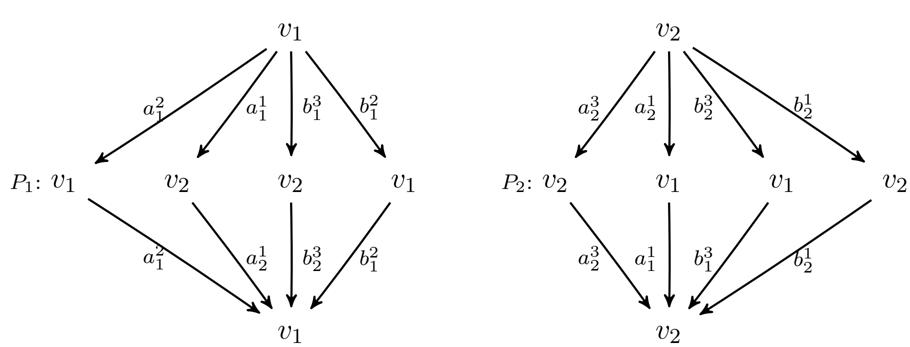


FIGURE 4.13. Indecomposable projective Λ_{Γ_2}-modules of Dyck-Brauer configuration algebra.

4.2.2 Dimension of a Dyck-Brauer Configuration Algebra and the Dimension of its Center

We define a family of integer sequences that are in relationship with the Catalan triangle. We also show that the Catalan triangle allows us to establish the dimension of Dyck-Brauer configuration algebras and the dimension of its corresponding centers.

Let $t^n_{i,j}$ be the integer numbers such that

$$\begin{aligned} t^1_{i,j} &= 1, \\ t^n_{i,j} &= \sum_{\substack{r-s=i-j \\ n-1\leq r\leq i}} t^{n-1}_{r,s} \text{ if } n>1, \\ t^n_{i,j} &= 0 \text{ if } j\leq 0, \end{aligned} \tag{4.33}$$

for $i \geq n-1$ and $j \leq i+1$. For example, Table A.4 shows integer sequences $t^n_{i,j}$ for $n = 2, \ldots, 5$ (see Appendix).

The following results describe some properties of the integer numbers $t^n_{i,j}$ and the Catalan triangle.

Proposition 4.9. $t^n_{i,j} = t^n_{i,j-1} + t^{n-1}_{i-1,j}$, *for* $i \geq 1$, $j \leq i$, *and* $1 < n < i+1$.

Proof. By induction. If $i = 1$, $t^2_{1,1} = t^1_{0,1} + t^2_{1,0} = 1$. Suppose that the proposition holds for $i = k$ and $1 < n < k+1$. Then for $i = k+1$, if $n = 2$,

$$t^2_{k+1,j} = \sum_{\substack{k+1-j=r-s \\ 1\leq r\leq k+1}} t^1_{r,s} = (k-j)+1 = \sum_{\substack{k-j=r-s \\ 1\leq r\leq k+1}} t^1_{r,s} + t^1_{k,j} = t^2_{k+1,j-1} + t^1_{k,j},$$

suppose that the assertion is true for $n = p-1 < k+1$, then

$$\begin{aligned} t^p_{k+1,j} &= \sum_{\substack{k+1-j=r-s \\ p-1\leq r\leq k+1}} t^{p-1}_{r,s} \\ &= \sum_{\substack{k+2-j=r-s \\ p-1\leq r\leq k+1}} t^{p-1}_{r,s} + \sum_{\substack{k-j=r-s \\ p-2\leq r\leq k}} t^{p-2}_{r,s} \\ &= t^p_{k+1,j-1} + t^{p-1}_{k,j}. \end{aligned}$$

□

Proposition 4.10. $t^n_{i,i+1} = t^n_{i,i}$, *for* $i \geq 1$ *and* $1 < n \leq i+1$.

Proof. We proceed by induction. If $i = 1$, $t^2_{1,2} = 1 = t^2_{1,1}$. Suppose that the assertion is true for $i = k$ and $1 < n < k+1$. Then for $i = k+1$, if $n = 2$,

$$t^2_{k+1,k+2} = \sum_{\substack{-1=r-s \\ 1\leq r\leq k+1}} t^1_{r,s} = \sum_{\substack{0=r-s \\ 1\leq r\leq k+1}} t^1_{r,s} = t^2_{k+1,k+1},$$

suppose that the property is true for $n = p-1 < k+2$, then

$$t^{p}_{k+1,k+2} = \sum_{\substack{1=r-s\\ p-1\leq r\leq k+1}} t^{p-1}_{r,s} = \sum_{\substack{0=r-s\\ p-1\leq r\leq k+1}} t^{p-1}_{r,s} = t^{p}_{k+1,k+1}.$$

□

Lemma 4.6. $f_{n,n+1-m} = \sum_{j=1}^{n} t^{m}_{n-1,j}$, *for* $n \geq 1$, $1 \leq m \leq n$.

