Algorithms of Differentiation for Posets with an Involution

Verónica Cifuentes Vargas

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, D.C.
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Verónica Cifuentes Vargas

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Advisor
Raymundo Bautista Ramos, Ph.D.
Emeritus Professor, National Autonomous University of Mexico

Coadvisor
Agustín Moreno Cañadas, Ph.D.
Associate Professor, National University of Colombia

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Title in English

Algorithms of differentiation for posets with an involution.

Titulo en español:
Algoritmos de diferenciación para poset con involución.

Abstract: In the last decades, the study and classification of finite-dimensional algebras with respect to their representation type has been one of the main aims in the theory of representations of algebras. Nazarova, Roiter, Zavadskij and Bondarenko have introduced and studied several classes of representations associated to partially ordered sets (posets). Here we are interested, on the one hand, in the category of representations of a poset with an equivalence relation, where the equivalence sets have at most two elements; these kind of posets are called posets with an involution. We give a natural exact structure for the category of representations of this kind of posets, describe the projective, injective objects and prove the existence of almost split sequences. On the other hand, we study the categorical properties of the differentiation algorithms DI and DIII introduced by Zavadskij in 1991.

Resumen. En las últimas décadas, el estudio y clasificación de álgebras de dimensión finita con respecto a su tipo de representación ha sido uno de los principales objetivos en la teoría de representaciones de álgebras. Nazarova, Roiter, Zavadskij y Bondarenko introdujeron y estudiaron distintas clases de representaciones asociadas a conjuntos parcialmente ordenados (posets). Aquí estamos interesados, de una parte, en la categoría de representaciones de conjuntos parcialmente ordenados con una relación de equivalencia, donde el conjunto de clases de equivalencia tienen a lo más dos elementos; esta clase de posets se denominan poset con involución. Damos una estructura natural exacta para la categoría de representaciones de esta clase de posets, describimos los objetos proyectivos e inyectivos y probamos la existencia de sucesiones que casi se dividen.Por otro parte, estudiamos las propiedades categóricas de los lagoritmos de diferenciación DI y DIII introducidos por Zavadskij en 1991.

Keywords: Representation theory of partially ordered sets, Auslander-Reiten theory, Matrix problem, Vector Space Representation, differentiation algorithms DI, differentiation algorithms DIII.

Palabras clave: teoría de representación de conjuntos parcialmente ordenados, Teoría de Auslander-Reiten, problema matricial,representación vectorial, algoritmos de diferenciación DI, algoritmo de diferenciación DIII.

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Advisor
Raymundo Bautista Ramos

Coadvisor
Agustín Moreno Cañadas

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Dedicated to

Hermeht, Camilo y Laura.

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Introduction

The theory of representation of posets was introduced and developed by Nazarova, Roiter and their students in Kiev in the 1970's. One of their ideas was to use it as a way of giving a solution of the second Brauer-Thrall conjecture for finite dimensional k-algebras [21, 22, 29].

The main tool to classify posets both ordinary and with additional structures has been the algorithms of differentiation, which are functors defined to reduce dimension of the objects of the categories involved in the procedure. The first of these algorithms of differentiation is known as the algorithm with respect to a maximal point, it was introduced by Nazarova and Roiter in 1972. It was used by Kleiner to obtain a criterion to classify posets of finite representation type and by Nazarova in order to classify posets of tame representation type in 1977 [18, 23]. In 1977, Zavadskij introduced the algorithm of differentiation with respect to a suitable pair of points which was used by Nazarova and Zavadskij himself in 1981 to classify posets of finite growth representation type [24, 29, 32].

In the early 1990's the research regarding classification of posets pointed to posets with an additional structure [9, 25, 32]. We recall that in 1991 Zavadskij introduced an apparatus of differentiation for posets consisting of the algorithms of differentiation DI, DII, DIII, DIV and DV. This apparatus was used by Bondarenko and Zavadskij himself to classify posets with an involution of tame and finite growth representation type [8]. Afterwards, Zavadskij and Zabarilo, who was one of his students, introduced equipped posets and classified one-parameter equipped posets. To do that, he introduced algorithms of differentiation VII-XVII in order to establish criteria to classify equipped posets of tame and finite growth representation type in 2003 and 2005 respectively [31, 34–36].

Categorical properties of the main algorithms of differentiation have been studied by Gabriel, Zavadskij, Cañadas et al. Gabriel established this line of research in 1973 and gave the categorical properties of the algorithm of differentiation with respect to a maximal point establishing a bijection between indecomposable representations of the corresponding categories. Zavadskij proved categorical properties of the algorithm of differentiation with respect to a suitable pair of points in 1991 describing also the Auslander-Reiten quiver of posets of finite growth representation type. He also gave categorical properties of his generalization of DI to posets with relations in 2005. In the same direction, Zavadskij and Cañadas gave the categorical properties of the algorithm of differentiation DII in 2006 [11] and Cañadas et al. described categorical properties of some versions of the algorithms of differentiation DVII-DIX for equipped posets in 2013 [12–14].

On the other hand, Auslander-Reiten theory was introduced by Auslander and Reiten [4] in 1975; their work deals with problems in the representation theory considered directly with module theoretical techniques. Additionally to the classical module theory available, including homological methods, they introduced the notion of almost split sequences. Although they initially developed their ideas in the case of the category mod A of finitely generated modules over an Artin algebra A, this theory has been extended to a number of other categories including categories of representations of ordinary posets [5, 29, 33] and of posets with additional structures [7, 29].

In this sense, the almost split sequences are one of the most important tools for classification of finitely generated modules over finite-dimensional k-algebras with tame representation type. The most important theorem regarding almost split sequences was given by Bautista and Martinez [5]. This result claims that there are almost split sequences in $\mathrm{Rep}(\mathcal{P})$.

The purpose of this work is the study of the categorical properties of two of the Zavadskij reduction algorithms by using Auslander-Reiten quivers. We recall that the general idea of the reduction algorithms for posets goes as follows: given some suitable conditions take a small portion $\mathcal{K}$ of the poset $\mathcal{P}$ and replace it for some other $\mathcal{K}'$ obtaining a new poset $\mathcal{P}'$. Then given a representation of $\mathcal{P}$, this is changed only in $\mathcal{K}$ obtaining a representation of $\mathcal{P}'$. In our approach we study the Auslander-Reiten quivers of the representations of $\mathcal{K}$ and $\mathcal{K}'$ and see how the reduction algorithms behave and then extend the properties obtained in this way to the whole partially ordered sets $\mathcal{P}$ and $\mathcal{P}'$.

Contributions

The following are the main contributions of this dissertation:

1. In Section 1.3, we describe the relationship between matrix representations and vector space representations of poset with involution. We present two algorithms that allow us to associate a matrix representation $\mathcal{M} = \{M_x\}_{x\in\mathcal{P}}$ of $(\mathcal{P}, \leq, \theta)$ with a vector space representation $V = (V_0, V_z)_{z\in\theta}$ of $(\mathcal{P}, \leq, \theta)$ and conversely.

2. In Section 1.4, we introduce the additivisation $\mathrm{Mat}^{ad}_{(\mathcal{P},\leq,\theta)}$ of the matrix problem for posets with involution $(\mathcal{P}, \theta)$ following Simson's ideas. An embedding of categories $q : \mathrm{Mat}_{(\mathcal{P},\theta)} \to \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ is defined and a dense and full functor $F : \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)} \to \mathrm{Rep}(\mathcal{P}, \theta)$ is introduced. This functor preserves the representation type and vanishing only on a finite set of isomorphism classes of indecomposable objects. In Proposition 1 we prove that the functor F induces the equivalence of categories
$$\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}/\mathcal{I} \to \mathrm{Rep}(\mathcal{P}, \theta) \tag{1}$$
where $\mathcal{I}$ is the ideal generated by objects $I_z = ((I_z)_0, (I_z)_w)_{w\in\theta}$ with $(I_z)_0 = 0$ and $(I_z)_w = 0$ if $z \neq w$ and $(I_z)_z = k$. As consecuence from above, Proposition 4 proves that $\mathrm{Rep}(\mathcal{P}, \theta)$ is an exact category.

3. In Chapter 2, we give an exact strcuture for the category of vector space representations of poset with an involution. Furthermore, we describe the almost split sequences for this category. The definitions and propositions presented in this chapter have been results obtained during the research process.

4. In Chapter 3, the Auslander-Reiten quiver for some posets with involution that we will call of type D_n is given. Moreover, this result is generalized to posets of type D_∞ and the Auslander Reiten quiver for these posets is presented. As in the previous chapter, the results presented there, are part of the research process.

5. In Chapter 4 and Chapter 5, we present some categorical properties of differentiation algorithm DI and differentiation algorithm DIII for poset with an involution introduced by Zavadskij in 1991, by using module theoretical approach.

Conferences

The main results of this research have been presented in the following conferences.

1. Coloquio Latinoamericano de Álgebra-PUCE. Quito-Ecuador, 08-2017.
2. IV Jornada de Algebra no Amazonas. Tabatinga-Brasil, 09-2019.
3. 4rd International Colloquium on Representations of Algebras and Its Applications; Alexander Zavadskij. Bogotá -Colombia, 11-2020.

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Outline

This thesis is distributed as follows:

In Chapter 1, in sections 1.1 and 1.2 we recall some definitions and well known facts in representation theory of both ordinary posets and posets with an involution which will be used in the work; particularly, it deals with matrix representations and vector space representations. Also, following the ideas presented by Simson in [29] for ordinary posets, in section 1.3 we describe the correspondence between matrix representations and vector space representations for posets with an involution. Here we present two algorithms that allow us to associate a matrix representation $\mathcal{M} = \{M_x\}_{x\in\mathcal{P}}$ of $(\mathcal{P}, \leq, \theta)$ a vector space representation $V = (V_0, V_z)_{z\in\theta}$ of $(\mathcal{P}, \theta)$ and conversely. In section 1.4, we introduce the additivisation $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ of the matrix problem for posets with involution $(\mathcal{P}, \theta)$. We define an embedding of categories $q : \mathrm{Mat}_{(\mathcal{P},\theta)} \to \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ and a dense and full functor $F : \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)} \to \mathrm{Rep}(\mathcal{P}, \theta)$ preserving the representation type and vanishing only on a finite set of isomorphism classes of indecomposable objects, which induces the equivalence of categories

$$\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}/\mathcal{I} \to \mathrm{Rep}(\mathcal{P}, \theta) \tag{2}$$

where $\mathcal{I}$ is the ideal generated by objects $I_z = ((I_z)_0, (I_z)_w)_{w\in\theta}$ with $(I_z)_0 = 0$ and $(I_z)_w = 0$ if $z \neq w$ and $(I_z)_z = k$. We illustrate the existence of a categorical equivalence between the categories $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ and the category of representations of the tensor differential algebra defined by quiver algebra kQ where Q is a quiver that has as many points as the cardinality of θ plus one. Since this last category has exact structure and almost split sequences, then $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ has exact structure and almost split sequences. By using this results together with the equivalence (2) and the Proposition 1.8 of Liu [19], we prove that $\mathrm{Rep}(\mathcal{P},\theta)$ is an exact category. Finally, in section 1.5 we present the main classifications theorems for posets with an involution.

In Chapter 2, in section 2.1 we introduce a collection of sequences ε with some properties which will play the role of exact sequences and we give a natural exact structure for the category of representations of posets with an involution $(\mathrm{Rep}(\mathcal{P},\theta),\varepsilon)$ following a different technique from the one presented in the previous chapter. For this category, we describe the projective objects and prove that it has enough projectives. Besides, we introduce the injective objects and in order to show that this category has enough injectives we introduce a category $\mathrm{Repq}(\mathcal{P},\theta)$ which is equivalent to $\mathrm{Rep}(\mathcal{P}^{op},\theta)$. Further, we obtain an equivalence of categories between $\mathrm{Repq}(\mathcal{P},\theta)$ and $\mathrm{Rep}(\mathcal{P},\theta)$ so by duality, we obtain the desired result. In section 2.2 we define a functor $H : \mathrm{Rep}(\mathcal{P},\theta) \to \mathrm{mod}\, A$ where $A = \mathrm{End}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P})$ which sends ε-sequences in exact sequences and for all $L \in \mathrm{Rep}(\mathcal{P},\theta)$, $\mathrm{soc}H(L)$ is projective. Thus, this functor H induces an equivalence of categories:

$$H : \mathrm{Rep}(\mathcal{P},\theta) \to \mathrm{mod}_{sp}(A).$$

This result is the main tool used in Chapter 4 to describe the Auslander-Reiten quiver of posets $(\mathcal{K},\theta)$.

In Chapter 3, following the results of the previous chapters, we describe the Auslander-Reiten quiver for some posets with involution that we will call of type D_n. These results are generalized to posets of type D_∞ and the Auslander Reiten quiver for these posets is presented. This result is a fundamental tool in the study of the categorical properties of differentiation algorithm DIII that will be presented in Chapter 5, since the posets $(\mathcal{K}',\theta')$ under certain conditions can be seen as a poset of type D_n.

In Chapter 4, we present a new proof of categorical equivalence of the differentiation algorithm DI introduced by Zavadskij [36] by using module theoretical approach. We recall that this result allows us to establish the categorical equivalence between $\mathrm{Rep}(\mathcal{P})/\mathcal{I}$ and $\mathrm{Rep}(\mathcal{P}')/\mathcal{I}'$ where $\mathcal{P} = a^{\nabla} + b_{\vartriangle} + \{c_1 < c_2 < \cdots < c_n\}$ is a poset with a suitable pair of points (a,b) and $\mathcal{I} = \langle k(a), k(a,c_1), \ldots, k(a,c_n)\rangle$ and $\mathcal{I}' = \langle k(a)\rangle$. For this purpose, we construct the Auslander-Reiten quiver of a subposet $\mathcal{K}$ of $\mathcal{P}$ and the subposet $\mathcal{K}'$ of $\mathcal{P}'$ and show the categorical equivalence that exists between $\mathrm{Rep}(\mathcal{K})/\mathcal{I}$ and $\mathrm{Rep}(\mathcal{K}')/\mathcal{I}'$. Taking into account the definition of the differentiation functor induced by the differentiation algorithm DI, we can prove this result to the general case.

In the last chapter, we study the categorical properties of differentiation algorithm DIII. In contrast to the case of differentiation algorithm DI, the functor

$$' : \mathcal{R} \to \mathcal{R}'$$

with $\mathcal{R} = \{U \in \mathrm{Rep}(\mathcal{P},\theta) \mid U_a^+ \subset U_b^+; U_a^- = 0\}$, and $\mathcal{R}' = \{U \in \mathrm{Rep}(\mathcal{P}',\theta') \mid U_{a_1}^+ \subset U_B^+; U_{a_n}^- = U_{a_{n+1}}^+\}$, induces a dense and full functor, but in general not faithful:

$$F : \mathcal{R}/\langle \phi(\lambda, n)_{\lambda \neq 0},\ K(A, b)\rangle \to \mathcal{R}'/\langle K(A, b_1)\rangle.$$

CHAPTER 1

Preliminaries

1.1 Matrix Problems

The matrix problems was introduced by several authors [15, 21, 28], with the purpose of solving classification problems, which consists in classifying the indecomposable objects of a given additive category $\mathcal{C}$ having a finite unique descomposition property in the sense that every object X of $\mathcal{C}$ has a direct sum descomposition $X = X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n$, where $X_1, X_2, \ldots, X_n$ are indecomposable objects of $\mathcal{C}$ and every such a descomposition is unique up to permutation and isomorphism.

Roiter and Gabriel [28] introduced a definition of matrix problem of size $m \times n$ as a pair $(\mathcal{M}, \mathcal{G})$ formed by an underlying set $\mathcal{M} \subset k^{m\times n}$ and a group $\mathcal{G} \subset GL_m \times GL_n$ such that $XAY^{-1} \in \mathcal{M}$ whenever $A \in \mathcal{M}$, and $(X, Y) \in \mathcal{G}$. The question raised by the matrix problem is to classify the orbits of $\mathcal{M}$ under the action of $\mathcal{G}$ defined by $(X, Y)A = XAY^{-1}$. In other words, it consists of a set $\mathcal{M}$ of finite matrices together with a set $\mathcal{G}$ of admissible transformations in rows and columns which determines an equivalence relation, and the goal is to find a canonical form, i.e. determine a set of canonical matrices such that each $\mathcal{G}$-equivalence class contains exactly one canonical matrix.

Taking into account that the matrices $M \in M_{m,n}(k)$ describe linear transformations, if V and W are k-vector space such that $\dim_k V = n$ and $\dim_k W = m$ and we choose basis in V and W, then we have an isomorphism of k-vector space

$$M_{m,n}(k) \cong \mathrm{Hom}_k(V, W).$$

In case that n and m are non-zero, in order for the above isomorphism to be true in any case, we introduce the empty matrices $I_{m,0}$, $I_{0,n}$, $I_{0,0}$. That is, matrices with zero number of rows or columns, for which is satisfied

$$aI_{n,0} + bI_{n,0} = I_{n,0}; \qquad aI_{0,n} + bI_{0,n} = I_{0,n},$$

for all $a, b \in k$, and for $A \in M_{m,n}(k)$:

$$I_{0,m}A = I_{0,n}; \qquad AI_{n,0} = I_{m,0}; \qquad I_{m,0}I_{0,n} = 0 \in M_{m,n}(k); \qquad I_{0,n}I_{n,0} = I_{0,0}.$$

The matrix $I_{0,0}$ corresponds to the identity of the trivial vector space 0. Henceforth, we will consider $I_{0,0}$ to be a nonsingular matrix.

We will put $M_{n,0}(k) = I_{n,0}$, $M_{0,n}(k) = I_{0,n}$, $M_{0,0}(k) = I_{0,0}$. Each one of these spaces are isomorphic to the trivial vector space.

Now, if $f : V \to W_1$ and $0 : 0 \to W_2$ are linear transformations, then their direct sum is

$$\begin{pmatrix} f \\ 0 \end{pmatrix} : V \bigoplus 0 \to W_1 \bigoplus W_2.$$

Thus, if $M \in M_{m,n}(k)$ and $I_{m_1,0}$ are the matrices corresponding to the transformations f and 0 respectively, then the matrix that corresponds to the direct sum of these transformations is

$$M \bigoplus I_{m_1,0} = \begin{pmatrix} M \\ 0_{m_1,n} \end{pmatrix},$$

where $0_{m_1,n}$ is the matrix $m_1 \times n$ of zeros. Similarly

$$I_{m_1,0} \bigoplus M = \begin{pmatrix} 0_{m_1,n} \\ M \end{pmatrix},$$

and

$$M \bigoplus I_{0,n_1} = (M \quad 0_{m,n_1}),$$
$$I_{0,n_1} \bigoplus M = (0_{m,n_1} \quad M).$$

1.1.1 Matrix Representations of Posets.

There are many useful matrix problems for which there exist constructive methods for the classification of their indecomposable objects. One of them is the classification of matrix representations of partially ordered sets (*poset*) as will be shown in this section.

Definition 1. *A partially ordered set (poset) is a pair ordered $(\mathcal{P}, \leq)$ which consists of a not empty set $\mathcal{P}$ and a binary relation contained in $\mathcal{P} \times \mathcal{P}$, called order, such that*

1. *$\leq$ is reflexive, which means that $x \leq x$ for all $x \in \mathcal{P}$;*
2. *$\leq$ is antisymmetric, that is $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in \mathcal{P}$;*
3. *$\leq$ is transitive, meaning that $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in \mathcal{P}$.*

Two elements x, y of a given poset $\mathcal{P}$ are comparable if $x \leq y$ or $y \geq x$.

A poset is finite (infinite, respectively) if and only if the underlying set is finite (infinite, respectively).

A poset can be visualized through its Hasse diagram, which is the graphical representation that represents each element of $\mathcal{P}$ as a vertex in the plane and draws a line segment or curve that goes upward from x to y whenever y covers x . These curves may cross each other but must not touch any vertices other than their endpoints. Such a diagram, with labeled vertices, uniquely determines its partial order.

Example 1. *Figure 1.1 is a Hasse diagram representing the poset* $(\mathcal{P},\leq)$ *with* $\mathcal{P} = \{a,b,c,d\}$ *and* $a<c, a<d, b<c, b<d$.

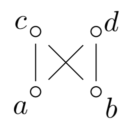


Figure 1.1. Hasse diagram of a poset.

If $(\mathcal{P},\leq)$ is a poset and $a\in\mathcal{P}$, then we denote the subsets of $\mathcal{P}$, $a^{\triangledown}$, $a_{\vartriangle}$, $a^{\blacktriangledown}$, and $a_{\blacktriangle}$, in such a way that:

$$\begin{aligned} a^{\triangledown} &= \{x\in\mathcal{P} \mid a\leq x\}\,, \\ a_{\vartriangle} &= \{x\in\mathcal{P} \mid x\leq a\}\,, \\ a^{\blacktriangledown} &= a^{\triangledown}\setminus\{a\}, \\ a_{\blacktriangle} &= a_{\vartriangle}\setminus\{a\}. \end{aligned} \tag{1.1}$$

Subset $a^{\triangledown}$ ($a_{\vartriangle}$,respectively) is called the *ordinary up-cone (down-cone,respectively)*, associated to the point $a\in\mathcal{P}$. Whereas subsets $a^{\blacktriangledown}$ and $a_{\blacktriangle}$ are called *truncated cones* (up and down, respectively) associated to the point $a\in\mathcal{P}$.

For a poset $(\mathcal{P},\leq)$ and $A\subset\mathcal{P}$, we define the subsets, $A^{\triangledown}$ and $A_{\vartriangle}$ such that

$$\begin{aligned} A^{\triangledown} &= \bigcup_{a\in A} a^{\triangledown}, \\ A_{\vartriangle} &= \bigcup_{a\in A} a_{\vartriangle}. \end{aligned} \tag{1.2}$$

An ordered set C is called a *chain* (or a *totally ordered set* or a *linearly ordered set*) if and only if for all $p,q\in C$ we have $p\leq q$ or $q\leq p$ (i.e., p and q are comparable). On the other hand, an ordered set $\mathcal{P}$ is called an *antichain* if $x\leq y$ in $\mathcal{P}$ only if $x=y$. The maximal cardinality of antichains in a poset $\mathcal{P}$ is called the *width* of $\mathcal{P}$.

Definition 2. *A matrix representation of* $(\mathcal{P},\leq)$ *is a collection of matrices* $\mathcal{M}=\{M_x\}_{x\in\mathcal{P}}$ *with* $M_x\in M_{d_0,d_x}(k)$. *The dimension vector of* $\mathcal{M}$ *is* $\dim(\mathcal{M})=d=(d_0,d_x)_{x\in\mathcal{P}}$.

Definition 3. *Two representations* $\mathcal{M} = \{M_x\}_{x\in\mathcal{P}}$ *and* $\mathcal{N} = \{N_x\}_{x\in\mathcal{P}}$ *are equivalent if*

1. $\dim(\mathcal{M}) = \dim(\mathcal{N})$.
2. *There are nonsingular matrices* $S_0 \in M_{d_0}(k)$, $S_x \in M_{d_x}(k)$ *for each* $x \in \mathcal{P}$ *and for each pair* $y < x$ *in* $\mathcal{P}$ *exists a matrix* $S_{y,x} \in M_{d_y,d_x}(k)$ *such that*

$$S_0M_x = N_xS_x + \sum_{y<x} N_yS_{y,x}, \tag{1.3}$$

for all $x \in \mathcal{P}$.

In this way we have defined a matrix problem $(\mathcal{M}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}})$.

Definition 4. *If* $\mathcal{M} = (M_x)_{x\in\mathcal{P}}$ *and* $\mathcal{N} = (N_x)_{x\in\mathcal{P}}$ *are two representations, the direct sum is*

$$\mathcal{M} \bigoplus \mathcal{N} = (M_x \bigoplus N_x)_{x\in\mathcal{P}}.$$

Definition 5. *A representation* $\mathcal{M} = (M_x)_{x\in\mathcal{P}}$ *is indecomposable if it is not equivalent to a direct sum* $\mathcal{L} \bigoplus \mathcal{N}$ *where* $\dim(\mathcal{L}) \neq 0$ *and* $\dim(\mathcal{N}) \neq 0$.

Example 2. *If* $\mathcal{P} = \{x\}$ *then a matrix representation* $\mathcal{M}$ *with dimension* $d = (d_0, d_x)$ *consists of one vertical stripe* $\mathcal{M} = [\ M_x\ ]$, $M_x \in k^{d_0\times d_x}$. *In this case, two representations* $\mathcal{M} = [\ M_x\ ]$ *and* $\mathcal{N} = [\ N_x\ ]$ *are equivalent if there exists invertible matrices* U *and* S *such that* $\mathcal{N} = U\mathcal{M}S^{-1}$.

Remark 1. *We can observe that the matrix* M_x *can be transformed by elementary row and column transformations into*

$$\mathcal{M} = \begin{bmatrix} 1_r & 0 \\ 0 & \mathbf{0} \end{bmatrix}, \tag{1.4}$$

where 1_r *stands for an identity matrix of size* r *and* $\mathbf{0}$ *for the zero matrix of size* $(d_0 - r) \times (d_x - r)$. *In case* $\mathcal{M}$ *has all its linearly independent rows then the right blocks in the reduced form (1.4) are matrices* $M_{d_0,0}$.

Example 3. *The two subspace problem is the matrix problem associated with a poset consisting of two incomparable elements, which consists of the set* $\mathcal{M}$ *of all pairs of matrices with the same number of rows under the equivalence relation:* $(M, N) \sim (M', N')$ *if there exists invertible matrices* U, S, *and* T *such that* $M' = UMS^{-1}$ *and* $N' = UNT^{-1}$.*The solution for this matrix problem (see,* [28]*) is given for the following normal form:*

$$M = \begin{bmatrix} 1_r & 0 & 0 & | & 0 & 1_r & 0 \\ 0 & 1_t & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 1_s & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \end{bmatrix} = [1_r|1_r] \bigoplus [1_t|] \bigoplus [|1_s] \bigoplus [0|0], \tag{1.5}$$

where $[1_r|1_r] = \underbrace{[1|1]\bigoplus[1|1]\bigoplus\cdots\bigoplus[1|1]}_{r} = [1|1]^r$. *Similarly* $[1_t|] = [1|]^t$, $[|1_s] = [|1]^s$ *and the* $[0|0] = [|]^r \bigoplus [|]^{s'} \bigoplus [|]^{s''}$, *where* r *is the number of rows of* $[0|0]$ *and* $s'(s'')$ *is the number of columns of the left (right) stripe.*

1.1.2 The Kronecker Problem

There are some classification problems whose matrix problem can be formulated in terms of matrix representations of poset with additional structures. Among them, is the Kronecker problem. This problem is the matrix problem of pairs of matrices (M, N) of the same size under the equivalence relation: $(M, N) \sim (M', N')$ if there exist invertible matrices U, S such that $M' = UMS^{-1}$ and $N' = UNS^{-1}$. So, the problem of classification of indecomposable Kronecker modules, as we will see next, is equivalent to this matrix problem and was solved by Kronecker in 1890 for the complex number field k. Due to its importance in our research, we will dedicate this section to its study following Simson's ideas [29].

Consider the *Kronecker algebra*

$$\Lambda = \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix},$$

where k is a field and the multiplication is given by the formula

$$\begin{pmatrix} d & u \\ 0 & c \end{pmatrix}\begin{pmatrix} f & v \\ 0 & e \end{pmatrix} = \begin{pmatrix} df & dv+ue \\ 0 & ce \end{pmatrix}.$$

Finite dimensional right $\Lambda-$modules are called Kronecker modules. Every such a module X can be identified with the quadruple

$$X = X_1 \overset{a}{\underset{b}{\rightrightarrows}} X_2,$$

where X_1, X_2 are the vector spaces $X(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$, $X(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})$ respectively, and a, b are linear maps defined by

$$a(x) = x(\begin{smallmatrix} 0 & i \\ 0 & 0 \end{smallmatrix}), \quad b(x) = (\begin{smallmatrix} 0 & j \\ 0 & 0 \end{smallmatrix}),$$

for $x \in X_1$, and $\{i, j\}$ is the standard basis of k^2. Any Λ-homomorphism $c : X_1' \to X$ can be identified with a pair (c_1, c_2) of linear maps

$$c_1 : X_1' \to X_1, \qquad c_2 : X_1' \to X_2,$$

such that $c_2a' = ac_1$ and $c_2b' = bc_1$. It follows that the category of Kronecker modules is equivalent to the category of pairs (A, B) of matrices A, B over k of the same size, where the map from (A', B') to (A, B) is a pair (C_1, C_2) of matrices over k such that $C_2A' = AC_1$ and $C_2B' = BC_1$.

This problem is equivalent to the problem of finding canonical Jordan form of pairs (A, B) of matrices with respect to the following elementary transformations:

(i) all elementary transformations on rows of the block matrix (A, B),

(ii) all elementary transformations made simultaneously on columns of A and B having the same index number.

We recall that, if k is an algebraically closed field, then up to isomorphism every indecomposable Kronecker module belongs to one of the following three classes [29],[21]:

$$\mathrm{I} = \mathrm{I}^*: \ \text{(a)} \ \boxed{\begin{array}{c|c} \mathrm{E}_n & \mathrm{J}_{(\lambda,n)} \end{array}},$$

$$\text{(b)} \ \boxed{\begin{array}{c|c} \mathrm{J}_{(\lambda,n)} & \mathrm{E}_n \end{array}},$$

where $J_{(\lambda,n)} \in \{J^+_{(\lambda,n)}, J^-_{(\lambda,n)}\}$ and $J^{\pm}_{(\lambda,n)}$ denotes a corresponding upper or lower Jordan block. Whereas, I^* denotes the dual case defined by the classification problem.

$$\mathrm{II} = \mathrm{III}^*: \ \boxed{\begin{array}{c|c} (\mathrm{E}_n, 0_{n,1}) & (0_{n,1}, \mathrm{E}_n) \end{array}},$$

$$\mathrm{III} = \mathrm{II}^*: \ \boxed{\begin{array}{c|c} \begin{pmatrix} 0_{1,n} \\ E_n \end{pmatrix} & \begin{pmatrix} E_n \\ 0_{1,n} \end{pmatrix} \end{array}}.$$

The cases II and III constitute the non-regular cases of this classification, whereas cases I constitute the regular one.

1.1.3 Matrix Representations of Posets with an Involution

Representations of poset with equivalence relation were introduced by Nazarova and Roiter with help of the matrix language [22].

Definition 6. *A partially ordered set with an equivalence relation is a triple $(\mathcal{P}, \leq, \theta)$, where $(\mathcal{P}, \leq)$ is a partially ordered set and in $\mathcal{P}$ there is an equivalence relation whose equivalence classes is θ. If the cardinality of each equivalence class is less than or equal to two, we will say that triple $(\mathcal{P}, \leq, \theta)$ is a partially ordered set with an involution. If $x \in \mathcal{P}$ we will denote by $[x]$ its equivalence class.*

Remark 2. *From now on we will omit the order relation in the notation for poset with involution, that is, we will write $(\mathcal{P}, \theta)$ instead of $(\mathcal{P}, \leq, \theta)$.*

If the class $[x]$ is a unit set, in this case we say that x is a *small point* and in the Hasse diagrams it will be designated with $\circ$ whereas if the class $[x]$ has two elements (we will write $x \sim x^*$, if this is the situation) we will say that x is a *large point.* The large points are noted in the Hasse diagrams with the symbol $\bullet$.

If $\mathcal{P}$ consists of only small points, then we get representations of partially ordered sets in the ordinary case as in the Definition 1. Henceforth, frequently a unit class $[x] = \{x\}$ will be identified with x and a class with two elements $[x] = \{x, x^*\}$ could be considered as an ordered pair (x, x^*) (the order chosen for points x, x^* will be done according to the context).

Example 4. *Let $(\mathcal{P}, \theta)$ be a poset with involution where $\mathcal{P}$ is as in Figure 1.2 with $a < b$, $c < b$, $c < a^*$ and $\theta = \{(a, a^*), b, c\}$.*

$(\mathcal{P}, \theta)=$

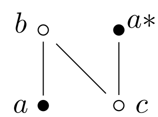


FIGURE 1.2. Diagram of a poset with an involution

Definition 7. *A matrix representation of $(\mathcal{P}, \theta)$ is a collection of matrices $\mathcal{M} = \{M_x\}_{x\in\mathcal{P}}$ with $M_x \in M_{d_0,d_x}(k)$ such that if $x \sim x^*$ then $d_x = d_{x^*}$. The dimension vector of $\mathcal{M}$ is* $\dim(\mathcal{M}) = (d_0, d_x)_{x\in\mathcal{P}}$.

Definition 8. *Two representations M and N are equivalent if*

1. $\dim(\mathcal{M}) = \dim(\mathcal{N})$.
2. *There are nonsingular matrices $S_0 \in M_{d_0}(k)$, $S_x \in M_{d_x}(k)$ for each $x \in \mathcal{P}$ and for each pair $y < x$ in $\mathcal{P}$ there exists a matrix $S_{y,x} \in M_{d_y,d_x}(k)$ such that $S_x = S_y$ if $[x] = [y]$ and*
$$S_0 M_x = N_x S_x + \sum_{y<x} N_y S_{y,x},$$
for all $x \in \mathcal{P}$.

Definition 9. *If $\mathcal{M} = (M_x)_{x\in\mathcal{P}}$ and $\mathcal{N} = (N_x)_{x\in\mathcal{P}}$ are two representations, the direct sum is*
$$\mathcal{M} \bigoplus \mathcal{N} = (M_x \bigoplus N_x)_{x\in\mathcal{P}}.$$

Definition 10. *A representation $\mathcal{M} = (M_x)_{x\in\mathcal{P}}$ is indecomposable if it is not equivalent to a direct sum $\mathcal{L} \bigoplus \mathcal{N}$ where* $\dim(\mathcal{L}) \neq 0$ *and* $\dim(\mathcal{N}) \neq 0$.

1.2 k-linear Representations

Another way to approach classification problems for linear transformation systems may be formulated in terms of a quiver and its representation introduced by Gabriel in 1973 [16]. For this purpose, he introduced the concept of a filtered k-linear representation of a poset $\mathcal{P}$ which is presented in this section.

1.2.1 Vector Space Representations of Posets

Definition 11. *A representation (or filtered k-linear representations or $\mathcal{P}$-space) U of a poset $\mathcal{P}$ is a system of finite-dimensional k-vector spaces of the form*

$$U = (U_0, U_x \mid x \in \mathcal{P}), \tag{1.6}$$

where U_0 is a finite-dimensional k-space and U_x is a subspace in U_0 for each $x \in \mathcal{P}$, such that $U_x \subseteq U_y$ provided that $x \leq y$.

Definition 12. *A morphism $\varphi : U \to V$ between two representations U and V is a k-linear transformation*

$$\varphi : U_0 \to V_0,$$

such that $\varphi(U_x) \subseteq V_x$, for each $x \in \mathcal{P}$.

Definition 13. *The radical of a representation U is the representation $\operatorname{rad} U = (U_0, \underline{U_x} \mid x \in \mathcal{P})$ where $\underline{U_x} = \sum_{y<x} U_y$ is the radical subspace of U_x.*

Definition 14. *The vector $\underline{\mathrm{cdim}}\, U = (d_0, d_x \mid x \in \mathcal{P})$, where $d_0 = \dim_k U_0$ and $d_x = \dim_k U_x/\underline{U_x}$ is called the coordinate vector of the representation U.*

Definition 15. *The direct sum between two representations U, V is a representation*

$$U \bigoplus V = W = (W_0, W_x \mid x \in \mathcal{P}),$$

such that $W_0 = U_0 \bigoplus V_0$ and $W_x = U_x \bigoplus V_x$, for any $x \in \mathcal{P}$.

Definition 16. *A representation $U \in \operatorname{rep} \mathcal{P}$ is said to be indecomposable provided that in a decomposition of the form $U = U_1 \bigoplus U_2$ either $U_1 = 0$ or $U_2 = 0$, otherwise U is a decomposable representation.*

Given a representation U of a poset $\mathcal{P}$ over a field k such that $\dim_k U_0 = 1$ then U is a *trivial representation.* For instance, if $A \subset \mathcal{P}$ then $k(A)$ is the indecomposable representation of $\mathcal{P}$, where $U_0 = k$ and

$$U_x = \begin{cases} k, & \text{if } x \in A^{\triangledown}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

In particular, the representation $k(\varnothing)$ has the field k as the ground vector space U_0 and $U_x = 0$ for any point $x \in \mathcal{P}$. We write $k(a_1, \dots, a_s)$ instead of $k(A)$ when $A = \{a_1, \dots, a_s\}$.

1.2.2 Vector Space Representations for Posets with an Involution

Following the ideas presented by Gabriel, Zavadskij introduced filtered k-linear representation of posets with an involution $(\mathcal{P}, \theta)$ [38]. Here we introduce an equivalent definition to the one given by him. For this, we consider $(\mathcal{P}, \theta)$ a poset with involution. We take V_0 a k-vector space and $z \in \theta$, take V_0^z the k-vector space consisting of all functions $h : z \to V_0$. For $x \in z$, we have the inclusion: $i_x : V_0 \to V_0^z$, defined by

$$i_x(v)(y) = \begin{cases} 0, & \text{if} \qquad y \neq x, \\ v, & \text{otherwise.} \end{cases}$$

and the projection in the summand x of V_0^z, $\pi_x : V_0^z \to V_0$, that is, for $h \in V_0^z$, $\pi_x(h) = h(x)$.

In the following, if V is a k-vector subspace of V_0^z and $x \in z$,

$$\begin{aligned} V_x^- &= i_x^{-1}(V) = \{v \in V_0 \mid i_x(v) \in V\}, \\ V_x^+ &= \pi_x(V) = \{h(x) \mid h \in V\}. \end{aligned}$$

Definition 17. *A vector space representation $V = (V_0, V_z)_{z\in\theta}$ of $(\mathcal{P}, \theta)$ is given by:*

1. *a finite-dimensional k- vector space V_0,*
2. *for each $z \in \theta$, a vector subspace V_z of V_0^z such that if $y < x$ then*

$$V_y^+ \subset V_x^-.$$

Example 5. *Let $(\mathcal{P}, \theta)$ be a poset with an involution where $\mathcal{P}$ is as in Figure 1.3 with $a < b^*$, $a^* < b$, $a^* < b^*$ and $\theta = \{(a, a^*), (b, b^*)\}$.*

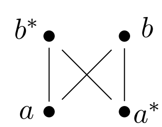
$(\mathcal{P}, \theta)=$ 

FIGURE 1.3. Diagram of a poset with an involution.

We will show that $V = (V_0, V_{(a,a^*)}, V_{(b,b^*)})$ is a vector space representation of $(\mathcal{P}, \theta)$, where $V_0 = \mathbb{R}^3$, $\mathcal{B} = \{e_1, e_2, e_3\}$ is the canonical basis of V_0 and $V_{(a,a^*)} = \langle h \rangle$, with

$$\begin{array}{rccc} h : & (a, a^*) & \to & \mathbb{R}^3 \\ & a & \mapsto & e_1 \\ & a^* & \mapsto & e_2 \end{array}$$

and $V_{(b,b^*)} = \langle h_1, h_2, h_3, h_4 \rangle$, with

$$\begin{array}{rccc} h_1: & (b,b^*) & \to & \mathbb{R}^3 \\ & b & \mapsto & e_2 \\ & b^* & \mapsto & 0 \end{array} \qquad \begin{array}{rccc} h_2: & (b,b^*) & \to & \mathbb{R}^3 \\ & b & \mapsto & 0 \\ & b^* & \mapsto & e_1 \end{array}$$

$$\begin{array}{rccc} h_3: & (b,b^*) & \to & \mathbb{R}^3 \\ & b & \mapsto & 0 \\ & b^* & \mapsto & e_2 \end{array} \qquad \begin{array}{rccc} h_4: & (b,b^*) & \to & \mathbb{R}^3 \\ & b & \mapsto & e_3 \\ & b^* & \mapsto & e_3 \end{array}$$

Indeed, since

$$\begin{array}{ll} V_a^+ = \pi_a(V_{(a,a^*)}) = \mathbb{R}\{e_1\}; & V_{a^*}^+ = \pi_{a^*}(V_{(a,a^*)}) = \mathbb{R}\{e_2\}; \\ V_b^- = i_b^{-1}(V_{(b,b^*)}) = \mathbb{R}\{e_2\}; & V_{b^*}^- = i_{b^*}^{-1}(V_{(b,b^*)}) = \mathbb{R}\{e_1,e_2\}; \end{array}$$

then, for $a < b^*$ is obtained that $V_a^+ \subset V_{b^*}^-$. For $a^* < b^*$, it is satisfied that $V_{a^*}^+ \subset V_{b^*}^-$ and for $a^* < b$, it is true that $V_{a^*}^+ \subset V_b^-$.

Definition 18. *If $V = (V_0, V_z)_{z\in\theta}$ and $W = (W_0, W_z)_{z\in\theta}$ are two representations of $(\mathcal{P},\theta)$, and $\varphi : V_0 \to W_0$ is a morphism of vector spaces, such that for each $z \in \theta$, we have the morphism $\varphi^z : V_0^z \to W_0^z$ and for $h : z \to V_0$, $\varphi^z(h) = \varphi h$. Then a morphism $V \to W$ consists of a morphism of vector space $\varphi : V_0 \to W_0$ such that*

$$\varphi^z(V_z) \subset W_z,$$

for all $z \in \theta$.

Definition 19. *If $V = (V_0, V_z)_{z\in\theta}$ and $W = (W_0, W_z)_{z\in\theta}$ are two representations of $(\mathcal{P},\theta)$ then their direct sum is*

$$V \bigoplus W = (V_0 \bigoplus W_0, V_z \bigoplus W_z)_{z\in\mathcal{P}}.$$

Now, for a vector space representation $(V_0, V_z)_{z\in\theta}$ of $(\mathcal{P},\theta)$, and for $z \in \theta$, we define

$$\underline{V}_z = \sum_{x\in z}\sum_{y<x} i_x(V_y^+) \subset V_z.$$

If $(V_0, V_z)_{z\in\theta}$ is a representation of a poset with an involution $(\mathcal{P},\theta)$ over a field k and $A \subset \mathcal{P}$, then we define the subspaces of V_0, denoted V_A^+ and V_A^-, in such a way that

$$\begin{aligned} V_A^+ &= \sum_{a\in A} V_a^+, \\ V_A^- &= \bigcap_{a\in A} V_a^-. \end{aligned}$$

For $A = \varnothing$, by definition $V_\varnothing^+ = 0$ and $V_\varnothing^- = V_0$.

A representation $(V_0, V_z)_{z\in\theta}$ of the poset with an involution $(\mathcal{P},\theta)$ is called *trivial* if $\dim_k V_0 = 1$. If $A \subset \mathcal{P}$, we define the trivial representation that we denote by $k(A)$

in such a way that $k(A) = (k; V_z)_{z\in\theta}$, where $V_z = k^{z\cap A^{\triangledown}}$, which means that for each $x \in \mathcal{P}$ we will have

$$V_x^+ = V_x^- = \begin{cases} k, & \text{if} \quad x \in A^{\triangledown}, \\ 0, & \text{in otherwise.} \end{cases}$$

Notice that, by definition $k(A) = k(A^{\triangledown}) = k(\min A)$. For simplicity we often write $k(X_1, ..., X_n)$ instead of $k(X_1 \cup \cdots \cup X_n)$ and in the case that we have a unit set $X_i = \{x_i\}$, we will make the identification $X_i = x_i$. For instance, $k(A, b) = k(A \cup \{b\})$.

1.3 Vector Space Representations Vs Matrix Representations for Posets with an Involution

Zavadskij in [38] states that the differentiation algorithms introduced by him were conceived using the matrix language but a strict foundation required a vector language; for this reason it is very important to describe the relationship between these two ways of representing the posets with an involution. For this, we take $V = (V_0, V_z \mid z \in \theta)$ be a vector space representation of $(\mathcal{P}, \theta)$ as above.

