UNIVERSIDAD NACIONAL DE COLOMBIA

Scenarios of evolutionary branching in the adaptive dynamics framework:
theory and applications to technological change

Escenarios de ramificaciones evolutivas en el marco de las dinámicas adaptativas:
teoría y aplicaciones al cambio tecnológico

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Abstract

In this thesis, we have reviewed the theory of Adaptive Dynamics, a theoretical background originated in evolutionary biology linking demographic dynamics to evolutionary changes, allowing it to describe evolutionary dynamics in the long-term when considering innovations as small and rare events in the market time scale. From this perspective, three mathematical models have been formulated to describe evolutionary branching: the coexistence between resident and similar innovative technologies and their further divergence in the evolutionary space. The first model addresses the problem of determining conditions on the energy market diversification from adaptive dynamics and the impact the imposition/allocation of taxes/subsidies may have on controlling market diversification. The second model explores the Coffee Berry Borer (*Hypothenemus hampei*) and its role in the evolutionary diversification of the coffee market; the influence that consumer's preference and control practices have on diversification is studied in detail, and correspond to the main source of insights. Finally, the third model in the fifth chapter, describes the competition among public transport systems, considering the number of transported passengers as the differentiation attribute is presented, the analysis allows to answer the question of under what condition the market diversifies, and which are the levels of transported passengers that will be reached in the long term depending on the budget allocation rate destined to increase the number of users. Adaptive dynamics describes evolution through an ordinary differential equation known as the canonical equation, which smooths on a continuous path the successive processes of innovation and substitution. This approach considers interactions to be the evolutionary driving force and considers the feedback between evolutionary change and the selection forces that agents undergo. One of the main (general) contributions of this thesis is to illustrate in detail how the theory of adaptive dynamics is very useful in areas of knowledge quite distant from evolutionary biology, in particular for engineering, given that its results predict the systems' long-term dynamics, as well as to control in the demographic/market timescale and to influence the long-term behavior of the evolving attributes in the evolutionary timescale.

Keywords: Technological Change, Adaptive Dynamics, Evolutionary Branching, Dynamic Systems, Energy Market, Coffee Market; Market Diversification; Coffee Berry Borer; Public Transportation; Simulation Modeling.

Resumen

En la ejecución de esta tesis, hemos revisado la teoría de la dinámicas adaptativas, un trasfondo teórico que se origina en la biología evolutiva, que vincula la dinámica demográfica con los cambios evolutivos, y permite describir la dinámica evolutiva a largo plazo al considerar las innovaciones como eventos pequeños y raros en la escala de tiempo del mercado. Desde esta perspectiva, se han formulado tres modelos matemáticos que permiten describir la ramificación evolutiva, es decir, la coexistencia entre tecnologías innovadoras residentes y similares y su posterior divergencia en el espacio evolutivo. El primer modelo aborda el problema de determinar las condiciones para la diversificación del mercado energético a partir de las dinámicas adaptativas y el impacto que la imposición/asignación de impuestos/subsidios puede tener en el control de la diversificación del mercado. El segundo modelo explora la broca del café (*Hypothenemus hampei*) y su papel en la diversificación evolutiva del mercado cafetero; la influencia que las preferencias de los consumidores y las prácticas de control tienen sobre la diversificación se estudia en detalle y corresponde a la principal fuente de información. Además, en el quinto capítulo, se presenta un modelo para la competencia entre los sistemas de transporte público, considerando el número de pasajeros transportados como el atributo de diferenciación; el análisis permite responder a la pregunta bajo qué condiciones se diversifica el mercado y cuáles son los niveles de pasajeros transportados que se alcanzarán a largo plazo dependiendo de la tasa de asignación presupuestaria destinada a aumentar el número de usuarios. La teoría de las dinámicas adaptativas describe la evolución a través de una ecuación diferencial ordinaria conocida como ecuación canónica, que suaviza en una trayectoria continua los procesos sucesivos de innovación y sustitución. Este enfoque considera las interacciones como la fuerza impulsora de la evolución y tiene en cuenta la retroalimentación entre el cambio evolutivo y las fuerzas de selección que sufren los agentes. Una de las principales contribuciones mas generales de esta tesis es ilustrar en detalle cómo la teoría de las dinámicas adaptativas es útil en áreas de conocimiento bastante distantes de la biología evolutiva, en particular para la ingeniería, dado que sus resultados permiten predecir el comportamiento de los sistemas a largo plazo, así como controlar dicho comportamiento en la escala de tiempo demográfica/de mercado e influir en la dinámica a largo plazo de los atributos en evolución en la escala de tiempo evolutiva.

Palabras clave: Cambio Tecnológico, Dinámicas Adaptativas, Ramificación Evolutiva, Sistemas Dinámicos, Mercado Energético, Mercado del Café; Diversificación del mercado; Broca del Café; Transporte público; Simulación de modelos.

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1. Introduction

The formation of new species, called speciation, is one of the central points of evolutionary theory. It occurs through the genetic and phenotypic divergence of populations of the same species, which adapt to different environmental niches, either within the same, or in different habitats. In allopatric speciation, two populations are geographically separated by natural or artificial barriers, while in parapatric speciation, the two populations evolve toward geographic isolation, through the exploitation of different environmental niches in contiguous habitats. In either of these two cases, geographical isolation constitutes an exogenous cause of speciation, instead of an evolutionary sequence [8, 31].

On the other hand, sympatric speciation considers a population in a single geographical location. As such, it is disruptive selection that exerts selection pressures, which favor extreme characteristics over average characteristics. This phenomenon may result, for example, from competition for alternative environmental niches, in which specializing may be more advantageous than being a generalist. Consequently, the population divides into two groups which are initially similar, but which later diverge on separate evolutionary paths (branches), each driven by their own mutations, undergoing what is called evolutionary branching. In Figure **1-1**, the evolutionary branching point concept, a product of sympatric speciation, is shown [8, 36].

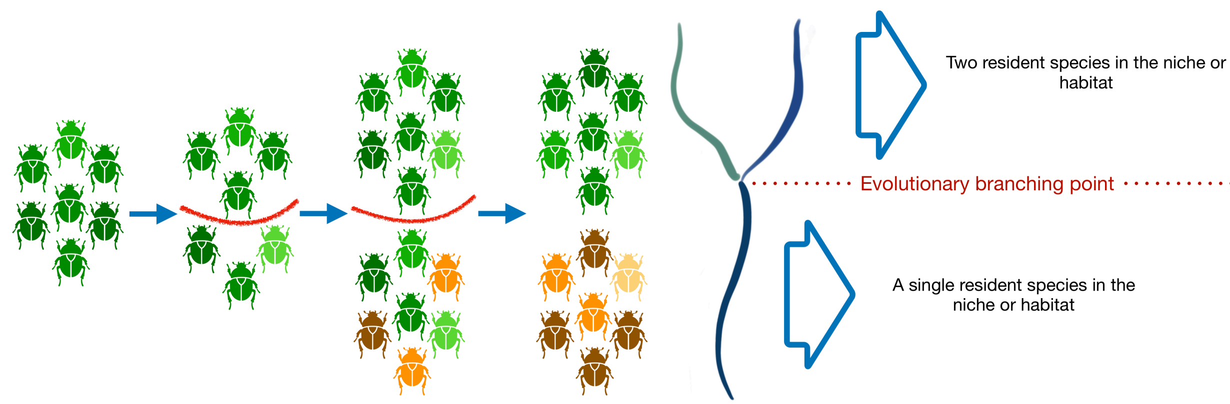

Figure 1-1.: **Left:** Different selection pressures and different genetic derivations can act in different environments, isolated populations can eventually become separate species. **Right:** Influenced by disruptive selection, a monomorphic population may become dimorphic in certain relevant attributes. Image elaborated by the author.

Human evolution shows empirical evidence of this evolutionary phenomenon. Humans form part of the *hominidae* family, which includes great apes (bonobos, chimpanzees, gorillas, and orangutans) and other extinct humanoid species. Since Darwin and the publication of The Descent of Man (1871), countless fossils have been found and dated, which show that humans and great apes shared a common ancestor approximately six or seven million years ago. The causes of the evolutionary branching which led to humans are a source of great debate. However, one of the most intriguing potential causes is the evolution of articulated language, thanks to fine control of the larynx or the mouth, which is regulated by a particular gene [31, 57].

Generally speaking, the basic units capable of evolution through innovation and competition processes are not limited to living organisms. Multiple examples of evolutionary branching can, in fact, be found in material products, ideas, and social norms [23, 29, 58]. In particular, commercial products are replicated each time that a product is bought, and services each time they are used. They go extinct whenever they are abandoned by users. Thus, improved versions are occasionally introduced, which are characterized by small innovations. These interact in the market with the prior established versions. Said interactions are usually competitive, and involve rivalry between products from both the same and different categories.

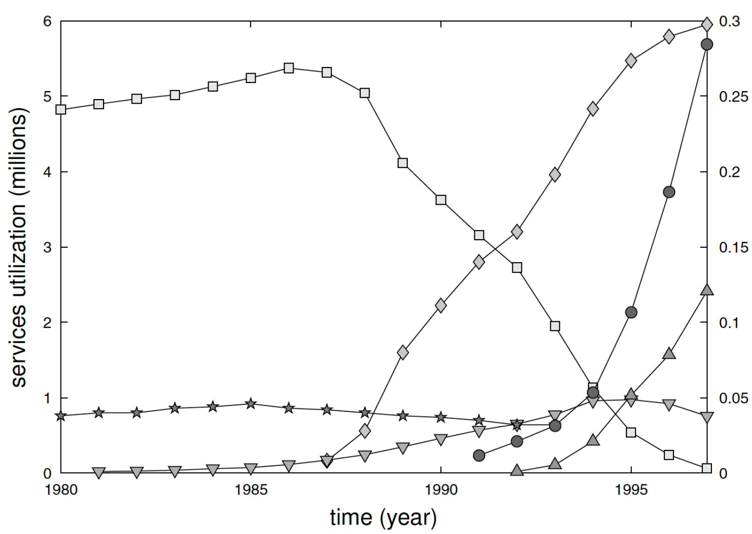

Figure 1-2.: Use of telecommunication services in Switzerland. Squares: analog telephones. Diamonds: digital telephones. Inverted triangles: subscribers to analog mobile phone services. Triangles: subscribers to digital mobile phone services. Stars: public payphones. Circles: internet hosts. Reprinted from [31].

One example of this is shown in Figure **1-2**, in which the evolution of different communication services in Switzerland, between 1980 and 2000, is shown. The arrival of digital telephones was a successful innovation, which has led to the substitution of analog telephones. This phenomenon is reported as attribute substitution. On the other hand, observe that internet hosts (circles) seem to coexist with both digital telephones and digital mobile telephone service subscribers [31].

With the information discussed up to this point, it is possible to answer to the question of what constitutes the theory of adaptive dynamics (AD). In general, it is a theoretical background which originates in evolutionary biology, and links demographic dynamics to evolutionary changes. It further permits the description of evolutionary dynamics in the long term, considering innovations to be small and rare events [31, 34, 45, 46]. This theory focuses on the evolutionary dynamic of quantitative adaptation attributes in the long term, and disregards genetic details, through the use of asexual demographic models. Among the most relevant aspects is that it recognizes interactions as the driving evolutionary force, and considers feedback between evolutionary change and the forces of selection experienced by the agents [31, 32, 36].

One of the reasons to consider, is pointed out from the work of Shummpeter [82], who establishes the need to differentiate between invention, innovation and diffusion. In its definition, innovation refers to the economic application of an invention (the development of a new "product"), while diffusion refers to the generalization of its use by buyers and production by different firms [71]. While this concept of associating innovation with the emergence of new products is valid, and in fact has been fundamental in the growth of global industries, it must be considered that economic development is not limited to the industrial sector, but the service sector as a source of great business opportunities. However, the services sector is usually considered as "intangible" and "interactive", so the concept of innovation has been more difficult to define. In that sense, Gallouj [43] considers the fact that innovations in services are influenced by a set of forces (driving forces) that he identifies as incentives or obstacles to the innovation process, divided into what he calls "trajectories", and corresponding to the professional, technological and social management and that are composed by different agents in each level, such as clients, competitors, the government, etc. [60].

Using the theoretical framework of adaptive dynamics, the canonical equation, corresponding to an ordinary differential equation, is presented to describe the behavior over time of the characteristic attribute as a result of innovation processes. The theoretical framework of adaptive dynamics has been used recently to model a varied spectrum of situations involving innovations or genetic variations; in particular, technological innovations: [23, 29], social interactions: [58], mutualistic interactions: [19, 34, 36, 40], competition: [24, 36, 53], predator-prey dynamics [1, 2, 17, 27, 32, 34, 59, 61], evolution of dispersal: [13,28,73] dynamics in allele space: [12,55], canibalistic interactions: [18,30], and food chains: [32] among many others.

One of the most important feature of the AD approach is to describe evolutionary branching, that is the coexistence between a resident and a similar mutant type and their further divergence in phenotypic space. In other words, the development of two different forms of organisms with a common ancestor [31]. Recently, many authors have used this approach to study different branching situations; however, there are still interesting theoretical and applicative unanswered questions in relation to degenerate branching situation.

In the second chapter of this thesis, from the genetic point of view, the Adaptive Dynamics approach is briefly described, the AD Canonical Equation is heuristically deduced and conditions on evolutionary stability, and specifically on evolutionary branching, are also presented. It should be clarified at this point, that it is not the objective of this work to make a theoretical/mathematical contribution, this chapter has been included in order to provide the reader with the theoretical tools necessary to understand the applications development in the subsequent chapters, and it also justifies the inclusion of the expression "theory" in the title of this thesis. For a more precise information on theoretical matters, the interested reader is encouraged to visit [31]. In addition, detailed mathematical developments of the theory can be found in documents such as: [4,10,17,20,31,34,44–46].

In the third chapter, we study a mathematical model based on ordinary differential equations to describe the dynamic interaction in the market of two types of energy called standard and innovative. The development of better tools to promote the validation of expert knowledge and facilitate the analysis of historical data to quantify the effects of future events on energy markets are well paid in terms of accuracy, transparency, reproducibility and learning, as well as understanding the market [86]. In this context, is necessary to carry out methodological advances in the description of energy markets and how intrinsic characteristics, such as the source of generation, emissions reduction, final price to the consumer, generation technologies, generation capacity, costs of generation or any other related characteristic, influence competition conditions (and viceversa) and the rise of diversity in markets with stablished energy agents been forced to compete with innovative technologies, raised from variations in one or several of those characteristics.

The model consists of an adaptation of the generalized Lotka-Volterra system in which the parameters are assumed to depend on a quantitative and continuous attribute characteristic of energy generation. Using the analysis of the model the fitness function for the innovative energy is determined, from which conditions of invasion can be established in a market dominated by the conventional generation technology. The canonical equation of the adaptive dynamics is studied to know the long-term behavior of the characteristic attribute and its impact on the market. Then we establish conditions under which evolutionary ramifications occur, that is to say, the requirements of coexistence and divergence of the characteristic attributes, whose occurrence leads to the origin of diversity in the energy market.

In the fourth chapter, an agroecological application is shown. The Coffee Berry Borer (CBB) is the main pest affecting coffee crops around the world. It causes major economic losses and diminishes beverage quality. Herein, a mathematical model, from the perspective of the Adaptive Dynamics, is studied. This allows for the establishment of conditions for diversity in the coffee market, with quality considered a differentiating attribute. The implementation of this study involves three stages: first, a deterministic model is formulated, based on differential equations, with coffee production and CBB population as the agroecological context, prior to the processing of different quality coffees, based upon damage caused by the pest. Second, the long term dynamics of quality traits, from the perspective of adaptive dynamics, are studied, so as to establish conditions in which competition between standard and special coffee results in invasion, coexistence, substitution, or extinction of these products. Finally, we established the conditions for the occurrence of evolutionary branching points in the adaptive dynamics of quality traits, in order to relate the proportional catch rate by the farmer (harvesting) to demographic parameters of CBB population and to get insights on the impact that control strategies available to regulate CBB population may have on the long-term evolution of coffee quality and the possibility of diversification into different and coexisting qualities.

In the fifth chapter a generalized model has been formulated for the competition between transport systems in a city, considering that the interaction occurs under the same market platform and competition is determined by the proportion of users adopting each transport system and, additionally, a measure of the amount of budget that the investor makes available in order to promote the expansion of the transportation system among users is considered. Later, under the assumptions of stability, a generalized model is formulated, to describe competition in the market when that stability is disturbed by the entry of an innovative transport system. From the perspective of the adaptive dynamics, it is possible to determine general conditions that must be met to guarantee or not the success of the innovation as the one managing to penetrate and expand into the market. This information is obtained from study of the sign of the fitness function for specific model coefficients. Additionally, the approach through adaptive dynamics is used to establish the long term dynamics of the quantitative attribute and permits the classification of the evolutionary equilibra as terminal points, those in which the evolution definitively halts, like the points where substitution takes place.

Chapter six correspond to conclusions and recommendations; in the appendix, a glossary is included to facilitate the reading for those who are not familiar with the jargon, and to precise genetic definitions in context of the AD approach. Finally, the cited bibliography can be found.

2. Theory

2.1. Introduction

Evolution by natural selection, has been described as "one of the oldest and most astounding and complex dynamical processes on Earth" [32]. Indeed, understanding the origin of new species remains one of the core problems in evolutionary biology. Traditionally, there are two basic approaches to understand the actual mechanisms by which a phenotypic cluster of individuals splits into two distinct descendant clusters with restricted gene flow between them [32,36]:

- **Allopatric speciation**: the subpopulations of a given species are thought to become geographically isolated, after which they follow separate evolutionary paths, eventually leading to different species that are reproductively isolated even after secondary contact.

- **Sympatric speciation**: it is assumed that there are different habitats favoring different genotypes, but the different genotypes occur sympatrically (occupying the same or overlapping geographic area) and are pooled for mating. One then studies the necessary conditions for reproductive isolation to evolve between the genotypes that are favored in the different habitats.

Three biological ingredients are considered in the Darwinian evolution of a quantitative trait: (i) reproduction passes the trait through generations altering heritable characteristics of individual agents, (ii) trait values variations are generated by mutations processes and, (iii) selection as a result from interactions between individuals and their environment, then selecting the best performances. In this context, Adaptive dynamics (AD) makes a link between ecological and evolutionary theory, it defines *fitness* as a quantity to measure the long-term per capita growth rate of a rare mutant in an environment that is determined by externally fixed parameters on the one hand and by the population density and the phenotype of the resident population on the other [36]. Then, to this usual influence of environment on the population, it adds the reverse influence of the population on the environment by considering how the ecological interactions modify fitness, i.e., ecological interactions are the evolutionary driving force, and the feedback between evolutionary change and the ecological conditions experienced by individuals is considered [10,36,44].

AD framework describes evolution through an ordinary differential equation known as the AD Canonical Equation ([31] and references therein); for deriving analytical results, in the Canonical Equation formulation, other assumptions arises, for example, on the demographic timescale mutations are sufficiently rare and small to have small effect on the mutant trait, so that mutants encounter monomorphic resident populations that are at their ecological equilibrium. This corresponds to assuming a separation of ecological and evolutionary timescales, with the ecological dynamics occurring faster than the evolutionary dynamics [31,34,36,46,65]. Each condition is briefly described below.

Condition 1. Mutations are rare. If mutations are rare on the demographic timescale (so-called mutation-limited evolution), the resident populations are challenged by one mutation at a time and are at equilibrium when a mutant appears; then, population genetics supports the idea that sexual reproduction can be neglected, so that one can look at long-term evolutionary dynamics of quantitative traits by using, at least for qualitative purposes, asexual demographic models, where individuals are characterized by their phenotypes. Since the demographic and evolutionary timescale are kept separated, each time a mutation occurs in one of the relevant traits, the resident and the mutant population have plenty of time to interact and define the new structure of the community before the next mutation occurs. In particular, if the mutant population replaces the resident population, the trait undergoes an evolutionary step (on the evolutionary timescale), while if mutants go extinct the trait does not change.

Condition 2. Mutations are small. Invading mutants generically replaces the corresponding residents, so that the final of a successful mutation is a small variation of the trait. If mutations are small (as is the case most of the time, in particular for quantitative traits), evolution proceeds by small steps in trait space and, in the limit of infinitesimal mutations, one can pretend to describe evolutionary trajectories by means of ODE's.

Condition 2 allows to consider the limit case of infinitesimal mutations and to approximate a stochastic sequence of successful mutations by a smooth evolution of traits governed by a system of ODE's, one ODE for each trait: the AD Canonical Equation, whose evolutionary trajectories may lead to:

- **Stationary evolutionary regimes:** which may be terminal points of evolutionary dynamics or sources of diversity (Evolutionary Branching).
- **Non stationary evolutionary regimes:** Red Queen Dynamics.
- **Evolutionary extinction:** correspond to three different types: evolutionary runaway, evolutionary murder and evolutionary suicide.

Natural selection is assumed to be described by means of deterministic demographic models where sex, space, and physiological details as age, stage and energy reserves, do not appear. Thus, all individuals of a population are identical and uniformly distributed, the habitat is homogeneous and each population is identified by its abundance and by the values of all its characteristic traits.

Mutations with small effect. It is considered that a mutation have a small effect only in one trait: the case in which the traits characterizing the population are controlled by non overlapping sets of genes, each with small effect on the controlled trait; if this is the case, their concomitant mutations are so strongly correlated that is possible to define a one-dimensional equivalent trait and the results for resident populations characterized by mutations which only influences a single (scalar) trait, can be applied. In such situations, conditions under which invasion implies substitution are not valid at an equilibrium of the canonical equation. Thus, once the evolutionary dynamics have found a halt at the stable equilibrium of the canonical equation, a further investigation of the resident-mutant model is required.

Therefore, the mutant population is identified by the same trait values carried by the resident population in which the mutation has occurred, except for the trait affected by the mutation, which is slightly different. After each mutation, the mutant population is very scarce, but it has the potential to spread and replace the resident population. Since mutations are rare events, demography has plenty of time between successive mutations to define the resident population.

2.2. Resident-mutant model

We study the interaction of two similar densities $n_1 = n_1(t)$ and $n_2 = n_2(t)$ at a time t, and p additional densities or state variables packed in the vector:

$$\mathbf{N}(t) = (N_1(t), \dots, N_p(t)) \in \mathbb{R}^p.$$

These p states could correspond to other resident "groups" of same species as densities n_1 and n_2, or could describe environmental states corresponding to the problem's context, not necessarily from the same species. To be consistent, consider n_1, n_2 and every density in $\mathbf{N}$ of the same species, to differ from each other in the value of a characteristic trait x; in particular, taking the value x_1 in the density n_1, referred here as *resident trait* and *resident density*, respectively; similarly, x takes the value x_2 in the density n_2, referred as *mutant trait* and *mutant density* respectively. Now it is defined the simplest possible notation by sacrificing the detailed description of the community structure; i.e., no discrimination between species, characteristic traits, and morphs within each species, to avoid a cumbersome notation (see Table **2-1**).

The biotic environment is a whole community identified by the abundances n_1, n_2 and $\mathbf{N}$ of resident and mutant densities and by a set of parameters characterizing the interactions between agents of same or different states. Usualy it is assumed a constant

Table 2-1.: Notation used in the resident-mutant model.

Symbol	*Description*
x_1	Characteristic trait affected by mutation: resident value
x_2	Characteristic trait affected by mutation: mutant value
n_1	Abundance of the resident population
n_2	Abundance of the mutant population
$\mathbf{N}$	Vector of other resident populations abundance or environmental states.

environment to focus on evolutionary dynamics generated by mutations and interaction processes.

Denote by $\dot{n}_1/n_1 = g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)$ the per-capita growth rate of resident density in a time t, resulting form the balance of $b(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)$ and $d(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)$, the per-capita birth and death rates, respectively; i.e.,

$$g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1) = b(n_1, n_2, \mathbf{N}, x_1, x_2, x_1) - d(n_1, n_2, \mathbf{N}, x_1, x_2, x_1), \tag{2-1}$$

where the last argument x_1 in $g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)$, stands to indicate the g function for the density n_1. Been n_1 and n_2 interacting densities of the same species, it is assumed a symmetric per-capita growth rate for the mutant density n_2 in a time t; thus, the per-capita growth rate for the mutant density n_2 is obtained by exchanging the final argument in the definition of $g-$function to x_2, i.e., $\dot{n}_2/n_2 = g(n_1, n_2, \mathbf{N}, x_1, x_2, x_2)$ with $b(n_1, n_2, \mathbf{N}, x_1, x_2, x_2)$ and $d(n_1, n_2, \mathbf{N}, x_1, x_2, x_2)$ the corresponding per-capita birth and death rates. Notice either one of the two densities can be considered as mutant, provided that the other one is considered the resident. For the remaining states in $\mathbf{N}$, one can define their per-capita growth rate $\dot{N}_i/N_i = g_i(n_1, n_2, \mathbf{N}, x_1, x_2)$ where i covers the dimension of the vector $\mathbf{N}$, to avoid this notational inconvenience, we pack the growth rates in the vector function $\mathbf{G}(n_1, n_2, \mathbf{N}, x_1, x_2)$; thus, the dynamics of the community is described by the following $(p+2)-$dynamic system:

$$\begin{cases} \dot{n}_1 = n_1 g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1) \\ \dot{n}_2 = n_2 g(n_2, n_1, \mathbf{N}, x_2, x_1, x_2) \\ \dot{\mathbf{N}} = \mathbf{G}(n_1, n_2, \mathbf{N}, x_1, x_2). \end{cases} \tag{2-2}$$

Some important properties can be observed from this formulation:

P1: $g(n_1, 0, \mathbf{N}, x_1, x_2, x_1) = g(n_1, 0, \mathbf{N}, x_1, \cdot, x_1)$ and $\mathbf{G}(n_1, 0, \mathbf{N}, x_1, x_2) = \mathbf{G}(n_1, 0, \mathbf{N}, x_1, \cdot)$, i.e., if the second argument vanishes $(n_2 = 0)$, then g and $\mathbf{G}$ do not depend anymore on their fifth argument, since the growth rate of a density n_1 cannot be affected by the trait of a non existing density n_2.