Proof. By induction. If $n = 1$, $f_{1,1} = 1 = t^{1}_{0,1}$. Suppose that satisfies for $n = k$ and $1 \leq m \leq k$. Then for $n = k+1$, if $m = 1$, $\sum_{j=1}^{k+1} t^{1}_{k,j} = k+1 = f_{k+1,k+1}$, suppose that the property holds for $m = p-1 < k+1$, then

$$\begin{aligned}
\sum_{j=1}^{k+1} t^{p}_{k,j} &= \sum_{\substack{r-s=k-1\\ p-1\leq r\leq k}} t^{p-1}_{r,s} + \cdots + \sum_{\substack{r-s=1\\ p-1\leq r\leq k}} t^{p-1}_{r,s} \\
&= \sum_{\substack{r-s=k-1\\ p-1\leq r\leq k-1}} t^{p-1}_{r,s} + \cdots + \sum_{\substack{r-s=1\\ p-1\leq r\leq k-1}} t^{p-1}_{r,s} + \sum_{j=1}^{k+1} t^{p-1}_{k,j} \\
&= \sum_{j=1}^{k} t^{p}_{k-1,j} + \sum_{j=1}^{k+1} t^{p-1}_{k,j} \\
&= f_{k,k+1-p} + f_{k+1,k+3-p} \\
&= f_{k,k+1-p} + \sum_{i=k+2-p}^{k+1} f_{k,i} \\
&= \sum_{i=k+1-p}^{k+1} f_{k,i} = f_{k+1,k+2-p}.
\end{aligned}$$

□

Proposition 4.11. $f_{n,n+1-m} = t^{1+m}_{n,n+1}$, *for* $n \geq 1$ *and* $1 \leq m \leq n$.

Proof. We proceed by induction. If $n = 1$, $f_{1,1} = 1 = t^{2}_{1,2}$. Suppose that the assertion is true for $n = k$ and $1 \leq m \leq k$. Then for $n = k+1$, if $m = 1$,

$$f_{k+1,k+1} = k+1 = \sum_{\substack{-1=r-s\\ 1\leq r\leq k+1}} t^{1}_{r,s} = t^{2}_{k+1,k+2},$$

suppose that the property holds for $m = p-1 < k+1$, then

$$\begin{aligned}
f_{k+1,k+2-p} &= \sum_{j=1}^{k+1} t^{p}_{k,j} \\
&= \sum_{\substack{r-s=k-1\\ p-1\le r\le k}} t^{p-1}_{r,s} + \cdots + \sum_{\substack{r-s=1\\ p-1\le r\le k}} t^{p-1}_{r,s} \\
&= \sum_{j=1}^{k+1} t^{p-1}_{k,j} + \cdots + \sum_{j=1}^{p} t^{p-1}_{p-1,j} \\
&= f_{k+1,k+1-p} + \cdots + f_{p,1} \qquad \text{(Lemma 4.6)} \\
&= t^{p}_{k+1,k+2} + \cdots + t^{p}_{p,p+1} \\
&= t^{p+1}_{k+1,k+2}.
\end{aligned}$$

□

Given a matrix $C = (c_{i,j})$ of $n \times m$, $M(C)$ is the column vector $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where $c_i = \sum_{j=1}^{m} c_{i,j}$.

