Thermodynamics of Black Holes in maximally symmetric spacetimes in $f(R)$ theories of gravity

By

ROGER ANDERSON HURTADO MOJICA

Department of Physics
UNIVERSIDAD NACIONAL DE COLOMBIA

Advisor
JOSÉ ROBEL ARENAS SALAZAR

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Abstract

In this work exact solutions of the field equations in the metric formalism of $f(R)$ theory are found for a spherical non-rotating and electrically charged mass distribution within the framework of the non-linear Born-Infeld theory. From these solutions the Black Hole temperature, entropy and specific heat are found and it was demonstrated that they coincide with the analogous quantities for the Reissner-Nordström Black Hole of General Relativity with cosmological constant. It is also found a hypergeometric model of cosmologically viable $f(R)$, whose main characteristic is to generalize the well-known Starobinsky and Hu-Sawicki models.
In Chapter 2 there is a review of the metric formalism of $f(R)$ theory, the the field equations are found and since the $f(R)$ theory of gravity can be expressed as a scalar-tensor theory with a scalar degree of freedom ϕ, by a conformal transformation, the action and its Gibbons-York-Hawking boundary term are written in the Einstein frame and the field equations in this frame are written. An effective potential is defined from part of the trace of the field equations in such a way that it can be calculated as an integral of a purely geometric term. This potential as well as the scalar potential are found, plotted and analyzed for some viable models of $f(R)$ and for two other proposed new, shown viable, models.
In Chapter 3, a cosmologically viable hypergeometric model in the modified gravity theory $f(R)$ is found from the need for asintoticity towards ΛCDM, the existence of an inflection point in the $f(R)$ curve, and the conditions of viability given by the phase space curves (m,r), where m and r are characteristic functions of the model. To analyze the constraints associated with the viability requirements, the models were expressed in terms of a dimensionless variable, i.e. $R \to x$ and $f(R) \to y(x) = x + h(x) + \lambda$, where $h(x)$ represents the deviation of the model from General Relativity. Using the geometric properties imposed by the inflection point, differential equations were constructed to relate $h'(x)$ and $h''(x)$, and the solutions found were Starobinsky (2007) and Hu-Sawicki type models, nonetheless, it was found that these differential equations are particular cases of a hypergeometric differential equation, so that these models can be obtained from a general hypergeometric model. The parameter domains of this model were analyzed to make the model viable.
Solutions of the field equations in $f(R)$ theory of gravity are found in Chapter 4 for a spherically symmetric and static spacetime in the non-linear electrodynamic theory of Born-Infeld (BI). It is found that the models allowed under these conditions must have the parametric form $f'(R)|_r = m + nr$, where m and n are constants, whose values and signs have a strong impact on the solutions, as well as on the form and range of the function $f(R)$. When $n = 0$, $f(R) = mR + m_0$ and Einstein-BI solution is found. When $m \neq 0$ and $n \neq 0$, the theory $f(R)$ is asymptotically equivalent to General Relativity (GR), so that the solutions of Schwarzschild and $f(R)$-Reissner-Nordström can be written in some limits. Similarly, if $n > 0$ and $r \gg 1$, the form of $f(R)$ can be approximated by an expansion in series and as a particular case, when $R_S = -\frac{m^2}{3n}$, can be found explicitly $f(R) = mR + 2n\sqrt{R} + m_0$. Finally, the solutions, scalar curvature and parametric form of

the function $f(r)$ in the non-linear regime ($m=0$) of the $f(R)$ theory are found, and some models are plotted for specific values of m and n.
In Chapter 5 it is used the conformal transformation between Jordan and Einstein frames in the formalism of the scalar-tensor theory, and the definitions of scalar field potentials, to determine in which cases the exact solutions shown here evade some generalized non-hair theorems for $f(R)$ theory. Also, the Starobinsky quadratic model is linearized using Green functions.
Some relevant Black Hole thermodynamic properties, namely entropy, temperature and specific heat are described and in some cases plotted, depending on the parameters m, n, q and Λ, of the $f(R)$ model, for the solutions found in Chapter 4. The technique used to calculate the Black Hole entropy is the Wald method and the symplectic potentials are calculated. It is found that the Black Hole entropy in this theory is no longer proportional to the square of the radius of the horizon, but that its expression changes according to the value of m and n.
Finally, the results are discussed in Chapter 7.

Works derived from this thesis

Paper **Spherically symmetric and static solutions in $f(R)$ gravity coupled with EM fields**: published in the Phys. Rev. D. Journal, (November 2020) [89].

Paper **Hypergeometric viable models in $f(R)$ gravity**: submitted in November to the Phys. Rev. D. (under refereeing review). [87].

Paper **Scalar-field potential for viable models in $f(R)$ theory**: accepted for publication in the International Journal of Astronomy and Astrophysics (IJAA) [88].

Lecture Linearized Field Equations in $f(R)$ theory for maximally symmetric spaces, Cartagena, oct 3-6, 2017, XXVII Congreso Nacional de Física.

Dedication

To my son, Roger, who has taught me the absoluteness of the true love.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

At this time, General Relativity (GR) is the most powerful and even beautiful known theory of gravity and space-time. GR was postulated by Albert Einstein around 1916, and ever since, it has explained in a very precise and amazing way many astrophysical phenomena, among which stand out the precession of perihelia, gravitational lensing and the astonishing success of predicting the existence of Black Holes, only to name a few [47, 119, 141, 164]. However, these experimental tests involve weak gravitational fields[1], and, at the date of this dissertation, GR continues without any test in the strong regime field.

Despite this, there are important observational facts that suggest the need to turn our eyes towards the foundations of GR or any other theory that pretends to predict the dynamics of the Universe. At the cosmological level, there are open questions like the accelerated expansion of the Universe, which plays an important role since its possible explanation requires introducing a new hypothetical type of matter and energy with the unearthly characteristic of having negative pressure. Such a supposition can be interpreted in the Einstein Field Equations as a cosmological constant; however, there is no definitive proof that allows us to assert this hypothesis, and at the moment, Energy and Dark matter are only amendments made to Relativity.

The question is, should we redefine our physical concepts about energy and matter or suppose that GR is not the final theory? There is not yet a final word on this subject, nevertheless there are two ways to face it. The first way is to trust that this type of matter really exists, and wait for the development in a foreseeable future of modern techniques and measurement systems that can track their gravitational signatures with high precision; the second way, modify GR so

[1]So that they can be expressed as small perturbations over the Minkowski space-time background

that we can understand the accelerated expansion of the Universe as an intrinsic phenomenon of another more general theory of gravity.

Those reasons make it necessary to think that GR cannot be considered as the ultimate theory of gravity, and one possible way to address this issue, taking advantage of all right predictions of GR, is redefining some of its structural characteristics, trying then to retake GR as an approximation, e.g. at first order for weak fields. Therefore, regardless of the new theory, its predictions must also agree with observations in the strong field regime in order to be considered as a more general theory than GR. In this sense objects such as Black Holes and Neutron stars would be used as perfect scenarios to see the behaviour of strong gravitational fields with high curvatures[149]. Such astronomical objects will be crucial to determine the underlying link in their nature between Quantum Mechanics and General Relativity.

This point is of great importance, since on the one hand, Black Holes behaves as a thermodynamical systems, and on the other, Quantum Mechanics describe the thermodynamics as a purely atomic effect and quantities as temperature or entropy are statistical averages over the ensembles with which the system is modeled. So it is worth asking again the following: is gravity a fundamental interaction in physics? Or it can be considered as a manifestation of phenomena such as heat or entropy? In these direction, papers have been written that claim to be able to calculate gravitational parameters as an effective cosmological constant from a modified Friedman equation, or even the Field Equations derived from an entropic formulation of gravity [13, 60, 167, 190].

The problem of making a super theory that unifies GR and Quantum Mechanics has no solution to the date; however, there are enormous efforts in theoretical physics in this sense [51, 104, 168, 183], and we expect that a full understanding of Black Holes will solve this problem, of course, linked to the hope that in the near future accurate measures close to the event horizon of the black hole (BH) at center of the Milky Way can be made [94, 95, 158]. This could also offer the possibility to take valuable information about the viability of the new theories and/or to impose restrictions on General Relativity, as well as to discard those theories that are not in agreement with observations.

The task of finding the new theory of gravity has had several proposals: as gravitational scalar fields [130, 166]; extra spatial dimensions [33, 49, 105]; higher order terms of the curvature scalar R, or of the trace of the stress-energy tensor T in the action [56, 75]. The last one constitutes the most natural generalization of the General Relativity, since the field equations in this theory result in the Einstein field equations as a particular case. Concretely, this work is based on the last model, i.e. $f(R)$ theory, where the gravitational action is a generalization of the Einstein-Hilbert action written in terms of a function [173] of the scalar curvature R, this is, higher-order curvature invariants of the Ricci scalar are introduced.

As a mathematical tool, $f(R)$ gravity admits any function of R, however physically there are a few models that can be considered as real candidates to be taken into account, and there is no reliable observational tests to discard or reaffirm the Dark matter model over $f(R)$ and vice versa. In spite of this, $f(R)$ gravity has the interesting advantage to address from a purely geometric perspective those problems that GR has not been solved without having to enter ad-hoc hypothesis, thus providing new insights into the concept of gravity at different scales; for example $f(R) = R^{-n}$ with $n > 0$ [48] describes the accelerated expansion of the Universe without using Dark matter, and the same argument to galactic scales to explain the profiles of its rotation curves [22, 42, 116].

We can summarize the conceptual advantages of $f(R)$ gravity as

- It is a theory that does not compete or put aside GR, but quite opposite, it takes advantage of the fundamentals of GR and generalizes it.
- Different terms involving powers of R may dominate the dynamics of different stages of the Universe evolution, as Inflation or the currently acceleration expansion.
- $f(R)$ gravity is an intrinsically geometric alternative to Dark Energy and Dark Matter. Even, without ruling out the Dark Matter model, $f(R)$ could offer another perspective that helps to understand the nature of the type of matter, for example through the galaxy rotation curves.
- Some models have passed the observational tests arising from large scale structure to Solar System scales, thus becoming true viable models.

1.1.1 Thermodynamics of the Black Holes beyond General Relativity

As in General Relativity, the simplest solution of the field equations in $f(R)$ theory (2.28) is the de Sitter (Anti de Sitter [AdS]) spacetime, which represents a sourceless maximally symmetric 4D-manifold with a positive (negative) cosmological constant Λ, and Hawking and Page [82] were the first to study the Thermodynamics of BH in the AdS spacetime. They found the relation between entropy and event horizon area from the Euclidean action of a Schwarzschild AdS BH solution through the canonical ensemble, and found that such black holes in asymptotically AdS spacetime have positive specific heat at high temperatures and can be in equilibrium stable with the thermal radiation at a fixed temperature. Then, Bousso and Hawking studied the quantum evaporation of Schwarzschild AdS BH [25], and found that for BH whose size is comparable to that of the cosmological horizon, this process differs from the asymptotically flat BH, moreover, they found that these quantum Schwarzschild AdS BH can anti-evaporate. Alternatively, in [57] studied the problem of finding static and spherically symmetric BH solutions in Schwarzschild AdS BH. Equally important was the expression for the Schwarzschild AdS BH entropy for $f(R) = 1/R$ gravity derived by Brevik et. al. [31], who used the Noether charge method [90] to find the modified entropy

$$S = A_H/3G, \tag{1.1}$$

then extend it to any spacetime of dimension D to find

$$S = \frac{D}{2+D}\frac{A}{2G}. \tag{1.2}$$

Important theorems of GR have been extended to high-order theories of gravity, for example Whitt found that the Birkhoff's theorem is still valid for $f(R) = R + \alpha R^2$, that is, the Schwarzschild metric is the only solution of the spacetime outside a spherically and static object [197]. Jacobson and Myers derived a general formula for the entropy of stationary BH in Lovelock theory of gravity[2], and they found that this expression is not simply one quarter of the surface area of the horizon, but contains additional intrinsic curvature invariants integrated over the horizon [92], and in [91] presented two cases in higher curvature gravity where the entropy is always increasing: (i) quasi-stationary processes[3] in which a BH accretes positive energy matter, and (ii) for $f(R) = R + P(R)$ the entropy satisfies the Second Law in any processes involving matter fields that satisfy the null energy condition. The expression for the entropy that they found is

$$S = \frac{1}{4G}\int_H d^{D-2}x\sqrt{|h|}[1+P'(R)], \tag{1.3}$$

where $P(R)$ stands for a polynomial of the Ricci Scalar, that is

$$P(R) = \sum_{n=2} a_n R^n, \tag{1.4}$$

$P'(R)$ is its derivative and D is the arbitrary dimension of the spacetime. Similarly Cañate et. al. showed analytically and numerically the absence of the geometric hair for asymptotically flat, static and spherically symmetric BH solutions, and for several classes of $f(R)$ models [38].

On the other hand, $f(R)$ theory (metric and Palatini formalism) is equivalent to the Scalar-Tensor Gravity (ST-G) because the action of $f(R)$ theory can be written as the action of the Brans-Dicke theory (details can be found in the next chapter). Important research of BH in Brans-Dicke theory was developed by Hawking in 1972, since he found that a stationary spacetime containing a BH is a solution in the Brans-Dicke theory if and only if it is a solution in GR [77]. This result was extended (with some conditions) by Sotiriou and Faraoni, to the fairly general class of ST-G (included the $f(R)$ theory) [174]. It was shown the BH entropy in the BD-Maxwell theory by Cai and Myung [34] from the Brans-Dicke action, where potential

$$U(\phi) \propto F_{\mu\nu}F^{\mu\nu}, \tag{1.5}$$

and $F_{\mu\nu}$ is the Maxwell field. They found that the entropy is proportional to the horizon area A_H, but mediated by the scalar function φ

$$S = \frac{A_H}{4G}\varphi, \tag{1.6}$$

[2] For a more concise description of the Thermodynamics of the BH in Lovelock theories read [129].

[3] Process in which the spacetime of background is only slightly perturbed by the infalling matter

where φ is evaluated at the outer horizon, so due to the scalar field, the area formula is no longer valid in BD theory, for related discussion read [35, 64, 194].
In 2005 Cognola et. al. reported the entropy associated with BH solutions with constant curvature in $f(R)$ gravity [53],

$$S = \frac{A_H}{4G} f'(R), \tag{1.7}$$

which is found making use of the Noether charge method, see also [7, 53, 64, 74, 191].
In reference [127], Mureika et. al. analysed the thermodynamics of the static non-rotating, rotating and regular Black Holes in the Modified Gravity (MOG). They started from the general action of the Scalar Tensor Vector Gravity (STVG) theory that contains the Einstein-Hilbert action (plus cosmological constant), the action of a massive vector field, the action of scalar fields and the action for pressureless matter, for details see [120]. Mureika et. al. derived the temperature and entropy for static and non-rotating BH from standard thermodynamics, and concluded that the Bekenstein-Hawking bound is modified in MOG. Recently, Sourshfar et. al. studied the thermodynamic behaviour, stability conditions and phase transition of BH in $f(R)$ gravity in three types, static, static charged and rotating charged [172], using the thermodynamic geometry methods, introduced by Weinhold[196], Ruppeiner [161] and Quevedo [152]. They assumed a model for $f(R)$ and started from an action for each type of BH, and concluded that the changes in the parameters of the spacetime affect the number of phase transitions of each type of BH.
Another important progress in the research of the BH Thermodynamics in $f(R)$ gravity for static spherically symmetric BH spacetime, was made by Akbar and Cai, since they found the field equations (2.28) can be written in the form of the first law $dE = TdS - PdV + Td\bar{S}$, where T is the Hawking temperature, S is the horizon entropy of the black hole, E is the horizon energy of the black hole, P is the radial pressure of matter, V is the volume of black hole horizon, and they interpreted $d\bar{S}$ as an entropy production term due to nonequilibrium thermodynamics of spacetime [8].

1.1.2 Conventions

Throughout this work we are going to use

- $(-,+,+,+)$ as the signature of the metric tensor $g_{\mu\nu}$.
- The Greek indices $(\mu,\nu,\dots)$ running over $0,1,2,3$.
- Planck units, where $c = G = \hbar = k_e = k_B = 1$.

Chapter 2

Modified gravity

Just like GR, $f(R)$ gravity describes a dynamical tensorial field $g_{\mu\nu}$ that depends on the distribution of matter and energy and relates it to the geometry of the space-time through the Field equations, which are found as the critical points of an action written from a lagrangian density.
In general there are three ways to vary the lagrangian, called formalisms, as described bellow; however, we only will focus on the Metric formalism. In this chapter it is described the equivalence between $f(R)$ gravity and the scalar tensor theory for some viable models. [140] strogradsky As usual, the Lagrangian characterising a system is chosen such that it does not depend on higher time derivatives of the positions than two. This is because the state of the system is univocally determined by the initial conditions of position and velocity and the equations of motion are second order differential equations relating the positions, velocities and accelerations.

2.1 Introduction

The constant efforts to modify the General Relativity in order to apply it at large scales started about 1933, when F. Zwicky concluded from observational data that the dynamical behaviour of clusters of galaxies did not correspond to the theoretical predictions, so he assumed that the content of non-luminous matter of the Universe must be greater than the content of luminous matter [202], which can be considered as the first hyphotesis of the existence of Dark matter (DM). Likewise observational cosmology[103, 109, 146, 155, 169, 177] has indicated that the Universe has experimented two phases of cosmic acceleration:

- Inflation [73, 110, 113] (before the radiation dominated epoch): this phase is required to solve the flatness and horizon problems, and explains part of the spectrum of temperature anisotropies observed in the Cosmic Microwave Background (CMB).

- Dark energy [54, 68, 182](after matter dominated epoch): the current acceleration expansion of the Universe.

However, due to the attractive gravitational effects of the baryonic matter, it is impossible to explain these acceleration phases only with its positively contributions of pressure and energy density to the dynamic of the Universe. Although this is an open problem, and these phases of acceleration are not fully understood, there are models that can account for them, such as: String Theory[16, 17, 96, 101], Big Bounce model[100, 138, 193], Loop quantum gravity[23, 189], among others for the case of Inflation; and Quintessence[27, 97, 128] and Cosmological constant[47, 143, 155, 195], among others for Dark energy. The last model, Cosmological constant (represented in the Einstein field equations by Λ), interpreted as a new component of energy with negative pressure, together with the model of DM, is known as Λ-Cold Dark Matter model (ΛCDM).
Now, almost 90 years after the publication of Zwicky, the nature of the constituents of the DM has not been fully explained or detected experimentally, and there are a major problems to test the Dark Energy model, because the vacuum energy density strongly depends on the high-scale physics scenario, so that the required corrections are perturbatively unstable [115, 195, 199]. Furthermore, ΛCDM does not explain inflation epoch because the radiation dominated phase started when inflation ended.
Alternatively, some models of Modified Gravity as the Scalar Tensor Gravity has been able to describe the galaxy rotation curves without the dark matter component [121–123]; and recent progress has been made in the $f(R)$ theory to explain the accelerated expansion of the Universe without the necessity to introduce the cosmological constant [170, 184, 188].
The ΛCDM model is a paradigm in physics due to the problems mentioned before; however, another solution for them could be the Gravity modified compared with GR in the context of $f(R)$. The power of this theory relies on a higher order curvature gravity, and therefore it is expected to reproduce the Einstein Field equations with cosmological constant (ΛCDM) when $f(R) = R - 2\Lambda$. Thus $f(R)$ theory allows, for example, to generalize the functional form for the ΛCDM model, to $f(R) = a_1 R + a_2 R^2$, with a_1, a_2 constants. The effects of expansion traced in the term $a_2 R^2$, have been studied first by Starobinsky, which constituted the first model for inflation [178]; and, as indicated by De Felice and Tsujikawa [56], this model is a very good alternative to scalar fields because it is well consistent with the temperature anisotropies observed in CMB.
However, in spite of the progress in the modified gravity, it is not a widely accepted theory that satisfies the gravity constrictions [9, 171, 185] of the cosmological purposes at large structure as well as to the scales of BH, and instead of this it is not easy to find a $f(R)$ model [14, 63, 99, 118, 162, 186].

2.2 Formalisms of the modified gravity

The $f(R)$ gravity emerges directly by a generalization of the Einstein-Hilbert action written as a function of the Ricci scalar. Equations of motion are derived from the principle of least action varying the same gravitational lagrangian with respect to the metric and/or the connection, the three formalisms [66, 173, 175, 176] to derive the field equations in this theory are:

1. Metric formalism: connection or Christoffel symbols are dependent on the metric tensor $g_{\mu\nu}$ and the variation of the action is made with respect to $g_{\mu\nu}$.

2. Palatini formalism: connection and metric tensor are independent variables, and the variation of the action is made with respect to both. Action matter is independent of the connection.

3. Metric-affine formalism: the same for Palatini formalism, but in this case the action matter depends on the connection.

Obviously, the Palatini formalism is obtained from Metric-affine one, as a particular case. Thus, basically there are only two formalisms. It is important to mention that both formalisms lead to the Einstein field equations when $f(R) = R$, although this does not hold for more general density Lagrangian functions.
In this dissertation we are going to work only with the Metric formalism, in which equations of motion are found varying the lagrangian only with respect to the metric.
The Einstein Equations or Field Equations in GR can be obtained in the metric formalism from the variational principle [47]

$$\delta I = 0, \tag{2.1}$$

where the action is related to the fields ϕ^μ by the Lagrangian density $\mathscr{L}(\phi^\nu, \phi^\nu_{,\mu})$,

$$I = \int d^4x \mathscr{L}(\phi^\nu, \phi^\nu_{,\mu}), \tag{2.2}$$

where $\phi_{,\mu}$ stands for derivative with respect to the x^μ coordinate. In General relativity, the dynamical variable is the metric[1] $g_{\mu\nu}$, the Lagrangian must be written in terms of it. However, the Riemann tensor, $R^\alpha_{\mu\beta\nu}$, contains all the information about the geometry of space-time, and it is defined in terms of the metric tensor through the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ [47, 164]

$$R^\alpha_{\mu\beta\nu} = \Gamma^\alpha_{\mu\nu,\beta} + \Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\mu\beta,\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta}, \tag{2.3}$$

where

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda} \right). \tag{2.4}$$

[1] Here it is used the metric signature $(-,+,+,+)$.

From the Riemman tensor, the Ricci scalar (R) is defined. R is the only independent scalar constructed from the Riemman tensor. The Ricci scalar is obtained from $R = g^{\mu\nu}R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor, defined as $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$, thus

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\alpha\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\alpha}. \tag{2.5}$$

General Relativity is grounded on the Einstein-Hilbert action, proposed as

$$I_H = \int d^4x \mathscr{L}_H(g^{\mu\nu}, R), \tag{2.6}$$

where the lagrangian density is defined as

$$\mathscr{L}_H(g^{\mu\nu}, R) = \sqrt{-g}R. \tag{2.7}$$

The full action, containing the terms associated to matter fields is written as

$$I = \frac{1}{2\kappa} I_H + I_M, \tag{2.8}$$

with the constant[2] $\kappa = 8\pi$.

2.3 Field equations

The $f(R)$ theory is based on a modification of the Einstein-Hilbert (E-H) action (2.6), that is to say the Lagrangian density is an arbitrary function of the Ricci scalar. The modified action is supposed to be the sum of a term that constitutes the generalization of the E-H action defined over a hypervolume Σ [56, 173] plus a Gibbons-York-Hawking boundary ($\partial\Sigma$) term [59, 71]

$$I_m = I'_m + I_{GYH}, \tag{2.9}$$

where we will suppose the modified action in the so called Jordan frame as

$$I'_m = \int_{\Sigma} d^4x \sqrt{-g} f(R) \tag{2.10}$$

and

$$I_{GYH} = 2 \int_{\partial\Sigma} d^3x \varepsilon \sqrt{|h|} F(R) K, \tag{2.11}$$

where $f = f(R)$ and $F = F(R) = f'(R) = df(R)/dR$, and with h the determinant of the induced metric, defined as

$$g^{\mu\nu} = h^{\mu\nu} + \varepsilon n^{\mu} n^{\nu}, \tag{2.12}$$

[2] In this dissertation $c = \hbar = G = k_B = 1$.

and

$$
\begin{aligned}
K =& n^{\mu}_{;\mu} \\
=& h^{\mu\nu} n_{\nu;\mu} \\
=& h^{\mu\nu}(\partial_\mu n_\nu - \Gamma^{\alpha}_{\mu\nu} n_\alpha),
\end{aligned}
\tag{2.13}
$$

is the trace of the extrinsic curvature at the boundary $\partial\Sigma$, n_μ is the unit normal to $\partial\Sigma$, and $\varepsilon = n_\mu n^\mu$ is equal to 1 (-1) if $\partial\Sigma$ is timelike (spacelike) hypersurface, and with no contribution of the variation of the metric tensor at the boundary

$$
\delta g_{\mu\nu}|_{\partial\Sigma} = 0. \tag{2.14}
$$

Variation of the modified action produces

$$
\begin{aligned}
\delta I_m =& \delta I'_m + \delta I_{GYH} \\
=& \int_\Sigma d^4x \delta\left(\sqrt{-g} f(R)\right) + 2\int_{\partial\Sigma} d^3x \varepsilon \sqrt{|h|}\, \delta(F(R)K) \\
=& \int_\Sigma d^4x \left(\sqrt{-g}\delta f + f\delta\sqrt{-g}\right) + 2\int_{\partial\Sigma} d^3x \varepsilon \sqrt{|h|}\, (F\delta K + K\delta F),
\end{aligned}
\tag{2.15}
$$

where it has been taken into account in the second line that the induced metric $h_{\mu\nu}$ is constant on $\partial\Sigma$, and the variation of K

$$
\begin{aligned}
\delta K =& h^{\mu\nu}\delta(\partial_\mu n_\nu - \Gamma^{\alpha}_{\mu\nu} n_\alpha) + (\partial_\mu n_\nu - \Gamma^{\alpha}_{\mu\nu} n_\alpha)\delta h^{\mu\nu} \\
=& -h^{\mu\nu}\delta(\Gamma^{\alpha}_{\mu\nu}) n_\alpha \\
=& \frac{1}{2} h^{\mu\nu}\left(\delta g_{\mu\nu;\lambda}\right) n^\lambda
\end{aligned}
\tag{2.16}
$$

where we have used the variation of the connection Eq. (A.14) in the Appendix A. From the identity

$$
\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.17}
$$

it is found

$$
\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.18}
$$

and noting that the variation of the Ricci scalar is given by Eq. (A.17), thus

$$
\begin{aligned}
F\delta R =& F R_{\mu\nu}\delta g^{\mu\nu} - F\left(\delta g^{\mu\nu}\right)_{;\mu\nu} + F g_{\mu\nu}\left(\delta g^{\mu\nu}\right)^{;\alpha}_{;\alpha} \\
=& \left(F R_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu} F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu} - F\left(\delta g^{\mu\nu}\right)_{;\mu\nu} + F_{;\nu\mu}\delta g^{\mu\nu} + g_{\mu\nu} F\left(\delta g^{\mu\nu}\right)^{;\alpha}_{;\alpha} - g_{\mu\nu} F^{;\alpha}_{;\alpha}\delta g^{\mu\nu},
\end{aligned}
\tag{2.19}
$$

and as we will later in the Wald method, this corresponds to an expansion using the chain rule

$$
\begin{aligned}
F\delta R =& \left(F R_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu} F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu} - \left[F\left(\delta g^{\mu\nu}\right)_{;\mu}\right]_{;\nu} + \left(F_{;\nu}\delta g^{\mu\nu}\right)_{;\mu} + \\
& g_{\mu\nu}\left[F\left(\delta g^{\mu\nu}\right)_{;\alpha}\right]^{;\alpha} - g_{\mu\nu}\left(F_{;\alpha}\delta g^{\mu\nu}\right)^{;\alpha}
\end{aligned}
\tag{2.20}
$$

therefore

$$
\begin{aligned}
\delta I_m &= \int_\Sigma d^4x\sqrt{-g}\left(F\delta R - \frac{1}{2}f g_{\mu\nu}\delta g^{\mu\nu}\right) + \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}\left(n^\lambda F h^{\mu\nu}\left(\delta g_{\mu\nu;\lambda}\right) + 2KF'\delta R\right)\\
&= \int_\Sigma d^4x\sqrt{-g}\left\{\left(FR_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu}F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu} - \left[F\left(\delta g^{\mu\nu}\right)_{;\mu}\right]_{;\nu} + \left(F_{;\nu}\delta g^{\mu\nu}\right)_{;\mu} + \right.\\
&\qquad \left. g_{\mu\nu}\left[F\left(\delta g^{\mu\nu}\right)_{;\alpha}\right]^{;\alpha} - g_{\mu\nu}\left(F_{;\alpha}\delta g^{\mu\nu}\right)^{;\alpha}\right\} + \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}\left(n^\lambda F h^{\mu\nu}\delta g_{\mu\nu;\lambda}\right)\\
&= \int_\Sigma d^4x\sqrt{-g}\left(FR_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu}F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu} - \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}\left[n_\nu F\left(\delta g^{\mu\nu}\right)_{;\mu} - \right.\\
&\qquad \left. n_\mu F_{;\nu}\delta g^{\mu\nu} - g_{\mu\nu}n^\alpha F\left(\delta g^{\mu\nu}\right)_{;\alpha} + g_{\mu\nu}n^\alpha F_{;\alpha}\delta g^{\mu\nu}\right] + \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\lambda h^{\mu\nu}\delta g_{\mu\nu;\lambda}\\
&= \int_\Sigma d^4x\sqrt{-g}\left(FR_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu}F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu} + \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}\left[g_{\mu\nu}n^\alpha F\left(\delta g^{\mu\nu}\right)_{;\alpha}\right] + \\
&\qquad \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\lambda h^{\mu\nu}\delta g_{\mu\nu;\lambda}
\end{aligned}
\tag{2.21}
$$

where $\delta f = F\delta R$, $\delta F = F'\delta R$, and it was assumed in the second line that $\delta R = 0$, and the variation of the metric tensor equal to zero at the boundary, i.e. $\delta g^{\mu\nu}|_{\partial\Sigma} = 0$, at the third line. Now, using the Stokes theorem,

$$
\begin{aligned}
\int_\Sigma d^4x\sqrt{-g}g_{\mu\nu}\left[F\left(\delta g^{\mu\nu}\right)_{;\alpha}\right]^{;\alpha} &= \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}g_{\mu\nu}n^\alpha F\left(\delta g^{\mu\nu}\right)_{;\alpha}\\
&= \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}(h_{\mu\nu} + \varepsilon n_\mu n_\nu)n^\alpha F\left(\delta g^{\mu\nu}\right)_{;\alpha}\\
&= \int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\alpha h_{\mu\nu}\left(\delta g^{\mu\nu}\right)_{;\alpha},
\end{aligned}
\tag{2.22}
$$

and $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$,

$$
\begin{aligned}
\int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\lambda h^{\mu\nu}\delta g_{\mu\nu;\lambda} &= -\int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\lambda h^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}\left(\delta g^{\alpha\beta}\right)_{;\lambda}\\
&= -\int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}Fn^\lambda h_{\alpha\beta}\left(\delta g^{\alpha\beta}\right)_{;\lambda},
\end{aligned}
\tag{2.23}
$$

consequently

$$
\delta I_m = \int_\Sigma d^4x\sqrt{-g}\left(FR_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - F_{;\nu\mu} + g_{\mu\nu}F^{;\alpha}_{;\alpha}\right)\delta g^{\mu\nu}. \tag{2.24}
$$

If a contribution of matter I_M is also taken into the total action [47]

$$
I = \frac{2}{\kappa}I_m + I_M, \tag{2.25}
$$

from (2.24), the stationary points are those for which $\delta S/\delta\phi^\mu = 0$,

$$
\frac{2}{\kappa}\left(FR_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} + g_{\mu\nu}F^{;\alpha}_{;\alpha} - F_{;\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}} = 0, \tag{2.26}
$$

where the definition of the energy-momentum tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}}, \tag{2.27}$$

leads to the field equations in the $f(R)$ theory

$$\boxed{f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[f'(R)\right]_{;\mu\nu} + g_{\mu\nu}\left[f'(R)\right]^{;\alpha}_{;\alpha} = \kappa T_{\mu\nu}}, \tag{2.28}$$

or written in terms of operators

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \kappa T_{\mu\nu}, \tag{2.29}$$

where the D'Alembertian is defined by $\Box f'(R) = [f'(R)]^{;\alpha}_{;\alpha}$ and covariant derivatives $\nabla_\mu\nabla_\nu f'(R) = \left[f'(R)\right]_{;\mu\nu}$. Field equations (2.29) are a set of fourth order partial differential equations from which Einstein field equations are obtained when $f(R) = R$.

The trace of the field equations (2.28), is obtained by multiplying by the metric tensor

$$\begin{aligned} Fg^{\mu\nu}R_{\mu\nu} - \frac{1}{2}fg^{\mu\nu}g_{\mu\nu} - g^{\mu\nu}F_{;\mu\nu} + 4F^{;\alpha}_{;\alpha} &= \kappa g^{\mu\nu}T_{\mu\nu} \\ FR - 2f - F^{;\mu}_{;\mu} + 4F^{;\alpha}_{;\alpha} &= \kappa T \\ f'(R)R - 2f(R) + 3[f'(R)]^{;\alpha}_{;\alpha} &= \kappa T \end{aligned} \tag{2.30}$$

where T is defined as $T = g^{\mu\nu}T_{\mu\nu}$. Even though Eq. (2.30) is a differential equation, as in GR, usually it is taken as an algebraic equation to relate R, $f(R)$ and $F(R)$. In GR, $T = 0$ implies that $R = 0$ only, but this does not hold in $f(R)$ theory.

On the other hand, the field equations (2.28) can be expressed in terms of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2.31}$$

so

$$G_{\mu\nu} = \frac{1}{F}\left[\kappa T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(FR - f) + F_{;\mu\nu} - g_{\mu\nu}F^{;\alpha}_{;\alpha}\right], \tag{2.32}$$

or

$$G_{\mu\nu} = \kappa\left[T^{(M)}_{\mu\nu} + T^{(G)}_{\mu\nu}\right], \tag{2.33}$$

where it has been defined

$$T^{(M)}_{\mu\nu} = \frac{1}{F}T_{\mu\nu} \tag{2.34}$$

and

$$T^{(G)}_{\mu\nu} = \frac{1}{2\kappa}g_{\mu\nu}\left(\frac{f}{F} - R\right) + \frac{1}{\kappa F}\left(F_{;\mu\nu} - g_{\mu\nu}F^{;\alpha}_{;\alpha}\right). \tag{2.35}$$

so, it is possible to imagine $T^{(G)}_{\mu\nu}$ as an effective energy-momentum tensor that depends only on the scalar curvature, and therefore has a geometric meaning: its components can be interpreted as flux of curvature.
This reinforces the idea of using scenarios with high curvatures as BH or neutron stars, where this term can play an outstanding role in the dynamics of the space-time, and so allow to see the behaviour of $f(R)$ theory.

2.3.1 Stability of $f(R)$ theories.

Generalising Einstein's theory of gravity by means of an arbitrary function of R has the advantage of sheltering its observational achievements by assuming functions of the form $R+\tilde{f}(R)$ when $\tilde{f}(R)$ tends to zero in some limit, but with profound gravitational or even quantum implications (e.g. the quadratic term in the Starobinsky model) in some other limit. However, care must be taken to ensure that this theory does not possess anomalies irreconcilable with the "good behaviour" that a theory must have in order to be considered a serious candidate to replace General Relativity, i.e. - be in agreement with observations at the cosmological and solar system level, - generate an effective mechanism that does not violate the causal structure of spacetime (next section), and - not possess ghost-like instabilities. And and since this field theory is described by a Lagrangian, it is worth reviewing Ostrogradsky's instability arguments in the formalism of Lagrangian mechanics, as described below.

In general, a Lagrangian describing any system, L, that depends up to the first time derivative of the coordinates, q, will possess an associated Hamiltonian, H, bounded below in terms of the conjugate momenta p, i.e., by means of Hamilton's equations, it is possible to invert the phase space to obtain a function of velocity in terms of position and momentum, $v=v(q,p)$, whereby

$$H = pv(q,p) - L, \tag{2.36}$$

this clear independence of the Hamiltonian with respect to a linear momentum term imposes very strong conditions on the stability of the system, since the Hamiltonian will be positively defined. An underlying condition for the inversion process in phase space is that the Lagrangian is non-degenerate, i.e. $\frac{\partial^2 L}{\partial \dot{q}^2} \neq 0$. The generalisation of this Hamiltonian stability to theories whose Lagrangians are non-degenerate and depend on higher orders derivatives is not possible, this result, known as the Ostrogradsky instability, is shown in detail in Appendix E, and can be seen explicitly in the linear dependence of the Hamiltonian on the $m-1$ canonically conjugate moments,

$$H = \sum_{n=1}^{m-1} p_n d_t^{(n)} q_1 + p_n d_t^{(m)} q_1(q_1,...,q_m,p_m) - L, \tag{2.37}$$

associated to a Lagrangian that depends on derivatives of order m, with the condition of non-degeneracy, read as

$$\frac{\partial^2 L}{\partial \left(d_t^{(m)} q\right)^2} \neq 0, \tag{2.38}$$

such that phase space becomes invertible and $q = q_1(q_1, ..., q_m, p_m)$. At this point it is evident that the dependence of the moments on time implies a Hamiltonian which could reach arbitrarily negative values.

2.3.2 The Cauchy problem

The physical viability of any theory is determined by the evolutive character that its equations of motion give to an initial condition of the fields and/or observables, in other words, the state of the system at some instant of time will be univocally determined by the initial conditions of the system through the dynamics imposed by the equations of motion. This behaviour is considered causal and the theory must correspond to a well-posed Cauchy problem. In General Relativity it implies that the definition of the time evolution of the fields in some particular configuration on a space-like hypersurface generates a unique final state through the field equations, which are hyperbolic.

The Cauchy problem also poses another important property that is necessary for the theory to be predictive: the stability of the states under small perturbations of the initial value problem, i.e. the field equations are not unstable under small perturbations of the initial fields and therefore the solutions cannot diverge from the solutions of the unperturbed case.

A well-known fact is that O'Hanlon's theory has a well-posed initial value problem [45]. In this theory, the Lagrangian can be expressed in general form as

$$I = \int d^4x \sqrt{-g} \left[R\phi - V(\phi)\right], \tag{2.39}$$

varying this action are obtained the field equations

$$\phi G_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} + \phi_{;\alpha}^{;\alpha} g_{\mu\nu} - \phi_{;\mu\nu} = \kappa T_{\mu\nu}, \tag{2.40}$$

which are equivalent to the field equations in $f(R)$ gravity when

$$\phi = F(R) \qquad \text{and} \qquad R = \frac{dV(\phi)}{d\phi}, \tag{2.41}$$

this means that mathematically the theory $f(R)$ in the metric formalism can be interpreted as an O'Hanlon theory, provided that $F(R)$ can be invertible in order to obtain the scalar field ϕ, in this way it can be concluded that $f(R)$ in the metric formalism has a well-posed initial value problem [45].

2.3.3 Conservation of energy

Any physical theory that describes any system in the Universe must be in accordance with the fundamental laws of local conservation of energy and momentum. For example, the laws of Thermodynamics that characterize the behaviour of an atomic system, at the background, are nothing more than conservation of energy and momentum laws described from a statistical (quantum) point of view, imposing relations between the characteristic quantities of the system as well as prohibiting some behaviours.

A necessary but not sufficient condition to state the laws of Thermodynamics in the context of Black Holes is that GR locally conserves energy and momentum, and since $f(R)$ satisfies this law, this reasoning could be extended at the generalization of GR to $f(R)$ theory. The law of conservation of energy and momentum for $f(R)$ theory can be observed as follows.

Taking the divergence of the field equations (2.28),

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} + FR^{\mu\nu}{}_{;\nu} - \frac{1}{2}f_{;\nu}g^{\mu\nu} - F^{;\mu\nu}_{;\nu} + g^{\mu\nu}F^{;\alpha}_{;\alpha\nu}, \tag{2.42}$$

but

$$f_{;\nu} = \frac{\partial f}{\partial x^{\nu}} = \frac{\partial f}{\partial R}\frac{\partial R}{\partial x^{\nu}} = FR_{;\nu} \tag{2.43}$$

so

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} + FR^{\mu\nu}_{;\nu} - \frac{1}{2}FR_{;\nu}g^{\mu\nu} - g^{\alpha\mu}F^{;\nu}_{;\alpha\nu} + g^{\mu\nu}F^{;\alpha}_{;\alpha\nu}, \tag{2.44}$$

rearranging some terms and relabel dummy indices

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} + FG^{\mu\nu}{}_{;\nu} - g^{\mu\nu}\left(F^{;\alpha}_{;\nu\alpha} + F^{;\alpha}_{;\alpha\nu}\right), \tag{2.45}$$

where the Einstein tensor, from Bianchi identities, is divergence free [47, 164]

$$G^{\mu\nu}{}_{;\nu} = 0 \tag{2.46}$$

thus

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} - g^{\mu\nu}\left(F^{;\alpha}_{;\nu\alpha} - F^{;\alpha}_{;\alpha\nu}\right), \tag{2.47}$$

The Ricci tensor and the second covariant derivatives are related by

$$F^{;\alpha}_{;\nu\alpha} - F^{;\alpha}_{;\alpha\nu} = R_{\sigma\nu}F^{;\sigma}, \tag{2.48}$$

thus

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} - g^{\mu\nu}F^{;\sigma}R_{\sigma\nu}, \tag{2.49}$$

$$\kappa T^{\mu\nu}{}_{;\nu} = F_{;\nu}R^{\mu\nu} - F_{;\nu}R^{\mu\nu}, \tag{2.50}$$

that is, $f(R)$ theory satisfies local conservation of energy and momentum

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{2.51}$$

this equation involves an important conclusion: $f(R)$ theory is consistent with the Einstein Equivalence Principle which establishes local conservation of energy, which constitutes the necessary but not sufficient condition for establishing the relation between $f(R)$ gravity and Thermodynamics.

2.4 Modified gravity as scalar tensor theory

One of the most studied alternative theories to General Relativity is undoubtedly Brans-Dicke theory [26], which is a scalar-tensor theory of gravity, described by the scalar field ϕ responsible for mediate the gravitational interaction. There is a lot of literature that has been written on this subject [36, 50, 78, 117, 160], and it is one of the few theories that remains valid since it has been able to overcome all the available observational tests [198]. Brans-Dicke theory is defined from an action of the form

$$I_{BD} = \frac{1}{2\kappa}\int_{\Sigma} d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right] + \frac{1}{\kappa}\int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}n^{\mu}_{;\mu}\phi + I_M, \tag{2.52}$$

where I_M does not depend on the scalar field ϕ, the parameter ω is called the dimensionless Brans-Dicke coupling constant, $V(\phi)$ is the scalar-field potential. An important fact of $f(R)$ gravity is that through some changes, its action can be written in the form of the Brans Dicke theory and in this way, it can be expressed as a scalar tensor theory [64, 98, 173]. This can be easily shown if we write the action (2.10) as

$$I = \frac{1}{2\kappa}\int_{\Sigma} d^4x\sqrt{-g}\,\{F(R)R - [F(R)R - f(R)]\} + \frac{1}{\kappa}\int_{\partial\Sigma} d^3x\varepsilon\sqrt{|h|}F(R)n^{\mu}_{;\mu} + I_M, \tag{2.53}$$

so, in the case where the parameter ω vanishes, comparing the two actions it is immediately seen that

$$\phi(R) = F(R), \qquad \text{and} \qquad V(R) = F(R)R - f(R), \tag{2.54}$$

so, we think of ϕ as the degree of freedom scalar field, while the field ψ is directly associated to the scalar curvature. In this way, we arrive to the modified action in the Jordan-Helmholtz frame

$$I_{m'} = \int \sqrt{-g}\,\{\phi R - V(\phi)\}\,d^4x. \tag{2.55}$$

Taking the variation of (2.52) with respect to $g^{\mu\nu}$ and taking into account the contribution of matter, it is found the field equations

$$G_{\mu\nu}\phi + g_{\mu\nu}\phi_{;\alpha}^{;\alpha} - \phi_{;\mu\nu} + \frac{\omega}{\phi}\left(\frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} - \phi_{,\mu}\phi_{,\nu}\right) + \frac{1}{2}g_{\mu\nu}V(\phi) = \kappa T_{\mu\nu}, \tag{2.56}$$

which reproduce the field equations (2.28) when relations (2.54) are fulfilled and $\omega = 0$. Note that Jordan and Jordan-Helmholtz frames are exactly the same because they have the same set of variables and observable, nevertheless if we perform a conformal transformation of the metric $\tilde{g}_{\mu\nu} = F(\psi)g_{\mu\nu} = \phi(\psi)g_{\mu\nu}$, the scalar curvature is expressed as

$$R = \phi(\psi)\left[\bar{R} + 3\frac{\phi'(\psi)}{\phi(\psi)}\bar{\psi}^{;\mu}_{;\mu} + 3\left(\frac{\phi''(\psi)}{\phi(\psi)} - \frac{\phi'(\psi)^2}{2\phi(\psi)^2}\right)\psi_{,\mu}\bar{\psi}^{,\mu}\right], \tag{2.57}$$

where $\bar{\psi}^{;\mu}_{;\mu} = \bar{g}^{\mu\nu}\psi_{;\mu\nu}$, $\bar{\psi}^{,\mu} = \bar{g}^{\mu\nu}\psi_{,\nu}$, and the action (2.53) is transformed as

$$\begin{aligned} I = \frac{1}{2\kappa}\int_\Sigma d^4x\sqrt{-\bar{g}}\left[\bar{R} + 3\left(\frac{\phi''(\psi)}{\phi(\psi)} - \frac{\phi'(\psi)^2}{2\phi(\psi)^2}\right)\psi_{,\mu}\bar{\psi}^{,\mu} + 3\left(\frac{\phi'(\psi)}{\phi(\psi)}\right)_{;\mu\nu}\bar{g}^{\mu\nu}\psi - \bar{V}(\psi)\right] + \\ \frac{3}{2\kappa}\int_{\partial\Sigma} d^3x\epsilon\sqrt{|\bar{h}|}\frac{1}{\phi(\psi)}\left\{\left[1 + \psi\left(\frac{\phi'(\psi)}{\phi(\psi)} - \frac{\phi''(\psi)}{\phi'(\psi)}\right)\right]\phi'(\psi)\bar{\psi}^{,\mu}n_\mu + \frac{2}{3}n^\mu_{;\mu}\right\} + I_M, \end{aligned} \tag{2.58}$$

and with the potential defined as

$$\bar{V}(\psi) = \frac{V(\psi)}{\phi(\psi)^2}, \tag{2.59}$$

action (2.58) it is said to be written in the Einstein frame of the theory. In particular, the transformation

$$\phi(\psi) = ce^{\sqrt{\frac{2\kappa}{3}}\psi}, \tag{2.60}$$

with c some constant, produces the action

$$I = \frac{1}{2\kappa}\int_\Sigma d^4x\sqrt{-\bar{g}}\left[\bar{R} + \kappa\psi_{,\mu}\bar{\psi}^{,\mu} - \bar{V}(\psi)\right] + \int_{\partial\Sigma} d^3x\epsilon\sqrt{|\bar{h}|}\left(\sqrt{\frac{3}{2\kappa}}\bar{\psi}^{,\mu}n_\mu + \frac{1}{\kappa}n^\mu_{;\mu}\right) + I_M, \tag{2.61}$$

in which the scalar field ψ is coupled minimally with matter [173]. These fields redefinitions only affect the form and not the background structure of the action itself; however, how can we be sure that it is indeed the same theory with different representation? The answer could escape the scope of this work, nevertheless a necessary condition, but maybe not sufficient, is the fact that both theories must describe equivalently the dynamics of any system, which means that the field equations are the same from a mathematical point of view, and as can be seen, variation of action (2.58) produces

$$\phi(\psi)\bar{G}_{\mu\nu} + \frac{3\phi'(\psi)^2}{2\phi(\psi)}\left(\frac{1}{2}\psi_{,a}\bar{\psi}^{,a}\bar{g}_{\mu\nu} - \psi_{,m}\psi_{,n}\right) + \frac{1}{2}\bar{V}\phi(\psi)\bar{g}_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.62}$$

which are exactly the same as Eq. (2.28) under the inverse transformation $g_{\mu\nu} = \phi^{-1}\bar{g}_{\mu\nu}$.