P2: $g(n_1, n_2, \mathbf{N}, x_1, x_2, \tilde{x}) = g(n_2, n_1, \mathbf{N}, x_2, x_1, \tilde{x})$, for any trait value $\tilde{x}$, i.e. the order in which both densities are considered does not matter. By the same consideration, $\mathbf{G}(n_1, n_2, \mathbf{N}, x_1, x_2) = \mathbf{G}(n_2, n_1, \mathbf{N}, x_2, x_1)$.

2.2.1. Resident model

In the absence of mutation the resident-mutant model (2-2) degenerates into the so-called *resident model*, obtained when $n_2 = 0$ and consisting of the $(p+1)-$dynamic system:

$$\begin{cases} \dot{n}_1 = n_1 g(n_1, 0, \mathbf{N}, x_1, \cdot, x_1) \\ \dot{\mathbf{N}} = \mathbf{G}(n_1, 0, \mathbf{N}, x_1, \cdot). \end{cases} \tag{2-3}$$

It is assumed that in suitable ranges of the trait x_1, the model (2-3) has a stable and positive equilibrium where the resident density n_2 and the environment defined by $\mathbf{N}$ are constant. This steady state is given by,

$$(n_1, \mathbf{N}) = (\overline{n}_1(x_1), \overline{\mathbf{N}}(x_1)), \tag{2-4}$$

and satisfies the system of $p+1$ equations,

$$g(P, x_1) = g(\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot, x_1) = 0, \quad \mathbf{G}(P) = \mathbf{G}(\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot) = \mathbf{0}, \tag{2-5}$$

where $P = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot)$ and $(P, x_1) = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot, x_1)$. It is assumed that the equilibrium (2-4) is the only strictly positive attractor of model (2-3), i.e., it is the only attractor at which resident density can share the environment (this assumption can be relaxed), thus, the evolutionary dynamics can be defined only in the open set $\mathcal{X}$ of the trait space x_1 in which (2-4) exist. The set $\mathcal{X}$ is called the *evolution set*.

The equilibrium (2-4) is a point in the demographic state space $(n, \mathbf{N})$ that is stable and strictly positive for all $x_1 \in \mathcal{X}$, but it may present degeneracies that can be studied directly from the eigenvalues of the Jacobian matrix $\mathbf{J}_r(x_1)$ of (2-3), recalling $g(P, x_1) = 0$ by (2-5), then,

$$\mathbf{J}_r(x_1) = \begin{bmatrix} \overline{n}_1 g_{n_1}(P, x_1) & \overline{n}_1 g_{N_1}(P, x_1) & \cdots & \overline{n}_1 g_{N_p}(P, x_1) \\ F^1_{n_1}(P) & F^1_{N_1}(P) & \cdots & F^1_{N_p}(P) \\ \vdots & \vdots & & \vdots \\ F^p_{n_1}(P) & F^p_{N_1}(P) & \cdots & F^p_{N_p}(P) \end{bmatrix}, \tag{2-6}$$

where $P = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot)$ as defined in (2-5), and the subscript r stands to denote that the Jacobian matrix correspond to the "resident" model.

2.2.2. Invasion and substitution in the resident-mutant model

Invasion refers to the case when a mutant density spreads in the environment, while *substitution* occur when the mutant density replaces the resident density, leading to a step in the evolution of the trait affected by the mutation (from x_1 to x_2). Just after the mutation, the resident and mutant densities are at (or close to) the equilibrium of resident-mutant model (2-2),

$$(n_1, n_2, \mathbf{N}) = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1)). \tag{2-7}$$

Notice that the face $n_2 = 0$ (an invariant set of the resident-mutant model) of the demografic state space $(n_1, n_2, \mathbf{N})$ degenerates into the resident model (2-3), and the face $n_1 = 0$ (also an invariant set of the resident-mutant model) degenerates into the mutant model:

$$\begin{cases} \dot{n}_2 = n_1 g(0, n_2, \mathbf{N}, \cdot, x_2, x_2) \\ \dot{\mathbf{N}} = \mathbf{G}(0, n_2, \mathbf{N}, \cdot, x_2). \end{cases} \tag{2-8}$$

By the property P2 above, model (2-8) is the same model (2-3) with n_1 and x_1 replaced by n_2 and x_2. Thus, provided $x_2 \in \mathcal{X}$, the equilibrium $(n_2, \mathbf{N}) = (\overline{n}_2(x_2), \overline{\mathbf{N}}(x_2))$ is the only strictly positive attractor of model (2-8) and its associated eigenvalues are those of the Jacobian matrix $\mathbf{J}_r(x_1)$ from (2-6) replacing x_1 by x_2. Moreover, the point

$$(n_1, n_2, \mathbf{N}) = (0, \overline{n}_2(x_2), \overline{\mathbf{N}}(x_2)), \tag{2-9}$$

is another equilibrium of the resident-mutant model (2-2).

The stability of equilibria (2-7) can be studied through linearization of mutant-resident model (2-2). To write the Jacobian matrix $\mathbf{J}(x_1, x_2) = (J_{ij})$, for $i, j = 1, \dots, p+2$ is enough to take partial derivatives respect to n_1, N_k (for $k = 1, \dots, p$) and n_2 (in that suitable order), substitute $(n_1, n_2, \mathbf{N}) = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1))$, and recall P1 and (2-5) to have,

$$J_{1,1} = \tfrac{\partial}{\partial n_1}(n_1 g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)) = 1 \cdot g(P, x_1) + \overline{n}_1 \tfrac{\partial g}{\partial n_1}(P, x_1),$$

$$J_{1,2} = \tfrac{\partial}{\partial N_1}(n_1 g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)) = \overline{n}_1 \tfrac{\partial g}{\partial N_1}(P, x_1),$$

$$J_{1,p+1} = \tfrac{\partial}{\partial N_p}(n_1 g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)) = \overline{n}_1 \tfrac{\partial g}{\partial N_p}(P, x_1),$$

$$J_{1,p+2} = \tfrac{\partial}{\partial n_2}(n_1 g(n_1, n_2, \mathbf{N}, x_1, x_2, x_1)) = \overline{n}_1 \tfrac{\partial g}{\partial n_2}(P, x_1),$$

$$J_{2,1} = \frac{\partial}{\partial n_1} F^1(n_1, n_2, \mathbf{N}, x_1, x_2) = \frac{\partial F^1}{\partial n_1}(P),$$

$$J_{2,2} = \frac{\partial}{\partial N_1} F^1(n_1, n_2, \mathbf{N}, x_1, x_2) = \frac{\partial F^1}{\partial N_1} F(P),$$

$$J_{2,p+1} = \frac{\partial}{\partial N_p} F^1(n_1, n_2, \mathbf{N}, x_1, x_2) = \frac{\partial F^1}{\partial N_p} F(P),$$

$$J_{2,p+2} = \frac{\partial}{\partial n_2} F^1(n_1, n_2, \mathbf{N}, x_1, x_2) = \frac{\partial F^1}{\partial n_2}(P),$$

$$J_{3,1} = \frac{\partial}{\partial n_1}(n_2 g(n_2, n_1, \mathbf{N}, x_2, x_1, x_2)) = 0,$$

$$J_{3,2} = \frac{\partial}{\partial N_1}(n_2 g(n_2, n_1, \mathbf{N}, x_2, x_1, x_2)) = 0,$$

$$J_{3,p+1} = \frac{\partial}{\partial N_p}(n_2 g(n_2, n_1, \mathbf{N}, x_2, x_1, x_2)) = 0,$$

$$J_{3,p+2} = \frac{\partial}{\partial n_2}(n_2 g(n_2, n_1, \mathbf{N}, x_2, x_1, x_2) = 1 \cdot g(0, \overline{n}_1(x_1), \overline{\mathbf{N}}(x_1), x_2, x_1, x_2),$$

thus, the jacobian matrix of (2-2) takes the form,

$$J = \left[\begin{array}{cccc|c} \overline{n}_1 g_{n_1}(P, x_1) & \overline{n}_1 g_{N_1}(P, x_1) & \cdots & \overline{n}_1 g_{N_p}(P, x_1) & \overline{n}_1 g_{n_2}(P, x_1) \\ F^1_{n_1}(P) & F^1_{N_1}(P) & \cdots & F^1_{N_p}(P) & F^1_{n_2}(P) \\ \vdots & \vdots & & \vdots & \vdots \\ F^p_{n_1}(P) & F^p_{N_1}(P) & \cdots & F^p_{N_p}(P) & F^p_{n_2}(P) \\ \hline 0 & 0 & \cdots & 0 & g(0, \overline{n}_1(x_1), \overline{\mathbf{N}}(x_1), x_2, x_1, x_2) \end{array}\right],$$

where $P = (\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, \cdot)$ as defined before. This matrix can be written as,

$$\mathbf{J}(x_1, x_2) = \left[\begin{array}{c|c} \mathbf{J}_r(x_1) & \cdots \\ \hline 0 & g(0, \overline{n}_1(x_1), \overline{\mathbf{N}}(x_1), x_2, x_1, x_2) \end{array}\right] \tag{2-10}$$

Due to the block structure of (2-10), the eigenvalues are given by those of $\mathbf{J}_r(x_1)$ which have negative real part by the stability of equilibria (2-7), and the eigenvalue,

$$\lambda(x_1, x_2) = g(0, \overline{n}_1(x_1), \overline{\mathbf{N}}(x_1), x_2, x_1, x_2), \tag{2-11}$$

which correspond to the growth rate of a very scarce mutant density, and is called the *invasion eigenvalue*, *invasion fitness* or *fitness function* of the mutant density, and gives the initial exponential growth rate of the mutant density appeared in the environment dominated by the resident density.

Similarly, the eigenvalue $\lambda(x_2, x_1)$ correspond to the equilibrium (2-9); since $\mathbf{J}_r(x_1)$ and $\mathbf{J}_r(x_2)$ have eigenvalues with negative real part, hence, the local stability of equilibria (2-7) and (2-9) are related only to the sign of their invasion eigenvalues. The invasion eigenvalues $\lambda(x_2, x_1)$ and $\lambda(x_1, x_2)$, generically have opposite sign, so that if equilibrium (2-7) is stable, then equilibrium (2-9) is unstable, and viceversa.

To derive conditions over the sign of $\lambda(x_1, x_2)$, first consider the case when the resident and mutant densities are identical ($x_1 = x_2$), then, by virtue of P2 and (2-5),

$$\lambda(x_1, x_1) = g(0, \overline{n}_1(x_1), \overline{\mathbf{N}}(x_1), x_1, x_1, x_1) = g(\overline{n}_1(x_1), 0, \overline{\mathbf{N}}(x_1), x_1, x_1, x_1) = 0. \quad \text{(2-12)}$$

On the other hand, the assumption $x_2 = x_1$ (identical resident and mutant traits), implies that there is actually one density $n_1 + n_2$, therefore,

$$(\dot{n}_1 + \dot{n}_2) = (n_1 + n_2) g(n_1 + n_2, 0, \mathbf{N}, x_1, x_1, x_1),$$

which implies that the two densities are characterized by the same growth rate

$$g(n_1, n_2, \mathbf{N}, x_1, x_1, x_1) = g(n_2, n_1, \mathbf{N}, x_1, x_1, x_1),$$

or equivalently, for any $0 \leq \phi \leq 1$,

$$g(n_1, n_2, \mathbf{N}, x_1, x_1, x_1) = g\left((1-\phi)(n_1 + n_2), \phi(n_1 + n_2), \mathbf{N}, x_1, x_1, x_1\right). \quad \text{(2-13)}$$

The last statement can be generalized for any trait value x as in the following property:

P3: For any $0 \leq \phi \leq 1$ and any $x \in \mathcal{X}$,

$$g(n_1, n_2, \mathbf{N}, x, x, \tilde{x}) = g\left((1-\phi)(n_1 + n_2), \phi(n_1 + n_2), \mathbf{N}, x, x, \tilde{x}\right),$$

i.e., any partition of the total density $n_1 + n_2$ into two categories with the same strategy x must result in the same growth rate for strategy $\tilde{x}$. Analogously, for $\mathbf{G}$,

$$\mathbf{G}(n_1, n_2, \mathbf{N}, x, x) = \mathbf{G}\left((1-\phi)(n_1 + n_2), \phi(n_1 + n_2), \mathbf{N}, x, x\right).$$

Moreover, by P1, P3 and (2-5), all the points of the segment,

$$(n_1, n_2, \mathbf{N}) = \left((1-\phi)\overline{n}_1(x_1), \phi\overline{n}_1(x_1), (1-\phi)\overline{\mathbf{N}}(x_1) + \phi\overline{\mathbf{N}}(x_1)\right),$$

connecting (2-7) with (2-9), are equilibria of the resident-mutant model (2-2) when $x_1 = x_2$. Those equilibria are neutrally stable (they are not unstable since $(n_1 + n_2, \mathbf{N})$ converges to the equilibrium (2-4) of the resident model (2-3), but do not attract all nearby trajectories), and hence have a vanishing eigenvalue, $\lambda(x_1, x_1) = 0$, and those from $\mathbf{J}_r(x_1)$.

Second, notice that the function $\lambda(x_1, x_2)$ has opposite sign for $x_2 > x_1$ or $x_2 < x_1$, with x_2 close to x_1. In deed, excluding nongeneric cases we have $\frac{\partial}{\partial x_2}\lambda(x_1, x_2) \neq 0$, and expanding in Taylor series around $x_2 = x_1$, one gets,

$$\lambda(x_1, x_2) = \lambda(x_1, x_1) + (x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1) + \mathcal{O}(|x_2 - x_1|^2).$$

Since $\lambda(x_1, x_1) = 0$ by virtue of (2-12), thus if $(x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1)$ is positive, i.e, when:

$$\frac{\partial \lambda}{\partial x_2}(x_1, x_1) > 0, \quad x_2 > x_1,$$

or,

$$\frac{\partial \lambda}{\partial x_2}(x_1, x_1) < 0, \quad x_2 < x_1,$$

the invasion eigenvalue $\lambda(x_1, x_2)$ is positive and the equilibrium (2-7) is unstable, while the equilibrium (2-9) is stable (mutant density invades), and vice versa if:

$$(x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1),$$

is negative (mutant density goes extinct). From now on the quantity:

$$\frac{\partial \lambda}{\partial x_2}(x_1, x_1), \tag{2-14}$$

is going to be called *selection derivative*.

The question of whether invasion implies substitution of the resident density, requieres the study of the global behavior of the resident-mutant model (2-2). In Appendix B of [31], it is proved the next theorem.

Theorem 2.2.1 (Invasion implies substitution) *Given x_1 in the evolution set $\mathcal{X}$, if $(x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1) > 0$ and $|x_2 - x_1|$ and $||(n_1(0) + n_2(0) - \overline{n}(x_1),\ \mathbf{N}(0), \overline{\mathbf{N}}(x_1))||$ are sufficiently small, then the trajectory $(n_1(t), n_2(t), \mathbf{N}(t))$ of the resident-mutant model (2-2) tends toward equilibrium (2-9) for $t \to \infty$.*

2.3. Adaptive dynamics canonical equation

Consider x and $\tilde{x}$ two similar values for the qualitative trait, an X a set of other non mutating traits characterizing the community. From the previous section it follows that if the *selection derivative*:

$$\frac{\partial \lambda}{\partial \tilde{x}}(x, x, X), \tag{2-15}$$

is positive, then the mutant population characterized by $\tilde{x} > x$ replace the resident population, while mutations characterized by $\tilde{x} < x$ leave no trace in the community; a similar analysis can be made in the case where (2-15) is negative. Consider x as a collection of traits x_i for $i = 1, 2, \ldots$, then the selection derivative relative to each trait is given by

$$\frac{\partial}{\partial \tilde{x}_i} \lambda(x_1, x_2, \ldots \tilde{x}_i)\bigg|_{\tilde{x}_i = x_i}. \tag{2-16}$$

The evolutionary change is a stochastic process determined by both the process of mutation and demographic stochasticity; i.e., even advantageous mutations may fail to invade due to accidental extinction. Thus the rate of evolutionary change $\dot{x}$ can not be described by a deterministic model, and can only be interpreted as the average evolutionary change among all possible realizations of the mutation process and demographic stochasticity, i.e.,

$$\dot{x} = \lim_{dt \to 0} \frac{E[x(t + dt) - x(t)]}{dt}, \tag{2-17}$$

where $E[\cdot]$ is the standard expected value operator and t spans the evolutionary timescale. Denoting $P(x, \tilde{x}, X, dt)d\tilde{x}$ the probability distribution that a community traits $(x, X) \in \mathcal{X}$ at a time t will be characterized by traits $(\tilde{x}, X)$ and $(\tilde{x} + d\tilde{x}, X)$ at time $t + dt$, then (2-17) becomes,

$$\dot{x} = \lim_{dt \to 0} \frac{1}{dt} \int_{-\infty}^{\infty} (\tilde{x} - x) P(x, \tilde{x}, X, dt) d\tilde{x}, \tag{2-18}$$

where $P(x, \tilde{x}, X, dt)$ is defined as the product of three different probabilities (Dercole F. and Rinaldi S. in [31] computes separately each probability):

- P_m, the probability that a mutation occurs in the time interval $[t + dt]$, given by,

  $$P_m(x, X, dt) = \mu(x) b(\overline{n}(x, X), 0, \overline{N}(x, X), x, \cdot, X)\overline{n}(x, X)dt + \mathcal{O}(dt^2), \tag{2-19}$$

  where $\mu(x)$ is the frequency of mutations in the population, $b(\overline{n}, 0, \overline{N}, x, \cdot, X)dt$ is the probability of a single birth and $\mathcal{O}(dt^2)$ is the probability of more than one mutation (but gives no contribution to the limit (2-18)).

- P', the probability that a mutant trait is between $\tilde{x}$ and $\tilde{x} + d\tilde{x}$, given by,

  $$P'(x, \tilde{x} - x) = \frac{1}{\epsilon} D\left(x, \frac{x - \tilde{x}}{\epsilon}\right), \tag{2-20}$$

D denotes a suitable probability distribution for the mutational step $(\tilde{x}-x)$ and variance given by $E[(\tilde{x}-x)^2] = \epsilon^2\sigma^2(x)$. Here ϵ is a timescaling factor used to separate the demographic and evolutionary timescales by considering $\epsilon \to 0$, thus a small amount dt of evolutionary time correspond to a large amount dt/ϵ of demographic time.

- P_s, the probability that the mutant substitutes the resident,

$$P_s(x,\tilde{x},X) = \begin{cases} \dfrac{\lambda(x,\tilde{x},X)}{\lambda_b(x,\tilde{x},X)}, & \text{if } \dfrac{\partial}{\partial \tilde{x}}\lambda(x,\tilde{x},X)\Big|_{\tilde{x}=x}(\tilde{x}-x) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2-21}$$

where λ is the fitness eigenvalue defined in the previous section, which splits into

$$\lambda(x,\tilde{x},X) = \lambda_b(x,\tilde{x},X) - \lambda_d(x,\tilde{x},X).$$

Under this results, is posible to get $P = P_m P' P_s$ as,

$$P(x,\tilde{x},X,dt) = \mu(x)\lambda_b(x,\tilde{x},X)\overline{n}(x,X)P'(x,\tilde{x}-x)P_s(x,\tilde{x},X)d\tilde{x}dt, \tag{2-22}$$

and substituting (2-22) into the limit (2-18), one gets,

$$\dot{x} = \mu(x)\lambda_b(x,\tilde{x},X)\overline{n}(x,X)\int_{-\infty}^{\infty}(\tilde{x}-x)P'(x,\tilde{x}-x)P_s(x,\tilde{x},X)d\tilde{x}. \tag{2-23}$$

If the selection derivative (2-15) is positive then, (2-23) correspond to,

$$\dot{x} = \mu(x)\lambda_b(x,\tilde{x},X)\overline{n}(x,X)\int_{x}^{\infty}(\tilde{x}-x)P'(x,\tilde{x}-x)\frac{\lambda(x,\tilde{x},X)}{\lambda_b(x,\tilde{x},X)}d\tilde{x}. \tag{2-24}$$

Now, considering that $\lambda(x,x,X) = 0$, by the first order expansion,

$$\frac{\lambda(x,\tilde{x},X)}{\lambda_b(x,\tilde{x},X)} = \frac{1}{\lambda_b(x,\tilde{x},X)}\frac{\partial}{\partial \tilde{x}}\lambda(x,\tilde{x},X)\Big|_{\tilde{x}=x}(\tilde{x}-x) + \mathcal{O}(|\tilde{x}-x|^2)$$

which is justified in the limit $\epsilon \to 0$ where $(\tilde{x}-x)$ becomes infinitesimal, one gets,

$$\dot{x} = \mu(x)\overline{n}(x,X)\frac{\partial}{\partial \tilde{x}}\lambda(x,\tilde{x},X)\Big|_{\tilde{x}=x}\int_{x}^{\infty}(\tilde{x}-x)^2P'(x,\tilde{x}-x)d\tilde{x} \tag{2-25}$$

Due to the symmetry of P', the integral is nothing but half of the variance $\epsilon^2\sigma^2(x)$ of the probabiliy distribution P', Then:

$$\dot{x} = \frac{1}{2}\mu(x)\sigma^2(x)\overline{n}(x, X) \left.\frac{\partial}{\partial\tilde{x}}\lambda(x, \tilde{x}, X)\right|_{\tilde{x}=x} \tag{2-26}$$

The same result can be obtained from (2-23), if the selection derivative is negative. Equation (2-26) is called the *Adaptive Dynamics Canonical Equation (ADCE)*, and can be seen that the rate of change $\dot{x}$ is influenced by three factors, as summarized by F. Dercole and S. Rinaldi in [32]:

- How often a mutation occurs in population; in fact, $\mu\overline{n}$ is proportional to the number of mutations occurring in population per unit of evolutionary time.
- How large is the trait change entailed by the mutation, zero mean and variance $\epsilon^2\sigma^2$, ϵ being a scaling factor separating the demographic and evolutionary timescales in the limit $\epsilon \to 0$;
- How likely it is that the initially scarce mutant population invades and replaces the corresponding resident population. The probability of invasion consists of two factors:
  - First, if the selection derivative (2-15) does not vanish, only mutations with trait value either larger or smaller than the resident value can invade.
  - Second, mutations at selective advantage may be accidentally lost in the initial phase of invasion when they are present only in a few items. The probability of success in the initial phase of invasion is shown to be proportional to the selective advantage of the mutation, as measured by the selection derivative. Finally, successful invasion generically implies substitution.

2.3.1. Evolutionary equilibria

To understand the evolutionary dynamics complexity of the AD Canonical Equation in (2-26), recall this is a nonlinear continuous-time dynamical system defined in the evolutionary set $\mathcal{X}$, whose attractors (evolutionary attractors) can be of several types

- Evolutionary equilibria
- Evolutionary cycles.
- Strange attractors.

Evolutionary trajectories can therefore tend toward a point in the trait space or toward a periodic or aperiodic solution. However, evolutionary dynamics can also be characterized by unstable evolutionary equilibria, cycles or strange invariant sets, thus, long-term evolution can depend on the evolutionary paths followed in the past. In 2- or 3-dimensional trait space, graphical representations of the state portrait are useful in the dynamics understanding.

Evolutionary equilibria of (2-26) are points $(\overline{x}, \overline{X}) \in \mathcal{X}$ where the selection derivatives (2-16) vanish (recall $\mu(x)$, $\sigma(x)$ and $\overline{n}(x, X)$ are positive for all $(x, X) \in \mathcal{X}$),

$$\frac{\partial \lambda}{\partial \tilde{x}}(\overline{x}, \overline{x}, \overline{X}) = 0, \tag{2-27}$$

but in such case, the invasion implies substitution theorem 2.2.1 does not hold, and a deeper analysis of the resident-mutant model (2-2) is needed.

In a community characterized by a single trait, the stability of evolutionary equilibria can be studied through linearization of (2-26), the eigenvalue associated to $\overline{x} \in \mathcal{X}$ is

$$\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial \lambda}{\partial \tilde{x}}(x, x)\right)\bigg|_{x=\overline{x}} = \kappa(\overline{x})\left(\frac{\partial^2 \lambda}{\partial x \partial \tilde{x}}(\overline{x}, \overline{x}) + \frac{\partial^2 \lambda}{\partial \tilde{x}^2}(\overline{x}, \overline{x})\right)$$

where $\kappa(\cdot) = \frac{1}{2}\mu(\cdot)\sigma^2(\cdot)\overline{n}(\cdot)$. Therefore the evolutionary equilibria $\overline{x}$ is an attractor for the evolutionary dynamics if and only if

$$\frac{\partial^2 \lambda}{\partial x \partial \tilde{x}}(\overline{x}, \overline{x}) + \frac{\partial^2 \lambda}{\partial \tilde{x}^2}(\overline{x}, \overline{x}) < 0$$

or the evolutionary equilibria $\overline{x}$ is unstable if that quantity is greater than zero.