Also, $M(C)_r$ is the column vector $\begin{pmatrix} c^r_1 \\ \vdots \\ c^r_n \end{pmatrix}$ such that $c^r_j = \begin{cases} 0, & \text{if } 1 \le j \le r, \\ c_j, & \text{if } r+1 \le j \le n, \end{cases}$ and $\overline{M(C)}_r$ is the column vector $\begin{pmatrix} \overline{c^r_1} \\ \vdots \\ \overline{c^r_n} \end{pmatrix}$ with $\overline{c^r_j} = \begin{cases} 0, & \text{if } 1 \le j \le r, \\ c_{j-r}, & \text{if } r+1 \le j \le n. \end{cases}$ For example,

$$M(A_2)_0 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ M(A_2)_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \overline{M(A_2)}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}. \tag{4.34}$$

Henceforth, we introduce a formula for the valency of the vertices of Dyck-Brauer configuration Γ_n via the vectors $M(A_n)$ and $M(B_n)$, in the following way:

Lemma 4.7. *Let A_n and B_n be the matrices given by Table A.5. Then*

$$\text{(i)}\ \ M(A_n^{i,f_{n,i+1}}) = \begin{cases} M(A_n), & \text{if } i = 0, \\ M(A_n^{i-1,f_{n,i}})_1, & \text{if } i = 1, \\ M(A_n^{i-1,f_{n,i}})_i + \overline{M(A_{n-1}^{i-2,f_{(n-1,i-1)}})}_n, & \text{if } 2 \le i \le n-1, \end{cases}$$

$$\text{(ii)}\ \ M(B_n^{i,f_{n,i+2}}) = \begin{cases} M(B_n), & \text{if } i = 0 \text{ or } i = -1, \\ M(B_n^{i-1,f_{n,i+1}})_i + \overline{M(B_{n-1}^{i-2,f_{n-1,i}})}_n, & \text{if } 1 \le i \le n-1, \end{cases}$$

for $n > 0$.

Proof. (i) Let $A_n = (a^n_{u,w})$ be the matrix of $T_n \times f_{n,2}$, and let $A_n^{i,f_{n,i+1}} = (a^{n,i}_{u,w})$ be the matrix of $T_n \times f_{n,i+1}$ such that satisfying identity A.2:

(1.1) If $i = 0$, $A_n^{0,f_{n,1}}$ is a matrix with $f_{n,1}$ columns, since $f_{n,1} = f_{n,2}$, $a^{n,0}_{u,w} = a^n_{u,w}$, then $a^{n,0}_u = a_u$ for $1 \le u \le T_n$.

(1.2) If $i = 1$, $A_n^{1,f_{n,2}}$ is a matrix with $f_{n,2}$ columns, $a_{u,w}^{n,1} = a_{u,w}^{n}$ for $2 \leq u \leq T_n$, i.e., $a_u^{n,1} = a_u$ for $2 \leq u \leq T_n$, and $a_1^{n,1} = 0$.

(1.3) If $2 \leq i \leq n-1$, $A_n^{i,f_{n,i+1}}$ is a matrix with $f_{n,i+1}$ columns, this matrix is equal to matrix $A_n^{i-1,f_{n,i}}$ by removing the row $i-1$ and the columns between $f_{n,i+1}$ and $f_{n,i}$, matrix A_n implies that in these columns the elements between the rows $n+1$ and T_n are given by the matrix $A_{n-1}^{i-2,f_{n-1,i-1}}$, i.e.,

$$a_u^{n,i} = \sum_{j=1}^{f_{n,i}} a_{u,j}^{n,i-1} - \sum_{j=1}^{f_{n-1,i-1}} a_{u-n,j}^{n-1,i-2} = a_u^{n,i-1} - \overline{a_{u-j}^{n-1,i-2}},$$

for $n+1 \leq u \leq T_n$ and $a_u^{n,i} = 0$ otherwise.

The case (ii) is similar to case (i). □

Proposition 4.12. *Let A_n and B_n be the matrices given by Table A.5. Then:*

(i) $a_u^{n,i} = f_{k,1+u-(T_n-T_k)} t^{n+1-k}_{n-k+u-(T_n-T_k+1),n-k+u-(T_n-T_k+1)-(i-1)}$, *for* $0 \leq i \leq n-1$,

(ii) $b_u^{n,i} = f_{k-1,u-(T_n-T_k)} t^{n+1-k}_{n-k+u-(T_n-T_k),n-k+u-(T_n-T_k)-i}$, *for* $-1 \leq i \leq n-1$,

with $T_n - T_k \leq u \leq T_n - T_{k-1}$ *for* $1 \leq k \leq n$, *and* $n > 1$.