Once defined $f(\psi)$ the behaviour of $V(R)$ is determined by

$$\frac{dV(\psi)}{d\psi} = \phi(\psi), \tag{2.63}$$

thus

$$f(\psi) = \psi V'(\psi) - V(\psi), \tag{2.64}$$

however, the evolution of the potential must be determined by the distribution of mass, whose dependence is provided by the trace equation

$$\phi\psi - 2f(\psi) = \kappa T - 3\phi^{;\alpha}_{;\alpha}, \tag{2.65}$$

so

$$\frac{dV(\phi)}{d\phi} - \frac{2V(\phi)}{\phi} = \frac{1}{\phi}\left(3\phi^{;\alpha}_{;\alpha} - \kappa T\right), \tag{2.66}$$

this is a first order differential equation whose left hand side can be expressed as a total derivative by multiplying both sides for the integrating factor ϕ^{-2}, that is

$$\begin{aligned}\frac{1}{\phi^2}\frac{dV(\phi)}{d\phi} - \frac{2V}{\phi^3} &= \frac{1}{\phi^3}\left(3\phi^{;\alpha}_{;\alpha} - \kappa T\right)\\ \frac{d}{d\phi}\left(\frac{V(\phi)}{\phi^2}\right) &= \frac{1}{\phi^3}\left(3\phi^{;\alpha}_{;\alpha} - \kappa T\right),\end{aligned} \tag{2.67}$$

with the solution:

$$V = \phi^2 \int \frac{1}{\phi^3}\left(3\phi^{;\alpha}_{;\alpha} - \kappa T\right) d\phi, \tag{2.68}$$

this integral depends on the matter distribution as well as the d'Alembertian of the function ϕ, which in turn is function of R. In order to solve the integral, we define the effective potential $v = v(\phi)$ as

$$\frac{dv}{d\phi} = 3\phi^{;\alpha}_{;\alpha} - \kappa T, \tag{2.69}$$

so

$$V(\phi) = \phi^2 \int \frac{1}{\phi^3}\frac{dv}{d\phi} d\phi; \tag{2.70}$$

however, by the chain rule the derivative of the potential v is expressed in terms of ψ,

$$\frac{dv}{d\phi} = \frac{dv}{d\psi}\frac{d\psi}{d\phi}, \tag{2.71}$$

and, again, by the trace equation

$$\frac{dv}{d\phi} = (2f(\psi) - \phi\psi)\phi', \tag{2.72}$$

Eq. (2.65) gives the potential as an integral

$$v(\phi) = \int \left[2f(\psi) - \phi\psi\right]\phi' d\psi. \tag{2.73}$$

In practice, given any $f(R)$ model, Eq. (2.73) is evaluated replacing the values of R obtained from the equation $F-\phi=0$. However, let us looking for the particular case when $V(\phi)=v(\phi)$, which implies that

$$\int\left[2f(\psi)-\phi(\psi)\psi\right]\phi'(\psi)d\psi=\phi(\psi)\psi-f(\psi), \tag{2.74}$$

or

$$\left[2f(\psi)-\phi(\psi)\psi-\psi\right]\phi'(\psi)=0. \tag{2.75}$$

If $\phi'(\psi)\neq 0$, and $R=\psi$, the solution is $f(\psi)=\psi+\mu\psi^2$, μ is constant, which constitutes the Starobinsky model proposed around 1980 [12, 67, 125, 142, 178, 179], which describes inflation from the higher order gravitational term, $\psi^2=R^2$. For this model the scalar curvature increases linearly with respect to ϕ, $\mu R=(\phi-1)/2$ and the effective potential as a function of ϕ is a parabola with axis at $\phi=-1$, that is $v(\phi)=\mu R^2=(\phi-1)^2$.

In next section the potentials $V(\phi)$ and $v(\phi)$ will be found for some models of $f(R)$.

2.4.1 Asymptotically ΛCDM $f(R)$ models

It is known that any model of $f(R)$ gravity should satisfy the following observational tests (summarized in [85])

- Cosmic Microwave Background: the theory must be in asymptotic correspondence with Λ-Cold-Dark-Matter (ΛCDM) model for high-redshift regime.
- Accelerated expansion of the Universe without a cosmological constant.
- Low redshift regime: constraints from the Solar system and the Equivalence Principle

Fortunately, there are several viable models satisfying these tests, some have been fairly studied as alternative models to Dark Matter within the framework of $f(R)$ theory. For example $1/R$, R^n and $R^{-n}+R^m$ with $n,m>0$, describe accelerated expansion and early time inflation [32, 40, 53, 107, 134–137, 154, 170]; the first model pass the Solar System test [133, 135], while in the case of $R+(-R^\beta)$ it is constrained from the Wilkinson Microwave Anisotropy Probe (WMAP) data $\beta\sim 10^{-3}$ and from Supernova Lagacy Survey (SNLS) and Sloan Digital SkySurvey (SDSS) data $\beta\sim 10^{-6}$ [108] and $\beta\sim 3\cdot 10^{-5}$ [102]; logarithmic models $R+\ln(R/\alpha)$ constrained by the weak energy condition and the current values of the derivatives of the scale factor of the Friedmann-Robertson-Walker, giving $\alpha\sim 1.2x10^{-41}m^{-2}$ [144]. New works based on parameterized functions, point towards the reconstruction of models from observations, either of evolution of the Universe or of the large scale structure [106].

In next sections we investigate the behaviour of the scalar-field potential for some viable models, taking into account the aforementioned observational tests and according to the following conditions derived in [85], rewriting $f(R) = R + \tilde{f}(R)$

$$\lim_{R\to\infty} \tilde{f}(R) = const, \tag{2.76}$$

and

$$\lim_{R\to 0} \tilde{f}(R) = 0. \tag{2.77}$$

2.4.1.1 Starobinsky inflationary model (1980)

The first model to consider is the Starobinsky model, proposed around 1980 to explain the early inflationary era without any inflationary scalar field [178]

$$f(R) = R + \mu R^2, \tag{2.78}$$

where μ is constant. The field equations are written as

$$R\left(R_{\alpha\beta} - \frac{1}{4}R g_{\alpha\beta}\right) + g_{\alpha\beta}R^{;\sigma}_{;\sigma} - R_{;\alpha\beta} = 3m^2\left(G_{\alpha\beta} - \kappa T_{\alpha\beta}\right), \tag{2.79}$$

where $m^2 = -\frac{1}{6\mu}$, and the trace equation gives

$$\left(\Box + m^2\right)R = -m^2\kappa T, \tag{2.80}$$

which is none other than the Klein-Gordon equation with source, describing the degree of freedom scalar field, R, known as the scalaron. Then, the field equations are

$$R_{;\mu\nu} - R\left(R_{\mu\nu} - \frac{1}{4}R g_{\mu\nu}\right) + m^2\left(3R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = m^2\kappa\left(3T_{\mu\nu} - T\right). \tag{2.81}$$

From $F(R) - \phi = 0$

$$R = \frac{\phi - 1}{2\mu}, \tag{2.82}$$

so that the scalar curvature increases linearly with respect to ϕ, Fig. 2.1(a). At the same time, the effective potential, Eq. (2.73), will be

$$\begin{aligned} v(\phi) =& \mu R^2 \\ =& \frac{(\phi-1)^2}{4\mu} = \frac{1}{4\mu}\left(e^{\sqrt{\frac{2\kappa}{3}}\psi} - 1\right)^2, \end{aligned} \tag{2.83}$$

which is a parabola with axis at $\phi = -1$, Fig. 2.1(b). As mentioned above, the scalar field potential coincides with $v(\phi)$.

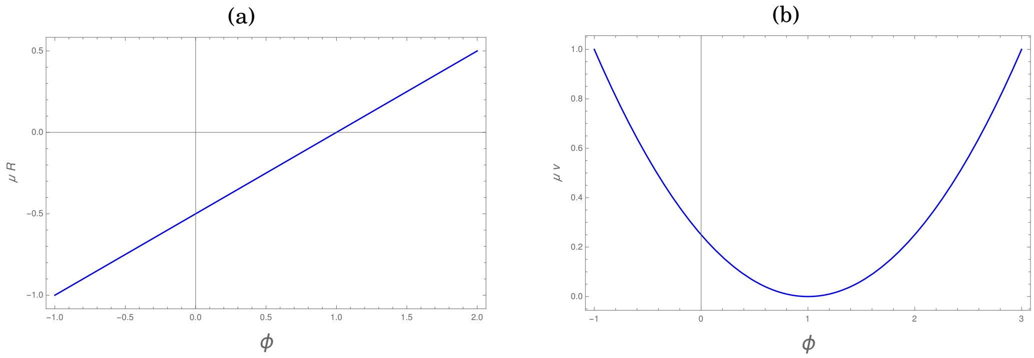


FIGURE 2.1. (a) Linear increase of the scalar curvature, and (b) parabolic behavior of the scalar-field potential as functions of ϕ for the Starobinsky model.

2.4.1.2 Starobinsky model (2007)

Now, let us consider the model proposed by Starobinsky [180] in order to satisfy the cosmological and Solar system constraints

$$f(R) = R + \mu R_0 \left[\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right], \tag{2.84}$$

with $n, \mu > 0$ parameters of the model, and R_0 is the constant characteristic curvature. The relation between R and ϕ

$$1 - \phi - 2\mu n x \left(1 + x^2\right)^{-n-1} = 0. \tag{2.85}$$

with $x = R/R_0$, cannot be inverted analytically for any value of the parameters, however in order to simplify the model we fix $R_0 = 1$. The behaviour of ϕ with respect to R is shown in Fig. 2.2 for some values of n and taking $\mu = 1$, and in Fig. 2.3 (a) are displayed some solutions of Eq. (2.85) for R as a function of the exponent n and for some values of ϕ.

The effective potential (2.73) for this model is calculated as

$$v(R,n,\mu) = \frac{\mu^2 n R}{\left(R^2+1\right)^{n+1}} \left(4 - \frac{(2n+1)R^2+1}{\mu n R} - \frac{[n(4n+7)+4]R^2+5n+4}{(2n+1)\left(R^2+1\right)^{n+1}} - \frac{3\mu^2 n^2 R}{2n+1} {}_2F_1\left(\frac{1}{2}, 2n+1; \frac{3}{2}; -R^2\right) \right), \tag{2.86}$$

which is plotted in Fig 2.4 (b) from those points that solve Eq. (2.85), shown in the same figure (a). In this plot it is possible to appreciate the mapping of points at infinity in plane $\phi - R$ to points in the plane $\phi - v$ with finite distances [69]. The position of the two maximal and the three inflection points in R do not depend on μ and are in $\pm R_0/\sqrt{2n+1}$, and $R = \pm R_0 \sqrt{3(n+1)/(2n^2+3n+1)}$ and $R = 0$ ($\phi = 1$), respectively, whose values ϕ do not depend on R_0

$$\phi = 1 \mp \frac{2\mu n}{\sqrt{2n+1}} \left(\frac{1}{2n+1} + 1 \right)^{-n-1} \quad \text{and} \quad \phi = 1 \mp \frac{\mu n \sqrt{6n+3}}{n+2} \left(\frac{3}{2n+1} + 1 \right)^{-n}. \tag{2.87}$$

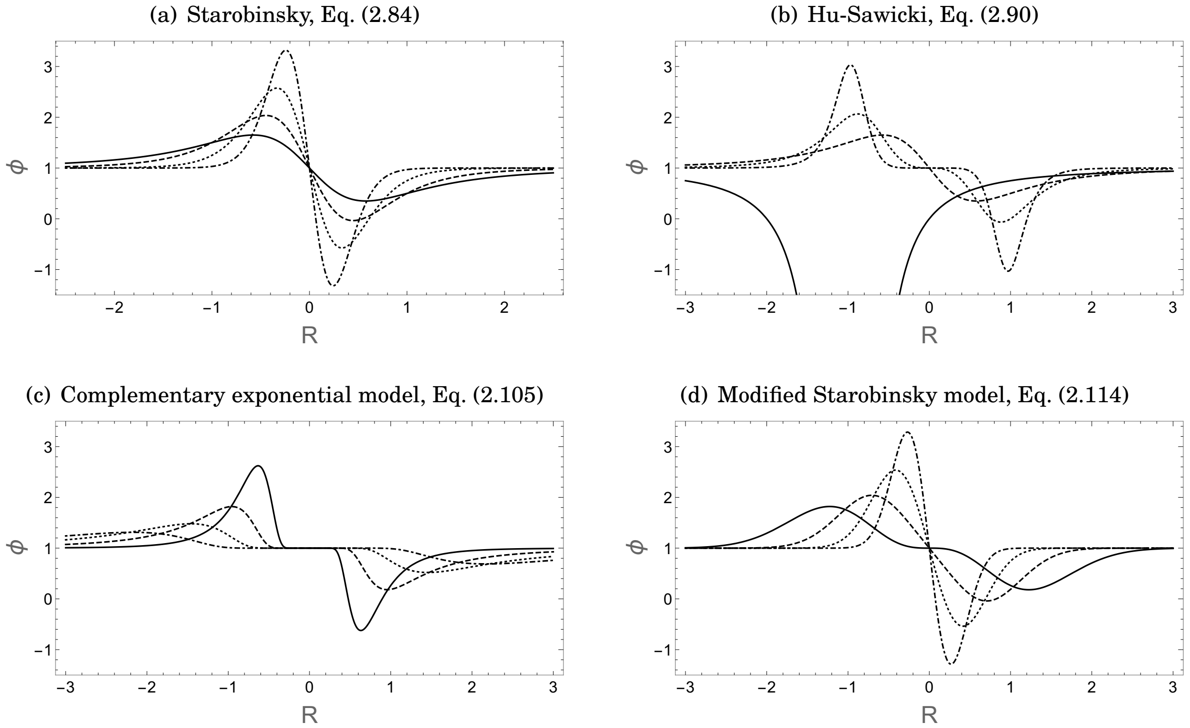


FIGURE 2.2. Scalar field as a function of the scalar curvature for some values of power $n = 1,2,4,8$, continuous, dotted, dashed, dot-dashed lines, respectively. All models, except Hu-Sawicki have continuos solutions with two maximum points and in (b), (c) and (d) there is at least one curve with zero slope transition region between those extreme points, this will have implications in the form of the effective and scalar potentials.

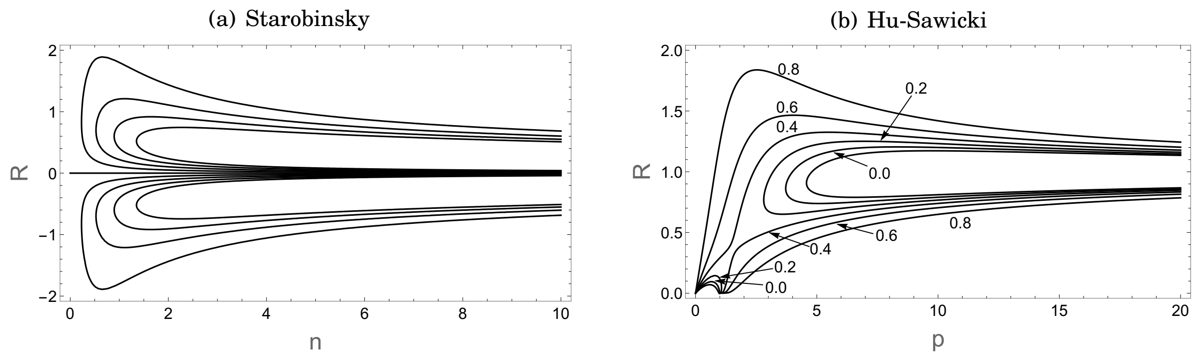


FIGURE 2.3. Scalar curvature as a function of the parameter n. In (a) from bottom to top and left to right ϕ =-1.2, -1.4, -1.6, -1.8, -0.8, -0.6, -0.4, -0.2, $R = 0$ is obtained when $\phi = 0$.

Starobinsky model has a fairly important physical content, since its expansion at $R = 0$ to order p reproduces the R^2 term of its own famous 1980 model [178]. From

$$f(R) = R - \frac{\mu n R^2}{R_0} + \mu n \sum_{m=2}^{p} \frac{(-1)^m (n+1)_{m-1}}{m!} \frac{R^{2m}}{R_0^{2m-1}}, \tag{2.88}$$

where $(n+1)_{m-1}$ is the Pochhammer symbol, it is possible to see the form of potential (2.59) in the Einstein frame, through transformation (2.60), but first we need to find the scalar field ψ as a function of R, that is

$$\psi(R) = \frac{3}{2} \ln \left[1 - 2\mu n \frac{R}{R_0} + 2\mu n \sum_{m=2}^{p} (-1)^m \frac{(n+1)_{m-1}}{\Gamma(m)} \left(\frac{R}{R_0} \right)^{2m-1} \right], \tag{2.89}$$

note that only for $p = 2$, the scalar curvature can be obtained analytically in terms of ψ, likewise, the Eq. (2.89) as well as its derived scalar potential (2.59) depend strongly on the parity of the order of the expansion, Fig. 2.5 (a) and (b) for odd p, an (c) and (d) for even p. When $p \to \infty$ the total form of model (2.84) is recovered, and the scalar field and potential are plotted in Fig. (2.6), where the similarity with the potential of Fig. 2.5 (d) can be observed.

Finally, in Fig. 2.8 it is shown the scalar potential (2.70) as a function of ϕ for $\mu = 1$.

2.4.1.3 Hu-Sawicki model

Hu and Sawicki [85] proposed a power law model to describe accelerated expansion without a cosmological constant and satisfies both, cosmological and Solar systems tests in the small-field limit. This model is given by

$$f(R) = R - \mu \frac{c_1 (R/\mu)^p}{1 + c_2 (R\mu)^p + 1}, \tag{2.90}$$

where $\mu, c_1, c_2, p > 0$ are the parameters of the model. Eq. $F - \phi = 0$ takes the form

$$\frac{c_1 \mu p \left(\frac{R}{\mu}\right)^p + R(\phi - 1)\left(c_2 \left(\frac{R}{\mu}\right)^p + 1\right)^2}{R\left(c_2 \left(\frac{R}{\mu}\right)^p + 1\right)^2} = 0 \tag{2.91}$$

it is simplified for $c1 = c2 = 1$, to

$$(1 - \phi)\left[\left(\frac{R}{\mu} \right)^p + 1 \right]^2 - p \left(\frac{R}{\mu} \right)^{p-1} = 0, \tag{2.92}$$

not for all values of R and p this equation can be solved, since it is observed that there will be divergences in $R = \mu(-1)^{1/p}$; however, some solutions are plotted in Fig. 2.2 (b), where it is noted the singularity at $R = -1$ when $n = 1$. The relation of R with respect to μ and ϕ, for $p = 1$ is given by

$$-\mu^2 \phi + R^2 (1 - \phi) + 2\mu R (1 - \phi) = 0, \tag{2.93}$$

and the solution for ϕ in terms of μ is a straight line with slope $\pm\left(1-(1-\phi)^{-1/2}\right)$ for $1>\phi$. Similarly, Eq. (2.92) can be solved for p with $\mu=1$, and for some values of ϕ, as depicted in Fig. 2.3 (b).

In Fig. 2.4 the points satisfying $F-\phi=0$ (c) are observed for function (2.90), and their image through the effective potential

$$v(R,p,\mu)=\frac{p}{4R\,(R^p+1)}\left(\frac{(1-4R)R^p-4R}{p\,(R^p+1)}+\frac{(4(2-R)R^p-4R+3)R^p+1}{(R^p+1)^2}-\frac{2pR^{2p}}{(R^p+1)^3}-\frac{1}{p^2}\right)+\frac{p}{4R}\left(\frac{1}{p^2}-1\right){}_2F_1\left(1,-\frac{1}{p};\frac{p-1}{p};-R^p\right), \tag{2.94}$$

are shown in the same Figure (d), and in Fig. 2.8 (b) is the numerical plot of the scalar potential.

For Hu-Sawicki and Starobinsky models in general, scalar potential can not be expressed analytically as a function of all parameters and the scalar curvature, however this is not be true for the next two models as will be seen below.

2.4.1.4 Tsujikawa model

In order to satisfy local gravity constraints as well as conditions of the cosmological scenario, Tsujikawa proposed the model [186]

$$f(R)=R-\mu R_T\tanh(R/R_T), \tag{2.95}$$

with $\mu, R_T>0$ the parameters of the model. For this model it is possible to invert $F-\phi=0$ analytically and find the potentials v and V. For this we see that

$$R=R_T\,\text{arccosh}\pm\sqrt{\frac{\mu}{1-\phi}}, \tag{2.96}$$

such that $1-\mu\le\phi<1$ to $R\in\Re$. The effective potential, Eq. (2.73), associated is

$$v(R)=\frac{\mu}{8\cosh^4(R/R_T)}\left[(1-\mu)R_T\sinh\left(\frac{4R}{R_T}\right)+2(1+2\mu)R_T\sinh\left(\frac{2R}{R_T}\right)-4R\cosh\left(\frac{2R}{R_T}\right)-4(1+\mu)R\right], \tag{2.97}$$

and in terms of the potential ϕ

$$v(\Phi)=\frac{\mu R_T}{4\Phi^4}\left\{\left[3\mu+2(1-\mu)\Phi^2\right]\sinh[2\,\text{arccosh}(\pm\Phi)]-2(\mu+2\Phi^2)\,\text{arccosh}(\pm\Phi)\right\}, \tag{2.98}$$

where $\Phi=\sqrt{\mu/(1-\phi)}$, with which is possible to write the scalar potential through the integral (2.70), giving

$$V(\Phi)=\frac{\mu R_T}{\Phi^2}\left[\Phi\sqrt{\Phi^2-1}-\text{arccosh}(\pm\Phi)\right]. \tag{2.99}$$

Figure 2.7 (a) and (c) shows the effective and scalar potentials for some values of μ.

2.4.1.5 Exponential model

Another interesting model to explain accelerated expansion constructed to mimic ΛCDM Universe and able to pass the Solar systems tests, was proposed in the middle of the last decade for different authors [18, 52, 201]. In this model, $f(R)$ takes the form

$$f(R) = R - \mu R_E \left(1 - e^{-R/R_E}\right), \tag{2.100}$$

where μ and R_E are the parameters of the model. For this model, equation $F(R) - \phi = 0$ is read as

$$1 - e^{R/R_E} = \phi, \tag{2.101}$$

whose solution is

$$R = R_E \ln\left(\frac{\mu}{\phi - 1}\right), \tag{2.102}$$

so, the potential (2.73) is

$$v(\phi) = \frac{1}{4} R_E (1-\phi)\left(2(\phi-3)\ln\left(\frac{\mu}{1-\phi}\right) + 8\mu + 5\phi - 9\right), \tag{2.103}$$

and the scalar potential and the scalar potential

$$V(\phi) = R_E \left[(\phi - 1)\ln\left(\frac{\mu}{1-\phi}\right) + \mu + \phi - 1\right]. \tag{2.104}$$

The functional form of these potentials can be seen in Fig. 2.7 as a function of ϕ and μ.

2.4.2 Complementary exponential model

Let us suppose the following model,

$$f(R) = R - \mu\left(1 + \frac{R_0^2}{R^2}\right) e^{-nR_0^2/R^2}, \tag{2.105}$$

where R_0 is the constant scalar curvature in vacuum $R_0 = 2\mu(3-2n)e^{-n}$, and μ, n parameters of the model, conditioned by (2.77) as $\mu \in \mathbb{R}$ and $n > 0$; and (2.76) implying

$$\lim_{R\to\infty} \tilde{f}(R) = -\mu. \tag{2.106}$$

As shown in [150], a model expressed in the form

$$\tilde{f}(R) = -2\Lambda y(R, b), \tag{2.107}$$

turns out to be an extension of the the ΛCDM model if the cosmological constant is reproduced at some limit of b, which is a characteristic parameter of each model. For (2.105) it is found

$$y(R, b) = \left[1 + \left(\frac{b\Lambda}{R}\right)^2\right] e^{-n(b\Lambda/R)^2}, \tag{2.108}$$

with $\Lambda = \mu/2$ and $b = 2R_0/\mu$, and the equivalency with the λCDM model seen through the limit (2.106) written as

$$\lim_{b\to 0} f(R) = R - 2\Lambda, \tag{2.109}$$

and

$$\lim_{b\to \infty} f(R) = R. \tag{2.110}$$

Model (2.105), as well as Starobinsky and Hu-Sawicki models can be expressed in the form (2.107) and therefore recover the viability of the ΛCDM model. For this model the relation between R and ϕ is

$$R^5(1-\phi)e^{n/R^2} - 2\left((n-1)R^2 + n\right) = 0, \tag{2.111}$$

which, in general has no analytical solutions, however Fig. 2.2 (c) shows ϕ as a function of R and the power n for $\mu = R_0 = 1$. By calculating the integral (2.73) it is found

$$\begin{aligned} v(x) = e^{-2n/x^2}\Bigg\{\frac{1}{R_0 x}\Bigg[-\frac{9(8n(2n+1)+11)}{128n^2} + \frac{8n(10n-19)-33}{32nx^2} - \left(n-\frac{15}{4}\right)\left(2n-\frac{3}{2}\right)\frac{1}{x^4} \\ + \left(\frac{13}{2}-4n\right)\frac{n}{x^6} - \frac{2n^2}{x^8}\Bigg] + \left(1+\frac{3-2n}{x^2}-\frac{2n}{x^4}\right)e^{n/x^2}\Bigg\} + 9\sqrt{2\pi}\,\frac{8n(2n+1)+11}{512R_0 n^{5/2}}\,\mathrm{Erf}\left(\frac{\sqrt{2n}}{x}\right) \end{aligned} \tag{2.112}$$

where $x = R/R_0$. Fig 2.4 (f) shows the relation between v and ϕ, from the solution of Eq. (2.111), plotted in the panel (e) of the same Figure, and in Fig. 2.8 (c) the shape of the scalar potential is observed in terms of the scalar field ϕ and n.

2.4.3 Modified Starobinsky model

Model (2.105), proposed in the previous section, meets the necessary conditions to be considered as an extension of the ΛCDM model, however, performing the expansion to second order at zR_0

$$\begin{aligned} f(R) = zR_0 - \mu e^{-n/z^2}\left(1+\frac{1}{z^2}\right) + \left\{R_0 - \frac{2\mu}{z^3}e^{-n/z^2}\left[n\left(\frac{1}{z^2}+1\right)-1\right]\right\}\left(\frac{R}{R_0}-z\right) \\ + \frac{\mu}{z^4}\left(-\frac{2n^2}{z^4}+\frac{(7-2n)n}{z^2}+3n-3\right)e^{-n/z^2}\left(\frac{R}{R_0}-z\right)^2 + O\left(\frac{R}{R_0}-a\right)^3, \end{aligned} \tag{2.113}$$

it is observed at the limit $z \to 0$ that this model does not contain the characteristic R^2 term present in Starobinsky expansion (2.88), and instead, Eq. (2.113) depends only on R. To avoid this, we will propose the function

$$f(R) = R - \mu\left[1 - \left(1+\frac{R^2}{R_0^2}\right)e^{-\frac{nR^2}{R_0^2}}\right], \tag{2.114}$$

which constitutes a modification of expression (2.84) where the role of the exponent n is played by the exponential function. Note that $R_0 = 2\mu e^{-n}(-2n + e^n - 1)$ is the constant scalar curvature and $m > 0$. Expansion of (2.114) at zR_0,

$$f(R) = zR_0 + \mu\left[e^{-nz^2}\left(1+z^2\right)-1\right] + \left[R_0 - 2\mu z e^{-nz^2}\left(nz^2+n-1\right)\right]\left(\frac{R}{R_0}-z\right) + e^{-nz^2}\left\{\mu+\mu n\left(z^2\left[2n\left(1+z^2\right)-5\right]-1\right)\right\}\left(\frac{R}{R_0}-z\right)^2 + O\left(\frac{R}{R_0}-z\right)^3, \tag{2.115}$$

yields the quadratic term in R when $z \to 0$, that is

$$f(R) \approx R + \frac{\mu(1-n)}{R_0^2}R^2, \tag{2.116}$$

which has the same functional form of the Starobinsky model of Starobinsky of 1980 [178]. This model satisfies (2.77) and (2.76) in the same way as (2.106) with $\mu \in \mathbb{R}$ and $n > 0$. Moreover, from Eq. (2.107) it is found

$$y(R,b) = 1 - \left[1 + \left(\frac{R}{b\Lambda}\right)^2\right]e^{-n(R/b\Lambda)^2}, \tag{2.117}$$

with $\Lambda = \mu/2$ and $b = 2R_0/\mu$. The equation between R and ϕ is

$$R_0^4(1-\phi)e^{nR^2/R_0^2} - 2\mu R\left(nR^2 + (n-1)R_0^2\right) = 0, \tag{2.118}$$

setting $\mu = R_0 = 1$, in Fig. 2.2 (d) the solutions of this equation for $n = 1, 2, 4, 8$ are shown. Integral (2.73) for this model leads to

$$v(x) = \frac{e^{-2nx^2}}{R_0}\left[-2n^2x^7 + n\left(\frac{3}{2}-4n\right)x^5 + \left(\frac{7}{8}-n(2n+1)\right)x^3 + \frac{(8n(13-10n)+21)x}{32n}\right] - e^{-nx^2}\left[2n\left(x^2+1\right)x\left(x-\frac{2}{R_0}\right) - x\left(x-\frac{4}{R_0}\right)+1\right] - 3\sqrt{2\pi}\,\frac{8(2n-1)n+7}{128n^{3/2}R_0}\,\mathrm{Erf}\left(\sqrt{2n}\,x\right), \tag{2.119}$$

where $x = R/R0$. Because this model is a modification of Eq. (2.84), the effective potential will have a similar shape of that of Starobinsky, as is depicted in Fig. 2.4 (h), drawn from those points that solve Eq. (2.118), shown in the same figure (g). As a part final of the analysis of this model, the graph of V vs ϕ and n is presented in Fig. (2.8).

2.4.4 Deviation from GR

All previous models of $f(R)$ (2.84), (2.90), (2.95), (2.100), (2.105) and (2.114) have a parameter μ that modulates the intensity of deviation of the model with respect to GR, so finally, it is interesting to see the plot of $\tilde{f}(R)/\mu = [f(R) - R]\mu$ for each model to compare them, Fig. 2.9.

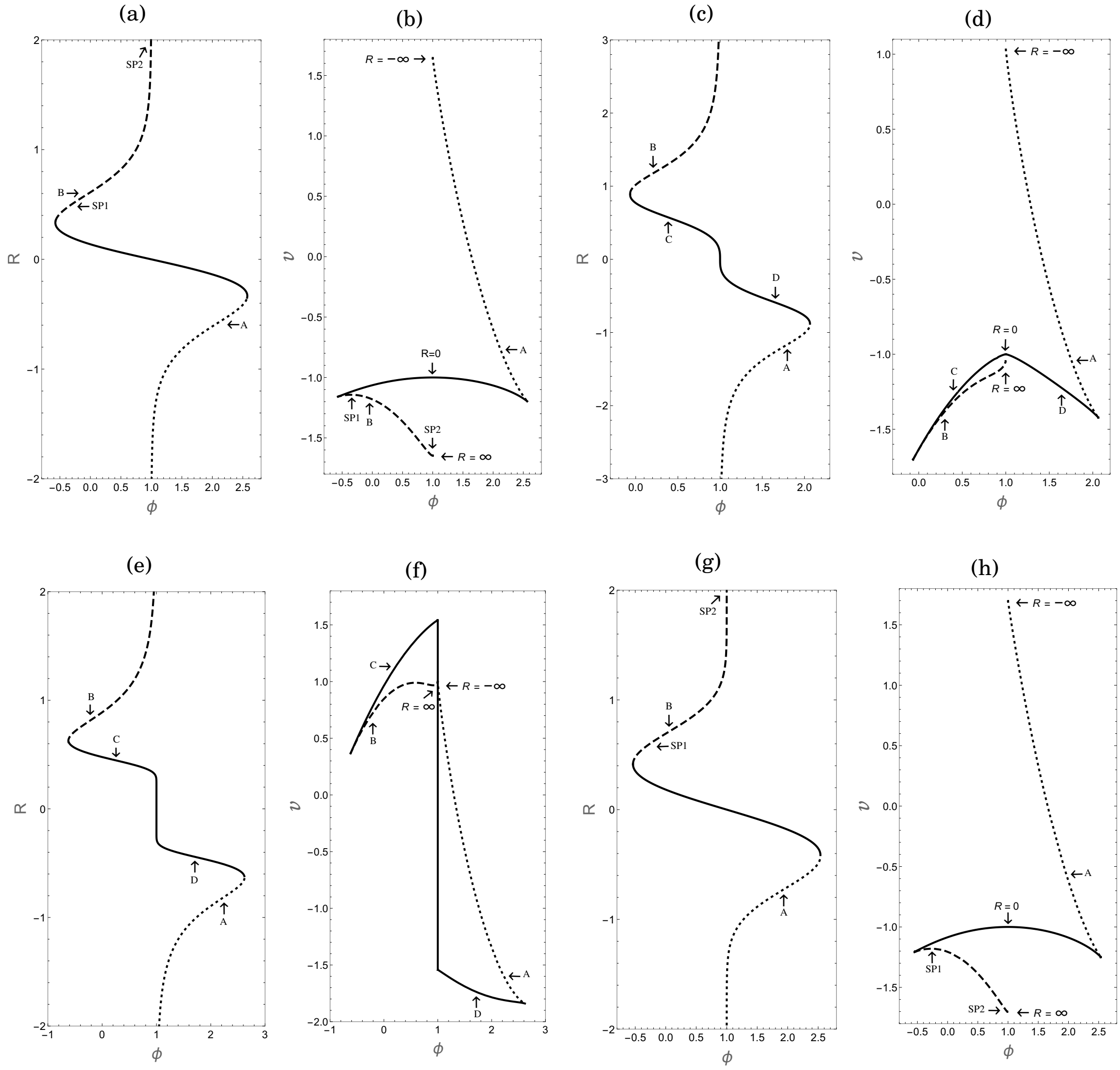


FIGURE 2.4. Scalar curvature (left) and effective potential (right) as a function of the scalar field for the Starobinsky, (a) and (b), Hu-Sawicki, (c) and (d), complementary exponential, (e) and (f), and Modified Starobinsky, (g) and (h), models; with $\mu = R_0 = 1$ and $n = 4$ ($n = 1$ in the complementary exponential), lines are matched for each pair of figures. For each model it is shown the mapping of points at $R \to \pm\infty$ to the points showed in the right panel. Starobinsky model exhibit other seven points of interest: absolute maximum and minimum $\phi = -0.574$ and $\phi = 2.574$ which are reached in $R = \pm 1/3$ respectively, ϕ=-0.096 (A) and 2.096 (B) at $R = \pm 0.057$, the de Sitter (in vacuum $T = 0$) non stable points SP1, $R = 0.484$ and $R = 0$, and the stationary point SP2, $R = 1.985$; $R = 0$ is an inflection point too in (a), de Sitter points can also be found as a local maxima of the effective potential. The scalar field in the Hu-Sawicki model (c) presents two maximum ϕ=-0.065 and 2.065 at $R = \pm 0.880112$, respectively; in this model, as well as the complementary exponential, two other inflection points appear, C and D, however, due to the exponent in this case $n = 4$, this model only presents one non stable de Sitter point $R = 0$ and five inflection points in the marks A, B, C, D, and in $R = 0$, which are ϕ=-2.377 and -2.37764 at $R = \pm 0.570017$, however, its effective potential have and the Hu-Sawicki model. Modified Starobinsky effective potential (h) has a geometry similar to (b), and thus the same number of characteristic points. Modified Starobinsky scalar field presents some region with $\phi = 0$ between maximal points like the Hu-Sawicki model for certains exponent values, Fig. 2.2, so that the effective potential will not have a continuos derivative at $R = 0$.

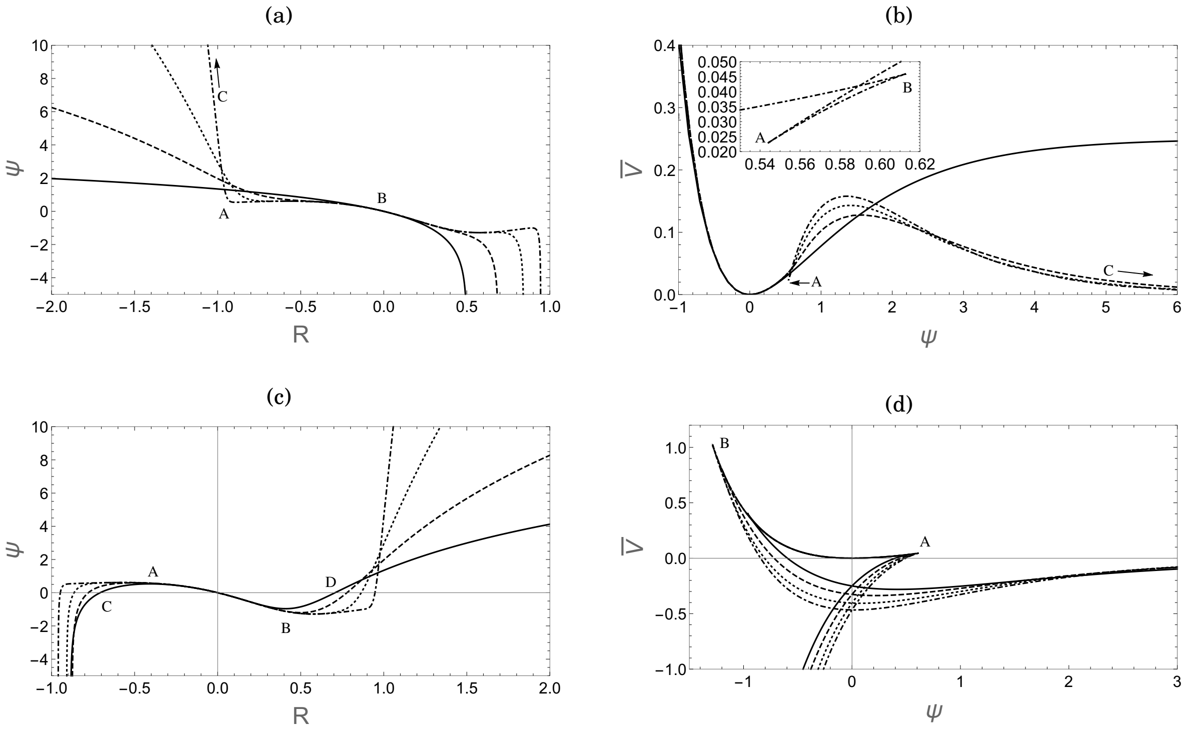


FIGURE 2.5. Scalar field ψ as a function of R (left) and scalar field potential $\tilde{V}$ as a function of ψ (right) for $p = 1, 3, 9, 39$ (a) and (b), and for $p = 2, 4, 10, 40$ (c) and (d), Black, dashed, dotted, dot-dashed lines; for the Starobinsky model expansion at $R = 0$ to order p. When $p = 1$ inflationary Starobinsky potential [178] is obtained. Uppercase letters show mapping between regions. In (a) it is observed that at A there is an inflection point which is translated into (b) that from points at infinity to A, the dominant term is R^2. In (c) and (d) there are four points of interest, A and B, maximal points, which imply a change of direction in the potential, and C and D, roots, which translate into the intersection points at $\psi = 0$ in (d). In all panels $\mu = n = 1$.

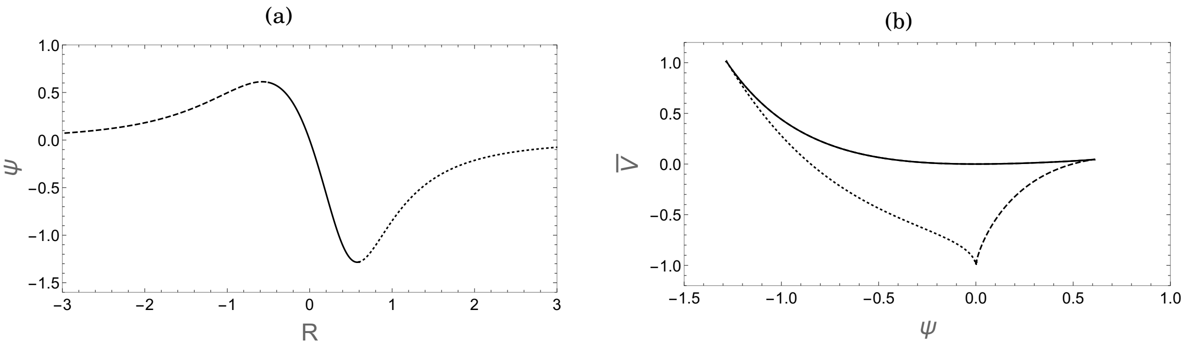


FIGURE 2.6. Scalar field ψ as a function of R (a) and scalar field potential $\tilde{V}$ as a function of ψ (b) for the Starobinsky model (2.84).

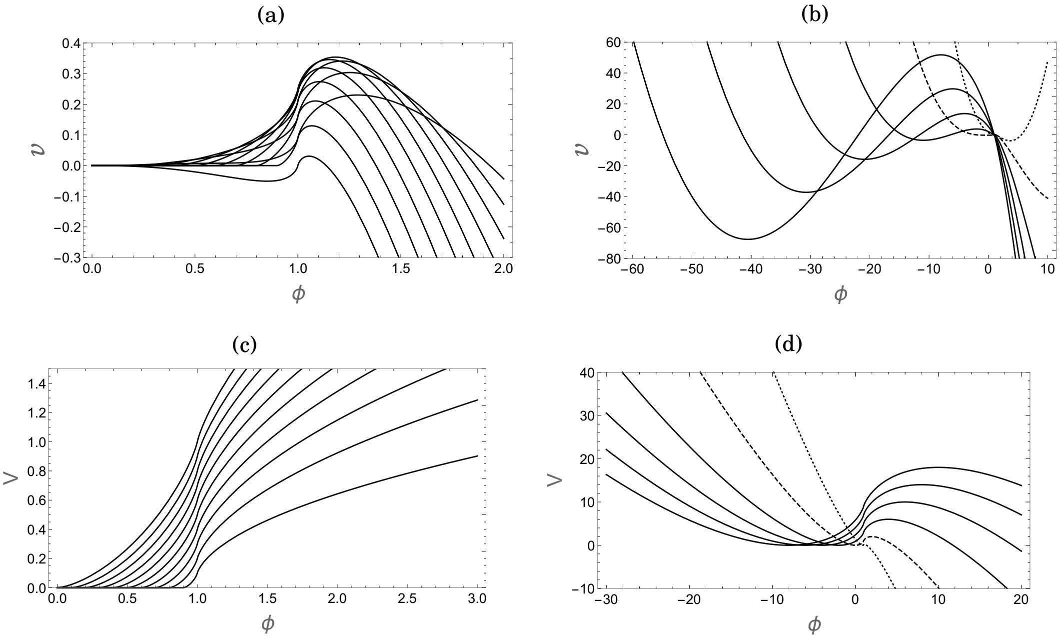


FIGURE 2.7. Potential $v(\phi)$ (a) and (b) and scalar potential $V(\phi)$, (c) and (d), for the Tsujikawa (2.95), (a) and (c) and exponential (2.100) models, (b) and (d). In (a) from left to right and bottom to top, and in (c) from left to right, $\mu = 0.1, 1, 3, 5, 7, 9$. In (b) and (d) from left to right $\mu = 9, 7, 5, 3, 1, 0.1$. Although all potential are continuous functions, in the interval $1 \le \phi$, both models have complex values and the real part has been plotted.