In a community characterized by more than one trait, the values of $\mu_i(\overline{x}_i)$, $\sigma_i(\overline{x}_i)$ and $\overline{n}_i(\overline{x}_1, \overline{x}_2, \ldots)$ depend on $i = 1, 2, \ldots$, and, thus affect the eigenvalues associated to the evolutionary equilibrium, then the functions λ_i do not determine anymore the stability of evolutionary equilibria.

Those points on the boundary of $\mathcal{X}$ are called **Boundary equilibria**. There, only the selection derivatives relative to nonvanishing resident populations annihilate. In those points different situations may occur:

- **Evolutionary murder**: the equilibrium abundance of one of the resident populations gradually vanishes along an evolutionary trajectory approaching the boundary of $\mathcal{X}$, when the boundary is reached (in a finite time), the vanishing population goes extinct and the other resident populations play the role of murderers.

- **Evolutionary runaway**: in the previous case, if there were no murderers, the extinction will be obtained asymptotically.

- **Evolutionary suicide**: the equilibrium abundances of the resident populations do not vanishes when the evolutionary trajectory approaches the boundary of $\mathcal{X}$, so that, the boundary is reached in finite time even in the absence of murderers.

In conclusion, reaching the boundary of the evolution set $\mathcal{X}$, implies the evolutionary extinction of one or more resident populations.

2.3.2. Evolutionary Branching

Evolutionary branching occurs when selection splits a phenotypically monomorphic population into two distinct phenotypic clusters. A prerequisite for evolutionary branching is that directional selection drives the population toward a fitness minimum in phenotype space. Therefore evolutionary branching offers a general basis for understanding adaptive speciation [36]. Evolutionary branching is a feature hard to account for in usual mutation-selection systems and is one of the privileged subjects of adaptive dynamics framework [44].

The invasion implies substitution Theorem 2.2.1 does not hold when selection derivative vanishes as in (2-27), then, an important question that arises is whether these points actually are evolutionary attractors. In the context of classical models of evolution, reaching such attractors implies that evolution comes to a halt (evolutionary attractors only occur at fitness maxima). However, Geritz et al. 1998 and Meszena et al. 2001, showed this is not necessarily true in the framework of adaptive dynamics; because frequency-dependent ecological interactions drive the evolutionary process, and then it is possible that an evolutionary attractor represents a fitness minimum at which the population experiences disruptive selection. Adaptive dynamics demonstrates that once the fitness minimum is reached, the population may split into two distinct and diverging in trait values. Thus, in adaptive dynamics, evolutionary convergence toward a fitness minimum can lead to evolutionary branching [36, 45, 63]. In [31], Dercole F. and Rinaldi S. describe in detail the results on the trait dynamics close to an evolutionary equilibria $\overline{x}$ (where the selection derivative vanishes), to show that,

- A mutant population can **coexist** with a resident population, giving rise to an extra population and an extra equation in the AD canonical equation. i.e. resident-mutant coexistence is possible for $X = \overline{X}$, x close to $\tilde{x}$ and if,

$$\frac{\partial^2 \lambda}{\partial x \partial \tilde{x}}(\overline{x}, \overline{x}, \overline{X}) < 0. \tag{2-28}$$

- Two coexisting resident and mutant populations diverge in trait values, and there is the so called **evolutionary branching**, if,

$$\frac{\partial^2}{\partial \tilde{x}^2}\lambda(\overline{x}, \overline{x}, \overline{X}) > 0. \tag{2-29}$$

Evolutionary equilibria can be partitioned in three types: **Branching Points**: a stable evolutionary equilibria at which at least one of the traits can branch, i. e., at which both branching conditions (2-28) and (2-29) hold with respect to at least one trait. **Terminal Points**: a stable evolutionary equilibria, where coexistence and divergence condition holds, i.e., non branching points where evolution has a halt, and finally, **Boundary Branching Points**: correspond to border points between the branching and terminal points. At all effects, this points correspond to bifurcation points in the AD canonical equation.

2.4. Previous modeling

A new class of models to incorporate Darwinian evolution of a quantitative trait have been presented by Hofbauer and Sigmund (1990), Marrow et al. (1992) and Metz et al. (1992). Later, in 1996, one important advance is due to Diekmann and Law, who derives an ordinary differential equation to describe the rate of change over time of the expected trait value in a monomorphic population, the so called *canonical equation of adaptive dynamics* [10].

Particularly, Hofbauer and Sigmund proposed a dynamics to model the effect of adaptation and relate it with the stability of equilibria; also, they discuss some examples concerning, in particular, iterated interactions and gamete sizes [49].

In 1992, Marrow et al. presented a model for the coevolution of body size of predators and their prey. Body sizes were assumed to affect the interactions between individuals, and the Lotka-Volterra population dynamics arising from these interactions provide the driving force for evolutionary change. The results point to a "loser wins" principle, in which the evolution leads to a weakening of the interaction between predator and prey [61].

Also in 1992, Metz et al. developed an study on the definition of "fitness" in ecological scenarios; their fundamental message is that the best fitness measure in a variable nonlinear world is one based on dominant Lyapunov exponentes; their study, also pointed out several issues like environmental variability and the effect of locally finite populations as important matters that should be considered by theoreticians in further investigations [65].

The origin of the canonical equation of adaptive dynamics is due to the paper of Diekmann and Law, back in 1996, they developed a dynamical theory of coevolution in ecological communities. The derivation explicitly accounts for the stochastic components of evolutionary change and is based on ecological processes at the level of the individual and discuss extensions of the derivation to more general ecological settings, in particular they allow for multi-trait coevolution and analyze coevolution under nonequilibrium population dynamics [34].

In 1996, Atamas studied the hypothesis that degenerate recognition with subsequent selection of recognizing elements can explain self-organization of these systems. An entirely numerical model was explored, using the cellular automata approach. Three intrinsic features of a common selective system were incorporated into this model: a large number of recognizing elements; degenerative recognition of stimuli by these elements; and subsequent selection. He conclude that systems with a large number of recognizing elements, degenerative recognition, and selection of recognizing elements can self organize based upon the pattern of the incoming stimuli. In his paper, the branching arises due to a competition between similar abstract "recognizers" to recognize some signals. In this framework, a monomorphic population may split into two distinct sub-populations in order to decrease this competition [6, 44].

Geritz in 1998, present a general framework for modeling adaptive trait dynamics in which they give a full classification of the singular strategies in terms of ESS-stability, convergence stability, the ability of the singular strategy to invade other populations if initially rare itself, and the possibility of protected dimorphisms occurring within the singular strategy's neighborhood. Of particular interest is a type of singular strategy that is an evolutionary attractor from a large distance, but once in its neighborhood a population becomes dimorphic and undergoes disruptive selection leading to evolutionary branching [45].

Genieys et al. in 2009 investigated an ingro-differental equation to study its pattern formation and branching capacity. The equation is related to the model presented in 1996 by Atamas and with Turing reaction-diffusion models for morphogenesis, except for the origin of structures, which in the model of Turing came from the competition of an activator and an inhibitor, whereas in the work of Genieys came from the competition inside of a single population [44].

In [36], Doebeli M. and Dieckmann U. use classical ecological models for symmetric and asymmetric competition, for mutualism, and for predator-prey interactions to describe evolving populations with continuously varying characters. For these models, they investigate the ecological and evolutionary conditions that allow for evolutionary branching and establish that branching is a generic and robust phenomenon. Also study the evolution of assortative mating as a quantitative character. They show that evolution under branching conditions selects for assortativeness and thus allows sexual populations to escape from fitness minima. They conclude that evolutionary branching offers a general basis for understanding adaptive speciation and radiation under a wide range of different ecological conditions.

In 2008 is published the book "Analysis of Evolutionary Processes" by Dercole F. and Rinaldi S. where the authors make a wide and deep description of the Adaptive

Dynamics framework from the theoretical and applicative point of view. They present the proofs of many of the results and present several applications to illustrate the most important issues in the theory. This book is of mandatory lecture for the interested reader [31].

In 2010, F. Dercole and S. Rinaldi, published a paper devoted to the presentation of the first chaotic evolutionary attractor, obtained through the AD approach. They considered a Lotka–Volterra tritrophic food chain composed of a resource, its consumer, and a predator species, each characterized by a single adaptive phenotypic trait, and showed that for suitable modeling and parameter choices the evolutionary trajectories approach a strange attractor in the three-dimensional trait space. The study is performed through the bifurcation analysis of the canonical equation of Adaptive Dynamics. In their conclusions they consider, first, that the study of evolving systems with more complex structures would reveal more complex chaotic regimes; second, briefly discuss about the possibility of identifying other chaotic evolutionary attractors through mathematical models, and about the possibility to dig into field and laboratory evolutionary time series and detect the footprint of deterministic chaos, as two major questions that arises from their study. Finally, they consider that chaotic Red Queen implies that evolutionary trajectories may not be predicted beyond a short evolutionary time, despite the forces of natural selection are strong and deterministic, which has important implications for questions of interest that include pathogen unpredictable evolution, the maintenance of genetic diversity in homogenized landscapes, and the process of speciation [32].

Since the first paper developing AD a wide range of applications and theoretical developments have been published; in particular, technological mutations: [23,29], social interactions: [58], mutualistic interactions: [19,34,36,40], competition: [24,36,53], predator-prey dynamics [1, 2, 17, 27, 32, 34, 59, 61], evolution of dispersal: [13, 28, 73] dynamics in allele space: [12, 55], canibalistic interactions: [18, 30], and food chains: [32] among many others. In addition, detailed mathematical developments of the theory can be found in documents such as: [4, 10, 17, 20, 31, 34, 44–46].

3. Conditions on the energy market diversification from adaptive dynamics

3.1. Introduction

The energy market is a complex system in a rapidly varying context in which decision-making is difficult. Its complexity is due to a large number of physical and economic factors involved. In particular, physical factors may be related to climatic conditions and have an unpredictable medium- and long-term behavior, as well as an unknown effect on aspects of the market such as supply, demand, and price. Market regulations and public policies generate causal relationships between all these elements producing highly complex interactions. Other factors associated with technological and social changes, such as innovations in energy generation or changes in consumption patterns, which are not predictable in the medium or long term, are also determinants [86].

In recent years there has been a significant development of alternative energy generation technologies, as reported in [9], who find that the EU ETS has increased low-carbon innovation among regulated firms by as much as 10 %, while not crowding out patenting for other technologies. They also find evidence that the EU ETS has not affected licensing beyond the set of regulated companies. These results imply that the ETS accounts for nearly a 1 % increase in European low-carbon patenting compared to a counterfactual scenario. In this context, it is necessary to study the energy market, and in particular, the dynamics that arise after the introduction of innovative technologies, using mathematical tools that help to describe the inherent complexity of the system. In the study of energy markets, it is necessary to take into account some intrinsic characteristics or attributes, such as generation source, emission reduction, final consumer price, generation technologies, generation capacity, level of investment, among many others. Also, it is essential to describe how its dynamics in the long-term influences the conditions of interaction between agents established in the market and those who consider themselves innovative. In [50] they define environmental innovations as a product, process, marketing and organizational changes leading to a noticeable reduction of environmental burdens. Positive ecological effects can be explicit goals or side-effects of innovations.

The Adaptive Dynamics constitute a theoretical background originating in evolutionary biology that link demographic dynamics to evolutionary changes and allows to describe evolutionary dynamics in the long-term when considering mutations as small and rare events in the demographic time scale [31, 34, 45, 46]. Adaptive dynamics describes evolution through an ordinary differential equation known as the canonical equation of the adaptive dynamics. This approach focuses on the long-term evolutionary dynamics of continuous (quantitative) adaptive traits and overlooks genetic detail through the use of asexual demographic models, which is justified under different demographic and evolutionary timescales. This approach considers interactions to be the evolutionary driving force and takes into account the feedback between evolutionary change and the selection forces that agents undergo [31, 32, 36, 41].

In the present chapter a mathematical model based on ordinary differential equations is studied, to describe the dynamics of a market dominated by a standard energy (SE) generation technology in interaction with an innovative energy (IE) generation technology. Initially, the model consists of an adaptation of the Lotka-Volterra equations under the consideration that interaction between both types of energy can occur in a market based on competition or cooperation as interaction strategies, as described by [37] in a cross-country study on the relationship between diffusion of wind and photovoltaic solar technology. In both cases, SE and IE are measured with the cumulative generation capacity (CGC) as a non-negative real number defining its level of penetration into the market. Under those scenarios, we determine conditions for EI to invade and establish into the market, giving rise to diversification. The model parameters are defined as functional coefficients depending on the values of a characteristic quantitative and continuous trait to determine some relevant aspects of energy generation. In general, adaptive dynamics theory allows us to study the long-term evolutionary dynamics of the quantitative attributes that characterize both energies CGCs and to describe how they affect the interaction dynamics in the short-term (market timescale). On the other hand, it also allows us to establish how the conditions of interaction in the market influence the evolutionary dynamics of the attributes, and ultimately, to determine which innovative characteristics can invade or which attributes disappear definitively.

Using the theoretical framework of adaptive dynamics, the canonical equation, corresponding to an ordinary differential equation, is presented to describe the behavior over time of the characteristic attribute as a result of innovation processes. The theoretical framework of adaptive dynamics has been used recently to model a varied spectrum of situations involving innovations or genetic variations; in particular, technological innovations, as in [23], where the authors explore the emergence of technological variety arising from market interaction and technological innovation. Particularly, existing products compete with the innovative ones resulting in a slow and continuous evolution of the underlying technological characteristics of successful products. Also in that context, in [29] is studied technological change and its impact on sustainable fisheries. The analysis is performed by means of Adaptive Dynamics and the results are qualitatively consistent with those obtained long ago through the principles of bioeconomics, it is fair to stress that the underlying assumptions are different. In fact, in the bioeconomic approach fleet technology does not evolve and fishing effort varies to produce economic optimization, while in the Adaptive Dynamics approach technological innovation is the key driver. A very interesting application of adaptive dynamics approach, came from social interactions, in [58], the authors propose a model to investigate the dynamics of fashion traits purely driven by social interactions. They assume that people adapt their style to maximize social success, and we describe the interaction as a repeated group game in which the payoffs reflect the social norms dictated by fashion.

Many other examples of the adaptive dynamics framework applied to biology can be found. Mutualistic interactions are studied in [40] where the authors show that asymmetrical competition within species for the commodities offered by mutualistic partners provides a simple and testable ecological mechanism that can account for the long-term persistence of mutualism. In the competition context, [44] present a model devoted to the study of an evolutionary system where similar individuals are competing for the same resources. Examples can be found in predator-prey dynamics, evolution of dispersal, dynamics in allele space, canibalistic interactions etc. In addition, detailed mathematical developments of the theory can be found, particularly in [31] a thorough review of theoretical aspects and applications is made.

In the second section of this chapter, the reader will find a description of the adaptation made to the Lotka-Volterra model to describe the interaction between two similar types of energy. Local stability is described and invasion conditions determined. In the third section, an explicit definition of the coefficients of the model according to the standard and innovative attributes is stated, to consider some particular aspects of the market and later to determine how they influence the conditions of invasion of the innovative energy. The canonical equation is described and, from this point, the long-term evolutionary dynamics of the characteristic attributes follows. In particular, there are conditions under which there is evolutionary branching that allow market diversification. We illustrate the situation with numerical simulations. Finally the conclusions and the references are shown.

3.2. Model description

3.2.1. Innovative-Standard model

Some technical assumptions on the model are the following: (a) we consider two types of energy generation, which are differentiated by the technology used (we refer to them in this paper as *energy generation technology*). (b) Each energy generation technology is characterized by the value of a given characteristic attribute quantifiable by means of a real number, i.e. a measure of the technology. It can be assumed that a higher attribute value is related to more advanced technologies, although innovations are not necessarily preferred by consumers. (c) In the absence of innovations, the generation of established energy reaches a specific equilibrium value on a time scale that we call the "market time scale". (d) Innovations are rare events on the market time scale, i.e., they occur on a much longer time scale that we call "evolutionary time scale." This separation of the scales allows us to assume that the market is in equilibrium when an innovation occurs and that the market is affected by a single innovation at the same time [31]. (e) Finally, it is assumed that the innovations are small, that is, the innovative attribute only differs somewhat from the established quality; this corresponds to consider marginal innovations that give origin to energies similar to those established.

Consider an energy market dominated by a *Standard Energy* generation technology (SE), with *cumulative generation capacity* (CGC) $n_1 = n_1(t)$ at any time t, and assume there is some standard characteristic trait x_1 to determine a suitable feature of SE generation. It can be, for example, the final price of energy to the consumer or other characteristics such as energy saving, emission reduction, or generation capacity or level of investment. Suppose a marginal innovation occur in this characteristic trait, slightly changing the value x_1 to x_2 and leading to the appearance of an *Innovative Energy* generation technology (IE), with CGC $n_2 = n_2(t)$, different from n_1, and characterized by the trait x_2, called innovative characteristic trait from now on.

Generation growth rate. Consider the CGC n of a given generation technology to grow at rate $r(x)$, as a function of the characteristic trait x. This function describes how fast n increases depending on the value of x. Growing rate r should be considered as a positive function $r(x) > 0$ for all $x \in \mathbb{R}$.

Maximum capacity. Let the function $K(x)$ to describe the maximum cumulative generation capacity that some generation technology can reach and allocate into the market, as a function of its characteristic trait x. As generation and demand grow, it is realistic to consider K as a nonnegative function of x, bounded above by some maximum value corresponding to technical limitations or imposed normative obeying public police.

Interaction coefficient. Define the function $c(x_i, x_j)$ to determine the interaction into the energy market between the i generation technology with CGC n_i and the j generation technology with CGC n_j. It correspond to the rate of increase/decrease of CGC suffered by n_i by the presence of n_j; we assume $c(x_i, x_i) = 1$ to indicate internal competition; i.e., $c(x_1, x_1) = 1$ correspond to internal competition between SE generation technologies, and similarly, $c(x_2, x_2) = 1$ correspond to internal competition between IE generation technologies.

Additional general situations can occur depending on the region of the $(x_1, x_2)-$plane where the point (x_1, x_2) is located:

- If $c(x_i, x_j) > 1$, for $x_i \neq x_j$, external competition predominates over internal competition; that is, $c(x_i, x_j) > 1$ implies that the competition between generation technology i and generation technology j is stronger than competition between systems generating the same type of energy.

- If $0 \leq c(x_i, x_j) \leq 1$, for $x_i \neq x_j$, then internal competition predominates over external competition. It has to be stronger competition between different SE generation technologies among each other than the competition between SE and IE generation technologies. In particular, if $c(x_i, x_j) = 0$, there is no interaction at all and if $c(x_i, x_j) = 1$, both competitions are equally strong.

- If $c(x_i, x_j) < 0$, for $x_i \neq x_j$, the interaction between generation technologies i and j does not correspond to competition but to cooperation, a situation that can describe the integration of systems. In this case, each one is rewarded by the presence of the other.

In general, it is assumed x_2 close to x_1, i.e., the innovation is small and it has a small effect. So doing, such an innovation always compete with the established one and, only after diversification, the market could turn cooperative. Additionally, there might be mixed cases. For instance, when $c(x_1, x_2) > 1$ and $c(x_2, x_1) < 1$ for $x_2 > x_1$, the low-tech energy generation suffers the high-tech more than itself, and conversely, when $c(x_1, x_2) < 1$ and $c(x_2, x_1) > 1$ for $x_2 > x_1$ the high-tech energy generation suffers the low-tech more than itself.

Under the assumptions described, we propose an interaction Lotka-Volterra model:

$$\begin{cases} \dot{n}_1 = n_1 r(x_1) \left(1 - \dfrac{n_1 + c(x_1, x_2) n_2}{K(x_1)}\right) = n_1 g(n_1, n_2, x_1, x_2, x_1) \\ \dot{n}_2 = n_2 r(x_2) \left(1 - \dfrac{n_2 + c(x_2, x_1) n_1}{K(x_2)}\right) = n_2 g(n_1, n_2, x_1, x_2, x_2), \end{cases} \tag{3-1}$$

defined on the set $\Omega = \{(n_1, n_2) : n_1 \geq 0, n_2 \geq 0\}$. Note that both *relative* growth rates $\dot{n}_1/n_1$ and $\dot{n}_2/n_2$ can be expressed by means of a single function g that in the AD framework is called *fitness generating function*

$$g(n_1, n_2, x_1, x_2, \tilde{x}) = r(\tilde{x}) \left(1 - \frac{c(x_1, \tilde{x}) n_1 + c(\tilde{x}, x_2) n_2}{K(\tilde{x})}\right).$$

Considering that we have assumed the condition $c(\tilde{x}, \tilde{x}) = 1$, for all $\tilde{x} \in \mathcal{X}$, in system (3-1),

$$g(n_1, n_2, x_1, x_2, x_1) = r(x_1) \left(1 - \frac{n_1 + c(x_1, x_2) n_2}{K(x_1)}\right),$$

represents the *relative* growth rate $\dot{n}_1/n_1$ of SE generation technology. Along the same lines, the relative growth rate of IE, $\dot{n}_2/n_2$, is given by $g(n_1, n_2, x_1, x_2, x_2)$. A more general description of state variables, functional coefficients and parameter description can be found in Table **3-1**.

Table 3-1.: Description of state variables and coefficients with their corresponding ranges. *CGC: cumulative generation capacity.

	State variables description	*Units*
$n_1(t)$	CGC* for Standard Energy, characterized by x_1	MW
$n_2(t)$	CGC for Innovative Energy, characterized by x_2	MW
	Parameter description	*Ranges*
x_1	Quantitative continuous characteristic trait defining SE	$x_1 \in \mathbb{R}$
x_2	Quantitative continuous characteristic trait defining IE	$x_2 \in \mathbb{R}$
$r(x_i)$	CGC growing rate as a function of x_i, for $i = 1,2$	$r > 0$
$K(x_i)$	Maximum CGC as function of x_i, for $i = 1,2$	$K > 0$ MW
$c(x_1, x_2)$	Interaction coefficient between both CGC as a function of x_1 and x_2	$c \in \mathbb{R}$

3.2.2. Innovative-Standard model local stability

The importance of the local stability analysis of the model is that it will provide us with relevant information regarding the dynamics of the market of the types of energy that interact, and will allow establishing conditions under which the coexistence is possible or the definitive disappearance of any of them. In particular, it is essential to know under what circumstances EI can invade and remain into the market.

By solving the system $\dot{n}_1 = 0$, $\dot{n}_2 = 0$ is possible to find four steady states of system (3-1) given by:

- $P_0(\overline{n}_1^0, \overline{n}_2^0) = (0,0)$, corresponding to the absence of SE and IE in the market.
- $P_1\left(0, \overline{n}_2^1(x_2)\right) = (0, K(x_2))$, corresponding to the exclusion of SE from the market and the IE is dominant.
- $P_2(\overline{n}_1^2(x_1), 0) = (K(x_1), 0)$, corresponding to the exclusion of IE from the market and the SE is dominant.
- $P_3\left(\overline{n}_1^3(x_1,x_2), \overline{n}_2^3(x_1,x_2)\right) = \left(\frac{c(x_1,x_2)K(x_2)-K(x_1)}{c(x_2,x_1)c(x_1,x_2)-1}, \frac{c(x_2,x_1)K(x_1)-K(x_2)}{c(x_2,x_1)c(x_1,x_2)-1}\right)$, corresponding to the case when SE and IE are both present and share the market.

Notice P_3 can be written as:

$$P_3(\overline{n}_1^3, \overline{n}_2^3) = \left(\frac{K(x_1)\left(H(x_1,x_2)-1\right)}{c(x_2,x_1)c(x_1,x_2)-1}, \frac{K(x_2)\left(H(x_2,x_1)-1\right)}{c(x_2,x_1)c(x_1,x_2)-1}\right),$$

where:

$$H(x_1,x_2) = \frac{c(x_1,x_2)K(x_2)}{K(x_1)}, \quad \text{and} \quad H(x_2,x_1) = \frac{c(x_2,x_1)K(x_1)}{K(x_2)}.$$

Therefore, $P_3 \in \Omega$ if and only if both of its coordinates are non-negative, this implies two different situations:

Case I: $c(x_2, x_1)c(x_1, x_2) < 1$, then $P_3 \in \Omega$ if and only if

$$H(x_1, x_2) < 1 \quad \text{and} \quad H(x_2, x_1) < 1.$$

In particular, when $H(x_1, x_2) = 1$ and $H(x_2, x_1) < 1$, P_3 coalesce with P_1, and when $H(x_1, x_2) < 1$ and $H(x_2, x_1) = 1$, P_3 coalesce with P_2.

Case II: $c(x_2, x_1)c(x_1, x_2) > 1$, then $P_3 \in \Omega$ if and only if

$$H(x_1, x_2) > 1 \quad \text{and} \quad H(x_2, x_1) > 1.$$

Analogously to the previous case, when $H(x_1, x_2) = 1$ and $H(x_2, x_1) > 1$, P_3 coalesce with P_1, and when $H(x_1, x_2) > 1$ and $H(x_2, x_1) = 1$, P_3 coalesce with P_2.

At this point, local stability analysis will bring some insights into the market dynamics and will help to answer a further question of under what conditions can IE spread into the market and interact or even substitute SE. Indeed, If we consider a market dominated exclusively by SE, the instability of P_2 is related to the possibility for an IE to invade the market, while the existence and stability of P_3 are related to the coexistence of both kinds of energy sharing the market, leading to diversification.