Proof. (i) (Induction) For $n = 2$. If $i = 0$, by Lemma 4.7, $M(A_2^{0,f_{2,1}}) = M(A_2)$, then

$$\begin{aligned} a_1^{2,0} &= 2 = f_{2,2} t^1_{0,1}, \\ a_2^{2,0} &= 1 = f_{2,3} t^1_{1,2}, \\ a_3^{2,0} &= 1 = f_{1,2} t^2_{1,2}. \end{aligned}$$

If $i = 1$, by Lemma 4.7, $M(A_2^{1,f_{2,2}}) = M(A_2)_1$, then

$$\begin{aligned} a_1^{2,1} &= 0 = f_{2,2} t^1_{0,0}, \\ a_2^{2,1} &= 1 = f_{1,2} t^1_{1,1}, \\ a_3^{2,1} &= 1 = f_{1,2} t^2_{1,1}, \end{aligned}$$

(see identity 4.34). Suppose that the assertion is true for $n = m$ and $0 \leq i \leq m$. Then for $n = m+1$, if $i = 0$, $M(A_{m+1}^{0,f_{m+1,1}}) = M(A_{m+1})$ (Lemma 4.7), if $0 < u < m+1$,

$$a_u^{m+1,0} = f_{m+1,i+u} = f_{m+1,i+u} t^1_{u-1,u}.$$

For the rows between $m+2$ and T_{m+1},

$$\begin{aligned} a_u^{m+1,0} &= \sum_{i=0}^{m-1} f_{k,1+u-(m+1)-(T_m-T_k)} t^{m+1-k}_{m-k+u-(m+1)-(T_m-T_{k-1}+1),m-k+u-(m+1)-(T_m-T_{k-1}+1)-(i-1)} \\ &= f_{k,1+u-(T_{m+1}-T_k)} \sum_{i=0}^{m-1} t^{m+1-k}_{m-k+u-(T_{m+1}-T_{k-1}+1),m-k+u-(T_{m+1}-T_{k-1}+1)-(i-1)} \\ &= f_{k,1+u-(T_{m+1}-T_k)} f_{m-k+u-(T_{m+1}-T_k),u-(T_{m+1}-T_k)} \qquad \text{(Lemma 4.6)} \\ &= f_{k,1+u-(T_{m+1}-T_k)} t^{m+2-k}_{m-k+u-(T_{m+1}-T_k),m-k+u-(T_{m+1}-T_k)+1} \qquad \text{(Proposition 4.11),} \end{aligned}$$

with $1 \leq k \leq m$.

If $i = 1$, $M(A_{m+1}^{1,f_{m+1,2}}) = M(A_{m+1}^{0,f_{m,1}})_1$ (Lemma 4.7), then, $a_1^{m+1,1} = 0 = f_{m+1,2}t_{0,0}^1$, and

$$\begin{aligned} a_u^{m+1,1} &= a_u^{m+1,0} \\ &= f_{k,1+u-(T_{m+1}-T_k)} t^{m+2-k}_{m-k+u-(T_{m+1}-T_k),m-k+u-(T_{m+1}-T_k)+1} \\ &= f_{k,1+u-(T_{m+1}-T_k)} t^{m+2-k}_{m-k+u-(T_{m+1}-T_k),m-k+u-(T_{m+1}-T_k)} \qquad \text{(Proposition 4.10)}, \end{aligned}$$

for $T_{m+1} - T_k < u \leq T_{m+1} - T_{k-1}$ with $u \neq 1$, and $1 \leq k \leq m+1$.

Suppose that the property is true for $i = p-1 < m$, then for $i = p$, then $M = (A_{m+1}^{p,f_{m+1,p+1}}) = M(A_{m+1}^{p-1,f_{m+1,p}})_p + M(A_m^{p-2,f_{(m,p-1)}})_{m+1}$ (Lemma 4.7), for $1 \leq u \leq p$

$$a_u^{m+1,p} = 0 = f_{m+1,1+u}t^1_{u-1,u-p},$$

for $p+1 \leq u \leq m+1$

$$a_u^{m+1,p} = a_u^{m+1,p-1} = f_{m+1,1+u}t^1_{u-1,u+1-p} = f_{m+1,1+u}t^1_{u-1,u-p},$$

for $u \geq m+1$,

$$\begin{aligned} a_u^{m+1,p} = \;& f_{k,1+u-(T_{m+1}-T_k)} t^{m+2-k}_{m+1-k+u-(T_{m+1}-T_k+1),m+1-k+u-(T_{m+1}-T_k+1)-(p-2)} \\ & -f_{k,i+u-(m+1)-(T_m-T_k)} t^{m+1-k}_{m-k+u-(m+1)-(T_m-T_k+1),m-k+u-(m+1)-(T_m-T_k+1)-(p-3)}, \end{aligned}$$

