Estudio de nuevos modelos epidemiológicos compartiméntales con inafectabilidad estocástica y movilidad

(Study of New Compartmental Epidemiological Models with Stochastic Infectivity and Mobility)

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Dedication

To my family.

Acknowledgments

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Abstract

Study of New Compartmental Epidemiological Models with Stochastic Infectivity and Mobility

Based on the study of recent and classical epidemiological models, we present a susceptible-infected-recovered (SIR) epidemiological compartment model in different regions encompassing the movement of individuals among such regions. In the first chapter, preliminaries of stochastic analysis are presented, which are needed to develop the theory. In the second chapter, we propose a stochastic model having as a starting point the SIR model. The feasibility of the model is demonstrated when assuring the existence and uniqueness of the solutions. Apart from showing a lack of explosion in the solutions and the positivity of the solutions, it is also shown a stability condition for the process of the sum of infected individuals in the regions. Also, we relate this result with the deterministic case and the extinction of the infection in a single region. In the third chapter, some numerical simulations were conducted explaining the implemented numerical method and comparing such solutions to the deterministic case.

Keywords: SIR epidemic model, Stochastic differential equation, transportation,multi-region extension.

Resumen

Estudio de nuevos modelos epidemiológicos compartiméntales con inafectabilidad estocástica y movilidad

Basándonos en el estudio de literatura reciente y clásica de los modelos epidemiológicos, presentamos un modelo epidemiológico compartimental (SIR) susceptible-infectado-recuperado con múltiples regiones y movimiento de individuos entre dichas regiones. En el primer capitulo se presentan los preliminares de análisis estocástico, los cuales son necesarios para desarrollar la teoría. En el segundo capitulo proponemos un modelo estocástico teniendo como punto de partida el modelo SIR. La viabilidad del modelo se demuestra al asegurar la existencia y unicidad de las soluciones. Además, de mostrar la falta de explosión de las soluciones y la positividad de las soluciones, también se muestra una condición de estabilidad para el proceso de la suma de los individuos infectados en las regiones. También, relacionamos este resultado con el caso determinístico y la extinción de la infección en una sola región. En el tercer capítulo, se presentan simulaciones numéricas, explicamos el método numérico implementado y se comparan las soluciones con el modelo determinístico.

Palabras clave: modelo SIR epidemiologico, ecuación diferencial estocástica, transporte, extensión multi-region.

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General Notation

a.s	almost surely or with probability 1.
$\mathbf{E}(X)$	the expectation of the random variable X.
1_A	Indicator function of the set A, i.e $1_A(x) = 1$ if $x \in A$ or 0 otherwise.
$a \vee b$	the maximum of a and b.
$a \wedge b$	the minimum of a and b
$f : A \to B$	the mapping f from A to B.
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of positive real numbers, i.e. $\mathbb{R}_+ = (0, \infty)$.
$\mathbb{R}^d$	the d-dimensional Euclidean space.
$\mathbb{R}^d_+$	the positive d-dimensional cone, i.e. $\{x \in \mathbb{R}^d : x_i > 0, 1 \le i \le d\}$.
$\mathbb{R}^{d\times m}$	the space of real $d \times m$-matrices
$\lvert x\rvert$	the euclidean norm of a vector x.
$\lvert\lvert x\rvert\rvert_p$	the p-norm of a vector x.
τ_i	the i-th stopping time τ_i see definition 19.
τ	the stopping time τ see definition 19.
τ_∞	the explosion time of a SDE, see definition 14.
$\mathrm{Tr}(A)$	the trace of a square matrix $A = (a_{ij})_{d\times d}$, i.e. $\sum_{i=1}^d a_{ii}$.
$A^\top$	the transpose of the matrix or vector A.
W_t	A d-dimensional Brownian motion see definition 1 and definition 2.

1. Introduction

The COVID-19 infectious process observed during the last year, which originated in Wuhan, China, is a complex phenomenon that has been very challenging for governments around the world. Therefore, the mathematical modeling of such a phenomenon could help to understand how this process occurs in a given region. Besides, it also provides an instrument to make predictions that can help stop the infection outbreak.

One of the main objectives of modeling the behavior of an infectious disease is to predict the evolution of the pathogen in a particular region. The SIR models are consistent with the observations made on different diseases [[Irwin, 2008]]. Notably, in the COVID-19 case, the SIR models with different regions and mobility among these regions, proposed by Luigi Brugnano and Felice Ivernaro [[Brugnano and Iavernaro, 2020]], have been helpful when predicting the evolution of the pathogen in Italy. Such predictions have been useful, especially in the case of COVID-19. Those have permitted governments to make more suitable decisions regarding the allocation of resources to medical staff or to choose the right moment to decree quarantines.

The stochastic modeling of epidemics is relevant when the number of infectious individuals is small or when the variability associated with transmission, recovery, deaths, and births affects the final result of the epidemic [[Allen, 2017]]. This document presents a model with different regions, and it also considers the variability associated with the transmissibility of the pathogen similar to the works of [[Mao et al., 2002, Dalal et al., 2008, Gray et al., 2011, Ji et al., 2011, Xu and Li, 2018]].

In the first chapter, we present all the necessary preliminaries of the stochastic analysis, including the definition of the Brownian motion. Kolmogorov's extension theorem and Kolmogorov's continuity criterion are also included and used to prove the existence and continuity of the Brownian motion. Subsequently, we define Itô's stochastic integral regarding the Brownian motion for some type of processes. We also enunciate some useful properties of Itô's integral. Then, we define the concept of a stochastic differential equation (SDE). We enunciate the existence and uniqueness theorem of the stochastic differential equations. We also present the definition of a local solution of an SDE apart from presenting the existence and uniqueness theorems on the existence and uniqueness of local solutions.

In the second chapter, we present some results. We have studied different deterministic and stochastic models found in the literature. Based on some deterministic models like [Brugnano and Iavernaro, 2020], [Chen et al., 2014], and [Godio et al., 2020], and the classical SIR model presented by [Kermack and McKendrick, 1927]; We show a deterministic extension to consider several regions. Also, based on recent literature for stochastic epidemiological models [Liu et al., 2019], [Liu and Jiang, 2019] and [Gray et al., 2011]; We propose to add stochasticity to the abovementioned deterministic model by considering the parameter of infectivity that it is modeled as a stochastic process and obtaining; as a result, a new system of stochastic differential equations. We demonstrate that the model is feasible in the sense that we are able to show the global existence and uniqueness of solutions. Besides, we also demonstrate that the solutions are positive. We present an asymptotic property for some components of the solutions of the model under certain conditions for the parameters. Also, We relate this result with the basic reproduction number for the SIR model for a single region.

In the third chapter, we present the Wong-Zakai method for stochastic differential equations in the sense of Stratonovich that is used as a numerical procedure to approximate the solution of the SDE. We also explain how to transform a stochastic differential equation in Itô's sense into a stochastic differential equation in Stratonovich's sense. Finally, we study some simulation results and compare the results with the deterministic models found in the literature.

2. Preliminaries

2.1. Brownian Motion

Definition 1. A (standard, one-dimensional) Brownian motion is a continuous, adapted, real-valued process $(W_t)_{t\geq 0}$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with the following properties:

- $W_0 = 0$ a.s.
- for $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$
- for $0 \leq s < t$, the increment $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.

The existence of a process with such properties is guaranteed by the following theorems, both due to Kolmogorov.

Theorem 2.1.1 ([Øksendal, 2003] p.p 11)**.** *(Kolmogorov's extension theorem). For all $t_1, ..., t_k \in T$, $k \in \mathbb{N}$ let $P_{t_1,...,t_k}$ be probability measures on $\mathbb{R}^{nk}$ such that*

$$P_{t\sigma(1),...,t\sigma(k)}(F_1 \times ... \times F_k) = P_{t_1,...,t_k}(F_{\sigma^{-1}(1)} \times ... \times F_{\sigma^{-1}(k)}) \tag{2-1}$$

for all permutations σ on $\{1, 2, ..., k\}$ and

$$P_{t_1,...,t_k}(F_1 \times ... \times F_k) = P_{t_1,...,t_k,t_{k+1},...,t_{k+m}}(F_1 \times ... \times F_k \times \mathbb{R}^n \times ... \times \mathbb{R}^n) \tag{2-2}$$

for all $m \in N$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $(X_t)_{t\in T}$ on Ω, $X_t : \Omega \to \mathbb{R}^n$, such that $P_{t_1,...,t_k}$

$$P_{t_1,...,t_k}(F_1 \times ... \times F_k) = P[X_{t_1} \in F_1, ..., X_{t_k} \in F_k] \tag{2-3}$$

for all $t_i \in T$, $k \in N$ and all Borel sets F_i.

Let $T = [0, \infty)$, $0 \leq t_1 \leq t_2 \cdots \leq t_k$ note that for $n = 1$ a suitable probability density function $f_{t_1,...,t_k}$ for the probability measure $P_{t_1,...,t_k}$ on $\mathbb{R}^k$ is

$$f_{t_1,...,t_k}(x_1, ..., x_k) = \prod_{i=1}^{k} (2\pi)^{-k/2} (t_i - t_{i-1})^{-1/2} \exp\left[-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right] \tag{2-4}$$

Where we define $t_0 = 0$ and $x_0 = 0$ and also we define $P_0 = \delta_0$ the unit point mass centered at 0. It follows that the family of probability measures $P_{t_1,\dots,t_k}$ with $k \in \mathbb{N}$ and every $t_i \in (0,\infty)$ meet the conditions of the theorem above; consequently, there is a process $(W_t)_{t\geq 0}$ with the required properties but the continuity. For the continuity of $(W_t)_{t\geq 0}$ we have the following theorem.

Theorem 2.1.2 ([Øksendal, 2003] p.p 14)**.** *(Kolmogorov's continuity criterion) Suppose that a process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the condition: for every $T > 0$ there exists positive constants α,β and C such that*

$$E|X_t - X_s|^{\alpha} \leq C|t-s|^{1+\beta}, 0 \leq s, t \leq T \tag{2-5}$$

Then there exists a continuous modification (see definitions 18 and17) $(\tilde{X}_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$.

By definition, $W_t - W_s$ is normally distributed with mean 0 and variance $t-s$, thus:

$$E\left(|W_t - W_s|^4\right) = 3|t-s|^2 \tag{2-6}$$

As a result, the Brownian motion has a continuous modification. Henceforth if we mention the Brownian motion, we mean the continuous version of it. Also, for the filtration of the definition, we set $\mathcal{F}_t = \sigma(\{W_s : s \leq t\})$, i.e., $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion up to time t. Lastly, we will work with a complete filtered probability space $\left(\Omega, \tilde{\mathcal{F}}, P\right)$ with filtration $(\tilde{\mathcal{F}}_t)$ satisfying the **usual conditions**, that is: $\tilde{\mathcal{F}}$ and every $\tilde{\mathcal{F}}_t$ contain all the P-null sets; Also, we require that $(\tilde{\mathcal{F}}_t)$ is a right-continuous filtration.
We set $\tilde{\mathcal{F}}$ to be the P-null augmentation of $\mathcal{F}$. Also, for the filtration, we set $(\tilde{\mathcal{F}}_t)$ to be the P-null augmentation of the filtration generated by the Brownian motion. It is a well-known fact that the procedure described above produces a probability space $\left(\Omega, \tilde{\mathcal{F}}, P\right)$ with filtration $(\tilde{\mathcal{F}}_t)$ satisfying the usual conditions [[Mao, 2008] p.p 16].
From now on, we omit the tilde, and when we mention a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$, we assume that it satisfies the usual conditions.

Definition 2. A d-dimensional process $(W_t)_{t\geq 0} = (W_t^1, \dots, W_t^d)_{t\geq 0}$, is called a d-dimensional Brownian motion if every $(W_t^i)_{t\geq 0}$ is a one-dimensional Brownian motion, and $(W_t^1)_{t\geq 0}, \dots, (W_t^d)_{t\geq 0}$ are independent.

2.2. Itô integral

First, we will define the class of processes for which the Itô integral is defined.

Definition 3. Let $(\Omega, \mathcal{F}, P)$ a complete probability space with filtration $(\mathcal{F}_t)$ satisfying the usual conditions. Define $\mathcal{V}([S,T];\mathbb{R})$ be the class of processes $(X_t)_{S\geq t\geq T}$ such that

- $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}[S,T] \otimes \mathcal{F}$-measurable, where $\mathcal{B}[S,T]$ denotes the Borel σ-algebra on $[S,T]$.
- $(X_t)_{S\geq t\geq T}$ is $(\mathcal{F}_t)$-adapted.
- $E[\int_S^T |X_t|^2 dt] < \infty$.

For processes $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$, we will show how to define the Itô integral

$$\mathcal{I}[X_t](\omega) = \int_S^T X_t(\omega) dW_t(\omega) \tag{2-7}$$

where W_t is a 1-dimensional Brownian motion.
The idea is natural: first, define $\mathcal{I}[\phi_t]$ for a simple class of processes. Then by some approximation procedure, show that each $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$ can be approximated in $\mathcal{V}([S,T];\mathbb{R})$ by simple processes $(\phi_t^i)_{S\geq t\geq T}$. Finally, define the integral $\mathcal{I}[X_t]$ as the limit in $L_2(P)$ of $\mathcal{I}[\phi_t^i]$ as $\phi_i \to X_t$ where the latter limit is in $L_2(P \times \mu_{[S,T]})$ (where $\mu_{[S,T]}$ is the Lebesgue measure in $[S,T]$).

Definition 4. A process $(\phi_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$ is called elementary or simple if there exists a partition $S = t_0 < ... < t_k = T$ of $[S,T]$ and random variables e_i, such that e_i is

$$\phi_t = \sum_{i=0}^{k-1} e_i 1_{[t_i,t_{i+1})}(t) \tag{2-8}$$

Definition 5. Let $(\phi_t)_{S\geq t\geq T}$ be an elementary process as in definition 4. The Itô integral for $(\phi_t)_{S\geq t\geq T}$ is defined as

$$\int_S^T \phi_t dW_t = \sum_{i=0}^{k-1} e_i [B_{t_{i+1}} - B_{t_i}] \tag{2-9}$$

The following result is crucial for the convergence of the integrals of the elementary processes to the Itô integral of $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$.

Lemma 2.2.1. *[[Mao, 2008] p.p 19, [Øksendal, 2003] p.p 26](The Itô isometry for elementary processes) If $(\phi_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R}$, then*

$$\mathbf{E}\left[\left(\int_S^T \phi_t dW_t\right)^2\right] = \mathbf{E}\left(\int_S^T \phi_t^2 dt\right) \tag{2-10}$$

Now we are in conditions to extend definition 5 for the Itô integral from elementary processes to arbitrary processes in $\mathcal{V}([S,T];\mathbb{R})$. As customary, this will be made in several steps. the details of this construction can be found in [[Mao, 2008] p.p 20-22 [Øksendal, 2003] p.p 27-28].