Proposition 3.2.1 *The steady state P_0 of system (3-1) is always unstable.*

Proof The Jacobian matrix of system (3-1) at P_0 is given by,

$$A(P_0) = \begin{bmatrix} r(x_1) & 0 \\ 0 & r(x_2) \end{bmatrix}.$$

Then, the corresponding eigenvalues are $r(x_1)$ and $r(x_2)$, both positive by definition; therefore, steady state P_0 is unstable.

Proposition 3.2.2 *The steady states P_1 and P_2 of system (3-1), are locally asymptotically stable if and only if $H(x_1, x_2) > 1$ and $H(x_2, x_1) > 1$ respectively.*

Proof The Jacobian matrix of system (3-1) at P_1 can be written as:

$$A(P_1) = \begin{bmatrix} r(x_1)\,(1 - H(x_1, x_2)) & 0 \\ -r(x_2)c(x_2, x_1) & -r(x_2) \end{bmatrix}.$$

Then, the corresponding eigenvalues are $r(x_1)\,(1 - H(x_1, x_2)) < 0$ if and only if $H(x_1, x_2) > 1$, as stated in the Proposition, and $-r(x_2) < 0$ by definition. On the other hand, the Jacobian matrix of system (3-1) at P_2 is given by,

$$A(P_2) = \begin{bmatrix} -r(x_1) & -r(x_1)c(x_1,x_2) \\ 0 & r(x_2)\left(1 - H(x_2,x_1)\right) \end{bmatrix}.$$

Then, the corresponding eigenvalues are $-r(x_0) < 0$ and $r(x_2)\left(1 - H(x_2,x_1)\right) < 0$ if and only if $H(x_2,x_1) > 1$. This proves the proposition.

Proposition 3.2.3 *Given the steady state P_3, when exists in Ω, its local stability is described in the following way:*

I. *If $c(x_2,x_1)c(x_1,x_2) < 1$, $H(x_1,x_2) < 1$ and $H(x_2,x_1) < 1$, then P_3 is locally asymptotically stable.*

II. *If $c(x_2,x_1)c(x_1,x_2) > 1$, $H(x_1,x_2) > 1$ and $H(x_2,x_1) > 1$, then P_3 is unstable.*

Proof The Jacobian matrix of system (3-1) at P_3 can be written as,

$$A(P_3) = \begin{bmatrix} -\dfrac{r(x_1)[H(x_1,x_2)-1]}{c(x_2,x_1)c(x_1,x_2)-1} & -\dfrac{[H(x_1,x_2)-1]\,r(x_1)c(x_1,x_2)}{c(x_2,x_1)c(x_1,x_2)-1} \\ -\dfrac{[H(x_2,x_1)-1]\,r(x_2)c(x_2,x_1)}{c(x_2,x_1)c(x_1,x_2)-1} & -\dfrac{r(x_2)[H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1} \end{bmatrix}.$$

Let Δ denote the determinant of $A(P_3)$; then, it can be written as:

$$\Delta = -\frac{r(x_1)r(x_2)[H(x_1,x_2)-1][H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1},$$

and similarly, let T be the trace of $A(P_3)$, then:

$$T = -\frac{r(x_1)[H(x_1,x_2)-1] + r(x_2)[H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1}.$$

To be consistent, consider the cases when $P_3 \in \Omega$. This implies two different situations:

Case I: $c(x_2,x_1)c(x_1,x_2) < 1$, then $P_3 \in \Omega$ if and only if

$$H(x_1,x_2) < 1 \quad \text{and} \quad H(x_2,x_1) < 1.$$

In this scenario, $\Delta > 0$ and $T < 0$. Then P_3 is locally asymptotically stable [75]. As stated above, when $H(x_1,x_2) = 1$ and $H(x_2,x_1) < 1$, P_3 coalesce with P_1 and transfers its stability to P_1 when $H(x_1,x_2) > 1$, case when $P_3 \notin \Omega$ although it exists and it is unstable (indeed, $H(x_1,x_2) > 1$ and $H(x_2,x_1) < 1$ implies $\Delta < 0$). Similarly, if $H(x_1,x_2) < 1$ and $H(x_2,x_1) = 1$, P_3 coalesce with P_2, and transfers its stability. In fact, $H(x_1,x_2) < 1$ and $H(x_2,x_1) > 1$ implies $\Delta < 0$ and $P_3 \notin \Omega$ and it is unstable. Both situations correspond to *transcritical bifurcations* [48, 75]. In Table **3-2** these results are summarized.

Case II: $c(x_2, x_1)c(x_1, x_2) > 1$, then $P_3 \in \Omega$ if and only if

$$H(x_1, x_2) > 1 \quad \text{and} \quad H(x_2, x_1) > 1.$$

Notice that, in this case, $\Delta < 0$, then $P_3 \in \Omega$ but it is unstable. Analogously to the previous case, when $H(x_1, x_2) = 1$ and $H(x_2, x_1) > 1$, P_3 collides with P_1, and when $H(x_1, x_2) > 1$ and $H(x_2, x_1) = 1$, P_3 meets with P_2. Both situations correspond to transcritical bifurcations also (see Table **3-2**).

Table 3-2.: Classification of local stability. Scenarios marked with an * correspond to impossible scenarios (see the text for further details). LAS: Locally Asymptotically Stable. U: Unstable.

Case	*Condition*	P_0	P_1	P_2	P_3
$c(x_2, x_1)c(x_1, x_2) < 1$	$H(x_1, x_2) > 1; H(x_2, x_1) < 1$	U	LAS	U	$\notin \Omega$
	$H(x_1, x_2) < 1; H(x_2, x_1) > 1$	U	U	LAS	$\notin \Omega$
	$H(x_1, x_2) > 1; H(x_2, x_1) > 1$*	U	LAS	LAS	$\notin \Omega$
	$H(x_1, x_2) < 1; H(x_2, x_1) < 1$	U	U	U	LAS
$c(x_2, x_1)c(x_1, x_2) > 1$	$H(x_1, x_2) > 1; H(x_2, x_1) < 1$	U	LAS	U	$\notin \Omega$
	$H(x_1, x_2) < 1; H(x_2, x_1) > 1$	U	U	LAS	$\notin \Omega$
	$H(x_1, x_2) > 1; H(x_2, x_1) > 1$	U	LAS	LAS	U
	$H(x_1, x_2) < 1; H(x_2, x_1) < 1$*	U	U	U	$\notin \Omega$

It is important to clarify that the last scenario in Table **3-2** is not possible. If $H(x_1, x_2) < 1$ and $H(x_2, x_1) < 1$, then $H(x_1, x_2)H(x_2, x_1) < 1$ implies $c(x_2, x_1)c(x_1, x_2) < 1$, contradicting the case hypothesis of being $c(x_2, x_1)c(x_1, x_2) > 1$. A similar situation occurs in the third scenario in Table **3-2**, if $H(x_1, x_2) > 1$ and $H(x_2, x_1) > 1$, then $H(x_1, x_2)H(x_2, x_1) > 1$ which implies $c(x_2, x_1)c(x_1, x_2) > 1$, contradicting the case of being $c(x_2, x_1)c(x_1, x_2) < 1$.

In Figure **3-1** the phase portrait of system (3-1) is shown, which corresponds to the case $c(x_2, x_1)c(x_1, x_2) > 1$, $H(x_1, x_2) > 1$ and $H(x_2, x_1) > 1$. As stated in Table **3-2**, P_0 and P_3 are unstable and P_1 and P_2 are both locally asymptotically stable. In this case, initial conditions determine which equilibria is going to attract a particular trajectory. Note that this is the unique scenario guaranteeing two simultaneous locally asymptotically stable equilibria. Thus the market final state will depend only on the initial conditions.

Note that condition $H(x_1, x_2) > 1$ implies $\frac{c(x_1,x_2)K(x_2)}{K(x_1)} > 1$ and then $c(x_1, x_2) > 0$. Similarly, condition $H(x_2, x_1) > 1$ implies $\frac{c(x_2,x_1)K(x_1)}{K(x_2)} > 1$ and thus $c(x_2, x_1) > 0$. Such situations can only occur in the case when we have competitive interactions (see the c function description in subsection 2.1). On the other hand, $H(x_1, x_2) < 1$ and $H(x_2, x_1) < 1$ implies $\frac{c(x_1,x_2)K(x_2)}{K(x_1)} < 1$ and $\frac{c(x_2,x_1)K(x_1)}{K(x_2)} < 1$ respectively. These conditions can be satisfied when c is positive or negative. Therefore, the corresponding interaction scenario can be competition or cooperation.

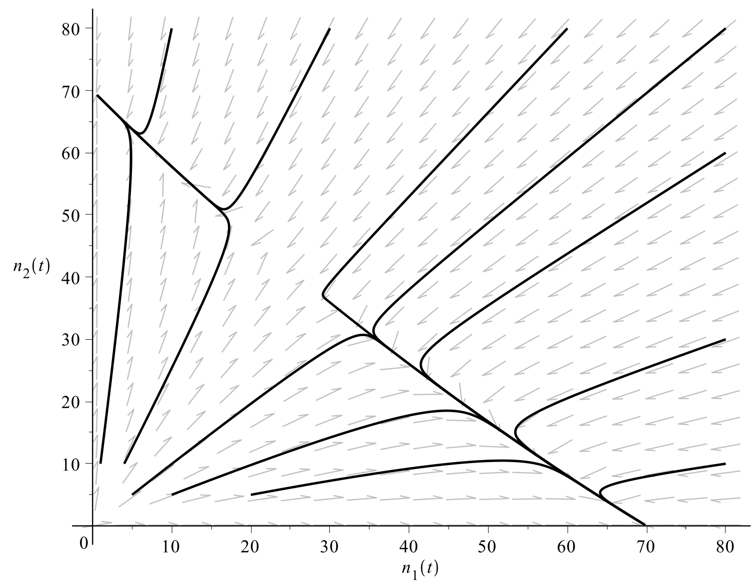

Figure 3-1.: Phase portrait corresponding to the 7^{th} scenario in Table **3-2**, where $c(x_2, x_1)c(x_1, x_2) > 1$, $H(x_1, x_2) > 1$ and $H(x_2, x_1) > 1$. As it can be deduced from the stability propositions and, as it is shown in the Table, P_0 and P_3 are unstable and P_1 and P_2 are both locally asymptotically stable. In this case, initial conditions determine which equilibria is going to attract a particular trajectory. For the simulations, we consider x_1 and x_2 in order to have $c(x_1, x_2) = 1.1$, $c(x_2, x_1) = 1.15$, $K(x_1) = K(x_2) = 70$ and $r(x_1) = r(x_2) = 0.3$. This is a scenario corresponding to competition favoring the SE; i.e., $c(x_1, x_2) > c(x_2, x_1)$, which could mean, for instance, a bigger taxes imposition on IE.

The local stability analysis implies that the energy market will not crash under any circumstances, guaranteeing a permanent energy supply from any (or both) generation technologies; i.e., there is at least one stable equilibria corresponding to dominance of SE, or IE or their coexistence to supply energy demand.

3.2.3. Standard energy model and invasion conditions

From the AD theory, invasion is ruled by the sign of the fitness function of the IE, as given by $\dot{n}_2/n_2$, from the g function at $P_2(\overline{n}_1^2(x_1), 0) = (K(x_1), 0)$. To describe this situation with more detail, we take into account that just before an innovation occurs, it is assumed that only SE is available to supply energy demand, that is $n_2 = 0$. For simplicity we denote $x_1 = x$, $n_1 = n$. Therefore the energy market is modeled by only one differential equation,

$$\dot{n} = nr(x)\left(1 - \frac{n}{K(x)}\right), \tag{3-2}$$

corresponding to the classical logistic equation. It is known that (3-2) has two equilibria given by $\overline{n}^0 = 0$ which are always unstable, and $\overline{n}^1 = K(x)$ is always asymptotically stable under the definitions given to r and K. Being SE the only generation technology available in the market, it is assumed that n reaches its maximum capacity $K(x)$ to satisfy the market demand.

Once an innovation occurs, it is interesting to determine whether or not the IE can invade and share the market (coexist) with the SE. Just after the innovation, it is assumed that system (3-1) is at equilibrium $P_2(\overline{n}_1^2(x_1), 0) = (K(x_1), 0)$. As discussed in the proof of Proposition 3.2.2, the Jacobian matrix at P_2 is given by:

$$A(P_2) = \begin{bmatrix} -r(x_1) & -r(x_1)c(x_1, x_2) \\ 0 & r(x_2)\left(1 - H(x_2, x_1)\right) \end{bmatrix}.$$

Define the *fitness function* of the IE as the innovative eigenvalue, also known as the *invasion eigenvalue* in the Adaptive Dynamics language, to have,

$$\lambda(x_1, x_2) = r(x_2)\left(1 - H(x_2, x_1)\right). \tag{3-3}$$

Clearly, P_2 stability is determined by the sign of $\lambda(x_1, x_2)$, i.e. if $\lambda(x_1, x_2) > 0$, then P_2 is unstable and therefore IE can invade the market. On the other hand, if $\lambda(x_1, x_2) < 0$, then P_2 is locally asymptotically stable and the IE is going to be excluded indefinitely from the market (see Proposition 3.2.2). For a further study of this situation, assume the non degenerate situation $\frac{\partial \lambda}{\partial x_2}(x_1, x_2) \neq 0$. Then the first order Taylor expansion of $\lambda(x_1, x_2)$ around $x_2 = x_1$ is,

$$\lambda(x_1, x_2) = \lambda(x_1, x_1) + (x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1) + O(|x_2 - x_1|^2). \tag{3-4}$$

Note that the term $\lambda(x_1, x_1)$ in the previous expansion,

$$\lambda(x_1, x_1) = r(x_1)\left(1 - c(x_1, x_1)\right) = 0,$$

therefore $\lambda(x_1, x_2)$, described as in (3-4), has opposite sign for $x_2 > x_1$ or $x_2 < x_1$, with x_2 close to x_1. Thus if $(x_2 - x_1)\frac{\partial \lambda}{\partial x_2}(x_1, x_1)$ is positive, i.e, if

$$\begin{aligned} &\frac{\partial \lambda}{\partial x_2}(x_1, x_1) > 0, \quad x_2 > x_1, \quad \text{or} \\ &\frac{\partial \lambda}{\partial x_2}(x_1, x_1) < 0, \quad x_2 < x_1, \end{aligned} \tag{3-5}$$

the invasion eigenvalue $\lambda(x_1, x_2)$ is positive and equilibria P_2 is unstable. In such case, IE invades into the market, and vice versa, if $(x_2 - x_1)\frac{\partial\lambda}{\partial x_2}(x_1, x_1)$ is negative, P_2 is locally asymptotically stable and IE goes extinct. From now on the quantity:

$$\frac{\partial\lambda}{\partial x_2}(x_1, x_1), \tag{3-6}$$

is going to be called *selection gradient* for the innovative energy. In the next section, specific coefficients are established according to the characteristic traits and the meaning and scope of the described invasion condition will be analyzed in more depth.

The question of whether invasion implies substitution of the standard energy, requires the study of the global behavior of the standard-innovative model. In Appendix B of [31], the following theorem is proved.

Theorem 3.2.4 (Invasion implies substitution) *Given x_1 in the evolution set $\mathcal{X}$, if $(x_2 - x_1)\frac{\partial}{\partial x'}\lambda(x_1, x_2) > 0$ and $|x_2 - x_1|$ and $|(n_1(0) + n_2(0) - \overline{n}(x_1))|$ are sufficiently small, then the trajectory $(n_1(t), n_2(t))$ of the standard-innovative model (3-1) tends toward equilibrium P_1 for $t \to \infty$.*

3.3. Evolutionary dynamics under cooperation and competition

3.3.1. Functional coefficients

Consider a market where the CGC growing rate r does not depend on the characteristic traits, and therefore it is constant.

To define the maximum capacity function K, we consider it as an increasing function of x, for $x \geq 0$, decreasing to zero if $x < 0$, and bounded above by some maximum value k_1 corresponding to technical limitations, imposed normative obeying public policies, or technical or financial restrictions. As an example, if we consider the amount of money invested in new technology as a measure of the technology of energy generation then, very large positive values of x (own resources) or negative values (resources coming from the indebtedness), would allow to increase the maximum generation capacity K. We consider the expression,

$$K(x) = \frac{k_1 x^2}{k_2^2 + x^2},$$

such that $K(x) \to k_1$ as $x \to \pm\infty$ as in Figure **3-2**-left. Note that $K(x)$ increases [decreases] rapidly when x is small and positive [negative], but at large positive [negative] values of

x (larger inversion from own resources [indebtedness], for instance), the maximum capacity grows up [decreases down] slowly to [from] its maximum k_1.

A large value of k_2 implies that it is necessary to invest more resources (large x) to reach the maximum value k_1, while a small value of k_2 implies that the maximum level k_1 is reached with smaller investments (smaller x). Geometrically, $K(x)$ increses rapidly for all $0 < x < \frac{\sqrt{3}k_2}{3}$, (rapidly decreases if $-\frac{\sqrt{3}k_2}{3} < x < 0$) and increases slowly to k_1, for all $x > \frac{\sqrt{3}k_2}{3}$ (decreases slowly from k_1 if $x < -\frac{\sqrt{3}k_2}{3}$). This correspond to the fact that the graph of K has two inflection points at $x = \pm\frac{\sqrt{3}k_2}{3}$. Figure **3-2**-left, shows the plot of K with the parameter values described in the caption.

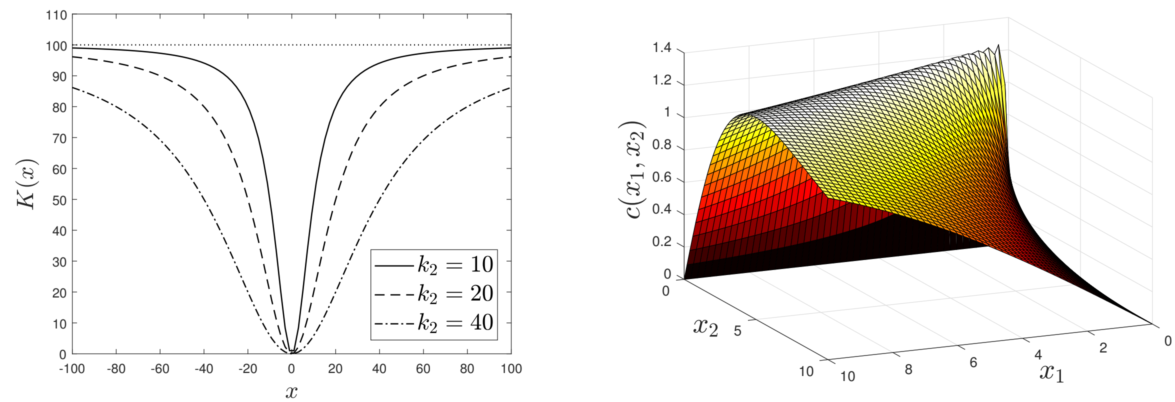

Figure 3-2.: Maximum capacity function $K(x) = \frac{k_1 x^2}{k_2^2 + x^2}$, plotted only for three different values of k_2 (left) and interaction function $c(x_1, x_2) = \frac{(c_1^2 + c_2^2)x_1 x_2}{c_1^2 x_1^2 + c_2^2 x_2^2}$, conveniently plotted only for positive values of x_1 and x_2 (right). Parameter values used are $r = 0.3$, $c_1 = 1$, $c_2 = 2$, $k_1 = 100$, $k_2 = 10$ (solid), $k_2 = 20$ (dashed) and $k_2 = 40$ (dash-dot).

Table 3-3.: Parameter description and the corresponding baseline values used at simulations.

	Parameter description	*Value*
k_1	Upper bound for the maximum capacity K, due to technical limitations or imposed public policies	100 MW
k_2	Measure of the speed at which maximum capacity can grow	10
c_1	Subsidies if positive/Taxes if negative or any other similar policy on SE	Varies
c_2	Subsidies if positive/Taxes if negative or any other similar policy on IE	Varies

In the formulation of the interaction function c, we want to consider the symmetry regarding line $x_2 = x_1$ as an important issue. In fact, by definition, $c(x_1, x_2)$ corresponds to the increasing/decreasing rate of CGC suffered by n_1 by the presence of n_2 and, conversely, $c(x_2, x_1)$ corresponds to the increasing/decreasing rate of CGC suffered by n_2 by the presence of n_1. If $c(x_i, x_j) < 0$, for $i, j = 1$ or 2, the interaction described by c corresponds to cooperation, and it corresponds to competition if $c(x_i, x_j) > 0$. If $c(x_1, x_2) = c(x_2, x_1)$ the interaction is called *fair*, and it is called *unfair* in any of the cases $c(x_1, x_2) > c(x_2, x_1)$ or $c(x_1, x_2) < c(x_2, x_1)$. Consider the interaction function between both kind of energies is given by the function:

$$c(x_1, x_2) = \frac{(c_1^2 + c_2^2)x_1 x_2}{c_1^2 x_1^2 + c_2^2 x_2^2}, \tag{3-7}$$

depicted in Figure **3-2**-right. Note that $c \in \mathbb{R}$, for all x_1 and x_2. Function c corresponds to competition if x_1 and x_2 have the same sign (first and third quadrants of the (x_1, x_2)-plane) and to cooperation if x_1 and x_2 have opposite signs (second and fourth quadrants of the (x_1, x_2)-plane). The coefficient c has a set of maximums on the line $x_2 = \frac{c_1}{c_2} x_1$, where the maximum competition takes place and its value is $\frac{c_1^2 + c_2^2}{2c_1 c_2}$. It has a set of minimums at the line $x_2 = -\frac{c_1}{c_2} x_1$ where the cooperation is maxima and its value is $-\frac{c_1^2 + c_2^2}{2c_1 c_2}$. Symmetric competition occurs when $c_1 = c_2$. In this case the lines of maxima and minima coincide with $x_2 = x_1$ and $x_2 = -x_1$ respectively. On the other hand, $c_2 > c_1$ [conversely $c_2 < c_1$] implies asymmetric interaction in favor of n_2 [conversely n_1].

Symmetric interaction is not likely to occur in almost any market. Therefore we will consider the asymmetric case by stating $c_1 \neq c_2$. Both parameters can be considered as the effect of market policies in the competition, such as subsidies awarded, or any other similar policy when $c_1, c_2 > 0$ or, some privative policy as taxes imposition when $c_1, c_2 < 0$. In general, whether an innovation is stimulated or unstimulated depends on if $x_1 > x_2$ or $x_2 > x_1$ and also on whether they are positive or negative. If $c_2 > c_1$ and $x_1 > 0$, a small innovation x_2 is stimulated by interaction if $x_2 < x_1$. Geometrically, the point (x_1, x_2) is below the diagonal (closer to the line of maxima) and $c(x_1, x_2) > 1$, while the point (x_2, x_1) is above the diagonal and $c(x_2, x_1) < 1$. It is unstimulated if $x_2 > x_1$.

3.3.2. Selection gradient and invasion conditions

A more detailed study of the invasion conditions will be discussed in this subsection. Under the definitions of r, K and c described above, the IE growing rate, also known as *fitness* function (5-14) takes the form:

$$\lambda(x_1, x_2) = r\left(1 - \frac{(c_1^2 + c_2^2)(k_2^2 + x_2^2)x_1^3}{(k_2^2 + x_1^2)(c_1^2 x_2^2 + c_2^2 x_1^2)x_2}\right),$$

and the selection gradient is explicitly given by:

$$\frac{\partial \lambda}{\partial x_2}(x_1, x_1) = \frac{\left[k_2^2(3c_1^2 + c_2^2) - (c_2^2 - c_1^2)x_1^2\right] r}{x_1(k_2^2 + x_1^2)(c_1^2 + c_2^2)}.$$

The invasion conditions were discussed in the previous section and established in (3-5). Now, with the explicit expressions for r, K and c, we will study invasion in the energy market in a more detailed way. Since $r > 0$ and $(k_2^2 + x_1^2)(c_1^2 + c_2^2) > 0$ in every case, then sign of the selection gradient is given by:

$$\frac{k_2^2(3c_1^2 + c_2^2) - (c_2^2 - c_1^2)x_1^2}{x_1}. \tag{3-8}$$

- If $c_2^2 - c_1^2 > 0$, then $\frac{\partial \lambda}{\partial x_2}(x_1, x_1) < 0$, implies two cases:

$$\begin{aligned} x_1 > 0 \iff & k_2^2(3c_1^2 + c_2^2) - (c_2^2 - c_1^2)x_1^2 < 0 \\ \iff & \frac{k_2^2(3c_1^2 + c_2^2)}{c_2^2 - c_1^2} - x_1^2 < 0 \\ \iff & x_1 \in (x_I, \infty). \end{aligned}$$

Similarly,

$$\begin{aligned} x_1 < 0 \iff & k_2^2(3c_1^2 + c_2^2) - (c_2^2 - c_1^2)x_1^2 > 0 \\ \iff & x_1 \in (-x_I, 0), \end{aligned}$$

where $x_I = k_2\sqrt{\frac{3c_1^2+c_2^2}{c_2^2-c_1^2}}$. In any of this cases, innovations with $x_2 > x_1$ invade.

On the other hand, $\frac{\partial \lambda}{\partial x_2}(x_1, x_1) > 0$ for $x_1 > 0$ implies $x_1 \in (0, x_I)$ and for $x_1 < 0$ implies $x_1 \in (-\infty, -x_I)$ in both cases, innovations with $x_1 > x_2$ invade.

In Figure **3-3**-left, its is shown the schematic structure of the invasion region in the $(x_1, x_2)-$plane when $c_2^2 - c_1^2 = 0.1025 > 0$ ($c_1 = 1$ and $c_2 = 1.05$ were used), blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients.