Then, for each $z \in \theta$, we choose V_z^0 a direct complement of $\underline{V}_z$, that is

$$V_z = V_z^0 \bigoplus \underline{V}_z.$$

Now, we choose a basis $\mathcal{L}$ for V_0 and for each $z \in \theta$ we choose a basis $\mathcal{U}(z)$ of V_z^0. Denote $B(V) = \mathcal{L} \bigcup \bigcup\limits_{z\in\theta} \mathcal{U}(z)$. In the first place, we will prove that the elements of the form $i_x(\pi_y(u))$ for $u \in \mathcal{U}([y])$ are a system of generators for $\underline{V}_z$, where $\mathcal{U}([y])$ is a basis for V_z^0, $[y]$ is a class in θ with $y \in [y]$ and $x \in z$, $y < x$. Indeed, we have for any $y \in \mathcal{P}$

$$\pi_y(\underline{V}_{[y]}) = \sum_{y_1 < y} \pi_{y_1}(V_{[y_1]}).$$

By definition of $\underline{V}_k$ we have

$$\pi_y(\underline{V}_{[y]}) = \pi_y \left(\sum_{x\in[y]} \sum_{y_1<x} i_x(V_{[y_1]}^+) \right) = \sum_{x\in[y]} \sum_{y_1<x} \pi_y i_x(V_{[y_1]}^+) \tag{1.8}$$

$$= \sum_{y_1<y} V_{y_1}^+ = \sum_{y_1<y} \pi_{y_1}(V_{[y_1]}). \tag{1.9}$$

Therefore, it is enough to prove that the space $i_x(V_y^+)$ with $x \in z$ and $y < x$ is generated by the elements of the form $i_x(\pi_{y_1}(u))$ with $u \in \mathcal{U}([y_1])$ and $y_1 < x$. We consider

$$S(z) = \{y \in \mathcal{P} \mid (\exists x \in z)(y < x)\},$$

and we suppose that y is minimal in $S(z)$, then by 1.8, $\pi_y(V_{[y]}) = \pi_y(V_{[y]}^0)$, thus the space $i_x(V_y^+) = i_x(\pi_y(V_{[y]})) = i_x(\pi_y)(V_y^0)$ is generated by elements of the form $i_x(\pi_y(u))$ with

$u \in \mathcal{U}([y])$.

Now suppose by induction that our claim holds true for all elements $y_1 \in S(z)$ with $y_1 < y$. By using 1.8 we obtain:

$$i_x(\pi_y(V_y)) = i_x(\pi_y(V^0_{[y]})) + \sum_{y_1 < y} i_x(\pi_{y_1}(V_{[y_1]})). \tag{1.10}$$

The space $i_x(\pi_y(V^0_{[y]}))$ is generated by elements of the form $i_x(\pi_y(u))$ with $u \in \mathcal{U}(y)$. By induction hypothesis the spaces $i_x(\pi_{y_1}(V_{[y_1]}))$, with $y_1 < y$, are generated by elements of the form $i_x(\pi_{y_2}(u))$ with $u \in \mathcal{U}[y_2])$ and $y_2 \in S(z)$. For $u \in \mathcal{U}(z)$ we have

$$u = \sum_{x \in z} \sum_{l \in \mathcal{L}} \alpha^x_{l,u} i_x(l),$$

where $\mathcal{L}$ is a basis for V_0. Therefore,

$$\pi_x(u) = \sum_{l \in \mathcal{L}} \alpha^x_{l,u} l.$$

Then, for each $x \in \mathcal{P}$ we obtain the matrix

$$M_x = (\alpha^x_{l,u}) \in \mathcal{M}_{d_0 \times d_z}(k),$$

where $d_0 = \dim_k V_0$ and $d_z = \dim_k(V_z/\underline{V}_z)$ for each $x \in z$.

Thus, we obtain a matrix representation

$$M(\mathcal{B}(V)) = (M_x)_{x \in \mathcal{P}}, \tag{1.11}$$

in terms of the basis $B(V) = \mathcal{L} \bigcup \bigcup_{k \in \theta} \mathcal{U}(k)$.

Finally, we will prove that the matrix representation is independent of the choice of the basis and the complementary subspace. For this we suppose that $\widetilde{V}^0_k$ is another complement of $\underline{V}_k$ in V_z and basis $\widetilde{\mathcal{L}}$ and $\widetilde{U}(k)$ of V_0 and $\widetilde{V}^0_k$ respectively. We take

$$\widetilde{B}(V) = \widetilde{\mathcal{L}} \bigcup \bigcup_{k \in \theta} \widetilde{\mathcal{U}}(k).$$

We will prove that $M(B(V))$ and $M(\widetilde{B}(V))$ are equivalent matrix representations.Indeed, for $\widetilde{u} \in \widetilde{U}(z)$ we have:

$$\widetilde{u} = \sum_{u \in \mathcal{U}(z)} \beta^z_{u,\widetilde{u}} u + \lambda(u),$$

where $\lambda(\widetilde{u}) \in \underline{V}_z$. Then by using the system of generators for $\underline{V}_k$ we obtain:

$$\begin{aligned}\lambda(\widetilde{u}) &= \sum_{x\in z}\sum_{y<x}\sum_{u\in\mathcal{U}([y])} \beta^{y,x}_{u,\widetilde{u}} i_x(\pi_y(u))\\ &= \sum_{x\in z}\sum_{y<x}\sum_{u\in\mathcal{U}([y])}\sum_{l\in\mathcal{L}} \beta^{y,x}_{u,\widetilde{u}} i_x \alpha^y_{l,u}(i_x(l)).\end{aligned}$$

We have $l = \sum\limits_{i\in\mathcal{L}} s_{\widetilde{l},l}\widetilde{l}$, where $S = (s_{l,l})$ are a non singular square matrices. Therefore:

$$\widetilde{u} = \sum_{u,l,\widetilde{l},x\in z} s_{l,\widetilde{l}}\alpha^x_{l,u}\beta^z_{u,\widetilde{u}} i_x(\widetilde{l}) + \sum_{x\in z, y<x, u\in\mathcal{U}([y])} s_{l,\widetilde{l}}\alpha^x_{l,u}\beta^{y,x}_{u,\widetilde{u}} i_x(\widetilde{l}).$$

It follows the equality

$$M(\widetilde{B}(V))_x = SM(\widetilde{B}(M))_x T_z + \sum_{y<x} SM(\widetilde{B}(V))_y T_{y,x},$$

where T^z is the non-singular square matrix $(\beta^z_{u,\widetilde{u}})$ and $T^{y,x} = (\beta^{y,x}_{u,\widetilde{u}})$.

From the above we conclude that $M(\widetilde{B}(V))$ and $M(B(V))$ are equivalent matrix representations.

Now, given $\mathcal{M} = (M_x)_{x\in\mathcal{P}}$ a matrix representation of $(\mathcal{P},\theta)$ with $M_x \in \mathcal{M}_{d_0\times d_{[x]}}(k)$, we can construct a vector space representation $V = (V_0; V_z \mid z \in \theta)$ of $(\mathcal{P},\theta)$ as follows:

we take $V_0 = k^{d_0}$, and for each $z \in \theta$, let $\mathcal{U}(z)$ be the set of vectors of the form

$$U(i) = \sum_{x\in z}\sum_{j=1}^{d_0} \alpha^x_{j,i} i_x(e_j),$$

where $\{e_1, \ldots, e_{d_0}\}$ is the canonical basis of V_0 and $i = 1, 2, \ldots, d_{[x]}$. We define

$$V_z = \sum_{u\in\mathcal{U}(z)} uk + \sum_{x\in z}\sum_{y<x}\sum_{v\in\mathcal{U}([y])} i_x(\pi_y(v))k, \tag{1.12}$$

so, by definition V_z is generated by the vectors $u \in \mathcal{U}(z)$ and the elements $i_x(\pi_y(v))$ with $x \in z, y < x$ and $v \in \mathcal{U}([y])$.

Finally, we take now $y < x$ in $\mathcal{P}$, in order to check that $V^+_y \subset V^-_x$. It is enough to prove that for any of the above non-zero generators w of $V_{[y]}$ we have that $\pi_y(w) \in V^-_x$.
In case $w = \pi_y(v)$ with $v \in \mathcal{U}([y])$ we have that $i_x(\pi_y(v))$ is one of the generators of $V_{[x]}$, therefore $w \in V^-_x$. In case $w = i_{y_1}(\pi_{y_2}(v))$

$$V_{[y]} = \sum_{u\in\mathcal{U}([y])} uk + \sum_{y\in[y]}\sum_{y_2<y_1}\sum_{v\in\mathcal{U}([y_2])} i_{y_1}(\pi_{y_2}(v))k,$$

then since $\pi_y(w)$ is non zero, we must have $y = y_1, \pi_y(w) = \pi_{y_2}(v)$ with $y_2 < x$ and $v \in \mathcal{U}([y_2])$ therefore

$$i_x(w) = i_x(\pi_{y_2}(v)) \in V_{[x]},$$

this implies $\pi_y(w) \in V_x^-$.

The previous constructions allows to describe algorithms as follows:

Algorithm 1
Input: a vector space representation $V = (V_0, V_z \mid z \in \theta)$ of $(\mathcal{P}, \theta)$. **Output:** a matrix representation $M = (M_x)_{x\in\mathcal{P}}$ of $(\mathcal{P}, \theta)$.
1. For each $z \in \theta$, calculate $$\underline{V}z = \sum_{x\in z}\sum_{y<x} i_x(V_y^+) \subset V_z.$$ 2. For each $z \in \theta$, choose V_z^0 a direct complement of $\underline{V}_z$ in V_z, that is $$V_z = V_z^0 \bigoplus \underline{V}_z.$$ 3. Choose a basis $\mathcal{L}$ for V_0. 4. For each $z \in \theta$, choose a basis $\mathcal{U}(z)$ of V_z^0. 5. For each $u \in \mathcal{U}(z)$, $\quad u = \sum_{x\in z}\sum_{l\in\mathcal{L}} \alpha_{l,u}^x i_x(l)$. 6. For each $x \in \mathcal{P}$ we obtain the matrix $M_x = (\alpha_{l,u}^x) \in \mathcal{M}_{d_0\times d_z}(k)$.

Example 6. *We consider the vector space representation for the poset with involution given in Example 5.*

1. *For $z = (a, a^*)$ we have $\underline{U}_{(a,a^*)} = 0$, and for $z = (b, b^*)$ we have $\underline{U}_{(b,b^*)} = \langle h_1\rangle + \langle h_2\rangle + \langle h_3\rangle$.*
2. *If $z = (a, a^*)$ then $U_z^0 = \langle h\rangle$, and if $z = (b, b^*)$ then $U_z^0 = \langle h_4\rangle$.*
3. *Choose $\mathcal{L} = \langle e_1, e_2, e_3\rangle$ a basis for $U_0 = \mathbb{R}^3$.*
4. *For $z = (a, a^*)$ we choose a basis $\mathcal{U}(z) = \langle h\rangle$, and for $z = (b, b^*)$ we choose a basis $\mathcal{U}(z) = \langle h_4\rangle$.*
5. *As $\mathcal{U}(z) = \langle h_4\rangle + \langle h\rangle$ we have*

$$\begin{aligned} h_4 &= \sum_{l\in\mathcal{L}} \alpha_{l,h_4}^b i_b(l) + \sum_{l\in\mathcal{L}} \alpha_{l,h_4}^{b^*} i_{b^*}(l) \\ &= 0\cdot i_b(e_1) + 0\cdot i_b(e_2) + 1\cdot i_b(e_3) + 0\cdot i_{b^*}(e_1) + 0\cdot i_{b^*}(e_2) + 1\cdot i_{b^*}(e_3), \end{aligned}$$

and

$$\begin{aligned} h &= \sum_{l\in\mathcal{L}} \alpha_{l,h_4}^a i_a(l) + \sum_{l\in\mathcal{L}} \alpha_{l,h_4}^{a^*} i_{a^*}(l) \\ &= 1\cdot i_a(e_1) + 0\cdot i_a(e_2) + 0\cdot i_a(e_3) + 0\cdot i_{a^*}(e_1) + 1\cdot i_{a^*}(e_2) + 0\cdot i_{a^*}(e_3). \end{aligned}$$

6. *Therefore, we obtain the matrix representation for* $(\mathcal{P}, \theta)$

$$\begin{array}{cc}
 & \begin{array}{cccccc} a & a^* & & b & b^* \end{array} \\
M = & \begin{pmatrix} 1 & 0 & | & 0 & 0 \\ 0 & 1 & | & 0 & 0 \\ 0 & 0 & | & 1 & 1 \end{pmatrix}
\end{array}$$

The next algorithm allows us to compute a vector space representation $V = (V_0, V_z \mid z \in \theta)$ of $(\mathcal{P}, \theta)$ from a matrix representation $M = (M_x)_{x \in \mathcal{P}}$ of $(\mathcal{P}, \theta)$.

Algorithm 2.

Input: a matrix representation $M = (M_x)_{x \in \mathcal{P}}$ of $(\mathcal{P}, \theta)$ with $M_x \in \mathcal{M}_{d_0 \times d_{[x]}}(k)$
Output: a vector space representation $V = (V_0, V_z \mid z \in \theta)$ of $(\mathcal{P}, \theta)$.

1. Take $V_0 = k^{d_0}$
2. For each $z \in \theta$, take $\mathcal{U}(z)$, the vectors set of the form
$$U(i) = \sum_{x \in z} \sum_{j=1}^{d_0} \alpha^x_{j,i} i_x(e_j),$$
where $\{e_1, \ldots, e_{d_0}\}$ is the canonical basis of V_0 and $i = 1, 2, \ldots, d_{[x]}$.
3. We define
$$V_z = \sum_{u \in \mathcal{U}(z)} uk + \sum_{x \in z} \sum_{y < x} \sum_{v \in \mathcal{U}([y])} i_x(\pi_y(v))k.$$
4. $V = (V_0, V_z)$ is a representation for $(\mathcal{P}, \theta)$.

Example 7. *Let* $(\mathcal{P}, \theta)$ *be the partially ordered set with an involution as the following figure*

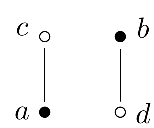
$(\mathcal{P}, \theta)=$


FIGURE 1.4. Diagram of a poset with an involution.

where $\theta = \{(a, b), c, d\}$. *Consider the next matrix representation for* $(\mathcal{P}, \theta)$

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccc} & & a & & & c & & d & & & & b & & \end{array} \\
M= & \begin{pmatrix}
1 & 0 & 0 & 0 & 0| & 0 & | & 1 & | & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0| & 0 & | & 0 & | & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0| & 0 & | & 0 & | & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1| & 0 & | & 0 & | & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0| & 1 & | & 1 & | & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

We can give a vector space representation following the ideas given above.

1. $V_0 = k^5$.

2. *For $z = c$ or $z = d$, $U(1) = \{e_5\}$ and $U(1) = \{e_1 + e_5\}$ respectively, for $z = (a, b)$, we find that*

$$U(1) = (e_1, e_1), U(2) = (e_2, e_2), U(3) = (0, e_3), U(4) = (e_3, e_4), U(5) = (e_4, e_5).$$

3. *For $z = c, z = d$ and $z = (a, b)$ we obtain*

$$\begin{aligned} V_{(a,b)} &= k\{(e_1, e_1), (e_2, e_2), (0, e_3), (e_3, e_4), (e_4, e_5), (0, e_1 + e_5)\}. \\ V_{(c)} &= k\{e_5, e_1, e_2, e_3, e_4\}, \\ V_{(d)} &= k\{e_1 + e_4\}. \end{aligned}$$

4. *So, $V = (k^5; V_{(a,b)}, V_{(c)}, V_{(d)})$ is a vector space representation of $(\mathcal{P}, \theta)$.*

1.4 Representations of Posets and Categories

Categories are an important element in representation theory. They provide a language as well as objects of investigation. They arise not only as natural generalizations of algebras but also as generalizations of various categories of modules. Categories are indispensable for the combinatorial description of algebras and modules which we shall produce. In the present section we introduce the basic categorical notions and describe their relations with classical algebra.

In this section we present the basic categorical notions, [see, [2, 27, 30]], and the category of representations of posets with involution [32].

Definition 20. *A category $\mathcal{R}$ is a class of objects together with the following data* [30]*:*

1. *a rule which assigns to any pair (U, V) of objects in $\mathcal{R}$ a set $\mathcal{R}(U, V)$, whose elements are called the morphisms from U to V;*

2. *for any triplet (U, V, W) a composition map*

$$\begin{aligned} \mathcal{R}(V, W) \times \mathcal{R}(U, V) &\longrightarrow \mathcal{R}(U, W) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

which is associative in the sense that $f \circ (g \circ h) = (f \circ g) \circ h$ and which admits identity elements in the sense that each set $\mathcal{R}(V, V)$ contains an element 1_V such that $1_V \circ f = f$ for all $f \in \mathcal{R}(U, V)$ and $g \circ 1_V = g$ for all $g \in \mathcal{R}(V, W)$.

Definition 21. *Let $\mathcal{R}$ be a category. A category $\mathcal{R}'$ is a subcategory of $\mathcal{R}$ if the following four conditions are satisfied:*

1. *the class of object of $\mathcal{R}'$ is a subclass of the class of objects of $\mathcal{R}$;*

2. *if $U,\ V \in \mathcal{R}'$ then $\mathcal{R}'(U, V) \subseteq \mathcal{R}(U, V)$;*

3. *the composition of morphisms in $\mathcal{R}'$ is the same as in $\mathcal{R}$;*

4. *for each object $U \in \mathcal{R}$, the identity morphism $1'_U \in \mathcal{R}'(U,U)$ coincides with the identity morphism $1_U \in \mathcal{R}(U,U)$.*

A subcategory $\mathcal{R}'$ of $\mathcal{R}$ is called *full* if $\mathcal{R}'(U,V) = \mathcal{R}(U,V)$ for all $U, V \in \mathcal{R}'$.

In a category $\mathcal{R}$ there are special types of morphisms which we define below.

Definition 22. *Let U and V be objects of a category $\mathcal{R}$ and $f : U \longrightarrow V$ be a morphism from U to V.*

1. *We call f a monomorphism if $fg = fh$ for all morphisms $g, h : W \longrightarrow U$ and all objects W implies that $g = h$.*
2. *We call f an epimorphism if $gf = hf$ for all morphisms $g, h : V \longrightarrow W$ and all objects W implies that $g = h$.*
3. *We call f an isomorphism if there exists a unique morphism $g : V \longrightarrow U$, such that $gf = 1_U$ and $fg = 1_V$. It is denoted by $U \cong V$.*

Definition 23. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two categories.*

1. *A covariant functor $F : \mathcal{R} \longrightarrow \mathcal{R}'$ is a rule which assigns to each object U of $\mathcal{R}$ an object $F(U)$ of $\mathcal{R}'$ and to each morphism $f : U \longrightarrow V$ in $\mathcal{R}$ a morphism $F(f) : F(U) \longrightarrow F(V)$ in $\mathcal{R}'$ in such a way that always $F(1_U) = 1_{F(U)}$ and $F(gf) = F(g)F(f)$.*
2. *A contravariant functor $F : \mathcal{R} \longrightarrow \mathcal{R}'$, is a rule which assigns to each object U of $\mathcal{R}$ an object $F(U)$ of $\mathcal{R}'$ and to each morphism $f : U \longrightarrow V$ a morphism $F(f) : F(V) \longrightarrow F(U)$ in such a way that always $F(1_U) = 1_{F(U)}$ and $F(gf) = F(f)F(g)$.*
3. *A functor $F : \mathcal{R} \longrightarrow \mathcal{R}'$ is an equivalence, if F admits a quasi-inverse, i.e. a functor $E : \mathcal{R}' \longrightarrow \mathcal{R}$ such that EF is isomorphic to $1_{\mathcal{R}}$ and FE to $1_{\mathcal{R}'}$.*

It is useful to have another way of describing when two categories are equivalent. We say that a functor F between categories is faithful if the morphism $F_{U,V} : \mathrm{Hom}_{\mathcal{R}}(U,V) \rightarrow \mathrm{Hom}_{\mathcal{R}'}(F(U), F(V))$ given by F is a monomorphism for all U, V in $\mathcal{R}$. The functor F is full if this morphism is an epimorphism. The functor F is dense if for each V in $\mathcal{R}'$ there is some U in $\mathcal{R}$ with $F(U) \cong V$. In this way, the following characterization of an equivalence of categories is obtained.

Theorem 1 (Theorem 1.2, [3])**.** *A covariant functor F between categories is an equivalence if and only if it is full, faithful and dense.*

If there exists an equivalence F of categories $\mathcal{R}$ and $\mathcal{R}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalence categories* and we note this $\mathcal{R} \cong \mathcal{R}'$.

In this work, we will refer to a covariant functor only as a *functor*. Now we show the concept of k-category and ideal in a k-category.

Definition 24. *Let k be a fixed commutative ring, a k-category is a category $\mathcal{R}$ whose morphism sets $\mathcal{R}(U,V)$ are endowed with k-module structures such that the composition maps are k-bilinear. A k-functor between two k-categories $\mathcal{R}$ and $\mathcal{S}$ is a functor $F : \mathcal{R} \longrightarrow \mathcal{S}$ whose defining maps $F(U,V) : \mathcal{R}(U,V) \longrightarrow \mathcal{S}(FU,FV)$ are k-linear for all $U,V \in \mathcal{R}$.*

Definition 25. *An object U of a k-category $\mathcal{R}$ will be called indecomposable if the endomorphism algebra $\mathcal{R}(U,U) = \mathrm{End}\,(U)$ has precisely two idempotents, namely 0 and $1_U \neq 0$.*

Each k-algebra A gives rise to a k-category which has one object Ω such that $\mathrm{Hom}\,(\Omega,\Omega) = A$. In the sequel, we shall identify k-algebras with the associated k-categories. Homomorphisms of algebras then correspond to k-functors. In view of this, k-categories generalize k-algebras. The generalization carries over to (two-sides) ideals, which are defined as follows [17].

Definition 26. *An ideal $\mathcal{I}$ of a k-category $\mathcal{R}$ is a family of subgroups $\mathcal{I}(U,V) \subset \mathcal{R}(U,V)$ such that $f \in \mathcal{I}(U,V)$ implies $gfh \in \mathcal{I}(Z,W)$ for all $h \in \mathcal{R}(Z,U)$ and $g \in \mathcal{R}(V,W)$. Each such ideal $\mathcal{I}$ gives rise to a k-quotient category $\mathcal{R}/\mathcal{I}$ which has the same objects as $\mathcal{R}$ and satisfies $(\mathcal{R}/\mathcal{I})(U,V) = \mathcal{R}(U,V)/\mathcal{I}(U,V)$ for all $U,V \in \mathcal{R}$.*

We recall the definition of the direct sum of two objects $U,V \in \mathcal{R}$ as in [17]: let $\mathcal{R}$ be a k-category. First we call summation of U and V in $\mathcal{R}$ a quintuplet (S,i,j,p,q) consisting of an object $S \in \mathcal{R}$ and of morphisms $U \underset{p}{\overset{i}{\rightleftarrows}} S \underset{q}{\overset{j}{\leftrightarrows}} V$ such that $pi = 1_U$, $qj = 1_V$ and $ip + jq = 1_S$. Such summations are known to be "unique up to uniquely determined isomorphisms". Therefore, whenever a summation of U and V exists, we will suppose that a "canonical" one has been chosen. The object S is then called the *sum* (or *coproduct*) of U and V in $\mathcal{R}$ and it is denoted by $U \bigoplus V$; the morphisms p,q are called *projections*, the morphisms i,j are called *immersions*.

For a category $\mathcal{R}$, we let $\langle U_i \mid i \in I \rangle_{\mathcal{R}}$ denote the ideal consisting of all morphisms passed through finite direct sums of the objects U_i. That is, if $\varphi : U \to V \in \langle U_i \mid i \in I \rangle_{\mathcal{R}}$ then there exist morphisms $f,g \in \mathcal{R}$ such that $\varphi = U \xrightarrow{f} \bigoplus_i U_i^{m_i} \xrightarrow{g} V$ with $m_i = 0$ for almost all i.

Definition 27. *The k-category $\mathcal{R}$ is called additive if $U \bigoplus V$ exists for all $U,V \in \mathcal{R}$ and if $\mathcal{R}$ contains a null object, i.e. an element 0 such that $1_0 = 0$.*

Let $\mathcal{R}$ be an additive category, a non-null object $U \in \mathcal{R}$ is said to be *indecomposable* provided $U \simeq V \bigoplus W$ implies $V = 0$ or $W = 0$ [27].

If $[X_1], \ldots, [X_k]$ are equivalence classes of indecomposable objects of a given category then we let $[X_1, X_2, \ldots, X_k]$ denote the union $\bigcup_{i=1}^{k} [X_i]$.

Definition 28. *Let $\mathcal{R}$ be an additive k-category and $U, V \in \mathcal{R}$. An idempotent is a morphism $e \in \mathcal{R}(U,U)$ such that $e^2 = e$. An idempotent $e \in \mathcal{R}(U,U)$ splits if there are morphisms $f : V \longrightarrow U$, $g : U \longrightarrow V$ with $gf = 1_V$ and $fg = e$.*

Definition 29. *Let k be a commutative ring. An additive k-category $\mathcal{K}$ is called Krull-Schmidt category provided that all idempotents split and the endomorphism ring* $\mathrm{End}(U)$, *of any object $U \in \mathcal{K}$, is a semi-perfect ring.*

Theorem 2 (Krull-Schmidt)**.** *Let $\mathcal{K}$ be a Krull-Schmidt category, let U_i, V_j be indecomposable objects in $\mathcal{K}$ with $1 \leq i \leq s$, $1 \leq j \leq t$, such that $\bigoplus\limits_{i=1}^{s} U_i \simeq \bigoplus\limits_{j=1}^{t} V_j$, Then $s = t$, and there is a permutation π of $\{1, \ldots, s\}$ such that $U_i \simeq V_{\pi(i)}$ for all i.*

Henceforth it is very useful to look at the matrix problems $(\mathcal{M}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}})$ of poset as the category whose objects are the matrices $M \in \mathcal{M}_{\mathcal{P}}$ and whose morphisms are pairs of matrices (S_0, S_x) where $S_0 \in GL(d_0, k)$ and $S_x \in GL(\underline{d}, k)$, where $\underline{d} = \sum\limits_{x \in \mathcal{P}} d_x$ which is a composition of elementary matrices corresponding to elementary transformation as in 1.3. This category will be denoted by $\mathrm{Mat}_{\mathcal{P}}$. In case of poset with involution $(\mathcal{P}, \theta)$ the category of matrix representations denoted $\mathrm{Mat}_{(\mathcal{P},\theta)}$ is a subcategory of $\mathrm{Mat}_{\mathcal{P}}$, whose morphisms satisfy the same conditions as the morphisms in $\mathrm{Mat}_{\mathcal{P}}$, but taking into account that if $[x], [y] \in \theta$ are such that $[x] = [y]$ then $S_x = S_y$. In [29], it is proved that the categories $\mathrm{Mat}_{\mathcal{P}}$ and $\mathrm{Mat}_{(\mathcal{P},\theta)}$ are Krull-Schmidt.

A fundamental role in the theory of representations of partially ordered sets is played by the category $\mathrm{Mat}_{\mathcal{P}}^{ad}$, which consists of additive enlargement of $\mathrm{Mat}_{\mathcal{P}}$ (called an additivisation of $\mathrm{Mat}_{\mathcal{P}}$), defined in [29] as the category whose objects are systems

$$V = (V_x, V_0, t_x)_{x \in \mathcal{P}},$$

of finite dimensional k-vector space V_x together with k-linear maps $t_x : V_x \to V_0$ for each $x \in \mathcal{P}$. If $V = (V_x, V_0, t_x)_{x \in \mathcal{P}}$ and $V' = (V'_x, V_0, t'_x)_{x \in \mathcal{P}}$ are objects, a morphism $V \to V'$ in $\mathrm{Mat}_{\mathcal{P}}^{ad}$ is a pair (g, g_0) of k-linear maps such that the following diagram is commutative

$$\begin{tikzcd}
\bigoplus\limits_{x \in \mathcal{P}} V_x \arrow[r, "t_x"] & V_0 \\
\bigoplus\limits_{x \in \mathcal{P}} V'_x \arrow[r, "t'_x"] \arrow[u, "g"] & V'_0 \arrow[u, "g_0"']
\end{tikzcd}$$

The maps $g := (g_{ij})$ have the upper triangular matrix form where $g_{ij} : V'_{x_j} \to V_{x_i}$ are k-linear maps and $g_{ij} = 0$ if $x_i \npreceq x_j$ holds in $\mathcal{P}$. The composition of maps in $\mathrm{Mat}_{\mathcal{P}}^{ad}$ is defined in a natural way. The integral vector $\mathrm{cdn}(V) = (\mathrm{dim}V_0, \mathrm{dim}V_x)_{x \in \mathcal{P}}$ is called the coordinate vector of V.

An embedding of categories

$$q : \mathrm{Mat}_{\mathcal{P}} \to \mathrm{Mat}_{\mathcal{P}}^{ad} \tag{1.13}$$

is defined in [29] as follows: if $M = (M_x)_{x\in\mathcal{P}}$ is a matrix in $\mathrm{Mat}_{\mathcal{P}}$ and $\mathrm{cdn}(M) = (d_0, d_x)$ then $q(M) := q((M_x)_{x\in\mathcal{P}}) = (k^{d_{x_j}}, k^{d_0}, M_{x_j})$ where, $M_{x_j} : k^{d_j} \to k^{d_0}$ is the k-linear map induced by the matrix M_{x_j} in the standard basis.

Lemma 1 (Lemma 2.2, [29]). *The map q has the following properties*

(a) Block matrices M, N in $\mathrm{Mat}_{\mathcal{P}}$ are $\mathcal{G}_{\mathcal{P}}$-equivalent if and only if $q(M) \cong q(N)$.

(b) A matrix M in $\mathrm{Mat}_{\mathcal{P}}$ is indecomposable if and only if $q(M)$ is indecomposable in $\mathrm{Mat}^{ad}_{\mathcal{P}}$.

(c) The map q establishes a one to one correspondence between $\mathcal{M}_{\mathcal{P}}$-equivalence classes of the indecomposables objects in $\mathrm{Mat}_{\mathcal{P}}$ and the isomorphism classes of indecomposable objects in $\mathrm{Mat}^{ad}_{\mathcal{P}}$. Morever, $\mathrm{cdn}(M) = \mathrm{cdn}(q(M))$ for all $M \in \mathrm{Mat}_{\mathcal{P}}$.

The vector space representations of a poset can also be viewed categorically. We denote by $\mathrm{Rep}\mathcal{P}$ and $\mathrm{Rep}(\mathcal{P}, \theta)$ the categories of representations of ordinary poset and poset with an involution respectively. If the involution is trivial then the category $\mathrm{Rep}(\mathcal{P}, \theta)$ coincides with category $\mathrm{Rep}\mathcal{P}$.

An important role in the study of matrix representations and vector space representations of a poset $\mathcal{P}$ is played by the following reduction functor

$$H : \mathrm{Mat}^{ad}_{\mathcal{P}} \to \mathrm{Rep}\,\mathcal{P}, \tag{1.14}$$

which assigns to $U = (U_{x_i}, U_0, t_{x_i})_{x\in\mathcal{P}}$ in $\mathrm{Mat}^{ad}_{\mathcal{P}}$ the representation $H(U) = (V_0, V_x \mid x \in \mathcal{P})$ where $V_0 = U_0$ and

$$V_x = \mathrm{Im}\left(\bigoplus_{x_i \le x_j} U_{x_i} \quad \overset{t_{x_i}}{\to} \quad U_0\right),$$

for each $x_j \in \mathcal{P}$. If $(g, g_0) : U' \to U$ is a map in $\mathrm{Mat}^{ad}_{\mathcal{P}}$, then $g_0(U'_{x_j}) \subseteq U_{x_j}$, for each $x_j \in \mathcal{P}$ and we put $H(g, g_0) = g_0$.

Theorem 3 (Theorem 3.1,[29]). *Let $\mathcal{P} = \{x_i \mid 1 \le i \le t\}$ be a poset and let H be the functor* 1.14*. Then the following staments hold.*

a. *The functor H is full and dense.*

b. *If V is indecomposable in $\mathrm{Mat}^{ad}_{\mathcal{P}}$, then $H(V) = 0$ if and only if $V \cong k(x_j \nearrow 0)$ for some $x_j \in \mathcal{P}$, where $k(x_j \nearrow 0)$ are indecomposable objects in $\mathrm{Mat}^{ad}_{\mathcal{P}}$ defined by*

$$k(x_j \nearrow 0) = (U_{x_i}, U_0, t_{x_i} | x_i \in \mathcal{P}),$$

with $U_0 = 0, U_{x_j} = k$ and $U_{x_i} = 0$, if $x_i \neq x_j$. The linear maps t_{x_j} are such that $t_{x_j} : k \to 0 \quad$ if $\quad x_i = x_j$, and $t_{x_i} : 0 \to 0 \quad$ if $\quad x_i \neq x_j$.

c. *If* $(\mathrm{Mat}^{ad}_{\mathcal{P}})_0$ *denotes the full subcategory of* $\mathrm{Mat}^{ad}_{\mathcal{P}}$*, which consists of objects which have not direct summand of objects in* X*, where* $X = \{k(x_j \nearrow 0) | x_j \in \mathcal{P}\}$ *then* H *induces a representation equivalence* $H_0 : (\mathrm{Mat}^{ad}_{\mathcal{P}})_0 \to \mathrm{Rep}\,\mathcal{P}$ *and an equivalence categories*

$$\mathrm{Mat}_{\mathcal{P}}/\mathcal{I} \cong \mathrm{Rep}\,\mathcal{P},$$

where $\mathcal{I} = \langle k(x_j \nearrow 0) | x_j \in \mathcal{P}\rangle$

From 1.13 and 1.14 it is obtained the functor $Hq : \mathrm{Mat}_{\mathcal{P}} \to \mathrm{Rep}\,\mathcal{P}$ which induces a one to one correspondence between the equivalence classes of indecomposables in $(\mathrm{Mat}^{ad}_{\mathcal{P}})_0$ and the isomorphism classes of indecomposables in Rep$\mathcal{P}$. If $M = (M_{x_i})_{x_i \in \mathcal{P}}$ is a matrix in $\mathrm{Mat}_{\mathcal{P}}$ and $cdn(M) = (d_0, d_{x_i})_{x_i \in \mathcal{P}}$ then $Hq(M) = (U_0, U_{x_j} \mid x_j \in \mathcal{P})$, where $U_0 = k^{d_0}$ and

$$U_{x_j} = \sum_{x_i \le x_j} \sum_{v \in c(M_{x_i})} kv \subset k^{d_0},$$

and $v \in c(M_{x_i})$ means that v is a column of M_{x_i}, considered as a vector in k^{d_0}. The inverse correspondence $U \mapsto M_U$, associates to each representation $U \in \mathrm{Rep}\,\mathcal{P}$ the matrix block of the linear application

$$(r_{x_1}, r_{x_2}, \ldots, r_{x_t}) : \bigoplus_{i=1}^{t} W_{x_i} \to U_0,$$

with respect to fixed basis of $W_{x_1}, \ldots, W_{x_t}, U_0$ and $W_{x_j} \subset U_{x_j}$ is a subspace of U_{x_j}, such that $\underline{U_{x_j}} \bigoplus W_{x_j} = U_{x_j}$ and r_{x_j} is the monomorphism composition $W_{x_j} \to U_{x_j} \to U_0$, and

$$\underline{U_{x_j}} = \sum_{x_i \le x_j} U_{x_i} = \sum_{x_i \in x_j{\blacktriangle}} U_{x_i}.$$

Following the ideas given above, we will now introduce the additivisation $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ of matrix problem for posets with an involution $\mathrm{Mat}(\mathcal{P}, \theta)$.

The objects $\underline{V} = (V_0, V_z)_{z \in \theta} \xrightarrow{\{t_a\}_{a \in z}} V_0$ of the category $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ consist of a k- vector space V_0 and a function that assigns to each element $z \in \theta$ a finite dimensional k-vector space V_z together with a linear transformation $t_a : V_z \to V_0$ for each $a \in z$.

If $\underline{V}' = (V'_0, V'_z)_{z \in \theta} \xrightarrow{\{t'_a\}_{a \in z}} V'_0$ is another object of this category, then a morphism of $\underline{V}$ to $\underline{V}'$ consists of linear transformations $f_z : V_z \to V'_z$ for $z \in \theta$, $f_0 : V_0 \to V'_0$ and for each relation $a < b$ a linear transformation $f_{a,b} : V_{[b]} \to V_{[a]}$ such that the following equation is satisfied:

$$f_0 t_a = t'_a f_z + \sum_{b<a} t'_b f_{b,a},$$

for all $a \in \mathcal{P}$.

If $g : \underline{V}' \to \underline{V}''$ is another morphism then gf is given by $(gf)_0 = g_0 f_0, (gf)_z = g_z f_z$ and for each relation $a < b$,

$$(gf)_{a,b} = g_{[a]} f_{a,b} + g_{a,b} f_{[b]} + \sum_{a<c<b} g_{a,c} f_{c,b}.$$

It can be proved that gf is indeed a morphism.

The category $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ is equivalent to the category of representations of the tensor differential algebra defined by quiver algebra kQ where Q has as many points as the cardinality of θ plus one. For every z in θ we take a vertex v_z and an additional vertex v_0. For each x in $\mathcal{P}$ we take a continuous arrow $\alpha_x : v_{[x]} \to v_0$, and for each relation $x < y$ a dashed arrow $\gamma_{x,y} : v_{[y]} \to v_{[x]}$, where for $x \in \mathcal{P}$ we denote by $[x]$ the class in θ containing x and a differential $\delta : kQ \to kQ$ such that

$$\begin{aligned} \delta(\alpha_x) &= -\sum_{y<x} \alpha_y \gamma_{y,x}, \\ \delta(\gamma_{y,x}) &= \sum_{y<z<x} \gamma_{y,z} \gamma_{z,x}. \end{aligned}$$

For a result independently obtained with different methods by Bautista and Kleiner on the one hand and by Butler and Burt on the other, in the category of representations of kQ the idempotents split. This category has an exact structure and almost split sequences. Thus, in the category $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ the idempotents split, there are an exact structure and almost split sequences. As in Lemma 1 there is a functor

$$q : \mathrm{Mat}_{(\mathcal{P},\theta)} \to \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}, \tag{1.15}$$

with the properties $(a), (b), (c)$.

Also, we have a functor

$$F : \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)} \to \mathrm{Rep}(\mathcal{P}, \theta), \tag{1.16}$$

defined as follows: let $\underline{V} = (V_0, V_z) \xrightarrow{\{t_x\}_{x \in z}} V_0$ be an object of $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$. We have a linear transformation $u_z : V_z \to V_0^z$ given by

$$u_z(v) = \sum_{x \in z} i_x(t_x(v)),$$

let $u(z) = u_z(V_z) \subset V_0^z$, then

$$F(\underline{V}) = (V_0, \hat{V}_z)_{z \in \theta},$$

where $\hat{V}_z$ is the subspace of V_0^z generated by $u(z)$ and the vectors of the form $i_x(\pi_y(v))$, where $y < x$ and $v \in \mathcal{U}([y])$, as in (1.10).

Let now $f = (f_0, f_z, f_{x<y})$ be a morphism $\underline{V} \to \underline{V}'$ in $\mathrm{Rep}(\mathcal{P}, \theta)$. We consider the morphism $f_0 : V_0 \to V_0'$ and for each $z \in \theta$ the morphism $f_0^z : V_0^z \to (V_0')^z$. This morphism sends $\hat{V}_z$

in $\hat{V}'_z$ therefore f_0 is a morphism

$$F(f) : (V_0, \widetilde{V}_z)_{z\in\theta} \to (V'_0, \widetilde{V}'_z)_{z\in\theta}.$$

Then if $(M_x)_{x\in\mathcal{P}}$ is a matrix representation of $(\mathcal{P}, \theta)$ and it is obtained that $Fq((M_x)_{x\in\mathcal{P}})$ coincides with the vector space representations associated to $(M_x)_{x\in\mathcal{P}}$ as in (1.12). Therefore, it is obtained that the functor is dense.

For each $z \in \theta$, we will consider the object I_z of $\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}$ with $(I_z)_0 = 0$ and $(I_z)_w = 0$ for $z \neq w$ and $(I_z)_z = k$. Then the following result is obtained.

Proposition 1. *The functor F (1.16) induces an equivalence of categories:*

$$\mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}/\mathcal{I} \to \mathrm{Rep}(\mathcal{P}, \theta),$$

where $\mathcal{I}$ is the ideal generated by the objects I_z.

Proof. From definition of F it follows that for all $z \in \theta$, $F(I_z) = 0$, therefore if f is a morphism in $\mathcal{I}$, then $F(f) = 0$. So, F induces a functor

$$\underline{F} : \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}/\mathcal{I} \to \mathrm{Rep}(\mathcal{P}, \theta).$$

Since F is dense, then $\underline{F}$ is dense. Let us prove that $\underline{F}$ is full. For this, let $\underline{V}$, $\underline{V}' \in \mathrm{Mat}^{ad}_{(\mathcal{P},\theta)}/\mathcal{I}$ and $h : V_0 \to V'_0$ a morphism of $F(\underline{V}) \to F(\underline{V}')$. We will define a morphism $f : \underline{V} \to \underline{V}'$ such that $F(f) = h$. We put $f_0 = h : V_0 \to V'_0$. We now define $f_z : V_z \to V'_z$ for $z \in \theta$ and for each $y < x \in z$ a morphism $f_{y,x} : V_z \to V'_{[y]}$. In order to do this, for each pair $y < x \in z$, we take $Z(y, x) = V'_{[y]}$ and

$$\phi(y, x) = i_x \pi_y u([y]) : Z(y, x) \to \hat{V}'_z.$$

We take the morphism of k-vector space:

$$\rho = V'_z \bigoplus_{y<x\in z} Z(y, x) \xrightarrow{(u(z),(\phi(y,x)))_{y<x\in z}} \hat{V}'_z,$$

by definition of $\hat{V}'_z$ this morphism is an epimorphism. Also, $h^z_0(\hat{V}_z) \subset \hat{V}'_z$ therefore there exists a morphism

$$\Psi : V_z \to V'_z \bigoplus_{y<x\in z} Z(y, x),$$

such that

$$h^z_0 u(z) = \rho\Psi, \tag{1.17}$$

then, if the component of Ψ are the morphisms $f_z : V_z \to V'_z$ and for $y < x \in z$, $f_{y,x} : V_z \to V_{[y]}$, the equality (1.17) implies that

$$t'_x f_z + \sum_{y<x\in z} t'_y f_{y,x} = h_0 t_x.$$

Therefore, the family of morphisms $(f_0, f_z, f_{y,x})_{z\in\theta, y<x\in\theta}$ determine a morphism $f : \underline{V} \to \underline{V}'$ such that $F(f) = h$. This proves that functor F is full and therefore $\underline{F}$ is full.

Suppose now that we have a morphism $f : \underline{V} \to \underline{V}'$ such that $F(f) = 0$. So, if f is given by the morphisms family $(f_0, f_z, f_{y,x})_{z\in\theta, y<x\in\theta}$ it is obtained that $F(f) = f_0 = 0$. Let $\underline{W}$ be given by the family of linear transformations $V_z \to 0$ for $z \in \theta$. Let $g : \underline{V} \to \underline{W}$ given by the family of morphisms $g_z : id_{V_z} : V_z \to V_z = W_i$ and $g_0 = 0 : V_0 \to W_0 = 0$ and for $y < x \in z$, $g_{y,x} = 0$.

Let $h : \underline{W} \to \underline{V}'$ given by the family of morphisms $h_0 = 0 : 0 \to V_0'$ for $z \in \theta$, $h_z = f_z : W_z = V_z \to V_z'$. For $y < x \in z$, $h_{y,x} = f_{y,x} : W_z = V_z \to V_{[y]}'$. Then g is in effect a morphism and $hg = f$ and $\underline{W}$ is a direct sum of objects of the form I_z. So $f \in \mathcal{I}$. This proves that the functor $\underline{F}$ is faithful. Consequently $\underline{F}$ is an equivalence of categories.