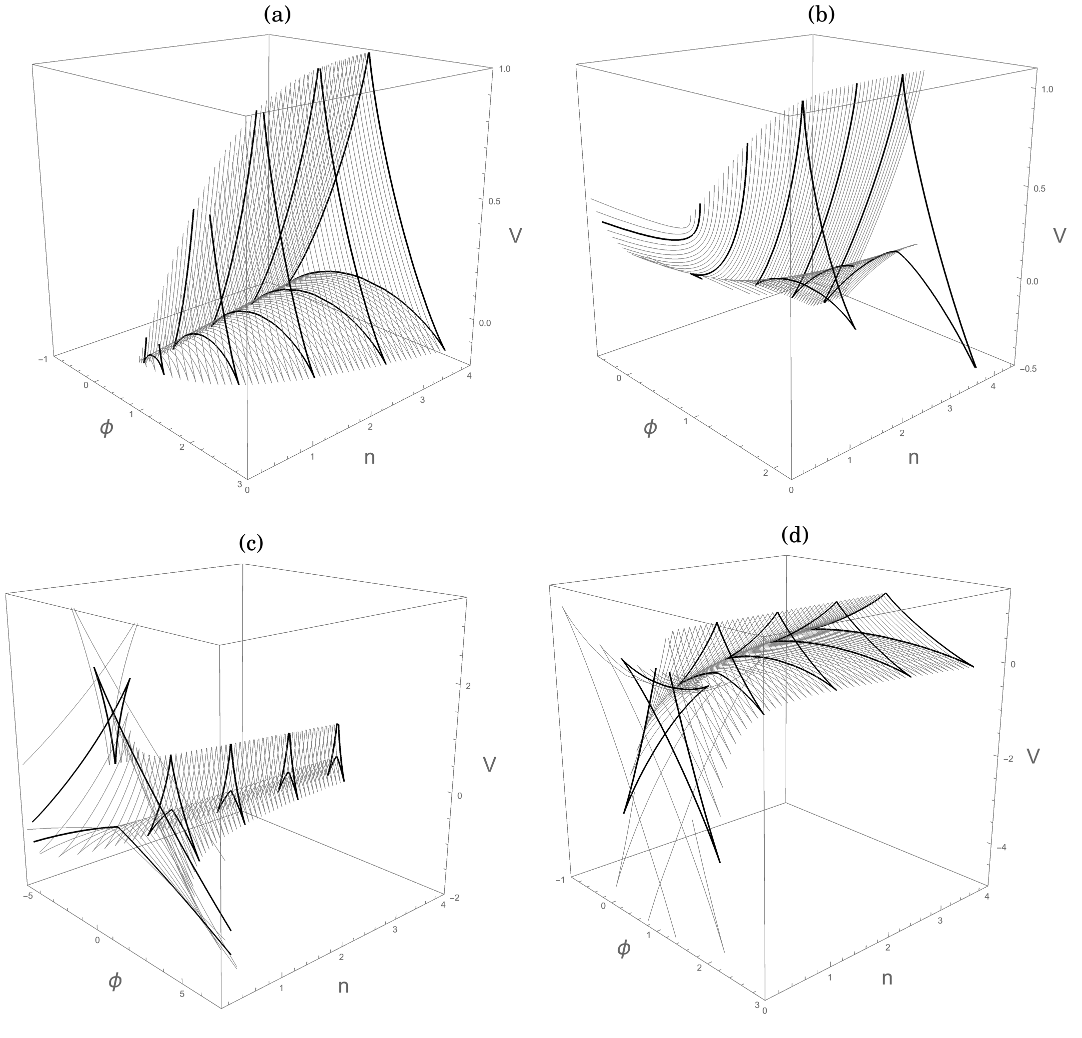


FIGURE 2.8. Scalar potential V as a function of the scalar field ϕ and the parameter n for the models studied in this work: Starobinsky (a), Hu-Sawicki (b), complementary exponential (c) and Modified Starobinsky model (d). Black thick lines represents $n = 0.25, 1, 2, 3, 4$.

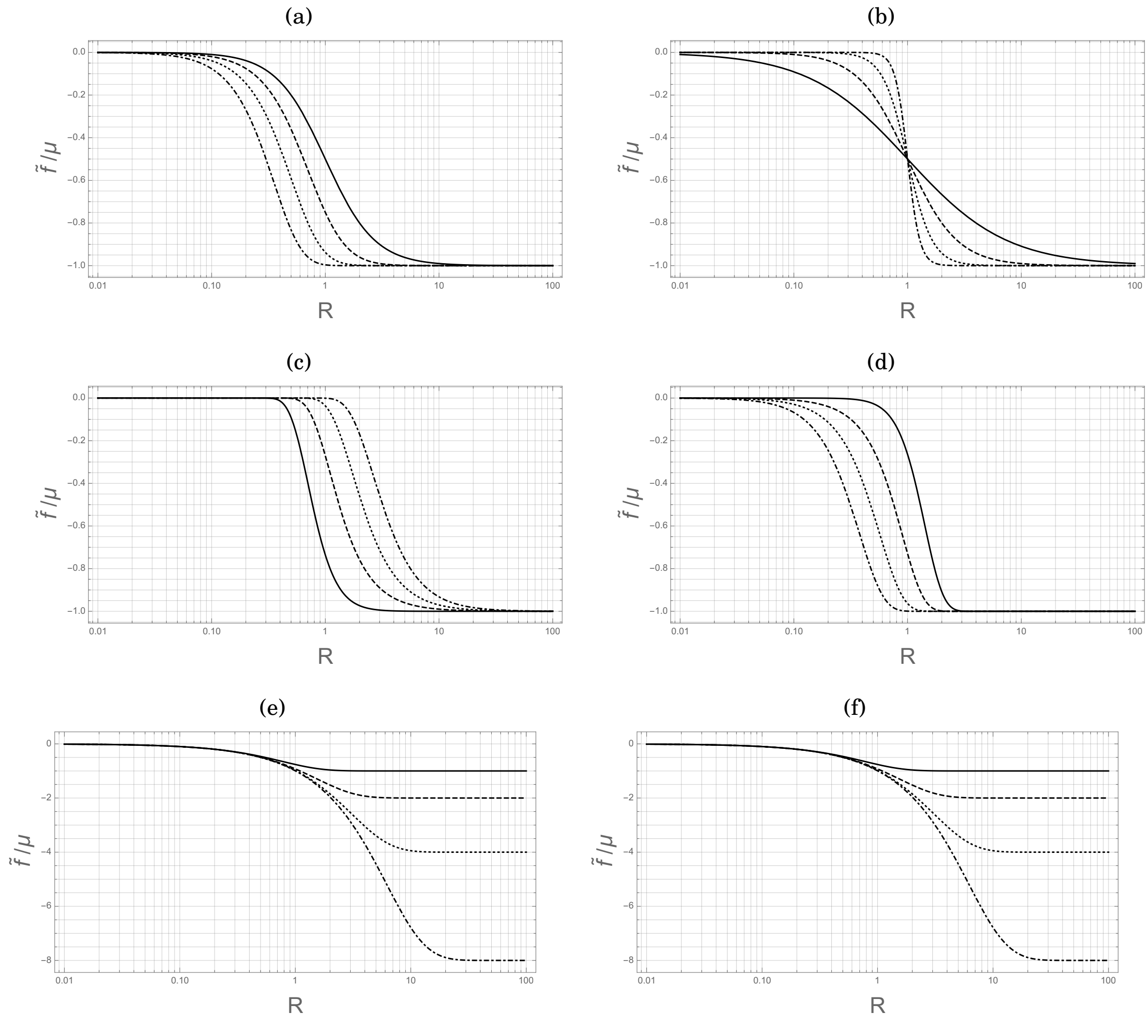


FIGURE 2.9. Behaviour of the net difference between each of the models considered in this work with respect to to GR, measured in the function $\tilde{f}(R)$. Starobinsky (a), Hu-Sawicki (b), proposed complementary exponential model (2.105) (c), proposed modified Starobinsky (2.114) (d), Tsujikawa (e) and exponential (f) for $n = R_T = R_E = 1,2,4,8$ (black, dashed, dotted and dot-dashed lines). Only in (b) lines intersect at three points: zero, $R = 1$ and infinite. For the same $R_T = R_E$ (e) and (f) tend to the same negative limit when $R \to \infty$, however the other models tend to -1, regardless the value of the exponent.

CHAPTER 3

HYPERGEOMETRIC VIABLE MODELS IN $f(R)$ GRAVITY

The task of finding a viable $f(R)$ model, which reproduces inflation, radiation-dominated phase followed by the matter-dominated phase and late-time accelerated expansion, while being able to pass the tests of the Solar system, is not at all easy, however in Ref [10] the general conditions for a model to be cosmologically acceptable are found, and examples of viable models are Starobinsky [181], Hu-Sawicki [86], Tsujikawa [187] and exponential models [112]. In particular the first two models have been tested using redshift of SN Ia data, their cosmological and free parameters were calculated using a Markov chain Monte Carlo simulation, and it was concluded that these models fit the data with high accuracy [84]. One characteristic of these models is that they present an inflection point, this property will be discussed in this work, focussing on the conditions that $f(R)$ models must possess in order to be considered cosmologically valid.

This chapter is devoted to analyze the conditions of viability of $f(R)$ models together with the assumptions of the existence of an inflection point in the function. In sections 3.3 and 3.4 a differential equation will be constructed from the geometric properties imposed by the conditions mentioned above and the solutions will be shown and generalized as a hypergeometric model in section 3.5.

3.1 Model constraints and inflection point

The general form of the function $f(R)$ can be expressed explicitly as the sum of the linear term R which reproduces GR plus a perturbation,

$$f(R) = R + \tilde{f}(R) + \lambda R_0, \tag{3.1}$$

where $\tilde{f}(R)$ represents the deviation of the model from GR, and the ΛCDM model can be obtained as a special case with $\tilde{f}(R) = 0$, where $\lambda = -2\Lambda/R_0$, and Λ is the cosmological constant. Thus, when defining the dimensionless coordinate by making $x = R/R_0$,

$$y(x) = x + h(x) + \lambda, \tag{3.2}$$

where $y(x) = f(R_0x)/R_0$, $\tilde{f}(R_0x) = R_0h(x)$, and with the definition of the characteristic functions [10]

$$m = \frac{Rf''(R)}{f'(R)} = \frac{xh''(x)}{1+h'(x)}, \tag{3.3}$$

and

$$r = -\frac{Rf'(R)}{f(R)} = -\frac{x\left[1+h'(x)\right]}{x+h(x)+\lambda}. \tag{3.4}$$

Now, let us suppose that $h(x)$ is a continuous function, with continuous derivatives in a domain I, such that $x_i \in I$ is an inflection point of $h(x)$, not stationary nor of infinite slope. That is, $h''(x_i) = 0$, which means that $h'(x_i)$ is a maximal point. The existence of an inflection point, together with the conditions of asintoticity towards ΛCDM [86],

$$\lim_{x\to 0} h(x) = -\lambda, \tag{3.5}$$

and

$$\lim_{x\to\infty} h(x) = \frac{k}{R_0} - \lambda, \tag{3.6}$$

where k is a constant that depends on the model; restrict the form of the function $h(x)$ to only two possibilities: decreasing concave (increasing convex) at $0 < x < x_i$ and decreasing convex (increasing concave) at $x_i < x$. Motivated by the results of Starobinsky model [84] whose virtue lies in the quadratic term, which reproduces accelerated expansion of the Universe without the need to introduce Dark Matter, and for $h(x)$ to contain only quadratic terms when expanded in Maclaurin series, it must be an even function; this implies that $x = 0$ is a maximal point and therefore

$$\lim_{x\to 0} h'(x) = 0, \tag{3.7}$$

and at the other hand at infinity the curve is flattened according to Eq. (3.6), thus

$$\lim_{x\to\infty} h'(x) = 0, \tag{3.8}$$

and in turn

$$\lim_{x\to 0} h''(x) = c, \tag{3.9}$$

	$h(x)$		
	Decreasing	Increasing	Domain
$h'(x)$	$-$	$+$	$x > 0$
$h''(x)$	$-$	$+$	$0 < x < x_i$
$h''(x)$	$+$	$-$	$x > x_i$
$h'(x) - h'(x_i)$	$+$	$-$	$x_i > 0$
$h'''(x_i)$	$+$	$-$	

Table 3.1: Sign of the first derivatives of the function $h(x)$ according to its monotonicity in the domain given in the last column.

where c is a constant, and

$$\lim_{x\to\infty} h''(x) = 0, \tag{3.10}$$

The two possible behaviors of the function affect the sign of its derivatives, as discussed in Table 3.1. Only models with a characteristic function, (3.3), $m \geq 0$ and close to ΛCDM can be considered cosmologically viable [10], thus there are two options for $h(x)$ in which this can be fulfilled, $h''(x) \geq 0$ and $h'(x) > -1$, or $h''(x) \leq 0$ and $h'(x) < -1$; so if $h(x)$ is a decreasing function, for $x \geq x_i$,

$$-1 < h'(x) < 0, \tag{3.11}$$

or for $0 \leq x \leq x_i$,

$$h'(x) < -1. \tag{3.12}$$

From the second option, when $h(x)$ is an increasing function, it is only possible to choose

$$h'(x) > 0, \tag{3.13}$$

for $0 < x \leq x_i$.

Due to condition (3.7), option (3.12) is discarded and to avoid singularities in the characteristic functions $m(x)$ and $r(x)$, option (3.13) will also be discarded. The inflection point x_i leads to the appearance of a minimum (maximum) in $h'(x)$ if $h(x)$ is a decreasing (increasing) function, and by Eq. (3.10), there is an inflection point, x_m, on the curve of $h'(x)$ and therefore $h''(x_m)$ will be a maximal at $x > x_i$ and $h''(x)$ will be a decreasing (increasing) function for $x > x_m$. The function $h''(x)$ is integrable in all its domain, that is, by limits (3.7) and (3.8), $\int_0^\infty h''(x)dx = 0$, thus the decreasing monotonicity of $h''(x)$ from x_m is the property that allows to consider the option (3.11) as the most viable, because if $x_m < \frac{x}{2} \leq t \leq x$, then

$$0 \leq h''(x) \leq h''(t), \tag{3.14}$$

so that

$$0 \leq \int_{x/2}^{x} h''(x)dt \leq \int_{x/2}^{x} h''(t)dt, \tag{3.15}$$

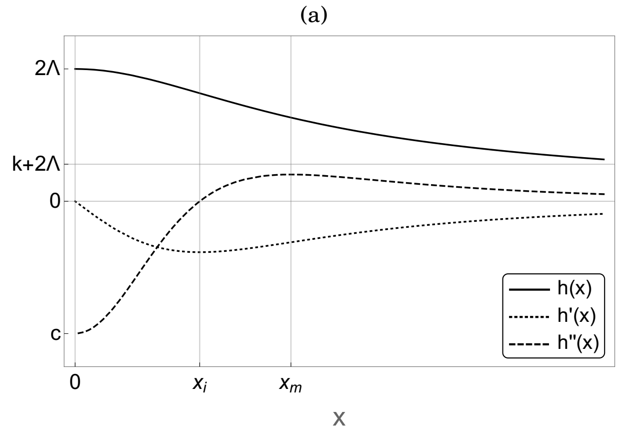


FIGURE 3.1. Behaviour of the function h and its derivatives as a function of x, with the necessary conditions for the model to be viable, equations (3.5) to (3.10), together with the existence of an inflection point, x_i. The image also shows the existence of the maximum of $h''(x)$, x_m. In this diagram $R_0 = 1$ was chosen.

or

$$0 \leq x h''(x) \leq 2\left[h'(x) - h'(x/2)\right]. \tag{3.16}$$

Similarly, by integrating Eq. (3.11), it is obtained

$$0 < x + h(x) + \lambda. \tag{3.17}$$

Conditions (3.16) and (3.17) are useful for calculating the limits of the r function (3.4), as will be seen below.

3.2 Characteristic functions

A model that reproduces a matter-dominated era with a corresponding transition to accelerated expansion must satisfy [10]

$$m(r) \approx +0, \quad \text{and} \quad m'(r) > -1 \quad \text{at} \quad r \approx -1, \tag{3.18}$$

where

$$\left.\frac{dm}{dr}\right|_x = \frac{x m'(x)}{r(x)[1 + m(x) + r(x)]}. \tag{3.19}$$

There are three points x for which $r(x) \approx -1$, these are $x_1 \to 0$,

$$\lim_{x \to 0} \frac{x\left[1 + h'(x)\right]}{x + h(x) + \lambda} = 1, \tag{3.20}$$

where we have used Eq. (3.5) and (3.7);

$$x_2 \to \frac{h(x_2)+\lambda}{h'(x_2)}; \tag{3.21}$$

and $x_3 \to \infty$, since

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{1+h'(x)+xh''(x)}{1+h'(x)} = 1, \tag{3.22}$$

where L'Hospital's rule and condition (3.16) were used. Now, since $m(x_1) = -0$, and note that the derivative of $m(r)$ can be expressed as

$$m'(r)|_x = -\frac{(\lambda+x+h(x))^2}{\lambda+h(x)} \frac{\big(1+h'(x)-xh''(x)\big)h''(x)+\big(1+h'(x)\big)xh'''(x)}{(1+h'(x))^2\left[(1+h'(x))\left(1-\frac{xh'(x)}{\lambda+h(x)}\right)+xh''(x)+\frac{x^2h''(x)}{\lambda+h(x)}\right]}, \tag{3.23}$$

such that, by Eq. (3.5), (3.7) and (3.9),

$$\lim_{x\to 0} \frac{(\lambda+x+h(x))^2}{\lambda+h(x)} = \lim_{x\to 0} \frac{2\big(1+h'(x)\big)^2+2(\lambda+x+h(x))h''(x)}{h''(x)} = \frac{2}{c}, \tag{3.24}$$

$$\lim_{x\to 0} \frac{xh'(x)}{\lambda+h(x)} = \lim_{x\to 0} \frac{2h''(x)+xh'''(x)}{h''(x)} = 2, \tag{3.25}$$

and

$$\lim_{x\to 0} \frac{x^2h''(x)}{\lambda+h(x)} = \lim_{x\to 0} \frac{2h''(x)+4xh'''(x)+x^2h^{(4)}(x)}{h''(x)} = 2, \tag{3.26}$$

therefore

$$\lim_{x\to 0} m'(r)|_x = -2. \tag{3.27}$$

hence

$$m'(r)|_{x_1} \overset{x\to 0}{=} -2, \tag{3.28}$$

we discard x_1, and noting that $m(x)$ is directly proportional to $h''(x)$, x_i is also a root of $m(x)$, i.e $x_2 \to x_i$, but to be a valid point, x_2 must tend to x_i on the right, x_{i+}, satisfying $h(x_{i+}) = x_{i+}h'(x_{i+})-\lambda$, however the last term of Eq. (3.19) diverges when $x \to x_i$ and $h'''(x_{i+}) > 0$ because $h'(x_{i+})$ is a minimum, so point x_2 is also discarded. On the contrary, x_3 is in itself a valid point that gives viability to the model, since by Eq. (3.8) and (3.16),

$$\lim_{x\to\infty} m = +0, \tag{3.29}$$

and

$$m'(r) \overset{x\to\infty}{=} 0. \tag{3.30}$$

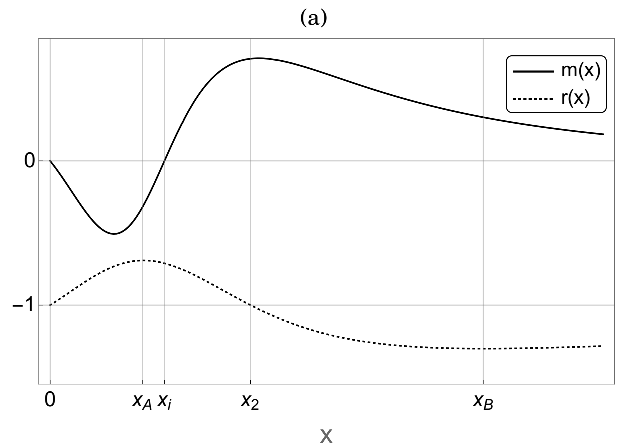


FIGURE 3.2. Characteristic functions m and r versus x. The inflection point, x_i, the x_2 point, in which $r = -1$, and the maximum points of $r(x)$, x_A and x_B, are appreciated.

Since $0 < 1 + h'(x)$, for $0 < x < x_i$, $h''(x) < 0$, and $m(x) < 0$, and simultaneously for $x > x_i$, $m(x) > 0$, therefore Eq. (3.29) expresses that $m(x)$ should be flattened towards zero at infinity. On the other hand, $r(x) > -1$ for $0 < x < x_2$, then it will have a minimum and will tend to -1 at infinity, Fig. (3.2).

The behavior of $m(r)$ in the phase space can now be drawn as shown in Fig. 3.3, where $m'(r)$ is also observed. It should be noted that $r'(x) = r(1 + m + r)/x$, so that the maximal points of r are found when $m + r + 1 = 0$, that is, the points where $m'(r)|_x$ diverges. Since $r = -1$ in x_1, x_2 and x_3, it has two maximum points, x_A and x_B in Fig. 3.2, and by definition, in these points the derivative is infinite, shown in Fig. 3.3, as (r_A, m_A) and (r_B, m_B).

In the next section a differential equation for $h(x)$ will be constructed from its geometry, considering the parity of the functions $h'(x)$ and $h''(x)$ and multiplying them by functions so that their roots coincide.

3.3 Starobinsky type models

Since $h(x)$ is an even function, $h'(x)$ and $h''(x)$ are odd and even functions respectively, as can be seen in Fig. 3.1, so it can be inferred that the function $xh''(x)$ will be odd with roots in $x = 0$ and $x = x_i$. Simultaneously, when multiplying $h'(x)$ by the factor $(x_i^2 - x^2)$, the same intervals of increase and decrease of $xh''(x)$ are obtained, besides the same roots, for both negative and positive x. Let us assume that $h'(x)$ and $h''(x)$ can be related by

$$q(1 + p(x))xh''(x) = \left(x_i^2 - x^2\right) h'(x), \tag{3.31}$$

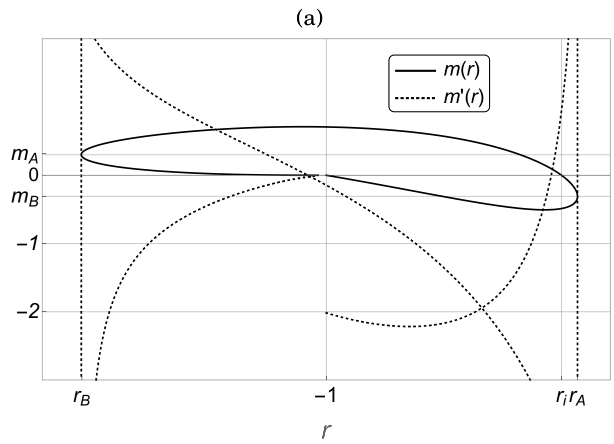


FIGURE 3.3. Phase space in the plane (m,r) as well as the corresponding derivative $(m'(r))$. It can be seen that in the points r_A and r_B the derivative is infinite. Note the behaviour of $m'(r)$ at $r(x=0)=-1$, given by Eq. (3.28), and at $r(x\to\infty)=-1$, the latter allowing the viability of the model.

where q is some constant and $p(x)$ is a function that is linearly independent of $h(x)$ and also even so that the left member of Eq. (3.31) is odd, moreover

$$\lim_{x\to 0} p(x) = 0, \tag{3.32}$$

and

$$\lim_{x\to\infty} \frac{1}{1+p(x)} = \lim_{x\to\infty} \frac{q}{x^2 - x_i^2} = 0. \tag{3.33}$$

since q is a constant, it is possible to evaluate it at some limit, for example

$$q = \lim_{x\to 0} \frac{(x_i^2 - x^2)h'(x)}{(1+p(x))xh''(x)} = \lim_{x\to 0} \frac{x_i^2}{1+p(x)} = x_i^2, \tag{3.34}$$

thus Eq. (3.31) can be rewritten as

$$(1+p(x))\frac{1}{x}h''(x) + \left(\frac{1}{x_i^2} - \frac{1}{x^2}\right)h'(x) = 0, \tag{3.35}$$

and integrating it

$$\frac{h(x)}{x_i^2} + \frac{h'(x)}{x} + \int \frac{1}{x}p(x)h''(x)dx = 0, \tag{3.36}$$

where it can be seen why $1+p(x)$ was chosen in Eq. (3.31) rather than $p(x)$. The last integral can be evaluated by parts twice, so that it does not depend on $h''(x)$, but on $h(x)$ and $h'(x)$, or equivalently for the purpose of reducing the order of the differential Eq. (3.31).

$$\int \frac{1}{x} p(x) h''(x) dx = \frac{1}{x} p(x) h'(x) + h(x) \left(\frac{p(x)}{x^2} - \frac{p'(x)}{x} \right) + \int \frac{h(x)}{x^3} \left(x^2 p''(x) - 2x p'(x) + 2p(x) \right) dx, \tag{3.37}$$

then $p(x)$ can be chosen as the solution of the differential equation

$$x^2 p''(x) - 2x p'(x) + 2p(x) = 0, \tag{3.38}$$

that is

$$p(x) = x(\alpha + \beta x), \tag{3.39}$$

with α and β constants, but for $p(x)$ to be an even function, $\alpha = 0$, so the equation is reduced to

$$\left(\frac{1}{x} + \beta x \right) h'(x) + \left(\frac{1}{x_i^2} - \beta \right) h(x) + c_1 = 0, \tag{3.40}$$

whose solution is

$$h(x) = \frac{k}{R_0} \left[1 - \left(1 + \beta x^2 \right)^{\frac{1}{2} - \frac{1}{2\beta x_i^2}} \right] - \lambda, \tag{3.41}$$

where the integration constants were found by Eq. (3.5) and (3.6) for $\beta > 0$ and $\beta x_i^2 < 1$, which, in turn, requires that the power must be negative,

$$\frac{1}{2} \left(1 - \frac{1}{\beta x_i^2} \right) = -m, \tag{3.42}$$

with $m > 0$, or analogously, when $x_i = (2m-1)^{-1/2}$ and $\beta = 1$,

$$h(x) = \frac{k}{R_0} \left[1 - \left(1 + x^2 \right)^{-m} \right] - \lambda, \tag{3.43}$$

so without loss of generality, it can be concluded that Eq. (3.41) actually represents Starobinsky's model [181]. In this way it is easy to find the value of the constant c, Eq. (3.9),

$$c = \frac{2km}{R_0}, \tag{3.44}$$

and therefore

$$y(x) = x + \frac{k}{R_0} \left[1 - \left(1 + x^2 \right)^{-m} \right], \tag{3.45}$$

where m, k, R_0 are parameters. In the next section, a generalized Hu-Sawicki model will be established through a similar procedure.

3.4 Hu-Sawicki type models

Equation (3.31) relates $h'(x)$ and $h''(x)$ in a geometric way through the inflection point x_i, that is by the root of $h''(x)$, provided by the difference $x_i^2 - x^2$ in the right member of the equation. However, in the more general case it is possible to write

$$ts(x)xh''(x) = (x_i^r - x^r)h'(x), \tag{3.46}$$

where t is constant, $r > 0$ is an even number and $s(x)$ is a continuous even function that, as in the previous section, will be requested linearly independent of $h(x)$, satisfying

$$\lim_{x\to x_i} s(x) = -\frac{rx_i^{r-2}h'(x_i)}{th'''(x_i)}, \tag{3.47}$$

and

$$\lim_{x\to\infty} \frac{1}{s(x)} = 0. \tag{3.48}$$

Note that Eq. (3.46) can be integrated as

$$\frac{1}{1+r}x\left[x^r - (1+r)x_i^r\right]h'(x) + \int\left[ts(x) - \frac{x^r}{1+r} + x_i^r\right]xh''(x)dx + c_1 = 0, \tag{3.49}$$

and integrating by parts twice the last integral

$$\begin{aligned}\frac{1}{1+r}x\left[x^r - (1+r)x_i^r\right]h'(x) + \left(ts(x) - \frac{x^r}{1+r} + x_i^r\right)xh'(x) + \left[x^r - x_i^r - t\left(s(x) + xs'(x)\right)\right]h(x) - \\ \int\left[rx^{r-1} - t\left(2s'(x) + xs''(x)\right)\right]h(x)dx + c_1 = \quad 0,\end{aligned} \tag{3.50}$$

for a null integrand,

$$s(x) = \frac{x^r}{t(1+r)} - \frac{\alpha}{x} + \beta, \tag{3.51}$$

with $\alpha = 0$ to allow the function to be even, thus Eq. (3.46) can be written as

$$\left(\frac{x^r}{1+r} + t\beta\right)xh'(x) - (x_i^r + t\beta)h(x) + c_1 = 0, \tag{3.52}$$

whose solution is

$$h(x) = \frac{k}{R_0}x^{1+\frac{x_i^r}{t}}\left[x^r + (1+r)t\right]^{-\frac{x_i^r+t}{rt}} - \lambda, \tag{3.53}$$

where β was absorbed in t, the constants were established from conditions (3.5) and (3.6), and by (3.11), $k < 0$. However, for this function to be even, the ratio x_i^r/t must be an odd number, and since the parameter r is even, they can be related by means of

$$x_i^r = (nr - 1)t, \tag{3.54}$$

where n is a natural number, and consequently, the function can be expressed as

$$h(x) = \frac{k}{R_0}\left[1+(1+r)tx^{-r}\right]^{-n} - \lambda, \tag{3.55}$$

with which for $nr > 2$ and $t > 0$, $c = 0$. In this scenario Eq. (3.2) is now expressed as

$$y(x) = x + \frac{k}{R_0}\left[1+(1+r)tx^{-r}\right]^{-n}. \tag{3.56}$$

As a particular case, when $n = 1$ and defining

$$c_1 = -\frac{c_2 k}{R_0}, \qquad \text{and} \qquad c_2 = \frac{1}{(1+r)t}, \tag{3.57}$$

the inflection point is obtained at

$$x_i^r = \frac{r-1}{c_2(r+1)}, \tag{3.58}$$

and

$$h(x) = -\lambda - \frac{c_1 x^r}{1+c_2 x^r}, \tag{3.59}$$

so

$$y(x) = x - \frac{c_1 x^r}{1+c_2 x^r}, \tag{3.60}$$

which is the Hu-Sawicki model [86], where $c = 0$. Nevertheless, because $r > 0$, it is not possible to obtain Starobinsky's model from Eq. (3.55) and therefore Eq. (3.41) and (3.55) represent different models, however, these models are part of a more general class of models, as will be seen in the next section.

3.5 Hypergeometric models

The similarities of models given by equations (3.41) and (3.55) can be found in the form of their differential equations, as well as in the possible values of their respective parameters. To see this, Eq. (3.40) is rewritten as

$$\frac{1+\beta x^2}{(1+2m-\beta)x}h'(x) + h(x) + \lambda - \frac{k}{R_0} = 0, \tag{3.61}$$

and Eq. (3.31)

$$\left(1+\beta x^2\right)xh''(x) + \left[(1+2m)x^2 - 1\right]h'(x) = 0, \tag{3.62}$$

multiplying Eq. (3.61) for nr and making $\beta = 1$, $v = -1$ and $r = -2$, these equations can be combined as

$$(v - x^r)x^2 h''(x) + \left[v\left(1-(m+n)r\right) + (1+nr)x^r\right]xh'(x) + mnr^2 v h(x) + 2mnr\left(\lambda - \frac{k}{R_0}\right) = 0. \tag{3.63}$$

In the the same manner, it is possible to express Eq. (3.52) and (3.46), respectively, as

$$\left(\frac{x^r}{t(1+r)}+1\right)xh'(x)-nrh(x)-nr\lambda=0, \tag{3.64}$$

and

$$\left(\frac{x^r}{t(1+r)}+1\right)xh''(x)+\left(\frac{x^r}{t}+1-nr\right)h'(x)=0, \tag{3.65}$$

or combined as

$$(v-x^r)x^2h''(x)+\left[v(1-(m+n)r)+((m-1)r-1)x^r\right]xh'(x)+mnr^2vh(x)=0, \tag{3.66}$$

where $v=-t\beta(1+r)$ and $\lambda=0$. Therefore, a generalization of Eq. (3.63) and (3.66) can be made as follows,

$$(v-x^r)x^2h''(x)+\left[v(1-(m+n)r)+((u-1)r-1)x^r\right]xh'(x)+mnr^2v\left(h(x)+\lambda-\frac{c}{2m}\right)=0, \tag{3.67}$$

where u is a parameter that can be adjusted according to the type of model, when $u=n$ and $r=-2$, the Starobinsky type model is obtained, Eq. (3.43), and when $u=m$, the Hu-Sawicki one, Eq. (3.55), is found. Now with the variable change

$$z=vx^{-r}, \tag{3.68}$$

it is realized that Eq. (3.67) is in effect the hypergeometric equation

$$(1-z)zg''(z)+(u-(1+m+n)z)g'(z)-mng(z)=0, \tag{3.69}$$

where

$$g(z)=h(z)+\lambda-\frac{c}{2m}, \tag{3.70}$$

by choosing the constants appropriately, according to equations (3.43) and (3.55), the solution can be written as (for $m,n,u,v,r\neq 0$)

$$h(z)=\left(\frac{k}{R_0}-\frac{c}{m}\right){}_2F_1(m,n,u;z)+\frac{c}{2m}-\lambda, \tag{3.71}$$

together with the condition of existence of the inflection point, given by the algebraic equation

$$\frac{{}_2F_1(m+1,n+1;u+1;z_i)}{{}_2F_1(m+2,n+2;u+2;z_i)}=\frac{(m+1)(n+1)r}{(r+1)(u+1)z_i}. \tag{3.72}$$

If this equation ensures the existence of a single point of inflection, it remains to analyze the domain of the parameters in which $h(x)$ is viable. When $r>0$, the model naturally satisfies limits Eq. (3.8) and Eq. (3.10), however to satisfy the limits (3.5) and (3.6), $c=0$, and at the same time, by the series expansion of $h'(x)$ and $h''(x)$, it is observed that $m>2/r$, $n>2/r$, $v\neq 0$, $m-n\notin\mathbb{Z}$ and $u\neq w$ with $w\in\mathbb{Z}$ and $w\leq 0$, for the model to meet limits (3.7) and (3.9). In addition, Euler's integral representation of the hypergeometric function allows to further restrict the parameters of the model according to Eq. (3.11), since for $x>0$, $-1<n<u$ and $v<0$ (with $R_0=1$)

$$h'(x) = -\frac{kmrvx^{mr-1}\Gamma(u)}{\Gamma(n)\Gamma(u-n)}\int_0^1 \frac{(1-t)^{u-n-1}t^n}{(x^r-vt)^{m+1}}dt, \tag{3.73}$$

where $\Gamma(x)$ is the Gamma function. Due to the positive integrand and the integral definition interval, the integral is positive, and because

$$\frac{\Gamma(u)}{\Gamma(n)\Gamma(u-n)} > 0, \tag{3.74}$$

for $n<u$, then for $h'(x)<0$, $k<0$. Simultaneously, in order that $h'(x)>-1$,

$$\frac{u(r+1)(u+1)x_i^{2r+1}}{|k|r^2v^2mn(m+1)(n+1)} > {}_2F_1\left(m+2,n+2;u+2;\frac{v}{x_i^r}\right), \tag{3.75}$$

where x_i is the inflection point, obtained from Eq. (3.72). Numerically it is found that for $0<r<2$, $m\geq 1$, $n\geq 1$, $n<u<n+1$, $v<-1$,

$$x^{r+1} > {}_2F_1\left(m+1,n+1;u+1;vx^{-r}\right), \tag{3.76}$$

so that if $u>kmnrv$, then $h'(x)>-1$.

Alternatively, a sufficient condition for $h'(x)>-1$ is

$$\frac{\left(1+\frac{1}{r}\right)_{w-1}(u)_w}{r\,|kv^w|\,(m)_w(n)_w} > \frac{{}_2F_1\left(m+w,n+w;u+w;\frac{v}{x_i^r}\right)}{x_i^{rw+1}}, \tag{3.77}$$

where $(a)_w$ is the Pochhammer symbol.

On the other hand, when $r<0$, for limit Eq. (3.5) to be satisfied, $c=2km/R_0$, whereas for limits (3.6), (3.8) and (3.10), $m>0$, $n>0$ and, as in the previous case, $u\neq w$. Likewise, when $r\leq -2$, limit Eq. (3.7) is satisfied [1], however when $r=-2$,

$$\lim_{x\to 0} h''(x) = -\frac{2kmnv}{uR_0}, \tag{3.78}$$

so that $u=-nv$, which in turn implies that $v<-1$. When $r<-2$, limit given by Eq. (3.9) is fulfilled. Note that although the value of the constant c depends on the sign of r, the Eq. (3.73) is still valid in this case, $r\leq -2$, so the restrictions on the parameters that were made previously, i.e. $n<u$ and $k<0$, remain valid.

Finally, the hypergeometric model can be expressed using Eq. (3.2),

$$y(x) = x + \left(\frac{k}{R_0}-\frac{c}{m}\right){}_2F_1\left(m,n,u;\frac{v}{x^r}\right)+\frac{c}{2m}, \tag{3.79}$$

and contains the generalized Starobinsky type (3.45) and Hu-Sawicki (3.56) type models, since when $u=n$, $r=-2$, $v=-1$ and $c=2km/R_0$, the first one is obtained, and when $u=m$, $v=-(1+r)t$ and $c=0$, the second one is obtained, therefore Eq. (3.79) can be considered as a generalization of these models.

[1] When $r\in(0,2)$, $r\neq -1$, both limits Eq. (3.7) and Eq. (3.9) are indeterminate, and for $r=-1$, $\lim_{x\to 0} h'(x) = -\frac{kmnv}{uR_0}\neq 0$

CHAPTER 4

SPHERICALLY SYMMETRIC VACUUM SOLUTION

The solutions of the field equations in spherically symmetric and static (SSS) spacetime have been widely investigated in GR and therefore constitute the starting point for analysing the new physics introduced by the non-linear terms of the scalar curvature in $f(R)$ theory. Different works have been done on this topic, including Schwarzschild-type solutions of constant R and $R = R(r)$ [44, 165], even using the Noether symmetry approach [43], charged SSS Black Hole (BH) solutions [131], and for particular models in cylindrically symmetric spaces [124, 126]. All these solutions are of paramount importance because of the differences and similarities they can offer with respect to the case of GR. For example the non-hair theorem (NHT) can be generalized into $f(R)$ for models that meet certain conditions (except $f(R) = R^2$) see [37], however models can be found that lead to *hairy* Asymptotically-Flat-SSS BH solutions for which R is not a constant and thus evade the theorem [38]. Similarly, solutions in $f(R)$ gravity have been found with R coupled to non-linear electromagnetic sources, which turn out to be generalizations of regular BH in GR [159].

This chapter explores the solutions of field equations in SSS spaces in $f(R)$ gravity coupled with EM fields in the BI theory[1] [24], first, the field equations in this theory and for a SSS spacetime are written and the conditions for function $f(R)$ expressed parametrically in terms of the variable r are determined. then, the GR-BI solution is shown as a particular case. The Schwarzschild-BI-type and $f(R)$-Maxwell solutions are presented as limit cases of the $f(R)$-BI solution and the constraints on the forms of the models $f(R)$ generated by those solutions are shown. Finally we show a particular solution of the $f(R)$-BI theory and its restrictions on the function $f(R)$.

[1]The Born-Infeld theory is a proposal of nonlinear electrodynamics with the objective of eliminating the singularity of electromagnetic fields present in the Maxwell's theory.

4.1 Maximally symmetric space

One important and simple solution to the field equations (2.28) is the Schwarzschild-like solution for maximally symmetric spaces, which are manifolds characterized by

1. Constant scalar curvature: $R = constant$

2. Ricci tensor proportional to the scalar curvature: $R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$

hereafter we will call $R = R_0$, so that

$$[f'(R_0)]_{,\mu} = \frac{\partial f'(R_0)}{\partial x^\mu} = \frac{\partial R_0}{\partial x^\mu}\frac{\partial f'(R_0)}{\partial R_0} = 0. \tag{4.1}$$

thus, field equations in vacuum ($T_{\mu\nu} = 0$), takes the form

$$f'(R_0)R_{\mu\nu} - \frac{1}{2}f(R_0)g_{\mu\nu} = 0, \tag{4.2}$$

and the algebraic equation for the trace (2.30) gives

$$f(R_0) = \frac{1}{2}f'(R_0)R_0, \tag{4.3}$$

by replacing in the equation (4.2),

$$f'(R_0)\left(R_{\mu\nu} - \frac{1}{4}R_0 g_{\mu\nu}\right) = 0. \tag{4.4}$$

The last condition for the Ricci tensor is stated as a property for the maximally symmetric spaces, however here it is obtained from the intrinsic dynamic imposed by the field equations in $f(R)$ theory. Also note that for the case $R_0 = 0$ the Einstein field equations in vacuum are obtained, that is $R_{\mu\nu} = 0$.

Besides, the spacetime surrounding a static and spherically symmetric object of mass M is determined by the condition of homogeneity and isotropy, which can be interpreted as the only form of the metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \tag{4.5}$$

with the angular element $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ as usual, and the exponential are chosen so the signature of the metric does not change. The non-vanishing components of the Ricci tensor, (2.5), are

$$R_{tt} = e^{2[\alpha(r)-\beta(r)]}\left[A(r) + \frac{2\alpha'(r)}{r}\right], \tag{4.6}$$

$$R_{rr} = -A(r) + \frac{2\beta'(r)}{r}, \tag{4.7}$$

$$R_{\theta\theta} = -e^{-2\beta(r)}\left(1 + r\left[\alpha'(r) - \beta'(r)\right]\right) + 1, \tag{4.8}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}, \tag{4.9}$$

where we have defined the function

$$A(r) = \alpha''(r) + \alpha'^2(r) - \alpha'(r)\beta'(r). \tag{4.10}$$

With these components the Ricci scalar can be obtained as

$$\begin{aligned}
R_0 =& g^{\mu\nu}R_{\mu\nu} \\
=& -e^{-2\beta}\left[A(r) + \frac{2\alpha'}{r}\right] + e^{-2\beta}\left[-A(r) + \frac{2\beta'}{r}\right] + \frac{2}{r^2}\left(-e^{-2\beta}\left[1 + r\left(\alpha' - \beta'\right)\right] + 1\right) \\
=& -2e^{-2\beta}\left[A(r) + \frac{2}{r}\left(\alpha' - \beta'\right) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right],
\end{aligned} \tag{4.11}$$

However, the four components of the Ricci tensor are not independent

$$\begin{aligned}
R_{tt} + e^{2(\alpha-\beta)}R_{rr} =& \frac{2}{r}e^{2(\alpha-\beta)}\left(\alpha' + \beta'\right) \\
R_{rr} =& \frac{2}{r}\left(\alpha' + \beta'\right) - e^{-2(\alpha-\beta)}R_{tt},
\end{aligned} \tag{4.12}$$

thus, the field equation (4.4) for the rr component is

$$\begin{aligned}
4R_{rr} - g_{rr}R_0 =& 0 \\
\frac{8}{r}\left(\alpha' + \beta'\right) - 4e^{-2(\alpha-\beta)}R_{tt} - e^{2\beta}R_0 =& 0 \\
\frac{8}{r}\left(\alpha' + \beta'\right) - 4\left[A(r) + \frac{2\alpha'}{r}\right] + 2\left[A(r) + \frac{2}{r}\left(\alpha' - \beta'\right) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right] =& 0 \\
\frac{2}{r}\left(\alpha' + \beta'\right) - A(r) + \frac{1}{r^2}\left(1 - e^{2\beta}\right) =& 0,
\end{aligned} \tag{4.13}$$

so, the equation of motion is

$$A(r) = \frac{2}{r}\left(\alpha' + \beta'\right) + \frac{1}{r^2}\left(1 - e^{2\beta}\right). \tag{4.14}$$

On the other hand, the component $\theta\theta$ provides the equation

$$\begin{aligned}
4\theta R_{\theta\theta} - R_0 r^2 =& 0 \\
2\left[1 + r\left(\alpha' - \beta'\right)\right] - 2e^{2\beta} - \left[A(r) + \frac{2}{r}\left(\alpha' - \beta'\right) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right]r^2 =& 0 \\
1 - e^{2\beta} - r^2 A(r) =& 0,
\end{aligned} \tag{4.15}$$

and again we obtain an equation for $A(r)$

$$A(r) = \frac{1}{r^2}\left(1 - e^{2\beta}\right). \tag{4.16}$$

Thus, from equations (4.14) and (4.16)

$$\alpha'(r) = -\beta'(r), \tag{4.17}$$

so that when replacing in either of two equations (4.14) and (4.16), we get the differential equation for $\beta(r)$

$$r^2\beta''(r) - 2r^2\beta'^2(r) - e^{2\beta(r)} = -1, \tag{4.18}$$

whose solution is

$$\beta(r) = -\frac{1}{2}\ln\left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right), \tag{4.19}$$

where c_1 and c_2 are constants to be determined. Thereby

$$\alpha(r) = \frac{1}{2}\ln\left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right) + c_3, \tag{4.20}$$

where, again c_3 is the constant of integration, and the metric (4.5) can be written as

$$ds^2 = -e^{c_3}\left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right)dt^2 + \left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{4.21}$$

Finally by rescaling the temporal coordinate $t \to e^{c_3}t$, the metric for a points at exterior of a spherically symmetric body in $f(R)$ theory takes the form

$$ds^2 = -\left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right)dt^2 + \left(1 + \frac{2c_1}{3}r^2 - \frac{c_2}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{4.22}$$

It is important to make clear that spite of the functional form of the Ricci Scalar with respect to the r coordinate, as can be seen in the equation (4.11), this dependence has to be cancelled so that curvature remains constant, by equations (4.16), (4.19) and (4.17)

$$\begin{aligned} R_0 &= -2e^{-2\beta}\left[A(r) + \frac{2}{r}\left(\alpha' - \beta'\right) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right] \\ &= -\frac{4}{r^2}\left(e^{-2\beta} - 1 - 2re^{-2\beta}\beta'\right) \\ &= -\frac{4}{r^2}\left(\frac{2c_1}{3}r^2 - \frac{c_2}{r} + \frac{4c_1}{3}r^2 + \frac{c_2}{r}\right) \\ &= -8c_1, \end{aligned} \tag{4.23}$$

therefore the Ricci scalar is constant, $R_0 = -8c_1$, as it must be.