- If $c_2^2 - c_1^2 < 0$ (equivalently $c_1^2 - c_2^2 > 0$), we can rewrite (3-8) as:

$$\frac{k_2^2(3c_1^2 + c_2^2) + (c_1^2 - c_2^2)x_1^2}{x_1},$$

therefore, $x_1 > 0$ implies $\frac{\partial \lambda}{\partial x_2}(x_1, x_1) > 0$ and the innovations with $x_1 > x_2$ invade; similarly, $x_1 < 0$ implies $\frac{\partial \lambda}{\partial x_2}(x_1, x_1) < 0$ and the innovations with $x_2 > x_1$ invade.

In Figure **3-3**-right its is shown the schematic structure of the invasion region in the $(x_1, x_2)-$plane when $c_2^2 - c_1^2 = -0.1025 < 0$ ($c_1 = 1.05$ and $c_2 = 1$ were used), as in the previous case, blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients.

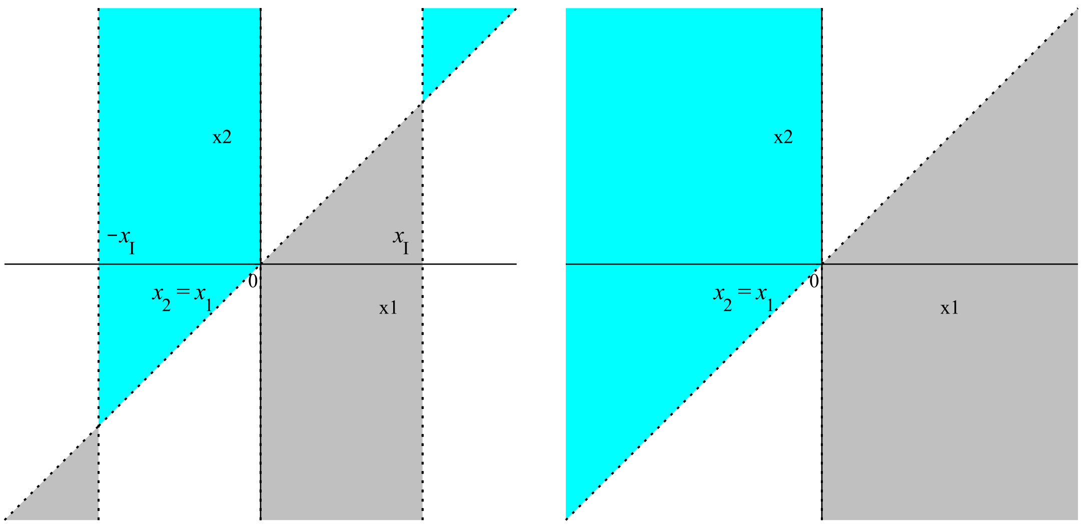

Figure 3-3.: Different regions in the $(x_1, x_2)-$plane where invasion conditions given in (3-5) are satisfied. Blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients. **Left:** $c_2^2 - c_1^2 = 0.1025 > 0$ ($c_1 = 1$ and $c_2 = 1.05$ were used). **Right:** $c_2^2 - c_1^2 = -0.1025 < 0$ (with $c_1 = 1.05$ and $c_2 = 1$). Note that a big innovation is required to have a cooperative market just after an innovation in a market dominated by SE generation technology.

Note that, although functional parameters r, K and c are defined for all x_1 and x_2 in $\mathbb{R}$, and also the interaction dynamics from system (3-1) is well defined for both strategies (cooperation and competition), the invasion conditions determine configurations (specific regions of the $(x_1, x_2)-$plane) under which the invasion of the innovative attribute is possible, and configurations that lead to its disappearance. Additionally, note that a big innovation is required to have a cooperative market just after an innovation in a market dominated by SE generation technology.

3.3.3. Adaptive dynamics canonical equation

The behavior and long-term evolution of the attribute x_1 that characterizes energy market is now described as a result of advantageous innovations on this attribute allowing for survival of the respective CGC in the market. The goal in this section is to describe the ADCE briefly. The reader is invited to review [10,26,31,46], to expand the information shown, in particular, those regarding to the equation deduction.

The dynamics of x_1 is given by the ordinary differential equation:

$$\dot{x_1} = \frac{1}{2}\mu(x_1)\sigma^2(x_1)\overline{n}(x_1)\frac{\partial\lambda}{\partial x_2}(x_1, x_1). \qquad (3\text{-}9)$$

In [31], there it is a full deduction of this equation. A parameter $\epsilon \to 0$ is considered, as a scaling factor separating the market timescale (the time considered above in all the derivatives of n_1 and n_2), from the evolutionary timescale for x_1. In fact, a small amount dt of time on the evolutionary timescale corresponds to a large amount of time dt/ϵ on the market timescale. This fact allows affirming that between one innovation and the next, the market has time enough to find an equilibrium configuration. It is worth clarifying that while n_1 and n_2 are on the market timescale, x is on the evolutionary timescale. All the derivatives concerning the time are represented with *dot* notation.

Equation (3-9) is known as the ADCE. In the context of this work, $\mu(x_1)$ is proportional to the probability that an IE entering the market corresponds to an innovation. $\sigma(x_1)$ is proportional to the standard deviation of the measure of the change in the attribute in which innovation occurs. $\overline{n}(x_1)$ represents the market equilibrium before innovation (i.e., $\overline{n}(x_1) = K(x_1)$), and $\frac{\partial\lambda}{\partial x_2}(x_1, x_1)$ is the selection gradient of the x_1 attribute on which the innovation is performed.

Denoting $x = x_1$ for simplicity, and considering $\mu(x) = \mu$ and $\sigma^2(x) = \sigma^2$ (i.e. they do not depend upon the characteristic trait), the ADCE is given by:

$$\dot{x} = \frac{\mu\sigma^2 k_1 r}{2(c_1^2 + c_2^2)} \frac{[(c_1 - c_2)(c_1 + c_2)x^2 + k_2^2(3c_1^2 + c_2^2)]x}{(k_2^2 + x^2)^2}. \qquad (3\text{-}10)$$

To study this non linear differential equation, it is necessary to find the equilibrium points (*evolutionary equilibria* from now on) by solving $\dot{x} = 0$, to find:

$$\overline{x}_0 = 0, \quad \overline{x}_1 = k_2\sqrt{\frac{3c_1^2 + c_2^2}{c_2^2 - c_1^2}} = x_I \quad \text{and} \quad \overline{x}_2 = -k_2\sqrt{\frac{3c_1^2 + c_2^2}{c_2^2 - c_1^2}} = -x_I,$$

which are real values when $c_2^2 - c_1^2 > 0$. We obtain the region R

$$R = \{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 > 0\}.$$

Now, to study the stability of equilibria $\overline{x}_i$, for $i = 0, 1, 2$, define

$$f(x) = \frac{\mu\sigma^2 k_1 r}{2(c_1^2 + c_2^2)} \frac{[(c_1 - c_2)(c_1 + c_2)x^2 + k_2^2(3c_1^2 + c_2^2)]x}{(k_2^2 + x^2)^2},$$

as the right hand of (3-10); then, linearizing,

$$\frac{df}{dx}(x) = -\frac{\mu\sigma^2 k_1 r}{2(c_1^2+c_2^2)} \frac{x^4(c_1^2-c_2^2)+6x^2(c_1^2+c_2^2)k_2^2-(3c_1^2+c_2^2)k_2^4}{(k_2^2+x^2)^3},$$

and we get:

$$\frac{df}{dx}(\overline{x}_0) = \frac{\sigma^2\mu k_1 r(3c_1^2+c_2^2)}{2k_2^2(c_1^2+c_2^2)} > 0 \quad \text{for all } \; c_1, c_2,$$

and,

$$\frac{df}{dx}(\overline{x}_1) = \frac{df}{dx}(\overline{x}_2) = -\frac{\mu\sigma^2 k_1 r(c_1^2-c_2^2)^2(3c_1^2+c_2^2)}{4k_2^2(c_1^2+c_2^2)} < 0 \quad \text{for all } \; c_1, c_2.$$

Note that $\overline{x}_0$ is always unstable and $\overline{x}_i$, for $i = 1, 2$ is always locally asymptotically stable. Thus, in the market, repeated innovations and replacements of generation technologies with new ones, drives the attribute x toward any of the equilibrium values $\overline{x}_1$ or $\overline{x}_2$. In Figure **3-4** some numeric solutions of the canonical equation 3-10 are shown, with the parameters described in the corresponding caption.

An important result at this point is that condition $c_2^2 - c_1^2 > 0$ not only determines scenarios for evolutionary equilibria to exist but also to be locally asymptotically stable.

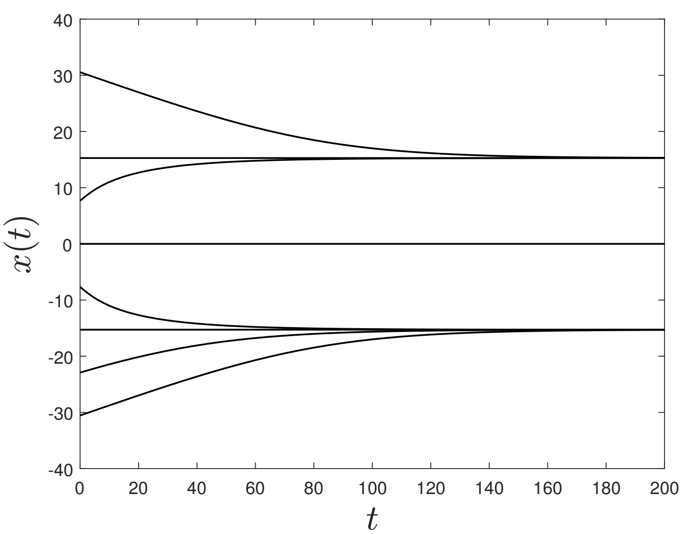

Figure 3-4.: Numeric simulation of evolutionary dynamics of the characteristic trait x described by the ADCE (3-10), considering $r = 0.3$, $c_1 = 1$, $c_2 = 2$, $k_1 = 100$, $k_2 = 10$, $\mu = 1$, and $\sigma = 1$.

Now, it is necessary to study evolutionary dynamics in a neighborhood of the evolutive equilibria $\overline{x}_i$, for $i = 1, 2$. Since in the vicinity of the singular strategy, $\frac{\partial \lambda}{\partial x_2}(\overline{x}_1, \overline{x}_1) = 0$, then the market and evolutionary dynamics are dominated by the second derivatives of the fitness function.

3.3.4. Coexistence and divergence

Geritz *et al.* [45, 46] showed that if the coexistence condition holds, innovative and standard energies mutually invade each other. This situation implies the instability of both "single trait" equilibria P_1 and P_2, i.e., coexistence describes the situation when the values of the characteristic traits of the IE and the SE are in the vicinity of the equilibrium $\overline{x}$, defining energies that are similar to each other, and sharing the market, that is, "coexist". On the other hand, if the coexistence condition does not hold, both energies mutually exclude and any of the "single trait" equilibria gains stability. In particular, Dercole and Rinaldi in [31], proves that IE-SE coexistence is possible if:

$$\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_i, \overline{x}_i) < 0, \quad i = 1, 2. \tag{3-11}$$

Explicitly we have,

$$\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_i, \overline{x}_i) = -\frac{4r(c_2^2 - c_1^2)c_2^2 c_1^2}{k_2^2(c_1^2 + c_2^2)^2(3c_1^2 + c_2^2)} < 0, \quad i = 1, 2.$$

Note that coexistence condition holds when $c_2^2 - c_1^2 > 0$. This situation corresponds to $(c_1, c_2) \in R$, which was defined above for the existence of $\overline{x}_i$, for $i = 1, 2$ in $\mathbb{R}$. This result can be stated as follows: *evolutionary stability, implies coexistence of IE and SE characteristic traits.*

An equally important question as coexistence is whether it can be guaranteed that the two attributes that coexist after the invasion of IE are indeed similar and not identical. That is, if it is not possible to differentiate x_1 from x_2, then the condition of coexistence would only mean that, in practice, there is only one type of energy that has been "virtually" separated into two classes. In this way, the divergence is understood as the dissimilarity between the values of the SE and IE characteristic attributes. This situation allows differentiating one from the other, implying the "Origin of Diversity" in the market. It is shown in [31] that x_1 and x_2 attributes diverge from each other, when:

$$\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_i, \overline{x}_i) > 0, \quad i = 1, 2. \tag{3-12}$$

Explicitly,

$$\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_i, \overline{x}_i) = \frac{(c_2^2 - c_1^2)(3c_1^2 - c_2^2)r}{k_2^2(3c_1^2 + c_2^2)(c_1^2 + c_2^2)} > 0, \quad i = 1, 2,$$

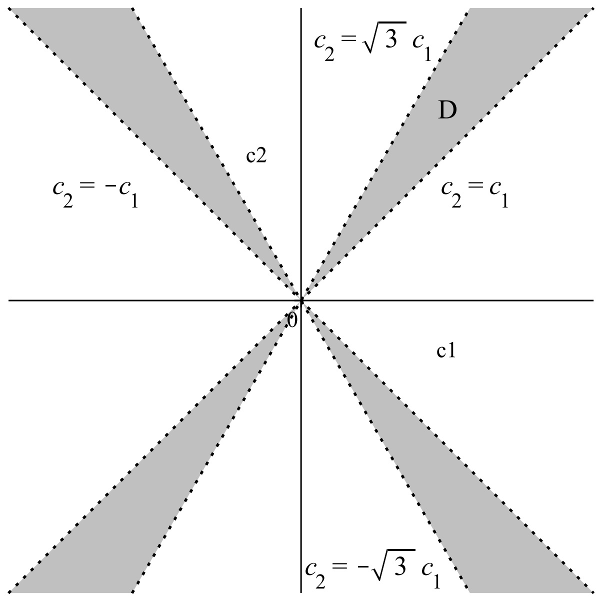

Figure 3-5.: Region D of attribute divergence in the $(c_1, c_2)-$plane

then, divergence is possible when $(c_1, c_2) \in D$, being D the portion of the $(c_1, c_2)-$plane, described by:

$$D = \{(c_1, c_2) \in \mathbb{R}^2 : (c_2^2 - c_1^2)(3c_1^2 - c_2^2) > 0\},$$

as illustrated in Figure (**3-5**) in gray areas. Then we can classify evolutionary equilibria in three categories:

- **Branching points (BP).** They are locally asymptotically stable evolutionary equilibria in which the attribute can branch, which occurs when both conditions (3-11) and (3-12) are satisfied. This implies that the BP occur when $(c_1, c_2) \in D$, as illustrated in Figure **3-5** and in the gray area in Figure **3-6** (labeled BP), where $\overline{x}_i \in \mathbb{R}$ is locally asymptotically stable and, in addition, conditions (3-11) and (3-12) are satisfied.

- **Terminal Points (TP).** They are locally asymptotically stable evolutionary equilibria, but they are not branching points. At these points the evolution and diversification is not possible. We have this situation when any of the conditions (3-11) or (3-12) fails. In this case it corresponds to $(c_1, c_2) \in T$, where

$$T = \{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 > 0, 3c_1^2 - c_2^2 < 0\}.$$

This region is shown in Figure **3-6** (in blue and labelled with TP).

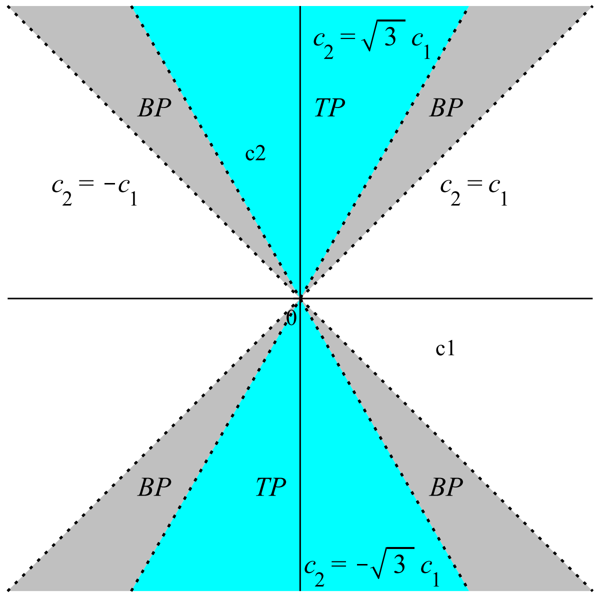

Figure 3-6.: Classification of stable evolutionary equilibria as BP, TP or BBP in the $(c_1, c_2)-$plane

- **Bifurcation Branching Points (BBP).** This situation corresponds to the border points between branch and terminal points. In this case we obtain the set of straight lines:

$$BBP = \{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 = 0, \vee, 3c_1^2 - c_2^2 = 0\}.$$

This bifurcation is unfolded in detail in [16, 22].

An example of competitive market dynamics under asymmetric interaction $c_2 > c_1$ is shown in Figure **3-7**-left. It illustrates the market dynamics under the influence of trait dependent maximum capacity K and interaction function c. The left panel shows the market dynamics considering $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, $x_1 = \overline{x}_1 = 31.7662$ and $x_2 = 1.1x_1$. Since $x_1 > 0$ and $x_2 > 0$, then the interaction corresponds to competition in the market, according to the model (3-1). The initial conditions $n_1(0) = K(\overline{x}_1) = 90.9836$ and $n_2(0) = 10$ were used. This scenario considers $x_2 > x_1$ and gives a coexistence condition $\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_1, \overline{x}_1) = -2.8763 \times 10^{-4} < 0$ and a divergence condition $\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_2, \overline{x}_2) = 1.9008 \times 10^{-4} > 0$. Hence, it corresponds to the case when both conditions (3-11) and (3-12) hold and then $\overline{x}_1$ is a Branching Point (BP). This market dynamics describes a case when IE invades the market but does not substitute SE. Thus they share the market.

By the other hand, Figure **3-7**-right illustrates an escenario when the evolutionary equilibrium $\overline{x}_1$ is a terminal point. The same parameters are considered but $c_2 = 2$. Therefore $x_1 = \overline{x}_1 = 15.2753$ and $x_2 = 1.1x_1$ and the initial conditions $n_1(0) = K(\overline{x}_1) = 70$

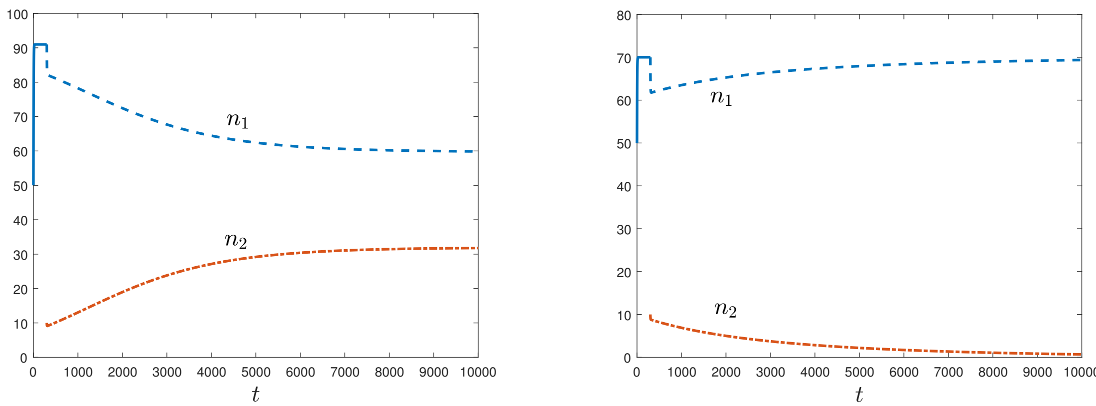

Figure 3-7.: Numeric simulation of market dynamics under trait dependent maximum capacity K and interaction function c. **Left**: shows the market dynamics considering $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, $x = x_1 = \overline{x}_1 = 31.7662$ and $x_2 = 1.1x_1$. Since $x_1 > 0$ and $x_2 > 0$, it correspond to competition in the market. Before the innovation occurs (solid line), the simulation corresponds to the resident model (3-2) with the initial condition $n(0) = 50$. Under the absence of competition, the equilibrium $\overline{n} = K(x) = 90.9836$ is reached. After the innovation, the simulation corresponds to system (3-1) with initial conditions $n_1(0) = K(\overline{x}_1) = 90.9836$ (dashed line) and $n_2(0) = 10$ (dash-dot line). Note that $(c_1, c_2) \in R$, thus the evolutionary equilibrium is a branching point (BP) and market diversification arises. This market dynamics describes a case when IE invades the market but does not substitute SE. Then they share the market. **Right**: corresponds to the same parameter configuration, but with $c_2 = 2$. In this case $x_1 = \overline{x}_1 = 15.2753$ and $x_2 = 1.1x_1$. Then the initial conditions are $n_1(0) = K(\overline{x}_1) = 70$ and $n_2(0) = 10$. Note that $(c_1, c_2) \in T$. Thus the evolutionary equilibrium is a terminal point (TP), and therefore diversification is not possible.

(recall $\overline{x}_1$ depends on c_2) and $n_2(0) = 10$ were used. Since $x_1 > 0$ and $x_2 > 0$, then the interaction corresponds also to competition. The coexistence condition is $\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_2, \overline{x}_2) = -8.2286 \times 10^{-4} < 0$ and the divergence condition $\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_2, \overline{x}_2) = -2.5714 \times 10^{-4} < 0$. The last one does not hold as stated by (3-12). In fact, as $(c_1, c_2) \in T$, the evolutionary equilibrium x_1 corresponds to a terminal point (TP) and evolution has a halt. Thus no market diversification is possible.

After the branching has occurred (i.e., both, coexistence and divergence conditions hold), the IE and SE share the market at the strictly positive equilibrium on the market space $P_3 = (\overline{n}_1(x_1, x_2), \overline{n}_2(x_1, x_2))$. Thus the IE becomes standard (i.e., there are two similar SE generation technologies with CGC n_1 and n_2 and characterized by the trait values x_1 and x_2 respectively). Now, it is possible to consider a new innovation to occur in any of the traits x_1 or x_2 leading to the appearance of a new (similar but slightly different) trait

x_1' or x_2'. This situation will be shown in the next two 3×3 systems:

$$\begin{cases} \dot{n}_1 = n_1 r(x_1) \left(1 - \frac{n_1 + c(x_1,x_2)n_2 + c(x_1,x_1')n_1'}{K(x_1)}\right) = n_1 g(n_1, n_2, n_1' x_1, x_2, x_1) \\ \dot{n}_2 = n_2 r(x_2) \left(1 - \frac{c(x_2,x_1)n_1 + n_2 + c(x_2,x_1')n_1'}{K(x_2)}\right) = n_2 g(n_1, n_2, n_1' x_1, x_2, x_2) \\ \dot{n}_1' = n_1' r(x_1') \left(1 - \frac{c(x_1',x_1)n_1 + c(x_1',x_2)n_2 + n_1'}{K(x_1')}\right) = n_2' g(n_1, n_2, n_1' x_1, x_2, x_1'), \end{cases} \tag{3-13}$$

and

$$\begin{cases} \dot{n}_1 = n_1 r(x_1) \left(1 - \frac{n_1 + c(x_1,x_2)n_2 + c(x_1,x_2')n_2'}{K(x_1)}\right) = n_1 g(n_1, n_2, n_1' x_1, x_2, x_1) \\ \dot{n}_2 = n_2 r(x_2) \left(1 - \frac{c(x_2,x_1)n_1 + n_2 + c(x_2,x_2')n_2'}{K(x_2)}\right) = n_2 g(n_1, n_2, n_1' x_1, x_2, x_2) \\ \dot{n}_2' = n_2' r(x_2') \left(1 - \frac{c(x_2',x_1)n_1 + c(x_2',x_2)n_2 + n_2'}{K(x_2')}\right) = n_1' g(n_1, n_2, n_1' x_1, x_2, x_2'). \end{cases} \tag{3-14}$$

After branching, it is irrelevant which one of the SE is called x_1 or x_2, and systems above are equivalent. The AD canonical equation governing the interaction of both SE's characteristic traits can be derived by repeating the analysis shown above. The invasion fitness of the IE's n_1' and n_2' are given by

$$\lambda(x_1, x_2, x_1') = g(0, \overline{n}_1(x_1, x_2), \overline{n}_2(x_1, x_2), x_1', x_1, x_2),$$

and,

$$\lambda(x_1, x_2, x_2') = g(0, \overline{n}_1(x_1, x_2), \overline{n}_2(x_1, x_2), x_2', x_1, x_2).$$

The canonical equation reads, respectively,

$$\dot{x_1} = \frac{1}{2}\mu_1 \sigma_1^2 \overline{n}_1(x_1, x_2) \left.\frac{\partial \lambda_1}{\partial x_1'}(x_1, x_2, x_1')\right|_{x_1' = x_1}, \tag{3-15}$$

and,

$$\dot{x_2} = \frac{1}{2}\mu_2 \sigma_2^2 \overline{n}_2(x_1, x_2) \left.\frac{\partial \lambda_2}{\partial x_2'}(x_1, x_2, x_2')\right|_{x_2' = x_2}, \tag{3-16}$$

where $\overline{n}_1(x_1, x_2)$ and $\overline{n}_2(x_1, x_2)$ are the coordinates corresponding to the coexistence equilibria P_3; i.e.,

$$\overline{n}_1(x_1, x_2) = \frac{c(x_1, x_2)K(x_2) - K(x_1)}{c(x_2, x_1)c(x_1, x_2) - 1}, \quad \text{and,} \quad \overline{n}_2(x_1, x_2) = \frac{c(x_2, x_1)K(x_1) - K(x_2)}{c(x_2, x_1)c(x_1, x_2) - 1}.$$

The explicit expressions of Equations (3-15) and (3-16) are omitted since they are very long. Nevertheless, they can be generated and handled by means of symbolic computation as in [27].