■

The ideal $\mathcal{I}$ of Proposition 1 is an admissible ideal in the Shiping Liu sense; therefore, by using the Proposition 1.8 of [19] the following result is obtained. It is worth mentioning that we will obtain this result in Chapter 2 using another technique.

Theorem 4. *The category Rep$(\mathcal{P}, \theta)$ has almost split sequences.*

Let $(\mathcal{P}, \theta)$ be a poset with involution and let us consider ε as the family of sequences of morphisms:

$$(V_0, V_z) \xrightarrow{u} (E_0, E_z) \xrightarrow{v} (W_0, W_z),$$

in the category of representations Rep$(\mathcal{P}, \theta)$ such that:

1. The sequence $0 \to V_0 \xrightarrow{u} E_0 \xrightarrow{v} W_0 \to 0$ is exact.
2. For all $z \in \theta$, the sequence $0 \to V_z \xrightarrow{u_z} E_z \xrightarrow{v_z} W_z \to 0$ is exact.

Definition 30. *A sequence*

$$(V_0, V_z) \xrightarrow{u} (E_0, E_z) \xrightarrow{v} (W_0, W_z),$$

is an almost split sequence if u is a source morphism, g is a sink morphism and $(u, v) \in \varepsilon$.

Proposition 2. *Let*

$$(V_0^1, V_z^1)_{z\in\theta} \xrightarrow{f} (V_0^2, V_z^2)_{z\in\theta} \xrightarrow{g} (V_0^3, V_z^3)_{z\in\theta}, \tag{1.18}$$

an ε-sequence in Rep$(\mathcal{P}, \theta)$, then this is an almost split sequence if and only if f and g are irreducible morphisms.

Proof. We suppose that f and g are irreducible. Since g is irreducible, and $\operatorname{Rep}(\mathcal{P},\theta)$ has almost split sequences, then there is an almost split sequence in the form:

$$(L_0, L_z)_{z\in\theta} \xrightarrow{u} (W_0, W_z)_{z\in\theta} \oplus (V_0^2, V_z^2)_{z\in\theta} \xrightarrow{(v,g)} (V_0^3, V_z^3)_{z\in\theta}, \tag{1.19}$$

here u and (v,g) are irreducible morphisms.

We consider the morphism:

$$(0,f)^t : (V_0^1, V_z^1)_{z\in\theta} \to (W_0, W_z)_{z\in\theta} \bigoplus (V_0^2, V_z^2)_{z\in\theta},$$

so, it is obtained that $(v,g)(0,f)^t = gf = 0$, and by (1.19) it follows that there exists

$$h : (V_0^1, V_z^1)_{z\in\theta} \to (L_0, L_z)_{z\in\theta},$$

such that $uh = (0,f)$, where $u = (u_1, u_2)^t$ with $u_1 : (L_0, L_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ and $u_2 : (L_0, L_z)_{z\in\theta} \to (V_0^2, V_z^2)_{z\in\theta}$. Since u is irreducible then u_2 is irreducible and $f = u_2 h$, here u_2 is a morphism which is not split epimorphism, as f is irreducible then h is an isomorphism, so $\dim(V_0) = \dim(L_0)$.
As (1.18) is exact it is obtained that

$$\dim(V_0^1) + \dim(V_0^3) = \dim(V_0^2),$$

and as (1.19) is exact it follows that

$$\dim(V_0^1) + \dim(V_0^3) = \dim(V_0^2) + \dim(W_0);$$

therefore, $\dim(W_0) = 0$. Then the sequence (1.18) coincides with the sequence (1.19); therefore the sequence 1.18 is almost split. Conversely, if (1.19) is an almost split sequence, then f and g are irreducible morphisms. ■

1.5 The Classification Theorems

The theory of representations of finite dimensional algebras and other algebraic structures had a rapid development. The investigations in this direction began in the second half of the last century being stimulated initially by the investigations of Brauer and Thrall in the 1940s. In these works, Brauer and Thrall made two famous conjectures called B-T, I and B-T, II.

Brauer-Thrall I. *If A is a k-algebra of infinite type, there is no bound on the k-dimension of indecomposable finitely generated A-modules.*

Brauer-Thrall II. *If A is a k-algebra of infinite type (over an infinite field k), then for an infinite number of dimensions there is an infinite number of indecomposable modules of this dimension.*

The first conjecture was proved by Roiter in [28] and later on generalized for Artin algebras by Auslander in [4]. A proof of the second Brauer Thrall conjecture for finite-dimensional

k-algebras, k algebraically closed field appears in [21]. The first proof by different methods was given by Bautista for k algebraically closed with characteristic different from 2 in [6] and for the case of characteristic 2 Bongartz proved it in [10].

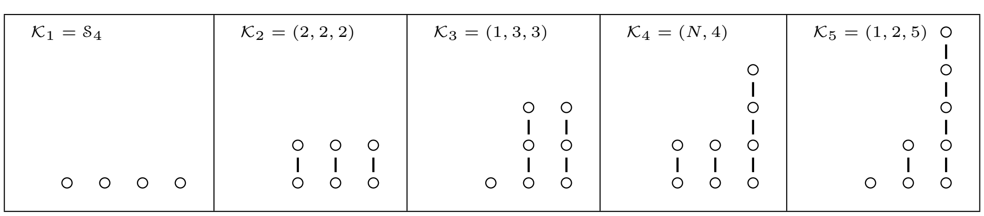
In the last decades, the study and classification of finite-dimensional algebras with respect to their representation type has been one of the main aims in the theory of representations of algebras. We recall that an algebra is said to be of finite representation type if there are only many finite isomorphism classes of indecomposable finite-dimensional modules. Later, the study was generalized to classify the algebras of infinite representation type. Special attention has been given to the so-called tame algebras, which are characterized by having for any fixed dimension only finitely many 1-parameter families of isomorphism classes of indecomposable modules. By a theorem of Drozd algebras which are not tame have to be wild which means that the classification of their indecomposable modules is as difficult as the classification of pairs of square matrices under simultaneous conjugation [15].

Theorem 5 (Theorem 14.14,[29]). *Every finite-dimensional k-algebra A over an algebraically closed field k is representation-finite, representation-tame or representation-wild and these types are mutually exclusive.*

The representations of ordinary posets were introduced by Nazarova and Roiter in 1972 to study of the representations of finite dimensional algebras [20]. The corresponding theory was developed during the 70s and 80s when in particular the main criteria were obtained for finite type representation [18], tame [23, 25] and finite growth [26].

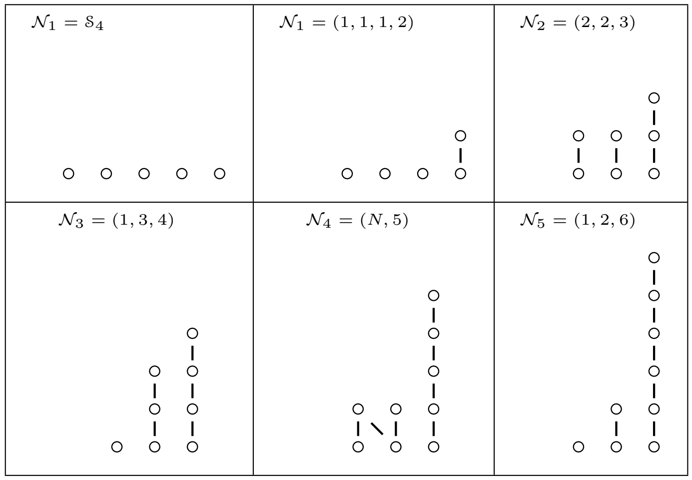
Kleiner [18] found out the following finite type criterion by using an algorithm of differentiation known as *differentiation with respect to a maximal point.*

Theorem 6 (Theorem 10.1,[29]). *A finite poset $\mathcal{P}$ is of finite representation type if and only if the poset $\mathcal{P}$ does not contain as a full subposet any of the following Kleiner's hypercritical posets*

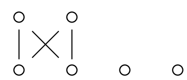


Nazarova [23] used differentiation with respect to a maximal point to prove the following tameness criterion.

Theorem 7 (Theorem 15.3, [29]). *A finite poset $\mathcal{P}$ of infinite representation type is tame if and only if $\mathcal{P}$ does not contain as a full subposet the following critical posets*



Theorem 8. *(Nazarova-Zavadskij 1981). A poset $\mathcal{P}$ of tame representation type is of finite growth representation type if and only if*

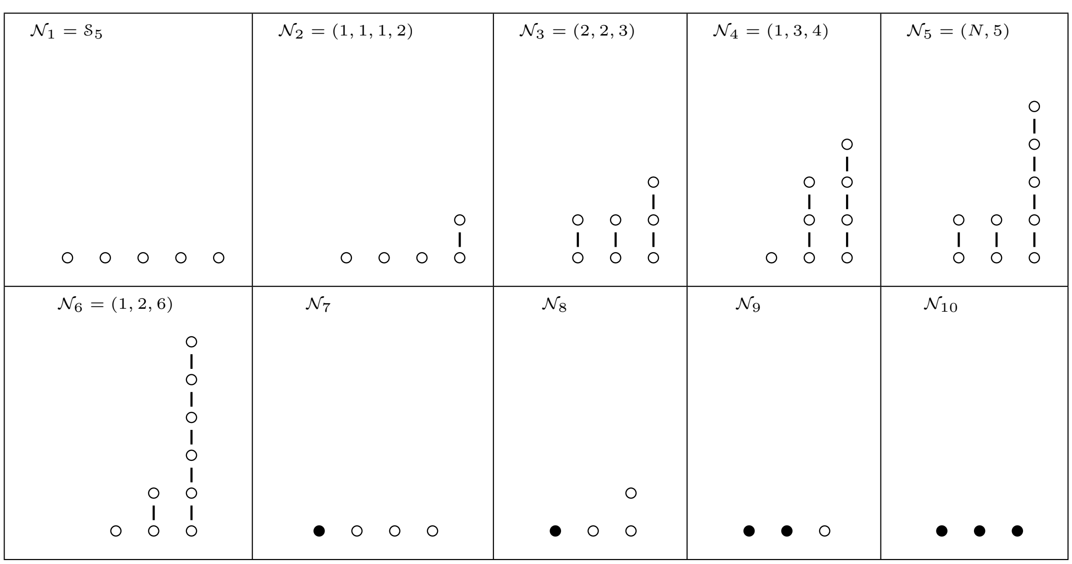
$\mathcal{P} \supsetneq$ 

In 1990, Nazarova et al [9] gave a generalization to the criterion shown in the Theorem 6 for the case of poset with involution.

Theorem 9 (Theorem, [9])**.** *For a poset with an involution $\mathcal{P}$ with an equivalence relation the following statements are equivalent:*

(a) $\mathcal{P}$ is tame (over an arbitrary field k);

(b) $\mathcal{P}$ does not contain subsets of the form:



(c) $\widetilde{\mathcal{P}}$ does not contain subsets $N_1, \ldots, N_6$.

Where $\widetilde{\mathcal{P}}$ *denoted the ordinary poset obtained from a poset with an involution by replacing every big point* x *by a pair of small points* x', x'' *inheriting all relations of the point* $x \in \mathcal{P}$.

The following is the finite growth criterion given by Bondarenko and Zavadskij [8]:

Theorem 10 (Theorem 2, [8])**.** *A tame poset* $\mathcal{P}$ *with involution is of finite growth if and only if all chains and nonperiodical cycles in* $\mathcal{P}$ *have no repetitions and do not intersect mutually (i.e., have no common vertices as graphs).*

We recall that if $x \sim y$ stands for equivalent points x and y and symbol $x||y$ is used to denote that points x and y are incomparable then a graph Q in a poset $\mathcal{P}$ is called a *cycle* if it has the form:

$$x_1 \sim x_2||x_3 \sim x_4|| \dots ||x_{2n-1} \sim x_{2n}, \text{ with } x_1 \sim x_{2n}, \quad n \geq 1,$$

besides a graph Q is said to be a *chain* in $\mathcal{P}$ if it has the form $A||B$, where $a||b$ for all $a \in A$ and $b \in B$ or $A||x_1 \sim x_2||x_3 \sim x_4|| \dots ||x_{2n-1} \sim x_{2n}||B$, where the set of vertices $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ are two-element small points subsets satisfying the conditions $a_1||a_2$ and $b_1||b_2$.

CHAPTER 2

An Exact Structure and Almost Split Sequences for the Category of Vector Space Representations of Posets with an Involution.

The Auslander-Reiten theory was introduced by Auslander and Reiten in 1975 [4] and has become a central tool in the theory of representations of finite-dimensional algebras. This theory has been extended to other categories including categories of representations of ordinary posets [[5], [29], [33]]. The most important theorem about almost split sequences in $\operatorname{Rep}(\mathcal{P})$ is that they exist. This theorem was given by Bautista and Martinez [1]. Taking into account that the category of representations of poset with an involution $(\mathcal{P},\theta)$ is not abelian, it is convenient to introduce a collection of sequences ε with some properties which will play the role of exact sequences.

In this chapter we prove that the category $(\operatorname{Rep}(\mathcal{P},\theta),\varepsilon)$ is exact and has enough injectives and projectives. Later we prove the existence of a categorical equivalence between $(\operatorname{Rep}(\mathcal{P},\theta),\varepsilon)$ and $\operatorname{mod}_{sp}(A)$ of socle projectives modules.

2.1 Exact Structure of $\operatorname{Rep}(\mathcal{P},\theta)$.

Let $\mathcal{A}$ be an additive category in which all idempotents split, and let ε be a collection of pairs of morphisms $M \xrightarrow{u} E \xrightarrow{v} N$. A morphism $u : M \xrightarrow{u} E$ is called an *ε-inflation* if there exists a morphism $v : E \to N$ such that $(u,v) \in \varepsilon$. A morphim $v : E \xrightarrow{v} N$ is called an *ε-deflation* if there exists a morphism $u : M \to E$ such that $(u,v) \in \varepsilon$.

The pair $(\mathcal{A};\varepsilon)$ will be called *exact structure* if the following conditions are satisfied:

1. The family ε is closed under isomorphisms; that is, if there exists a commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{u} & E & \xrightarrow{v} & N \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle r} \\ M_1 & \xrightarrow{u_1} & E_1 & \xrightarrow{v_1} & N_1 \end{array}$$

where s, t, r are isomorphisms and the top row is in ε, then the bottom row belongs to ε.

2. If $(u, v) \in \varepsilon$, then u is a kernel of v and v is a cokernel of u.

3. $\mathrm{id}_M : M \to M$ is both ε-inflation and ε- deflation.

4. **a.** For each ε-sequence $M \xrightarrow{f} E \xrightarrow{g} N$ and each morphism $w : X \to N$ there are morphisms $\beta : F \to X$ and $\lambda : F \to E$ such that the pair (λ, β) is a pullback of the pair (g, w) and β is an ε-deflation.

   b. For each ε-sequence $M \xrightarrow{f} E \xrightarrow{g} N$ and each morphism $u : M \to X$ there are morphisms $\alpha : X \to F$ and $\lambda : E \to F$ such that the pair (α, λ) is a pushout of the pair (u, f) and α is an ε-inflation.

5. The composition of ε-inflations (ε-deflations, respectively) is again an ε-inflation (ε-deflation, respectively).

6. If u_2u_1 is an ε-inflation then u_1 is an ε-inflation. If v_2v_1 is an ε-deflation then v_2 is an ε-deflation.

Example 8. *1. The category* mod$A$ *of modules over a ring A is an exact structure.*

2. The family of split exact sequences in mod$A$ *is an exact structure.*

In our case, let $(\mathcal{P}, \theta)$ be a poset with involution and let us consider ε the family of sequences of morphisms:

$$(V_0, V_z) \xrightarrow{u} (E_0, E_z) \xrightarrow{v} (W_0, W_z),$$

in the category of representations $\mathrm{Rep}(\mathcal{P}, \theta)$ such that:

1. The sequence $0 \to V_0 \xrightarrow{u} E_0 \xrightarrow{v} W_0 \to 0$ is exact.

2. For all $z \in \theta$, the sequence $0 \to V_z \xrightarrow{u_z} E_z \xrightarrow{v_z} W_z \to 0$ is exact.

Definition 31. *A morphism $f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ will be called a proper epimorphism if $f : V_0 \to W_0$ is an epimorphism and for each $z \in \theta$, $f^z : V_0^z \to W_0^z$ induces an epimorphism $f^z : V_z \to W_z$.*

Proposition 3. *Let $f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ be a proper epimorphism. If $U_0 = \ker(f)$ and $U_z = \ker(f^z) \cap V_z$ then $(U_0, U_z)_{z\in\theta}$ is a representation of $(\mathcal{P}, \theta)$.*

Proof. In the first place, we observe that for each $z \in \theta$, $\ker(f^z) = U_0^z$, then $U_z = U_0^z \cap V_z$. We suppose now that $a < a_1$ and $(x, y) \in U_{(a,b)}$ so, $(x, y) \in V_{(a,b)}$ then if $z_1 = (a_1, b_1), (x, 0) \in V_{(a_1,b_1)}$ as, $(x, y) \in U_{(a,b)}$, $(f(x), f(y)) = (0, 0)$ then $f(x) = 0$; therefore $(x, 0) \in U_{(a_1,b_1)}$. This proves that in effect $(U_0, U_z)_{z\in\theta}$ is a representation of $(\mathcal{P}, \theta)$. ■

Corollary 1. *If $f:(V_0,V_z)_{z\in\theta}\to(W_0,W_z)_{z\in\theta}$ is a proper epimorphism and $(U_0,U_z)_{z\in\theta}$ is as in the previous proposition, then an ε-sequence*

$$(U_0,U_z)_{z\in\theta}\xrightarrow{g}(V_0,V_z)_{z\in\theta}\xrightarrow{f}(W_0,W_z)_{z\in\theta},$$

is obtained. Therefore, f is an ε-deflation if and only if f is a proper epimorphism.

Definition 32. *A morphism $f:(U_0,U_z)_{z\in\theta}\to(V_0,V_z)_{z\in\theta}$ will be called a proper monomorphism if $f:U_0\to V_0$ is a monomorphism, $f^z:U_z\to V_z$ is a monomorphism for all $z\in\theta$ and for each $z=(a,b)\in\theta$ is satisfied that $(x,y)\in f^z(U_z)$ if and only if $y\in f(U_0)$ and $(x,y)\in V_z$.*

Proposition 4. *If $f:(U_0,U_z)_{z\in\theta}\to(V_0,V_z)_{z\in\theta}$ is a proper monomorphism then there exists a sequence in ε:*

$$(U_0,U_z)_{z\in\theta}\xrightarrow{f}(V_0,V_z)_{z\in\theta}\xrightarrow{g}(W_0,W_z)_{z\in\theta},$$

Proof. Let $g:V_0\to W_0$ be the cokernel of f. For $z\in\theta$ we define $W_z=f^z(V_z)$. We will check that $(W_0,W_z)_{z\in\theta}$ is a representation of $(\mathcal{P},\theta)$. Indeed, let $(x,y)\in W_{(a,b)}$ and $(a_1,b_1)\in\theta$ with $a<a_1$. Then $x=g(x_1)$, $y=g(y_1)$ with $(x_1,y_1)\in V_{(a,b)}$ so, $(x,0)\in V_{(a_1,b_1)}$, therefore $(x,0)=g(x_1,0)\in W_{(a_1,b_1)}$. This proves that $(W_0,W_z)_{z\in\theta}$ is a representation. We prove now that for each $z\in\theta$, the sequence:

$$0\to U_z\xrightarrow{f^z}V_z\xrightarrow{g^z}W_z\to 0,$$

is exact. Since f^z is a monomorphism, g^z is an epimorphism and $g^zf^z=0$. It only remains to prove that if $(x,y)\in V_z$ is such that $(g(x),g(y))=(0,0)$ then $(x,y)\in f^z(U_z)$. Since the sequence

$$0\to U\xrightarrow{f}V\xrightarrow{g}W\to 0,$$

is exact, then $(x,y)\in f(U)$. Also $(x,y)\in V_z$ and f is a proper monomorphism, it follows that $(x,y)\in f^z(U_z)$. This proves our affirmation. ■

From the above, it follows that f is an ε-inflation if and only if f is a proper monomorphism.

Proposition 5. *The pair $(\mathrm{Rep}(\mathcal{P},\theta),\varepsilon)$ is an exact category.*

Proof. The conditions 1, 2 and 3 are verified directly; the conditions 5 and 6 are followed by our characterization of ε-deflations and ε-inflations. Thus, it remains to prove the condition 4.
Let us show condition 4a. Let

$$(U_0,U_z)_{z\in\theta}\xrightarrow{u}(E_0,E_z)_{z\in\theta}\xrightarrow{v}(V_0,V_z)_{z\in\theta},$$

be an ε-sequence and let $f:(W_0,W_z)_{z\in\theta}\to(V_0,V_z)_{z\in\theta}$ be a morphism. We consider the direct sum $(E_0\bigoplus W_0,E_z\bigoplus W_z)_{z\in\theta}$ and the morphism:

$$\phi=(v,f):(E_0\bigoplus W_0,E_z\bigoplus W_z)\to(V_0,V_z)_{z\in\theta}.$$

Since $v^z : E_z \to V_z$ is an epimorphism, then $\phi_z = (v^z, f^z) : E_z \bigoplus W_z \to V_z$ is an epimorphism for all $z \in \theta$. Therefore, we obtain a sequence in ε:

$$(L_0, L_z)_{z\in\theta} \xrightarrow{(-h_1,h_2)^t} (E_0 \bigoplus W_0, E_z \bigoplus W_z)_{z\in\theta} \xrightarrow{\phi} (V_0, V_z)_{z\in\theta}, \tag{2.1}$$

and a commutative diagram:

$$\begin{array}{ccc}
(L_0, L_z)_{z\in\theta} & \xrightarrow{h_2} & (W_0, W_z)_{z\in\theta} \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f} \\
(E_0, E_z)_{z\in\theta} & \xrightarrow{v} & (V_0, V_z)_{z\in\theta}
\end{array}$$

We will prove that h_2 is an ε-deflation; for this purpose, we need to prove that $h_2 : L_0 \to W_0$ is an epimorphism, and that for all $z \in \theta$, $h_2^z : L_z \to W_z$ is an epimorphism. h_2 is an epimorhism since if $w \in W_0$ and we take $f(w) \in V_0$; as v is an epimorphism, then there exists $e \in E_0$ such that $f(w) = v(e)$ so, $\phi(w, -e) = f(w) - v(e) = 0$. As (2.1) is an ε-sequence, the exact sequence

$$0 \to L_0 \xrightarrow{(-h_1,h_2)^t} E_0 \bigoplus W_0 \xrightarrow{\phi} V_0 \to 0$$

is obtained. Therefore, there exists $x \in L_0$ such that $(-h_1(x), h_2(x)) = (-e, w)$, so $w = h_2(x)$; which proves that h_2 is an epimorphism.

For all $z \in \theta$, we have the exact sequence

$$0 \to L_z \xrightarrow{(-h_1^z,h_2^z)^t} (E_z \bigoplus W_z) \xrightarrow{\phi^z} V_z \to 0, \tag{2.2}$$

and by a similar argument to the previous one it is proved that $h_2^z : L_z \to W_z$ is an epimorphism. This allows us to conclude that h_2 is an ε-deflation and since (2.1) is an ε-sequence then the pair (h_1, h_2) is a pullback of the pair (v, f).

Now, we will prove the part (b) of the condition 4. We take an ε-sequence as in (2.1) and a morphism $f : (U_0, U_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$. We consider the morphism

$$\psi = (u, f)^t : (U_0, U_z)_{z\in\theta} \to (E_0 \bigoplus W_0, E_z \bigoplus W_z)_{z\in\theta},$$

and we prove that this is a proper monomorphism. Since u is a monomorphism then

$$\psi = (u, f)^t : U_0 \to E_0 \bigoplus W_0,$$

is a monomorphism. For $z \in \theta$ the morphism $u^z : U_z \to E_z$ is a monomorphism too; therefore $\psi^z = (u^z, f^z) : U_z \to E_z \bigoplus W_z$ is a monomorphism. In particular, its restriction to U_z is a monomorphism.

We suppose that $z = (a, b)$ and $(u(x), f(x), u(y), f(y)) \in E_z \bigoplus W_z$ then $(u(x), u(y)) \in E_z$. Since u is a proper monomorphism then there exists $(x_0, y_0) \in U_z$ such that $(u(x_0), u(y_0)) = (u(x), u(y))$, and since u is a monomorphism then $x = x_0$, $y = y_0$ and

$\psi((x,y)) = (u(x), f(x), u(y), f(y))$ with $(x,y) \in U_z$; therefore, φ is a proper monomorphism. Then the ε-sequence

$$(U_0, U_z)_{z\in\theta} \xrightarrow{\psi} (E_0 \bigoplus W_0, E_z \bigoplus W_z)_{z\in\theta} \xrightarrow{(g_1,-g_2)} (N_0, N_z)_{z\in\theta},$$

and the commutative diagram

$$\begin{array}{ccc}
(U_0, U_z)_{z\in\theta} & \xrightarrow{u} & (E_0, E_z)_{z\in\theta} \\
\downarrow f & & \downarrow g_1 \\
(W_0, W_z)_{z\in\theta} & \xrightarrow{g_2} & (N_0, N_z)_{z\in\theta}
\end{array}$$

are obtained.

In a similar way to case (a) it can be proved that g is a proper monomorphism and therefore an ε-inflation. From the construction of $(N_0, N_z)_{z\in\theta}$ we have that the pair (g_1, g_2) is a pushout of (u, f). ■

2.1.1 The ε-projectives

Definition 33. *A representation $(P_0, P_z)_{z\in\theta}$ will be called ε-projective if given a ε-deflation $g : (E_0, E_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$ and a morphism $f : (P_0, P_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$, there exists a morphism $h : (P_0, P_z)_{z\in\theta} \to (E_0, E_z)_{z\in\theta}$ such that $gh = f$.*

Remark 3. *The representation $\underline{S} = (k, S_z)_{z\in\theta}$ with $S_z = 0$ for all $z \in \theta$, is a projective representation.*

Let $w = (a, b) \in \theta$, we will define the representation $\underline{P}(w) = (P(w)_0, P(w)_w)$, where $P(w)_0 = k\langle e_1, e_2\rangle$ the vector space of dimension two with bases e_1, e_2. If a and b are incomparable $P(w)_w = \langle (e_1, e_2)\rangle$, while if $a < b$ then $P(w)_w = \langle (0, e_1), (e_1, e_2)\rangle$.

Henceforth, we will use the following notation, if $d_1, d_2 \in \mathcal{P}$ then

$$\lambda(d_1, d_2) = \begin{cases} 1 & \text{if} \quad d_1 < d_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $z = (a_1, b_1)$, the space $P(w)_z$ is the vector space generated by the vectors $(\lambda(a, a_1)e_1, 0), (0, \lambda(a, b_1)e_1), (\lambda(b, a_1)e_2, 0), (0, \lambda(b, b_1)e_2)$. In case that, $w = \{a\}$ then $P_0(w) = k\langle e\rangle$ and for $z = (a_1, b_1)$

$$P(w)_z = \langle (\lambda(a, a_1)e, 0), (0, \lambda(a, b_1)e)\rangle.$$

It can be verified that $\underline{P}(w) = (P_0(w), P(w)_z)_{z\in\theta}$ is in effect a representation.

Definition 34. *The element $(e_1, e_2) \in P(w)_w$ will be called the generator of the representation $(P(w)_0, P(w)_z)_{z\in\theta}$, when $w = (a, b)$ while the element $e \in P(w)_w$ is the generator when w consists of a single point.*

Proposition 6. *Let $(V_0, V_z)_{z\in\theta}$ be a representation of $(\mathcal{P}, \theta)$, then if $w = (a, b) \in \theta$ and $v \in V_w$ there exists an unique morphism $f : P(w) \to (V_0, V_z)_{z\in\theta}$ such that $f((e_1, e_2)) = v$. If $w = \{a\}$ and $v \in V_w$ there exists an unique morphism as before such that $f(e) = v$.*

Proof

1. If $w = (a, b)$ and $a < b$. Let $v = (v_1, v_2) \in V_{(a,b)}$ and $f : P(w)_0 \to V_0$ with $f(e_1) = v_1$; $f(e_2) = v_2$. Since $(0, v_1) \in V_{(a,b)}$, then $f_w(0, e_1) = (0, v_1)$ and $f_w((e_1, e_2)) = (v_1, v_2) \in V_{(a,b)}$; therefore $f_w(P(w)_w) \in V_w$.

   Let $(a_1, b_1) \in \theta$, then $P(w)_{(a_1,b_1)}$ is generated by the vectors $(\lambda(a, a_1)e_1, 0), (0, \lambda(a, b_1)e_1), (\lambda(b, a_1)e_2, 0), (0, \lambda(b, b_1)e_2)$. If $\lambda(a, a_1) \neq 0$ then $a < a_1$ and therefore $(v_1, 0) \in V_{(a_1,b_1)}$ and $f_z(\lambda(a, a_1)e_1, 0) = (v_1, 0) \in V_{(a,b)}$. In the same way, it is seen that f_z, sends each generator from $P(w)_{(a_1,b_1)}$ into $V_{(a_1,b_1)}$. The uniqueness of f is clear.

2. If $w = (a, b)$ and a, b are incomparable. In this case, $P(w)_{(a,b)} = \langle(e_1, e_2)\rangle$ and $f_w((e_1, e_2)) = (v_1, v_2) \in V_{(a,b)}$. Therefore, $f_w(P(w)(a, b)) \in V_{(a,b)}$. For the rest it is checked as in the previous case.

3. If $w = \{a\}$, the proof is similar to the previous cases.

■

Proposition 7. *The representations $\underline{P}(w) = (P(w)_0, P(w)_z)_{z\in\theta}$ have the following properties:*

1. *$\underline{P}(w)$ is an ε-projective representation.*

2. *$\mathrm{End}(\underline{P}(w)) \cong k$ if $w = (a, b)$ with a and b incomparable or when w consists of a single element. If $w = (a, b)$ with $a < b$ then $\mathrm{End}(\underline{P}(w)) \cong k[x]/x^2$. Therefore $\underline{P}(w)$ is indecomposable for all $w \in \theta$.*

3. *For any representation $(V_0, V_z)_{z\in\theta}$, there exists an ε-deflation $g : (Q_0, Q_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$, where $(Q_0, Q_z)_{z\in\theta}$ is ε-projective.*

4. *If $(Q_0, Q_z)_{z\in\theta}$ is an indecomposable projective representation of* Rep($\mathcal{P}, \theta$)*, with $Q_z \neq 0$ for some $z \in \theta$, then $(Q_0, Q_z)_{z\in\theta} \cong \underline{P}(w)$ for some $w \in \theta$.*

Proof

1. Let $f : (E_0, E_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$ an ε-deflation and $g : (P(w)_0, P(w)_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$ be a morphism. We take $f^w(\underline{e}) \in V_w$, where $\underline{e}$ is the generator of $(P(w)_0, P(w)_z)_{z\in\theta}$. Since f^w is surjective there exists $v_1 \in E_w$ such that $f^w(v_1) = v$. By Proposition 6 there exists a morphism $h : (P(w)_0, P(w)_z)_{z\in\theta} \to (E_0, E_z)_{z\in\theta}$ such that $h(\underline{e}) = v_1$, so $fh(\underline{e}) = g(\underline{e})$. By the uniqueness in the Proposition 6 is obtained that $fh = g$. Therefore, $(P(w)_0, P(w)_z)_{z\in\theta}$ is a projective representation.

2. If $w = (a, b)$ or $w = \{a\}$ then $P(w)_w = \langle\underline{e}\rangle$ with $\underline{e}$ the generator of $(P(w)_0, P(w)_z)_{z\in\theta}$; therefore, if $f : (P(w)_0, P(w)_z)_{z\in\theta} \to (P(w)_0, P(w)_z)_{z\in\theta}$ then $f(\underline{e}) = c\underline{e}$ with $c \in k$. Hence, $f = c(id_{P(w)_0})$. This proves that

$$\mathrm{End}((P(w)_0, P(w)_z)_{z\in\theta} = k(id_{P(w)}) \cong k.$$

We suppose now that $w = (a, b)$ with $a < b$, then if $\underline{e} = (e_1, e_2)$ is the generator of $P(w)_0$. We have that $P(w)_w = \langle (e_1, e_2), (0, e_1) \rangle$. Let f be an endomorphism of $(P(w)_0, P(w)_z)_{z\in\theta}$, then $f_w((e_1, e_2)) = c(e_1, e_2) + d(0, e_1)$ with $c, d \in k$. Therefore $f(e_1) = ce_1, f(e_2) = ce_2 + de_1$. In view of Proposition 6 the morphism f is completely determined by the matrix $M(f) = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$. If f_1 is another automorphism of $(P(w)_0, P(w)_z)_{z\in\theta}$, then $M(f_1 f) = M(f_1)M(f)$. Hence,

$$\mathrm{End}((P(w)_0, P(w)_z)_{z\in\theta}) \cong \left\{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \mid c, d \in k \right\} \cong k[x]/(x^2).$$

3. For V we choose a basis $B(0)$ and for each $z \in \theta$ such that $V_z \neq 0$, we choose $B(z)$ a k-basis of V_z. For each $v \in B(0)$ we take the morphism $f_v : \underline{S} \to (V_0, V_z)_{z\in\theta}$ which sends $1 \in k$ in $v \in V$ and for $v \in B(z)$ we have a morphism $f_v : (P(w)_0, P(w)_z)_{z\in\theta} \to (V, V_z)_{z\in\theta}$, such that $f_v(\underline{e}) = v$ where $\underline{e}$ is the generator of $(P(w)_0, P(w)_z)_{z\in\theta}$. Let $B = \bigcup_z B(z)$, then we have a morphism

$$f = (f_v)_{v\in B} : \bigoplus_{v\in B(0)} \underline{S} \bigoplus_z \bigoplus_{v\in B(z)} \underline{P}(z) \to (V_0, V_z)_{z\in\theta};$$

clearly this morphism is an ε-deflation and the representation

$$\bigoplus_{v\in B(0)} \underline{S} \bigoplus_z \bigoplus_{v\in B(z)} \underline{P}(Z),$$

is ε-projective.

4. Let $(Q_0, Q_z)_{z\in\theta}$ be a projective representation, such that for some $z \in Q_z \neq 0$. From the above, we have a deflation:

$$\underline{P} \xrightarrow{f} (Q_0, Q_z)_{z\in\theta},$$

then there exists a morphism $h : (Q_0, Q_z)_{z\in\theta} \to \underline{P}$ such that $fh = id_Q$. This implies that $(Q_0, Q_z)_{z\in\theta}$ is a direct sum of $\underline{P}$. The last representation is a direct sum of representations $\underline{S}$ and $\underline{P}(z)$; therefore, our representation is isomorphic to one of these, and as for some $z \in \theta$, $Q_z \neq 0$, then $(Q_0, Q_z)_{z\in\theta} \cong \underline{P}(w)$ for some $w \in \theta$.

■

Remark 4. *An exact category is said to have enough projectives if it satisfies property 3 of Proposition* 7.

Definition 35. *A representation $(I_0, I_z)_{z\in\theta}$ is called ϵ-injective if given an ε-inflation $f : (V_0, V_z)_{z\in\theta} \to (E_0, E_z)_{z\in\theta}$ and a morphism $g : (V_0, V_z)_{z\in\theta} \to (I_0, I_z)_{z\in\theta}$, there exists a morphism $h : (E_0, E_z)_{z\in\theta} \to (I_0, I_z)_{z\in\theta}$ such that $hf = g$.*

Henceforth it is convenient to use the following notation to represent poset with involution: the pair $(V_0, V_z)_{z\in\theta}$ where V_0 is a k-vector space and $V_z \subset V_0^z$, is a representation of $(\mathcal{P}, \theta)$

if and only if for each $x \leq y$ in $\mathcal{P}$, there exists a linear transformation $\tau : V_{[x]} \to V_{[y]}$ such that the following diagram is commutative

$$\begin{array}{ccc} V_{[x]} & \xrightarrow{i_{[x]}} & V_0^{[x]} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle i_y\pi_x} \\ V_{[y]} & \xrightarrow{i_{[y]}} & V_0^{[y]} \end{array}$$

where $i_{[x]} : V_{[x]} \to V_0^{[x]}$ and $i_y : V_0 \to V_0^{[y]}$ are the inclusions and $\pi_x : V_0^{[x]} \to V_0$ is the projection.

2.1.2 Representations by Quotients

Let $(\mathcal{P},\theta)$ be a poset with an involution and k be a field. A *representation by quotient* $(V_0, j_z)_{z\in\theta}$, consists of a k-vector space V_0 and for each $z \in \theta$ an epimorphism $j_z : V_0^z \to V_z$ such that if $a_1 < a$ and $z = (a,b)$, $z_1 = (a_1,b_1)$ then there exists a morphism $\tau : V_z \to V_{z_1}$ such that

$$\tau j_z = j_{z_1} i_{a_1} \pi_a.$$

A morphism $f : (V_0, j_z)_{z\in\theta} \to (V_0', j_z')_{z\in\theta}$ consists of a linear transformation $f_0 : V_0 \to V_0'$ and for each $z \in \theta$ a linear transformation $f^z : V_z \to V_z'$ such that the following diagram commutes

$$\begin{array}{ccc} V_0^z & \xrightarrow{j_z} & V_z \\ \downarrow{\scriptstyle f_0^z} & & \downarrow{\scriptstyle f_z} \\ (V_0')^z & \xrightarrow{j_z'} & V_z' \end{array}$$

We denote by $\mathrm{Repq}(\mathcal{P},\theta)$ the category of quotient representations.

Proposition 8. *There are functors*

$$C : \mathrm{Rep}(\mathcal{P},\theta) \to \mathrm{Repq}(\mathcal{P}^{op},\theta),$$

defined by $C((V_0, V_z)_{z\in\theta}) = (V_0, \mathrm{Coker}(i_z))_{z\in\theta}$, *where* $i_z : V_z \to V_0^z$ *is the inclusion and*

$$K : \mathrm{Repq}(\mathcal{P}^{op},\theta) \to \mathrm{Rep}(\mathcal{P},\theta),$$

where $K((V_0, j_z))_{z\in\theta} = (V_0, \mathrm{Ker}(j_z)_{z\in\theta})$. *Further,* $CK \cong id_{\mathrm{Repq}(\mathcal{P}^{op},\theta)}$ *and* $KC \cong id_{\mathrm{Rep}(\mathcal{P},\theta)}$ *therefore* $\mathrm{Rep}(\mathcal{P},\theta)$ *is equivalent to* $\mathrm{Repq}(\mathcal{P}^{op},\theta)$.

Proof Let $(V_0, V_z)_{z\in\theta}$ be an object of $\mathrm{Rep}(\mathcal{P},\theta)$ and we take $j_z : V_0^z \to V_z'$ the cokernel of i_z. We suppose that $x \in z$ and $y \in z_1$ with $x < y$ then we obtain the morphism $i_y\pi_x : V_0^z \to V_0^{z_1}$ and a morphism $\tau : V_z \to V_{z_1}$ such that $i_{z_1}\tau = i_y\pi_x i_z$. Therefore there exists a morphism $\tau' : V_z' \to V_{z_1}'$ such that the following diagram is commutative

$$A: \begin{tikzcd} V_z \arrow[r, "i_z"] \arrow[d, "\tau"] & V_0^z \arrow[r, "j_z"] \arrow[d, "i_y\pi_x"] & V'_z \arrow[d, "\tau'"] \\ V_{z_1} \arrow[r, "i_{z_1}"] & V_0^{z_1} \arrow[r, "j_{z_1}"] & V'_{z_1} \end{tikzcd}$$

This proves that $(V_0, j_z)_{z\in\theta} \in \text{Repq}(\mathcal{P}^{op}, \theta)$.

Now, let $f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ be a morphism in Rep$(\mathcal{P}, \theta)$; we denote $r_z : W_z \to W_0^z$ the inclusion and by $r'_z : W_0^z \to W'_z$ its cokernel. The morphism $g_z : V_z \to W_z$ is obtained, and it is such that $f_0^z i_z = r_z g_z$. Therefore there exists a morphism $f_z : V'_z \to W'_z$ such that the following diagram is commutative

$$B: \begin{tikzcd} V_z \arrow[r, "i_z"] \arrow[d, "g_z"] & V_0^z \arrow[r, "i'_z"] \arrow[d, "f_0^z"] & V'_z \arrow[d, "f_z"] \\ W_z \arrow[r, "r_z"] & W_0^z \arrow[r, "r'_z"] & W'_z \end{tikzcd}$$

thus, f_0^z is a morphism of $C((V_0, V_z)_{z\in\theta})$ in $C((W_0, W_z)_{z\in\theta})$. We define $C(f) = f_0^z$.

Now, if $(V_0, j_z)_{z\in\theta}$ is an object of Repq$(\mathcal{P}^{op}, \theta)$, by using diagram A, is obtained that $K((V_0, j_z)_{z\in\theta}) \in \text{Rep}(\mathcal{P}, \theta)$. If $f : (V_0, j_z)_{z\in\theta} \to (W_0, r'_z)_{z\in\theta}$ is a morphism in Repq$(\mathcal{P}^{op}, \theta)$ such that $f_0 : V_0 \to W_0$ then by using B is obtained that f_0 produces a morphism of $K((V_0, j_z)_{z\in\theta})$ in $K((W_0, r'_z)_{z\in\theta})$. The rest of the proof is clear. ∎

Henceforth, if W is a k-vector space $D(W) = \text{Hom}_k(W, k)$.

Proposition 9. *There are contravariant functors*

$$D_1 : \text{Rep}(\mathcal{P}, \theta) \to \text{Repq}(\mathcal{P}, \theta),$$

with $D_1((V_0, V_z)_{z\in\theta}) = (D(V_0), D(i_z))_{z\in\theta}$ *where* $i_z : V_z \to V^z$ *is the inclusion and* $D(i_z) : D(V^z) = D(V)^z \to D(V_z)$ *and*

$$D_2 : \text{Repq}(\mathcal{P}, \theta) \to \text{Rep}(\mathcal{P}, \theta),$$

with $D_2((V, j_z)_{z\in\theta}) = (D(V), im(D(j_z)))$*, further* $D_2D_1 \cong id_{\text{Rep}(\mathcal{P},\theta)}$ *and* $D_1D_2 \cong id_{\text{Repq}(\mathcal{P},\theta)}$.

Proof. We identify $D(V_0^Z) = D(V_0)^z$. Let $(V_0, V_z)_{z\in\theta} \in \text{Rep}(\mathcal{P}, \theta)$, then $D_1((V_0, V_z)_{z\in\theta}) = (D(V_0), D(i_z))_{z\in\theta}$ where $i_z : V_z \to V_0^z$ is the inclusion. Then if $a \in z$, $a_1 \in z_1$ with $a_1 < a$. Hence there exists a morphism $\tau : V_{z_1} \to V_z$ such that

$$i_z\tau = i_a\pi_{a_1}i_{z_1};$$

therefore,

$$D(\tau)D(i_z) = D(i_{z_1})D(\pi_{a_1})D(i_a),$$

We observe that $D(i_{a_1}) : D(V_0)^{z_1} \to D(V_0)$ is equal to π_{a_1} and $D(\pi_a) : D(V_0) \to D(V_0)^z$ is equal to i_a; therefore

$$D(\tau)D(i_z) = D(i_{z_1})i_{a_1}\pi_a.$$

The above implies that $D_1((V_0, V_z)_{z\in\theta}) \in \mathrm{Rep}(\mathcal{P},\theta)$. It is clear that

$$f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$$

is a morphism in $\mathrm{Rep}(\mathcal{P},\theta)$, then $D(f_0) : D(W_0) \to D(V_0)$ determines a morphism $D_1(f) : D_1((W_0, W_z)_{z\in\theta}) \to D_1((V_0, V_z)_{z\in\theta})$. The rest of the proposition proceeds in a similar way. ■

Definition 36. *We consider ε_q the class of sequences in* $\mathrm{Repq}(\mathcal{P},\theta)$ *which have the form*

$$(V_0^1, j_z^1)_{z\in\theta} \xrightarrow{f} (V_0^2, j_z^2)_{z\in\theta} \xrightarrow{g} (V_0^3, j_z^3)_{z\in\theta},$$

such that

$$0 \to V_0^1 \xrightarrow{f_0} V_0^2 \xrightarrow{g_0} V_0^3 \to 0,$$

and

$$0 \to V_z^1 \xrightarrow{f_z} V_z^2 \xrightarrow{g_z} V_z^3 \to 0,$$

are exact, where $j_z^i : (V_0^i)^z \to V_z^i$.