To see the value of c_1, we turn back to field equations (4.2) using the value found for the scalar curvature, and the Ricci component tensor (4.4)

$$c_1 = -\frac{1}{4}\frac{f(R_0)}{f'(R_0)}. \tag{4.24}$$

Notice also that if $f(R) = R$ GR is retaken, $c_1 = R_0$, and $R_0 = 0$, so the field equations (4.2) turn into the Einstein equations for vacuum and the metric (4.22) becomes the Scwarzschild metric as usual.

$$ds^2 = -\left(1 - \frac{c_2}{r}\right) dt^2 + \left(1 - \frac{c_2}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{4.25}$$

in that case, for correspondence with the Schwarzschild metric, if $c_1 = 0$ then $c_2 = R_S = 2M$, where M is the mass of the object. Finally, we can write the line element 4.22, as

$$\boxed{ds^2 = -\left(1 - \frac{1}{6}\frac{f(R_0)}{f'(R_0)} r^2 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{1}{6}\frac{f(R_0)}{f'(R_0)} r^2 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}. \tag{4.26}$$

Metric (4.26) shows that $f(R)$ reduces to (Anti) de Sitter spacetime when it comes a Maximally Symmetric Space.

4.1.1 Born-Infeld BH solution in GR

Metric (4.26) was constructed under the assumptions of constant curvature and staticity of the BH, however we can extend the characteristics of the BH in the $f(R)$ gravity by assuming that it has an net electric and magnetic charge, this class of solutions are called Einstein-Maxwell-BH (EMBH), which, through the appropriate replacements, results equivalent to the Reissner-Nordström BH (RNBH) in the AdS Spacetime. In this section we are going to extend again the EMBH solutions to the charged BH in the so called Einstein-Born-Infeld theory for $f(R)$ gravity.

Maxwell's classical electromagnetism is based on the definition of the electromagnetic tensor[2] F

$$F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu} \tag{4.27}$$

with the electromagnetic four-potential $A^\mu = (\phi, \mathbf{A})$. Electric and magnetic fields are expressed as $\mathbf{E} = -\nabla\phi - \mathbf{A}_{,t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, whose dynamics is determined by the Maxwell equations

$$\partial_\mu F^{\mu\nu}_{,\mu} = J^\nu, \tag{4.28}$$

and

$$F_{[\mu\nu,\sigma]} = 0; \tag{4.29}$$

where $J = (\rho, \mathbf{J})$ is the four current density (ρ and $\mathbf{J}$ are charge and current densities). The Einstein-Maxwell theory is a modification of the Einstein field equations in which matter is coupled with the fields through the traceless electromagnetic stress-energy tensor

$$T_{\mu\nu} = F_{\mu\sigma} F^{\sigma}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}, \tag{4.30}$$

[2]In order to not confuse this notation with $F(R) = f'(R)$, in the remainder of this chapter the dependency of such derivative with respect to R will be made explicit.

whose solutions associated to field equations in GR

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.31}$$

for a charged, static and spherically symmetric body, are called Reissner-Nordström BH [47]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2+p^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2+p^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{4.32}$$

where M is interpreted as the mass, q and p are the electric and magnetic charge of the BH.

4.2 Field equations in BI-$f(R)$ theory

The dynamics of the spacetime in $f(R)$ theory of gravity is determined by the field equations (2.28), which are found from the least action principle applied to the action

$$I = \frac{1}{2\kappa}\left(\int_\Sigma d^4x\sqrt{-g}f(R) + I_{GYH}\right) + \int d^4x\sqrt{-g}L_M, \tag{4.33}$$

defined over a hypervolume Σ, where I_{GYH}, represents the Gibbons-York-Hawking term that corrects the boundary action value problem, and L_M is the stress-energy Lagrangian contribution, which in the BI formulation for an electromagnetic field in vacuum [29, 30], is written as

$$L_M(\mathrm{F}) = b^2\left(1 - \sqrt{1 + \frac{2\mathrm{F}}{b^2}}\right), \tag{4.34}$$

where b is the BI parameter, which can be chosen in such a way that the classical fields of Maxwell's theory are obtained as an approximation of the BI theory, i.e. in the limit $b \to \infty$, while when $b \to 0$, finite values of the electromagnetic fields are obtained. At the same time, $\mathrm{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell's classic Lagrangian density, with the electromagnetic field tensor $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and A_μ is the electromagnetic four-potential. Under conditions of spherical symmetry and staticity[3] the only nonzero components of the field tensor are, without sources of magnetic fields,

$$F_{tr} = -F_{rt} = \frac{q}{\sqrt{r^4 + q^2/b^2}}. \tag{4.35}$$

and the stress-energy tensor

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu}L_M - F_{\mu\sigma}F^{\sigma}_{\nu}L'_M \\ &= \frac{F_{\mu\sigma}F^{\sigma}_{\nu}}{\sqrt{1 + \frac{1}{2b^2}F_{\alpha\beta}F^{\alpha\beta}}} + b^2 g_{\mu\nu}\left(1 - \sqrt{1 + \frac{1}{2b^2}F_{\alpha\beta}F^{\alpha\beta}}\right), \end{aligned} \tag{4.36}$$

[3] In Gaussian units.

where the comma means total derivative with respect to F. It should be noted that in Maxwell's theory the stress-energy tensor is traceless, however tensor (4.36) in the BI frame has $T = T^{\mu}_{\mu} \neq 0$, which means that $R = R_0$ does not imply that $R_{\mu\nu} = \frac{1}{4}R_0 g_{\mu\nu}$, and instead, by the trace equation

$$f(R) = \frac{1}{2}\left(F(R)R + 3F(R)^{;\alpha}_{;\alpha} - \kappa T\right), \tag{4.37}$$

can be expressed as

$$F(R)R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\left[F(R)R - F(R)^{;\alpha}_{;\alpha} - \kappa T\right] - F(R)_{;\mu\nu} = \kappa T_{\mu\nu}, \tag{4.38}$$

or rearranging some terms the field equations are,

$$4\left[F(R)R_{\mu\nu} - \kappa T_{\mu\nu}\right] - \left[F(R)R - \kappa T - F(R)^{;\alpha}_{;\alpha}\right]g_{\mu\nu} - 4F(R)_{;\mu\nu} = 0. \tag{4.39}$$

This equation depends on the second covariant derivatives of the scalar function $f(R)$, which are combinations of partial derivatives of the metric, and to simplify the equations we will assume a SSS spacetime, defined by the metric

$$ds^2 = -a(r)dt^2 + a^{-1}(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.40}$$

from which the scalar curvature gives

$$R = \left[2 - 2a(r) - 4ra'(r)\right]r^{-2} - a''(r), \tag{4.41}$$

and the covariant derivatives of the function are

$$F(R)_{;tt} = -\frac{1}{2}a(r)a'(r)\partial_r F(R) = a^2(r)\left(\partial_r^2 F(R) - F(R)_{;rr}\right), \tag{4.42}$$

$$F(R)_{;\theta\theta} = ra(r)\partial_r F(R) = \csc^2\theta F(R)_{;\phi\phi}, \tag{4.43}$$

and

$$F(R)^{;\alpha}_{;\alpha} = \left[a'(r) + \frac{2a(r)}{r}\right]\partial_r F(R) + a(r)\partial_r^2 F(R). \tag{4.44}$$

In this way the components tt and rr of the field equations are (the other two components are indeed the same equation)

$$4\frac{F(R)R_{tt} - \kappa T_{tt}}{a(r)^2} + \frac{F(R)R - \kappa T}{a(r)} + \left[\frac{a'(r)}{a(r)} - \frac{2}{r}\right]\partial_r F(R) = \partial_r^2 F(R),$$

$$4(F(R)R_{rr} - \kappa T_{rr}) - \frac{F(R)R - \kappa T}{a(r)} - \left[\frac{a'(r)}{a(r)} - \frac{2}{r}\right]\partial_r F(R) = 3\partial_r^2 F(R),$$

but since $R_{tt} = a^2(r)R_{rr}$ and $T_{tt} = a^2(r)T_{rr}$, adding the equations it must be fulfilled that

$$\partial_r^2 F(R) = 0. \tag{4.45}$$

this condition imposes a strong restriction on the form of the function $f(R)$, since it must satisfy

$$\left.\frac{df(R)}{dR}\right|_{R=R(r)} = m + nr, \tag{4.46}$$

and by the fundamental theorem of calculus

$$f(R) = \left[\int dr\, f'(R)|_{R=R(r)}\partial_r R(r)\right]_{R(r)=R}, \tag{4.47}$$

or

$$f(R) = \left[\int dr (m+nr)\partial_r R(r)\right]_{R(r)=R}, \tag{4.48}$$

with m and n constants. At first glance, one might try to find a solution to the field equations by proposing a function $f(R)$ that satisfies them simultaneously with definition of the scalar curvature (4.41) and finally, if necessary, adjust the constants associated with the integrals by replacing $a(r)$ in the field equations. However proposing a viable model $f(R)$ that meets (4.48) and satisfies both the cosmological and the Solar system tests in the so called chameleon mechanism is not a simple task [28], indeed, if the model satisfies the observational tests it means that it is at some limit indistinguishable from the Λ-Cold Dark Matter model (ΛCDM) [151], which can be used to rule out some models. On the contrary, finding the general solutions of the field equations will allow us to observe the behavior of R and thus restrict the functions $f(R)$ according to the constants m and n, in three ways: (i) $m \neq 0$ and $n = 0$, (ii) $m \neq 0$ and $n \neq 0$ and (iii) $m = 0$ and $n \neq 0$.

4.3 Einstein-BI solutions in vacuum

As expected, GR must be recovered when $f(R) = R$ or equivalently if $m \neq 0$ and $n = 0$, in which case the equations (4.39) become

$$4\left(mR_{\mu\nu} - \kappa T_{\mu\nu}\right) - (mR - \kappa T)g_{\mu\nu} = 0, \tag{4.49}$$

From the definition of R,

$$R = \frac{2 - 2a(r) - 4ra'(r)}{r^2} - a''(r), \tag{4.50}$$

and the rr component of the Ricci tensor

$$R_{rr} = -\frac{2a'(r) + ra''(r)}{2ra(r)}, \tag{4.51}$$

field equations in the BI theory, are reduced

$$r^2 a''(r) - 2a(r) - \frac{16b\pi q^2}{m\sqrt{b^2r^4 + q^2}} + 2 = 0, \tag{4.52}$$

integrating this equation results in

$$r\left[ra'(r)-2a(r)+2\right]-\frac{16\pi bq^2}{m}\int\frac{1}{\sqrt{b^2r^4+q^2}}dr+c_1=0, \tag{4.53}$$

or

$$r\left[ra'(r)-2a(r)+2\right]+\frac{16\pi q^2}{mr}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)+c_1=0 \tag{4.54}$$

multiplying by r^{-4} and integrating again

$$\frac{a(r)}{r^2}-\frac{1}{r^2}-\frac{c_1}{3r^3}+c_2+\frac{16\pi q^2}{m}\int\frac{{}_2F_1}{r^5}dr=0, \tag{4.55}$$

and

$$a(r)=1+\frac{c_1}{3r}-c_2r^2+\frac{8\pi b^2}{3m}r^2\left(1-\sqrt{1+\frac{q^2}{b^2r^4}}+\frac{2q^2}{b^2r^4}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)\right), \tag{4.56}$$

this solution is also obtained when $R=constant$ is assumed in the field equations, resulting $f'(R)=m$. Solution (4.56) describes an electrically charged RN-AdS BH solution in GR-BI theory [83, 153], and the scalar curvature defines constant hypersurfaces

$$R=\frac{16\pi b^2}{m}\left[\left(2+\frac{q^2}{b^2r^4}\right)\left(1+\frac{q^2}{b^2r^4}\right)^{-1/2}-2\right]+12c_2, \tag{4.57}$$

such that $R_{b\to 0,\infty}=4\Lambda$, as it must be. Now, by (4.48), the function can be obtained directly

$$f(R)=mR+m_0, \tag{4.58}$$

where m_0 is the integration constant, obtained from the trace equation, $m_0=-6c_2m$. Constants can be identified at the limit $b\to\infty$, when the solution is compared with the RN metric, i.e. $c_1=-3R_S$ and $c_2=0$. Solution Eq. (4.56) satisfies the field equations,

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{\kappa}{m}T_{\mu\nu}, \tag{4.59}$$

as well as the trace equation,

$$mR+\kappa T=0. \tag{4.60}$$

However if we compare the solution at $b\to\infty$ with the RN(A)dSitter metric, $c_2=-\Lambda/3$, and the solution satisfies the field equations in GR with cosmological constant in the BI theory and $\kappa\to\kappa/m$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=\frac{\kappa}{m}T_{\mu\nu}, \tag{4.61}$$

and the trace equation

$$m(4\Lambda-R)=\kappa T. \tag{4.62}$$

It is concluded that in this case, the models are reduced to the ΛCDM model with $\kappa\to\kappa/m$.

4.4 $f(R)$-BI solutions

In the most general case, when $m \neq 0$ and $n \neq 0$,

$$f(R) = \left[\int (m+nr)\frac{\partial R(r)}{\partial r}\,dr\right]_{R(r)=R} = mR + n\left[\int r\frac{\partial R(r)}{\partial r}\,dr\right]_{R(r)=R}, \tag{4.63}$$

and $F(r) = m + nr$, so that

$$F(r)_{;\alpha}^{;\alpha} = \left[a'(r) + \frac{2a(r)}{r}\right]\frac{\partial F(r)}{\partial r}, \tag{4.64}$$

and the field equations, for the rr component, take the form

$$r^2\left[(m+nr)a''(r) + na'(r)\right] - 2a(r)(m+2nr) + 2(m+nr) = \frac{16\pi b q^2}{\sqrt{b^2r^4+q^2}}, \tag{4.65}$$

integrating this equation it is found that

$$2mr + nr^2 - 2ra(r)(m+nr) + r^2a'(r)(m+nr) + c_1 = \int \frac{16\pi b q^2}{\sqrt{b^2r^4+q^2}}\,dr, \tag{4.66}$$

or

$$2mr + nr^2 - 2ra(r)(m+nr) + r^2a'(r)(m+nr) + c_1 = -\frac{16\pi q^2}{r}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right), \tag{4.67}$$

multiplying by $(m+nr)^{-1}r^{-4}$ and integrating again

$$\frac{a(r)}{r^2} + \frac{c_1n - 2m^2}{2m^2r^2} + \left(\frac{c_1n}{m^2} - 1\right)\ln\left[n + \frac{m}{r}\right]^{\frac{n^2}{m^2}} + \frac{n\left(m^2 - c_1n\right)}{m^3r} - \frac{c_1}{3mr^3} + c_2 = -16\pi q^2\int \frac{{}_2F_1}{r^5(m+nr)}\,dr, \tag{4.68}$$

thus

$$a(r) = 1 - c_2r^2 - \frac{c_1}{m}\left(\frac{n}{2m} - \frac{1}{3r}\right) + \frac{n}{m}\left(1 - \frac{c_1n}{m^2}\right)\left(\ln\left[n + \frac{m}{r}\right]^{\frac{n}{m}} - \frac{1}{r}\right)r^2 - 16\pi q^2r^2\int \frac{{}_2F_1}{r^5(m+nr)}\,dr, \tag{4.69}$$

although the solution cannot be found analytically but expressed in terms of an integral, with $c_1 = -3R_S$ and $c_2 = 0$,

$$a(r) = 1 + \frac{R_S}{m}\left(\frac{3n}{2m} - \frac{1}{r}\right) + \frac{n}{m}\left(1 + \frac{3nR_S}{m^2}\right)\left(\ln\left[\frac{m}{r} + n\right]^{\frac{n}{m}} - \frac{1}{r}\right)r^2 - 16\pi q^2r^2\int dr\frac{{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)}{r^5(m+nr)} \tag{4.70}$$

This solution leads to the scalar curvature

$$R(r) = \frac{1}{m+nr}\left(\frac{6n}{m+nr}\left[\frac{m}{r} + \frac{19n}{6} + \frac{2n^2r}{m} - \frac{m+2nr}{m}\left(\frac{1}{2r^2} - \frac{3n}{mr} - \frac{3n^2}{m^2}\right)R_S\right] - \frac{16\pi b q^2}{r^2\sqrt{b^2r^4+q^2}}\right)$$
$$-\left(1 + \frac{3nR_S}{m^2}\right)\ln\left[n + \frac{m}{r}\right]^{\frac{12n^2}{m^2}} + \frac{16\pi q^2(4m+3nr)}{r^4(m+nr)^2}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right) + 192\pi q^2\int dr\frac{{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)}{r^5(m+nr)},$$

and by Eq. (4.48)

$$f(r) = 16\pi b^2\left(\frac{br^2}{\sqrt{b^2r^4+q^2}}-1\right)+\frac{n}{m+nr}\left[8n+\frac{6m}{r}+3R_S\left(\frac{6n^2}{m^2}+\frac{3n}{mr}-\frac{1}{r^2}\right)\right]-\left(1+\frac{3nR_S}{m^2}\right)$$
$$\ln\left[n+\frac{m}{r}\right]^{\frac{6n^2}{m}}+\frac{32\pi mq^2}{r^4(m+nr)}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)+96\pi mq^2\int dr\frac{{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{q^2}{b^2r^4}\right)}{r^5(m+nr)}. \tag{4.71}$$

The solution Eq. (4.70) together with the function Eq. (4.71) satisfy the field equations Eq. (2.28), e.g. for the rr component, with

$$F_{;rr} = \frac{na'(r)}{2a(r)} \tag{4.72}$$

and

$$T_{rr} = \frac{b\left(br^2-\sqrt{q^2+b^2r^4}\right)}{r^2a(r)}, \tag{4.73}$$

$$r(m+nr)a''(r)+(2m+nr)a'(r)-4na(r)+rf(r) = \frac{2\kappa b}{r}\left(\sqrt{b^2r^4+q^2}-br^2\right), \tag{4.74}$$

and also the trace equation,

$$2f(r) = (m+nr)R(r)+3\left[a'(r)+\frac{2a(r)}{r}\right]n-\kappa\left[4b^2-\frac{2b(q^2+2b^2r^4)}{r^2\sqrt{b^2r^4+q^2}}\right]. \tag{4.75}$$

In spite of the solution in this case is given in terms of an non-analytical integral, it is still possible to obtain information about the form of the functions because the logarithm must be defined in the following intervals depending on the signs of the constants, i.e.

- I_1: $r>0$ if $m,n>0$,
- I_2: $r>|\frac{m}{n}|$ if $m<0$ and $n>0$,
- I_3: $0<r<|\frac{m}{n}|$ if $m>0$ and $n<0$.

We are modeling the stress-energy tensor with the electromagnetic Lagrangian density in the BI theory, so there are two natural cases of physical interest, $b=0$ (or equivalently $q=0$): $f(R)$ gravity in a SSS vacuum spacetime without the presence of electromagnetic fields. And $b\to\infty$: $f(R)$ gravity coupled with classic electromagnetic fields, which are described below.

4.4.1 $f(R)$-Schwarzschild-type solution

In the first case ($b=0$) the solution is

$$a(r) = 1+\frac{R_S}{m}\left(\frac{3n}{2m}-\frac{1}{r}\right)+\frac{n}{m}\left(1+\frac{3nR_S}{m^2}\right)\left(\ln\left[\frac{m}{r}+n\right]^{\frac{n}{m}}-\frac{1}{r}\right)r^2, \tag{4.76}$$

this correspond to the Schwarzschild-type solution in $f(R)$ theory for a non-constant scalar curvature, which gives

(4.77)
$$R = \frac{mn}{(m+nr)^2}\left[\frac{6}{r}+\frac{19n}{m}+\frac{12n^2}{m^2}r-\frac{3R_S}{m}\left(\frac{2n}{m}+\frac{1}{r}\right)\left(\frac{1}{r}-\frac{6n}{m}-\frac{6n^2}{m^2}r\right)\right]-\left(1+\frac{3nR_S}{m^2}\right)\ln\left[\frac{m}{r}+n\right]^{\frac{12n^2}{m^2}},$$

and the function

$$f(r) = \frac{n}{m+nr}\left[\frac{6m}{r}+8n-9Rs\left(\frac{1}{3r^2}-\frac{n}{mr}-\frac{2n^2}{m^2}\right)\right]-\left(1+\frac{3nR_S}{m^2}\right)\ln\left[n+\frac{m}{r}\right]^{\frac{6n^2}{m}}. \tag{4.78}$$

scalar (4.77) cannot be analytically inverted for all m and n, so it is not possible to express the function f in terms of R, moreover it presents singularities at the limits of the regions mentioned above, $r \to 0$ in I_1, $r \to |\frac{m}{n}|$ in I_2 and the combination of the previous two in I_3. Notwithstanding the foregoing, we can qualitatively analyze the behavior of the scalar with respect to r in each of the three regions and draw $f(R)$ by numerically inverting Eq. (4.77) when assigning values to m and n according to each of the regions. For this purpose we must consider the critical points of $R(r)$ and $f(R)$ since these will determine the domain R in which $f(R)$ can be defined. Note that by Eq. (4.48)

$$\partial_r f(r) - (m+nr)\partial_r R(r) = 0, \tag{4.79}$$

both functions share the same maximum points. If case I_1 or I_3 (if $n > -\frac{m^2}{3R_S}$) there is one absolute maximum and minimum, respectively at

$$r_c = \frac{m}{n}\left(h^{1/3}-1\right), \tag{4.80}$$

of value

$$R_c = n\frac{1-h}{R_S}\left(\frac{2}{3}+\frac{1-4h+2h^{2/3}}{h^{1/3}-1}+4h\ln\left[\frac{nh^{1/3}}{h^{1/3}-1}\right]\right), \tag{4.81}$$

where

$$h = 1+\frac{3R_S n}{m^2}. \tag{4.82}$$

which is precisely the term accompanying the logarithm. In I_1 for $0 < r < r_c$, $f(R)$ is an increasing function, and in $r = r_c$ the function folds and when $r \to \infty$, $R(r)$ and $f(r)$ take the constant values

$$R|_{r\to\infty} = R_\infty = -\frac{12n^2}{m^2}h\ln n, \tag{4.83}$$

valid for I_1 and I_2 since r is bounded in I_3, and

$$f(R)|_{r\to\infty} = \frac{m}{2}R, \tag{4.84}$$

which is valid not only for I_1, but in all cases, so the solution of GR is recovered in the spatial infinity. However, at $r \to \infty$ the series expansion of $R(r)$ and $f(r)$ produces, for $n > 0$

$$R(r) = R|_{r\to\infty} + \frac{1}{r^2} + O\left(\frac{1}{r}\right)^5, \tag{4.85}$$

and

$$f(r) = f(R)|_{r\to\infty} + \frac{2n}{r} + \frac{m}{r^2} - \frac{hm^3}{2n^2r^4} + O\left(\frac{1}{r}\right)^5, \tag{4.86}$$

thus for large r, function can be approximated to

$$f(R) \approx m\left(R - \frac{1}{2}R_\infty\right) + 2n\sqrt{R - R_\infty} - \frac{hm^3}{2n^2}(R - R_\infty)^2. \tag{4.87}$$

In Fig. (4.1) (a) it is shown $f(R)$ for some values of the constants according to I_1, where it should be noted that as r tends to r_c in the interval $(0, r_c)$, the behavior of $f(R)$ is almost linear. In case I_2, $f(r)$ and $R(r)$ are decreasing and do not present maximum or minimum, so that $f(R)$ is a monotonically increasing function whose domain is $R > R|_{r\to\infty}$. In case I_3, if $h > 0$, functions $f(r)$ and $R(r)$ have an absolute minimum at r_c, Eq. (4.80), and $f(R)$ will present a fold, Fig. (4.1) (b), while for $h < 0$ functions are monotonically decreasing and since Eq. (4.83) is defined for $n > 0$, the domain of $f(R)$ is $(-\infty, \infty)$, in such a way that Eq. (4.84) is fulfilled and

$$\left.\frac{f(R)}{R}\right|_{r\to 0} = m, \tag{4.88}$$

which implies that the solutions are asymptotically similar to GR.

However there is a third sub-case of special interest because the scalar is invertible and therefore the function $f(R)$ can be expressed directly. This is the case when $h = 0$, the logarithmic term vanishes and the solution (4.76) is simplified to

$$a(r) = \frac{1}{2} + \frac{m}{3nr}, \tag{4.89}$$

the scalar

$$R = \frac{1}{r^2}, \tag{4.90}$$

and by (4.48), the function could only be

$$f(R) = mR + 2n\sqrt{R} + m_0, \tag{4.91}$$

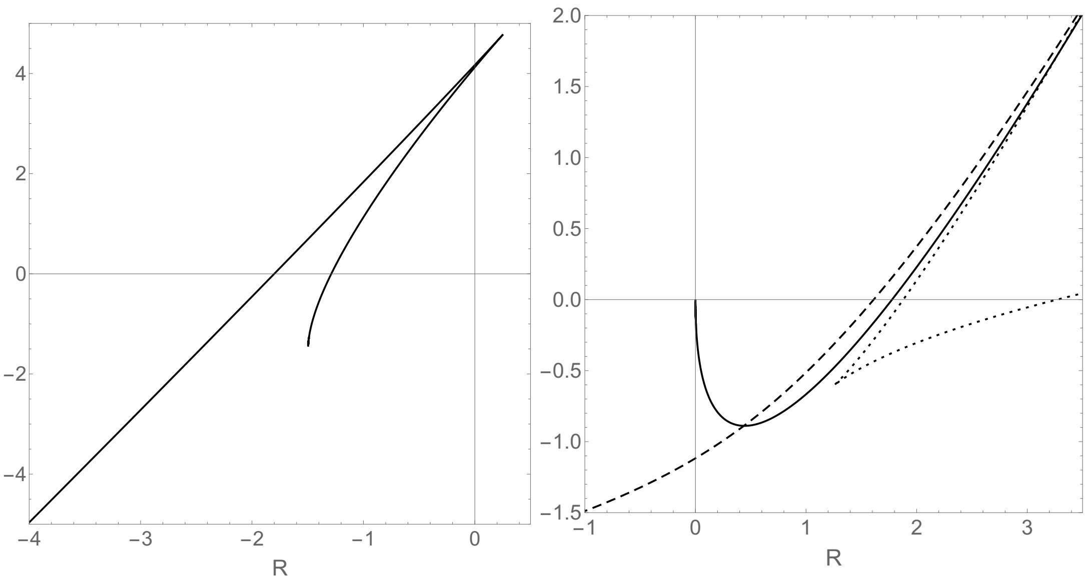
Eq. (4.87) reproduces this function when $h = 0$ and its behaviour can be seen in Fig. (4.1) (b) for $m_0 = 0$, $m = 2$, $n = -4/3$ and $R_S = 1$. Model (4.91) can be interpreted as a a perturbation of the Ricci scalar in GR around the vacuum solution, moreover, it is a particular case ($m_0 = 0$, $m = 1$, $n = -\alpha/2$) of the family models for DE [10, 11, 48]

$$f(R) = R - \alpha^{2(\beta+1)}R^{-\beta}, \tag{4.92}$$

with $\beta = 1/2$. Likewise, the BH thermodynamic properties and stability in this models are studied in [62]. Note that (4.91) is the first order expansion at $R = 0$ of

$$f(R) = \frac{2n}{\bar{n}\left(1 - \bar{n}\sqrt{R}\right)}, \tag{4.93}$$

when $m_0 = \frac{4n^2}{m}$ and $\bar{n} = \frac{m}{2n}$, moreover Eq. (4.93) is a good approximation of model (4.91) for $|n| \gg |m|$.



(a) $n = 1.2$.

(b) $n = -1.383, -1.333, -1.283$ (dashed, continuous, dotted).

FIGURE 4.1. Models $f(R)$ allowed for the Schwarzschild-type space and some values of m and n. In (a), when $r_c = 0.397$, $R = 0.253$ and $f(R) = 4.780$, which is the absolute maximum, and the spatial infinity is mapped to the point (-1.496,-1.496). In (b), the three possible sub-cases depending on h Eq. (4.82). In both panels $m = 2$ and $R_S = 1$.

4.4.2 $f(R)$-RN solution

The second case of interest, which actually contains the Schwarzschild solution, is when $b \to \infty$, i.e. $f(R)$-Maxwell theory for a SSS spacetime, and the solution is given by

$$a(r) = 1 + \frac{n}{m}\left[\frac{3R_S}{2m} + \frac{4\pi q^2}{n}\left(\frac{2n^2}{m^2} + \frac{1}{r^2}\right) - \left(\frac{16\pi q^2}{3m} + \frac{R_S}{n}\right)\frac{1}{r} + \left(1 + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3}\right)\left(\ln\left[\frac{m}{r} + n\right]^{\frac{n}{m}} - \frac{1}{r}\right) r^2\right], \tag{4.94}$$

with the scalar curvature

$$R = \frac{2n^2}{(m+nr)^2}\left\{\frac{19}{2} + \frac{27nR_S}{m^2} + \frac{144\pi n^2 q^2}{m^3} - \left(\frac{3R_S}{2n} + \frac{8\pi q^2}{m}\right)\frac{1}{r^2} + \frac{2m}{n}\left(\frac{3}{2} + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3}\right)\frac{1}{r} + \frac{6n}{m}\left(1 + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3}\right)\left(r - \left(\frac{m}{n} + r\right)^2 \ln\left[\frac{m}{r} + n\right]^{\frac{n}{m}}\right)\right\}, \tag{4.95}$$

and the function is determined by

$$f(r) = \frac{6n^2}{m+nr}\left[\frac{4}{3} + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3} - \left(\frac{R_S}{2n} + \frac{8\pi q^2}{3m}\right)\frac{1}{r^2} + \left(\frac{m}{n} + \frac{3R_S}{2m} + \frac{8\pi n q^2}{m^2}\right)\frac{1}{r}\right] - \left(1 + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3}\right)\ln\left[\frac{m}{r} + n\right]^{\frac{6n^2}{m}}, \tag{4.96}$$

from where it is recognized the term that modulates the logarithm

$$\bar{h} = 1 + \frac{3nR_S}{m^2} + \frac{16\pi n^2 q^2}{m^3}, \tag{4.97}$$

in this sense, the classification that was made according to the value of the constants is also useful to find the critical points, and Eq. (4.80) remains valid, as well as the limit (2.106) but with $h \to \bar{h}$, such that Eq. (4.84) and (4.87) are also fulfilled. Note that the last term of $\bar{h}$ is quadratic in q and n, so it does not affect the number of critical points when $m > 0$ and the analysis made in the previous section is still valid for I_1, however in I_2, when $\bar{h} < 0$, there will be an absolute maximum at

$$r_c = -\frac{m}{n}\left[(-\bar{h})^{1/3} + 1\right], \tag{4.98}$$

this critical point will produce a bend in the plot $f(R)$ at R_c, therefore its domain will be $R \leq R_c$. At the other hand, if $\bar{h} > 0$, $R(r)$ and $f(r)$ are monotonically decreasing, so $f(R)$ will be increasing with domain $R > R|_{r\to\infty}$. In I_3 the appearance of the critical point occurs when $\bar{h} < 1$, that is when $m < \frac{9R_S^2}{64\pi q^2}$ and at the same time

$$n \geq \frac{m}{32\pi q^2}\left(\sqrt{9R_S^2 - 64m\pi q^2} - 3R_S\right) \tag{4.99a}$$

or

$$-\frac{3m}{16\pi q^2} < n \leq -\frac{m}{32\pi q^2}\left(\sqrt{9R_S^2 - 64m\pi q^2} + 3R_S\right), \tag{4.99b}$$

or when

$$m \geq \frac{9R_S}{64\pi q^2} \qquad \text{and} \qquad n \geq -\frac{3mR_S}{16\pi q^2}. \tag{4.100}$$

In those cases, $f(R)$ will be defined for $R_c \leq R$, otherwise the domain of $f(R)$ will be $R < R|_{r\to\infty}$.

When observing the scalar (4.95), four possibilities to choose the constants when $n > 0$ are highlighted and their importance consists in the form and simplicity that the solution takes and correspondingly the function $f(R)$.

1. If $\bar{h} = 0$, the solution does not depend on the r and r^2 terms, including the logarithmic one,

$$a(r) = \frac{1}{2} + \frac{4\pi q^2}{mr^2} + \frac{m}{3nr}, \tag{4.101}$$

   the scalar is just eq. (4.90) and the function is given by eq. (4.91), this fact is explained because any solution of the form $a(r) = \frac{1}{2} + \frac{\alpha}{r} + \frac{\beta}{r^2}$, produces $R = r^{-2}$, with α and β some constants.

2. If $\bar{h}=1$, the term r disappears in the solution, which is given by

$$a(r)=1+\frac{4\pi q^2}{mr^2}-\frac{nr}{m}+\frac{n^2r^2}{m^2}\ln\left[\frac{m}{r}+n\right], \tag{4.102}$$

the scalar and the function are respectively

$$R=\frac{6n}{mr}+\frac{n^2}{(m+nr)^2}\left(7+\frac{6n}{m}r\right)-\frac{12n^2}{m^2}\ln\left[\frac{m}{r}+n\right], \tag{4.103}$$

$$f(r)=2n\left(\frac{3}{r}+\frac{n}{m+nr}-\frac{3n}{m}\ln\left[\frac{m}{r}+n\right]\right). \tag{4.104}$$

3. If $\bar{h}=-1/2$, the solution is

$$a(r)=\frac{1}{4}+\frac{4\pi q^2}{mr^2}+\frac{m}{2nr}+\frac{n}{2m}r-\frac{n^2}{2m^2}r^2\ln\left[\frac{m}{r}+n\right], \tag{4.105}$$

but the scalar does not depend on r^{-1} term

$$R=\frac{n^2}{(m+nr)^2}\left(\frac{3m^2}{2n^2r^2}-\frac{6nr}{m}-8\right)+\frac{6n^2}{m^2}\ln\left[\frac{m}{r}+n\right], \tag{4.106}$$

and the function

$$f(r)=\frac{3m}{2r^2}-\frac{n^2}{m+nr}+\frac{3n^2}{m}\ln\left[\frac{m}{r}+n\right]. \tag{4.107}$$

4. If $\bar{h}=-1/18$, the solution has a similar form to the previous Eq. (4.105)

$$a(r)=\frac{17}{36}+\frac{4\pi q^2}{mr^2}+\frac{19m}{54nr}+\frac{nr}{18m}-\frac{n^2r^2}{18m^2}\ln\left[\frac{m}{r}+n\right], \tag{4.108}$$

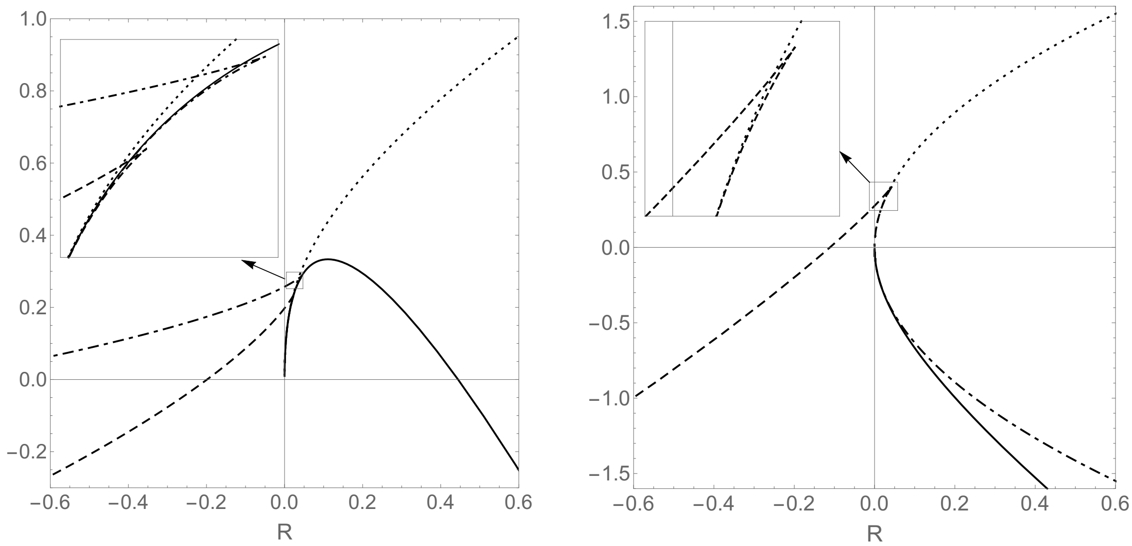
however the scalar contains a r^{-1} term that (4.106) does not have

$$R=\frac{m^2}{3(m+nr)^2}\left(\frac{19}{6r^2}+\frac{16n}{3mr}-\frac{2n^3r}{m^3}\right)+\frac{2n^2}{3m^2}\ln\left[\frac{m}{r}+n\right], \tag{4.109}$$

and function

$$f(r)=\frac{1}{9}\left(\frac{19m}{2r^2}+\frac{16n}{r}-\frac{n^2}{m+nr}\right)+\frac{3n^2}{m}\ln\left[\frac{m}{r}+n\right]. \tag{4.110}$$

Note that the value of R_S is determined by constants m and n according of each case, however, because the limit (4.84) only depends on m, functions $f(R)$ are expected to coincide when $r\to\infty$. This can be seen in Fig. (4.2) (a), where n was established from $R|_{r\to\infty}$, since although this limit depends on $\bar{h}$, it also depends on n, so when $n=1$, the limits coincide.



(a) $\bar{h} = 0, 1, -1/2, -1/18$, Eq. (4.97) (continuous, dotted, dashed and dotdashed lines) for $m = -3$, $n = 1$ and $q = 1$.

(b) $b = 0$ ($q = 0$) and $n = -1, 1$ (dotdashed and dotted lines), and $b \to \infty$ ($q = 1$) for $n = -1, 1$ (continuous and dashed lines).

FIGURE 4.2. Functions $f(R)$ found numerically for some values of m and n according to the $f(R)$-RN solution (4.94), (a), and $f(R)$-non-linear-BI (4.114), (b). The convergence of the functions is observed when r goes from the maximal point (if any) to infinity, or equivalently for $0 < R < R_c$, this fact is explained by Eq. (4.87). In (a) each curve has a different Schwarzschild radius. Although the solutions are not equal because they depend on the presence ($b \to \infty$) or not ($b \to 0$) of the Maxwell fields as well as the m and n constants, dotted and dashed lines in both panels have similar form, moreover $f(R)$ represented by dashed lines do not have continuous derivative in R_c

4.5 $f(R)$-non-linear-BI model

When $m = 0$ and $n \neq 0$ the models $f(R)$ do not have the linear term R, and thus are separated from GR. the solution cannot be obtained directly, as might be supposed, from Eq. (4.70) in the most general case, so it is necessary to write the field equations,

$$r^2\left[ra''(r) + a'(r)\right] - 4ra(r) - \frac{16\pi b q^2}{n\sqrt{b^2r^4 + q^2}} + 2r = 0, \tag{4.111}$$

integrating

$$r^2\left[ra'(r) - 2a(r) + 1\right] + \frac{16\pi q^2}{nr}\,{}_2F_1 + c_1 = 0 \tag{4.112}$$

multiplying this equation by r^{-5} and integrating again, noting that

$${}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{q^2}{b^2r^4}\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{9}{4}; -\frac{q^2}{b^2r^4}\right) - \frac{2q^2}{45b^2r^4}\,{}_2F_1\left(\frac{5}{4}, \frac{3}{2}; \frac{13}{4}; -\frac{q^2}{b^2r^4}\right), \tag{4.113}$$

it is obtained

$$a(r) = \frac{1}{2} + c_2 r^2 + \frac{c_1}{4r^2} + \frac{16\pi q^2}{5nr^3}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{9}{4}; -\frac{q^2}{b^2 r^4}\right), \tag{4.114}$$

where c_1 and c_2 are constants to be determined in some limit. However the scalar curvature does not depend on c_1

$$R = -12c_2 + \frac{1}{r^2} + \frac{16\pi b^2}{nr}\left[\left(1 + \frac{q^2}{b^2 r^4}\right)^{-1/2} - {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{q^2}{b^2 r^4}\right)\right], \tag{4.115}$$

and by means the Eq. (4.48) and (2.28) the function takes the form

$$f(R) = n\left[\int r\partial_r R(r) dr\right]_{R(r)=R}, \tag{4.116}$$

thus

$$f(r) = -16\pi b^2 + \frac{2n}{r} + \frac{16\pi b^3 r^2}{\sqrt{b^2 r^4 + q^2}}. \tag{4.117}$$

The solution (4.114), together with scalar curvature and the function (4.117), satisfy the trace equation

$$f = \frac{1}{2} nrR(r) + 3n\left[\frac{1}{2}a'(r) + \frac{1}{r}a(r)\right] - 16\pi b^2\left(1 - \frac{2b^2 r^4 + q^2}{2br^2\sqrt{b^2 r^4 + q^2}}\right). \tag{4.118}$$

It is worth noting that when $r \gg 1$,

$$R(r) = -12c_2 + \frac{1}{r^2} + O\left(\frac{1}{r}\right)^5, \tag{4.119}$$

and

$$f(r) = \frac{2n}{r} - \frac{8\pi q^2}{r^4} + O\left(\frac{1}{r}\right)^5 \tag{4.120}$$

thus the function can be approximated to

$$f(R) \approx 2n\sqrt{12c_2 + R} - 8\pi q^2 (12c_2 + R)^2, \tag{4.121}$$

this function has a similar form to Eq. (4.87) with exception to the linear term R, both expressions contain the $2n\sqrt{R}$ term, however they are linearly independent, because even if $c_2 = R_\infty$ could be the case, when $m = 0$, Eq. (4.121) cannot be obtained from Eq. (4.87). In addition $f(R)$, Eq. (4.121) can be expressed in the limits $b = 0$ or SSS vacuum space without Maxwell fields ($q = 0$) and $b = \infty$ or classical electromagnetism, in the first case the solution is reduced to

$$a(r) = \frac{1}{2} + \frac{c_1}{4r^2} + c_2 r^2, \tag{4.122}$$

the scalar is

$$R = -12c_2 + \frac{1}{r^2}, \tag{4.123}$$

and function

$$f(R) = 2n\sqrt{12c_2 + R}, \tag{4.124}$$

which can be found from Eq. (4.121) when $q = 0$. At the other hand, $b \to \infty$, leads to the solution

$$a(r) = \frac{1}{2} + \frac{16\pi q^2}{5nr^3} + \frac{c_1}{4r^2} + c_2 r^2, \tag{4.125}$$

with the scalar

$$R = -12c_2 + \frac{1}{r^2} - \frac{32\pi q^2}{5nr^5}, \tag{4.126}$$

and the function

$$f(r) = \frac{2n}{r} - \frac{8\pi q^2}{r^4}. \tag{4.127}$$

Since $m = 0$, the scalar curvature as well as the function $f(R)$ will be determined by the sign of n as can be seen in Fig. (4.2) (b) where are plotted the 4 possible forms of the functions. When $n > 0$, $R(r)$ has a similar form to the $f(R)$-RN models when $\bar{h} = 1, -1/2$, and for Maxwell fields $R(r)$ has a maximal point and $f(R)$ a corresponding fold at R_c while for $n < 0$ the function will be monotonically decreasing.

CHAPTER 5

HAIRY SOLUTIONS AND LINEARIZATION OF FIELD EQUATIONS IN $f(R)$

Birkhoff's (Jebsen-Birkhoff) theorem states that the only Asymptotically Flat (AF), spherically symmetric and static solution in the vacuum, of the field equations in GR is the Schwarzschild's solution, which only depends on a single parameter, the mass M of the BH. If the BH also has a net charge Q, the solution is the well-known Reissner-Nordström one, while the Kerr-Newman solution also takes into account the angular momentum J. This is the most general AF and stationary three-parametric solution possible in the vacuum. Such a proposition is the so-called Wheeler's non-hair conjecture, which points out that all AF and stationary BH are fully described by M, Q and J.

Since $f(R)$ is a generalization of GR, not all the theorems established in the latter will be applicable in the former. For instance, Birkhoff's theorem is not valid in $f(R)$ theory, see Ref. [65]. However, because $f(R)$ is in effect a scalar-tensor theory, some characteristics can be extracted from the theorems established in GR formalism as scalar-tensor theory, to be applied in $f(R)$. In this respect, it is important to mention that the solutions of the field equations in $f(R)$ with constant curvature scalar, are in fact equivalent to the solutions in GR with cosmological constant, so that Wheeler's conjecture and other associated theorems, will be applicable in this type of solutions and it is expected that such black holes do not possess hair. However, the solutions obtained in the previous chapter exhibit a dependence of R with r, eventually evading such theorems and therefore being hairy.

5.1 Scalar field potential

In Chapter 2 it was seen that action (2.8) in the $f(R)$ theory can be expressed as a special case of the Brans-Dicke scalar tensor theory, in which the scalar field potential V in the Jordan frame, is defined as

$$V(\phi) = \phi R(\phi) - f(\phi), \tag{5.1}$$

and the potential

$$\bar{V} = \frac{\phi R(\phi) - f(\phi)}{\phi^2}, \tag{5.2}$$

which is expressed in the Einstein frame through a conformal transformation $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$. The relation between ϕ and R is determined by the trace equation, which in the vacuum takes the form

$$\phi_{;\alpha}^{;\alpha} = \frac{2f(\phi) - R(\phi)\phi}{3}, \tag{5.3}$$

let us define the potential $U(\phi)$ by

$$U'(\phi) = \phi_{;\alpha}^{;\alpha}, \tag{5.4}$$

hence

$$U''(\phi) = \frac{d\phi_{;\alpha}^{;\alpha}}{d\phi} = \frac{dR}{d\phi}\frac{d\phi_{;\alpha}^{;\alpha}}{dR} = \frac{1}{3}\left(\frac{\phi}{\phi'} - R(\phi)\right). \tag{5.5}$$

This potential is used to determine whether a solution of a particular model satisfies or evades a Non-Hair Theorem (NHT), stated as follows [37]: Given a model $f(R)$ ($f \not\propto R^2$) of class $\mathscr{C}^2$ with $f(0) = 0$ and with $U''(R) \geq 0$, the only AF and SSS BH solution in vacuum is $R(r) = 0$. In the same way, by the conformal transformation $\phi(\psi) = e^{\sqrt{\frac{2\kappa}{3}}\psi}$, and bringing up the conditions discussed in Ref. [38], to verify if a solution obeys the non-hair theorem when investigating the sign of the potential $\mathscr{U}(\psi) = \bar{V}(\phi(\psi))$, for a specific model.