Step 1. Let $(g_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$ be bounded and $(t,\omega) \to g_t(\omega)$ continuous for each ω. Then there exist a sequence of elementary processes $(\phi_t^n)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$ such that

$$\mathbf{E}\left(\int_S^T (g_t - \phi_t^n)^2 dt\right) \to 0 \text{ as } n \to \infty \tag{2-11}$$

Step 2. Let $(h_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$ be bounded. Then there exist a sequence of bounded processes $(g_t^n)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$ such that $(t,\omega) \to g_t^n$ is continuous for all ω and n, and we have

$$\mathbf{E}\left(\int_S^T (h_t - g_t^n)^2 dt\right) \to 0 \text{ as } n \to \infty \tag{2-12}$$

Step 3. Let $(X_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$. Then there exists a sequence of processes $(h_t^n)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$ such that $(h_t^n)_{S\geq t\geq T}$ is bounded for each n and

$$\mathbf{E}\left(\int_S^T (X_t - h_t^n)^2 dt\right) \to 0 \text{ as } n \to \infty \tag{2-13}$$

We are now in conditions to define the Itô integral

Definition 6. Let $(X_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$. then the Itô integral of $(X_t)_{S\leq t\leq T}$ is defined by

$$\int_S^T X_t dW_t = \lim_{n\to\infty} \int_S^T \phi_t^n dW_t \text{ limit in } L^2(P) \tag{2-14}$$

where $(\phi_t^n)_{S\leq t\leq T}$ is a sequence of elementary processes such that

$$\mathbf{E}\left[\int_S^T (X_t - \phi_t^n)^2 dt\right] \to 0 \text{ as } n \to \infty \tag{2-15}$$

Remark. The existence of the elementary processes $(\phi_t^n)_{S\geq t\geq T}$ converging to $(X_t)_{S\leq t\leq T}$ is guaranteed by steps 1-3 above. Also, the existence and uniqueness of the limit in the equation 2-14 is a consequence of the Itô isometry for elementary processes (lemma 2.2.1) and the fact that $L^2(P)$ and $\mathcal{V}([S,T];\mathbb{R})$ are complete metric spaces.

For completeness purposes, we mention the Itô isometry

Corollary 2.2.1.1. *(The Itô isometry) for all $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$, then*

$$\mathbf{E}\left[\left(\int_S^T X_t dW_t\right)^2\right] = \mathbf{E}\left[\int_S^T X_t^2 dt\right] \tag{2-16}$$

We now define the process $(I_t)_{S\leq t\leq T}$ for some integrable process $(X_t)_{S\leq t\leq T}$

Definition 7. Let $(X_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$ and let $S < t' \leq T$ define

$$I_{t'} = \int_S^{t'} X_s dW_s, \text{ for } S < t' \leq T \tag{2-17}$$

where, by definition, $I_S = \int_S^S X_s dW_s = 0$. Note that the latter defines a stochastic process $(I_t)_{S\leq t\leq T}$ for every integrable process $(X_t)_{S\leq t\leq T}$.

Now we mention some important properties of the Itô integral.

Theorem 2.2.2. *[[Mao, 2008] p.p 23] If $(X_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$, then the Itô integral $(I_t)_{S\leq t\leq T}$ of $(X_t)_{S\leq t\leq T}$ is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)$.*

Theorem 2.2.3. *Let $(X_t)_{S\leq t\leq T}, (Y_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$ and let $S < U < T$. Then*

1. $\int_S^T X_t dW_t = \int_S^U X_t dW_t + \int_U^T X_t dW_t$ *a.s*
2. $\int_S^T (cX_t + Y_t) dW_t = c\int_S^T X_t dW_t + \int_S^T Y_t dW_t$
3. $\mathbf{E}\left(\int_S^T X_t dW_t\right) = 0$
4. $\int_S^T X_t dW_t$ *is $\mathcal{F}_T$-measurable*

Definition 8. [[Mao, 2008] p.p 25] Let $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R})$, and let τ be a stopping time such that $S \leq \tau \leq T$ a.s. then it follows that $(1_{[S,\tau]}(t)X_t)_{S\leq t\leq T} \in \mathcal{V}([S,T];\mathbb{R})$ and we define

$$\int_S^\tau X_s dW_s = \int_S^T 1_{[S,\tau]}(s) X_s dW_s \tag{2-18}$$

Furthermore, if ρ is another stopping time with $S \leq \rho \leq \tau \leq T$ a.s., we define

$$\int_\rho^\tau X_s dW_s = \int_S^\tau X_s dW_s - \int_S^\rho X_s dW_s \tag{2-19}$$

It is easy to see from the definition and the linearity of the Itô integral that

$$\int_\rho^\tau X_s dW_s = \int_S^T 1_{[\rho,\tau]}(s) X_s dW_s \tag{2-20}$$

Remark. Note that definition 8 agrees with the process $(I_t)_{S\leq t\leq T}$ defined in 7. This means that $\int_S^\tau X_s dW_s = I_\tau$ [[Mao, 2008] p.p 26, [Nualart, 2011] p.p. 46].

We now extend the definition of the Itô integral to multidimensional processes, we define

Definition 9. Let $\mathcal{V}([S,T];\mathbb{R}^{d\times m}))$ be the class of $d \times m$-matrix-valued stochastic processes $(X_t)_{S\geq t\geq T}$ such that

- $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}[S,T] \otimes \mathcal{F}$-measurable, where $\mathcal{B}[S,T]$ denotes the Borel σ-algebra on $[S,T]$.
- $(X_t)_{S\geq t\geq T}$ is $(\mathcal{F}_t)$-adapted.
- $\mathbf{E}[\int_S^T \|X_t(\omega)\|_2^2 dt] < \infty$.

where $\|\sigma\|_2 = \sqrt{\sum_{i=1}^d \sum_{j=1}^m |\sigma_{ij}|^2}$.

Definition 10 ([Mao, 2008] p.p 28)**.** Let $(X_t)_{S\geq t\geq T} \in \mathcal{V}([S,T];\mathbb{R}^{d\times m})$ be a $d \times m$-matrix-valued stochastic process. Using matrix notation, we define the multi-dimensional indefinite Itô integral

$$\int_S^t X_s dW_s = \int_S^t \begin{pmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{d1} & \dots & X_{dm} \end{pmatrix} \begin{pmatrix} dW_s^1 \\ \vdots \\ dW_s^m \end{pmatrix} \tag{2-21}$$

to be the d-dimensional process whose ith component is the following sum of 1-dimensional Itô integrals

$$\sum_{j=1}^m \int_0^t X_{ij}(s) dW_s^j \tag{2-22}$$

It is usual to extend the definition of the Itô integral $\int_S^T X_s dW_s$ to processes not in $\mathcal{V}([S,T];\mathbb{R}^{d\times m})$. We define:

Definition 11. Define $\mathcal{W}^p([S,T];\mathbb{R}^{d\times m})$ as the class of $d\times m$-matrix-valued stochastic processes $(X_t)_{S\geq t\geq T}$ such that

- $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}([S,T]) \otimes \mathcal{F}$-measurable, where $\mathcal{B}[S,T]$ denotes the Borel σ- algebra on $[S,T]$.
- $(X_t)_{S\geq t\geq T}$ is $(\mathcal{F}_t)$-adapted.
- $\int_S^T \|X_s\|_p^p ds < \infty$ a.s

where $\|\sigma\|_p = \sqrt[p]{\sum_{i=1}^d \sum_{j=1}^m |\sigma_{ij}|^p}$.

It is possible to define the Itô integral for processes in $\mathcal{W}^2([S,T];\mathbb{R}^{d\times m})$ [[Mao, 2008] p.p 30, [Kloeden and Platen, 1992] p.p 90]. Clearly $\mathcal{V}([S,T];\mathbb{R}^{d\times m}) \subset \mathcal{W}^2([S,T];\mathbb{R}^{d\times m})$. This extension is advantageous because it enables us to consider the Itô integral for all continuous $(\mathcal{F}_t)-$adapted processes that might not be in $\mathcal{V}([S,T];\mathbb{R}^{d\times m})$ [[Steele, 2001] p.p 95]. We finish this section by mentioning the definition of an Itô process and the Itô formula.

Definition 12. [[Mao, 2008] p.p 31] Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)$ and let $(W_t)_{0\leq t}$ be an m-dimensional Brownian motion on it. A d-dimensional Itô process is an $\mathbb{R}^d$ valued continuous adapted process $(X_t)_{S\leq t\leq T}$ of the form

$$X_t = X_S + \int_S^t b(s)ds + \int_S^t \sigma(s)dW_s \tag{2-23}$$

Where $b(t) \in \mathcal{W}^1([S,T]; \mathbb{R}^{d\times 1})$ and $\sigma(t) \in \mathcal{W}^2([S,T]; \mathbb{R}^{d\times m})$ a common notation is

$$dX_t = b(t)dt + \sigma(t)dW_t \tag{2-24}$$

Theorem 2.2.4. *[[Mao, 2008] p. 36](The multi-dimensional Itô formula) let $(X_t)_{S\leq t\leq T}$ a d-dimensional Itô process, as in definition 12, i.e., satisfies equation (2-23) or in differential form satisfies equation (2-24). Let V be a continuous real-valued function with up to 2 partial spatial continuous derivatives and one partial continuous derivative in t then $V(X_t, t)$ is again an Itô process given by*

$$\begin{aligned} &V(X_t,t) - V(X_S,S) = \\ &\int_S^t \frac{\partial V}{\partial t}(X_u,u)du + \int_S^t \frac{\partial V}{\partial x}(X_u,u)b(u)du + \\ &\int_S^t \frac{1}{2}\operatorname{Tr}(\sigma^\top(u)\frac{\partial^2 V}{\partial x^2}(X_u,u)\sigma(u))du + \int_S^t \frac{\partial V}{\partial x}(X_u,u)\sigma(u)dW_u \ a.s \end{aligned} \tag{2-25}$$

Where $\frac{\partial V}{\partial x}$ is the Jacobian matrix of $V(x,t)$ with respect to the spatial variable x; $\frac{\partial^2 V}{\partial x^2}$ is the Jacobian matrix of the function $\frac{\partial V}{\partial x}(x,t)$, with respect to the spatial variable x; Finally, the meaning of $\frac{\partial V}{\partial t}$ is apparent.
We present the definition of the Stratonovich integral and the formula to change from an SDE in Itô's sense to an SDE in Stratonovich's sense later in section 4.1.

2.3. Stochastic differential equations

By a stochastic differential equation, we mean an integral equation of the form

$$X_t = X_{t_0} + \int_{t_0}^t b(X_s,s)ds + \int_{t_0}^t \sigma(X_s,s)dW_s \tag{2-26}$$

such that $b(s, X_s)$ and $\sigma(s, X_s)$ satisfy the integrability conditions that we mention in the following definition.

Definition 13 ([Mao, 2008] p.p 48)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)$ and let $(W_t)_{0\leq t}$ be an m-dimensional Brownian motion on it. A solution for the stochastic differential equation 2-26 is a continuous $(\mathcal{F}_t)$-adapted process $(X_t)_{t_0\leq t\leq T}$ such that $b(X_t,t) \in \mathcal{W}^1([t_0,T]; \mathbb{R}^{d\times 1})$, $\sigma(X_t,t) \in \mathcal{W}^2([t_0,T]; \mathbb{R}^{d\times m})$ and equation 2-26 holds almost surely.

Usually, the equation (2-26) is written in differential form as

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t \tag{2-27}$$

For stochastic differential equations, we have a similar existence and uniqueness theorem similar to that of ordinary differential equations, which we state below

Theorem 2.3.1. *[[Mao, 2008] p.p 51] Suppose a stochastic differential equation as in (2-26) and assume that there exist two positive constants $\tilde{K}$ and K such that*

- *(Lipschitz condition) For all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$*

$$|b(x,t) - b(y,t)|^2 \vee \|\sigma(x,t) - \sigma(y,t)\|^2 \leq K|x-y|^2 \tag{2-28}$$

- *(Linear growth condition) For all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$*

$$|b(x,t)|^2 \vee \|\sigma(x,t)\|^2 \leq \tilde{K}(1 + |x|^2) \tag{2-29}$$

Then there exists a unique solution $(X_t)_{t_0 \leq t \leq T}$ to the SDE, and the solution is in $\mathcal{V}([t_0, T]; \mathbb{R}^m)$.

Note that the linear growth condition can be implied by the Lipschitz condition as long as we have $\sup_{t_0 \leq t \leq T} |b(0,t)|^2 \vee \|\sigma(0,t)\|^2 < \infty$. Also, we can relax the conditions of the last theorem and still get a solution defined in $[t_0, T]$ [[Mao, 2008] p.p. 56-59]. Here we are interested in the concept of local solution for a stochastic differential equation which we state below.

Definition 14. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)$ and let $(W_t)_{0 \leq t}$ be an m-dimensional Brownian motion on it. Fix $X_{t_0} \in \mathbb{R}^d$. Let $(X_t^n)_{t_0 \leq t \leq T}$ a sequence of stochastic processes indexed by $n = 1, 2, ...$ We say that the sequence $(X_t^n)_{t_0 \leq t \leq T}$ is a local solution if there exists a non-decreasing sequence of stopping times τ_n such that

$$X_{t \wedge \tau_n}^n = X_{t_0} + \int_{t_0}^{t \wedge \tau_n} b(X_s^n, s)ds + \int_{t_0}^{t \wedge \tau_n} \sigma(X_s^n, s)dW_s \tag{2-30}$$

We say that the solution is defined in the interval $[t_0, \tau_e)$ where $\tau_e = \lim_{n \to \infty} \tau_n$. Also, we say that the solution $(X_t^n)_{t \geq t_0}$ is **globally defined (is a global solution)** if $\tau_e = \infty$ a.s. [[Mao, 1991] p. 162].

Remark. It is important to note that the sequence of processes $(X_t^n)_{t_0 \leq t \leq T}$ is a consistent sequence of stochastic processes for the sequence of stopping times τ_n, which means that $X_t^n = X_t^{n+1}$ if $t_0 \leq t \leq \tau_n$. Because of that, we can just write $X_t = X_t^n$ when $t_0 \leq t \leq \tau_n$ for some n, so for a local solution, we adopt the notation $(X_t)_{t_0 \leq t < \tau_e}$.

We finish this section by stating the existence of local solutions for stochastic differential equations; this is a consequence of theorem 2.3.1 by performing a truncation procedure over the coefficients of the SDE.

Theorem 2.3.2. *[[Mao, 2008] p. 57, [Arnold, 1974] p. 112, [Markus, 2012] p. 40]Suppose a stochastic differential equation as in (2-26), let $X_{t_0} \in \mathbb{R}^d$ and assume that $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are continuous functions with the following property:*
(Local Lipschitz condition) there exists constants K_n such that, for all $t \in [t_0, T]$ and all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq n$

$$|b(x,t) - b(y,t)|^2 \vee \|\sigma(x,t) - \sigma(y,t)\|^2 \leq K_n|x-y|^2 \tag{2-31}$$

Then the stochastic differential equation (2-26) admits a unique local solution $(X_t)_{t_0\leq t<\tau_e}$ in the stochastic interval $t \in [t_0, \tau_e)$ as defined in 14.

Remark. If the requirements of theorem 2.3.1 are meet for every $T \geq t_0$. We will be able to construct a solution $(X_t)_{t_0\leq t\leq T} \in \mathcal{V}([t_0, T]; \mathbb{R}^m)$ for every $T \geq t_0$. Consequently, the explosion time $\tau_e = \infty$ a.e., which means that the solution is global.

