# *Categorification of Some Integer Sequences and Its Applications*

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July, 2020.

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Thesis work to obtain the degree of
Doctor in Mathematics

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Research line
Representation theory of algebras

Research group
terenufia-unal

National University of Colombia
Faculty of Science
Department of Mathematics
Bogotá, D.C.
July, 2020.

**Title**

Categorification of Some Integer Sequences and Its Applications.

**Abstract:** Categorification of real valued sequences, and in particular of integer sequences is a novel line of investigation in the theory of representation of algebras. In this theory introduced by Ringel and Fahr, numbers of a sequence are interpreted as invariants of objects of a given category. The categorification of the Fibonacci numbers via the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver is an example of this kind of identifications.

In this thesis, we follow the ideas of Ringel and Fahr to categorify several integer sequences but instead of using the 3-Kronecker quiver, we deal with a kind of algebras introduced recently by Green and Schroll called Brauer configuration algebras. Relationships between these algebras, some matrix problems and rational knots are used to interpret numbers in some integer sequences as invariants of indecomposable modules over path algebras of the 2-Kronecker quiver and the four subspace quiver.

The results enable us to define the message of a Brauer Configuration and labeled Brauer configurations in order to give an interpretation of the number of perfect matchings of snake graphs, the number of homological ideals of some Nakayama algebras, and the number of $k$-paths linking two fixed points (associated to the Lindström problem) in a quiver as specializations of indecomposable modules over suitable Brauer configuration algebras. Actually, this setting can be also used to define the Gutman index of a tree (or the trace norm of a digraph, which is a fundamental notion in the topological index theory), magic squares, and different parameters of traffic flow models in terms of this kind of algebras.

This research was partially supported by COLCIENCIAS convocatoria doctorados nacionales 785 de 2017.

**Keywords:** Brauer configurations, Brauer configuration algebra, categorification of integer sequences, energy of a graph, four subspace problem, homological ideals, indecomposable module, integer specialization, Kronecker problem, magic squares, OEIS, perfect matching, snake graph, tangles, theory of representation of algebras, traffic flow.

## Acceptation Note

Thesis Work

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Bogotá, July, 2020.

# Dedicated to

God, for allow me to end this process. God continues to make the impossible possible.

The memory of my three grandfathers Pedro Elías Fernández Fernández (1934-1979), Lisandro Espinosa Gamez (1944-2011), and Alexander Zavadskij (1946-2012), they will always live in my soul.

My parents Yolanda Espinosa García, Pedro Elías Fernández García for their unconditional support, for their excellent example throughout my life and for their invaluable effort.

My professor Agustín Moreno Cañadas, who is the wisest person that I have never seen and who was crucial in my life.

My twin sisters Leidy Johana Fernández Espinosa and Laura Camila Fernández Espinosa for give color to my life.

My friend Motas who died in the last part of this process.

# Acknowledgments

First and foremost, I would like to thank my professor, Agustin Moreno Cañadas Ph.D. for being such a real mentor and example for me. His continued advice, knowledge, patience, encouragement, and wisdom were crucial for me throughout these ten years.

I would like to thanks my family for everything; words cannot express how grateful I am with my parents Yolanda and Pedro, thanks a lot for all the sacrifices, efforts, and love that you have given me. I will always be proud of you. Thanks also my sisters Leidy and Laura for teaching me a lot with their happiness.

I would like to express my gratitude to Dr. Henning Krause for the great opportunity to discuss some ideas of my work during my visit to Universität Bielefeld in 2019. Also, a special thanks to Dr. José Antonio de la Peña, Dr. Hernan Giraldo Salazar, Dr. José Velez Marulanda and Dr. William Crawley-Boevey for the useful discussions, courses, emails and time that I had the opportunity to shared with them.

I especially express my regards to my partners of the TERENUFIA-UNAL research group for the discussions in the seminar, for their feedback, for their contributions, and mainly for encouraging me along the way. I also want to express my deep gratitude to my close friends Dr. Isaías David Marín Gaviria, Veronica Cifuentes Vargas, and Gabriel Bravo Rios for being such a second family for me. Thanks for teaching me a lot, for always motivating me even in the most difficult moments of this process and also for providing discussion spaces to improve this research.

I would like to thank some of my math friends. I had the pleasure to spend time and discuss math with them: Dr. Marco Armenta Armenta, Dr. Kaveh Mousavand, Dr. Magnus Hellstrøm-Finnsen and Dr. Kwstas Liamphs. Thanks a lot also my close friends and familiars: Jesus Adolfo Caicedo, Sandra Chavarro, Adriana Fonce, Melissa Martínez, Diana Rojas, Heber Sarmiento and Cesar Andres Walteros all your support and love were really important to me.

Last but not least, I would like to thank Professor Alexander Zavadskij, who inspired us to investigate representation theory and its applications. His legacy and mathematical results were an inspiration to me. Finally, I thank God, for all the blessings I have received during these years. Thanks, My Lord.

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# Introduction

According to Ringel and Fahr [20] a categorification of a sequence of numbers means to consider instead of these numbers suitable objects in a category (for instance, representation of quivers) so that the numbers in question occur as invariants of the objects, equality of numbers may be visualized by isomorphisms of objects functional relations by functorial ties. The notion of this kind of categorification arose from the use of suitable arrays of numbers to obtain integer partitions of dimensions of indecomposable preprojective modules over the 3-Kronecker algebra (see Figure (1) where it is shown the 3-Kronecker quiver and a piece of the oriented 3-regular tree or universal covering $(T, E, \Omega_t)$ as described by Ringel and Fahr in [19]). Firstly, they noted that the vector dimension of these kind of modules consists of even-index Fibonacci numbers (denoted $f_i$ and such that $f_i = f_{i-1} + f_{i-2}$, for $i \geq 2$, $f_0 = 0$, $f_1 = 1$) then they used results from the universal covering theory developed by Gabriel and his students to identify such Fibonacci numbers with dimensions of representations of the corresponding universal covering.

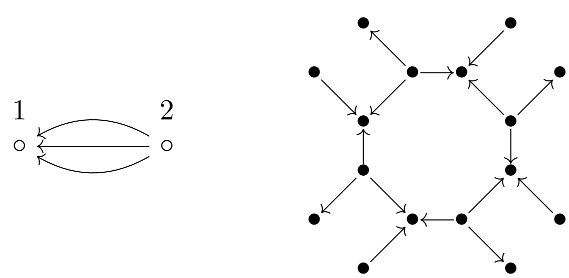



FIGURE 1. The 3-Kronecker quiver and an illustration of its corresponding universal covering.

For the sake of clarity, we give here a brief insight into the program of Ringel and Fahr.

First of all, we note that the road to a categorification of the Fibonacci numbers has several stops some of them dealing with diagonal (lower) arrays of numbers of the form $D = (d_{i,j})$ with $0 \leq j \leq i \leq n$, (columns numbered from right to the left, see Figure 2) for some $n \geq 0$ fixed and such that:

$$\begin{aligned}
d_{i,i} &= 1, \quad \text{for all } i \geq 0,\\
d_{i,j} &= 0 \quad \text{for all } j > i\\
d_{2k+i,i-1} &= 0, \quad \text{if } i \geq 1,\ k \geq 0,\\
d_{2k,0} &= 3d_{2k-1,1} - d_{2(k-1),0}, \quad k \geq 1,\\
d_{i+1,j-1} &= 2d_{i,j} + d_{i,j-2} - d_{i-1,j-1}, \quad i, j \geq 2.
\end{aligned} \tag{1}$$

Besides, if $i \geq 4$ then the following identity (hook rule) holds;

$$\sum_{k=0}^{i-2} d_{i+k,i-k} + d_{2i-2,0} = d_{2i-1,1}. \tag{2}$$

Note that to each entry $d_{i,i-j}$ it is possible to assign a weight $w_{i,i-j}$ such that:

$$w_{i,i-j} = \begin{cases} 3 \cdot 2^{2\lfloor \frac{i-j}{2} \rfloor + a}, & \text{if } j \text{ is even}, \ i \neq j, \\ 0, & \text{if } j \text{ is odd}, \ i \neq j, \\ 1, & \text{if } i = j = 2h \text{ for some } h \geq 0. \end{cases}$$

Where $\lfloor x \rfloor$ is the greatest integer number less than $x$, $a \in \{0, -1\}$, $a = -1$ if $i$ is even, it is 0 otherwise.

The first stop consists of defining partitions of the even-index Fibonacci numbers in the following form:

$$f_{2i+2} = \sum_{j=0}^{i} (w_{i,i-j})(d_{i,i-j}), \tag{3}$$

to do that, Ringel and Fahr interpreted weights $w_{i,i-j}$ as distances in a 3-regular tree $(T, E)$ (with $T$ a vertex set and $E$ a set of edges) from a fixed point $x_0 \in T$ to any point $y \in T$. They define sets $T_r$ whose points have distance $r$ to $x_0$, in such a case $T_0 = \{x_0\}$, $T_1$ are the neighbors of $x_0$ and so on. A given vertex $y$ is said to be even or odd according to this parity [19].

Any vertex $y \in T$ yields a suitable reflection $\sigma_y$ on the set of functions $T \to \mathbb{Z}$ with finite support, denoted $\mathbb{Z}[T]$, and some reflection products denoted $\Phi_0$ and $\Phi_1$ according to the parity of $y$ are introduced in [19]. Then some maps $a_t : \mathbb{N}_0 \to \mathbb{Z} \in \mathbb{Z}[T]$ are defined in such a way that if $a_0$ is the characteristic function of $T_0$ then $a_0(x) = 0$ unless $x = x_0$ in which case $a_0(x_0) = 1$, and $a_t = (\Phi_0\Phi_1)^t a_0$, for $t \geq 0$, with $a_t[r] = a_t(x)$, for $r \in \mathbb{N}_0$ and $x \in T_r$, these maps $a_t$ give the values $d_{i,j}$ of the array (1). The following table is an example of such array with $n = 12$. Rows are giving by the values of $t$, $P_t$ is a notation for a 3-Kronecker preprojective module with dimension vector $[f_{2t+2} \ \ f_{2t}]$ (see [21]).

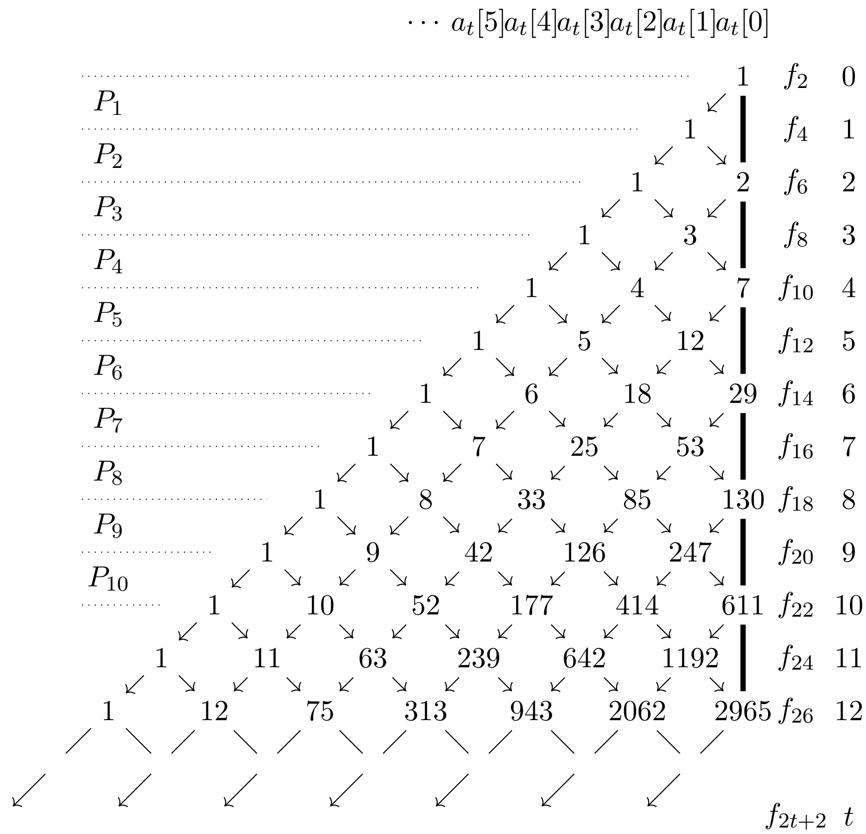



FIGURE 2. The even-index Fibonacci partition triangle.

For example for $t = 3$ and $t = 4$, we compute $f_8$ and $f_{10}$ as follows;

$$\begin{aligned} 21 = f_8 &= 0 + 3(3 \cdot 2^0) + 0 + 1(3 \cdot 2^2), \\ 55 = f_{10} &= 1 \cdot 7 + 0 + 4(3 \cdot 2^1) + 0 + 1(3 \cdot 2^3), \end{aligned} \tag{4}$$

sequences $a_t[0] = d_{2i,0}$ and $a_t[1] = d_{2i+1,1}$ are encoded respectively as A132262 and A110122 in the OEIS (On-Line Encyclopedia of Integer Sequences). Actually, sequence $a_t[0]$ had not been registered in the OEIS before the publication of Ringel and Fahr.

A second stop of the trip to a categorification of Fibonacci numbers consists of giving a generalization of the results obtained in [19]. In order to reach such a goal Ringel and Fahr adopted in [20] a notation $\sum^x$ ($\underline{\sum^x}$) for the composition of all the reflections $\sigma_y$ with $\{x, y\} \in E$ and $d(x, y)$ being even (the composition of all reflections $\sigma_y$ with $d(x, y)$ being odd). In particular, $\Phi_0 = \sum^{x_0}$ and $\Phi_1 = \underline{\sum^{x_0}}$. Some Fibonacci vectors $s_t(x)$, $t \geq 0$ and $r_t(x, y)$, $t \in \mathbb{Z}$ are defined recursively as follows; $s_0(x) = s(x)$ which is a generator associated to a vertex $x$ of the free abelian group $K_0(T)$ generated by $T$ and $r_0(x, y) = r(x, y) = s(x) + s(y)$

$$s_{t+1}(x) = \begin{cases} \underline{\sum^x} s_t(x), & \text{if } t \text{ is even,} \\ \sum^x s_t(x), & \text{if } t \text{ is odd.} \end{cases}$$

$$r_{t+1}(x) = \begin{cases} \sum^x r_t(x,y), & \text{if } t \text{ is even,} \\ \underline{\sum}^x r_t(x,y), & \text{if } t \text{ is odd.} \end{cases}$$

Note that, $s_{2t}(x_0) = a_t$, $s_{2t+1}(x_0) = \Phi_1 a_t$ and

$$\begin{aligned} s_t(x)_- &= \sum_{d(x,z)\not\equiv t \bmod 2} s_t(x)_z = f_{2t}, \\ s_t(x)_+ &= \sum_{d(x,z)\equiv t \bmod 2} s_t(x)_z = f_{2t+2}, \\ r_t(x,y)_- &= \sum_{d(x,z)\not\equiv t \bmod 2} r_t(x,y)_z = f_{2t-1}, \\ r_t(x,y)_+ &= \sum_{d(x,z)\equiv t \bmod 2} r_t(x,y)_z = f_{2t+1}. \end{aligned} \tag{5}$$

Ringel and Fahr [20] proved that if $x \in T$ with neighbors $y, y', y''$ and for an integer $t \geq 1$ $P_t(x)$ is an indecomposable representation of the quiver $Q = (T, E, \Omega_t^x)$ ($\Omega_t^x$ is a bipartite orientation such that $x$ is a sink in case $t$ is even and a source in case $t$ is odd) with dimension vector $s_t(x)$ and $R_t(x,y)$ is an indecomposable representation of $Q$ with dimension vector $r_t(x,y)$, assuming that for even $t$ the vertex $x$ is a sink and for $t$ odd the vertex $x$ is a source then there are exact sequences:

$$\begin{aligned} 0 \to P_{t-1}(y) \to P_t(x) \to R_t(x,y) \to 0 \\ 0 \to P_{t-1}(y') \to R_t(x,y) \to R_{t-1}(y'',x) \to 0. \end{aligned} \tag{6}$$

They also proved that if $x_0, x_1, \ldots, x_t$ is a path with $x_0$ a sink. Then $P_t(x_t)$ has a filtration

$$\begin{aligned} P_0(x_0) \subset P_1(x_1) \subset \cdots \subset P_t(x_t) \text{ with factors} \\ P_i(x_i)/P_{i-1} = R_i(x_i, x_{i-1}), \quad 1 \leq i \leq t. \end{aligned} \tag{7}$$

Another result regarding categorification of integer sequences found out by Ringel and Fahr states that if $x_0$ is a source and $x_{-1}, x_0, \ldots, x_t, x_{t+1}$ is a path with $z_i$ a neighbor of $x_i$ different from $x_{i-1}$ and $x_{i+1}$ then there is an exact sequence

$$0 \to P_0(z_0) \oplus \cdots \oplus P_t(z_t) \to R_{t+1}(x_t, X_{t+1}) \to R_0(x_{-1}, x_0). \tag{8}$$

Exact sequences (6) and (8) and filtration (7) are respectively categorifications of the identities

$$\begin{aligned} f_{t+1} &= f_{t-1} + f_t, \\ f_{2t+1} &= 1 + \sum_{i=1}^{t} f_{2i} \text{ and} \\ f_{2t} &= \sum_{i=1}^{t} f_{2i-1}. \end{aligned} \tag{9}$$

Finally, we also recall that the Auslander-Reiten sequences

$$\begin{aligned} &0 \to P_{n-1} \to P_n^3 \to P_{n+1} \to 0 \text{ and} \\ &0 \to R_{n-1,\lambda} \to E(n,\lambda) \to R_{n+1,\lambda} \to 0 \end{aligned} \tag{10}$$

where $E(n,\lambda)$ is an indecomposable module having dimension vector $3(\dim R(n,\lambda))$ are categorifications of the identity

$$f_{t-2} + f_{t+2} = 3f_t. \tag{11}$$

In a third stop of the road to a categorification of Fibonacci numbers Ringel and Fahr [21] named the array (1) a Fibonacci triangle and stated that its entries (nonzero entries) are categorified by the modules $P_n = P_n(x)$ (called Fibonacci modules) provided that such entries give the Jordan-Hölder multiplicities of these modules.

We also remind that Ringel in [54] made a discussion regarding the role of the representation theory of representation-finite hereditary artin algebras in the categorification of the Catalan combinatorics, which consists of all the enumerating problems whose solution is given by the sequence of Catalan numbers with the form $\frac{1}{n+1}\binom{2n}{n}$, for instance, if $\Lambda_n$ denotes the path algebra of the linearly ordered quiver of type $\mathbb{A}$ then the lattice of exceptional subcategories of mod $\Lambda_n$ denoted $\mathbf{A}(\text{mod}\,\Lambda_n)$ may be identified with the lattice of non-crossing partitions as introduced by Kreweras in [43], Fomin and Zelevinsky [22] also observed that there is a bijection between these partitions and the number of clusters for the cluster algebra associated to a certain orientation of a Dynkin diagram which is a Catalan number. An axiomatic point of view of the categorification of these combinatorial data and additional bijections associated to Catalan objects are given respectively by Hubery and Krause in [39] and Ingalls and Thomas in [40].

The idea of using the theory of representation of algebras to categorify Catalan numbers goes back at least 30 years, for example Gabriel and De la Peña proved in [24] that Catalan numbers count the number of discrete subsets contained in the set of representatives of isoclasses of indecomposable finite-dimensional modules over $\Lambda_n$.

Ringel argues in [54] that formulations concerning counting of modules are meant as a short form for counting isomorphism classes of modules. In particular he introduces Dynkin functions, which are functions attaching to any Dynkin diagram an integer (or more generally a real number, sometimes even a set or a sequence of real numbers). A Dynkin function $f$ has associated four sequences $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ and five single values $f(\mathbb{E}_6)$, $f(\mathbb{E}_7)$, $f(\mathbb{E}_8)$, $f(\mathbb{F}_4)$, and $f(\mathbb{G}_2)$. The number $r(\Delta_n)$ of indecomposable modules over hereditary artin algebras $\Lambda$ of Dynkin type $\Delta_n$, $c(\Delta_n)$ the number of complete exceptional sequences or $t_n(\Delta_n)$ the number of tilting modules (multiplicity free) are examples of Dynkin functions. Worth noting that $t_n(\mathbb{A}_n)$ is the $n$th Catalan number, thus tilting modules over algebras of type $\mathbb{A}_n$ categorify in the sense of Ringel and Fahr the Catalan numbers as well. Ringel also proposes to create an On-Line Encyclopedia of Dynkin Functions with the same purposes as the famous OEIS.

Regarding results in combinatorics, we recall that recently several mathematicians have studied snake graphs, which are combinatorial objects arising from the research of cluster algebras. They allowed to Çanakçi and Schiffler to compute the Laurent expansions of the cluster variables in cluster algebras of surface type. The terms in the Laurent polynomial of such variables are parametrized by the perfect matchings of the associated snake graph [53, 56–61]. Such graphs were studied by Prop [53] in the context of the investigation of the Laurent phenomenon, which is a problem of paramount importance in the theory of cluster algebras. Prop proved that two examples of rational recurrences, the two-dimensional frieze patterns of Conway and Coxeter and the tree of Markoff numbers-relate to one another and to the Laurent phenomenon. In the program of Prop perfect matchings of snake graphs derived from triangulations of polygons are linked with frieze patterns of Conway and Coxeter.

Prop in [53] also reported an interesting connection between snake graphs and continued fractions, according to him, work of Benjamin and Quinn in the context of the strip tiling model, shows how combinatorial models can illuminate facts about continued fractions. In [56–60] Çanakçi and Schiffler go beyond Prop by proving that each snake graph $\mathcal{G}$ has associated a unique continued fraction whose numerator is given by the number of perfect matchings of a suitable snake graph. They report that snake graphs provide a new combinatorial model for continued fractions allowing to interpret the numerators and denominators of positive continued fractions as cardinals of combinatorially defined sets.

Regarding applications of the theory of snake graphs we recall that recently Çanakçi and Schroll [61] defined abstract string modules associating to each of such modules a suitable snake graph, whose lattice of perfect matchings is in bijective correspondence with the lattice of submodules of such abstract module. In this work, the number of perfect matchings of a snake graph is interpreted as the message of a labeled Brauer configuration. The same is done for the number of $k$-paths connecting two fixed points $u$ and $v$ in an acyclic finite digraph.

Concerning categorification of integer sequences we follow the ideas of Ringel et al. to categorify some integer sequences but instead of using finite-representation algebras or the 3-Kronecker algebra, we use the 2-Kronecker algebra (or simply the Kronecker algebra) and Brauer configuration algebras (introduced recently by Green and Schroll [31]) to obtain categorifications of sequences A005258 and A100705 in the OEIS. Such Brauer configuration algebras are defined by configurations of some multisets called polygons.

Let us point out more clearly some differences and similarities between the work of Ringel and Fahr and our approach. Firstly, we note that in the scenario of Ringel and Fahr, Fibonacci numbers (sequence A000045 in the OEIS) are categorified by identifying information arising from the preprojective (preinjective and regular) component of the Auslander-Reiten quiver of the 3-Kronecker quiver with information arising from indecomposable representations of an oriented 3-regular quiver by using the theory of universal covering, whereas in our proposal numbers in the sequences $n!(\frac{((-1)^n+2n+3)}{4})$ (A052558 in the OEIS) and $n^3+(n+1)^2$ (A100705 in the OEIS) are categorified by identifying information arising from the preprojective components of the Auslander-Reiten quiver of the 2-Kronecker quiver and the category rep $\mathcal{P}$ of $k$-linear representations of four incomparable points with information arising from indecomposable modules over some suitable Brauer configuration algebras. We recall here that the Kronecker problem is equivalent to the problem of determining the indecomposable representations over a field $k$ of the following quiver $Q$ (2-Kronecker quiver):

$$Q = \begin{tikzcd} \underset{0}{\circ} & \underset{1}{\circ} \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] \end{tikzcd} \tag{12}$$

whereas to determine the indecomposable representations of four incomparable points (a tetrad) is a very well known matrix problem named the four subspace problem (FSP). The solution of this problem is equivalent to determine all of the indecomposable representations of the four subspace quiver with the following form [28, 49, 65, 74]:

$$\begin{tikzcd} & & \overset{5}{\circ} & & \\ \underset{1}{\circ} \arrow[urr] & \underset{2}{\circ} \arrow[ur] & & \underset{3}{\circ} \arrow[ul] & \underset{4}{\circ} \arrow[ull] \end{tikzcd}$$

We will see that indecomposable representations of the quivers associated to Brauer configuration algebras are categorifications of polygons and if to each polygon it is assigned a number of an integer sequence then such indecomposable representations are categorifications of the corresponding numbers in the sequence. In fact, since polygons in Brauer configuration algebras are multisets, we will often assume that such polygons consists of words of the form

$$w = x_1^{f_1} x_2^{f_2} \dots x^{f_k - 1} x_k^{f_k} \tag{13}$$

where for each $i$, $1 \leq i \leq k$, $x_i$ is an element of the polygon called vertex and $f_i$ is the number of times that the vertex $x_i$ occurs in the polygon. In particular, if vertices $x_i$ in a polygon $V$ of a Brauer configuration are integer numbers then the corresponding word $w$ will be interpreted as a partition of an integer number $n_V$ associated to the polygon $V$ where it is assumed that each vertex $x_i$ is a part of the partition and $f_i$ is the number of times that the part $x_i$ occurs in the partition and $w = n_V = \sum_{i=1}^{n} x_i^{f_i}$.

Finally, we remind that in [6] Auslander, Platzeck and Todorov introduced homological ideals or strong idempotent ideals. These ideals arise from the research of heredity ideals and quasi-hereditary algebras. For these ideals the corresponding quotient map induces a full and faithful functor between derived categories. Recently, homological ideals have been studied in different contexts, for instance Gatica, Lanzilotta and Platzeck and independently Xu and Xi established some relationships with the so called finitistic dimension conjecture and the Igusa-Todorov functions [26]. Furthermore, De la Peña and Xi in [17] and Armenta in [4] studied the impact of these ideals in the context of Hochschild cohomology and one point extensions. In this work, via the integer specialization of a suitable Brauer configuration algebra and its corresponding message we give a categorification of Fibonacci numbers by using homological ideals.

As an example of the practical applications of our results, we define parameters in traffic flow models by using, in particular, the dimension of the center of some Brauer configuration algebras.

**Main results, contributions, papers and conferences**

This research regards the categorification of some integer sequences and its applications.

**Contributions**

The following are the main contributions:

1. The notions of message of a Brauer configuration and labeled Brauer configuration are introduced.
2. It is given an explicit formula for the number of perfect matchings of snake graphs via the message of some Brauer configurations algebras.
3. Suitable labeled Brauer configurations are used to define determinants, and as a consequence, solutions of some very well known problems, as the paths problem solved by Lindström, Gessel and Viennot can be interpreted as specializations of Brauer configurations.
4. It is introduced the notion of Kronecker snake graph, these kind of snake graphs allow us to build non-regular modules of the Kronecker algebra. Preprojective Kronecker tangles and the group structure of the preinjective Kronecker snake graphs are also introduced in this work.
5. A categorification in the sense of Ringel and Fahr is given to some integer sequences arising from some matrix problems as the Kronecker problem and the four subspace problem.
6. It is given the number of homological ideals associated to some Nakayama algebras via the integer specialization of a suitable Brauer configuration algebra and its corresponding message. Moreover, we use the number of homological ideals to establish an alternative partition formula for even-index Fibonacci numbers.
7. As a practical application, Brauer configuration algebras are used to define parameters of traffic flow models.

**Papers**

Results of this research allowed us to submit the following manuscripts for possible publication:

1. Brauer Configuration Algebras and Matrix Problems to Categorify Integer Sequences, 2021. Submitted.
2. On Some Relationships Between Snake Graphs and Brauer Configuration Algebras, 2020. Accepted in Algebra and Discrete Mathematics.

3. Homological Ideals as Integer Specializations of Some Brauer Configuration Algebras, 2020. Accepted in Ukrainian Mathematical Journal.

**Conferences**

The main results of this research have been presented in the following conferences.

1. Maurice Auslander Distinguished Lectures and International Conference, Woods Hole Oceanographic Institute. Woods Hole MA-USA, 04-2017.
2. Coloquio Latinoamericano de Álgebra-PUCE. Quito-Ecuador, 08-2017.
3. Isfahan School and Conference on Representations of Algebras. Isfahan-Irán, 04-2019.
4. Primer Encuentro de Estudiantes de Posgrado en Matemáticas, Medellín-Colombia, 02-2020.

**Research stays**

The author is indebted with the following institutions and academics for his warm hospitality during his several research stays.

1. Bielefeld Representation Theory Group and seminar (BIREP), Faculty of Mathematics, Universität Bielefeld, Bielefeld- Germany, Professor Henning Krause.
2. Algebra seminar IMERL at Instituto de Matemática y Estadística Rafael Laguardia (IMERL), Montevideo-Uruguay, Professor Marcelo Lanzilotta Mernies.
3. Representation theory seminar at the Centro de Investigación en Matemáticas A.C., CIMAT, Guanajuato, México, Professor José Antonio Stephan De la Peña Mena.
4. Algebra seminar at the Instituto de Matemáticas, Universidad de Antioquia, Medellín-Colombia, Professor Hernán Giraldo.

This thesis is distributed as follows:

Chapter 1 aims to present a theoretical introduction to the Brauer configuration algebras, matrix problems, snake graphs, binomial trees and tangles, as well as, definitions and notation to be used throughout the work.

In Chapter 2, message of a Brauer configuration and labeled Brauer configurations are introduce. These concepts allow us to obtain an explicit formula for the number of perfect matchings of a snake graph and establish a connection between Brauer configuration algebras and path problems as the Lindström problem. Besides, we introduce the notion of Kronecker snake graphs, which are useful to describe in an alternative fashion the non-regular Kronecker indecomposable modules. Regarding the preprojective Kronecker snake graphs, the message of a suitable Brauer configuration algebras allow us to obtain the preprojective Kronecker tangles and the group structure of preinjective Kronecker snake graphs is described as well.

In Chapter 3, some integer sequences arising from some flat matrix problems are categorified, to do that, we define Brauer configuration algebras whose indecomposable projective modules are in bijective correspondence with some solutions of these matrix problems. In this chapter, the energy value of some preprojective trees is given via the message of some suitable Brauer configuration algebras. Moreover, we define parameters in traffic flow models by using the dimension of the center of some Brauer configuration algebras.

In Chapter 4, an introduction to homological ideals is presented. Besides, we prove some combinatorial conditions which allow to establish whether an idempotent ideal in some Nakayama algebras is homological or not. We also give the number of homological ideals associated to these algebras via the integer specialization of a suitable Brauer configuration. Moreover, we use the number of homological ideals to establish an alternative version of the partition formula for even-index Fibonacci numbers given by Ringel and Fahr in [19] attaining in this way a new algebraic interpretation for these numbers.

Finally, in Chapter 5 we present an application of Brauer configuration algebras in combinatorics. In this part of the work following the ideas described by Ringel and Fahr the Brauer configuration algebras and some suitable specializations are used to categorify the magic squares of order $n$ for $3 \leq n \leq 9$.

# CHAPTER 1

# Preliminaries

In this chapter, we present basic definitions and notations to be used throughout this thesis. Section 1.1 is devoted to matrix problems, in particular we remind the Kronecker problem and the four subspace problem. Some facts regarding quivers of finite representation type and Brauer configuration algebras are given in sections 1.2 and 1.3, respectively. Binomial trees, cluster algebras and snake graphs are described in sections 1.4, 1.5, 1.6. Finally, some classical definitions and results regarding tangles are also given in section 1.7 [3, 5, 22, 23, 25, 41, 53, 57, 58, 65].

## 1.1 Matrix problems

The role of matrix problems in the research of the theory of representation of algebras has been remarkable. We recall that according to Simson [65] matrix problems were in the road map of the solution proposed by Roiter of the second Brauer-Thrall conjecture. Roiter's idea consisted in transforming the original problem to a suitable matrix problem therefore to a problem of classification of posets. Worth noting, that according to Gabriel and Roiter [25] if a matrix involved in a classification problem is partitioned into vertical stripes and some admissible row and column transformations are allowed between them then the corresponding matrix problem gives rise to a representation of a poset which can be of finite, tame or wild representation type according to the Drozd's theorem.

We also recall that a *matrix problem* is a pair $(\mathcal{G}, \mathcal{M})$ formed by an underlying set $\mathcal{M} \subset k^{m\times n}$ and a group $\mathcal{G} \subset GL_m \times GL_n$ such that $XAY^{-1} \in \mathcal{M}$ whenever $A \in \mathcal{M}$ and $(X, Y) \in \mathcal{G}$ (here $k$ is a field, $k^{m\times n}$ is the space of all $m \times n$-matrices and $GL_n$ is the corresponding general linear group). Among all the matrix problems there are the linear matrix problems $(\mathcal{G}, \mathcal{M})$ where $\mathcal{G}$ is the group $D^l$ of invertible elements of a subalgebra $D \subset k^{m\times m} \times k^{n\times n}$. The aim of the study of a given matrix problem is classifying the orbits of $\mathcal{M}$ under the action of $\mathcal{G}$ defined by $(X, Y)A = XAY^{-1}$ [25, 65].

The Kronecker problem which consists of classifying all pairs of linear maps between two finite-dimensional vector spaces over a field $k$, and the four subspace problem (FSP) of giving a complete classification of the indecomposable representations of four incomparable points (a tetrad or quadruple) over $k$ are examples of linear matrix problems. The

Kronecker problem was solved by Weierstrass in 1867 for some particular cases and by Kronecker for the complex numbers field in 1890, whereas FSP was solved by Gelfand and Ponomarev in 1970 for an algebraically closed field $k$ and by Nazarova (1967-1973) for the arbitrary case. Afterwards, in 2004, an elementary solution of this problem was given by Medina and Zavadskij [25, 28, 49, 50, 52, 74]. Zavadskij himself and Djoković found out a solution of the semi-linear Kronecker problem, which consists of classifying indecomposable pair of matrices $(A, B)$ of rectangular matrices of equal size over a division ring $k$ with an automorphism $\sigma$ with respect to transformations of the form $(A, B) \to (X^{-1}AY, X^{-1}BY^{\sigma})$ where $X$ and $Y$ are non-singular square matrices [18, 76].

Another classification of matrix problems was inspired by I.M. Gelfand who in the International Congress of Mathematicians which was held at Nice in 1970 presented the talk, *Cohomology of Infinite Dimensional Lie Algebras, Some Problems of Integral Geometry.* In such a talk Gelfand proposed to obtain a description of the indecomposable representations of the quiver;

$$\begin{tikzcd} \circ \arrow[r, bend right, "B"'] & \circ \arrow[l, bend right, "A"'] \arrow[r, bend left, "C"] & \circ \arrow[l, bend left, "D"] \end{tikzcd} \tag{1.1}$$

$$AB = CD$$

The solution of this problem corresponds to the classification of the Harish-Chandra modules for $\mathrm{Sl}_2(\mathbb{R})$. Nazarova and Roiter [51] found out a solution of (1.1) in 1973. Another solution and generalizations of this problem were given by Bondarenko [9], to do that, he defined a suitable flat matrix problem of type Gelfand these kind of problems are in fact of tame representation type. Since the Kronecker problem and the FSP are flat matrix problems of type Gelfand, we can deduce that these problems are both of tame representation type.

### 1.1.1 The Kronecker problem and the four subspace problem

In this section, we give detailed descriptions of the solutions of the Kronecker problem, which consists of classifying all pairs of linear maps between two finite-dimensional vector spaces over a field $k$, and the four subspace problem (FSP), which consists of giving a complete classification of the indecomposable representations of four incomparable points (a tetrad or quadruple) over $k$.

#### 1.1.1.1 Kronecker problem

The classification of indecomposable Kronecker modules was solved by Weierstrass in 1867 for some particular cases and by Kronecker in 1890 for the complex number field case. This flat problem of type Gelfand is equivalent to the problem of finding canonical Jordan form of pairs $(A, B)$ of matrices with respect to the following elementary transformations:

(i) All elementary transformations on rows of the block matrix $(A, B)$.

(ii) All elementary transformations made simultaneously on columns of $A$ and $B$ having the same index number.

If $k$ is an algebraically closed field then up to isomorphism every indecomposable Kronecker module belongs to one of the following three classes:

$$0: \quad \begin{array}{|c|c|}\hline \mathrm{I}_n & \mathrm{F}_n \\ \hline \end{array}$$

where $F_n$ is a Frobenius matrix or companion matrix of a minimal polynomial $p^s(t)$ with $n = s\partial p(t)$, $\partial p(t)$ denotes the degree of the polynomial $p(t)$.

$$\mathrm{I} = \mathrm{I}^*\text{: (a)} \quad \begin{array}{|c|c|}\hline \mathrm{I}_n & \mathrm{J}_n(0) \\ \hline \end{array}$$

$$\text{(b)} \quad \begin{array}{|c|c|}\hline \mathrm{J}_n(0) & \mathrm{I}_n \\ \hline \end{array}$$

where $J_n(0) \in \{J_n^+(0), J_n^-(0)\}$ and $J_n^{\pm}(0)$ denotes a corresponding upper or lower Jordan block. Whereas, $\mathrm{I}^*$ denotes the dual case defined by the classification problem.

$$\mathrm{II} = \mathrm{III}^*: \quad \begin{array}{|c|c|}\hline \overset{\rightarrow}{\mathrm{I}}_n & \overset{\leftarrow}{\mathrm{I}}_n \\ \hline \end{array}$$

$$\mathrm{III} = \mathrm{II}^*: \quad \begin{array}{|c|c|}\hline \mathrm{I}_n^{\uparrow} & \mathrm{I}_n^{\downarrow} \\ \hline \end{array}$$

In this case, $\overset{\rightarrow}{\mathrm{I}}_n$ ($\overset{\leftarrow}{\mathrm{I}}_n$, respectively) denotes an $n \times (n+1)$ matrix obtained from the identity $\mathrm{I}_n$ by adding a column of zeroes in fact the last column (the first column, respectively) in these matrices consists only of zeroes. In the same way, $\mathrm{I}_n^{\uparrow}$ ($\mathrm{I}_n^{\downarrow}$, respectively) denotes an $(n+1) \times n$ matrix obtained from the identity $\mathrm{I}_n$ by adding a row of zeroes.

If $n = 0$ then $U_0 = k$, and $V_0 = 0$. Cases II and III constitute the non-regular case of this classification, whereas cases 0 and I constitute the regular case.

Figure 1.1 shows the preprojective component of the Auslander-Reiten quiver of the 2-Kronecker quiver, which has as vertices indecomposable representations of type III. The preinjective component has indecomposable representations of type $\mathrm{III}^*$ as vertices.

$$\begin{tikzcd}
 & {[2\ 1]} \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & {[4\ 3]} \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & {[6\ 5]} & \cdots \\
{[1\ 0]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & {[3\ 2]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & {[5\ 4]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & {}
\end{tikzcd}$$

FIGURE 1.1. Preprojective component of the Auslander-Reiten quiver of the 2-Kronecker quiver.

Henceforth, we let $(n+1, n)$ $((n, n+1))$ denote a representative of the class of all indecomposable preprojective (preinjective) Kronecker modules obtained from a representation of type III (II) via elementary transformations of type (ii).

*Remark* 1. Several generalizations of the Kronecker problem have been posed and solved throughout the years. Perhaps, the semilinear Kronecker problem is one of the most remarkable generalizations of such a matrix problem [27, 76].

#### 1.1.1.2 Four subspace problem (FSP)

The four subspace problem is another example of a flat matrix problem of type Gelfand, in this case a *quadruple* of finite-dimensional $k$ vector spaces is a system of the form $U = (U_0, U_1, U_2, U_3, U_4)$ where $U_0$ is a finite- dimensional $k$ vector space and $U_1, \ldots, U_4$ is an ordered collection of four subspaces of $U_0$. Two quadruples are said to be isomorphic if there exists a $k$-space isomorphism $\varphi : U_0 \to V_0$ such that $\varphi(U_i) = V_i$ for all $i$. And a quadruple $U$ is *decomposable* if $(U = U' \oplus U'')$ the identity $U_i = (U_i \cap U_0) \oplus (U_i \cap U_0')$ holds for any $i$ and direct sum $U_0 \oplus U_0'$.

The four subspace problem consists of classifying all indecomposable quadruples up to isomorphism.

FSP was solved by Gelfand and Ponomarev in 1970 for $k$ algebraically closed and by Nazarova (1967-1973) for the arbitrary case. An advance to this problem was given by Brenner who described the indecomposable quadruples with non-zero defect $\partial(U) = \sum_{i=1}^{4} \dim U_i - 2\dim U_0$ (called non-regular) in particular she extended the results of Gelfand and Ponomarev to the case of an arbitrary field $k$. Afterwards, in 2004 Medina and Zavadskij gave an elementary solution of this problem. According to them all the indecomposable matrix representations of the quadruple can be reduced up to isomorphism and block permutations to one of the following six types of representations [10, 11, 28, 49, 50, 52, 74].

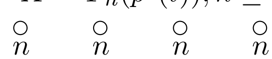
Representations III-V with negative defect are preprojective, whereas representations III*-V* with positive defect are preinjective.

Regular Component

$0 = 0^*$

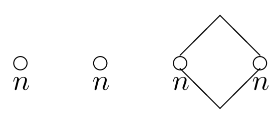
| | | | |
|---|---|---|---|
| $\mathrm{I}_n$ | $0$ | $\mathrm{I}_n$ | $X$ |
| $0$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ |

$X = F_n(p^s(t)), n \geq 1, \quad p(t) \neq t, t-1$



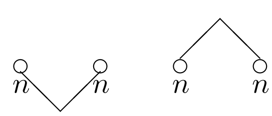

$\mathrm{I} = \mathrm{I}^*$

| | | | |
|---|---|---|---|
| $\mathrm{I}_n$ | $\mathrm{I}_n$ | $0$ | $J_n^+(0)$ |
| $0$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ |




$\mathrm{II} = \mathrm{II}^*$

| | | | |
|---|---|---|---|
| $\mathrm{I}_n$ | $\mathrm{I}_n$ | $0$ | $J_n^+(0)$ |
| $0$ | $\mathrm{I}_n^{\leftarrow}$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ |

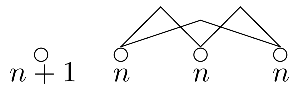



$n \geq 0,\ d_0 = 2n$

Non − Regular Component

III

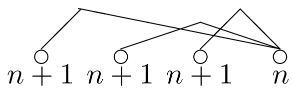
| | | | |
|---|---|---|---|
| $\mathrm{I}_{n+1}$ | 0 | $\mathrm{I}_n^{\uparrow}$ | $\mathrm{I}_n^{\downarrow}$ |
| 0 | $\mathrm{I}_n$ | $\mathrm{I}_n$ | $\mathrm{I}_n$ |




$n \geq 0,\ d_0 = 2n+1$

IV

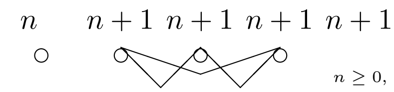
| | | | |
|---|---|---|---|
| $\mathrm{I}_{n+1}$ | 0 | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\uparrow}$ |
| 0 | $\mathrm{I}_{n+1}$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\downarrow}$ |




$n \geq 0,\ d_0 = 2n+2$

III*

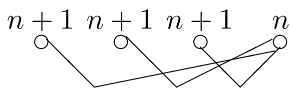
| | | | |
|---|---|---|---|
| $\mathrm{I}_n$ | 0 | $\mathrm{I}_n^{\leftarrow}$ | $\mathrm{I}_n^{\rightarrow}$ |
| 0 | $\mathrm{I}_{n+1}$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_{n+1}$ |




$n \geq 0,\ d_0 = 2n+1$

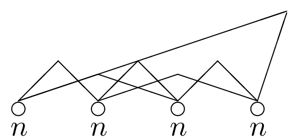
IV*

| | | | |
|---|---|---|---|
| $\mathrm{I}_{n+1}$ | 0 | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\leftarrow}$ |
| 0 | $\mathrm{I}_{n+1}$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\rightarrow}$ |




V

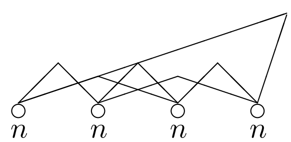
| | | | |
|---|---|---|---|
| $\mathrm{I}_n$ | 0 | $\mathrm{J}_n^{+}(0)$ | $\mathrm{I}_n$ |
| 0 | $\mathrm{I}_n$ | $\mathrm{I}_n$ | $\mathrm{J}_n^{+}(0)$ |
| $00\cdots0$ | $00\cdots0$ | $10\cdots0$ | $10\cdots0$ |




$n \geq 0,\ d_0 = 2n+1$

V*

| | | | |
|---|---|---|---|
| $\mathrm{I}_n^{\leftarrow}$ | $\mathrm{I}_n^{\leftarrow}$ | $\mathrm{I}_n^{\rightarrow}$ | 0 |
| 0 | $\mathrm{I}_n^{\rightarrow}$ | $\mathrm{I}_n^{\leftarrow}$ | $\mathrm{I}_n^{\leftarrow}$ |
| $10\cdots0$ | $10\cdots0$ | $10\cdots0$ | $10\cdots0$ |




$n \geq 0,\ d_0 = 2n+1$

## 1.2 Quiver representations

In this section, we recall some facts regarding quivers and its representations [5].

A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets $Q_0$ whose elements are called *vertices* and $Q_1$ whose elements are called arrows, $s$ and $t$ are maps $s, t : Q_1 \to Q_0$ such that if $\alpha$ is an arrow then $s(\alpha)$ is called the *source* of $\alpha$, whereas $t(\alpha)$ is called the *target* of $\alpha$ [5]. A *path* of length $l \geq 1$ with source $a$ and target $b$ is a sequence $(a \mid \alpha_1, \alpha_2, \ldots, \alpha_l \mid b)$ where $t(\alpha_i) = s(\alpha_{i+1})$ for any $1 \leq i < l$. Vertices are paths of length 0 [5, 7, 65].

If $Q$ is a quiver and $k$ is an algebraically closed field then the *path algebra* $kQ$ of $Q$ is the $k$-algebra whose underlying $k$-vector space has as basis the set of all paths of length $l \geq 0$ in $Q$, the natural graph concatenation is the product of two paths [5, 7, 54].

A $k$-algebra $A$ is said to be *basic* if it has a complete set $\{e_1, e_2, \ldots, e_l\}$ of primitive orthogonal idempotents such that $e_iA \ncong e_jA$ for all $i \neq j$.

A *relation* for a quiver $Q$ is a linear combination of paths of length $\geq 2$ with same starting points and same end points, not all coefficients being zero [5, 7, 54].

Let $Q$ be a finite and connected quiver. The two sided ideal of the path algebra $kQ$ generated by the arrows of $Q$ is called the *arrow ideal* of $kQ$ and is denoted by $R_Q$, $R_Q^l$ is the ideal of $kQ$ generated as a $k$-vector space, by the set of all paths of length $\geq l$. A two sided ideal $\mathcal{I}$ of the path algebra $kQ$ is said to be *admissible* if there exists $m \geq 2$ such that $R_Q^m \subseteq \mathcal{I} \subset R_Q^2$.

If $\mathcal{I}$ is an admissible ideal of $kQ$, the pair $(Q, \mathcal{I})$ is said to be a *bound quiver*. The quotient algebra $kQ/\mathcal{I}$ is said to be a *bound quiver algebra*.

The following theorems regarding the classification of quivers of finite representation type were proved by Gabriel [5]:

**Theorem 1.** *Suppose that $k$ is an algebraically closed field. Let $A$ be a basic and connected finite dimensional $k$-algebra. There exists an admissible ideal $\mathcal{I}$ of $kQ_A$ such that $A \cong kQ_A/\mathcal{I}$.*

A *$k$-linear representation* or *representation* $M$ of a quiver $Q$ is a system of the form:

$$M = ((M_x, \varphi_\alpha) \mid x \in Q_0, \alpha \in Q_1),$$

where $M_x$ is a $k$-vector space for each $x \in Q_0$ and $\varphi : M_a \to M_b$ is a $k$-linear map associated to each arrow $\alpha : a \to b \in Q_1$.

**Theorem 2.** *Let $Q$ be a finite, connected, and acyclic quiver; $k$ be an algebraically closed field; and $A = kQ$ be the path $k$-algebra of $Q$.*

1. *The algebra $A$ is representation-finite if and only if the underlying graph $\overline{Q}$ of $Q$ is one of the Dynkin diagrams $\mathbb{A}_n$ with $n \geq 2$, $\mathbb{D}_n$ with $n \geq 4$, $\mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$.*

2. *If $\overline{Q}$ is a Dynkin graph, then the mapping $\mathbf{dim} : M \to \mathbf{dim}\, M$ induces a bijection between the set of isomorphism classes of indecomposable $A$-modules and the set of positive roots of the quadratic form $q_Q$ of $Q$.*

3. *The number of the isomorphism classes of indecomposable $A$-modules equals $\frac{n(n+1)}{2}$, $n^2 - n$, 36, 63 and 120, if $\overline{Q}$ is the Dynkin graph $\mathbb{A}_n$ with $n \geq 2$, $\mathbb{D}_n$ with $n \geq 4$, $\mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$.*

## 1.3 Brauer configuration algebras

In this section, we recall the definition of a Brauer configuration algebra as Green and Schroll defined in [31].

Brauer configuration algebras were introduced by Green and Schroll in [31] as a generalization of Brauer graph algebras, which are biserial algebras of tame representation type and whose representation theory is encoded by some combinatorial data based on graphs. Actually, underlying every Brauer graph algebra is a finite graph with acyclic orientation of the edges at every vertex and a multiplicity function [61]. The construction of a Brauer graph algebra is a special case of the construction of a Brauer configuration algebra in the sense that every Brauer graph is a Brauer configuration with the restriction that every polygon is a set with two vertices. In the sequel, we give precise definitions of a Brauer configuration and a Brauer configuration algebra.

A *Brauer configuration* $\Gamma$ is a quadruple of the form $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ where:

$(B1)$ $\Gamma_0$ is a finite set whose elements are called *vertices*,

$(B2)$ $\Gamma_1$ is a finite collection of multisets called *polygons*. In this case, if $V \in \Gamma_1$ then the elements of $V$ are vertices possibly with repetitions, $\mathrm{occ}(\alpha, V)$ denotes the frequency of the vertex $\alpha$ in the polygon $V$ and the *valency* of $\alpha$ denoted $val(\alpha)$ is defined in such a way that:

$$val(\alpha) = \sum_{V\in\Gamma_1} \mathrm{occ}(\alpha, V). \tag{1.2}$$

$(B3)$ $\mu$ is an integer valued function such that $\mu : \Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers, it is called the *multiplicity function*,

$(B4)$ $\mathcal{O}$ denotes an orientation defined on $\Gamma_1$ which is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which $\alpha$ occurs as a vertex, including repetitions, we denote $S_\alpha$ such collection of polygons. More specifically, if $S_\alpha = \{V_1^{(\alpha_1)}, V_2^{(\alpha_2)}, \dots, V_t^{(\alpha_t)}\}$ is the collection of polygons where the vertex $\alpha$ occurs with $\alpha_i = \mathrm{occ}(\alpha, V_i)$ and $V_i^{(\alpha_i)}$ meaning that $S_\alpha$ has $\alpha_i$ copies of $V_i$ then an orientation $\mathcal{O}$ is obtained by endowing a linear order $\leq$ to $S_\alpha$ and adding a relation $V_t \leq V_1$, if $V_1 = \min S_\alpha$ and $V_t = \max S_\alpha$,

$(B5)$ Every vertex in $\Gamma_0$ is a vertex in at least one polygon in $\Gamma_1$,

$(B6)$ Every polygon has at least two vertices,

$(B7)$ Every polygon in $\Gamma_1$ has at least one vertex $\alpha$ such that $\mu(\alpha)val(\alpha) > 1$.

The set $(S_\alpha, \leq)$ is called the *successor sequence* at the vertex $\alpha$.

A vertex $\alpha \in \Gamma_0$ is said to be *truncated* if $val(\alpha)\mu(\alpha) = 1$, that is, $\alpha$ is truncated if it occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$. A vertex is *non-truncated* if it is not truncated.

### The Quiver of a Brauer Configuration Algebra

The quiver $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1)$ of a Brauer configuration algebra is defined in such a way that the vertex set $(Q_\Gamma)_0 = \{v_1, v_2, \dots, v_m\}$ of $Q_\Gamma$ is in correspondence with the set of polygons $\{V_1, V_2, \dots, V_m\}$ in $\Gamma_1$, noting that there is one vertex in $(Q_\Gamma)_0$ for every polygon in $\Gamma_1$.

Arrows in $Q_\Gamma$ are defined by the successor sequences. That is, there is an arrow $v_i \xrightarrow{s_i} v_{i+1} \in (Q_\Gamma)_1$ provided that $V_i \leq V_{i+1}$ in $(S_\alpha, \leq) \cup \{V_t \leq V_1\}$ for some non-truncated vertex

$\alpha \in \Gamma_0$. In other words, for each non-truncated vertex $\alpha \in \Gamma_0$ and each successor $V'$ of $V$ at $\alpha$, there is an arrow from $v$ to $v'$ in $Q_\Gamma$ where $v$ and $v'$ are the vertices in $Q_\Gamma$ associated to the polygons $V$ and $V'$ in $\Gamma_1$, respectively.