So in order to show that some solutions in the BI-$f(R)$ theory, previously described, can evade the non-hair theorem, we are going to plot the potentials $\mathscr{U}(\phi)$ and $U''(R)$, by numerically inverting $R = R(r)$ and $f = f(r)$ for some particular models and according to the classification (I_1, I_2 and I_3) of constants given in Sec. 4.4 of Chapter 4.

When $n = 0$, $f(R) = m(R - 6c_2)$, $F = m$ and $\mathscr{U}(\psi) = \frac{3c_2}{\kappa}e^{-\sqrt{\frac{2\kappa}{3}}\psi}$, which depends on the sign of constant c_2, furthermore, in order to satisfy $f(0) = 0$ and to be able to apply the Theorem, $c_2 = 0$, however if this is the case, $F' = 0$ and the potential (5.5) cannot be defined.

When $n \neq 0$ and $m \neq 0$, $F = m + nr$, therefore

$$F'(R)|_r = \frac{n}{R'(r)}, \tag{5.6}$$

and

$$3U''(\phi)|_r = \left(\frac{m}{n} + r\right) R'(r) - R(r). \tag{5.7}$$

The plots for $f(R)$, $\mathscr{U}(\psi)$ and $U''(R)$ are shown in Fig. 5.1 (left, center and right respectively) for some values of m and n ($m, n > 0$ up, $m < 0$ and $n > 0$ center, and $m > 0$ and $n < 0$, bottom) and with $R_S = 1$, when R and f are given by Eq. (4.77) and (4.78) respectively. Since in I_1 and I_2 it is fulfilled that

$$f(R) \overset{r\to\infty}{=} \frac{m}{2} R \overset{r\to\infty}{=} -\frac{12n^2}{m^2}\left(1 + \frac{3nR_S}{m^2}\right) \ln n, \tag{5.8}$$

the potential is

$$U''(R) \overset{r\to\infty}{=} -R, \tag{5.9}$$

which is represented in the U'' vs R graph, (c) and (f), as a straight dotted line. In addition when $m, n > 0$ (a) and $m > 0$ and $-\frac{m^2}{3R_S} < n$ (g); if $f(0) = 0$, $U''(0) = 0$ (yellow curve), a maximum (minimum in [g]) can be seen in $f(R_c)$, in (a) and (g) it is observed that this maximum is formed at the point where the bend or cusp occurs, indicating that for those m and n function $f(R)$ fails to be differentiable at those locations, this is the case for a function that is continuous, but not differentiable. Despite this, the maximum points correspond to the minimum of potential

$$U''(R_c) = -R_c = \frac{3n^2}{m^2}\left(\frac{2}{3} + \frac{2h^{2/3} - 4h + 1}{h^{1/3} - 1} + 4h \ln\left[\frac{nh^{1/3}}{h^{1/3} - 1}\right]\right), \tag{5.10}$$

where $h = 1 + \frac{3R_S n}{m^2}$. Note that Eq. (5.10) is equivalent to Eq. (4.81). For $0 < n \leq 1$, $U''(R_c) < 0$, this implies that condition $U''(R) \geq 0$ is not satisfied for all R and thus the associated solution (4.76) can evade the Theorem.

Whether or not the solutions satisfy Theorem depends strongly on the value of the constants m, n and R_S, as well as on their relation. In the case I_3 ($m > 0$ and $n < 0$), if $n = -\frac{m^2}{3R_S}$, it is possible to invert $R(r)$ and obtain

$$f(R) = mR - \frac{2m^2}{3R_S}\sqrt{R}, \tag{5.11}$$

which satisfies $f(0) = 0$ and presents a minimum in

$$R_c = \frac{m^2}{9R_S^2}, \tag{5.12}$$

with value $f(R_c) = -mR_c$ and specifically the potential evaluated at this point yields $U''(R_c) = -R_c < 0$. Also in Fig. 5.1 it can be seen that the potential $\mathscr{U}(\psi)$ is negative for some values of the constants, evading the NHT, for example the red and green curves in (e) ($m < 0$ and $n > 0$) and in (h) ($m > 0$ and $n < 0$) diagrams. However, for some values of the constants ($m = 1$, $R_S = 1$ and

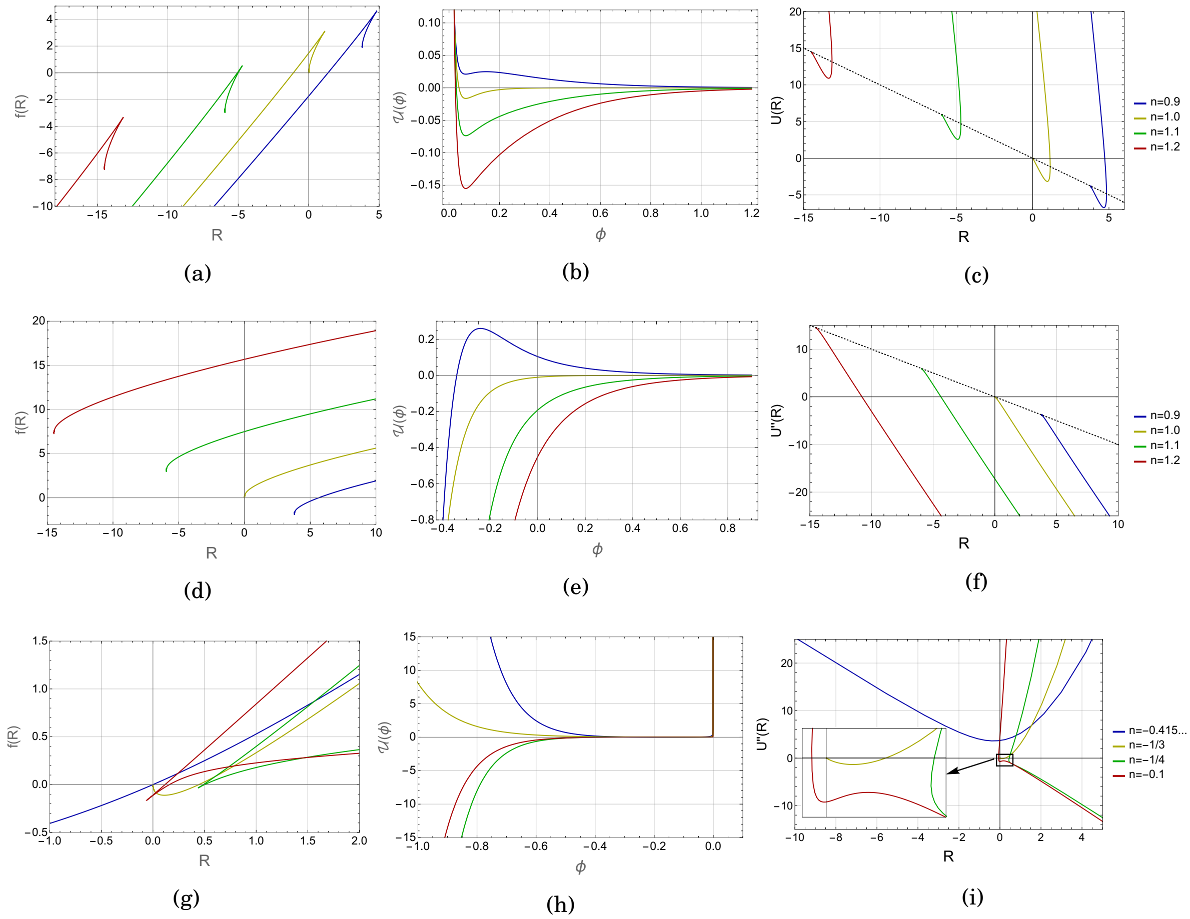


FIGURE 5.1. From left to right: $f(R)$ and potentials $\mathscr{U}(\psi)$ and $U''(R)$, obtained from solution (4.76) when $R_S = 1.0$, $q = 0.1$, and for some values of m and n, in three cases: $m = 1.0$ ($n > 0$), plots (a), (b) and (c), when $m = -1.0$ ($n > 0$), (d), (e) and (f), and when $m = 1.0$ ($n < 0$), plots (g), (h) and (i). A maximum of the function is shown in diagrams (a) and (g) (red and green lines on the last one), but they make the function not continuous, so the Theorem cannot be applied, however, of the two models represented by the red and blue line in (g) (continuous and with continuous derivative and such that $f(0) = 0$), only the blue model, $m = 1.0$ and $n = -0.4158011...$, has $U''(R) > 0$, since the yellow model has a minimum $U''(R_c) < 0$.

$n = -0.415...$) $f(0) = 0$, and at the same time $U''(R) > 0$ for all R, see the blue curve in (g) and (i) diagrams.

When $m = 0$, solution is given by Eq. (4.114), we can write the potential in the parametric way

$$\mathscr{U}(r) = -\frac{1}{16\pi n r^3} + \frac{b^2}{n^2 r^2}\left[1 - {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{q^2}{b^2 r^4}\right)\right], \tag{5.13}$$

and the potential

$$U''|_r = \frac{16\pi b^2}{nr}\left[-\frac{3n}{16\pi b^2 r} + \left(\frac{q^2}{b^2r^4}-1\right)\left(\frac{q^2}{b^2r^4}+1\right)^{-3/2} + \left(\frac{q^2}{b^2r^4}+1\right)^{1/2} {}_2F_1\left(\frac{3}{4},1;\frac{5}{4};-\frac{q^2}{b^2r^4}\right)\right], \tag{5.14}$$

we can observe the behaviour of these potential at some limits, at first, $b \to \infty$,

$$\mathcal{U}(\psi) = \frac{n^2}{16\pi} e^{-4\sqrt{3\pi}\psi}\left(\frac{8\pi n^2 q^2}{5} e^{-4\sqrt{3\pi}\psi} - 1\right), \tag{5.15}$$

which satisfies $\mathcal{U}(\psi) < 0$ for $\psi > -\frac{1}{4\sqrt{3\pi}}\ln\left(\frac{5}{8n^2\pi q^2}\right)$, and at the critical point $\psi_c = \frac{1}{4\sqrt{3\pi}}\ln\left(\frac{16}{5}n^2\pi q^2\right)$, the second derivative of the potential $\mathcal{U}''(\psi_c) = \frac{15}{16\pi q^2} > 0$ indicates that $\mathcal{U}(\psi_c) = -\frac{5}{512\pi^2 q^2} < 0$, is a minimum that does not depend on n. This behaviour can bee seen in Fig. 5.2 (b); where $f(R)$ (left), the potentials $\mathcal{U}(\psi)$ (middle) and $U''(R)$ (right) are plotted, in the upper part when $b \to \infty$, and in the lower part when $b \to 0$. Similarly, despite the fact that $f(0) = 0$, the potential $U''(R)$ (c) presents a minimum at $R_c = \frac{1}{10}\left(\frac{n^2}{2\pi^2 q^4}\right)^{1/3} > 0$, of value $U''(R_c) = -\frac{9}{4}R_c < 0$. In this case also the function $f(R)$ is not differentiable for all R, so the Theorem cannot be applied. At the other hand, when $b \to 0$, $f(R)$ is a continuous and differentiable function for all R, moreover, $f(0) = 0$ for all n, see Fig. 5.2 (d), and the potential

$$\mathcal{U}(\psi) = -\frac{n^2}{16\pi} e^{-4\sqrt{3\pi}\psi}, \tag{5.16}$$

which is negative for all ψ, (e) in the same figure, and in panel (f) it is plotted the potential $U''(R) = -3R$, so this solution can evade the NHT.

5.2 Linearization of field equations in f(R) theory

Since Einstein Field Equations (2.29) have a great mathematical complexity, the dynamical solutions for evolving systems are usually explored in a numerical way, so it is to expect that field equations (2.28) should be treated in the same way. However it is important to note that in GR there are only six equations for six quantities among the $g_{\mu\nu}$ [164] and it is possible to choose a nearly Lorentz coordinates system in which metric tensor can be expressed in terms of the Minkowski tensor $\eta_{\mu\nu}$, which is the simplest form of the flat-space metric. Then by means of the gauge and transformations and weak fields it is possible to simplify the field equations enormously and find the physical significance of the constants obtained in any solution of the field equations, which is called linearized theory and is described below.

For weak fields it is possible to express the metric tensor as a sum of the Minkowski $\eta_{\mu\nu}$ tensor plus a small perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.17}$$

so that the Christoffel symbols (2.4) can be written as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu} + h^{\alpha}_{\nu,\mu} - h^{,\alpha}_{\mu\nu}\right), \tag{5.18}$$

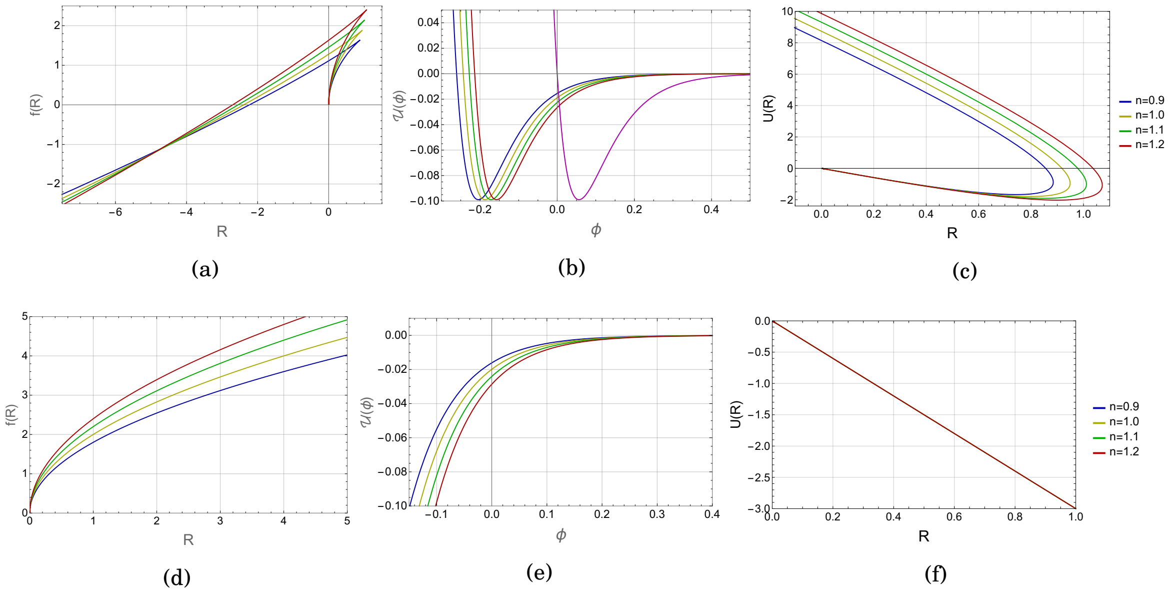


FIGURE 5.2. From left to right: $f(R)$ and potentials $\mathscr{U}(\psi)$ and $U''(R)$, when $m=0$, $q=0.1$, and for some values of n, in two cases: $b\to\infty$, plots (a), (b) and (c), and when $b\to 0$, plots (d), (e) and (f). Since $\mathscr{U}(\psi)<0$ for $\psi > -\frac{1}{4\sqrt{3\pi}}\ln\left(\frac{5}{8\pi n^2 q^2}\right)$, when $n^2=\frac{5}{8\pi q^2}$, $\mathscr{U}(\psi)>0$, for $\psi>0$, magenta line in (b), there it can be seen that all the potential lines have different critical points, but the same minimum value $\mathscr{U}_c=-\frac{5}{512\pi^2 q^2}<0$. Although the models meet the conditions mentioned in Theorem 1, as seen in (d), the potential $\mathscr{U}(\psi)$ is less than zero for all ψ (e) and the sign of U'' will depend on that of R (f).

where we have neglected terms higher than $h_{\mu\nu}$, and $h^{,\alpha}_{\mu\nu}=\eta^{\alpha\lambda}h_{\mu\nu,\lambda}$. With this we can construct the Ricci tensor

$$R_{\mu\nu}=\frac{1}{2}\left(h^{\alpha}_{\nu,\mu\alpha}-h^{,\alpha}_{\mu\nu,\alpha}-h_{,\mu\nu}+h^{,\alpha}_{\mu\alpha,\nu}\right), \tag{5.19}$$

and then the Ricci scalar

$$R=h^{,\mu\alpha}_{\mu\alpha}-h^{,\mu}_{,\mu} \tag{5.20}$$

where the trace of the perturbation is $h=\eta^{\mu\nu}h_{\mu\nu}$.

5.2.1 Starobinsky quadratic model

As a particular case, we are going to restrict the theory to the well known Starobinsky model, which constitutes a viable model that takes into account quadratic terms in the function $f(R)$, proposed by Starobinsky [178]

$$f(R)=a_0+a_1R+a_2R^2, \tag{5.21}$$

where a_i and $i = 0, 1, 2$ are constant. The field equations (2.28) can be written as

$$(a_1 + 2a_2 R)R_{\mu\nu} - \frac{1}{2}\left(a_0 + a_1 R + a_2 R^2\right) g_{\mu\nu} - 2a_2\left(R_{,\mu\nu} - g_{\mu\nu}R^{;\alpha}_{,\alpha}\right) = \kappa T_{\mu\nu}, \tag{5.22}$$

with the metric (5.17), the field equations are

$$\begin{aligned}
(a_1 + 2a_2 R)R_{\mu\nu} - \frac{1}{2}\left(a_0 + a_1 R + a_2 R^2\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right) - 2a_2\left[R_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right)R^{;\alpha}_{,\alpha}\right] &= \kappa T_{\mu\nu}\\
a_1 R_{\mu\nu} - \frac{1}{2}\left[a_0\left(\eta_{\mu\nu} + h_{\mu\nu}\right) + a_1 R\eta_{\mu\nu}\right] - 2a_2\left[R_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right)R^{;\alpha}_{,\alpha}\right] &= \kappa T_{\mu\nu}\\
a_1 G_{\mu\nu} - \frac{1}{2}a_0\left(\eta_{\mu\nu} + h_{\mu\nu}\right) - 2a_2\left[R_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right)R^{;\alpha}_{,\alpha}\right] &= \kappa T_{\mu\nu}
\end{aligned} \tag{5.23}$$

where we have neglected quadratic terms in $h_{\mu\nu}$, $RR_{\mu\nu}$, R^2, and the Einstein tensor $G_{\mu\nu}$ in the Minskwoski space is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R, \tag{5.24}$$

which can be linearized also by replacing the Ricci tensor and scalar, (5.19) and (5.20) respectively

$$\begin{aligned}
G_{\mu\nu} =& \frac{1}{2}\left(h^{\alpha}_{\nu,\mu\alpha} - h^{;\alpha}_{\mu\nu,\alpha} - h_{,\mu\nu} + h^{;\alpha}_{\mu\alpha,\nu}\right) - \frac{1}{2}\eta_{\mu\nu}\left(h^{;\alpha\beta}_{\alpha\beta} - h^{;\alpha}_{,\alpha}\right)\\
=& \frac{1}{2}\left(h^{\alpha}_{\nu,\mu\alpha} - h^{;\alpha}_{\mu\nu,\alpha} - \frac{1}{2}h_{,\mu\nu} - \frac{1}{2}h_{,\mu\nu} + h^{;\alpha}_{\mu\alpha,\nu} - \eta_{\mu\nu}h^{;\alpha\beta}_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}h^{;\alpha}_{,\alpha} + \frac{1}{2}\eta_{\mu\nu}h^{;\alpha}_{,\alpha}\right)\\
=& \frac{1}{2}\left[\left(h^{\alpha}_{\nu,\mu\alpha} - \frac{1}{2}\eta^{\alpha}_{\nu}h_{,\mu\alpha}\right) - \left(h^{;\alpha}_{\mu\nu,\alpha} - \frac{1}{2}\eta_{\mu\nu}h^{;\alpha}_{,\alpha}\right) + \left(h^{;\alpha}_{\mu\alpha,\nu} - \frac{1}{2}\eta_{\mu\alpha}h^{;\alpha}_{,\nu}\right) - \eta_{\mu\nu}\left(h^{;\alpha\beta}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h^{;\alpha\beta}\right)\right]\\
=& \frac{1}{2}\left[\left(h^{\alpha}_{\nu} - \frac{1}{2}\eta^{\alpha}_{\nu}h\right)_{,\mu\alpha} - \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right)^{;\alpha}_{,\alpha} + \left(h_{\mu\alpha} - \frac{1}{2}\eta_{\mu\alpha}h\right)^{;\alpha}_{,\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right)^{;\alpha\beta}\right],
\end{aligned}$$

now, through the definition of the trace reverse perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{5.25}$$

whose name is due to

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h = -h, \tag{5.26}$$

the Einstein tensor to first order in the metric is

$$G_{\mu\nu} = \frac{1}{2}\left[\bar{h}^{\alpha}_{\nu,\mu\alpha} - \bar{h}^{;\alpha}_{\mu\nu,\alpha} + \bar{h}^{;\alpha}_{\mu\alpha,\nu} - \eta_{\mu\nu}\bar{h}^{;\alpha\beta}_{\alpha\beta}\right]. \tag{5.27}$$

At the same time, it is necessary to write the derivatives of the Ricci scalar. Eq. (5.20) in terms of the perturbation and so get the linearized field equations, that is

$$
\begin{aligned}
R_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right) R^{;\alpha}_{,\alpha} &= \left(h^{;\sigma\rho}_{\sigma\rho} - h^{;\sigma}_{,\sigma}\right)_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right)\left(h^{;\sigma\rho}_{\sigma\rho} - h^{;\sigma}_{,\sigma}\right)^{;\alpha}_{,\alpha} \\
&= \left(h^{;\sigma\rho}_{\sigma\rho} - h^{;\sigma}_{,\sigma}\right)_{,\mu\nu} - \eta_{\mu\nu}\left(h^{;\sigma\rho}_{\sigma\rho} - h^{;\sigma}_{,\sigma}\right)^{;\alpha}_{,\alpha} \\
&= h^{;\sigma\rho}_{\sigma\rho,\mu\nu} - h^{;\sigma}_{,\sigma\mu\nu} - \eta_{\mu\nu} h^{;\sigma\rho\alpha}_{\sigma\rho,\alpha} + \eta_{\mu\nu} h^{;\sigma\alpha}_{,\sigma\alpha} \\
&= \bar{h}^{;\sigma\rho}_{\sigma\rho,\mu\nu} - \frac{1}{2}\eta_{\sigma\rho}\bar{h}^{;\sigma\rho}_{,\mu\nu} + \bar{h}^{;\sigma}_{,\sigma\mu\nu} - \eta_{\mu\nu}\bar{h}^{;\sigma\rho\alpha}_{\sigma\rho,\alpha} + \frac{1}{2}\eta_{\mu\nu}\eta_{\sigma\rho}\bar{h}^{;\sigma\rho\alpha}_{,\alpha} - \eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha} \\
&= \bar{h}^{;\sigma\rho}_{\sigma\rho,\mu\nu} - \frac{1}{2}\bar{h}^{;\sigma}_{,\sigma\mu\nu} + \bar{h}^{;\sigma}_{,\sigma\mu\nu} - \eta_{\mu\nu}\bar{h}^{;\sigma\rho\alpha}_{\sigma\rho,\alpha} + \frac{1}{2}\eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha} - \eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha} \\
&= \bar{h}^{;\sigma\rho}_{\sigma\rho,\mu\nu} + \frac{1}{2}\bar{h}^{;\sigma}_{,\sigma\mu\nu} - \eta_{\mu\nu}\bar{h}^{;\sigma\rho\alpha}_{\sigma\rho,\alpha} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha}.
\end{aligned} \tag{5.28}
$$

Let us suppose that by the freedom of Gauge

$$\bar{h}^{\mu\nu}_{,\nu} = 0, \tag{5.29}$$

which is known as the Lorentz Gauge, in a similar way to the electromagnetic case. In this Gauge, the Einstein tensor (5.27) can be reduced to the simple form

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}^{;\alpha}_{\mu\nu,\alpha}, \tag{5.30}$$

and expression (5.28)

$$R_{,\mu\nu} - \left(\eta_{\mu\nu} + h_{\mu\nu}\right) R^{;\alpha}_{,\alpha} = \frac{1}{2}\bar{h}^{;\sigma}_{,\sigma\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha}. \tag{5.31}$$

With equations (5.30) and (5.31) the linearized field equations (5.23) take the form

$$-\frac{1}{2}a_1\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \frac{1}{2}a_0\left(\eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) - a_2\left(\bar{h}^{;\sigma}_{,\sigma\mu\nu} - \eta_{\mu\nu}\bar{h}^{;\sigma\alpha}_{,\sigma\alpha}\right) = \kappa T_{\mu\nu}, \tag{5.32}$$

which lead to the linearized Einstein field equations when $a_0 = a_2 = 0$ and $a_1 = 1$.

5.2.1.1 Green's Function to solve the equation

In contrast with GR, where the evolution of the perturbation tensor $\bar{h}_{\mu\nu}$ is defined only by the proportionality of its d'alambertian with the stress-energy tensor, field equations (5.32) relate $\bar{h}_{\mu\nu}$ even with the second d'alambertian operator of its trace, which implies more complexity in the form of the equations. However we can take advantage of this relation taking the trace of the equations, that is

$$3a_2\bar{h}^{;\sigma\rho}_{,\sigma\rho} - \frac{1}{2}a_1\bar{h}^{;\sigma}_{,\sigma} - \frac{1}{2}a_0\left(4 - \bar{h}\right) = \kappa T, \tag{5.33}$$

thus we are dealing with a fourth-order partial differential equation and our goal is to determine $\bar{h}$ using the Green's functions method [58]. Note that this equation can be written in terms of operators

$$\left(3a_2\Box^2 - \frac{1}{2}a_1\Box + \frac{1}{2}a_0\right)\bar{h}(x^\sigma) = \kappa T(x^\sigma) + 2a_0, \tag{5.34}$$

where we have reordered some terms and the operator $\Box^2$ represents $\Box\Box$. Unfortunately find the Green's function or propagator associated to the operator $6a_2\Box^2 - a_1\Box + a_0$ is not an easy task, however we can decompose it in two factors, as follows

$$3a_2\Box^2 - \frac{1}{2}a_1\Box + \frac{1}{2}a_0 = (3a_2\Box - A)(\Box - B), \tag{5.35}$$

where the constans A and B satisfy

$$2AB = a_0 \qquad \text{and} \qquad 2A + 6a_2B = a_1, \tag{5.36}$$

or

$$A = \frac{a_1 \mp \sqrt{a_1^2 - 24a_0a_2}}{4}, \tag{5.37}$$

and

$$B = \frac{a_1 \pm \sqrt{a_1^2 - 24a_0a_2},}{12a_2}. \tag{5.38}$$

In this way, Eq. (5.34) could be written as

$$(3a_2\Box - A)(\Box - B)\bar{h}(x^\sigma) = \kappa T(x^\sigma) + 2a_0. \tag{5.39}$$

which has the form of a composition of two Klein-Gordon equations. Let us suppose that the right hand side of the equation can be written as an integral using the sifting property of the Dirac Delta

$$\kappa T(x^\sigma) + 2a_0 = \int \left[\kappa T(y^\sigma) + 2a_0\right] \delta^{(4)}(x^\sigma - y^\sigma) d^4y, \tag{5.40}$$

and we will define $G_1(x^\sigma, y^\sigma)$ as the Green's function that solves

$$(3a_2\Box - A)G_1(x^\sigma, y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma), \tag{5.41}$$

thus

$$\kappa T(x^\sigma) + 2a_0 = (3a_2\Box - A)\int \left[\kappa T(y^\sigma) + 2a_0\right] G_1(x^\sigma, y^\sigma) d^4y, \tag{5.42}$$

where the operator has left the integral because it is acting over x^σ. Making further use of the sifting property of the Dirac Delta we can write the Green's function as

$$G_1(x^\sigma, y^\sigma) = \int G_1(z^\sigma, y^\sigma)\delta^{(4)}(x^\sigma - z^\sigma) d^4z, \tag{5.43}$$

with which we arrive to

$$\kappa T(x^\sigma) + 2a_0 = (3a_2\Box - A)\int\int \left[\kappa T(y^\sigma) + 2a_0\right] G_1(z^\sigma, y^\sigma)\delta^{(4)}(x^\sigma - z^\sigma) d^4z d^4y, \tag{5.44}$$

and defining $G_2(x^\sigma, z^\sigma)$ through

$$(\Box - B)G_2(x^\sigma, z^\sigma) = \delta^{(4)}(x^\sigma - z^\sigma), \tag{5.45}$$

then replacing we get

$$\kappa T(x^\sigma) + 2a_0 = (3a_2\Box - A)(\Box - B)\int\int \left[\kappa T(y^\sigma) + 2a_0\right] G_1(z^\sigma, y^\sigma) G_2(x^\sigma, z^\sigma) d^4z d^4y, \tag{5.46}$$

now by getting rid of the operators we readily find

$$\bar{h}(x^\sigma) = \int\int \left[\kappa T(y^\sigma) + 2a_0\right] G_1(z^\sigma, y^\sigma) G_2(x^\sigma, z^\sigma) d^4z d^4y. \tag{5.47}$$

Of course, we have only moved the problem to find $\bar{h}$ to solve equations (5.43) and (5.45), however this could be an easier task if we take the Fourier transform, defined as

$$\tilde{\phi}(k^\sigma) = \frac{1}{(2\pi)^2}\int \phi(x^\sigma) e^{-ikx} d^4x, \tag{5.48}$$

and the inverse

$$\phi(x^\sigma) = \frac{1}{(2\pi)^2}\int \tilde{\phi}(k^\sigma) e^{ikx} d^4k, \tag{5.49}$$

where $kx = -k_0x_0 + \vec{k}\cdot\vec{x}$. So, in the Fourier space Eq. (5.45) takes the algebraic form

$$\left(-k^2 - B\right)\tilde{G}_2(k^\sigma, z^\sigma) = \frac{1}{4\pi^2} e^{-ikz}, \tag{5.50}$$

or

$$\tilde{G}_2(k^\sigma, z^\sigma) = -\frac{1}{4\pi^2}\frac{e^{-ikz}}{k^2 + B}, \tag{5.51}$$

where $k^2 = -k_0^2 + \mathbf{k}^2$. Thus in the position space we have

$$G_2(x^\sigma, z^\sigma) = -\frac{1}{16\pi^4}\int \frac{e^{ik(x-z)}}{k^2 + B} d^4k, \tag{5.52}$$

It is clearly observed the care that must be taken when performing the integral, since there are singularities or poles in the integrand, i.e. those surfaces where $-k_0^2 + \mathbf{k}^2 + B = 0$. This can be

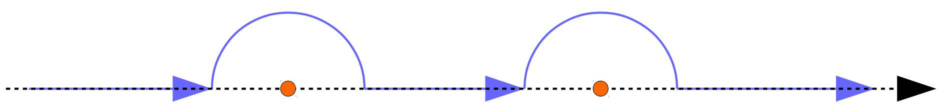


FIGURE 5.3. The singularity of the integrand is removed choosing an appropriate path in the complex plane. Move up from the real line implies a clockwise contour integral and gives the retarded Green's function.

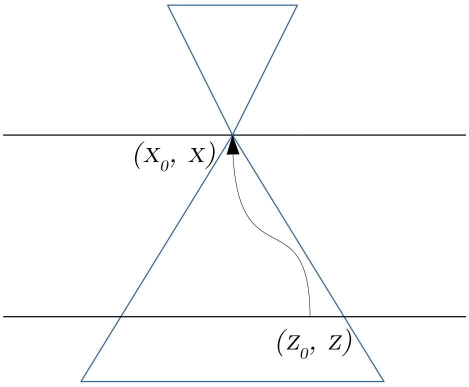


FIGURE 5.4. Field $\bar{h}(x_0,\vec{x})$ is the result of the perturbation from any source at position $(z_0,\vec{z})$ in its past light cone.

seen better if the integral is first made over k_0,

$$G_2(x^\sigma,z^\sigma) = -\frac{1}{16\pi^4}\int e^{i\vec{k}\cdot(\vec{x}-\vec{z})}\int \frac{e^{-ik_0(x_0-z_0)}}{-k_0^2+\mathbf{k}^2+B}dk_0 d^3k, \tag{5.53}$$

which implies that there are poles in $k_0 = \pm\sqrt{\mathbf{k}^2+B}$, and integration must be regularized by shifting the pole, Fig. 5.3, so that there will be different Green's functions or propagators depending on the type of pole movement that is made [147].

On the other hand, we are interested in the field h at a point $(x_0,\vec{x})$, which results from all the contributions of sources in its past light cone, that is to say, $h(x_0,x)$ is the sum of the fields emitted by the sources at any point $\vec{z}$ and at the time z_0 in the causal past of the point $(x_0,\vec{x})$, such that $z_0 < x_0$, Fig. (5.4). From this we will call the retarded Green's Function to

$$G_2(x^\sigma,z^\sigma) = -\frac{1}{16\pi^4}\lim_{\varepsilon\to 0}\int e^{i\vec{k}\cdot(\vec{x}-\vec{z})}\int \frac{e^{-ik_0(x_0-z_0)}}{-(k_0+i\varepsilon)^2+\mathbf{k}^2+B}dk_0 d^3k, \tag{5.54}$$

integrated over a contour clockwise region as depicted in Fig. 5.3.

Defining the dispersion relation as

$$\omega_{2\mathbf{k}}^2 = \mathbf{k}^2 + B, \tag{5.55}$$

integral over k_0 gives

$$\begin{aligned}\int \frac{e^{-ik_0(x_0-z_0)}}{(k_0+i\varepsilon)^2-\mathbf{k}^2-B}dk_0 &= \int \frac{e^{-ik_0(x_0-z_0)}}{2\omega_{2\mathbf{k}}}\left(\frac{1}{k_0+i\varepsilon-\omega_{2\mathbf{k}}} - \frac{1}{k_0+i\varepsilon+\omega_{2\mathbf{k}}}\right)dk_0 \\ &= \frac{i\pi}{\omega_{2\mathbf{k}}}\left(e^{-i(\omega_{2\mathbf{k}}-i\varepsilon)(x_0-z_0)} - e^{i(\omega_{2\mathbf{k}}+i\varepsilon)(x_0-z_0)}\right),\end{aligned} \tag{5.56}$$

where we have used the residue theorem. Thus the Green's Function gives, taking the limit

$$\begin{aligned}G_2(x^\sigma,z^\sigma) &= \frac{i}{16\pi^3}\int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{z})}}{\omega_{2\mathbf{k}}}\left(e^{-i\omega_{2\mathbf{k}}(x_0-z_0)} - e^{i\omega_{2\mathbf{k}}(x_0-z_0)}\right)d^3k \\ &= \frac{1}{8\pi^3}\int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{z})}}{\omega_{2\mathbf{k}}}\sin(\omega_{2\mathbf{k}}(x_0-z_0))d^3k,\end{aligned} \tag{5.57}$$

in spherical coordinates

$$
\begin{aligned}
G_2(x^\sigma, z^\sigma) &= \frac{1}{8\pi^3}\int_0^\infty \int_0^{2\pi}\int_0^\pi \frac{e^{i\mathbf{k}|\vec{x}-\vec{z}|\cos\theta}}{\omega_{2\mathbf{k}}}\sin[\omega_{2\mathbf{k}}(x_0-z_0)]\mathbf{k}^2\sin\theta d\theta d\phi d\mathbf{k} \\
&= \frac{1}{4\pi^2}\int_0^\infty \int_0^\pi \frac{e^{i\mathbf{k}|\vec{x}-\vec{z}|\cos\theta}}{\omega_{2\mathbf{k}}}\sin[\omega_{2\mathbf{k}}(x_0-z_0)]\mathbf{k}^2\sin\theta d\theta d\mathbf{k} \\
&= \frac{1}{4\pi^2}\int_0^\infty \frac{\mathbf{k}^2}{\omega_{2\mathbf{k}}}\sin[\omega_{2\mathbf{k}}(x_0-z_0)]\int_0^\pi e^{i\mathbf{k}|\vec{x}-\vec{z}|\cos\theta}\sin\theta d\theta d\mathbf{k} \\
&= \frac{1}{2\pi^2}\frac{1}{|\vec{x}-\vec{z}|}\int_0^\infty \frac{\mathbf{k}}{\omega_{2\mathbf{k}}}\sin[\omega_{2\mathbf{k}}(x_0-z_0)]\sin(\mathbf{k}|\vec{x}-\vec{z}|)d\mathbf{k} \\
&= -\frac{1}{2\pi^2}\frac{1}{|\vec{x}-\vec{z}|}\frac{d}{d|\vec{x}-\vec{z}|}\int_0^\infty \frac{1}{\omega_{2\mathbf{k}}}\sin[\omega_{2\mathbf{k}}(x_0-z_0)]\cos(\mathbf{k}|\vec{x}-\vec{z}|)d\mathbf{k} \\
&= -\frac{1}{4\pi^2}\frac{1}{|\vec{x}-\vec{z}|}\frac{d}{d|\vec{x}-\vec{z}|}\left(\int_0^\infty \frac{1}{\omega_{2\mathbf{k}}}\sin\left[\omega_{2\mathbf{k}}(x_0-z_0)+\mathbf{k}|\vec{x}-\vec{z}|\right]d\mathbf{k} + \right. \\
&\qquad \left. \int_0^\infty \frac{1}{\omega_{2\mathbf{k}}}\sin\left[\omega_{2\mathbf{k}}(x_0-z_0)-\mathbf{k}|\vec{x}-\vec{z}|\right]d\mathbf{k}\right),
\end{aligned} \tag{5.58}
$$

and noting that

$$
\omega_{2\mathbf{k}}(x_0-z_0)+\mathbf{k}|\vec{x}-\vec{z}| = \sqrt{B}\left[(x_0-z_0)\sqrt{\frac{\mathbf{k}^2}{B}+1}+|\vec{x}-\vec{z}|\frac{\mathbf{k}}{\sqrt{B}}\right] \tag{5.59}
$$

it is helpful to use the variable change

$$
\mathbf{k} = \sqrt{B}\sinh v, \tag{5.60}
$$

furthermore, since we are looking for causal relation between two events defined by x^σ and z^σ, they must be separated by a timelike interval, i.e.,

$$
(x_0-z_0)^2 > |\vec{x}-\vec{z}|^2, \tag{5.61}
$$

with $x_0 > z_0$, which allows us to express

$$
|\vec{x}-\vec{z}| = (x_0-z_0)\tanh u, \tag{5.62}
$$

consequently

$$
\begin{aligned}
\omega_{2\mathbf{k}}(x_0-z_0)+\mathbf{k}|\vec{x}-\vec{z}| &= \sqrt{B}(x_0-z_0)[\cosh v+\tanh u\sinh v] \\
&= \sqrt{B}\frac{x_0-z_0}{\cosh u}[\cosh u\cosh v+\sinh u\sinh v] \\
&= \sqrt{B}\sqrt{-|x-z|^2}\,[\cosh u\cosh v+\sinh u\sinh v] \\
&= \sqrt{B}\sqrt{-|x-z|^2}\cosh(u+v),
\end{aligned} \tag{5.63}
$$

and $|x-z|^2 = -(x_0-z_0)^2+|\vec{x}-\vec{z}|^2$, thus

$$
\int_0^\infty \frac{1}{\omega_{2\mathbf{k}}}\sin\left[\omega_{2\mathbf{k}}(x_0-z_0)+\mathbf{k}|\vec{x}-\vec{z}|\right]d\mathbf{k} = \int_0^\infty \sin\left[\sqrt{B}\sqrt{-|x-z|^2}\cosh(u+v)\right]dv, \tag{5.64}
$$

and noting that the Bessel function of the first kind admits an integral representation of the form

$$J_0(x) = \frac{2}{\pi}\int_0^{\infty} \sin(x\cosh t)dt, \tag{5.65}$$

we arrive at

$$\int_0^{\infty} \frac{1}{\omega_{2\mathbf{k}}} \sin\left[\omega_{2\mathbf{k}}(x_0 - z_0) \pm \mathbf{k}|\vec{x} - \vec{z}|\right] d\mathbf{k} = \frac{\pi}{2}\Theta\left(-|x-z|^2\right) J_0\left(\sqrt{B}\sqrt{-|x-z|^2}\right), \tag{5.66}$$

where the Heaviside step function denotes the timelike framework that the Green's function must have. A finally step to write the solution is make use of the chain rule

$$\frac{d}{d|\vec{x}-\vec{z}|} = -2|\vec{x}-\vec{z}|\frac{d}{d(-|x-z|^2)}, \tag{5.67}$$

Now, it is possible to obtain finally the propagator

$$\begin{aligned} G_2(x^\sigma, z^\sigma) &= \frac{1}{2\pi}\frac{d}{d(-|x-z|^2)}\left[\Theta\left(-|x-z|^2\right) J_0\left(\sqrt{B}\sqrt{-|x-z|^2}\right)\right] \\ &= \frac{1}{2\pi}\left[\delta\left(-|x-z|^2\right) - \frac{\sqrt{B}}{2\sqrt{-|x-z|^2}}\Theta\left(-|x-z|^2\right) J_1\left(\sqrt{-B|x-z|^2}\right)\right], \end{aligned} \tag{5.68}$$

however, since $-|x-z|^2 = (x_0 - z_0)^2 - |\vec{x}-\vec{z}|^2 = (x_0 - z_0 - |\vec{x}-\vec{z}|)(x_0 - z_0 + |\vec{x}-\vec{z}|)$, which is positive for $x_0 - z_0 > |\vec{x}-\vec{z}| > 0$, Heaviside function takes the simpler form $\Theta(x_0 - z_0 - r_{xz})$, with $r_{xz} = |\vec{x}-\vec{z}|$. Likewise Dirac delta function can be expressed as

$$\begin{aligned} \delta\left(-|x-z|^2\right) &= \frac{1}{2r_{xz}}\left(\delta(x_0 - z_0 - r_{xz}) + \delta(x_0 - z_0 + r_{xz})\right) \\ &= \frac{1}{2r_{xz}}\delta(x_0 - z_0 - r_{xz}), \end{aligned} \tag{5.69}$$

where we have taken into account the distributional concept of the Dirac delta, since it will only make sense within an integral and because its argument is greater than zero, i.e. $x_0 - z_0 > 0$, $r_{xz} > 0$, such an integral will vanish over the whole domain. With this we have that the retarded Green's function is

$$G_2(x^\sigma, z^\sigma) = \frac{1}{4\pi}\left[\frac{\delta(x_0 - z_0 - r_{xz})}{r_{xz}} - \frac{\sqrt{B}\,\Theta(x_0 - z_0 - r_{xz})}{\sqrt{(x_0 - z_0)^2 - r_{xz}^2}} J_1\left(\sqrt{B}\sqrt{(x_0 - z_0)^2 - r_{xz}^2}\right)\right], \tag{5.70}$$

and by the same procedure for solving Eq. (5.41), we obtain

$$G_1(z^\sigma, y^\sigma) = \frac{1}{12\pi a_2}\left[\frac{\delta\left(z_0 - y_0 - r_{zy}\right)}{r_{zy}} - \frac{\sqrt{A}\,\Theta\left(z_0 - y_0 - r_{zy}\right)}{\sqrt{3a_2}\sqrt{(z_0 - y_0)^2 - r_{zy}^2}} J_1\left(\sqrt{\frac{A}{3a_2}}\sqrt{(z_0 - y_0)^2 - r_{zy}^2}\right)\right]. \tag{5.71}$$

So, perturbation (5.47) is

$$\bar{h}(x^\sigma) = \int \left[\kappa T(y^\sigma) + 2a_0\right] \int G_2(x^\sigma, z^\sigma) G_1(z^\sigma, y^\sigma) d^4z d^4y, \tag{5.72}$$

which can be performed by making the integral over z_0, so we are going to focus on the integral

$$\int G_2(x^\sigma, z^\sigma) G_1(z^\sigma, y^\sigma) dz_0 = \frac{1}{48\pi a_2}(I_1 + I_2 + I_3 + I_4). \tag{5.73}$$

where

$$\begin{aligned} I_1 &= \frac{1}{r_{xz} r_{zy}} \int_{-\infty}^{\infty} \delta(x_0 - z_0 - r_{xz}) \delta\left(z_0 - y_0 - r_{zy}\right) dz_0 \\ &= \frac{\delta(r_{xz} + r_{zy} - x_0 + y_0)}{r_{xz} r_{zy}}, \end{aligned} \tag{5.74}$$

and

$$\begin{aligned} I_2 &= -\sqrt{\frac{A}{3a_2}} \frac{1}{r_{xz}} \int_{-\infty}^{\infty} \delta(x_0 - z_0 - r_{xz}) \frac{\Theta\left(z_0 - y_0 - r_{zy}\right)}{\sqrt{(z_0 - y_0)^2 - r_{zy}^2}} J_1\left(\sqrt{\frac{A}{3a_2}} \sqrt{(z_0 - y_0)^2 - r_{zy}^2}\right) dz_0 \\ &= -\sqrt{\frac{A}{3a_2}} \frac{1}{r_{xz}} \frac{\Theta\left(x_0 - y_0 - r_{xz} - r_{zy}\right)}{\sqrt{(x_0 - y_0 - r_{xz})^2 - r_{zy}^2}} J_1\left(\sqrt{\frac{A}{3a2}} \sqrt{(x_0 - y_0 - r_{xz})^2 - r_{zy}^2}\right), \end{aligned} \tag{5.75}$$

and

$$\begin{aligned} I_3 &= -\frac{\sqrt{B}}{r_{zy}} \int_{-\infty}^{\infty} \delta\left(z_0 - y_0 - r_{zy}\right) \frac{\Theta(x_0 - z_0 - r_{xz})}{\sqrt{(x_0 - z_0)^2 - r_{xz}^2}} J_1\left(\sqrt{B}\sqrt{(x_0 - z_0)^2 - r_{xz}^2}\right) dz_0 \\ &= -\frac{\sqrt{B}}{r_{zy}} \frac{\Theta\left(x_0 - y_0 - r_{xz} - r_{zy}\right)}{\sqrt{(x_0 - y_0 - r_{zy})^2 - r_{xz}^2}} J_1\left(\sqrt{B}\sqrt{(x_0 - y_0 - r_{zy})^2 - r_{xz}^2}\right), \end{aligned} \tag{5.76}$$

and

$$\begin{aligned} I_4 &= \frac{\sqrt{AB}}{3a_2} \int_{-\infty}^{\infty} \frac{\Theta(x_0 - z_0 - r_{xz})}{\sqrt{(x_0 - z_0)^2 - r_{xz}^2}} \frac{\Theta\left(z_0 - y_0 - r_{zy}\right)}{\sqrt{(z_0 - y_0)^2 - r_{zy}^2}} J_1\left(\sqrt{B}\sqrt{(x_0 - z_0)^2 - r_{xz}^2}\right) \\ &\qquad J_1\left(\sqrt{\frac{A}{3a_2}} \sqrt{(z_0 - y_0)^2 - r_{zy}^2}\right) dz_0 \\ &= \frac{\sqrt{AB}}{3a_2} \int_{-\infty}^{\infty} \frac{J_1\left(\sqrt{B}\sqrt{(x_0 - z_0)^2 - r_{xz}^2}\right)}{\sqrt{(x_0 - z_0)^2 - r_{xz}^2}} \frac{J_1\left(\sqrt{\frac{A}{3a_2}} \sqrt{(z_0 - y_0)^2 - r_{zy}^2}\right)}{\sqrt{(z_0 - y_0)^2 - r_{zy}^2}} dz_0 \\ &= 0. \end{aligned} \tag{5.77}$$

Finally we can write the perturbation as

$$\bar{h}(x^\sigma) = \frac{1}{48\pi a_2} \int\int \left[\kappa T(y^\sigma) + 2a_0\right] \left[\frac{\delta(r_{xz} + r_{zy} - x_0 + y_0)}{r_{xz} r_{zy}} + I_2 + I_3\right] dy_0 d^3 z d^3 y. \tag{5.78}$$

5.2.1.2 Newtonian limit

In the Newtonian limit ($|v| \ll 1$) $|T^{00}| \gg |T^{ii}|$, and $|\bar{h}^{00}| \gg |\bar{h}^{ii}|$, so $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} \approx -\bar{h}_{00}$. Furthermore if the system is static time derivatives are equally zero [47, 164]

$$-\frac{1}{2}a_1\bar{h}^{00,i}_{,i} + \frac{1}{2}a_0\left(1-\frac{1}{2}\bar{h}^{00}\right) - a_2\bar{h}^{00,ij}_{,ij} = \kappa T^{00}$$

$$-\frac{1}{2}a_1\nabla^2\bar{h}^{00} + \frac{1}{2}a_0\left(1-\frac{1}{2}\bar{h}^{00}\right) - a_2\nabla^2\nabla^2\bar{h}^{00} = \kappa T^{00}, \tag{5.79}$$

and noting that

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = \frac{1}{2}\bar{h}_{00}, \tag{5.80}$$

thus the linearized field equations are

$$-a_1\nabla^2 h^{00} + \frac{1}{2}a_0\left(1-h^{00}\right) - 2a_2\nabla^2\nabla^2 h^{00} = \kappa T^{00}, \tag{5.81}$$

knowing that the potentials are only functions of r, as in the case of the metric (4.22), the equation reads as

$$-a_1\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dh^{00}}{dr}\right) + \frac{1}{2}a_0\left(1-h^{00}\right) - 2a_2\frac{1}{r^2}\frac{d}{dr}\left\{r^2\frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dh^{00}}{dr}\right)\right]\right\} = \kappa T^{00}. \tag{5.82}$$

This is a fourth order differential equation for $h^{00}(r)$, whose singular solution is

$$h^{00}(r) = 1 - \frac{2\kappa}{a_0}T^{00}, \tag{5.83}$$

and assuming a solution of the related homogeneous equation in the form

$$h^{00}(r) = \frac{1}{r}e^{mr}, \tag{5.84}$$

it is found that m satisfies

$$4a_2m^4 + 2a_1m^2 + a_0 = 0, \tag{5.85}$$

thus the general solution is a four-parameter family of solutions

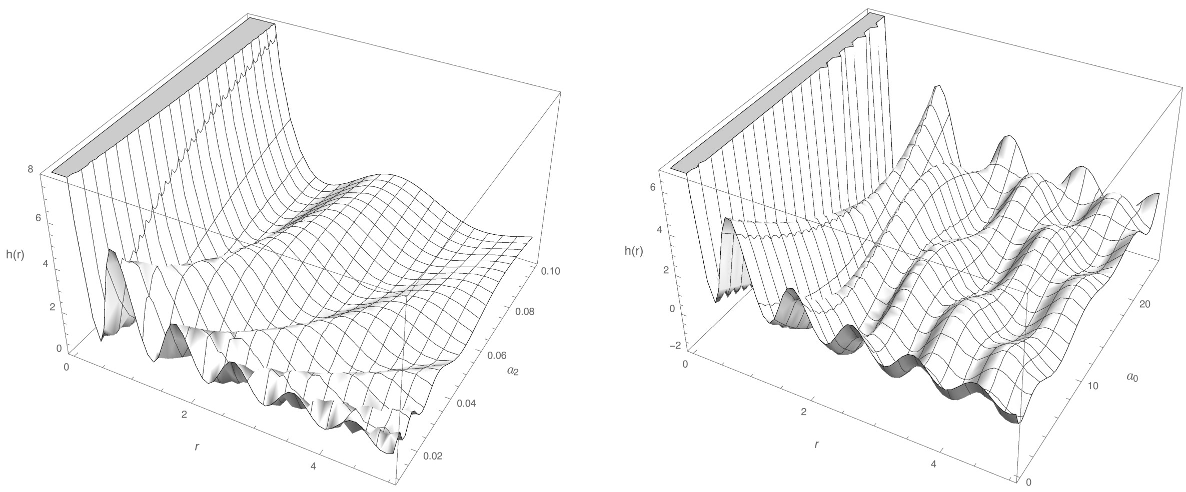
$$h^{00}(r) = 1 - \frac{2\kappa}{a_0}T^{00} + \frac{1}{r}\sum_{i=1}^{4} c_i e^{m_i r}, \tag{5.86}$$

where the m_i are the roots of Eq. (5.85).

$$m_i = \frac{(-1)^i}{2}\sqrt{-\frac{a_1 + (-1)^i\sqrt{a_1^2 - 4a_0a_2}}{a_2}}, \tag{5.87}$$

for $i = 1, \ldots, 4$.