3. Model

3.1. Model definition

We want to model the spread of an infectious disease in M regions. The spread of the disease within each region will be modeled by a basic SIR model as proposed by Kermack and McKendrick in [Kermack and McKendrick, 1927] hence for the community i we get:

$$\begin{aligned} \frac{dS_i}{dt} &= -\beta_i S_i I_i \\ \frac{dI_i}{dt} &= \beta_i S_i I_i - \gamma_i I_i \\ \frac{dR_i}{dt} &= \gamma_i I_i \end{aligned} \tag{3-1}$$

We want to propose a model that considers the transportation of individuals from one region into another. We begin by taking a discrete approximation of model (3-1)

$$\begin{aligned} \Delta S_i &= -\beta_i S_i I_i \Delta t \\ \Delta I_i &= \beta_i S_i I_i \Delta t - \gamma_i I_i \Delta t \\ \Delta R_i &= \gamma_i I_i \Delta t \end{aligned} \tag{3-2}$$

We will introduce the parameters $\lambda_{ij}\Delta t$ to describe the number of individuals going from region i to region j by unit time. Also, we assume that the proportion of susceptible individuals going from region i to j is the same as the current proportion of susceptibles in region i, namely $\frac{S_i}{N_i}$. Note that as customary, we denote $N_i = S_i + I_i + R_i$ [Kermack and McKendrick, 1927]. Giving us that the total number of susceptibles going from i to j by unit time will be $\lambda_{ij}\frac{S_i}{N_i}\Delta t$ the last implies that the total number of susceptibles leaving region i by unit time will be $\sum_{j=1}^{M} \lambda_{ij}\frac{S_i}{N_i}$. We use the convention $\lambda_{ii} = 0$ because, in this context, transportation only makes sense between different regions. Another consideration about the parameters λ_{ij} is that $\sum_{j=1}^{M} \lambda_{ij} = \sum_{j=1}^{M} \lambda_{ji}$. This is a reasonable approximation to the reality given the low variability in population size for each region over a short period of time, like days, weeks, or months. Also, we can assert that the number of Susceptibles entering region i can be written as $\sum_{j=1}^{M} \lambda_{ji}\frac{S_j}{N_j}$. A similar analysis can be made for the infected and the recovered compartments, changing S_i by I_i or R_i, respectively.

Summing up the discrete scheme for a discrete SIR model with multiple regions with transportation will be

$$
\begin{aligned}
\Delta S_i &= -\beta_i S_i I_i \Delta t + \sum_{j=1}^{M} \lambda_{ji} \frac{S_j}{N_j} \Delta t - \sum_{j=1}^{M} \lambda_{ij} \frac{S_i}{N_i} \Delta t \\
\Delta I_i &= \beta_i S_i I_i \Delta t - \gamma_i I_i \Delta t + \sum_{j=1}^{M} \lambda_{ji} \frac{I_j}{N_j} \Delta t - \sum_{j=1}^{M} \lambda_{ij} \frac{I_i}{N_i} \Delta t \\
\Delta R_i &= \gamma_i I_i \Delta t + \sum_{j=1}^{M} \lambda_{ji} \frac{R_j}{N_j} \Delta t - \sum_{j=1}^{M} \lambda_{ij} \frac{R_i}{N_i} \Delta t
\end{aligned} \tag{3-3}
$$

We assume that the change of susceptible, infected, and recovered individuals is caused only by the city's infectious process or individuals' transportation from one region to another. Taking the limit when $\Delta t \to 0$, We get from 3-3 the following system of differential equations.

$$
\begin{aligned}
\frac{dS_i}{dt} &= -\beta_i S_i I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{S_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{S_i}{N_i} \\
\frac{dI_i}{dt} &= \beta_i S_i I_i - \gamma_i I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{I_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{I_i}{N_i} \\
\frac{dR_i}{dt} &= \gamma_i I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{R_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{R_i}{N_i}
\end{aligned} \tag{3-4}
$$

This model is a particular case of that studied in [Chen et al., 2014], assuming that infections do not occur during travel. Also, this procedure to include multiple regions has been used with other compartmental models like the SEIR model in [[Kiran et al., 2020]]. We want to introduce randomness to the model (3-4) similar to that presented by [Gray et al., 2011]. We let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ a complete filtered probability space satisfying the usual conditions (i.e., $(\mathcal{F}_t)$ is right continuous, and $\mathcal{F}_0$ contains all the P-null sets) and let $(W_t)_{t\geq 0} = (W_t^1, ..., W_t^M)_{t\geq 0}$ be an M-dimensional Brownian motion defined on the probability space. We rewrite the infective part of (3-4) naturally as:

$$
dI_i = \left(\beta_i S_i I_i - \gamma I + \sum_{j=1}^{M} \lambda_{ji} \frac{I_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{I_i}{N_i} \right) dt \tag{3-5}
$$

Here $[t, t+dt)$ is a small-time interval, and we use the notation $d\cdot$ for the small change in any quantity over this time interval, in fact, $dI_i = I(t+dt) - I(t)$, and the change in dI_i for every i is described by (3-5). Consider the disease transmission coefficient β_i for the deterministic model. We can interpret this parameter as the rate at which each individual of the region i makes potentially infectious contacts. A potentially infectious contact will transmit the disease if an infectious individual makes contact with a susceptible individual. Thus from the analysis made that lead to (3-5), we can assert that the total number of new infections for the region i in the small-time interval $[t, t+dt)$ is:

$$\beta_i S_i I_i dt \tag{3-6}$$

thus a single infected individual makes

$$S_i \beta_i dt \tag{3-7}$$

new infectious contacts with all the susceptible individuals in region i in the time interval $[t, t+dt)$. As a result, a single infected individual makes

$$\beta_i dt \tag{3-8}$$

Potentially infectious contacts with each susceptible individual in $[t, t+dt)$. Now, as in [Gray et al., 2011] suppose that some stochastic environmental factor acts simultaneously on each individual in each region. In this case, $\beta_i dt$ changes to a random variable $\tilde{\beta}_i$. More precisely, each individual makes

$$\tilde{\beta}_i = \beta_i dt + \sigma_i dW_i \tag{3-9}$$

Potentially infectious contacts with each susceptible individual in $[t, t+dt)$. Therefore we replace $\beta_i dt$ in (3-5) by $\beta_i dt + \sigma_i dW_i$ we get

$$dI_i = \left(\beta_i S_i I_i - \gamma I + \sum_{j=1}^{M} \lambda_{ji} \frac{I_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{I_i}{N_i} \right) dt + \sigma_i S_i I_i dW_i \tag{3-10}$$

performing the same change to the susceptible compartment in each region, we get the SDE:

$$\begin{aligned}
dS_i &= \left(-\beta_i S_i I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{S_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{S_i}{N_i} \right) dt - \sigma_i S_i I_i dW_i \\
dI_i &= \left(\beta_i S_i I_i - \gamma I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{I_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{I_i}{N_i} \right) dt + \sigma_i S_i I_i dW_i \\
dR_i &= \left(\gamma_i I_i + \sum_{j=1}^{M} \lambda_{ji} \frac{R_j}{N_j} - \sum_{j=1}^{M} \lambda_{ij} \frac{R_i}{N_i} \right) dt
\end{aligned} \tag{3-11}$$

The last model is biologically realistic; as explained in [Gray et al., 2011], there have been many studies for the single region model; for example, [Lin and Jiang, 2013, Xu, 2017].

Remark. Note that $N_i(t)$ is the population size for the region i at time t and is defined as $N_i = S_i + I_i + R_i$, because we are assuming that at every time each individual belongs to the category susceptible, infected, or recovered; we can see easily that $dN_i = 0$, as a consequence N_i is constant a.s.

We also write the system of stochastic differential equations (3-11) as a vector SDE:

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW$$

We write the system of stochastic differential equations that defines the dynamics of susceptible infected and Recovered as a multidimensional stochastic differential equation where:

$$X_t = \begin{pmatrix} S_1(t) \\ I_1(t) \\ R_1(t) \\ \vdots \\ R_M(t) \end{pmatrix} \tag{3-12}$$

$$b(X_t, t) = \begin{pmatrix} -\beta_1 S_1 I_1 + \sum_{j=1}^{M} \lambda_{j1} \frac{S_j}{N_j} - \sum_{j=1}^{M} \lambda_{1j} \frac{S_1}{N_1} \\ \beta_1 S_1 I_1 - \gamma I + \sum_{j=1}^{M} \lambda_{j1} \frac{I_j}{N_j} - \sum_{j=1}^{M} \lambda_{1j} \frac{I_1}{N_1} \\ \gamma_1 I_1 + \sum_{j=1}^{M} \lambda_{j1} \frac{R_j}{N_j} - \sum_{j=1}^{M} \lambda_{1j} \frac{R_1}{N_1} \\ \vdots \\ \gamma_M I_M + \sum_{j=1}^{M} \lambda_{jM} \frac{R_j}{N_j} - \sum_{j=1}^{M} \lambda_{Mj} \frac{R_M}{N_M} \end{pmatrix} \tag{3-13}$$

$$\sigma(X_t, t) = \begin{pmatrix} -\sigma_1 S_1(t) I_1(t) & 0 & \dots & 0 \\ \sigma_1 S_1(t) I_1(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & -\sigma_2 S_2(t) I_2(t) & \dots & 0 \\ 0 & \sigma_2 S_2(t) I_2(t) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\sigma_M S_M(t) I_M(t) \\ 0 & 0 & \dots & \sigma_M S_M(t) I_M(t) \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{3-14}$$

3.2. Existence and uniqueness of the solutions

Theorem 3.2.1. *For any initial value $(S_1(0), I_1(0), R_1(0), ..., R_M(0)) \in {\mathbb{R}_+}^{3M}$, there exist a unique global solution $(X_t)_{t\geq 0}$ for the SDE (3-11) and on $t \geq 0$, the solution remains in ${\mathbb{R}_+}^{3M}$ with probability 1, namely $X_t \in \mathbb{R}_+^{3M}$ for all $t \geq 0$ almost surely.*

Proof. We will give an argument similar to the one found in the literature for the single region case [Mao et al., 2002, Dalal et al., 2008, Gray et al., 2011, Ji et al., 2011, Xu and Li, 2018]. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)$ and let $(W_t)_{0\leq t}$ be an M-dimensional Brownian motion on it. We know from theorem 2.3.2 that for any initial values $(S_1(0), I_1(0), R_1(0), ..., R_M(0)) \in {\mathbb{R}_+}^{3M}$, there exists a unique local solution $(X_t)_{0\leq t<\tau_e}$, because the coefficients of the model (3-11) are locally Lipschitz continuous, we need to show $\tau_e = \infty$ a.s to show that the solution is globally defined on $\mathbb{R}_+^{3M}$. First, define a sequence of stopping times by

$$\begin{aligned} \tau_k = \inf\{t \geq 0 : \min\{S_1(t), I_1(t), R_1(t), ..., R_M(t)\} \leq \frac{1}{k} \text{ or} \\ \max\{S_1(t), I_1(t), R_1(t), ..., R_M(t)\} \geq k\} \end{aligned} \tag{3-15}$$

It is clear that $\tau_i \leq \tau_j$ as long as $i \leq j$; let $\lim_{m\to\infty} \tau_m = \tau_\infty$ a.s., we have that $\tau_i \leq \tau_e$ for every i, as a consequence $\tau_\infty \leq \tau_e$. We will show that $\tau_\infty = \infty$ with probability 1, to get the required $\tau_e = \infty$ with probability 1; Also, we will have that $X_t \geq 0$ for every $t \geq 0$.

The proof will be made by contradiction. Let $(S_1(0), I_1(0), R_1(0), ..., R_M(0)) \in {\mathbb{R}_+}^{3M}$ be a positive initial condition if we have $P(\tau_\infty = \infty) \neq 1$ there will be 2 constants T y $0 < \epsilon < 1$ such that $P(\tau_\infty \leq T) \geq \epsilon$. Note also that since $\tau_k \leq \tau_\infty$ for every k, we have that $\{\tau_\infty \leq T\} \subseteq \{\tau_k \leq T\}$ as a consequence $P(\tau_k \leq T) \geq \epsilon$.

Since the initial condition $(S_1(0), I_1(0), R_1(0), ..., R_M(0))$ is assumed positive we have that for every k, $P(\tau_k \geq 0) = 1$. It does not matter whether the process stops at 0; in fact, this may be the case for the first stopping times. We only require that $X_{t\wedge\tau_k}$ remains positive which, is indeed the case. We make the following remark to show that it makes sense to work with such stopping times.

Remark. Given that the initial condition is positive, there will be a constant L such that for any $k \geq L$, every component of the initial condition will be between $1/k$ and k. Consequently, for every $k \geq L$, we will have that $\mathbb{P}(\tau_k > 0) = 1$. Thus, without loss of generality from now on, when we mention k it will be any arbitrary $k \geq L$.

Using the fact that the process $X_{t\wedge\tau_k}$ stops before any of its components become negative, the following function is well defined for each $0 \leq t < \infty$ and any t_k with

$k \geq L$.

$$
\begin{aligned}
&V(X_{t\wedge\tau_k}, t\wedge\tau_k) = \\
&\sum_{i=1}^{M}[S_i(t\wedge\tau_k) - 1 - \ln(S_i(t\wedge\tau_k))] + \\
&\sum_{i=1}^{M}[I_i(t\wedge\tau_k) - 1 - \ln(I_i(t\wedge\tau_k))] + \\
&\sum_{i=1}^{M}[R_i(t\wedge\tau_k) - 1 - \ln(R_i(t\wedge\tau_k))]
\end{aligned}
\tag{3-16}
$$

Remark. We mention some important properties of the function $f(x) = x-1-\ln(x)$. The domain of f is the set of positive real numbers, and the range of f is the set of non-negative real numbers. Also we have that $f(0) = 0$, f is decreasing in $(0, 1]$, $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$ since f is increasing in $[1, \infty)$ the later implies that $\lim_{x\to\infty} f(x) \wedge f(1/x) = \infty$. The verifications of these facts can be done by simple calculations.