### The Ideal of Relations and Definition of a Brauer Configuration Algebra

Fix a polygon $V \in \Gamma_1$ and suppose that $\mathrm{occ}(\alpha, V) = t \geq 1$ then there are $t$ indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. Then the *special $\alpha$-cycles* at $v$ are the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$ where $v$ is the vertex in the quiver of $Q_\Gamma$ associated to the polygon $V$. If $\alpha$ occurs only once in $V$ and $\mu(\alpha) = 1$ then there is only one special $\alpha$-cycle at $v$.

Let $k$ be a field and $\Gamma$ a Brauer configuration. The *Brauer configuration algebra associated to* $\Gamma$ is defined to be the bounded path algebra $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where $Q_\Gamma$ is the quiver associated to $\Gamma$ and $I_\Gamma$ is the *ideal* in $kQ_\Gamma$ generated by the following set of relations $\rho_\Gamma$ of type I, II and III.

1. **Relations of type I**. For each polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of non-truncated vertices $\alpha_i$ and $\alpha_j$ in $V$, the set of relations $\rho_\Gamma$ contains all relations of the form $C^{\mu(\alpha_i)} - C'^{\mu(\alpha_j)}$ where $C$ is a special $\alpha_i$-cycle and $C'$ is a special $\alpha_j$-cycle.
2. **Relations of type II**. Relations of type II are all paths of the form $C^{\mu(\alpha)}a$ where $C$ is a special $\alpha$-cycle and $a$ is the first arrow in $C$.
3. **Relations of type III**. These relations are quadratic monomial relations of the form $ab$ in $kQ_\Gamma$ where $ab$ is not a subpath of any special cycle unless $a = b$ and $a$ is a loop associated to a vertex of valency 1 and $\mu(\alpha) > 1$.

As an example consider a configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

1. $\Gamma_0 = \{1, 2, 3, 4\}$,
2. $\Gamma_1 = \{U = \{1, 1, 4\}, V = \{1, 2, 2\}, W = \{2, 3, 3\}, X = \{3, 4, 4\}\}$,
3. At vertex 1, it holds that; $U < U < V, \quad val(1) = 3$,
4. At vertex 2, it holds that; $V < V < W, \quad val(2) = 3$,
5. At vertex 3, it holds that; $W < W < X, \quad val(3) = 3$,
6. At vertex 4, it holds that; $X < X < U, \quad val(4) = 3$,
7. $\mu(\alpha) = 1$ for any vertex $\alpha$.

The ideal $I$ of the corresponding Brauer configuration algebra $\Lambda_\Gamma$ is generated by the following relations (see Figure 1.2), for which it is assumed the following notation for the special cycles:

$$\begin{aligned}
C_1^{U,1} &= a_1^1 a_2^1 a_3^1,\\
C_1^{U,2} &= a_2^1 a_3^1 a_1^1,\\
C_1^{V,1} &= a_3^1 a_1^1 a_2^1,\\
C_2^{V,1} &= a_1^2 a_2^2 a_3^2,\\
C_2^{V,2} &= a_2^2 a_3^2 a_1^2,\\
C_2^{W,1} &= a_3^2 a_1^2 a_2^2,\\
C_3^{W,1} &= a_1^3 a_2^3 a_3^3,\\
C_3^{W,2} &= a_2^3 a_3^3 a_1^3,\\
C_3^{X,1} &= a_3^3 a_1^3 a_2^3,\\
C_4^{X,1} &= a_1^4 a_2^4 a_3^4,\\
C_4^{X,2} &= a_2^4 a_3^4 a_1^4,\\
C_4^{U,1} &= a_3^4 a_1^4 a_2^4,
\end{aligned} \tag{1.3}$$

1. $a_i^h a_r^s$, if $h \neq s$, for all possible values of $i$ and $r$,
2. $(a_1^1)^2$, $(a_1^2)^2$, $(a_1^3)^2$, $(a_1^4)^2$, $a_3^1 a_2^1$, $a_3^2 a_2^2$, $a_3^3 a_2^3$, $a_3^4 a_2^4$,
3. $C_j^{U,i} - C_l^{U,k}$, for all possible values of $i, j, k$ and $l$,
4. $C_j^{V,i} - C_l^{V,k}$, for all possible values of $i, j, k$ and $l$,
5. $C_j^{W,i} - C_l^{W,k}$, for all possible values of $i, j, k$ and $l$,
6. $C_j^{X,i} - C_l^{X,k}$, for all possible values of $i, j, k$ and $l$,
7. $C_i^{U,j} a$ $(C_i^{V,j} a')$ , with $a$ $(a')$ being the first arrow of $C_i^{U,j}$ $(C_i^{V,j})$ for all $i, j$,
8. $C_i^{W,j} a$ $(C_i^{X,j} a')$ , with $a$ $(a')$ being the first arrow of $C_i^{W,j}$ $(C_i^{X,j})$ for all $i, j$.

The following Figures (1.2-1.5) show the quiver $Q_\Gamma$ associated to this configuration and the corresponding indecomposable projective modules $P_U$, $P_V$, $P_W$ and $P_X$. The corresponding heart and radical square of these modules are described as well.

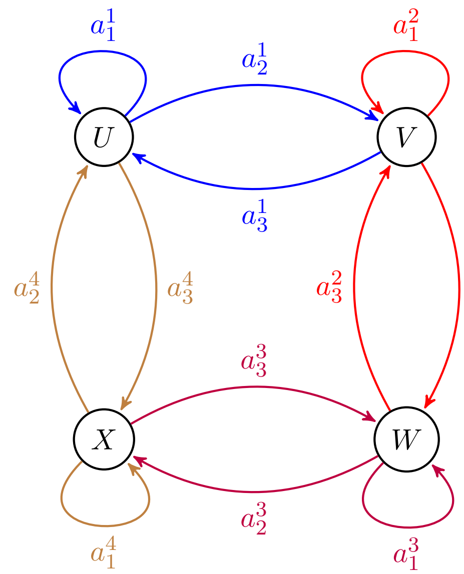



FIGURE 1.2. The quiver $Q_\Gamma$ associated to this configuration.

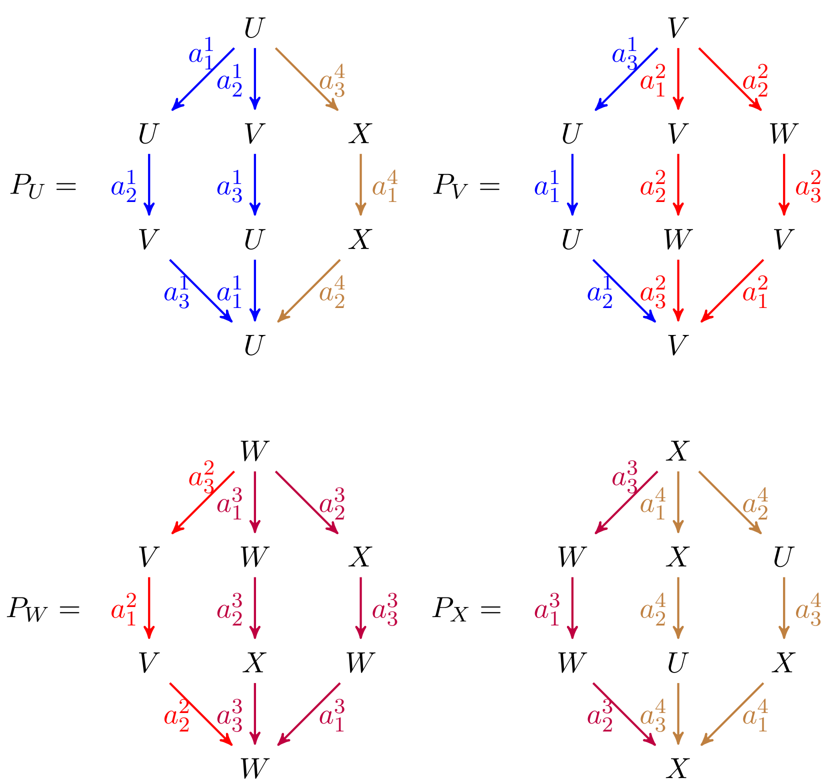



FIGURE 1.3. Indecomposable projective modules $P_U$, $P_V$, $P_W$, and $P_X$.

$$\mathrm{Heart}(P_U) = \begin{array}{ccc} U & V & X \\ \downarrow a_2^1 & \downarrow a_3^1 & \downarrow a_1^4 \\ V & U & X \end{array} \qquad \mathrm{Heart}(P_V) = \begin{array}{ccc} U & V & W \\ \downarrow a_1^1 & \downarrow a_2^2 & \downarrow a_3^2 \\ U & W & V \end{array}$$

$$\mathrm{Heart}(P_W) = \begin{array}{ccc} V & W & X \\ \downarrow a_1^2 & \downarrow a_2^3 & \downarrow a_3^3 \\ V & X & W \end{array} \qquad \mathrm{Heart}(P_X) = \begin{array}{ccc} W & X & U \\ \downarrow a_1^3 & \downarrow a_2^4 & \downarrow a_3^4 \\ W & U & X \end{array}$$

FIGURE 1.4. Heart of projective indecomposable modules $P_U$, $P_V$, $P_W$, and $P_X$.

$$\mathrm{Rad}^2(P_U) = \begin{tikzcd} V \arrow[dr, "a_3^1"'] & U \arrow[d, "a_1^1"'] & X \arrow[dl, "a_2^4"] \\ & U & \end{tikzcd} \qquad \mathrm{Rad}^2(P_V) = \begin{tikzcd} U \arrow[dr, "a_2^1"'] & W \arrow[d, "a_3^2"'] & V \arrow[dl, "a_1^2"] \\ & V & \end{tikzcd}$$

$$\mathrm{Rad}^2(P_W) = \begin{tikzcd} V \arrow[dr, "a_2^2"'] & X \arrow[d, "a_3^3"'] & W \arrow[dl, "a_1^3"] \\ & W & \end{tikzcd} \qquad \mathrm{Rad}^2(P_X) = \begin{tikzcd} W \arrow[dr, "a_2^3"'] & U \arrow[d, "a_3^4"'] & X \arrow[dl, "a_1^4"] \\ & X & \end{tikzcd}$$

FIGURE 1.5. $\mathrm{Rad}^2$ of projective modules $P_U$, $P_V$, $P_W$, and $P_X$.

The following results give some description of the structure of Brauer configuration algebras [31, 64].

**Theorem 3.** *Let $\Lambda$ be a Brauer configuration algebra with Brauer configuration $\Gamma$.*

1. *There is a bijective correspondence between the set of projective indecomposable $\Lambda$-modules and the polygons in $\Gamma$.*
2. *If $P$ is a projective indecomposable $\Lambda$-module corresponding to a polygon $V$ in $\Gamma$, then* rad $P$ *is a sum of $r$ indecomposable uniserial modules, where $r$ is the number of (non-truncated) vertices of $V$ and where the intersection of any two of the uniserial modules is a simple $\Lambda$-module.*
3. *A Brauer configuration algebra is a multiserial algebra.*
4. *The number of summands in the heart of an indecomposable projective $\Lambda$-module $P$ such that* $\mathrm{rad}^2\, P \neq 0$ *equals the number of non-truncated vertices of the polygons in $\Gamma$ corresponding to $P$ counting repetitions.*
5. *If $\Lambda'$ is a Brauer configuration algebra obtained from $\Lambda$ by removing a truncated vertex of a polygon in $\Gamma_1$ with $d \geq 3$ vertices then $\Lambda$ is isomorphic to $\Lambda'$.*

**Proposition 1.** *Let $\Lambda$ be the Brauer configuration algebra associated to the Brauer configuration $\Gamma$. For each $V \in \Gamma_1$ choose a non-truncated vertex $\alpha$ and exactly one special $\alpha$-cycle $C_V$ at $V$ then*

$\{\overline{p} \mid p\, is\, a\, proper\, prefix\, of\, some\, C^{\mu(\alpha)}\, where\, C\, is\, a\, special\, \alpha - cycle\} \bigcup \{\overline{C^{\mu(\alpha)}} \mid V \in \Gamma_1\}$ *is a $k$-basis of $\Lambda$.*

**Proposition 2.** *Let $\Lambda$ be a Brauer configuration algebra associated to the Brauer configuration $\Lambda$ and let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that for $i = 1, \ldots, t$, $C_i$ is a special $\alpha_i$-cycle where $\alpha_i$ is a non-truncated vertex in $\Gamma$. Then*

$$\dim_k \Lambda = 2|Q_0| + \sum_{C_i \in \mathcal{C}} |C_i|(n_i|C_i| - 1),$$

*where $|Q_0|$ denotes the number of vertices of $Q$, $|C_i|$ denotes the number of arrows in the $\alpha_i$-cycle $C_i$ and $n_i = \mu(\alpha_i)$.*

**Proposition 3.** *Let $\Lambda$ be the Brauer configuration algebra associated to a connected Brauer configuration $\Gamma$. The algebra $\Lambda$ has a length grading induced from the path algebra $kQ$ if and only if there is an $N \in \mathbb{Z}_{>0}$ such that for each non-truncated vertex $\alpha \in \Gamma_0$ $val(\alpha)\mu(\alpha) = N$.*

Sierra [64] proved the following result regarding the center of a Brauer configuration algebra.

**Theorem 4.** *Let $\Gamma$ be a reduced (i.e, without truncated vertices) and connected Brauer configuration and let $Q$ be its induced quiver and let $\Lambda$ be the induced Brauer configuration algebra such that $\operatorname{rad}^2 \Lambda \neq 0$ then the dimension of the center of $\Lambda$ denoted $\dim_k Z(\Lambda)$ is given by the formula:*

$$\dim_k Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(Loops\ Q) - |\mathfrak{C}_\Gamma|, \tag{1.4}$$

*where $|\mathfrak{C}_\Gamma| = \{\alpha \in \Gamma_0 \mid val(\alpha) = 1,\ and\ \mu(\alpha) > 1\}$.*

As an example the following is the numerology associated to the algebra $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$ with $Q_\Gamma$ as shown in Figure (1.2) and special cycles given in (1.3), ($|r(Q_\Gamma)|$ is the number of indecomposable projective modules, $r_U$, $r_V$, $r_W$ and $r_X$ denote the number of summands in the heart of the indecomposable projective modules $P(U)$, $P(V)$ , $P(W)$ and $P(X)$. Note that, $|C_i| = val(i)$):

$$
\begin{aligned}
|r(Q_\Gamma)| &= 4,\\
r_U = 3, \quad r_V &= 3, \quad r_W = 3, \quad r_X = 3,\\
|C_1| = 3, \quad |C_2| &= 3, \quad |C_3| = 3, \quad |C_4| = 3,\\
\sum_{\alpha\in\Gamma_0}\sum_{X\in\Gamma_1} \operatorname{occ}(\alpha, X) &= 12, \quad \text{the number of special cycles,}\\
\dim_k \Lambda_\Gamma &= 2(4) + 3(2) + 3(2) + 3(2) + 3(2) = 32,\\
\dim_k Z(\Lambda_\Gamma) &= 1 + 4 + (4 - 4) + 4 - 0 = 9.
\end{aligned}
\tag{1.5}
$$

*Remark* 2. Note that, the Brauer configuration algebra $\Lambda_\Gamma$ with quiver $Q_\Gamma$ shown in Figure 1.2 has a length grading induced by the path algebra $kQ_\Gamma$ as Green and Schroll describe in [31] section 3.3.

Green and Schroll in [30] proved the following result regarding relationships between Brauer configuration algebras, its multiplicity function and the trivial extension of almost gentle algebras.

**Corollary 1.** *Every symmetric special multiserial algebra with multiplicity function identically equal to one in its defining pair is a trivial extension of an almost gentle algebra. Equivalently, we have that every Brauer configuration algebra with multiplicity function identically equal to one is the trivial extension of an almost gentle algebra.*

## 1.4 Binomial trees and integer partitions

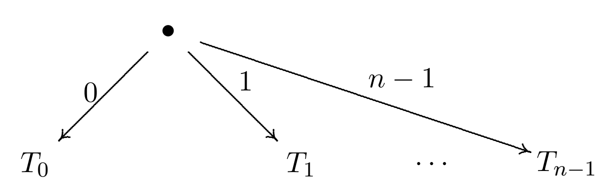
In this section, we recall definitions of integer partitions and binomial trees as given in [3] and [42].

### 1.4.1 Partitions

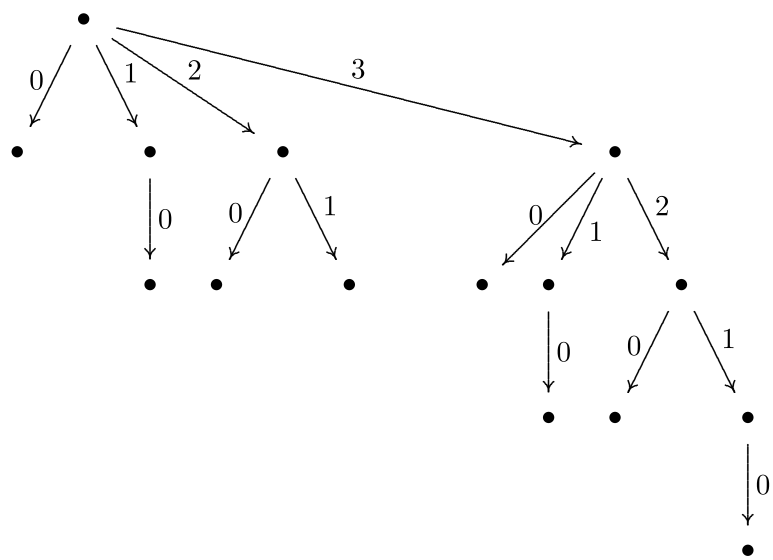
A *partition* of a positive integer $n$ is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum\limits_{i=1}^{r} \lambda_i = n$. The $\lambda_i$ are called the *parts* of the partition [3]. Often, $n$ is called the *weight* of the partition $\lambda$ and the symbol $|\lambda|$ is used to denote the *size* of the partition. A *composition* is a partition in which the order of the summands is considered.

### 1.4.2 Binomial trees

Binomial trees appear in many fields of the mathematics, they are binary trees with the shape [42]:



As an example $T_4$ has the following form:



Note that, at each level $T_4$ gives integer partitions of numbers 1, 2 and 3 (without taking into account 0 as a part). Often, these types of trees are said to be *partition trees* which can be used to store partitions of a given positive integer $n$ or of all positive integers $\leq n$ [45]. In [46], P. Luschny describes partition trees for different integer numbers and use them to define some orders on the set of integer partitions.

*Remark* 3. Let $\mathcal{F}(n, r)$ be the set of partitions $\lambda$ of a fixed positive integer $n$ into $r$ parts, we write $\lambda = \{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r\} \in \mathcal{F}(n, r)$ with $n = \sum\limits_{i=1}^{r} \lambda_i$.

$\mathcal{F}(n, r)$ is endowed with an order $\preceq$ such that if $\lambda, \lambda' \in \mathcal{F}(n, r)$ then $\lambda \preceq \lambda'$ provide that either $\lambda_i = \lambda'_i$ for any $i$, $1 \leq i \leq r$ or there exists an index $j$, such that $\lambda_i = \lambda'_i$ for any $i$, $1 \leq i < j$ and $\lambda_j < \lambda'_j$ [3, 14].

We let $T_{(S,n)}$ denote a partition tree whose last level contains only partitions of a positive integer $n$ into parts belonging to a given set $S$.

A subset of positive integers $\mathcal{G}_S = \{c_1, c_2, \ldots, c_h\}$ is said to be a *generator* of a set of partitions $(S, n)$ of a positive integer $n$ whose parts belong to a set $S$ of positive integers, if there exists a set of indexes $1 \leq i_1, i_2, i_3, \ldots, i_r \leq h$, $i_j = i_{j-1} + 1$, $2 \leq j \leq h$ such that any partition $\lambda \in (S, n)$ can be written in the form:

$$\lambda = \{c_1, c_2, \ldots, c_{i_1 - 1}, \sum_{s=1}^{r} c_{i_s}, c_{i_{r+1}}, \ldots, c_h\}. \tag{1.6}$$

Often, in order to generate restricted partitions of type $\mathcal{F}(n, r)$ several sums as defined in (1.6) must be applied to the components of $\mathcal{G}_S$ in such a way that $\lambda$ is obtained by a suitable association of the components without alter its order.

## 1.5 Cluster algebras

In this section, we make a short introduction to cluster algebras, which allow to give a categorification of Catalan numbers via the clusters associated to Dynkin diagrams of type $\mathbb{A}_n$ [22, 23, 44].

Cluster algebras were conceived by Fomin and Zelevinsky [22, 23] in the spring of 2000 as a tool for studying total positivity and dual canonical bases in Lie theory. However, the theory of cluster algebras has since taken on a life of its own, as connections and applications have been discovered to diverse areas of mathematics including quiver representations, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

Let us $\mathcal{F}$ be a field extension of $\mathbb{Q}$. Typically we have $\mathcal{F} = \mathbb{Q}(u_1, u_2, \dots, u_n)$ for some algebraically independent variables $u_1, u_2, \dots, u_n$. The field $\mathcal{F}$ is called the ambient field.

A *cluster* is a sequence $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{F}^n$ of algebraically independent elements of length $n$. We refer to the elements in a cluster $\mathbf{x} \in \mathcal{F}^n$ as cluster variables.

If $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{F}^n$ is a cluster, then the field $\mathcal{F}$ must contain the field $\mathbb{Q}(x_1, x_2, \dots, x_n)$. Thus, if we have a distinguished cluster $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{F}^n$, then the smallest possible field, namely $\mathbb{Q}(x_1, x_2, \dots, x_n)$, is a natural choice of an ambient field.

A *seed* is a pair $(\mathbf{x}, Q)$ where $\mathbf{x} \in \mathcal{F}^n$ is a cluster and $Q$ is a quiver with vertices $Q_0 = \{1, 2, \dots, n\}$ without loops and 2-cycles.

Assume that $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are two seeds given by clusters $\mathbf{x}, \mathbf{x}' \in \mathcal{F}^n$ and quivers $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$. We say that the seeds are isomorphic, if there exists a quiver isomorphism given by two bijections $\sigma : Q_0 \mapsto Q'_0$ and $\tau : Q_1 \mapsto Q'_1$ such that $x_i = x'_{\sigma(i)}$for all indices $i \in \{1, 2, \dots, n\}$. In other words, two seeds are isomorphic if they are obtained from each other by a simultaneous reordering of cluster variables and quiver vertices. In this case we write $(\mathbf{x}, Q) \cong (\mathbf{x}', Q')$. Often we identify isomorphic seeds [44].

Let $(\mathbf{x}, Q)$ be a seed and $k \in \{1, 2, \dots, n\}$ an index. The *mutation* of $(\mathbf{x}, Q)$ at $k$ is a seed $\mu_k(\mathbf{x}, Q) = (\mu_k(\mathbf{x}), \mu_k(Q))$ where $\mu_k(Q)$ is the mutation of the quiver $Q$ at vertex $k$ and $\mu_k(x) = (x'_1, x'_2, \dots, x'_k) \in \mathcal{F}^n$ is the cluster with $x'_l = x_l$ if $l \neq k$ and

$$x'_k = \frac{1}{x_k} \left( \prod_{\alpha : i \to k} x_i + \prod_{\beta : k \to j} x_j \right) \in \mathcal{F}. \tag{1.7}$$

Here the product is taken over all arrows in $\alpha \in Q_1$ that start or terminate in vertex $k$, respectively, counted possibly with multiplicity. Of course, the product is understood to be 1 if there are no such arrows.

Let $B = B(Q)$ is the signed adjacency matrix of the quiver $Q$ in a seed $(\mathbf{x}, Q)$, then we can rewrite equation 1.7 as

$$x_k x_k' = \prod_{\alpha:i\to k} x_i + \prod_{\beta:k\to j} x_j = \prod_{i\in\{1,2,\ldots,n\}:b_{ik}>0} x_i^{b_ik} + \prod_{i\in\{1,2,\ldots,n\}:b_{ik}<0} x_i^{-b_ik} \tag{1.8}$$

Equation (1.8) is also called *exchange relation.*

It is easy to see that the mutation is well-defined, i.e. the mutation of a seed at an index is again a seed. Moreover, mutation is involutory, i.e. for all seeds $(\mathbf{x}, Q)$ and all indices $k \in \{1, 2, \ldots, n\}$ we have $(\mu_k \circ \mu_k)(\mathbf{x}, Q) \cong (\mathbf{x}, Q)$.

Mutation equivalence defines an equivalence relation on the class of all quivers without loops and 2-cycles.

We say that two seeds $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are *mutation equivalent* if there exists a sequence $(k_1, k_2, \cdots, k_r) \in Q_0^r$ of indices of length $r \geq 0$ such that the seed $(\mu_{k_1} \circ \mu_{k_2} \circ \ldots \circ \mu_{k_r})(\mathbf{x}, Q)$ is isomorphic to $(\mathbf{x}', Q')$. In this case, we also write $(\mathbf{x}, Q) \sim (\mathbf{x}', Q')$.

Let $(\mathbf{x}, Q)$ be a seed. The *cluster algebra* $A(\mathbf{x}, Q)$ attached to the seed is the subalgebra of the ambient field $\mathcal{F}$ generated by the set

$$\chi(\mathbf{x}, Q) = \bigcup_{(\mathbf{x}',Q')\sim(\mathbf{x},Q)} \{x_1' x_2', \ldots, x_n'\},$$

a cluster algebra is of *finite type* if it has finitely many seeds. The following is the finite type criterion as described by Fomin and Zelevinsky in [23].

**Theorem 5.** *For a cluster algebra $A$ the following are equivalent:*

*(i) $A$ is of finite type,*

*(ii) The set $\chi$ of all cluster variables is finite.*

*(iii) For every seed $(\boldsymbol{x}, \boldsymbol{p}, B)$ in $A$ the entries of the matrix $B = (b_{xy})$ satisfy the inequalities $\mid b_{xy} b_{yx} \mid \leq 3$, for all $x, y \in \boldsymbol{x}$.*

*(iv) $A = A(B_0, \boldsymbol{p}_0)$ for some sign-skew-symmetric matrix $B_0 = (b_{ij})$ such that $A = A(B_0)$ and $b_{ij} b_{ik} \geq 0$ for all $i, j, k$ and $\boldsymbol{p}_0$ is a $2n$-tuple of elements in $\mathbb{P}$ satisfying the normalization conditions,*

*where if $\mathcal{F}$ is isomorphic to the field of rational functions (as described above) then $\boldsymbol{x}$ is an $n$-element subset of $\mathcal{F}$, $\mathbb{P}$ is a torsion-free semifield, $\mathcal{F}$ is a transcendence basis over the field of fractions of $\mathbb{ZP}$. $\boldsymbol{p} = (p_x^{\pm})_{x\in\boldsymbol{x}}$ is a $2n$-tuple of elements of $\mathbb{P}$, satisfying the normalization condition $p_x^+ \oplus p_x^- = 1$ for all $x \in \boldsymbol{x}$ and $B = (b_{xy})_{x,y\in\boldsymbol{x}}$ is an $n \times n$-integer matrix with rows and columns indexed by $\boldsymbol{x}$, which is sign-skew-symmetric (i.e., for any $x, y \in \boldsymbol{x}$, either $b_{xy} = b_{yx} = 0$ or $b_{xy} b_{yx} < 0$).*

## 1.6 Snake graphs

In this section, some classical theorems (without proof) of snake graphs, continued fractions and cluster variables are reminded [53, 56–61].

Snake graphs are combinatorial objects arising from the research of cluster algebras, they allowed to Çanaçi, Schiffler et al to compute the Laurent expansions of the cluster variables in cluster algebras of surface type. The terms in the Laurent polynomial of such variables are parametrized by the perfect matchings of the associated snake graph [53, 56–61]. Such graphs were studied by Prop [53] in the context of the investigation of the Laurent phenomenon, which is a problem of paramount importance in the theory of cluster algebras, Prop proved that two examples of rational recurrences, the two-dimensional frieze patterns of Conway and Coxeter and the tree of Markoff numbers-relate to one another and to the Laurent phenomenon. In the program of Prop perfect matchings of snake graphs derived from triangulations of polygons are linked with frieze patterns of Conway and Coxeter.

Prop [53] also reported an interesting connection between snake graphs and continued fractions, according to him, work of Benjamin and Quinn in the context of the strip tiling model, shows how combinatorial models can illuminate facts about continued fractions. In [56–60] Çanaçi and Schiffler go beyond Prop by proving that each snake graph $\mathcal{G}$ has associated a unique continued fraction whose numerator is given by the number of perfect matchings of a suitable snake graph. They report that snake graphs provides a new combinatorial model for continued fractions allowing to interpret the numerators and denominators of positive continued fractions as cardinalities of combinatorially defined sets.

A *tile* $G$ is a square in the plane whose sides are parallel or orthogonal to the elements in the standard orthonormal basis of the plane (as in [60] in this work a tile $G$ is considered as a graph with four vertices and four edges in the obvious way).

A *snake graph* $\mathcal{G}$ is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \dots, G_d$ such that $G_i$ and $G_{i+1}$ share exactly one edge $e_i$ and this edge is either the north edge of $G_i$ and the south edge of $G_{i+1}$ or the east edge of $G_i$ and the west edge of $G_{i+1}$ [56–60].

Denote by $\mathrm{Int}(\mathcal{G}) = \{e_1, e_2, \dots, e_{d-1}\}$ the set of interior edges of the snake graph $\mathcal{G}$. We will use the natural ordering of the set of interior edges of $\mathcal{G}$, so that $e_i$ is the edge shared by tiles $G_i$ and $G_{i+1}$.

A snake graph is called *straight* if all its tiles lie in one column or one row, and a snake graph is called *zigzag* if no three consecutive tiles are straight. Two snake graphs are *isomorphic* if they are isomorphic as graphs.

A *labeled* snake graph is a snake graph in which each edge and each tile carries a label or weight. For example, for snake graphs from cluster algebras of surface type, these labels are cluster variables. Formally, a labeled snake graph is a snake graph $\mathcal{G}$ together with two functions

$$\{\text{tiles in } \mathcal{G}\} \to \{\text{edges in } \mathcal{G}\} \to \mathcal{F},$$

where $\mathcal{F}$ is a set. Throughout this document we only consider labels over the tiles.

For positive integers $n_1, n_2, \ldots, n_k$, we let $\mathcal{G}_f(n_1, n_2, \ldots, n_k)$ denote a snake graph with $n_1 \geq 2$ tiles in the first row, $n_2 \geq 2$ in the first column, $n_3 \geq 2$ tiles in the second row and so on up to $n_k \geq 2$, in this case the last tile in a given row is the first tile in the next column (if it exists) vice versa the last tile in a given column coincides with the first tile in the next row. As an example, in Figure 1.6, it is shown the snake graph $\mathcal{G}_f(5,3,3,2,5,4,2)$.

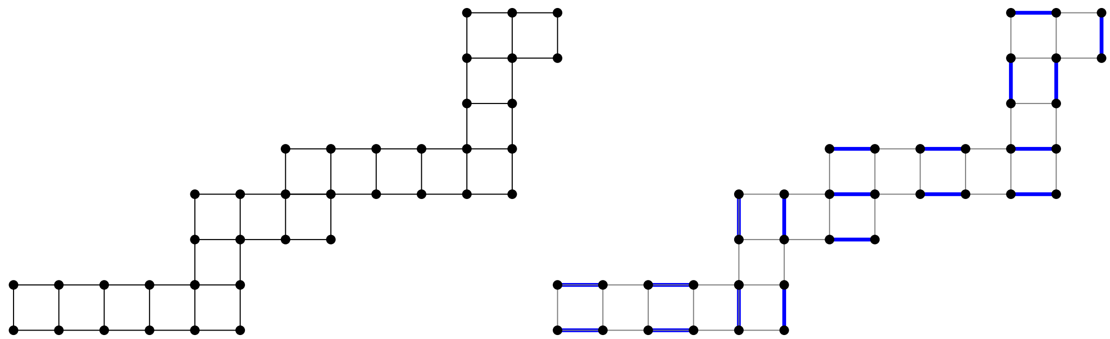


FIGURE 1.6. Snake graph $\mathcal{G}_f(5,3,3,2,5,4,2)$ (left) and an example of its perfect matchings.

A *perfect matching* $P$ of a graph $G$ is a subset of the edges of $G$ such that every vertex of $G$ is incident to exactly one edge in $P$. We denote by $\mathrm{Match}(G)$ the set of perfect matchings of $G$.

A *sign function* $f$ of a snake graph $\mathcal{G}$ is a map $f$ from the set of edges of $\mathcal{G}$ to $\{+,-\}$ such that on every tile in $\mathcal{G}$ the north and the west edge have the same sign and the sign on the north edge is opposite to the sign on the south edge.

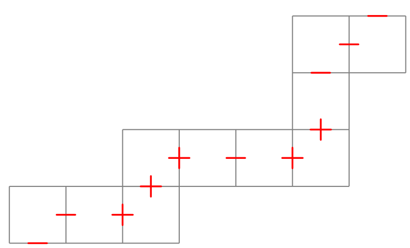



FIGURE 1.7. Example of a sign function defined on the set of edges of a snake graph.

Note that, on every snake graph there are exactly two sign functions. A snake graph is determined up to symmetry by its sequence of tiles together with a sign function on its interior edges $\{e_1, e_2, \ldots, e_{d-1}\}$. Henceforth, it will be assumed the notation $e_0 = sw(\mathcal{G})$ (the edge at the southwest of the first tile).

If $e_d \in ne(\mathcal{G})$ (the edge at the northeast of the last tile) the sign function can be extended in a unique way to all edges in $\mathcal{G}$ and it is obtained a sign sequence

$sgn(\mathcal{G}) = \{f(e_0), f(e_1), f(e_2), \ldots, f(e_{d-1}), f(e_d)\}$ this sequence uniquely determines the snake graph and a choice of a north east edge $e_d \in ne(\mathcal{G})$.

R. Schiffler et al [48] obtained the following result giving a formula for cluster variables by using perfect matchings of suitable snake graphs:

**Theorem 6.** *If $\gamma$ is an arc in a triangulated surface $(S, M)$, the cluster variable $x_\gamma$ is given by the formula:*

$$x_\gamma = \frac{1}{\text{cross}G_\gamma} \sum_{P \in \text{Match } \mathrm{G}_\gamma} x(P). \tag{1.9}$$

Where the sum runs over all perfect matchings of $G_\gamma$, $x(P)$ is the weight of the perfect matching $P$ and cross$(G_\gamma)$ is the product (with multiplicities) of all initial cluster variables whose arcs are crossed by $\gamma$.

### 1.6.1 Relationships between snake graphs and continued fractions

A positive finite *continued fraction* is a function

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}} \tag{1.10}$$

on $n$ variables $a_1, a_2, \ldots, a_n$, $a_i \in \mathbb{Z}_{\geq 1}$ [56, 60].

Now let $[a_1, a_2, \ldots, a_n]$ be a positive continued fraction and let $d = a_1 + a_2 + \cdots + a_n - 1$ and consider the sign sequence:

$$(\underbrace{-\varepsilon, \ldots, -\varepsilon}_{a_1}, \underbrace{\varepsilon, \ldots, \varepsilon}_{a_2}, \ldots, \underbrace{\pm\varepsilon, \ldots, \pm\varepsilon}_{a_n}) \tag{1.11}$$

where $\varepsilon \in \{+, -\}$,

$$-\varepsilon = \begin{cases} +, & \text{if } \varepsilon = -, \\ -, & \text{if } \varepsilon = + \end{cases}$$

$$sgn(a_i) = \begin{cases} -\varepsilon, & \text{if } i \text{ is odd,} \\ \varepsilon, & \text{if } i \text{ is even.} \end{cases}$$

Thus each integer $a_i$ corresponds to a maximal subsequence of constant sign $sgn(a_i)$ in the sequence (1.11).

The snake graph $\mathcal{G}[a_1, a_2, \ldots, a_n]$ of the positive continued fraction $[a_1, a_2, \ldots, a_n]$ is the snake graph with $d$ tiles determined by the sign sequence (1.11). In particular, $\mathcal{G}[1]$ is a single edge and the continued fraction of the graph in Figure 1.7 is $[2, 3, 1, 2, 3]$.

Schiffler et al report the following results regarding snake graphs and their relationships with continued fractions [56–60].:

**Theorem 7.** *The number of snake graphs with exactly $N$ perfect matchings is $\phi(N)$ where $\phi$ is the totient Euler function.*

**Theorem 8.**
1. *The number of perfect matchings of $\mathcal{G}[a_1, a_2, \ldots, a_n]$ is equal to the numerator of the continued fraction $[a_1, a_2, \ldots, a_n]$.*
2. *The number of perfect matchings of $\mathcal{G}[a_2, a_3, \ldots, a_n]$ is equal to the denominator of the continued fraction $[a_1, a_2, \ldots, a_n]$.*
3. *If* $\mathrm{Match}(\mathcal{G})$ *denotes the number of perfect matchings of the snake graph $\mathcal{G}$ then $[a_1, a_2, \ldots, a_n] = \frac{\mathrm{Match}(\mathcal{G})[a_1,a_2,\ldots,a_n]}{\mathrm{Match}(\mathcal{G})[a_2,a_3,\ldots,a_n]}$.*

For instance the snake graph $\mathcal{G}[2, 3, 1, 2, 3]$ shown in Figure 1.6 has 84 perfect matchings. For $[a_1, a_2, \ldots, a_n] = [1, 1, \ldots, 1]$ we recall that the straight snake graph $\mathcal{G} = \mathcal{G}[1, 1, \ldots, 1]$ with $n - 1$ tiles has $F_{n+1}$ perfect matchings where $F_{n+1}$ denotes the $(n + 1)$st Fibonacci number.

A continued fraction $[a_1, a_2, \ldots, a_n]$ is said to be of *even length* if $n$ is even. It is called *palindromic* if the sequences $(a_1, a_2, \ldots, a_n)$ and $(a_n, a_{n-1}, \ldots, a_2, a_1)$ are equal.

A snake graph $\mathcal{G}$ is called palindromic if it is the snake graph of a palindromic continued fraction.

**Theorem 9.** *Let $\mathcal{G} = \mathcal{G}[a_1, a_2, \ldots, a_n]$ be a snake graph and $\mathcal{G}_{\leftrightarrow}$ its palindromification. Let $\mathcal{G}' = \mathcal{G}[a_2, a_3, \ldots, a_n]$ then* $\mathrm{Match}(\mathcal{G}_{\leftrightarrow}) = (\mathrm{Match}(\mathcal{G}))^2 + (\mathrm{Match}(\mathcal{G}'))^2$.

**Corollary 2.**
1. *If $N = p^2 + q^2$ with $(p, q) = 1$ (i.e., $N$ is a sum of two relatively prime squares) then there exists a palindromic snake graph of even length such that* $\mathrm{Match}(\mathcal{G}) = N$.
2. *For each positive integer $N$, the number of ways we can write $N$ as a sum of two relatively prime numbers is equal to one half of the number of palindromic snake graphs of even length with $N$ perfect matchings.*
3. *For each positive integer $N$, the number of ways one can write $N$ as a sum of two relatively prime squares is equal to one half of the number of palindromic continued fractions of even length with numerator $N$.*

Regarding Markoff numbers we recall the following results:

A triple of positive integers $(m_1, m_2, m_3)$ is called a *Markoff triple* if it is a solution of the markoff equation

$$x^2 + y^2 + z^2 = 3xyz \tag{1.12}$$

An integer is called a Markoff number if it is a member of a Markoff triple.

Frobenius conjectured that the *largest number in a Markoff triple determines the other two* (1913). This is known as the uniqueness conjecture for Markoff numbers.

It is also known that *every Markoff number except 1 and 2 is a sum of two relatively prime squares.* And that *every Markoff number is the numerator of an even palindromic continued fraction. And except 1, it is a sum of two relatively prime squares.*

## 1.7 Tangles

In this section, we recall some basic concepts regarding rational knots [16, 41].

Rational knots and links, also known in the literature as four-plots, Viergeflechte and 2-bridge knots, are a class of alternating links of one or two unknotted components and they are the easiest knots to make (also for nature). The first twenty five knots except $8_5$, are rational. Furthermore all knots up to ten crossings are either rational or are obtained from rational knots by certain simple operations.

A *rational tangle* is the result of consecutive twists on neighbouring end points of two trivial arcs. Rational knots are obtained by taking numerator closures of rational tangles, which form a basis for their classification. Rational knots and rational tangles are of fundamental importance in the study of DNA recombination.

A rational tangle is associated in a canonical manner with a unique reduced rational number or $\infty$, called the fraction of the tangle.

Rational tangles are classified by their fractions by means of the following theorem

**Theorem 10** (Conway, 1970)**.** *Two rational tangles are isotopic if and only if they have the same fraction.*

More than one rational tangle can yield the same or isotopic rational knots and the equivalence relation between the rational tangles is reflected into an arithmetic equivalence of their corresponding fractions. This is marked by a theorem due originally to Schubert [62] and reformulated by Conway [16] in terms of rational tangles.

**Theorem 11** (Schubert, 1956)**.** *Suppose that rational tangles with fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given ($p$ and $q$ are relatively prime. Similarly for $p'$ and $q'$.) If $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are topologically equivalent if and only if*

1. *$p = p'$ and*
2. *either $q \equiv q' \pmod{p}$ or $qq' \equiv 1 \pmod{p}$.*

An $(m, n)$ tangle is an embedding of a finite collection of arcs (homeomorphics [0,1]) and circles into the three dimensional euclidean space such that the end points of the arcs go to a specific set of $m+n$ points on the surface of a ball $B^3$ standard embedded in $S^3$ so that, the $m$ points lie on the upper hemisphere and the $n$ points lie in the lower hemisphere.

An $(n, n)-$tangle will be called an $n$-tangle knots and links are 0-tangles and braids on $n$ strands are the most well known example of $n$-tangles

One can define a diagram of an $(m, n)$- tangle to be a regular projection of the tangle in the plane of this great circle.

**Definition 1.** *Let $t$ be a pair of unoriented arcs properly embedded in a 3-ball $B$. A 2-tangle is rational if there exists an orientation preserving homeomorphisms of pairs*

$$g : (B, t) \longleftrightarrow ((D^2) \times I, \{x, y\} \times I) \quad \textit{(a trivial tangle).}$$

The last definition is equivalent to saying that rational tangles can be obtained by applying a finite number of consecutive twists of neigbouring endpoints to the elementary tangles [0] or [$\infty$].

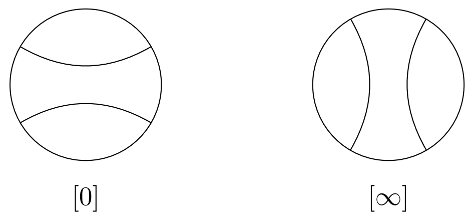



FIGURE 1.8. The trivial tangles [0] and [$\infty$].

If $T_{(m,n)}$ denote the set of all $(m, n)$ tangles among all tangles, the class $T_{(2,2)}$ of 2-tangles is particularly interesting fo various reasons. For one, it is closed under addition (+) and start ($*$) product as illustrated in Figures 1.9 and 1.10.

Addition is accomplished by placings the tangles side-by-side and attaching the NE strand of the left tangle to the NW strand of the right tangle, while attaching the SE strand of the left tangle to the SW strand of the right tangle (see Figure 1.9).

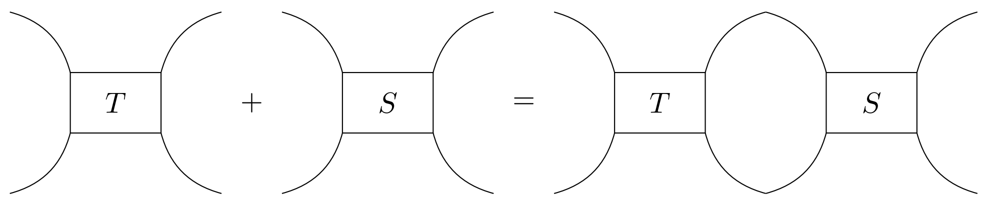



FIGURE 1.9. Addition of 2-tangles.

The star ($*$) product is accomplished by placing one tangle underneath the other and attaching the upper strands of the lower tangle to the lower strands of the upper tangle (see Figure 1.10).

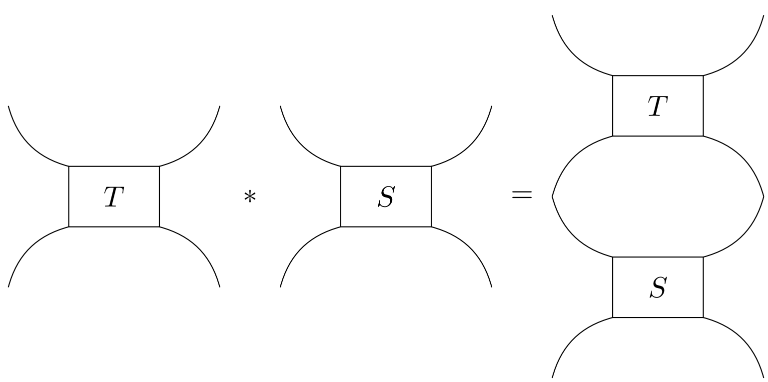



FIGURE 1.10. Product of 2-tangles.

The mirror image of a tangle $T$ is denoted by $-T$ and it is obtained by switching all the crossings in $T$.

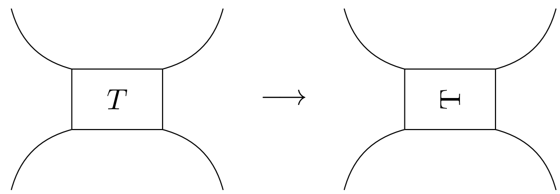



FIGURE 1.11. Inversion of 2-tangles.

A third operation illustrated is inversion accomplished by turning the tangle counter clockwise 90 in the plane it is denoted $T^i$. We note that all operations in $T_{(2,2)}$ can be generalized appropriately to operations in $T_{(m,n)}$ [41].

We defined rational tangles as being obtained by applying a finite number of consecutive twists of neighbouring end points to the elementary tangles $[0]$ and $[\infty]$. Clearly the simplest rational tangles are the $[0]$, the $[\infty]$ the $[+1]$ and the $[-1]$ tangles while the next simplest ones are:

(i) The integer tangles, denoted by $[n]$ made of $n$ horizontal twists, $n \in \mathbb{Z}$.

(ii) The vertical tangles denoted by $\frac{1}{[n]}$ made of $n$ vertical twists, $n \in \mathbb{Z}$

These are the inverses of the integer tangles, see Figure 1.12. This terminology explains the need for mirror imaging in the definition of inversion.

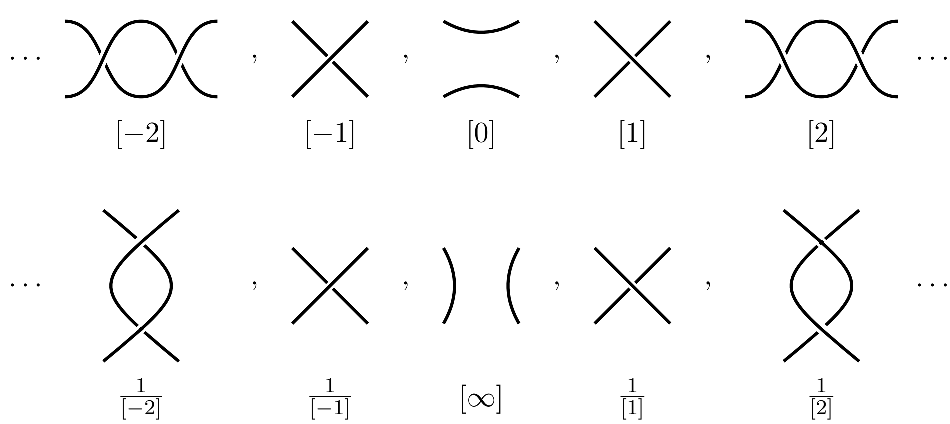



FIGURE 1.12. The elementary rational tangles.

**Example 1.** *Figure 1.13 shows an example of a rational tangle $\mathcal{T}_1$, which is the sum of the tangles $\mathcal{W}(3,2)$ and $\mathcal{W}(4,3)$. In Figure 1.13 on the left hand side the tangle $\mathcal{W}(3,2)$ is obtained by the start product of the integer tangles* $[3], [-3], [-2], [2]$ *and* $[3]$*, on the right* *hand side the tangle $\mathcal{W}(4,3)$, which is obtained by the multiplication of the integer tangles $[4], [-4], [-2], [2], [4], [2]$ and $[-6]$.*

*The general case of this example i.e. for $n \geq 1$ the tangle $\mathcal{T}_n = \mathcal{W}(3,2) + \mathcal{W}(4,3) + \cdots + \mathcal{W}(n+3, n+2)$, where $\mathcal{W}(n+3,n+2) = [n+3] * [-(n+3)] * [-2] * [2] * [n+3] * [2] * [-(n+5)] * [2] * [n+3] * [2] * [-(n+5)] * [2] * [n+3] * \cdots$ with $l(\mathcal{W}(n+3,n+2)) = 2n+5$ is said to be the $n$-th preprojective Kronecker tangle, which can be obtained as the message of the Brauer configuration algebra introduced in Section 2.4.1 (see Theorem 19).*

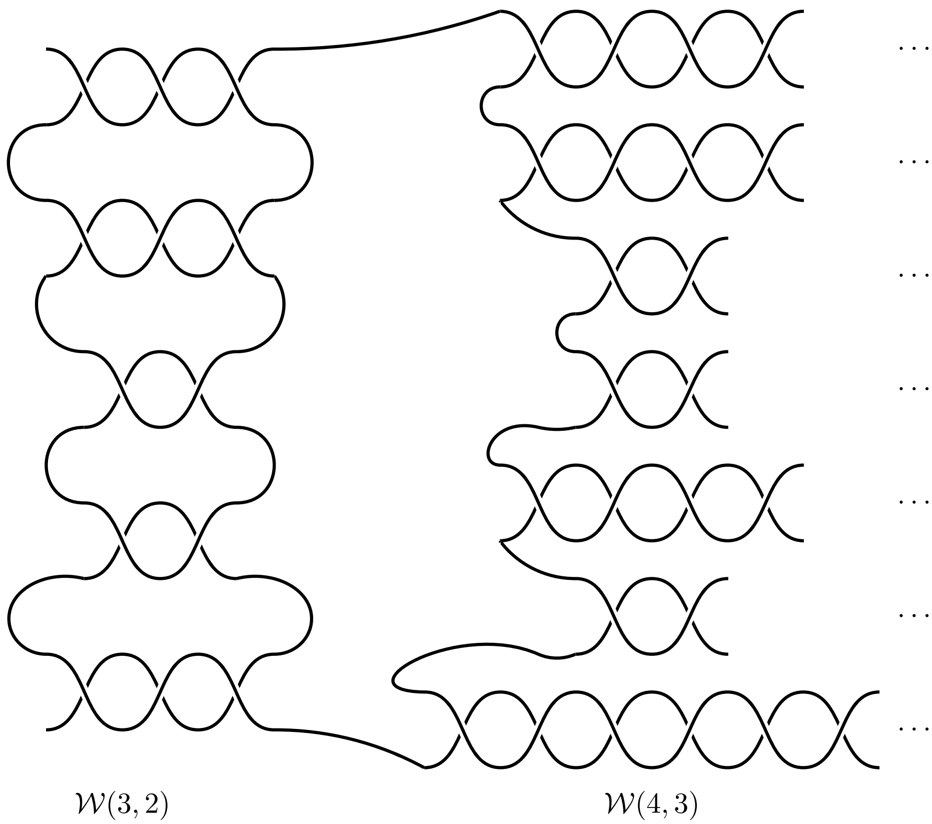



FIGURE 1.13. Preprojective Kronecker tangle $\mathcal{T}_1$.

# CHAPTER 2

# Categorification of continued fractions and Brauer configuration algebras

In this chapter, the notions of the message of a Brauer configuration and labeled Brauer configurations are introduced. These concepts allow us to establish unexpected connections between different fields of mathematics and categorify some integer sequences. In particular, in section 2.2 we show how some suitable Brauer configurations and these notions can be used to give an explicit formula for the number of perfect matchings of a snake graph. Also, in section 2.3 some relationships between Brauer configuration algebras with path problems as the Lindström problem are described. In section 2.4, we introduce Kronecker snake graphs and we use them to describe the non-regular Kronecker indecomposable modules inspired mainly in some ideas described in [75]. Regarding the preprojective Kronecker snake graphs in section 2.4.1 we introduce the preprojective Kronecker tangles as the message of a labeled Brauer configuration. Finally, in section 2.4.2 we study the group structure of preinjective Kronecker snake graphs.