Proposition 10. *The functor D_1 sends ε-sequences to ε_q-sequences and the functor D_2 sends ε_q-sequences in ε-sequences. In particular, a morphism $f : (V_0, V_z)_{z\in\theta} \to (V_0', V_z')_{z\in\theta}$ in* $\mathrm{Rep}(\mathcal{P},\theta)$ *is an ε-inflation (ε-deflation, respectively) if and only if $D_1(f)$ is an ε-deflation (ε-inflation, respectively).*

Corollary 2. *The class of morphisms ε_q is an exact structure. Further the category* $\mathrm{Repq}(\mathcal{P},\theta)$ *has enough injectives.*

Corollary 3. *The exact category* $(\mathrm{Rep}(\mathcal{P},\theta),\varepsilon)$ *has enough injectives. The indecomposable injectives of this category have the form $KD_1(P_z)$ for $z \in \theta$ and $KD_1((k, 0_z)_{z\in\theta})$, where P_z and $(k, 0_z)_{z\in\theta}$ are projectives in* $\mathrm{Rep}(\mathcal{P}^{op},\theta)$.

Proof The indecomposable injectives of $\mathrm{Repq}(\mathcal{P}^{op},\theta)$ have the form $D_1(P_z)$ and $D_1((k, 0_z))_{z\in\theta}$. Since the functor K is an equivalence of categories such that sends ε_q-sequences in ε-sequences then the injectives indecomposables of $\mathrm{Rep}(\mathcal{P},\theta)$ are the form $KD_1(P_z)$ for $z \in \theta$ and $KD_1((k, 0_z)_{z\in\theta})$. ■

Remark 5. *$KD_1((k, 0_z)_{z\in\theta}) = J$ is the representation $J = (J_0, J_z)_{z\in\theta}$ such that $J_0 = k$ and $J_z = k^z$.*

2.2 The Endomorphism Algebra

Let $(\mathcal{P},\theta)$ be a poset with an involution. We know that in the exact category $(\mathrm{Rep}(\mathcal{P},\theta),\varepsilon)$ a system of representatives of isomorphism classes of indecomposable projectives is given by $\underline{P}_z$ for $z\in\theta$ and $\underline{S}=\underline{P}_0$. We take $\underline{P}=\bigoplus_{z\in\theta}\underline{P}_z\bigoplus\underline{S}$ and $A=\mathrm{End}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P})$. We have the functor $H:\mathrm{Rep}(\mathcal{P},\theta)\to \mathrm{mod}A$ given by

$$H(L)=\mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P},L).$$

The functor H sends ε-sequences in exact sequences, because $\underline{P}$ is ε-projective. We observe that $H(\underline{P})=A$. Let $\underline{P}_j$ with $j=z\in\theta$ or $j=0$. If we consider the projection $\pi_j:\underline{P}\to\underline{P}_j$ and the inclusion $\sigma_j:\underline{P}_j\to\underline{P}$, we obtain the idempotent $e_j=\sigma_j\pi_j\in\mathrm{End}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P})=A$. We have

$$1_A=\sum_{z\in\theta}e_z+e_0,$$

and

$$A=\bigoplus_{z\in\theta}e_zA\bigoplus e_0A.$$

Lemma 2. *$H(P_z)\cong e_zA$, $H(\underline{S})=e_0A$.*

Proof. We have the morphism $H(\pi_j):H(\underline{P})\to H(\underline{P}_j)$ and $H(\sigma_j):H(\underline{P}_j)\to H(\underline{P})$. It is obtained that $H(\pi_j)H(\sigma_j)=H(1_{\underline{P}_j})=1_{H(\underline{P}_j)}$. Therefore $H(\sigma_j)$ is a monomorphism and induces an isomorphism of $H(\underline{P}_j)$ in $\mathrm{Im}H(\sigma_j)$. Since

$$H(\sigma_j)=H(e_j)H(\sigma_j),\quad H(e_j)=H(\sigma_j)H(\pi_j),$$

then $\mathrm{Im}H(\sigma_j)=\mathrm{Im}H(e_j)=e_jA$. ■

Lemma 3. *For $L=(L_0;L_z)_{z\in\theta}\in\mathrm{Rep}(\mathcal{P},\theta)$ is obtained*

$$\mathrm{soc}H(L)\cong(e_0A)^l,$$

with $l=\dim_k(L_0)$.

Proof. Let $I=\theta\cup\{0\}$

$$\mathrm{rad}(A)=\bigoplus_{i,j\in I}e_i\mathrm{rad}Ae_j,$$

with $e_i\mathrm{rad}Ae_j=\mathrm{rad}(P_j,P_i)=\{f:P_i\to P_j\mid f\ \text{ is not an isomorphism}\}$. Since $e_0\mathrm{rad}(A)=\bigoplus_{j\in I}e_0\mathrm{rad}Ae_j=e_0\mathrm{rad}Ae_0=0$.
Then $(H(L)e_0)\mathrm{rad}A=H(L)(e_0\mathrm{rad}A)=0$. Therefore, $H(L)e_0\subset\mathrm{soc}H(L)$. On the other hand, $\mathrm{soc}H(L)=\bigoplus_{i\in I}\mathrm{soc}H(L)e_i$ and $\mathrm{soc}H(L)e_i=\mathrm{soc}H(L)\cap H(L)e_i$. Now, we will prove that for $i\neq 0$, $\mathrm{soc}H(L)\cap H(L)e_i=0$. We have

$$H(L)e_i=\mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}_i,L).$$

We suppose that $s\in\mathrm{soc}H(L)\cap H(L)e_i$ then if $s\neq 0$, $s:(\underline{P}_i)_0\to L_0$ is a non zero morphism, so there exists $x\in(\underline{P}_i)_0$ such that $s(x)\neq 0$. On the other hand, there exists a morphism $t : \underline{S} \to \underline{P}_i$ such that if y is a generator of $\underline{S}$, $t(y) = x$, so $st \neq 0$, the morphism $t : \underline{S} \to \underline{P}_i$ is not an isomorphism because $i \neq 0$, then $t \in \mathrm{rad}A$, therefore $s = 0$. From this it follows that

$$H(L)e_0 \subset \mathrm{soc}H(L) \subset H(L)e_0.$$

Consequently, $H(L)e_0 = \mathrm{soc}H(L)$. This implies that there is an epimorphism $(e_0A)^m \to H(L)e_0$. Since e_0A is simple then $H(L)e_0 \cong (e_0A)^l$ for some integer l.

Since, $e_0A = e_0Ae_0 \cong \mathrm{End}_A(e_0A) \cong \mathrm{End}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{S}) \cong k$. Therefore,

$$\dim_k H(L)e_0 = l\dim_k(e_0A) = l,$$

$$\dim_k H(L)e_0 = \dim_k(\underline{S}, L) = \dim_k L_0,$$

■

Definition 37. *Let* $f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ *be a morphism in* $\mathrm{Rep}(\mathcal{P}, \theta)$. *We define* $\mathrm{Im}(f) = (f(V_0), f^z(V_z))_{z\in\theta}$.

We can see that $\mathrm{Im}(f) \in \mathrm{Rep}(\mathcal{P}, \theta)$, also the morphism f induces a proper epimorphism

$$f : (V_0; V_z)_{z\in\theta} \to \mathrm{Im}(f).$$

Proposition 11. *1. If* $f : L \to L'$ *is a proper epimorphism in* $\mathrm{Rep}(\mathcal{P}, \theta)$ *then* $H(f) : H(L) \to H(L')$ *is an epimorphism.*

2. If $f : (V_0, V_z)_{z\in\theta} \to (W_0, W_z)_{z\in\theta}$ *is such that* $f : V_0 \to W_0$ *is a monomorphism then* $H(f)$ *is a monomorphism.*

Proof.

1. Since f is a proper epimorphism there is an ε-sequence $L'' \xrightarrow{g} L \xrightarrow{f} L'$, and since H is exact, the following sequence in $\mathrm{mod}A$ is obtained

$$0 \to H(L'') \xrightarrow{H(g)} H(L) \xrightarrow{H(f)} H(L') \to 0,$$

and therefore our statement is true.

2. Let $V = (V_0, V_z)_{z\in\theta}$, $W = (W_0, W_z)_{z\in\theta}$ and $\underline{P} = (\underline{P}_0, \underline{P}_z)_{z\in\theta}$. Now

$$H(f) = \mathrm{Hom}(1, f) : \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}, V) \to \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}, \theta, W),$$

then, if $0 \neq s \in \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}, V)$ and as $f : V_0 \to W_0$ is a monomorphism, then $fs : \underline{P}_0 \to W_0$ is non zero, so fs is non zero and therefore $H(f)$ is a monomorphism.

■

Proposition 12. *For all* $L \in \mathrm{Rep}(\mathcal{P},\theta)$*,* $\mathrm{soc}H(L)$ *is projective, in particular* $\mathrm{soc}A = \mathrm{soc}H(\underline{P})$ *is projective. Therefore,*

1. *A is a right peak algebra.*
2. *For $L \in \mathrm{Rep}(\mathcal{P},\theta)$, $H(L)$ is socle projective, that is, $H(L) \in \mathrm{mod}_{sp}(A)$.*
3. *Let $J = (k, k^z)_{z\in\theta}$, then $H(J) \cong E$, where E is the injective envelope of the simple e_0A.*
4. *The functor H induces an equivalence of categories:*

$$H : \mathrm{Rep}(\mathcal{P},\theta) \to \mathrm{mod}_{sp}(A),$$

Proof. Items (1) and (2) are obtained from the previous lemmas.
Now we will check item (3): since $\mathrm{soc}H(J) = e_0A$, therefore there is a monomorphism

$$H(J) \to E,$$

where E, the injective envelope of e_0A is $D(Ae_0)$.
It follows that for each $z \in \theta$, $D(Ae_0)e_z = D(e_zAe_0)$ and

$$e_zAe_0 = \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{S}, \underline{P}_z) \cong (\underline{P}_z)_z \cong k^z.$$

Therefore,

$$\dim_k D(e_zAe_0) = \mathrm{card(z)},$$

then for all $z \in \theta$, $\dim_k(J_z) = \dim_k Ee_z$. This implies that the monomorphism $H(J) \to E$ is an isomorphism.
To check item (4), we first observe that there is an isomorphism

$$H : \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}_i, \underline{P}_j) \to \mathrm{Hom}_A(e_iA, e_jA),$$

Indeed, the composition of morphism H with monomorphism $\mathrm{Hom}_A(e_iA, e_jA) \cong e_jAe_i$ is an isomorphism, so H is too. This implies that $H : \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(Q, Q') \to \mathrm{Hom}_A(H(Q_1), H(Q_2))$ is an isomorphism when Q_1, Q_2 are ε-projectives in $(\mathrm{Rep}(\mathcal{P},\theta), \varepsilon)$.
Let now, $L \in \mathrm{Rep}(\mathcal{P},\theta)$, by (3) of Proposition 7 we have ε-sequences

$$K(L) \to Q_1(L) \to L,$$
$$K_1(L) \to Q_2(L) \to K(L).$$

As H is an exact functor the exact sequences in $\mathrm{mod}_{sp}(A)$

$$0 \to H(K(L)) \to H(Q_1(L)) \to H(L) \to 0,$$
$$0 \to H(K_1(L)) \to H(Q_2(L)) \to H(K(L)) \to 0,$$

are obtained. Therefore, if f_L is the composition of $Q_2(L) \to K(L)$ with $K(L) \to Q_1(L)$ and $g_L : Q_1(L) \to L$ the projective presentation of L

$$Q_2(L) \xrightarrow{f_L} (Q_1(L)) \xrightarrow{g_L} L,$$

and the projective presentation of $H(L)$

$$H(Q_2(L)) \xrightarrow{H(f_L)} H(Q_1(L)) \xrightarrow{H(g_L)} H(L).$$

are obtained.

Let $\alpha : L \to L'$ be a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$, then there exist morphisms $h_1 : Q_1(L) \to Q_1(L')$, $h_2 : Q_2(L) \to Q_2(L')$ and commutative diagrams

$$\begin{tikzcd}
Q_2(L) \arrow[r, "f_L"] \arrow[d, "h_2"] & Q_1(L) \arrow[r, "g_L"] \arrow[d, "h_1"] & L \arrow[d, "\alpha"] \\
Q_2(L') \arrow[r, "f_{L'}"] & Q_1(L') \arrow[r, "g_L"] & L'
\end{tikzcd}$$

$$\begin{tikzcd}
H(Q_2(L)) \arrow[r, "H(f_L)"] \arrow[d, "H(h_2)"] & H(Q_1(L)) \arrow[r, "H(g_L)"] \arrow[d, "H(h_1)"] & L \arrow[d, "\alpha"] \\
H(Q_2(L')) \arrow[r, "H(f_{L'})"] & H(Q_1(L')) \arrow[r, "H(g_L)"] & H(L')
\end{tikzcd}$$

The functor H is faithful; indeed if $H(\alpha) = 0$ then there exists a morphism $t : H(Q_1(L)) \to H(Q_2(L'))$ such that $H(f_{L'})t = H(h_1)$. Here $t = H(s)$ for some $s : Q_1(L) \to Q_2(L')$, therefore $f_{L'}s = h_1$, which implies that $\alpha = 0$.

Now, we prove that H is full: let $\beta : H(L) \to H(L')$ be a morphism, then there exist morphisms $t_1 : H(Q_1(L)) \to H(Q_1(L'))$,and $t_2 : H(Q_2(L)) \to H(Q_2(L'))$ such that $H(g_{L'})t_1 = \beta H(g_L)$, $H(f_{L'})t_2 = t_1 H(f_L)$. Then there exist morphisms $h_1 : Q_1(L) \to Q_1(L')$, $h_2 : Q_2(L) \to Q_2(L')$ such that $f_{L'}h_2 = h_1 f_L$ and due to the properties of projective presentations, there is a morphism $\alpha : L \to L'$ such that $g_{L'}h_1 = \alpha g_L$. Applying functor H we obtain the equalities

$$H(g_{L'})H(h_1) = H(\alpha)H(g_L) = \beta H(g_L).$$

Since $H(g_L)$ is an epimorphism then $H(\alpha) = \beta$.

Finally, we prove that H is a dense functor. Let $M \in \mathrm{mod}_{sp}(A)$, so $\mathrm{soc(M)} \cong (e_0 A)^l$ for some natural l. Therefore, an injective envelope

$$u : M \to H(J)^l$$

and a projective cover

$$v : H(Q) \to M$$

are obtained, where Q is an ε-projective. We take $f = uv : H(Q) \to H(J)$, then there exists $g : Q \to J$ such that $H(g) = f$. We have a proper epimorphism $v' : Q \to \mathrm{Im}(g)$ and an inclusion $u' : \mathrm{Im}(g) \to J^l$, and $g = u'v'$. Then $f = H(g) = H(u')H(v')$ with $H(v')$ an epimorphism and $H(u')$ a monomorphism. Therefore

$$M = \mathrm{Im} f \cong H(\mathrm{Im}(g)).$$

■

In [29] it was proved that category $\mathrm{mod}_{sp}(A)$ has almost split sequences, therefore as consequence from (4) of Proposition 12 we obtain the following proposition.

Theorem 11. *The category* $\mathrm{Rep}(\mathcal{P}, \theta)$ *has almost split sequences.*

Proposition 13. *Let* $\mathcal{P} = \{1, 2, \ldots, m\}$ *be a poset and* $\hat{\mathcal{P}} = \mathcal{P} \cup \{0\}$ *with* $i < 0$*, for all* $i \in \mathcal{P}$*. Then the algebra* $A = \mathrm{End}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}) \cong I(\hat{\mathcal{P}})$*, the incidence algebra of* $\hat{\mathcal{P}}$*.*

Proof. For $i, j \in \hat{\mathcal{P}} : e_j A e_i = \mathrm{Hom}_{\mathrm{Rep}(\mathcal{P},\theta)}(\underline{P}_i, \underline{P}_j) \cong k$ if $j \leq i$, in other cases $e_j A e_i = 0$.
■

CHAPTER 3

The Auslander-Reiten Quiver of Posets with an Involution of Type D_n.

In this chapter, by using the results from the previous chapter, we construct the Auslander-Reiten quiver for a poset type that we will denote by D_n. This result is of interest to us since it will be used in Chapter 5 where we will describe the categorical properties of the DIII differentiation algorithm. In this chapter we will assume that k is an algebraically closed field.

3.1 Poset with an Involution of Type D_n

We denote by D_n to the poset with an involution $(\mathcal{P}, \leq, \theta)$ where $(\mathcal{P}, \leq) = \{a_n < a_{n-1} < \cdots < a_1 < b_1 < b_2 < \cdots < b_{n-1} < b_n\}$ and $\theta = \{(a_i, b_i)\}_{i=1,\ldots,n}$. We denote by $\mathrm{Rep}(\mathrm{D}_n)$ the category of representations of the poset D_n.

The Hasse diagram of the poset D_n is as follows

$$(\mathcal{P}, \theta) = \begin{array}{rc} b_n & \bullet \\ & | \\ b_{n-1} & \bullet \\ & \vdots \\ b_1 & \bullet \\ & | \\ a_1 & \bullet \\ & \vdots \\ a_{n-1} & \bullet \\ & | \\ a_n & \bullet \end{array}$$

FIGURE 3.1. Hasse diagram of poset with an involution of type D_n

We consider the following representations of D_n :

a. $\mathcal{L}_{1,i} = (\mathcal{L}_0, \mathcal{L}_{(a_j,b_j)})_{j\geq 1}$ where $\mathcal{L}_0 = k\{e\}$ and

$$\mathcal{L}_{(a_j,b_j)} = \begin{cases} (0,0), & \text{if} \quad j < i, \\ \langle (0,e) \rangle, & \text{if} \quad j \geq i. \end{cases} \tag{3.1}$$

b. $\mathcal{L}_{2,i} = (\mathcal{L}_0, \mathcal{L}_{(a_j,b_j)})_{j\geq 1}$ where $\mathcal{L}_0 = k\{e\}$ and

$$\mathcal{L}_{(a_j,b_j)} = \begin{cases} \langle (0,e),(e,0) \rangle, & \text{if} \quad j \leq i, \\ \langle (0,e) \rangle, & \text{if} \quad j > i. \end{cases} \tag{3.2}$$

c. $\mathcal{L}_{3,i} = (\mathcal{L}_0, \mathcal{L}_{(a_j,b_j)})_{j\geq 1}$ where $\mathcal{L}_0 = k\{e_1, e_2\}$ and

$$\mathcal{L}_{(a_j,b_j)}) = \begin{cases} \langle (0,e_1),(e_1,0) \rangle, & \text{if} \quad j < i, \\ \langle (0,e_1),(e_1,e_2) \rangle, & \text{if} \quad j = i, \\ \langle (0,e_1),(0,e_2) \rangle, & \text{if} \quad j > i. \end{cases} \tag{3.3}$$

It is clear that each one of the previous representations are indecomposable.

Remark 6. *The representation $\underline{S} = (k, S_z)_{z\in\theta}$ with $S_z = 0$ for all $z \in \theta$, is called a trivial indecomposable representation of* D_n.

Proposition 14. *The representations above is the complete list of non trivial indecomposable representations of* D_n.

Proof. We will prove by induction on n that any representation of D_n can be written as a direct sum of some representations isomorphic to those presented in the previous list.

Let $\mathcal{V} = (V_0, V_{(a_1,b_1)})$ be a representation of D_1. We consider the corresponding matrix representation (M_{a_1}, M_{b_1}), so there are nonsingular matrices S, T such that

$$SM_{a_1}T = N_{a_1}, \qquad SM_{b_1}T = N_{b_1},$$

where (N_{a_1}, N_{b_1}) is the direct sum of pairs of matrices as follows:

(a) $\begin{pmatrix} E_n \\ 0_{1,n} \end{pmatrix}, \begin{pmatrix} 0_{1,n} \\ E_n \end{pmatrix}$.

(b) $\begin{pmatrix} 0_{1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} E_n \\ 0_{1,n} \end{pmatrix}$.

(c) $(E_n, 0_{n,1}), (0_{n,1}, E_n)$.

(d) $(0_{n,1}, E_n), (E_n, (0_{n,1})$.

(e) $J_{\lambda,n}, E_n$.

(f) $E_n, J_{\lambda,n}$.

By Definition 8, the matrix representation (M_{a_1}, M_{b_1}) is equivalent to the matrix representation (N_{a_1}, N_{b_1}). Therefore the vector space representation $\mathcal{V}$ is isomorphic to the direct sum of the matrix representations listed above.

We will check in each case the corresponding vector space representation. For this, we will remember from section 1.3 and by using the notation of this section that for $\mathcal{V} = (k^m, V_{(a_1,b_1)})$, taking $U = U_{(a_1,b_1)}$ and $U^+ = \pi_{a_1}(U)$ it is obtained that U and $(0, U^+)$ generate $V_{(a_1,b_1)}$.

(a) In this case the vector space representation is given by $(k^{n+1}, V_{(a_1,b_1)})$ where the space $V_{(a_1,b_1)} \subset k^{n+1} \bigoplus k^{n+1}$, with

$$U(1) = (e_1, e_2), U(2) = (e_2, e_3), \ldots, U(n) = (e_n, e_{n+1}),$$

thus, $U^+ = \{e_1, \ldots, e_n\}$ and

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (e_1, e_2), (e_2, e_3), \ldots, (e_n, e_{n+1})\rangle \cup \langle (0, e_1), (0, e_2), \ldots, (0, e_n)\rangle \\ &= \langle (e_1, 0), (0, e_1), (e_2, 0), (0, e_2), \ldots, (0, e_n), (e_n, e_{n+1})\rangle, \end{aligned}$$

from here it follows

$$(k^{n+1}, V_{(a_1,b_1)}) = \mathcal{L}_{2,1}(e_1) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_{n-1}) \bigoplus \mathcal{L}_{3,1}(e_n, e_{n+1}).$$

(b) Here the vector space representation is $(k^{n+1}, V_{(a_1,b_1)})$ and

$$U(1) = (e_2, e_1), \quad U(2) = (e_3, e_2), \quad \ldots \quad U(n) = (e_{n+1}, e_n),$$

then $U^+ = \{e_2, e_3, \ldots, e_{n+1}\}$, and

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (e_2, e_1), (e_3, e_2), \ldots, (e_{n+1}, e_n)\rangle \cup \langle (0, e_2), (0, e_3), \ldots, (0, e_{n+1})\rangle \\ &= \langle (0, e_2), (e_2, e_1), (e_3, 0), (0, e_3), \ldots, (e_{n+1}, 0), (0, e_{n+1})\rangle. \end{aligned}$$

Therefore,

$$(k^{n+1}, V_{(a_1,b_1)}) = \mathcal{L}_{3,1}(e_2, e_1) \bigoplus, \mathcal{L}_{2,1}(e_3) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_{n+1}).$$

(c) The corresponding vector space representation in this case is $(k^n, V_{(a_1,b_1)})$, where

$$U(1) = (e_1, 0), \quad U(2) = (e_2, e_1), \quad \ldots \quad U(n) = (e_n, e_{n-1}), \quad U(n+1) = (0, e_n),$$

here, $U^+ = \{e_1, e_2, \ldots, e_n\}$ and

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (e_1, 0), (e_2, e_1), \ldots, (e_n, e_{n-1}), (0, e_n)\rangle \cup \langle (0, e_1), (0, e_2), \ldots, (0, e_n)\rangle \\ &= \langle (e_1, 0), (0, e_1), (0, e_2), (e_2, 0), \ldots, (e_n, 0), (0, e_n)\rangle. \end{aligned}$$

Therefore,

$$(k^n, V_{(a_1,b_1)}) = \mathcal{L}_{2,1}(e_1) \bigoplus \mathcal{L}_{2,1}(e_2) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_n),$$

(d) This case is similar to the previous one.

(e) Here the vector space representation is $(k^n, V_{(a_1,b_1)})$ and

$$U(1) = (\lambda e_1, e_1), \qquad U(2) = (\lambda e_2 + e_1, e_2), \qquad \dots, \qquad U(n) = (\lambda e_n + e_{n-1}, e_n),$$

then, if $\lambda \neq 0$, $U^+ = \{e_1, e_2, \dots, e_n\}$ and

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (\lambda e_1, e_1), , (\lambda e_2 + e_1, e_2), \dots, (\lambda e_n + e_{n-1}, e_n)\rangle \cup \langle (0, e_1), (0, e_2), \dots, (0, e_n)\rangle \\ &= \langle (e_1, 0), (0, e_1), (0, e_2), (e_2, 0), \dots, (e_n, 0), (0, e_n)\rangle, \end{aligned}$$

therefore,

$$(k^n, V_{(a_1,b_1)}) = \mathcal{L}_{2,1}(e_1) \bigoplus \mathcal{L}_{2,1}(e_2) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_n).$$

Now, if $\lambda = 0$ and $n = 1$, $U(1) = (0, e_1)$, in this case

$$(k, V_{(a_1,b_1)}) = \mathcal{L}_{1,1}(e_1).$$

For $n \geq 2$,

$$U(1) = (0, e_1), \quad U(2) = (e_1, e_2), \quad \dots \quad U(n) = (e_{n-1}, e_n),$$

so, $U^+ = \langle e_1, e_2, \dots, e_{n-1}\rangle$ then

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (0, e_1), (e_1, e_2), \dots, (e_{n-1}, e_n)\rangle \cup \langle (0, e_1), (0, e_2), \dots, (0, e_{n-1})\rangle \\ &= \langle (e_1, 0), (0, e_1), (0, e_2), (e_2, 0), \dots, (e_{n-2}, 0), (0, e_{n-2}), (0, en-1), (e_{n-1}, e_n)\rangle, \end{aligned}$$

therefore,

$$(k^n, V_{(a_1,b_1)}) = \mathcal{L}_{2,1}(e_1) \bigoplus \mathcal{L}_{2,1}(e_2) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_{n-2}) \bigoplus \mathcal{L}_{3,1}(e_{n-1}, e_n).$$

(f) Here the vector space representation is $(k^n, V_{(a_1,b_1)})$ and

$$U(1) = (\lambda e_1, \lambda e_1), \quad U(2) = (e_2, e_1 + \lambda e_2), \quad \dots \quad U(n) = (e_n, e_{n-1} + e_{n-1} + \lambda e_n),$$

as $U^+ = \langle e_1, e_2, \dots, e_n\rangle$ then

$$\begin{aligned} V_{(a_1,b_1)} &= \langle (e_1, \lambda e_1), (e_2, e_1 + \lambda e_2), \dots, (e_n, e_{n-1} + \lambda e_n)\rangle \cup \langle (0, e_1), (0, e_2), \dots, (0, e_n)\rangle \\ &= \langle (e_1, 0), (0, e_1), (0, e_2), (e_2, 0), \dots, (e_n, 0), (0, e_n)\rangle. \end{aligned}$$

So,

$$(k^n, V_{(a_1,b_1)}) = \mathcal{L}_{2,1}(e_1) \bigoplus \mathcal{L}_{2,1}(e_2) \bigoplus \cdots \bigoplus \mathcal{L}_{2,1}(e_n).$$

This shows our result for case $n = 1$.

We assume the result is true for D_{n-1}. Let $\mathcal{V} = (V_0, V_{(a_j,b_j)})_{j\geq 1}$ be a representation of D_n. We take $\mathrm{D}_1 = (a_n < b_n; (a_n, b_n))$ and $\mathcal{V} = (V_0, V_{(a_n,b_n)})$ a representation of D_1. Then

$$V_{(a_n,b_n)}) = (0, V_1) \bigoplus (V_2, 0) \bigoplus (0, V_2) \bigoplus \bigoplus_{i=1}^{l} \mathcal{L}_{3,1}(e_i, f_i),$$

where, $V_0 = V_1 \bigoplus V_2 \bigoplus V_3 \bigoplus V_3' \bigoplus W$, with $V_3 = \langle e_1, e_2, \ldots, e_l \rangle$ and $V_3' = \langle f_1, f_2, \ldots, f_l \rangle$. Let $\phi : V_3 \to V_3'$ be an isomorphism such that $\phi(e_i) = f_i$, then

$$\bigoplus_{i=1}^{l} \mathcal{L}_{3,1}(e_i, f_i) = (0, V_3) \bigoplus H_\phi(V_3, V_3'),$$

where, $H_\phi(V_3, V_3') = \{(u, \phi(u)) \mid u \in V_3\}$. Thus,

$$V_{(a_n,b_n)} = (0, V_1) \bigoplus (V_2, 0) \bigoplus (0, V_3) \bigoplus H_\phi(V_3, V_3'),$$

Let $\mathrm{D}_{n-1} = \{a_{n-1} < \cdots < a_1 < b_1 < \cdots < b_{n-1}; (a_1, b_1), \ldots, (a_{n-1}, b_{n-1})\}$ and $\underline{V} = (V_0, V_{(a_i,b_i)})_{i \le n-1}$ be the restriction of the representation V to D_{n-1}.

Remark 7. *If $u \in V_2 \bigoplus V_3$ then $(u, 0)$ and $(0, u)$ are in $V_{(a_i,b_i)}$ for all $i < n$. Indeed, if $u \in V_2$, $(u, 0) \in V_{(a_n,b_n)}$. As $a_n < a_i$, $(u, 0) \in V_{(a_i,b_i)}$ and as $a_i < b_i$, $(0, u) \in V_{(a_i,b_i)}$ too. If $u \in V_3$ then $(u, \phi(u)) \in V_{(a_n,b_n)}$. As $a_n < a_i$, $(u, 0) \in V_{(a_i,b_i)}$; therefore $(0, u) \in V_{(a_i,b_i)}$.*

Remark 8. *If $(u, v) \in V_{(a_i,b_i)}$ with $i < n$ then u and v are in $V_1 \bigoplus V_2 \bigoplus V_3$. Indeed, as $a_i < b_n$, $(0, u) \in (a_n, b_n)$, therefore $u \in V_1 \bigoplus V_2 \bigoplus V_3$. Analogously, as $b_i < b_n$ then $(0, v) \in V_{(a_n,b_n)}$, thus $v \in V_1 \bigoplus V_2 \bigoplus V_3$.*

We consider $V' = (V_1, V_{(a_i,b_i)})_{i \le n-1}$ where

$$V'_{(a_i,b_i)} = \{(u, v) \in V_{(a_i,b_i)} \mid u, v \in V_1\},$$

and $V'' = (L, V''_{(a_i,b_i)})_{i \le n-1}$, with $L = V_2 \bigoplus V_3 \bigoplus V_3' \bigoplus W$ and

$$V''_{(a_i,b_i)} = \{(u, v) \in V_{(a_i,b_i)} \mid u, v \in L\}.$$

Both V' and V'' are representations of D_{n-1}.

Affirmation. $\underline{V} = V' \bigoplus V''$.
We just have to prove that for all $i < n$,

$$V_{(a_i,b_i)} = V'_{(a_i,b_i)} \bigoplus V''_{(a_i,b_i)}.$$

Let $(u, v) \in V_{(a_i,b_i)}$, due to Remark 8, $u = u_1 + u_2$, $v = v_1 + v_2$ where v_1, v_2 are in V_1, and u_2, v_2 are in $V_2 \bigoplus V_3$; thus $(u_2, v_2) = (u_2, 0) + (0, v_2) \in V_{(a_i,b_i)}$, so $(u_1, v_1) \in V_{(a_i,b_i)}$ and since u_1, v_1 are in V_1, it is obtained that $(u_1, v_1) \in V'_{(a_1,b_1)}$ and clearly $(u_2, v_2) \in V''_{(a_i,b_i)}$. This proves our claim.

Now, by application of the induction hypothesis

$$V' = \left(\bigoplus_s \mathcal{L}_{1,s}(e_{1,s}) \right) \bigoplus \left(\bigoplus_t \mathcal{L}_{2,t}(e_{2,t}) \right) \bigoplus \left(\bigoplus_r \mathcal{L}_{3,r}(e_{3,r}, f_{3,r}) \right),$$

where $V_0 = \langle e_{1,s}, e_{2,t}, e_{3,r}, f_{3,r}\rangle$.

Let $\hat{\mathcal{L}}_{1,s}(e_{1,s}) = (ke_{1,s}, (\hat{\mathcal{L}}_{1,s})_{(a_i,b_i)})_{i\leq n}$ such that its restriction to D_{n-1} coincides with $\mathcal{L}_{1,s}$ and $\hat{\mathcal{L}}_{a_n,b_n} = k(0, e_{1,s})$.

The representation $\hat{\mathcal{L}}_{2,t}(e_{2,t}) = (ke_{2,t}, \hat{\mathcal{L}}_{2,t})_{(a_i,b_i)}$ is such that its restriction to D_{n-1} coincides with $\mathcal{L}_{2,t}(e_{2,t})$ and $(\hat{\mathcal{L}}_{2,t})_{(a_n,b_n)} = k(0, e_{2,t})$. Similarly, $\hat{\mathcal{L}}_{3,r}(e_{3,r}, f_{3,r}) = (\langle e_{3,r}, f_{3,r}\rangle, (\hat{\mathcal{L}}_{3,r})_{(a_i,b_i)})$ is the representation such that its restriction a D_{n-1} is $\mathcal{L}_{3,r}(e_{3,r}, f_{3,r})$ and $(\hat{\mathcal{L}}_{3,r})_{(a_n,b_n)} = \langle (0, e_{3,r}), (0, f_{3,r})\rangle$.

We consider $h_1, h_2, \ldots, h_m$ a basis for V_2 then

$$V = \left(\bigoplus_s \hat{\mathcal{L}}_{1,s}(e_{1,s})\right) \bigoplus \left(\bigoplus_t \hat{\mathcal{L}}_{2,t}(e_{2,t})\right) \bigoplus \left(\bigoplus_r \hat{\mathcal{L}}_{3,r}(e_{3,r}, f_{3,r})\right) \bigoplus \left(\bigoplus_{i=1}^{m} \mathcal{L}_{2,n}(h_i)\right) \bigoplus \left(\bigoplus_{i=1}^{l} \mathcal{L}_{3,n}(e_i, f_i)\right).$$

Clearly, $\hat{\mathcal{L}}_{1,s}(e_{1,s}) = \mathcal{L}_{1,s}(e_{1,s})$ as representations of D_n, similarly $\hat{\mathcal{L}}_{2,t}(e_{2,t}) = \mathcal{L}_{2,t}(e_{2,t})$ as representations of D_n and $\hat{\mathcal{L}}_{3,r}(e_{3,r}, f_{3,r}) = \mathcal{L}_{3,r}(e_{3,r}, f_{3,r})$ as representations of D_n. From here our result is obtained. ■

Henceforth, if $\varphi : U \to V$ be a morphism, we denote $\mathcal{B} = \{e\}$ or $\mathcal{B}_1 = \{e_1, e_2\}$ a canonical bases of U_0 or V_0 when $\dim_k U_0 = 1 = \dim_k V_0$, or $\dim_k U_0 = 2 = \dim_k V_0$ respectively.

Lemma 4. $\mathrm{Hom}(\mathcal{L}_{1,i}, \mathcal{L}_{s,j}) = \begin{cases} 0 & \text{if} \quad s = 1 \quad \textit{and} \quad i < j, \\ k & \text{if} \quad (s = 1 \wedge i \geq j) \vee (s = 3 \wedge i \leq j) \vee s = 2, \\ k^2 & \text{if} \quad s = 3 \wedge i > j. \end{cases}$

Proof Let $\varphi \in \mathrm{Hom}(\mathcal{L}_{1,i}, \mathcal{L}_{s,j})$. If $s = 1$ and $i < j$, then $\varphi : k\{e\} \to k\{e\}$ is such that $\varphi((0, e)) \subset (0, 0)$ so, $\varphi = 0$, whereas if $i \geq j$ it is obtained that $\varphi((0, e)) \subset (0, e)$, so $\varphi(e) = \lambda e$, with $\lambda \in k$. Now, if $s = 2$ then $\varphi : k\{e\} \to k\{e_1\}$ is such that $\varphi(0, e) \subset \langle (e_1, 0), (0, e_1)\rangle$ and $\varphi((0, e)) \subset (0, e)$ therefore, $\varphi(e) = \lambda e_1$, with $\lambda \in k$. Finally, if $s = 3$ and $i \leq j$, then $\varphi : k\{e\} \to k\{e_1, e_2\}$ is such that $\varphi(0, e) \subset \langle (e_1, 0), (0, e_1)\rangle$, $\varphi(0, e) \subset \langle (0, e_1), (e_1, e_2)\rangle$ and $\varphi(0, e) \subset \langle (0, e_1), (0, e_2)\rangle$, so, $\varphi(e) = \lambda e_1$, with $\lambda \in k$, whereas if $i > j$ $\varphi : k\{e\} \to k\{e_1, e_2\}$ is such that $\varphi(0, e) \subset \langle (0, e_1), (0, e_2)\rangle$, so $\varphi(e) = \lambda_1 e_1 + \lambda_2 e_2$, with $\lambda_1, \lambda_2 \in k$. ■

Lemma 5. $\mathrm{Hom}(\mathcal{L}_{2,i}, \mathcal{L}_{s,j}) = \begin{cases} k & \text{if} \quad (s = 2 \wedge i \leq j) \vee (s = 3 \wedge i < j), \\ 0 & \text{otherwise}. \end{cases}$

Proof Let $\varphi \in \mathrm{Hom}(\mathcal{L}_{2,i}, \mathcal{L}_{s,j})$. If $s = 1$ then $\varphi : k\{e\} \to k\{e\}$ is such that $\varphi\langle (0, e), (e, 0)\rangle \subset (0, 0)$ when $i < j$, and $\varphi\langle (0, e), (e, 0)\rangle \subset \langle (0, e)\rangle$ when $i \geq j$ so, in both cases it is obtained $\varphi = 0$. If $s = 2$ and $i > j$ then $\varphi : k\{e\} \to k\{e\}$ is such that $\varphi\langle (0, e), (e, 0)\rangle \subset (0, 0)$, so $\varphi = 0$, whereas if $i \leq j$, $\varphi\langle (0, e), (e, 0)\rangle \subset \langle (0, e), (e, 0)\rangle$, then $\varphi(e) = \lambda e$ with $\lambda \in k$. Finally, let $s = 3$. If $i < j$ then $\varphi : k\{e\} \to k\{e_1, e_2\}$ is such that $\varphi\langle (0, e), (e, 0)\rangle \subset \langle (0, e_1), (e_1, 0)\rangle$ and $\varphi\langle (0, e)\rangle \subset \langle (0, e_1), (e_1, e_2)\rangle$, then $\varphi(e) = \lambda e_1$, with $\lambda \in k$, whereas if $i \geq j$, $\varphi\langle (0, e), (e, 0)\rangle \subset \langle (0, e_1), (e_1, e_2)\rangle$ and $\varphi\langle (0, e)\rangle \subset \langle (0, e_1), (0, e_2)\rangle$, so $\varphi = 0$. ■

Lemma 6. $\mathrm{Hom}(\mathcal{L}_{3,i}, \mathcal{L}_{s,j}) = \begin{cases} k^2, & \text{if} \quad (s = 2 \wedge i \leq j) \vee (s = 3), \\ k, & \text{if} \quad (s = 1 \wedge i \geq j) \vee (s = 2 \wedge i > j), \\ 0, & \text{if} \quad s = 1 \wedge i < j. \end{cases}$

Proof. Let $\varphi \in \mathrm{Hom}(\mathcal{L}_{3,i}, \mathcal{L}_{s,j})$. If $s = 1$ and $i \geq j$ then $\varphi : k\{e_1, e_2\} \to k\{e\}$ is such that $\varphi\langle(0, e_1), (e_1, 0)\rangle \subset \langle(e, 0)\rangle$, $\varphi\langle(0, e_1), (0, e_2)\rangle \subset \langle(e, 0)\rangle$, and $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(e, 0)\rangle$, so, $\varphi(e_1) = 0$ and $\varphi(e_2) = \lambda e$, with $\lambda \in k$, whereas if $i < j$, $\varphi = 0$, since $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset (0, 0)$. Now if $s = 2$ and $i \leq j$, it is obtained that $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(e, 0)\rangle$, so $\varphi(e_1, e_2) = (\varphi(e_1), \varphi(e_2)) = \lambda_1 e_1 + \lambda_2 e_2$, with $\lambda_1, \lambda_2 \in k$; now if $i < j$, $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(0, e)\rangle$, so $\varphi(e_1) = 0$, and $\varphi(e_2) = \lambda e_2$. When $s = 3$, $\varphi : k\{e_1, e_2\} \to k\{e_1, e_2\}$ is such that if $i = j$, then $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(0, e_1), (e_1, e_2)\rangle$, so $\varphi(e_1) = \lambda e_1$ and $\varphi(e_2) = \beta e_1 + \lambda e_2$ and if $i > j$, then $\varphi\langle(0, e_1), (e_1, 0)\rangle \subset \langle(0, e_1), (e_1, e_2)\rangle$ and $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(0, e_1), (0, e_2)\rangle$, so $\varphi(e_1) = 0$ and $\varphi(e_2) = \lambda_1 e_1 + \lambda_2 e_2$. Finally, if $i < j$ then $\varphi\langle(0, e_1), (e_1, e_2)\rangle \subset \langle(0, e_1), (e_1, 0)\rangle$ and $\varphi\langle(0, e_1), (0, e_2)\rangle \subset \langle(0, e_1), (e_1, e_2)\rangle$, so $\varphi(e_1) = \lambda e_1$ and $\varphi(e_2) = \beta e_1$. ■

Proposition 15. *Let $\varphi : U \to V$ be a non zero morphism in $\mathrm{Rep}\,\mathrm{D}_n$ with $U, V \in \mathrm{Rep}\,\mathrm{D}_n$ then the following statements hold.*

1. *φ is a monomorphism if and only if it satisfies one the following conditions:*
   - **(a)** *$U = \mathcal{L}_{1,i}$ and $V = \mathcal{L}_{s,j}$ where $s = 1$ and $i \geq j$ or, $s = 2$, or $s = 3$.*
   - **(b)** *$U = \mathcal{L}_{2,i}$ and $V = \mathcal{L}_{s,j}$ where $s = 2$ and $j \geq i$ or, $s = 3$ and $i < j$.*
   - **(c)** *$U = \mathcal{S}$ and $V = \mathcal{L}_{s,j}$ where $s \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, n\}$.*
2. *φ is an epimorphism if and only if it satisfies one the following conditions:*
   - **(a)** *$U = \mathcal{L}_{1,i}$ and $V = \mathcal{L}_{s,j}$ where $s = 1$ and $i \geq j$ or, $s = 2$ or, $s = 3$ and $i > j$.*
   - **(b)** *$U = \mathcal{L}_{2,i}$ and $V = \mathcal{L}_{s,j}$ where $s = 2$ and $j \geq i$.*
   - **(c)** *$U = \mathcal{L}_{3,i}$ and $V = \mathcal{L}_{s,j}$ where $s = 1$ and $i \geq j$ or $s = 2$ or $s = 3$ and $i = j$.*

Proof. Let $\varphi : U \to V$ be a morphism. We denote $\mathcal{B} = \{e\}$ or $\mathcal{B}_1 = \{e_1, e_2\}$ a canonical basis of U_0 or V_0 when $\dim_k U_0 = 1 = \dim_k V_0$, or $\dim_k U_0 = 2 = \dim_k V_0$ respectively.