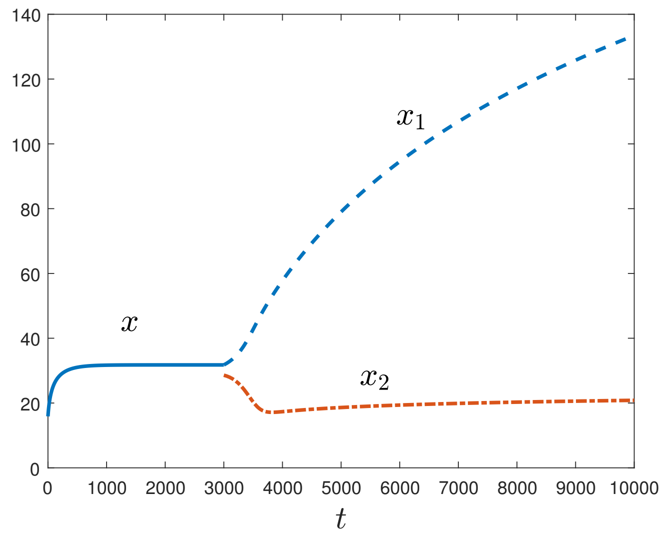

Figure 3-8.: Characteristic traits considering $x_2 < x_1$. It shows the trait dynamics with $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, corresponding to the branching point $\bar{x}_1 = 31.7662$ shown in Figure **3-7**-left. The first part of the curve (before the innovation occurs) corresponds to the simulation of equation (3-10) with initial condition $x(0) = 15.8831$. After the innovation, the curves correspond to the simulation of Eqs. (3-15) and (3-16) with initial conditions $x_1(0) = \bar{x}_1$ and $x_2(0) = 0.9\bar{x}_1$.

Figure **3-8** shows the evolutionary dynamics of the characteristic traits under asymmetric interaction ($c_2 > c_1$) and considering $x_2 < x_1$. The first part of the curve (before the innovation occurs) corresponds to the simulation of Equation (3-10) with initial condition $x(0) = 15.8831$. After the innovation, the curves correspond to the simulation of Equations (3-15) and (3-16) considering the parameter configuration described in the corresponding caption. Initially, x grows towards $\bar{x}_1$ until the branching occurs. Then, the dynamics is the result of the interaction between the innovative energy IE and the standard energy SE. The attribute x_1 permanently grows, while x_2 initially decreases. This makes perfectly sense since each of them is being governed by a different canonical equation, and therefore coexist under different selection pressures.

3.4. Degenerated scenarios in the energy market

The analysis of the proposed model allows us to observe that degenerate evolutionary branching arises when the derivatives in (3-11) or (3-12) vanishes. This scenarios correspond to the region

$$BBP = \{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 = 0, \vee, 3c_1^2 - c_2^2 = 0\},$$

corresponding to the Boundary Branching Points. This points correspond to the usual bifurcation points of the canonical equation. Evidently when $c_2^2 - c_1^2 = 0$, the equilibrium solutions $\overline{x}_1$ and $\overline{x}_2$ of the canonical equation (3-10) are not defined, and also, as has been already proven, $\overline{x}_0$ is always unstable, therefore in this case, the market does not have evolutionarily stable strategies and diversification is not possible. This is an undesirable scenario for the energy market. In deed, under the assumptions of this model, it implies that the investment (indebtedness, if it is considered a negative x) in power generation will grow without bound. This result is artificial, since in part it responds to the fact that the model does not consider restrictions imposed by demand or the budget that can be invested.

On the other hand, if $3c_1^2 - c_2^2 = 0$; that is, if $c_2 = \sqrt{3c_1}$, we obtain that the equilibrium points of the canonical equation are $\overline{x}_1 = 17.3205$ and $\overline{x}_2 = -17.3205$ (certainly the selection gradient $\frac{\partial \lambda}{\partial x_2}(\overline{x}_i, \overline{x}_i)$ vanishes for $i = 1$ or 2). Both equilibria are LAS, therefore correspond to evolutionary strategies, in which an evolutionary branch could occur. The conditions of coexistence and divergence in this case result in: $\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_1, \overline{x}_1) = -7.5000 \times 10^{-04} < 0$ and $\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_1, \overline{x}_1) = 0$ respectively. What it indicates, that although it is possible to obtain two sources of generation that coexist in the market, really, in the long term, it is not possible to differentiate one from the other. Equivalent results were obtained for $\overline{x}_2$.

In the Figure **3-9**, this situation is illustrated, in which it can be observed, additionally, that the innovative energy, although it manages to coexist with the established energy, does not really manage to efficiently penetrate into the market to establish itself in it.

3.5. Results and Conclusions

Local stability analysis of model (3-1) brings information on the market dynamics and helps to answer a further question of under what conditions can IE spread into the market and interact or even substitute SE. Indeed, in a market dominated exclusively by SE, the instability of P_2 is related to the possibility for an IE to invade the market, while the existence and stability of P_3 are related to the coexistence of both kinds of energy sharing the market, leading to diversification. Additionally, stability analysis implies that energy market will not crash under any circumstances, guaranteeing a permanent energy supply from any (or both) of the generation technologies. This situation means that there is at least one stable equilibrium corresponding to the dominance of SE, or IE or their coexistence, to satisfy energy demand. Even more, it shows that both cooperative and competitive strategies are adequate to guarantee market stability.

The interaction function c describes both, competition and cooperation, depending on the values of the characteristic traits: competition on the first and third quadrants of

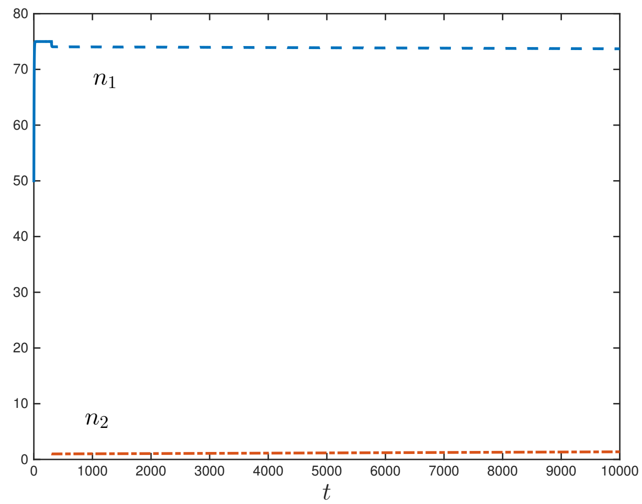

Figure 3-9.: Numeric simulation of market dynamics under the influence of trait dependent maximum capacity K and interaction function c. Shows the market dynamics considering $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, $x = x_1 = \overline{x}_1 = 17.3205$ and $x_2 = 1.1x_1$. Since $x_1 > 0$ and $x_2 > 0$, it correspond to competition in the market. Before the innovation occurs (solid line), the simulation corresponds to the resident model (3-2) with the initial condition $n(0) = 50$. Under the absence of competition, the equilibrium $\overline{n} = K(x) = 75$ is reached. After the innovation, the simulation corresponds to system (3-1) with initial conditions $n_1(0) = K(\overline{x}_1) = 75$ (dashed line) and $n_2(0) = 10$ (dash-dot line). Note that (c_1, c_2) belongs to the border between branching and terminal points regions. The conditions of coexistence and divergence in this case result in: $\frac{\partial^2\lambda}{\partial x_1 \partial x_2}(\overline{x}_i, \overline{x}_i) = \times 10^{-04} < 0$ and $\frac{\partial^2\lambda}{\partial x_2^2}(\overline{x}_i, \overline{x}_i) = 0$, thus the evolutionary equilibrium is a boundary branching point (BBP) and market diversification is an artifact, since it is possible to obtain two sources of generation that coexist in the market, but really, in the long term, it is not possible to differentiate one from the other and therefore diversification is not possible.

the $(x_1, x_2)-$plane and cooperation at the second and fourth quadrants of the (x_1, x_2)-plane. However, although functional parameters r, K and c are defined for all x_1 and x_2 in $\mathbb{R}$, and also the interaction dynamics defined by (3-1) is well defined for both market configurations, the invasion conditions determine specific regions of the $(x_1, x_2)-$plane under which the invasion of the innovative attribute is possible, and configurations that lead to its disappearance. Furthermore, it was proven that under convenient configurations of subsidies awarded (or taxes imposed) to both energy generation technologies, it is possible to determine scenarios in order to evolutionary equilibria to exist, to be locally asymptotically stable and also, it was shown that evolutionary stability implies coexistence.

Under the assumptions of our analysis, evolutionary equilibria of the ADCE can be terminal points, where no marginal innovation can invade into the market. However, evolutionary equilibria can also be branching points, where innovative energy can penetrate, coexist and diversify the market, concerning the previously established. In this context, although both parameters c_1 and c_2 describe the dynamics in the market time scale, they finally make a difference in the evolutionary time scale. In fact, the expressions $c_2^2 - c_1^2$ and $3c_1^2 - c_2^2$ are a measure of the strength of diversification through innovation. Taking into account the geometric characteristics of the interaction function c, we can say that diversification occurs in markets that are at least slightly asymmetric and in which IE is stimulated over SE, either by the allocation of subsidies or by the imposition of lower taxes.

In order to understand the evolution of the system after the second branch, it is necessary to repeat the analysis obtaining a $3-$dimensional canonical equation and then try to verify if the attributes converge to evolutionary solutions where the conditions of coexistence and divergence are satisfied. Due to the complexity of the expressions, it is necessary to perform the verification by computational methods. Finally, repeated process of innovation can give origin to a rich variety of different kinds of energy generation technologies. However, it is important to note that, this processes of emergence and disappearance of energy generation technologies is influenced by a wide range of external and internal factors, which may exert additional selection processes on innovations. Specific situations should be studied in greater depth and detail in order to achieve an informed decision making.

It has been proved for the model formulated in this chapter, that there are two conditions under which the degenerate evolutionary branching scenarios occur. The first one corresponds to the case in which the value of taxes (or subsidies) assigned to each type of energy are equal, a scenario that is undesirable for the energy market. In deed, under the assumptions of this model, it implies that the investment (indebtedness, if it is considered x as a negative number) in power generation will grow without bound. The second scenario corresponds to the case in which the condition of divergence is vanished, a situation in which the innovative energy, although it manages to coexist with the established energy, does not really manage to enter the market to establish itself in it. Additionally, a vanishing divergence condition indicates, that although it is possible to obtain two sources of generation that coexist in the market, really, in the long term, it may not be possible to differentiate one from the other.

The conditions established in this study to classify evolutionary equilibria as branching points, terminal points or degenerate branching points, can be used as control strategies that allow to reach precise objectives in relation to the long-term behavior of the energy market. It is inferred from the analysis of the model, under the assumptions considered here, that for the energy market to function in a “healthy” manner, it is necessary to exercise strict control over the taxes or subsidies that it is decided to apply to the sources of energy generation depending on the objectives that the regulatory agent wants to achieve. If the objective is to promote market diversification, it must establish interaction rules that locate the system in the branching points region, but if, on the contrary, the regulatory agent wants to avoid that new generation technologies have the possibility of entering the market, it is in the region of terminal points, or even the degenerate evolutionary branching region, where the system must be located.

4. Coffee Berry Borer (Hypothenemus hampei) and its role in the evolutionary diversification of the coffee market

4.1. Introduction

There are many elements which combine to make colombian coffee a unique product. Firstly, coffee farmers manually harvest only ripe coffee cherries, which requires great effort due to the topography of the colombian Andes. Secondly, farmers carry out post-harvest processes, which entail the elimination of defective grains, pulping, washing, and drying. Later, the coffee is threshed to obtain parchment coffee, the raw roasting material. The economic importance of coffee as an export good is well justified by the impact it has on thousands of colombian families and millions of consumers around the world [38].

According to the National Federation of Coffee Growers[1], (FNC for the acronym in Spanish), about 560000 families grow coffee in Colombia on farms of less than 5 hectares and they are responsible for 69 % of production. Of the 940000 hectares of coffee grown in Colombia, around 780000 correspond to technified crops, which means they are planted with improved coffee varieties, such as rust resistant trees (about 80 % compared to only 35 % that were in 2010). The annual production of Colombian coffee from 1956 to 2018, reported in Fig. **4-1**A, shows a positive trend, with turnarounds typically related to climatic phenomena (such as El Niño). It is however evident from Fig. **4-1**B that the revenue from exportations is not directly related to production. Global economic factors, such as the interaction with other international coffee markets, and even changes in the currency exchange rates, definitely influence the revenue, but there is a growing consensus on the role played by the coffee quality attributes, with respect to which the production is nowadays increasingly diversified. Indeed, a wide variety of specialty coffees are currently produced, and consumers around the world are learning the value of high quality coffee, for which they are willing to pay higher prices. The FNC defines the special coffees as

[1] Retrieved from `https://www.federaciondecafeteros.org`

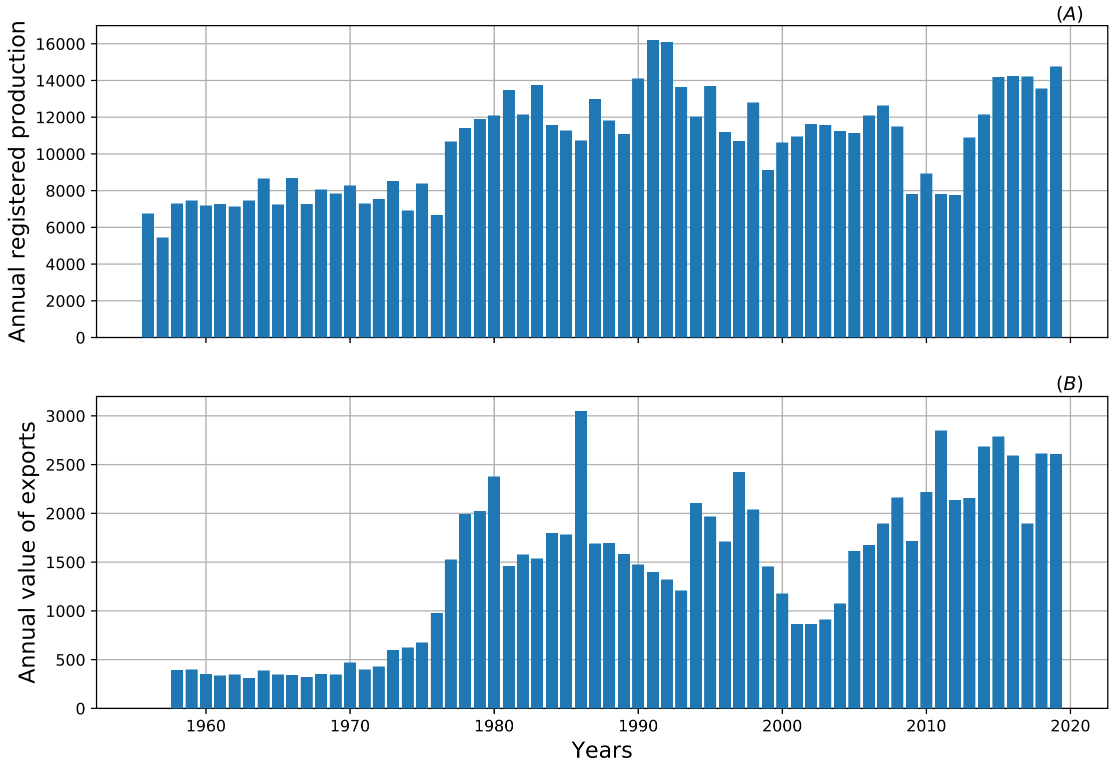

Figure 4-1.: (A) Anual registered Colombian coffee production in thousands of bags of 60 kg of green coffee. (B) Value of exports to all destinations - Anual total. Unit: Millions of dollars [38].

those obtained through (i) the development of new equipment and new forms of preparation, which guarantee better beverage quality, (ii) the association of coffee production with concepts such as sustainability, (iii) caring for the environment, (iv) social responsibility, and (v) economic equity. Special coffees are becoming a source of income for small producers who mainly market their product locally or through alternative trade shops, or who manage to export their coffee with certificated origin and production. According to the report "Global Specialty Coffee Market Size by Grade (80-84.99, 85-89.99, 90-100) by Application (Home, Commercial) by Region and Forecast 2019 to 2025"[2], the world market for coffee reached revenues up to USD 35.9 billion in 2018, with a prospect of USD 83.6 billion by 2025. Unfortunately, in Colombia, the systematic collection, analysis, and dissemination of accurate information on the production, processing, and sales of special coffees is not yet consolidated, mainly because special coffees are often produced in small quantities and marketed directly by the producer. Currently, the FNC statistical records of coffee exports are only available by type: green, decaffeinated green, roasted in beans,

[2]Retrieved from `https://www.adroitmarketresearch.com`

roasted and ground, and extract and soluble [38].

Beyond the main denominations (i)-(v) of special coffees, one of the factors that most impacts the quality of coffee is the cultivation protocols, that must be optimized to minimize the attack of pests and diseases, eliminate outer fruit layers, and control humidity [67]. The most widespread coffee pest worldwide is a beetle, the coffee berry borer (CBB) *Hypothenemus hampei* (Coleoptera: Curculionidae: Scolytinae) [72]. CBB adult females bore into coffee beans through the navel and into the endosperm, where it makes galleries to deposit its eggs. CBB causes different types of damage to the product, which are caused by (i) boring and feeding habits of adult and immature insects, which cause a reduction in yield and final product quality, and (ii) physical damage, which causes bored beans to become vulnerable to infection and further pest attacks [15]. Montoya's investigation [67] strove to define what CBB infestation percentages and what levels of damage permit to obtain of a coffee drink of acceptable quality. Specifically, the investigation supports the hypothesis that, beyond the damage, the CBB directly impacts the coffee market, by introducing a variety of different coffee types/qualities, classified depending on the proportion of bored grains that pass to production.

As a result of this situation, significant efforts have been made to identify the best CBB control strategies, as biological control, carried out via different types of interaction, mainly parasitism and predation [42, 51, 66, 68, 77]. Chemical control methods, use a series of insecticides to kill insects in adult states [72]. In addition, cultural control refers to agricultural practices, the safest of all possible, which consist, among others, in collecting adult, overripe, and dry grains (black grains no longer useful), in order to prevent adult CBB insects from finding refuge, and therefore preventing their reproduction [3,72]. Institutions supporting coffee production in Colombia, such as the FNC, recommend a combination of the three strategies, known as the Integrated CBB Control [14, 72].

This calls for a more comprehensive view of the CBB phenomenon, that includes the market dynamics and its feedback on coffee production and on the whole agro-ecological context. It is indeed the final coffee consumer who exerts the selection to determine which new products invade the market and which get established or eliminated. In turn, strategic decisions on the production and commercialization sides are source of innovation for coffee types and qualities, innovations that are then filtered by the market competition. In this sense, it is important to study this feedback loop between market and production of coffee, when CBB damage is considered a cause of quality differentiation. Such a study should be able to determine the conditions under which an innovative special coffee, characterized by some quality attribute that differentiates it from standard coffee, can invade the market and either replace or coexist with standard coffee. In particular, conditions for coexistence open up the possibility for further diversification, even of two initially similar products. Under this comprehensive view, the role CBB control strategies will be particularly highlighted, not only limited at the pest containment, but extended to control the emergence of market niches for different coffee products.

These kinds of questions are addressed by evolutionary modeling approaches, and particularly by the Adaptive Dynamics (AD) framework [31,34,45,46,64]. AD is a theoretical modeling framework which originated in evolutionary biology and describes the long-term evolution of quantitative traits (i.e., continuous attributes determined by the cumulative contributions of many genes). The key feature of AD is to explicitly consider the feedback loop that binds demographic and evolutionary change. In biology, demography selects the traits who win the competition and evolution proceeds through sequences of genetic mutations that are selected by demography. In economics, the role of mutations and selection are played by innovations on the production side and market competition on the consumer side. By focusing on incremental innovations, put forward at a frequency that is low compared to the market dynamics, AD describes the evolution of the products' traits (the coffee quality in our case) in terms of a differential equation, called the AD canonical equation, thus characterizing evolutionary equilibria as well as transients and non-stationary regimes [17,25,27,32,35]. Most importantly, AD endogenously integrates the changing system's diversity, as the number of coexisting product types increases when innovative and established products coexist and further differentiate, evolutionary branching, and is pruned when evolution eliminates outcompeted products, evolutionary extinction [22,36,45,46] (see [11,21,26,44] for further theoretical developments). This is the most important added value of AD to the economic literature on diversification. As extensively discussed in [23], product diversity is both a means and a result of economic development and growth in variety is crucial and not independent from growth in production [47,80,81,83]. In the AD framework, product diversity indeed emerges as a result of the feedback between innovation and competition processes.

Since its introduction, a wide range of applications have been published. Biologically oriented applications have addressed competition [24, 36, 53], and predator–prey interactions [1,2,17,27,32,35,59], food chains [32], mutualistic [19,34,36,40] and cannibalistic interactions [18,30], evolution of dispersal [13,28,73], and even evolution at the genetic level [12,55]. In the socio-economic context, technological innovations [23,29,85] and the evolution of fashion traits [58] have been investigated with the tools of AD. However, to the best of our knowledge, no application have addressed an agro-industrial phenomena, such as coffee production.

Here, a stylized AD model is formulated and analyzed to describe the evolution of coffee quality. An agro-ecological model describes the growth and harvest of the coffee plantation coupled with the demography of a CBB population, the latter explicitly structured into immature and adult individuals to reflect the damage caused by their reproduction and feeding habits. A market model describes the competition of different coffee types, defined by the proportion of healthy versus bored grains used in their production. Consumer preference favoring high or low quality is considered in competition describe their budget limitations, therefore, the model indirectly considers the rol of coffee prices. The model is, e.g., useful to derive the conditions under which a special coffee invades a market dominated by standard one; and conditions under which the special coffee eventually eliminates the standard one from the market, product substitution, or the two types coexist by sharing the market. Linking this conditions to the consumers' preference for low or hight quality coffees is a way to consider their budget limitations and the roles of coffee price in market diversification. Finally, the AD canonical equation describes the evolution of coffee quality, closing the feedback loop between the introduction of new coffee types and their competition in the market. The analysis of the model provides insights on the impact of CBB population on the evolving structure of the coffee market. The major result is that, independently of the consumers' preference for high or low quality, a mild control of the CBB population allows the emergence of several coffee types/qualities through evolutionary branching, whereas a strong (and expensive) pest control would impoverish the market diversity and could therefore lead to an economic loss.

4.2. Methods

4.2.1. Standard-special coffee model

We consider a coffee plantation of H hectares and n trees per hectare, each tree with the average productivity ρ (kilos of mature coffee beans on a healthy, unharvested tree, on average [5]). We do not consider seasonality, so the product $k = nH\rho$ gives us the biomass of coffee beans reached at equilibrium by the healthy, unexploited plantation (factors such as soil, climate, care, affecting production, are included in the parameter ρ, see [5]). This simplification allows us to use the logistic equation to describe the growth of the healthy coffee biomass $C(t)$ available in the plantation on a daily basis, with net growth rate r (the difference between daily production rate and loss of overripe and dry grains at low density) and carrying capacity k.

To include the effect of CBB on coffee production, we model the CBB population with two classes, according to the state of maturity of individuals. Let $I(t)$ be the density of immatures (eggs, larvae, and pupae) and $M(t)$ be the density of adult females in the plantation. Because adult females cause damage, by boring healthy beans to oviposit, we consider an average daily production of bored coffee given by the term $\beta C(t)M(t)$. Bored coffee beans translate into a new class of unhealthy coffee, denoted by $C_b(t)$, for which we use a loss rate d possibly higher than the one included in the net growth rate r of healthy coffee.

Assuming that coffee growers harvest coffee (healthy and bored) proportionally to the available biomass (constant harvesting effort h), we get the following two equations for the growth of coffee:

$$\begin{aligned} \dot{C} &= rC\left(1-\frac{C}{k}\right) - \beta CM - hC, && \text{(4-1a)} \\ \dot{C}_b &= \beta CM - (d+h)C_b. && \text{(4-1b)} \end{aligned}$$

At the same time, the demography of the CBB population is described by

$$\begin{aligned} \dot{I} &= \epsilon\beta CM - (\omega+\delta)I, && \text{(4-2a)} \\ \dot{M} &= \omega I - \mu M, && \text{(4-2b)} \end{aligned}$$

where parameter ω is the maturation rate ($1/\omega$ is the average duration of the immature stage), ϵ is the CBB reproductive efficiency (the food-to-oviposition conversion rate), and δ and μ are the mortality rates (from natural causes or as a result of control strategies by the farmer) in the two classes. Equations (4-1) and (4-2) constitute the agro-ecological model.

Harvested coffee is used for the production of parchment coffee and subsequent commercialization. Depending on the mix of healthy and bored coffee beans, we consider parchment coffee of two different qualities, which share the market with sold quantities (kg per day, on average) denoted by $N_1(t)$ and $N_2(t)$, respectively. These will henceforth be referred to as "standard coffee" for N_1, of quality q_1, and as "special coffee" for N_2, of quality q_2. The quality q is assumed to be a continuous attribute that controls the mix of healthy and bored coffee beans. Specifically, we use the smooth sigmoid function

$$Q(q) = \frac{q^\alpha}{q_0^\alpha + q^\alpha}, \quad \alpha > 1, \tag{4-3}$$

from 0 (at $q = 0$) to 1 (when $q \to \infty$), for the fraction of the harvest of healthy coffee to be used in production, the complementary fraction $1 - Q(q)$ taken from the harvest of bored coffee. The resulting quality therefore depends of the quality attribute q, but also on the agro-ecological context. The threshold parameter q_0 separates low quality coffee ($q_i < q_0$, so that $Q(q_i) < 1/2$), promoting the use of bored coffee beans, from high quality coffee ($q_i > q_0$, $Q(q_i) > 1/2$), promoting the use of healthy beans, $i = 1$ or 2.