## 2.1 Labeled Brauer configuration algebras

In this section, we give the notion of labeled Brauer configurations, which is helpful to define suitable specializations of some Brauer configuration algebras. Besides, the message of a Brauer configuration is introduced.

Let $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathcal{O}\}$ be a Brauer configuration and let $U \in \Gamma_1$ be a polygon such that $U = \{\alpha_1^{f_1}, \alpha_2^{f_2}, \dots, \alpha_n^{f_n}\}$, where $f_i = \mathrm{occ}(\alpha_i, U)$. The term

$$w(U) = \alpha_1^{f_1}\alpha_2^{f_2}\dots\alpha_n^{f_n} \tag{2.1}$$

is said to be the *word associated to* $U$. The sum

$$M(\Gamma) = \sum_{U\in\Gamma_1} w(U) \tag{2.2}$$

is said to be the *message of the Brauer configuration* $\Gamma$.

An *integer specialization* of a Brauer configuration $\Gamma$ is a Brauer configuration $\Gamma^e = (\Gamma_0^e, \Gamma_1^e, \mu^e, \mathcal{O}^e)$ endowed with a preserving orientation map $e : \Gamma_0 \to \mathbb{N}$, such that

$$\begin{aligned}\Gamma_0^e &= \operatorname{Img} e \subset \mathbb{N},\\ \Gamma_1^e &= e(\Gamma_1), \quad \text{if } H \in \Gamma_1 \text{ then } e(H) = \{e(\alpha_i) \mid \alpha_i \in H\} \in e(\Gamma_1),\\ \mu^e(e(\alpha)) &= \mu(\alpha), \text{ for any } \alpha \in \Gamma_0.\end{aligned} \tag{2.3}$$

Besides $e(U) \preceq e(V)$ in $\Gamma_1^e$ provided that $U \preceq V$ in $\Gamma_1$.

*Remark* 4. A real valued sequence $S = \{s_i\}$ is said to be built by a specialization $e$ of a Brauer configuration $\Gamma$, if for any $s_i \in S$ there is a subset $\mathcal{G} \in \Gamma_0$ such that $e(\mathcal{G}) = s_i$.

We let $w^e(U) = (e(\alpha_1))^{f_1}(e(\alpha_2))^{f_2} \cdots (e(\alpha_n))^{f_n}$ denote the specialization under $e$ of a word $w(U)$. In such a case, $M(\Gamma^e) = \sum\limits_{U \in \Gamma_1^e} w^e(U)$ is the *specialized message* of the Brauer configuration $\Gamma$ with the usual integer sum and product (in general with the sum and product associated to Img $e$).

A Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ is said to be *labeled* if each polygon is labeled by an element of $\mathbb{N}^s$ for some $s \geq 1$. In such a case we write

$$\Gamma_1 = \{(U_1, n_1), (U_2, n_2), \ldots, (U_k, n_k)\}, \quad n_j \in \mathbb{N}^s,$$

with $(U_i, n_i) \prec (U_{i+1}, n_{i+1})$, for $1 \leq i \leq k-1$ if $U_i \prec U_{i+1}$ in $\Gamma$, i.e., the labeling do not alter the orientation $\mathcal{O}$.

As an example, we define the following labeled Brauer configuration $\mathcal{K} = (\mathcal{K}_0, \mathcal{K}_1, \mu, \mathcal{O})$, where:

$$\begin{aligned}\mathcal{K}_0 &= \{\alpha_w^i \mid 1 \leq i \leq k, w \in \{0,1\}^{k-1}, k \geq 2 \text{ fixed}\},\\ \mathcal{K}_1 &= \{(U_w, n) \mid \alpha_w^i \in (U_w, n), n = (n_1, n_2, \ldots, n_k), \text{ fixed}, n_j \geq 2\}.\end{aligned} \tag{2.4}$$

Vertices $\alpha_w^i \in (U_w, n) \in \mathcal{K}_1$ are given by the following formula bearing in mind that $w$ is of the form $w = (w_1, w_2, \ldots, w_{k-1})$.

$$\alpha_w^i = n_i - g(w_{i-1}, i) - g(w_i, i) + 2, \tag{2.5}$$

where $g$ is a map $g := \{0,1\} \times \mathbb{Z}^+ \to \{1,2\}$ defined by

$$g(0,i) = \begin{cases} 2, & \text{if } i \text{ is even;} \\ 1, & \text{if } i \text{ is odd;} \end{cases} \quad \text{and} \quad g(1,i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd.} \end{cases}$$

In particular, $g(w_0, 1) = g(w_k, k) = 0$. The definition of $g$ can be reformulated by the rule $g(x, n) = 2 - (x + n \pmod 2)$.

In this case, $\mu(\alpha) = 2$, for any vertex $\alpha \in \mathcal{K}_0$ and the orientation $\mathcal{O}$ is given by the relation $\prec$.

## 2.2 On the number of perfect matchings of snake graphs via Brauer configuration algebras

In this section, formulas for the number of perfect matchings of snake graphs $\mathcal{G}_f(n_1, n_2, \ldots, n_k)$ are given by specializations of the labeled Brauer configuration 2.4 (the results described in this section were obtained in a joint work with J.L. Ramírez, J.F. González, J. P. Herran and A.M. Cañadas).

Firstly, we note that;

$$\mathrm{Match}(\mathcal{G}_f((n)) = F_{n+2} \tag{2.6}$$

where $F_n$ denotes the $n$-th Fibonacci number.

**Theorem 12.** $\mathrm{Match}(\mathcal{G}_f((n)) = F_{n+2}$, *for all integer number* $n \geq 1$.

*Proof.* For any perfect matching of $\mathcal{G}_f((n))$ there are two options: either the vertical right edge of the last square is contained in the matching or the horizontal edges of the last square are contained in the matching, see Figure 2.1.

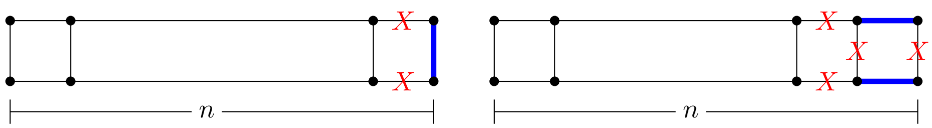



FIGURE 2.1. A perfect matching of $\mathcal{G}_f(n)$.

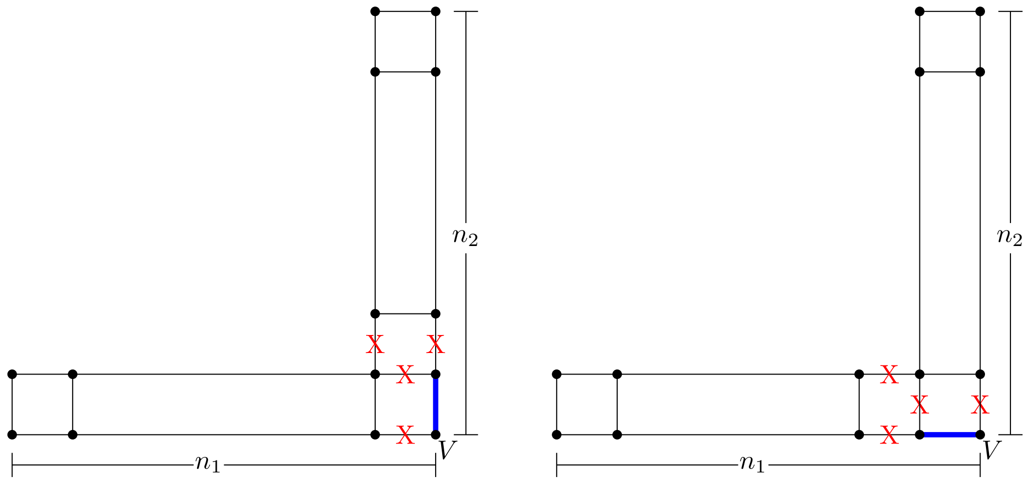
From the definition of perfect matching it is clear that the edges labelled with red X cannot be used. Therefore we have the recurrence relation

$$\mathrm{Match}(\mathcal{G}_f(n)) = \mathrm{Match}(\mathcal{G}_f(n-1)) + \mathrm{Match}(\mathcal{G}_f(n-2)).$$

Since $\mathrm{Match}(\mathcal{G}_f(1)) = 2$ and $\mathrm{Match}(\mathcal{G}_f(2)) = 3$, we conclude that $\mathrm{Match}(\mathcal{G}_f(n)) = F_{n+2}$ for all $n \geq 2$. □

**Corollary 3.** $\mathrm{Match}(\mathcal{G}_f(n_1, n_2)) = F_{n_1+1}F_{n_2} + F_{n_1}F_{n_2+1}$ *for all* $n_1, n_2 \geq 2$.

*Proof.* Let $V$ be the vertex on the lower right corner of $\mathcal{G}_f(n_1, n_2)$. We consider the adjacent edges to the vertex $V$. So, we have the following possible configurations:




Therefore, it holds that

$$\mathrm{Match}(\mathcal{G}_f(n_1,n_2)) = \mathrm{Match}(\mathcal{G}_f(n_1-1))\mathrm{Match}(\mathcal{G}_f(n_2-2)) + \mathrm{Match}(\mathcal{G}_f(n_1-2))\mathrm{Match}(\mathcal{G}(n_2-1)).$$

Theorem 12 allows to obtain the desired result. We are done. $\square$

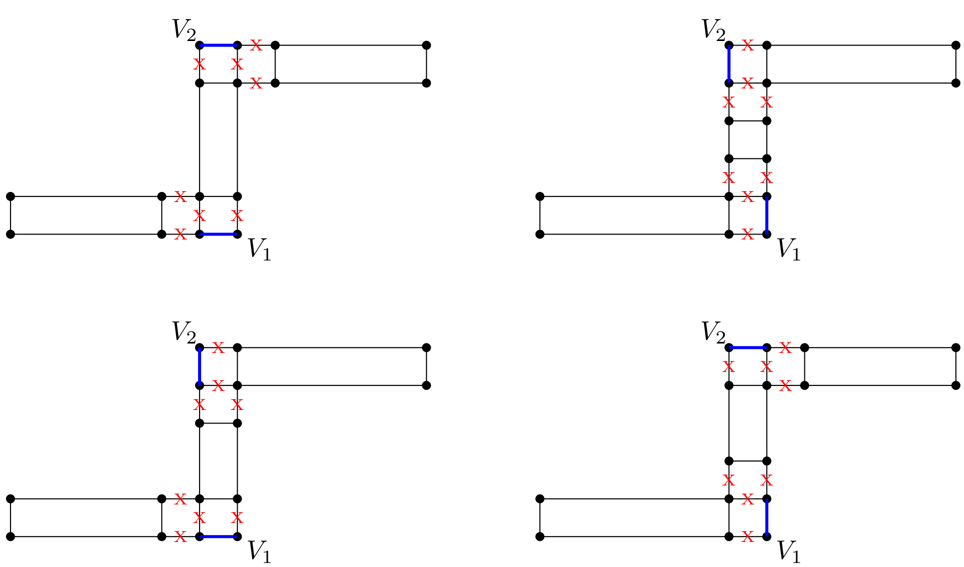
The following result corresponds to the case of a Kronecker snake graph with three straight subsnake graphs.

**Corollary 4.**

$$\mathrm{Match}(\mathcal{G}_f(n_1,n_2,n_3)) = F_{n_1}F_{n_2}F_{n_3} + F_{n_1+1}F_{n_2-2}F_{n_3+1} + F_{n_1}F_{n_2-1}F_{n_3+1} + F_{n_1+1}F_{n_2-1}F_{n_3}$$

*for all integer numbers* $n_1, n_2, n_3 \geq 2$*.*

*Proof.* Firstly, let us suppose that $n_2 \geq 4$. For the cases $n_2 = 2, 3$, we can use a similar argument. Let $V_1$ and $V_2$ be the vertices in the lower right corner and the upper left corner, respectively. By considering the adjacent edges with the vertices $V_1$ and $V_2$, we obtain the following four options:



From the above decomposition, we obtain that

$$\begin{aligned}\mathrm{Match}(\mathcal{G}_f(n_1,n_2,n_3)) &= \mathrm{Match}(\mathcal{G}_f(n_1-2))\mathrm{Match}(\mathcal{G}_f(n_2-2))\mathrm{Match}(\mathcal{G}_f(n_3-2))\\ &+ \mathrm{Match}(\mathcal{G}_f(n_1-1))\mathrm{Match}(\mathcal{G}_f(n_2-4))\mathrm{Match}(\mathcal{G}_f(n_3-1))\\ &+ \mathrm{Match}(\mathcal{G}(n_1-2))\mathrm{Match}(\mathcal{G}_f(n_2-3))\mathrm{Match}(\mathcal{G}_f(n_3-1))\\ &+ \mathrm{Match}(\mathcal{G}_f(n_1-1))\mathrm{Match}(\mathcal{G}_f(n_2-3))\mathrm{Match}(\mathcal{G}_f(n_3-2)).\end{aligned}$$

Theorem 12 allows us to conclude the desired result. □

The following result gives the number of perfect matchings of a snake graph of type $\mathcal{G}_f(n_1,n_2,\ldots,n_k)$ as a specialized message of the Brauer configuration (2.4). In this case, words concatenation arising from the configuration is specialized by the usual product of natural numbers.

**Theorem 13.** *For all integers $n_1,n_2\ldots,n_k\geq 2$, we have*

$$\mathrm{Match}(\mathcal{G}_f(n_1,n_2,\ldots,n_k)) = M(\mathcal{K}^e),$$

*where $\mathcal{K}$ is the Brauer configuration given by identities* (2.4) *and* (2.5)*, $M(\mathcal{K})$ defined as in* (2.2)*. And $e$ is an integer specialization of $\mathcal{K}$ with associated map $e$ of the form $e:\mathcal{K}_0\rightarrow\mathbb{N}$ such that $e(\alpha^i_w)=F_{\alpha^i_w}$ with $F_j$ being the $j$-th Fibonacci number.*

*Proof.* The definition of the Brauer configuration $\mathcal{K}$ and the corresponding specialization $e$ allow us to infer that it suffices to see that

$$\mathrm{Match}(\mathcal{G}_f(n_1,n_2,\ldots,n_k)) = \sum_{w\in\{0,1\}^{k-1}}\prod_{\ell=1}^{k}F_{n_\ell-g(w_{\ell-1},\ell)-g(w_\ell,\ell)+2}.$$

Note that, for $l$ fixed a product of the form $\prod_{w_l}F_{n_\ell-g(w_{\ell-1},\ell)-g(w_\ell,\ell)+2}$ is a specialized message $w^e((U_l,n_l))$ of the labeled polygon $(U_l,n_l)$. Now, we proceed to prove the proposed identity.

Let $V_1,V_2,\ldots,V_{k-1}$ be the vertices on the $k-1$ corners of the snake graph $\mathcal{G}_f(n_1,n_2,\ldots,n_k)$, see Figure 2.2. There are $2^{k-1}$ ways to choose the adjacent edges with the vertices $V_1,V_2,\ldots,V_{k-1}$.

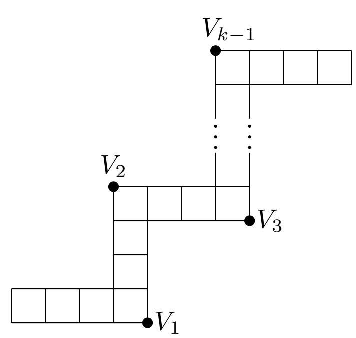



FIGURE 2.2. Snake graph $\mathcal{G}_f(n_1,n_2,\ldots,n_k)$.

Let $e_i$ be one of the incident edge with $V_i$, for $i = 1, 2, \ldots, k-1$. For each $V_i$, there are two options for $e_i$: either $e_i$ is vertical or horizontal. If $e_i$ is vertical and $i$ is odd, we have to consider the number of perfect matchings for the snake graphs $\mathcal{G}_f(n_i - 1)$ and $\mathcal{G}_f(n_{i+1} - 2)$. Note that for the first case we do not consider the last tile of the row that contains the vertex $V_i$, and for the second case we do not consider the first two tiles of the column that contains the vertex $V_i$. Analogously, if $e_i$ is horizontal and $i$ is odd, we have to consider the perfect matching for the snake graphs $\mathcal{G}_f(n_i - 2)$ and $\mathcal{G}_f(n_{i+1} - 1)$. Similarly, for the case when $i$ is even.

Finally, we can encode this situation with binary words. We use 0 for vertical edges and 1 for horizontal edges. So, it is clear that the function $g(x, n)$ encodes the subtraction of the tiles that we must apply to each vertex $V_i$.

Identity (2.6), the multiplication principle and the definition of the message $M(\mathcal{K}^e)$ of the Brauer configuration $\mathcal{K}^e$ allow us to conclude that

$$\begin{aligned}\mathrm{Match}(\mathcal{G}_f(n_1, n_2, \ldots, n_k)) &= \sum_{w\in\{0,1\}^{k-1}} \prod_{\ell=1}^{k} \mathrm{Match}(\mathcal{G}_f(n_\ell - g(w_{\ell-1}, \ell) - g(w_\ell, \ell)) \\ &= \sum_{w\in\{0,1\}^{k-1}} \prod_{\ell=1}^{k} F_{n_\ell - g(w_{\ell-1},\ell) - g(w_\ell,\ell)+2} = M(\mathcal{K}^e).\end{aligned}$$

□

**Example 2.** *In this example we define a Brauer configuration algebra induced by the Brauer configuration $\mathcal{K}$ for $k = 3$ (see,* (2.3), (2.4) *and* (2.5)*). The relations defined here can be adapted for all the distinct values of $k$ in order to define the corresponding Brauer configuration algebras, in this particular case, we have that $w \in \{0,1\}^2 = \{(0,0), (1,0), (0,1), (1,1)\}$, $n = (n_1, n_2, n_3)$ and*

$$\begin{aligned}\mathcal{K}_0 &= \{\alpha^1_{(0,0)}, \alpha^2_{(0,0)}, \alpha^3_{(0,0)}, \alpha^1_{(1,0)}, \alpha^2_{(1,0)}, \alpha^3_{(1,0)}, \alpha^1_{(0,1)}, \alpha^2_{(0,1)}, \alpha^3_{(0,1)}, \alpha^1_{(1,1)}, \alpha^2_{(1,1)}, \alpha^3_{(1,1)}\},\\ \mathcal{K}_1 &= \{(U_{(0,0)}, n), (U_{(1,0)}, n), (U_{(0,1)}, n), (U_{(1,1)}, n),\ \textit{with } n = (n_1, n_2, n_3), n_j \geq 2\}.\end{aligned}$$

*In Table 2.1 we compute all the vertices and polygons of $\mathcal{K}$ by using the values of $i$ and $w$.*

| $i$ \ $w$ | | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|---|---|---|---|---|---|
| 1 | $n_1 - g(w_0, 1) - g(w_1, 1) + 2$ | $n_1 + 1$ | $n_1$ | $n_1 + 1$ | $n_1$ |
| 2 | $n_2 - g(w_1, 2) - g(w_2, 2) + 2$ | $n_2 - 2$ | $n_2 - 1$ | $n_2 - 1$ | $n_2$ |
| 3 | $n_3 - g(w_2, 3) - g(w_3, 3) + 2$ | $n_3 + 1$ | $n_3 + 1$ | $n_3$ | $n_3$ |

TABLE 2.1. In this table entries correspond to the vertices and columns correspond to polygons of the Brauer configuration $\mathcal{K}$.

*Explicitly,* $(U_{(0,0)}, n) = \{n_1+1, n_2-2, n_3+1\}$, $(U_{(1,0)}, n) = \{n_1, n_2-1, n_3+1\}$, $(U_{(0,1)}, n) = \{n_1+1, n_2-1, n_3\}$, $(U_{(1,1)}, n) = \{n_1, n_2, n_3\}$ *and*

$$\begin{aligned}
w((U_{(0,0)}, n)) &= n_1 + 1 \cdot n_2 - 2 \cdot n_3 + 1, \\
w((U_{(1,0)}, n)) &= n_1 \cdot n_2 - 1 \cdot n_3 + 1, \\
w((U_{(0,1)}, n)) &= n_1 + 1 \cdot n_2 - 1 \cdot n_3, \\
w((U_{(1,1)}, n)) &= n_1 \cdot n_2 \cdot n_3.
\end{aligned} \tag{2.7}$$

*Now, by using the specialization $e(\alpha_w^i) = F_{\alpha_w^i}$ defined in Theorem 13 with $F_j$ being the $j$-th Fibonacci number, we have:*

$$\begin{aligned}
w^e((U_{(0,0)}, n)) &= F_{n_1+1}F_{n_2-2}F_{n_3+1}, \\
w^e((U_{(1,0)}, n)) &= F_{n_1}F_{n_2-1}F_{n_3+1}, \\
w^e((U_{(0,1)}, n)) &= F_{n_1+1}F_{n_2-1}F_{n_3}, \\
w^e((U_{(1,1)}, n)) &= F_{n_1}F_{n_2}F_{n_3}.
\end{aligned} \tag{2.8}$$

*Considering the specialized message $M(\mathcal{K}^e) = \sum\limits_{U \in \Gamma_1^e} w^e(U)$ of the Brauer configuration $\mathcal{K}$.*

$$\begin{aligned}
M(\mathcal{K}^e) &= F_{n_1+1}F_{n_2-2}F_{n_3+1} + F_{n_1}F_{n_2-1}F_{n_3+1} + F_{n_1+1}F_{n_2-1}F_{n_3} + F_{n_1}F_{n_2}F_{n_3} \\
&= \mathrm{Match}(\mathcal{G}_f(n_1, n_2, n_3))
\end{aligned}$$

*For $k = 3$, the Brauer configuration algebra associated to $\mathcal{K}$ is defined as follows:*

1. $\mathcal{K}_0 = \{n_1 + 1, n_2 - 2, n_3 + 1, n_2 - 1, n_1, n_2, n_3\}$,
2. $\mathcal{K}_1 = \{(U_{(0,0)}, n), (U_{(1,0)}, n), (U_{(0,1)}, n), (U_{(1,1)}, n),$ *with* $n = (n_1, n_2, n_3)\}$,
3. *At vertex $n_1 + 1$, it holds that;* $(U_{(0,0)}, n) < (U_{(0,1)}, n), \quad val(n_1 + 1) = 2$,
4. *At vertex $n_2 - 2$, it holds that;* $(U_{(0,0)}, n), \quad val(n_2 - 2) = 1$,
5. *At vertex $n_3 + 1$, it holds that;* $(U_{(0,0)}, n) < (U_{(1,0)}, n), \quad val(n_3 + 1) = 2$,
6. *At vertex $n_2 - 1$, it holds that;* $(U_{(1,0)}, n) < (U_{(0,1)}, n), \quad val(n_2 - 1) = 2$,
7. *At vertex $n_1$, it holds that;* $(U_{(1,0)}, n) < (U_{(1,1)}, n), \quad val(n_1) = 2$,
8. *At vertex $n_2$, it holds that;* $(U_{(1,1)}, n), \quad val(n_2) = 1$,
9. *At vertex $n_3$, it holds that;* $(U_{(0,1)}, n) < (U_{(1,1)}, n), \quad val(n_3) = 2$,
10. $\mu(\alpha) = 2$ *for any vertex $\alpha$.*

*The ideal $I$ of the corresponding Brauer configuration algebra $\Lambda_{\mathcal{K}}$ is generated by the following relations (see Figure 2.3), for which it is assumed the following notation for the* *special cycles:*

$$\begin{aligned}
&C^{U_{(0,0)},1}_{n_1+1} = a_1^{n_1+1}a_2^{n_1+1}, \quad C^{U_{(0,1)},1}_{n_1+1} = a_2^{n_1+1}a_1^{n_1+1}, \quad C^{U_{(0,0)},1}_{n_2-2} = a_1^{n_2-2},\\
&C^{U_{(0,0)},1}_{n_3+1} = a_1^{n_3+1}a_2^{n_3+1}, \quad C^{U_{(1,0)},1}_{n_3+1} = a_2^{n_3+1}a_1^{n_3+1}, \quad C^{U_{(1,0)},1}_{n_2-1} = a_1^{n_2-1}a_2^{n_2-1},\\
&C^{U_{(0,1)},1}_{n_2-1} = a_2^{n_2-1}a_1^{n_2-1}, \quad C^{U_{(1,0)},1}_{n_1} = a_1^{n_1}a_2^{n_1}, \quad C^{U_{(1,1)},1}_{n_1} = a_2^{n_1}a_1^{n_1},\\
&C^{U_{(1,1)},1}_{n_2} = a_1^{n_2}, \quad C^{U_{(0,1)},1}_{n_3} = a_1^{n_3}a_2^{n_3}, \quad C^{U_{(1,1)},1}_{n_3} = a_2^{n_3}a_1^{n_3},
\end{aligned} \tag{2.9}$$

1. $a_i^h a_r^s$, *if* $h \neq s$, *for all possible values of* $i$ *and* $r$,
2. $\left(C_j^{U_{(0,0)},i}\right)^2 - \left(C_l^{U_{(0,0)},k}\right)^2$, *for all possible values of* $i, j, k$ *and* $l$,
3. $\left(C_j^{U_{(0,1)},i}\right)^2 - \left(C_l^{U_{(0,1)},k}\right)^2$, *for all possible values of* $i, j, k$ *and* $l$,
4. $\left(C_j^{U_{(1,0)},i}\right)^2 - \left(C_l^{U_{(1,0)},k}\right)^2$, *for all possible values of* $i, j, k$ *and* $l$,
5. $\left(C_j^{U_{(1,1)},i}\right)^2 - \left(C_l^{U_{(1,1)},k}\right)^2$, *for all possible values of* $i, j, k$ *and* $l$,
6. $\left(C_i^{U_{(0,0)},j}a\right)^2$ *(*$\left(C_i^{U_{(0,1)},j}a'\right)^2$*)* , *with* $a$ *(*$a'$*) being the first arrow of* $C_i^{U_{(0,0)},j}$ *(*$C_i^{U_{0,1},j}$*)* *for all* $i, j$,
7. $\left(C_i^{U_{(1,0)},j}a\right)^2$ *(*$\left(C_i^{U_{(1,1)},j}a'\right)^2$*)* , *with* $a$ *(*$a'$*) being the first arrow of* $C_i^{U_{(1,0)},j}$ *(*$C_i^{U_{(1,1)},j}$*)* *for all* $i, j$.

*Figure 2.3 shows the quiver* $Q_{\mathcal{K}}$ *associated to this configuration.*

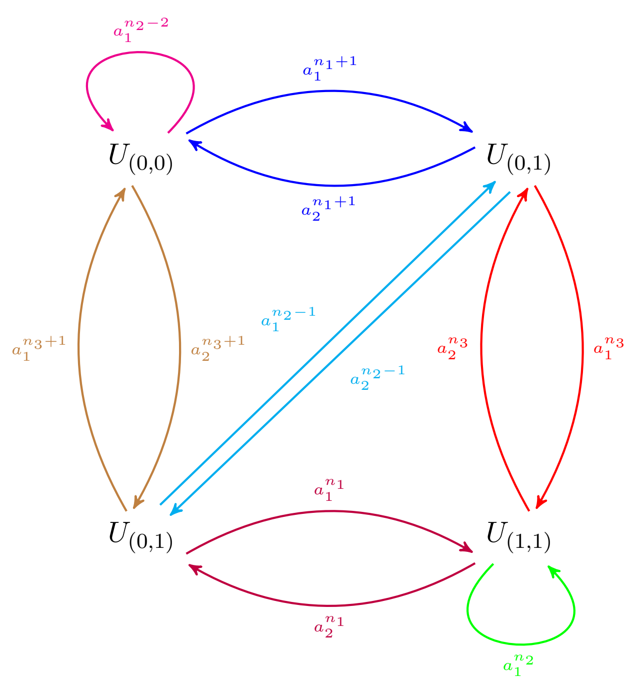



FIGURE 2.3. The quiver $Q_{\mathcal{K}}$ associated to this configuration.

*The following is the numerology associated to the algebra $\Lambda_{\mathcal{K}} = kQ_{\mathcal{K}}/I$ with $Q_{\mathcal{K}}$ as shown in Figure 2.3 and special cycles given in (2.9), ($|r(Q_{\mathcal{K}})|$ is the number of indecomposable projective modules, $r_{U_{(0,0)}}$, $r_{U_{(0,1)}}$, $r_{U_{(1,0)}}$ and $r_{U_{(1,1)}}$ denote the number of summands in the heart of the indecomposable projective modules $P(U_{(0,0)})$, $P(U_{(0,1)})$, $P(U_{(1,0)})$ and $P(U_{(1,1)})$. Note that, $|C_i| = val(i)$):*

$$
\begin{aligned}
|r(Q_{\mathcal{K}})| &= 4,\\
r_{U_{(0,0)}} &= 3, \quad r_{U_{(0,1)}} = 3, \quad r_{U_{(1,0)}} = 3, \quad r_{U_{(1,1)}} = 3,\\
|C_{n_1+1}| &= 2, \quad |C_{n_2-2}| = 1, \quad |C_{n_3+1}| = 2, \quad |C_{n_2-1}| = 2,\\
|C_{n_1}| &= 2, \quad |C_{n_2}| = 1, \quad |C_{n_3}| = 2,\\
\sum_{\alpha\in\mathcal{K}_0}\sum_{X\in\mathcal{K}_1} \mathrm{occ}(\alpha, X) &= 12, \quad \textit{the number of special cycles},\\
\dim_k \Lambda_{\mathcal{K}} &= 8 + 2(3) + 1(1) + 2(3) + 2(3) + 2(3) + 1(1) + 2(3) = 40,\\
\dim_k Z(\Lambda_{\mathcal{K}}) &= 1 + 14 + 4 - 7 + 2 - 2 = 12.
\end{aligned}
$$

As another example of Theorem 13 consider the following snake graph of type $\mathcal{G}_f(5, 3, 3, 2, 5, 4, 2)$

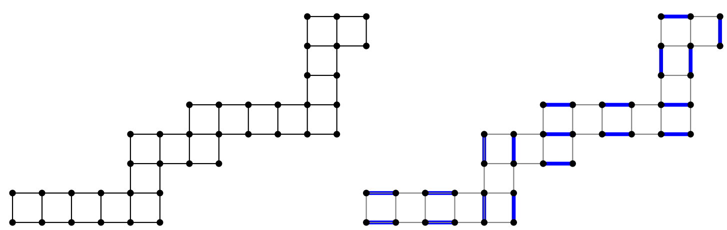


FIGURE 2.4. The snake graph $\mathcal{G}_f(5, 3, 3, 2, 5, 4, 2)$.

In this case,

$$
\begin{aligned}
&\mathrm{Match}(\mathcal{G}_f(5, 3, 3, 2, 5, 4, 2)) = 3221\\
&\quad = 4F_3F_4F_5F_2^4 + 12F_1F_3F_4F_6F_2^3 + 16F_1F_3^2F_4F_5F_2^2 + 12F_1^2F_3^2F_4F_6F_2 + 4F_1^2F_3^3F_4F_5.
\end{aligned}
$$

Note that, sequences (Fibonacci words) $F_3F_4F_5F_2^4$, $F_1F_3F_4F_6F_2^3, \ldots$ are specialized polygons of the Brauer configuration (2.4).

## 2.3 Determinants and paths problems via Brauer configurations

In this section, we describe the way that specializations of suitable Brauer configurations (or Brauer configuration algebras) can be used to define determinants, thus solutions of some very well known problems, as the paths problem solved by Lindström, Gessel and Viennot can be interpreted as a specialization of a Brauer configuration and as a consequence of such interpretation the message described in Theorem 13 can be viewed as a product of specialized Brauer configurations.

Let us consider a labeled Brauer configuration $\mathcal{D}(k) = \{\mathcal{D}_0(k), \mathcal{D}_1(k), \nu, \mathcal{O}\}$ obtained from the labeled Brauer configuration $\mathcal{K}$ defined by identities (2.4), and (2.5) by redefining vertices labels and polygons as follows:

$$\begin{aligned}
\mathcal{D}_0(k) &= \{\alpha_\pi^i = \alpha_{(i,\pi(i))} \in G \mid 1 \leq i \leq k, \pi \in S_k, k > 2 \text{ fixed}\},\\
\mathcal{D}_1(k) &= \{(U_\pi, \pi) \mid \pi \in S_k\}, \quad (U_\pi, \pi) = \{\alpha_{(i,\pi(i))} \mid \pi \in S_k \text{ fixed}\},\\
\nu(\alpha_{(i,\pi(i))}) &= 1, \quad \text{for any vertex } \alpha_{(i,\pi(i))} \in \mathcal{D}_0(k),
\end{aligned} \tag{2.10}$$

where $G$ is a field, $\pi$ is an element of the group $(S_k, \preceq)$ of permutations of $k$ elements endowed with a linear order $\preceq$, the labels in this case have the form $(\pi(1), \pi(2), \dots, \pi(k))$, $\nu$ is a multiplicity function. And the orientation $\mathcal{O}$ is defined in such a way that labeled polygons $(U_{\pi_j}, \pi_j)$ and $(U_{\pi_{j+1}}, \pi_{j+1})$ are consecutive in $\mathcal{D}_1(k)$ provided that $\pi_j$ and $\pi_{j+1}$ are consecutive in $(S_k, \preceq)$.

For the sake of accuracy in this case, to each word $w(U_\pi, \pi)$ associated to the polygon $(U_\pi, \pi)$ it is defined $\operatorname{sign}(w(U_\pi, \pi)) = \operatorname{sign}(\pi)$ and the message $M(\mathcal{D})$ of the Brauer configuration $\mathcal{D}(k)$ is given by the identity:

$$M(\mathcal{D}(k)) = \sum_{(U_\pi,\pi)\in\mathcal{D}_1} \operatorname{sign}(w) w(U_\pi, \pi). \tag{2.11}$$

The following result follows immediately from the definitions (2.10) and (2.11).

**Theorem 14.** *$M(\mathcal{D}(k)) = |\alpha_{(i,j)}|$ where $|\alpha_{(i,j)}|$ is the determinant with entries $\alpha_{(i,j)} \in \mathcal{D}_0(k)$.*

Now several specializations can be defined for the message (2.11).

Henceforth, we let $M(\mathcal{D}^{e^k_{\mathcal{F}}}(k))$ denote the specialization of the message (2.11) with an associated function of the form $e^k_{\mathcal{F}} : \mathcal{D}_0(k) \to \mathbb{C}$ such that

$$e^k_{\mathcal{F}}(\alpha_{(r,s)}) = \begin{cases} i = \sqrt{-1}, & \text{if } \ s = r+1, \ \ r \text{ fixed}, \ \ 1 \leq r \leq k-1,\\ i, & \text{if } \ s = r-1, \ \ r \text{ fixed}, \ \ 2 \leq r \leq k,\\ 1, \ \text{if } \ s = r, \ \ 1 \leq r \leq k, & \\ 0, \ \text{elsewhere.} & \end{cases}$$

Then the following result holds (see [12] for the calculus of this family of determinants).

**Corollary 5.** *$M(\mathcal{D}^{e^k_{\mathcal{F}}}(k)) = F_{k+1}$ where $F_j$ is the $j$th Fibonacci number.*

*Proof.* $M(\mathcal{D}^{e^k_{\mathcal{F}}}(k))$ is a $k \times k$-determinant whose entries are given by identities $e^k_{\mathcal{F}}(\alpha_{(r,s)})$ then column transformations of the form $C'_{j+1} \leftrightarrow -\frac{F_j}{F_{j+1}} C_j i + C_{j+1}$, for $1 \leq j \leq k-1$ reduce $|e^k_{\mathcal{F}}(\alpha_{(r,s)})|$ to a determinant with entries of the form:

$$T(e^{k}_{\mathcal{F}}(\alpha_{(r,s)})) = \begin{cases} \frac{F_{j+1}}{F_j}, & \text{if } s=r, \ 1\leq r\leq k, \\ i, & \text{if } s=r-1, \ r \text{ fixed}, \ 2\leq r\leq k, \\ 0, \ \text{elsewhere}. \end{cases}$$

Thus, $T(e^{k}_{\mathcal{F}}(\alpha_{(r,s)}))$ is a diagonal determinant such that $|T(e^{k}_{\mathcal{F}}(\alpha_{(r,s)}))| = \prod\limits_{j=1}^{k} \frac{F_{j+1}}{F_j} = F_{k+1}$.

□

The following result is proved by Theorem 13 and Corollary 5.

**Corollary 6.** *For all* $n_1, n_2 \dots, n_k \geq 2$, *we have that* $\mathrm{Match}(\mathcal{G}_f(n_1, n_2, \dots, n_k)) = \sum_{w\in\{0,1\}^{k-1}} \prod_{\ell=1}^{k} M(\mathcal{D}^{e^{h}_{\mathcal{F}}}(k))$ *where* $h = n_\ell - g(w_{\ell-1}, \ell) - g(w_\ell, \ell) + 1$.

### 2.3.1 The Lindström's theorem

Specializations of the Brauer configuration $\mathcal{D}(k)$ allow us to interpret the Lindström's theorem as a message $M(\mathcal{D}(k))$. To do that, let us recall such a result as Gessel and Viennot described in [29].

If $Q$ is an acyclic digraph with finitely many paths between any two vertices. Let $k$ be a fixed positive integer. A $k$-vertex is a $k$-tuple of vertices of $Q$, if $u = (u_1, u_2, \dots, u_k)$ and $v = (v_1, v_2, \dots, v_k)$ are $k$-vertices of $Q$ then a $k$-path from $u$ to $v$ is a $k$-tuple $A = (A_1, A_2, \dots, A_k)$ such that $A_i$ is a path from $u_i$ to $v_i$. The $k$-path $A$ is disjoint if the paths $A_i$ are vertex disjoint. Let $S_k$ be the set of permutations of $\{1, 2, \dots, k\}$ then for $\pi \in S_k$, by $\pi(v)$ we mean the $k$-vertex $(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$.

Let us assign a weight to every edge of $Q$, we define the weight of a path to be the product of the weights of its edges and the weight of a $k$-path to be the product of the weights of its components.

Let $\mathcal{P}(u_i, v_j)$ be the set of paths from $u_i$ to $v_j$ and $P(u_i, v_j)$ be the sum of their weights. Define $\mathcal{P}(u, v)$ and $P(u, v)$ analogously for $k$ paths from $u$ to $v$.

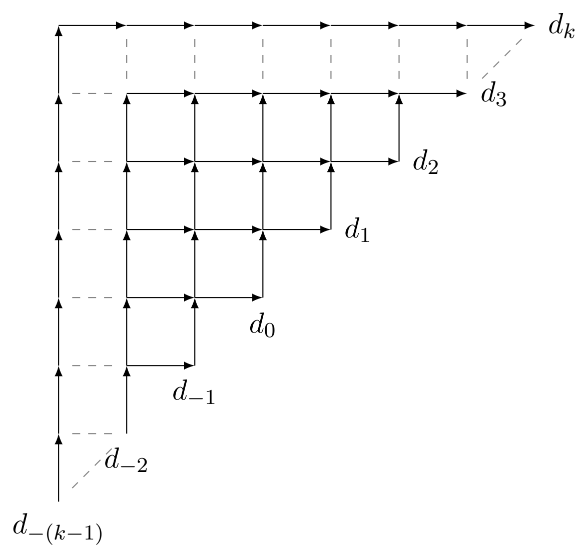
Let $\mathcal{N}(u, v)$ be the subset of $\mathcal{P}(u, v)$ of disjoint paths and let $N(u, v)$ be the sum of their weights, then it is clear that for any permutation $\pi \in \{1, 2, \dots, k\}$, it holds that $P(u, \pi(v)) = \prod\limits_{i=1}^{k} P(u_i, v_{\pi(i)})$. Thus the specialization with associated function of the form $h : \mathcal{D}_0(k) \to \mathbb{N}$ such that $h(\alpha_{(i,\pi(i))}) = P(u_i, v_{\pi(i)})$ and words defined by the specialized polygons $h(U_\pi, \pi) = \{P(u_i, v_{\pi(i)}) \mid 1 \leq i \leq k, \pi \in S_k \text{ fixed}\}$ of the form $w(h(U_\pi, \pi)) = \mathrm{sign}(\pi) P(u, \pi(v))$ build the following Brauer configuration version of the theorem of Lindström [29].

**Theorem 15.** $M(\mathcal{D}^h(k)) = \sum\limits_{\pi\in S_k} \mathrm{sign}(\pi) N(u, \pi(v))$.

The following results are well known consequences of Theorem 15 giving values of $n \times n$, $t$-Catalan determinants. For which, we define specialized messages $M(\mathcal{D}^{h_t}(n))$ with $P(u_{1+h}, v_{j-h}) = P(u_1, v_j) = C_{t-1+j}$, $t \geq 1$ fixed, $0 \leq h \leq j-1$, $1 \leq j \leq n$, $j-h > 0$, and $P(u_{k+l}, v_{n-l}) = P(u_k, v_n) = C_{t+n+k-2}$, for $2 \leq k \leq n$ and $0 \leq l \leq n-k$, $C_s$ denotes the $s$th Catalan number.

**Corollary 7.** $M(\mathcal{D}^{h_1}(k)) = 1$.

*Proof.* Consider the infinite directed graph $G$ with $\mathbb{Z} \times \mathbb{Z}$ as the set of vertices and directed edges from $(i,j)$ to $(i+1,j)$ and to $(i,j+1)$ for every $i,j \in \mathbb{Z}$. Let $d_i$ denote the vertex $(i,i)$ in $G$, $i \in \mathbb{Z}$. Note that the number of directed paths in $G$ from $d_i$ to $d_j$, with $j \geq i$ is equal to the Catalan number $C_{j-i}$. Let $Q_k^1$ be the family consisting of all $k$ pairwise vertex disjoint directed paths $(A_0, A_1, \ldots, A_{k-1})$ in $G$ such that $A_i$ joins $d_{-i}$ with $d_{i+1}$, $i = 0, 1, \ldots, k-1$ then $M(\mathcal{D}^{h_1}(k)) = |Q_k^1| = 1$, where $|Q_k^1|$ is the number of vertices of the graph $Q_k^1$, see the diagram below.




□

The following is a more general result obtained via specializations $M(\mathcal{D}^{h_t}(k))$ and digraphs $Q_k^t$ (as described in the proof of Corollary 7) where the system of $k$-paths $(A_0, A_1, \ldots, A_{k-1})$ and $A_i$ connect vertices $d_{-i}$ and $d_{t+i}$ [47].

**Corollary 8.** $M(\mathcal{D}^{h_t}(k)) = |Q_k^t|$.

For example $M(\mathcal{D}^{h_2}(k)) = k+1$ and $M(\mathcal{D}^{h_3}(k)) = \frac{(k+1)(k+2)(2k+3)}{6}$.

### 2.3.2 On the Brauer configuration algebra $\Lambda_{\mathcal{D}(k)}$ induced by the Brauer configuration $\mathcal{D}(k)$

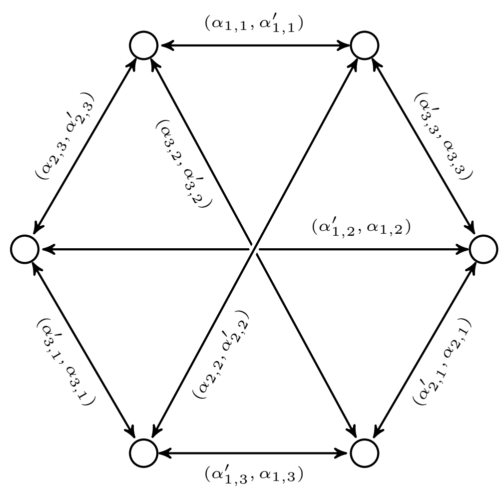
Note that each vertex $\alpha_{(i,j)} \in \mathcal{D}_0(k)$ has associated a successor sequence of the form

$$S_{(i,j)} = (U_{\pi_{i_1}}, \pi_{i_1}) \preceq (U_{\pi_{i_2}}, \pi_{i_2}) \preceq \cdots \preceq (U_{\pi_{i_k}}, \pi_{i_k}), \tag{2.12}$$

$(i,j) \in \pi_{i_j}$ with $\pi_{i_j}$ being a $k$-set permutation and for any $j$ it holds that $1 \leq i \leq j$. Successor sequences $S_{(i,j)}$ define the corresponding special cycles $C_{(i,j)}$. Then the following are relations generating the admissible ideal $I$ of $\Lambda_{\mathcal{D}(k)}$.

1. If $x_i = (i, \pi(i))$ and $x_j = (j, \pi(j))$ are elements of $\pi \in S_k$ then a relation of the form $C_{(i,\pi(i))} - C_{(j,\pi(j)))}$ has place,
2. If $a$ is the first arrow of a special cycle $C_{(i,j)}$ then a relation $C_{(i,j)}a$ has also place,
3. If $\gamma$ is an arrow of a given special cycle $C_{(i,j)}$ and $\beta$ is arrow of a special cycle $C'_{(i',j')}$ with the final vertex $e(\alpha)$ being the initial vertex $s(\beta)$ and $C_{(i,j)} \neq C'_{(i,j)}$ then a relation of the form $\alpha\beta$ holds in $I$,
4. The Brauer quiver $Q_{\mathcal{D}(k)}$ has no loops.

The following is the Brauer quiver $Q_{\mathcal{D}(3)}$:

$Q_{\mathcal{D}(3)} =$ 


For the sake of clarity any cycle of the form $\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet$ is written as $\xleftrightarrow{(b,a)}$

Note that in this case relations of the following form have place $\alpha'_{(i,j)}\alpha_{(j+n) \text{ mod}3,r}$ and $\alpha_{(i,j)}\alpha'_{(j+n) \text{ mod}3,s}$, besides $\alpha_{(1,1)}\alpha_{(2,3)}$, $\alpha_{(2,3)}\alpha_{(3,1)}$, $\alpha_{(3,1)}\alpha_{(1,3)}$, $\alpha_{(1,3)}\alpha_{(2,1)}$, $\alpha_{(1,3)}\alpha_{(3,2)}$, $\alpha_{(3,2)}\alpha_{(2,3)}$, $\alpha_{(2,3)}\alpha_{(1,2)}$, $\alpha_{(2,2)}\alpha_{(1,3)}$, $\alpha'_{(2,3)}\alpha'_{(1,1)}$, $\alpha'_{(2,1)}\alpha'_{(1,3)}$, $\alpha'_{(1,3)}\alpha'_{(2,2)}$, $\alpha'_{(1,3)}\alpha'_{(3,1)}$, $\alpha'_{(3,1)}\alpha'_{(2,3)}$, $\alpha'_{(3,2)}\alpha'_{(1,3)}$, $\alpha'_{(1,2)}\alpha'_{(2,3)}$ for all possible values of $r$ and $s$. Thus in general the following result holds:

**Theorem 16.** *For the Brauer configuration $\Lambda_{\mathcal{D}(k)}$ induced by the Brauer configuration $\mathcal{D}(k)$ the following statements hold:*

1. *$\Lambda_{\mathcal{D}(k)}$ has $k!$ indecomposable projective modules.*
2. *If $\alpha_{(i,j)} \in \mathcal{D}_0(k)$ the $\mathrm{val}(\alpha_{(i,j)}) = (k-1)!$.*
3. *The number of summands in the heart of an indecomposable projective module given by a polygon of the form $(U_\pi, \pi)$ is $k$.*
4. *$\dim \Lambda_{\mathcal{D}(k)} = 2(k! + k^2 t_{((k-1)!-1)})$ where $t_s$ denotes the sth triangular number.*
5. *$\dim_k Z(\Lambda_{\mathcal{D}(k)}) = 1 + k!$.*

*Proof.* 1. The assertion follows from Theorem 3 (item 1) and the fact that $|\mathcal{D}_1(k)| = |S_k| = k!$.

2. By definition of a $k \times k$-determinant it holds that each entry-vertex $\alpha_{(i,\pi(i))}$ occurs in $(k-1)!$ summands-polygons of the form $\alpha_{(1,\pi(1))}\alpha_{(2,\pi(2))}\cdots\alpha_{(k,\pi(k))}$.
3. We note that if $P$ is an indecomposable projective $\Lambda_{\mathcal{D}(k)}$-module corresponding to a polygon $(U_\pi, \pi)$ then $\mathrm{rad}^2 P \neq 0$ and the result follows bearing in mind that any polygon $(U_\pi, \pi)$ has $k$ vertices each of them occurring in $(k-1)!$ polygons (i.e., all vertices in a given polygon are non-truncated).
4. Proposition 2 allows to conclude that

$$\dim_k \Lambda_{\mathcal{D}(k)} = 2k! + \sum_{\alpha_{(i,j)} \in \mathcal{D}_0} |C_{\alpha_{(i,j)}}|(|C_{\alpha_{(i,j)}}| - 1)$$

   where for each $\alpha_{(i,j)}$, $|C_{\alpha_{(i,j)}}| = val(\alpha_{(i,j)}) = (k-1)!$. Thus, the statement holds taking into account that for any $j \geq 2$, $j(j-1) = 2t_{j-1}$.
5. Since $\mathrm{rad}^2 \Lambda_{\mathcal{D}(k)} \neq 0$, the statement is a consequence of Theorem 4 with $\nu(\alpha_{(i,j)}) = 1$, for all $\alpha_{(i,j)} \in \mathcal{D}_0(k)$, $|\mathcal{D}_0(k)| = k^2$, $|\mathcal{D}_1(k)| = k!$, $\#(\mathit{Loops}\ Q_{\mathcal{D}(k)}) = 0$ and $|\mathcal{C}_{\mathcal{D}(k)}| = 0$.

□

**Corollary 9.** *For $n > 2$ the algebra $\Lambda_{\mathcal{D}(n)}$ associated to the Brauer configuration $\mathcal{D}(n)$ has a length grading induced from the path algebra $kQ_{\mathcal{D}(n)}$.*

*Proof.* Since $\mathcal{D}(n)$ is connected by definition, then the corollary holds as a consequence of Proposition 3 bearing in mind that for any $\alpha_{(i,j)} \in \mathcal{D}_0(n)$, $\nu(\alpha_{(i,j)}) = 1$ and $val(\alpha_{(i,j)}) = (n-1)!$.

□

## 2.4 Kronecker snake graphs

In this section, categorification in the sense of Ringel and Fahr [19–21] is given to sequences of continued fractions. To do that, to each non-regular indecomposable Kronecker module, it is associated a suitable snake graph, we named *Kronecker snake graphs* all those graphs associated to the non-regular components of the Auslander-Reiten quiver of the Kronecker algebra.

The following theorem defines Kronecker snake graphs associated to indecomposable preprojective Kronecker modules.

**Theorem 17.** *For $n \geq 2$ fixed, the $(2n+1)-$terms snake graph*

$$\mathcal{G}_{pk}(n+1, n+1, 2, 2, n+1, 2, n+3, 2, n+1, 2, n+3, \dots)$$

*builds the indecomposable Kronecker preprojective module $(n+1, n)$. Moreover the corresponding continued fraction of $\mathcal{G}_{pk}$ has the following form*

$$[2, \underbrace{1, 1, \dots, 1}_{n-2}, 2, \underbrace{1, 1, \dots, 1}_{n-2}, 4, \Delta, 2],$$

*where*

$$\Delta = \underbrace{1, 1, \dots, 1}_{n-2}, 3, \underbrace{1, 1, \dots, 1}_{n}, 3, \underbrace{1, 1, \dots, 1}_{n-2}, 3 \underbrace{1, 1, \dots, 1}_{n}, \dots$$

*and the length of $\Delta$, $l(\Delta)$ is*

$$l(\Delta) = \begin{cases} n(n-1) - 1, & \text{if } n \text{ is odd,} \\ n(n-1) - 2, & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* Let us recall that an indecomposable preprojective Kronecker module can be represented as a matrix block of the form $U = (A, B)$ with $n$ columns and $n+1$ rows, which can be defined by straight subsnake graphs of $\mathcal{G}_{pk}$ in alternative fashion (horizontal, vertical, etc). First horizontal subsnake graph corresponds to entries of the first row of the matrix block in such a way that the first tile and the remaining interior tiles are labeled with an entry in the first row of $A$, whereas the entry $b_{1,1} = 1$ is chosen to label the last tile of this subsnake graph. Second subsnake graph is labeled by entries in the first column of the matrix block $B$ its last tile is labeled with the entry $b_{n+1,1}$ the label of the last tile in the next horizontal subsnake graph is $a_{n+1,n}$ and the label of the next vertical subsnake graph with starting tile $a_{n+1,n}$ is $a_{n,n}$ and so on until all the starting and final tiles of all straight subsnake graphs are labeled by entries of the matrix block $U$.

Assuming natural ordering for the straight subsnake graphs, we apply to even horizontal subsnake graphs and the first vertical subsnake graph an additional special labeling dealing with orientation of row and columns, in this case we take into account that if the last tile of a given row $r_i$ has not a special labeling, then the first tile of the next column $c_i$ has not a special labeling. Moreover, these special labels determine the way that rows and columns of a matrix block must be constructed. The procedure goes as follows:

Special labeled horizontal straight snake graphs indicates that each tile corresponds to an entry of a row of the matrix block developed from the right to the left. Whereas, a special labeled vertical straight snake graph indicates that the tiles correspond to the entries of a column of the matrix block developed from the top to the bottom.