For $0 \leq t < \infty$ we will use the Itô formula to compute $dV(X_{t\wedge\tau_k}, t\wedge\tau_k)$. Since the function $V(x, t)$ is continuous, real-valued of class C^∞ in the spatial and temporal variables. First, we compute the necessary derivative and jacobians needed to use the Itô formula:

$$
\begin{aligned}
&\frac{\partial V}{\partial x}(X_{t\wedge\tau_k}, t\wedge\tau_k) = \\
&\left(1 - \frac{1}{S_1(t\wedge\tau_k)}, 1 - \frac{1}{I_1(t\wedge\tau_k)}, 1 - \frac{1}{R_1(t\wedge\tau_k)}, \ldots, 1 - \frac{1}{R_M(t\wedge\tau_k)}\right)
\end{aligned}
\tag{3-17}
$$

also,

$$
\begin{aligned}
&\frac{\partial^2 V}{\partial x^2}(X_{t\wedge\tau_k}, t\wedge\tau_k) = \\
&\begin{pmatrix}
\frac{1}{S_1^2(t\wedge\tau_k)} & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{I_1^2(t\wedge\tau_k)} & 0 & \ldots & 0 \\
0 & 0 & \frac{1}{R_1^2(t\wedge\tau_k)} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \frac{1}{R_M^2(t\wedge\tau_k)}
\end{pmatrix}
\end{aligned}
\tag{3-18}
$$

finally

$$
\frac{\partial V}{\partial t}(X_{t\wedge\tau_k}, t\wedge\tau_k) = 0 \tag{3-19}
$$

now we compute:

$$
\begin{aligned}
&\frac{\partial V}{\partial x}(X_{t\wedge\tau_k}, t\wedge\tau_k)b(X_{t\wedge\tau_k}, t\wedge\tau_k) \\
&= \sum_{i=1}^{M}\left(-\beta_i S_i(t\wedge\tau_k)I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{S_j(t\wedge\tau_k)}{N_j}\right. \\
&\left.- \sum_{j=1}^{M}\lambda_{ij}\frac{S_i(t\wedge\tau_k)}{N_i}\right)\left(1-\frac{1}{S_i(t\wedge\tau_k)}\right) \\
&+ \sum_{i=1}^{M}\left(\beta_i S_i(t\wedge\tau_k)I_i(t\wedge\tau_k) - \gamma_i I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{I_j(t\wedge\tau_k)}{N_j}\right. \\
&\left.- \sum_{j=1}^{M}\lambda_{ij}\frac{I_i(t\wedge\tau_k)}{N_i}\right)\left(1-\frac{1}{I_i(t\wedge\tau_k)}\right) \\
&+ \sum_{i=1}^{M}\left(\gamma_i I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{R_j(t\wedge\tau_k)}{N_j} - \sum_{j=1}^{M}\lambda_{ij}\frac{R_i(t\wedge\tau_k)}{N_i}\right)\left(1-\frac{1}{R_i(t\wedge\tau_k)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{M}\left(-\beta_i S_i(t\wedge\tau_k)I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{S_j(t\wedge\tau_k)}{N_j}\right. \\
&\left.- \sum_{j=1}^{M}\lambda_{ij}\frac{S_i(t\wedge\tau_k)}{N_i}\right)\left(-\frac{1}{S_i(t\wedge\tau_k)}\right) \\
&+ \sum_{i=1}^{M}\left(\beta_i S_i(t\wedge\tau_k)I_i(t\wedge\tau_k) - \gamma_i I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{I_j(t\wedge\tau_k)}{N_j}\right. \\
&\left.- \sum_{j=1}^{M}\lambda_{ij}\frac{I_i(t\wedge\tau_k)}{N_i}\right)\left(-\frac{1}{I_i(t\wedge\tau_k)}\right) \\
&+ \sum_{i=1}^{M}\left(\gamma_i I_i(t\wedge\tau_k) + \sum_{j=1}^{M}\lambda_{ji}\frac{R_j(t\wedge\tau_k)}{N_j} - \sum_{j=1}^{M}\lambda_{ij}\frac{R_i(t\wedge\tau_k)}{N_i}\right)\left(-\frac{1}{R_i(t\wedge\tau_k)}\right) \\
&= \sum_{i=1}^{M}\left(\beta_i I_i(t\wedge\tau_k) - \sum_{j=1}^{M}\lambda_{ji}\frac{S_j(t\wedge\tau_k)}{N_j S_i(t\wedge\tau_k)} + \sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right) + \\
&\sum_{i=1}^{M}\left(-\beta_i S_i(t\wedge\tau_k) + \gamma_i - \sum_{j=1}^{M}\lambda_{ji}\frac{I_j(t\wedge\tau_k)}{N_j I_i(t\wedge\tau_k)} + \sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right) + \\
&\sum_{i=1}^{M}\left(-\gamma_i\frac{I_i(t\wedge\tau_k)}{R_i(t\wedge\tau_k)} - \sum_{j=1}^{M}\lambda_{ji}\frac{R_j(t\wedge\tau_k)}{N_j R_i(t\wedge\tau_k)} + \sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)
\end{aligned}
\tag{3-20}
$$

finally we compute:

$$
\begin{aligned}
&\left(\frac{\partial V}{\partial x}(X_{t\wedge\tau_k}, t\wedge\tau_k)\sigma(X_{t\wedge\tau_k}, t\wedge\tau_k)\right)^{\top} \\
=&\begin{pmatrix} \left[\left(1-\frac{1}{I_1(t\wedge\tau_k)}\right)-\left(1-\frac{1}{S_1(t\wedge\tau_k)}\right)\right]\sigma_1 S_1(t\wedge\tau_k)I_1(t\wedge\tau_k) \\ \vdots \\ \left[\left(1-\frac{1}{I_M(t\wedge\tau_k)}\right)-\left(1-\frac{1}{S_M(t\wedge\tau_k)}\right)\right]\sigma_1 S_M(t\wedge\tau_k)I_M(t\wedge\tau_k) \end{pmatrix} \\
=&\begin{pmatrix} \sigma_1 I_1(t\wedge\tau_k)-\sigma_1 S_1(t\wedge\tau_k) \\ \vdots \\ \sigma_M I_M(t\wedge\tau_k)-\sigma_M S_M(t\wedge\tau_k) \end{pmatrix}
\end{aligned}
\tag{3-21}
$$

by the fact that $X_{\tau_k\wedge T}\geq 0$ and using (3-20), it is clear that:

$$
\begin{aligned}
&\frac{\partial V}{\partial x}(X_{t\wedge\tau_k}, t\wedge\tau_k)b(X_{t\wedge\tau_k}, t\wedge\tau_k) \\
\leq&\sum_{i=1}^{M}\left(\beta_i I_i(t\wedge\tau_k)+\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)+\sum_{i=1}^{M}\left(\gamma_i+\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)+ \\
&\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right) \\
\leq&\sum_{i=1}^{M}\left(\beta_i N_i+\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)+\sum_{i=1}^{M}\left(\gamma_i+\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)+ \\
&\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\lambda_{ij}\frac{1}{N_i}\right)
\end{aligned}
\tag{3-22}
$$

The last term of the inequalities in (3-22) is a constant, which we will call C_1. Note also that the latter inequalities do not depend on which k we are considering. Using (3-14) and (3-18), we compute:

$$
\begin{aligned}
&\mathrm{Tr}\left(\sigma^{\top}(X_{t\wedge\tau_k}, t\wedge\tau_k)\frac{\partial^2 V}{\partial x^2}(X_{t\wedge\tau_k}, t\wedge\tau_k)\sigma^{\top}(X_{t\wedge\tau_k}, t\wedge\tau_k)\right) \\
=&\sum_{i=1}^{M}\frac{\sigma_i^2 S_i^2(t\wedge\tau_k)I_i^2(t\wedge\tau_k)}{S_i^2(t\wedge\tau_k)}+\frac{\sigma_i^2 S_i^2(t\wedge\tau_k)I_i^2(t\wedge\tau_k)}{I_i^2(t\wedge\tau_k)} \\
=&\sum_{i=1}^{M}\sigma_i^2 I_i^2(t\wedge\tau_k)+\sigma_i^2 S_i^2(t\wedge\tau_k) \\
\leq&2\sum_{i=1}^{M}\sigma_i^2 N_i^2
\end{aligned}
\tag{3-23}
$$

The last term of the inequalities in (3-23) is a constant, which we will call C_2. Note also that the latter inequalities do not depend on which k we are considering. Also, from (3-21), the square of each component of

$$\left(\frac{\partial V}{\partial x}(X_{t\wedge\tau_k}, t\wedge\tau_k)\sigma(X_{t\wedge\tau_k}, t\wedge\tau_k)\right)^{\top} \tag{3-24}$$

is bounded by $2\sigma_i N_i$. As a consequence the expectation of component i squared

$$\mathbf{E}\left(\left(\frac{\partial V}{\partial x}(X_{t\wedge\tau_M}, t\wedge\tau_M)\sigma(X_{t\wedge\tau_M}, t\wedge\tau_M)\right)_i^2\right) < \infty \tag{3-25}$$

which implies that for every $k \geq L$

$$\int_0^{t\wedge\tau_k} \frac{\partial V}{\partial x}(X_u, u)\sigma(X_u, u)dW_u \tag{3-26}$$

is a sum of square-integrable martingales with zero expectation by theorem 2.2.2. Thus, taking expectation at $t = T$, we get

$$\begin{aligned} &\mathbf{E}(V(X_{T\wedge\tau_k}, T\wedge\tau_k)) \\ \leq &V(X_0, 0) + K_1\mathbf{E}(T\wedge\tau_k) + \frac{K_2}{2}\mathbf{E}(T\wedge\tau_k) \\ \leq &V(X_0, 0) + (C_1 + \frac{C_2}{2})T \end{aligned} \tag{3-27}$$

Note that the last term of the inequalities in (3-27) is a constant that we will call C, which only depends on T, C_1, C_2, and the initial condition X_0. Also, C is independent of k and ω. For every k, consider the set $\Omega_k^T = \{\tau_k \leq T\}$. At the beginning of the proof, we showed that $P(\tau_k \leq T) = P(\Omega_k^T) \geq \epsilon$ using this fact and (3-27) we get

$$\begin{aligned} C \geq &\mathbf{E}(V(X_{T\wedge\tau_k}, T\wedge\tau_k)) \\ \geq &\mathbf{E}(1_{\Omega_k^T} V(X_{T\wedge\tau_k}, T\wedge\tau_k)) \\ \geq &\mathbf{E}\left(1_{\Omega_k^T}\left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right)\wedge(k - 1 - \ln k)\right) \\ = &\epsilon\left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right)\wedge(k - 1 - \ln k) \end{aligned} \tag{3-28}$$

we get the last inequality from the fact that if $\omega \in \Omega_k^T$ there should be a variable of the vector $X_{T\wedge\tau_k}$ is either k or $1/k$ when $k \geq L$ then $V(X_{T\wedge\tau_k}, T\wedge\tau_k)$ is at least $(\frac{1}{k} - 1 - \ln\frac{1}{k}) \wedge (k - 1 - \ln k)$, finally we get the contradiction $\infty > C \geq \epsilon\left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right) \wedge (k - 1 - \ln k) = \infty$ when $k \to \infty$. □

3.3. Stability

First, we recall some definitions from the theory of dynamical systems. Consider a d-dimensional ordinary differential equation:

$$\dot{x}(t) = f(x(t), t) \tag{3-29}$$

Assume that for every initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ there exits a unique solution on $[t_0, \infty)$ which we denote $x(t; t_0, x_0)$, assume furthermore that $f(0, t) = 0$ for every $t \geq t_0$, the solution $x(t; t_0, 0) = 0$ is called a **trivial solution** or **equilibrium position**.
A trivial solution is said to be **stable** if for every $\epsilon > 0$, there exits a $\delta > 0$ such that for all $t \geq t_0$ and every initial condition x_0 such that $|x_0| < \delta$ we have

$$|x(t; t_0, x_0)| \leq \epsilon \tag{3-30}$$

Otherwise, the trivial solution is said to be **unstable**.
The trivial solution is said to be **asymptotically stable** if it is stable and there exits a $\delta > 0$ such that for every initial condition x_0 such that $|x_0| < \delta$ we have

$$\lim_{t \to \infty} x(t; t_0, x_0) = 0 \tag{3-31}$$

There are several definitions for the stability of the solutions of SDE's. More information can be found in [[Khasminskii, 2012]p.p 22] and in [[Mao, 2008]p.p 110-111]. In this case, we will be interested in a definition of stability of processes due to Mao, which we state below.

Definition 15 ([Mao, 2008]p.p 119)**.** Let $(X_t)_{t \geq t_0}$ be a process; we say that $(X_t)_{t \geq t_0}$ is almost surely exponentially stable if

$$\limsup_{t \to \infty} \frac{1}{t} \ln(|X_t|) < 0 \text{ a.s.} \tag{3-32}$$

It is important to remark that almost surely exponential stability implies that the process trajectories tend to the equilibrium exponentially fast [[Mao, 2008]p.p. 120]. More precisely, let

$$\limsup_{t \to \infty} \frac{1}{t} \ln(|X_t|) = -c \tag{3-33}$$

where $c > 0$ we will have that for ϵ such that $0 < \epsilon < c$ there exits $\xi > 0$ such that

$$|X_t| \leq \xi \exp(\epsilon - c)t \text{ for all } t \geq t_\epsilon(\omega). \tag{3-34}$$

Theorem 3.3.1. *If* $\frac{\beta_i N_i}{\gamma_i} < 1$ *for every* i*, then the process* $\left(\sum_{i=1}^{M} I_i(t) \right)_{t \geq 0}$ *is almost surely exponentially stable*

Proof. We have already proved that for every initial positive initial condition, the solution of the SDE (3-11) will remain positive, so the function $\ln(I_i(t))$ is well defined for every i, note also that by the monotonicity of the function $\ln(\cdot)$ we have $\ln(I_i(t)) \leq \ln(\sum_{i=1}^M I_i(t))$ for every i, consider the last function as a function of the solution of the SDE as follows:

$$V(X_t, t) = \ln\left(\sum_{i=1}^{M} I_i(t)\right) \tag{3-35}$$

we will use the Itô formula to calculate $dV(X_t, t)$; first we compute:

$$\frac{\partial V}{\partial x}(X_t, t) = \left(0, \frac{1}{\sum_{i=1}^M I_i(t)}, 0, 0, \frac{1}{\sum_{i=1}^M I_i(t)}, 0, \ldots, 0, \frac{1}{\sum_{i=1}^M I_i(t)}, 0\right) \tag{3-36}$$

$$\frac{\partial^2 V}{\partial x^2}(X_t, t) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \frac{-1}{\left(\sum_{i=1}^M I_i(t)\right)^2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{-1}{\left(\sum_{i=1}^M I_i(t)\right)^2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{3-37}$$

$$\frac{\partial V}{\partial t}(X_t, t) = 0 \tag{3-38}$$

it is clear that

$$\begin{aligned} &\frac{\partial V}{\partial x}(X_t, t) b(X_t, t) \\ =& \sum_{i=1}^{M} \left(\frac{\beta_i S_i(t) I_i(t) - \gamma_i I_i(t) + \sum_{j=1}^M \lambda_{ji} \frac{I_j(t)}{N_j} - \sum_{j=1}^M \lambda_{ij} \frac{I_i(t)}{N_i}}{\sum_{i=1}^M I_i(t)}\right) \\ =& \sum_{i=1}^{M} \left(\frac{\beta_i S_i(t) I_i(t) - \gamma_i I_i(t)}{\sum_{i=1}^M I_i(t)}\right) \end{aligned} \tag{3-39}$$

we also compute

$$\operatorname{Tr}\left(\sigma^\top(X_t, t) \frac{\partial^2 V}{\partial x^2}(X_t, t) \sigma(X_t, t)\right) = -\frac{\left(\sum_{i=1}^M \sigma_i S_i I_i\right)^2}{\left(\sum_{i=1}^M I_i(t)\right)^2} \tag{3-40}$$

finally

$$\frac{\partial V}{\partial x}(X_t, t)\sigma(X_t, t) = \begin{pmatrix} \frac{\sigma_1 S_1(t) I_1(t)}{\left(\sum_{i=1}^M I_i(t)\right)} & \cdots & \frac{\sigma_M S_M(t) I_M(t)}{\left(\sum_{i=1}^M I_i(t)\right)} \end{pmatrix} \tag{3-41}$$

by the Itô formula, we have that

$$
\begin{aligned}
&dV(X_t,t)\\
=&\sum_{i=1}^{M}\left(\frac{\beta_i S_i(t)I_i(t)-\gamma_i I_i}{\sum_{i=1}^{M} I_i(t)}-\frac{(\sigma_i S_i(t)I_i(t))^2}{2\left(\sum_{i=1}^{M} I_i(t)\right)^2}\right)dt\\
+&\sum_{i=1}^{M}\left(\sigma_i S_i(t)I_i(t)\frac{1}{\left(\sum_{i=1}^{M} I_i(t)\right)}dW_i\right)
\end{aligned}
\tag{3-42}
$$