1. **(a)** If $U = \mathcal{L}_{1,i}$, then $\dim_k U_0 = 1$. Thus, if $V = \mathcal{L}_{s,j}$ with $s = 1$ or $s = 2$, then $\dim_k V_0 = 1$. Therefore if $\varphi \neq 0$, it is a monomorphism. It is worth noting, that if $s = 1$ and $i \geq j$ then $\varphi = 0$. If $s = 3$, i.e., $V = \mathcal{L}_{3,j}$, $\dim_k V_0 = 2$, when $i < j$, $\varphi(e) = \lambda e_1$ with, $\lambda \in k$ and if $i \geq j$ then $\varphi(e) = \lambda_1 e_1 + \lambda_2 e_2$ with $\lambda_1, \lambda_2 \in k$. Therefore, φ is a monomorphism.

   (b) If $U = \mathcal{L}_{2,i}$, then $\dim_k U_0 = 1$. Thus, if $V = \mathcal{L}_{s,j}$ with $s = 2$ then $\dim_k V_0 = 1$, when $j \geq i$, $\varphi \neq 0$, therefore it is a monomorphism. In case that $V = \mathcal{L}_{3,j}$ then $\dim_k V_0 = 2$ and $\varphi(e) = \lambda e_1$ when $i < j$, which is a monomorphism. Otherwise $\varphi = 0$. Finally, if $s = 1$ then $\varphi = 0$.

   (c) If $U = \mathcal{L}_{3,i}$, then $\dim_k U_0 = 2$. Thus if $V = \mathcal{L}_{3,j}$, $\dim_k U_0 = 2$, when $i = j$ it is obtained that $\varphi(e_1, e_2) = (\lambda e_1, \beta e_1 + \lambda e_2)$ therefore φ is a monomorphism, whereas if $i \neq j$ then φ is not a monomorphism since $\varphi(e_1, e_2) = (\lambda e_1, \beta e_1 + \lambda e_2)$, or $\varphi(e_1, e_2) = (\lambda e_1, \beta e_1 + \lambda e_2)$. Now, if $s = 2$ or $s = 1$ and $\varphi \neq 0$, then φ is not monomorphism since $\varphi(e_1) = 0$ and $\varphi(e_2) = \lambda e$.

2. **(a)** If $U = \mathcal{L}_{1,i}$ and $V = \mathcal{L}_{s,j}$ with $s = 1$ and $i \geq j$ or $s = 2$, then $\dim_k V_0 = 1$ and $\varphi(e) = \lambda e$, where $\lambda \in k$ so $\mathrm{Im}(\varphi) \cong k$. It is worth noting, that if $s = 1$ and $i \geq j$ then $\varphi = 0$. If $s = 3$ then $\dim_k V_0 = 2$ and $\varphi(e) = \lambda e_1$, or $\varphi(e) = \lambda_1 e_1 + \lambda_2 e_2$, if $i \leq j$ or $i > j$ respectively, then φ is an epimorphism when $i > j$.

(b) If $U = \mathcal{L}_{2,i}$ and $V = \mathcal{L}_{s,j}$ with $s = 2$ and $i \geq j$ then φ is an epimorphism since $\varphi(e) = \lambda e$ and $\mathrm{Im}(\varphi) \cong k$. If $s = 3$ and $i < j$, $\varphi e = \lambda e_1$ where $\lambda \in k$ then $\mathrm{Im}(\varphi) \cong k$, but $\dim_k V_0 = 2$, thus φ is not an epimorphism. If $s = 1$, we obtained that $\varphi = 0$.

(c) If $U = \mathcal{L}_{2,i}$ and $V = \mathcal{L}_{s,j}$ with $s = 1$ and $i < j$ or $s = 2$ and $i > j$, then $\varphi(e_1) = 0$ and $\varphi(e_2) = \lambda e_2$, so $\mathrm{Im}(\varphi) \cong k$ and therefore φ is an epimorphism since $\dim_k V_0 = 1$. Analogously, we obtain that if $s = 2$, φ is an epimorphism. Finally, if $s = 3$, $\dim_k V_0 = 2$ and $\varphi(e_1) = \lambda e_1$ and $\varphi(e_2) = \beta e_1 + \lambda e_2$, when $i = j$ therefore φ is an epimorphism, whereas when $i \neq j$ we obtain that $\varphi(e_1) = 0$ and $\varphi(e_2) = \lambda_1 e_1 + \lambda_2 e_2$, or $\varphi(e_1) = \lambda e_1$ and $\varphi(e_2) = \beta e_1$ so $\mathrm{Im}(\varphi) \cong k$ and thus is not an epimorphism.

■

Corollary 4. *Let $\varphi : \mathcal{L}_{k,i} \to \mathcal{L}_{s,j}$ be a non zero morphism in* RepD_n *with $k, s \in \{1, 2, 3\}$ and $i, j \in \{1, 2, \ldots, n\}$ then the following statements hold.*

(a) $\ker\varphi \cong \begin{cases} k, & \text{if} \quad k = 3 \wedge (s = 1 \vee s = 2 \vee (s = 3 \wedge i \neq j)), \\ 0, & \text{otherwise.} \end{cases}$

(b) $\mathrm{Im}\varphi \cong \begin{cases} k^2, & \text{if} \quad k = 3 \wedge i = j, \\ k, & \text{otherwise.} \end{cases}$

Proof It follows from the previous proposition. ■

Proposition 16. *The following is a complete list of all the ε-sequences that there exist in* RepD_n.

(a) $\mathcal{L}_{1,k} \to \mathcal{L}_{3,j} \to \mathcal{L}_{1,s}$, *where $s \leq j$ and $k, j, s \in \{1, 2, \ldots, n\}$.*

(b) $\mathcal{L}_{1,k} \to \mathcal{L}_{3,j} \to \mathcal{L}_{2,s}$, *where $k, j, s \in \{1, 2, \ldots, n\}$.*

(c) $\mathcal{L}_{2,k} \to \mathcal{L}_{3,j} \to \mathcal{L}_{1,s}$, *where $k < j$, $s \leq j$, and $k, j, s \in \{1, 2, \ldots, n\}$.*

(d) $\mathcal{L}_{2,k} \to \mathcal{L}_{3,j} \to \mathcal{L}_{2,s}$, *where $k < j$, and $k, j, s \in \{1, 2, \ldots, n\}$.*

(e) $\mathcal{S} \to \mathcal{L}_{3,n} \to \mathcal{L}_{2,n}$.

Proof. It is a consequence of Corollary 4 and Proposition 15. ■

Proposition 17. *For the morphisms in the ϵ-sequences of the previous proposition we can affirm the following:*

(a) $\mathcal{L}_{1,k} \to \mathcal{L}_{3,j}$ *is an irreducible morphism if and only if $k = j = 1$ or $k = j + 1$.*

(b) $\mathcal{L}_{3,j} \to \mathcal{L}_{1,s}$ *is an irreducible morphism if and only if $j = s$.*

(c) $\mathcal{L}_{2,k} \to \mathcal{L}_{3,j}$ *is an irreducible morphism if and only if $j = k + 1$.*

(d) *$\mathcal{L}_{3,j} \to \mathcal{L}_{2,s}$ is an irreducible morphism if and only if $j = s$.*

Proof

(a) If $k = j = 1$, then, $\varphi : \mathcal{L}_{1,1} \to \mathcal{L}_{3,1}$, where $\varphi = \langle (1,0)^t \rangle$. φ is neither a section nor a retraction since $\varphi_1 : \mathcal{L}_{3,1} \to \mathcal{L}_{1,1}$ is such that $\varphi_1 = \langle (0,1) \rangle$ and $\varphi\varphi_1 \neq id_{\mathcal{L}_{3,1}}$ and $\varphi_1\varphi \neq id_{\mathcal{L}_{1,1}}$. The unique representation $U \in \mathrm{Ind}(\mathrm{D}_n)$, such that $\varphi = \varphi_1\varphi_2$ with $\varphi_1 : \mathcal{L}_{1,1} \to U$ and $\varphi_2 : U \to \mathcal{L}_{3,1}$ is $U = \mathcal{L}_{1,1}$. In this case, $\varphi_1 = id$ which is a section, so, φ is irreducible. Now, if $k = j+1$ then $\varphi : \mathcal{L}_{1,j+1} \to \mathcal{L}_{3,j}$, where $\varphi = \langle (0,1)^t \rangle$. φ is neither a retraction nor a section since $\mathrm{Hom}(\mathcal{L}_{3,j}, \mathcal{L}_{1,j+1}) = 0$. In this case there is no $U \in \mathrm{Ind}(\mathrm{D}_n)$ such that $\varphi_1 : \mathcal{L}_{1,1} \to U$, $\varphi_2 : U \to \mathcal{L}_{3,1}$ and $\varphi = \varphi_2\varphi_1$. Therefore φ is an irreducible morphism. Reciprocally, we suppose that φ is irreducible and $k \neq 1$ or $j \neq 1$ and $k \neq j+1$. If we consider $U = \mathcal{L}_{3,1}$ then we will obtain that $\varphi = \varphi_2\varphi_1$ with $\varphi_1 : \mathcal{L}_{1,1} \to U$, $\varphi_2 : U \to \mathcal{L}_{3,1}$, where $\varphi_1 = \langle (1,0)^t \rangle$ is not a section and $\varphi_2 = \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$ is not a retraction, which contradicts that φ is an irreducible morphism.

(b) If $j = s$, then, $\varphi : \mathcal{L}_{3,s} \to \mathcal{L}_{1,s}$, where $\varphi = \langle (0,1) \rangle$. φ is neither a section nor a retraction since $\mathrm{Hom}(\mathcal{L}_{1,s}, \mathcal{L}_{3,s}) = \langle (1,0)^t \rangle$ and $\varphi\varphi_1 \neq id_{\mathcal{L}_{3,1}}$ and $\varphi_1\varphi \neq id_{\mathcal{L}_{1,1}}$. The unique representation $U \in \mathrm{Ind}(\mathrm{D}_n)$, such that $\varphi = \varphi_1\varphi_2$ with $\varphi_1 : \mathcal{L}_{3,1} \to U$ and $\varphi_2 : U \to \mathcal{L}_{1,1}$ is obtained when $U = \mathcal{L}_{1,1}$. In this case $\varphi_2 = id$ and therefore is a section, so φ is irreducible. Reciprocally, if we suppose that φ is irreducible and $j \neq s$ then if we consider $U = \mathcal{L}_{1,s}$ then we will obtain that $\varphi = \varphi_2\varphi_1$ with $\varphi_1 : \mathcal{L}_{3,1} \to U$ and $\varphi_2 : U \to \mathcal{L}_{1,s}$, where $\varphi_1 = \langle (0,1) \rangle$ is not a retraction and $\varphi_2 = \langle 1 \rangle$ is not a section, which contradicts that φ is an irreducible morphism.

(c) If $j = k+1$, then $\varphi : \mathcal{L}_{2,k} \to \mathcal{L}_{3,k+1}$, where $\varphi = \langle (1,0)^t \rangle$. φ is neither a section nor a retraction since if $\varphi' \in \mathrm{Hom}(\mathcal{L}_{3,k+1}, \mathcal{L}_{2,k}) = \langle (0,1), (1,0) \rangle$ then $\varphi'\varphi \neq id_{\mathcal{L}_{2,k}}$ and $\varphi\varphi' \neq id_{\mathcal{L}_{3,k+1}}$. The unique representation $U \in \mathrm{Ind}(\mathrm{D}_n)$, such that $\varphi = \varphi_1\varphi_2$ with $\varphi_1 : \mathcal{L}_{2,k} \to U$ and $\varphi_2 : U \to \mathcal{L}_{2,j}$ is obtained when $U = \mathcal{L}_{2,k}$. In this case, $\varphi_1 = \langle id \rangle$ and $\varphi_2 = \langle (1,0) \rangle$. Thus, $\varphi = \varphi_2\varphi_1$, and φ_1 is a retraction, so φ is irreducible. Reciprocally, if we suppose that φ is irreducible and $j > k+1$ then if we consider $U = \mathcal{L}_{3,k+1}$ then we will obtain that $\varphi = \varphi_2\varphi_1$ where $\varphi_1 : \mathcal{L}_{2,k} \to U$ and $\varphi_2 : U \to \mathcal{L}_{3,j}$, with $\varphi_1 = \langle (1,0)^t \rangle$ is not a retraction and $\varphi_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle$ is not a section, which contradicts that φ is an irreducible morphism.

(d) If $j = s$, then, $\varphi : \mathcal{L}_{3,j} \to \mathcal{L}_{2,j}$, where $\varphi = \langle (1,0) \rangle$. φ is neither a section nor a retraction since $\mathrm{Hom}(\mathcal{L}_{2,j}, \mathcal{L}_{3,j}) = 0$. The unique representation $U \in \mathrm{Ind}(\mathrm{D}_n)$, such that $\varphi = \varphi_1\varphi_2$ with $\varphi_1 : \mathcal{L}_{2,k} \to U$ and $\varphi_2 : U \to \mathcal{L}_{2,j}$ is obtained when $U = \mathcal{L}_{2,j}$. In this case $\varphi_1 = \langle (0,1), (1,0) \rangle$ and $\varphi_2 = \langle id \rangle$, so φ_2 is a section. Reciprocally, if we suppose that φ is irreducible and $j \neq s$ then if we consider $U = \mathcal{L}_{3,s}$ then we will obtain that $\varphi = \varphi_2\varphi_1$ where $\varphi_1 : \mathcal{L}_{3,j} \to U$ is not a retraction and $\varphi_2 : U \to \mathcal{L}_{2,s}$ is not a section, with $\varphi_1 = \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$ and $\varphi_2 = \langle (1,0) \rangle$, which contradicts that φ is an irreducible morphism. ■

Corollary 5. *The following is a complete list of all almost split sequences that there exist in* $\mathrm{Rep}\,\mathrm{D}_n$.

1. $\mathcal{L}_{1,1} \to \mathcal{L}_{3,1} \to \mathcal{L}_{1,1}$.
2. $\mathcal{L}_{1,i+1} \to \mathcal{L}_{3,i} \to \mathcal{L}_{2,i}$ *for* $0 < i < n$.
3. $\mathcal{L}_{2,i} \to \mathcal{L}_{3,i+1} \to \mathcal{L}_{1,i+1}$ *for* $0 < i < n$.
4. $\mathcal{S} \to \mathcal{L}_{3,n} \to \mathcal{L}_{2,n}$.

In this way we obtain that the Auslander-Reiten quiver for D_n is given by

$$
\begin{tikzcd}
 & \mathcal{L}_{1,1} \arrow[d, bend left] & \\
 & \mathcal{L}_{3,1} \arrow[u, bend left] \arrow[dr] & \\
\mathcal{L}_{1,2} \arrow[ur] \arrow[rr, equal, dashed] & & \mathcal{L}_{2,1} \arrow[dl] \\
 & \mathcal{L}_{3,2} \arrow[ul] \arrow[dr] & \\
\mathcal{L}_{1,3} \arrow[ur] \arrow[rr, equal, dashed] & & \mathcal{L}_{2,2} \arrow[dl] \\
 & \mathcal{L}_{3,3} \arrow[ul] & \\
\vdots & \vdots & \vdots \\
 & \mathcal{L}_{3,n-1} \arrow[dr] & \\
\mathcal{L}_{1,n-1} \arrow[ur] \arrow[rr, equal, dashed] & & \mathcal{L}_{2,n-1} \arrow[dl] \\
 & \mathcal{L}_{3,n} \arrow[ul] \arrow[dr] & \\
\mathcal{S} \arrow[ur] & & \mathcal{L}_{2,n}
\end{tikzcd}
$$

FIGURE 3.2. Auslander-Reiten quiver of poset with involution of type D_n.

The previous results can be extended to posets of type D_∞ in the following way. We define the functor

$$\Xi_n : \mathrm{Rep}(\mathrm{D}_n) \to \mathrm{Rep}(\mathrm{D}_\infty)$$

such that if $V = (V_0, V_{(a_i,b_i)})_{1\le i\le n}$ then $\Xi(V) = (V_0, V_{(a_i,b_i)})_{1\le i}$ with $V_{(a_j,b_j)} = (0, V_0)$ for $j > n$.

Clearly if $f : V \to W$ is a morphism in $\mathrm{Rep}(\mathrm{D}_n)$ determined by the morphism $f : V_0 \to W_0$ then this morphism is the same in the category $\mathrm{Rep}(\mathrm{D}_\infty)$. So, taking $\Xi_n(f) = f$ we have defined the functor Ξ_n. Analogously we define a functor: $\Theta_n : \mathrm{Rep}(\mathrm{D}_n) \to \mathrm{Rep}(\mathrm{D}_{n+1})$ and it is obtained that $\Xi_{n+1}\Theta_n = \Xi_n$.

Proposition 18. *The indecomposable representations of* D_∞ *are the representations* $\hat{L}_{s,i} = \Xi_i(L_{s,i})$ *plus the simple trivial representation* S.

Proof. Let $V = (V_0, V_{(a_i,b_i)})_{1\le i}$ be an indecomposable representation of D_∞. We suppose that $V_{(a_1,b_1)} = 0$, then for each n the restriction of V to D_n is the form $(V)_{\mathrm{D}_n} = \bigoplus_{i=1}^{s} L_s$ with $L_s = \mathcal{L}_{1,i}$ with $i > 1$ or the trivial representation. Since V_0 is finite dimensional there exists n and finite sum $W = \bigoplus_{i=1}^{s} L_s$ in D_n such that for all $m > n$, V restricted to D_m coincides with the restriction of $\Xi_n(W)$ to D_m, therefore $V = \Xi_n(W)$ and since V is an indecomposable, then $V = \Xi_n(\mathcal{L}_{1,n})$.

Now we suppose that $V_{(a_1,b_1)} \neq 0$ then V restricted to D_n is the form $\bigoplus_{i=1}^{s} L_s$ where each L_s is the form $\mathcal{L}_{j,i}$ with $j = 1, 2, 3$ and at least one L_s has the form $\mathcal{L}_{2,i}$ or $\mathcal{L}_{3,i}$. As before, there exists n such that for all $m \ge n$ the restriction from V to D_m coincides with the restriction of $\Xi_n(W)$ to D_m therefore $V = \Xi_n(\mathcal{L}_{j,n})$ with $j = 2$ or $j = 3$. ■

Proposition 19. *Let* $a : X \xrightarrow{u} Y \xrightarrow{v} Z$ *be an almost split sequence in* $\mathrm{Rep}(\mathrm{D}_n)$ *with* X *different from the trivial representation. Then*

$$b : \Xi_n(X) \xrightarrow{\Xi_n(u)} \Xi_n(Y) \xrightarrow{\Xi_n(v)} \Xi_n(Z),$$

is an almost split sequence in $\mathrm{Rep}(\mathrm{D}_\infty)$.

Proof. The sequence b is a nontrivial ε-sequence in $\mathrm{Rep}(\mathrm{D}_\infty)$ whose extremes are indecomposable. Let $h : Y \to \Xi_n(Y)$ be a morphism that is not a retraction in $\mathrm{Rep}(\mathrm{D}_\infty)$ with Y indecomposable, then $Y = \Xi_m(W)$ with W indecomposable in $\mathrm{Rep}(\mathrm{D}_m)$ for some $m > n$. We have that $\Xi_m(a)$ is an almost split sequence in $\mathrm{Rep}(\mathrm{D}_m)$, so we can suppose that $m = n$ and then $h = \Xi_m(w)$ where $w : W \to Z$ is a morphism that is not a retraction. Therefore, there exists $g : W \to Y$ with $vg = w$, thus $h = \Xi_m(v)\Xi_m(g)$. This proves our assertion. ■

3.2 Tube Deformation.

To describe the Auslander-Reiten quiver for posets of type D_∞, it is necessary to introduce functors $H_T : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$, $H_S : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$ and $H_{T,S} : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$ with $\mathcal{V}$ the category whose objects are the pairs $U = (U_0, \phi_U)$ where $\phi_U : U_0 \to U_0$ is a k-application such that $\phi^m_{(U)} = 0$ for some $m \ge 1$, as it is shown in the following.

We observe that the category $\mathcal{V}$ is equivalent to the subcategory of $k[x]$-modules, whose objects are the $k[x]$-modules M such that $x^m M = 0$ for some integer m. Then if the pair $U = (U_0, \phi_U)$ is considered as a $k[x]-$module is obtained that $\mathrm{rad}(U) = \phi_U(U_0)$. A system of representants of the indecomposable objects of $\mathcal{V}$ is given by the objects $V^n = (V_0^n, \phi_{V^n})$ such that $V_0^n = \langle e_n, \phi(e_n), \ldots, \phi^n(e_n)\rangle$ with $\phi^{n+1}(e_n) = 0$. In this case $V^n \cong k[x]/(x^n+1)$. The irreducible morphisms of this category are given by $\lambda_i : V^i \to V^{i+1}$ such that $\lambda_i(e_i) = \phi(e_{i+1})$ and $\nu_i : V^{i+1} \to V^i$ such that $\nu_i(e_{i+1}) = e_i$.

We will define a functor $H_T : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$ as follows: for $V^n = (V_0^n, \phi_{V^n}) \in \mathcal{V}$ we define $H_T(V^n) = \mathcal{L}_{1,n}(\mathrm{top}(V^n))$ and we can see that if $f : V^n \to V^m$ is a morphism then f induces a morphism $\hat{f} : \mathrm{top}(V^n) \to \mathrm{top}(V^m)$ such that $H_T(f) = \hat{f} : H_T(V^n) \to H_T(V^m)$ is a morphism in $\mathrm{Rep}(\mathrm{D}_\infty)$. It is clear that $H_T(f) \neq 0$ if and only if f is an epimorphism. $H_S : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$ is defined as zero for $n = 1$ and for $n \geq 2$, as $H_S(V^n) = \mathcal{L}_{2,n-1}(\mathrm{soc}(V^n))$. As before, if $f : V^n \to V^m$ is a morphism, then f induces a morphism $f : \mathrm{soc}(V^n) \to \mathrm{soc}(V^m)$ which is the morphism $H_S(f) : H_S(V^n) \to H_S(V^m)$, then $H_S(f)$ is different from zero if and only if f is a monomorphism.

Finally, the functor $H_{T,S} : \mathcal{V} \to \mathrm{Rep}(\mathrm{D}_\infty)$ is defined as $H_{T,S}(V^1) = V^1$ and for $m > 1$:

$$H_{T,S}(V^m) = \mathcal{L}_{3,m-1}(\mathrm{soc}(V^m), \mathrm{top}(V^m)).$$

Thus, natural morphisms of functors are obtained:

$$\eta : H_T \to H_{T,S}; \quad \rho : H_{T,S} \to H_S,$$

for each n, the ε- sequence:

$$H_T(V^n) \xrightarrow{\eta_{V^n}} H_{T,S}(V^n) \xrightarrow{\rho_{V^n}} H_S(V^n)$$

is obtained. For $n \geq 2$, the sequence previous is an almost split sequence.

The morphism $\lambda_n : V^n \to V^{n+1}$ induces a morphism

$$\underline{\lambda}_n : H_S(V^n) \to H_{T,S}(V^{n+1}).$$

The epimorphism $\nu_n : V^{n+1} \to V^n$ induces a morphism $\underline{\nu}_n : H_{T,S}(V^{n+1}) \to H_T(V^n)$.
For $n \geq 2$, the almost split sequence:

$$H_S(V^n) \xrightarrow{\underline{\lambda}_n} H_{T,S}(V^{n+1}) \xrightarrow{\underline{\nu}_n} H_T(V^n)$$

is obtained. The morphism $\nu_1 : V^2 \to V^1$ induces a morphism $\underline{\nu}_1 : H_{T,S}(V^2) \to H_T(V^1)$.
The sequence:

$$H_T(V^1) \xrightarrow{\underline{\lambda}_1} H_{T,S}(V^2) \xrightarrow{\underline{\nu}_1} H_T(V^1),$$

is an almost split sequence. In this way, we obtain that the Auslander-Reiten quiver for D_∞ is given by

$$
\begin{tikzcd}[column sep=small]
\mathcal{S} & H_T(V^1) \arrow[d, bend left] & \\
& H_{T,S}(V^2) \arrow[u, bend left] \arrow[dr] & \\
H_T(V^2) \arrow[ur] \arrow[rr, dashed, no head, shift left] \arrow[rr, dashed, no head, shift right] & & H_S(V^2) \arrow[dl] \\
& H_{T,S}(V^3) \arrow[ul] \arrow[dr] & \\
H_T(V^3) \arrow[ur] \arrow[rr, dashed, no head, shift left] \arrow[rr, dashed, no head, shift right] & & H_S(V^3) \arrow[dl] \\
& H_{T,S}(V^4) \arrow[ul] & \\
\vdots & \vdots & \vdots \\
& H_{T,S}(V^n) \arrow[dr] & \\
H_T(V^n) \arrow[ur] \arrow[rr, dashed, no head, shift left] \arrow[rr, dashed, no head, shift right] & & H_S(V^n) \arrow[dl] \\
& H_{T,S}(V^{n+1}) \arrow[ul] \arrow[dr] & \\
H_T(V^{n+1}) \arrow[ur] & & H_S(V^{n+1}) \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

FIGURE 3.3. Auslander-Reiten quiver of a poset with an involution of type D_∞.

CHAPTER 4

Categorical Properties of Algorithm of Differentiation DI

The algorithm of differentiation with respect to a suitable pair of points (called algorithm DI too) was introduced by Zavadskij in 1977. It can be seen as a generalization of the algorithm of differentiation with respect to a maximal point introduced by Nazarova and Roiter in 1972, which can be applied to posets with width at least two [21, 22, 32, 36]. Afterwards in 1991 he described the categorical properties of this algorithm. That is, Zavadskij proved that DI induces a categorical equivalence between quotient categories and a corresponding relationship between the number of indecomposable representations in the original category and the category of representations of the derived poset [37].

In this chapter, we present a new proof of such categorical equivalence by using module theoretical approach, which allows us to give the indecomposable objects and the irreducible morphisms explicitly. For this purpuse, we follow the ideas of Bautista et al [5, 7]. That is, we can extend a poset $\mathcal{P}$, to obtain an algebra Λ, which is right peak, left peak and 1-Gorenstein. Moreover, the category of representations of posets is equivalent to the category $\mathcal{U}$, of right modules with projective socle, which do not have the projective injective module as a direct summand as presented in the section 2.2.

4.1 Differentiation with Respect to a Suitable Pair of Points (DI)

The following is the definition of the algorithm of differentiation DI with respect to a suitable pair of small points (a, b) for a poset with involution and it is corresponding theorems as in [32].

Definition 38. *Let $\mathcal{P}$ be a finite poset then a pair of small points $(a, b) \in \mathcal{P}$ is said to be suitable (suitable for differentiation* DI*) if*

$$\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + C,$$

where $C = \{c_1 < c_2 < \cdots < c_n\}$ is a small-point chain (eventually empty) and, moreover, the points a, b, c_i are mutually incomparable. The derivative poset $(\mathcal{P}', \theta')$ is obtained from the poset $(\mathcal{P}, \theta)$ with respect to the pair (a, b) as follows:

1. *The chain C is changed by two chains $C^+ = \{c_1^+ < \cdots < c_n^+\}$ and $C^- = \{c_1^- < \cdots < c_n^-\}$.*
2. *The following relations are added: $c_i^- < c_i^+, a < c_i^+, c_i^- < b$, for all $1 \leq i \leq n$.*
3. *Any of the points c_i^-, c_i^+ inherit all relations of the order which the point c_i had before with points of a subset $\mathcal{P} \backslash C$.*

It is supposed that the two element classes are not changed. That is $\theta' = \theta$. That is, if $\mathcal{L} = \mathcal{L}(\mathcal{P})$ is the modular lattice generated by $\mathcal{P}$ then the derived poset of $\mathcal{P}$ is a subposet of $\mathcal{L}$ such that

$$\mathcal{P}' = \mathcal{P}'_{(a,b)} = \mathcal{P} \backslash\, C + \{a + c_1, \ldots, a + c_n\} + \{c_1 b, \ldots c_n b\}, \quad \textit{with } a + c_i = c_i^+,\ c_i b = c_i^-.$$

The following figure illustrates this differentiation.

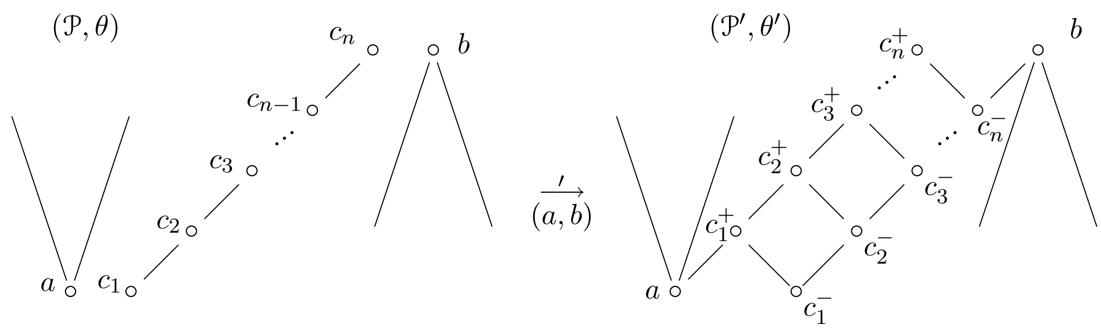


FIGURE 4.1. The diagram of a poset with involution $(\mathcal{P}, \theta)$ and its corresponding derivate poset.

Denote the categories by $\mathcal{R} = \mathrm{Rep}(\mathcal{P}, \theta)$ and $\mathcal{R}' = \mathrm{Rep}(\mathcal{P}', \theta')$. The *differentiation functor* $D_{(a,b)} : \mathcal{R} \to \mathcal{R}'$ also denoted $' : \mathcal{R} \to \mathcal{R}'$ is defined in the following way,

- To each object $U = (U_0, U_z)_{z \in \theta} \in \mathcal{R}$ an object $U' = (U'_0, U'_z)_{z \in \theta'} \in \mathcal{R}'$ is assigned as follows:

$$\begin{aligned}
U'_0 &= U_0, \\
U'_a &= U_a, \\
U'_b &= U_b, \\
U'_{c_i^+} &= U_a + U_{c_i}, \quad 1 \leq i \leq n, \\
U'_{c_i^-} &= U_b \cap U_{c_i}, \quad 1 \leq i \leq n \\
U'_z &= U_z, \;\; \text{for remaining classes } z \in \theta'.
\end{aligned} \tag{4.1}$$

- If $\varphi : U \to V$ is a morphism in $\mathcal{R}$ then $\varphi' = \varphi$.

Remark 9. *We note that* $k'(a) = k'(a, c_i) = k(a)$, $1 \leq i \leq n$. *Indeed, we recall that* $k(a) = (U_0, U_x)_{x \in \mathcal{P}}$, *where* $U_0 = k$, $U_x = k$ *if* $x \geq a$ *and* $U_x = 0$ *otherwise. By applying the functor* 4.1 *we obtain* $U'_0 = k$, $U'_a = k$, $U'_{c_i^+} = U_a + U_{c_i} = k$, $U'_{c_i^-} = U_{c_i} \cap U_b = 0$ *and* $U'_b = U_b = 0$. *So,* $k(a)' = k(a)$. *Analogously,* $k(a, c_i) = (U_0, U_x)_{x \in \mathcal{P}}$, *where* $U_0 = k$, $U_x = k$ *si* $x \geq c_i$ *or* $x \geq a$, *in otherwise* $U_x = 0$. *To applying the functor* 4.1 *we obtained* $U'_0 = k$, $U'_a = k$, $U'_{c_i^+} = U_a + U_{c_i} = k$, $U'_{c_i^-} = U_{c_i} \cap U_b = 0$ *and* $U'_b = U_b = k$. *So,* $k(a, c_i)' = k(a)$.

Example 9. *Let* $\mathcal{P}$ *the poset as in the Figure 4.2.*

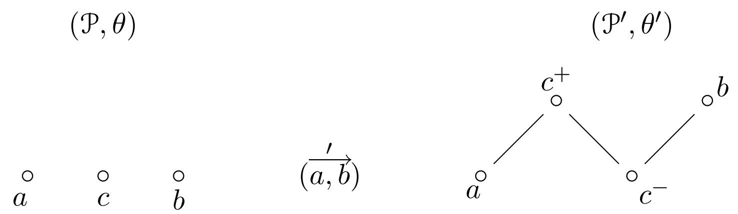


FIGURE 4.2. The diagram of a poset $(\mathcal{P}, \theta)$ and its corresponding derivate poset $(\mathcal{P}', \theta')$.

We consider $U = (U_0, U_a, U_c, U_b) = (k \oplus k, k \oplus 0, 0 \oplus k, (1+1)k)$, *with* $(1+1)k = \{(\lambda, \lambda) \mid \lambda \in k\}$ *a representation of* $\mathcal{P}$. *Applying the functor* 4.1 *we obtain* $U' = (U_0, U_a, U_{c^+}, U_{c^-}, U_b)$ *where*

$$\begin{aligned} U'_0 &= U_0 = k \oplus k, \\ U'_a &= U_a = k \oplus 0, \\ U'_b &= U_b = (1+1)k \\ U'_{c^+} &= U_a + U_c = (k \oplus 0) + (0 \oplus k) = k \oplus k, \\ U'_{c^-} &= U_b \cap U_c = (1+1)k \cap (0 \oplus k) = 0, \end{aligned}$$

Therefore, $U' = (U_0, U_a, U_{c_1^+}, U_{c_1^-}, U_b) = (k \oplus k, k \oplus 0, k \oplus k, 0, (1+1)k)) = (k, k, k, 0, 0) \oplus (k, 0, k, 0, (1+1)k)) = k(a) \oplus k(c_1^+, b)$.

Now, we consider $k(a) = (U_0, U_a, U_c, U_b) = (k, k, 0, 0)$. *By applying the functor* 4.1 *we obtain* $U'_0 = k$, $U'_a = k$, $U'_{c^+} = U_a + U_c = k$, $U'_{c^-} = U_c \cap U_b = 0$ *and* $U'_b = k$. *So,* $k(a)' = (U_0, U_a, U_{c^-}, U_{c^+}, U_b) = (k, k, 0, k, k) = k(a)$. *Analogously,* $k(a, c) = (U_0, U_a, U_c, U_b) = (k, k, k, 0)$. *To applying the functor* 4.1 *we obtained* $U'_0 = k$, $U'_a = k$, $U'_{c^+} = U_a + U_{c_i} = k$, $U'_{c^-} = U_{c_i} \cap U_b = 0$ *and* $U'_b = k$. *So,* $k(a, c_i)' = k(a)$.

We consider the subposet $\mathcal{K}$ of $\mathcal{P}$, with $\mathcal{K} = \mathcal{P} \backslash \{a^{\blacktriangledown} + b_{\blacktriangle}\} = a + b + \{c_1 < c_2 < \cdots < c_n\}$. In this case, the Hasse diagrams of the poset $\mathcal{K}$ and $\mathcal{K}'$ by applying the differentiation functor (4.1), are described in the following figure:

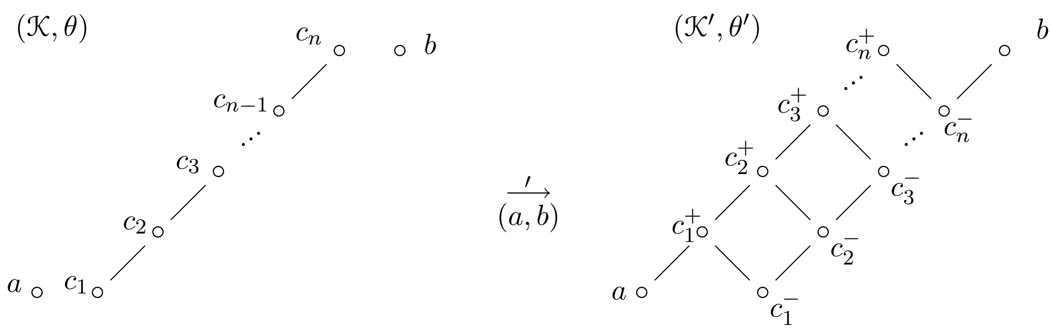


FIGURE 4.3. Diagrams of a poset $\mathcal{K}$ and its corresponding derivative poset $\mathcal{K}'$.

From now on, in this chapter we denote by $\mathcal{I}$ the ideal in the category $\text{Rep}(\mathcal{K})$ formed by the morphisms that are factored through finite direct sums of objects in the set $A = \{k(a), k(a, c_1), \ldots, k(a, c_n)\}$ and by $\mathcal{I}'$ the ideal in $\text{Rep}(\mathcal{K}')$ formed by the morphisms that are factored through $k(a)$. Besides we denote by $\mathcal{C}$ ($\mathcal{C}'$, respectively) the Auslander-Reiten Quiver of $\text{Rep}(\mathcal{K})$ ($\text{Rep}(\mathcal{K})'$, respectively) and we designate by $\underline{\mathcal{C}}$($\underline{\mathcal{C}}'$, respectively) the Auslander-Reiten Quiver of $\text{Rep}(\mathcal{K})/\mathcal{I}(\text{Rep}(\mathcal{K})'/\mathcal{I}'$, respectively).

We consider the following representations of $\mathcal{K} : P(0) = k(0), P(a) = k(a), P(b) = k(b)$, $k(a, b)$ and for $1 \leq i \leq n$, $P(c_i) = k(c_i)$, $k(a, c_i), k(c_i, b), k(a, c_i, b)$, $R(c_{n-i})(e_1, e_2) = (U_0, U_a, U_{c_1}, \ldots, U_{c_j}, \ldots, U_{c_n}, U_b)$, where:

$$\begin{aligned}
U_0 &= \langle e_1, e_2 \rangle, \\
U_a &= \langle e_1 \rangle, \\
U_b &= \langle e_2 \rangle, \\
U_{c_j} &= \begin{cases} \langle e_1 + e_2 \rangle, & \text{if} \quad n - i \leq j \leq n, \\ 0, & \text{if} \quad j < n - i. \end{cases}
\end{aligned} \tag{4.2}$$

and $R^l(c_{n-i})(e_1, e_2) = (U_0, U_a, U_{c_1}, \ldots, U_{c_i}, \ldots, U_{c_n}, U_b)$, for $1 \leq l \leq i$ where

$$\begin{aligned}
U_0 &= \langle e_1, e_2 \rangle, \\
U_a &= \langle e_1 \rangle, \\
U_b &= \langle e_2 \rangle, \\
U_{c_{n-j-1}} &= \langle e_1, e_2 \rangle \quad \text{for} \quad j = 1, \ldots, l, \\
U_{c_{n-l}} &= \langle e_1 + e_2 \rangle, \\
U_{c_{n-j}} &= 0, \quad \text{for} \quad j > i.
\end{aligned} \tag{4.3}$$

Henceforth, we will put $R^0(c_{n-i}) = R(c_n)$.

Analogously, we consider the representations for $\mathcal{K}'$: $k(\emptyset); k(a); k(b); k(a, b)$; for $1 \leq i \leq n$ $k(c_i^-); k(c_i^+); k(c_i^+, b); k(a, c_i^-)$ and for $1 \leq i < j$, $k(c_i^+, c_j^-)$.

Proposition 20. *The differentiation functor* $' : \mathcal{R} \to \mathcal{R}'$ *defined by formulas* (4.1) *induces a quiver isomorphism of* $\underline{\mathcal{C}}$ *in* $\underline{\mathcal{C}}'$. *Consequently, the above list is the complete list of indecomposable representations of* $\text{Rep}\,(\mathcal{K})$ *and* $\text{Rep}\,(\mathcal{K})'$.

Proof. In Rep($\mathcal{K}$) we obtain the irreducible morphisms between projectives:

$$k(\emptyset) \to k(a), \quad k(\emptyset) \to k(b), \quad k(\emptyset) \to k(c_n) \to k(c_{n-1}) \to \cdots \to k(c_1).$$

We have the following commutative diagram $D(i)$:

$$\begin{array}{ccc} k(c_{n-i})(e_1+e_2) & \xrightarrow{f_1} & R(c_{n-i})(e_1,e_2) \\ \downarrow f_2 & & \downarrow g_1 \\ k(c_{n-i-1})(e_1+e_2) & \xrightarrow{g_2} & R(c_{n-i-1})(e_1+e_2) \end{array}$$

such that the sequences $S(D(i))$:
$k(c_{n-i})(e_1+e_2) \xrightarrow{(f_1,-f_2)^T} R(c_{n-i})(e_1,e_2) \bigoplus k(c_{n-i-1})(e_1+e_2) \xrightarrow{(g_1,g_2)^T} R(c_{n-i-1})(e_1,e_2)$
are ε-sequences.

Also, we have the ε-sequence $k(c_1)(e_1+e_2) \to R(c_1)(e_1,e_2) \to k(a,b)(e_2)$,.

We have the commutative diagram

$$\begin{array}{ccc} k(\emptyset) & \xrightarrow{(\lambda_1,\lambda_2)^T} & k(a)(e_1) \bigoplus k(b)(e_2) \\ \downarrow \lambda_3 & & \downarrow (\nu_1,\nu_2) \\ k(c_n)(e_1+e_2) & \xrightarrow{\nu_3} & R(c_n)(e_1,e_2) \end{array}$$

and the ε-sequence

$$k(\emptyset)(f) \xrightarrow{(\lambda_1,\lambda_2,-\lambda_3)} k(a)(e_1) \bigoplus k(b)(e_2) \bigoplus k(c_n)(e_1+e_2) \xrightarrow{(\nu_1,\nu_2,\nu_3)} R(c_n)(e_1,e_2).$$

As $k(\emptyset)$ is a simple object, if $k(\emptyset) \to V$ is a irreducible morphism then V is projective. Therefore $V = k(a), k(b)$ or, $k(c_n)$, hence the sequence $SD(0)$ is an almost split sequence and then $\tau(R(c_n)) = k(\emptyset)$ and

$$\nu_3 : k(c_n)(e_1+e_2) \to R(c_n)(e_1,e_2),$$

is an irreducible morphism. If $k(c_n)(e_1+e_2) \to V$ is an irreducible morphism and V is not projective then there exists an irreducible morphism $\tau(V) \to k(c_n)(e_1+e_2)$, then $\tau V = k(\emptyset)$ and $V = R(c_n)(e_1,e_2)$; therefore the only irreducible ones up to multiples non trivial scalars that come out of $k(c_n)(e_1+e_2)$ are $f_1, -f_2$. Therefore $SD(0)$ is an almost split sequence.

By induction it is proved that $SD(1), \ldots, SD(n-2)$ are almost split sequences. In particular the irreducible morphism $k(c_1)(e_1+e_2) \to R(c_1)(e_1,e_2)$ is obtained and this is up to multiple non-trivial scalars the only irreducible coming out of $k(c_1)(e_1+e_2)$. This implies that there is an almost split sequence

$$k(c_1)(e_1+e_2) \to R(c_1)(e_1,e_2) \to k(a,b)(e_2).$$

We have the irreducible morphism $k(b)(e_1+e_2) \to R(c_n)(e_1,e_2)$. If $k(b)(e_1+e_2) \to V$ is an irreducible morphism, where V is not projective; therefore there exists an irreducible morphism $\tau(V) \to k(b)(e_1+e_2)$, so $\tau V = k(\emptyset)$; hence $V = R(c_n)(e_1+e_2)$. Thus we obtained the almost split sequence

$$k(b)(e_1+e_2) \to R(c_n)(e_1,e_2) \to k(a,c_n).$$

Analogously, we obtained the almost split sequence

$$k(a)(e_1+e_2) \to R(c_n)(e_1,e_2) \to k(b,c_n).$$

We have the commutative diagrams $D(i,j)$:

$$\begin{tikzcd}
R^i(c_{n-j})(e_1,e_2) \arrow[r, "f_1"] \arrow[d, "f_2"] & R^{i+1}(c_{n-j})(e_1,e_2) \arrow[d, "g_1"] \\
R^i(c_{n-j-1})(e_1,e_2) \arrow[r, "g_2"] & R^{i+1}(c_{n-j-1})(e_1,e_2)
\end{tikzcd}$$

and the ε-sequence $SD(i,j)$:

$$R^i(c_{n-j})(e_1,e_2) \xrightarrow{(f_1,f_2)^T} R^{i+1}(c_{n-j})(e_1,e_2) \bigoplus R^i(c_{n-j-1})(e_1,e_2) \xrightarrow{(g_1,g_2)} R^{i+1}(c_{n-j-1})(e_1,e_2).$$

Let now $R(c_n)(e_1,e_2) \to V$ be an irreducible morphism. In this case V can not be projective; therefore there exists an irreducible morphism $\tau(V) \to R(c_n)(e_1,e_2)$ then $\tau(V)$ is $k(a), k(b)$ or $k(c_n)$, hence V is equal to $k(b,c_n)(e_2)$ or $k(a,c_n)(e_1)$ or $R(c_{n-1})(e_1,e_2)$, from here the almost split sequence:

$$R(c_n)(e_1,e_2) \to k(b,c_n) \bigoplus k(a,c_n) \bigoplus R(c_n)(e_1,e_2) \to R^1(c_{n-1}).$$

is obtained. So, $SD(0,1)$ is an almost split sequence and by induction all sequences $SD(0,j)$ with $j = 1,\ldots,n-2$.