In this situation, like GR, $f(R)$ gravity must take the same predictions as Newtonian theory, and it is possible to find the constants according to the gravitational conditions at the weak fields, for example the Newtonian laws of motion. In fact, as GR represents a particular case of $f(R)$, Newtonian gravity is expected to be a limiting case of $f(R)$.



(a)

(b)

FIGURE 5.5. Perturbation tensor behaviour as a function of r and of parameters a_0 and a_2: (a) $a_0 = 0.01$ and (b) $a_2 = 0.01$. In both graphs the other constants were taken as 1 and $T = 0$. It is shown that h oscillates with decreasing amplitude as the radius vector increases as expected due the flat asymptoticity of the potential.

CHAPTER 6

THE NOETHER CHARGE AND THE BH ENTROPY

Of course, the aim of this work is not going into detail about the particularities of deriving the laws of Thermodynamics in General Relativity, so they only will be mentioned in the next section due to their great importance for the thesis and because the objectives are directed towards the generalization of these laws in the $f(R)$ theory.

6.1 Laws of Black Hole Thermodynamics in GR

The research in the BH theory at the beginning of 70's brought an important change in the understanding of GR, on the one hand, at classical level the gravitational behaviour of BH obeyed laws that had a mathematically similarity with the usual laws of thermodynamic, and on the other, it was discussed by Bekenstein[19, 20] from the point of view of information theory the concept of black-hole entropy and was showed by Hawking [76, 79] that BH radiated as black bodies, that was a surprising quantum effect not expected within the GR framework, opening a very intriguing relationship between GR, Thermodynamics and Quantum Mechanics.
This issue arises to consider that, as is usual in Thermodynamics for a thermal system, the properties as entropy, temperature and energy are obtained from the statistical mechanics of its microstates, that is, from its intrinsic quantum nature, but for BH Thermodynamics, these quantities are given in terms of the gravitational characteristics of the BH, the surface gravity κ_s, related to the Hawking temperature

$$T = \frac{\kappa_s}{2\pi}, \tag{6.1}$$

and in terms of the area of the event horizon A_H, related to the entropy by

$$S = \frac{A_H}{4G}. \tag{6.2}$$

Since the first works of Hawking and Bekenstein, the relations between temperature of the BH and its surface gravity Ec. (6.1), and between its entropy and area (6.2), have aroused much interest in the scientific community and there has been an enormous research in the thermodynamics of the BH, summarized in the following statements.
For a stationary BH, with energy $E = M$, angular momentum J, charge Q, and described with a metric asymptotically flat, the four laws of thermodynamics[15] are as follows

0. The surface gravity is constant over the event horizon

$$\delta\kappa_s \overset{H}{=} 0 \tag{6.3}$$

1. The change in energy dE induced by small changes in δA, δJ and δQ, when a BH varies from one stationary state to another is

$$\delta E = \frac{\kappa}{8\pi G}\delta A_H + \Omega\delta J + \Phi\delta Q, \tag{6.4}$$

   where Ω and Φ are the angular velocity and electrostatic potential at the horizon.

2. The area of the event horizon of a BH never decreases

$$\delta A_H \geq 0. \tag{6.5}$$

3. It is impossible to reach $\kappa = 0$ with a finite number of steps, which means that it is not possible that exists a BH without surface gravity.

Black Holes in GR obey the Laws of Thermodynamics, classically they absorb everything and do not emit anything, so their absolute temperature is zero [192], which is not true in quantum theory since the Hawking radiation implies that they radiate with a perfect thermal spectrum and therefore decreasing the size of the BH by evaporation, even disappearing [80, 81]. This effect must be compensated by quantum fluctuations in order to maintain the second law of the Black Holes correct. In this regard, a fundamental achievement of any theory must be linked to quantum mechanics through explaining the entropy and temperature of a BH, however despite the greats advances it is a major problem that has not yet been explained at all, which constitutes a big goal to be reached for any theory of modified gravity as $f(R)$.

6.2 Noether charge

Classical field theory relates the physical fields, ϕ, to matter through the Field Equations, which are obtained from the critical points of an action S, expressed as an integral of a Lagrange density evaluated over the n-dimensional space

$$I = \int d^n x \mathscr{L}. \tag{6.6}$$

We assume a gravitational Lagrangian density defined over a manifold M in a local coordinate system $x_1,\dots,x_d$, that depends on the Lorentz signature metric $g_{\mu\nu}$ and other dynamical fields ψ, which we will refer simply as $\phi=(g_{\mu\nu},\psi)$.

$$\mathscr{L}=\mathscr{L}(g_{\alpha\beta},R_{\alpha\beta\mu\nu},R_{\alpha\beta\mu\nu;\sigma_1},\dots,R_{\alpha\beta\mu\nu;(\sigma_1,\dots,\sigma_m)},\psi,\psi_{;\sigma_1},\dots,\psi_{;(\sigma_1,\dots,\sigma_n)}), \tag{6.7}$$

a first variation of the Lagrangian with respect to a some parameter such that $\phi=\phi(\lambda)$, gives

$$\begin{aligned}\delta\mathscr{L}=\frac{\partial\mathscr{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta}+\frac{\partial\mathscr{L}}{\partial R_{\alpha\beta\mu\nu}}\delta R_{\alpha\beta\mu\nu}+\frac{\partial\mathscr{L}}{\partial R_{\alpha\beta\mu\nu;\sigma_1}}\delta R_{\alpha\beta\mu\nu;\sigma_1}+\dots+\frac{\partial\mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1,\dots,\sigma_m)}}\delta R_{\alpha\beta\mu\nu;(\sigma_1,\dots,\sigma_m)}+\\ \frac{\partial\mathscr{L}}{\partial\psi}\delta\psi+\frac{\partial\mathscr{L}}{\partial\psi_{;\sigma_1}}\delta\psi_{;\sigma_1}+\dots+\frac{\partial\mathscr{L}}{\partial\psi_{;(\sigma_1,\dots,\sigma_n)}}\delta\psi_{;(\sigma_1,\dots,\sigma_n)}.\end{aligned} \tag{6.8}$$

We want to express this variation as a linear sum over the dynamical fields ϕ plus the exterior derivative of a differential $(n-1)$-form $\tilde{\theta}$, known in the literature as the *symplectic potential*[1]

$$\delta\mathscr{L}=A^{\alpha\beta}\delta g_{\alpha\beta}+B^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu}+C\delta\psi+d\tilde{\theta}, \tag{6.9}$$

where $d\tilde{\theta}$ is understood by the exterior derivative, i.e., in a coordinate system, $\tilde{\theta}^{\mu}_{;\mu}$. To accomplish this objective we write the symmetrized covariant derivative of the Riemann tensor [47]

$$\begin{aligned}R_{\alpha\beta\mu\nu;(\sigma_1,\dots,\sigma_m)}=&\frac{1}{m!}\left(R_{\alpha\beta\mu\nu;\sigma_1\dots\sigma_m}+\text{sum over permutations of indices }\sigma_1\dots\sigma_m\right)\\ =&\frac{1}{m!}\left(R_{\alpha\beta\mu\nu,\sigma_1\dots\sigma_m}+R_{\alpha\beta\mu\nu,\sigma_m\sigma_1\dots\sigma_{m-1}}+\cdots\right)\\ =&\frac{1}{m!}\left(R_{\alpha\beta\mu\nu,\sigma_1\dots\sigma_m}m!+\text{terms proportional to }g_{\alpha\beta}\right)\\ =&R_{\alpha\beta\mu\nu,\sigma_1\dots\sigma_m}+\text{terms proportional to }g_{\alpha\beta},\end{aligned} \tag{6.10}$$

thus, the variation of this tensor could be written as

$$\begin{aligned}\delta R_{\alpha\beta\mu\nu;(\sigma_1\dots\sigma_m)}=&\delta R_{\alpha\beta\mu\nu,\sigma_1\dots\sigma_m}+\text{terms proportional to }\delta g_{\alpha\beta}\\ =&\left(\delta R_{\alpha\beta\mu\nu,\sigma_2\dots\sigma_m}\right)_{,\sigma_1}+\Gamma^{\rho}_{\alpha\sigma_1}\delta R_{\rho\beta\mu\nu,\sigma_2\dots\sigma_m}-\Gamma^{\rho}_{\alpha\sigma_1}\delta R_{\rho\beta\mu\nu,\sigma_2\dots\sigma_m}+\cdots\\ =&\left(\delta R_{\alpha\beta\mu\nu,\sigma_2\dots\sigma_m}\right)_{;\sigma_1}+\text{terms proportional to }\delta g_{\alpha\beta}\\ =&\left(\delta R_{\alpha\beta\mu\nu;\sigma_2\dots\sigma_m}+\Gamma^{\rho}_{\alpha\sigma_m}\delta R_{\rho\beta\mu\nu;\sigma_2\dots\sigma_{m-1}}+\cdots\right)_{;\sigma_1}+\text{terms proportional to }\delta g_{\alpha\beta}\\ =&\left(\delta R_{\alpha\beta\mu\nu;\sigma_2\dots\sigma_m}\right)_{;\sigma_1}+\text{terms proportional to }g_{\alpha\beta}\text{and }\delta g_{\alpha\beta},\end{aligned} \tag{6.11}$$

[1]Strictly speaking a symplectic form is a 2-form that satisfies two conditions, nondegeneracy and closedness [39].

and multiplying by the partial derivative of the Lagrangian density to construct the typical term

$$\begin{aligned}\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\delta R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)} =& \frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\left(\delta R_{\alpha\beta\mu\nu;\sigma_2\ldots\sigma_m}\right)_{;\sigma_1} + \text{terms prop to } g_{\alpha\beta} \text{and } \delta g_{\alpha\beta}\\ =& \left(\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\delta R_{\alpha\beta\mu\nu;\sigma_2\ldots\sigma_m}\right)_{;\sigma_1} - \left(\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\right)_{;\sigma_1}\delta R_{\alpha\beta\mu\nu;\sigma_2\ldots\sigma_m}\\ &+ \text{terms prop to } g_{\alpha\beta} \text{and } \delta g_{\alpha\beta}\\ =&(-1)^m\left(\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\right)_{;\sigma_m\ldots\sigma_1}\delta R_{\alpha\beta\mu\nu} + \text{terms prop to } g_{\alpha\beta} \text{and } \delta g_{\alpha\beta},\end{aligned} \tag{6.12}$$

in which the last line has been found by doing the same procedure iteratively m times. The variation of the Lagrangian Eq. (6.8), then is

$$\begin{aligned}\delta \mathscr{L} =& \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \left[\frac{\partial \mathscr{L}}{\partial R_{\alpha\beta\mu\nu}} - \left(\frac{\partial \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;\sigma_1}}\right)_{;\sigma_1} + \cdots + (-1)^m\left(\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\right)_{;\sigma_m\ldots\sigma_1}\right]\delta R_{\alpha\beta\mu\nu} +\\ &\left[\frac{\partial \mathscr{L}}{\partial \psi} - \left(\frac{\partial \mathscr{L}}{\partial \psi_{;\sigma_1}}\right)_{;\sigma_1} + \cdots + (-1)^n\left(\frac{\delta \mathscr{L}}{\partial \psi_{;(\sigma_1\ldots\sigma_n)}}\right)_{;\sigma_n\ldots\sigma_1}\right]\delta\psi + \text{terms prop to } g_{\alpha\beta} \text{and } \delta g_{\alpha\beta}.\end{aligned} \tag{6.13}$$

In this way it is possible to determine the tensors that appear in Eq. (6.9)

$$A^{\alpha\beta} = \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}}, \tag{6.14}$$

$$B^{\alpha\beta\mu\nu} = \frac{\partial \mathscr{L}}{\partial R_{\alpha\beta\mu\nu}} - \left(\frac{\partial \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;\sigma_1}}\right)_{;\sigma_1} + \cdots + (-1)^m\left(\frac{\delta \mathscr{L}}{\partial R_{\alpha\beta\mu\nu;(\sigma_1\ldots\sigma_m)}}\right)_{;\sigma_m\ldots\sigma_1}, \tag{6.15}$$

$$C = \frac{\partial \mathscr{L}}{\partial \psi} - \left(\frac{\partial \mathscr{L}}{\partial \psi_{;\sigma_1}}\right)_{;\sigma_1} + \cdots + (-1)^n\left(\frac{\delta \mathscr{L}}{\partial \psi_{;(\sigma_1\ldots\sigma_n)}}\right)_{;\sigma_n\ldots\sigma_1}. \tag{6.16}$$

Notice that the principle of least action implies that equations of motion of the theory are found when $A^{\alpha\beta} = 0$, $B^{\alpha\beta\mu\nu} = 0$ and $C = 0$.

We can construct an antisymmetrized quantity derived from the symplectic potential by taking the mixed partial derivatives of the Lagrangian variation (6.9), with respect two parameters (λ_1, λ_2), that is

$$\begin{aligned}\delta_2\delta_1\mathscr{L} =& \delta_2 A^{\alpha\beta}\delta_1 g_{\alpha\beta} + A^{\alpha\beta}\delta_2\delta_1 g_{\alpha\beta} + \delta_2 B^{\alpha\beta\mu\nu}\delta_1 R_{\alpha\beta\mu\nu} + B^{\alpha\beta\mu\nu}\delta_2\delta_1 R_{\alpha\beta\mu\nu} + \delta_2 C\delta_1\psi +\\ & C\delta_2\delta_1\psi + \delta_2 d\tilde{\theta}_1,\end{aligned} \tag{6.17}$$

and subtracting the two partial derivatives

$$0 = \delta_2 A^{\alpha\beta}\delta_1 g_{\alpha\beta} - \delta_1 A^{\alpha\beta}\delta_2 g_{\alpha\beta} + \delta_2 B^{\alpha\beta\mu\nu}\delta_1 R_{\alpha\beta\mu\nu} - \delta_1 B^{\alpha\beta\mu\nu}\delta_2 R_{\alpha\beta\mu\nu} + \delta_2 C\delta_1\psi - \delta_1 C\delta_2\psi + d\tilde{\omega}, \tag{6.18}$$

where it is defined the symplectic current $\tilde{\omega}$ as

$$\tilde{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_2\tilde{\theta}(\phi, \delta_1\phi) - \delta_1\tilde{\theta}(\phi, \delta_2\phi), \tag{6.19}$$

It is worth noting that on shell

$$d\tilde{\omega} = 0, \tag{6.20}$$

which allows us to relate the symplectic current with a $(n-2)$-form $\tilde{\Omega}(\phi, \delta_1\phi, \delta_2\phi)$, called the *pre-symplectic form*, by

$$\tilde{\omega} = d\tilde{\Omega}. \tag{6.21}$$

6.3 Noether current

Now, from the symplectic potential it is defined the *Noether density current*

$$\mathscr{J} = \tilde{\theta} - \imath_\xi \mathscr{L}, \tag{6.22}$$

which, of course, is a symplectic $(n-1)$-form, and where $\imath_\xi \mathscr{L}$ is the interior product[2] of the lagrangian density and the infinitesimal generator of the symmetries, i.e. the killing vector field ξ^μ. Note that when equations of motion are satisfied the symplectic character of this form is tangible through its closedness, $d\mathscr{J} = 0$, which can be seen by the divergence

$$\begin{aligned}\mathscr{J}^\mu_{;\mu} =& \tilde{\theta}^\mu_{;\mu} - \xi^\mu \mathscr{L}_{;\mu} \\ =& \delta\mathscr{L} - A^{\alpha\beta}\delta g_{\alpha\beta} - B^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu} - C\delta\psi - \xi^\mu \mathscr{L}_{;\mu}\end{aligned} \tag{6.23}$$

and knowing that for a diffeomorphism covariant lagrangian the infinitesimal gauge transformations implies

$$\mathfrak{L}_\xi \mathscr{L}(\phi) = \frac{\partial \mathscr{L}}{\partial \phi}\mathfrak{L}_\xi \phi, \tag{6.24}$$

where $\mathfrak{L}_\xi$ is the Lie derivative along the vector field ξ^μ on M, and

$$\delta\phi = \mathfrak{L}_\xi \phi = \xi^\mu \phi_{,\mu}, \tag{6.25}$$

then

$$\begin{aligned}\delta\mathscr{L} =& \frac{\partial \mathscr{L}}{\partial \phi}\delta\phi \\ =& \frac{\partial \mathscr{L}}{\partial \phi}\mathfrak{L}_\xi \phi \\ =& \mathfrak{L}_\xi \mathscr{L} \\ =& \xi^\mu \mathscr{L}_{;\mu},\end{aligned} \tag{6.26}$$

therefore the Noether current density results

$$\mathscr{J}^\mu_{;\mu} = -A^{\alpha\beta}\delta g_{\alpha\beta} - B^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu} - C\delta\psi, \tag{6.27}$$

[2] Interior product is a mapping from n-form to a $(n-1)$-form.

hence the Noether current is conserved on shell, which means that it can be expressed as

$$\mathscr{J} = d\mathscr{Q}, \tag{6.28}$$

that is, it could be expressed as the divergence of a second rank tensor quantity, $\mathscr{Q}^{(\mu\nu)}$, similarly called the Noether charge, so

$$\mathscr{J}^{\mu} = \mathscr{Q}^{\mu\nu}_{;\nu}. \tag{6.29}$$

Using the identity

$$\mathfrak{L}_{\xi}\theta = \imath_{\xi} d\theta + d(\imath_{\xi}\theta), \tag{6.30}$$

it is possible to write the variation (on shell) of the current density as

$$\begin{aligned} \delta\mathscr{J} =& \delta\tilde{\theta} - \imath_{\xi}\delta\mathscr{L} \\ =& \delta\tilde{\theta} - \imath_{\xi} d\tilde{\theta} \\ =& \delta\tilde{\theta} - \mathfrak{L}_{\xi}\tilde{\theta} + d(\imath_{\xi}\tilde{\theta}), \end{aligned} \tag{6.31}$$

and noting that $\mathfrak{L}_{\xi}\theta = \delta\theta$, the symplectic current, Eq. (6.19), can be written as

$$\tilde{\omega}(\phi, \mathfrak{L}_{\xi}\tilde{\theta}, \delta\tilde{\theta}) = \delta\tilde{\theta}(\phi) - \mathfrak{L}_{\xi}\tilde{\theta}(\phi) \tag{6.32}$$

thus the variation of the Noether current could be written as

$$\delta\mathscr{J} = \tilde{\omega}(\phi, \mathfrak{L}_{\xi}\tilde{\theta}, \delta\tilde{\theta}) + d(\imath_{\xi}\tilde{\theta}). \tag{6.33}$$

In spite of the notation we have used so far, from now on we will use the scalar and tensor quantities related to their corresponding densities by [70],

$$L = \frac{1}{\sqrt{-g}}\mathscr{L}, \tag{6.34}$$

$$j^{\mu} = \frac{1}{\sqrt{-g}}\mathscr{J}^{\mu}, \tag{6.35}$$

$$Q^{\mu\nu} = \frac{1}{\sqrt{-g}}\mathscr{Q}^{\mu\nu}. \tag{6.36}$$

and in the same way to $(\tilde{\theta}, \tilde{\omega}, \tilde{\Omega}) \rightarrow (\theta, \omega, \Omega)$. With this quantities the Noether charge is

$$j = dQ, \tag{6.37}$$

and its variation

$$\delta j = \omega(\phi, \mathfrak{L}_{\xi}\theta, \delta\theta) + d(\imath_{\xi}\theta). \tag{6.38}$$

integrating on an asymptotically flat surface Σ, with boundary $\partial\Sigma$,

$$\begin{aligned}\int_\Sigma \delta j &= \int_\Sigma \omega(\phi,\mathfrak{L}_\xi\theta,\delta\theta)+\int_\Sigma d(\imath_\xi\theta)\\ &= \int_{\partial\Sigma}\Omega(\phi,\mathfrak{L}_\xi\theta,\delta\theta)+\int_{\partial\Sigma}\imath_\xi\theta\end{aligned} \tag{6.39}$$

where Eq. (6.19) and the Stokes theorem

$$\int_\Sigma d(\imath_\xi\theta)=\int_{\partial\Sigma}\imath_\xi\theta, \tag{6.40}$$

have been considered on both terms of the right side of the equation. At this point stands out the importance of defining the Hamiltonian of the system associated to the symplectic current, since interior product can be considered as an antiderivation, from Eq. (B.1)

$$dH=\imath_{X_H}\omega, \tag{6.41}$$

allows to write a variation of the Hamiltonian in terms of the pre-symplectic form

$$\delta H=\int_{\partial\Sigma}\Omega(\phi,\mathfrak{L}_\xi\theta,\delta\theta), \tag{6.42}$$

thus Eq. (6.39) implies that Noether current, together the boundary term $\int_{\partial\Sigma}\imath_\xi\theta$, are part of the Hamiltonian

$$\delta H=\delta\int_\Sigma j-\int_{\partial\Sigma}\imath_\xi\theta, \tag{6.43}$$

this equation could be written in a simpler way if we assuming there is a $(n-1)$-form, Θ, that satisfies

$$\delta\int_{\partial\Sigma}\imath_\xi\Theta=\int_{\partial\Sigma}\imath_\xi\theta, \tag{6.44}$$

then, Hamiltonian associated to the dynamics imposed for ξ^μ, is

$$H=\int_\Sigma j-\int_{\partial\Sigma}\imath_\xi\Theta, \tag{6.45}$$

and by the definition of charge and the Stokes theorem again,

$$\begin{aligned}H&=\int_\Sigma dQ-\int_{\partial\Sigma}\imath_\xi\Theta\\ &=\int_{\partial\Sigma}\left(Q-\imath_\xi\Theta\right).\end{aligned} \tag{6.46}$$

Let us suppose a Black Hole solution in a stationary but non static and asymptotically flat spacetime Σ, whose the two-dimensional boundary $\partial\Sigma$ is given by surface $\partial\mathfrak{B}$ describing the event horizon and $\partial\Sigma_\infty$ describing the spatial infinity [70]. The event horizon is described by a killing vector field written as a linear combination of the vector field $\xi^\mu_{(t)}$ associated to time translations at infinity and a rotational vector field $\xi^\mu_{(\phi)}$, i.e.

$$\xi^\mu=\xi^\mu_t+\Omega_H\xi^\mu_\phi, \tag{6.47}$$

where Ω_H is the constant angular velocity of the horizon. From these Killing vector fields we can now define a canonical energy E corresponding to time translation and associated to the value of the Hamiltonian, as

$$E = \int_{\partial\Sigma_\infty} \left(Q[\xi_{(t)}] - \imath_{\xi_{(t)}}\Theta\right), \tag{6.48}$$

and in the same way it is defined the canonical angular momentum associated to the asymptotic rotation

$$J = -\int_{\partial\Sigma_\infty} \left(Q[\xi_{(\phi)}] - \imath_{\xi_{(\phi)}}\Theta\right), \tag{6.49}$$

where in both definitions it has been taken into account that on $\partial\mathfrak{B}$, $\xi^\mu = 0$, moreover the surface boundary $\partial\Sigma$ is chosen in such a way that the asymptotic rotation $\xi^\mu_{(\phi)}$ is tangent to it, so that $\imath_{\xi_{(\phi)}}\Theta$ does not contribute to the total angular momentum, thus

$$J = -\int_{\partial\Sigma_\infty} Q[\xi_{(\phi)}]. \tag{6.50}$$

From the before definitions it is possible to found a similar expression for the First Law, if we assume that $\delta H = 0$, Eq. (6.43) becomes

$$\delta\int_\Sigma j = \int_{\partial\Sigma} \imath_\xi\theta, \tag{6.51}$$

replacing the Noether current by the charge, Eq. (6.29) and using the Stokes Theorem

$$\delta\int_{\partial\Sigma} Q[\xi] = \int_{\partial\Sigma} \imath_\xi\theta, \tag{6.52}$$

and the killing vector field (6.47)

$$\begin{aligned} \delta\int_{\partial\Sigma} Q &= \int_{\partial\Sigma} \imath_{\xi_t}\theta + \Omega_H\int_{\partial\Sigma} \imath_{\xi_\phi}\theta \\ \delta\int_{\partial\Sigma_\infty} Q - \delta\int_{\partial\mathfrak{B}} Q &= \int_{\partial\Sigma_\infty} \imath_{\xi_t}\theta + \Omega_H\int_{\partial\Sigma_\infty} \imath_{\xi_\phi}\theta, \end{aligned} \tag{6.53}$$

assuming that Noether charge is a linear mapping, that is

$$Q[\xi_t + \Omega_H\xi_\phi] = Q[\xi_t] + \Omega_H Q[\xi_\phi] \tag{6.54}$$

rearranging some terms and noting that by Eq. (6.52) and Eq. (6.44)

$$\begin{aligned} \delta\int_{\partial\mathfrak{B}} Q[\xi] =& \delta\int_{\partial\Sigma_\infty} Q[\xi_t + \Omega_H\xi_\phi] - \delta\int_{\partial\Sigma_\infty} \imath_{\xi_t}\Theta - \Omega_H\delta\int_{\partial\Sigma_\infty} \imath_{\xi_\phi}\Theta \\ =& \delta\int_{\partial\Sigma_\infty} Q[\xi_t] + \Omega_H\delta\int_{\partial\Sigma_\infty} Q[\xi_\phi] - \delta\int_{\partial\Sigma_\infty} \imath_{\xi_t}\Theta - \Omega_H\delta\int_{\partial\Sigma_\infty} \imath_{\xi_\phi}\Theta, \end{aligned} \tag{6.55}$$

thus by the definitions of energy (6.48) and (6.49)

$$\delta E = \delta\int_{\partial\mathfrak{B}} Q[\xi] + \Omega_H\delta J, \tag{6.56}$$

this expression shows that a small change in energy is produced by variations of the Noether charge along the event horizon and variations of the angular momentum, resembling the first law of the Black Hole Thermodynamics, Eq. (6.4), describing changes from stationary states induced only by entropy δS and angular momentum δJ, that is

$$\delta E = \frac{\kappa_s}{2\pi}\delta S + \Omega_H \delta J, \tag{6.57}$$

whose correspondence leads to

$$\boxed{\int_{\partial\mathfrak{B}} Q[\xi] = \frac{\kappa_s}{2\pi} S}, \tag{6.58}$$

note that the denominator should have a constant G which is set to 1 because of the choice of the system of natural units. Relation (6.58) is an outstanding result demonstrating that Noether charge is a measure of the Black Hole entropy at the event horizon.

This theory is described below applied to the more general case of $f(R)$ gravity and in Appendix D to General Relativity.

6.4 Modified gravity

Modified Lagrangian density

$$\mathscr{L}(g_{\alpha\beta}, R, T_{\alpha\beta}) = \frac{1}{2\kappa}\mathscr{L}_f + \mathscr{L}_{\text{matter}}, \tag{6.59}$$

where $\mathscr{L}_{\text{matter}} = \mathscr{L}_m(T_{\alpha\beta})$, the constant $\kappa = 8\pi G$, and the modified Lagrangian density is defined in a similar way of the Einstein-Hilbert density

$$\mathscr{L}_f = \mathscr{L}_f(g_{\alpha\beta}, R) = \sqrt{-g}\, f(R). \tag{6.60}$$

The first variation of this Lagrangian is

$$\begin{aligned}
\delta\mathscr{L}_f =& \frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_f}{\partial f}\delta R \\
=& \frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_H}{\partial f}R_{\alpha\beta}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_f}{\partial f}g^{\alpha\beta}\delta R_{\alpha\beta} \\
=& \frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\alpha\beta}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_f}{\partial f}\left[g_{\alpha\beta}g^{\mu\nu}\left(\delta g^{\alpha\beta}\right)_{;\mu\nu} - \left(\delta g^{\alpha\beta}\right)_{;\alpha\beta}\right]
\end{aligned} \tag{6.61}$$

where $f = f(R)$,

$$F = F(R) = \frac{df(R)}{dR}, \tag{6.62}$$

and it has been used the variation of the Ricci scalar, Eq. (A.17). So, applying the method proposed by Wald [191], and described by the decomposition of the Lagrangian partial derivatives using the chain rule in Eq. (6.11)

$$F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha\beta} = \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} - \left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta}\left(\delta g^{\alpha\beta}\right)_{;\alpha}$$

$$= \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} - \left[\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta}\delta g^{\alpha\beta}\right]_{;\alpha} + \left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta\alpha}\delta g^{\alpha\beta}, \tag{6.63}$$

thus

$$\begin{aligned}\delta\mathscr{L}_f =& \frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\alpha\beta}\delta g^{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} - g_{\alpha\beta}g^{\mu\nu}\left[\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\nu}\delta g^{\alpha\beta}\right]_{;\mu} + \\ & g_{\alpha\beta}g^{\mu\nu}\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\nu\mu}\delta g^{\alpha\beta} - \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} + \left[F\left(\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta}\delta g^{\alpha\beta}\right]_{;\alpha} - \left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta\alpha}\delta g^{\alpha\beta} \\ =& \left[\frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\nu\mu} - \left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta\alpha}\right]\delta g^{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} \\ & - g_{\alpha\beta}g^{\mu\nu}\left[\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\nu}\delta g^{\alpha\beta}\right]_{;\mu} - \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} + \left[\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\beta}\delta g^{\alpha\beta}\right]_{;\alpha}\end{aligned} \tag{6.64}$$

covariant derivative of the partial derivative of this Lagrangian with respect to $f(R)$ function gives

$$\left(F\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\mu} = F_{;\mu}\frac{\partial\mathscr{L}_f}{\partial f} + F\left(\frac{\partial\mathscr{L}_f}{\partial f}\right)_{;\mu}$$

$$= F_{;\mu}\frac{\partial\mathscr{L}_f}{\partial f}, \tag{6.65}$$

hence the variation results

$$\begin{aligned}\delta\mathscr{L}_f =& \left[\frac{\partial\mathscr{L}_f}{\partial g^{\alpha\beta}} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}F_{;\nu\mu}\frac{\partial\mathscr{L}_f}{\partial f} - F_{;\beta\alpha}\frac{\partial\mathscr{L}_f}{\partial f}\right]\delta g^{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} - \\ & g_{\alpha\beta}g^{\mu\nu}\left[F_{;\nu}\frac{\partial\mathscr{L}_f}{\partial f}\delta g^{\alpha\beta}\right]_{;\mu} - \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} + \left[F_{;\beta}\frac{\partial\mathscr{L}_f}{\partial f}\delta g^{\alpha\beta}\right]_{;\alpha},\end{aligned} \tag{6.66}$$

this variations allows to write the total Lagrangian (6.59), as

$$\delta\mathscr{L} = \frac{1}{2\kappa}\delta\mathscr{L}_f + \delta\mathscr{L}_m, \tag{6.67}$$

where is immediately recognized from Eq. (6.9) the tensor

$$A_{\mu\nu} = \frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_f}{\partial g^{\mu\nu}} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\mu\nu} + g_{\mu\nu}F^{;\alpha}_{;\alpha}\frac{\partial\mathscr{L}_f}{\partial f} - F_{;\nu\mu}\frac{\partial\mathscr{L}_f}{\partial f}\right) + \frac{\partial\mathscr{L}_m}{\partial g^{\mu\nu}}, \tag{6.68}$$

and the symplectic potential density

$$\begin{aligned}\tilde{\theta}^{\nu}_{;\nu} =& \frac{1}{2\kappa}\left\{ g_{\alpha\beta}g^{\mu\nu}\left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} - g_{\alpha\beta}g^{\mu\nu}\left[F_{;\nu}\frac{\partial\mathscr{L}_f}{\partial f}\delta g^{\alpha\beta}\right]_{;\mu} - \left[F\frac{\partial\mathscr{L}_f}{\partial f}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} + \right. \\ & \left. \left[F_{;\beta}\frac{\partial\mathscr{L}_f}{\partial f}\delta g^{\alpha\beta}\right]_{;\alpha}\right\} \\ =& \frac{1}{2\kappa}\frac{\partial\mathscr{L}_f}{\partial f}\left[g_{\alpha\beta}g^{\mu\nu}F\left(\delta g^{\alpha\beta}\right)_{;\mu} - g_{\alpha\beta}g^{\mu\nu}F_{;\mu}\delta g^{\alpha\beta} - F\left(\delta g^{\mu\nu}\right)_{;\mu} + F_{;\mu}\delta g^{\mu\nu}\right]_{;\nu},\end{aligned} \tag{6.69}$$

from where

$$\tilde{\theta}^{\nu} = \frac{1}{2\kappa}\frac{\partial\mathscr{L}_f}{\partial f}\left[Fg^{\mu\nu}(\delta g)_{;\mu} - F_{;\mu}g^{\mu\nu}\delta g - F\left(\delta g^{\mu\nu}\right)_{;\mu} + F_{;\mu}\delta g^{\mu\nu}\right], \tag{6.70}$$

where we have used the notation $\delta g = g_{\mu\nu}\delta g^{\mu\nu}$, and knowing that

$$\frac{\partial\mathscr{L}_f}{\partial f} = \sqrt{-g} \qquad \text{and} \qquad \frac{\partial\mathscr{L}_f}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,f g_{\mu\nu}, \tag{6.71}$$

equations of motion are satisfied when $A_{\mu\nu}$, Eq. (6.68), is equating to zero, that is

$$\begin{aligned}\frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_f}{\partial g^{\mu\nu}} + F\frac{\partial\mathscr{L}_f}{\partial f}R_{\mu\nu} + F^{;\alpha}_{;\alpha}g_{\mu\nu}\frac{\partial\mathscr{L}_f}{\partial f} - F_{;\nu\mu}\frac{\partial\mathscr{L}_f}{\partial f}\right) + \frac{\partial\mathscr{L}_m}{\partial g^{\mu\nu}} =& 0 \\ \frac{1}{2\kappa}\left(-\frac{1}{2}\sqrt{-g}fg_{\mu\nu} + \sqrt{-g}FR_{\nu\mu} + \sqrt{-g}F^{;\alpha}_{;\alpha}g_{\mu\nu} - \sqrt{-g}F_{;\nu\mu}\right) - \frac{1}{2}\sqrt{-g}T_{\mu\nu} =& 0 \\ \frac{1}{\kappa}\left(-\frac{1}{2}fg_{\mu\nu} + FR_{\nu\mu} + F^{;\alpha}_{;\alpha}g_{\mu\nu} - F_{;\nu\mu}\right) - T_{\mu\nu} =& 0\end{aligned} \tag{6.72}$$

where it has been used the definition of the stress-energy tensor [47]

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_m}{\delta g^{\mu\nu}}, \tag{6.73}$$

Eq. (6.72) are precisely the equations of motion, or field equations, for the $f(R)$ theory of gravity

$$\boxed{FR_{\nu\mu} + F^{;\alpha}_{;\alpha}g_{\mu\nu} - F_{;\nu\mu} - \frac{1}{2}fg_{\mu\nu} = \kappa T_{\mu\nu}}. \tag{6.74}$$

On the other hand, the symplectic potential associated to the modified Lagrangian density results

$$\begin{aligned}\theta^{\mu} =& \frac{1}{2\kappa}F\left[g^{\mu\nu}(\delta g)_{;\nu} - \left(\delta g^{\mu\nu}\right)_{;\nu}\right] - \frac{1}{2\kappa}F_{;\nu}\left[g^{\mu\nu}\delta g - \delta g^{\mu\nu}\right] \\ =& \frac{1}{2\kappa}\left[F\left(g^{\mu\nu}\delta g - \delta g^{\mu\nu}\right)_{;\nu} - F_{;\nu}\left(g^{\mu\nu}\delta g - \delta g^{\mu\nu}\right)\right] \\ =& \frac{1}{2\kappa}\left(F\zeta^{\mu\nu}{}_{;\nu} - F_{;\nu}\zeta^{\mu\nu}\right),\end{aligned} \tag{6.75}$$

where function $\zeta^{\mu\nu}$ depends entirely on the metric and has been defined as

$$\zeta^{\mu\nu} = g^{\mu\nu}\delta g - \delta g^{\mu\nu}. \tag{6.76}$$

Consider now the Lie derivative of the metric along the killing vector field generating the isometries, ξ_μ, which satisfies

$$\mathfrak{L}_\xi g^{\mu\nu} = \delta g^{\mu\nu} = -\xi^{\mu;\nu} - \xi^{\nu;\mu}, \tag{6.77}$$

with which is possible to express the function $\zeta^{\mu\nu}$ as

$$\begin{aligned}
\zeta^{\mu\nu} =& g^{\mu\nu} g_{\alpha\beta}\delta g^{\alpha\beta} - \delta g^{\mu\nu} \\
=& -g^{\mu\nu} g_{\alpha\beta}\left(\xi^{\alpha;\beta} + \xi^{\beta;\alpha}\right) + \xi^{\mu;\nu} + \xi^{\nu;\mu} \\
=& -g^{\mu\nu}\left(\xi^{\alpha}{}_{;\alpha} + \xi^{\beta}{}_{;\beta}\right) + \xi^{\mu;\nu} + \xi^{\nu;\mu} \\
=& -2g^{\mu\nu}\xi^{\alpha}{}_{;\alpha} + \xi^{\mu;\nu} + \xi^{\nu;\mu},
\end{aligned} \tag{6.78}$$

thus

$$\begin{aligned}
\zeta^{\mu\nu}{}_{;\nu} =& -2\xi^{\alpha\ ;\mu}_{;\alpha} + \xi^{\mu;\nu}{}_{;\nu} + \xi^{\nu;\mu}{}_{;\nu} \\
=& 2\xi_\alpha R^{\alpha\mu} - 2\xi^{\alpha;\mu}{}_{;\alpha} + \xi^{\mu;\nu}{}_{;\nu} + \xi^{\nu;\mu}{}_{;\nu} \\
=& 2\xi_\alpha R^{\alpha\mu} - \xi^{\nu;\mu}{}_{;\nu} + \xi^{\mu;\nu}{}_{;\nu},
\end{aligned} \tag{6.79}$$

where the non-commutativity of the covariant derivatives has been used, Eq. (C.32),

$$\xi^{\alpha\ ;\mu}_{;\alpha} = -\xi_\alpha R^{\alpha\mu} + \xi^{\alpha;\mu}{}_{;\alpha} \tag{6.80}$$

hence

$$\begin{aligned}
\theta^\mu =& \frac{1}{2\kappa}\left[F\left(2\xi_\alpha R^{\alpha\mu} - \xi^{\nu;\mu}{}_{;\nu} + \xi^{\mu;\nu}{}_{;\nu}\right) + F_{;\nu}\left(2g^{\mu\nu}\xi^{\alpha}{}_{;\alpha} - \xi^{\mu;\nu} - \xi^{\nu;\mu}\right)\right] \\
=& \frac{1}{\kappa}\left[F\xi_\alpha R^{\alpha\mu} + F\xi^{[\mu;\nu]}{}_{;\nu} + F^{;\mu}\xi^{\alpha}{}_{;\alpha} - F_{;\nu}\xi^{\mu;\nu} + \frac{1}{2}F_{;\nu}\left(\xi^{\mu;\nu} - \xi^{\nu;\mu}\right)\right] \\
=& \frac{1}{\kappa}\left[F\xi_\alpha R^{\alpha\mu} + \left(F\xi^{[\mu;\nu]}\right)_{;\nu} + F^{;\mu}\xi^{\alpha}{}_{;\alpha} - F_{;\nu}\xi^{\mu;\nu}\right],
\end{aligned} \tag{6.81}$$