Proposition 4.9 implies that

$$a_u^{m+1,p} = f_{k,i+u-(m+1)-(T_m-T_k)} t^{m+2-k}_{m+1-k+u-(T_{m+1}-T_k+1),m+1-k+u-(T_{m+1}-T_k+1)-(p-1)},$$

for $1 \leq k \leq m$. The case (ii) is similar to case (i). □

For notation $\omega : \mathbb{N} \to \{1, 2\}$ is a map where

$$\omega(n) = \begin{cases} 1, & \text{if } n \neq 2, \\ 2, & \text{if } n = 2. \end{cases}$$

The following result regards dimension of Λ_{Γ_n} and its corresponding center.

Theorem 4.4. *Let Λ_{Γ_n} be a Dyck-Brauer configuration algebra. Then*

(i) $dim_k(\Lambda_{\Gamma_n}) = 2(C_n + \omega(n)) + \sum\limits_{u=1}^{T_n} (a_u^{n,0})^2 + (b_u^{n,0})^2 - (a_u^{n,0} + b_u^{n,0})$,

(ii) $dim_k(Z(\Lambda_{\Gamma_n})) = 1 + 2\omega(n) + C_n$,

for $n > 0$.

Proof. (i) Firstly, we note that the number of vertices in Q_{Γ_n} is the $n-$th Catalan number. Secondly, we note that val$(\alpha_{n_1}^{n_2})$ (resp. val$(\beta_{n_1}^{n_2})$) is given by $a^{n,0}_{n(n_1)+n_2}$ (resp.

$a^{n,0}_{n(n_1)+n_2}$). As a consequence of Proposition 4.12, we have that $a_n^{n,0} = 1 = b_n^{0,n}$ for any n. In particular case, when $n = 2$, also, $a_2^{0,1} = b_2^{0,1} = 1$. Finally, recall that identity 4.27 describes the multiplicity function. (ii) The number of loops in Q_{Γ_n} is equal to the number of elements in the set $\mathfrak{C}_{\Gamma_n}$. □

For example, for $n = 2$, $a_1^{2,0} = 2 = b_3^{2,0}$, and $a_2^{2,0} = a_3^{2,0} = b_1^{2,0} = b_2^{2,0} = 1$, then

$$\begin{aligned} \dim_k(\Lambda_{\Gamma_2}) &= 2(C_2 + \omega(2)) + 12 - 8 \\ &= 2(2+2) + 4 = 12, \end{aligned}$$

and

$$\dim(Z(\Lambda_{\Gamma_2})) = 1 + 2\omega(2) + C_2 = 7.$$

APPENDIX A

Appendix

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	0																		
3	0	1																	
4	0	1	2	3															
5	0	1	2	4	4	5	6	7											
6	0	1	2	5	4	6	8	11	8	9	10	12	12	13	14	15			
7	0	1	2	6	4	7	10	16	8	10	12	17	16	19	22	26	16	17	18

TABLE A.1. Elements of the sequence C_n^m.

$n\backslash r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	2															
3	4	5														
4	8	11	12	11												
5	16	23	26	26	26	29	28	23								
6	32	47	54	57	56	64	64	57	54	65	68	64	64	65	60	47
7	64	95	110	120	116	135	138	130	116	140	148	145	144	149	142	120

TABLE A.2. Number of sections in the case $\mathbb{A}_n$.

Table A.2. The number $S_{(\mathbb{A}_n)^{i_0 i_1 \cdots i_k}_{j_0 j_1 \cdots j_m}}$ (see, formula 2.4 and Remark 2.5) of sections in the Auslander-Reiten quiver of the path algebra $k\overrightarrow{\mathbb{A}_n}$ where $\overrightarrow{\mathbb{A}_n}$ is an oriented Dynkin diagram of type $\mathbb{A}_n$ with $i_r < n$ sinks.

n and n-1 are sources																
$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	17															
5	45	46														
6	109	118	119	112												
7	253	278	287	284	284	290	283	260								
8	573	630	659	664	660	686	683	652	649	684	690	668	671	670	643	588
n is a source and n-1 is a sink (viceversa)																
4	14															
5	38	42														
6	94	106	110	108												
7	222	250	262	266	264	276	274	258								
8	510	570	602	616	608	642	646	628	606	648	660	650	650	656	638	592
n and n-1 are sinks																
4	14															
5	36	43														
6	88	103	110	112												
7	208	239	254	263	260	278	280	269								
8	480	543	578	598	588	629	638	628	592	642	660	659	656	670	658	618

TABLE A.3. Number of sections in the case $\mathbb{D}_n$.