In general, we have the ε- sequences $H(i)$:

$$R^i(c_{n-i})(e_1,e_2) \to k(b,c_{n-i}) \bigoplus k(a,c_{n-i}) \bigoplus R^i(c_{n-i-1)(e_1,e_2) \to R^{i+1}(c_{n-i-1})}.$$

As the sequence $SD(0,1)$ is an almost split sequence, we obtain the irreducible morphisms

$$R^1(c_{n-1})(e_1,e_2) \to R^1(c_{n-2})(e_1,e_2);$$

therefore the irreducible morphism

$$R^1(c_{n-2})(e_1,e_2) \to R^2(c_{n-2})(e_1,e_2)$$

is obtained.

With the previous procedures it is proved that $H(1)$ is an almost split sequence and as before it is proven that the sequences $SD(i,j)$ with $i \le n-2$ are almost split sequences.

Following in this way, it is proved that for $i \leq n-2$ the sequences $O(b,i)$:

$$k(b, c_{n-i})(e_1 + e_2) \to R^{i+1}(c_{n-i-1})(e_1 + e_2) \to k(a, c_{n-i-1}),$$

and $O(a,i)$:

$$k(a, c_{n-i})(e_1 + e_2) \to R^{i+1}(c_{n-i-1})(e_1, e_2) \to k(b, c_{n-i-1}),$$

are almost split sequences, then we have irreducible morphisms:

$$k(c_1)(e_1 + e_2) \to R(c_1)(e_1, e_2) \to R^1(c_1)(e_1, e_2) \to \cdots \to R^{n-1}(c_1)(e_1, e_2) \to k(b, c_1)(e_1).$$

Then the ε-sequence:

$$k(c_1)(e_1 + e_2) \to R(c_1)(e_1, e_2) \to k(a, b)(e_1),$$

is obtained, and by induction the ε- sequences

$$L(0) : R(c_1)(e_1, e_2) \to R^1(c_1)(e_1, e_2) \bigoplus k(a, b)(e_1) \to k(a, b, c_n)(e_1),$$

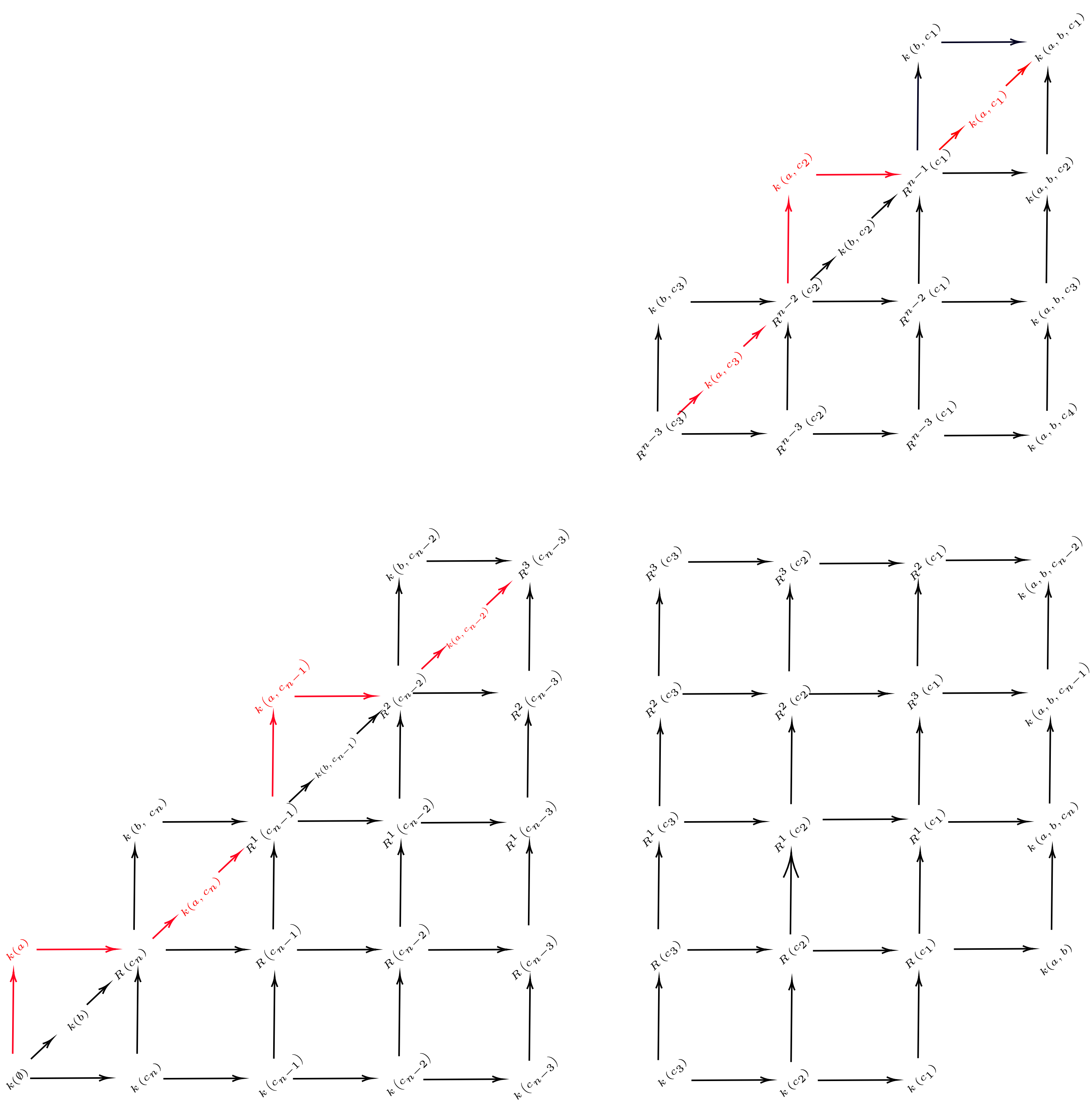
and for $1 \leq i \leq n-2$,

$$L(i) : R^i(c_1)(e_1, e_2) \to R^{i+1}(c_1)(e_1, e_2) \bigoplus k(a, b, c_{n-i+1})(e_1) \to k(a, b, c_{n-i})(e_1)$$

are obtained.

As the representations $k(a, b), k(a, b, c_n), \ldots, k(a, b, c_1)$ and $k(a, c_1), k(b, c_1)$ are injectives, the construction of Auslander-Reiten quiver of $\mathcal{K}$ is completed.

Figure 4.4. The Auslander-Reiten quiver of a poset $\mathcal{K}$



Analogously, in $\mathrm{Rep}(\mathcal{K}')$ we obtain the irreducible morphisms between projectives:

$$k(\emptyset) \to k(b) \quad k(\emptyset) \to k(c_n^+) \to k(c_{n-1}^+) \to \cdots \to k(c_1^+).$$

We have the following commutative diagram $D(i)$:

$$\begin{array}{ccc} k(c_{n-i}^+)(e) & \xrightarrow{f_1} & k(c_{n-i}^+, b)(e) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ k(c_{n-i-1}^+)(e) & \xrightarrow{g_2} & k(c_{n-i-1}^+, b)(e) \end{array}$$

such that the sequences $S(D(i))$:

$$k(c_{n-i}^+)(e) \xrightarrow{(f_1,f_2)^T} k(c_{n-i}^+, b)(e) \bigoplus k(c_{n-i-1}^+)(e) \xrightarrow{(g_1,g_2)} k(c_{n-i-1}^+, b)(e),$$

are an ε-sequences.

We have the diagram:

$$\begin{array}{ccc} k(\emptyset) & \xrightarrow{\lambda_1} & k(b)(e) \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \nu_1} \\ k(c_n^+)(e) & \xrightarrow{\nu_2} & k(c_n^+, b)(e) \end{array}$$

and the ε-sequence

$$k(\emptyset)(e) \xrightarrow{(\lambda_1,\lambda_2)} k(b)(e) \bigoplus k(c_n^+)(e) \xrightarrow{(\nu_1,\nu_2)} k(c_n^+, b)(e).$$

As $k(\emptyset)$ is a simple object, if $k(\emptyset) \to V$ is an irreducible morphism then V is a projective; therefore $V = k(b)$ or $V = k(c_n^+)$; hence the sequence $SD(0)$ is an almost split sequence and then $\tau(k(c_n^+, b)) = k(\emptyset)$ and

$$\nu_2 : k(c_n^+)(e) \to k(c_n^+, b)(e),$$

is an irreducible morphism. If $k(c_n^+)(e) \to V$ is an irreducible morphism and V is not a projective, then there exists an irreducible morphism $\tau(V) \to k(c_n^+)(e)$. According to $\tau(V) = k(\emptyset)$ and $V = k(c_n^+, b)(e)$, the only irreducible ones up to multiple non trivial scalars that come out of $k(c_n^+)(e)$ are f_1, f_2. Thus, $SD(0)$ is an almost split sequence.

By induction it is proved that $SD(1), \ldots, SD(n-2)$ are almost split sequences. In particular, since the irreducible morphisms $k(c_1^+)(e) \to k(a)(e)$ and $k(c_1^+)(e) \to k(c_1^+, b)(e)$ are obtained, these are up to multiple non-trivial scalars the only irreducibles coming out of $k(c_1)(e)$. This implies that there is an almost split sequence

$$k(c_1^+)(e) \to k(c_1^+, b)(e) \bigoplus k(a)(e) \to k(a, b)(e).$$

We have the irreducible morphism $k(b)(e) \to k(c^+, b)(e)$. If $k(b)(e) \to V$ is an irreducible morphism, where V is not projective. Therefore there exists an irreducible morphism $\tau(V) \to k(b)(e)$, so $\tau V = k(\emptyset)$; hence $V = k(c^+, b)(e)$, so we obtain the almost split sequence

$$k(b)(e) \to k(c_n^+, b)(e) \to k(c_n^-).$$

We have the commutative diagrams $\bar{D}(j) : (1 \leq j \leq n-2)$

$$\begin{array}{ccc} k(c_{n-j}^+, b)(e) & \xrightarrow{f_1} & k(c_{n-j}^+, c_n^-)(e) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ k(c_{n-j-1}^+)(e) & \xrightarrow{g_2} & k(c_{n-j-1}^+, c_n^+)(e) \end{array}$$

and the ε-sequences $S\bar{D}(j)$:

$$k(c_{n-j}^+, b)(e) \xrightarrow{(f_1,f_2)^T} k(c_{n-j}^+, c_n^-)(e) \bigoplus k(c_{n-j-1}^+, b)(e) \xrightarrow{(g_1,g_2)} k(c_{n-j-1}^+, c_n^+)(e).$$

Let $k(c_n^+, b) \to V$ be an irreducible morphism, in this case V cannot be projective; therefore there exists an irreducible morphism $\tau(V) \to k(c_n^+, b)$ then $\tau(V)$ is $k(b)$ or $k(c_n^+)$; hence V is equal to $k(c_n^-)(e)$ or $k(c_{n-1}^+, b)(e)$, from here the almost split sequence:

$$k(c_n^+, b) \to k(c_n^-)(e) \bigoplus k(c_{n-1}^+, b)(e) \to k(c_{n-1}^+, c_n^-)(e),$$

is obtained. So, $S\bar{D}(1)$ is an almost split sequence and by induction all sequences $S\bar{D}(j)$ with $j = 1, \ldots, n-2$ are too.

In general, we have the ε- sequences $H(i)$:

$$k(c_{n-j}^+, c_{n-j+1}^-)(e) \to k(c_{n-j-1}^+, c_{n-j+1}^-) \bigoplus k(c_{n-j}^-) \to k(c_{n-j-1}^+, c_{n-j}^-)(e).$$

As the sequence $S\bar{D}(1)$ is an almost split sequence, we obtain the irreducible morphisms

$$k(c_{n-1}^+, c_n^-)(e) \to k(c_{n-2}^+, c_n^-)(e).$$

Therefore the irreducible morphism

$$k(c_{n-2}^+, c_n^-)(e) \to k(c_{n-2}^+, c_{n-1}^-)(e),$$

is obtained.

With the previous procedures, it is proved that $H(1)$ is an almost split sequence and as before, it is proven that the sequences $S\bar{D}(j)$ with $1 \leq j \leq n-2$ are almost split sequences.

Now, for $i = n, n-1, \ldots, 4$, we have the commutative diagrams $\hat{D}(i,j) : (n-i+2 \leq j \leq n-2)$

$$\begin{array}{ccc} k(c_{n-j}^+, c_i^-)(e) & \xrightarrow{f_1} & k(c_{n-j}^+, c_{i-1}^-)(e) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ k(c_{n-j-1}^+, c_i^-)(e) & \xrightarrow{g_2} & k(c_{n-j-1}^+, c_{i-1}^-)(e) \end{array}$$

and the ε-sequences $S\hat{D}(i,j)$:

$$k(c^+_{n-j},c^-_i)(e) \xrightarrow{(f_1,f_2)^T} k(c^+_{n-j},c^-_{i-1})(e) \bigoplus k(c^+_{n-j-1},c^-_i)(e) \xrightarrow{(g_1,g_2)} k(c^+_{n-j-1},c^-_{i-1})(e),$$

Let now $k(c^+_{n-2},c^-_n)(e) \to V$ be an irreducible morphism. In this case V cannot be projective; therefore there exists an irreducible morphism $\tau(V) \to k(c^+_{n-2},c^-_n)(e)$ then $\tau(V)$ is $k(c^+_{n-1},c^-_n)(e)$ or $k(c^+_{n-2},b)(e)$; hence V is equal to $k(c^+_{n-2},c^-_{n-1})(e)$ or $k(c^+_{n-3},c^-_n)(e)$, from here the almost split sequence:

$$k(c^+_{n-2},c^-_n)(e) \to k(c^+_{n-2},c^-_{n-1})(e) \bigoplus k(c^+_{n-3},c^-_n)(e) \to k(c^+_{n-3},c^-_{n-1})(e),$$

is obtained. Thus, $S\hat{D}(2)$ is an almost split sequence and by induction all sequences $S\hat{D}(j)$ with $j = 2,\ldots,n-2$ are too. As before it is proven that the sequences $S\hat{D}(j)$ with $j \leq n-2$ are almost split sequences.

Following in this way it is proved that for $i \leq n-2$, the sequences

$$O(b,i) : k(b,c_{n-i})(e_1+e_2) \to R^{i+1}(c_{n-i-1})(e_1+e_2) \to k(a,c_{n-i-1}),$$

are almost split sequences, then we have irreducible morphisms:

$$k(c^+_1)(e) \to k(c^+_1,b)(e) \to k(c^+_1,c^-_n)(e) \to \cdots \to k(c^+_1,c^-_2)(e) \to k(c^-_1)(e_1),$$

Then the ε-sequence

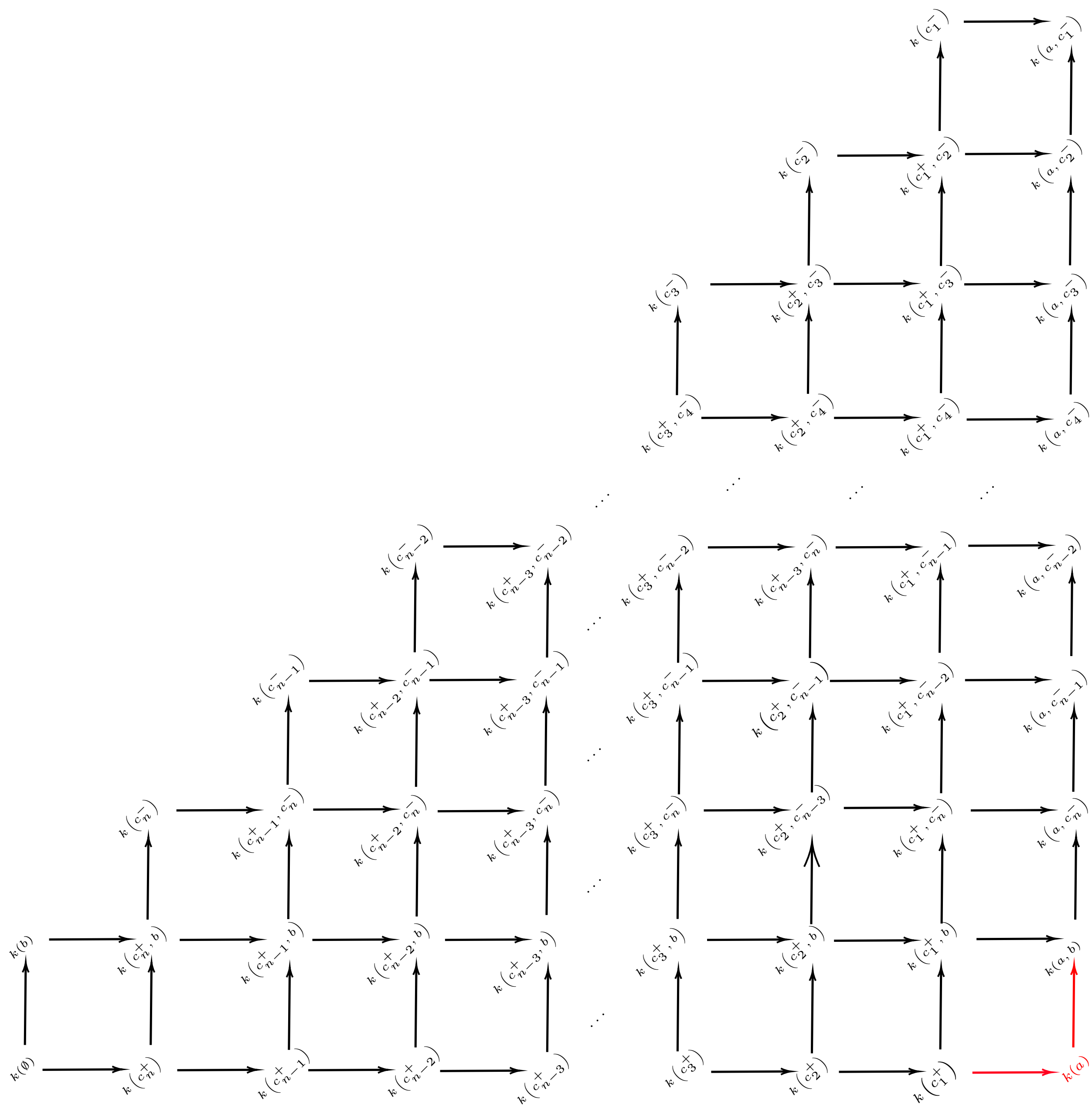
$$L(0) : k(c^+_1,b)(e) \to k(c^+_1,c^-_n) \bigoplus k(a,b)(e) \to k(a,c_n)(e),$$

is obtained, and by induction for $0 \leq i \leq n-2,$ the sequences

$$L(i) : k(c^+_1,c^-_{n-i}) \to k(c^+_1,c^-_{n-i-1}) \bigoplus k(a,c^-_{n-i})(e) \to k(a,c_{n-i-1})(e).$$

are obtained. As the representations $k(a,b), k(a,c^-_n),\ldots,k(a,c^-_1)$ and $k(c^-_1)$ are injective, the construction of Auslander-Reiten quiver of $\mathcal{K}'$ is completed.

FIGURE 4.6. The Auslander-Reiten quiver of a poset $\mathcal{K}'$



Now, we consider the derivation $' : \mathcal{R} \to \mathcal{R}'$ as in (4.1). For $1 \leq l \leq i$ we have that

$$R^l(c_{n-i})(e_1, e_2)' = k(a)(e_1) \bigoplus k(c^+_{n-i}, c^-_{n-i+1})(e_2),$$

$$R(c_i)(e_1, e_2)' = k(a)(e_1) \bigoplus k(c^+_i, b)(e_2),$$

further,

$$\begin{array}{cccc} k(a)' = k(a), & k(b)' = k(b), & k(a,b)' = k(a,b), & k(c_i)' = k(c^+_i), \\ k(a,c_i)' = k(a), & k(b,c_i)' = k(c^-_i), & k(a,c_i,b)' = k(a,c^-_i). & \end{array}$$

Finally, if $X \xrightarrow{u} Y \xrightarrow{v} Z$ is an almost split sequence in $\mathrm{Rep}(\mathcal{K})$ with $1_X, 1_Y \notin \mathcal{I}$, then the sequence $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z$ in $\mathrm{Rep}(\mathcal{K})/\mathcal{I}$ has the following properties:

1. $\underline{vu} = 0$.
2. If $h : W \to Y$ is a morphism such that $\underline{v}h = 0$, then h is factored by $\underline{u}$.
3. If $g : W \to Z$ is a morphism in $\mathrm{Rep}(\mathcal{K})/\mathcal{I}$ which is not a retraction then is factored by $\underline{v}$.

The first and third paragraphs are clear. For the second we consider $h = \underline{h}_1$ with $h_1 : W \to Y$ being a morphism in $\mathrm{Rep}(\mathcal{K})$, so $vh_1 = \mu\nu$ with $\nu : W \to L$ and $\mu : L \to Z$ where L is the sum of some of the indecomposable ones that generate $\mathcal{I}$.

Since $1_Z \notin \mathcal{I}$, then μ is not retraction, so it is factored by v and it is obtained that $\mu = v\mu_1$, thus $vh = v\mu_1\nu$ that is, $v(h - \mu_1\nu) = 0$. Therefore, there exists $g : W \to X$ such that $ug = h - \mu_1\nu$ thus $\underline{ug} = \underline{h}$.

Let $F : \mathrm{Rep}(\mathcal{K})/\mathcal{I} \to \mathrm{Rep}(\mathcal{K})'/\mathcal{I}'$ be the functor induced por (4.1), then $F(X)$ is projective in $\mathrm{Rep}(\mathcal{K}')$ if and only if X is projective in $\mathrm{Rep}(\mathcal{K})$.
Let $u : X \to Y$ be a sink morphism en $\mathrm{Rep}(\mathcal{K})$ with $Y \notin \mathcal{I}$ then if Y is projective and $h : Z \to X$ is a morphism such that $\underline{uh} = 0$ then $\underline{h} = 0$. Indeed, if this happens then $uh = \nu\lambda$, with $\lambda : Z \to W$ and $\nu : W \to Y$ and where $W \in \mathcal{I}$. Since $Y \notin \mathcal{I}$, there exists a morphism $s : W \to X$ such that $\nu = us$. Therefore $uh = us\lambda$, so $u(h - s\lambda) = 0$ and as u is injective we obtain that $h = s\lambda$, thus $\underline{h} = 0$.

Let $\underline{f} : X \to Y$ be a morphism in $\mathrm{Rep}(\mathcal{K})/\mathcal{I}$ such that $F(\underline{f}) = \underline{f}' = 0$. We will prove that $\underline{f} = 0$. If $X = Y$ or there exists an arrow from X to Y it is clear. Let us observe that if there are two paths from X to Y in $\mathrm{Rep}(\mathcal{K})/\mathcal{I}$ these have the same length, the same occurs in $\mathrm{Rep}(\mathcal{K}')/\mathcal{I}'$.

We will prove our result by induction on the length of the paths from X to Y. Our result is true when the length is one. Suppose that the result be true for lengths less than n and suppose that the length of the path from X to Y is n. Then we have a sink morphism

$$(F(u_1), \ldots, F(u_r), \lambda)^T : F(Z_1) \bigoplus \cdots \bigoplus F(Z_r) \bigoplus L \to F(Y),$$

with $L \in \mathcal{I}$ and a sink morphism

$$(u_1, \ldots, u_r)^T : Z_1 \bigoplus \cdots \bigoplus Z_r \to Y,$$

with $f = \sum_{s=1}^{r} u_s v_s$. Therefore $F(f) = \sum_{s=1}^{r} F(u_s v_s) = 0$. If Y is projective, $F(Y)$ is projective, and thus $(F(v_1), \ldots, F(v_r)) = 0$, which implies $F(v_i) = 0$, for all i, $1 \leq i \leq r$ and by induction hypothesis $v_1 = 0, \ldots, v_r = 0$, so $f = 0$. Now, we suppose that Y is not projective, then there exists an almost split sequence in $\mathrm{Rep}(\mathcal{K})$

$$W \xrightarrow{(a_1,\ldots,a_r,\nu)^T} Z_1 \bigoplus \cdots \bigoplus Z_r \bigoplus L \xrightarrow{(u_1,\ldots,u_r,\mu_1)} Y,$$

and we have the almost split sequence in $\mathrm{Rep}(\mathcal{K}')$:

$$F(W) \xrightarrow{(F(a_1),\ldots,F(a_r),F(\nu))^T} f(Z_1) \bigoplus \cdots \bigoplus F(Z_r) \bigoplus F(L) \xrightarrow{(F(u_1),\ldots,F(u_r),F(\mu_1))} F(Y),$$

in the category $\mathrm{Rep}(\mathcal{K}')/\mathcal{I}'$ we have $\sum_{j=1}^{r} F(u_j v_j) = 0$ then by property 2 above, there exists $\rho : X \to W$ such that $F(a_j)F(\rho) = F(v_j)$. Since the length of the paths from X to Z_j are strictly less than n, by induction hypothesis $v_j = a_j$, thus $\underline{f} = \sum_j \underline{u_j a_j \rho} = 0$. ■

Proposition 21. *Let $\mathcal{L} = k(a, c_i)$ or $\mathcal{L} = k(a)$ in $\mathrm{Rep}(\mathcal{P}, \theta)$. Then*

1. *If $f : V|_{\mathcal{K}} \to \mathcal{L}|_{\mathcal{K}}$ is a morphism in $\mathrm{Rep}(\mathcal{K})$, then $f : V \to \mathcal{L}$ is a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$.*

2. *If $g : \mathcal{L}|_{\mathcal{K}} \to V|_{\mathcal{K}}$ is a morphism in $\mathrm{Rep}(\mathcal{K})$ then $g : \mathcal{L} \to V$ is a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$.*

Proof Let $f : V|_{\mathcal{K}} \to k(a, c_i)(e)|_{\mathcal{K}}$ is a morphism in $\mathrm{Rep}(\mathcal{K})$. We will see that $f : V \to k(a, c_i)$ is a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$. Let $z \in \theta$, such that $z \neq a, b, c_1, \ldots, c_n$. Let $(\lambda, \mu) \in V_z$ and let $z = (x, y)$. If $x < b$, $y < b$ then λ, $\mu \in V_b$ and since $k(a, c_i)_b = 0$ and $f(V_b) \subset k(a, c_i)_b$ then $f(\lambda) = 0, f(\mu) = 0$; therefore $(f(\lambda), f(\mu)) = (0, 0) \subset k(a, c_i)_z = 0$. If $x > a$, $y > a$, $f(\lambda) = ce, f(\mu) = c'e$ and $(f(\lambda), f(\mu)) = (ce, c'e) \in k(a, c_i)_z$. Analogously if $x > a, y < b$, then $f(\lambda) = 0, f(\mu) = ce$ and $f(\lambda, \mu) = (ce, 0) \in k(a, c_i)(e)_z$. Therefore, f is a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$.

Now, we prove that $g : k(a, c_i) \to V$ is a morphism in $\mathrm{Rep}(\mathcal{P}, \theta)$. Let $z \in \theta$ different from $a, b, c_1, \ldots, c_n$. We suppose that $z = (x, y)$ if $x < b, y < b$. We have $k(a, c_i)_z = (0, 0)$, therefore $g^z k(a, c_i)_z \in V_z$. We suppose that $x > a, y < b$, Then if $(\lambda, \mu) \in k(a, c_i)_z$ it is obtained that $\lambda = ce, \mu = 0$, $ce \in k(a, c_i)_z$, therefore $g(ce) \in V_a$ and since $x > a, (g(ce), 0) \in V_z$, so that $g^z(k(a, c_i)_z) \subset V_z$. Similarly it is proved when $x > a, y > a$. For $\mathcal{L} = k(a)$, the proof is similar. ■

Corollary 6. *Let $\mathcal{L}$ be equal to $k(a, c_i)$ or $k(a)$ such that $\mathcal{L}|_{\mathcal{K}}$ is a direct summand of $V|_{\mathcal{K}}$, then $\mathcal{L}$ is a direct summand of V in $\mathrm{Rep}(\mathcal{P}, \theta)$.*

Proof If $\mathcal{L}|_{\mathcal{K}}$ is a direct summand of $V|_{\mathcal{K}}$, then there exists a morphism in $\mathrm{Rep}(\mathcal{K})$, $f : V|_{\mathcal{K}} \to \mathcal{L}|_{\mathcal{K}}$ and $g : \mathcal{L}|_{\mathcal{K}} \to V|_{\mathcal{K}}$ such that $fg = 1_{\mathcal{L}}$. By previous proposition f and g are morphisms in $\mathrm{Rep}(\mathcal{P}, \theta)$; hence $\mathcal{L}$ is direct summand of V in $\mathrm{Rep}(\mathcal{P}, \theta)$. ■

Proposition 22. *Let W be a representation of* $\mathrm{Rep}(\mathcal{P}', \theta')$. *The following statements hold:*

1. *If $f : W|_{\mathcal{K}'} \to k(a)|_{\mathcal{K}'}$ is a morphism in* $\mathrm{Rep}(\mathcal{K}')$ *then $f : W \to k(a)$ is a morphism in* $\mathrm{Rep}(\mathcal{P}', \theta')$.

2. *If $g : k(a)|_{\mathcal{K}'} \to W|_{\mathcal{K}'}$ is a morphism in* $\mathrm{Rep}(\mathcal{K})'$, *then $g : k(a) \to W$ is a morphism in* $\mathrm{Rep}(\mathcal{P}', \theta')$.

Proof.

1. If $z \in \theta$ is such that $z \neq a, b, c_1^+, \dots, c_n^+, c_1^-, \dots, c_n^-$. We take $(\lambda, \mu) \in W_z$, if $x < b$ then $\lambda \in W_b$ so, $f(\lambda) \in k(a)_b = 0$. Hence, if $x < b, y < b, f(\lambda) = f(\mu) = 0$ and in this case $(f(\lambda), f(\mu)) = (0, 0) \in k(a_z)$ then $f^z(W_z) \subset k(a)_z$. If $x > a, y < b, f(\lambda) = ce, f(\mu) = 0$ and $(f(\lambda), f(\mu)) = (ce, 0) \in k(a)_z$. If $x > a, y > a, f(\mu) = c'e$ then $(f(\lambda), f(\mu)) = (ce, c'e) \in k(a)_z$. So, in any case $f^z(W_z) \subset k(a_z)$.

2. We have $k(a)(e)a = \langle e \rangle$ so $g(e) \in W_a$. Let $z \in \theta$ such that $z \neq a, b, c_1^+, \dots, c_n^+, c_1^-, \dots, c_n^-$. If $x < b, y < b, k(a)(e)z = 0$; hence $g^z(k(a)(e)_z) \in W_z$ if $x < b, y > a, k(a)_z = \{(0, ce) \mid c \in k\}$‘. Here, $y > ayg(e) \in W_a$; therefore, $(0, g(e)) \in W_z$. If $x > a, y > a, k(a)(e)_z = \{f(ce, c'e) \mid (c, c') \in k\}$, so $(g(ce), g(c'e)) = c(g(e), 0) + c'(0, g(e)) \in W_z$. ■

As a consequence of the previous proposition we obtain the following corollary.

Corollary 7. *If $k(a)|_{\mathcal{K}'}$ is a direct summand of $W|_{\mathcal{K}'}$ in* $\mathrm{Rep}\,(\mathcal{K}')$ *then $k(a)$ is a direct summand of W in* $\mathrm{Rep}\,(\mathcal{P}', \theta')$.

Theorem 12. *Let $\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \{c_1 < c_2 < \cdots < c_n\}$ be a poset with a suitable pair of points (a, b). Then the differentiation functor*

$$' : \mathcal{R} \to \mathcal{R}',$$

defined by formulas (4.1) *induces an equivalence between quotient categories*

$$\mathcal{R}/\langle k(a), k(a, c_1), \dots, k(a, c_n) \rangle \xrightarrow{\sim} \mathcal{R}'/\langle k(a) \rangle.$$

Proof. Let us prove that the functor is dense. Let $M \in \mathcal{R}'$ be an object without direct summand $k(a)$. We consider $M|_{\mathcal{K}'} = \bigoplus M^u$ where M^u is indecomposable in $\mathrm{Rep}(\mathcal{K}')$. By Corollary 7 , each M^u is not isomorphism to $k(a, c_i)$ or $k(a)$. By Proposition 20 , for each M^u there exists an object V^u in $\mathrm{Rep}(\mathcal{K})$ such that $(V^u)' = T^u \bigoplus M^u$ where $T^u = k(a)$ or $T^u = 0$. Let $V = \bigoplus V^u \in \mathrm{Rep}(\mathcal{K})$, and we consider for each $z \in \theta$, with $z \neq a, b, c_i$ the subspace $V_z = (\bigoplus\limits_u T^u)_z \bigoplus M_z$ of V_0^z and for a, c_i, b, $V_a = \bigoplus\limits_u V_a^u$, $V_{c_i} = \bigoplus\limits_u V_{c_i}^u$ and $V_b = \bigoplus V_b^u$ respectively.

We will prove that $V = (V_0, V_z)_{z\in\theta}$ is a representation of $(\mathcal{P}, \theta)$. Indeed, $(\bigoplus_u T^u) \bigoplus M$ is a representation of $(\mathcal{P}', \theta')$. We suppose that z and z_1 are different from a, b, c_i such that z and z_1 is in θ'; if $z = (x, y)$, $z_1 = (x_1, y_1)$ with $x_1 < x$, $\nu \in V^+_{x_1}$ then $(\nu, 0) \in V_z$.
Let $z = (x, y)$ and $a < x$, then $V_a = \bigoplus V^u_a$. Let $\lambda \in V^u_a = (V^u)'_a = (T_u)_a \bigoplus M_a$, so $(\lambda, 0) \in (T_u \bigoplus V^u)_z \subset V_z$. Similarly, if $b < x$ and $\lambda \in (V^u)_b$ it is proved that $(\lambda, 0) \in V_z$. In case that $c_i < x$, and $\lambda \in (V^u)_{c_i}$, it is obtained that $(V^u)'_{c_i^+} = V^u_{c_i} + V^u_a = (T_u \bigoplus M)_{c_i^+}$. By definition $c_i^+ < x$; therefore as $\lambda \in (Tu \oplus M)_{c_i^+}$, then $(\lambda, 0) \in (T_u \bigoplus M)_z \subset V_z$.

Now, we suppose $x < a$ and $(\lambda, \nu) \in (\bigoplus_u T_u \bigoplus M)_z$, then $\lambda \in (\bigoplus T_u \oplus M)_a = \bigoplus_u (V^u)'_a = \bigoplus_u V^u_a = V_a$. Similarly, if $x < b$, then for $(\lambda, \nu) \in V_z$, $\lambda \in V_b$. When $x < c_i$, it is obtained that $x < c_i^-$ and $(\lambda, \nu) \in V_z = (\bigoplus_u T_u \oplus M^u)_z$. Thus

$$\lambda \in (\bigoplus_u T_u \bigoplus M)_{c_i^-} = \bigoplus_u (V^u)'_{c_i^-} = \bigoplus_u (V^u_{c_i} \cap V^u_b) \subset V_{c_i},$$

and therefore, $V \in \operatorname{Rep}(\mathcal{P}, \theta)$.

Now we have

$$(V_{\mathcal{K}'})' = \bigoplus_u (V^u)'|_{\mathcal{K}'} = \bigoplus_u (T_u \bigoplus M^u)|_{\mathcal{K}'} = \bigoplus_u (T_u \bigoplus M)|_{K'},$$

and for $z \in \theta'$, $z \neq a, b, c_i^+, c_i^-$ it is obtained that $V'_z = V_z = (\bigoplus_u T_u \bigoplus M)_z$; therefore $V' = \bigoplus_u T_u \bigoplus M$.

Finally, we will prove that the functor is faithful and full. Let $f : U \to V$ be a morphism in $\operatorname{Rep}(\mathcal{P}, \theta)$ and we suppose that $f' = h_2 h_1 : U' \to V'$, where $h_1 : U' \to k(a)$, $h_2 : k(a) \to V'$ are morphisms in $\operatorname{Rep}(\mathcal{P}', \theta')$. Therefore, we have that $f'|_{\mathcal{K}'}$ is factored by $k(a)$. By Proposition 20 , $f_{\mathcal{K}} : U|_{\mathcal{K}} \to V|_{\mathcal{K}}$ is factored through the objects $k(a, c_i), k(a)$, and by Proposition 21 , f is factored through the objects $k(a, c_i), k(a)$ in $\operatorname{Rep}(\mathcal{P}, \theta)$. Thus the functor is faithful.
Let now $f : U' \to V'$ be a morphism in $\operatorname{Rep}(\mathcal{P}', \theta')$. We will consider the morphism

$$f|_{\mathcal{K}'} : U'|_{\mathcal{K}'} = (U|_{\mathcal{K}})' \to V'|_{\mathcal{K}'} = (V|_{\mathcal{K}})'$$

By Proposition 20 , there exists $h : U|_{\mathcal{K}} \to V|_{\mathcal{K}}$ such that $h' = g = f + \lambda_2\lambda_1$, where $\lambda_1 : U'|_{\mathcal{K}'} \to k(a)(X)$ and $\lambda_2 : k(a)(X) \to U'|_{\mathcal{K}'}$ for a finite dimensional k-vector space X. By (1) and (2) of Proposition 22 , λ_1 and λ_2 are morphisms in $\operatorname{Rep}(\mathcal{P}', \theta')$; therefore g is a morphism in $\operatorname{Rep}(\mathcal{P}', \theta')$.
By definition $h = g : U_0 \to V_0$ and for $z \in \theta$, $z \neq a, b, c_i$, we have that $h^z(U_z) = g^z(U'_z) \bigoplus V'_z = V_z$. Therefore, h is a morphism in the category $\operatorname{Rep}(\mathcal{P}, \theta)$ and $h' = g = f + \lambda_2\lambda_1$. This proves our result. ∎

CHAPTER 5

Categorical Properties of Algorithm of Differentiation III

In this chapter we will study the categorical properties of the differentiation algorithm DIII following the ideas used for the differentiation algorithm DI presented in the previous chapter. It is worth mentioning that this algorithm is essentially different from such an algorithm, in the sense that the most basic case that can be presented is a poset $\mathcal{P}$ consisting of two big incomparable equivalent points a, b which is of the tame representation type (see, [32]).

5.1 Algorithm of Differentiation III

Definition 39. *A pair of points (a, b) of a poset $(\mathcal{P}, \theta)$ is said to be suitable for Differentiation III if $\mathcal{P} = a^{\triangledown} + b_{\vartriangle}$, where a, b are big incomparable equivalent points. In this situation, the derivative poset $(\mathcal{P}', \theta')$ is obtained from the poset $(\mathcal{P}, \theta)$ as follows:*

1. *the point a is replaced by an infinite decreasing chain $a_1 > a_2 > a_3 \cdots$ and the point b by an infinite increasing chain $b_1 < b_2 < b_3 \cdots$;*

2. *a relation $a_1 < b_1$ is added with its induced relations;*

3. *θ' is obtained from θ by deleting the class $\{a, b\}$ and adding the classes $\{a_n, b_n\}$, $n \geq 1$.*

The following Figure 5.1 illustrates the Hasse diagram for this differentiation.

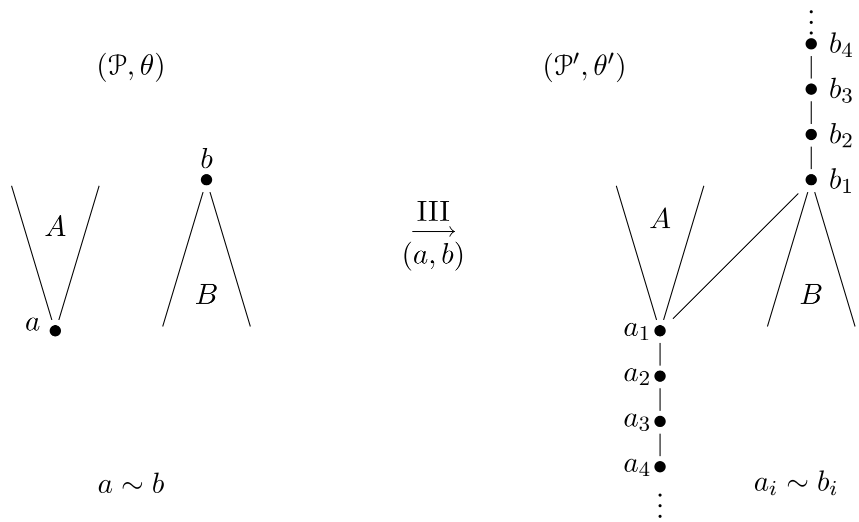


FIGURE 5.1. The diagram of a poset with involution $(\mathcal{P}, \theta)$ and its corresponding derivate.

Let the subsets $A, B \subseteq \mathcal{P}$ be $A = a^{\triangledown}\backslash\{a\}$; $B = b_{\vartriangle}\backslash\{b\}$. Then $\mathcal{R}$ and $\mathcal{R}'$ denote the following subcategories of $\operatorname{Rep}(\mathcal{P}, \theta)$ and $\operatorname{Rep}(\mathcal{P}', \theta')$ respectively for $n \geq 1$:

$$\mathcal{R} = \{U \in \operatorname{Rep}(\mathcal{P}, \theta) \mid U_a^+ \subset U_b^+; U_a^- = 0\}, \tag{5.1}$$

$$\mathcal{R}' = \{U \in \operatorname{Rep}(\mathcal{P}', \theta') \mid U_{a_1}^+ \subset U_B^+; U_{a_n}^- = U_{a_{n+1}}^+\}. \tag{5.2}$$

The differentiation DIII induces the functor $\;' : \mathcal{R} \to \mathcal{R}'$ which is defined as follows:

- To each object $U = (U_0, U_z)_{z\in\theta} \in \mathcal{R}$ an object $U' = (U_0', U_z')_{z\in\theta'} \in \mathcal{R}'$ is assigned such that

$$\begin{aligned} U_0' &= U_0, \\ U_z' &= U_z, \quad \text{for } z \neq (a_n, b_n), \\ U'_{(a_n,b_n)} &= (U_B^+, U_0) \cap U_{(a,b)}^{[n]} + (0, U_a^+), \end{aligned} \tag{5.3}$$

  where $U_{(a,b)}^{[n]} = \{(t_0, t_n) \mid (t_0, t_1) \in U_{(a,b)}, (t_1, t_2) \in U_{(a,b)}, \ldots, (t_{n-1}, t_n) \in U_{(a,b)}\}$.

- If $\phi : (U_0, U_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$ is a morphism in $\mathcal{R}$, then $\phi' := \phi : U_0 = U_0' \to V_0' = V_0$.

Remark 10. *It follows from the definition above that* $K(A, b)' = K(A, b_1)$.