Noether current can be expressed as

$$\begin{aligned}
j^\mu =& \theta^\mu - \xi^\mu L \\
=& \theta^\mu - \frac{1}{2\kappa}\xi^\mu L_f - \xi^\mu L_m \\
=& \frac{1}{\kappa}\left[F\xi_\alpha R^{\alpha\mu} + \left(F\xi^{[\mu;\nu]}\right)_{;\nu} + F^{;\mu}\xi^{\alpha}{}_{;\alpha} - F_{;\nu}\xi^{\mu;\nu} - \frac{1}{2}\xi^\mu f\right] - \xi^\mu L_m \\
=& \frac{1}{\kappa}\left[\left(F\xi^{[\mu;\nu]}\right)_{;\nu} + F^{;\mu}\xi^{\alpha}{}_{;\alpha} - F^{;\alpha}\xi^{\mu}_{;\alpha} + \kappa\xi_\nu T^{\mu\nu} - F^{;\alpha}_{;\alpha}\xi^\mu + F^{;\mu\nu}\xi_\nu\right] - \xi^\mu L_m \\
=& \frac{1}{\kappa}\left[\left(F\xi^{[\mu;\nu]}\right)_{;\nu} + \left(F^{;\mu}\xi^{\alpha}\right)_{;\alpha} - \left(F^{;\alpha}\xi^{\mu}\right)_{;\alpha}\right] + \xi_\nu T^{\mu\nu} - \xi^\mu L_m \\
=& \frac{1}{\kappa}\left(F\xi^{[\mu;\nu]} + F^{;\mu}\xi^{\nu} - F^{;\nu}\xi^{\mu}\right)_{;\nu} + \xi_\nu T^{\mu\nu} - \xi^\mu L_m,
\end{aligned} \tag{6.82}$$

where we have used in the third line the Einstein field equation, moreover we know from the stress-energy tensor definition that

$$
\begin{aligned}
T_{\mu\nu} &= -2\frac{\delta L_m}{\delta g^{\mu\nu}} - 2L_m \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \\
&= -2\frac{\delta L_m}{\delta g^{\mu\nu}} + L_m g_{\mu\nu},
\end{aligned} \tag{6.83}
$$

so if we assume that the matter fields do not depend on geometry, Noether current can be written as

$$
\begin{aligned}
j^{\mu} &= \frac{1}{\kappa}\left(F\xi^{[\mu;\nu]} + F^{;\mu}\xi^{\nu} - F^{;\nu}\xi^{\mu}\right)_{;\nu} + \xi_{\nu}L_m g^{\mu\nu} - \xi^{\mu}L_m \\
&= \frac{1}{\kappa}\left(F\xi^{[\mu;\nu]} + F^{;\mu}\xi^{\nu} - F^{;\nu}\xi^{\mu}\right)_{;\nu} \\
&= \frac{1}{\kappa}\left(F\xi^{[\mu;\nu]} + 2F^{[;\mu}\xi^{\nu]}\right)_{;\nu}.
\end{aligned} \tag{6.84}
$$

Since the Noether charge is defined in terms of the current, Eq. (6.29)

$$
j^{\mu} = Q^{\mu\nu}{}_{;\nu}, \tag{6.85}
$$

therefore

$$
Q^{\mu\nu} = \frac{1}{\kappa}\left(F\xi^{[\mu;\nu]} + 2F^{[;\mu}\xi^{\nu]}\right), \tag{6.86}
$$

however, the Black Hole entropy as given by Eq. (6.58), is defined as the integration over the event horizon, which is described by a killing vector field ξ^{μ}, so by using the killing equation, the Noether charge takes the form

$$
Q^{\mu\nu} = \frac{1}{\kappa}F\xi^{\mu;\nu} \tag{6.87}
$$

and the Black Hole entropy is found by integrating the Noether charge

$$
\begin{aligned}
S &= \frac{2\pi}{\kappa_s}\int_{\partial\mathfrak{B}} d\sigma_{\mu\nu}Q^{\mu\nu} \\
&= \frac{1}{4\kappa_s}F\int_{\partial\mathfrak{B}} d\sigma_{\mu\nu}\xi^{\mu;\nu},
\end{aligned} \tag{6.88}
$$

using the null complex tretad $(l, m, \bar{m}, n)$, defined over the BH event horizon

$$
\xi^{\mu} \overset{H}{=} l^{\mu}, \tag{6.89}
$$

such that $l_{\mu}n^{\mu} = -1$, and by the Eq. (C.15)

$$
\begin{aligned}
l_{\mu;\nu}l^{\nu} &\overset{H}{=} \kappa_s l_{\mu} \\
l_{\mu;\nu}l^{\nu}n^{\mu} &\overset{H}{=} \kappa_s l_{\mu}n^{\mu} \\
l_{\mu;\nu}l^{\nu}n^{\mu} &\overset{H}{=} -\kappa_s.
\end{aligned} \tag{6.90}
$$

$$\begin{aligned} S &= \frac{1}{4\kappa_s} F \int_\sigma d^2\sigma l^{\mu;\nu} l_\mu n_\nu \\ &= -\frac{1}{4\kappa_s} F \int_\sigma d^2\sigma l^{\nu;\mu} l_\mu n_\nu \\ &= \frac{1}{4} F \int_\sigma d^2\sigma, \end{aligned} \tag{6.91}$$

Finally it is found the relation for the Black Hole entropy and its area

$$\boxed{S = \frac{1}{4} F A}. \tag{6.92}$$

6.5 Thermodynamic quantities

Now that the entropy of the BH has been defined in terms of the function $f(R)$, we are able to calculate some of the most relevant characteristics associated with the thermodynamics of the BH. Let us start considering the solution (4.70),

(6.93)

$$a(r) = 1 + \frac{\Lambda}{3} r^2 + \frac{R_S}{m}\left(\frac{3n}{2m} - \frac{1}{r}\right) + \frac{n}{m}\left(1 + \frac{3nR_S}{m^2}\right)\left(\ln\left(n + \frac{m}{r}\right)^{\frac{n}{m}} - \frac{1}{r}\right) r^2 - 16\pi(qr)^2 \int \frac{{}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{q^2}{b^2 r^4}\right)}{r^5(m+nr)} dr$$

where we have include the constant $c_2 = -\Lambda/3$. The Hawking temperature on the BH horizon[3], will be

$$T(r_+) = \frac{1}{4S}\left[3R_S - mr_+ - \frac{S}{\pi r_+} - \frac{16\pi q^2}{r_+}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{\pi\Phi^2}{b^2 S}\right)\right], \tag{6.94}$$

where the entropy is given by

$$S = \pi r_+^2 (m + nr_+), \tag{6.95}$$

and the electrostatic potential, defined as

$$\Phi = \frac{q\sqrt{m + nr_+}}{r_+}. \tag{6.96}$$

When $b \to 0$ ($q = 0$), the BH mass can be calculated as

$$R_S = \frac{2m^2 r_+}{3} \frac{m\left(m\left(\Lambda r_+^2 + 3\right) - 3nr_+\right) + 3n^2 r_+^2 \ln\left[\frac{m}{r_+} + n\right]}{m\left(2m^2 - 3mnr_+ + 6n^2 r_+^2\right) - 6n^3 r_+^3 \ln\left[\frac{m}{r_+} + n\right]}, \tag{6.97}$$

and

$$T(r_+) = -\frac{1}{2S}\left(\frac{mr_+}{2} + \frac{S}{2\pi r_+} + \frac{3nr_+ - m\left(3 + \Lambda r_+^2\right) - \frac{3n^2}{m} r_+^2 \ln\left[n + \frac{m}{r_+}\right]}{\frac{2}{r_+} - \frac{3n}{m^2}(m - 2nr_+) - \frac{6n^3}{m^3} r_+^2 \ln\left[n + \frac{m}{r_+}\right]}\right), \tag{6.98}$$

[3]See Appendix C

which reproduce the BH temperature in the GR with cosmological constant case when $n = 0$,

$$T(r_+) = \frac{\pi m + \Lambda S}{4m\pi^{3/2}\sqrt{S}}. \tag{6.99}$$

At this point the classification made in Section 4.4 of Chapter 4 is useful: $r > 0$ if $m, n > 0$; $r > |\frac{m}{n}|$ if $m < 0$ and $n > 0$; and $0 < r < |\frac{m}{n}|$ if $m > 0$ and $n < 0$. However, Hawking temperature will have a singularity when

$$w(r_+) := \frac{2}{r_+} - \frac{3n}{m^2}(m - 2nr_+) - \frac{6n^3}{m^3} r_+^2 \ln\left[n + \frac{m}{r_+}\right] = 0, \tag{6.100}$$

if this equation has a real solution for $r_+ > 0$, then the maximal point will be r_{c+}, given by

$$\frac{6n^3}{m^3} \ln\left[n + \frac{m}{r_{c+}}\right] r_{c+} - \frac{3n^2(m + 2nr_{c+})}{m^2(m + nr_{c+})} + \frac{1}{r_{c+}^2} = 0, \tag{6.101}$$

of value

$$w(r_{c+}) = \frac{3m}{r_{c+}(m + nr_{c+})}, \tag{6.102}$$

so, if $m, n > 0$, $0 < w(r_{c+}) < w(r_+)$, and $T(r_+)$ will be a continuous function for all $r > 0$. If $m < 0$ and $n > 0$, and there is a maximum point given by Eq. (6.101), then $w(r_+) < w(r_{c+}) < 0$, and again $T(r_+)$ is continuous. However for $0 < r_{c+} < |\frac{m}{n}|$ with $m > 0$ and $n < 0$, it is possible to achieve $w(r_+) = 0$. These temperature behaviors can be seen in Fig. 6.1, where T vs S has been plotted for some values of the constants m, n and Λ. When $m = 1$ and $n = 0$, the minimum temperature will be reached in $S = \pi/\Lambda$, and will have a value $T = \frac{\sqrt{\Lambda}}{2\pi}$. Note that when $m = 1$, for $0 < n < 1$, $w(r_+)$ has no root, however if $n = 0$

$$\lim_{r_+ \to \infty} w(r_+) = \lim_{r_+ \to \infty} \left(\frac{2}{r_+} - 3n(1 - 2nr_+) - 6n^3 r_+^2 \ln\left[n + \frac{1}{r_+}\right]\right) = 0, \tag{6.103}$$

and for $n > 1$, $w(r_+)$ will have a root and therefore the temperature will not be continuous, as shown in panel (a) of the Figure. The second law of BH thermodynamics states that the change in the area of the event horizon is always greater than or equal to zero, in analogy to the entropy of an isolated system. Moreover an infinitesimal change of entropy of the BH is given by the transfer of energy, δQ divided by temperature,

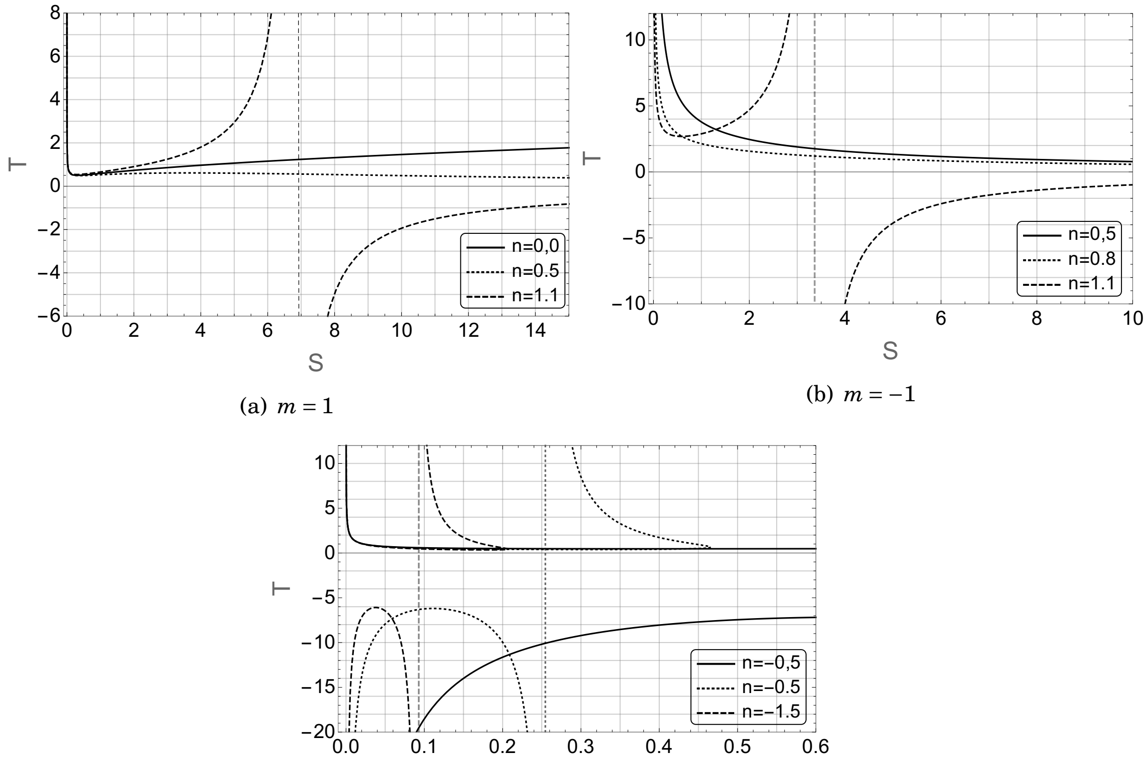
$$dS = \frac{\delta Q}{T}, \tag{6.104}$$

with $\delta Q = CdT$, or

$$dS = \frac{C}{T} dT, \tag{6.105}$$

where C is the specific heat of the BH, defined by

$$C = T\frac{\partial S}{\partial T}. \tag{6.106}$$



(a) $m = 1$

(b) $m = -1$

(c) $m = 1$

FIGURE 6.1. Hawking temperature as a function of the BH Entropy for the $f(R)$-Schwarzschild-type space and some values of m and n. In all panels $\Lambda = 10$.

The analytical expression for specific heat is quite complicated, so it will be more useful to graph it for some particular values of m and n, as shown in Fig. 6.2, where C vs S is shown. When $n = 0$, the specific heat is reduced to

$$C(r_+) = \frac{2\pi r^2\left(\Lambda r_+^2 + 1\right)}{\Lambda r_+^2 - 1} \tag{6.107}$$

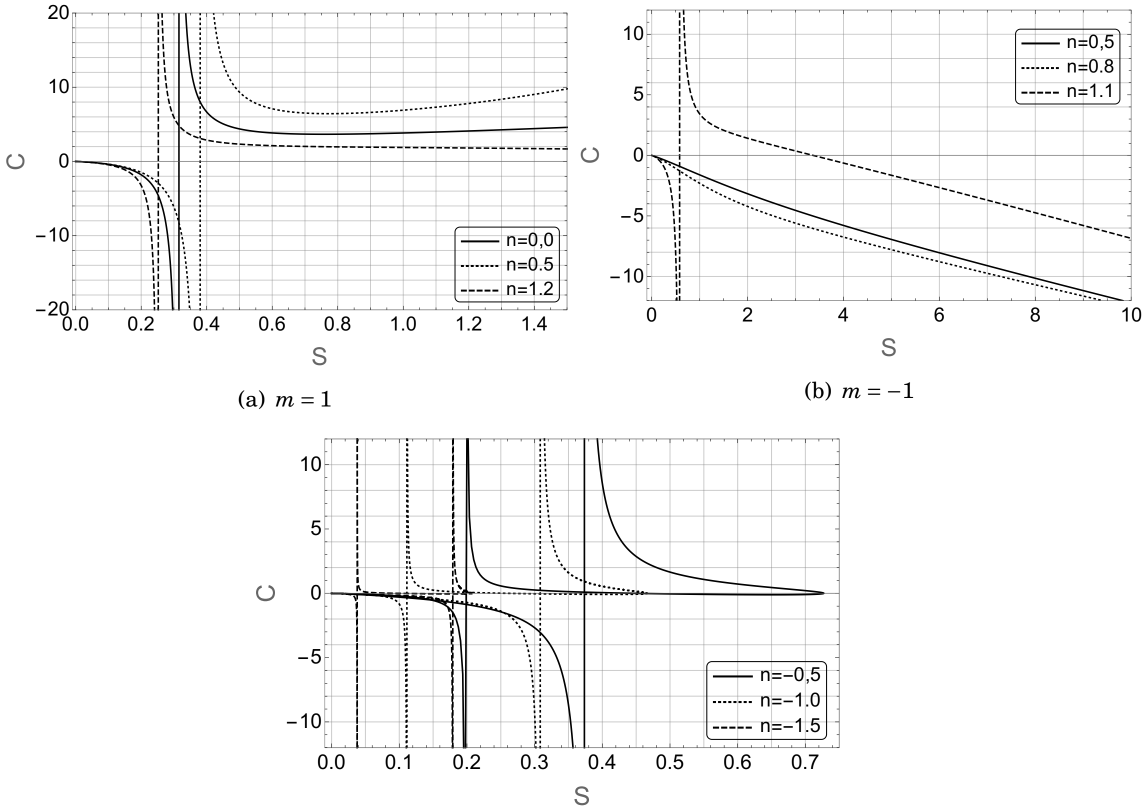
or int terms of the entropy

$$C(S) = \frac{2S(\Lambda S + \pi)}{\Lambda S - \pi}, \tag{6.108}$$

this expression reproduces the result for the Schwarzschild Black Hole in Anti-De Sitter Space.

In the more general case, when $b \to \infty$, or BI-$f(R)$ theory, the mass can be written as

$$R_S = -\frac{1}{3}\left(\frac{m^2}{n} + \frac{16\pi n q^2}{m}\right) + \frac{\frac{m^2}{3n} + \frac{4\pi q^2}{r} + mr\left(\frac{1}{2} + \frac{\Lambda}{3}r^2\right)}{1 - \frac{3n}{2m}r + \frac{3n^2}{m^2}r^2 - \frac{3n^3}{m^3}r^3 \ln\left[\frac{m}{r} + n\right]}, \tag{6.109}$$



(a) $m = 1$

(b) $m = -1$

(c) $m = 1$

FIGURE 6.2. Specific heat as a function of the BH Entropy for the $f(R)$-Schwarzschild-type space and some values of m and n. In all panels $\Lambda = 10$.

and the Hawking temperature in terms of the BH horizon radius is

$$T(r_+) = -\frac{1}{4nS^2}\left(\frac{S^3}{\pi^2 r_+^4} + \frac{16\pi n r_+ S\Phi^2}{m} - \frac{mS\left(\frac{2m}{r} + n\left(3 + 2\Lambda r_+^2\right)\right) + 24\pi^2 n r_+^2 \Phi^2}{\frac{2}{r_+} - \frac{3n}{m^2}(m - 2nr_+) - \frac{6n^3}{m^3} r_+^2 \ln\left[\frac{m}{r_+} + n\right]}\right), \tag{6.110}$$

which when $n = 0$ and $m = 1$, reproduces Hawking temperature for the RN-(A)dS black hole

$$T(S,\Phi) = \frac{\pi + \Lambda S - 4\pi^2\phi^2}{4\pi^{3/2}\sqrt{S}}, \tag{6.111}$$

The Hawking temperature as a function of the entropy is shown in Fig. 6.3, and because the continuity of temperature is determined by the same function $w(r_+)$, Eq. (6.100), the analysis made above is still valid in this case. Now, the specific heat can be calculated for a constant potential,

$$C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_\Phi, \tag{6.112}$$

in Fig. 6.4 the behaviour of C_Φ is plotted as a function of S for some particular values of the parameters and charge. It is found that when $m = 1$ and $n = 0$, the specific heat as a function of

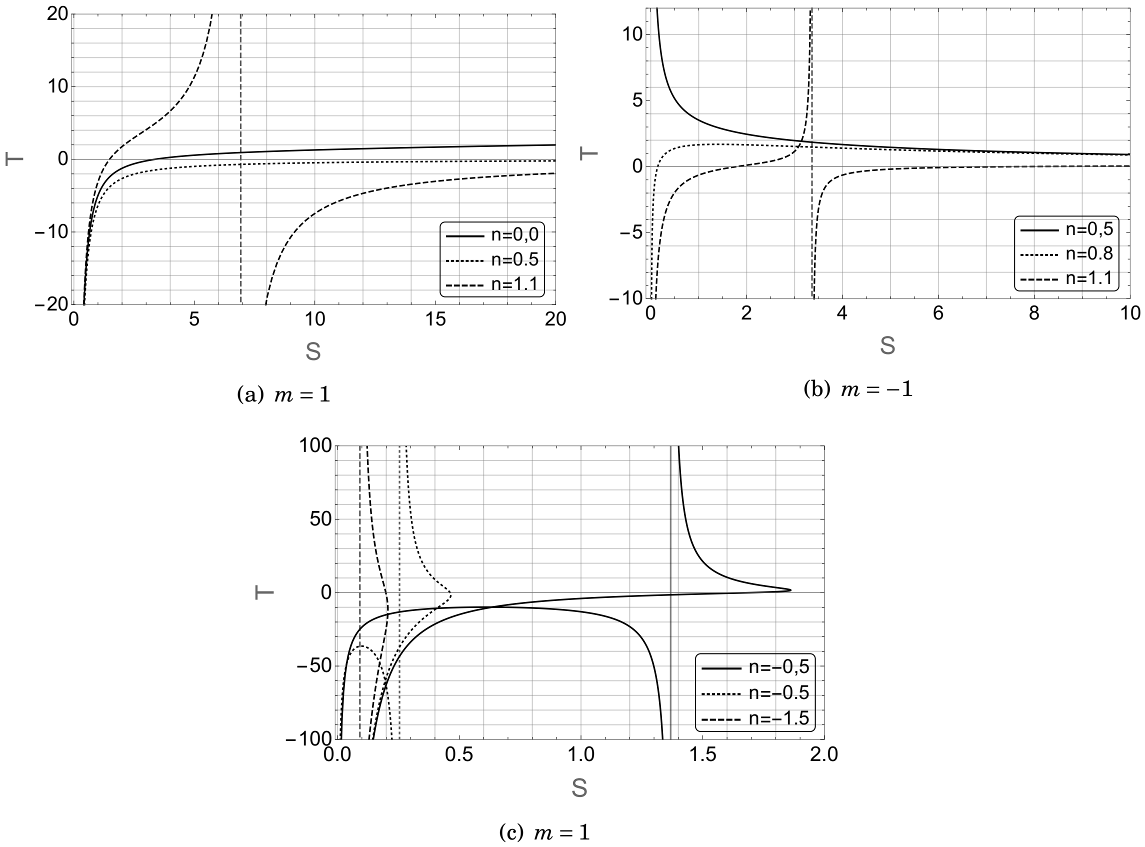


FIGURE 6.3. Hawking temperature as a function of the BH Entropy for the RN-$f(R)$ in the (A)de Sitter space and some values of m and n. In all panels $\Lambda = 10$ and $q = 1$. The monotonicity, continuity and existence of the critical point in the T-S plane depends on the value of the parameters m and n, as well as the charge of the BH.

the BH horizon, is given by

$$C_\Phi(r_+) = \frac{2\pi r_+^2\left(\Lambda r_+^4 + r_+^2 - 4\pi q^2\right)}{r_+^2\left(\Lambda r_+^2 - 1\right) + 12\pi q^2}, \tag{6.113}$$

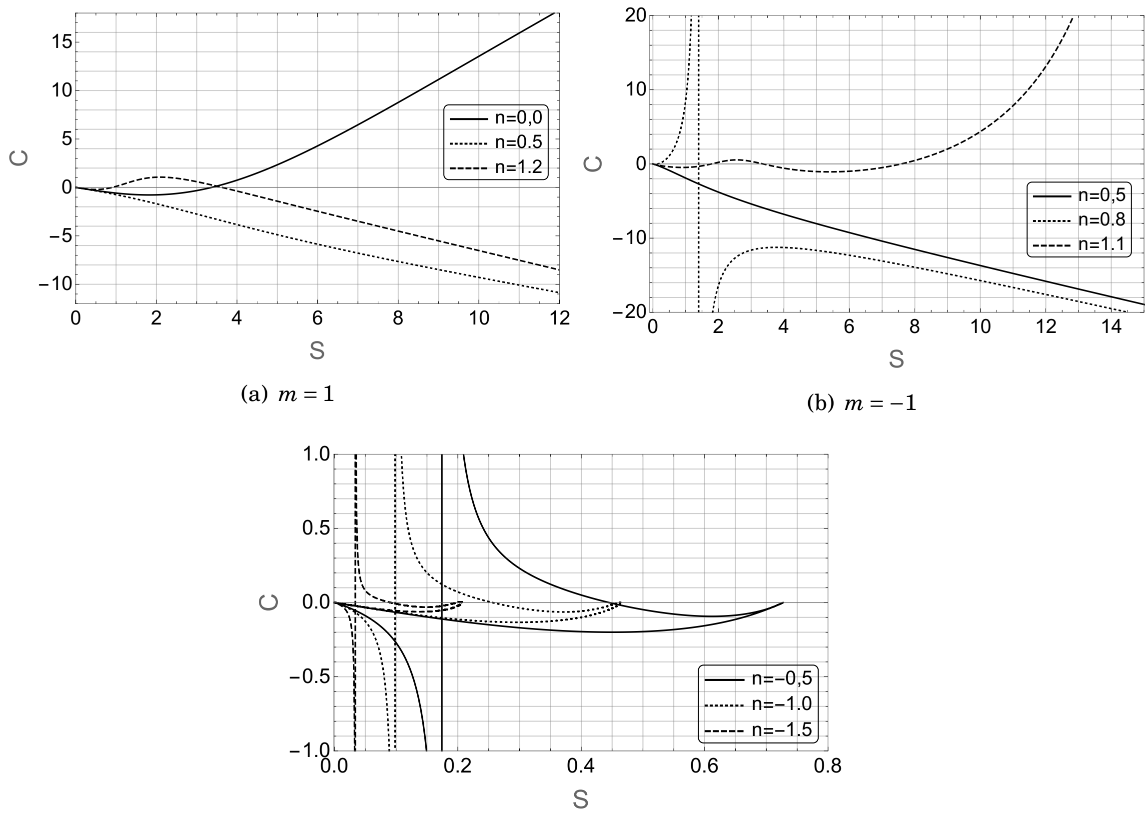
or in terms of the entropy and potential

$$C_\Phi(S,\Phi) = \frac{2S\left(\Lambda S + \pi\left(1 - 4\pi\phi^2\right)\right)}{\Lambda S - \pi\left(1 - 12\pi\phi^2\right)}, \tag{6.114}$$

this is the specific heat for RN-(A)dS BH, which is plotted in panel (a) of Fig. 6.4.

6.6 Non-linear BI

This section will describe the temperature of the BH in a non-linear model of $f(R)$ in the BI theory. From solution (4.114) of the Section 4.5, the associated Hawking temperature as a function of the



(a) $m = 1$ (b) $m = -1$ (c) $m = 1$

FIGURE 6.4. Specific heat as a function of the BH Entropy for the RN-$f(R)$ in the (A)de Sitter space and some values of m and n. In all panels $\Lambda = 10$ and $q = 1$. The monotonicity, continuity and existence of the critical point in the T-S plane depends on the value of the parameters m and n, as well as the charge of the BH.

BH horizon radius, is

$$T(r_+) = \frac{1}{4\pi r_+} + \frac{c_2 r}{\pi} - \frac{4q^2}{5nr_+^4}\,{}_2F_1\left(\frac{1}{2},\frac{5}{4};\frac{9}{4};-\frac{q^2}{b^2 r_+^4}\right) \tag{6.115}$$

or

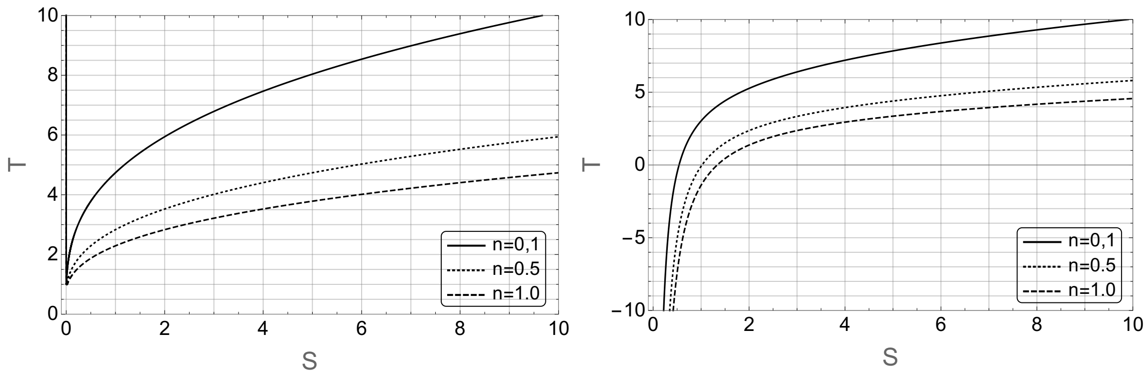
$$T(S,\Phi) = c_2\left(\frac{S}{\pi^4 n}\right)^{1/3} + \frac{1}{4}\left(\frac{n}{\pi^2 S}\right)^{1/3} - \frac{4\pi\Phi^2}{5nS}\,{}_2F_1\left(\frac{1}{2},\frac{5}{4};\frac{9}{4};-\frac{\pi\Phi^2}{b^2 S}\right), \tag{6.116}$$

where the entropy is defined as

$$S = n\pi r_+^3, \tag{6.117}$$

and the electrostatic potential

$$\Phi = q\left(\frac{n}{r_+}\right)^{1/2}, \tag{6.118}$$



(a) $q = 0$ (b) $q = 1$

FIGURE 6.5. Temperature vs Entropy in the Non-linear-BI-$f(R)$ theory for some values of n and with $b \to \infty$. In both panels $c_2 = 10$. Note that when the black hole is not loaded, the minimum temperature is $T = 1$.

Taking the limit at $b \to \infty$

$$T(r_+) = \frac{1}{4\pi r} + \frac{c_2 r}{\pi} - \frac{4q^2}{5nr^4}, \tag{6.119}$$

this temperature is depicted in Fig. 6.5 as a function of the entropy

$$T(S,\Phi) = c_2\left(\frac{S}{\pi^4 n}\right)^{1/3} + \frac{1}{4}\left(\frac{n}{\pi^2 S}\right)^{1/3} - \frac{4\pi\phi^2}{5nS}. \tag{6.120}$$

The specific heat with a fixed potential, in the non-linear-BI-$f(R)$ theory will be

$$C_\Phi = \frac{3\pi bnr^5\sqrt{b^2r^4+q^2}\left(5nr^3\left(4\mathrm{c}2r^2+1\right) - 16\pi q^2\, {}_2F_1\left(\frac{1}{2},\frac{5}{4};\frac{9}{4};-\frac{q^2}{b^2r^4}\right)\right)}{5br^4\left(nr\left(4\mathrm{c}2r^2-1\right)\sqrt{b^2r^4+q^2} + 16\pi bq^2\right) - 16\pi q^2\left(b^2r^4+q^2\right){}_2F_1\left(1,\frac{7}{4};\frac{9}{4};-\frac{q^2}{b^2r^4}\right)}, \tag{6.121}$$

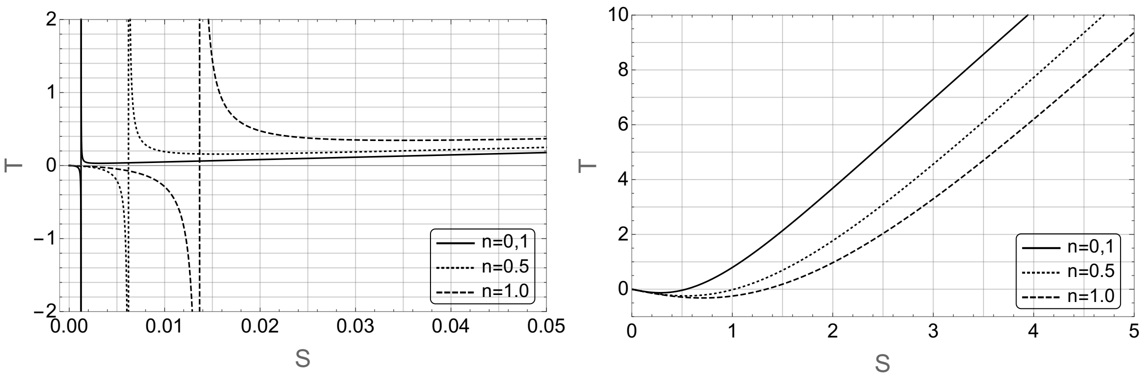
taking the limit $b \to \infty$

$$C_\Phi(r_+) = \frac{3\pi nr^3\left(5nr^3\left(4c_2r^2+1\right) - 16\pi q^2\right)}{5nr^3\left(4\mathrm{c}2r^2-1\right) + 64\pi q^2} \tag{6.122}$$

or in terms of the entropy and potential

$$C_\Phi(S) = \frac{3}{4}\left(\frac{5\left(20c_2 n^{2/3}S^{7/3} + 3\pi^{2/3}n^{4/3}S^{5/3}\right)}{20c_2 n^{2/3}S^{4/3} - 5\pi^{2/3}n^{4/3}S^{2/3} + 64\pi^{7/3}\phi^2} - S\right), \tag{6.123}$$

which can be seen in Fig.



(a) $q=0$

(b) $q=1$

FIGURE 6.6. Specific heat vs Entropy in the Non-linear-BI-$f(R)$ theory for some values of n and with $b \to \infty$. In both panels $c_2 = 10$. The BH charge avoids the singularities in $C_\Phi(S)$.

Chapter 7

Conclusions

In spite of that GR is the most widely accepted theory of gravity: it predicts the expansion of the Universe and the associated redshifts of the galaxies as a dynamic consequence of its evolution from the so-called Big Bang; as well as other remarkable phenomena such as the gravitational lensing effect [4, 163], black holes and gravitational waves [61], recently detected [1]; the conclusions of the observational data of Supernovae type Ia (SN Ia) [145, 157], showed that the Universe experiences an accelerated expansion phase, this fact has no clear interpretation in the framework of GR and it is necessary to introduce a type of unknown negative pressure force, called Dark Energy, whose action dominates gravitational attraction on large scales [46, 145, 156]. This model, known as Lambda Cold Dark Matter (ΛCDM), also takes into account a new and strange type of gravitating matter, but not interacting with radiation, called Dark Matter [132, 203], whose effect is to correct the discrepancy between the theory and the observed flat rotation curves of spiral galaxies [55, 139].

ΛCDM fits very well within a wide spectrum of cosmological observations, [3–6, 93], however, the nature of DM and DE is unknown, even though according to observations made by the ESA's Planck satellite in 2013 [2], within the theoretical scope of ΛCDM, their content is 27% and 68% respectively, i.e. what is known of the Universe comprises only 5% of its energy density, which raises even more questions than answers.

ΛCDM is indeed a paradigm, and in the course of the last century new ideas were added to complement it, such is the case of the theory of cosmological inflation, which accounts for the homogeneity and isotropy of the Universe at large scale from the accelerated expansion of the early Universe [5], solving, among others, the flatness problem [72] and the magnetic monopole problem [111]; however, to date there is no generally accepted model for inflation and likewise the standard model of cosmology has some problems (see [114] for a synthesis on this subject) that make it necessary to reconsider our understanding of GR on cosmological scales. One such alternative, motivated mainly by the search for a geometrical explanation for the late-time acceleration, is the $f(R)$ theory, which was the focus of this work, and whose dynamics is obtained from an action written in terms of a general function of the scalar curvature, R. The main motivation for this work was that there are several $f(R)$ models for solving the DM [22, 41, 200], DE [10, 11, 41] problems and even the inflationary phase, whose first model in the context of $f(R)$ theory, was proposed by Starobinsky in 1980 [178], which is constructed by adding to Einstein-Hilbert (EH) action a quadratic term for the curvature scale, i.e. $f(R) = R + \alpha R^2$, with α constant, this model has been carefully studied and is in agreement with the data recently observed by the Planck satellite [148].

7.1 Concluding remarks

Modified gravity in the context of an arbitrary function $f(R)$ is indeed a scalar tensor theory with a scalar degree of freedom, and particularly a Brans-Dicke theory with a null parameter ω. In Chapter 2 we have made the conformal transformation as general as possible to show the action with the Gibbons-York-Hawking boundary term and the related field equations in the Einstein frame, which are in agreement with the usual field equations in $f(R)$ theory under the inverse transformation. In addition we have defined the effective potential which depends on both geometry and matter-energy, but through trace of the field equations, it turns out to be an integral of a pure geometric term depending only on $f(R)$. With this potential it is possible to find the scalar potential. Those potentials were calculated, plotted and analyzed for different models recorded in the literature and able to pass the observational test, namely, Starobinsky, Hu-Sawicki, Tsujikawa, Exponential models, and for two new models proposed, which, although look like mathematical toy models, we show that pass the test mentioned above.

A cosmologically viable hypergeometric model in the $f(R)$ gravity theory has been constructed in Chapter 3 from the assumption of the existence of an inflection point of the $f(R)$ curve, the viability conditions in the (m,r) plane and such that it reproduces ΛCDM at some limit. This last quality was used to express the limits of the model, written in terms of the dimensionless variable x, as $y(x) = x + h(x) + \lambda$, where $h(x)$ represents the deviation of the model from GR. From the geometric point of view, the existence of the inflection point, x_i, besides the decreasing monotonicity of $h(x)$, ensure that limits (3.5) and (3.6) are satisfied, allowing to consider the model as a perturbation around ΛCDM, and at the same time when $x \to \infty$, the conditions $r = -1$, $m = +0$ and $m'(-1) = 0$ are met, see figures (3.2) and (3.3), enabling $y(x)$ to have a matter domination epoch. The physical interpretation of x_i is indeed to allow the model to have an asymptotic behaviour towards ΛCDM.

The geometrical conditions imposed by x_i, both on the function $h(x)$ and its derivatives, was used to construct a differential equation in such a way that the roots of $xh''(x)$, modulated by a function $p(x)$, coincide with a term $h'(x)$ multiplied by the factor $(x^2 - x_i^2)$. This differential equation was integrated and the function $p(x)$ was chosen to allow the integrand to be expressed as an exact differential. The solution, Eq. (3.45) corresponds to Starobinsky's 2007 model [181]. Through a similar procedure, but this time expressing the factor as $(x^r - x_i^r)$, being r a parameter of the model, a differential equation was constructed whose solution, Eq. (3.56), corresponded to a generalization of Hu-Sawicki model [86].

It was found that the differential equations of each model in effect belonged to a particular case of the hypergeometric differential equation, and as a result the hypergeometric model, Eq. (3.79), could be established. This model depends on five parameters m, n, r, u, and v, however, the equation for the inflection point, (3.72), represents a constriction of the model, since it is a necessary condition for its viability, reducing the number of parameters to four. Moreover the constant $k < 0$ and the value of c, in a concrete way, can be generally determined from the classification in the two sub-models, Hu-Sawicki type ($r > 0$): $c = 0$; and Starobinsky type ($r < 0$): $c = 2km/R_0$.

When $r > 0$ and $x > 0$, for the model to satisfy limits (3.5) to (3.10), as well as condition Eq. (3.11), the parameters must fulfill $m > 2/r$, $u > n > 2/r$, $v < 0$, and $m - n \notin \mathbb{Z}$, in addition to Eq. (3.75). For example, when $0 < r < 2$, $m \geq 1$, $n \geq 1$, $n < u < n+1$, $v < -1$, $u > kmnrv$, the hypergeometric model is cosmologically viable.

At the other hand, only for $r \leq 2$, $m > 0$ and $u > n > 0$, the hypergeometric model is viable. Specifically when $r = -2$, it is found that $u = -nv$ and $v < -1$.

The main quality of the hypergeometric model is that it encompasses a family of functions that have an inflection point and at the same time mimics the ΛCDM model, examples of which are the well-known Starobinsky and Hu-Sawicki models. The hypergeometric model proposed here depends on four free parameters, offering the possibility of having greater freedom of adjustment according to the restrictions offered by observational data at both cosmological and local scales. To carry out this objective, it should be noted, the appropriate computational tools are needed as indicated in Ref. [84], however, the outlook for achieving this goal is encouraging, since in the near future modified gravity theory could be tested by major advances in observational techniques in high curvature scenarios such as black holes or neutron stars, where $f(R)$ can play an important role in the dynamics of spacetime, and its effects could be appreciated.

Among all the models analyzed, we delve a little deeper in the Starobinsky model due to its physical relevance, and we show that basically, it consist of a dominant linear and quadratic terms plus 4th-order terms in R, and from this expression it was calculated the scalar field ψ, which in turn allowed us to plot the Starobinsky potential in the Einstein frame.

The metric for a spherically symmetric and static object was calculated in Chapter 4, and the extension to charged BH was made to obtain the Reissner-Nordström metric in an (A)dS space, and we have studied the solutions of the field equations in spherically symmetric and static spaces in the $f(R)$ theory of gravity with non-linear electromagnetic fields in the BI frame, this theory depends of the value of parameter b, which in turn exhibits two limits with classical physical meaning, $f(R)$ Schwarzschild-type ($b \to 0$) and $f(R)$-RN ($b \to \infty$) solutions. We found that the only models allowed in this framework must have the parametric form $\mathscr{F} = m + nr$, with m and n constants. From this condition it is possible to determine the form, domain and range of the supported $f(R)$ functions, in some cases expressed in a parametric way $f(r)$, while the classification of the solutions from the values of the constants in three possible instances: GR ($m \neq 0$ and $n = 0$), $f(R)$-BI ($m \neq 0$ and $n \neq 0$) and $f(R)$-non-linear-BI ($m = 0$ and $n \neq 0$), allowed to find a variety of sub-solutions and therefore models.

In the first case, when $n = 0$, GR is recovered with the rescaling $\kappa \to \kappa/m$ and $f(R) = m(R + 2\Lambda)$, where Λ can be interpreted as an integration constant and obtained by comparing the solution in the limit $b \to \infty$ with the RN-AdS metric.

When $m \neq 0$ and $n \neq 0$, the field equations cannot be integrated analytically, however the solution, R and $f(R)$ are expressed in terms of a hypergeometric integral of the radial coordinate r, similarly, the domain and range of the models can be determined by the signs of m and n. When $b = 0$, $f(R)$ is asymptotically ($r \to \infty$) equivalent to GR, and from the definition of the parameter h Eq. (4.82), which modulates to the logarithmic term of the solution, it is possible to distinguish the following cases

- $n, m > 0$: $f(R)$ is defined for $R < R_c$, where R_c is given by (4.81), however it has no continuous derivative and thus does not represent a viable physical model.
- $m < 0$ and $n > 0$: $f(R)$ is monotonically increasing with $R > -\frac{12n^2}{m^2} h \ln n$.
- $m > 0$ and $n < 0$: when $-3nR_S < m^2$, $f(R)$ has no continuous derivative and $R > R_c$. When $-3nR_S > m^2$, $f(R)$ is monotonically increasing for all $R \in \Re$. When $-3nR_S = m^2$, $f(R) = mR + 2n\sqrt{R} + m_0$, which is a model for the expansion of the Universe without DE.

Since the $f(R)$-RN case is a generalization of the previous one with electric charge, the above arguments remain valid, but now with the definition of the parameter $\bar{h}$ Eq. (4.97). However the existence of q implies that the function $f(R)$ has no longer continuous derivative for all $m < 0$ and $n > 0$, and thus defined only for $R \leq R_c$ or $R > -\frac{12n^2}{m^2} \bar{h} \ln n$, or vice versa in the opposite case, $m > 0$ and $n < 0$. When $n > 0$ and depending on $\bar{h}$, different solutions are found, which affects the form of the functions, although for $r \gg 1$ all these coincide according to Eq. (4.87). Some graphs were made for particular values of m and n and showing the shape and characteristics of the models.

When $m = 0$ we depart from the lineal term of GR, although this does not mean that the model has to be discarded, on the contrary, the curves of the models show that there are indeed similarities with some functions in the $f(R)$-RN solution and this can have physical implications beyond just serving as a mathematical toy. In this case the solution is written in terms of a new hypergeometric function Eq. (4.114), and cannot be obtained from solution Eq. (4.70), so they are linearly independent solutions, however $R(r)$ is expressed in terms of the same hypergeometric function of the solution Eq. (4.56). and since it is not possible to solve f in terms of R for all r, when $r \gg 1$, $f(R)$ can be given as an approximation, Eq. (4.121), which is linearly independent to Eq. (4.87) and thus representing different models.

The solutions shown here (with the exception of $n = 0$ or GR) imply that R is a non-trivial function of r outside the BH, which would imply the existence of hairy solutions. We have discussed in Chapter 5 the possible cases in which the no-hair theorems (NHT) can be evaded from the analysis of the potential $\mathscr{U}(\psi)$, defined in the conformal transformation between the Jordan and the Einstein Frames, in addition to the conditions imposed by Theorem 1 of Ref. [37]. It was found that the models depend strongly on the sign and value of the constants m and n, and the potentials were plotted in the different cases, as well as the form of the function $f(R)$. The way in which these solutions evade the NHT in $f(R)$ gravity are the subject of a paper in preparation.

The field equations were linearized for the Starobinsky quadratic model and the tensor perturbation was expressed as a sum of integrals of the Bessel functions, however, the perturbation field was found in the Newtonian approximation, which can be used to constrain the model in the Solar system range. This topic is also being prepared for publication.

Using the Wald method, the entropy of the Black Hole in $f(R)$ theory was found, and the symplectic potentials associated were calculated. We finally investigated the behavior of a BH in $f(R)$-BI plus cosmological constant, the BH temperature, entropy and specific heat was found in terms of the horizon radius and these quantities were plotted and it was found that they reproduce their analogous values within the framework of the theory of RN Black Holes in GR with cosmological constant. These same quantities were found and plotted for a non-linear model of $f(R)$.

A paper was extracted from Chapter 2 and accepted for publication at the IJAA. Similarly, Chapter 3 was the basis for a paper submitted to the PRD, and from Chapter 4 a paper was extracted and published in November of this year in the PRD.

7.2 Challenges of the BH physics

In general, there are no direct observational tests of BH thermodynamics, however its foundations seem to be mathematically firm enough to be considered the most promising candidates to explain the nature of the gravitationally-quantum behaviour of BH.

Black Holes are among the most fascinating gravitational system in the Universe, and many people around the world are constantly working in this area. Although there has been an important progress in solving key questions, there are still open modern questions as "BH information paradox." How is the evolution process of a pure quantum state to a mixed state, or what are the degrees of freedom responsible for BH entropy [192].

Without going too far, it is possible to assume that there are even a lot more basic questions than

can be redirected with a higher order gravitational theory as $f(R)$ modified gravity.
The amount of work and papers that have been done and published about the Thermodynamics of BH in higher order gravity is not insignificant, however, it is not clear yet how the Laws of Thermodynamics for BH change and how such a theory could relate GR to QM. Therefore, there is currently a lot of work to be done in the search of general solutions for BH in $f(R)$ theory and for those models or families of $f(R)$ that explain cosmological phenomena, like DE or DM. Moreover $f(R)$ is a theory that could have distinctive observational features and in turn could provide evidence decisive to determine the existence of a BH, for example describing the orbit and behaviour of its accretion disk or its event horizon.