Table A.3. Rows give the value of n and columns give the location of the underlying graph $\mathbb{A}'_{n-2}$, with

$$r = \begin{cases} \displaystyle\sum_{t=w-r+2}^{w} 2^{j_t-1} - \sum_{t=1}^{r-1} 2^{i_t-1}, & \text{if } j_w > 1, \\ 0, & \text{if } j_w = 0. \end{cases} \tag{A.1}$$

n	$i \backslash j$	1	2	3	4	5	6	n	1	2	3	4	5	6
2	1	1	1					4						
	2	1	2	2										
	3	1	2	3	3				1	3	5	5		
	4	1	2	3	4	4			1	4	9	14	14	
	5	1	2	3	4	5	5		1	4	10	19	28	28
3	2	1	2	2				5						
	3	1	3	5	5									
	4	1	3	6	9	9			1	4	9	14	14	
	5	1	3	6	10	14	14		1	5	14	28	42	42

TABLE A.4. Examples of integer numbers $t^n_{i,j}$.

$$
\begin{array}{c}
\begin{array}{c|cccccccc|}
 & f_{n,2} & f_{n,3} & & f_{n,i} & f_{n,i+1} & & f_{n,n} & f_{n,n+1} \\
\hline
1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1 \\
2 & & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1 \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots \\
i-1 & & & & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1 \\
i & & & & & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1 \\
 & & & & & & & \vdots & \vdots \\
n-1 & & & & & & & 1\cdots1 & 1 \\
n & & & & & & & & 1 \\
n+1 & & & & & & & & \\
\hline
T_n & A_{n-1}^{0,f_{(n-1,1)}} & A_{n-1}^{1,f_{(n-1,2)}} & \cdots & A_{n-1}^{i-2,f_{(n-1,i-1)}} & A_{n-1}^{i-1,f_{(n-1,i)}} & \cdots & A_{n-1}^{n-2,f_{(n-1,n-1)}} & \\
\hline
\end{array}
\\[2em]
\begin{array}{c|cccccccc|}
 & f_{n,2} & f_{n,3} & & f_{n,i} & f_{n,i+1} & & f_{n,n} & f_{n,n+1} \\
\hline
1 & 1\cdots1 & & & & & & & \\
2 & & 1\cdots1 & & & & & & \\
 & & & 1\cdots1 & & & & & \\
i-1 & & & & 1\cdots1 & & & & \\
i & & & & & 1\cdots1 & & & \\
 & & & & & & 1\cdots1 & & \\
n-1 & & & & & & & 1\cdots1 & \\
n & & & & & & & & 1 \\
n+1 & & & & & & & & \\
\hline
T_n & B_{n-1}^{-1,f_{(n-1,1)}} & B_{n-1}^{0,f_{(n-1,2)}} & \cdots & B_{n-1}^{i-3,f_{(n-1,i-1)}} & B_{n-1}^{i-2,f_{(n-1,i)}} & \cdots & B_{n-1}^{n-2,f_{(n-1,n-1)}} & B_{n-1}^{n-2,f_{(n-1,n)}} \\
\hline
\end{array}
\end{array}
$$

TABLE A.5. Matrix A_n (left) and matrix B_n (right).

Table A.5. $A_n^{i,f_{(n,i+1)}} = (a'_{u,w})$ (resp. $B_n^{i,f_{(n,i+2)}} = (b'_{u,w})$) is a matrix of $T_n \times f_{n,i+1}$ (resp. $T_n \times f_{n,i+2}$) such that

$$a_{u,w}^{n,i} \text{ (resp. } b_{u,w}^{n,i}) = \begin{cases} a_{u,w} \text{ (resp. } b_{u,w}) & \text{if } u > i, \\ 0 & \text{if } u \le i, \end{cases} \tag{A.2}$$

for $0 \leq i \leq n-1$ (resp. $-1 \leq i \leq n-1$), $A_n = (a_{u,w})_{T_n \times f_{n,2}}$ (resp. $B_n = (b_{u,w})_{T_n \times f_{n,2}}$). The number $f_{i,j}$ belongs to the triangle described in (4.9) with $f_{0,1} = 1$.

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