integrating from 0 to t leads to

$$
\begin{aligned}
&V(X_t,t)\\
=&\sum_{i=1}^{M}\int_0^t\left(\left(\frac{\beta_i S_i(t)I_i(t)-\gamma_i I_i(t)}{\sum_{i=1}^{M} I_i(t)}\right)-\frac{(\sigma_i S_i(t)I_i(t))^2}{2\left(\sum_{i=1}^{M} I_i(t)\right)^2}\right)du\\
+&\sum_{i=1}^{M}\int_0^t\left(\frac{\sigma_i S_i(t)I_i(t)}{\sum_{i=1}^{M} I_i(t)}dW_i\right)+V(X_0,0)
\end{aligned}
\tag{3-43}
$$

if we divide for $t>0$ and taking lim sup, we get

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{V(X_t,t)}{t}\\
=&\limsup_{t\to\infty}\left[\frac{1}{t}\sum_{i=1}^{M}\int_0^t\left(\frac{\beta_i S_i(t)I_i(t)-\gamma_i I_i(t)}{\sum_{i=1}^{M} I_i(t)}-\frac{(\sigma_i S_i(t)I_i(t))^2}{2\left(\sum_{i=1}^{M} I_i(t)\right)^2}\right)du\right.\\
&\left.+\frac{1}{t}\sum_{i=1}^{M}\int_0^t\frac{\sigma_i S_i(t)I_i(t)}{\sum_{i=1}^{M} I_i(t)}dW_i+\frac{1}{t}V(X_0,0)\right]\\
\leq&\limsup_{t\to\infty}\sum_{i=1}^{M}\frac{1}{t}\int_0^t\left(\frac{\beta_i S_i(t)I_i(t)-\gamma_i I_i(t)}{\sum_{i=1}^{M} I_i(t)}-\frac{(\sigma_i S_i(t)I_i(t))^2}{2\left(\sum_{i=1}^{M} I_i(t)\right)^2}\right)du\\
&+\limsup_{t\to\infty}\sum_{i=1}^{M}\frac{1}{t}\int_0^t\frac{\sigma_i S_i(t)I_i(t)}{\sum_{i=1}^{M} I_i(t)}dW_i+\limsup_{t\to\infty}\frac{1}{t}V(X_0,0)\\
=&\limsup_{t\to\infty}\sum_{i=1}^{M}\frac{1}{t}\int_0^t\left(\frac{\beta_i S_i(t)I_i(t)-\gamma_i I_i(t)}{\sum_{i=1}^{M} I_i(t)}-\frac{(\sigma_i S_i(t)I_i(t))^2}{2\left(\sum_{i=1}^{M} I_i(t)\right)^2}\right)du\\
&+\limsup_{t\to\infty}\sum_{i=1}^{M}\frac{1}{t}\int_0^t\frac{\sigma_i S_i(t)I_i(t)}{\sum_{i=1}^{M} I_i(t)}dW_i
\end{aligned}
\tag{3-44}
$$

note that

$$\mathbf{E}\left(\left(\frac{\sigma_i S_i I_i}{\sum_{i=1}^M I_i(t)}\right)^2\right) \leq \mathbf{E}\left((\sigma_i N_i)^2\right) = (\sigma_i N_i)^2 < \infty \tag{3-45}$$

which implies that

$$\int_0^t \frac{\sigma_i S_i(t) I_i(t)}{\sum_{i=1}^M I_i(t)} dW_i \tag{3-46}$$

is a martingale by theorem 2.2.2 and using the law of large numbers for local martingales [[Mao, 2008]p.p 12]

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\sigma_i S_i I_i}{\sum_{i=1}^M I_i(t)} dW_i = 0 \tag{3-47}$$

on the other hand

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{i=1}^M \frac{1}{t} \int_0^t \left(\frac{\beta_i S_i(t) I_i(t) - \gamma_i I_i(t)}{\sum_{i=1}^M I_i(t)} - \frac{(\sigma_i S_i(t) I_i(t))^2}{2\left(\sum_{i=1}^M I_i(t)\right)^2}\right) du \\
\leq &\limsup_{t\to\infty} \sum_{i=1}^M \frac{1}{t} \int_0^t \left(\frac{\beta_i S_i(t) I_i(t) - \gamma_i I_i(t)}{\sum_{i=1}^M I_i(t)}\right) du \\
= &\limsup_{t\to\infty} \sum_{i=1}^M \frac{1}{t} \int_0^t \left((\beta_i S_i - \gamma_i)\left(\frac{I_i(t)}{\sum_{i=1}^M I_i(t)}\right)\right) du \\
\leq &\limsup_{t\to\infty} \sum_{i=1}^M \frac{1}{t} \int_0^t \left(\max_{i\in\{1,\dots,M\}} (\beta_i S_i - \gamma_i)\left(\frac{I_i(t)}{\sum_{i=1}^M I_i(t)}\right)\right) du \\
\leq &\limsup_{t\to\infty} M \max_{i\in\{1,\dots,M\}} (\beta_i S_i - \gamma_i) \frac{1}{t} \int_0^t \sum_{i=1}^M \left(\frac{I_i(t)}{\sum_{i=1}^M I_i(t)}\right) du \\
= &\limsup_{t\to\infty} M \max_{i\in\{1,\dots,M\}} (\beta_i S_i - \gamma_i) \frac{1}{t} \int_0^t 1 du \\
= &M \max_{i\in\{1,\dots,M\}} (\beta_i S_i - \gamma_i) < 0
\end{aligned} \tag{3-48}$$

The last inequality is just the hypothesis of the theorem, if we have that for every $i = 1, \dots, M$ $\frac{\beta_i N_i}{\gamma_i} < 1$ we can rewrite those inequalities as $\beta_i N_i - \gamma_i < 0$ for every i in particular $\max_{i\in\{1,\dots,N\}}(\beta_i N_i - \gamma_i) < 0$. □

Remark. Note that for a single region, the condition of the last theorem for stability coincides with the condition for the extinction of the pathogen based on the basic reproduction number $\frac{\beta N}{\gamma}$ [[Perasso, 2018]p.p 128] from deterministic epidemiology.

4. Numerical Exploration

4.1. Stratonovich Integral

In order to explain the method used to approximate the paths of the model (3-11), we have to define the Stratonovich integral in terms of the Itô integral defined in chapter 2.

Definition 16. Assume that for $i = 1, \ldots, m$

$$dY_i(t) = b_i(Y_i(t), t)dt + \sigma_i(Y_i(t), t)dW \tag{4-1}$$

where $b_i(t, Y_i(t)) \in \mathcal{W}^1([t_0, T]; \mathbb{R}^{d\times 1})$ and $\sigma_i(t, Y_i(i)) \in \mathcal{W}^2([t_0, T]; \mathbb{R}^{d\times m})$. We define the Stratonovich integral $\int_{t_0}^t Y_t \circ dW$, where Y_t is the $d \times m$ matrix-valued process whose i column is $Y_i(t)$, as the $\mathbb{R}^d$-valued process defined as

$$\int_{t_0}^t Y_s \circ dW_s = \int_{t_0}^t Y_s dW_s + \frac{1}{2}\int_{t_0}^t \sum_{i=1}^m (\sigma_i)_i ds \tag{4-2}$$

where $(\sigma_i)_i$ is the i-column of $\sigma_i(Y_i, t)$.

Also, a definition of a stochastic differential equation in Stratonovich's sense can be made. Now we present the formula to change from an SDE in Itô's sense to an SDE in Stratonovich's sense and vice versa. Assume that X_t is a global solution of a stochastic differential equation in Itô's sense

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW, \qquad X_{t_0} = x_0 \tag{4-3}$$

where $b(t, X_t) \in \mathcal{W}^1([t_0, T]; \mathbb{R}^{d\times 1})$ and $\sigma(t, X_t) \in \mathcal{W}^2([t_0, T]; \mathbb{R}^{d\times m})$, moreover assume that each entry of the function $\sigma(x, t)$ has up to 2 spatial continuous derivatives and 1 partial continuous derivative in t such that we can use theorem 2.2.4 to get $d(\sigma(X_t, t))_i$ for $i = 1, \ldots, n$ then using definition 16 we get that if $(X_t)_{t_0 \le t \le T}$ is a solution of equation (4-3) in Ito's sense, then $(X_t)_{t_0 \le t \le T}$ is also a solution of the following SDE in Stratonovich's sense

$$dX_t = \bar{b}(X_t, t)dt + \sigma(X_t, t) \circ dW, \qquad X_{t_0} = x_0 \tag{4-4}$$

where

$$\bar{b}(X_t, t) = b(X_t, t) - \frac{1}{2}\sum_{j=1}^d \frac{\partial}{\partial x_j}(X_t, t)(\sigma^\top)_j(X_t, t) \tag{4-5}$$

A similar formula can be obtained to get an SDE in Itô's from an SDE in Statonovich's sense. There is also a theorem about the existence and uniqueness of the solution for stochastic differential equations in Stratonovich's sense, which we state below

Theorem 4.1.1. *Consider the following SDE in Statonovich's sense*

$$dX_t = b(X_t, t)ds + \sigma(X_t, t) \circ dW_t \qquad X_{t_0} = x_0 \tag{4-6}$$

If $b : \mathbb{R}^n \times [t_0, t] \to \mathbb{R}^n$ is Lipschitz continuous in the spatial variable and $\sigma : \mathbb{R}^n \times [t_0, t] \to \mathbb{R}^{n \times d}$ is twice continuously differentiable in the spatial variable and continuously differentiable in the time variable. Also assume that $||\partial\sigma/\partial x_j(x, t)||_2$ are bounded for all $j = 1, 2, ..., n$ then the equation has a unique solution on $[t_0, T]$.

More information about the Stratonovich integral can be found in [[Londoño, 2020]p.p 50] and in [[Karatzas and Shreve, 1998]p.p 156].

4.2. The Wong-Zakai Method

Now we can describe the approximation procedure that we are going to use. We consider the following Stratonovich SDE

$$dX_t = b(X_t, t)ds + \sigma(X_t, t) \circ dW_t \qquad X_{t_0} = x_0 \tag{4-7}$$

for this equation we will approximate the solution at points $t_0 < \ldots < t_k = T$ of the interval $[t_0, T]$ as follows, let $\hat{X}_j$ the numerical approximation of X_{t_j}. For $\hat{X}_0 = X_{t_0}$ and for each sub interval $[t_j, t_{j+1}]$, $j = 0, \ldots, k-1$, $\hat{X}_{j+1}$ will be calculated as the solution at time t_{j+1} of the following initial value problem

$$\frac{d\hat{X}(t)}{dt} = b(\hat{X}(t), t) + \frac{1}{\Delta_j}\sigma(\hat{X}(t))\Delta W_j \qquad \hat{X}(t_j) = \hat{X}_j \tag{4-8}$$

where $\Delta_j = t_{j+1} - t_j$ and $\Delta W_j = W_{t_{j+1}} - W_{t_j}$.

The main feature of this method is that it enables us to use robust methods already developed for ODEs. More information about this method and evaluation of its numerical performance can be found in [[Londoño and Villegas, 2016]].

4.3. Implementation of the Wong-Zakai Method

We first convert the Itô stochastic SDE (3-11) to Stratonovich form to implement the numerical method described. To do that, we use equation (4-5) to get the Stratonovich SDE:

$$dX_t = \bar{b}(X_t, t)dt + \sigma(X_t, t) \circ dW, \qquad X_{t_0} = x_0 \tag{4-9}$$

where

$$\bar{b}(X_t,t) = \begin{pmatrix} -\beta_1 S_1(t)I_1(t) + \sum_{j=1}^{M}\lambda_{j1}\frac{S_j(t)}{N_j} - \sum_{j=1}^{M}\lambda_{1j}\frac{S_1(t)}{N_1} + \frac{\sigma_1^2 S_1^2(2)I_1(t) - \sigma_1^2 S_1(t)I_1^2(t)}{2} \\ \beta_1 S_1(t)I_1(t) - \gamma I_1(t) + \sum_{j=1}^{M}\lambda_{j1}\frac{I_j(t)}{N_j} - \sum_{j=1}^{M}\lambda_{1j}\frac{I_1(t)}{N_1} + \frac{\sigma_1^2 S_1(t)I_1^2(t) - \sigma_1^2 S_1^2(t)I_1(t)}{2} \\ \gamma_1 I_1(t) + \sum_{j=1}^{M}\lambda_{j1}\frac{R_j(t)}{N_j} - \sum_{j=1}^{M}\lambda_{1j}\frac{R_1(t)}{N_1} \\ \vdots \\ \gamma_M I_M(t) + \sum_{j=1}^{M}\lambda_{jM}\frac{R_j(t)}{N_j} - \sum_{j=1}^{M}\lambda_{Mj}\frac{R_M(t)}{N_M} \end{pmatrix} \tag{4-10}$$

$$\sigma(X_t,t) = \begin{pmatrix} -\sigma_1 S_1(t)I_1(t) & 0 & \dots & 0 \\ \sigma_1 S_1(t)I_1(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & -\sigma_2 S_2(t)I_2(t) & \dots & 0 \\ 0 & \sigma_2 S_2(t)I_2(t) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\sigma_M S_M(t)I_M(t) \\ 0 & 0 & \dots & \sigma_M S_M(t)I_M(t) \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{4-11}$$

We implement the Wong-Zakai method as described in section 4.2 on the Julia programming language [Bezanson et al., 2017] due to its performance and the availability of a robust package to solve differential equations [Rackauckas and Nie, 2017]. We also use the package Distributions [Besançon et al., 2019] to get a probability distribution for the simulated data; finally, we use Plotly [Plotly Technologies, 2015] to plot the trajectories of the solution. The code used can be found in appendix B.

Now we present some simulations. We will use the official reported data for the covid-19 in France, Germany, and Italy. Also, we use parameters reported in the literature for the SIR model, calibrated for each country, and reported tourism data between each country for the mobility parameters. First, we perform some deterministic simulations for the model (3-4). Then, we show the simulations for the proposed stochastic model (3-11), for which we calculated the parameters controlling the randomness of the model σ_i by choosing those that give the maximum likelihood with respect to the reported data.

Now we present the parameters used for the simulations

Table 4-1.: Parameters for the infectious process for Italy, Germany, and France before the first lockdown as reported by [Simha et al., 2020].

Country	N	β	γ
Italy	6.055×10^7	4.5×10^{-9}	0.04
Germany	8.28×10^7	4.6×10^{-9}	0.005
France	6.7×10^7	4.4×10^{-9}	0.04

Table 4-2.: Transition parameters calculated from the total number of tourists between countries [DIRECTION GÉNÉRALE DES ENTREPRISES, 2018, Sistema statistico nazionale Istituto nazionale di Statistica, 2019, WORLD TOURISM ORGANIZATION, 2021].

	Italy	Germany	France
Italy	0	44227	32157
Germany	44227	0	43776
France	32157	43776	0

Table 4-3.: Initial conditions at $t = 0$, which represents the date 24-02-20 as reported by [Johns Hopkins University of Medicine, 2020].

Country	S_0	I_0	R_0
Italy	60549771	220	9
Germany	82799987	2	11
France	66999991	1	8

We used subscripts to maintain the usual notation for stochastic processes. Thus I_0 represents the initial value of infectives for each region. Also, R_0 denotes the number of individuals who are no longer infective to other individuals; this includes the recovered and dead individuals as in [[Simha et al., 2020]]; in this case, do not confuse R_0 with the basic reproductive number for the infection in each region. Finally, S_0 is the initial value for the susceptible individuals, calculated as $S_0 = N - I_0 - R_0$ where N total number of individuals as reported by [[Simha et al., 2020]].

First, we show the deterministic simulations. It is important to note that we plot the natural logarithm of the solutions given by the numerical method, and we compare it with the natural logarithm of the reported data to compensate for the exponential nature of the solutions.