An indecomposable preprojective module $(n+1, n)$ is obtained from $\mathcal{G}_{pk}$ by assigning alternatively either 0 or a 1 to the ends of the straight snake graphs constituting $\mathcal{G}_{pk}$.

In this case, a 0 is assigned to the first tile in the first row, then a 1 is assigned to the corresponding last tile, which is the first tile of the next straight snake graph, which has assigned a 0 in its last tile, and the procedure goes on. Numbers 1's are entries of the identities in the matrix block, which can be completed by definition. The remaining results hold immediately from the construction described above and definitions (1.10) and (1.11). □

The following example illustrates step by step the arguments posed in the proof of Theorem 17. Considering the labeled snake graph $\mathcal{G}$ presented in Figure 2.5. In Figure 2.6, 0 and 1 are assigned alternately to the ends of each straight snake subgraph, in Figure 2.7, $\mathcal{G}$ is rolled up into the matrix block and finally by using the definition of a preprojective module we complete each matrix block (see Figure 2.8).

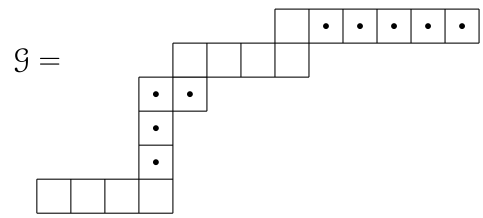


FIGURE 2.5. Labeled snake graph $\mathcal{G}$.

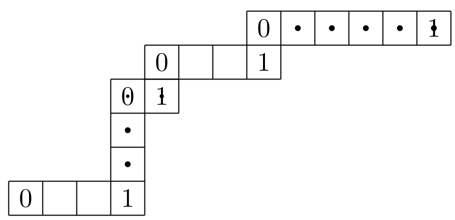


FIGURE 2.6. Assigned a 0 or 1 to the ends of each straight snake subgraph.

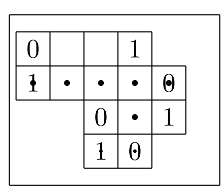


FIGURE 2.7. The snake graph is rolled up into the matrix block.

$$\begin{array}{|ccc|ccc|}
\hline
0 & \bullet & \bullet & 1 & \bullet & \bullet \\
1 & \bullet & \bullet & \bullet & \bullet & 0 \\
\bullet & \bullet & 0 & \bullet & \bullet & 1 \\
\bullet & \bullet & 1 & 0 & \bullet & \bullet \\
\hline
\end{array}$$

FIGURE 2.8. The preprojective module is completed by definition.

$$\begin{array}{|ccc|ccc|}
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}$$

The following theorem is the preinjective version of Theorem 17.

**Theorem 18.** *For $n \geq 3$ fixed. The $(2n+1)-$terms snake graph $\mathcal{G}_{ik}(3, n, 4, 2, n+2, 2, n+4, 2, n+2, 2, n+4, \dots)$ defines the indecomposable Kronecker preinjective module $(n+1, n)$. Moreover the corresponding continued fraction of $\mathcal{G}_{ik}$ has the following form*

$$[2, 2, \underbrace{1, 1, \dots, 1}_{n-3}, 2, 1, 3, \underbrace{1, 1, \dots, 1}_{n-1}, \Delta, 2],$$

*where* $\Delta = 3, \underbrace{1, 1, \dots, 1}_{n+1}, 3, \underbrace{1, 1, \dots, 1}_{n-1}, 3, \underbrace{1, 1, \dots, 1}_{n+1}, 3\, \underbrace{1, 1, \dots, 1}_{n-1}, \dots$ *and the length of* $\Delta$, $l(\Delta)$ *is*

$$l(\Delta) = \begin{cases} n(n-2) - 3, & \text{if } n \text{ is odd;} \\ n(n-2) - 2, & \text{if } n \text{ is even;} \end{cases}$$

The snake graphs $\mathcal{G}_{pk}(n+1, n+1, 2, 2, \dots)$ are said to be *preprojective Kronecker snake graphs*. Whereas the snake graphs $\mathcal{G}_{ik}(3, n, 4, 2, n+2, 2, \dots)$ are said to be *preinjective Kronecker snake graphs*.

Regarding snake graphs $\mathcal{G}_{pk}$ and $\mathcal{G}_{ik}$, it is easy to see the following result.

**Proposition 4.** *If $l(\mathcal{G})$ denotes the length of a snake graph $\mathcal{G}$, then*

$$l(\mathcal{G}_{pk}) = \begin{cases} (n+1)(n+2) - 2, & \text{if } n \text{ is odd,} \\ n(n+3) - 1, & \text{if } n \text{ is even,} \end{cases}$$

*and*

$$l(\mathcal{G}_{ik}) = \begin{cases} n(n+2) - 2, & \text{if } n \text{ is odd,} \\ n(n+2) - 1, & \text{if } n \text{ is even.} \end{cases}$$

Now, we study some interesting connections of preprojective and preinjective Kronecker snake graphs with knot theory and Auslander algebras. On one hand, in section 2.4.1 we use the message of suitable Brauer configurations algebras to construct the rational tangles introduced in Example 1 and a bijective correspondence between rational tangles of type $\mathcal{W}(i+1, i)$ and preprojective Kronecker snake graphs is described as well. On the other hand, in section 2.4.2 the set of preinjective Kronecker snake graphs is endowed of the group structure and we proof a bijection between the elements in that group and the full exceptional sequences of some Auslander algebras.

### 2.4.1 Preprojective Kronecker tangles

In this section, we describe the preprojective Kronecker tangles introduced in Example 1 as the message of the following Brauer configuration algebra.

Let $\Gamma_{\mathfrak{T}_k} = kQ_{\mathfrak{T}_k}/M$ be the Brauer configuration algebra induced by the Brauer configuration $\mathfrak{T}_k$ such that for $k \geq 2$ fixed, $\mathfrak{T}_k = (\mathfrak{T}_0, \mathfrak{T}_1, \mu, \mathcal{O})$ with:

1. $$\begin{aligned}
&\mathfrak{T}_0 = \{(-(k+5), k+3) \setminus \{-1, 0, 1\}\}, \text{where the interval is consider over } \mathbb{Z}\\
&\mathfrak{T}_1 = \{P_0, P_1, P_2, \ldots, P_k\}, \quad \text{where}\\
&P_i = \{i+3, -(i+3), -2, 2, i+3, 2, -(i+5), 2, i+3, 2, -(i+5), 2, i+3, \ldots\},\\
&\quad \text{for } \ 0 \leq i \leq k \ \text{ and } \ |P_i| = 2i+5.
\end{aligned}$$

2. *The orientation $\mathcal{O}$ is defined in such a way that:*

$$\begin{aligned}
\text{At vertex } \ 2;&\ P_0 < P_1^{(2)} < P_2^{(3)} < \cdots < P_k^{(k+1)}.\\
\text{At vertex } \ n;&\ P_{n-3}^{\lceil \frac{n}{2} \rceil}, \text{with } 3 \leq n \leq k+3.\\
\text{At vertex } \ -2;&\ P_0 < P_1 < P_2 < P_3 < P_4 < P_5 < \cdots < P_k.\\
\text{At vertex } \ -3;&\ P_0.\\
\text{At vertex } \ -4;&\ P_1.\\
\text{At vertex } \ -5;&\ P_2.\\
\text{At vertex } \ -n;&\ P_{(n-5)}^{\lceil \frac{n-3}{2} \rceil - 1} < P_{n-3}, \text{with } 6 \leq n \leq k+3.\\
\text{At vertex } \ -n;&\ P_{(n-5)}^{\lceil \frac{n-5}{2} \rceil}, \text{with } k+4 \leq n \leq k+5.
\end{aligned}$$

3. *the multiplicity function $\mu$ is such that if* $k \geq 4$ then,

$$\mu(\alpha) = \begin{cases} 2, & \text{if } \alpha = -5, -4, -3;\\ 1, & \text{otherwise.} \end{cases}$$

4. *the multiplicity function $\mu$ is such that if* $2 \leq k \leq 3$ then,

$$\mu(\alpha) = \begin{cases} 2, & \text{if } \alpha = -7, -6, -5, -4, -3 \text{ and } k = 2,\\ 2, & \text{if } \alpha = -7, -5, -4, -3 \text{ and } k = 3,\\ 1, & \text{otherwise.} \end{cases}$$

The following is the Brauer quiver $Q_{\mathfrak{T}_k}$ associated to this configuration $\mathfrak{T}_k$ when $k \geq 4$, for the sake of clarity we will divide the labeling of the quiver in two parts: In the first one (Figure 2.9) we use the symbol $[x_j; y]$ to denote that the vertex $x_j$ occurs $y$ times at the corresponding polygon (see identity (2.1)) and also draw the successor sequence at the vertex 2 (in blue), at the vertex $-2$ (in red), at the vertex $-k$ (in green),etc. In the second one, $a_i^j$ is the set of loops at the corresponding polygon and $j$ is the vertex. For instance $a_1^{-3}$ denotes the loop associated to the vertex $-3$ in the polygon $P_0$ and $a_{h_{k-3}}^{(-k+2)}$ with $1 \leq h_n \leq \lceil \frac{n+3}{2} \rceil$ denotes the set of loops associated to the vertex $-(k+2)$ in the polygon $P_{k-3}$. All these labels are useful to establish relations of type I, II and III.

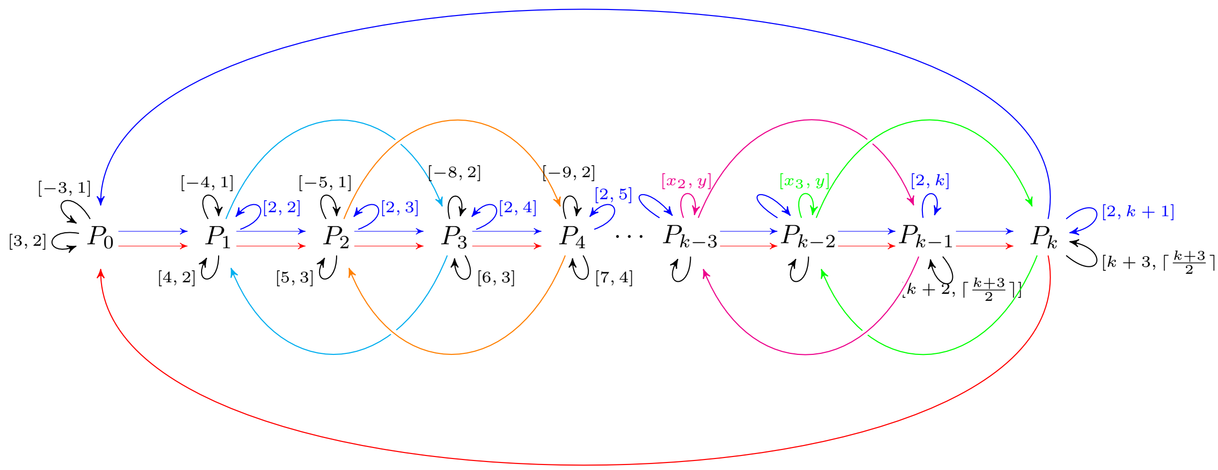


FIGURE 2.9. Quiver associated to the Brauer configuration $\mathfrak{T}_k = (\mathfrak{T}_0, \mathfrak{T}_1, \mu, \mathcal{O})$.

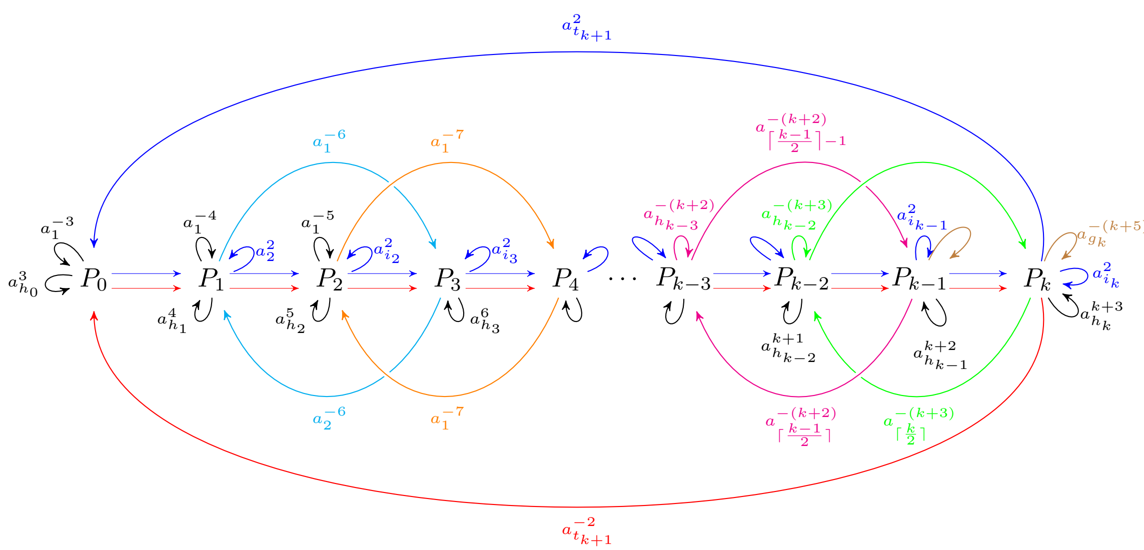


FIGURE 2.10. Quiver associated to the Brauer configuration $\mathfrak{T}_k = (\mathfrak{T}_0, \mathfrak{T}_1, \mu, \mathcal{O})$ with cycles.

The ideal $M$ of the corresponding Brauer configuration algebra $\Lambda_{\mathfrak{T}_k}$ is generated by the following relations (see Figure 2.10), for which it is assumed the following notation for the special cycles:

$$
\begin{aligned}
&C_{-3}^{P_0,1} = a_1^{-3}, \quad C_{-4}^{P_1,1} = a_1^{-4}, \quad C_{-5}^{P_2,1} = a_1^{-5},\\
&C_{-2}^{P_i,1} = \begin{cases} a_1^{-2}\cdots a_{k+1}^{-2} & \text{if } i=0\\ a_{i+1}^{-2}\cdots a_i^{-2} & \text{otherwise}\end{cases}, \text{with } 0\le i\le k,\\
&C_{-n}^{P_{n-5},t} = \begin{cases} a_1^{-n}\cdots a_{\lceil\frac{n-3}{2}\rceil}^{-n} & \text{if } t=1\\ a_t^{-n}\cdots a_{t-1}^{-n} & \text{otherwise}\end{cases}, \text{with } 6\le n\le k+3 \quad \text{and} \quad 2\le t\le \left\lceil\frac{n-3}{2}\right\rceil -1,\\
&C_{-n}^{P_{n-3},1} = a_{\lceil\frac{n-3}{2}\rceil}^{-n} a_1^{-n}\cdots a_{\lceil\frac{n-3}{2}\rceil-1}^{-n} \text{ where } 6\le n\le k+3,\\
&C_{-n}^{P_{n-5},r} = \begin{cases} a_1^{-n}\cdots a_{\lceil\frac{n-5}{2}\rceil}^{-n} & \text{if } r=1\\ a_r^{-n}\cdots a_{r-1}^{-n} & \text{otherwise}\end{cases}, \text{with } k+4\le n\le k+5 \quad \text{and} \quad 2\le r\le \left\lceil\frac{n-5}{2}\right\rceil,\\
&C_{n}^{P_{n-3},s} = \begin{cases} a_1^{n}\cdots a_{\lceil\frac{k}{2}\rceil}^{n} & \text{if } s=1 \quad \text{and} \quad n\ge 3\\ a_s^{n}\cdots a_{s-1}^{n} & \text{otherwise}\end{cases}, \text{with } 2\le s\le \left\lceil\frac{n}{2}\right\rceil,\\
&C_{2}^{P_j,u} = \begin{cases} a_1^{2}\cdots a_{t_{k+1}}^{2} & \text{if } j=0\\ a_{j+u}^{2}\cdots a_{j+u-1}^{2} & \text{otherwise}\end{cases}, \text{with } 1\le u\le j+1, \quad 0\le j\le k,
\end{aligned}
\tag{2.13}
$$

where $t_n$ denotes the triangular number $n$.

1. $\left(C_{-3}^{P_0,1}\right)^2 - C_{-2}^{P_0,1}$, $C_3^{P_0,1} - \left(C_{-3}^{P_0,1}\right)^2$, $C_2^{P_0,1} - \left(C_{-3}^{P_0,1}\right)^2$,
2. $\left(C_{-4}^{P_1,1}\right)^2 - C_{-2}^{P_1,1}$, $\left(C_{-4}^{P_1,1}\right)^2 - C_{-6}^{P_1,1}$, $\left(C_{-4}^{P_1,1}\right)^2 - C_{2}^{P_1,u}$, $\left(C_{-4}^{P_1,1}\right)^2 - C_{4}^{P_1,s}$ for all possible values of $s$ and $u$,
3. $\left(C_{-5}^{P_2,1}\right)^2 - C_{-2}^{P_2,1}$, $\left(C_{-5}^{P_2,1}\right)^2 - C_{-7}^{P_2,1}$, $\left(C_{-5}^{P_2,1}\right)^2 - C_{5}^{P_2,s}$, $\left(C_{-5}^{P_2,1}\right)^2 - C_{2}^{P_2,u}$ for all possible values of $s$ and $u$,
4. For $k\ge 3$; $C_{-2}^{P_k,1} - C_{-(k+5)}^{P_k,t}$, $C_{-2}^{P_k,1} - C_{k+3}^{P_k,s}$, $C_{-2}^{P_k,1} - C_{-(k+3)}^{P_k,1}$, $C_{-2}^{P_k,1} - C_{2}^{P_k,u}$, $C_{-2}^{P_k,1} - C_{-n}^{P_k,r}$ for all possible values of $s$, $u$ and $r$,
5. $(a_1^{-3})^2 a_1^{-3}$, $(a_1^{-4})^2 a_1^{-4}$, $(a_1^{-5})^2 a_1^{-5}$,
6. $C_{-2}^{P_i,1} a$ ($C_{-n}^{P_{n-5},t} a'$), with $a$ ($a'$) being the first arrow of $C_{-2}^{P_i,1}$ ($C_{-n}^{P_{n-5},t}$) for all possible values of $i$ and $n$,
7. $C_{-n}^{P_{n-3},1} a$ ($C_{n}^{P_{n-3},s} a'$), with $a$ ($a'$) being the first arrow of $C_{-n}^{P_{n-3},1}$ ($C_{n}^{P_{n-3},s}$) for all possible values of $n$ and $s$,
8. $C_{-n}^{P_k,1} a$ ($C_{n}^{P_k,r} a'$), with $a$ ($a'$) being the first arrow of $C_{-n}^{P_k,1}$ ($C_{n}^{P_k,r}$) for all possible values of $n$ and $r$,
9. $C_2^{P_j,u} a$, with $a$ being the first arrow of $C_2^{P_j,u}$ for all possible values of $j$ and $u$,
10. For $3\le n\le k$ we have $a_{\lceil\frac{n}{2}\rceil-1}^{-(n+3)} a_{i_n}^2$, $a_{\lceil\frac{n}{2}\rceil-1}^{-(n+3)} a_{t_{n+1}}^2$ with $t_n+1\le i_n\le t_{n+1}-1$,
11. $a_{\lceil\frac{n}{2}\rceil-1}^{-(n+3)} a_{n+1}^{-2}$, $a_{\lceil\frac{n}{2}\rceil-1}^{-(n+3)} a_{h_n}^{n+3}$ with $1\le h_n\le \lceil\frac{n+3}{2}\rceil$,

12. $a^{-(n+3)}_{\lceil\frac{n}{2}\rceil-1}a^{-(n+5)}_{j_n}$, $a^{-(n+3)}_{\lceil\frac{n}{2}\rceil-1}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$ , for $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

13. $a^{-2}_n a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$, $a^{-2}_n a^{n+3}_{h_n}$, with $1 \leq h_n \leq \lceil\frac{n+3}{2}\rceil$,

14. $a^{-2}_n a^2_{t_{n+1}}$, $a^{-2}_n a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$, $a^{-2}_n a^{-(n+5)}_{j_n}$, with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

15. $a^{-2}_n a^2_{i_n}$, $a^{-2}_{n+1}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}$, with $t_n + 1 \leq i_n \leq t_{n+1} - 1$,

16. $a^2_{t_n}a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$, $a^2_{t_n}a^2_{t_{n+1}}$, $a^2_{t_n}a^2_{i_n}$, with $t_n + 2 \leq i_n \leq t_{n+1} - 1$,

17. $a^2_{t_n}a^{n+3}_{h_n}$, $a^2_{t_n}a^{-2}_{n+1}$, $a^2_{t_n}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$, with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

18. $a^2_{t_n}a^{-(n+5)}_{j_n}$, $a^2_{t_{n+1}}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}$ with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

19. $a^2_{i_n}a^{-(n+5)}_{j_n}$, $a^2_{i_n}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$, $a^2_{i_n}a^{-2}_{n+1}$ with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil-2$ and $t_n+2 \leq i_n \leq t_{n+1}-1$,

20. $a^2_{i_n}a^{n+3}_{h_n}$, $a^2_{i_n}a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$ with $1 \leq h_n \leq \lceil\frac{n+3}{2}\rceil$ and $t_n + 2 \leq i_n \leq t_{n+1} - 1$,

21. $a^2_{i_n}a^2_{t_{n+1}}$, $a^{-(n+5)}_{j_n}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$ with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil + 3$ and $t_n + 1 \leq i_n \leq t_{n+1} - 2$,

22. $a^{-(n+5)}_{j_n}a^2_{t_{n+1}}$, $a^{-(n+5)}_{j_n}a^{-2}_{n+1}$, $a^{-(n+5)}_{j_n}a^{n+3}_{h_n}$ with $1 \leq h_n \leq \lceil\frac{n+3}{2}\rceil$ and $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

23. $a^{-(n+5)}_{j_n}a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$, $a^{-(n+5)}_{j_n}a^2_{i_n}$, with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$ and $t_n + 1 \leq i_n \leq t_{n+1} - 1$,

24. $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^{n+3}_{h_n}$, $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^2_{i_n}$ with $1 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$ and $t_n + 1 \leq i_n \leq t_{n+1} - 1$,

25. $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$, $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^{-(n+5)}_{j_n}$ with $2 \leq j_n \leq \lceil\frac{n+2}{2}\rceil - 2$,

26. $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^2_{t_{n+1}}$, $a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}a^{-2}_{n+1}$,

27. $a^{n+3}_{h_n}a^{-2}_{n+1}$, $a^{n+3}_{h_n}a^2_{t_{n+1}}$, $a^{n+3}_{h_n}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil-1}$ with $1 \leq h_n \leq \lceil\frac{n+3}{2}\rceil$,

28. $a^{n+3}_{h_n}a^{-(n+5)}_{j_n}$, $a^{n+3}_{h_n}a^2_{i_n}$, $a^{n+3}_{h_n}a^{-(n+3)}_{\lceil\frac{n}{2}\rceil}$ with $1 \leq h_n \leq \lceil\frac{n+3}{2}\rceil$,

29. $a^{-n}_{g_n}a^2_{i_n}$, $a^{-n}_{g_n}a^2_{t_{n+1}}$, $a^{-n}_{g_n}a^{-2}_{n+1}$, $a^{-n}_{g_n}a^{n+3}_{h_n}$, $a^{-n}_{g_n}a^{-(n+5)}_{\lceil\frac{n+2}{2}\rceil}$ with $1 \leq g_n \leq \lceil\frac{n-5}{2}\rceil$,

30. $a^{-(n+3)}_{\lceil\frac{n}{2}\rceil-1}a^{-n}_{g_n}$, $a^{-2}_n a^{-n}_{g_n}$, $a^{n+3}_{h_n}a^{-n}_{g_n}$, $a^2_{t_n}a^{-n}_{g_n}$, $a^2_{i_n}a^{-n}_{g_n}$ with $1 \leq g_n \leq \lceil\frac{n-5}{2}\rceil$,

31. $a^{-3}_1 a^2_1$, $a^{-3}_1 a^{-2}_1$, $a^{-3}_1 a^3_{h_0}$, $a^2_{t_{k+1}}a^{-3}_1$, $a^3_{h_0}a^{-3}_1$, $a^{-2}_{k+1}a^{-3}_1$, $a^{-4}_1 a^{-6}_1$, $a^{-4}_1 a^4_{h_1}$,

32. $a^{-4}_1 a^2_2$, $a^{-4}_1 a^2_3$, $a^{-4}_1 a^{-2}_2$, $a^{-6}_2 a^{-4}_1$, $a^4_{h_1}a^{-4}_1$, $a^2_1 a^{-4}_1$, $a^{-2}_1 a^{-4}_1$, $a^2_2 a^{-4}_1$, $a^2_2 a^{-6}_1$,

33. $a^{-6}_2 a^4_{h_1}$, $a^{-6}_2 a^2_2$, $a^{-6}_2 a^2_3$, $a^{-6}_2 a^{-2}_2$, $a^{-5}_1 a^{-7}_1$, $a^{-5}_1 a^5_{h_2}$, $a^{-5}_1 a^2_{i_2}$, $a^2_1 a^{-6}_1$, $a^{-2}_1 a^{-6}_1$,

34. $a^{-5}_1 a^{-2}_3$, $a^2_3 a^{-5}_1$, $a^2_{i_2}a^{-5}_1$, $a^5_{h_2}a^{-5}_1$, $a^{-7}_2 a^{-5}_1$, $a^{-2}_2 a^{-5}_1$,$a^{-5}_1 a^2_6$, $a^4_{h_1}a^{-6}_1$.

The following result regards the Brauer configuration algebra $\Lambda_{\mathfrak{T}_k}$.

**Theorem 19.** *For the specialization $e(i) = [i]$ with the concatenation defined by the usual tangle product it holds that $M(\mathfrak{T}_k^e) = \mathcal{T}_n$ (see Example 1).*

*Proof.* For $k \geq 2$ fix, we have $k+1$ polygons $P_i$ with $0 \leq i \leq k$. According to (2.1) $w(P_i) = (i+3)^{\lceil \frac{i+3}{2} \rceil}(-(i+5))^{\lceil \frac{i}{2} \rceil}(-(i+3))^1(2)^{i+1}(-2)^1 = (i+3)(-(i+3))(-2)(2)(i+3)(2)(-(i+5))(2)(i+3)(2)(-(i+5))(2)(i+3)\cdots$ where the length of $w(P(i))$ is equal to $2i+5$.

Now, applying the specialization we have:

$$\begin{aligned} w^e(P_i) &= [i+3] * [-(i+3)] * [-2] * [2] * [i+3] * [2] * [-(i+5)] * [2] * [i+3] * [2] * [-(i+5)] \cdots \\ &= \mathcal{W}(i+3, i+2) \end{aligned}$$

Thus, the specialized message is given by the following formula

$$M(\mathfrak{T}_k^e) = \sum_{i=0}^{k} w^e(P_i) = \sum_{i=0}^{k} \mathcal{W}(i+3, i+2) = \mathcal{T}_k$$

□

**Corollary 10.** *There is a bijective correspondence between rational tangles of type $\mathcal{W}(i+1, i)$ and the preprojective Kronecker snake graphs.*

*Proof.* It is enough to associate each $\mathcal{W}(i+1, i)$ the corresponding representative in the class of the helices associated to the preprojective Kronecker module $(i+1, i)$ defined in [13] or in Section 3.1. The corollary holds as a consequence of Theorem 22. □

The following results are consequences of Theorem 4 and Proposition 2.

**Corollary 11.** *For $k \geq 4$ fixed,*

$$\dim_k \Lambda_{\mathfrak{T}_k} = \begin{cases} \frac{12n^4+40n^3+60n^2+32n+21}{3}, & \text{if } k = 2n, \\ \frac{12n^4+64n^3+138n^2+128n+57}{3}, & \text{if } k = 2n+1. \end{cases}$$

*Proof.* It is enough to observe that for $k \geq 4$ fixed, it holds that

$$val(\alpha) = \begin{cases} \lceil \frac{-\alpha-5}{2} \rceil, & \text{if } -(k+5) \leq \alpha \leq -(k+4); \\ \lceil \frac{-\alpha-3}{2} \rceil, & \text{if } -(k+3) \leq \alpha \leq -6; \\ 1, & \text{if } -5 \leq \alpha \leq -3; \\ k+1, & \text{if } \alpha = -2; \\ t_{k+1}, & \text{if } \alpha = 2; \\ \lceil \frac{\alpha}{2} \rceil, & \text{if } 3 \leq \alpha \leq k+3. \end{cases}$$

According to Proposition 2 the following identity holds:

$$\begin{aligned}\dim_k \Lambda_{\mathfrak{T}_k} &= 2(k+1) + \sum_{\alpha=-(k+5)}^{-(k+4)} 2t_{\lceil \frac{-\alpha-5}{2} \rceil -1} + \sum_{\alpha=-(k+3)}^{-6} 2t_{\lceil \frac{-\alpha-3}{2} \rceil -1} \\ &+ 3 + 2t_k + 2t_{t_{k+1}-1} + \sum_{\alpha=3}^{k+3} 2t_{\lceil \frac{\alpha}{2} \rceil -1},\end{aligned} \tag{2.14}$$

where $t_j$ denotes the $jth$ triangular number. The result holds as a consequence of the following identities:

$$\sum_{\alpha=-(k+5)}^{-(k+4)} 2t_{\lceil \frac{-\alpha-5}{2} \rceil -1} = \begin{cases} 4t_{n-1}, & \text{if } k = 2n; \\ 2t_{n-1} + 2t_n, & \text{if } k = 2n+1; \end{cases}$$

$$\sum_{\alpha=-(k+3)}^{-6} 2t_{\lceil \frac{-\alpha-3}{2} \rceil -1} = \begin{cases} 2\left(\sum\limits_{i=1}^{n-1} 2t_i\right) = \frac{2n(n^2-1)}{3}, & \text{if } k = 2n; \\ 2\left(\sum\limits_{i=1}^{n} 2t_i\right) - 2t_n = \frac{n(n+1)(2n+1)}{3}, & \text{if } k = 2n+1; \end{cases}$$

$$\sum_{\alpha=3}^{k+3} 2t_{\lceil \frac{\alpha}{2} \rceil -1} = \begin{cases} 2\left(\sum\limits_{i=1}^{n+1} 2t_i\right) - 2t_{n+1} = \frac{(n+2)(2n^2+5n+3)}{3}, & \text{if } k = 2n; \\ 2\left(\sum\limits_{i=1}^{n+1} 2t_i\right) = \frac{2(n+2)(n^2+4n+3)}{3}, & \text{if } k = 2n+1. \end{cases}$$

We are done. □

**Corollary 12.** *For $k \geq 4$ fixed, it holds that:*

$$\dim_k Z(\Lambda_{\mathfrak{T}_k}) = \begin{cases} 7n^2 + 2n + 5, & \text{if } k = 2n; \\ 7n^2 + 7n + 7, & \text{if } k = 2n+1. \end{cases}$$

*Proof.* Since $\operatorname{rad}^2 \Lambda_{\mathfrak{T}_k} \neq 0$, the result is a consequence of Theorem 4 with $\mu(-5) = \mu(-4) = \mu(-3) = 2$, $|\mathfrak{T}_0| = 2k+6$, $|\mathfrak{T}_1| = k+1$, $|\mathcal{C}_{\mathfrak{T}_k}| = 3$, $val(2) = t_k$ and $\#(Loops\ Q)$ given by the following formula:

$$\#(Loops\ Q) = \begin{cases} 7n^2 + 3, & \text{if } k = 2n; \\ 7n^2 + 5n + 4, & \text{if } k = 2n+1. \end{cases} \tag{2.15}$$

Formula (2.15) can be obtained taking into account the following identities:

$$\sum_{i=3}^{k+3} \left\lceil \frac{i}{2} \right\rceil = \begin{cases} 2(2n^2-1), & \text{if } k = 2n; \\ 2(2n^2+n-1), & \text{if } k = 2n+1; \end{cases}$$

$$\sum_{i=-(k+3)}^{-6} \left\lceil \frac{-i-3}{2} \right\rceil - 2 = \begin{cases} (n-1)(n-2), & \text{if } k = 2n; \\ (n-1)^2, & \text{if } k = 2n+1; \end{cases}$$

$$\sum_{i=-(k+5)}^{-(k+4)} \left\lceil \frac{-i-5}{2} \right\rceil = \begin{cases} 2n, & \text{if } k = 2n; \\ 2n+1, & \text{if } k = 2n+1; \end{cases}$$

We are done. □

### 2.4.2 The group of the preinjective Kronecker snake graphs

In this section, the set $\mathcal{S}^{w}_{(n,n+1)}$ of preinjective Kronecker snake graphs of a fixed preinjective Kronecker module is endowed with an operation, which makes of that set a finite group.

A Kronecker snake graph $\mathcal{G}_{(n,n+1)k}$ has associated a unique word $w^{h}_{\mathcal{G}} = \left(a_{p_{r_1}q_{s_1}}b_{p_{r_1}q_{s_2}}, b_{p_{r_2}q_{s_2}}a_{p_{r_2}q_{s_3}}, a_{p_{r_3}q_{s_3}}b_{p_{r_3}q_{s_4}}, \dots, l_{p_{r_n}q_{s_n}}l_{p_{r_n}q_{s_{n+1}}}\right)$ given by the corresponding entries, such correspondence makes of the Kronecker graph associated to the fixed preinjective Kronecker module $(n, n+1)$ a finite group with a multiplication

$$* : \mathcal{S}^{w}_{(n,n+1)} \times \mathcal{S}^{w}_{(n,n+1)} \longrightarrow \mathcal{S}^{w}_{(n,n+1)}$$

defined as follows for $h, h' \in \mathcal{S}^{w}_{(n,n+1)}$:

$$h * h' = \left(a_{p_{r_1}q_{s_1}}b_{p_{r_1}q_{s_2}}, b_{p_{r_2}q_{s_2}}a_{p_{r_2}q_{s_3}}, a_{p_{r_3}q_{s_3}}b_{p_{r_3}q_{s_4}}, \dots, l_{p_{r_n}q_{s_n}}l_{p_{r_n}q_{s_{n+1}}}\right) * \left(a'_{p'_{r_1}q'_{s_1}}b'_{p'_{r_1}q'_{s_2}}, b'_{p'_{r_2}q'_{s_2}}a'_{p'_{r_2}q'_{s_3}}, a'_{p'_{r_3}q'_{s_3}}b'_{p'_{r_3}q'_{s_4}}, \dots, l'_{p'_{r_n}q'_{s_n}}l'_{p'_{r_n}q'_{s_{n+1}}}\right)$$

$$h * h' = \left(l_{p_{r_{p'_{r_1}}}q_{s_{p'_{r_1}}}}l_{p_{r_{p'_{r_1}}}q_{s_{p'_{r_1+1}}}} * a'_{p'_{r_1}q'_{s_1}}b'_{p'_{r_1}q'_{s_2}}, l_{p_{r_{p'_{r_2}}}q_{s_{p'_{r_2}}}}l_{p_{r_{p'_{r_2}}}q_{s_{p'_{r_2+1}}}} * b'_{p'_{r_2}q'_{s_2}}a'_{p'_{r_2}q'_{s_3}}, \dots, l_{p_{r_{p'_{r_n}}}q_{s_{p'_{r_n}}}}l_{p_{r_{p'_{r_n}}}q_{s_{p'_{r_n+1}}}} * l'_{p'_{r_n}q'_{s_n}}l'_{p'_{r_n}q'_{s_{n+1}}}\right)$$

$$h * h' = \left(a'_{p_{r_{p'_{r_1}}}q_{s_1}}b'_{p_{r_{p'_{r_1}}}q_{s^{1}(*)}}, b'_{p_{r_{p'_{r_2}}}q_{s^{1}(*)}}a'_{p_{r_{p'_{r_2}}}q_{s^{2}(*)}}, \dots, l'_{p_{r_{p'_{r_n}}}q_{s^{n}(*)}}l'_{p_{r_{p'_{r_n}}}q_{s^{n+1}(*)}}\right) \quad (1)$$

with $l' \in \{a, b\}$ and $p_{r_{p'_{r_n}}} - q_{s^{n+1}(*)} = p'_{r_n} - q_{s_n+1}$ or equivalently

$$q_{s^{n+1}(*)} = p_{r_{p'_{r_n}}} - p'_{r_n} + q_{s_n+1}.$$

**Theorem 20.** $(\mathcal{S}^{w}_{(n,n+1)}, *)$ *is a finite group.*

*Proof.* Closure and associativity are trivial from (1). The identity is given by:

$$id(n) = \begin{cases} (a_{1n+1}b_{12}, b_{22}a_{22}, a_{32}b_{34}, \dots, l_{nn-1}l_{nn+1}) & \text{if } n \text{ is odd} \\ (a_{1n+1}b_{12}, b_{22}a_{22}, a_{32}b_{34}, \dots, l_{nn}l_{nn}) & \text{if } n \text{ is even} \end{cases}$$

Finally, each helix $h$ has an inverse $h^{-1}$ uniquely determined by

$$h^{-1} = \left(a'_{p'_{r_1}q'_{s_1}} b'_{p'_{r_1}q'_{s_2}}, b'_{p'_{r_2}q'_{s_2}} a'_{p'_{r_2}q'_{s_3}}, a'_{p'_{r_3}q'_{s_3}} b'_{p'_{r_3}q'_{s_4}}, \dots, l'_{p'_{r_n}q'_{s_n}} l'_{p'_{r_n}q'_{s_{n+1}}}\right)$$

such that $p_{r_{p'_{r_i}}} = i$ for all $1 \leq i \leq n$. $\square$

**Example 3.** *The elements of the group* $(\mathcal{S}^w_{(3,4)}, *)$ *are the following:*

$$h_1 = (a_{14}b_{12}, b_{22}a_{22}, a_{32}b_{34})$$
$$h_2 = (a_{14}b_{12}, b_{32}a_{33}, a_{23}b_{23})$$
$$h_3 = (a_{24}b_{23}, b_{13}a_{11}, a_{31}b_{34})$$
$$h_4 = (a_{24}b_{23}, b_{33}a_{33}, a_{13}b_{12})$$
$$h_5 = (a_{34}b_{34}, b_{24}a_{22}, a_{12}b_{12})$$
$$h_6 = (a_{34}b_{34}, b_{14}a_{11}, a_{21}b_{23}).$$

*Some products are*

$$\begin{aligned} h_1 * h_2 &= (a_{14}b_{12}, b_{22}a_{22}, a_{32}b_{34}) * (a_{14}b_{12}, b_{32}a_{33}, a_{23}b_{23}) \\ &= (a_{14}b_{12} * a_{14}b_{12}, a_{32}b_{34} * b_{32}a_{33}, b_{22}a_{22} * a_{23}b_{23}) \\ &= (a_{14}b_{12}, b_{32}a_{33}, a_{23}b_{23}) = h_2 \end{aligned}$$

$$\begin{aligned} h_3 * h_3 &= (a_{24}b_{23}, b_{13}a_{11}, a_{31}b_{34}) * (a_{24}b_{23}, b_{13}a_{11}, a_{31}b_{34}) \\ &= (b_{13}a_{11} * a_{24}b_{23}, a_{24}b_{23} * b_{13}a_{11}, a_{31}b_{34} * a_{31}b_{34}) \\ &= (a_{14}b_{12}, b_{22}a_{22}, a_{32}b_{34}) = h_1 \end{aligned}$$

$$\begin{aligned} h_5 * h_3 &= (a_{34}b_{34}, b_{24}a_{22}, a_{12}b_{12}) * (a_{24}b_{23}, b_{13}a_{11}, a_{31}b_{34}) \\ &= (b_{24}a_{22} * a_{24}b_{23}, a_{34}b_{34} * b_{13}a_{11}, a_{12}b_{12} * a_{31}b_{34}) \\ &= (a_{24}b_{23}, b_{33}a_{33}, a_{13}b_{12}) = h_4 \end{aligned}$$

*The Cayley table of* $(\mathcal{S}^w_{(3,4)}, *)$ *has the following form:*

| $*$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ |
|---|---|---|---|---|---|---|
| $h_1$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ |
| $h_2$ | $h_2$ | $h_1$ | $h_6$ | $h_5$ | $h_4$ | $h_3$ |
| $h_3$ | $h_3$ | $h_4$ | $h_1$ | $h_2$ | $h_6$ | $h_5$ |
| $h_4$ | $h_4$ | $h_3$ | $h_5$ | $h_6$ | $h_2$ | $h_1$ |
| $h_5$ | $h_5$ | $h_6$ | $h_4$ | $h_3$ | $h_1$ | $h_2$ |
| $h_6$ | $h_6$ | $h_5$ | $h_2$ | $h_1$ | $h_3$ | $h_4$ |

**Corollary 13.** $(\mathcal{S}^w_{(n,n+1)}, *)$ *is isomorphic to the symmetric group* $(S_n, \circ)$.

*Proof.* The result follows directly from Theorem 20 and Theorem 24. $\square$

#### 2.4.2.1 Auslander algebras

In this section, we present a connection between the group $(\mathcal{S}^{w}_{(n,n+1)}, *)$ and the full exceptional sequences of the Auslander algebras $\mathcal{A}_t$ of $k[t]/(x^t)$ [38].

In this section, we consider that the modules are left modules over a suitable Auslander algebra, and that compositions of arrows are made from right to left. This family of finite-dimensional algebras is well-known in representation theory. It also occurs for certain matrix problems, i.e. actions of linear groups on flags [38].

The algebra $\mathcal{A}_t$ is defined as the path algebra of the quiver with $t$ vertices and $2t-2$ arrows

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 3 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdots \underset{\beta}{\overset{\alpha}{\rightleftarrows}} t-1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} t$$

bound by a zero relation $\beta\alpha = 0$ at 1, and commutativity relations $\alpha\beta = \beta\alpha$ at intermediate vertices $2, \ldots, t-1$. We emphasise that there is no relation at $t$, this distinguishes $\mathcal{A}_t$ from the preprojective algebra of the $A_t$- quiver.

It is well known, $\mathcal{A}_t$ occurs as the Auslander algebra of the ring $R := \mathbf{k}[x]/(x^t)$ [36]. This means that $R$ has finitely many indecomposable (finitely generated) modules, these are $M(i) := R/(X^t) = \mathbf{k}[X]/(X^t)$ for $i = 1, \ldots, t$ and that $\mathcal{A}_t$ is the endomorphism algebra of their direct sum: $\mathcal{A}_t = \mathrm{End}_R(M(1) \oplus \cdots \oplus M(t))$.

Moreover, $\mathcal{A}_t$ also occurs as the endomorphism algebra of a very special tilting object of geometric nature (see Appendix B in [36] and [37]).

A non-zero, indecomposable thin representation of $\mathcal{A}_t$ is a sequence of maps $\mathbf{k} \xrightarrow{\alpha} \mathbf{k}$ and $\mathbf{k} \xleftarrow{\beta} \mathbf{k}$. Therefore, it is uniquely encoded by a word in the letters $\alpha$ and $\beta$, together with the last index of a non-zero vector space in the representation. For exceptional modules, the encoding is particularly simple, since the index of the last non-zero vector space is always $t$.

We will depict these modules using the following convention: we read the word in the letters $\alpha$ and $\beta$ from left to right, and $\alpha$ is drawn as a line going right and $\beta$ is drawn as a line going up. This we call a *worm*.

**Example 4.** *The seven exceptional $\mathcal{A}_3$- modules, as representations and worms:*

$S(3) = \nabla(1) = \Delta(1) = [\,0 \quad 0 \quad \boldsymbol{k}\,]$

$\Delta(2) = [\,0 \quad \boldsymbol{k} \xrightarrow{\alpha} \boldsymbol{k}\,]$ $\qquad$ $\nabla(3) = [\,\boldsymbol{k} \xleftarrow{\beta} \boldsymbol{k} \xleftarrow{\beta} \boldsymbol{k}\,]$

$\Delta(3) = [\boldsymbol{k} \xrightarrow{\alpha} \boldsymbol{k} \xrightarrow{\alpha} \boldsymbol{k}\,]$ $\qquad$ $\nabla(2) = [\,0 \quad \boldsymbol{k} \xleftarrow{\beta} \boldsymbol{k}\,]$

$[\boldsymbol{k} \xrightarrow{\alpha} \boldsymbol{k} \xleftarrow{\beta} \boldsymbol{k}\,]$ $\qquad$ $[\boldsymbol{k} \xleftarrow{\alpha} \boldsymbol{k} \xrightarrow{\beta} \boldsymbol{k}\,]$

Now, we define worm diagrams as certain collections of worms. Worms will always be conflated with exceptional modules. We consider $\mathbb{Z} \times \mathbb{Z}$ as a lattice grid in the obvious way.

**Definition 2.** *A worm diagram of size $t$ is a graph with the following properties:*

*(1) the vertices exhaust the triangle $\{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m+n \leq t$ and $m,n \geq 1\}$,*

*(2) the edges lie on the lattice grid,*

*(3) the connected components are $t$ worms of lengths $1, 2, \dots, t$, respectively.*

**Proposition 5.** *Every worm diagram of size $t$ gives rise to a full exceptional sequence of $\mathcal{A}_t$-modules.*

L. Hille and D. Ploog proved in [36] the following result:

**Theorem 21.** *For fixed $t \in \mathbb{N}$, there are bijections between the following sets:*

1. *Full exceptional sequences of $\mathcal{A}_t$-modules.*
2. *Worm diagrams of size $t$.*
3. *The symmetric group $(S_t, \circ)$.*

The following result follows from Corollary 13 and Theorem 21.

**Corollary 14.** *There is a bijection between the full exceptional sequences of $\mathcal{A}_t$-modules and the elements of the group $\mathcal{S}^w_{(n,n+1)}$.*

*Proof.* The assertion follows from Theorem 20 and Theorem 21. $\square$

# CHAPTER 3

# Categorification via some matrix problems and Brauer configuration algebras

In this chapter, we categorify some integer sequences arising from some matrix problems. In section 3.1, we give an explicit formula for the number of Kronecker snake graphs defined in Chapter 2, these numbers correspond to the sequence A052558 in the On-Line Encyclopedia of Integer Sequences (OEIS). In section 3.2, we categorify the numbers of the sequence A052558, to do that, we define Brauer configuration algebras whose indecomposable projective modules are in bijective correspondence with preprojective Kronecker modules, formulas for the dimension of this type of algebras and its corresponding centers are given as well. In section 3.3, we categorify the sequence A100705, which appears in the context of the Bert Konstant's game and in the four subspace problem. As in the case of the sequence A052558, we define Brauer configuration algebras whose indecomposable projective modules are also in bijective correspondence with preprojective representations of the tetrad via some cycles associated to these indecomposable representations. Throughout the categorification process of the sequences A052558 and A100705 we associate some partition trees to the preprojective solutions of its corresponding matrix problem, in section 3.4, we introduce the general notion of a preprojective tree and the explicit value of the energy of these kind of trees is computed by using the message of a suitable Brauer configuration algebra.

## 3.1 On the number of Kronecker snake graphs

In this section, some relationships between preprojective snake graphs, trees and helices defined by the author et al. [13] are described as well.

In accordance with the digression made in Chapter 2 Kronecker snake graphs define indecomposable preprojective and preinjective modules. Actually, the notion of Kronecker snake graph can be generalized by admitting that rows and columns of a given snake graph are given by rows and columns of a matrix Kronecker block $(A, B)$ applying the same rules as in the generic case, e.g., entries $a_{1,1}, \ldots, a_{1,n}$ and $b_{1,1}$ are used to label the first row of the snake graph, then tiles in the first column are labeled with entries $b_{1,1}, \ldots, b_{i_1,1}$, for some $1 < i_1 \leq n+1$, then tiles in the second row are labeled with entries $b_{i_1,1}, \ldots, b_{i_{i_1},i_1}$ the second column is labeled with entries of the form $b_{i_1,i_1},\dots,b_{i_2,i_1}$, for some $1 < i_2 \leq n+1$, $(i_2 \notin \{1, i_1\})$. Tiles in the third row are labeled with entries of the form $b_{i_2,i_1},\dots,b_{i_2,i_2}$, tiles in third column are labeled with entries of the form $b_{i_2,i_2},\dots,b_{i_3,i_2}$ $(i_3 \notin \{1, i_1, i_2\})$ and so on, bearing in mind that there is not a row in the snake graph whose tiles are labeled by entries of the form $a_{n+1,j}$, $j \neq n+1$, and $b_{n+1,j}$, $1 \leq j \leq n$. Thus, generalized Kronecker snake graphs give place to helices associated to indecomposable Kronecker modules defined by the author et al. as follows:

An *helix* $h$ defined on the set of entries of an indecomposable non-regular Kronecker module is a path (oriented graph) whose vertices are entries of the matrix blocks $A$ and $B$. Such that, arrows connect an alternating sequence of the form $\{a_{1,j}, b_{1,1}, b_{r_1,1}, a_{r_1,s_1}, a_{r_2,s_1}, b_{r_2,s_2}, b_{r_3,s_2}, a_{r_3,s_3},\dots,l_{r_t,s_t}\}$ where starting vertices are entries in the null row of matrix $A$ (although, starting vertices in the matrix block $B$ can be also considered to build a sequence according to this selection), the $r_i$'s visit all the rows of the indecomposable, $r_i \neq r_j$ if $i \neq j$, $a_{r_i,s_j} \neq a_{r_{i'},s_j}$, $l \in \{a,b\}$, and $b_{r_h,s_k} \neq b_{r_{h'},s_k}$. In particular, each horizontal arrow in a helix $h$ visits a given row in a matrix block $(A, B)$ just once.

Some cases of helices are given in the following example:

**Example 5.** *Helices associate to the Kronecker modules* $(3,2)$, $(4,3)$ *and* $(3,4)$.

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Two preprojective (preinjective) Kronecker snake graphs in the same Kronecker module are said to be equivalent. Therefore, the main problem regarding Kronecker snake graphs consists of giving the cardinality of the corresponding equivalence classes.

The following result follows from the definition of helices and Kronecker snake graphs:

**Theorem 22.** *For $n \geq 2$ fixed, there is a bijective correspondence between preprojective snake graphs and helices associated to the indecomposable preprojective (preinjective) Kronecker module $(n+1, n)$ $((n, n+1))$.*

Theorem 22 allows us to reinterpret Problem 1 as a labeling problem for Kronecker snake graphs. That is, the number of helices associated to a given non-regular indecomposable Kronecker module is equal to the number of ways that a Kronecker snake graph can be labeled by the entries of the corresponding matrix block.

The following results solve Problem 1 for indecomposable preprojective (preinjective) Kronecker modules. We recall that some advances to this problem have been proposed by the author et al. in [13].

**Theorem 23.** *If $(n+1,n)$ denotes an indecomposable preprojective Kronecker module then the number of helices associated to $(n+1,n)$ is $h_n^p = n!\lceil \frac{n}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$. In fact:*

$$(n+1,n) \simeq (n'+1,n') \text{ if and only if } h_n^p = h_{n'}^p.$$

*Proof.* We note that there is only one helix associated to the indecomposable preprojective module $(2,1)$ and two for the indecomposable projective module $(3,2)$. And the vertices sequence of helices associated to the indecomposable $(4,3)$ with $a_{1,j}$ fixed are:

$$\begin{aligned} hl_1 &= \{a_{1,j}, b_{1,1}, b_{2,1}, a_{2,1}, a_{3,1}, b_{3,3}, b_{4,3}, a_{4,3}\}, \\ hl_2 &= \{a_{1,j}, b_{1,1}, b_{3,1}, a_{3,2}, a_{2,2}, b_{2,2}, b_{4,2}, a_{4,3}\}, \\ hl_3 &= \{a_{1,j}, b_{1,1}, b_{4,1}, a_{4,3}, a_{3,3}, b_{3,3}, b_{2,3}, a_{2,1}\}, \\ hl_4 &= \{a_{1,j}, b_{1,1}, b_{4,1}, a_{4,3}, a_{2,3}, b_{2,2}, b_{3,2}, a_{3,2}\}. \end{aligned} \tag{3.1}$$

The number of helices is given by the number of vertices at the last level of the following associated tree:

$$\begin{tikzcd} & & (a_{1,j}, b_{1,1}) \arrow[dll] \arrow[d] \arrow[drr] & & & \\ b_{2,1} \arrow[d] & & b_{3,1} \arrow[d] & & b_{4,1} \arrow[dl] \arrow[dr] & \\ a_{3,1} \arrow[d] & & a_{2,2} \arrow[d] & a_{3,3} \arrow[d] & & a_{2,3} \arrow[d] \\ b_{4,3} & & a_{4,2} & b_{2,3} & & b_{3,2} \end{tikzcd} \tag{3.2}$$

Suppose now that the result is true for any indecomposable preprojective Kronecker module $(k+1,k)$, $1 \leq k < n$ then we can see that in general the rooted tree $T_n$ associated to the indecomposable preprojective Kronecker module $(n+1,n)$ has the following characteristics bearing in mind that vertex $b_{1,1}$ gives the root node $a_1^0$:

$(a)$ $a_1^0$ has $n$ children enumerated from the left to the right as $(a_1^1, a_2^1, \ldots, a_n^1)$,

($b$) For $1 \leq i \leq n-1$ each vertex $a_i^1$ has $n-2$ children enumerated from the left to the right as $(a_{i,1}^1, a_{i,2}^1, \dots, a_{i,n-2}^1)$, whereas vertex $a_n^1$ has $n-1$ children of the form $(a_{n,1}^1, a_{n,2}^1, \dots, a_{n,n-1}^1)$, each children of a vertex $a_{n,l_1}^1$, $1 \leq l_1 \leq n-1$ has $n-2$ children $a_{n,l_1,l_2}^1$ with $1 \leq l_2 \leq n-2$, in general for this particular tree a vertex $a_{n,l_1,l_2,l_3,\dots,l_k}^1$ has $n-(k+1)$ children, $1 \leq k \leq n-2$. Note that the number of vertices at the last level of the rooted tree $T'_n$ with $a_n^1$ as root node is $(n-1)!$,

($c$) For each $h$, $1 \leq h \leq n-2$, vertex $a_{i,h}^1$ is a root node of the tree $T_{n-2}$.