Example 10. *We consider the poset with an involution* $(\mathcal{P}, \theta)$ *where* $\mathcal{P}$ *is as in Figure 5.2, with* $a < c$*,* $d < b$ *and* $\theta = \{(a, b), (c, d)\}$ *are the equivalence class.*

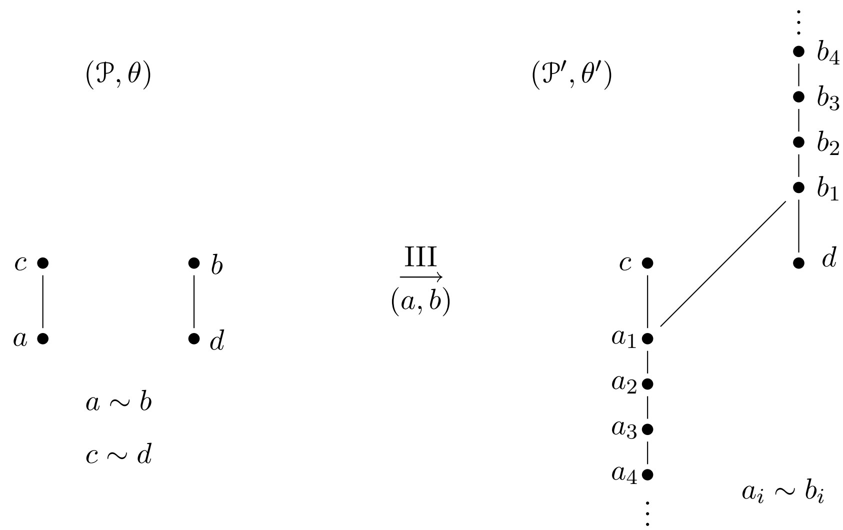


FIGURE 5.2. The diagram of a poset $(\mathcal{P}, \theta)$ and its corresponding derivate poset $(\mathcal{P}', \theta')$.

We consider the representation of $(\mathcal{P}, \theta)$, $U = (U_0, U_{(a,b)}, U_{(c,d)})$, *where*

$$\begin{aligned} U_0 &= \mathbb{R}^4, \\ U_{(a,b)} &= \langle (0, e_1), (e_1, e_2), (0, e_3), (e_3, e_4) \rangle \\ U_{(c,d)} &= \langle (0, e_1), (e_1, e_2), (0, e_3), (e_3, e_4) \rangle. \end{aligned}$$

As $U_a^+ = \{e_1, e_3\}$ *and* $U_b^+ = \{e_1, e_2, e_3, e_4\}$ *then,* $U_a^+ \subseteq U_b^+$, *also* $U_a^- = 0$. *Therefore* $U \in \mathcal{R}$. *By applying the functor 5.3 we obtained that*

$$\begin{aligned} U'_0 &= U_0 = k^n, \\ U'_{(a_1,b_1)} &= (U_B^+, U_0) \cap U_{(a,b)} + (0, U_a^+) = \langle (0, e_1), (e_1, e_2), (0, e_3) \rangle, \\ U'_{(a_2,b_2)} &= (U_B^+, U_0) \cap U^2_{(a,b)} + (0, U_a^+) = \langle (0, e_2), (0, e_4), (0, e_1), (0, e_3) \rangle, \qquad (5.4) \\ \text{and for} \quad n \geq 3, & \\ U'_{(a_3,b_3)} &= (U_B^+, U_0) \cap {U^3}_{(a,b)} + (0, U_a^+) = \langle (0, e_2), (0, e_4), (0, e_1), (0, e_3) \rangle, \end{aligned}$$

Proposition 23. $U' = (U'_0, U'_z)_{z \in \theta'}$ *defined by formulae in 5.3 is an object of the category* $\mathcal{R}'$.

Proof. It is enough to check with the classes (a_i, b_i) of $\theta' = \theta \backslash \{(a, b)\} \cup \{(a_i, b_i)\}$. In the first place, we prove that $(U'_{b_n})^+ \subseteq (U'_{b_{n+1}})^-$: let $z \in (U'_{b_n})^+$, then there exists $w \in U_0$ such that $(w, z) \in U'_{(a_n, b_n)}$, so $(w, z) \in (U_B^+, U_0) \cap U^{[n]}_{(a,b)} + (0, U_a^+)$. Therefore $(w, z) = (w, z_1) + (0, z_2)$ with $z_1 \in U_b^+, z_2 \in U_a^+$. As $U_B^+ \subset U_b^-$ we have $(0, w) \in U_{(a,b)}$ so, coupling this with $(w, z_1) \in U^{[n]}_{(a,b)}$ we obtain that $(0, z_1) \in U^{[n+1]}_{(a,b)}$ and as $(0, z_2) \in (0, U_a^+)$ then $(0, z_1 + z_2) = (0, z) \in U^{[n+1]}_{(a,b)}$; therefore $z \in (U'_{b_{n+1}})^-$.
In the second place, we prove that $(U'_{a_n})^- = (U'_{a_{n+1}})^+$, which is equivalent to proving that $(U'_{a_n})^- = \pi_a((U_B^+, U_a^+) \cap U^{[n]}_{(a,b)}) = (U'_{a_{n+1}})^+$.

- If $z \in (U'_{a_n})^-$ then $(z,0) \in U'_{(a_n,b_n)}$; that is, $(z,0) \in (U_B^+, U_0) \cap U_{(a,b)}^{[n]} + (0, U_a^+)$ so, $(z,0) = (z,w) + (0,v)$, with $w \in U_b^+, v \in U_a^+$, and $w + v = 0$; thus $(z,0) \in (U_B^+, U_a^+) \cap U_{(a,b)}^{[n]}$, and therefore $z \in \pi_a((U_B^+, U_a^+) \cap U_{(a,b)}^{[n]})$. Conversely, if $z \in \pi_a((U_B^+, U_a^+) \cap U_{(a,b)}^{[n]})$ then there exists $w \in U_0$ such that $(z,w) \in (U_B^+, U_a^+) \cap U_{(a,b)}^{[n]}$ so, $(z,w) \in U'_{(a_n,b_n)}$, also $(0,w) \in U'_{(a_n,b_n)}$ since $w \in U_a^+$ then $(z,0) \in U'_{(a_n,b_n)}$; therefore $z \in (U'_{a_n})^-$.

- $(U'_{a_{n+1}})^+ = \pi_a((U_B^+, U_a^+) \cap U_{(a,b)}^{[n]})$. If $z \in (U'_{a_{n+1}})^+$ then there exists w such that $(z,w) \in U'_{(a_{n+1},b_{n+1})}$; that is, $(z,w) \in (U_B^+, U_0) \cap U_{(a,b)}^{[n+1]} + (0, U_a^+)$, so $(z,w) = (z,w_1) + (0,w_2)$, then $(z, w - w_2) \in (U_B^+, U_a) \cap U_{(a,b)}^{[n]}$; therefore $z \in \pi_a((U_B^+, U_a) \cap U_{(a,b)}^{n})$. Conversely, if $z \in \pi_a((U_B^+, U_a^+) \cap U_{(a,b)}^{[n]})$ then there exists $w \in U_0$ such that $(z,w) \in (U_B^+, U_a^+) \cap U_{(a,b)}^{[n]}$ with $w \in U_a^+ \cap U_b^+ = U_a^+$ since $U_a^+ \subset U_b^+$ by definition of $\mathcal{R}$; therefore there exists $t \in U_0$ such that $(w,t) \in U_{(a,b)}$ and as $(z,w) \in U_{(a,b)}^{[n]}$, then $(z,t) \in U_{(a,b)}^{[n+1]}$ and therefore $z \in (U'_{a_{n+1}})^+$. ∎

Proposition 24. *If $\phi : (U_0, U_z)_{z\in\theta} \to (V_0, V_z)_{z\in\theta}$ is a morphism in $\mathcal{R}$ then $\phi' = \phi : (U_0, U'_z)_{z\in\theta'} \to (V_0, V'_z)_{z\in\theta'}$ is a morphism in $\mathcal{R}'$.*

Proof Taking into account that $\theta' = \theta \backslash \{(a,b)\} \cup \{(a_i,b_i)\}$, then for all $z \in \theta$, $\phi'(U'_z) = \phi'(U_z) = \phi(U_z) \subset V_z = V'_z$. Let $z = (a_i,b_i)$, we will prove that $\phi(U'_z) \subset V'_z$. If $(x,y) \in U_{(a_i,b_i)}$ then $(x,y) \in (U_B^+, U_0) \cap U_{(a,b)}^i + (0, U_a^+)$; that is, $x \in U_B^+ \cap U_a^+$ and $y \in U_b^+ + U_a^+$ and therefore $\phi(x) \subseteq V_B^+ \cap V_a^+$ and $\phi(y) \subseteq V_b^+ + V_a^+$; thus $\phi(x,y) = (\phi(x), \phi(y)) \subseteq (V_B^+, V_0) \cap V_{(a,b)}^n + (0, V_a^+) = V'_{(a_n,b_n)}$. ∎

Lemma 7. *Let $V = (V_0, V_z)_{z\in\theta}$ be a representation of $(\mathcal{P}, \theta)$ and let W_0 be a vector subspace of V_0, for each big point $z = (x,y) \in \theta$ we consider the subspace $\tilde{W}_z$ of V_z consisting of $(u,v) \in V_z$, such that $u, v \in W_0$ and for each small point $z = [x] \in \theta, \tilde{W}_z = V_z \cap W_0$. Then $(W_0, \tilde{W}_z)_{z\in\theta}$ is a representation of $(\mathcal{P}, \theta)$.*

Proof We suppose that $y < x$ in $\mathcal{P}$ and $u \in \tilde{W}_y^+ \subset V_y^+$. Then if the class x consists of (x, x_1) then $(u,0) \in V_{[x]}$. As $u \in W$, then $(u,0) \in \tilde{W}$, if $[x]$ consists only of x, then $u \in V_{[x]} \cap W = \tilde{W}_{[x]}$. Therefore, $\tilde{W}$ is a representation of $(\mathcal{P}, \theta)$. ∎

Let $(U_0, U_z)_{z\in\theta}$ be an object of $\mathcal{R}$, by hypothesis $U_a^+ \subset U_b^+$. For each $v \in U_b^+$ there exists $u \in U_0$ such that $(u,v) \in U_{(a,b)}$, if $(u_1, v) \in U_{(a,b)}$, then $(u - u_1, 0) \in U_{(a,b)}$, but $U_a^- = 0$, then $u = u_1$. Therefore, if $v \in U_b^+$, then there exists an unique $\phi(v)$ such that $(\phi(v), v) \in U_{(a,b)}$. As $U_a^+ \subset U_b^+$, thus we obtain the map $\phi : U_b^+ \to U_b^+$ such that $(\phi(v), v) \in U_{(a,b)}$. It is clear that ϕ is a lineal transformation and $U_{(a,b)} = \{(\phi(v), v) \mid v \in U_b^+\}$.

By Fitting lemma there exist subspaces W_1, W_2 of U_b^+ such that $V_0 = W_1 \bigoplus W_2$ and $\phi = \phi_1 \bigoplus \phi_2$ where $\phi_1 : W_1 \to W_1$ is not singular and $\phi_2 : W_2 \to W_2$ is nilpotent. We have that $U_0 = W_1 \bigoplus W_2 \bigoplus W$ for a subspace W of V_0. We consider $L = W_2 \bigoplus W$ and by using the notation of Lemma 7, we consider the representations: $\tilde{W_1} = ((W_1)_0, (\tilde{W_1})_z)_{z\in\theta}, \tilde{L} = (L_0, \tilde{L}_z)_{z\in\theta}$.

Proposition 25. $(U_0, U_z)_{z\in\theta} = \tilde{W}_1 \bigoplus \tilde{L}$.

Proof. First we will describe $\tilde{W}_1$. If $x, y \in a^{\blacktriangledown}$ and $u \in (W_1)_0$, $(u, \phi_1^{-1}(u)) \in U_{(a,b)}$, as $a < x$, then $(u, 0)$ and $(0, u) \in V_{(x,y)}$ and it is in $(\tilde{W}_1)_{(x,y)}$. This implies that $(\tilde{W}_1)_{(x,y)} = ((W_1)_0, 0) \bigoplus (0, (W_1)_0)$.
If $x \in a^{\blacktriangledown}$ and $y \in b_{\blacktriangle}$ and $(u, v) \in (\tilde{W}_1)_{(x,y)}$, then as $y < b$, $(0, v) \in U_{(a,b)}$, so $\phi(v) = \phi_1(v) = 0$; therefore $v = 0$, and as before $(u, 0) \in (\tilde{W}_1)_{(x,y)}$ for all $u \in (W_1)_0$, so in this case $(\tilde{W}_1)_{(x,y)} = ((W_1)_0, 0)$, and analogously if $x, y \in b_{\blacktriangle}$ then $(\tilde{W}_1)_{(x,y)} = \langle(0, 0)\rangle$. If $[x]$ consists only of x, and $x \in a^{\blacktriangledown}$, then $(\tilde{W}_1)_{[x]} = W_1$, if $x \in b_{\blacktriangle}$ then $(\tilde{W}_1)_{[x]} = 0$.
So, we have $U_0 = (W_1)_0 \oplus L_0$.
Now, let $(u, v) \in U_{(x,y)}$, with $u = u_1 + u_2; v = v_1 + v_2, u_1, v_1 \in W_1; u_2; v_2 \in L$. We suppose that $x, y \in a^{\blacktriangledown}$, then $(u, v) = (u_1, 0) + (0, v_1) + (u_2, v_2)$; as $(u_1, 0)$ and $(0, v_1) \in U_{(x,y)}$, then $(u_2, v_2) \in U_{(x,y)}$ consequently is in $\tilde{L}_{(x;y)}$ and $(u_1, v_1) \in (\tilde{W}_1)_{(x,y)}$.

Now we suppose that $x \in a^{\blacktriangledown}$ and $y \in b_{\blacktriangle}$. As $y < b$ then $(0, u_2 + v_2) \in U_{(a,b)}$, and $0 = \phi(u_2 + v_2) = \phi_1(u_2) + \phi_2(v_2)$; thus $\phi_1(u_2) = 0$ so $u_2 = 0$. As $x \in a^{\blacktriangledown}$ then $(u_1, 0) \in U_{(a,b)}$ then, $(u, v) = (u_1, 0) + (v_1, v_2)$ thus $(v_1, v_2) \in U_{(a,b)}$ and therefore is in $\tilde{L}_{(a,b)}$. If $x, y \in b_{\blacktriangle}$ then if $(u, v) \in U_{(a,b)}$ it is obtained as before that $u, v \in L$. The same reasoning is done if x or y consist of a single element. Finally, it is clear that $U_{(a,b)} = (\tilde{W}_1)_{(a,b)} \oplus \tilde{L}_{(a,b)}$. ■

Definition 40. *If $u \neq 0 \in U_b^+$ we denote by m_u the integer such that $\phi_2^{m_u}(u) \neq 0$ and $\phi_2^{m_u+1}(u) = 0$.*

Definition 41. *A subset $\mathcal{B}$ of U_b^+ is called a strict system of generators of U_b^+ if the set*

$$\tilde{\mathcal{B}} = \{\phi_2^i(b) \mid b \in \mathcal{B} \quad and \quad 0 \leq i \leq m_i\},$$

is a k-basis of U_b^+.

Proposition 26. *The space U_b^+ has a strict system of generators.*

Proof. The space U_b^+ has a structure of $k[z]$-module where for $u \in U_b^+$, $zu = \phi_2(u)$. Since U_b^+ is finite dimensional over k and ϕ_2 is nilpotent then $U_b^+ = \bigoplus\limits_{i\in I} L(i)$, where I is a finite set and for each $i \in I$ there is an isomorphism of $k[z]$-modules $\psi_i : k[z]/(z^{m_i+1}) \to L(i)$. Let $\underline{1}_i$ the image of $1 \in k[z]$ in $L(i)$. It is clear that the set $\mathcal{B}$ formed by the elements $b(i) = \psi_i(\underline{1}_i)$ is a strict system of generators for U_b^+. ■

If $\mathcal{B}$ is a strict system of generators for U_b^+, then there are numbers $m_1 > m_2 > \cdots > m_L = 0$ such that $\mathcal{B} = \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \cdots \bigcup \mathcal{B}_L$ with $\mathcal{B}_i = \{b \in \mathcal{B} \mid m(b) = m_i\}$.
Let $X_i = \langle \mathcal{B}_i \rangle$ then

$$U_b^+ = X_1 \bigoplus \phi_2(X_1) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1) \bigoplus X_2 \bigoplus \cdots \bigoplus \phi_2^{m_2}(X_2) \bigoplus \cdots \bigoplus X_L.$$

From now on we will put $\underline{m}(\mathcal{B}) = (m_1, \ldots, m_L)$.

Lemma 8. **(i)** *If x is a nonzero element of $\phi_2^i(X_i)$ with $0 < i \leq m_i$, then $m(x) = m_i - i$.*

(ii) *Let $\mathcal{B}'_i$ be a k-basis for X_i, then $\mathcal{B}' = \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \cdots \bigcup \mathcal{B}_{i-1} \bigcup \mathcal{B}'_i \bigcup \mathcal{B}_{i+1} \bigcup \cdots \bigcup \mathcal{B}_L$ is a strict system of generators for U_b^+.*

(iii) *Let $b \in \mathcal{B}_i$ and $y \in \phi_2^{m_{i+1}-m_i}(X_{i+1}) \bigoplus \cdots \bigoplus \phi_2^{m_1-m_i}(X_1)$. If we put $b' = b + y$, it is obtain $m(b') = m(b) = m_i$. If $\mathcal{B}'_i = (\mathcal{B}_i \backslash \{b\}) \bigcup \{b'\}$, then $\mathcal{B}' = \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \cdots \bigcup \mathcal{B}_{i-1} \bigcup \mathcal{B}'_i \bigcup \cdots \bigcup \mathcal{B}_L$ is a strict system of generators for U_b^+.*

(iv) *$Ker(\phi_2) = \phi_2^{m_1}(X_1) \bigoplus \cdots \bigoplus \phi_2^{m_{L-1}}(X_{L-1}) \bigoplus X_L$.*

(v) *The morphism $\phi_2^{m_i}$ induces an isomorphism of $X_i \bigoplus \phi_2^{m_{i-1}-m_i}(X_{i-1}) \bigoplus \cdots \bigoplus \phi_2^{m_1-m_i}(X_1)$ in $\phi_2^{m_i}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1)$.*

Proof. **(i)** It is clear.

(ii) Since $\mathcal{B}$ is a strict system generators of U_b^+ the set $\tilde{\mathcal{B}} = \{\phi_2^i(b) \mid b \in \mathcal{B}, 0 \leq i \leq m(b)\}$ is a k-basis of U_b^+. We observe the equality

$$\mathrm{Card}(\tilde{\mathcal{B}}) = \sum_{i=1}^{l} (m_i + 1)\mathrm{Card}(\mathcal{B}_i).$$

Let $\mathcal{B}'_i$ be a basis of $X_i = \langle \mathcal{B}_i \rangle$, then $\mathrm{Card}(\mathcal{B}'_i) = \mathrm{Card}(\mathcal{B}_i)$. We take $\tilde{\mathcal{B}}' = \{\phi_2^i(b) \mid b \in \mathcal{B}', 0 \leq i \leq m(b)\}$. It is obtained

$$\mathrm{Card}(\tilde{\mathcal{B}}') = \sum_{s \neq i} (m_s + 1)\mathrm{Card}(\mathcal{B}_s) + (m_i + 1)\mathrm{Card}(\mathcal{B}'_i) = \mathrm{Card}(\tilde{\mathcal{B}}) = \dim(U_b^+).$$

Since the elements of $\phi_2^j(\mathcal{B}_i)$ can be written as linear combinations of elements in $\phi_2^j(\mathcal{B}'_i)$ then the elements of $\tilde{\mathcal{B}}'$ are a system of generators of U_b^+ and besides the number of these elements coincides with the dimension of U_b^+ thus $\tilde{\mathcal{B}}'$ is a basis for U_b^+. This implies that $\mathcal{B}'$ is a strict system of generators.

(iii) It is true that $y = u_{i+1} + \cdots + u_1$, with

$$u_{i+1} \in \phi_2^{m_{i+1}-m_1}(X_{i+1}), \quad u_i \in \phi_2^{m_i - m_{i+1}}(X_i), \quad \ldots \quad u_1 \in \phi_2^{m_1 - m_{i+1}}(X_1).$$

By (i), $m(u_{i+1}) = m(u_i) = \cdots = m(u_1) = m_i$. Therefore

$$\phi_2^{m_i}(b + y) = \phi_2^{m_i}(b) + \sum_{s=1}^{i+1} \phi_2^{m_i}(u_s) \neq 0,$$

and $\phi_2^{m_i+1}(b + y) = 0$, thus $m(b + y) = m_i$. As in (ii) let us take $\tilde{\mathcal{B}}_i = \{\phi_2^i(b) \mid b \in \mathcal{B}, 0 \leq i \leq m(b)\}$, $\tilde{\mathcal{B}}' = \{\phi_2^i(b) \mid b \in \mathcal{B}', 0 \leq i \leq m(b)\}$. Since $m(b + y) = m_i$, the cardinality of $\tilde{\mathcal{B}}$ is equal to the cardinality of $\tilde{\mathcal{B}}'$. Additionally all elements of $\tilde{\mathcal{B}}$ can be written as linear combinations of elements in $\tilde{\mathcal{B}}'$, so the latter is a set of generators of U_b^+ with the cardinality equal to $\dim(U_b^+)$, therefore $\tilde{\mathcal{B}}'$ is a basis for U_b^+ and thus a strict system of generators.

(iv) Let $Y_1 = X_1 \bigoplus \phi_2(X_1) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1)$, $Y_2 = X_2 \bigoplus \phi_2(X_2) \bigoplus \cdots \bigoplus \phi_2^{m_2}(X_2), \ldots, Y_L = X_L$. We take $y = y_1 + y_2 + \cdots + y_L$ with $y_i \in Y_i$, then $\phi_2(y) = \phi_2(y_1) + \phi_2(y_2) + \phi_2(y_{L-1})$ where $\phi_2(y_i) \in Y_i$; therefore $\phi_2(y) = 0$ if and only if $\phi_2(y_i) = 0$ for each $i = 1, \ldots, L$. Each y_i has the form

$$y_i = z_{i,1} + \cdots + z_{i,m_i},$$

where $z_{i,1} \in X_i, \ldots, z_{i,m_i} \in \phi_2^{m_i}(X_i)$, thus

$$\phi_2(y_i) = \phi_2(z_{i,1}) + \cdots + \phi_2^{m_i+1}(z_{i,m_i}) \in \phi_2(X_i) \bigoplus \phi_2^2(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_i}(X_i).$$

It is obtained from (i) that if $z_{i,j} \neq 0$ for $j < m_i$, then $\phi_2(z_{i,j}) \neq 0$, so $\phi_2(y) = 0$ if and only if $\phi_2(y_i) = 0$ for all i, if and only if $z_{i,j} = 0$ for $j < m_i$, therefore $\phi_2(y) = 0$ if and only if $z_{i,j} = 0$ for $j < m_i$, thus $\phi_2(y) = 0$ if and only if $y \in \phi_2^{m_1}(X_1) \bigoplus \phi_2^{m_2}(X_2) \bigoplus \cdots \bigoplus X_L$.

(v) The morphism $\phi_2^{m_i}$ induces an epimorphism

$$X_i \bigoplus \phi_2^{m_{i-1}-m_i}(X_{i-1}) \bigoplus \cdots \bigoplus \phi_2^{m_1-m_i}(X_1) \longrightarrow \phi_2^{m_i}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1),$$

where the kernel is zero. Indeed, let $y = y_1 + y_2 + \cdots + y_{m_i}$ with $y_1 \in X_i$, $y_2 \in \phi_2^{m_{i-1}-m_i}(X_{i-1}), \ldots, y_i \in \phi_2^{m_1-m_i}(X_1)$. From (i), if $y_s \neq 0$ then $\phi_2^{m_i}(y_s) \neq 0$ therefore if $y \neq 0$ then $\phi_2^{m_i}(y) \neq 0$. This proves our claim. ■

Proposition 27. *There exists a strict system of generators $\mathcal{B}$ of U_b^+, where $m(\mathcal{B}) = (m_1, \ldots, m_L)$ and subsets $\mathcal{B}'_i \subset \mathcal{B}_i = \{b \in \mathcal{B} \mid m(b) = m_i\}$ such that*

$$U_B^+ = \langle \mathcal{B}'_L \rangle \bigoplus \phi_2^{m_{L-1}}(\langle \mathcal{B}'_{L-1} \rangle) \bigoplus \cdots \bigoplus \phi_2^{m_1}(\langle \mathcal{B}'_1 \rangle).$$

Proof. Since there exists a strict system of generators $\mathcal{B}$ for U_b^+, by suitable changes using (ii) and (iii) of Lemma 8, we will find a strict system of generators such that our proposition is satisfied.
Let $X_i = \langle \mathcal{B}_i \rangle$ and $\mathcal{H}_i$, $i = 1, \ldots, L$ defined as follows,

$$\mathcal{H}_L = X_L \bigoplus \phi_2^{J_{L-1}}(X_{L-1}) \bigoplus \phi_2^{J_{L-2}}(X_{L-2}) \bigoplus \cdots \bigoplus \phi_2^{J_1}(X_1) = \ker \phi, \tag{5.5}$$

$$\mathcal{H}_i = \phi_2^{m_i}(X_i) \bigoplus \phi_2^{m_{i-1}}(X_{i-1}) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1) \quad \text{for} \quad 1 \leq i < l, \tag{5.6}$$

and we consider $U_{L-1}, U_{L-2}, \ldots, U_1$ such that

$$\begin{aligned}
U_B^+ &= U_{L-1} \bigoplus (\mathcal{H}_{L-1} \cap U_B^+), \\
\mathcal{H}_{L-1} \cap U_B^+ &= U_{L-2} \bigoplus (\mathcal{H}_{L-2} \cap U_B^+), \\
\mathcal{H}_{L-2} \cap U_B^+ &= U_{L-3} \bigoplus (\mathcal{H}_{L-3} \cap U_B^+), \\
&\vdots \\
\mathcal{H}_2 \cap U_B^+ &= U_1 \bigoplus (\mathcal{H}_1 \cap U_B^+).
\end{aligned}$$

To prove our result, it is enough to show that there is a strict system of generators $\mathcal{B}$ and subsets $\mathcal{B}'_i \subset \mathcal{B}_i$ such that $U_{i-1} = \phi_2^{m_i}(\langle \mathcal{B}'_i \rangle)$ for $i = 1, \ldots, L$. For this, we will prove by induction on i the following affirmation:

Affirmation 1. *There is a strict system of generators* $\mathcal{B}$ *such that for each* $j \leq i$, $U_{j-1} = \phi_2^{m_j}(\mathcal{B}'_j)$ *with* $\mathcal{B}'_j \subset \mathcal{B}_j$.

Proof. We suppose $i = 0$, then $U_0 = \mathcal{H}_1 \cap U_B^+ = \phi_2^{m_1}(X_1) \cap U_B^+ \subset \phi_2^{m_1}(X_1)$. Here, $\phi_2^{m_1}$ induces an isomorphism of X_1 in $\phi_2^{m_1}(X_1)$, so there exists a subspace $Z \subset X_1$ such that $\phi_2^{m_1}(Z) = U_0$. Let $\mathcal{B}'_1$ be a k-basis of Z and $\mathcal{B}''_1$ a complementary basis of X_1, then $\mathcal{B}'_1 \cup \mathcal{B}''_1$ is a k-basis of X_1 and from (ii) of Lemma 8, changing $\mathcal{B}_1$ by $\mathcal{B}'_1 \cup \mathcal{B}''_1$ if it is necessary, we can suppose that $\mathcal{B}'_1 \subset \mathcal{B}_1$ and $\phi_2^{m_1}(\langle \mathcal{B}'_1 \rangle) = \phi_2^{m_1}(Z) = U_0$.
We suppose that our afirmattion is true for i and we will prove for $i+1$.
Since $\mathcal{H}_{i+1} \cap U_B^+ = U_i \bigoplus (\mathcal{H}_i \cap U_B^+)$ and $\mathcal{H}_{i+1} \cap U_B^+ \subset \mathcal{H}_{i+1}$, that is,

$$\mathcal{H}_{i+1} \cap U_B^+ \subset \phi_2^{m_{i+1}}(X_{i+1}) \bigoplus \phi_2^{m_i - m_{i+1}}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1 - m_{i+1}}(X_1)).$$

By (v) of Lemma 8, there exists

$$Z \subset X_{i+1} \bigoplus \phi_2^{m_i - m_{i+1}}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1 - m_{i+1}}(X_1)$$

such that $\phi_2^{m_{i+1}}(Z) = U_i$.
We consider a basis $l_1, l_2, \ldots, l_t$ of Z, then

$$\begin{array}{rcl} l_1 & = & x_1 + y_1, \\ l_2 & = & x_2 + y_2, \\ & \vdots & \\ l_t & = & x_t + y_t, \end{array}$$

where for $1 \leq s \leq t$, $x_s \in X_{i+1}$, and each $y_s \in \phi_2^{m_i - m_{i+1}}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1 - m_{i+1}}(X_1)$.
We will check that the set $\{x_1, x_2, \ldots, x_t\}$ is linearly independent. Indeed, let $c_1, \ldots, c_t \in k$ such that $c_1 x_1 + c_2 x_2 + \cdots + c_t x_t = 0$, then

$$c_1 l_1 + c_2 l_2 + \cdots + c_t l_t = c_1 y_1 + c_2 y_2 + \cdots + c_t y_t.$$

Thus,

$$w = c_1 \phi_2^{m_{i+1}}(l_1) + \cdots + c_t \phi_2^{m_{i+1}}(l_t) = c_1 \phi_2^{m_{i+1}}(y_1) + \cdots + c_t \phi_2^{m_{i+1}}(y_t), \tag{5.7}$$

it is obtained that,

$$w \in \phi_2^{m_{i+1}}(Z) \cap \phi_2^{m_i}(X_i) \bigoplus \cdots \bigoplus \phi_2^{m_1}(X_1) = U_i \cap \mathcal{H}_i = 0.$$

Therefore,

$$\phi_2^{m_i}(c_1 l_1 + c_2 l_2 + \cdots + c_t l_t) = 0,$$

and by (v) of Lemma 8

$$c_1 l_1 + c_2 l_2 + \cdots + c_t l_t = 0,$$

which implies that

$$c_1 = c_2 = \cdots = c_t = 0,$$

thus $x_1, \ldots, x_t$ are linearly independent. By (ii) of Lemma 8, we can suppose that $x_1, \ldots, x_t \in \mathcal{B}_i$ and by applying several times (iii) of Lemma 8, we can change $\mathcal{B}_i$ in such a way that $l_1, l_2, \ldots, l_t \in \mathcal{B}_i$. Thus we obtain a strict system of generators $\mathcal{B}$ such that $Z = \langle \mathcal{B}'_i \rangle$ with $\mathcal{B}'_i \subset \mathcal{B}_i$. This follows that $\mathcal{B}$ satisfies our affirmation for $i+1$. ■

We consider the subposet $\mathcal{K}$ of $\mathcal{P}$ where $\mathcal{K} = \mathcal{P} \backslash (A+B) = \{a, b \mid a || b\}$ and $\theta = \{(a,b)\}$ as the only equivalence class. The Hasse diagram in this case is presented in the following Figure 5.2:

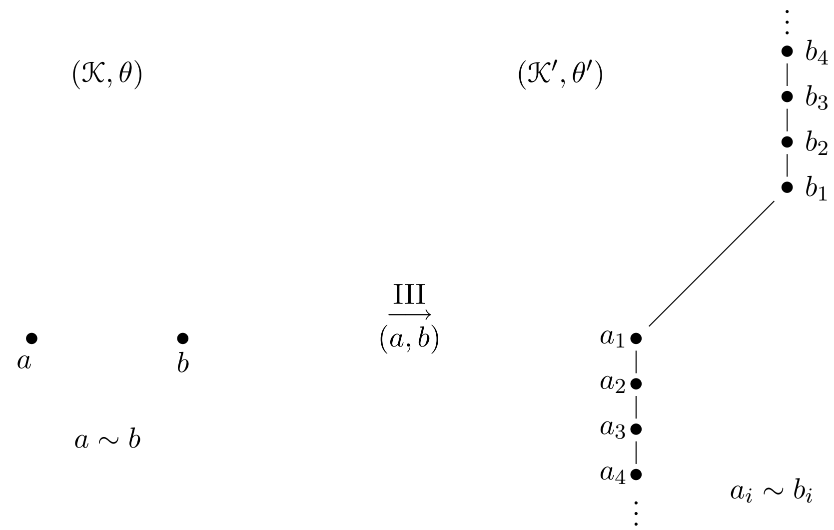


FIGURE 5.3. The diagram of a poset $(\mathcal{K}, \theta)$ and its corresponding derivate poset $(\mathcal{K}', \theta')$.

From now on we will assume for a representation $U = (U_0, U_z)_{z \in \theta}$ of $(\mathcal{P}, \theta)$ that there exists a strict generator system $\mathcal{B}$ for U_b^+ which satisfies the conditions of Proposition 27. With the notation of such proposition we will put $\mathcal{B}' = \bigcup_{i=1}^{L} \mathcal{B}'_i$, $\mathcal{B}''_i$ is the complement of $\mathcal{B}'_i$ in $\mathcal{B}_i$ and $\mathcal{B}'' = \bigcup_{i=1}^{L} \mathcal{B}''_i$.

We will also use the following notation $\mathcal{B}' = \{e_u\}_{u \in \mathcal{B}'}, \mathcal{B}'' = \{e_u\}_{u \in \mathcal{B}''}$ and $\mathcal{B} = \mathcal{B}' \bigcup \mathcal{B}''$. If $e_u \in \mathcal{B}_i$, we will put $m_u = m_i$. For $e_u \in \mathcal{B}$ we will denote by $\mathcal{L}^{m_u+1}(e_u)$ to the representation of $\mathcal{K}$ defined by

$$\begin{aligned}
\mathcal{L}_0^{m_u+1} &= \langle e_u, \phi(e_u), \ldots, \phi^{m_u}(e_u) \rangle, \\
\mathcal{L}_{(a,b)}^{m_u+1} &= \langle (0, \phi^{m_u}(e_u)), (\phi^{m_u}(e_u), \phi^{m_u-1}(e_u)), \ldots, (\phi(e_u), e_u) \rangle.
\end{aligned}$$

Lemma 9. *Let $U \in \mathcal{R}$ and $\{e_u\}_{u \in \mathcal{B}}$ be a strict system of generators for U_b^+ such that the properties of Proposition 27 are satisfied. Then*

a. *If* $u \in \mathcal{B}'$ *and* $m_u = 0$ *then* $(\mathcal{L}^{m_u+1}(e_u))' = \mathcal{L}_{1,1}(e_u)$ *and if* $m_u \geq 1$ *then* $(\mathcal{L}^{m_u+1}(e_u))' = \mathcal{L}_{1,1}(X(u)) \bigoplus \mathcal{L}_{3,m_u}(\phi^{m_u}(e_u), e_u)$ *where,*

$$X(u) = \begin{cases} 0, & \text{if} \quad m_u = 1, \\ \langle \phi^{m_u-1}(e_u), \ldots, \phi(e_u) \rangle, & \text{if} \quad m_u \geq 2, \end{cases}$$

b. *If* $u \in \mathcal{B}''$ *and* $m_u \geq 0$ *then* $(\mathcal{L}^{m_u+1}(e_u))' = \mathcal{L}_{1,1}(X(u)) \bigoplus \mathcal{L}_{1,m_u+1}(e_u)$ *where,*

$$X(u) = \begin{cases} 0, & \text{if} \quad m_u = 0, \\ \langle \phi^{m_u}(e_u), \ldots, \phi(e_u) \rangle, & \text{if} \quad m_u \geq 1, \end{cases}$$

Proof. **a**.

$$(\mathcal{L}_0^{m_u+1})' = \langle e_u, \phi(e_u), \ldots, \phi^{m_u}(e_u) \rangle$$

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})' = \begin{cases} \langle (0, \phi^{m_u+1-i}(e_u)), (\phi^{m_u}(e_u), 0), (0, \phi^{m_u}(e_u)), \ldots, (0, \phi(e_u)) \rangle, & \text{if} \quad 1 \leq i < m_u - 1, \\ \langle (0, \phi(e_u)), (\phi^{m_u}(e_u), e_u), (0, \phi^{m_u-1}(e_u)), \ldots, (0, \phi(e_u)) \rangle, & \text{if} \quad i = m_u, \\ \langle (0, \phi^{m_u}(e_u)), (0, \phi^{m_u-1}(e_u)), \ldots (0, e_u) \rangle, & \text{if} \quad i > m_u. \end{cases}$$

By considering $X(U) = \langle \phi(e_u), \ldots, \phi^{m_u-1}(e_u) \rangle$ then we obtain that

$$\begin{aligned} (\mathcal{L}_0^{m_u+1})' &= \langle e_u, \phi^{m_u}(e_u) \rangle \bigoplus X(U), \\ (\mathcal{L}_{(a_i,b_i)}^{m_u+1})' &= (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1 \bigoplus (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2. \end{aligned}$$

where,

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1 = \begin{cases} \langle (0, \phi^{m_u}(e_u)), (\phi^{m_u}(e_u), 0) \rangle, & \text{if} \quad 1 \leq i < m_u - 1, \\ \langle (\phi^{m_u}(e_u), e_u), (0, \phi^{m_u}(e_u)), & \text{if} \quad i = m_u, \\ \langle (0, \phi^{m_u}(e_u)), (0, e_u) \rangle, & \text{if} \quad i > m_u. \end{cases}$$

and

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2 = \langle (0, \phi(e_u)), \ldots, (0, \phi^{m_u-1}(e_u)) \rangle$$

Therefore,

$$\begin{aligned} (\mathcal{L}^{m_u+1}(e_u))' &= (X(U), (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2) \bigoplus (\langle e_u, \phi^{m_u}(e_u) \rangle, (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1) \\ &= \mathcal{L}_{1,1}(X(U)) \bigoplus \mathcal{L}_{3,m_u}(\phi^{m_u}(e_u), e_u) \end{aligned}$$

b. If $m_u = 0$ then $\mathcal{L}^1(e_u) = (\mathcal{L}_0^1, \mathcal{L}_{(a,b)}^1) = (\langle e_u \rangle, \langle (0, e_u) \rangle$ thus

$$(\mathcal{L}^1(e_u))' = ((\mathcal{L}^1)'_0, (\mathcal{L}_{(a_i,b_i)}^1)' = (\langle e_u \rangle, \langle (0, e_u) \rangle) = \mathcal{L}_{1,1}(e_u)).$$

If $m_u \geq 1$ then $\mathcal{L}^{m_u+1}(e_u) = (\mathcal{L}_0^{m_u+1}, \mathcal{L}_{(a,b)}^{m_u+1})$, where

$$\begin{aligned} \mathcal{L}_0^{m_u+1} &= \langle e_u, \phi(e_u), \ldots, \phi^{m_u}(e_u) \rangle \\ \mathcal{L}_{(a,b)}^{m_u+1} &= \langle (0, \phi^{m_u}(e_u)), (\phi^{m_u}(e_u), \phi^{m_u-1}(e_u)) \ldots, (\phi(e_u), e_u) \rangle). \end{aligned}$$

Since $(\mathcal{L}^{m_u+1}(e_u))' = ((\mathcal{L}_0^{m_u+1})', (\mathcal{L}_{(a_i,b_i)}^{m_u+1})')$ with

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})' = (U_B^+ \cap \mathcal{L}_0^{m_u+1}(e_u), \mathcal{L}_0^{m_u+1}(e_u)) \cap (\mathcal{L}^{m_u+1}(e_u)_{(a,b)})^{[i]} + (0, (\mathcal{L}^{m_u+1}(e_u))_a^+),$$

here, $U_B^+ \cap \mathcal{L}_0^{m_u+1}(e_u) = 0$, $(\mathcal{L}^{m_u+1}(e_u))_a^+ = \langle \phi^{m_u}(e_u), \phi^{m_u-1}(e_u), \ldots, \phi(e_u)\rangle$ and $(\mathcal{L}^{m_u+1}(e_u)_{(a,b)})^{[i]} = \{(0, \phi^{m_u-(i+1)}(e_u)), (\phi^{m_u}(e_u), \phi^{m_u-i}(e_u)), \ldots, (\phi^i(e_u), e_u)\}$ then

$$(\mathcal{L}_0^{m_u+1})' = \langle e_u, \phi(e_u), \ldots, \phi^{m_u}(e_u)\rangle$$

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})' = \begin{cases} \langle (0, \phi^{m_u}(e_u)), (0, \phi^{m_u-1}(e_u)), \ldots, (0, \phi(e_u))\rangle & \text{if } \; 1 \le i \le m_u \\ \langle (0, \phi^{m_u}(e_u)), \ldots, (0, \phi(e_u)), (0, e_u)\rangle & \text{if } \; i > m_u \end{cases}$$

By considering $X(U) = \langle \phi(e_u), \ldots, \phi^{m_u}(e_u)\rangle$ then we obtain that

$$\begin{aligned} (\mathcal{L}_0^{m_u+1})' &= \langle e_u\rangle \bigoplus X(U), \\ (\mathcal{L}_{(a_i,b_i)}^{m_u+1})' &= (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1 \bigoplus (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2 \end{aligned}$$

where

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1 = \langle (0, \phi^{m_u}(e_u)), (0, \phi^{m_u-1}(e_u)), \ldots, (0, \phi(e_u))\rangle$$

and

$$(\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2 = \begin{cases} 0, & \text{if } \; 1 \le i \le m_u, \\ \langle (0, e_u)\rangle, & \text{if } \; i > m_u. \end{cases}$$

Therefore,

$$\begin{aligned} (\mathcal{L}^{m_u+1}(e_u))' &= (X(U), (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_1) \bigoplus (\langle e_u\rangle, (\mathcal{L}_{(a_i,b_i)}^{m_u+1})'_2) \\ &= \mathcal{L}_{1,1}(X(U)) \bigoplus \mathcal{L}_{1,m_u+1}(e_u). \end{aligned}$$

Proposition 28. *Let $U \in \mathcal{R}$ and $\{e_u\}_{u\in\mathcal{B}}$ be a strict system of generators for U_b^+ such that the properties of Proposition 27 are satisfied. Then*

$$U|_{\mathcal{K}} = \mathcal{L}^1(Z(U)) \bigoplus \bigoplus_{\substack{u\in B' \\ m_u>1}} \mathcal{L}^{m_u+1}(e_u) \bigoplus \bigoplus_{\substack{u\in B'' \\ m_u>0}} \mathcal{L}^{m_u+1}(e_u) \bigoplus S(U_s),$$

where $S(U_s) = (U_s, U_{(a,b)} = 0)$. Even more,

$$U'|_{\mathcal{K}'} = \mathcal{L}_{1,1}(Z(U)) \bigoplus \bigoplus_{\substack{u\in B' \\ m_u>0}} \mathcal{L}_{1,1}(X(u)) \bigoplus \mathcal{L}_{3,m_u}(\phi^{m_u}(e_u), e_u) \bigoplus \bigoplus_{\substack{u\in B'' \\ m_u>0}} \mathcal{L}_{1,1}(X(u)) \bigoplus \mathcal{L}_{1,1+m_u}(e_u) \bigoplus S(U_s),$$

where $S(U_s) = (U_s, U_{(a_i,b_i)} = 0)$.

Proof. The first item is clear. The second follows from the first item, since $U'|_{\mathcal{K}'} = (U|_{\mathcal{K}})'$and from Lemma 9.