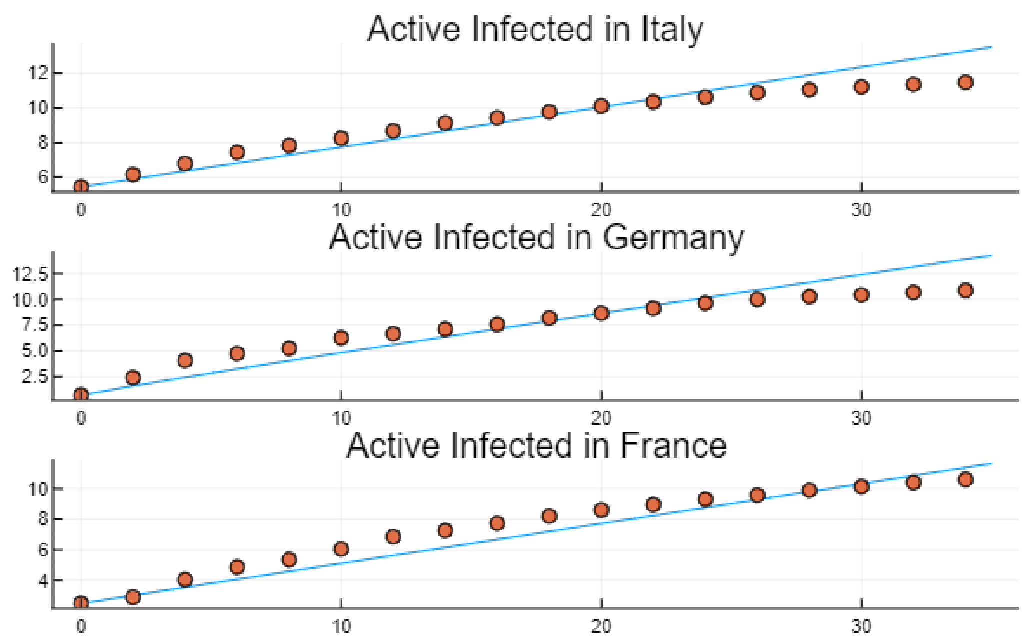

Figure 4-1.: Graphs of the natural logarithm of the deterministic case $\sigma_1 = \sigma_2 = \sigma_3 = 0$ orange dots represent the natural logarithm of the observed data [Johns Hopkins University of Medicine, 2020].

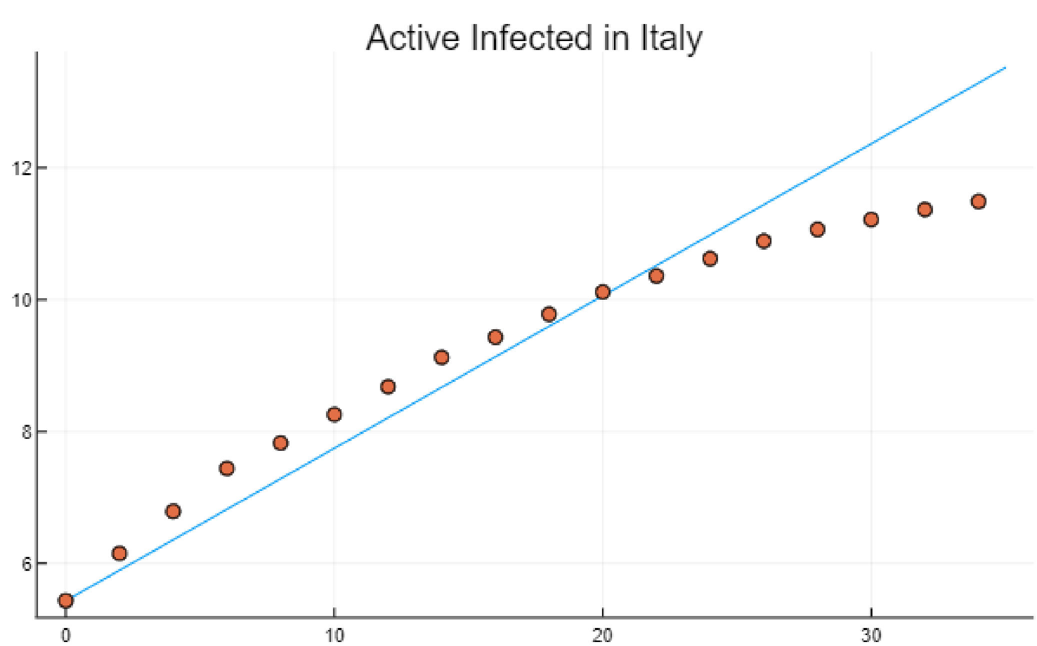

Figure 4-2.: Graph of the natural logarithm of the deterministic case for Italy.

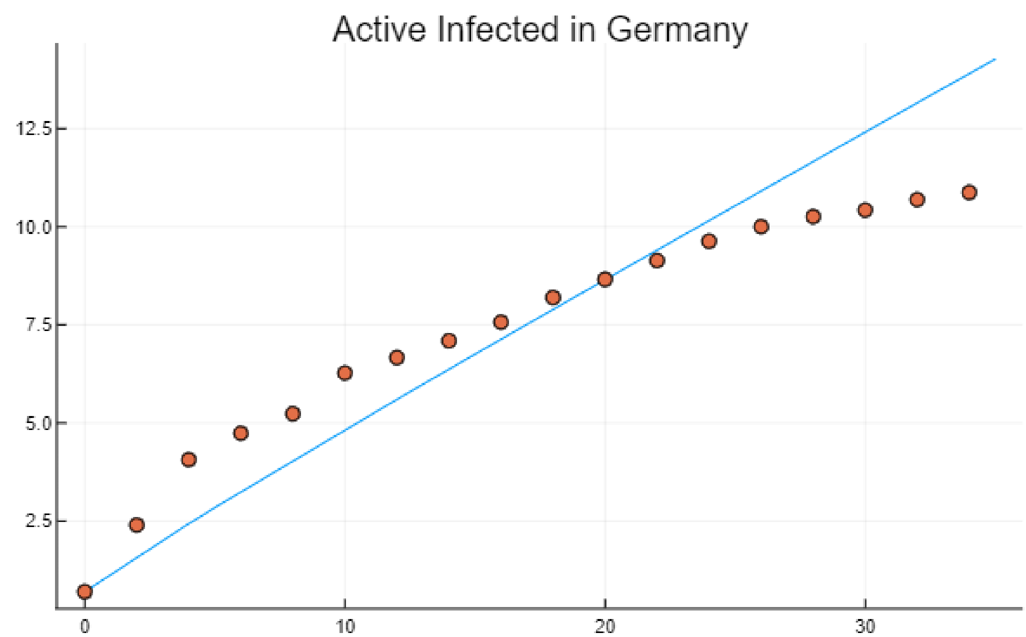

Figure 4-3.: Graph of the natural logarithm of the deterministic case for Germany.

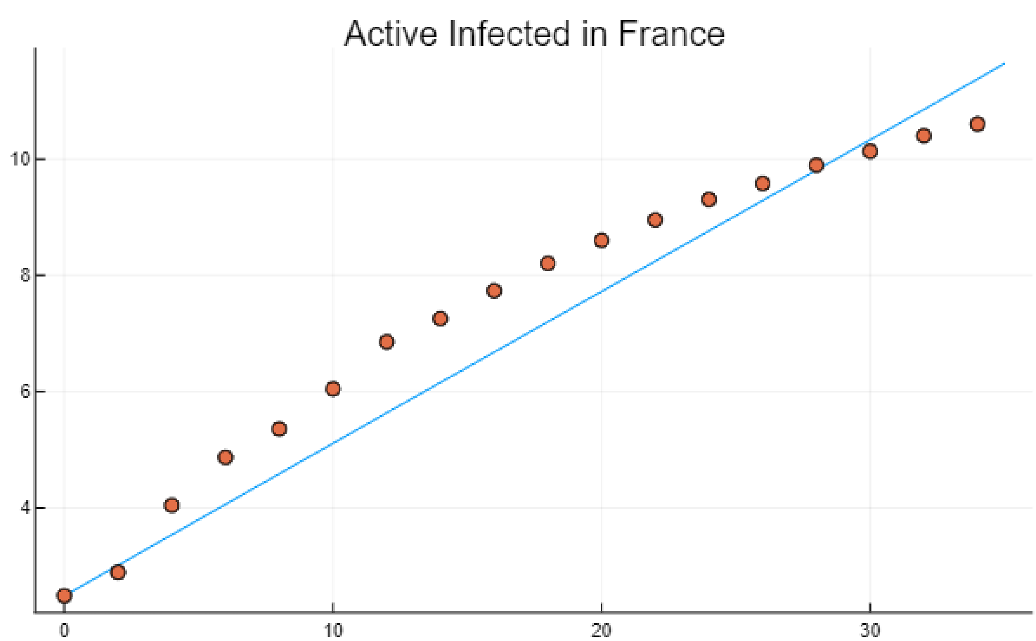

Figure 4-4.: Graph of the natural logarithm of the deterministic case for France.

Note that in figures **4-1**,**4-2**,**4-3** and **4-4**, the observed data and the simulations go in different directions at $t = 25$, which is the date 20-03-20; one explanation for that phenomenon is that the measures taken by the government started to be noticeable by those days. The first lockdown in Italy took place on the 9th of March; in France, it was on the 17th of March; in Germany, it was decreed on the 23rd of March. As a result, the parameters β_i have to be modified for each region to compensate for the lockdowns [Karnakov et al., 2020, Godio et al., 2020].

Now we present some simulations for the stochastic model, the parameters for mobility between countries are the same as in table **4-2**, and initial conditions are the same as in table **4-3**. We keep the recovery rates γ_i and infection rates β_i the same as in the deterministic case; we estimate the parameters σ_i, using the simulated maximum likelihood procedure described in [[Hurn et al., 2003]]. Also, we include the code used to estimate those parameters in appendix B.

Table 4-4.: Parameters for the infectious process for Italy, Germany, and France used for the stochastic model.

Country	N	β	γ	σ
Italy	6.055×10^7	4.5×10^{-9}	0.04	2×10^{-9}
Germany	8.28×10^7	4.6×10^{-9}	0.005	4×10^{-9}
France	6.7×10^7	4.4×10^{-9}	0.04	2×10^{-9}

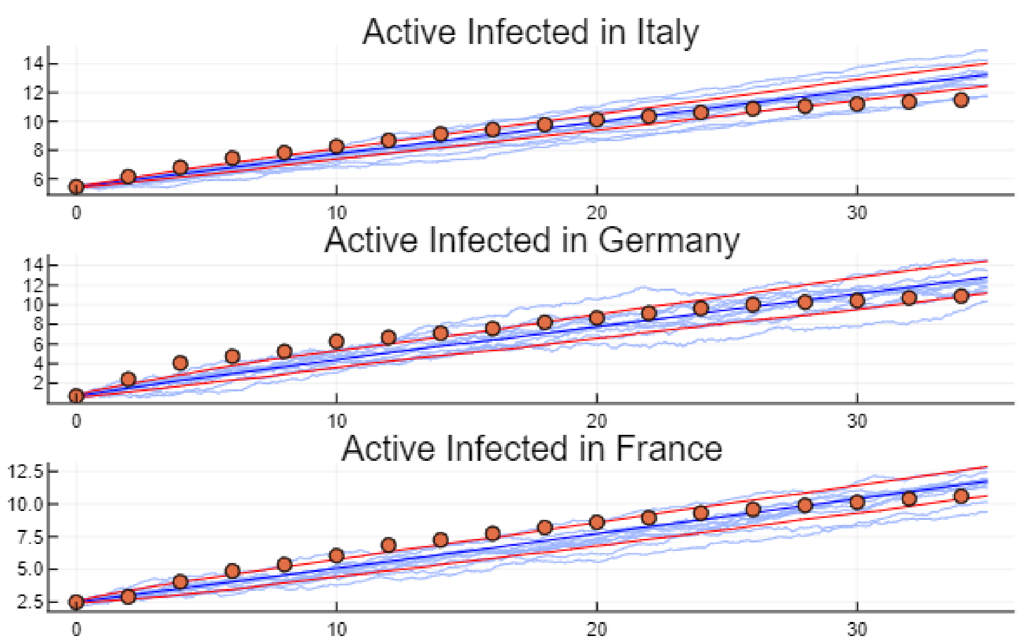

Figure 4-5.: Graphs of the natural logarithm of the stochastic case with parameters as in table **4-4** orange dots represent the natural logarithm of the observed data [Johns Hopkins University of Medicine, 2020].

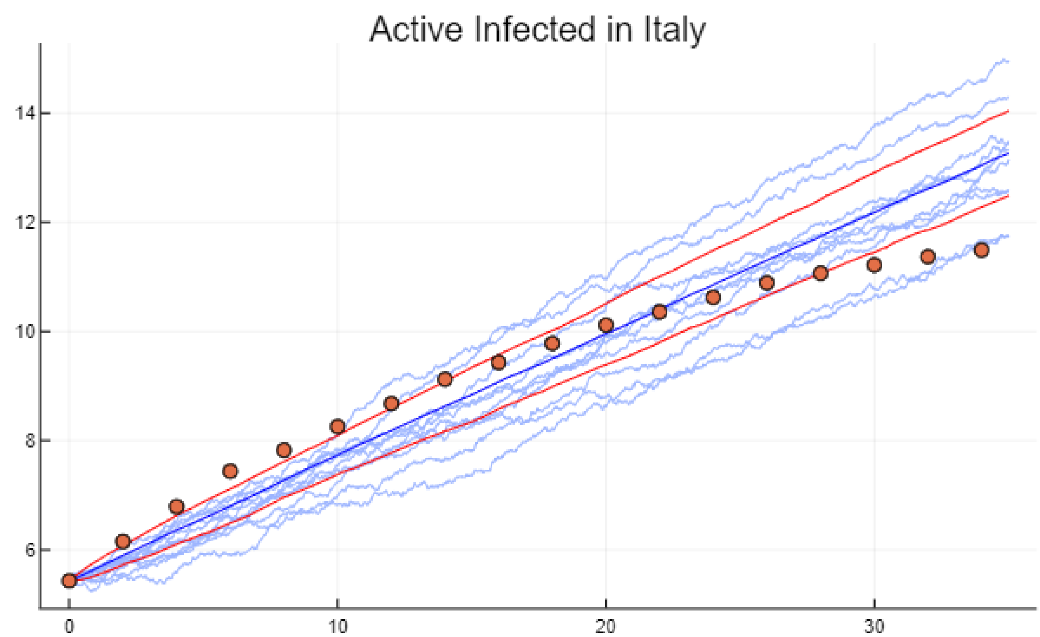

Figure 4-6.: Graph of the natural logarithm of the stochastic case for Italy.