The following diagram shows the general structure of the rooted tree $T_n$

$$\begin{tikzcd}
 & & & n\text{-children} \arrow[dll] \arrow[d] \arrow[dr] & & \\
 & (n-2) \arrow[dl] \arrow[d] \arrow[dr] & & \cdots(n-2)\cdots & (n-1) \arrow[dl] \arrow[d] \arrow[dr] & \\
T_{(n-2)} \cdots & T_{(n-2)} \cdots & T_{(n-2)} & (n-2)\cdots \arrow[d] & (n-2)\cdots \arrow[d] & (n-2) \arrow[d] \\
 & (n-2)\text{-children} & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & (1) & (1) & (1)
\end{tikzcd} \tag{3.3}$$

According to the rules $(a)-(c)$ the number of vertices $L_{T_n}$ at the last level of the tree $T_n$ is given by the formula

$$\begin{aligned}
L_{T_n} &= (n-1)(n-2)L_{T_{n-2}} + L(T'_n) = (n-1)(n-2)\frac{h_{n-2}^p}{n-2} + (n-1)! \\
&= (n-1)!\left\lceil \frac{n}{2} \right\rceil = \frac{h_n^p}{n}.
\end{aligned} \tag{3.4}$$

We are done. □

Henceforth, partition trees described in the proof of Theorem 23 will be called Kronecker trees and will be denoted $T_{(k+1,k)}$, in Section 3.4.1 we will give a bound for trace norm of this kind of trees.

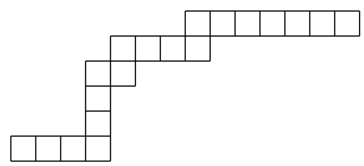
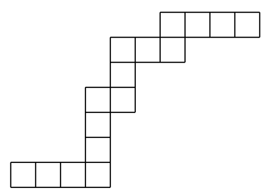
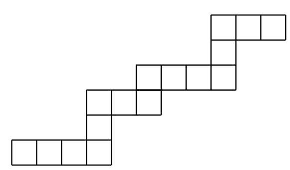
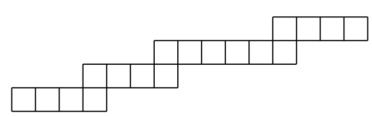
The number $h_n^i$ of helices whose starting points are entries $a_{i(n+1)} \in A$ defined in the same way as in the case for preprojective Kronecker modules are invariants for preinjective Kronecker modules. Thus we have the following result:

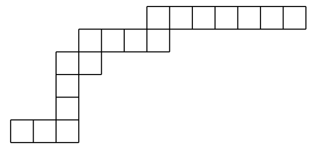
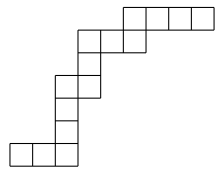
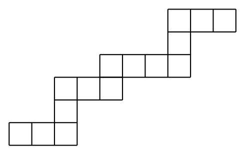
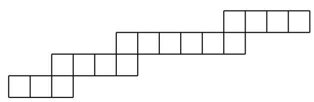
**Theorem 24.** *If $(n, n+1)$ denotes an indecomposable preinjective Kronecker module then the number of helices associated to $(n, n+1)$ is $h_n^i = n!$. In fact:*

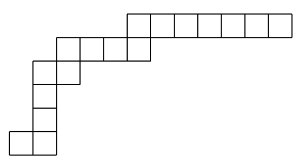
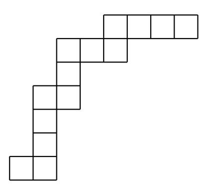
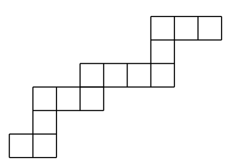
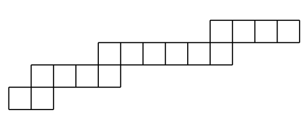
$$(n, n+1) \simeq (n', n'+1) \text{ if and only if } h_n^i = h_{n'}^i.$$

*Proof.* By definition, there is a bijection between the set of permutations of the no null columns of a representation $(n, n+1)$ and the set of all helices associated to it. Actually an helix $h = \left(a_{p_{r_1}q_{s_1}} b_{p_{r_1}q_{s_2}}, b_{p_{r_2}q_{s_2}} a_{p_{r_2}q_{s_3}}, a_{p_{r_3}q_{s_3}} b_{p_{r_3}q_{s_4}}, \dots, l_{p_{r_n}q_{s_n}} l_{p_{r_n}q_{s_{n+1}}}\right)$ defines the $n$-elements permutation $(p_{r_1}, p_{r_2}, p_{r_3}, \dots, p_{r_n})$. Moreover, it indicates the order that the helix follows to visit the rows of the matrix block. That is, the first row to be visited by the helix is $p_{r_1}$, the second row is $p_{r_2}$, and so on. $\square$

**Example 6.** *In this example, we give all the elements of the equivalence class of Kronecker snake graphs associated to the preprojective* $(4, 3)$*. In the following tables we show, the helix, the generic Kronecker snake graph and its corresponding number of perfect matchings.*

| Helices | Snake Graph | Perfect Matchings |
|---|---|---|
| $\left[\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}\right]$ |  | $\mathcal{G}_f(4,4,2,2,4,2,6) = 2243$ |
| $\left[\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}\right]$ |  | $\mathcal{G}_f(4,4,2,3,3,2,4) = 1146$ |
| $\left[\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}\right]$ |  | $\mathcal{G}_f(4,3,3,2,4,3,3) = 896$ |
| $\left[\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}\right]$ |  | $\mathcal{G}_f(4,2,4,2,6,2,4) = 2417$ |

| Helices | Snake Graph | Perfect Matchings |
|---|---|---|
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(3,4,2,2,4,2,6) = 1424$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(3,4,2,3,3,2,4) = 727$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(3,3,3,2,4,3,3) = 841$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(3,2,4,2,6,2,4) = 1551$ |

| Helices | Snake Graph | Perfect Matchings |
|---|---|---|
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(2,4,2,2,4,2,6) = 819$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(2,4,2,3,3,2,4) = 419$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(2,3,3,2,4,3,3) = 492$ |
| $\begin{array}{ccc\|ccc} 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{array}$ |  | $\mathcal{G}_f(2,2,4,2,6,2,4) = 866$ |

## 3.2 A categorification of the sequence A052558

In this section, we prove Corollary 15 which is useful to determine the number of summands in the heart of the indecomposable projective modules over some Brauer configuration algebras $\Lambda_{K^n} = kQ_{K^n}/I$. Such result and Theorem 23 allow us to give a categorification of the integer sequence A052558 in the OEIS via Kronecker modules and Brauer configuration algebras [71].

For $n \geq 3$ fixed, let $K^n = (K_0^n, K_1^n, \mu, \mathcal{O})$ be a Brauer configuration such that:

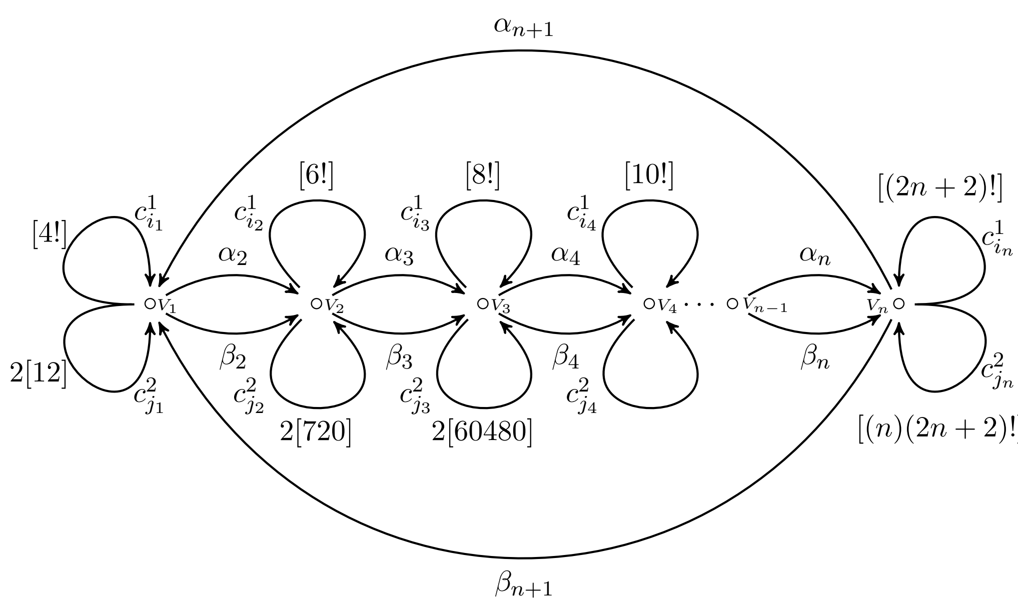
1.
$$\begin{aligned} K_0^n &= \{x_1, x_2\}, \\ K_1^n &= \{V_k = x_1^{(2k+2)!} x_2^{((k)(2k+2)!)}\}_{1 \leq k \leq n}. \end{aligned} \tag{3.5}$$

2. *The orientation $\mathcal{O}$ is defined in such a way that for $n \geq 1$*
$$\begin{aligned} &\text{At vertex } x_1;\ V_1^{(4!)} \leq V_2^{(6!)} \leq V_3^{(8!)} \leq \cdots \leq V_n^{((2n+2)!)}, \\ &\text{At vertex } x_2;\ V_1^{2(12)} \leq V_2^{2(720)} \leq V_3^{2(60480)} \leq \cdots \leq V_n^{(((n)(2n+2)!))}. \end{aligned} \tag{3.6}$$

3. *the multiplicity function $\mu$ is such that $\mu(x_1) = \mu(x_2) = 1$.*

Where the symbol $x_i^j$ in a given polygon $V_k$ means that $\mathrm{occ}(x_i, V_k) = j$.

The following is the Brauer quiver $Q_{K^n}$ associated to this configuration (numbers attached to the loops denote the occurrence of the vertex ($x_1$ above, $x_2$ below) in the corresponding polygon $V_k$, $1 \leq k \leq n$):



The ideal $I$ of relations is generated by the following relations (in this case, if there are associated $l_k^1$ ($l_k^2$) loops at the vertex $V_k$ associated to $x_1$ (associated to $x_2$) then we let $P_k^j$ denote the product of $j \leq l_k^t$ loops, $t \in \{1,2\}$), $c_{h_s}^t$ is a notation for a set of cycles $\{c^t_{h_{s,1}}, c^t_{h_{s,2}}, \dots, c^t_{h_{s,l_s^t}}, t \in \{1,2\}, h \in \{i,j\}, s \in \{1,2,\dots,n\}\}$:

1. $c^1_{i_{s,x}} c^1_{i_{s,y}} - c^1_{i_{s,y}} c^1_{i_{s,x}}$, for all possible values of $i, s, x, y$,
2. $c^2_{j_{s,x}} c^2_{j_{s,y}} - c^2_{j_{s,y}} c^2_{j_{s,x}}$, for all possible values of $i, s, x, y$,
3. $c^1_{i_{s,x}} c^2_{j_{s,y}}$ and $c^2_{j_{s,x}} c^1_{i_{s,y}}$, for all possible values of $i, s, x, y$,
4. $c^1_{i_{s,x}} \beta_{s+1}$; $c^2_{j_{s,y}} \alpha_{s+1}$; $\beta_s c^1_{i_{s,x}}$; $\alpha_s c^2_{j_{s,x}}$, for all possible values of $i, s, x, y$,
5. $(c^1_{i_{s,x}})^2$; $(c^2_{j_{s,y}})^2$, for all possible values of $i, s, x, y$,
6. $\alpha_k \alpha_{k+1}$; $\alpha_{n+1}\alpha_2$; $\beta_k \beta_{k+1}$; $\beta_{n+1}\beta_2$; $\alpha_k \beta_{k+1}$; $\beta_j \alpha_{j+1}$; $\alpha_{n+1}\beta_2$; $\beta_{n+1}\alpha_2$, for all possible values of $j, k$,
7. $\alpha_i P_i^j \gamma_{i+1}$; $\alpha_{n+1} P_1^j \gamma_2$; $\beta_k P_k^h \gamma_{k+1}$; $\beta_{n+1} P_1^h \gamma_2$; $0 < j < l_i^1$, $0 < h < l_k^2$, $1 \leq i, k \leq n$, $\gamma \in \{\alpha, \beta\}$,
8. For all the possible products (special cycles) of the form:

$$
\begin{aligned}
\varepsilon_1^1 &= \alpha_k P_k^{l_k^1} \alpha_{k+1} P_{k+1}^{l_{k+1}^1} \cdots \alpha_n P_n^{l_n^1} \alpha_{n+1} P_1^{l_1^1} \cdots \alpha_{k-1} P_{k-1}^{l_{k-1}^1}, \\
\varepsilon_1^2 &= P_{k-1}^j \alpha_k P_k^{l_k^1} \alpha_{k+1} P_{k+1}^{l_{k+1}^1} \cdots \alpha_n P_n^{l_n^1} \alpha_{n+1} P_1^{l_1^1} \cdots \alpha_{k-1} P_{k-1}^{l_{k-1}^1 - j}, \\
\varepsilon_2^3 &= \beta_k P_k^{l_k^2} \beta_{k+1} P_{k+1}^{l_{k+1}^2} \cdots \beta_n P_n^{l_n^2} \beta_{n+1} P_1^{l_1^2} \cdots \beta_{k-1} P_{k-1}^{l_{k-1}^2}, \\
\varepsilon_2^4 &= P_{k-1}^h \beta_k P_k^{l_k^2} \beta_{k+1} P_{k+1}^{l_{k+1}^2} \cdots \beta_n P_n^{l_n^2} \beta_{n+1} P_1^{l_1^2} \cdots \beta_{k-1} P_{k-1}^{l_{k-1}^2 - h},
\end{aligned}
\tag{3.7}
$$

   relations of the form $\varepsilon_i^r - \varepsilon_j^s$, $r, s \in \{1,2,3,4\}$, $i, j \in \{1,2\}$ take place. Note that, products of the form $P_{k-1}^0$ correspond to suitable orthogonal primitive idempotents $e_k$, $1 \leq k \leq n$,
9. $\varepsilon_1^1 \alpha_k$, $\varepsilon_2^3 \beta_k$.

The following result holds for indecomposable projective modules over the algebra $\Lambda_{K^n}$.

**Corollary 15.** *For $n \geq 3$ fixed and $1 \leq k \leq n$, the number of summands in the heart of the indecomposable projective representation $V_k$ over the Brauer configuration algebra $\Lambda_{K^n}$ is equals to the number of helices associated to the preprojective Kronecker module $(2k+3, 2k+2)$, $1 \leq k \leq n$.*

*Proof.* Firstly, we note that for any $k$, $\operatorname{rad}^2 V_k \neq 0$. Thus according to the Theorem 3 the number of summands in the heart of any of the indecomposable projective modules $V_k$ equals $\operatorname{occ}(x_1, V_k) + \operatorname{occ}(x_2, V_k) = (2k+2)! + k(2k+2)! = h^p_{2k+2} = h^{(2k+3,2k+2)}_{2k+2}$, which is the number of helices associated in a unique form to the indecomposable preprojective Kronecker module $(2k+3, 2k+2)$. We are done. $\square$

The following results regard the dimension of algebras of type $\Lambda_{K^n}$.

**Corollary 16.** *For $n \geq 3$ fixed, it holds that* $\frac{1}{2}(\dim_k \Lambda_{K^n}) = n + t_{\gamma_n - 1} + t_{\delta_n - 1}$, *where* $\gamma_n = \sum\limits_{k=1}^{n} k(2k+2)!$, $\delta_n = \sum\limits_{k=1}^{n} (2k+2)!$, *and $t_h$ denotes the hth triangular number.*

*Proof.* Proposition 2 allows to conclude that $\dim_k \Lambda_{K^n}/I = 2n + \sum\limits_{i=1}^{n} |C_i|(|C_i| - 1)$ where for each $i = 1, 2$, $|C_i| = val(x_i)$. The theorem holds taking into account that for any $j \geq 2$, $j(j-1) = 2t_{j-1}$. □

**Corollary 17.** *For $n \geq 3$ fixed, it holds that* $\dim_k Z(\Lambda_{K^n}) = -n + 1 + \sum\limits_{k=1}^{n} h^p_{2k+2}$.

*Proof.* Since $\mathrm{rad}^2 \Lambda_{K^n} \neq 0$, the result is a consequence of Theorem 4 with $\mu(x_1) = \mu(x_2) = 1$, $|K_0^n| = 2$, $|K_1^n| = n$ and $\mathrm{occ}(x_1, V_k) + \mathrm{occ}(x_2, V_k) = h^p_{2k+2}$. □

*Remark* 5. Similar results as in Corollaries 15-17 can be obtained for preprojective Kronecker modules of the form $(4k+2, 4k+1), k \geq 1$ by considering in the original Brauer configuration that

$$\begin{aligned} K_0^n &= \{x_1, x_2\}, \\ K_1^n &= \{V_k = x_1^{(4k+1)!} x_2^{2k(4k+1)!)}\}_{1 \leq k \leq n}. \end{aligned} \tag{3.8}$$

and keeping the relations in the quiver without changes (bearing in mind of course the new occurrences of the vertices for the different products). In particular, it holds that $\dim_k Z(\Lambda_{K^n}) = n + 1 + \sum\limits_{k=1}^{n} h^p_{4k+1}$.

## 3.3 A categorification of the integer sequence A100705

In this section, elements of the integer sequence $h_n = n^3 + (n+1)^2$ are interpreted as polygons of some Brauer configurations, such interpretation allows to categorify the number of cycles associated to some indecomposable preprojective representations of the tetrad.

Firstly, we establish an identity between the number of some invariants associated to indecomposable preprojective representations of type IV (see Figure 3.1) of the tetrad and an integer number (in the sense of (2.1)) defined by the indecomposable projective modules (polygons) over some Brauer configuration algebras. The following is the matrix presentation of such preprojective representations where $I_n$ is an $n \times n$ identity matrix. $n$ is said to be the *order* of the representation.

| | | | |
|---|---|---|---|
| $\mathrm{I}_{n+1}$ | $0$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\uparrow}$ |
| $0$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_{n+1}$ | $\mathrm{I}_n^{\downarrow}$ |

FIGURE 3.1. Representation of type IV and order $n$ of the tetrad.

As in solutions of the Kronecker problem (see Section 3.1) to each indecomposable representation of type IV it is possible to associate a finite family of directed graphs called *cycles* in such a case if

$$U_n = \begin{array}{|c|c|c|c|} \hline A & B & C & D \\ \hline A' & B' & C' & D' \\ \hline \end{array}$$

is a representation of type IV, then it is associated to $U_n$ a unique family of cycles constituted by arrows connecting the following entries as vertices:

$$\{a_{1,1}, a'_{1,1}, b'_{1,1}, b_{(n+1),1}, c_{(n+1),(n+1)}, c'_{i,(n+1)}, d'_{i,i}, d_{j,i}, c_{j,j}, c'_{h,j}, b'_{h,h}, b_{1,h}, a_{1,1}\},$$

$h$, $i$ and $j$ are fixed integers, $1 \leq i \leq n$, $1 \leq j \leq n$, $h \in \{2, 3, 4, \ldots, n+1\}$. In this case, no cycle has entries of the form $d_{(n+1),s}$, $1 \leq s \leq n$ as vertices.

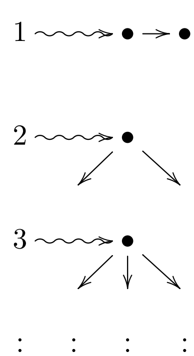
The following is an example of a cycle associated to a preprojective representation of the tetrad for $n = 4$.

$$\begin{array}{|ccccc|ccccc|ccccc|cccc|} \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Secondly, we note that the Brauer configuration (3.11) allows to see each polygon $V_n$ as a partition of the number $h_n$ into two parts of the form $\{n, n+1\}$ where $n$ occurs $(n)^2$ times and $n+1$ occurs $n+1$ times. Assuming the classical notation for partitions [3] each number $h_n$ can be expressed as follows:

$$h_n = (n)^{(n^2)}(n+1)^{(n+1)}, \quad n \geq 1. \tag{3.9}$$

we let $P_n$ denote such a partition. The partition tree $T_{P_n}$ associated to each partition of the form $P_n$ is obtained by assuming the notation:




(3.10)

In this case, $T_{P_n}$ has a root node with $n+1$ children, $n$ of them have $n$ children and the last one has $n+1$ children in such a way that in the last level of $T_{P_n}$, $n$ of these children represent a partition of the form $(n)^{(n-1)}(n+1)^{(1)}$ and the last one represents a partition of the form $(n)^{(n)}(n+1)^{(1)}$. Partition trees of the form $T_{P_n}$ are used in the proof of theorem 25.

Now we consider Brauer configuration algebras of the form $\Lambda_{\Gamma_n} = kQ_{\Gamma_n}/J$ induced by the Brauer configuration $\Gamma_n$ such that For $n \geq 2$ fixed, $\Gamma_n = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ with

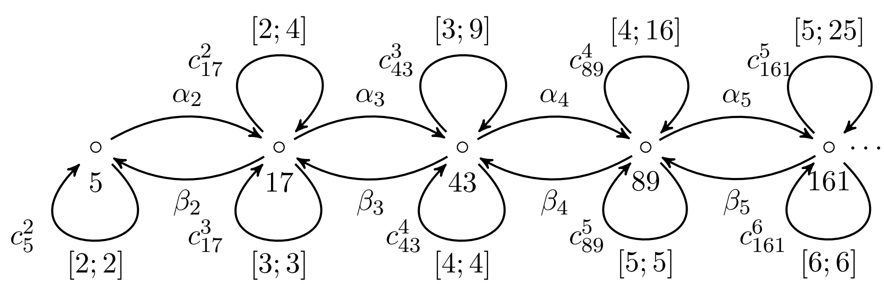
1.
$$\begin{aligned}
&\Gamma_0 = \{1, 2, 3 \ldots, n, n+1\},\\
&\Gamma_1 = \{V_k = k^{(k^2)}(k+1)^{(k+1)}\}_{1\leq k\leq n},\ i.e., \mathrm{occ}(k, V_k) = k^2, \mathrm{occ}(k+1, V_k) = k+1.
\end{aligned} \tag{3.11}$$

2. *The orientation $\mathcal{O}$ is defined in such a way that*

   for $2 \leq i \leq n$ *at vertex* $i$, $V_{i-1}^{(i,<)} < V_i^{(i^2,<)}$, where $V_x^{(y,<)}$ means that the polygon $V_x$ occurs $y$ times in the successor sequence of the corresponding vertex, in particular, $V_{i-1} < V_i$.

3. The multiplicity function $\mu$ is such that $\mu(j) = 1$, for any $j \in \Gamma_0$.

The following is the quiver $Q_{\Gamma_n}$ associated to the Brauer configuration $\Gamma_n$, worth noting that there is no arrow connecting vertex 1 with any other vertex provided that it is truncated (see Theorem 3, item 5), besides we use the symbol $[x_j; y_j]$ to denote that the vertex $x_j$ occurs $y_j$ times at the polygon $h_j = j^3 + (j+1)^2$ (see identity (2.1)). And $c_j^i$ is a set of loops $\{c_{j_y}^i \mid 1 \leq y \leq \mathrm{occ}(x_j, h_j) - 1, 2 \leq i \leq n+1\}$. For instance, at 17 there are associated the loops, $c_{17_1}^2, c_{17_2}^2, c_{17_3}^2$ and $c_{17_1}^3, c_{17_2}^3$.




The following are examples of polygons in a Brauer configuration $\Gamma_n$.

$$\begin{aligned}
5 &= (1) + (2+2) = (1)^{(1)}(2)^{(2)},\\
17 &= (2+3) + (2+3) + (2+2+3) = (2)^{(4)}(3)^{(3)},\\
43 &= (3+3+4) + (3+3+4) + (3+3+4) + (3+3+3+4) = (3)^{(9)}4^{(4)},\\
89 &= (4+4+4+5) + (4+4+4+5) + (4+4+4+5) + (4+4+4+5) + (4+4+4+4+5)\\
\vdots &= \vdots
\end{aligned} \tag{3.12}$$

The ideal $J$ is generated by the following relations where for a fixed $2 \leq l \leq n+1$, $P^{i,l}_{h_j}$ is the product of $i$ loops of type $l$ ($1 \leq i \leq \operatorname{occ}(l, h_j) - 1$) attached to the polygon $h_j$ with $y_j - 1$ being the total number of such loops ($y_j \in \{j^2, j\}$):

1. $c^u_{j_x} c^v_{j_y}$, if $u \neq v$, for all the possible values of $u$, $v$, $x$, $y$ and $j$,

2. $c^t_{j_x} c^t_{j_y} = c^t_{j_y} c^t_{j_x}$, for all the possible values of $x$, $y$, $t$, and $j$,

3. $(c^t_{j_x})^2$ for all the possible values of $j$, $t$ and $x$,

4. $c^h_{j_x}\alpha_{h+1}$; $\alpha_h c^{h+1}_{(j+1)_x}$ $c^h_{j_x}\beta_{h-1}$; $\beta_h c^{h-1}_{(j-1)_x}$, $\alpha_j\beta_j$ for all the possible values of $h, j$ and $x$,

5. $\alpha_i\alpha_{i+1}$; $\beta_{j+1}\beta_j$, $2 \leq i \leq n-1$, $2 \leq j \leq n-1$,

6. If

$$\begin{aligned}
\varepsilon^1_j &= P^{u,j}_{h_j}\alpha_j P^{y_{j+1}-1,j}_{h_{j+1}}\beta_j P^{y_j-(1+u),j}_{h_j},\\
\varepsilon^2_j &= \alpha_j P^{y_{j+1}-1,j}_{h_{j+1}}\beta_j P^{y_j-1,j}_{h_j},\\
\varepsilon^3_j &= P^{u,j}_{h_{j+1}}\beta_j P^{y_j-1,j}_{h_j}\alpha_j P^{y_{j+1}-(1+u),j}_{h_{j+1}},\\
\varepsilon^4_j &= \beta_j P^{y_j-1,j}_{h_j}\alpha_j P^{y_{j+1}-1,j}_{h_{j+1}},\\
\varepsilon^5_{j+1} &= P^{v,j+1}_{h_{j+1}}\alpha_{j+1} P^{y_{j+2}-1,j+1}_{h_{j+2}}\beta_{j+1} P^{y_{j+1}-(1+v),j+1}_{h_{j+1}},\\
\varepsilon^6_{j+1} &= \alpha_{j+1} P^{y_{j+2}-1,j+1}_{h_{j+2}}\beta_{j+1} P^{y_{j+1}-1,j+1}_{h_{j+1}},\\
\varepsilon^7_{j+1} &= P^{v,j+1}_{h_{j+2}}\beta_{j+1} P^{y_{j+1}-1,j+1}_{h_{j+1}}\alpha_{j+1} P^{y_{j+2}-(1+v),j+1}_{h_{j+2}},\\
\varepsilon^8_{j+1} &= \beta_{j+1} P^{y_{j+1}-1,j+1}_{h_{j+1}}\alpha_{j+1} P^{y_{j+2}-1,j+1}_{h_{j+2}},
\end{aligned} \tag{3.13}$$

   then there are relations of the form $\varepsilon^r_s - \varepsilon^{r'}_{s'}$ where $r, r' \in \{1, \dots, 8\}$, $r \neq r'$ and $s, s' \in \{j, j+1\}$, for all the possible values of $u$, $v$ and $j$,

7. $\varepsilon^2_j\alpha_j$, $\varepsilon^4_j\beta_j$, $\varepsilon^6_{j+1}\alpha_{j+1}$, $\varepsilon^8_j\beta_{j+1}$.

The following result regards the Brauer configuration algebra $\Lambda_{\Gamma_n}$. Recall that the notation $n_V$ (see (2.1)) is adopted for the integer number associated to the polygon $V$, in this case $V$ is interpreted as an integer partition of $n_V$.

**Theorem 25.** *For $n \geq 2$ fixed and $2 \leq i \leq n$ the number $n_{V_i} = i^3 + (i+1)^2$ associated to the polygon $V_i = h_i \in \Gamma_1$ (see Figure (3.1) and formulas (3.11)) is the number of cycles associated to the indecomposable preprojective representation of type* IV *and order $i+1$. And such identity defines a bijection between indecomposable projective modules over the Brauer configuration algebra $\Lambda_{\Gamma_n}$ and preprojective representations of type* IV *of the tetrad.*

*Proof.* According to Theorem 3, in order to prove that there exists the required bijection, it suffices to find out the number of cycles associated to a given preprojective representation of type IV. To do that, we fix a representation $U_n$ of this type of order $n \geq 2$, and denote its different blocks as follows:

$$U_n = \begin{array}{|c|c|c|c|}\hline A & B & C & D \\ \hline A' & B' & C' & D' \\ \hline \end{array}.$$

We note that all the cycles associated to $U_n$ can be seen as trees $T_{c_{(n+1),(n+1)}}$, which have the entry $c_{(n+1),(n+1)}$ as root node with $n$ branches whose successors are given by entries

$$c'_{1,(n+1)}, c'_{2,(n+1)}, \dots, c'_{n,(n+1)}.$$

Each entry $c'_{i,(n+1)}$ has $n-1$ branches if $i \neq n$, whereas $c'_{n,(n+1)}$ has $n$ branches. Besides, all of these entries give rise to an arrow

$$c'_{i,(n+1)} \to d_{j,i},$$

for some entry $d_{j,i} \in D$. Actually, $d_{j,i}$ is a successor root of $c'_{i,(n+1)}$ with $(n-1)$ branches in the tree whenever $j \in \{1, \dots, n\}$ and $i \neq 1$. If $i = 1$ then $d_{1,j}$ has by construction $n$ branches in $C'$. Therefore, the structure of $T_{c_{(n+1),(n+1)}}$ has the following shape:

$$\begin{tikzcd}[column sep=small]
& & & & & c_{(n+1),(n+1)} \arrow[dlll] \arrow[d] \arrow[drrr] & & & & & \\
& & c'_{1,(n+1)} \arrow[dll] \arrow[d] \arrow[drr] & & & \cdots c'_{i,(n+1)} \cdots & & & c'_{n,(n+1)} \arrow[dll] \arrow[d] \arrow[drr] & & \\
d_{n,1} & \cdots & d_{i,1} & \cdots & d_{1,1} & & d_{n,n} & \cdots & d_{i,n} & \cdots & d_{1,n}
\end{tikzcd} \tag{3.14}$$

Which corresponds to the partition tree $T_{P_{(n-1)}}$ of $h_{(n-1)} = (n-1)^3 + (n)^2$, thus the correspondence $T_{P_{(n-1)}} \to T_{c_{(n+1),(n+1)}}$ is a bijection between indecomposable preprojective representations of type IV of the tetrad and polygons of the Brauer configuration (3.11). $\square$

As an example the following is the diagram of $T_{c_{4,4}}$ such that the number of vertices in the last level gives the number of associated cycles (described in the proof of Theorem 3) to the indecomposable representation of the tetrad $U_3$:

$$\begin{tikzcd}[column sep=tiny]
& & & & & & & & & & c_{44} \arrow[dlllllll] \arrow[d] \arrow[drrrrrrrr] \\
& & & c'_{14} \arrow[dll] \arrow[dr] & & & & & & & c'_{24} \arrow[dll] \arrow[dr] & & & & & & & & c'_{34} \arrow[dlll] \arrow[d] \arrow[drrr] \\
& d_{31} \arrow[dl] \arrow[dr] & & & d_{11} \arrow[dl] \arrow[d] \arrow[dr] & & & & d_{22} \arrow[dl] \arrow[dr] & & & d_{12} \arrow[dl] \arrow[d] \arrow[dr] & & & & d_{33} \arrow[dl] \arrow[dr] & & & d_{23} \arrow[dl] \arrow[dr] & & & d_{13} \arrow[dl] \arrow[d] \arrow[dr] \\
\bullet & & \bullet & \bullet & \bullet & \bullet & & \bullet & & \bullet & \bullet & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet
\end{tikzcd}$$

The number of cycles associated to the indecomposable preprojective representation of the tetrad $U_3$ equals the second term of the integer sequence A100705. Actually, the number of cycles associated to $U_n$ is given by $h_{(n-1)} = (n-1)^3 + (n)^2$, $n \geq 2$, which corresponds to the $(n-1)$th term of this sequence. Black arrows denote the common part of all these cycles.

$$U_3 = \left|\begin{array}{cccc|cccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & & 0\,1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right|$$

The following results are consequences of Theorems 3, 4 and Proposition 2.

**Corollary 18.** *For $n \geq 2$ fixed and $2 \leq i \leq n$, the number of summands in the heart of the indecomposable projective representation $V_i$ over the algebra $\Lambda_{\Gamma_n}$ is $i^2 + i + 1$.*

*Proof.* Since for any indecomposable projective representation $V_i$, it holds that $\mathrm{rad}^2\, V_i \neq 0$ then the theorem follows from Theorem 3 and the definition of the polygon $V_i$, which has $i^2 + i + 1$ non-truncated vertices counting repetitions. $\square$

**Corollary 19.** *For $n \geq 2$ fixed, $\dim_k \Lambda_{\Gamma_n} = \sum\limits_{k=2}^{n} (k(k+1))^2 - \frac{1}{3}(n-3)(n+1)(n+2)$. And $\dim_k \Lambda_{\Gamma_{n+1}} - \dim_k \Lambda_{\Gamma_n} = 2(1 - t_n) + [(n+1)(n+2)]^2$, where for $j \geq 1$, $t_j$ denotes the $j$th triangular number.*

*Proof.* It is enough to observe that for $n \geq 2$ and $2 \leq j < n+1$, it holds that $val(j) = j^2 + j$, whereas $val(n+1) = n+1$. The corollary holds as a consequence of Proposition 2. $\square$

**Corollary 20.** *For $n \geq 2$ fixed, it holds that $\dim_k Z(\Lambda_{\Gamma_n}) = \frac{n(n+1)(n+2)}{3} + 2$.*

*Proof.* Since $\mathrm{rad}^2\, \Lambda_{\Gamma_n} \neq 0$, the result is a consequence of Theorem 4 with $\mu(i) = 1$, for any $2 \leq i \leq n+1$, $|\Gamma_0| = n$, $|\Gamma_1| = n$, $\mathrm{occ}(i, h_i) + \mathrm{occ}(i+1, h_i) = i^2 + i + 1$, $2 \leq i \leq n$, and $\mathrm{occ}(2, h_1) = 2$. $\square$

*Remark* 6. Note that Corollaries 18-20 are categorifications of the integer sequences $n^2 + n + 1$ (encoded in the OEIS as A002061), $\sum\limits_{k=2}^{n} (k(k+1))^2 - \frac{1}{3}(n-3)(n+1)(n+2)$, and $\frac{n(n+1)(n+2)}{3} + 1$ (which is the sequence A064999). Elements of the sequence A064999 appear as coefficients (in the case $k = 3$) of the generating polynomial of a $n$-twist knot with the form $P_n(x) = \sum\limits_{k \geq 0} a_{n,k} x^k$.

Sequence $\sum\limits_{k=2}^{n} (k(k+1))^2 = \sum\limits_{1 \leq i < j \leq n} (j-i)^3$ is encoded A024166 in the OEIS. Worth noting that sequence $a(n) = \frac{1}{6}(n-3)(n+1)(n+2)$ , $n \geq 4$ has the code A005586, which has an interesting relation with Dyck paths. Regarding Catalan objects, we observe that the sequence A005586 counts the number of walks on the square lattice and the number of left factors of Dyck paths from $(0,0)$ to $(n+5, n-1)$, e.g., $a(1) = 5$, $UDUDUD$, $UDUUDD$, $UUDDUD$, $UUDUDD$ and $UUUDDD$, where $U = (1,1)$ and $D = (1,-1)$.

*Remark* 7. The integer sequence A100705=$\{1, 5, 17, 43, 89, \ldots, n^3 + (n+1)^2\}$ is very interesting for itself, for instance it can be used to build a family of directed trees with an explicit value of energy (see Section 3.4). Sequence A100705 can be also used to define directed graphs whose vertices are either happy or excited in the sense of the Bert Konstant's game. Such a game can be defined as follows [15].

Let $G = (G_0, G_1)$ be a simple graph and set $G_0 = \{1, 2, \ldots, n\}$. For $i \in G_0$, let $N(i)$ denote the set of neighbors of $i$.

Suppose now that chips will be distributed among vertices of $G$ in such a way that for $i \in G_0$ we have $c_i \geq 0$ chips, the vector $(c_i \mid 1 \leq i \leq n)$ is said to be a *configuration*, we say that a vertex $i$ is:

1. *Happy*, if $c_i = \frac{1}{2} \sum\limits_{j \in N(i)} c_j$,
2. *Excited*, if $c_i > \frac{1}{2} \sum\limits_{j \in N(i)} c_j$,
3. *Unhappy*, if $c_i < \frac{1}{2} \sum\limits_{j \in N(i)} c_j$.

**Goal of the game:** Make every one happy or excited

A well known result regarding the game of Bert Konstant establishes that *it is finite if and only $G$ is a Dynkin diagram $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.*

### 3.3.1 On the game of Bert Konstant

The game of Bert Konstant finishes for a graph $G$ if and only if $G$ is of finite representation type as we recall before. Let us describe how the game is developed.

Initially no chips are present (i.e. $c_i = 0$, for all $i$, and all vertices are happy).

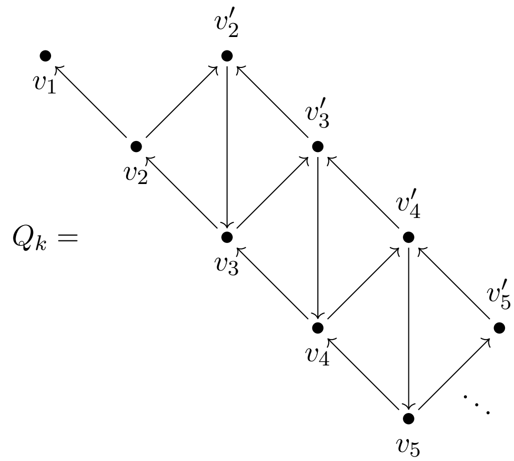
Then we place at vertex $v_{i_0} = 1$, so $i_0$ is excited but neighbors of $i_0$ are unhappy.

Subsequently, do the following reflection.

Pick any unhappy vertex $i$ and replace $c_i \longrightarrow -c_i + \sum\limits_{j \in N(i)} c_j$.

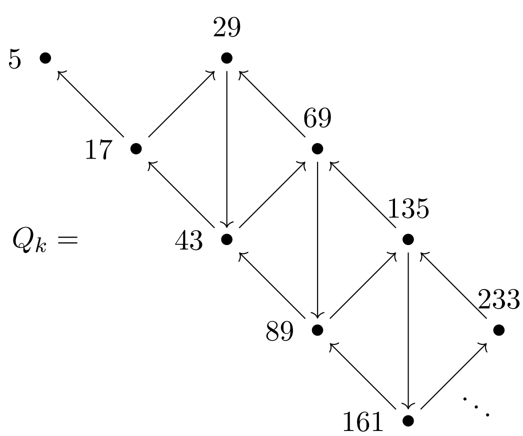
Now, we define an oriented version of the game described above, to do that, we say that if an arrow $\alpha : v_1 \to v_2 \in Q_1$ where $Q_1$ is the set of arrows of a given quiver $Q$ then $v_2$ is a neighbor of $v_1$, whereas $v_1$ is not a neighbor of $v_2$ (in other words, we are interested in the out-degree $deg^+(v)$ of vertices $v \in Q_0$).

Actually, the following result holds for the following quiver $Q_k$:




**Theorem 26.** *For the quiver $Q_k$ there exist integer sequences $c_i(c_i')$ associated to each vertex $v_i(v_i') \in (Q(\Gamma))_0$ and an integer sequence $f(i)$ such that the transformation $c_i \longrightarrow c_i - f(i) = c_i'$ makes vertices $v_i$ happy and vertices $v_i'$ excited for any $i \geq 2$.*

*Proof.* It suffices to define $c_i$ as $c_i = i^3 + (i+1)^2$, $i \geq 1$ the theorem follows if $f(j)$ is defined in such a way that for $j \geq 3$ it holds that, $f(j) = 2(p_j^5 - p_{j-1}^5)$ and $c_h' = c_{h+1} - f(h+1)$, for $h \geq 2$, where $p_j^5$ denotes the $j$st pentagonal number, in this way $Q_k$ has the following shape:




$\square$

## 3.4 Energy of preprojective partition trees

The arguments in the proof of Theorems 23 and 25 includes the introduction of some suitable partition trees associated to a preprojective Kronecker module and some indecomposable preprojective representations of type IV of the tetrad, respectively. In section 3.4.1 we recall briefly the definition of the energy of a finite graph, we also use the concept of the message of a Brauer configuration algebra introduced in section 2.1 in order to compute the energy (in the sense of I. Gutman) of some suitable trees, which appear in the context of matrix problems.

### 3.4.1 Energy of a graph

The definition of the energy of a finite graph $G = G(V, E)$ as a topological index was introduced by I. Gutman in 1978 [33]. According to him the energy of a graph $\mathcal{E}(G)$ of the graph $G$ is given by the identity:

$$\mathcal{E}(G) = \sum_{\lambda \in E(M(G))} |\lambda|, \tag{3.15}$$

where $\lambda$ denotes an eigenvalue of the adjacency matrix $M(G)$ of the graph $G$ and $E(M)$ is the corresponding characteristic space [33].

The energy can be defined for quivers by using the *trace norm* $M(G)$ of the adjacency matrix of $G$, also known as the Schatten 1-norm, Ky Fan norm or nuclear form defined as follows:

$$||M(G)||_* = \sum_{\sigma \in E(M(G)M^t(G))} |\sigma|, \tag{3.16}$$

where $\sigma$ denotes a singular value of $M(G)$.

*One of the main problems in the theory of topological indexes consists of studying extremal values of the energy of significant classes of graphs.*

For instance, I. Gutman proved that for an arbitrary tree $T_n$ with $n$ vertices, it holds that

$$\mathcal{E}(S_n) \leq \mathcal{E}(T_n) \leq \mathcal{E}(A_n),$$

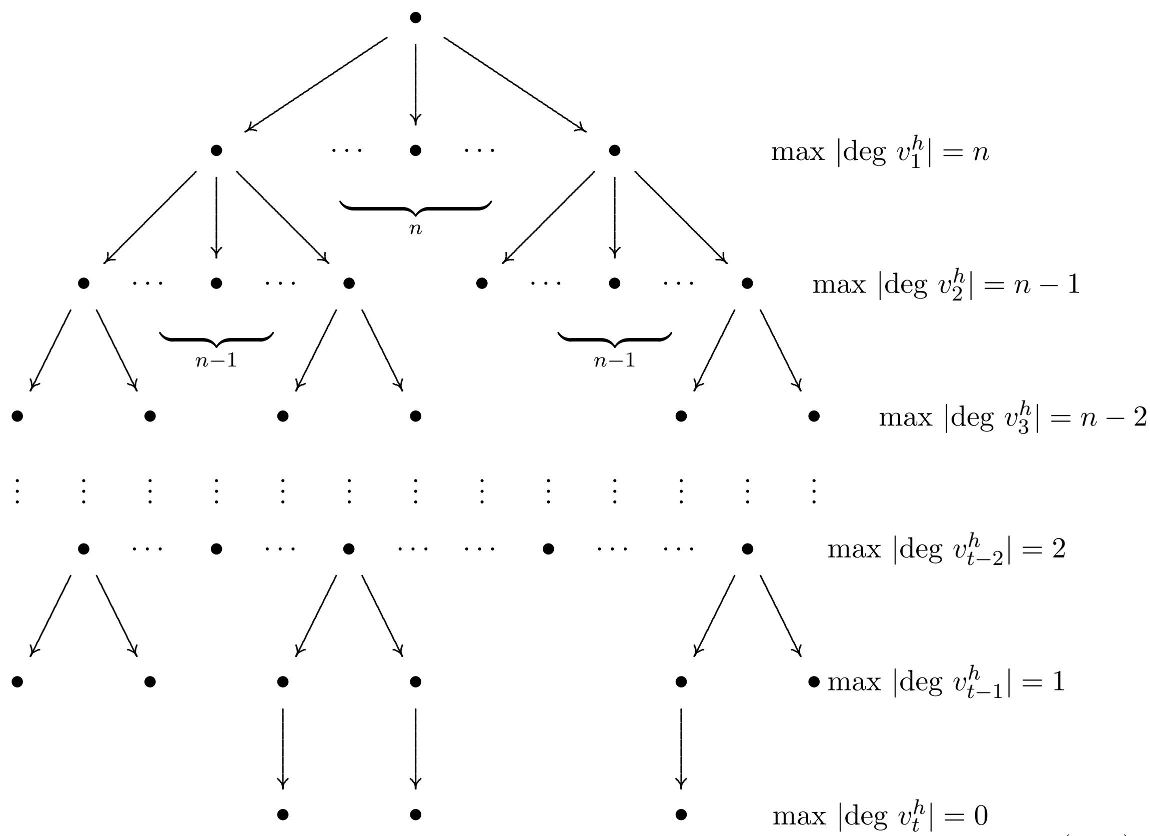
where $S_n$ is the corresponding star with $n+1$ vertices and $A_n$ is a Dynkin diagram. N. Agudelo et al. in [1] have investigated extremal values of the energy of the family $\Omega(n, i)$ consisting of trees with $n$ vertices and $i$ ramifications [1].

The arguments in the proof of Theorem 23 includes the introduction of some suitable partition trees associated to a preprojective Kronecker module. In this work, we compute the energy (in the sense of I. Gutman [33]) of some suitable trees that appear in the context of some matrix problems.

In fact, there are many examples of matrix problems whose preprojective solutions have associated both helices and partition trees (e.g., the four subspace problem among others).

Henceforth we called preprojective (preinjective) partition tree of a given tree associated to a preprojective solution of a matrix problem.

Often, preprojective (preinjective) trees are defined by a set of suitable helices, cycles or directed graphs (digraphs). Thus, they can be described as follows: A preprojective (preinjective) tree is a rooted tree with a fixed number of children (say $n$-children). Each of these children has at most $(n-1)$-children and so on until reaching a generation whose vertices have at most two children. The following diagram shows an example of this construction:




(3.17)

**Theorem 27.** *If $\|M(T)\|_*$ is the trace norm of a preprojective partition tree $T$ then for $n \geq 6$:*

$$\frac{(n-1)(n)(4n+1)}{6} + d\left\lfloor \sqrt{n} \right\rfloor \leq \|M(T)\|_* \leq \frac{2n_2(n-2)^{3/2} + 6\sqrt{n}}{3}$$

*where $n_2$ is the number of bifurcations of $T$ and $d = n - \lfloor \sqrt{n} \rfloor$.*

*Proof.* By construction, we note that there is a bijection $f : B \to S$ where $B$ is the set of ramifications of $T$ of the form $b_n = \begin{array}{c}\bullet\\ \swarrow\ \downarrow\ \searrow\\ \underbrace{\bullet\ \ \bullet\ \ \bullet}_{n-\text{ vertices}}\end{array}$ and the set of singular values of $M(T)$ given by the rule $f(b_n) = \sqrt{n}$. Actually, the characteristic polynomial has the form $P(\lambda) = \lambda^{n_0}(\lambda - 1)^{n_1}(\lambda - \sqrt{2})^{n_2} \cdots (\lambda - \sqrt{n})^{n_k}$, with $n_k = 1$. We note that the most frequent singular values is $n_2$ therefore

$$\begin{aligned}
\sum_{j=0}^{n} \sqrt{j} = \|M(T)\|_* &\leq n_2 \left( \sum_{i=1}^{n-2} \sqrt{i} \right) + 2\sqrt{n} \\
&\leq n_2 \frac{2(n-2)^{3/2}}{3} + 2\sqrt{n} \\
&= \frac{2n_2(n-2)^{3/2}}{3} + 2\sqrt{n} \\
&= \frac{2n_2(n-2)^{3/2} + 6\sqrt{n}}{3}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|M(T)\|_* &\geq \sum_{j=0}^{n} \sqrt{j} \\
&= \sum_{i=1}^{n} i(2i+1) + d \left\lfloor \sqrt{n} \right\rfloor \\
&= \frac{(n-1)(n)(4n+1)}{6} + d \left\lfloor \sqrt{n} \right\rfloor .
\end{aligned}$$

And with this argument we are done. □

The following results give the explicit value of the energy of some oriented partition trees with $((n-1)^3 + (n)^2) + (n^2 + 2)$ vertices and $n^2 + 2$ ramifications.

**Corollary 21.** *For each $n \geq 3$, the trace norm $||M(T_{P_{(n-1)}})||_*$ of a partition tree $T_{P_{(n-1)}}$ is given by the formula $||M(T_{P_{(n-1)}})||_* = (n+2)\sqrt{n} + n(n-1)\sqrt{n-1}$.*

*Proof.* For each $n \geq 3$, $T_{P_{(n-1)}} \in \Omega(h_n + i_n, i_n)$ with $i_n = n^2 + 2$. Thus, for each $n$, $T_{P_{(n-1)}}$ has the shape:

$$
\begin{tikzcd}
 & & & (n) \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & (n-1) \arrow[dl] \arrow[d] \arrow[dr] & & \cdots (n-1) \cdots & & (n) \arrow[dl] \arrow[d] \arrow[dr] & \\
 (n-1)\cdots & (n-1)\cdots & (n) & & (n-1)\cdots & (n-1)\cdots & (n)
\end{tikzcd}
\tag{3.18}
$$

where $(x)$ denotes that the corresponding vertex has $x$ children. Therefore the associated characteristic polynomial $P_n(\lambda)$ of the oriented tree has the form $P_n(\lambda) = (\lambda - n)^{n+2}(\lambda - (n-1))^{n(n-1)}\lambda^{n^3+(n+1)^2}$. And the corresponding singular values have the form $\sigma_1 = 0$, $\sigma_2 = \sqrt{n}$ and $\sigma_3 = \sqrt{n-1}$ up to multiplicities. $\square$

There is another way to compute the energy of a partition tree $T_{P_{(n-1)}}$ as the message of a Brauer configuration algebra (see 2.2). In this case, words concatenation arising from the configuration is specialized by the usual sum of real numbers.

Consider the Brauer configuration algebras $\Gamma_{\Delta_k} = kQ_{\Delta_k}/L$ induced by the Brauer configuration $\Delta_k$ such that for $n \geq 3$ fixed, $\Delta_k = (\Delta_0, \Delta_1, \mu, \mathcal{O})$ with

1.
$$
\begin{aligned}
&\Delta_0 = \{x_0, x_{k-1}, x_k\},\\
&\Delta_1 = \{P_1 = \{x_0, x_k\}, P_2 = \{x_{k-1}^{(k-1)}, x_k\}, P_3 = \{x_{k-1}^{(k^2-2k+1)}, x_k^{(k)}\}\},
\end{aligned}
\tag{3.19}
$$

   Where the symbol $x_i^{(j)}$ in a given polygon $P_k$ means that $\mathrm{occ}(x_i, P_k) = j$

2. *The orientation $\mathcal{O}$ is defined in such a way that for $n \geq 3$*

$$
\begin{aligned}
\text{At vertex } x_0;\ & P_0,\\
\text{At vertex } x_{k-1};\ & P_2^{(k-1)} \leq P_3^{(k^2-2k+1)},\\
\text{At vertex } x_k;\ & P_1 \leq P_2 \leq P_3^{(k)}.
\end{aligned}
\tag{3.20}
$$

3. *The multiplicity function $\mu$ is such that $\mu(x_0) = 2$ and $\mu(x_{k-1}) = \mu(x_k) = 1$.*

The following is the Brauer quiver $Q_{\Delta_k}$ associated to this configuration, we use the symbol $[x_j; y_j]$ to denote that the vertex $x_j$ occurs $y_j$ times at the corresponding polygon $P_n$ with $1 \leq n \leq 3$. We also consider the notation $a_l^{x_{k-1}}$ with $1 \leq l \leq k-2$, $a_m^{x_{k-1}}$ with $k \leq m \leq k^2 - 2k - 1$ and $a_n^{x_k}$ with $3 \leq n \leq k+1$ to indicate the loops at $P_2$ and $P_3$.