We really hope that in the near future it will be possible to understand all of the major problems about BH physics, as well as that with the development of better telescopes and new theory capable to break boundaries of the mind, BH can serve as natural scenarios for understand the unification between General Relativity and Quantum Mechanics.

APPENDIX

A

RICCI SCALAR VARIATION

The Riemann tensor is defined by

$$R^{\mu}_{\alpha\nu\beta} = \Gamma^{\mu}_{\alpha\beta,\nu} + \Gamma^{\mu}_{\rho\nu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\alpha\nu}, \tag{A.1}$$

and its variation

$$\delta R^{\mu}_{\alpha\nu\beta} = \delta\Gamma^{\mu}_{\alpha\beta,\nu} + \Gamma^{\mu}_{\rho\nu}\delta\Gamma^{\rho}_{\alpha\beta} + \Gamma^{\rho}_{\alpha\beta}\delta\Gamma^{\mu}_{\rho\nu} - \delta\Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\rho\beta}\delta\Gamma^{\rho}_{\alpha\nu} - \Gamma^{\rho}_{\alpha\nu}\delta\Gamma^{\mu}_{\rho\beta}, \tag{A.2}$$

but by the definition of covariant derivative

$$\left(\delta\Gamma^{\mu}_{\alpha\beta}\right)_{;\nu} = \delta\Gamma^{\mu}_{\alpha\beta,\nu} + \Gamma^{\mu}_{\rho\nu}\delta\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\rho}_{\alpha\nu}\delta\Gamma^{\mu}_{\rho\beta} - \Gamma^{\rho}_{\beta\nu}\delta\Gamma^{\mu}_{\alpha\rho}, \tag{A.3}$$

and

$$-\left(\delta\Gamma^{\mu}_{\alpha\nu}\right)_{;\beta} = -\delta\Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\rho\beta}\delta\Gamma^{\rho}_{\alpha\nu} + \Gamma^{\rho}_{\alpha\beta}\delta\Gamma^{\mu}_{\rho\nu} + \Gamma^{\rho}_{\nu\beta}\delta\Gamma^{\mu}_{\alpha\rho}, \tag{A.4}$$

so

$$\delta R^{\mu}_{\alpha\nu\beta} = \left(\delta\Gamma^{\mu}_{\alpha\beta}\right)_{;\nu} - \left(\delta\Gamma^{\mu}_{\alpha\nu}\right)_{;\beta}, \tag{A.5}$$

and with the contraction of the first and third terms we arrive to the Palatini identity

$$\delta R_{\alpha\beta} = \left(\delta\Gamma^{\mu}_{\alpha\beta}\right)_{;\mu} - \left(\delta\Gamma^{\mu}_{\alpha\mu}\right)_{;\beta}. \tag{A.6}$$

Now, in order to find the variation of the Christoffel symbols we need to express them in terms of the metric tensor

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}\right), \tag{A.7}$$

whose variation is

$$\begin{aligned}\delta\Gamma^{\mu}_{\alpha\beta} =& \frac{1}{2}\left(g_{\alpha\lambda,\beta}+g_{\beta\lambda,\alpha}-g_{\alpha\beta,\lambda}\right)\delta g^{\mu\lambda}+\frac{1}{2}g^{\mu\lambda}\left(\delta g_{\alpha\lambda,\beta}+\delta g_{\beta\lambda,\alpha}-\delta g_{\alpha\beta,\lambda}\right)\\ =& g_{\mu\lambda}\Gamma^{\mu}_{\alpha\beta}\delta g^{\mu\lambda}+\frac{1}{2}g^{\mu\lambda}\left(\delta g_{\alpha\lambda,\beta}+\delta g_{\beta\lambda,\alpha}-\delta g_{\alpha\beta,\lambda}\right),\end{aligned} \tag{A.8}$$

but with the definition of the covariant derivative again

$$\delta g_{\alpha\lambda,\beta} = (\delta g_{\alpha\lambda})_{;\beta}+\Gamma^{\mu}_{\alpha\beta}\delta g_{\mu\lambda}+\Gamma^{\mu}_{\lambda\beta}\delta g_{\alpha\mu}, \tag{A.9}$$

$$\delta g_{\beta\lambda,\alpha} = \left(\delta g_{\beta\lambda}\right)_{;\alpha}+\Gamma^{\mu}_{\alpha\beta}\delta g_{\mu\lambda}+\Gamma^{\mu}_{\lambda\alpha}\delta g_{\beta\mu}, \tag{A.10}$$

and

$$-\delta g_{\alpha\beta,\lambda} = -\left(\delta g_{\alpha\beta}\right)_{;\lambda}-\Gamma^{\mu}_{\alpha\lambda}\delta g_{\mu\beta}-\Gamma^{\mu}_{\beta\lambda}\delta g_{\alpha\mu}, \tag{A.11}$$

thus

$$\begin{aligned}\delta g_{\alpha\lambda,\beta}+\delta g_{\beta\lambda,\alpha}-\delta g_{\alpha\beta,\lambda} =& (\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}+2\Gamma^{\mu}_{\alpha\beta}\delta g_{\mu\lambda}\\ =& (\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}-2\Gamma^{\mu}_{\alpha\beta}g_{\mu\nu}g_{\lambda\rho}\delta g^{\nu\rho},\end{aligned} \tag{A.12}$$

where we have used the relationship,

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}, \tag{A.13}$$

which follows in a direct calculation from $g^{\mu\alpha}g_{\alpha\beta}=\delta^{\mu}_{\beta}$, where δ^{μ}_{ν} is the Kronecker delta. With this result we can write the variation Eq. (A.8) as

$$\begin{aligned}\delta\Gamma^{\mu}_{\alpha\beta} =& g_{\mu\lambda}\Gamma^{\mu}_{\alpha\beta}\delta g^{\mu\lambda}+\frac{1}{2}g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}\right]-g^{\mu\lambda}g_{\mu\nu}g_{\lambda\rho}\Gamma^{\mu}_{\alpha\beta}\delta g^{\nu\rho}\\ =& g_{\mu\lambda}\Gamma^{\mu}_{\alpha\beta}\delta g^{\mu\lambda}-\delta^{\lambda}_{\nu}g_{\lambda\rho}\Gamma^{\mu}_{\alpha\beta}\delta g^{\nu\rho}+\frac{1}{2}g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}\right]\\ =& \Gamma^{\mu}_{\alpha\beta}\left(g_{\mu\lambda}\delta g^{\mu\lambda}-g_{\lambda\rho}\delta g^{\lambda\rho}\right)+\frac{1}{2}g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}\right]\\ =& \frac{1}{2}g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\beta}+\left(\delta g_{\beta\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\beta}\right)_{;\lambda}\right],\end{aligned} \tag{A.14}$$

so

$$\begin{aligned}-\left(\delta\Gamma^{\mu}_{\alpha\mu}\right)_{;\beta} =& \frac{1}{2}g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\mu}+\left(\delta g_{\mu\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\mu}\right)_{;\lambda}\right]_{;\beta}\\ =& \frac{1}{2}\left[g^{\mu\lambda}(\delta g_{\alpha\lambda})_{;\mu}+g^{\mu\lambda}\left(\delta g_{\mu\lambda}\right)_{;\alpha}-g^{\mu\lambda}\left(\delta g_{\alpha\mu}\right)_{;\lambda}\right]_{;\beta}\\ =& \frac{1}{2}\left[(\delta g_{\alpha\lambda})^{;\lambda}+g^{\mu\lambda}\left(\delta g_{\mu\lambda}\right)_{;\alpha}-\left(\delta g_{\alpha\mu}\right)^{;\mu}\right]_{;\beta}\\ =& \frac{1}{2}g^{\mu\lambda}\left(\delta g_{\mu\lambda}\right)_{;\alpha\beta},\end{aligned} \tag{A.15}$$

and replacing into the Palatini identity, Eq. (A.6),

$$\delta R_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}\left[(\delta g_{\alpha\lambda})_{;\beta} + \left(\delta g_{\beta\lambda}\right)_{;\alpha} - \left(\delta g_{\alpha\beta}\right)_{;\lambda}\right]_{;\mu} - \frac{1}{2} g^{\mu\lambda}\left(\delta g_{\mu\lambda}\right)_{;\alpha\beta} \tag{A.16}$$

Finally we can construct the variation of the Riccie scalar

$$\begin{aligned}
\delta R =& \delta(g^{\alpha\beta}R_{\alpha\beta}) \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta} \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{2}\left(g^{\mu\lambda}g^{\alpha\beta}(\delta g_{\alpha\lambda})_{;\beta\mu} + g^{\mu\lambda}g^{\alpha\beta}\left(\delta g_{\beta\lambda}\right)_{;\alpha\mu} - g^{\mu\lambda}g^{\alpha\beta}\left(\delta g_{\alpha\beta}\right)_{;\lambda\mu} - g^{\alpha\beta}g^{\mu\lambda}\left(\delta g_{\mu\lambda}\right)_{;\alpha\beta}\right) \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{2}\left(-\left(\delta g^{\beta\mu}\right)_{;\beta\mu} - \left(\delta g^{\alpha\mu}\right)_{;\alpha\mu} + \left(g^{\mu\lambda}g^{\alpha\beta}g_{\alpha\sigma}g_{\beta\rho}\delta g^{\sigma\rho}\right)_{;\lambda\mu} + \left(g^{\alpha\beta}g^{\mu\lambda}g_{\mu\sigma}g_{\lambda\rho}\delta g^{\sigma\rho}\right)_{;\alpha\beta}\right) \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{2}\left(-2\left(\delta g^{\alpha\mu}\right)_{;\alpha\mu} + \left(g^{\mu\lambda}\delta^{\alpha}_{\rho}g_{\alpha\sigma}\delta g^{\sigma\rho}\right)_{;\lambda\mu} + \left(g^{\alpha\beta}\delta^{\mu}_{\rho}g_{\mu\sigma}\delta g^{\sigma\rho}\right)_{;\alpha\beta}\right) \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} - \left(\delta g^{\alpha\mu}\right)_{;\alpha\mu} + \left(g^{\alpha\beta}g_{\sigma\mu}\delta g^{\sigma\mu}\right)_{;\alpha\beta} \\
=& R_{\alpha\beta}\delta g^{\alpha\beta} - \left(\delta g^{\alpha\beta}\right)_{;\alpha\beta} + g_{\alpha\beta}\left(\delta g^{\alpha\beta}\right)_{;\mu}^{\ \ ;\mu}
\end{aligned} \tag{A.17}$$

A.1 Covariant derivative and Riemann tensor

The curvature of the spacetime can be interpreted in terms of the non-commutativity of the covariant derivatives of a vector A^{α}. First covariant derivative is defined as

$$A^{\alpha}_{;\mu} = A^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}A^{\sigma}, \tag{A.18}$$

and the second covariante derivative

$$\begin{aligned}
A^{\alpha}_{;\mu\nu} &= \left(A^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}A^{\sigma}\right)_{;\nu} \\
&= \left(A^{\alpha}_{,\mu}\right)_{;\nu} + \left(\Gamma^{\alpha}_{\sigma\mu}A^{\sigma}\right)_{;\nu} \\
&= A^{\alpha}_{,\mu\nu} + \Gamma^{\alpha}_{\sigma\nu}A^{\sigma}_{,\mu} - \Gamma^{\sigma}_{\mu\nu}A^{\alpha}_{,\sigma} + \Gamma^{\alpha}_{\sigma\mu,\nu}A^{\sigma} + \Gamma^{\alpha}_{\sigma\mu}A^{\sigma}_{,\nu} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\sigma\mu}A^{\sigma} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}A^{\sigma},
\end{aligned} \tag{A.19}$$

when exchanging the indexes and subtracting stands out the fact of finding the Riemann tensor (A.1)

$$\begin{aligned}
A^{\alpha}_{;\mu\nu} - A^{\alpha}_{;\nu\mu} &= \left(\Gamma^{\alpha}_{\sigma\mu,\nu} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\sigma\mu} - \Gamma^{\alpha}_{\sigma\nu,\mu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\sigma\nu}\right)A^{\sigma} \\
&= R^{\alpha}_{\sigma\nu\mu}A^{\sigma}.
\end{aligned} \tag{A.20}$$

APPENDIX B

HAMILTONIAN VECTOR FIELD

A very important formulation of the Hamiltonian mechanics is through the theoretical framework of the symplectic spaces. A symplectic manifold (M,ω) is a manifold M together a closed and non-degenerate symplectic 2-form ω, and the Hamilton equations of motion can be obtained from a real valued and infinitely differentiable function H defined on M, that is $H : M \to \Re$, whose exact differential results from contract the symplectic form with a vector field X_H associated to H, that is [39]

$$\imath_{X_H}\omega = dH, \tag{B.1}$$

where the left side of the equation stands for interior product. In the phase space R^{2n} with canonical coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$, the Darboux theroem stablish [21] that the 2-form is given by

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i, \tag{B.2}$$

where the symbol $\wedge$ means exterior product. Since ω is non-degenerate, there must be an only vector field H that we suppose as

$$X_H = \sum_{i=1}^{n}\left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i}\right). \tag{B.3}$$

Remembering that interior product satisfies a Leibniz rule, that is, if α is a p-form and β is a q-form, then

$$\imath_X\left(\alpha\wedge\beta\right) = (\imath_X\alpha)\wedge\beta + (-1)^p\,\alpha\wedge\left(\imath_X\beta\right), \tag{B.4}$$

thus

$$\begin{aligned} \imath_{X_H}\omega &= \sum_{i=1}^{n}\left(\imath_{X_H}dq_i\right)\wedge dp_i - \sum_{i=1}^{n} dq_i\wedge(\imath_X dp_i) \\ &= \sum_{i=1}^{n}(a_i dp_i - dq_i b_i), \end{aligned} \tag{B.5}$$

but the differential of the 0-form H is the 1-form

$$dH = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right), \tag{B.6}$$

thus, equating similar terms in the equation (B.1)

$$a_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad b_i = -\frac{\partial H}{\partial q_i}, \tag{B.7}$$

and the associated vector field to H will be

$$X_H = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right). \tag{B.8}$$

Remembering that the Poisson bracket of two functions defined in the phase space, that is, $f = f(q_i, p_i, t)$ and $g = g(q_i, p_i, t)$, is

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \tag{B.9}$$

thus, X_H applied on a function $A = A(q_i, p_i, t)$ on the symplectic manifold, gives its time evolution

$$X_H A = \{A, H\} = \dot{A}, \tag{B.10}$$

but from

$$\dot{A} = \sum_{i=1}^{n}\left(\frac{\partial A}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial A}{\partial p_i}\frac{dp_i}{dt}\right), \tag{B.11}$$

Hamilton equations of motion are correspondingly

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q_i}. \tag{B.12}$$

Since Eq. (B.1) is equivalent to (B.12), X_H is called the Hamiltonian vector field or symplectic Hamiltonian gradient with Hamiltonian function H.

APPENDIX C

Killing vectors and Surface Gravity of the BH event horizon

Killing vectors have special importance in almost all branches in Physics, since they describe the symmetries of the system under consideration in an invariant way, that is, without taking into account the reference frame. In particular a Killing vector field is defined as the vector ξ generating diffeomorphisms[1] which are isometries[2] as well. This condition leads to the Lie derivative of the metric along the Killing vector field must be null

$$\mathfrak{L}_{\xi} g_{\mu\nu} = 0, \tag{C.1}$$

and since $g_{\mu\nu;\sigma} = 0$, the killing equation can be written as

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{C.2}$$

which implies that the killing vector covariant derivative is an antisymmetric tensor, that is

$$\begin{aligned} \xi_{[\mu;\nu]} =& \frac{1}{2}\left(\xi_{\mu;\nu} - \xi_{\nu;\mu}\right) \\ =& \xi_{\mu;\nu}. \end{aligned} \tag{C.3}$$

It is useful to define

$$\xi^2 = \xi_{\mu}\xi^{\mu}, \tag{C.4}$$

[1]A diffeomorphism is an invertible map ϕ between two manifolds, such that ϕ and ϕ^{-1} are C^{∞}.

[2]An isometry is a diffeomorphism ϕ that leaves invariant the metric tensor through the pullback, i.e. $\phi^* g_{\alpha\beta} = g_{\alpha\beta}$

such that

$$\begin{aligned}
\xi^2_{,\alpha} =& \xi^2_{;\alpha}\\
=& \xi_{\mu;\alpha}\xi^\mu + \xi_\mu \xi^\mu_{;\alpha}\\
=& \xi_{\mu;\alpha}\xi^\mu + g^{\mu\nu}\xi_\mu\xi_{\nu;\alpha}\\
=& 2\xi_{\mu;\alpha}\xi^\mu,
\end{aligned} \tag{C.5}$$

and

$$\begin{aligned}
\xi_{[\alpha}\xi_{\beta;\mu]} =& \frac{1}{6}\left(\xi_\alpha\xi_{\beta;\mu} + \xi_\mu\xi_{\alpha;\beta} + \xi_\beta\xi_{\mu;\alpha} - \xi_\alpha\xi_{\mu;\beta} - \xi_\mu\xi_{\beta;\alpha} - \xi_\beta\xi_{\alpha;\mu}\right)\\
=& \frac{1}{3}\left(\xi_\alpha\xi_{[\beta;\mu]} + \xi_\mu\xi_{[\alpha;\beta]} + \xi_\beta\xi_{[\mu;\alpha]}\right)
\end{aligned} \tag{C.6}$$

and we can construct the quantity

$$\begin{aligned}
\xi_{[\alpha}\xi^2{}_{;\beta]} =& \frac{1}{2}\left(\xi_\alpha\xi^2{}_{;\beta} - \xi_\beta\xi^2{}_{;\alpha}\right)\\
=& \left(\xi_\alpha\xi_{\mu;\beta}\xi^\mu - \xi_\beta\xi_{\mu;\alpha}\xi^\mu\right)\\
=& -\xi^\mu\left(\xi_\alpha\xi_{[\beta;\mu]} + \xi_\mu\xi_{[\alpha;\beta]} + \xi_\beta\xi_{[\mu;\alpha]}\right) + \xi^2\xi_{\alpha;\beta}\\
=& -3\xi^\mu\xi_{[\alpha}\xi_{\beta;\mu]} + \xi^2\xi_{\alpha;\beta}.
\end{aligned} \tag{C.7}$$

The event horizon (H) of a BH is a null surface with vector field ξ^μ normal to it, that is, ξ^μ is tangent to the null generators of the Killing horizon, and

$$\xi^2 \overset{H}{=} 0, \tag{C.8}$$

where the H over the equal represents an evaluation on the event horizon, thus the identity (C.7) will be

$$\xi_{[\alpha}\xi^2{}_{;\beta]} \overset{H}{=} -3\xi^\mu\xi_{[\alpha}\xi_{\beta;\mu]}, \tag{C.9}$$

however from the condition (C.8), we can construct the Eq. (C.6), by multiplying it for a scalar $\xi^\alpha\xi^\beta\xi_{\alpha;\beta}$, that is

$$\xi^2\xi^\alpha\xi^\beta\xi_{\alpha;\beta} \overset{H}{=} 0, \tag{C.10}$$

doing a permutation of the indices and adding

$$\begin{aligned}
\xi^2\xi^\beta\xi^\mu\xi_{\beta;\mu} + \xi^2\xi^\alpha\xi^\beta\xi_{\alpha;\beta} + \xi^2\xi^\mu\xi^\alpha\xi_{\mu;\alpha} \overset{H}{=}& 0\\
\xi^\alpha\xi^\beta\xi^\mu\left(\xi_\alpha\xi_{\beta;\mu} + \xi_\mu\xi_{\alpha;\beta} + \xi_\beta\xi_{\mu;\alpha}\right) \overset{H}{=}& 0,
\end{aligned} \tag{C.11}$$

from which it is obtained

$$\begin{aligned}
\xi_\alpha\xi_{[\beta;\mu]} + \xi_\mu\xi_{[\alpha;\beta]} + \xi_\beta\xi_{[\mu;\alpha]} \overset{H}{=}& 0\\
\xi_{[\alpha}\xi_{\beta;\mu]} \overset{H}{=}& 0,
\end{aligned} \tag{C.12}$$

and therefore

$$
\begin{aligned}
\xi_{[\alpha}\xi^{2}{}_{;\beta]} &\overset{H}{=} 0\\
\xi_{\alpha}\xi^{2}{}_{;\beta} - \xi_{\beta}\xi^{2}{}_{;\alpha} &\overset{H}{=} 0\\
\xi_{\alpha}\xi_{\mu;\beta}\xi^{\mu} - \xi_{\beta}\xi_{\mu;\alpha}\xi^{\mu} &\overset{H}{=} 0, \qquad \text{(C.13)}
\end{aligned}
$$

where we have used the identity (C.5). This relation shows that the killing vectors evaluated over the event horizon are proportional to its covariant derivatives, since using the killing equation

$$
\xi_{\alpha}\xi_{\beta;\mu}\xi^{\mu} \overset{H}{=} \xi_{\beta}\xi_{\alpha;\mu}\xi^{\mu}, \qquad \text{(C.14)}
$$

it is possible to make a "separation" because ξ_α and ξ_β could be thinking as independent variables, but with the same behaviour at the event horizon, and so they must vary independently, implying that

$$
\begin{aligned}
\xi_{\mu;\nu}\xi^{\nu} &\overset{H}{\propto} \xi_{\mu}\\
\xi_{\mu;\nu}\xi^{\nu} &\overset{H}{=} \kappa_s\xi_{\mu} \qquad \text{(C.15)}
\end{aligned}
$$

where κ is the proportionality constant[3]. In spite that the norm of the killing vector is zero at the event horizon, we are interested in knowing the behaviour of its covariant derivative, so using the killing equation and completing the square we could write the above equation as

$$
\boxed{\xi^{2}{}_{;\mu} \overset{H}{=} -2\kappa_s\xi_{\mu}}. \qquad \text{(C.16)}
$$

Using this equation the dependence of the constant κ in terms of the killing vectors could be seen, multiplying both sides of Eq. (C.12) by $\xi^{\beta;\mu}$ and using the killing equation

$$
\begin{aligned}
\xi_{\alpha}\xi_{\beta;\mu}\xi^{\beta;\mu} &\overset{H}{=} -\xi_{\mu}\xi_{\alpha;\beta}\xi^{\beta;\mu} - \xi_{\beta}\xi_{\mu;\alpha}\xi^{\beta;\mu}\\
&\overset{H}{=} -\kappa_s\xi_{\alpha;\beta}\xi^{\beta} + \kappa_s\xi_{\mu;\alpha}\xi^{\mu}\\
&\overset{H}{=} -2\kappa_s^2\xi_{\alpha}, \qquad \text{(C.17)}
\end{aligned}
$$

from where

$$
\boxed{\kappa_s^2 \overset{H}{=} -\frac{1}{2}\xi_{\mu;\nu}\xi^{\mu;\nu}}, \qquad \text{(C.18)}
$$

this expression in any particular case could be found from the metric elements

$$
ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}. \qquad \text{(C.19)}
$$

In next section we calculate explicitly the surface gravity at the event horizon of a Black Hole in the $f(R)$ gravity in the static and spherically symmetric case.

[3] This constant should not be confused with constant $\bar{\kappa}$ that appears in the Field Equations.

C.1 Surface Gravity

Let us consider a static and spherically symmetric vacuum solution of the Field Equations in $f(R)$ gravity without electromagnetic fields, described by the metric (4.26)

$$ds^2 = -a(r)dt^2 + a^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{C.20}$$

written in terms of the function a of the coordinate r

$$a(r) = 1 - \frac{1}{6}\frac{f(R_0)}{f'(R_0)}r^2 - \frac{R_S}{r}. \tag{C.21}$$

and consider the covariant derivatives of its timelike killing vectors[4], $\xi^\alpha = \delta^\alpha_t$ and $\xi_\alpha = g_{\alpha t}$, that is

$$\begin{aligned} \xi_{\alpha;\beta} &= \xi_{\alpha,\beta} - \Gamma^\sigma_{\alpha\beta}\xi_\sigma \\ &= g_{\alpha t,\beta} - \Gamma^t_{\alpha\beta} g_{tt}, \end{aligned} \tag{C.22}$$

with

$$\begin{aligned} \Gamma^t_{\alpha\beta} &= \frac{1}{2}g^{t\lambda}\left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right) \\ &= \frac{1}{2}g^{tt}\left(g_{t\alpha,\beta} + g_{t\beta,\alpha}\right) \end{aligned} \tag{C.23}$$

where it is supposed that the metric components are independet of time, $g_{\alpha\beta,t} = 0$, so

$$\begin{aligned} \xi^{\alpha;\beta} &= \xi^{\alpha,\beta} + \Gamma^\alpha_{\sigma\rho}\xi^\sigma g^{\rho\beta} \\ &= \Gamma^\alpha_{t\beta} g^{\beta\beta}, \end{aligned} \tag{C.24}$$

and since that metric components are only function of r coordinate,

$$\begin{aligned} \Gamma^\alpha_{t\beta} &= \frac{1}{2}g^{\alpha\lambda}\left(g_{\lambda t,\beta} - g_{t\beta,\lambda}\right) \\ &= \frac{1}{2}\left(g^{\alpha t}g_{tt,\beta} - g^{\alpha r}g_{t\beta,r}\right). \end{aligned} \tag{C.25}$$

Now we are enabled to write the next covariant derivative product

$$\begin{aligned} \xi_{\alpha;\beta}\xi^{\alpha;\beta} &= \left(g_{\alpha t,\beta} - \Gamma^t_{\alpha\beta}g_{tt}\right)\Gamma^\alpha_{t\beta}g^{\beta\beta} \\ &= \frac{1}{4}\left(g_{t\alpha,\beta} - g_{t\beta,\alpha}\right)\left(g^{\alpha t}g_{tt,\beta} - g^{\alpha r}g_{t\beta,r}\right)g^{\beta\beta} \\ &= \frac{1}{4}\left(g_{t\alpha,r}g^{r\beta} - g_{tt,\alpha}g^{t\beta}\right)\left(g^{\alpha t}g_{tt,\beta} - g^{\alpha r}g_{t\beta,r}\right) \\ &= \frac{1}{2}g^{rr}g^{tt}\left(g_{tt,r}\right)^2, \end{aligned} \tag{C.26}$$

[4] Any spherically symmetric vacuum space possesses a timelike killing vector [47].

where components of metric are $g_{tt} = -a(r)$, $g^{tt} = -a^{-1}(r)$ and $g^{rr} = a(r)$, thus replacing in (C.18)

$$\kappa_s \overset{H}{=} \frac{1}{2} a'(r), \tag{C.27}$$

where the comma stands for the total derivative with respect the coordinate. Thus

$$\kappa_s \overset{H}{=} \frac{1}{2}\frac{R_S}{r^2} - \frac{1}{6}\frac{f(R_0)}{f'(R_0)} r, \tag{C.28}$$

However the event horizon is defined when

$$1 - \frac{1}{6}\frac{f(R_0)}{f'(R_0)} r^2 - \frac{R_S}{r} \overset{H}{=} 0, \tag{C.29}$$

or

$$\frac{1}{2}\frac{R_S}{r^2} \overset{H}{=} \frac{1}{2}\frac{1}{r} - \frac{1}{12}\frac{f(R_0)}{f'(R_0)} r, \tag{C.30}$$

with which

$$\kappa_s \overset{H}{=} \frac{1}{2r} + \frac{1}{4}\frac{f(R_0)}{f'(R_0)} r. \tag{C.31}$$

C.2 Killing vector field and Riemann tensor

In section A.1 of appendix A is presented a relation between Riemann tensor and covariant derivatives of any vector, for example the killing vector ξ_μ, i.e.

$$\xi_{\mu;\alpha\beta} - \xi_{\mu;\beta\alpha} = \xi_\sigma R^\sigma_{\mu\alpha\beta}, \tag{C.32}$$

which arises from the non-commutativity of the covariant derivative. Whit this relation in mind and the fact that the sum over the permutation of the lower index of the Riemann tensor vanishes [47, 141]

$$R^\sigma_{[\mu\alpha\beta]} = 0, \tag{C.33}$$

or equivalently

$$\begin{aligned}
\xi_\sigma \left(R^\sigma_{\mu\alpha\beta} + R^\sigma_{\beta\mu\alpha} + R^\sigma_{\alpha\beta\mu} \right) &= 0 \\
\xi_\sigma R^\sigma_{\mu\alpha\beta} + \xi_{\beta;\mu\alpha} - \xi_{\beta;\alpha\mu} + \xi_{\alpha;\beta\mu} - \xi_{\alpha;\mu\beta} &= 0 \\
\xi_\sigma R^\sigma_{\mu\alpha\beta} + \xi_{\mu;\alpha\beta} - \xi_{\mu;\beta\alpha} - 2\xi_{\beta;\alpha\mu} &= 0 \\
2\xi_\sigma R^\sigma_{\mu\alpha\beta} - 2\xi_{\beta;\alpha\mu} &= 0,
\end{aligned} \tag{C.34}$$

where it has been used the killing equation in the third line. Thus the second covariant derivative of any killing vector can be understood in terms of the Riemann tensor and must satisfy the equation

$$\xi_\sigma R^\sigma_{\mu\alpha\beta} = \xi_{\beta;\alpha\mu}, \tag{C.35}$$

which allow us to write the mixed covariant derivatives (C.32) as

$$\xi_{\mu;\alpha\beta} - \xi_{\mu;\beta\alpha} = \xi_{\beta;\alpha\mu}. \tag{C.36}$$

Appendix D

Noether Charge and Entropy in General Relativity

General Relativity can be obtained from the general Lagrangian density

$$\mathscr{L}(g_{\alpha\beta}, R, T_{\alpha\beta}) = \frac{1}{2\kappa}\mathscr{L}_H + \mathscr{L}_{\text{matter}}, \tag{D.1}$$

where $\mathscr{L}_{\text{matter}} = \mathscr{L}_m(T_{\alpha\beta})$, the constant $\kappa = 8\pi$, and the Einstein Hilbert Lagrangian is defined as

$$\mathscr{L}_H = \mathscr{L}_H(g_{\alpha\beta}, R) = \sqrt{-g}R, \tag{D.2}$$

The first variation of this Lagrangian is

$$\delta\mathscr{L}_H = \frac{\partial\mathscr{L}_H}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + \frac{\partial\mathscr{L}_H}{\partial R}R_{\alpha\beta}\delta g^{\alpha\beta} + \frac{\partial\mathscr{L}_H}{\partial R}\left[g_{\alpha\beta}g^{\mu\nu}\left(\delta g^{\alpha\beta}\right)_{;\mu\nu} - \left(\delta g^{\alpha\beta}\right)_{;\alpha\beta}\right] \tag{D.3}$$

with Ricci scalar given by Eq. (A.17). So, applying the method proposed by Wald [191]

$$\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\alpha\beta} = \left[\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} - \left[\left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\beta}\delta g^{\alpha\beta}\right]_{;\alpha} + \left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\beta\alpha}\delta g^{\alpha\beta}, \tag{D.4}$$

thus

$$\begin{aligned}\delta\mathscr{L}_H = &\left[\frac{\partial\mathscr{L}_H}{\partial g^{\alpha\beta}} + \frac{\partial\mathscr{L}_H}{\partial R}R_{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\nu\mu} - \left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\beta\alpha}\right]\delta g^{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left[\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} - \\ &g_{\alpha\beta}g^{\mu\nu}\left[\left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\nu}\delta g^{\alpha\beta}\right]_{;\mu} - \left[\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta} + \left[\left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\beta}\delta g^{\alpha\beta}\right]_{;\alpha}\end{aligned} \tag{D.5}$$

this expression is simplified since for the Einstein Hilbert Lagrangian (6.60)

$$\left(\frac{\partial\mathscr{L}_H}{\partial R}\right)_{;\mu} = 0, \tag{D.6}$$

therefore

$$\delta\mathscr{L}_H = \left(\frac{\partial\mathscr{L}_H}{\partial g^{\alpha\beta}} + \frac{\partial\mathscr{L}_H}{\partial R}R_{\alpha\beta}\right)\delta g^{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\left[\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\mu}\right]_{;\nu} - \left[\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\alpha}\right]_{;\beta}, \tag{D.7}$$

whereas the variation of the total Lagrangian (D.1),

$$\delta\mathscr{L} = \frac{1}{2\kappa}\delta\mathscr{L}_H + \delta\mathscr{L}_m, \tag{D.8}$$

has the same form as Eq. (6.9), with

$$A_{\mu\nu} = \frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_H}{\partial g^{\mu\nu}} + \frac{\partial\mathscr{L}_H}{\partial R}R_{\mu\nu}\right) + \frac{\partial\mathscr{L}_m}{\partial g^{\mu\nu}}, \tag{D.9}$$

and the symplectic potential

$$\tilde{\theta}^{\nu}_{;\nu} = \frac{1}{2\kappa}\left[g_{\alpha\beta}g^{\mu\nu}\frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\alpha\beta}\right)_{;\mu} - \frac{\partial\mathscr{L}_H}{\partial R}\left(\delta g^{\mu\nu}\right)_{;\mu}\right]_{;\nu}, \tag{D.10}$$

hence

$$\tilde{\theta}^{\nu} = \frac{1}{2\kappa}\frac{\partial\mathscr{L}_H}{\partial R}\left(g^{\mu\nu}\delta g - \delta g^{\mu\nu}\right)_{;\mu}, \tag{D.11}$$

where $\delta g = g_{\mu\nu}\delta g^{\mu\nu}$, and by

$$\frac{\partial\mathscr{L}_H}{\partial R} = \sqrt{-g} \qquad \text{and} \qquad \frac{\partial\mathscr{L}_H}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}Rg_{\mu\nu}, \tag{D.12}$$

equations of motion are equivalent to $A_{\mu\nu}$,

$$\begin{aligned} \frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_H}{\partial g^{\mu\nu}} + \frac{\partial\mathscr{L}_H}{\partial R}R_{\mu\nu}\right) + \frac{\partial\mathscr{L}_m}{\partial g^{\mu\nu}} &= 0 \\ \frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - T_{\mu\nu} &= 0 \end{aligned} \tag{D.13}$$

thus we have arrived to the Einstein Field equations for the General Relativity Theory

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}}. \tag{D.14}$$

And the symplectic potential associated to the general relativity Lagrangian density is

$$\boxed{\theta^{\mu} = \frac{1}{2\kappa}\left(g^{\mu\nu}\delta g - \delta g^{\mu\nu}\right)_{;\nu}}. \tag{D.15}$$

And noting that

$$\mathfrak{L}_\xi g_{\mu\nu} = \delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}, \tag{D.16}$$

potential will be

$$\begin{aligned} \theta^{\mu} &= \frac{1}{2\kappa}\left[g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta} - \delta g^{\mu\nu}\right]_{;\nu} \\ &= \frac{1}{2\kappa}\left(-2\xi^{\alpha}_{;\alpha}{}^{;\mu} + \xi^{\mu;\nu}{}_{;\nu} + \xi^{\nu;\mu}{}_{;\nu}\right), \end{aligned} \tag{D.17}$$

but since covariant derivatives do not commute, Eq. (C.32),

$$\xi^{\alpha\ ;\mu}_{;\alpha} = -\xi_\alpha R^{\alpha\mu} + \xi^{\alpha;\mu}{}_{;\alpha} \tag{D.18}$$

hence

$$\theta^\mu = \frac{1}{2\kappa}\left(2\xi_\alpha R^{\alpha\mu} + \xi^{\mu;\nu}{}_{;\nu} - \xi^{\nu;\mu}{}_{;\nu}\right), \tag{D.19}$$

Now, Noether current is

$$\begin{aligned} j^\mu =& \theta^\mu - \xi^\mu L \\ =& \frac{1}{\kappa}\xi^{[\mu;\nu]}{}_{;\nu} + \xi_\nu T^{\mu\nu} - \xi^\mu L_m, \end{aligned} \tag{D.20}$$

where we have used in the second line the Einstein field equation, moreover we know from the stress-energy tensor definition that

$$T_{\mu\nu} = -2\frac{\delta L_m}{\delta g^{\mu\nu}} + L_m g_{\mu\nu}, \tag{D.21}$$

so if we assume that the matter fields do not depend on geometry, Noether current can be written as

$$\begin{aligned} j^\mu =& \frac{1}{\kappa}\xi^{[\mu;\nu]}{}_{;\nu} + \xi_\nu L_m g^{\mu\nu} - \xi^\mu L_m \\ =& \frac{1}{\kappa}\xi^{[\mu;\nu]}{}_{;\nu}. \end{aligned} \tag{D.22}$$

Since the Noether charge is defined in terms of the current, Eq. (6.29)

$$j^\mu = Q^{\mu\nu}{}_{;\nu}, \tag{D.23}$$

therefore

$$Q^{\mu\nu} = \frac{1}{\kappa}\xi^{[\mu;\nu]}, \tag{D.24}$$

so

$$Q^{\mu\nu} = \frac{1}{\kappa}\xi^{\mu;\nu} \tag{D.25}$$

and the Black Hole entropy

$$\begin{aligned} S_H =& \frac{2\pi G}{\kappa_s}\int_{\partial\mathfrak{B}} d\sigma_{\mu\nu} Q^{\mu\nu} \\ =& \frac{1}{4}\int_\sigma d^2\sigma, \end{aligned} \tag{D.26}$$

where we have used the Eq. (6.90). Finally it is found the relation for the Black Hole entropy and its area

$$\boxed{S_H = \frac{1}{4}A}. \tag{D.27}$$

Appendix E

Ostrogradsky instability

It would seem that nature relates the variables of physical systems by means of first or second order differential equations, for example Newton's second law, Maxwell's equations, Schrödinger equation, Einstein's field equations, etc. Perhaps this is because the dynamics of material particles are described simply in terms of their positions, velocities and accelerations, which are no more than low-order derivatives, it is precisely the knowledge of these positions at all time that constitutes physical determinism from the point of view of causality in spacetime. Although there is no law of nature that forbids higher-order differential equations to describe any physical system, manifest energy instabilities in the Hamiltonian of a system whose Lagrangian depends on derivatives of order higher than one, make it evident that such lagrangians are lacking in the literature. This statement is known as the Ostrogradsky theorem, which will be described in this appendix in the framework of Lagrangian mechanics since its validity can be extended to classical Lagrangian field theories.

The second order differential Euler-Lagrange equations of motion for a system of N particles and s degrees of freedom, governed by a Lagrangian depending on the generalised coordinates q_j and velocities $\dot{q}_j$, with $j = 1, ..., s$ and $s = 3N$, are given by

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \tag{E.1}$$

or equivalently written

$$\frac{\partial^2 L}{\dot{q}_i \dot{q}_j}\ddot{q}_i = \frac{\partial L}{\partial q_j} - \frac{\partial^2 L}{\partial t \partial \dot{q}_j} - \frac{\partial^2 L}{\partial \dot{q}_j \partial q_i}\dot{q}_i, \tag{E.2}$$

note that in order for these equations to be cast in the form of Newton's second law,

$$\ddot{q} = F(q, \dot{q}), \tag{E.3}$$

it is necessary that

$$\frac{\partial^2 L}{\dot{q}_i \dot{q}_j} \neq 0, \tag{E.4}$$

which is known as the non-degeneracy of the Lagrangian and implies that the kinetic term of L, whose form is quadratic, can be inverted in the phase space, (q_j, p_j) with

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \tag{E.5}$$

whereby $\dot{q}_j = \dot{q}_j(q_i, p_i)$. Eq. (E.3) further ensures that the state of the system can be found at any instant if the initial values of position q_0 and velocity $\dot{q}_0$ are known, i.e.,

$$q(t) = q(q_0, \dot{q}_0, t). \tag{E.6}$$

In the next section we will derive the Euler-Lagrange equations and review the instability of the Hamiltonian for theories whose Lagrangian depends on accelerations.

E.1 $L = L(q, \dot{q}, \ddot{q}, t)$

Let us consider the case when the Lagrangian is a function not only of positions and velocities, as in the previous case, but also of the generalised accelerations, $\ddot{q}$, however, for simplicity we will assume that the Lagrangian has only one degree of freedom, i.e. defined as $L = L(q, \dot{q}, \ddot{q}, t)$, with which the action is constructed as

$$I = \int_{t_1}^{t_2} dt L(q, \dot{q}, \ddot{q}, t), \tag{E.7}$$

and effecting the variation

$$\delta L = \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial \ddot{q}} \delta \ddot{q}, \tag{E.8}$$

with

$$\frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q, \tag{E.9}$$

and

$$\frac{\partial L}{\partial \ddot{q}} \delta \ddot{q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right) - \frac{d}{dt} \left(\frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \delta q \right) + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} \delta q, \tag{E.10}$$

where the commutation of the time derivative and variation operators was assumed. In this way

$$\delta I = \left[\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \right) \delta q + \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} \right] \delta q, \tag{E.11}$$

the integrated term cancels out when the conditions $\delta q|_{t_1,t_2} = \delta \dot{q}|_{t_1,t_2} = 0$ are satisfied and the Euler-Lagrange equations are found to be

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{E.12}$$

These are a fourth order differential equations and we need to know four initial value data to solve entirely the problem. Again, under the condition of non-degeneracy

$$\frac{\partial^2 L}{\partial \ddot{q}^2} \neq 0, \tag{E.13}$$

we can write a form similar to newton's second law,

$$\frac{d^4 q}{dt^4} = F(q, \dot{q}, \ddot{q}, \dddot{q}), \tag{E.14}$$

and

$$q = q(q_0, ..., \dddot{q}_0, t). \tag{E.15}$$

The total energy or Hamiltonian of the system, expressed in terms of the time dependent coordinates q_1 and q_2, and momenta p_1 and p_2, is related to the Lagrangian via a Legendre transformation

$$H(p_1, p_2, q_1, q_2, t) + L(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = p_1 \dot{q}_1 + p_2 \dot{q}_2, \tag{E.16}$$

from which the Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{E.17}$$

with $i = 1, 2$.

$$q_1 \equiv q, \qquad \text{and} \qquad q_2 \equiv \dot{q}, \tag{E.18}$$

and the conjugated momenta

$$\begin{aligned} \dot{p}_1 &= \frac{\partial L}{\partial q_i} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right), \end{aligned} \tag{E.19}$$

and

$$\begin{aligned} \dot{p}_2 &= -\frac{\partial H}{\partial \dot{q}_1} \\ &= -p_1 + \frac{\partial L}{\partial \dot{q}_1} \\ &= \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \end{aligned} \tag{E.20}$$

that is to say

$$p_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \qquad \text{and} \qquad p_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{E.21}$$

The fact that the Lagrangian is non-degenerate allows us to invert the equations (E.18) and (E.21) in phase space and thus determine $\ddot{q} = \ddot{q}(q_1, q_2, p_2)$, thus, the Hamiltonian can be written as

$$H(p_1, p_2, q_1, q_2, t) = p_1\dot{q}_1 + p_2\ddot{q}_1(q_1, q_2, p_2) - L(q_1, q_2, \dot{q}_1, \dot{q}_2, t), \tag{E.22}$$

the linear dependence of the Hamiltonian on the momentum p_2 is conditional on the form that the function $\ddot{q}(q_1, q_2, p_2)$ may take, however this same consideration cannot be taken into account for p_1, i.e., regardless of the form of $L(q, \dot{q}, \ddot{q})$, the Hamiltonian will be linearly dependent on p_1, therefore its value cannot be bounded below.

E.2 Ostrogradsky Theorem

Ostrogradsky Theorem: *A system governed by a non-degenerate Lagrangian, L, which depends on the derivatives of order greater than one of the q-coordinates, i.e.* $L = L\left(q, \dot{q}, ..., \frac{d^{(m)}q}{dt^{(m)}}, t\right)$ *with* $m > 1$*, has a Hamiltonian which is linearly dependent on the conjugate momentum p, and which is not bounded from below.*

The following outlines the arguments of this theorem by generalising the steps followed in the previous section.

In the general case of a non-degenerate Lagrangian that depends up to the m-th time derivative of the coordinates, $L = L(q, ..., d_t^{(m)}q, t)$, where $d_t^{(m)} = \frac{d^{(m)}}{dt^{(m)}}$, the Euler-Lagrange equations are

$$\sum_{n=0}^{m} (-1)^n d_t^{(n)} \frac{\partial L}{\partial\left(d_t^{(n)} q\right)} = 0, \tag{E.23}$$

and the Hamiltonian

$$H(p_1, ..., p_m, q_1, ..., q_m, t) = \sum_{n=1}^{m} p_n \dot{q}_n - L(q_1, ..., q_m, \dot{q}_1, ..., \dot{q}_m, t), \tag{E.24}$$

and defining

$$q_i = d_t^{(i-1)} q \tag{E.25}$$

with $i = 1, ..., m$, by means of the canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{E.26}$$

it is found

$$\dot{p}_i = \frac{\partial L}{\partial\left(d_t^{i-1} q_{i-1}\right)} - p_{i-1}, \tag{E.27}$$

therefore, the Euler-Lagrange equations lead to

$$p_i = \sum_{n=0}^{m-i} (-1)^n d_t^{(n)} \frac{\partial L}{\partial\left(d_t^{(n+i)} q\right)}. \tag{E.28}$$

Now, the non-degeneracy of the Lagrangian

$$\frac{\partial^2 L}{\partial\left(d_t^{(m)} q\right)^2} \neq 0, \tag{E.29}$$

implies that it is possible to invert the equations (E.25) and (E.28) in phase space and thus obtain $d_t^{(m)} q = d_t^{(m)} q(q_1, ..., q_m, p_m)$, and finally

$$\begin{aligned} H(p_1, ..., p_m, q_1, ..., q_m, t) &= \sum_{n=1}^{m} p_n d_t^{(n)} q_1 - L(q_1, ..., q_m, \dot{q}_1, ..., \dot{q}_m, t) \\ &= \sum_{n=1}^{m-1} p_n d_t^{(n)} q_1 + p_n d_t^{(m)} q_1(q_1, ..., q_m, p_m) - L(q_1, ..., q_m, \dot{q}_1, ..., \dot{q}_m, t), \end{aligned} \tag{E.30}$$

thus, the linear dependence of the hamiltonian on the $m-1$ moments cannot be removed and therefore it can take arbitrarily negative values when those momenta are function of time, producing instabilities. The only particular case where the Hamiltonian is positively defined is when $m = 1$.

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