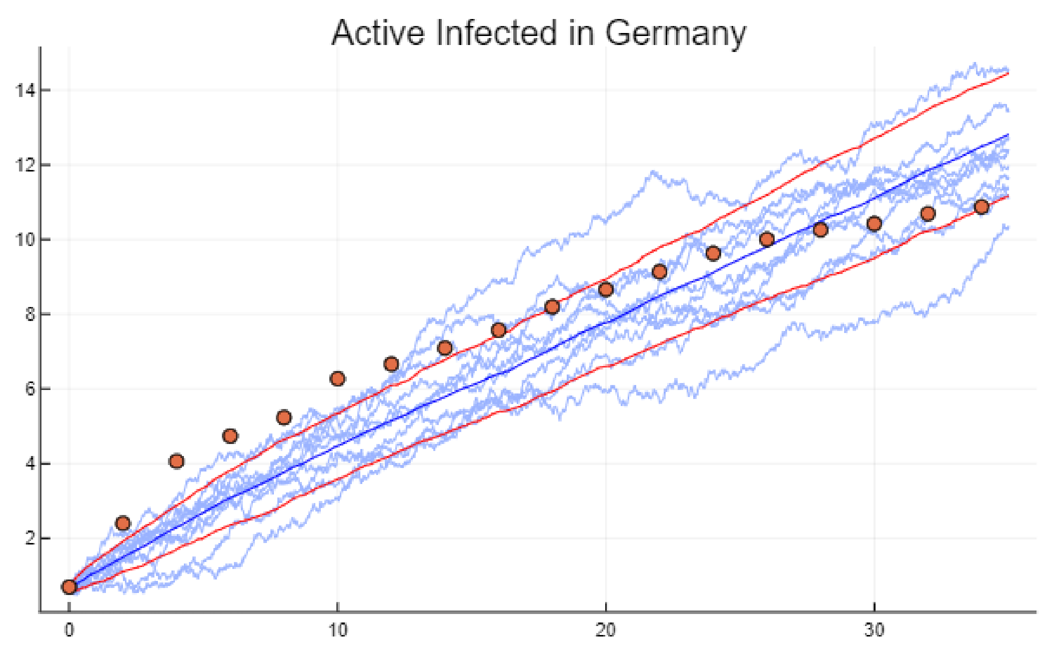

Figure 4-7.: Graph of the natural logarithm of the stochastic case for Germany.

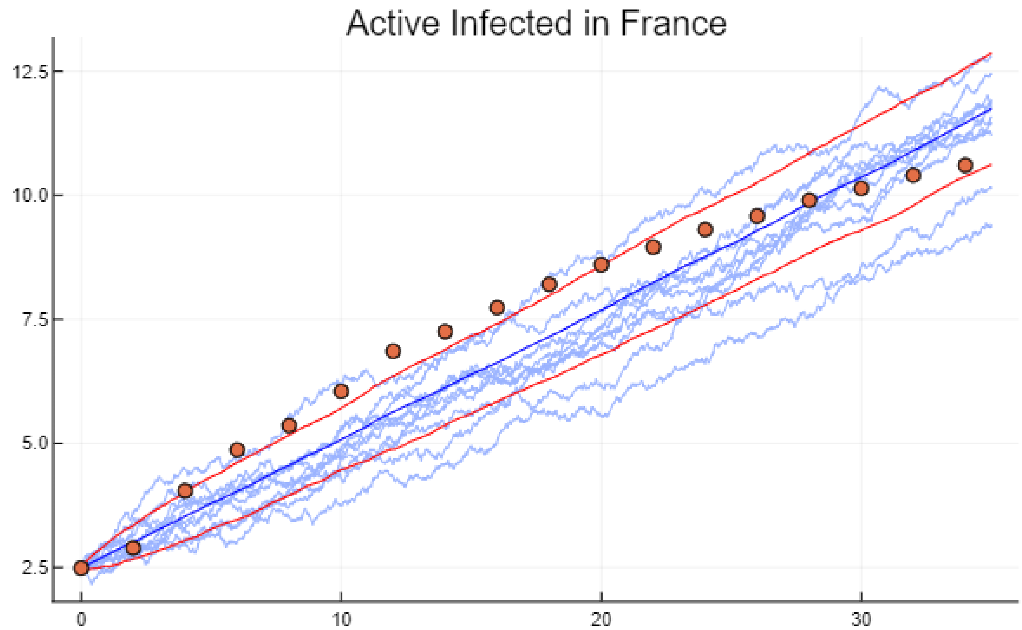

Figure 4-8.: Graph of the natural logarithm of the stochastic case for France.

Figures **4-5**, **4-6**, **4-7**, and **4-8** show the natural logarithm of the sample paths for multiple simulations using the Wong-Zakai method described in section 4.2. We highlighted the mean value of the natural logarithm of the simulated trajectories. Also, we plotted in red the mean value of the trajectories plus and minus the standard deviation of the natural logarithm of the sample phats. Note that most of the observations are located between the mean plus and minus the standard deviation of the sample phats. If we want to make predictions with this model, we can make various simulations to obtain a probability density for the values of the solution at a given time.

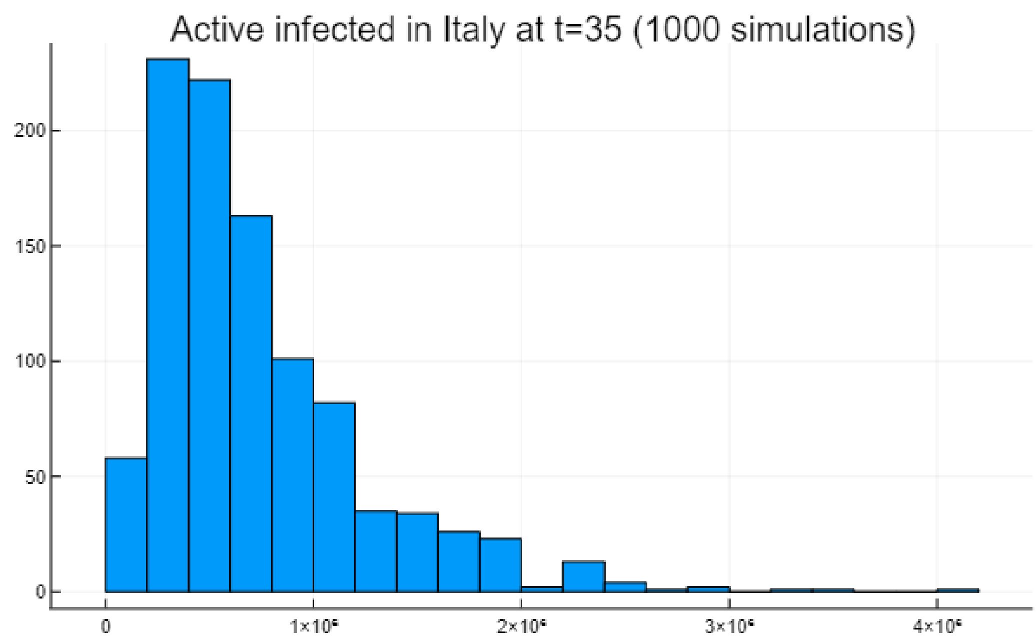

Figure 4-9.: Histogram for the number of active infections in Italy on the 30th March 2020

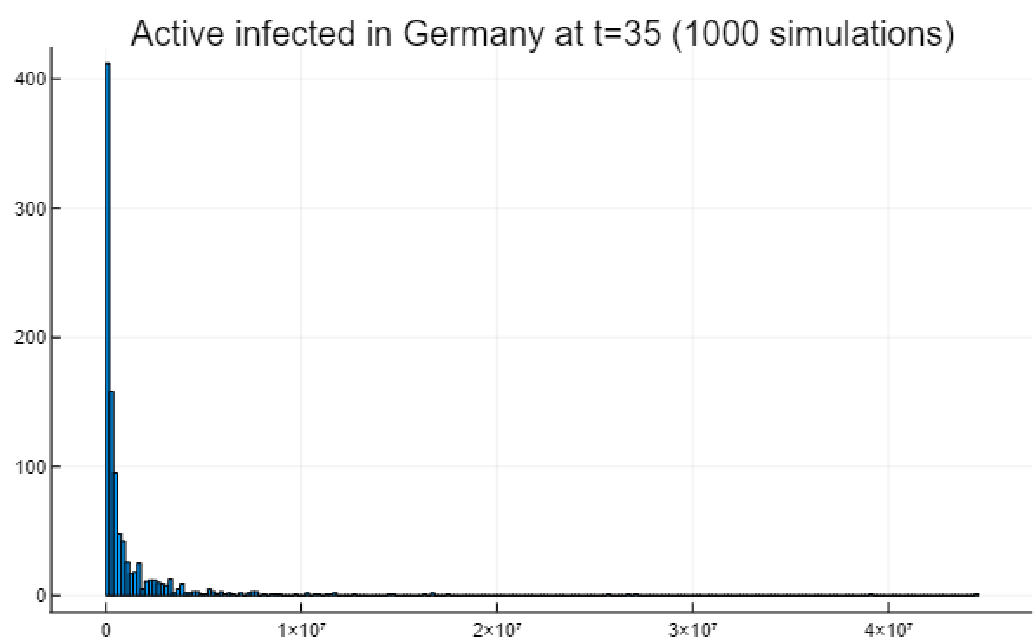

Figure 4-10.: Histogram for the number of active infections in Germany on the 30th March 2020

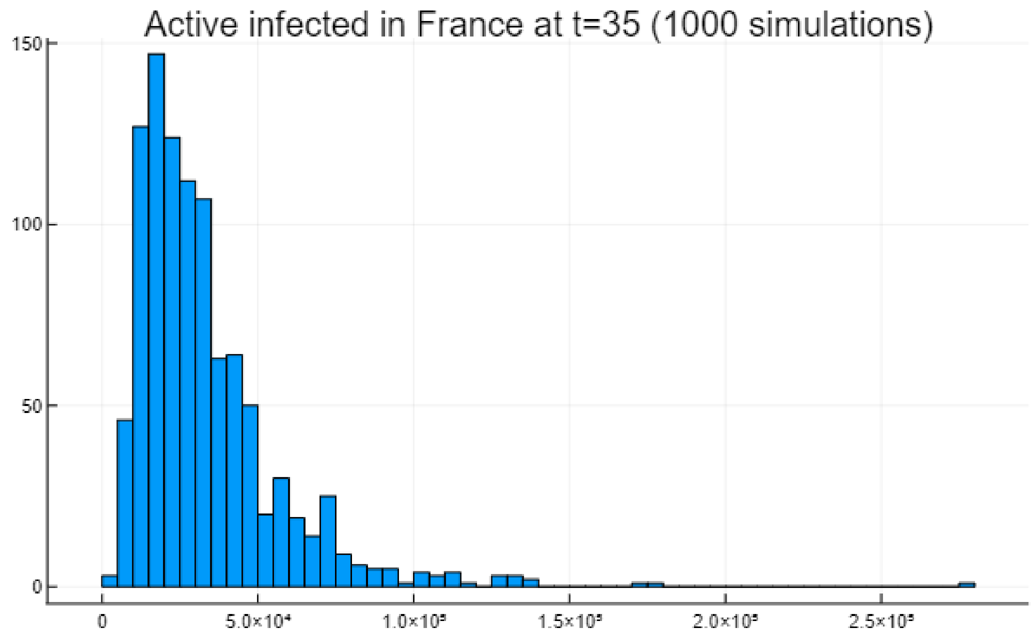

Figure 4-11.: Histogram for the number of active infections in France on the 30th March 2020

We use information from **4-9**, **4-10**, and **4-11** to fit a normal random variable for the number of infected individuals in each country. This information could be helpful to make predictions based on this model.

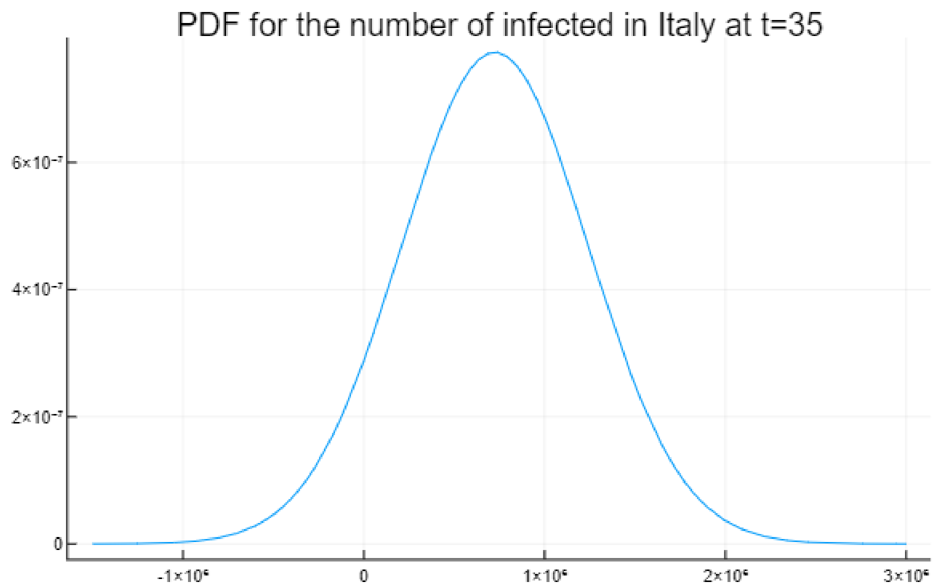

Figure 4-12.: Fitted PDF for Italy with parameters μ=724823.108, σ=515918.925

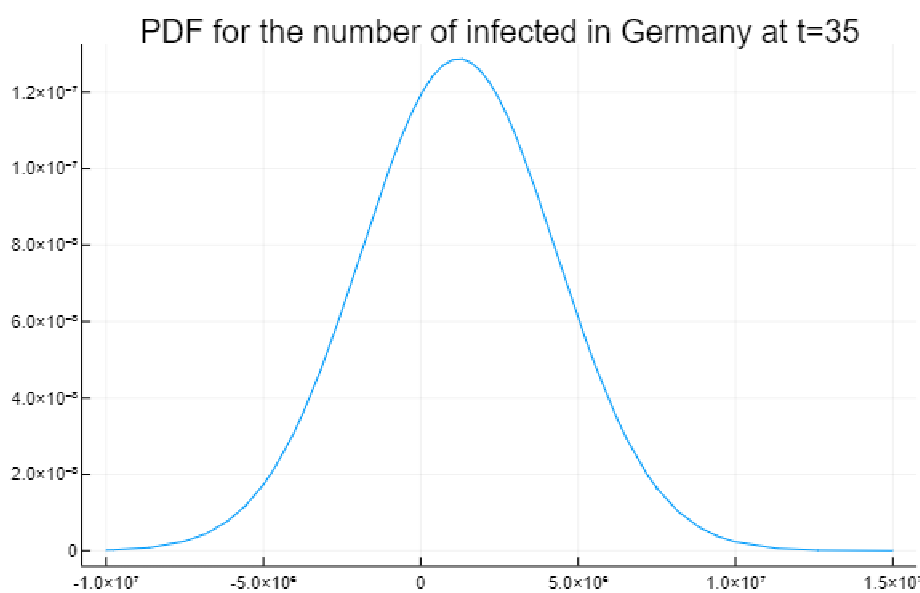

Figure 4-13.: Fitted PDF for Germany with parameters μ=1210347.391, σ=3099983.360

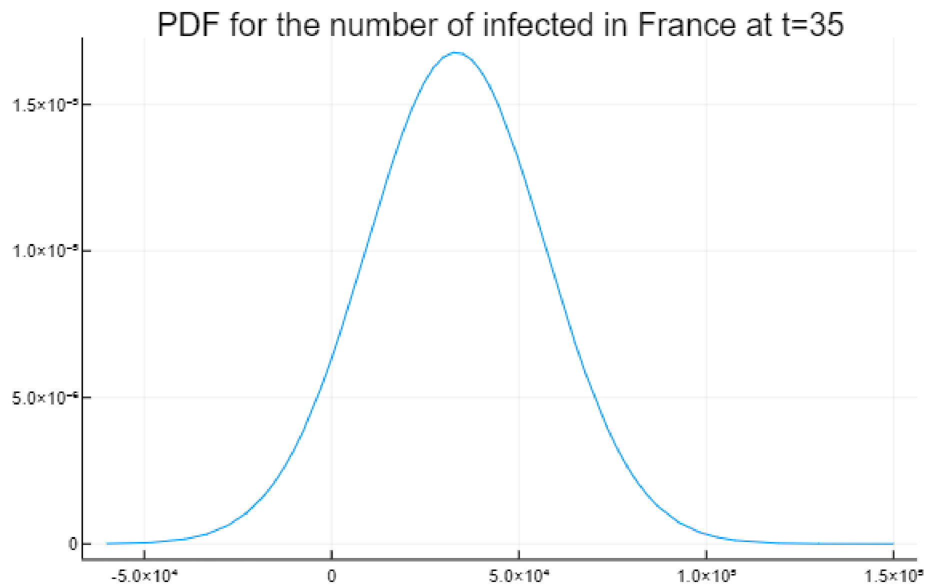

Figure 4-14.: Fitted PDF for France with parameters μ=33203.205, σ=23772.842

5. Conclusions

This study proposed a new epidemiological stochastic model on multiple regions with transport. We have demonstrated the feasibility of the model by demonstrating the existence, uniqueness, and positivity of the solution. Also, we presented an asymptotic property for the number of infected individuals in each region, and we related that result with the basic reproduction number in the single region case. Moreover, deterministic models found in the literature were compared. Likewise, from the simulations, it was possible to achieve a probability density function for the process variables, which can be used to make future predictions about the variables of the process when we calibrate with real data.