Consumers' demand is not explicitly considered in this model. However, consumers' budget constraints are translated into the preference for high or low quality coffees. Consumers' preference is a source of competition between the two types of coffee, that we include in the following Lotka-Volterra market competition model:

$$\begin{aligned} \dot{N}_1 &= N_1\big(Q(q_1)hC + (1-Q(q_1))hC_b - f(q_1,q_1)N_1 - f(q_1,q_2)N_2\big), && \text{(4-4a)} \\ \dot{N}_2 &= N_2\big(Q(q_2)hC + (1-Q(q_2))hC_b - f(q_2,q_1)N_1 - f(q_2,q_2)N_2\big), && \text{(4-4b)} \end{aligned}$$

where $P_i = Q(q_i)hC + (1-Q(q_i))hC_b$ is the production of coffee type i and $f(q_i, q_j)$ is the competition function, measuring the loss of market share for coffee type i for each

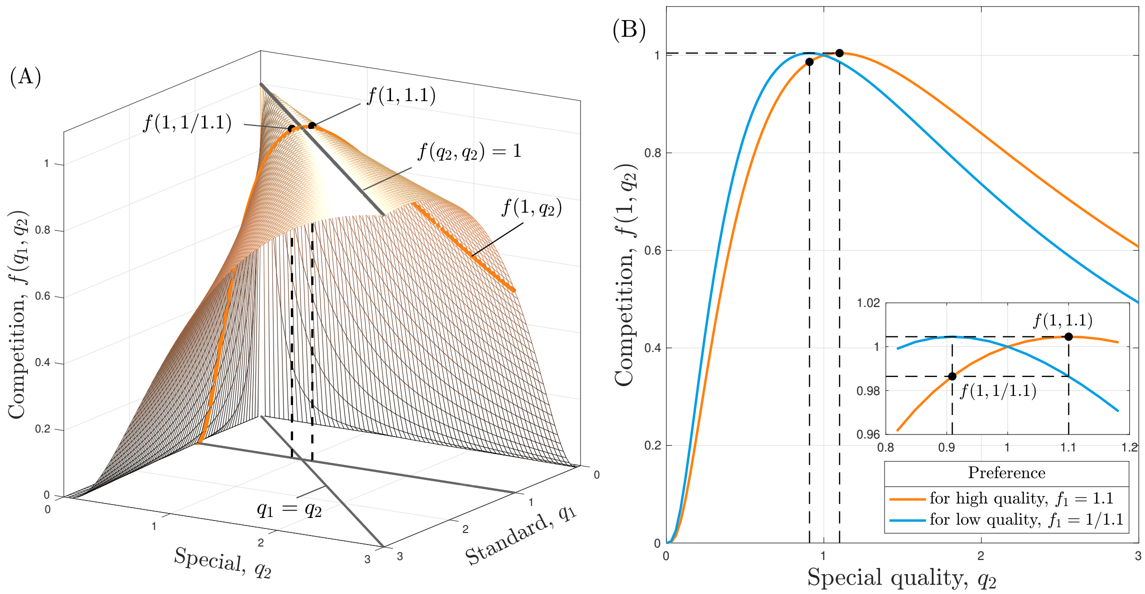

Figure 4-2.: (A) Competition function $f(q_1, q_2)$ with $f_1 = 1.1$ and $f_2 = 1$. The restriction to $q_1 = 1$ is shown by the orange curve. (B) Planar representation of the restriction to $q_1 = 1$ for both $f_1 = 1.1$ (orange) and $f_1 = 1/1.1$ (blue). As highlighted in the zoomed inset, for $f_1 = 1.1$ (consumers' preference for higher coffee quality; orange curve), the share loss for coffee type 1 (with quality $q_1 = 1$) is larger than 1 and maximum when $q_2 = f_1 = 1.1$ ($f(1, 1.1) = 1.0046$). At this value of q_2, the share loss $f(q_2, q_1)$ for coffee type 2 can be read on the orange curve, because $f(q_2, q_1) = f(1, q_1/q_2)$ ($f(1, 1/1.1) = 0.9865$). Similarly, for $f_1 = 1/1.1$ (consumers' preference for lower coffee quality; blue curve), the maximal share loss for coffee type 1 (again equal to 1.0046) is realized for $q_2 = 1/1.1 = 0.9090$, while the corresponding share loss for coffee type 2 can be read (on the blue curve) as $f(1, q_1/q_2) = f(1, 1.1)$ (again equal to 0.9865).

unit sold of coffee type j ($f(q_i, q_i) = 1$). In the absence of special coffee (type 2) and with a constant production P_i (at an equilibrium of the agro-ecological model (4-1, 4-2)), standard coffee (type 1) penetrates the market logistically, with initial rate ($\dot{N}_1/N_1$ at low N_1) assumed to be the production itself. Eventually, the sold amount reaches production (market clearing, i.e., $N_1 = P_1$), that sets a single-coffee market with quality q_1.

For the competition function, we use the log-normal formulation proposed in [23] in a context of technological change:

$$f(q_1, q_2) = \exp\left(\frac{\ln^2 f_1}{2 f_2^2}\right) \exp\left(-\frac{1}{2 f_2^2} \ln^2\left(\frac{f_1 q_1}{q_2}\right)\right). \tag{4-5}$$

See Fig. **4-2**A for a graphical representation. Coffee qualities with low or high ratio

q_1/q_2 are assumed to weakly compete (f tends to zero as the ratio goes to either zero or infinity), because targeted by customers with widely different budgets. On the contrary, similar coffees do compete. How competition fades as the ratio q_1/q_2 leaves 1 is controlled by parameter f_2, that plays the role of the log-normal standard deviation. A key competition parameter is f_1: it indicates the consumers' preference for higher ($f_1 > 1$) or lower ($f_1 < 1$) coffee quality, depending on budget constraints. Indeed, if $f_1 > 1$, the share loss $f(q_1, q_2)$ for coffee type 1 is larger/smaller than 1 if q_2 is larger/smaller than (and close to) q_1, while the loss $f(q_2, q_1)$ is reciprocally smaller/larger than 1 (see Figure **4-2**B, orange curve), which gives a competitive advantage to the larger quality. Vice-versa, $f_1 < 1$ gives a competitive advantage to the lower quality (see Figure **4-2**B, blue curve), while competition is symmetric for $f_1 = 1$.

The agro-ecological and market model (4-1, 4-2, 4-4) constitute our henceforth called "standard-special coffee model." It is subject to non-negative initial conditions. In Table **4-1**, we summarize the state variables and parameters, respectively indicating the initial conditions and the baseline values used in simulations.

4.2.2. Standard coffee model, invasion fitness, and invasion conditions

Before special coffee enters the market, standard coffee (type 1) is the only option, and the two eqs. (4-4a,b) of the standard-special coffee model degenerate into the single equation

$$\dot{N}_1 = N_1 \left(Q(q_1)hC + (1 - Q(q_1))\, hC_b - N_1 \right). \tag{4-6}$$

Equation (4-6), jointly with the agro-ecological eqs. (4-1, 4-2) constitute our henceforth called "standard coffee model." As already discussed in Section 4.2.1, the model directs the market dynamics to an equilibrium at which all the production is sold ($N_1 = P_1$). Including the variables of the agro-ecological model, and also the special coffee (type 2) with no sales ($N_2 = 0$), we denote this equilibrium with

$$E_0 : \big(C, C_b, I, M, N_1, N_2\big) = \big(\bar{C}, \bar{C}_b, \bar{I}, \bar{M}, \bar{N}(q_1), 0\big), \tag{4-7}$$

where

$$\bar{N}(q_1) = Q(q_1)h\bar{C} + (1 - Q(q_1))h\bar{C}_b \tag{4-8}$$

shortly denotes the "standard coffee equilibrium." Note that E_0 is also an equilibrium for the standard-special coffee model (4-1, 4-2, 4-4).

Invasion of a small amount of special coffee (type 2), arising from an innovation when the market is at (or close to) E_0 is possible only if the quality q_2 of the special coffee is such that E_0 is an unstable equilibrium for the standard-special coffee model. On the contrary, i.e., if E_0 is a locally asymptotically stable (LAS) equilibrium of the standard-special coffee model, then the initially small sales N_2 of special coffee will drop and the

Table 4-1.: State variables and parameters of the agro-ecological and market models, together with initializations and baseline values respectively employed in simulations. *brs = individuals of CBB in any state of maturation.

	State variable description		*Init. Cond.*	*Units*
C	Average healthy coffee biomass		1	kg
C_b	Average bored coffee biomass		0	kg
M	Average number of CBB adult females		1	brs*
I	Average number of immature CBB insects		0	brs
N_1	Average amount of standard coffee sold per day		1	kg
N_2	Average amount of special coffee sold per day		0.1	kg
	Parameter description	*Value*	*Units*	*Ref.*
q_1	Quality of standard coffee	varies	–	–
q_2	Quality of special coffee	varies	–	–
r	Net coffee growth rate	0.8	d^{-1}	[5]
H	Average number of cultivated hectares	1	ha	*ad hoc*
n	Average number of coffee trees per hectare	5484	tree·ha^{-1}	[5]
ρ	Average productivity per tree	0.3	kg	[5]
k	Coffee biomass reached at equilibrium	1645.2	kg	–
h	Harvesting rate	0.2	d^{-1}	–
d	Bored coffee loss rate	0.1	d^{-1}	[72]
β	Effective CBB boring rate	varies	d^{-1}	–
ϵ	CBB reproductive efficiency	0.02	kg^{-1}	*ad hoc*
ω	CBB maturation rate	1/7.95	d^{-1}	[39]
μ	Adult CBB death rate	varies	d^{-1}	–
δ	Immature CBB death rate	1/13.8	d^{-1}	[39]
q_0	Production-mix quality threshold	10	–	*ad hoc*
α	Production-mix sensitivity exponent	3	–	*ad hoc*
f_1	Quality consumers' preference (high > 1; low < 1)	varies	–	*ad hoc*
f_2	Quality width of competing coffees	1	–	*ad hoc*

the special coffee will exit the market soon after its introduction. The stability of E_0 is determined by the sing of the so-called "invasion fitness" of the innovation [31,65]

$$\lambda(q_1, q_2) = Q(q_2)h\bar{C} + (1 - Q(q_2))h\bar{C}_b - f(q_2, q_1)\bar{N}(q_1), \tag{4-9}$$

that is technically the eigenvalue of the system's Jacobian at the equilibrium E_0 associated with the eigenvector with nonzero N_2-component. More economically, the invasion fitness is the the per-unit penetration rate, in terms of initial sales (N_2 nearly zero), of the special coffee quality q_2, facing the established quality q_1 at its equilibrium $\bar{N}(q_1)$. Indeed, $\lambda(q_1, q_2)$ can be derived from eq. (4-4b) as $\dot{N}_2/N_2$, by setting $N_1 = \bar{N}(q_1)$ and $N_2 = 0$ in the right-hand side and by replacing C and C_b with their equilibrium values. Note that $\lambda(q_1, q_1) = 0$. This is economically obvious, because standard coffee is at the market equilibrium.

4.2.3. The AD canonical equation and conditions for branching in the quality attribute

The dynamics described by the standard-special coffee model (4-1, 4-2, 4-4) occur in the agro-ecological, industrial, and market timescales, simply "market timescale" in the following, where time t is measured in days. The main assumption of Adaptive Dynamics (AD) is that this timescale is faster than the one on which innovations in the production process are put forward in the market. In terms of the quality attribute here considered, if new types of coffee are put on sale on average every τ days, we assume τ is relatively large. The timescale t/τ, on which the unit represents the average time between two consecutive innovations, is called the "innovation timescale." This is the timescale on which AD describes the dynamics of the coffee quality attributes, that is indeed innovation-driven and henceforth called "innovation dynamics."

In the scenario in which only one type of coffee is established in the market, with quality q_1, and in the technical limit of rare (large τ) and small innovations, the expected innovation dynamics of the intrinsically stochastic path of q_1 is described by the following differential equation

$$\dot{q}_1 = \frac{1}{2}\sigma^2(q_1)\bar{N}(q_1)\ \frac{\partial}{\partial q_2}\lambda(q_1, q_2)\bigg|_{q_2=q_1}, \tag{4-10}$$

where the dot-notation here stands for the time-derivative on the innovation timescale. Eq. (5-18) is the AD canonical equation for the quality attribute q_1 [31, 34]. It describes the expected dynamics of q_1 resulting from a sequence of substitutions of the currently established coffee by an innovative one.

When an innovative type of coffee q_2 is put on sale, the established quality q_1 is at (or close to) its equilibrium $\bar{N}(q_1)$ (see eq. (4-8)), because of the large time τ elapsed since the previous innovation. The sign of the invasion fitness $\lambda(q_1, q_2)$ therefore determines whether the innovation invades or quickly disappears. Moreover, one of the theoretical pillar of AD, the "invasion implies substitution" theorem [26, 31], says that if q_2 is sufficiently close to q_1, invasion under a nonzero "selection gradient"

$$s(q_1) = \frac{\partial}{\partial q_2}\lambda(q_1, q_2)\bigg|_{q_2=q_1}, \tag{4-11}$$

implies the substitution of the former quality by the new one. Mathematically speaking, this means that the point of substitution

$$E_1 : \left(C, C_b, I, M, N_1, N_2\right) = \left(\bar{C}, \bar{C}_b, \bar{I}, \bar{M}, 0, \bar{N}(q_2)\right), \tag{4-12}$$

is also an equilibrium of the standard-special coffee model (4-1, 4-2, 4-4) and that the trajectories originating close to E_0 with low initial sales N_2 of the new quality coffee q_2 converge to E_1. After the substitution transient, the coffee quality q_1 is kicked out of the market and replaced by quality q_2, that can therefore be renamed q_1, i.e., the new established quality.

Note that the selection gradient determines the direction of the innovation process. Geometrically, it is the slope of the fitness landscape at (q_1, q_1) in the direction of the special coffee quality q_2. Considering the fitness first-order expansion w.r.t. q_2 at $q_2 = q_1$, i.e.,

$$\lambda(q_1, q_2) = \underbrace{\lambda(q_1, q_1)}_{0} + s(q_1)(q_2 - q_1) + \cdots, \tag{4-13}$$

one sees that under a positive selection gradient, the quality q_1 is replaced by innovative products with higher quality; vice-versa, under a negative selection gradient, lower quality coffees win the competition (in both cases, it results $s(q_1)(q_2 - q_1) > 0$ for q_2 sufficiently close to q_1).

Because of the timescale separation obtained for large τ, innovations can be considered one at a time and each substitution transient takes a small time on the innovation timescale. Assuming that innovations are randomly introduced into the market (at frequency $1/\tau$) with mean quality equal to the currently established q_1 and small standard deviation $\sigma(q_1)/\tau$, the expected quality dynamics become smooth on the innovation timescale and ruled by the AD canonical eq. (5-18). The name canonical follows from the fact that, in evolutionary biology, the selection gradient appears in other evolutionary models based on fitness landscapes, such as quantitative genetics [34].

The AD canonical eq. (5-18) can be used as long as the quality attribute q_1 is far from a stationary solution $\bar{q}$ that nullifies the selection gradient (4-11), a so-called "singular strategy" in the AD jargon. Indeed, close to a singular strategy, invasion does not necessarily imply substitution. Let us restrict the attention to a stable singular strategy $\bar{q}$, i.e., an attracting equilibrium of eq. (5-18), toward which the innovation process directs the quality attribute q_1. In particular, expanding the invasion fitness $\lambda(q_1, q_2)$ up to second-order w.r.t. both (q_1, q_2) at $(\bar{q}, \bar{q})$, one can see that $\lambda(q_1, q_2)$ and $\lambda(q_2, q_1)$ can both be positive close to $(\bar{q}, \bar{q})$, so that both quality attributes can invade a market established by the other. Without going into the details of the expansion (originally developed in [45, 46, 64] and also included in [31]), this occurs under the condition

$$\left.\frac{\partial^2}{\partial q_1 \partial q_2} \lambda(q_1, q_2)\right|_{q_1 = q_2 = \bar{q}} < 0 \tag{4-14}$$

in a region of the plane (q_1, q_2) that, locally to $(\bar{q}, \bar{q})$, is a cone with vertex in $(\bar{q}, \bar{q})$ and symmetric opening w.r.t. the anti-diagonal $(\bar{q} - q, \bar{q} + q)$. This is the region of "coexistence" of coffee types 1 and 2, because for (q_1, q_2) in this region, the trajectories of the standard-special coffee model (4-1, 4-2, 4-4) originating close to equilibria E_0 and E_1 converge to an internal equilibrium of coexistence, characterized by positive sales $\bar{N}_1(q_1, q_2)$ and $\bar{N}_2(q_1, q_2)$ of both coffee types [26],

$$\bar{N}_1(q_1, q_2) = \frac{h(Q(q_2)\bar{C} + (1 - Q(q_2))\bar{C}_b)}{f(q_2, q_1)}, \quad \text{(4-15a)}$$

$$\bar{N}_2(q_1, q_2) = \frac{h(Q(q_1)\bar{C} + (1 - Q(q_1))\bar{C}_b)}{f(q_1, q_2)}, \quad \text{(4-15b)}$$

The coexistence of two different, though very similar, types of coffee is the first step to go from a single-coffee market to a diversified one. However, to really generate two different products, the innovation process must be such that, after the coexistence is established, successive innovations move q_1 and q_2 in opposite directions. First of all, the invasion fitness of an innovative type q' facing two established coffee types with qualities q_1 and q_2 is a function of the three arguments (q_1, q_2, q'), that we denote by $\Lambda(q_1, q_2, q')$. Its expression is not important here and will be derived in the next section. Let us here consider $\Lambda(q_1, q_2, q')$ as a function of the innovative quality q', for given q_1 and q_2 (let us also assume $q_2 > q_1$, though this choice is irrelevant). Taking into account that $\Lambda(q_1, q_2, q')$ vanishes at both $q' = q_1$ and $q' = q_2$ (because of the market equilibrium), the quadratic expansion of Λ w.r.t. q' is shaped, locally to $\bar{q}$, as in Fig. **4-3**. Working out the details (see again [45,46,64] or [31]), it turns out that the discriminant between the two cases is linked to the invasion fitness in the single-coffee market (see eq. (4-9)). Specifically, if

$$\left.\frac{\partial^2}{\partial q_2^2}\lambda(\bar{q}, q_2)\right|_{q_2=\bar{q}} > 0, \quad \text{(4-16)}$$

innovations in the quality q_1 invade and replace the established coffee type 1 if $q' < q_1$, while the same occurs for the coffee type 2 if $q' > q_2$ (in both cases the invasion fitness $\Lambda(q_1, q_2, q')$ is positive, see Fig. **4-3**A). As a result, the quality attributes q_1 and q_2 get further diversified and the market selection is said to be "disruptive" at the singular strategy. On the contrary, if

$$\left.\frac{\partial^2}{\partial q_2^2}\lambda(\bar{q}, q_2)\right|_{q_2=\bar{q}} < 0, \quad \text{(4-17)}$$

only innovations that make q_1 and q_2 even more siw.r.t.milar win the competition. As a result, even if the model formally allows for the coexistence of two similar coffee types, in practice, the market remains single-product (see Fig. **4-3**B). Singular strategies characterized by condition (4-17) are said to be locally "evolutionarily stable" (ESS), that means protected from the invasion of similar strategies. Note that evolutionarily stability is a different concept from the dynamical stability of the equilibria of the canonical eq. (5-18).

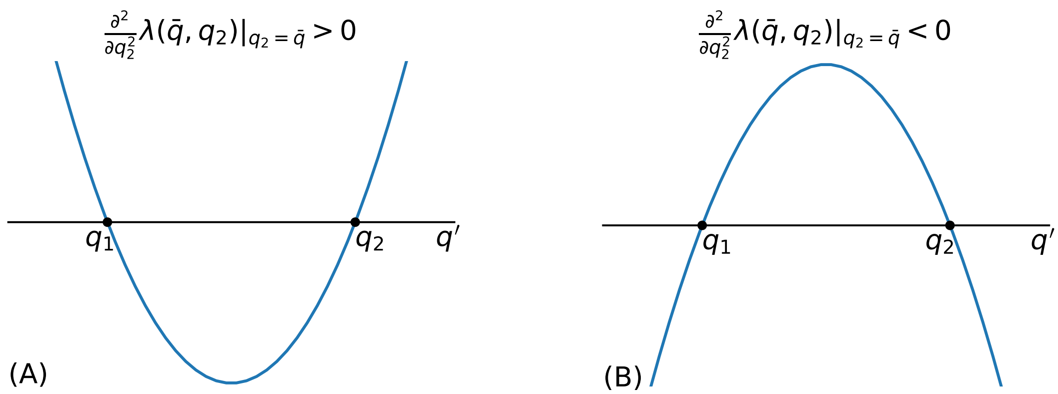

Figure 4-3.: (A) In a diversified market with strategies q_1 and q_2, an innovation q' in the coffee quality q_1 invade and substitute q_1 if $q' < q_1$; similarly, an innovation q' in the coffee quality q_2 invade and substitute q_2 if $q' > q_2$. (B) innovative strategies q_1 and q_2 evolves one toward each other, in this scenario further diversification is not posible.

To underline the difference, the latter concept is often called "convergence stability" in the AD jargon.

Convergence stable singular strategies characterized by conditions (4-14) and (4-16) are called "branching points" (BP) of the innovation process [45, 46, 64]. If $\bar{q}$ is a BP, the dynamics of the quality attribute q_1 is first attracted by $\bar{q}$, in a phase in which the market is single-product (there is only one type of established coffee that is at the market equilibrium for each value of q_1), but with q_1 evolving on the innovation timescale. Once close to $\bar{q}$, a second coffee type with quality q_2 gets established in the market (because of the coexistence condition (4-14)) and a second phase with two coexisting and evolving products begins. The disruptive condition (4-16) implies that the two quality attributes q_1 and q_2 initially diverge one from the other at the beginning of this second phase. To study the further innovation dynamics of q_1 and q_2, we need to derive a two-dimensional AD canonical equation, in which substitution sequences are considered for both the established coffee types. This is done in the following section.

Convergence stable singular strategies at which one or both of the conditions (4-14) and (4-16) hold with reversed inequality sign are called "terminal points" (TP) of the innovation process [31]. Indeed the innovation dynamics driven by rare and small innovations halt there. Cases with vanishing second fitness derivatives in (4-14) and (4-16) are bordering cases between BP and TP and are technically bifurcation points [16, 22].

4.2.4. Innovation dynamics after branching

After a branching at $\bar{q}$, the market sets at the equilibrium

$$E_c : \left(C, C_b, I, M, N_1, N_2\right) = \left(\bar{C}, \bar{C}_b, \bar{I}, \bar{M}, \bar{N}_1(q_1, q_2), \bar{N}_2(q_1, q_2)\right) \tag{4-18}$$

of the standard-special coffee model (4-1, 4-2, 4-4). The explicit expressions for the equilibrium sales $\bar{N}_1(q_1, q_2)$ and $\bar{N}_2(q_1, q_2)$ can be computed but are here omitted for brevity. Nevertheless, they have been handled by means of symbolic computation.

Similarly to the case of the single-coffee market, the invasion fitness $\Lambda(q_1, q_2, q')$ of an innovative coffee type with quality attribute q', facing the two established coffee types 1 and 2, is the per-unit penetration rate of the innovative coffee, in terms of initial sales N' (N' nearly zero), when coffee types 1 and 2 are at the market equilibrium E_c. To derive the per-unit penetration rate ($\dot{N}'/N'$), we need to extend the standard-special coffee model to a three-type model, including the standard, special, and innovative types. We here write only the differential equation for the innovative type:

$$\frac{\dot{N}'}{N'} = Q(q')hC + (1 - Q(q'))hC_b - f(q', q_1)N_1 - f(q', q_2)N_2 - N', \tag{4-19}$$

from which it immediately follows the expression of the invasion fitness:

$$\begin{aligned} \Lambda(q_1, q_2, q') = {} & Q(q')h\bar{C} + (1 - Q(q'))h\bar{C}_b \\ & - f(q', q_1)\bar{N}_1(q_1, q_2) - f(q', q_2)\bar{N}_2(q_1, q_2). \end{aligned} \tag{4-20}$$

Again assuming that innovations are rare events on the daily market time-scale that introduce small variations in coffee quality (i.e., quality q' is close to either q_1 or q_2), the expected innovation dynamics followed by the two established qualities q_1 or q_2 are ruled by the following two-dimensional AD canonical equation:

$$\dot{q}_i = \frac{1}{2}\sigma_i^2(q_i)\bar{N}_i(q_1, q_2) \left. \frac{\partial}{\partial q'}\Lambda(q_1, q_2, q') \right|_{q'=q_i}. \tag{4-21}$$

4.3. Results

4.3.1. The standard coffee model

An in-depth study of all stationary solutions of the standard coffee model (4-1, 4-2, 4-6) is not the goal of this paper, thus such a task remains available for a future work. However, the model has five stationary solutions $S_i = (\bar{C}, \bar{C}_b, \bar{M}, \bar{I}, \bar{N}_1)$, for $i = 0, \ldots, 5$, a brief discussion of them follows:

- $S_0 = (0, 0, 0, 0, 0)$. Constant solution always existing and corresponds to the absence of all magnitudes in the model. It lack practical interest. However, it is easy to prove that the eigenvalues of the linearization matrix, are: $\lambda_1 = r - h$, $\lambda_2 = -d - h$, $\lambda_3 = -\mu$, $\lambda_4 = -\delta - \omega$ and $\lambda_5 = 0$. Therefore, if $r - h > 0$ it is unstable, but if the contrary is true, it is not possible to study its stability with standard methods.