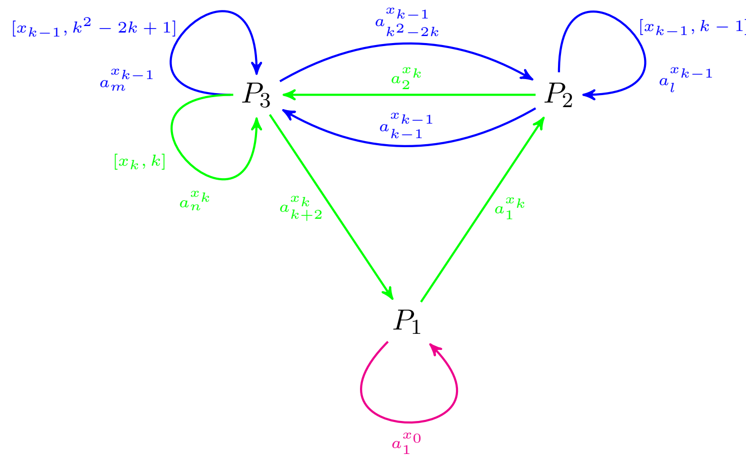



FIGURE 3.2. Quiver associated to the Brauer configuration $\Delta_k = (\Delta_0, \Delta_1, \mu, \mathcal{O})$.

The ideal $L$ of the corresponding Brauer configuration algebra $\Lambda_{\Delta_k}$ is generated by the following relations (see Figure 3.2), for which it is assumed the following notation for the special cycles:

$$
\begin{aligned}
C^{P_1,1}_{x_0} &= a^{x_0}_1,\\
C^{P_2,h}_{x_{k-1}} &= \begin{cases} a^{x_{n-1}}_1 \cdots a^{x_{k-1}}_{k^2-k} & \text{if } h=1 \\ a^{x_{k-1}}_h \cdots a^{x_{k-1}}_{h-1} & \text{otherwise} \end{cases}, \text{with } 1 \le h \le k-1,\\
C^{P_3,i}_{x_{k-1}} &= a^{x_{k-1}}_{i+(k-1)} a^{x_{k-1}}_{i+k} \cdots a^{x_{k-1}}_{i+(k-2)} \text{ where } 1 \le i \le k^2-2k+1,\\
C^{P_1,1}_{x_k} &= a^{x_k}_1 a^{x_k}_2 \cdots a^{x_k}_{k+2},\\
C^{P_2,1}_{x_k} &= a^{x_k}_2 a^{x_k}_3 \cdots a^{x_k}_{k+2} a^{x_k}_1,\\
C^{P_3,j}_{x_k} &= a^{x_k}_{j+2} a^{x_k}_{j+3} \cdots a^{x_k}_{j+1} \text{ where } 1 \le j \le k.
\end{aligned}
\tag{3.21}
$$

1. $\left(C^{P_1,1}_{x_0}\right)^2 - \left(C^{P_1,1}_{x_k}\right)$, $C^{P_2,1}_{x_k} - C^{P_2,h}_{x_{k-1}}$, $C^{P_3,i}_{x_{k-1}} - C^{P_3,j}_{x_k}$, for all possible values of $h, i$ and $j$.
2. $\left(C^{P_1,1}_{x_0}\right)^2 a$ $(C^{P_1,1}_{x_n} a')$, with $a$ $(a')$ being the first arrow of $\left(C^{P_1,1}_{x_0}\right)^2$ $(C^{P_1,1}_{x_n})$,
3. $C^{P_2,h}_{x_{k-1}} a$ $(C^{P_2,1}_{x_k} a')$, with $a$ $(a')$ being the first arrow of $C^{P_2,h}_{x_{k-1}}$ $(C^{P_2,1}_{x_k})$ for all $h$,
4. $C^{P_3,i}_{x_{k-1}} a$ $(C^{P_3,j}_{x_k} a')$, with $a$ $(a')$ being the first arrow of $C^{P_3,i}_{x_{k-1}}$ $(C^{P_3,j}_{x_k})$ for all $i, j$,
5. $a^{x_0}_1 a^{x_k}_1$, $a^{x_{k-1}}_l a^{x_k}_2$, with $1 \le l \le k-2$,
6. $a^{x_{k-1}}_l a^{x_{k-1}}_{k-1}$, with $1 \le l \le k-3$, $a^{x_{k-1}}_m a^{x_k}_{k+2}$, with $k \le m \le k^2-k$,
7. $a^{x_{k-1}}_m a^{x_k}_n$, for all possible values of $m$ and $n$,
8. $a^{x_{k-1}}_m a^{x_{k-1}}_{k^2-2k}$, with $k \le m \le k^2-k-1$, $a^{x_{k-1}}_{k-1} a^{x_{k-1}}_{k^2-2k}$,

9. $a_{k-1}^{x_{k-1}} a_m^{x_{k-1}}$, $k+1 \leq m \leq k^2-k$, $a_{k-1}^{x_{k-1}} a_n^{x_k}$, with $3 \leq n \leq k+1$,
10. $a_{k-1}^{x_{k-1}} a_{k+2}^{x_k}$, $a_1^{x_k} a_{k-1}^{x_{k-1}}$, $a_1^{x_k} a_m^{x_{k-1}}$, with $1 \leq m \leq k-2$,
11. $a_2^{x_k} a_{k^2-2k}^{x_k}$, $a_2^{x_k} a_m^{x_{k-1}}$, with $k \leq m \leq k^2-2k-1$, $a_2^{x_k} a_{k+2}^{x_k}$,
12. $a_2^{x_k} a_n^{x_k}$ with $4 \leq n \leq k+2$, $a_n^{x_k} a_m^{x_{k-1}}$, for all possible values of $m$ and $n$,
13. $a_n^{x_k} a_{k^2-2k}^{x_{k-1}}$, $a_n^{x_k} a_{k+2}^{x_k}$, with $3 \leq n \leq k$,
14. $a_{k+2}^{x_k} a_1^{x_0}$.

The following result describes the way that the message associated to the Brauer configuration algebra $\Lambda_{\Delta_k}$ gives the value of the energy of a partition tree.

**Theorem 28.** *For $k$ fixed, the energy of a partition tree $T_{P_{(k-1)}}$ see (3.18) is given by $M(\Delta_k^e)$.*

*Proof.* It is enough to observe that for $k \geq 3$, it holds that $val(x_{k-1}) = k(k-1)$, whereas $val(x_k) = k+2$. The theorem holds as a consequence of the specialization and Corollary 21. $\square$

As an example consider the following Brauer configuration algebra and the specialization described in Theorem 28.

$$
\begin{aligned}
\Delta_3 &= (\Delta_0, \Delta_1, \mu, \mathcal{O}) \\
\Delta_0 &= \{x_0, x_2, x_3\}. \\
\Delta_1 &= \{P_1 = \{x_0, x_3\}; \quad P_2\{x_2^{(2)}, x_3\}; \quad P_3\{x_2^{(4)}, x_3^3\}\} \\
\mu(x_0) &= 2, \quad \mu(x_2) = \mu(x_3) = 1
\end{aligned}
\tag{3.22}
$$

The orientation $\mathcal{O}$ is defined in such a way that:

$$
\begin{aligned}
&\text{At vertex } x_0;\, P_0, \\
&\text{At vertex } x_2;\, P_2^2 \leq P_3^{(4)}, \\
&\text{At vertex } x_k;\, P_1 \leq P_2 \leq P_3^{(3)}.
\end{aligned}
\tag{3.23}
$$

The following is the Brauer quiver $Q_{\Delta_k}$ associates to this configuration.

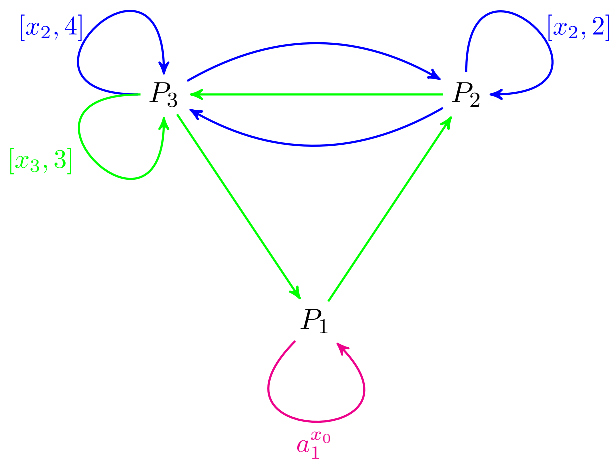



FIGURE 3.3. Quiver associated to the Brauer configuration $\Delta_3 = (\Delta_0, \Delta_1, \mu, \mathcal{O})$.

According to 2.3 we have the following identities:

$$w^e(P_1) = (\sqrt{0})^1(\sqrt{3})^1,$$
$$w^e(P_2) = (\sqrt{2})^2(\sqrt{3})^1,$$
$$w^e(P_3) = (\sqrt{2})^4(\sqrt{3})^3.$$

In accordance with the concatenation defined by the usual sum of real numbers, $M(\Delta_3^e)$ is given by:

$$\begin{aligned} M(\Delta_3^e) &= (\sqrt{3}) + (\sqrt{2}+\sqrt{2}+\sqrt{3}) + (\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{3}+\sqrt{3}+\sqrt{3}) \\ &= 6\sqrt{2}+5\sqrt{3}. \end{aligned} \tag{3.24}$$

The following results are consequences of Theorem 4 and Proposition 2.

**Corollary 22.** *For $k \geq 3$ fixed,* $\dim_k \Lambda_{\Delta_k} = 7+2(t_{k^2-k-1})+2(t_{k+1}) = k^4-2k^3+k^2+4k+9$. *And* $\dim_k \Lambda_{\Delta_{k+1}} - \dim_k \Lambda_{\Delta_k} = 4k^3+4$, *where for $j \geq 1$, $t_j$ denotes the $j$th triangular number.*

*Proof.* It is enough to observe that for $k \geq 3$, it holds that $val(x_{k-1}) = k(k-1)$, whereas $val(x_k) = k+2$. The theorem holds as a consequence of Proposition 2. □

**Corollary 23.** *For $n \geq 3$ fixed, it holds that* $\dim_k Z(\Lambda_{\Delta_k}) = k^2+2$.

*Proof.* Since $\operatorname{rad}^2 \Lambda_{\Delta_k} \neq 0$, the result is a consequence of Theorem 4 with $\mu(x_0) = 2$, $\mu(x_{k-1}) = \mu(x_k) = 1$, $|\Delta_0| = 3$, $|\Delta_1| = 3$, $\operatorname{occ}(x_{k-1}, P_2) + \operatorname{occ}(x_{k-1}, P_3) = k^2-k$, $\operatorname{occ}(x_k, P_3) = k$ and $\operatorname{occ}(x_0, P_1) = 1$. □

For preprojective Kronecker trees $T_{(n+1,n)}$ we have the following corollary:

**Corollary 24.**

$$\frac{(n-1)(n)(4n+1)}{6} + d\left\lfloor\sqrt{n}\right\rfloor \leq \left\|M(T_{(n+1,n)})\right\|_{*} \leq \frac{(n-1)!\left\lfloor\frac{n}{2}\right\rfloor(n-2)^{3/2}+6\sqrt{n}}{3} \quad (3.25)$$

*Proof.* By construction, we note that there is a bijection $f : B \to S$ where $B$ is the set of ramifications of $T$ of the form $b_n = \begin{array}{c} \bullet \\ \swarrow\ \downarrow\ \searrow \\ \underbrace{\bullet\ \ \bullet\ \ \bullet}_{n-\text{ vertices}} \end{array}$ and the set of singular values of $M(T_{(n+1,n)})$ given by the rule $f(b_n) = \sqrt{n}$. Actually, the characteristic polynomial has the form $P(\lambda) = \lambda^{n_0}(\lambda-1)^{n_1}(\lambda-\sqrt{2})^{n_2}\cdots(\lambda-\sqrt{n})^{n_k}$, with $n_k = n_{k-1} = 1$ we note that the coefficients in the recurrence

$$T_{n_2} = T_{n-1}(n-2)(n-1) + \sqrt{n} + (n-1)\sqrt{n-2} + F_n$$

where $F_n = \sqrt{n-1}+(n-1)\sqrt{n-2}+(n-1)(n-2)\sqrt{n-3}+\ldots+(n-1)(n-2)(n-3)\cdots 3\sqrt{2}$, gives the algebraic multiplicity of all singular values $\left\{0, 1, \sqrt{2}, \ldots, \sqrt{n}\right\}$. Thus, the most frequent singular values are $n_0 = (n-1)!\left\lceil\frac{n}{2}\right\rceil$ and $n_2 = \frac{(n-1)!}{2}\left\lceil\frac{n}{2}\right\rceil$ therefore

$$\begin{aligned}
\sum_{j=0}^{n}\sqrt{j} = \left\|M(T_{(n+1,n)})\right\|_{*} &\leq \frac{(n-1)!\left\lfloor\frac{n}{2}\right\rfloor}{2}\left(\sum_{i=1}^{n-2}\sqrt{i}\right) + \sqrt{n} \\
&\leq \frac{(n-1)!\left\lfloor\frac{n}{2}\right\rfloor}{2}\frac{2(n-2)^{3/2}}{3} + 2\sqrt{n} \\
&= \frac{(n-1)!\left\lfloor\frac{n}{2}\right\rfloor(n-2)^{3/2}}{3} + 2\sqrt{n} \\
&= \frac{(n-1)!\left\lfloor\frac{n}{2}\right\rfloor(n-2)^{3/2}+6\sqrt{n}}{3}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\|M(T_{(n+1,n)})\right\|_{*} &\geq \sum_{j=0}^{n}\sqrt{j} \\
&= \sum_{i=1}^{n} i(2i+1) + d\left\lfloor\sqrt{n}\right\rfloor \\
&= \frac{(n-1)(n)(4n+1)}{6} + d\left\lfloor\sqrt{n}\right\rfloor
\end{aligned}$$

And with this argument we are done. □

*Remark* 8. We note that, although the research on topological indexes has its origin in chemistry nowadays, researches regarding these two subjects are independent. However, results as those given in Corollary 24 are examples of the strong connection between these fields. Actually, it is worth noting that, the sequence in the lower bound of the inequality

3.25 corresponds to the number of zinc sulfide (ZnS) molecules in clusters of $n$ layers in zinc blende crystals.

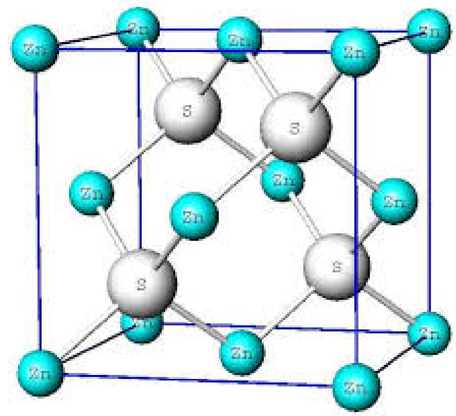



FIGURE 3.4. Crystal structure of zinc sulfide blende (ZnS).

*Remark* 9. According to Corollary 1 the Brauer configuration algebras studied in sections 3.1 and 3.3 are the trivial extension of an almost gentle algebra, whereas the algebras studied in sections 3.4.1 and 2.4.1 are not.

### 3.4.2 A practical application of the theory of Brauer configuration algebras; traffic flow

Nowadays one of the most important challenges faced by big cities is the control of traffic flow. Actually, in the contemporary society people spend much time stuck in traffic jams every day. A poor management of the vehicular flow implies serious economical problems for the cities. On one hand, some cities have considered to discourage the use of cars causing a minor number of taxpayers thus many problems to the sustainability of public finances.

On the other hand some cities have decided to improve the roads investing much money in an effort to keep countries moving and make journeys faster, better and more reliable for all the citizens. However, it means gridlocked cities and more pollution thus more respiratory diseases amongst the population.

Through this section we will use Brauer configuration algebras to model some aspects of the traffic flow. We assume that the route in this model is circular.

Traffic problems have been addressed for several researches by considering issues regarding traffic prediction. For instance, traffic speed, traffic density estimation, traffic flow, traffic congestion, accident prone-area, etc. We recall some of these concepts briefly [63].

- Vehicular Traffic consists of various type of vehicles, i.e., performances, various sizes, and characteristics.
- Traffic flow or volume $q$ is the number of vehicles ($n$) passing some designated roadway point in a given time interval ($t$); $q = \frac{n}{t}$.

- Traffic flow data is usually collected to obtain factual data concerning the movement of vehicles at selected points on the street, to indicate for example trends in volume.

We also have other measures in traffic prediction theory:

- Average Daily Traffic (ADT), Annual Average Daily Traffic (AADT) and Hourly Traffic (HT), to determine peak periods, capacity deficiencies, and to determine geometric design parameters.

- Density (D) or concentration ($k$) is the number of vehicles occupying a given length ($l$) of a lane or roadway at a particular instant, $k = \frac{n}{l} = \frac{q}{u}$; $q = u_f k - \frac{u_f}{k_f} k^2$, where $u$ is the speed.

There are two types of facilities:

- Uninterrupted flow (freeway, multilane highways, two-lane highways).

- Interrupted flow (signalized streets, un-signalized streets with stops signs, transit lanes, pedestrian walkways).

Regarding acquisition of traffic information in real time, several big cities as Rio de Janeiro have developed different tools to make such task.

For instance, BusesInRio [32] is a tool designed to carry out the acquisition of data arising from trajectories generated by an active GPS device installed in vehicles, in such a way that trajectories are understood as mobile traffic sensors.

A vehicle raw trajectory is considered as a continuous data stream generated by a GPS device.

BusesInRio not only offers the instantaneous data if not it queries data and stores its entries for future processing, data analysis and planning.

However, some problems have arisen from the implementation of this tool, for instance, Nasser et al solved an acquisition data problem generated by the original tool provided by the City Hall Web service on Dados Rio such a gap was generated due that the data provided by the GPS service has no information about which route the bus is following at that moment.

The authors solved the problem by considering two consecutive samples of a bus. These consecutive positions determine a vector $V$, these two positions are also projected on each route $R_i$ respecting the route direction, to create the vectors $V_{R_i}$, the chosen route is that with the smallest angle (as a kind of map matching).

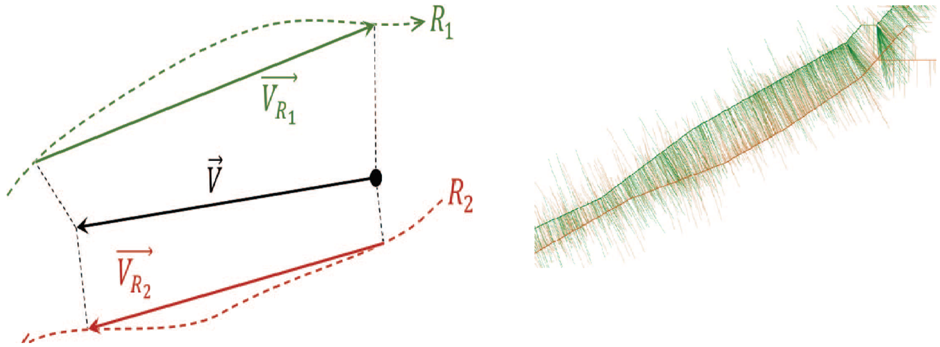



Figure 3.5. Projection vectors associated to two consecutive samples and trajectories (paths) constructed according to the algorithm of Nasser et al. for the tool BusesInRio.

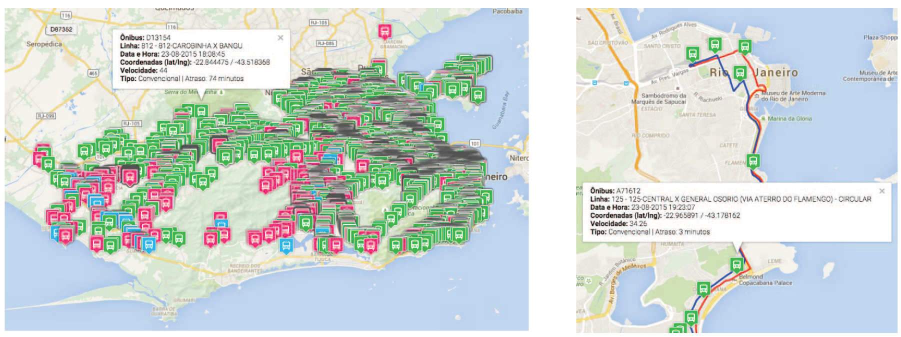


Figure 3.6. Route matching by direction analysis and data acquired by BusesInRio for two route lines.

### 3.4.3 Some models

Some traffic flow models consider the following aspects and questions to construct an accurate and reliable mirror of the real-world vehicular traffic:

- How many cars are on a road in a given time?
- How many of these cars have mechanical breakdowns?
- How much time is spent at a traffic light?
- What is the total time while driving in a given route?
- How much time does it take to complete a (e.g., circular) route?

Some of these questions can be addressed by defining the traffic flow, the average time or the density. However, in many cases the geometry of the route generated by the trajectories of the vehicles matter (e.g. BusesInRio).

In fact, the traffic flow defined in this way (via vehicular trajectories) describe a well known algebraic structure called path algebra, and the restrictions (as, stop-lights, vehicles with mechanical breakdowns or different obstacles) transform the traffic flow in a Brauer configuration algebra.

### 3.4.4 A Brauer configuration algebra defined by traffic flow

**Definition 3.** *Traffic flow defines a Brauer configuration algebra*

$$\Phi(t) = (\Phi_0(t), \Phi_1(t), \mu, \mathcal{O}) \text{ such that:}$$

- $\Phi_0(t)$ *consists of all vehicles on the road (or route) in a given time* $t$ *(including vehicles with mechanical breakdowns). We assume that the route is circular.*
- $\Phi_1(t)$ *interprets the observation points-polygons on the road where the counting process is carry out. In the data, each of these points gives information about the number and classes of vehicles detected.*
- $\mu(v) = 1$, *for any vehicle* $v$.
- *Observation points are numbered* $P_0, P_1, \ldots, P_n = P_0$ *according to the locations of the points of observation in the road. It is assumed that vehicles completing the route are non-truncated vertices and vehicles with mechanical breakdowns are truncated vertices (these vertices can be taken out from the route without affect the behavior the traffic flow) and that* $P_0$ *is the starting point. Thus,* $P_i < P_{i+1}$, $0 \leq i \leq n-1$, *and* $P_n = P_0$.

***The Ideal Interpretation***

- *Relations generating the ideal of the Brauer configuration algebra* $\Lambda_\Phi(t) = \Phi(t)/I$ *are defined in a natural fashion.*
- *For instance, if the end point of a vehicle-trajectory* $v_i$ *coincides with the starting point of other vehicle trajectory* $w_{i+1}$, *it holds that* $v_i w_{i+1} = 0$.
- *If the traffic is interrupted then it is assumed independence between two consecutive parts of the road (because all the vehicles should be stopped at each observation point). Thus,* $v_i v_{i+1} = 0$ *for each vehicle-trajectory* $v_i$.
- *If the traffic is uninterrupted it holds that* $v_i v_{i+1} \neq 0$, *for any* $i$, *and* $v_i w_{i+1} = 0$.
- *If the traffic is interrupted time spend at each observation point is represented by loops (one loop for each suitable unit of time) in the Brauer quiver. Thus,* $c^n_{i,v_i} \in P_k$ *means that the vehicle* $v_i$ *was* $n$ *time units at the observation point* $P_k$. *In such a case* $\left(\sum_{h=1}^{k} c^h_{i,v_i}\right)(v_{i+1}) = 0$, $k < n$, *and* $v_i w_{i+1} = 0$ *for any other trajectory-vehicle* $w_{i+1}$.
- *Data arising from the trajectory-vehicles at each observation point-polygon are identified. That is, if the traffic analysis allows they are considered as a cluster.*
- *Fixed a time* $t$, *only data arising from non truncated vehicles in just one round are used.*

The Brauer configuration algebra $\Lambda_\Phi(t)$ allows to give the following information:

1. How much time ($\tau_{\Phi}$) all vehicles were stopped at the observation points in a circular route?, we let $\tau_{v_{i,k}}$ denote the time that the trajectory-vehicle $v_i$ remains stopped at the observations point-polygon $P_k$ ($\tau_{\Phi} = \sum\limits_{v_{i,k}} \tau_{v_{i,k}}$). $\tau_{\Phi,v_i} = \sum\limits_{k} \tau_{v_{i,k}}$ is the time that a trajectory-vehicle $v_i$ was stopped in the route.

2. How many vehicles were observed without mechanical breakdowns at a given observation point ($v_{P_k}$) or through the entire route ($v_{\phi} = \sum\limits_{k=0}^{n-1} v_{P_k}$)?

Firstly, we observe that;

$$\tau_{\Phi} = \dim_k(Z(\Lambda_{\Phi}(t))) - n - 1,$$

where $Z(\Lambda_{\Phi}(t))$ denotes the center of $\Lambda_{\Phi}(t)$.

The number of summands in the radical square of the indecomposable projective module associated to $P_k$ is $v_{P_k}$.

The following are examples of the Definition 3.

**Example 7.** *Consider* $\Phi(t) = (\Phi_0(t), \Phi_1(t), \mu, \mathcal{O})$*, such that:*

1. $\Phi_0(t) = \{1, 2, 3\}$,
2. $\Phi_1(t) = \{V_1 = \{1, 1, 1, 2, 3\}, V_2 = \{1, 2, 2, 2, 3\}, V_3 = \{1, 2, 3, 3, 3\}, V_4 = \{1, 2, 3\}\}$,
3. *At vertex 1, it holds that;* $V_1 < V_1 < V_1 < V_2 < V_3 < V_4, \quad val(1) = 5$,
4. *At vertex 2, it holds that;* $V_1 < V_2 < V_2 < V_2 < V_3 < V_4, \quad val(2) = 5$,
5. *At vertex 3, it holds that;* $V_1 < V_2 < V_3 < V_3 < V_3 < V_4, \quad val(3) = 5$,
6. $\mathrm{occ}(1, V_1) = 3, \quad \mathrm{occ}(1, V_2) = 1, \quad \mathrm{occ}(1, V_3) = 1, \quad \mathrm{occ}(1, V_4) = 1$,
7. $\mathrm{occ}(2, V_1) = 1, \quad \mathrm{occ}(2, V_2) = 3, \quad \mathrm{occ}(2, V_3) = 1, \quad \mathrm{occ}(2, V_4) = 1$,
8. $\mathrm{occ}(3, V_1) = 1, \quad \mathrm{occ}(3, V_2) = 1, \quad \mathrm{occ}(3, V_3) = 3, \quad \mathrm{occ}(3, V_4) = 1$,
9. $\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1$.

   *The ideal $I$ of the corresponding Brauer configuration algebra $\Phi(t)$ is generated by the following relations (see Figure 3.7), for which it is assumed the following notation* *for the special cycles:*

$$\begin{aligned}
C_1^{V_1,1} &= a_1^1a_2^1a_3^1a_4^1a_5^1a_6^1, & C_1^{V_1,2} &= a_2^1a_3^1a_4^1a_5^1a_6^1a_1^1,\\
C_1^{V_1,3} &= a_3^1a_4^1a_5^1a_6^1a_1^1a_2^1, & C_1^{V_2,1} &= a_4^1a_5^1a_6^1a_1^1a_2^1a_3^1,\\
C_1^{V_3,1} &= a_5^1a_6^1a_1^1a_2^1a_3^1a_4^1, & C_1^{V_4,2} &= a_6^1a_1^1a_2^1a_3^1a_4^1a_5^1,\\
C_2^{V_1,1} &= a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2, & C_2^{V_2,1} &= a_2^2a_3^2a_4^2a_5^2a_6^2a_1^2,\\
C_2^{V_2,2} &= a_3^2a_4^2a_5^2a_6^2a_1^2a_2^2, & C_2^{V_2,3} &= a_4^2a_5^2a_6^2a_1^2a_2^2a_3^2,\\
C_2^{V_3,1} &= a_5^2a_6^2a_1^2a_2^2a_3^2a_4^2, & C_2^{V_4,1} &= a_6^2a_1^2a_2^2a_3^2a_4^2a_5^2,\\
C_3^{V_1,1} &= a_1^3a_2^3a_3^3a_4^3a_5^3a_6^3, & C_3^{V_2,1} &= a_2^3a_3^3a_4^3a_5^3a_6^3a_1^3,\\
C_3^{V_3,1} &= a_3^3a_4^3a_5^3a_6^3a_1^3a_2^3, & C_3^{V_3,2} &= a_4^3a_5^3a_6^3a_1^3a_2^3a_3^3,\\
C_3^{V_3,3} &= a_5^3a_6^3a_1^3a_2^3a_3^3a_4^3, & C_3^{V_4,1} &= a_6^3a_1^3a_2^3a_3^3a_4^3a_5^3.
\end{aligned} \tag{3.26}$$

(a) $a_i^h a_r^s$, *if* $h \neq s$, *for all possible values of* $i$ *and* $r$,

(b) $C_j^{V_1,i} - C_l^{V_1,k}$, *for all possible values of* $i, j, k$ *and* $l$,

(c) $C_j^{V_2,i} - C_l^{V_2,k}$, *for all possible values of* $i, j, k$ *and* $l$,

(d) $C_j^{V_3,i} - C_l^{V_3,k}$, *for all possible values of* $i, j, k$ *and* $l$,

(e) $C_j^{V_4,i} - C_l^{V_4,k}$, *for all possible values of* $i, j, k$ *and* $l$,

(f) $C_i^{V_1,j}a$ $(C_i^{V_2,j}a')$ , *with* $a$ $(a')$ *being the first arrow of* $C_i^{V_1,j}$ $(C_i^{V_2,j})$ *for all* $i, j$,

(g) $C_i^{V_3,j}a$ $(C_i^{V_4,j}a')$ , *with* $a$ $(a')$ *being the first arrow of* $C_i^{V_3,j}$ $(C_i^{V_4,j})$ *for all* $i, j$.

*The following is the quiver associated to the Brauer configuration* $\Phi(t)$*, the colors means the different special cycles associated to each vertex as follows: for vertex* $1$ *red, for vertex* $2$ *green, for vertex* $3$ *blue.*

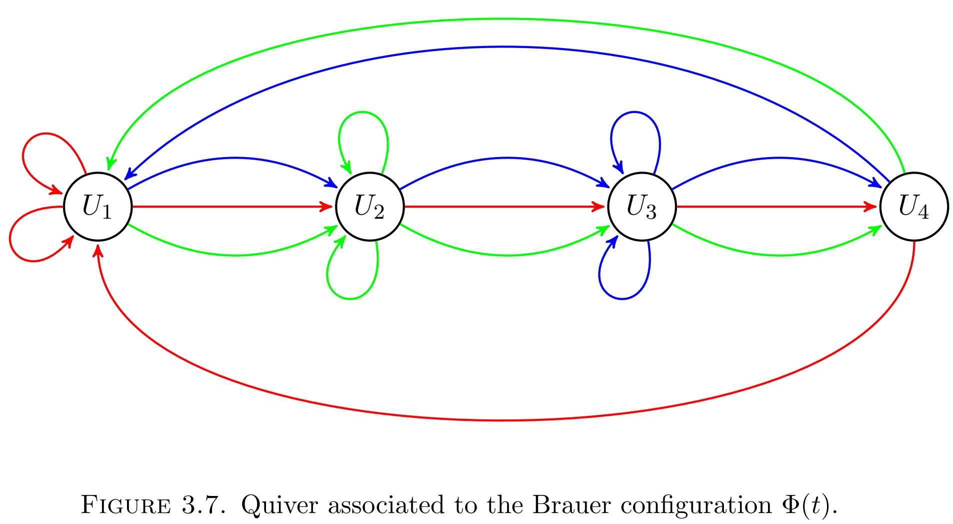



FIGURE 3.7. Quiver associated to the Brauer configuration $\Phi(t)$.

*For this example we have three kind of vehicles on the route, four observation points, in this case, the vehicles stopping time is given by the following formula:*

$$\begin{aligned}\tau_{\Phi} &= \dim_k(Z(\Lambda_{\Phi}(t))) - n - 1 \\ &= 11 - 4 - 1 = 6 \textit{ suitable units of time (SUT).}\end{aligned} \tag{3.27}$$

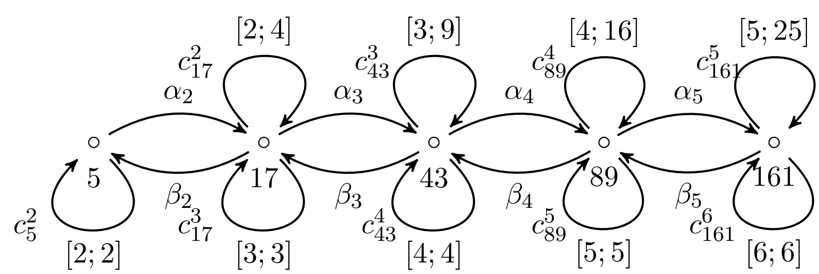
*In accordance with the model purpose in this work the number of vehicles observed without mechanical breakdowns through the entire route is*

$$v_{\phi} = \sum_{k=0}^{n-1} v_{P_k} = 5 + 5 + 5 + 3 = 18.$$

**Example 8.** *We consider the Brauer configuration algebra described in section 3.3*

1. $\Phi'_0(t) = \{1, 2, 3, 4, 5, 6\}$,
2. $\Phi'_1(t) = \{V_1 = \{1^{(1)}, 2^{(2)}\}, V_2 = \{2^{(4)}, 3^{(3)}\}, V_3 = \{3^{(9)}, 4^{(4)}\}, V_4 = \{4^{(16)}, 5^{(5)}\}, V_5 = \{5^{(25)}, 6^{(6)}\}\}$,
3. *At vertex 1, it holds that;* $V_1$, $\quad val(1) = 1$,
4. *At vertex 2, it holds that;* $V_1^{(2)} < V_2^{(4)}$, $\quad val(2) = 6$,
5. *At vertex 3, it holds that;* $V_2^{(3)} < V_3^{(9)}$, $\quad val(3) = 12$,
6. *At vertex 4, it holds that;* $V_3^{(4)} < V_4^{(16)}$, $\quad val(4) = 20$,
7. *At vertex 5, it holds that;* $V_4^{(5)} < V_5^{(25)}$, $\quad val(5) = 30$,
8. *At vertex 6, it holds that;* $V_5^{(6)}$, $\quad val(6) = 6$, *where* $V_x^{(y)}$ *means that the polygon* $V_x$ *occurs* $y$ *times in the successor sequence of the corresponding vertex*
9. $\mu(j) = 1$, *for any* $j \in \Gamma_0$.

*The ideal* $I$ *of the corresponding Brauer configuration algebra* $\Phi'(t)$ *is generated by the relations defined in (3.13). The following is the quiver associated to* $\Phi'(t)$




*In this example with six kind of vehicles moving on the route and five observation points, the corresponding vehicles stopping time can be obtained as follows:*

$$\begin{aligned}\tau_{\Phi'} &= \dim_k(Z(\Lambda_{\Phi'}(t))) - n - 1 \\ &= 72 - 5 - 1 = 65 \textit{ SUT.}\end{aligned} \tag{3.28}$$

*In accordance with the model purpose in this work the number of vehicles observed without mechanical breakdowns through the entire route is*

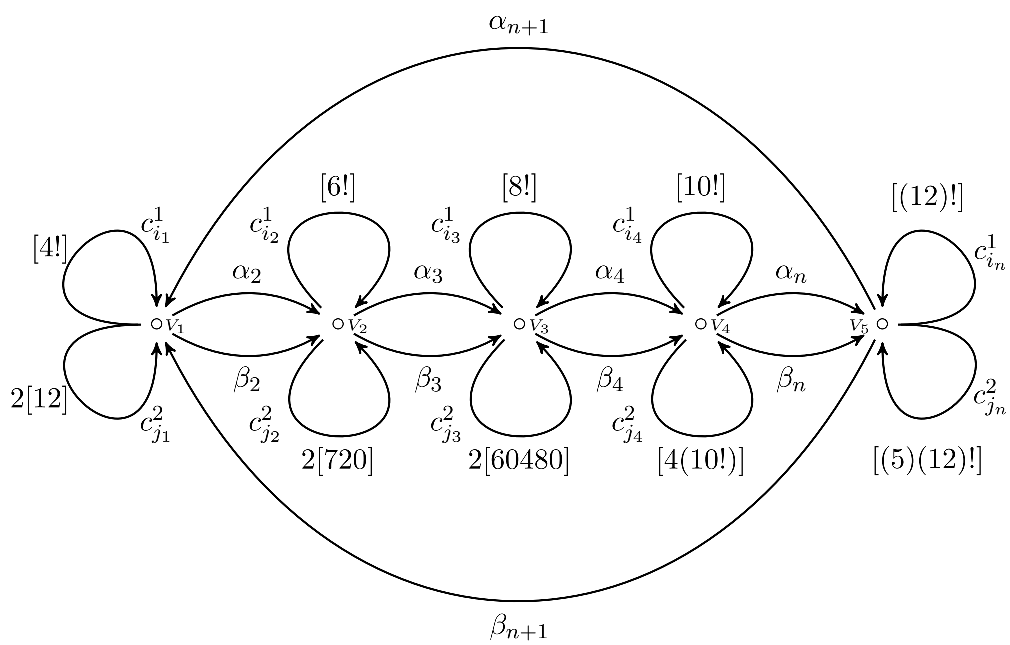
$$v_{\Phi'} = \sum_{k=0}^{n-1} v_{P_k} = 2 + 7 + 13 + 21 + 31 = 74,$$

*and there is only one vehicle with mechanical breakdowns.*

**Example 9.** *Consider* $\Phi''(t) = (\Phi''_0(t), \Phi''_1(t), \mu, \mathcal{O})$, *such that:*

1. $\Phi''_0(t) = \{x_1, x_2\}$,
2. $\Phi''_1(t) = \{V_1 = \{x_1^{(4!)}, x_2^{(4!)}\}, V_2 = \{x_1^{(6!)}, x_2^{2(6!)}\}, V_3 = \{x_1^{(8!)}, x_2^{3(8!)}\}, V_4 = \{x_1^{(10!)}, x_2^{4(10!)}\}, V_5 = \{x_1^{(12!)}, x_2^{5(12!)}\}\}$,
3. *At vertex* $x_1$; $V_1^{(4!)} \leq V_2^{(6!)} \leq V_3^{(8!)} \leq V_4^{(10!)} \leq V_5^{(12!)}$, $\quad val(x_1) = \sum_{k=1}^{5} k(2k+2)!$
4. *At vertex* $x_2$; $V_1^{2(12)} \leq V_2^{2(720)} \leq V_3^{2(60480)} \leq V_4^{2(10!)} \leq V_5^{2(12!)}$, $\quad val(x_2) = \sum_{k=1}^{5}(2k+2)!$,
5. $\mu(x_1) = \mu(x_2) = 1$.

*The ideal* $I$ *of the corresponding Brauer configuration algebra* $\Phi''(t)$ *is generated by the relations defined in (3.7). The following is the quiver associated to* $\Phi''(t)$



*In this new example with two kind of vehicles and five observation points the vehicles stopping time is*

$$
\begin{aligned}
\tau_{\Phi''} &= -n + 1 + \sum_{k=1}^{n} h^{p}_{2k+2} - n - 1 \\
&= -2n + \sum_{k=1}^{n} h^{p}_{2k+2} \\
&= 2892317078 \ SUT.
\end{aligned}
\tag{3.29}
$$

*In accordance with the model purpose in this work the number of vehicles observed without mechanical breakdowns through the entire route is:*

$$
\begin{aligned}
v_{\Phi''} = \sum_{k=0}^{n-1} v_{P_k} &= 48 + 2160 + 161280 + 18144000 + 2874009600 \\
&= 2892317088.
\end{aligned}
\tag{3.30}
$$

*Remark* 10. According to Corollary 1 the Brauer configuration algebras defined by traffic flow are always the trivial extension of an almost gentle algebra.

# CHAPTER 4

# Categorification of Fibonacci numbers via homological ideals and applications of Brauer configurations algebras

In this chapter, we give combinatorial conditions to determine whether an idempotent ideal associated to some Nakayama algebras is homological or not. We give the number of such ideals via the integer specialization of a suitable Brauer configuration algebra and its corresponding message. Moreover, we use the number of homological ideals to establish a partition formula for even-index Fibonacci numbers. Some interesting sequences in the On-line Encyclopedia of Integer Sequences (OEIS) [66] arising from these computations are described as well.

## 4.1 Homological ideals

Homological ideals or strong idempotent ideals are ideals of an algebra introduced by Auslander, Platzeck and Todorov in [6]. These ideals arise from the research of heredity ideals and quasi-hereditary algebras. For these ideals the corresponding quotient map induces a full and faithful functor between derived categories. Recently, homological ideals have been studied in different contexts, for instance Gatica, Lanzilotta and Platzeck and independently Xu and Xi established some relationships with the so called finitistic dimension conjecture and the Igusa-Todorov functions [26]. Furthermore, De la Peña and Xi in [17] and Armenta in [4] studied the impact of these ideals in the context of Hochschild cohomology and one point extensions.

This section deals with the combinatorial properties of homological ideals associated to some path algebras and their relationships with the novel Brauer Configuration algebras, which have been introduced recently by Green and Schroll in [31]. In particular, we use the notion of the message of a Brauer configuration introduced in Section 2.1, such messages enable to compute the number of homological ideals associated to some Nakayama algebras. Moreover, such number of ideals allow us to obtain an alternative version of the partition formula for even-index Fibonacci numbers given by Ringel and Fahr in [19] (see identity 3) attaining in this way a new algebraic interpretation for these numbers.

For an algebra $A$ we mean a finite dimensional basic and connected algebra over an algebraically closed field $k$. We denote the category of finite dimensional right $A$-modules as $mod(A)$, and the bounded derived category of $mod(A)$ as $D^b(A)$. We will assume that $A$ is a bounded path algebra of the form $kQ/I$ with $Q$ a finite quiver and $I$ an admissible ideal.

An epimorphism of algebras $\phi : A \to B$ is called *homological epimorphism* if it induces a full and faithful functor

$$D^b(\phi^*) : D^b(B) \to D^b(A).$$

Let $I$ be a two sided ideal of $A$. Since the quotient map $\pi : A \to A/I$ is an epimorphism then the induced functor $\pi^* : mod(A/I) \to mod(A)$ is full and faithful.

A two sided ideal $I$ of $A$ is *homological* if the quotient map $\pi : A \to A/I$ is an homological epimorphism.

The following results characterize homological ideals [6, 17].

**Proposition 6.** *Let $I$ be an ideal of $A$, then*

1. *$I$ is an homological ideal of $A$ if and only if $\mathrm{Tor}_n^A(I, A/I) = 0$ for all $n \geq 0$. In this case, $I$ is idempotent.*
2. *If $I$ is idempotent and $A$-projective, then $I$ is homological.*
3. *If $I$ is idempotent then $I$ is homological if and only if $\mathrm{Ext}_A^n(I, A/I) = 0$ for all $n \geq 0$.*

We denote the *trace* of an $A$-module $M$ in an $A$-module $N$ as

$$tr_M(N) := \sum_{f \in Hom_A(M,N)} Im(f) \subset N.$$

*Remark* 11. We recall that according to Auslander et al [6], if $P$ is an $A$-projective module then $tr_P(A)$ is an idempotent ideal of $A$ and one obtains all the idempotent ideals of $A$ this way.

*Remark* 12. Note that, since the homological ideals are idempotent ideals and the idempotent ideals are traces of projective modules over $A$ then there is always a finite number of homological ideals.

Following the assumption that $A$ is a bounded quiver algebra of the form $kQ/I$ and the number of vertices of $Q$ are finite for every subset $\{a_1, ..., a_m\} \subset Q_0$, we will assume the following notation for every idempotent ideal generated by the trace of $P(a_1) \oplus \cdots \oplus P(a_m)$ in $A$:

$$I_{a_1,...,a_m} = tr_{\left(P(a_1) \oplus \cdots \oplus P(a_m)\right)}(A). \tag{4.1}$$

In this section, we combine tools developed by Auslander et al. in [6], Xi and De la Peña in [17] and the integer specializations of some Brauer configuration (see Section 2.1) to establish an explicit formula for the number of homological ideals associated to some Nakayama algebras.

First of all, for $n \geq 4$ fixed, we consider a Brauer configuration $\Gamma_n = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

1. $\Gamma_0 = \{n-k-1 \in \mathbb{N} \mid 2 \leq k \leq n-1\} \cup \{n-2\}$,
2. $\Gamma_1 = \{U_k = \{n-2, n-k-1\} \mid 2 \leq k \leq n-1\}$.
3. The orientation $\mathcal{O}$ is defined in such a way that
   - $(a)$ Vertex $n-2$ has associated the successor sequence $U_2 < U_3 < \cdots < U_{n-1}$, in this case, $val(n-2) = n-2$,
   - $(b)$ If $2 \leq k \leq n-1$ then at vertex $n-k-1$, it holds that the corresponding successor sequence consists only of $U_k$, and for each $k$, $val(n-k-1) = 1$.
4. $\mu(n-2) = 1$,
5. $\mu(n-k-1) = n-2, \quad 2 \leq k \leq n-1$.

The ideal $I_{\Gamma_n}$ of the corresponding Brauer configuration algebra $\Lambda_{\Gamma_n}$ is generated by the following relations (see Figure 4.1), for which it is assumed the following notation for the special cycles:

$$\begin{aligned} C^{U_k}_{n-2} &= \begin{cases} a_1^{n-2} a_2^{n-2} \cdots a_{k-1}^{n-2}, & \text{if } k = 2, \\ a_{k-1}^{n-2} a_k^{n-2} \cdots a_{k-2}^{n-2}, & \text{otherwise}, \end{cases} \\ C^{U_k}_{n-k-1} &= a_1^{n-k-1}. \end{aligned} \tag{4.2}$$

1. $a_i^h a_r^s$, if $h \neq s$, for all possible values of $i$ and $r$ unless for the loops associated to the vertices $n-k-1$,
2. $C^{U_k}_{n-2} - \left(C^{U_k}_{n-k-1}\right)^{n-2}$, for all possible values of $k$,
3. $C^{U_k}_{n-2} a$ with $a$ being the first arrow of $C^{U_k}_{n-2}$ for all $k$,
4. $\left(C^{U_k}_{n-k-1}\right)^{n-2} a'$ with $a'$ being the first arrow of $C^{U_k}_{n-k-1}$ for all $k$.

Figure 4.1 shows the quiver $Q_{\Gamma_n}$ associated to this configuration.

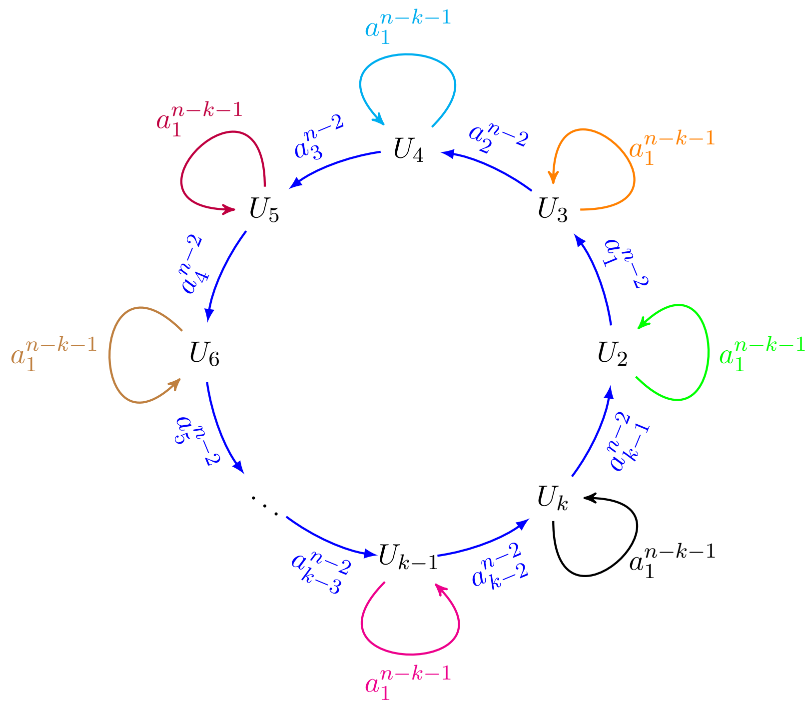



FIGURE 4.1. The quiver $Q_{\Gamma_n}$ defined by the Brauer configuration $\Gamma_n$.

As an example the following is the numerology associated to the algebra $\Lambda_{\Gamma_n} = kQ_{\Gamma_n}/I_{\Gamma_n}$ with $Q_{\Gamma_n}$ as shown in Figure 4.1 and special cycles given in (4.2), ($|r(Q_{\Gamma_n})|$ is the number of indecomposable projective modules. Note that, $|C_i| = val(i)$):

$$\begin{aligned}
|r(Q_{\Gamma_n})| &= n-2,\\
|C_{n-2}| &= n-2, \quad |C_{n-k-1}| = 1,\\
\sum_{\alpha\in\Gamma_0}\sum_{X\in\Gamma_1} \mathrm{occ}(\alpha, X) &= n-1, \quad \text{the number of special cycles},\\
\dim_k \Lambda_{\Gamma_n} &= 2(n-2) + (n-2)(n-3) + (n-3)(n-2) = 2(n-2)^2,\\
\dim_k Z(\Lambda_{\Gamma_n}) &= 1 + 1 + (n-2)^2 + (n-2) - (n-1) + (n-2) - (n-2) =\\
&= n^2 - 4n + 5.
\end{aligned}$$

*Remark* 13. $\Lambda_{\Gamma_n}$ is a Brauer graph algebra and according to Proposition 3, the Brauer configuration algebra $\Lambda_{\Gamma_n}$ with quiver $Q_{\Gamma_n}$ shown in Figure 4.1 has a length grading induced by the path algebra $kQ_{\Gamma_n}$, provided that for any $\alpha \in \Gamma_0$ it holds that $\mu(\alpha)val(\alpha) = n-2$.

**Example 10.** *For the Brauer configuration $\Gamma_n$ whose associated quiver is shown in Figure 4.1, we define the specialization $e(\alpha) = 2^\alpha$, $\alpha \in \Gamma_0$ with the concatenation in each word given by the difference of the specializations of the vertices belonging to a determined* *polygon, in such a case for n fixed, we have:*

$$\begin{aligned} w(U_k) &= (n-2)(n-k-1), \text{ for } 2 \leq k \leq n-1, \\ w^e(U_k) &= 2^{n-2} - 2^{n-k-1}, \text{ for } 2 \leq k \leq n-1, \\ M(\Gamma_n^e) &= \sum_{U_k \in \Gamma_1} w^e(U_k) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1}. \end{aligned} \tag{4.3}$$

## 4.2 Homological ideals associated to Nakayama algebras

In this section, we prove some combinatorial conditions which allow to establish whether an idempotent ideal in some Nakayama algebras is homological or not. We also give the number of homological ideals associated to these algebras via the integer specialization of the Brauer configuration $\Gamma_n$ defined in Example 10.

Let $Q$ be either a linearly oriented quiver with underlying graph $\mathbb{A}_n$ or a cycle $\widetilde{\mathbb{A}_n}$ with cyclic orientation. That is, $Q$ is one of the following quivers

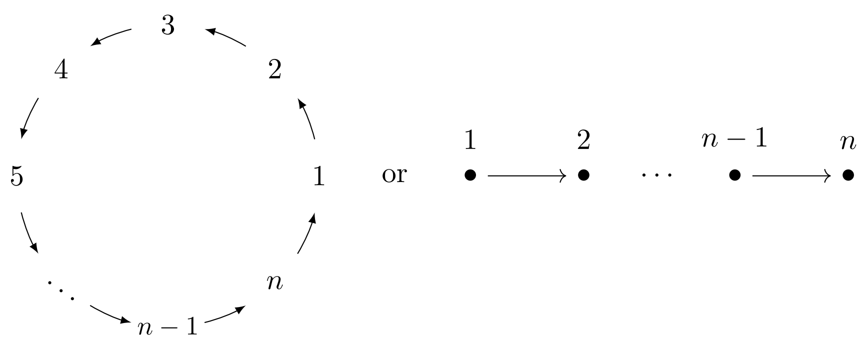



FIGURE 4.2. Quiver $\widetilde{\mathbb{A}_n}$ with cyclic orientation and Dynkin diagram $\mathbb{A}_n$ linearly oriented.

A quotient $A$ of $kQ$ by an admissible ideal $I$ is called a *Nakayama algebra* [35].

In this work, for $n \geq 3$ fixed, we consider the algebras $A_{R_{(i,j,k)}} = kQ/I$ where $Q$ is a Dynkin diagram of type $\mathbb{A}_n$ linearly oriented and $I$ is an admissible ideal generated by one relation $R_{(i,j,k)}$ of length $k$ starting at a vertex $i$ and ending at a vertex $j$ of the given quiver, $1 \leq i < j \leq n$. The following picture shows the general structure of quivers $Q$ which we are focused in this section.

$$\mathbb{A}_n = 1 \to \cdots \to i \to i+1 \to \cdots \to i+k = j \to j+1 \to \cdots \to n-1 \to n.$$

The following Lemmas 1-7 allow to determine which idempotent ideals of an algebra $A_{R_{(i,j,k)}}$ are also homological ideals. In this case, Lemmas 1 and 2 regard the case whenever the idempotent ideal is generated by the trace of just one projective module associated to a vertex of the quiver.