For further research, it is recommended to refine the condition of almost asymptotic stability of theorem 3.3.1, for instance, by providing a condition that considers the parameters controlling the randomness of the model. Furthermore, we recommend studying the stochastic extension of the deterministic model with different incidence rates like those presented in [[Kiran et al., 2020, Irwin, 2008]]. It is also suggested to continue the extension to several regions from previous studies conducted with stochastic models for a single region, e.g., including the compartments for individuals in quarantine and asymptomatic as mentioned in [[Liu et al., 2019, Aràndiga et al., 2020]], considering different age groups inside the same region with a similar model as proposed by [[Ji et al., 2011, Liu and Jiang, 2019, Cao et al., 2020]]. It may also be recommended to include dynamics of life such as births and deaths as presented in [[Anqi Miao et al., 2018]].

A. Additional Definitions

Definition 17. Let $(X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$. Then we say that $(X_t)_{t\geq 0}$ is continuous in probability if

$$P(\{\omega : \lim_{s\to t} |X_s - X_t| = 0\}) = 1 \text{ for all } t \tag{A-1}$$

Definition 18. Suppose that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $(X_t)_{t\geq 0}$ is a version of (or a modification of) $(Y_t)_{t\geq 0}$ if

$$P(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 \text{ for all } t \tag{A-2}$$

Definition 19. Consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$ let T be the index set for the time and $\overline{T}$ its closure on $[-\infty, \infty]$ we say that a random variable τ with values in $\overline{T}$ is called a stopping time if for each $t \in T$ the event $\{\tau \leq t\} \in \mathcal{F}_t$

B. Code Listings

gen-sim.jl

```
using DifferentialEquations
using Plots
include("fun-sim.jl")
tin = 0.0 #initial time
tfi = 30 #final time
delt = 0.001 #internal delta
delm = 0.01 #sampling delta

#initial condition vector S1,I1,R1,S2,I2,R2,S3,I3,R3
u0 = [60549771,220,9,82799987,2,11,66999991,1,8]
# model parameters
# beta1,gamma1,sigma1,lambda12,lambda21,
# beta2,gamma2,sigma2,lambda23,lambda32,
# beta3,gamma3,sigma3,lambda13,lambda31
Par = [0.0000000045,0.04,0.000000001,44227,44227,
       0.0000000046,0.005,0.000000001,43776,43776,
       0.0000000044,0.04,0.000000001,32157,32157]
times,solution = simul(tin,tfi,delt,delm,u0,Par)


plotly() # Using the Plotly backend

ps1=plot(times,solution[:,1])
pi1=plot(times,solution[:,2])
pr1=plot(times,solution[:,3])

ps2=plot(times,solution[:,4])
pi2=plot(times,solution[:,5])
pr2=plot(times,solution[:,6])

ps3=plot(times,solution[:,7])
pi3=plot(times,solution[:,8])
pr3=plot(times,solution[:,9])

l = @layout [
grid(3,1) grid(3,1) grid(3,1)
]
plot(ps1,pi1,pr1,ps2,pi2,pr2,ps3,pi3,pr3,linewidth=1,xaxis=" ",legend=
```

```
    false,title=["S1" "I1" "R1" "S2" "I2" "R2" "S3" "I3" "R3"],layout=l
    )
```

gen-stats.jl

```
using DifferentialEquations
using Plots
using Distributions
include("fun-sim.jl")
tin = 0.0 #initial time
tfi = 30.0 #final time
delt = 0.001 #internal delta
delm = 0.01 #sampling delta
Ntra = 1000 #number of simulations
#initial condition vector S1,I1,R1,S2,I2,R2,S3,I3,R3
u0 = [60549771,220,9,82799987,2,11,66999991,1,8]
# model parameters
# beta1,gamma1,sigma1,lambda12,lambda21,
# beta2,gamma2,sigma2,lambda23,lambda32,
# beta3,gamma3,sigma3,lambda13,lambda31
Par = [0.0000000045,0.04,0.000000002,44227,44227,
      0.0000000046,0.005,0.000000004,43776,43776,
      0.0000000044,0.04,0.000000002,32157,32157]

Finalsize = Array{Float64,2}(undef,3,Ntra)
for i=1:Ntra
      times,solution = simul(tin,tfi,delt,delm,u0,Par)
      Finalsize[1,i]=solution[Int(((tfi-tin)/delm)+1),:][3]
      Finalsize[2,i]=solution[Int(((tfi-tin)/delm)+1),:][6]
      Finalsize[3,i]=solution[Int(((tfi-tin)/delm)+1),:][9]
end
plotly()
histogram(Finalsize[1,:], label=false, title="Italy")
histogram(Finalsize[2,:], label=false, title="Germany")
histogram(Finalsize[3,:], label=false, title="France")
dit=fit_mle(Normal, Finalsize[1,:])
dge=fit_mle(Normal, Finalsize[2,:])
dfr=fit_mle(Normal, Finalsize[3,:])
pit=plot([x -> pdf(dit, x)],30000, 55000, title="Probability density
    function", label=false)
pge=plot([x -> pdf(dge, x)],30000, 55000, title="Probability density
    function", label=false)
pfr=plot([x -> pdf(dfr, x)],30000, 55000, title="Probability density
    function", label=false)
```

parameter-fit.jl

```
tin = 0.0 #initial time
tfi = 1 #final time
delt = 0.001 #internal delta
```

```
delm = 1 #sampling delta

Nit = 60550000
it =[220,310,455,593,822,1049,1578,1837,2265,2709,3299,
3919,5064,6391,7991,8518,10593,12842,14958,17753,
20607,23077,26066,28711,33191,37859,42672,46625,
50396,53995,57469,61956,66352,69997,73806,75444]

reit = [9,13,15,62,67,79,124,201,239,383,562,720,822,989,
1188,1638,1876,2280,2712,3416,4155,4920,5458,7021,
7865,9185,10926,12533,13545,15186,16910,18625,20136,
22464,23870,26279]

Nge = 82800000
ge = [2,3,11,32,58,63,114,149,187,246,528,652,782,1022,
1204,1545,1938,2714,3621,4544,5754,7188,9274,12194,
15161,19600,22071,24513,28480,29542,33570,37998,43862,
48781,52683,52740]

rege = [11,12,12,13,13,13,13,13,13,13,14,15,15,15,17,
17,25,28,51,52,56,81,90,130,156,245,290,357,573,
3446,3750,5937,7006,8911,9749,14142]

Nfr = 67000000
fr = [1,2,5,24,42,80,107,162,181,248,371,578,840,1073,
1247,1582,2019,2546,3242,3991,4795,5874,6254,7442,
8332,9431,10997,11689,14991,15894,17706,19857,
22268,26140,26708,29542]

refr = [8,8,9,9,9,10,10,11,12,12,15,17,24,27,38,41,56,
69,87,99,135,156,773,862,1663,2033,2145,2870,
3056,4377,5227,6640,7691,8010,9804,10947]


simulations = 1000
Finalsize = Array{Float64,2}(undef,(3,simulations))
Finalsum = 0
paramit = 0
paramge = 0
paramfr = 0
previoussum = -Inf
#loop for the parameters
for i in 1:9
    print(i)
    print("\n")
    for g in 1:9
        for f in 1:9
            si = i/1000000000
            sg = g/1000000000
```

```
            sf = f/1000000000
                 # model parameters
                 # beta1,gamma1,sigma1,lambda12,lambda21,
                 # beta2,gamma2,sigma2,lambda23,lambda32,
                 # beta3,gamma3,sigma3,lambda13,lambda31
            Par = [0.0000000045,0.04,si,44227,44227,
                   0.0000000046,0.005,sg,43776,43776,
                   0.0000000044,0.04,sf,32157,32157];
            Finalsum = 0
            #loop for the times
            for tiemp in 1:25
                #initialcondition vector
                #S1,I1,R1,S2,I2,R2,S3,I3,R3
                u0 = [60550000-it[tiemp]-reit[tiemp],it[tiemp],reit[
                    tiemp],82800000-ge[tiemp]-rege[tiemp],ge[tiemp],rege[
                    tiemp],67000000-fr[tiemp]-refr[tiemp],fr[tiemp],refr[
                    tiemp]];
                #for para las simulaciones
                for simulation in 1:simulations
                    times,solution = simul(tin,tfi,delt,delm,u0,Par);
                    Finalsize[:,simulation] = [solution[end,2],solution[
                        end,5],solution[end,8]]
                end
                d = kde!(Finalsize)
                current = log(d(reshape(float([it[tiemp+1],ge[tiemp+1],
                    fr[tiemp+1]]),3,1))[1])
                if current == -Inf
                    Finalsum = Finalsum - 1000
                else
                    Finalsum = Finalsum + log(d(reshape(float([it[tiemp
                        +1],ge[tiemp+1],fr[tiemp+1]]),3,1))[1])
                end
            end
            if Finalsum > previoussum
                previoussum = Finalsum
                paramit = i
                paramge = g
                paramfr = f
            end
        end
    end
end
print(paramit)
print(paramge)
print(paramfr)
```

fun-sim.jl

```
function simul(tin,tfi,delt,delm,u0,Par)
```

```
    times = Array(tin:delm:tfi)
    solution = Array{Float64,2}(undef,Int(((tfi-tin)/delm)+1),9)
    solution[1,:] = u0
    p=Array{Float64,1}(undef,19)
    p[1:16] = [Par[1],Par[2],Par[3],Par[4],Par[5],Par[6],Par[7],Par[8],
              Par[9],Par[10],Par[11],Par[12],Par[13],Par[14],Par[15],
                  delt]
    k = 2
    for j in tin:delm:tfi-delt
        for i in 0:delt:delm-delt
            p[17:19] = randn(3)
            tspan=(j+i,j+i+delt)
            prob = ODEProblem(f,u0,tspan,p)
            sol = solve(prob,Feagin14())#Here we can change the solver for the
                package differential equations
            u0=last(sol)
        end
    solution[k,:] = u0
    k = k+1
    end
    return times,solution
end
function f(du,u,p,t)
    S1,I1,R1,S2,I2,R2,S3,I3,R3 = u
    beta1,gamma1,sigma1,lambda12,lambda21,beta2,gamma2,sigma2,lambda23,
        lambda32,beta3,gamma3,sigma3,lambda13,lambda31,delt,Nor1,Nor2,
        Nor3 = p

    du[1]=(I1*S1^2*sigma1^2)/2 - (I1^2*S1*sigma1^2)/2 - I1*S1*beta1 - (
        S1*lambda12)/(I1 + R1 + S1) - (S1*lambda13)/(I1 + R1 + S1) + (S2
        *lambda21)/(I2 + R2 + S2) + (S3*lambda31)/(I3 + R3 + S3) - (I1*
        Nor1*S1*sigma1)/delt^(1/2)
    du[2]=(I1^2*S1*sigma1^2)/2 - (I1*S1^2*sigma1^2)/2 - I1*gamma1 + I1*
        S1*beta1 - (I1*lambda12)/(I1 + R1 + S1) - (I1*lambda13)/(I1 + R1
         + S1) + (I2*lambda21)/(I2 + R2 + S2) + (I3*lambda31)/(I3 + R3 +
         S3) + (I1*Nor1*S1*sigma1)/delt^(1/2)
    du[3]=I1*gamma1 - (R1*lambda12)/(I1 + R1 + S1) - (R1*lambda13)/(I1
        + R1 + S1) + (R2*lambda21)/(I2 + R2 + S2) + (R3*lambda31)/(I3 +
        R3 + S3)
    du[4]=(I2*S2^2*sigma2^2)/2 - (I2^2*S2*sigma2^2)/2 - I2*S2*beta2 + (
        S1*lambda12)/(I1 + R1 + S1) - (S2*lambda21)/(I2 + R2 + S2) - (S2
        *lambda23)/(I2 + R2 + S2) + (S3*lambda32)/(I3 + R3 + S3) - (I2*
        Nor2*S2*sigma2)/delt^(1/2)
    du[5]=(I2^2*S2*sigma2^2)/2 - (I2*S2^2*sigma2^2)/2 - I2*gamma2 + I2*
        S2*beta2 + (I1*lambda12)/(I1 + R1 + S1) - (I2*lambda21)/(I2 + R2
         + S2) - (I2*lambda23)/(I2 + R2 + S2) + (I3*lambda32)/(I3 + R3 +
         S3) + (I2*Nor2*S2*sigma2)/delt^(1/2)
    du[6]=I2*gamma2 + (R1*lambda12)/(I1 + R1 + S1) - (R2*lambda21)/(I2
        + R2 + S2) - (R2*lambda23)/(I2 + R2 + S2) + (R3*lambda32)/(I3 +
```

```
        R3 + S3)
    du[7]=(I3*S3^2*sigma3^2)/2 - (I3^2*S3*sigma3^2)/2 - I3*S3*beta3 + (
        S1*lambda13)/(I1 + R1 + S1) + (S2*lambda23)/(I2 + R2 + S2) - (S3
        *lambda31)/(I3 + R3 + S3) - (S3*lambda32)/(I3 + R3 + S3) - (I3*
        Nor3*S3*sigma3)/delt^(1/2)
    du[8]=(I3^2*S3*sigma3^2)/2 - (I3*S3^2*sigma3^2)/2 - I3*gamma3 + I3*
        S3*beta3 + (I1*lambda13)/(I1 + R1 + S1) + (I2*lambda23)/(I2 + R2
         + S2) - (I3*lambda31)/(I3 + R3 + S3) - (I3*lambda32)/(I3 + R3 +
         S3) + (I3*Nor3*S3*sigma3)/delt^(1/2)
    du[9]=I3*gamma3 + (R1*lambda13)/(I1 + R1 + S1) + (R2*lambda23)/(I2
        + R2 + S2) - (R3*lambda31)/(I3 + R3 + S3) - (R3*lambda32)/(I3 +
        R3 + S3)

end
```

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