Lemma 10. *Let $W \in \mathcal{R}'$, then $W|_{\mathcal{K}'} = \bigoplus_{u \in J} W^u$ where each W^u is trivial or $W^u = \mathcal{L}_{1,l_u+1}(e_u)$, or $W^u = \mathcal{L}_{3,l_u}(f_u, e_u)$, with $f_u \in W_B^+$. If $W^u = \mathcal{L}_{1,l_u+1}(e)$ with $l_u \geq 1$, then $e \notin U_B^+$.*

Proof. Since each W^u such that is not trivial has the form $\mathcal{L}_{1,l_u+1}(e_u)$ or $\mathcal{L}_{2,l_u}(e_u)$ or $\mathcal{L}_{3,l_u}(f_u, e_u)$. In the first case if $l_u \geq 1$ then $W^u_{(a_1,b_1)} = 0$. Then if $e_u \in U_B^+, e_u = \sum_s v_s$ with $v_s \in W^+_{z_s}$, $z_s = (x, y)$ with $x < b_1$, then $(0, v_s) \in W_{(a_1,b_1)}$; therefore $(0, e_u) \in W^u_{(a_1,b_1)}$ which is a contradiction; so $e_u \notin W_B^+$. If $W^u = \mathcal{L}_{2,l_u}(e_u)$, it is obtained that $(W^u)^+_{a_{l_u-1}} = \langle e_u \rangle$ and $(W^u)^+_{a_{l_u}} = 0$ which does not happen because $W \in R'$. If $W^u = \mathcal{L}_{3,l_u}(f_u, e_u)$ it is obtained that $f \in (W^u)^+_{a_1} \subset W^+_{a_1} \subset W_B^+$. This proves our affirmation. ■

Proposition 29. *Let $M \in \mathcal{R}'$ and $f : M_0 \to ke$ be a k-linear transformation. The following two conditions are equivalent:*

1. *$f : M|_{\mathcal{K}'} \to k(A, b_1)(e)|_{\mathcal{K}'}$ is a morphism in* $\operatorname{Rep}\mathcal{K}'$ *and $f(M_B^+) = 0$.*
2. *$f : M \to k(A, b_1)$ is a morphism in $\mathcal{R}'$.*

Proof. We will prove that 1. implies 2. For this, it is enough to prove that for $z \in \theta', z \neq (a_i, b_i)$ it is obtained that $f^z(M_z) \subset k(A, b_1)_z$.
Let $z = (x, y), (\lambda, \mu) \in M_z$, we have that $f(\lambda) = ce$, $f(\mu) = c'e$. If both $x, y \in A$ $(f(\lambda), f(\mu)) = c(e, 0) + c'(0, e) \in k(A, b_1)_z$ if $x \in A$, $y \in B$, then $\mu \in M_B^+$. As by hypothesis $f(\mu) = 0$, hence $(f(\lambda), f(\mu)) = c(e, 0) \in k(A, b_1)_z$, if both $x, y \in B$ then it is obtained $(f(\lambda), f(\mu)) = (0, 0) \in k(A, b_1)_z$. Therefore 1. implies 2. Conversely, is clear. ■

Proposition 30. *Let M be a representation of $\mathcal{R}'$ and $g : ke \to M_0$ be a k-linear transformation. Then the following statements are equivalent:*

1. *$g : k(A, b_1)(e)|_{\mathcal{K}'} \to M|_{\mathcal{K}'}$ is a morphism* $\operatorname{Rep}(\mathcal{K}')$ *and $g(e) \in M_A^-$.*
2. *$g : k(A, b_1) \to M$ is a morphism in $\mathcal{R}'$.*

Proof. It is similar to the previous one. ■

Let $M \in \mathcal{R}'$ with

$$M|_{\mathcal{K}'} = \bigoplus_{u \in \mathcal{B}_1} M^u \bigoplus_{u \in \mathcal{B}_2} M^u \bigoplus_{u \in \mathcal{B}_3} M^u \bigoplus_{u \in \mathcal{B}_4} M^u \bigoplus S(M_s),$$

such that for $u \in \mathcal{B}_1, M^u = \mathcal{L}_{3,m_u}(f_u, e_u)$, for $u \in \mathcal{B}_2$, $M^u = \mathcal{L}_{1,m_u+1}(e_u)$ with $m_u > 0$, for $u \in \mathcal{B}_3$, $M_u = \mathcal{L}_{1,1}(e_u)$ with $e_u \in M_B^+$, for $u \in \mathcal{B}_4$, $M^u = \mathcal{L}_{1,1}(e_u)$ with $e_u \notin M_B^+$.

Proposition 31. *Let M in $\mathcal{R}'$ such that*

$$M|_{\mathcal{K}'} = \mathcal{L}_{1,1}(Z) \bigoplus \bigoplus_{\substack{u \in \mathcal{B}_1 \\ m_u > 0}} \mathcal{L}_{3,m_u}(f_u, e_u) \bigoplus \bigoplus_{\substack{u \in \mathcal{B}_2 \\ m_u > 0}} \mathcal{L}_{1,m_u+1}(e_u) \bigoplus S(M_s).$$

The following statements are equivalent:

1. *$k(A, b_1)(e)$ is a direct summand of M in $\mathcal{R}'$.*

2. *There exists an element $x \in Z \bigcap M_A^- \backslash M_B^+$.*

Proof. We suppose that 1. is true, then $M = k(A, b_1)(e) \bigoplus L$ so, $M|_{\mathcal{K}'} = \mathcal{L}_{1,1}(e) \bigoplus L|_{\mathcal{K}'}$. If $\lambda \in M_y^+$ with $y \in B$, then since $b_2 > y$, $(0, \lambda) \in M_{(a_2,b_2)}$. For $y \in B$,

$$M_y^+ = k(A, b_1)_y^+ \bigoplus L_y^+ = L_y^+ \subset L_0,$$

consequently $M_B^+ \subset L_0$, thus $e \notin M_B^+$ and $e \in M_A^-$.

Here $(\mathcal{L}_{1,m_u+1})_{(a_1,b_1)} = 0$ for $m_u \geq 1$, so

$$(0, e) \in M_{(a_1,b_1)} = (0, Z) \bigoplus \bigoplus_{u \in \mathcal{B}_1} \langle (0, f_u) \rangle,$$

therefore $e = x + \sum\limits_{u \in \mathcal{B}_1} c_u f_u$ with $x \in Z$. Each $f_u \in M_A^-$ and by Lemma 10, $f_u \in M_B^+$ then $x \in M_A^-$ and since $e \notin M_B^+$, then $x \notin M_B^+$.

Now we suppose that 2. is true. It has

$$M|_{\mathcal{K}'} = \mathcal{L}_{1,1}(Z) \bigoplus_{u \in B_1} \mathcal{L}_{3,m_u}(f_u, e_u) \bigoplus_{\substack{u \in B_2 \\ m_u > 0}} \mathcal{L}_{1,1+m_u}(e_u) \bigoplus S(M_s),$$

with $M_B^+ = Z \bigcap M_B^+ \bigoplus \bigoplus\limits_{\substack{u \in B_1 \\ m_u > 0}} \langle f_u \rangle$. We have $x \notin M_B^+$, so a vector space Z_1 containing x is obtained and such that $Z = Z \bigcap M_B^+ \bigoplus Z_1$. Therefore, we can put

$$M|_{\mathcal{K}'} = \mathcal{L}_{1,1}(x) \bigoplus N$$

with $M_B^+ \subset N$ and $x \in M_A^-$. Then by Propositions 29 and 30, $k(A, b_1)$ is a direct summand of M. ■

Proposition 32. *Let U be a representation of $\mathcal{R}$.*

a. *Let $f : U_0 \to ke$ be a linear transformation, then $f : U \to k(A, b)(e)$ is a morphism in $\mathcal{R}$ if and only if $f : U|_{\mathcal{K}} \to \mathcal{L}^1(e)$ is a morphism in* Rep($\mathcal{K}$) *and $f(U_B^+) = 0$.*

b. *Let $g : ke \to U_0$ be a linear transformation, then $g : k(A, b) \to U$ is a morphism in $\mathcal{R}$ if and only if $g : \mathcal{L}^1(e) \to U$ is also a morphism in* Rep($\mathcal{K}$) *and $g(e) \in U_A^-$.*

Proof. a. Let $y \in B$ and $z = (x, y) \in \theta$. When $(\lambda, \mu) \in U_z$, it is obtained that $(f(\lambda), f(\mu)) \in k(A, b)_z$; therefore $f(\mu) = 0$, so $f(U_y^+) = 0$ for all $y \in B$. This implies that $f(U_B^+) = 0$. Conversely, we suppose that $f : U|_{\mathcal{K}} \to k(A, b)|_{\mathcal{K}} = \mathcal{L}^1(e)$ is a morphism in Rep($\mathcal{K}$) and $f(U_B^+) = 0$. Then if $z = (x, y) \neq (a, b)$ and $(\lambda, \mu) \in U_z$, with $x \in B$, then $\lambda \in M_B^+$ and $f(\lambda) = 0$. Now, if both $x, y \in B$, then $f^z(U_z) = 0 = k(A, b)(e)_z$. By contrast, if both $x, y \in A$, then $f(\lambda) = ce$ and $f(\mu) = c'e$. Thus $(f(\lambda), f(\mu)) = c(e, 0) + c'(0, e) \in k(A, b)(e)_z$. Finally, if $x \in A$, $y \in B$ it is clear that $f(U_z) \subset k(A, b)_z$.

b. If $g : k(A,b) \to U$ is a morphism then $f(e) \in U_A^-$ and the restriction to $\mathcal{K}$ is a morphism. Conversely, if $g : \mathcal{L}^1(e) \to U|_{\mathcal{K}}$ is a morphism such that $g(e) \in U_A^-$, then for $z \neq (a,b)$ where $z = (x,y)$ with $x \in A$ and $y \in B$, it is obtained that for $(\lambda,\mu) \in k(A,b)(e)_z$, with $\lambda = ce$, $\mu = 0$; hence $(g(\lambda), g(\mu)) = (g(ce), 0) \in U_z$. The remaining cases are similar and it follows that $g(k(A,b)_z) \subset U_z$ for all $z \in \theta$. ■

Proposition 33. *Let $U \in \mathcal{R}$ and $Z(U) = \bigoplus_{\substack{u\in\mathcal{B}\\ m_u=0}} \langle e_u\rangle$. Then $k(A,b)$ is a direct summand of U if and only if there exists an element $x \in Z(U) \cap U_A^- \backslash U_B^+$.*

Proof. We suppose that $k(A,b)(e)$ is a direct summand of U. We have:

$$U = k(A,b)(e) \bigoplus W;$$

therefore $e \in U_A^-$ and $U_{(a,b)} = \langle (0,e)\rangle \bigoplus W_{(a,b)}$. For $y < b$:

$$U_y^+ = k(A,b)(e)_y^+ \bigoplus W_y^+ = W_y^+ \subset W_0;$$

therefore $U_B^+ \subset W_0$. We have $e = \sum_{\substack{u\in B(U)\\ m_u>0}} c_u \phi^{m_u}(e_u) + x$ with $x \in Z(U)$. Since each $\phi^{m_u}(e_u) \in U_A^-$ and $e \in U_A^-$, then $x \in U_A^-$. If $\pi : U|_{\mathcal{K}} \to \mathcal{L}^1(e)$ be the projection,

$$e = \pi(e) = \sum_u c_u \phi^{m_u}(\pi(e_u)) + \pi(x).$$

Therefore, $\pi(x) \neq 0$, which implies that $x \notin U_B^+$.

Conversely, we suppose that there exists an element $x \in Z(U) \cap U_A^- \backslash U_B^+$. We have

$$U = \mathcal{L}^1(Z(U)) \bigoplus \bigoplus_{\substack{u\in\mathcal{B}'\\ n_u\geq 1}} \mathcal{L}^{n_u+1}(e_u) \bigoplus \bigoplus_{\substack{u\in\mathcal{B}''\\ n_u\geq 1}} \mathcal{L}^{n_u+1}(e_u),$$

where $U_B^+ = \bigoplus_{\substack{u\in\mathcal{B}'\\ n_u>0}} \langle \phi^{n_u}(e_u)\rangle \bigoplus Z(U) \cap U_B^+$. Since $x \notin Z(U) \cap U_B^+$ we can choose a vector space Z_1 containing x such that $Z(U) = Z(U) \cap U_B^+ \bigoplus Z_1$, then

$$\mathcal{L}^1(Z(U)) = \mathcal{L}^1(Z(U) \cap U_B^+) \bigoplus \mathcal{L}^1(Z_1).$$

Therefore $\mathcal{L}^1(x)$ is a direct summand of $\mathcal{L}^1(Z(U))$ and it follows that $U|_{\mathcal{K}} = \mathcal{L}^1(x) \bigoplus W$ with $U_B^+ \subset W_0$. Since $x \in U_B^+$ by Proposition 32, it is obtained that $k(A,b)(x)$ is a direct summand of U. ■

Remark 11. *Suppose $M, N \in \mathcal{R}'$, with $M = (M_0, M_z)_{z\in\theta}$, $N = (N_0, N_z)_{z\in\theta}$ such that $M_0 = N_0 \bigoplus L$, $N_z \subset M_z$ for all $z \in \theta$. Suppose that $\pi : M_0 \to N_0$, the projection is a morphism in $\mathcal{R}$, then $M = N \bigoplus \tilde{L}$, where $\tilde{L}$ is the representation associated to the space L according to Lemma 7. Indeed, for $(\mu,\nu) \in M_z$, $\mu = \mu_1 + \nu_1$, $\nu = \nu_1 + \nu_2$ with $\mu_1, \nu_1 \in N_0$, $\mu_2, \nu_2 \in L$, since $(\mu,\nu) = (\mu_1,\nu_1) + (\mu_2,\nu_2)$ and $\mu_1, \nu_1 = (\pi(\mu), \pi(\nu)) \in M_z$, then $(\mu_2,\nu_2) \in \tilde{L}_z$.*

Proposition 34. *Let $U \in \mathcal{R}$ and let $\{e_u\}_{u\in\mathcal{B}}$ be a strict system of generators with the conditions of Proposition* 27. *We take $X(U) = \bigoplus_{\substack{u\in\mathcal{B}\\ m_u\geq 1}} X(u)$, then*

1. *$U' = k(A,b_1)(X(U)) \bigoplus (U')^{\downarrow}$, with*

$$(U')^{\downarrow}|_{\mathcal{K}'} = \mathcal{L}_{1,1}(Z(U)) \bigoplus \bigoplus_{\substack{u\in\mathcal{B}'\\ m_u\geq 1}} \mathcal{L}_{3,m_u}(\phi^{m_u}(e_u), e_u) \bigoplus \bigoplus_{\substack{m_u\geq 1\\ u\in\mathcal{B}''}} \mathcal{L}_{1,m_u+1}(e_u) \bigoplus S(U_s).$$

2. *$k(A,b)$ is a direct summand of U if and only if $k(A,b_1)$ is a direct summand of $(U')^{\downarrow}$.*

Proof.

1. Let X the basis of $X(U)$ formed by the elements of the form $\phi^i(e_u)$. Since

$$X(U) = \bigoplus_{x\in X} \langle x\rangle,$$

then $x \in U_A^- = (U')_A^+$ and $x \notin U_B^+ = (U')_B^+$. Then by second part of Proposition 31, the projection π_x of U_0 in $\langle x\rangle$ is a morphism in $\mathcal{R}$ of U' in $k(A,b_1)(x)$; thus the projection of U_0 in $X(U)$ is a morphism in $\mathcal{R}$ of U' in $k(A,b_1)(X(U))$. Therefore by Remark 11 and by using the notation of Lemma 7, is obtained that

$$U' = k(A,b_1)(X(U)) \bigoplus \tilde{L},$$

where

$$L = \bigoplus_{\substack{m_u>0\\ u\in\mathcal{B}}} \langle(\phi^{m_u}(e_u), e_u)\rangle \bigoplus \bigoplus_{\substack{m_u>0\\ u\in\mathcal{B}''}} \langle e_u\rangle;$$

therefore by defining $\tilde{L} = (U')^{\downarrow}$ the first part of our affirmation is obtained.

2. Let $M = (U')^{\downarrow}$, since

$$M = \mathcal{L}_{1,1}(Z(U)) \bigoplus \bigoplus_{\substack{u\in B'\\ m_u>1}} \mathcal{L}_{3,m_u}(\phi^{m_u}, e_u) \bigoplus \bigoplus_{\substack{u\in B''\\ m_u>0}} \mathcal{L}_{1,m_u+1}(e_u) \bigoplus S(M_s).$$

By Proposition 32, if $k(A,b)$ is a direct summand of U, there exists $x \in Z(U) \bigcap U_A^- \backslash U_B^-$. We have $U_A^- = (U')_A^-$, where

$$(U')_A^- = \bigoplus_{\substack{u\in\mathcal{B}'\\ 0<j<m_u}} \langle\phi^j(e_u)\rangle \bigoplus \bigoplus_{\substack{u\in\mathcal{B}''\\ j\geq 1}} \langle\phi^j(e_u)\rangle \bigoplus M_A^-.$$

Therefore $x \in M_A^-$ and since $U_B^+ = (U')_B^+ = M_B^+$, it is obtained that $x \in Z(U) \bigcap M_A^- \backslash M_B^+$. By Proposition 29, $k(A,b_1)$ is a direct summand of M.

Conversely, if $k(A,b_1)$ is a direct summand of M, there exists $x \in Z(U) \bigcap M_A^- \backslash M_B^+ = Z(U) \bigcap U_A^- \backslash M_B^+$. Therefore $k(A,b)$ is a direct summand of U.

■

We consider the representation $\mathcal{L}^{n+1}(e)$ of $\mathcal{K}$, which given by

$$\mathcal{L}_0^{n+1} = V^n(e)$$
$$\mathcal{L}_{(a,b)}^{n+1} = \langle((0, \phi_n^n(e)), (\phi_n^n(e), \phi_n^{n-1}(e)), \ldots, (\phi_n(e), e))\rangle = \{(\phi_n(v), v) \mid v \in V^n(e)\}$$

where $\phi_n : V^n(e) \to V^n(e)$ is a k linear map with $\phi_n^{n+1} = 0$. The k vector space $V^n(e)$ is a $k[x]-$module where for $v \in V^n(e)$, $xv = \phi_n(v)$. If $\mathcal{L}_{3,n}(\phi_n(e), e) = M$ then $V^n(e) = M_0(e) \bigoplus X(e)$ where

$$X(e) = \begin{cases} 0, & \text{if} \quad n = 1, \\ \langle \phi_n(e), \ldots, \phi_n^{n-1}(e)\rangle, & \text{if} \quad n \geq 2, \end{cases}$$

and if $\mathcal{L}_{1,n+1}(e) = M$ then $V^n(e) = M_0 \oplus X(e)$, where

$$X(e) = \begin{cases} 0, & \text{if} \quad n = 0, \\ \langle \phi_n(e), \ldots, \phi_n^{n}(e)\rangle, & \text{if} \quad n \geq 1, \end{cases}$$

Proposition 35. *Let $h : \mathcal{L}_{3,n}(\phi^n(e), e) \to \mathcal{L}_{3,m}(\phi^m(f), f)$ be a morphism in* $\mathrm{Rep}(\mathcal{K}')$ *then there exists a morphism of $k[x]$- modules $\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix} : V^n(e) \to V^m(f)$ where $\beta : M_0 \to X(f), \alpha : X(e) \to N_0$ and $\gamma : X(e) \to X(f)$ and the following statements are satisfied:*

a. *if $n = m$, then $\alpha = 0$, $\beta = 0$.*

b. *If $n > m$ then $\beta = 0$,$\alpha(\phi^i(e)) = 0$ for $1 \leq i < m$ and $\alpha(\phi^m(e)) = c_1\phi^m(f)$ for some $c_1 \in k$.*

c. *If $n < m$, then $\alpha = 0$ and $\beta(e) = \phi^{n-m}(e)$, $\beta(\phi^n(e)) = 0$.*

Proof. a. If $m = n$, a morphism $h : \mathcal{L}_{3,m}(\phi^m(e), e) \to \mathcal{L}_{3,m}(\phi^m(f), f)$ is such that $h(e) = c_1 f + c_2\phi^m(f)$. Then if we define $\hat{h} : V^m(e) \to V^m(f)$ in such a way that $\hat{h}(\phi^i(e)) = \phi^i f$, we get our result for this case.

b. In this case $h(e) = c_1 f + c_2\phi^m(f)$, so we define $\hat{h} : V^n(e) \to V^m(f)$ such that $\hat{h}(\phi^j(e)) = \phi^j(h(e))$. Hence $\beta = 0$ and $\alpha(\phi^j(e)) = 0$ for $1 \leq j < m$ and $\alpha(\phi^m(e)) = c_1\phi^m(f)$.

c. We suppose that $n < m$. In this case $h(e) = c_1\phi^m(f), h(\phi^n(e)) = c_2\phi^m(f)$. For this case we define $\hat{h} : V^n(e) \to V^m(f)$ such that $\hat{h}(e) = c_1\phi^{m-n}(f) + c_2\phi^m(f)$ and $\hat{h}(\phi^i(e)) = \phi^i(\hat{h}(e))$. Here $\alpha = 0$ and $\beta : M \to X(f)$ is such that $\beta(e) = c_1\phi^{m-n}(f)$, $\beta(\phi^n(e)) = 0$. ■

Proposition 36. *Let $h : \mathcal{L}_{3,n}(\phi^n(e), e) = M \to \mathcal{L}_{1,m+1}(f) = N$ be a morphism in* $\mathrm{Rep}(\mathcal{K}')$ *then there exists a morphism of $k[x]$- modules*

$$\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix} : M_0 \bigoplus X(e) \to N_0 \bigoplus X(f)$$

such that:

a. *if* $n \geq m+1$ *then* $\alpha = 0, \beta = 0,$

b. *if* $n < m$ *then* $h = 0$ *and* $\hat{h} = 0.$

Proof. **a.** In this case $h(e) = cf$, so we put $\hat{h}(e) = cf$ and for $j \geq 1, \hat{h}(\phi^j(e)) = \phi^j(\hat{h}(e)$. Then $\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix}$, where $\alpha = 0$ and $\beta = 0$.
b. It is clear. ∎

Proposition 37. *Let* $h : \mathcal{L}_{1,n+1}(e) \to \mathcal{L}_{1,m+1}(f)$ *be a morphism in* $\mathrm{Rep}(\mathcal{K}')$ *then there exists a morphism of* $k[x]$*- modules*

$$\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix} : M_0 \bigoplus X(e) \to N_0 \bigoplus X(f)$$

such that:

a. *if* $n < m$, $h = 0$, $\hat{h} = 0$,

b. *if* $n \geq m, \alpha = 0$ *and* $\beta = 0.$

Proof. The first case is clear and for the second, it is obtained that $h(e) = cf$. Then we put $\hat{h}(e) = cf$ and $\hat{h}(\phi^i(e)) = c\phi^i(f)$, so $\alpha = 0$ and $\beta = 0$. ∎

Proposition 38. *Let* $h : M = \mathcal{L}_{1,n+1}(e) \to \mathcal{L}_{3,m}(\phi^m(f), f) = N$ *be a morphism in* $\mathrm{Rep}(\mathcal{K}')$*, then there exists a morphism of* $k[x]$*- modules*

$$\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix} : V^n(e) \to V^m(f)$$

such that:

a. *if* $n+1 \leq m$, *then* $\alpha = 0, \beta = 0,$

b. *if* $n+1 > m$, *then* $\alpha : X(e) \to N_0$ *is such that* $\alpha(\phi^i(e)) = 0$ *for* $i \neq m$ *and* $\alpha(\phi^m(e)) = \phi^m(f), \beta = 0.$

Proof. a. It is clear.

b. In this case, we have $h(e) = c\phi^n(f)$, then $\hat{h} : V^n \to V^m$ is such that $\hat{h}(\phi^i(e)) = 0$. ∎

5.1.1 Integration

Let W be a representation in $\mathcal{R}'$ such that

$$W|_{\mathcal{K}'} = \bigoplus_{u \in J} W^u \bigoplus S(W_s).$$

with each $W^u = \mathcal{L}_{1,l_u+1}(e_u)$ or $W^u = \mathcal{L}_{3,l_u}(f_u, e_u)$ and $S(W_s)$ is the trivial representation with $S(W_s)_0 = W_s$. We consider $J_i = \{u \in J \mid l_u = l_i\}$, with $l_1 > l_2 > \cdots > l_{L_1} > l_L = 1$ and $J = \bigcup_{i=1}^{L} J_i$. Let $J'_i = \{u \in J_i \mid W^u = \mathcal{L}_{3,l_i}(f_u, e_u)\}$ and $J''_i = \{u \in J_i \mid W^u = \mathcal{L}_{1,l_i+1}(e_u)\}$. For $u \in J'_i$ we take a k-vector space Z_u as follows:

$$Z_u = \begin{cases} 0, & \text{if } \ l_i = 0, \\ \langle e_u^1, \ldots, e_u^{l_u-1} \rangle, & \text{if } \ l_i \geq 1 \end{cases} \tag{5.8}$$

Now, for $u \in J''_i$ we put

$$Z_u = \begin{cases} 0, & \text{if } \ l_i = 0, \\ \langle e_u^1, \ldots, e_u^{l_u} \rangle, & \text{if } \ l_i \geq 1. \end{cases} \tag{5.9}$$

We consider the representation of $\mathcal{R}'$

$$\hat{W} = V \bigoplus \bigoplus_{u \in J} K(A, b_1)(Z_u).$$

Affirmation 2. *There exists $V \in \mathcal{R}$ such that $V' = \hat{W}$.*

Proof. We have that $\hat{W} = (\hat{W}_0, \hat{W}_z)_{z \in \theta}$ with $\hat{W}_0 = W_0 \bigoplus \bigoplus_{u \in J} Z_u$. We define $V_0 = \hat{W}_0$ and for $z \neq (a, b)$ we put $V_z = \hat{W}_z$. For $u \in J'_i$ we define

$$V_u = \begin{cases} \langle (0, f_u), (f_u, e_u^{l_u}), \ldots, (e_u^2, e_u^1), (e_u^1, e_u) \rangle, & \text{if } \quad l_i \geq 1, \\ \langle (0, e) \rangle, & \text{if } \quad l_i = 0, \end{cases}$$

and for $u \in J''_i$ we define

$$V_u = \begin{cases} \langle (0, e^{l_u}), \ldots, (e_u^1, e_u) \rangle, & \text{if } \quad l_i \geq 1, \\ \langle (0, e) \rangle, & \text{if } \quad l_i = 0, \end{cases}.$$

We define $V_{(a,b)} = \bigoplus_{u \in J} V_u$. We have to prove that V is a representation of $(\mathcal{P}, \theta)$. Let $(x, y) \in \theta$, with $a < x$. We have to prove that $V_a^+ \subset V_x^-$. We consider $(\lambda, \nu) \in V_{(a,b)}$, so $(\lambda, \nu) = \sum_{u \in J} (\lambda_u, \nu_u)$, with $(\lambda_u, \nu_u) \in V_u$. If $l_u = 1, \lambda_u = 0$. For $u \in J'_u, \lambda_u = c_0 f_u + \sum_{j=1}^{l_u-1} c_j \mathfrak{e}_u^j$. we have that $f_u \in W_{a_{l_u}}^+$. Here $x \in A$ and $a_{l_u} < x$ in $\mathcal{P}'$ and $f_u \in W_{a_u}^+$. Therefore $(\lambda, 0) \in V_{(x,y)}$, thus $\lambda \in V_x^-$.

We suppose that $a > x$. Let $\lambda \in V_x^+$ and we take $N > l_1$; since $aN > x$, then $\lambda \in V_x^+ = \hat{W}_x^+ = W_x^+$; therefore $\lambda \in W_{a_N}^-$. Here, $W(a_N, b_N) = \bigoplus_{u \in J} W_{a_N, b_N}^u$. As $N > l_1 > l_2 > \cdots > l_L$ then $(W_{(a_N, b_N)}^-) = 0$. This implies that $\lambda = 0$ so, $\lambda \in V_a^-$. Now, let $b > x$. It is obtained that $V_x^+ = \hat{W}_x^+ = W_x^+$ then in $\mathcal{R}'$, $b_1 > x$; therefore $W_x^+ \subset W_{b_1}^-$. We have $W_{b_1}^- = \bigoplus_{u \in J} (W^u)_{b_1}^-$. If $W^u = \mathcal{L}_{3,l_u}(f_u, e_u)$, then $(W^u)_{b_1}^- = \langle f_u \rangle$. If $W^u = \mathcal{L}_{1,l_u+1}(e_u)$ then for $l_u \geq 1$, $W^u_{(a_1,b_1)} = 0$ and for $l_u = 0$ it is obtained that $(W^u)^-_{b_1} = \langle e_u \rangle$. Therefore if $\lambda \in W^+_{b_1}$, then $\lambda = \sum_{u\in J'} c_u f_u + \sum_{u \in J''} l_u = 0$. Therefore $\lambda \in V^-_b$.

■

We consider $\mathcal{R}'_r$ the full subcategory $\mathcal{R}'$ consisting of the objects without direct summand $k(A, b_1)$. If $M \in \mathcal{R}'$, we choose $M^{\downarrow}$ without direct summand of the form $k(A, b_1)$, such that $M = M^{\downarrow} \bigoplus k(A, b_1)^l$ for some l. In this way we obtain a functor

$$(^{\downarrow})' : \mathcal{R}' \to \mathcal{R}'_r.$$

Let U be a object in $\mathcal{R}$, then $U_0 = U_r \bigoplus U_s$ with $U_r = U^+_b$. We recall that there exists a k-linear transformation $\phi_U : U^+_b \to V^+_b$ such that $V_{(a,b)} = \{(\phi_U(x), x) \mid x \in V^+_b\}$.

We have $U|_{\mathcal{K}} = \bigoplus_{u\in B(U)} \mathcal{L}^{m_u+1}(e_u) \bigoplus S(U_s)$ with $B(U)$ being a strict system of generators for U with the conditions of the Proposition 27 and $S(U_s) = (U_s, U_{(a,b)} = 0)$.

Let $f : U \to V$ be a morphism in $\mathcal{R}$ then we have $V|_{\mathcal{K}} = \bigoplus_{u\in B(V)} \mathcal{L}^{m_u+1}(e_u) \bigoplus S(V_s)$ with $B(V)$ being a strict system of generators for V with the conditions of the Proposition 27 and $S(V_s) = (V_s, V_{(a,b)} = 0)$ and $V_0 = V_r \oplus V_s, V_r = V^+_b$. We observe that $f\phi_U = \phi_V f|_{U_r}$. Indeed, $(\phi_{U(x)}, x) \in U_{(a,b)}$, then $(f(\phi_{U(x)}), f(x)) \in V_{(a,b)}$; hence $f\phi_U(x) = \phi_V f(x)$ for all $x \in U_r$.

Theorem 13. **a.** *The functor* $(^{\downarrow})' : \mathcal{R} \to \mathcal{R}'_r$ *induces a dense and full functor, but in general not faithful:*

$$F : \mathcal{R}/\langle \phi(\lambda, n)_{\lambda\neq 0},\, k(A, b)\rangle \to \mathcal{R}'/\langle k(A, b_1)\rangle.$$

b. *Let* $U \in \mathcal{R}$ *without a direct summand* $k(A, b)$ *or* $\phi(\lambda, n)$ *and let* $f : U \to U$ *be a morphism in* $\mathcal{R}$ *such that* $F(f) = 0$, *then* f *is nilpotent.*

Proof. **a.** Let $U, V \in \mathcal{R}$. We take $\mathcal{B}(U) = \{e_u\}_{u\in\mathcal{B}(U)}$ ($\mathcal{B}(V) = \{e_u\}_{u\in\mathcal{B}(V)}$) be a strict system of generators with the conditions of Proposition 27 for U^+_b (V^+_b, respectively). We have

$$U|_{\mathcal{K}} = \bigoplus_{u\in B(U)} \mathcal{L}^{m_u+1}(e_u) \bigoplus S(U_s),$$

and

$$V|_{\mathcal{K}} = \bigoplus_{u\in B(V)} \mathcal{L}^{m_u+1}(e_u) \bigoplus S(V_s),$$

then

$$U' = M \bigoplus \bigoplus_{u\in B(U)} k(A, b_1)(X(u)), \quad V' = N \bigoplus \bigoplus_{u\in B(V)} k(A, b_1)(X(u)),$$

with $M = (U')^{\downarrow}, N = (V')^{\downarrow}$. We take $\mathcal{C}(U) = \mathcal{B}(U) \bigcup \{z_1\}$, $\mathcal{C}(V) = \mathcal{B}(V) \bigcup \{z_2\}$. We define $M^{z_1} = S(U_s)$, $M^{z_2} = S(V_s)$, $X(z_1) = 0$ and $X(z_2) = 0$. We have

$$M|_{\mathcal{K}'} = \bigoplus_{u\in\mathcal{C}(U)} M^u, \quad N|_{\mathcal{K}'} = \bigoplus_{u\in\mathcal{C}(V)} M^u.$$

By Propositions 35 , 36 , 37 , 38, for each pair $u \in B(U), u' \in B(V)$, we have a k linear map:

$$\hat{h}_{u',u} = \begin{pmatrix} h_{u',u} & \alpha_{u',u} \\ \beta_{u',u} & \gamma_{u',u} \end{pmatrix} : M_0^u \bigoplus X(u) \to M_0^{u'} \bigoplus X(u')$$

such that

$$\hat{h}_{u',u} = \mathcal{L}^{m_u+1}(e_u) \to \mathcal{L}^{m_{u'}+1}(e_{u'})$$

is a morphism in $\mathrm{Rep}(\mathcal{K})$. Also this linear transformation is a morphism in $\mathrm{Rep}(\mathcal{K}')$:

$$\hat{h}_{u',u} = \begin{pmatrix} h_{u',u} & \alpha_{u',u} \\ \beta_{u',u} & \gamma_{u',u} \end{pmatrix} : M^u \bigoplus \mathcal{L}_{1,1}(X(u)) \to M^{u'} \bigoplus \mathcal{L}_{1,1}(X(u'))$$

For $u = z_1$ and $u_0 \in B(V)$ we have

$$\hat{h}_{u_1,z_1} : \begin{pmatrix} h_{u',z_1} \\ \beta_{u',z_1} \end{pmatrix} : U_s = M_0^{z_1} \to M_0^{u'} \bigoplus X(u'),$$

with $\beta_{u',z_1} = 0$. This k-linear map gives a morphism $S(U_s) \to \mathcal{L}^{m'_u+1}(e_{u'})$ in the category $\mathrm{Rep}(\mathcal{K})$ and also gives a morphism $S(U_s) \to M^{u'} \bigoplus \mathcal{L}_{1,1}(X(u')$ in the category $\mathrm{Rep}(\mathcal{K}')$.

For $u = z_1, u' = z_2$ the morphism

$$\hat{h}_{z_2,z_1} : U_s = M_0^{z_1} \to M_0^{z_2} = V_s$$

gives a morphism $S(U_s) \to S(V_s)$ in $\mathrm{Rep}(\mathcal{K})$ and in $\mathrm{Rep}(\mathcal{K}')$. For $u \in B(U)$ and $u' = z_2$ the morphism:

$$\hat{h}_{z_2,u} = h_{z_2,u} : M_0^u \to M_0^{z_2} = V_s$$

is a morphism in $\mathrm{Rep}(\mathcal{K}')$ from M^u to $S(V_s)$; therefore $\hat{h}_{z_2,u} = 0$. We have $U_0 = M_0 \oplus X(U)$ and $V_0 = N_0 \bigoplus X(V)$ with $M_0 = \bigoplus_{u \in \mathcal{C}(U)} M_0^u$, $N_0 = \bigoplus_{u \in \mathcal{C}(V)} M_0^v$, $X(U) = \bigoplus_{u \in \mathcal{C}(U)} X(u)$, $X(V) = \bigoplus_{u \in \mathcal{C}(V)} X(u)$. Thus we obtain that the morphism $\hat{h} : U_0 \to V_0$ gives a morphism from $U|_{\mathcal{K}}$ to $V|_{\mathcal{K}}$.

Let

$$\hat{h} = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix} : M_0 \bigoplus X(M) \to N_0 \bigoplus X(N)$$

where $\alpha = (\alpha_{u',u}) : \bigoplus_{u \in B(U)} X(u) \to \bigoplus_{u \in B(V)} M^u$ and $\beta = (\beta_{u',u}) : \bigoplus_{u \in B(U)} M^u \to \bigoplus_{u \in B(V)} X(u)$.

It is clear that

$$\gamma : k(A, b_1)(X(M)) \to k(A, b_1)(X(N)),$$

is a morphism in $\mathcal{R}'$. We will prove that $\alpha : k(A, b_1)(X(M)) \to N$ and $\beta : M \to k(A, b_1)(X(N))$ are morphisms in this category.

We consider the morphism $\hat{\alpha}_{u',u} : \mathcal{L}_{1,1}(X(u)) \to N$ defined by

$$\alpha_{u',u} : \mathcal{L}_{1,1}(X(u)) \to N^{u'},$$

followed by the inclusion $N^{u'}$ in N. If $\alpha_{u',u} \neq 0$, then by Propositions 35 , 36 , 37 , 38, $u \in B(U)'$ with $m_u > 0$ and also $u' \in B(V)'$ with $m_{u'} > 0$ and $m_u > m_{u'}$, or $u \in B''(U), u' \in B(V)'$ with $m_u + 1 > m_{u'}$. In both cases, $\hat{\alpha}_{u',u}$ is the composition of the projection of

$$\mathcal{L}_{1,1}(X(u)) \to \mathcal{L}_{1,1}(\phi^{m_{u'}}(e_u)),$$

followed by morphism $s : \mathcal{L}_{1,1}(\phi^{m_{u'}}(e_u)) \to N$ such that sends $\phi^{m_{u'}}(e_u)$ in $c_1\phi^{m_{u'}}(e_{u'}) \in N_A^+$; therefore by Proposition 30, it is obtained that s and therefore $\hat{\alpha}_{u',u} : k(A,b_1)(X(u)) \to N$ are morphisms in $\mathcal{R}'$. Hence, $\alpha : k(A,b_1)(X(M)) \to N$ is a morphism in $\mathcal{R}'$.

We consider the morphism $\beta : M_{\mathcal{K}'} \to \mathcal{L}_{1,1}(X(N))$ which is the sum of the morphisms

$$\hat{\beta}_{u',u} : M \to \mathcal{L}_{1,1}(X(N))$$

with $\hat{\beta}_{u',u} = i_{u'}\beta_{u',u}\pi_u$ where $\pi_u : M \to M^u$ is the projection and $i_{u'} : \mathcal{L}_{1,1}(X(u')) \to \mathcal{L}_{1,1}(X(N))$ is the inclusion. If $\beta_{u',u} \neq 0$, then by Propositions 35 , 36 , 37 , 38, it is obtained that

$$m_u < m_{u'}, \quad m_u > 0, \quad \text{and} \quad \beta_{u',u}(e_u) = \phi^{m_{u'}-m_u}(e_{u'}), \quad \beta_{u',u}(\phi^{m_u}(e_u)) = 0,$$

therefore $\hat{\beta}_{u',u}(M_B^+) = 0$, and by Proposition 29, this morphism is a morphism $\hat{\beta}_{u',u} : M \to k(A,b_1)(X(N))$.
From here, it is obtained that $\beta : M \to k(A,b_1)(X(N))$ is a morphism in $\mathcal{R}'$. From the above it follows that the morphism $\hat{h} : M \bigoplus k(A,b_1)(X(M)) \to N \bigoplus k(A,b_1)(X(N))$ is a morphism in $\mathcal{R}'$.
We will prove that $\hat{h}$ is a morphism of U in V in the category $\mathcal{R}$. We know that $\hat{h}$ restricted to $\mathcal{K}$ is a morphism of $U|_{\mathcal{K}}$ in $V|_{\mathcal{K}}$. Let now $z \in \theta$, $z \neq \{a,b\}$, then

$$\hat{h}^z(U_z) = \hat{h}^z(U'_z) \subset V'_z = V_z.$$

Therefore, $\hat{h}$ is a morphism in $\mathcal{R}$ and $F(\hat{h}) = h$. So, F is full.

b. We suppose that $U \in \mathcal{R}$ without direct summand of the form $k(A,b)$ or $\phi(\lambda,n)$. By Proposition 34, $M = (U')^{\downarrow}$ does not have a direct summand $k(A,b_1)$.

Now, let $f : U \to U$ be a morphism in $\mathcal{R}$ such that $F(f) = 0$, since

$$U' = k(A,b_1)(X(M)) \bigoplus M$$

where $M = (U')^{\downarrow}$, then

$$f = \begin{pmatrix} h & \alpha \\ \beta & \gamma \end{pmatrix}$$

where $h : M \to M$ is a morphism in $\mathcal{R}'$ which is factored through $k(A,b_1)(W)$ for some finite dimensional k-vector space W.
We have then $h(M_B^+) = 0$, $h(M_0) \subset M_A^-$ and

$$M|_{\mathcal{K}'} = \mathcal{L}_{1,1}(Z) \bigoplus \bigoplus_{\substack{u \in B' \\ m_u > 0}} \mathcal{L}_{3,m_u}(\phi^{m_u}(e_u), e_u) \bigoplus \bigoplus_{\substack{u \in B'' \\ m_u > 0}} \mathcal{L}_{1,m_u+1}(e_u)$$

and U does not have a direct summand $k(A,b)$, then by Proposition 31, $Z \bigcap M_A^- \subset M_B^+$. Since h is factored through $k(A,b_1)(W)$, for $x \in M_0$, $h(x) = z + z_1$ with $z \in Z$ and z_1 in the vector space generated by the elements $\phi^{m_u}(e_u)$. Here, $h(x)$ and z_1 is in M_A^-; therefore $z \in M_A^- \bigcap Z \subset M_B^+$; consequently $h^2(x) = h(z) + h(z_1) = h(z_1) \subset \phi(U_0)$. From the above

$$(f^2)(U_0) = \begin{pmatrix} h^2 + h\beta & h\alpha + \alpha\gamma \\ \beta h + \gamma\beta & \beta\alpha + \gamma^2 \end{pmatrix} (U_0) \subset \phi(U_0),$$

is obtained. Therefore $(f^2)^L = 0$, where L is the maximum of the numbers m_u. ■

Corollary 8. *Let $U \in \mathcal{R}$ without direct summand $k(A,b)$ then $(U')^{\downarrow}$ is indecomposable if and only if U is indecomposable.*

Proof. Since U does not have a direct summand $k(A,b)$, then $M = (U')^{\downarrow}$ does not have a direct summand $k(A,b_1)$; therefore the endomorphisms of M which are factored by sums of $k(A,b_1)$ are in $\operatorname{radEnd}_{\mathcal{R}'}(M)$. If U is indecomposable and $f : U \to U$ is such that $F(f) = 0$ then $f \in \operatorname{radEnd}_{\mathcal{R}}(U)$; therefore F induces an isomorphism

$$\operatorname{End}_{\mathcal{R}}(U)/\operatorname{radEnd}_{\mathcal{R}}(U) \to \operatorname{End}_{\mathcal{R}'}(M)/\operatorname{radEnd}_{\mathcal{R}'}(M);$$

therefore M is indecomposable.
If U is not indecomposable $U \cong \bigoplus\limits_i U^i$ where U_i are indecomposable; therefore

$$M = (U')^{\downarrow} = \bigoplus_i ((U^i)')^{\downarrow}.$$

From the above, each $((U^i)')^{\downarrow}$ is indecomposable. So, if M is indecomposable, U is too. ■

Corollary 9. *Let M and N be in $\mathcal{R}'$ and suppose that both objects do not have direct summand isomorphic to $k(A,b_1)$. Suppose that $f : U \to V$ is a morphism in $\mathcal{R}$ such that $F(f) : M \to N$ is an irreducible morphism, then f is irreducible.*

Proof. First let us prove that f is neither a retraction nor a section. Suppose it were a retraction, then there exists $g : V \to U$ such that $fg = 1_V$, thus $F(f)F(g) = 1_N$ which does not happen because $F(f)$ is irreducible. In the same way, it is proved that f is not a section. Now, we suppose that $f = vu$ with $u : U \to W$ and $v : W \to V$ therefore $F(f) = F(v)F(u)$, then either $F(u)$ is a section or $F(v)$ is a retraction. If the first case occurs, there exists $\alpha : F(W) \to F(U)$ such that $\alpha F(u) = 1_M$. Since the functor F is full, there exists $a : W \to U$ such that $\alpha = F(a)$; therefore $F(a)F(u) = F(au) = F(1_U)$ so $F(au - 1_U) = 0$; so by **c**. of Theorem 13, $\rho = au - 1_U$ is nilpotent. Therefore $au = 1_U + \rho$ is an isomorphism and therefore u is a section. In a similar way, it is proved that if $F(v)$ is a retraction, then v is a retraction. ■

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