**Lemma 1.** *Every idempotent ideal $I_r$ of an algebra $A_{R_{(i,j,k)}}$ (see (4.1)) with $j \leq r$ or $r \leq i$ is homological.*

*Proof.* For $r \leq i$, we have the following cases:

1. $tr_{P(r)}(P(t)) = 0$ if $t > r$.
2. $tr_{P(r)}(P(t)) = P(r)$ if $t \leq r$, where $P(r)$ denotes the $r$-th projective module.

If $r \geq j$, we consider the following cases:

1. $tr_{P(r)}(P(t)) = P(r)$ if $i < t \leq r$, where $P(r)$ denotes the $r$-th projective module.
2. $tr_{P(r)}(P(t)) = 0$.

In all cases $tr_{P(r)}(A_{R_{(i,j,k)}}) = P(r)^l$ for some $l \in \mathbb{N}$. The result follows as a consequence of Proposition 6, item 2. We are done. $\square$

**Lemma 2.** *Every idempotent ideal $I_t$ of an algebra $A_{R_{(i,j,k)}}$, with $i+1 \leq t \leq j-1$ is not homological.*

*Proof.* Consider $L_t = tr_{P(t)}P(i) = P(i)/S(i) \oplus \cdots \oplus S(t-1)$ where $S(k)$ denote the $k$-th simple module, also note that there are not morphisms from $P(t)$ to $P(j)$ if $t \neq j$ which means that $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(L_t, P(j))$ is a direct summand of $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(I_t, A_{R_{(i,j,k)}}/I_t)$, provided that $L_t$ is a direct summand of $I_t$ and $P(j)$ is a direct summand of $A_{R_{(i,j,k)}}/I_t$. Applying the functor $\mathrm{Hom}_{A_{R_{(i,j,k)}}}(-, P(j))$ to a projective resolution of $L_t$ with the form

$$0 \to P(j) \to P(t) \to L_t \to 0$$

it is obtained the sequence

$$0 \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), P(j)) \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), P(j)) \to 0.$$

Thus, $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(L_t, P(j)) \cong k$ and $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(I_i, A_{R_{(i,j,k)}}/I_i) \neq 0$. Then the idempotent ideal $I_t$ is not an homological ideal as a consequence of Proposition 6, item 3. $\square$

**Lemma 3.** *If each idempotent ideal $I_{\alpha_w}$ of an algebra $A_{R_{(i,j,k)}}$ is not homological then every idempotent ideal of the form $I_{\alpha_1,\ldots,\alpha_l}$ is not homological for $2 \leq l \leq k-1$.*

*Proof.* For $l$ fixed, we start by computing $I_{\alpha_1,\ldots,\alpha_l}$,

$$I_{\alpha_1,\ldots,\alpha_l} = tr_{P(\alpha_1)\oplus\cdots\oplus P(\alpha_l)}(A_{R_{(i,j,k)}}) = \sum_{w=1}^{l} tr_{P(\alpha_w)}(A_{R_{(i,j,k)}})$$

In accordance with the hypothesis $\alpha_w \in [i+1, j-1]$ and taking into account that

$$tr_{P(\alpha_w)}(A_{R_{(i,j,k)}}) = \underbrace{L_{\alpha_w}}_{i-times} \oplus \underbrace{P(\alpha_w)}_{\alpha_w - i-times} \oplus \underbrace{0}_{n-\alpha_w-times} \tag{4.4}$$

$$tr_{P(\alpha_1)\oplus\cdots\oplus P(\alpha_l)}(A_{R_{(i,j,k)}}) = \underbrace{L_{\alpha_1}}_{i-times} \oplus \bigoplus_{w=1}^{l} P(\alpha_w) \oplus \underbrace{0}_{n-i-l-times} \tag{4.5}$$

it holds that according to the identity (4.5), $P(j)$ is a direct summand of $A_{R_{(i,j,k)}}/I_{\alpha_1\ldots\alpha_l}$ and $L_{\alpha_1}$ has the following projective resolution

$$0 \to P(j) \to P(\alpha_1) \to L_{\alpha_1} \to 0$$

Applying the functor $\mathrm{Hom}_{A_{R_{(i,j,k)}}}(-, P(j))$, we have that $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(L_{\alpha_1}, P(j)) \neq 0$ and by Proposition 6 item 3, we conclude that the idempotent ideal $I_{\alpha_1\ldots\alpha_l}$ is not an homological ideal. $\square$

**Lemma 4.** *For $l$ fixed, if each idempotent ideal $I_{\alpha_w}$ of an algebra $A_{R_{(i,j,k)}}$ with $1 \leq w \leq l$ is homological then every idempotent ideal of the form $I_{\alpha_1,\ldots,\alpha_l}$ is also homological.*

*Proof.* It suffices to observe that $tr_{P(\alpha_w)}(A_{R_{(i,j,k)}}) = P(\alpha_w)^l$ for some $l \in \mathbb{N}$. $\square$

**Lemma 5.** *Every ideal $I_{i,t}$ or $I_{t,j}$ of an algebra $A_{R_{(i,j,k)}}$ is homological.*

*Proof.* In accordance with the previous Lemma we can conclude that if $I_t$ is homological then the result holds. If it is not the case then we consider the following cases:

1. For $I_t$ non homological we can compute $I_{i,t} = tr_{P(i)\oplus P(t)}(A_{R_{(i,j,k)}})$ (see identity (4.4)) since $r \leq i$ then $tr_{P(i)}P(r) = P(i)$ therefore ideal $I_{i,t}$ is projective and idempotent. Thus for Proposition 6, item 2. We conclude that ideal $I_{i,t}$ is homological.

2. We start by computing $I_{t,j}$ as follows:

$$I_{t,j} = tr_{P(t)\oplus P(j)}(A_{R_{(i,j,k)}}) = \underbrace{L_t}_{i-times} \oplus \underbrace{P(t)}_{t-i-times} \oplus \underbrace{P(j)}_{j-t-times} \oplus \underbrace{0}_{n-j-times}$$

$A_{R_{(i,j,k)}}/I_{t,j}$ is given by:

$$A_{R_{(i,j,k)}}/I_{t,j} = \frac{P(1) \oplus P(2) \oplus \cdots \oplus P(i) \oplus \cdots \oplus P(t) \oplus \cdots \oplus P(j) \oplus \cdots \oplus P(n)}{L_t \oplus \cdots \oplus L_t \oplus P(t) \oplus \cdots \oplus P(t) \oplus P(j) \oplus \cdots \oplus P(j) \oplus 0 \oplus \cdots \oplus 0}$$

In order to compute $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}} = (I_{t,j}, A_{R_{(i,j,k)}}/I_{t,j})$ we consider the projective resolution of $L_t$

$$0 \to P(j) \to P(t) \to L_t \to 0.$$

Applying the functor $\mathrm{Hom}_{A_{R_{(i,j,k)}}}(-, A_{R_{(i,j,k)}}/I_{t,j})$ we obtain:

$$0 \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), A_{R_{(i,j,k)}}/I_{t,j}) \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), A_{R_{(i,j,k)}}/I_{t,j}) \to 0$$

Taking into account that

$$\begin{cases} \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), \frac{P(z)}{L_t}) = 0 \quad if \quad 1 \leq z \leq i \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), \frac{P(y)}{P(t)}) = 0 \quad if \quad i+1 \leq y \leq t-1 \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), \frac{P(v)}{P(j)}) = 0 \quad if \quad t+1 \leq v \leq j-1 \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(t), P(u)) = 0 \quad if \quad j+1 \leq u \leq n \end{cases}$$

$$\begin{cases} \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), \frac{P(z)}{L_t}) = 0 \quad if \quad 1 \leq z \leq i \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), \frac{P(y)}{P(t)}) = 0 \quad if \quad i+1 \leq y \leq t-1 \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), \frac{P(v)}{P(j)}) = 0 \quad if \quad t+1 \leq v \leq j-1 \\ \mathrm{Hom}_{A_{R_{(i,j,k)}}}(P(j), P(u)) = 0 \quad if \quad j+1 \leq u \leq n \end{cases}$$

We conclude that $\mathrm{Ext}^n_{A_{R_{(i,j,k)}}}(I_{t,j}, A_{R_{(i,j,k)}}/I_{t,j}) = 0$ and that the idempotent ideal $I_{t,j}$ is an homological ideal as a consequence of Proposition 6, item 3.

□

*Remark* 14. If the non homological ideal $I_t$ has the form $I_{t_1,\ldots,t_n}$ the previous Lemma 5 also holds.

**Lemma 6.** *For $1 \leq h \leq i-1$, $1 \leq l \leq k-1$ and $1 \leq m \leq n-j$ fixed. Every idempotent ideal of the form $I_{z_1,\ldots,z_h,t_1,\ldots,t_l,y_1,\ldots,y_m}$ of an algebra $A_{R_{(i,j,k)}}$, where $z_a \in [1, i-1]$, $t_b \in [i+1, j-1]$, $y_c \in [j+1, n]$ is not homological.*

*Proof.* For $h$, $l$ and $m$ fixed, we start by computing $I_{z_1,\ldots,z_h,t_1,\ldots,t_l,y_1,\ldots,y_m}$,

$$\begin{aligned} I_{z_1,\ldots,z_h,t_1,\ldots,t_l,y_1,\ldots,y_m} &= tr_{P(z_1)\oplus\ldots\oplus P(z_h)\oplus P(t_1)\oplus\ldots\oplus P(t_l)\oplus P(y_1),\oplus\ldots\oplus P(y_m)}(A_{R_{(i,j,k)}}) \\ &= \underbrace{\sum_{a=1}^{h} tr_{P(z_a)}(A_{R_{(i,j,k)}})}_{(1)} + \underbrace{\sum_{b=1}^{l} tr_{P(t_b)}(A_{R_{(i,j,k)}})}_{(2)} + \underbrace{\sum_{c=1}^{m} tr_{P(y_c)}(A_{R_{(i,j,k)}})}_{(3)} \end{aligned} \tag{4.6}$$

The traces (1), (2), (3) can be written as follows:

$$\begin{aligned} \sum_{a=1}^{h} tr_{P(z_a)}(A_{R_{(i,j,k)}}) &= \bigoplus_{a=1}^{h} P(z_a) \oplus 0 \oplus \cdots \oplus 0, \\ \sum_{b=1}^{l} tr_{P(t_b)}(A_{R_{(i,j,k)}}) &= \underbrace{L_{t_1}}_{i-times} \oplus \bigoplus_{b=1}^{l} P(t_b) \oplus \underbrace{0}_{n-i-l-times}, \\ \sum_{c=1}^{m} tr_{P(y_c)}(A_{R_{(i,j,k)}}) &= \underbrace{0}_{i-times} \oplus \underbrace{P(y_1)}_{j-i-times} \oplus \bigoplus_{c=1}^{m} P(y_c) \oplus \underbrace{0}_{n-m-j-times}. \end{aligned} \tag{4.7}$$

Thus, the ideal $I_{z_1,\dots,z_h,t_1,\dots,t_l,y_1,\dots,y_m}$ has the following form:

$$\bigoplus_{a=1}^{h} P(z_a) \oplus \underbrace{L_{t_1}}_{i-h-times} \oplus \bigoplus_{b=1}^{l} P(t_b) \oplus \underbrace{P(y_1)}_{j-i-l-times} \oplus \bigoplus_{c=1}^{m} P(y_c) \oplus \underbrace{0}_{n-m-j-times} \tag{4.8}$$

In accordance with (4.8) we have that $\frac{P(j)}{P(y_1)}$ is a direct summand of the quotient $A_{R_{(i,j,k)}}/I_{z_1,\dots,z_h,t_1,\dots,t_l,y_1,\dots,y_m}$ and $L_{t_1}$ has the following projective resolution:

$$0 \to P(j) \to P(t_1) \to L_{t_1} \to 0. \tag{4.9}$$

Applying the functor $\mathrm{Hom}_{A_{R_{(i,j,k)}}}\left(-, \frac{P(j)}{P(y_1)}\right)$ to the resolution (4.9) we obtain the following exact sequence

$$0 \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}\left(P(t_1), \frac{P(j)}{P(y_1)}\right) \to \mathrm{Hom}_{A_{R_{(i,j,k)}}}\left(P(j), \frac{P(j)}{P(y_1)}\right) \to 0$$

Then $\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}\left(L_t, \frac{P(j)}{P(y_1)}\right) \cong k$ and

$$\mathrm{Ext}^1_{A_{R_{(i,j,k)}}}(I_{z_1,\dots,z_h,t_1,\dots,t_l,y_1,\dots,y_m}, A_{R_{(i,j,k)}}/I_{z_1,\dots,z_h,t_1,\dots,t_l,y_1,\dots,y_m}) \neq 0$$

by Proposition 6, item 3, we conclude that the idempotent ideal $I_{z_1,\dots,z_h,t_1,\dots,t_l,y_1,\dots,y_m}$ is not an homological ideal. □

**Lemma 7.** *For $1 \le h \le i-1$, $1 \le l \le k-1$ and $1 \le m \le n-j$ fixed. The idempotent ideals $I_{z_1,\dots,z_h,t_1,\dots,t_l}$ and $I_{t_1,\dots,t_l,y_1,\dots,y_m}$ of an algebra $A_{R_{(i,j,k)}}$ where $z_a \in [1, i-1]$, $t_b \in [i+1, j-1]$, $y_c \in [j+1, n]$ are not homological.*

*Proof.* It is enough to consider in (4.6) the trace $\sum_{a=1}^{h} tr_{P(z_a)}(A_{R_{(i,j,k)}}) = 0$ or the trace $\sum_{c=1}^{m} tr_{P(y_c)}(A_{R_{(i,j,k)}}) = 0$. □

### 4.2.1 On the number of homological ideals associated to some Nakayama algebras

The following results allow us to compute the number of homological and non homological ideals in a bounded algebra $A_{R_{(i,j,k)}}$ by using the integer specialization $e$ of the Brauer configuration $\Gamma_n$ introduced in Example 10.

**Theorem 29.** *For $n \ge 4$ fixed and $2 \le k \le n-1$ the number $|\mathbb{NHI}_n^k|$ of non homological ideals of an algebra $A_{R_{(i,j,k)}}$ is given by the identity $|\mathbb{NHI}_n^k| = w^e(U_k)$.*

*Proof.* We note that according to Lemmas 2 and 3 there are $2^{k-1} - 1$ non homological ideals associated only to the vertices inside the relation $R_{(i,j,k)}$, by Lemma 6 there are additional $2^{n-k-1}$ non homological ideals arising from the combination of vertices which are inside and outside of the relation. The theorem follows taking into account the product rule and Example 10. □

**Corollary 25.** *For $n \geq 4$ fixed and $2 \leq k \leq n-1$ the number of homological ideals $|\mathbb{HI}^k_n|$ of an algebra $A_{R_{(i,j,k)}}$ is given by the identity $|\mathbb{HI}^k_n| = 2^n - w^e(U_k) = 3 \cdot 2^{n-2} + 2^{n-k-1}$.*

*Proof.* Since there are $2^n$ idempotent ideals in $A_{R_{(i,j,k)}}$ then the result holds as a consequence of Theorem 29. □

The formula obtained in Theorem 29 induces the following triangle:

**Non homological triangle** $\mathbb{NHIT}$

| $n/k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | - | - | - | - | - | - | - |
| 4 | 2 | 3 | - | - | - | - | - | - |
| 5 | 4 | 6 | 7 | - | - | - | - | - |
| 6 | 8 | 12 | 14 | 15 | - | - | - | - |
| 7 | 16 | 24 | 28 | 30 | 31 | - | - | - |
| 8 | 32 | 48 | 56 | 60 | 62 | 63 | - | - |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Entries $|\mathbb{NHI}^k_n|$ of triangle $\mathbb{NHIT}$ can be calculated inductively as follows: we start by defining $|\mathbb{NHI}^2_n| = 2^{n-3}$ for all $n \geq 3$. Now, we assume that $|\mathbb{NHI}^k_n| = 0$ with $k \leq 1$ and for the sake of clarity we denote the specialization under $e$ of a word $w(U_k)$ of the polygon $U_k$ in the Brauer configuration $\Gamma_n$ as $w^e(U^n_k)$ (see Example 10). Then, for $k \geq 3$:

$$w^e(U_k) = w^e(U^n_k) = (w^e(U^n_{k-1}) + w^e(U^{n-1}_{k-1})) - w^e(U^{n-1}_{k-2}).$$

or equivalently,

$$|\mathbb{NHI}^k_n| = (|\mathbb{NHI}^{k-1}_n| + |\mathbb{NHI}^{k-1}_{n-1}|) - |\mathbb{NHI}^{k-2}_{n-1}|.$$

These arguments prove the following proposition.

**Proposition 7.** *$M(\Gamma^e_n)$ equals the sum of the elements in the $n$-th row of the non homological triangle $\mathbb{NHIT}$ (see Example (10)).*

*Remark* 15. The integer sequence generated by $M(\Gamma^e_n) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1} =$ $\{1, 5, 17, 49, 129, 321, 769, 1793, 4097, 9217, \ldots\}$ is encoded A000337 in the OEIS. Elements of the sequence A000337 also correspond to the sums of the elements of the rows of the Reinhard Zumkeller triangle.

*Remark* 16. The sum of entries in the diagonals of the non homological triangle is the sequence A274868 in the OEIS, and it is related with the number of set partitions of $[n]$ into exactly four blocks such that all odd elements are in blocks with an odd index, whereas all even elements are in blocks with an even index.

Similarly, for the homological ideals Corollary 25 induces the following triangle:

**Homological triangle $\mathbb{HIT}$.**

| $n/k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 7 | - | - | - | - | - | - | - |
| 4 | 14 | 13 | - | - | - | - | - | - |
| 5 | 28 | 26 | 25 | - | - | - | - | - |
| 6 | 56 | 52 | 50 | 49 | - | - | - | - |
| 7 | 112 | 104 | 100 | 98 | 97 | - | - | - |
| 8 | 224 | 208 | 200 | 196 | 194 | 193 | - | - |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

The elements of the homological triangle are closely related with the research of categorification of integer sequences. Particularly, these numbers deal with the work of Ringel and Fahr regarding categorification of Fibonacci numbers.

### 4.2.2 Categorification of the Fibonacci numbers

In this section, we give some relationships between the number of homological ideals of an algebra $A_{R_{(i,j,k)}}$ and the partition formula given by Ringel and Fahr for even-index Fibonacci numbers in [19].

We recall the array (2) mentioned in the introduction of this document, where $0 \leq j \leq i \leq n$, and

$$\begin{aligned} d_{i,i} &= 1, \quad \text{for all } i \geq 0,\\ d_{i,j} &= 0 \quad \text{for all } j > i\\ d_{2k+i,i-1} &= 0, \quad \text{if } i \geq 1,\ k \geq 0,\\ d_{2k,0} &= 3d_{2k-1,1} - d_{2(k-1),0}, \quad k \geq 1,\\ d_{i+1,j-1} &= 2d_{i,j} + d_{i,j-2} - d_{i-1,j-1}, \quad i,j \geq 2. \end{aligned} \tag{4.10}$$

Note that to each entry $d_{i,i-j}$ it is possible to assign a weight $h_{i,i-j}$ by using the numbers in the homological triangle $\mathbb{HIT}$ as follows:

$$h_{i,i-j} = \begin{cases} |\mathbb{HI}^k_{2s+2}| - 2^{2\cdot s-k+1}, & \text{if } j \text{ is even}, i \text{ is odd and } i \neq j+1,\\ |\mathbb{HI}^k_{2s+1}| - 2^{2\cdot s-k}, & \text{if } j \text{ is even}, i \text{ is even},\\ 3, & \text{if } i \text{ odd}, j \text{ even and } i = j+1,\\ 1, & \text{if } i = j = 2h \text{ for some } h \geq 0,\\ 0, & \text{if } j \text{ is odd}, \ i \neq j. \end{cases}$$

Where $s = \lfloor \frac{i-j}{2} \rfloor$ and $\lfloor x \rfloor$ is the greatest integer number less than $x$. If we consider the multiplication of the entry $d_{i,i-j}$ with its corresponding weight $h_{i,i-j}$, we can rewrite the identity (3) as follows:

**Theorem 30.**

$$
\begin{aligned}
\sum_{j=0}^{2t}(h_{2t,2t-j})(d_{2t,2t-j}) &= \sum_{r even}|T_r|\cdot a_t[r], \quad t\geq 0\\
\sum_{j=0}^{2t-1}(h_{2t-1,2t-1-j})(d_{2t-1,2t-1-j}) &= \sum_{r odd}|T_r|\cdot a_t[r], \quad t\geq 1.
\end{aligned}
\tag{4.11}
$$

# CHAPTER 5

## Categorification of magic squares

In this chapter, we use Brauer configuration algebras to give a categorification to magic squares of order $n$ for $3 \leq n \leq 9$.

### 5.1 Categorification of magic squares

In this section, magic squares are categorified following the ideas of Ringel and Fahr [21], who established that all entries in the Fibonacci triangle are categorified by Fibonacci modules, provided that such entries give the Jordan-Hölder multiplicities of these modules. In our case magic squares will be categorified by identifying information arising from the combinatorial properties of some Brauer configurations with the entries of magic squares of different orders.

A *magic square* of order $n$ is a square array $M$ such that to each entry $m_{i,j}$ it is assigned an integer number $n_{i,j}$ with $1 \leq n_{i,j} \leq n^2$ and $n_{i,j} \neq n_{i',j'}$ if $m_{i,j} \neq m_{i',j'}$. In these arrays the *magic sum* $S = \frac{n(n^2+1)}{2}$ of the entries along rows, columns and diagonal is the same. The Lo Shu array is a magic square of order 3, actually this array is the only (up to permutations and reflections) of such order, and perhaps the most known magic squares of order 4 are the Dürer magic square and the Jaina magic square. The OEIS encodes as A033812 , A080992 and A126710 the entries of the Lo Shu, Dürer and Jaina magic squares respectively (see tables 5.1, 5.2 and 5.3) [67, 68].

| | | | |
|---|---|---|---|
| 4 | 9 | 2 | =15 |
| 3 | 5 | 7 | =15 |
| 8 | 1 | 6 | =15 |
| 15 | 15 | 15 | $=\frac{3(9+1)}{2}$ |

Table 5.1. Lo Shu magic square.

| | | | | |
|---|---|---|---|---|
| 16 | 3 | 2 | 13 | =34 |
| 5 | 10 | 11 | 8 | =34 |
| 9 | 6 | 7 | 12 | =34 |
| 4 | 15 | 14 | 1 | =34 |
| 34 | 34 | 34 | 34 | $=\frac{4(16+1)}{2}$ |

Table 5.2. Dürer magic square.

| | | | | |
|---|---|---|---|---|
| 7 | 12 | 1 | 14 | =34 |
| 2 | 13 | 8 | 11 | =34 |
| 16 | 3 | 10 | 5 | =34 |
| 9 | 6 | 15 | 4 | =34 |
| 34 | 34 | 34 | 34 | $=\frac{4(16+1)}{2}$ |

Table 5.3. Jaina magic square.

Up to date, there are several unsolved problems regarding magic squares, which have encouraged many mathematicians both amateurs and professionals to still investigating many of its properties. For instance, determining the number of magic squares of an arbitrary order is an unsolved problem in number theory, besides, Guy encodes as A6 and D15 some open problems in number theory devoted to this subject [34].

The sequence A006052 lists the number of magic and semi-magic squares of some orders counted up to rotations and reflections and it is estimated that the number of magic squares of order 10 is $6{,}5\times10^{2056}$ [69]. Recently, Rippatti proposed that the number of semi-magic squares of order 6 is exactly 94,590,660,245,399,996,601,600 [55]. And also Ahmed, Stanley et al. have used algebraic combinatorics to tackle all these problems [2, 72, 73]. On that sense, one of the purposes of this work is looking for a categorification of magic squares by using Brauer configurations, which after a suitable specialization allow to obtain magic squares of different orders.

The following result gives a realization of the Lo Shu array (see Table 5.1) based on the structure of a suitable path algebra, which we have named *The Lo Shu algebra.*

**Theorem 31.** *The specialization $e(b_i) = i$ associated to the algebra $\mathcal{B} = kQ_{lsh}/\mathrm{I}$ generates the Lo Shu square in the sense of (1.6). In this case, $Q_{lsh}$ is the quiver shown in Figure 5.1 and the ideal $I$ is defined in such a way that $\mathrm{I} = \langle C \subset Q_{lsh} : C \text{ is an oriented cycle}\rangle$.*

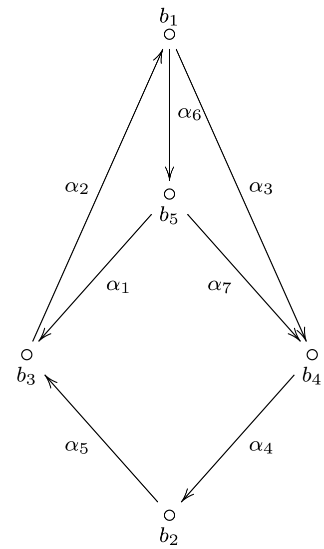



FIGURE 5.1. The Lo Shu quiver $Q_{lsh}$.

*The Lo Shu square is built (in the sense of Remark 4) by a specialization of the following Brauer configuration* $\Gamma = (\Gamma_0, \Gamma_1, \mathcal{O}, \mu)$*;*

$$\Gamma_0 = \{b_1, b_2, b_3, b_4, b_5\},$$
$$\Gamma_1 = \{U_1, U_2, U_3, U_4, U_5, U_6.\}$$

*The successor sequence at each vertex* $b_i \in \Gamma_0$ *is defined as follows:*

$$\begin{aligned}
&\text{At vertex } b_1;\, U_2^{(1)} \leq U_6^{(2)},\\
&\text{At vertex } b_2;\, U_2^{(2)} \leq U_5^{(2)} \leq U_6^{(1)},\\
&\text{At vertex } b_3;\, U_2^{(1)} \leq U_5^{(2)},\\
&\text{At vertex } b_4;\, U_2^{(3)} \leq U_5^{(1)} \leq U_6^{(1)},\\
&\text{At vertex } b_5;\, U_2^{(3)} \leq U_1^{(1)} \leq U_3^{(2)} \leq U_4^{(1)} \leq U_5^{(2)} \leq U_6^{(2)},\\
&\qquad \mu(b_i) = 1, \quad 1 \leq i \leq 5.
\end{aligned}$$

*The ideal* $I$ *of the corresponding Brauer configuration algebra* $\Lambda_\Gamma$ *is generated by the following relations (see Figure 5.2), for which it is assumed the following notation for the* *special cycles:*

$$\begin{aligned}
&C_{b_1}^{U_2,1} = a_1^{b_1}a_2^{b_1}a_3^{b_1}, \quad C_{b_1}^{U_6,1} = a_2^{b_1}a_3^{b_1}a_1^{b_1}, \quad C_{b_1}^{U_6,2} = a_3^{b_1}a_1^{b_1}a_2^{b_1},\\
&C_{b_2}^{U_2,1} = a_1^{b_2}a_2^{b_2}a_3^{b_2}a_4^{b_2}a_5^{b_2}, \quad C_{b_2}^{U_2,2} = a_2^{b_2}a_3^{b_2}a_4^{b_2}a_5^{b_2}a_1^{b_2}, \quad C_{b_2}^{U_5,1} = a_3^{b_2}a_4^{b_2}a_5^{b_2}a_1^{b_2}a_2^{b_2},\\
&C_{b_2}^{U_5,2} = a_4^{b_2}a_5^{b_2}a_1^{b_2}a_2^{b_2}a_3^{b_2}, \quad C_{b_2}^{U_6,1} = a_5^{b_2}a_1^{b_2}a_2^{b_2}a_3^{b_2}a_4^{b_2},\\
&C_{b_3}^{U_2,1} = a_1^{b_3}a_2^{b_3}a_3^{b_3}, \quad C_{b_3}^{U_5,1} = a_2^{b_3}a_3^{b_3}a_1^{b_3}, \quad C_{1}^{U_5,2} = a_3^{b_3}a_1^{b_3}a_2^{b_3}\\
&C_{b_4}^{U_2,1} = a_1^{b_4}a_2^{b_4}a_3^{b_4}a_4^{b_4}a_5^{b_4}, \quad C_{b_4}^{U_2,2} = a_2^{b_4}a_3^{b_4}a_4^{b_4}a_5^{b_4}a_1^{b_4}, \quad C_{b_4}^{U_2,3} = a_3^{b_4}a_4^{b_4}a_5^{b_4}a_1^{b_4}a_2^{b_4},\\
&C_{b_4}^{U_5,1} = a_4^{b_4}a_5^{b_4}a_1^{b_4}a_2^{b_4}a_3^{b_4}, \quad C_{b_4}^{U_6,1} = a_5^{b_4}a_1^{b_4}a_2^{b_4}a_3^{b_4}a_4^{b_4},\\
&C_{b_5}^{U_2,1} = a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5},\\
&C_{b_5}^{U_2,2} = a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5},\\
&C_{b_5}^{U_2,3} = a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5},\\
&C_{b_5}^{U_1,1} = a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5},\\
&C_{b_5}^{U_1,2} = a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5},\\
&C_{b_5}^{U_3,1} = a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5},\\
&C_{b_5}^{U_3,2} = a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5},\\
&C_{b_5}^{U_4,1} = a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5},\\
&C_{b_5}^{U_4,2} = a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5},\\
&C_{b_5}^{U_5,1} = a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5},\\
&C_{b_5}^{U_5,2} = a_{11}^{b_5}a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5},\\
&C_{b_5}^{U_6,1} = a_{12}^{b_5}a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5},\\
&C_{b_5}^{U_6,2} = a_{13}^{b_5}a_1^{b_5}a_2^{b_5}a_3^{b_5}a_4^{b_5}a_5^{b_5}a_6^{b_5}a_7^{b_5}a_8^{b_5}a_9^{b_5}a_{10}^{b_5}a_{11}^{b_5}a_{12}^{b_5}.
\end{aligned} \tag{5.1}$$

1. *$a_i^h a_r^s$, if $h \neq s$, for all possible values of $i$ and $r$,*
2. *$C_j^{U_1,i} - C_l^{U_1,k}$, for all possible values of $i, j, k$ and $l$,*
3. *$C_j^{U_2,i} - C_l^{U_2,k}$, for all possible values of $i, j, k$ and $l$,*
4. *$C_j^{U_3,i} - C_l^{U_3,k}$, for all possible values of $i, j, k$ and $l$,*
5. *$C_j^{U_4,i} - C_l^{U_4,k}$, for all possible values of $i, j, k$ and $l$,*
6. *$C_j^{U_5,i} - C_l^{U_5,k}$, for all possible values of $i, j, k$ and $l$,*
7. *$C_j^{U_6,i} - C_l^{U_6,k}$, for all possible values of $i, j, k$ and $l$,*
8. *$C_i^{U_1,j}a$ $(C_i^{U_2,j}a')$, with $a$ $(a')$ being the first arrow of $C_i^{U_1,j}$ $(C_i^{U_2,j})$ for all $i, j$,*
9. *$C_i^{U_3,j}a$ $(C_i^{U_4,j}a')$, with $a$ $(a')$ being the first arrow of $C_i^{U_3,j}$ $(C_i^{U_4,j})$ for all $i, j$,*
10. *$C_i^{U_5,j}a$ $(C_i^{U_6,j}a')$, with $a$ $(a')$ being the first arrow of $C_i^{U_5,j}$ $(C_i^{U_6,j})$ for all $i, j$.*

*The following is the quiver associated to the Brauer configuration $\Gamma$, the colors means the different special cycles associated to each vertex as follows: for vertex $b_1$ red, for vertex $b_2$ green, for vertex $b_3$ magenta, for vertex $b_4$ cyan and for vertex $b_5$ blue.*

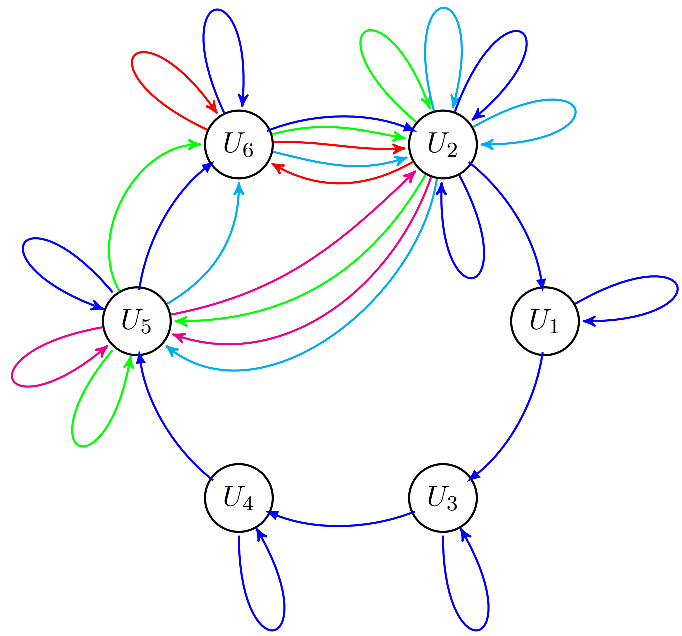



FIGURE 5.2. Quiver associated to the Brauer configuration $\Gamma$.

*Proof.* Since the path $P_5 = \alpha_1\alpha_2\alpha_3\alpha_4 \in \mathcal{B}$ visits all vertices in the form $(a_5, a_3, a_1, a_4, a_2)$ then the specialization $b_i = i$ defines the integer vector $v = (5, 3, 1, 4, 2)$, which is a generator of the Lo Shu array in the sense of Remark 3 and identity (1.6). Note that, in this case each partition $\lambda \in \mathcal{F}(15, 3)$ has three distinct parts $\in \{1, 2, \ldots, 9\}$. Thus, Lo Shu sums are generated by bracketing coordinates of vector $v$ as follows:

$$\begin{aligned}
U_1 &= 5 + 3 + (1 + 4 + 2) = 5 + 3 + 7,\\
U_2 &= 5 + (3 + 1) + 6 = 5 + 4 + 6,\\
U_3 &= 5 + (3 + 1 + 4) + 2 = 5 + 8 + 2,\\
U_4 &= (5 + 3) + 1 + 6 = 8 + 1 + 6,\\
U_3 &= (5 + 3) + (1 + 4) + 2 = 8 + 5 + 2,\\
U_5 &= (5 + 3 + 1) + 4 + 2 = 9 + 4 + 2.
\end{aligned} \tag{5.2}$$

The Lo Shu array can be built in the following way by adding vertices, which are visited by the corresponding path:

| | | |
|---|---|---|
| $\alpha_2$ | $\alpha_1\alpha_2$ | $e_{b_2}$ |
| $e_{b_3}$ | $e_{b_5} = \alpha_3$ | $\alpha_3\alpha_4$ |
| $\alpha_2\alpha_3$ | $e_{b_1}$ | $\alpha_4$ |

In order to construct the Brauer configuration $\Gamma$ the same procedure can be done with paths;

$$\begin{aligned} P_1 &= \alpha_6\alpha_7\alpha_4\alpha_5, \\ P_2 &= \alpha_5\alpha_2\alpha_6\alpha_7, \\ P_3 &= \alpha_2\alpha_6\alpha_7\alpha_4, \\ P_4 &= \alpha_4\alpha_5\alpha_2\alpha_6, \\ P_5' &= \alpha_7\alpha_4\alpha_5\alpha_2. \end{aligned}$$

Each of them visits all vertices of the quiver $Q_{lsh}$ as follows:

$$\begin{aligned} P_1 &= (1,5,4,2,3), \\ P_2 &= (2,3,1,5,4), \\ P_3 &= (3,1,5,4,2), \\ P_4 &= (4,2,3,1,5), \\ P_5' &= (5,4,2,3,1). \end{aligned}$$

Among all the Lo Shu sums $P_5'$ also generates $U_6 = 5+9+1$. And $U_1, \ldots, U_6$ are all the sums generated by the different paths (regardless permutations of the summands).

In this case, vertices $b_i \in \Gamma_0$ are given by ordered paths $P_i$ and polygons are the sums generated by such paths, for instance, polygons $U_1$, $U_2$, $U_3$, $U_4$ and $U_5$ contain all of them the vertex $b_5$ (see identities (5.2)). Actually, the following identifications have place:

$$\begin{aligned} b_i &= P_i, \quad 1 \leq i \leq 4, \\ b_5 &= P_5 + P_5'. \end{aligned}$$

Polygons $U_i$ are ordered according to the relation $\preceq$ defined in Remark 3, in fact, it holds that

$$U_2 \prec U_1 \prec U_3 \prec U_4 \prec U_5 \prec U_6.$$

For instance in $b_1 = P_1$, the successor sequence has a relation of the form $U_2 \preceq U_6$. Then the successor sequence at each vertex $b_i$ is obtained by ordering polygons via relation $\preceq$. And this is enough to define the Brauer configuration $\Gamma$ where vertices are defined by specializations $b_i = P_i$ and polygons consists of compositions of 15 generated by such paths.

□

The following results regard categorification of magic squares.

**Theorem 32.** *Any magic square of order $n$ for $3 \leq n \leq 9$ is built by a specialization of the Brauer configuration $\Delta = (\Delta_0, \Delta_1, \mathcal{O}, \mu)$ where:*

$$\begin{aligned} \Delta_0 &= \{a, b_i, d_i, e_i, g_j, h_j, x, y, z, w, v \mid 1 \leq i \leq 4, 0 \leq j \leq 2\}, \\ \Delta_1 &= \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9\}. \end{aligned} \tag{5.3}$$

*At each vertex the successor sequence has the form;*

$$
\begin{aligned}
&\text{At vertex } x;\, V_1^{(4)} \leq V_2^{(3)} \leq V_3^{(4)} \leq V_4^{(5)} \leq V_5^{(6)} \leq V_6^{(4)} \leq V_7^{(4)} \leq V_8^{(5)} \leq V_9^{(4)},\\
&\text{At vertex } y;\, V_1^{(2)} \leq V_2^{(2)} \leq V_3^{(2)} \leq V_4^{(1)} \leq V_5^{(2)} \leq V_6^{(2)} \leq V_7^{(2)} \leq V_8^{(2)} \leq V_9^{(2)},\\
&\text{At vertex } z;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } w;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } v;\, V_1^{(2)} \leq V_2^{(2)} \leq V_3^{(2)} \leq V_4^{(2)} \leq V_5^{(2)} \leq V_6^{(2)} \leq V_7^{(2)} \leq V_8^{(2)} \leq V_9^{(2)},\\
&\text{At vertex } a;\, V_1^{(8)} \leq V_2^{(7)} \leq V_3^{(7)} \leq V_4^{(7)} \leq V_5^{(10)} \leq V_6^{(8)} \leq V_7^{(7)} \leq V_8^{(9)} \leq V_9^{(8)},\\
&\text{At vertex } b_1;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_8^{(1)},\\
&\text{At vertex } b_2;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } b_3;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_8^{(1)},\\
&\text{At vertex } b_4;\, V_1^{(1)} \leq V_4^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_8^{(1)},\\
&\text{At vertex } d_1;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } d_2;\, V_1^{(1)} \leq V_4^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } d_3;\, V_5^{(1)},\\
&\text{At vertex } d_4;\, V_5^{(1)},\\
&\text{At vertex } e_1;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_7^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } e_2;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_6^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } e_3;\, V_1^{(1)} \leq V_3^{(1)} \leq V_4^{(1)} \leq V_7^{(1)} \leq V_8^{(1)},\\
&\text{At vertex } e_4;\, V_5^{(1)},\\
&\text{At vertex } g_0;\, V_5^{(1)},\\
&\text{At vertex } g_1;\, V_4^{(1)} \leq V_8^{(1)},\\
&\text{At vertex } g_2;\, V_1^{(1)} \leq V_2^{(1)} \leq V_3^{(1)} \leq V_6^{(1)} \leq V_7^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } h_0;\, V_1^{(1)} \leq V_2^{(1)} \leq V_5^{(1)} \leq V_6^{(1)} \leq V_8^{(1)} \leq V_9^{(1)},\\
&\text{At vertex } h_1;\, V_4^{(1)},\\
&\text{At vertex } h_2;\, V_3^{(1)} \leq V_7^{(1)},\\
&\qquad \mu(\alpha) = 1, \quad \textit{for any vertex}\, \alpha \in \Delta_0,
\end{aligned}
$$

*Actually, such integer arrays are given by specializations of elements of a suitable path algebra $kQ$ with $Q$ induced by polygons $V_1, \dots, V_9$.*

*Proof.* Brauer configuration (5.3) can be specialized by defining a set a words whose letters belong to sets $A = \{x, y, z, w, v\}$ and $B = \{a, b_i, d_i, e_i, g_m, h_m\}$. Then polygons are set of words of the form $ABABABABA\dots$, where elements of the set $B$ connect letters of the set $A$ in such a way that to each element of $A$ follows an element of $B$ and that any word should be ended with an element of $A$. We follow the syntactic rules $(i) - (viii)$ in order to connect letters $x, y, z, w$ and $v$.

$(i)$ Letters $x$ are always connected in the form $xaxaxa\dots$. In this case a linear numbering is defined on the set of letters $x$, assign a 1 to the first $x$ appearing in a word, 2 to the second $x$ and so on.

$(ii)$ $yb_ix$ and $xe_ky$ are the only ways to connect letters $x$ and $y$, where $b_i$ and $e_k$ denotes which copy of a letter $x$ is connected with $y$ via symbols $b$ and $d$ respectively.

$(iii)$ $yay$ is the only way to connect letters $y$.

$(iv)$ $waz$, $zay$, $zb_ix$, $xe_kz$ and $zd_lx$ are the ways to connect a letter $z$ with other letters, $d_l$ is defined as $b_i$ and $e_k$, $1 \leq i, j, k, l \leq 4$.

$(v)$ $wb_jx$, $wd_lx$, and $vaw$ are the other ways to connect letter $w$ with other letters.

$(vi)$ $vb_jx$, $vd_lx$, and $vav$ are the other ways to connect letter $v$.

$(vii)$ Since by definition a letter $v$ appears in the first place in a word containing it then $g_m$ is a shift function specifying which letter of a word $W$ containing letters only in the set $\{v, w, z, y\}$ connect the first letter of a sequence $S = xaxa\dots$, in this case, $g_m(S)$ means that the first letter $x$ in $S$ must be connected with the $m$th letter of $W$, with $0 \leq m \leq 2$, in particular, $g_0(S)$ means that the first letter of $S$ is connected with the first letter of $W$.

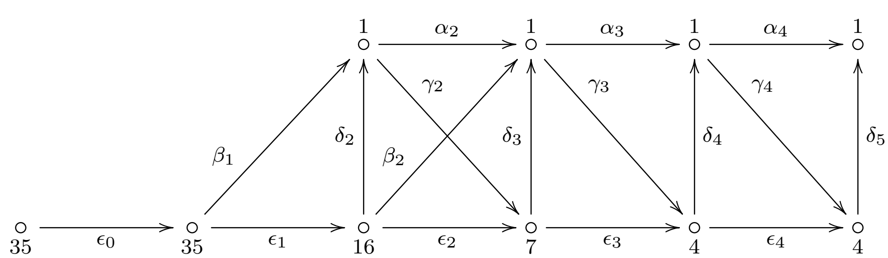
$(viii)$ Symbols $h_m$, $0 \leq m \leq 2$ are connection-maps, in this case, given a word $X$ we have that $h_0(X)$ means that $X$ must contain a word of the form $zayay$, whereas $h_1(X)$ means that $X$ does not contain $zay$ as subword. Finally, $h_2(X)$ means that there is not a sequence of the form $yay$ in $X$.

If we define an specialization of the form:

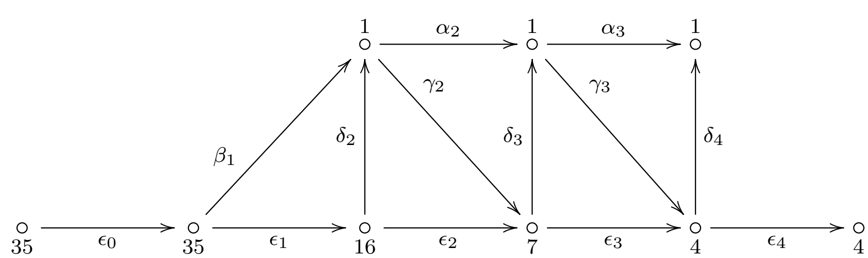
$$x = 1, y = 4, z = 7, w = 16, v = 35, a = \rightarrow, b = \uparrow, d = \nearrow, e = \searrow . \tag{5.4}$$

then polygons $V_1 - V_9 \in \Delta_1$ defined by configuration (5.3) have the following shapes by applying rules $(i) - (viii)$:

$V_1 = V_{1,9} =$




$V_2 = V_{2,9} =$



$$V_3 = V_{3,9} = \begin{tikzcd}
 & & 1 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] \arrow[dr, "\gamma_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] & 7 \arrow[r, "\epsilon_3"'] \arrow[u, "\delta_3"] & 4 \arrow[u, "\delta_4"] & 4
\end{tikzcd}$$

$$V_4 = V_{4,9} = \begin{tikzcd}
 & 1 \arrow[r, "\alpha_1"] & 1 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[u, "\delta_1"] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] & 7 \arrow[u, "\delta_3"] & 4 \arrow[u, "\delta_4"] &
\end{tikzcd}$$

$$V_5 = V_{5,9} = \begin{tikzcd}
1 \arrow[r, "\alpha_0"] & 1 \arrow[r, "\alpha_1"] & 1 \arrow[r, "\alpha_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] \arrow[u, "\delta_0"] & 35 \arrow[r, "\epsilon_1"'] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] \arrow[ur, "\beta_2"] & 7 \arrow[r, "\epsilon_3"'] \arrow[u, "\delta_3"] & 4 \arrow[r, "\epsilon_4"'] & 4
\end{tikzcd}$$

$$V_6 = V_{6,9} = \begin{tikzcd}
 & & 1 \arrow[r, "\alpha_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] \arrow[ur, "\beta_2"] & 7 \arrow[r, "\epsilon_3"'] & 4 \arrow[r, "\epsilon_4"'] \arrow[u, "\delta_4"] & 4 \arrow[u, "\delta_5"]
\end{tikzcd}$$

$$V_7 = V_{7,9} = \begin{tikzcd}
 & & 1 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & 1 \arrow[r, "\alpha_3"] & 1 \arrow[r, "\alpha_4"] \arrow[dr, "\gamma_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[ur, "\beta_2"] & 7 \arrow[r, "\epsilon_3"'] & 4 \arrow[u, "\delta_4"] & 4 \arrow[u, "\delta_5"]
\end{tikzcd}$$

$$V_8 = V_{8,9} = \begin{tikzcd}
 & 1 \arrow[r, "\alpha_1"] & 1 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[u, "\delta_1"] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] & 7 \arrow[r, "\epsilon_3"'] \arrow[u, "\delta_3"] & 4 \arrow[r, "\epsilon_4"'] \arrow[u, "\delta_4"'] & 4
\end{tikzcd}$$

$$V_9 = V_{9,9} = \begin{tikzcd}
 & & 1 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & 1 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & 1 \arrow[r, "\alpha_4"] & 1 \\
35 \arrow[r, "\epsilon_0"'] & 35 \arrow[r, "\epsilon_1"'] \arrow[ur, "\beta_1"] & 16 \arrow[r, "\epsilon_2"'] \arrow[ur, "\beta_2"] & 7 \arrow[r, "\epsilon_3"'] \arrow[u, "\delta_3"] & 4 \arrow[r, "\epsilon_4"'] & 4
\end{tikzcd}$$

The following quiver $Q$ is obtained by putting all polygons together:

$$Q = \begin{tikzcd}
a_0 \arrow[r, "\alpha_0"] & a_1 \arrow[r, "\alpha_1"] & a_2 \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & a_3 \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & a_4 \arrow[r, "\alpha_4"] \arrow[dr, "\gamma_4"] & a_5 \\
a'_0 \arrow[r, "\epsilon_0"'] \arrow[u, "\delta_0"] & a'_1 \arrow[r, "\epsilon_1"'] \arrow[u, "\delta_1"] \arrow[ur, "\beta_1"] & a'_2 \arrow[r, "\epsilon_2"'] \arrow[u, "\delta_2"] \arrow[ur, "\beta_2"] & a'_3 \arrow[r, "\epsilon_3"'] \arrow[u, "\delta_3"] & a'_4 \arrow[r, "\epsilon_4"'] \arrow[u, "\delta_4"'] & a'_5 \arrow[u, "\delta_5"']
\end{tikzcd} \tag{5.5}$$

Therefore, each polygon defines a magic sum of order 9 by using corresponding basic elements as follows:

$$\begin{aligned}
V_1 &= \{\varepsilon_2\delta_3\alpha_3\alpha_4, \beta_1, \beta_1\alpha_2, \beta_1\gamma_2\varepsilon_3, \varepsilon_1\beta_2\gamma_3\delta_4, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\delta_5, \varepsilon_3\varepsilon_4\delta_5, \varepsilon_0\beta_1\alpha_2\gamma_3\delta_4, \alpha_2\gamma_3\},\\
V_2 &= \{\varepsilon_0\beta_1\gamma_2, \varepsilon_1\delta_2\gamma_2\delta_3\gamma_3\varepsilon_4, \varepsilon_1\varepsilon_2, \beta_1\gamma_2\delta_3\gamma_3, \beta_1\alpha_2\alpha_3, \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_3\delta_4, \delta_2, e_{a'_3}\},\\
V_3 &= \{\varepsilon_0\beta_1\gamma_2\delta_3, \varepsilon_1\delta_2\gamma_2\delta_3\gamma_3\delta_4\gamma_4, \beta_1\gamma_2\delta_3\gamma_3\delta_4, \beta_1\alpha_2\alpha_3\alpha_4, \varepsilon_2\varepsilon_3\delta_4\alpha_4, \varepsilon_1\varepsilon_2\delta_3, \delta_2\alpha_2\alpha_3, \delta_2\alpha_2, \delta_3\},\\
V_4 &= \{\varepsilon_0\beta_1\gamma_2\delta_3\alpha_3, \varepsilon_0, \varepsilon_1\varepsilon_2\delta_3\alpha_3, \beta_1\gamma_2\delta_3\gamma_3\delta_4\alpha_4, \delta_1\alpha_1\alpha_2\alpha_3\alpha_4, \varepsilon_2\delta_3\gamma_3\delta_4\alpha_4, \delta_2\alpha_2\alpha_3\alpha_4, \delta_3\alpha_3\alpha_4, \delta_3\alpha_3\},\\
V_5 &= \{\varepsilon_0\beta_1\gamma_2\delta_3\alpha_3\alpha_4, \varepsilon_1\varepsilon_2\delta_3\alpha_3\alpha_4, \varepsilon_0\beta_1, \varepsilon_1, \delta_0\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4, \varepsilon_2\varepsilon_3\varepsilon_4, \beta_2\gamma_3, \varepsilon_3, e_{a_5}\},\\
V_6 &= \{\varepsilon_0\beta_1\alpha_2\alpha_3, \varepsilon_0\beta_1\alpha_2, \varepsilon_1\varepsilon_2\varepsilon_3, \varepsilon_1\delta_2, \beta_1\alpha_2\gamma_3\delta_4, \varepsilon_2\varepsilon_3\varepsilon_4\delta_5, \beta_2\gamma_3\delta_4, \varepsilon_3\delta_4, \alpha_4\},\\
V_7 &= \{\varepsilon_0\beta_1\alpha_2\alpha_3\alpha_4, \varepsilon_1\varepsilon_2\varepsilon_3\delta_4\alpha_4, \varepsilon_1\varepsilon_2\varepsilon_3\delta_4, \varepsilon_1\beta_2\alpha_3, \beta_1\gamma_2, \varepsilon_2\varepsilon_3\delta_4\gamma_4\delta_5, \varepsilon_2, \varepsilon_3\delta_4\alpha_4, \alpha_3\alpha_4\},\\
V_8 &= \{\varepsilon_0\delta_1\alpha_1\alpha_2\alpha_3\alpha_4, \varepsilon_1\delta_2\alpha_2\alpha_3\alpha_4, \varepsilon_1\delta_2\alpha_2\alpha_3, \beta_1\gamma_2\delta_3, \delta_2\gamma_2\delta_3\gamma_3\varepsilon_4\delta_5, \varepsilon_2\delta_3, \delta_3\gamma_3\delta_4\alpha_4, \varepsilon_1\varepsilon_2\delta_3\gamma_3\delta_4\alpha_4, e_{a'_5}\},\\
V_9 &= \{\varepsilon_0\beta_1\alpha_2\gamma_3, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_1\beta_2\gamma_3, \beta_1\gamma_2\delta_3\alpha_3\alpha_4, e_{a'_0}, \varepsilon_2\delta_3\alpha_3, \varepsilon_3\varepsilon_4, \gamma_3, \beta_1\gamma_2\delta_3\alpha_3\}.
\end{aligned} \tag{5.6}$$

Via the specialization (5.4) with $a = b = c = d = e = +$, we obtain the following integer sequences which are columns of a magic square of order 9:

$$
\begin{aligned}
V_1 &= \{37, 6, 47, 16, 57, 26, 67, 36, 77\},\\
V_2 &= \{78, 38, 7, 48, 17, 58, 27, 68, 28\},\\
V_3 &= \{29, 79, 39, 8, 49, 18, 59, 19, 69\},\\
V_4 &= \{70, 30, 80, 40, 9, 50, 10, 60, 20\},\\
V_5 &= \{21, 71, 31, 81, 41, 1, 51, 11, 61\},\\
V_6 &= \{62, 22, 72, 32, 73, 42, 2, 52, 12\},\\
V_7 &= \{13, 63, 23, 64, 33, 74, 43, 3, 53\},\\
V_8 &= \{54, 14, 55, 24, 65, 34, 75, 44, 4\},\\
V_9 &= \{5, 46, 15, 56, 25, 66, 35, 76, 45\}.
\end{aligned} \tag{5.7}
$$

The following are magic sums of order 8 according to the Brauer configuration $\Delta$:

$$
\begin{aligned}
V_{1,8} &= \{\varepsilon_1\varepsilon_2\varepsilon_3\delta_4\alpha_4, \beta_1\gamma_2\delta_3\gamma_3\delta_4, \delta_3, \beta_1\alpha_2\gamma_3, \varepsilon_2\delta_3\gamma_3\varepsilon_4, \delta_1\alpha_1\alpha_2\alpha_3\alpha_4, \delta_2, \gamma_3\varepsilon_4\},\\
V_{2,8} &= \{\varepsilon_1\varepsilon_2, \varepsilon_3\varepsilon_4, \varepsilon_2, \delta_2\gamma_2\delta_3\gamma_3\varepsilon_4\delta_5, \delta_2\gamma_2\delta_3\alpha_3, \beta_1\gamma_2\varepsilon_3, \varepsilon_1\delta_2\alpha_2\alpha_3\alpha_4, \alpha_4\},\\
V_{3,8} &= \{\varepsilon_1\varepsilon_2\delta_3, \delta_3\gamma_3\delta_4\alpha_4, \beta_2\gamma_3\delta_4, e_{a'_1}, \varepsilon_2\varepsilon_3, \beta_1\gamma_2\delta_3\alpha_3\alpha_4, \varepsilon_1\beta_2\alpha_3\alpha_4, \alpha_3\alpha_4\},\\
V_{4,8} &= \{\delta_4, \varepsilon_1\delta_2, \beta_1\gamma_2\delta_3, \delta_2\gamma_2\varepsilon_3\delta_4, \beta_1\alpha_2, \delta_2\alpha_2\alpha_3\alpha_4, \varepsilon_3\delta_4, \varepsilon_1\varepsilon_2\delta_3\alpha_3\alpha_4\},\\
V_{5,8} &= \{\alpha_2\alpha_3\alpha_4, \varepsilon_1\delta_2\alpha_2, \beta_1\gamma_2\delta_3\alpha_3, \varepsilon_2\varepsilon_3\delta_4, \beta_1, \beta_2\gamma_3, \gamma_2\delta_3\gamma_3, \varepsilon_1\varepsilon_2\delta_3\alpha_3\},\\
V_{6,8} &= \{\varepsilon_1\varepsilon_2\varepsilon_3, \varepsilon_3, \delta_2\alpha_2\alpha_3, \beta_1\alpha_2\alpha_3, \delta_2\gamma_2\delta_3\gamma_3\delta_4, \beta_1\gamma_2, \varepsilon_1, \gamma_3\delta_4\},\\
V_{7,8} &= \{\varepsilon_1\varepsilon_2\varepsilon_3\delta_4, \beta_1\gamma_2\delta_3\gamma_3\delta_4\alpha_4, \beta_1\alpha_2\gamma_3\delta_4, \beta_1\alpha_2\alpha_3\alpha_4, \delta_2\alpha_2, \delta_3\alpha_3\alpha_4, \alpha_2\alpha_3\gamma_4, \varepsilon_2\varepsilon_3\varepsilon_4\},\\
V_{8,8} &= \{\varepsilon_1\delta_2\alpha_2\gamma_3, \varepsilon_1\beta_2\gamma_3, \delta_2\gamma_2\delta_3, \delta_2\gamma_2\delta_3\gamma_3\varepsilon_4, \varepsilon_2\delta_3, \delta_3\gamma_3\varepsilon_4, e_{a_2}, \beta_1\gamma_2\epsilon_3\delta_4\}.
\end{aligned} \tag{5.8}
$$

In this case,

$$
\begin{aligned}
V_{1,8} &= \{8, 49, 41, 32, 40, 17, 9, 64\},\\
V_{2,8} &= \{58, 15, 23, 34, 26, 47, 55, 2\},\\
V_{3,8} &= \{59, 14, 22, 35, 27, 46, 54, 3\},\\
V_{4,8} &= \{5, 52, 44, 29, 37, 20, 12, 61\},\\
V_{5,8} &= \{4, 53, 45, 28, 36, 21, 13, 60\},\\
V_{6,8} &= \{62, 11, 19, 38, 30, 43, 51, 6\},\\
V_{7,8} &= \{63, 10, 18, 39, 31, 42, 50, 7\},\\
V_{8,8} &= \{1, 56, 48, 25, 33, 24, 16, 57\}.
\end{aligned} \tag{5.9}
$$

Magic sums of order 7 can be obtained via the following basic elements:

$$
\begin{aligned}
V_{1,7} &= \{\beta_2\gamma_3\delta_4, \delta_4, \varepsilon_2\delta_3\gamma_3\delta_4\alpha_4, \delta_3\gamma_3\delta_4, \beta_1\alpha_2\alpha_3, \beta_2\gamma_3, \beta_1\alpha_2\gamma_3\} = \{22, 5, 30, 13, 38, 21, 46\},\\
V_{2,7} &= \{\beta_1\gamma_2\varepsilon_3, \varepsilon_2, \delta_4\alpha_4, \varepsilon_2\varepsilon_3\varepsilon_4, \gamma_2\varepsilon_3\delta_4\alpha_4, \beta_1\alpha_2\alpha_3\alpha_4, \varepsilon_3\varepsilon_4\} = \{47, 23, 6, 31, 14, 39, 15\},\\
V_{3,7} &= \{e_{a'_2}, \beta_1\gamma_2\varepsilon_3\delta_4, \varepsilon_2\delta_3, e_{a'_3}, \delta_2\gamma_2\varepsilon_3\varepsilon_4, \varepsilon_4, \delta_1\alpha_1\alpha_2\alpha_3\alpha_4\} = \{16, 48, 24, 7, 32, 8, 40\},\\
V_{4,7} &= \{\beta_1\alpha_2\gamma_3, \delta_2, \beta_1\gamma_2\varepsilon_3\delta_4\alpha_4, \delta_2\gamma_2\delta_3, e_{a_5}, \delta_2\gamma_2\varepsilon_3\varepsilon_4\delta_5, \gamma_3\varepsilon_4\} = \{41, 17, 49, 25, 1, 33, 9\},\\
V_{5,7} &= \{\gamma_2\delta_3\alpha_3, \beta_1\alpha_2\gamma_3\delta_4, \delta_2\alpha_2, \beta_1\gamma_2, \delta_2\gamma_2\delta_3\alpha_3, \alpha_2, \delta_2\gamma_2\delta_3\gamma_3\varepsilon_4\delta_5\} = \{10, 42, 18, 43, 26, 2, 34\},\\
V_{6,7} &= \{e_{a'_1}, \varepsilon_3, \beta_1, \delta_2\alpha_2\alpha_3, \beta_1\gamma_2\delta_3, \varepsilon_2\varepsilon_3, \alpha_2\alpha_3\} = \{35, 11, 36, 19, 44, 27, 3\},\\
V_{7,7} &= \{\alpha_2\alpha_3\alpha_4, \delta_2\gamma_2\varepsilon_3\delta_4, \varepsilon_3\delta_4, \beta_1\alpha_2, \delta_2\alpha_2\alpha_3\alpha_4, \beta_1\gamma_2\delta_3\alpha_3, \varepsilon_2\varepsilon_3\delta_4\} = \{4, 29, 12, 37, 20, 45, 28\}.
\end{aligned}
\tag{5.10}
$$

The following are magic sums of order 6:

$$
\begin{aligned}
V_{1,6} &= \{\alpha_3\gamma_4, \alpha_2\alpha_3\gamma_4, \delta_2\alpha_2\alpha_3, \delta_2\alpha_2, \varepsilon_2\delta_3\alpha_3, \beta_1\} = \{6, 7, 19, 18, 25, 36\},\\
V_{2,6} &= \{\varepsilon_2\delta_3\gamma_3\varepsilon_4, \varepsilon_3, \delta_3\gamma_3\delta_4\gamma_4, \delta_2\alpha_2\alpha_3\alpha_4, \varepsilon_2\varepsilon_3\delta_4\alpha_4, \delta_4\} = \{32, 11, 14, 20, 29, 5\},\\
V_{3,6} &= \{\alpha_3\alpha_4, \varepsilon_2\varepsilon_3, \delta_3\gamma_3\varepsilon_4, \beta_2\gamma_3\delta_4, \delta_3\alpha_3\alpha_4, \varepsilon_2\delta_3\gamma_3\varepsilon_4\delta_5\} = \{3, 27, 16, 22, 10, 33\},\\
V_{4,6} &= \{\varepsilon_2\delta_3\gamma_3\delta_4\gamma_4\delta_5, \varepsilon_2\varepsilon_3\delta_4, \varepsilon_3\varepsilon_4, \beta_2\gamma_3, \gamma_3\varepsilon_4, e_{a'_4}\} = \{34, 28, 15, 21, 9, 4\},\\
V_{5,6} &= \{e_{a'_1}, \gamma_2, \varepsilon_2, \delta_2, \varepsilon_2\delta_3\alpha_3\alpha_4, \alpha_4\} = \{35, 8, 23, 17, 26, 2\},\\
V_{6,6} &= \{e_{a_5}, \varepsilon_2\delta_3\gamma_3\delta_4\alpha_4, \varepsilon_2\delta_3, \varepsilon_3\delta_4\alpha_4, \varepsilon_3\delta_4, \varepsilon_2\varepsilon_3\varepsilon_4\} = \{1, 30, 24, 13, 12, 31\}.
\end{aligned}
\tag{5.11}
$$

Magic sums of order 5 are given by the following elements in $kQ$:

$$
\begin{aligned}
V_{1,5} &= \{\varepsilon_3, e_{a'_4}, \beta_2, \delta_3\alpha_2\alpha_3, \varepsilon_3\} = \{11, 4, 17, 10, 23\},\\
V_{2,5} &= \{\varepsilon_3\delta_3, \delta_3\gamma_3, \delta_4, \beta_2\alpha_3, \alpha_3\gamma_4\} = \{24, 12, 5, 18, 6\},\\
V_{3,5} &= \{e_{a'_3}, \varepsilon_3\delta_3\alpha_3, \delta_3\gamma_3\delta_4, e_{a_4,\delta_2}\alpha_2\alpha_3\} = \{7, 25, 13, 1, 19\},\\
V_{4,5} &= \{\delta_2\alpha_2\alpha_3\alpha_4, \delta_3, \beta_2\gamma_3, \delta_3\gamma_3\delta_4\alpha_4, \alpha_4\} = \{20, 8, 21, 14, 2\},\\
V_{5,5} &= \{\alpha_3\alpha_4, \delta_3\gamma_3\varepsilon_4, \gamma_3\varepsilon_4, \delta_2\alpha_2\gamma_3, \varepsilon_3\varepsilon_4\} = \{3, 16, 9, 22, 15\}.
\end{aligned}
\tag{5.12}
$$

The following is a description of magic sums of order 4:

$$
\begin{aligned}
V_{1,4} &= \{\delta_3\gamma_3\varepsilon_4, \gamma_3, \gamma_3\varepsilon_4, e_{a'_5}\} = \{16, 5, 9, 4\},\\
V_{2,4} &= \{\alpha_3\alpha_4, \delta_3\alpha_3\alpha_4, \alpha_3\gamma_4, \varepsilon_3\varepsilon_4\} = \{3, 10, 6, 15\},\\
V_{3,4} &= \{\alpha_3, \varepsilon_3, e_{a'_3}, \delta_3\gamma_3\delta_4\alpha_4\} = \{2, 11, 7, 14\},\\
V_{4,4} &= \{\varepsilon_3\delta_4\alpha_4, \varepsilon_4, \varepsilon_3\delta_4, e_{a_5}\} = \{13, 8, 12, 1\}.
\end{aligned}
\tag{5.13}
$$

The Lo Shu magic square has the following interpretation in $kQ$:

$$
\begin{aligned}
V_{1,3} &= \{e_{a_5'}, \alpha_3\alpha_4, \varepsilon_4\} = \{4, 3, 8\}, \\
V_{2,3} &= \{\varepsilon_4\delta_5 = \delta_4\gamma_4, \delta_5 = \delta_4, e_{a_5}\} = \{9, 5, 1\}, \\
V_{3,3} &= \{\alpha_3, \alpha_3\gamma_4\delta_5, \alpha_3\gamma_4\} = \{2, 7, 6\}.
\end{aligned}
\tag{5.14}
$$

Note that, polygons (5.6) show that every summand of a magic sum can be partitioned into parts in the set $\{1, 4, 7, 16, 35\}$. Since the underlying graph built by the corresponding paths is connected then identities (5.6-5.14) prove the Theorem 32. We are done.

□

The following result is a direct consequence of the definition of the specialization given in Theorem 32.

**Corollary 26.** *If $Q$ is the quiver defined in Theorem 32 then any partition of a positive integer $n$ into parts $\leq 100$ is a specialization of an element of $kQ$.*

*Proof.* It suffices to observe that the following specialized quiver generates numbers 82 to 100.

$$
Q = \begin{tikzcd}
\overset{a_0}{\circ} \arrow[r, "\alpha_0"] & \overset{a_1}{\circ} \arrow[r, "\alpha_1"] & \overset{a_2}{\circ} \arrow[r, "\alpha_2"] \arrow[dr, "\gamma_2"] & \overset{a_3}{\circ} \arrow[r, "\alpha_3"] \arrow[dr, "\gamma_3"] & \overset{a_4}{\circ} \arrow[r, "\alpha_4"] \arrow[dr, "\gamma_4"] & \overset{a_5}{\circ} \\
\underset{a_0'}{\circ} \arrow[u, "\delta_0"] \arrow[r, "\epsilon_0"'] & \underset{a_1'}{\circ} \arrow[u, "\delta_1"] \arrow[ur, "\beta_1"] \arrow[r, "\epsilon_1"'] & \underset{a_2'}{\circ} \arrow[u, "\delta_2"] \arrow[ur, "\beta_2"] \arrow[r, "\epsilon_2"'] & \underset{a_3'}{\circ} \arrow[u, "\delta_3"] \arrow[r, "\epsilon_3"'] & \underset{a_4'}{\circ} \arrow[u, "\delta_4"] \arrow[r, "\epsilon_4"'] & \underset{a_5'}{\circ} \arrow[u, "\delta_5"']
\end{tikzcd}
$$

In particular, factors of paths $\varepsilon_0\varepsilon_1\varepsilon_2\delta_3\gamma_3\delta_4\alpha_4$ and $\varepsilon_0\beta_1\gamma_2\varepsilon_3\varepsilon_4\delta_5$ are specialized by numbers 82, 86, 87, 93, 94, 98, 99 and 100 as follows:

$$
\begin{aligned}
82 &= \varepsilon_0\beta_1\gamma_2\varepsilon_3, \\
86 &= \varepsilon_0\beta_1\gamma_2\varepsilon_3\varepsilon_4, \\
87 &= \varepsilon_0\beta_1\gamma_2\varepsilon_3\varepsilon_4\delta_5, \\
93 &= \varepsilon_0\varepsilon_1\varepsilon_2, \\
94 &= \varepsilon_0\varepsilon_1\varepsilon_2\delta_3, \\
98 &= \varepsilon_0\varepsilon_1\varepsilon_2\delta_3\gamma_3, \\
99 &= \varepsilon_0\varepsilon_1\varepsilon_2\delta_3\gamma_3\delta_4, \\
100 &= \varepsilon_0\varepsilon_1\varepsilon_2\delta_3\gamma_3\delta_4\alpha_4.
\end{aligned}
$$

□

The following result holds as a direct consequence of Corollary 26.

**Corollary 27.** *Any partition of a positive integer $n$ into parts $\leq 100$ can be rewritten as a partition $P = \{p_1, p_2, \ldots, p_n\}$ whose parts $p_j$ have the form*

$$p_j = (35)^{(t_1)}(16)^{(t_2)}(7)^{(t_3)}(4)^{(t_4)}(1)^{(t_5)},$$

*where $(x)^{(y)}$ means that the part $x$ occurs $y$ times in $p_j$ and $t_i$ is a nonnegative integer, $0 \leq t_1, t_4 \leq 2$, $0 \leq t_2, t_3 \leq 1$, $0 \leq t_5 \leq 6$.*

The following is a version of Theorem 32 for the case $n = 3$.

**Corollary 28.** *Let $\Delta_3$ be a Brauer configuration such that $\Delta_3 = (\Delta_{0,3}, \Delta_{1,3}, \mathcal{O}, \mu)$ with;*

$$\begin{aligned}\Delta_{0,3} &= \{a, c_i, x, y \mid 1 \leq i \leq 3\},\\ \Delta_{1,3} &= \{U_1, U_2, U_3\}.\end{aligned}$$

*At each vertex the successor sequence has the form:*

$$\begin{aligned}
&\text{At vertex } x;\ U_1^{(2)} \leq U_2^{(3)} \leq U_3^{(2)},\\
&\text{At vertex } y;\ U_1^{(2)} \leq U_2^{(1)} \leq U_3^{(2)},\\
&\text{At vertex } a;\ U_1^{(2)} \leq U_2^{(2)} \leq U_3^{(2)},\\
&\text{At vertex } c_1;\ U_1^{(1)},\\
&\text{At vertex } c_2;\ U_3^{(1)},\\
&\text{At vertex } c_3;\ U_2^{(1)},\\
&\mu(\alpha) = 1, \quad \textit{for } \alpha = a, x \textit{ and } y,\\
&\mu(\alpha) = 2, \quad \textit{for } \alpha = c_i.
\end{aligned}$$

*Then a specialization of $\Delta_3$ builds the Lo Shu square. The ideal $I$ of the corresponding Brauer configuration algebra $\Lambda_\Delta$ is generated by the following relations (see Figure 5.3), for which it is assumed the following notation for the special cycles:*

$$\begin{aligned}
&C_x^{U_1,1} = a_1^x a_2^x a_3^x a_4^x a_5^x a_6^x a_7^x, \quad C_x^{U_1,2} = a_2^x a_3^x a_4^x a_5^x a_6^x a_7^x a_1^x,\\
&C_x^{U_2,1} = a_3^x a_4^x a_5^x a_6^x a_7^x a_1^x a_2^x, \quad C_x^{U_2,2} = a_4^x a_5^x a_6^x a_7^x a_1^x a_2^x a_3^x,\\
&C_x^{U_2,3} = a_5^x a_6^x a_7^x a_1^x a_2^x a_3^x a_4^x, \quad C_x^{U_3,1} = a_6^x a_7^x a_1^x a_2^x a_3^x a_4^x a_5^x,\\
&C_x^{U_3,2} = a_7^x a_1^x a_2^x a_3^x a_4^x a_5^x a_6^x, \quad C_y^{U_1,1} = a_1^y a_2^y a_3^y a_4^y a_5^y,\\
&C_y^{U_1,2} = a_2^y a_3^y a_4^y a_5^y a_1^y, \quad C_y^{U_2,1} = a_3^y a_4^y a_5^y a_1^y a_2^y\\
&C_y^{U_3,1} = a_4^y a_5^y a_1^y a_2^y a_3^y, \quad C_y^{U_3,2} = a_5^y a_1^y a_2^y a_3^y a_4^y,\\
&C_a^{U_1,1} = a_1^a a_2^a a_3^a a_4^a a_5^a a_6^a, \quad C_a^{U_1,2} = a_2^a a_3^a a_4^a a_5^a a_6^a a_1^a,\\
&C_a^{U_2,1} = a_3^a a_4^a a_5^a a_6^a a_1^a a_2^a, \quad C_a^{U_2,2} = a_4^a a_5^a a_6^a a_1^a a_2^a a_3^a,\\
&C_a^{U_3,1} = a_5^a a_6^a a_1^a a_2^a a_3^a a_4^a, \quad C_a^{U_3,2} = a_6^a a_1^a a_2^a a_3^a a_4^a a_5^a,\\
&C_{c_1}^{U_1,1} = a_1^{c_1}, \quad C_{c_2}^{U_3,1} = a_1^{c_2}, \quad C_{c_3}^{U_2,1} = a_1^{c_3}.
\end{aligned} \tag{5.15}$$

1. *$a_i^h a_r^s$, if $h \neq s$, for all possible values of $i$ and $r$,*

2. *$C_j^{U_1,i} - C_l^{U_1,k}$, for all possible values of $i, j, k$ and $l$,*

3. *$C_j^{U_2,i} - C_l^{U_2,k}$, for all possible values of $i, j, k$ and $l$,*

4. *$C_j^{U_3,i} - C_l^{U_3,k}$, for all possible values of $i, j, k$ and $l$,*

5. *$C_i^{U_1,j}a$ $(C_i^{U_2,j}a')$ , with $a$ $(a')$ being the first arrow of $C_i^{U_1,j}$ $(C_i^{U_2,j})$ for all $i, j$.*

6. *$C_i^{U_3,j}a$, with $a$ being the first arrow of $C_i^{U_3,j}$ for all $i, j$.*

*The following is the quiver associated to the Brauer configuration $\Gamma$, the colors means the different special cycles associated to each vertex as follows: for vertex $b_1$ red, for vertex $b_2$ green, for vertex $b_3$ magenta, for vertex $b_4$ cyan and for vertex $b_5$ blue.*

*The following is the quiver associated to the configuration $\Delta_3$:*

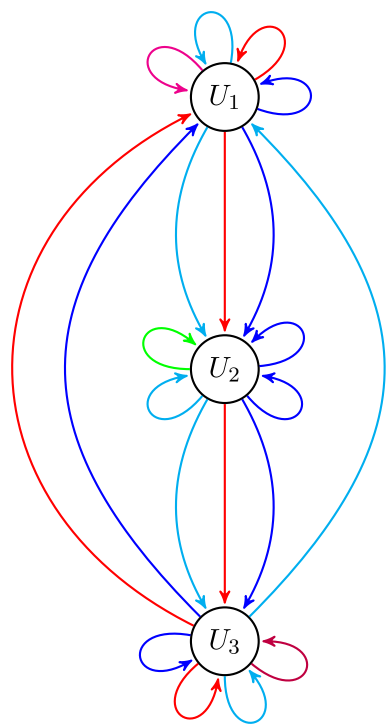



FIGURE 5.3. Quiver associated to the Brauer configuration $\Delta$.

*Proof.* Polygons of the Brauer configuration $\Delta_3$ consists of words whose letters are elements $A = \{x, y\} \subset \Delta_{0,3}$ connected by symbols $B = \{a, c_i\} \subset \Delta_{0,3}$ $c_i$ is defined as $b_i$ in Theorem 32 where $xc_iy$ is the only way to connect letters $x$ and $y$ and more than one letter $y$ are connected. Thus, if we define the specialization $a =\rightarrow$, $c =\downarrow$ then polygons in $\Delta_3$ have the following shapes:

$$\begin{tikzcd}
\overset{x}{\circ} \arrow[r, "\alpha_4"] \arrow[d, "\delta'_4"'] & \overset{x}{\circ} \\
\underset{y}{\circ} \arrow[r, "\epsilon_4"'] & \underset{y}{\circ}
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{x}{\circ} \arrow[r, "\alpha_3"] & \overset{x}{\circ} \arrow[r, "\alpha_4"] & \overset{x}{\circ} \arrow[d, "\delta'_5"'] \\
& & \underset{y}{\circ}
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{x}{\circ} \arrow[r, "\alpha_3"] & \overset{x}{\circ} \arrow[d, "\delta'_4"'] & \\
& \underset{y}{\circ} \arrow[r, "\epsilon_4"'] & \underset{y}{\circ}
\end{tikzcd}$$

The following are factors of polygons $U_1, U_2$ and $U_3$:

$$U_1 = \{\ x \ \rightarrow \ x\ ,\ \begin{array}{ccc} x & & \\ \downarrow & & \\ y & \rightarrow & y \end{array}\ ,\ y\ \},$$

$$U_2 = \{\ x \ \rightarrow \ x \ \rightarrow \ x\ ,\ \begin{array}{c} x \\ \downarrow \\ y \end{array}\ ,\ \begin{array}{ccccc} x & \rightarrow & x & \rightarrow & x \\ & & & & \downarrow \\ & & & & y \end{array}\ \},$$

$$U_3 = \{\ x\ ,\ \begin{array}{ccc} x & \rightarrow & x \\ & & \downarrow \\ & & y \end{array}\ ,\ y \ \rightarrow \ y\ \}.$$

Putting it all together, it is obtained the following subquiver $Q_3 \subset Q$ (see Figure (5.5) by replacing $x = 1$, $y = 4$:

$$Q_3 = \begin{tikzcd}
\overset{1}{\circ} \arrow[r, "\alpha_3"] & \overset{1}{\circ} \arrow[r, "\alpha_4"] \arrow[d, "\delta'_4"'] & \overset{1}{\circ} \arrow[d, "\delta'_5"'] \\
& \underset{4}{\circ} \arrow[r, "\epsilon_4"'] & \underset{4}{\circ}
\end{tikzcd}$$

In this case, polygons $U_1, U_2$ and $U_3$ can be described as follows ($\rightarrow = \downarrow = +$):

$$U_1 = \{2, 9, 4\},$$
$$U_2 = \{3, 5, 7\},$$
$$U_3 = \{1, 6, 8\}.$$

which are up to permutations the rows of the Lo Shu square. Actually, factors of $\alpha_3\delta_4'\varepsilon_4$ and $\alpha_3\alpha_4\delta_5'$ in $kQ_3$ generate the complete array as follows (see Figure (5.5)):

$$
\begin{aligned}
\{\alpha_3, \delta_4', \varepsilon_4\} &= \{2, 5, 8\},\\
\{\alpha_3\delta_4', \delta_4', e_{a_4'}\} &= \{4, 5, 6\},\\
\{\alpha_3, e_{a_4'}, \delta_4'\varepsilon_4\} &= \{2, 4, 9\},\\
\{\alpha_3\alpha_4, \delta_5', \alpha_3\alpha_4\delta_5'\} &= \{3, 5, 7\},\\
\{e_{a_3}, \alpha_3\delta_4', \varepsilon_4\} &= \{1, 6, 8\},\\
\{e_{a_3}, \delta_4', \delta_4'\varepsilon_4\} &= \{1, 5, 9\},\\
\{\alpha_3, \alpha_4\delta_5', \alpha_3\alpha_4\delta_5'\} &= \{2, 6, 7\}.
\end{aligned}
$$

□

**Corollary 29.** *Let $\Delta = \Delta_4 \cup \Delta_4'$ be a disconnected Brauer configuration such that $\Delta_4 = (\Delta_{0,4}, \Delta_{1,4}, \mathcal{O}, \mu)$, $\Delta_4' = (\Delta_{0,4}', \Delta_{1,4}', \mathcal{O}', \mu')$ with:*

$$
\begin{aligned}
\Delta_{0,4} &= \{a, b_i, c_j, g_k, h_l, x, y, z \mid i = 1, 3; j = 2, 3; k = 0, 2; l = 0, 2\},\\
\Delta_{0,4}' &= \{a', b_i', c_j', g_k', h_l', x', y', z' \mid i = 1, 2; j = 1, 2, 3; k = 0; l = 0, 1\},\\
\Delta_{1,4} &= \{U_1, U_2, U_3, U_4\},\\
\Delta_{1,4}' &= \{U_1', U_2', U_3'\}.
\end{aligned}
$$

*At each vertex the successor sequences have the form:*

$$
\begin{array}{ll}
\text{At vertex } x; U_1^{(1)} \le U_2^{(3)} \le U_3^{(3)} \le U_4^{(3)}; & x'; U_1'^{(2)} \le U_2'^{(3)} \le U_3'^{(3)},\\
\text{At vertex } y; U_1^{(2)} \le U_2^{(2)} \le U_3^{(2)} \le U_4^{(1)}; & y'; U_1'^{(2)} \le U_2'^{(1)} \le U_3'^{(2)},\\
\text{At vertex } z; U_1^{(1)} \le U_2^{(1)} \le U_3^{(1)} \le U_4^{(1)}; & z'; U_1'^{(1)} \le U_2'^{(1)} \le U_3'^{(1)},\\
\text{At vertex } a; U_1^{(2)} \le U_2^{(4)} \le U_3^{(3)} \le U_4^{(3)}; & a'; U_1'^{(3)} \le U_2'^{(3)} \le U_3'^{(3)},\\
\text{At vertex } b_1; U_1^{(1)} \le U_2^{(1)} \le U_3^{(1)}; & b_1'; U_3'^{(1)},\\
\text{At vertex } b_3; U_4^{(1)}; & b_2'; U_2'^{(1)},\\
\text{At vertex } c_2; U_4^{(1)}; & c_1'; U_1'^{(1)},\\
\text{At vertex } c_3; U_3^{(1)}; & c_2'; U_1'^{(1)},\\
\text{At vertex } g_0; U_1^{(1)} \le U_2^{(1)} \le U_3^{(1)}; & c_3'; U_3'^{(1)},\\
\text{At vertex } g_2; U_1^{(1)}; & g_0'; U_1'^{(1)} \le U_2'^{(1)} \le U_3'^{(1)},\\
\text{At vertex } h_0; U_1^{(1)} \le U_2^{(1)} \le U_4^{(1)}; & h_0'; U_1'^{(1)} \le U_2'^{(1)},\\
\text{At vertex } h_2; U_3^{(1)}; & h_1'; U_3'^{(1)},
\end{array}
$$

$$\mu(\alpha) = \mu'(\alpha') = 1, \quad \textit{for vertices } \alpha \in \Delta_{0,4}, \alpha' \in \Delta_{0,4}'.$$

*Then a specialization of $\Delta$ builds any magic sum of order 4. In particular, a specialization of $\Delta_4$ builds the Dürer magic square, whereas a specialization of $\Delta_4'$ builds the Jaina magic square.*

*Proof.* As in the previous case, we build a quiver $Q_4$ and a suitable specialization such that any magic sum of order 4 can be expressed by a specialization of a linear combination of paths in $Q_4$. Firstly, we construct the quiver induced by the Brauer configurations $\Delta_4$ and $\Delta_4'$ by applying the previous substitutions for the corresponding vertices. In this case, letters $z$ and $y$ are always connected in the form $zay$. Therefore $h_0(X)$ means that $X$ contains subwords of the form $yay$. Whereas $h_2(X)$ means that $X$ does not contain this type of sequence. Thus the polygons $U_1, \dots, U_4$ and $U_1', \dots, U_3'$ have the following shapes:

$$U_1 = \begin{tikzcd} & & x \\ z \arrow[r, "\epsilon_3"'] & y \arrow[r, "\epsilon_4"'] & y \arrow[u, "\delta_5"] \end{tikzcd}$$

$$U_2 = \begin{tikzcd} x \arrow[r, "\alpha_3"] & x \arrow[r, "\alpha_4"] & x \\ z \arrow[u, "\delta_3"] \arrow[r, "\epsilon_3"'] & y \arrow[r, "\epsilon_4"'] & y \end{tikzcd}$$

$$U_3 = \begin{tikzcd} x \arrow[r, "\alpha_3"] & x \arrow[r, "\alpha_4"] & x \arrow[d, "\delta_5'"] \\ z \arrow[u, "\delta_3"] \arrow[r, "\epsilon_3"'] & y & y \end{tikzcd}$$

$$U_4 = \begin{tikzcd} x \arrow[r, "\alpha_3"] & x \arrow[r, "\alpha_4"] \arrow[d, "\delta_4'"'] & x \\ & z \arrow[r, "\epsilon_4"'] & y \arrow[u, "\delta_5"'] \end{tikzcd}$$

$$U_1' = \begin{tikzcd} x \arrow[r, "\alpha_3"] \arrow[d, "\delta_3'"] & x \arrow[d, "\delta_4'"] & \\ z \arrow[r, "\epsilon_3"'] & y \arrow[r, "\epsilon_4"'] & y \end{tikzcd}$$

$$U_2' = \begin{tikzcd} x \arrow[r, "\alpha_3"] & x \arrow[r, "\alpha_4"] & x \\ z \arrow[r, "\epsilon_3"'] & y \arrow[u, "\delta_4"] & \end{tikzcd}$$

$$U_3' = \begin{tikzcd} x \arrow[r, "\alpha_3"] & x \arrow[r, "\alpha_4"] & x \arrow[d, "\delta_5'"] \\ z \arrow[u, "\delta_3"] & y \arrow[r, "\epsilon_4"'] & y \end{tikzcd}$$

The following are factors of polygons $U_1, \ldots, U_4$ and $U_1', \ldots, U_3' = U_4'$:

$$U_1 = \left\{ \begin{array}{ccccc} & & & & x \\ & & & & \uparrow \\ z & \to & y & \to & y \end{array}, \begin{array}{c} x \\ \uparrow \\ y \end{array}, \begin{array}{ccc} & & x \\ & & \uparrow \\ y & \to & y \end{array}, y \right\},$$

$$U_2 = \left\{ x \to x \to x, \begin{array}{ccccc} x & \to & x & \to & x \\ \uparrow & & & & \\ z & & & & \end{array}, \begin{array}{ccc} x & \to & x \\ \uparrow & & \\ y & & \end{array}, z \to y \to y \right\},$$

$$U_3 = \left\{ x \to x, z \to y, z, \begin{array}{ccccc} x & \to & x & \to & x \\ \uparrow & & & & \downarrow \\ z & & & & y \end{array} \right\},$$

$$U_4 = \left\{ \begin{array}{ccccc} x & \to & x & & \\ & & \downarrow & & \\ & & z & \to & y \end{array}, \begin{array}{c} x \\ \downarrow \\ z \end{array}, \begin{array}{ccc} & & x \\ & & \uparrow \\ z & \to & y \end{array}, x \right\},$$

$$U'_1 = \left\{ z, x \to x, \begin{array}{ccccc} x & & & & \\ \downarrow & & & & \\ z & \to & y & \to & y \end{array}, \begin{array}{ccc} x & & \\ \downarrow & & \\ y & \to & y \end{array} \right\},$$

$$U'_2 = \left\{ x \to x \to x, \begin{array}{ccc} & & x \\ & & \uparrow \\ z & \to & y \end{array}, \begin{array}{ccccc} & & x & \to & x \\ & & \uparrow & & \\ z & \to & y & & \end{array}, \begin{array}{ccc} x & \to & x \\ \uparrow & & \\ y & & \end{array} \right\},$$

$$U'_3 = \left\{ x, \begin{array}{c} x \\ \downarrow \\ z \end{array}, \begin{array}{ccccc} x & \to & x & & \\ & & \downarrow & & \\ & & y & \to & y \end{array}, z \to y \to y \right\},$$

$$U'_4 = \left\{ \begin{array}{ccccc} x & \to & x & \to & x \\ \uparrow & & & & \downarrow \\ z & & & & y \end{array}, z \to y, \begin{array}{c} x \\ \downarrow \\ y \end{array}, y \right\}$$

We obtain the following quiver $Q_4$ putting it all together and by applying the substitution $x = 1$, $y = 4$, $z = 7$:

$$Q_4 = \begin{tikzcd} \overset{1}{\circ} \arrow[r, "\alpha_3"] \arrow[d, shift left, "\delta'_3"] & \overset{1}{\circ} \arrow[r, "\alpha_4"] \arrow[d, shift left, "\delta'_4"] & \overset{1}{\circ} \arrow[d, shift left, "\delta'_5"] \\ \underset{7}{\circ} \arrow[r, "\epsilon_3"'] \arrow[u, shift left, "\delta_3"] & \underset{4}{\circ} \arrow[r, "\epsilon_4"'] \arrow[u, shift left, "\delta_4"] & \underset{4}{\circ} \arrow[u, shift left, "\delta_5"] \end{tikzcd}$$

Magic squares of order 4 can be obtained as linear combinations of factors of the different paths constituting polygons $U_1, \ldots, U_4$ or $U'_1, \ldots, U'_3$ and by substituting $x = 1$, $y = 4$ and $z = 7$ with $\delta_i \delta'_i = 0$. Thus, this specialization allows to see polygons $U_i$ and $U'_i$ as follows:

$$\begin{aligned}
U_1 &= \{16, 5, 9, 4\},\\
U_2 &= \{3, 10, 6, 15\},\\
U_3 &= \{2, 11, 7, 14\},\\
U_4 &= \{13, 8, 12, 1\}.
\end{aligned}$$

Which are the columns of the Dürer's magic square, whereas polygons

$$\begin{aligned}
U_1' &= \{7, 2, 16, 9\},\\
U_2' &= \{12, 13, 3, 6\},\\
U_3' &= \{1, 8, 10, 15\},\\
U_4' &= \{14, 11, 5, 4\}.
\end{aligned}$$

are columns of the Jaina magic square. Any other magic sum of order 4 can be obtained by associating the corresponding linear combination of paths in $Q_4$. □

*Remark* 17. Theorem 32 and Corollary 26 deal with the research of magic labelings which have been studied with great detail by Ahmed, Stanley and Stewart [2, 72, 73]. According to Ahmed [2], a *labeling* of a graph $G$ is an assignment of a nonnegative integer to each edge of $G$. A *magic labeling* of magic sum $r$ of $G$ is a labeling such that for each vertex $v$ of $G$ the sum of the labels of all edges incident to $v$ is $r$ (loops are counted as incident once). Graphs with a magic labeling are also called *magic graphs*.

A magic labeling for a quiver $Q$ is an assignment of a nonnegative integer to each arrow of $Q_1$ such that for each vertex $v_i \in Q_0$, the sum of the labels of all arrows with $v_i$ as the initial vertex is $r$, and the sum of the labels of all arrows with $v_i$ as the terminal vertex is also $r$. Cayley digraphs of order $n$ are examples of quivers with magic sum $\frac{n(n-1)}{2}$.

Since according to Ahmed [2] magic squares constitute a cone, then in order to obtain formulas for the number of magic squares of a given order, she considered to use a *Hilbert basis* $\mathrm{HB}(C)$, which is a finite set of points such that each element of the semigroup $S_C = C \cap \mathbb{Z}^n$ is a linear combination of elements from $\mathrm{HB}(C)$ with nonnegative integer coefficients. In this case, an integral point of a cone $C$ is *irreducible* if it is not a linear combination with integer coefficients of other integral points. For instance, the following is a linear combination for the Jaina magic square in terms of a Hilbert basis associated to magic squares of order 4.

$$\begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 0&1&1&0\\\hline 0&0&0&2\\\hline 1&0&1&0\\\hline\end{array}
+2\begin{array}{|c|c|c|c|}\hline 1&0&0&1\\\hline 0&0&1&1\\\hline 1&0&1&0\\\hline 0&2&0&0\\\hline\end{array}
+\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&0&1&0\\\hline 0&0&0&1\\\hline 0&1&0&0\\\hline\end{array}
+2\begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 1&0&1&0\\\hline 0&1&0&1\\\hline 0&0&1&1\\\hline\end{array}
+8\begin{array}{|c|c|c|c|}\hline 0&1&0&0\\\hline 0&0&0&1\\\hline 0&0&1&0\\\hline 1&0&0&0\\\hline\end{array}+$$

$$\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline 0&1&1&0\\\hline 2&0&0&0\\\hline 0&1&0&1\\\hline\end{array}
+\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&0&0&1\\\hline 0&1&0&0\\\hline 0&0&1&0\\\hline\end{array}
+11\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&1&0&0\\\hline 1&0&0&0\\\hline 0&0&1&0\\\hline\end{array}
+\begin{array}{|c|c|c|c|}\hline 0&1&0&0\\\hline 0&0&1&0\\\hline 1&0&0&0\\\hline 0&0&0&1\\\hline\end{array}
=\begin{array}{|c|c|c|c|}\hline 7&12&1&14\\\hline 2&13&8&11\\\hline 16&3&10&5\\\hline 9&6&15&4\\\hline\end{array}$$

In this work similar results are obtained but instead of Hilbert basis we use the basis of a suitable path algebra $kQ$ with $Q$ induced by the configuration (5.3).

We note that there are interesting relationships between magic labelings, perfect matchings and cluster variables. Indeed, if an $n$-matching of $G$ is a magic labeling of $G$ with magic sum at most $n$ and labels extracted from the set $\{0, 1, 2, \ldots, n\}$ then a perfect matching of $G$ is a 1-matching of $G$ with magic sum 1.

The following result describes a relationship between Hilbert bases and perfect matchings [2]:

**Proposition 8.** *The perfect matchings of a graph $G$ are the elements of the minimal Hilbert basis of $C_G$ of magic sum 1 and the number of perfect matchings of $G$ is $H_G(1)$.*

where $H_G(r)$ denotes the number of magic labelings of $G$ of magic sum $r$. We also recall that Stanley computed $H_{K_5}(r)$ where $K_n$ is the complete general graph on $n$ vertices [72].

# Bibliography


[1] N. Agudelo, J.A. de la Peña, and J.P. Rada, *Extremal values of the trace norm over oriented trees*, Linear Algebra Appl **505** (2016), 261-268.

[2] M. Ahmed, *Algebraic Combinatorics of Magic Squares*, California University, Dissertation, 2004. 1-92.

[3] G. Andrews, *The Theory of Partitions*, Cambridge University. Press, Cambridge, 1998. 1-255.

[4] M. Armenta, *Homological Ideals of Finite Dimensional Algebras*, CIMAT, Mexico, 2016. Master Thesis.

[5] I. Assem, D. Simson, and A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, Cambridge University Press, Cambridge UK, 2006. 1-457.

[6] M. Auslander, M. I. Platzeck, and G. Todorov, *Homological Theory of Idempotent Ideals*, Transactions of the American Mathematical Society **332** (1992), no. 2, 667-692.

[7] M. Auslander, I. Reiten, and O. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press, Cambridge UK, 1997. 1-425.

[8] D. J. Benson, *Representations and cohomology of finite groups I*, Vol. 30, Cambridge Studies in Advanced Mathematics, Cambridge, 1991. 1-246.

[9] V.M. Bondarenko, *Representations of bundles of semichained sets and their applications*, Algebra i Analiz **3** (1991), no. 5, 38-61. English Translation; W. Crawley-Boevey, U. Hansper, I. Voulis, 2018.

[10] S. Brenner, *Endomorphism algebras of vector spaces with distinguished sets of subspaces*, J. Algebra **6** (1967), 100-114.

[11] ———, *On four subspaces of a vector space*, J. Algebra **29** (1974), 587-599.

[12] N.D. Cahill, J.R. D'Errico, D.A. Narayan, and J.Y. Narayan, *Fibonacci determinants*, The college mathematics journal **33** (2002), no. 3, 221-225.

[13] A. M. Cañadas, P.F.F. Espinosa, and I.D.M. Gaviria, *Categorification of some integer sequences via Kronecker modules*, JPANTA **38** (2016), no. 4, 339-347.

[14] A. M. Cañadas, H. Giraldo, and G.B. Rios, *An algebraic approach to the number of some antichains in the powerset* $2^n$, JPANTA **38** (2016), no. 1, 45–62.

[15] E. Chen, *Topics in Combinatorics; Lecture Notes*, MIT, Massachusetts, 2017. 1-40.

[16] J.H. Conway, *An enumeration of knots and links and some of their algebraic properties, Proceedings of the conference on Computational problems in Abstract Algebra held at Oxford in 1967*, Pergamon Press, **J. Leech ed.** (1970), 329-358.

[17] J. A. De la Peña and Changchang Xi, *Hochschild Cohomology of Algebras with Homological Ideals*, Tsukuba J. Math. **30** (2006), no. 1, 61-79.

[18] D.Ž. Djoković, *Classification of pairs consisting of a linear and a semi-linear map*, Linear Algebra Appl. **20** (1978), 147-165.

[19] P. Fahr and C. M. Ringel, *A partition formula for Fibonacci numbers*, J. Integer Seq. **11** (2008), no. 08.14, 1-9.

[20] ———, *Categorification of the Fibonacci numbers using representations of quivers*, J. Integer Seq. **15** (2012), no. 12.2.1, 1-12.

[21] ———, *The Fibonacci triangles*, Advances in Mathematics. **230** (2012), 2513-2535.

[22] S. Fomin and A. Zelevinsky, *Cluster Algebras I. Foundations*, J. Amer. Math. Soc **15** (2002), no. 2, 497–529 (electronic). MR 1887642 (2003f:16050).

[23] ———, *Cluster Algebras II. Finite type classification*, Invent. Math **154** (2003), no. 1, 63–121. MR 2004457 (2004m:17011).

[24] P. Gabriel and J.A. Peña, *Quotients of representation-finite algebras*, Communications in Algebra **15** (1987), 279-307.

[25] P. Gabriel and A.V. Roiter, *Representations of Finite Dimensional Algebras*, Algebra VIII, Encyclopedia of Math. Sc., vol. 73, Springer-Verlag, 1992. 1-177.

[26] M.A. Gatica, M. Lanzilotta, and M.I. Platzeck, *Idempotent Ideals and the Igusa-Todorov Functions*, Algebr Represent Theory **20** (2017), 275-287.

[27] I.D. M. Gaviria, *The Auslander-Reiten Quiver of Equipped Posets of Finite Growth Representation Type, some Functorial Descriptions and Its Applications*, (PhD. Thesis) Universidad Nacional de Colombia (2020), 1-164.

[28] I.M. Gelfand and V.A. Ponomarev, *Problems of linear algebra and classification of quadruples of subspaces in a finite dimensional vector space*, Colloq. Math. Soc. János Bolyai, Hilbert Space Operators, Tihany **5** (1970), 163-237.

[29] I. Gessel and X.G. Viennot, *Determinants, paths and plane partitions*, preprint (1989).

[30] E.L. Green and S. Schroll, *Almost gentle algebras and their trivial extensions*, Proceedings of the Edinburgh Mathematical Society (2018), 1-16.

[31] ———, *Brauer configuration algebras: A generalization of Brauer graph algebras*, Bull. Sci. Math. **141** (2017), 539-572.

[32] B. Guberfain, R. Nasser, M. Casanova, and H. Lopes, *BusesInRio: buses as mobile traffic sensors Managing the bus GPS data in the City of Rio de Janeiro*, 17th IEEE International Conference on Mobile Data Management (2016), 369-372.

[33] I. Gutman, *The energy of a graph*, Ber. Math.-Statist. Sekt. Forschungszentrum Graz **103** (1978), 1-22.

[34] R.K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004. 1-161.

[35] D. Happel and D. Zacharia, *Algebras of finite global dimension*, 8, Springer, Heidelberg, 2013. In: Algebras, quivers and representations, Abel Symp.

[36] L. Hille and D. Ploog, *Exceptional sequences and spherical modules for the Auslander algebra of* $k[x]/(x^t)$, arXiv **1709.03618v2** (2017), 1-19.

[37] ———, *Tilting chains of negative curves on rational surfaces*, Nagoya Math Journal (2017), 1-16.

[38] L. Hille and G. Röhrle, *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transform. Groups **4** (1999), 35-52.

[39] A. Hubery and H. Krause, *A categorification of non-crossing partitions and representations of quivers*, arXiv **1310.1907** (2013), 1-34.

[40] C. Ingalls and H. Thomas, *Noncrossing partitions and representations of quivers*, Comp. Math **145** (2009), 1533-1562.

[41] L. H. Kauffman and S. Lambropoulou, *Classifying and applying rational knots and rational tangles*, 2002. 1-37.

[42] D. Knuth, *The Art of Computer Programming*, Vol. 4, Addison-Wesley, 2004. 1-300.

[43] G. Kreweras, *Sur les partitions non croisées d. un cycle*, Discrete Math **1** (1972), no. 4, 333–350.

[44] P. Lampe, *Cluster Algebras*, Preprint (2013), 1-64.

[45] R.B. Lin, *On the applications of partition diagrams for integer partitioning*, Proc. The 23rd workshop on combinatorial mathematics and computation theory (2006), 349-354.

[46] P. Luschny, *Counting with partitions*, 2011. http://www.luschny.de/math/seq/CountingWithPartitions.html.

[47] M.E. Mays and J. Wojciechowski, *A determinant property of Catalan numbers*, Discrete Mathematics **211** (2000), 125-133.

[48] G. Musiker, R. Schiffler, and L. Williams, *Positivity for cluster algebras*, Adv. Math **227** (2011), 2241-2308.

[49] L.A. Nazarova, *Representations of a tetrad*, Izv. AN SSSR Ser. Mat. **7** (1967), no. 4, 1361-1378 (in Russian). English transl. in: Math. USSR Izvestija 1 (1967) 1305-1321, 1969.

[50] ______, *Representations of quivers of infinite type*, Izv. AN SSSR Ser. Mat. **37** (1973), 752-791 (in Russian). English transl. in: Math. USSR Izvestija 7 (1973) 749-792.

[51] L.A. Nazarova and A.V. Roiter, *On the Problem of I.M. Gelfand*, Funct. Anal. Appl. **31** (1973), no. 6, 54-69.

[52] ______, *Representations of partially ordered sets*, Zap. Nauchn. Semin. LOMI **28** (1972), 5-31 (in Russian). English transl. in J. Sov. Math. 3 (1975) 585-606.

[53] J. Prop, *The combinatorics of frieze patterns and Markoff numbers*, arXiv **4** (2008), no. math/0511633, 1-12.

[54] C.M. Ringel, *The Catalan combinatorics of the hereditary artin algebras*, Contemporary Mathematics **673** (2016), 51-177.

[55] A. Ripatti, *On the number of semi-magic squares of order 6*, arXiv **1807.02983v1** (2017), 1-14.

[56] R. Schiffler and I. Çanackçi, *Snake graphs and continued fractions*, European J. Combin. **86** (2020), 1-19.

[57] ______, *Snake graphs calculus and cluster algebras from surfaces*, J. Algebra **382** (2013), 240-281.

[58] ______, *Snake graphs calculus and cluster algebras from surfaces II: Self-crossings snake graphs*, Math. Z. **281** (2015), no. 1, 55-102.

[59] ______, *Snake graphs calculus and cluster algebras from surfaces III: Band graphs and snake rings*, Int. Math. Res. Not. (IMRN) **157** (2017), 1-82.

[60] ______, *Cluster algebras and continued fractions*, Compositio Mathematica **154** (2018), no. 3, 565-593.

[61] S. Schroll and I. Çanackçi, *Lattice bijections for string modules snake graphs and the weak Bruhat order*, arXiv **1** (2018), no. 1811.06064.

[62] H. Schubert, *Knoten mit zwei Brücken*, Math. Zeitschrift **65** (1956), 133-170.

[63] W. Shi, Q. Kong, and Y. Liu, *A GPS/GIS Integrated System for Urban Traffic Flow Analysis*, Proceedings of the 11th International IEEE Conference on Intelligent Transportation Systems (2008), 844-849.

[64] A. Sierra, *The dimension of the center of a Brauer configuration algebra*, J. Algebra **510** (2018), 289-318.

[65] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, London, 1992.

[66] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, The OEIS Foundation, Available at https://oeis.org.

[67] ______, *On-Line Encyclopedia of Integer Sequences*, Vol. http://oeis.org/A033812, The OEIS Foundation.

[68] ______, *On-Line Encyclopedia of Integer Sequences*, Vol. http://oeis.org/A080992, The OEIS Foundation.

[69] ______, *On-Line Encyclopedia of Integer Sequences*, Vol. http://oeis.org/A006052, The OEIS Foundation.

[70] ______, *On-Line Encyclopedia of Integer Sequences*, Vol. http://oeis.org/A100705, The OEIS Foundation.

[71] ______, *On-Line Encyclopedia of Integer Sequences*, Vol. http://oeis.org/A052558, The OEIS Foundation.

[72] R.P. Stanley, *Magic labelings of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings*, Duke Mathematical Journal **43** (1976), no. 3, 511-531.

[73] B.M. Stewart, *Magic graphs*, Canad, J. Math **18** (1966), 1031-1059.

[74] A.G. Zavadskij and G. Medina, *The four subspace problem; An elementary solution*, Linear Algebra Appl. **392** (2004), 11-23.

[75] A.V. Zabarilo and A.G. Zavadskij, *One-Parameter Equipped Posets and Their Representations*, Functional Analysis and Its Applications **34** (2000), no. 2, 138–140.

[76] A.G. Zavadskij, *On the Kronecker problem and related problems of linear algebra*, Linear Algebra Appl. **425** (2007), 26-62.