- $S_1 = \left(\frac{k(r-h)}{r}, 0, 0, 0, 0\right)$. This solution makes sense only under the condition $r - h > 0$, and corresponds to the presence of healthy coffee and the absence of both CBB and standard coffee sales (no standard coffee is produced). This scenario reflects situations in which coffee grows naturally, without human or CBB intervention. The eigenvalues of the linearization matrix in the neighborhood of this equilibrium are:

$$\lambda_1 = -(r-h), \quad \lambda_2 = \frac{kh(r-h)Q(q_1)}{r}, \quad \lambda_3 = -(h+d),$$

$$\lambda_{4,5} = \frac{-r(\delta+\mu+\omega) \pm \sqrt{4\epsilon\beta\omega kr(r-h) + r^2(\delta-\mu+\omega)^2}}{2r},$$

Under the condition $r - h > 0$ and since $0 < Q(q_1) \leq 1$, $\lambda_2 > 0$ is obtained, and there is sufficient information to state that E_1 is always unstable. If $Q(q_1) = 0$, (only bored coffee is used), then $\lambda_2 = 0$. On the other hand, if $r - h = 0$, then the equilibrium E_1 collides with E_0, a situation in which several null eigenvalues are obtained.

- $S_2 = \left(\frac{k(r-h)}{r}, 0, 0, 0, \frac{kh(r-h)}{r}\frac{Q(q_1)}{b(q_1)}\right)$. With $r - h > 0$, this constant solution corresponds to the scenario with both coffee production and standard coffee sales, but in the absence of CBB. In this case, a collision occurs with E_0 when $r - h = 0$ and, provided $r - h > 0$ and $Q(q_1) = 0$, then E_2 collides with E_1.

In the description of the following constant solutions, for ease of reading, the following has been defined:

$$B_0 = \frac{\epsilon\beta\omega kr}{\epsilon\beta h\omega k + \mu r(\delta+\omega)}.$$

as a biological parameter, corresponding to the "net reproduction rate of CBB" [76,78,79].

- $S_3 = \left(\frac{\mu(\omega+\delta)}{\epsilon\beta\omega}, \frac{\mu r(\delta+\omega)}{\epsilon\beta\omega(h+d)}\frac{B_0-1}{B_0}, \frac{r}{\beta}\frac{B_0-1}{B_0}, \frac{\mu r}{\beta\omega}\frac{B_0-1}{B_0}, 0\right)$. This constant solution reflects the presence of healthy and bored coffee and CBB, but the absence standard coffee sales . Note that $B_0 > 1$ is a necessary and sufficient condition for $\bar{C}_b > 0$, $\bar{M} > 0$ and $\bar{I} > 0$, a case in which there will be CBB in both adult and immature stages in the coffee crop. Also, if $B_0 = 1$ then $\frac{\mu(\omega+\delta)}{\epsilon\beta\omega} = \frac{k(r-h)}{r}$, as can be proven by means of algebraic processes, then, $B_0 = 1$ implies that E_3 and E_1 collide.

- $S_4 = E_0 = \left(\frac{\mu(\omega+\delta)}{\epsilon\beta\omega}, \frac{\mu r(\delta+\omega)}{\epsilon\beta\omega(h+d)}\frac{B_0-1}{B_0}, \frac{r}{\beta}\frac{B_0-1}{B_0}, \frac{\mu r}{\beta\omega}\frac{B_0-1}{B_0}, \frac{\mu(\omega+\delta)}{\epsilon\beta\omega}\frac{B_0 Q(q_1)h+(B_0-1)(1-Q(q_1))r}{B_0 b(q_1)}\right)$. Finally, if $B_0 > 1$, this constant solution represents the only equilibrium in which there is a presence of both healthy and bored coffee, presence of CBB in both states of maturity and standard coffee sales. Again, $B_0 = 1$ implies that E_2 and E_4 collide.

Solving the condition $B_0 > 1$ for the CBB boring and death rates β and μ and for the harvesting rate h, gives

$$\beta > \beta^* = \frac{\mu r(\delta+\omega)}{\epsilon\omega k(r-h)}, \quad \mu < \mu^* = \frac{\epsilon\beta\omega k(r-h)}{r(\delta+\omega)}, \quad h < h^* = r\left(1 - \frac{\mu(\delta+\omega)}{\epsilon\beta\omega k}\right). \quad \text{(4-22)}$$

These three parameters are often considered in the following, because they allow to discuss the joint effects of coffee production and CBB control practices. For instance, an intense harvesting rate h greater than h^*, would be enough to achieve CBB elimination.

Equilibrium E_0 is the invasion equilibrium for the special coffee entering the market. Whose stability conditions determine the possibility that a small sold quantity of special coffee (arising from an innovation) can enter and share the market with standard coffee.

4.3.2. Innovation dynamics in the single-coffee market: stability and bifurcation analysis

The invasion fitness (4-9) can be explicitly determined and handled numerically to illustrate invasion regions in the $(q_1, q_2)-$plane. In Fig. **5-7** the contour map of $\lambda(q_1, q_2)$ is shown for consumers' preference for low quality coffee ($f_1 = 1/1.1$ in panels A and C) and when consumers prefer high quality coffee ($f_1 = 1.1$ in panels B and D), the other parameters are as described in the caption. Each panel illustrates in blue the region where a small sold amount of special coffee N_2, can invade in a single-coffee market dominated by standard coffee N_1. The solid green curve and the solid black line (the diagonal $q_2 = q_1$) correspond to points where the invasion fitness λ vanishes; geometrically, two zero-level curves of λ intersect transversally, therefore the partial derivatives of λ are zero in any direction, particularly, the selection gradient vanishes, thus the intersection points $(\bar{q}_i, \bar{q}_i)$ for $i = 1$ or 2 correspond to singular strategies, i.e., the $\bar{q}_i$ are equilibria of the AD canonical equation, presented in Sect. 4.2.3).

The adult CBB death rate μ has been decreased from $\mu = 0.6$ in Fig. **5-7**A, where there are two intersections points, to $\mu = 0.3$ in Fig. **5-7**C without intersection points. Similarly, the adult death rate is increased from $\mu = 0.2$ in Fig. **5-7**B to $\mu = 0.5$ in Fig. **5-7**D to illustrate how, under the consumers' preference for high quality coffee, increasing CBB control leads to disappearance of both intersection points. Notice that when $\bar{q}_i$ do not exist, the consumer preference is the dominating force driving the innovation processes, so that the quality attribute diverge for $f_1 > 1$ and go to zero for $f_1 < 1$. Denote these as "trivial solutions", that correspond to a single coffee with top quality (100 % healthy

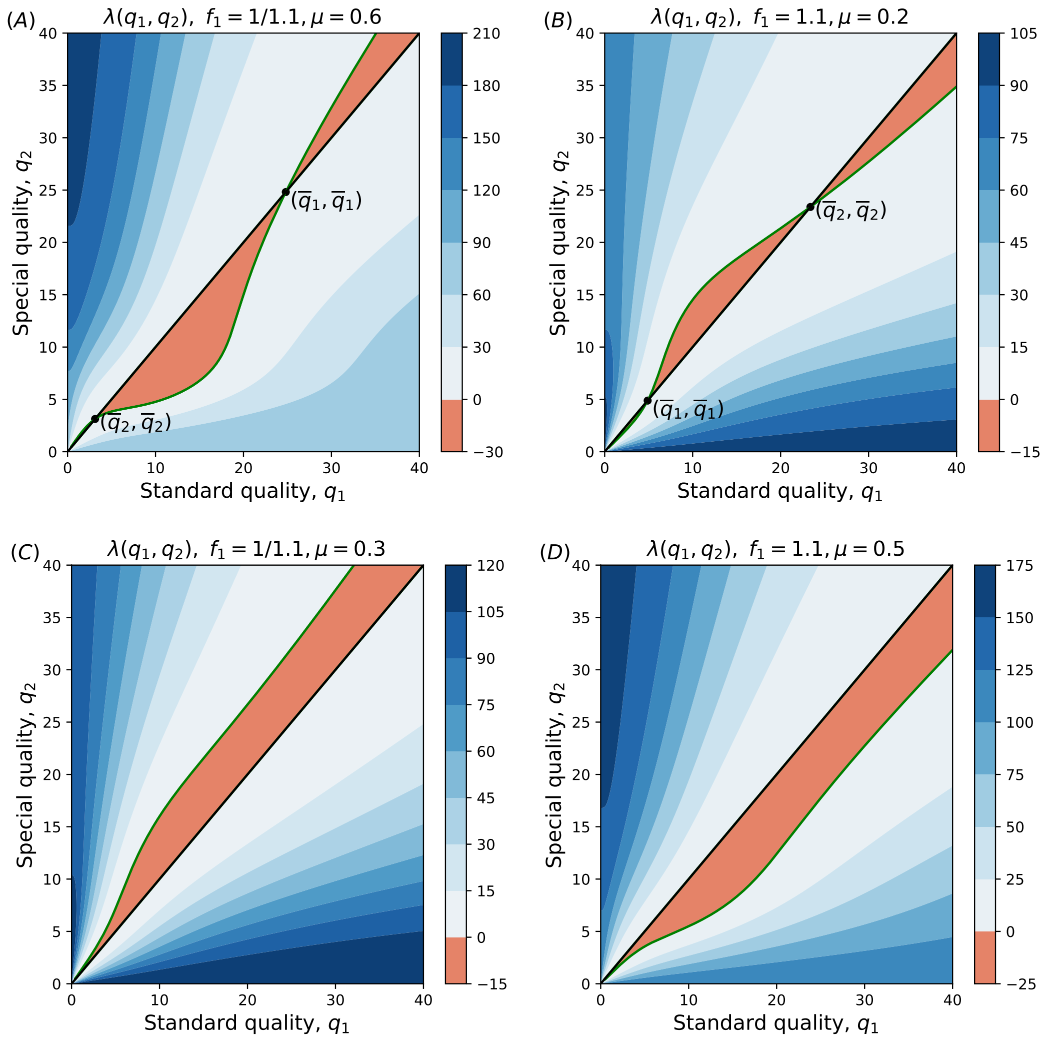

Figure 4-4.: The contour map defined by the invasion fitness $\lambda(q_1, q_2)$ is shown to illustrate four different scenarios, the invasion regions correspond to the points on the $(q_1, q_2)-$plane for which the invasion fitness is positive (blue regions). Parameters are shown in Table **4-1** with $\beta = 0.05$, $h = 0.2$ and f_1 and μ as indicated in the panel title.

coffee beans) for $f_1 > 1$, and to the worst-quality (100 % bored coffee beans) for $f_1 < 1$. This occurs for large μ under $f_1 > 1$ (Fig. **5-7**D), because with a low CBB population, the harvest is mainly composed of healthy coffee beans, so that increasing the quality does not reduce the production. Vice-versa, for $f_1 < 1$, the equilibria $\bar{q}_i$ disappear for low μ (Fig. **5-7**C), because only with a well-developed pest the harvest can sustain a low quality production. Interestingly, for intermediate μ a stable equilibrium $\bar{q}_1$ exists and the unstable solution $\bar{q}_2$ separates its basin of attraction from the one of the trivial solution.

To have an explicit expression for the AD canonical eq. (5-18), the selection gradient $s(q_1)$ is calculated, obtained from the expression (4-9) and equilibrium coordinates of E_0: $\bar{C}$, $\bar{C}_b$ and $\bar{N}(q_1)$, it is obtained the canonical eq. (5-18) as,

$$\dot{q} = F(q) = \frac{1}{2}m\sigma^2 \frac{h^2\mu^2(\omega+\delta)^2\left[F_2(q^\alpha)^2 + F_1 q^\alpha + F_0\right]}{(q_0^\alpha + q^\alpha)^3\beta^3\epsilon^3\omega^3 k(h+d) f_2^2 q}\left(q^\alpha + \frac{r q_0^\alpha}{h+d}\frac{B_0-1}{B_0}\right)$$

where,

$$F_2 = -\epsilon\beta\omega k(h+d)\ln f_1,$$

$$F_1 = -q_0^\alpha\left[\alpha\epsilon\beta\omega k\,(2h+d-r)\,f_2^2 + \mu r(\omega+\delta)\left(\alpha f_2^2 - \ln f_1\right) + \epsilon\beta\omega k(d+r)\ln f_1\right],$$

$$F_0 = -\frac{B_0-1}{B_0}\epsilon\beta\omega k r q_0^{2\alpha}\ln f_1.$$

This equation makes no sense for $B_0 < 1$, because the equilibrium coordinates $\bar{M}$ and $\bar{I}$ would be negative. Then, the singular strategies for $B_0 > 1$, satisfying $\dot{q}_1 = 0$, are given by,

$$\bar{q}_{1,2} = \left[\frac{-F_1 \pm \sqrt{F_1^2 - 4F_2F_0}}{2F_2}\right]^{1/\alpha}, \tag{4-23}$$

denoting $\bar{q}_1$ the "plus" solution, and $\bar{q}_2$ the "minus" solution. Note equilibria $\bar{q}_1$ and $\bar{q}_2$ coincide when $F_1^2 - 4F_2F_0 = 0$ (both equilibria collide through a fold bifurcation, indeed, there is no equilibria when $F_1^2 - 4F_2F_0 < 0$). To illustrate this bifurcation through simulations, we consider two scenarios: firstly, the effective CBB boring rate β to vary as a function of the harvesting rate h (considering the other parameters as fixed constants), and secondly, the adult CBB death rate μ to vary also as a function of the harvesting rate h. The bifurcation curve establishes a threshold that will be used in the next subsection to establish CBB control policies that promote or impede market diversification, according to the interests of the decision-maker.

Clearing β from the vanishing square root in (4-23), two explicit expressions for β defining bifurcation curves $\beta_k(h)$ and $\beta_b(h)$ are obtained,

$$\beta_b(h) = \frac{\mu r(\delta+\omega)}{\epsilon\omega k(r-d-2h)},$$

and,

$$\beta_k(h) = \frac{\mu r(\delta+\omega)(2\ln f_1 \alpha f_2^2 - \alpha^2 f_2^4 - \ln^2 f_1)}{\epsilon\omega k\left(2\ln f_1\alpha f_2^2(d+r) - (r-d-2h)(\alpha^2 f_2^4 + \ln^2 f_1)\right)}. \tag{4-24}$$

Subscript k is related to the curve color used in Fig. **4-5** (k stands for solid black, while subscript b is used to differentiate both expressions). Although not reported here, we have verified that provided $f_1 < 1$, the curve β_k is at the right of the curve β_b, and vice-versa, if $f_1 > 1$. Indeed, when $f_1 = 1$ both curves coalesce, it merits note that no singular strategies are obtained in the thin region defined by the two expressions.

An equivalent situation occurs when the adult CBB death rate μ is related to the harvesting rate by the farmer h; in this case, clearing μ from the bifurcation curve $F_1^2 - 4F_2F_0 = 0$ the following expressions are obtained:

$$\mu_b(h) = -\frac{2\epsilon\beta\omega k}{r(\delta+\omega)}h + \frac{\epsilon\beta\omega k(r-d)}{r(\delta+\omega)},$$

and,

$$\mu_k(h) = \left[-\frac{2\epsilon\beta\omega k}{r(\delta+\omega)}h + \frac{\epsilon\beta\omega k(r-d)}{r(\delta+\omega)}\right]\frac{\alpha^2 f_2^4 + \ln^2 f_1}{\left(\ln f_1 - \alpha f_2^2\right)^2} + \frac{\epsilon\beta\omega k(r-d)}{r(\delta+\omega)}\frac{2\alpha f_2^2(d+r)\ln f_1}{\left(\ln f_1 - \alpha f_2^2\right)^2(d-r)}, \quad (4\text{-}25)$$

corresponding to a pair of lines in the $(h,\mu)-$plane. With $\beta = 0.05$, Fig. **4-5** shows the graph of $\mu_k(h)$ in solid black, and the graph of $\mu_b(h)$ is omitted since it plays no role in the analysis.

Similarly, if h is cleared from $F_1^2 - 4F_2F_0 = 0$, two expressions are obtained for the bifurcation curve, given explicitly by:

$$h_b(\mu) = -\frac{\beta d\epsilon\omega k - \epsilon\beta\omega k r + \delta\mu r + \mu\omega r}{2\epsilon\beta\omega k}$$

and,

$$h_k(\mu) = -\frac{\left[\epsilon\beta\omega k(d-r) + (\delta+\omega)\mu r\right]\left[\alpha^2 f_2^4 + \ln(f_1)^2\right] + 2\ln(f1)\alpha f_2^2\left[\epsilon\beta\omega k(d+r) - (\delta+\omega)\mu r\right]}{2\epsilon\beta\omega k\left[\alpha^2 f_2^4 + \ln(f_1)^2\right]}. \quad (4\text{-}26)$$

In Table **4-2** these expressions are used to establish regions where the singular strategy of the canonical equation is a branching point.

Fig. **4-5** (panels A and B for $\mu = 0.2$) illustrates the value reached by the singular strategy $\bar{q}_1$ when the effective CBB boring rate and the harvesting rate are varied: for consumers' preference favoring low quality coffee $f_1 = 1/1.1$ in Fig. **4-5**A, and for consumers' preference favoring high quality coffee $f_1 = 1.1$ in Fig. **4-5**B. Each panel shows the curve $B_0 = 1$ in solid red and the fold bifurcation curve $\beta_k(h)$ in solid black.

Considering that only $B_0 > 1$ makes sense is clear that the blue shaded region between the black and red curves allows for estimating the values of h and β required to obtain non-negative singular strategies. Similarly, panels C and D in Fig. **4-5** (for $\beta = 0.05$), illustrate the singular strategy $\bar{q}_1$ when the adult CBB death rate and the harvesting rate are varied. For consumers' preference favoring low quality $f_1 = 1/1.1$ (see Fig. **4-5**C), the singular strategies are obtained in the shaded region between the black bifurcation

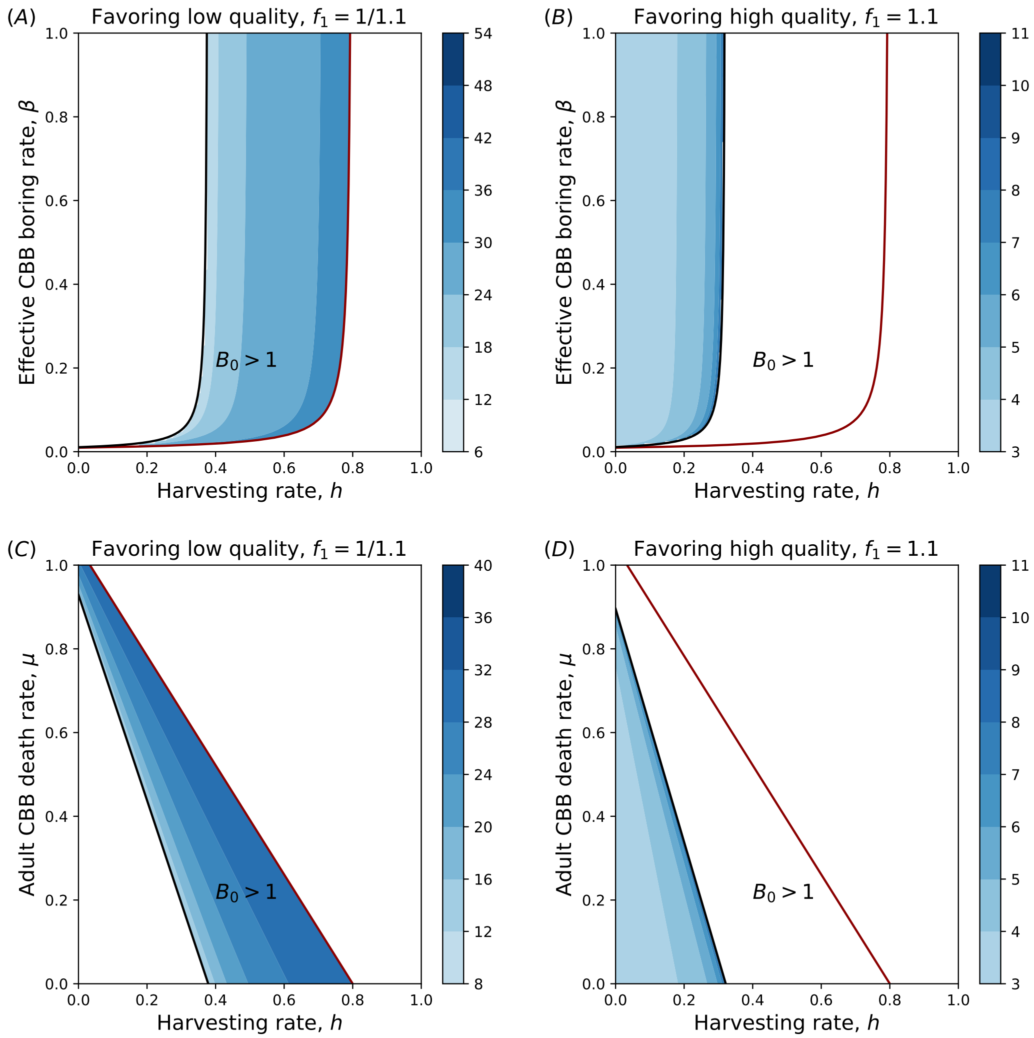

Figure 4-5.: Regions of definition of $\bar{q}_1$ (when real and non negative) are shown in dashed blue. The region where $B_0 > 1$ is at the left of the red curve ($B_0 = 1$). The bifurcation curve is shown in solid black. The parameter values used are those shown in Table **4-1**. (A) considering $\mu = 0.2$, notice $\bar{q}_1$ is non-negative for pairs (h, β) such that $B_0 > 1$ (points at the left of the red curve and at the right of the black bifurcation curve). (B) for $\mu = 0.2$, in this case, $\bar{q}_1$ is non-negative for pairs (h, β) at the left of the black bifurcation curve. Panels (C) and (D) illustrate the same situation, but considering the $(h, \mu)-$plane and $\beta = 0.05$.

line $\mu_k(h)$ and the red line $B_0 = 1$. For consumers' preference for high quality $f_1 = 1.1$ (Fig. **4-5**D) the definition region is located at the left of the black bifurcation line.

The stability of a singular strategy $\bar{q}$ is determined by the sign of the associated eigenvalue $\frac{ds(q_1)}{dq_1}|_{q_1=\bar{q}}$, where $s(q_1)$ is the selection gradient $s(q_1)$ defined in (4-11), when

Table 4-2.: Conditions on β, μ and h guaranteeing that $\bar{q}_1$ is a stable nontrivial singular solution. The threshold values β^*, μ^* and h^* were defined respectively in (4-22). The bifurcation thresholds β_k, μ_k and h_k were derived in (4-24), (4-25) and (4-26) respectively.

Consumers' preference	*Condition on* β	*Condition on* μ	*Condition on* h
Preference for low quality ($f_1 < 1$)	$\beta^* < \beta < \beta_k$	$\mu_k < \mu < \mu^*$	$h_k < h < h^*$
Preference for high quality ($f_1 > 1$)	$\beta_k < \beta$	$\mu < \mu_k$	$h < h_k$

negative the singular strategy is stable, and it is unstable if the associated eigenvalue is positive. Considering the parameter definitions and baseline values in Table **4-1**, the region of stability of the singular strategy $\bar{q}_1$ correspond to the same shaded blue regions in Fig. **4-5**, in which the singular strategy $\bar{q}_1$ is defined; on the other hand, in those scenarios $\bar{q}_2$ is unstable.

In general, considering the thresholds in (4-22), (4-24) and (4-25), under preference for low quality coffee ($f_1 < 1$), it is required an effective CBB boring rate satisfying $\beta^* < \beta < \beta_k$ and $\mu_k < \mu < \mu^*$ to guarantee that q_1 is a stable singular strategy, but when consumers prefer high quality coffee ($f_1 > 1$) it is required $\beta_k < \beta$ and $\mu < \mu_k$. Regarding the harvesting rate (observe Fig. **4-5**A), when consumers prefer low quality coffee, stability is achieved for intermediate harvesting rates h (between 0.3 and 0.8 approximately), while when consumers favor high quality coffees as in Fig. **4-5**B, stability occurs for small values of h (below 0.3 approximately); harvesting rates in the white region between the black bifurcation and red line (still $B_0 > 1$) can be used, but in this case we lose the singular strategy and therefore the coffee quality becomes top (100 % healthy beans). As mentioned before, intense harvesting rates lead to $B_0 < 1$, the region at the right of the red curve, and (theoretically) to CBB elimination, an escenario lacking of meaning for our model. Similarly in Fig. **4-5**C, when consumers' preference favors low quality coffee, a low adult CBB death rate μ permits intermediate harvesting rates ($h_k < h < h^*$, with h_k as described in (4-26)) and viceversa, a high adult CBB death rate allows for smaller harvesting rates ($h < h_k$). These analytical results are presented in Table **4-2**.

4.3.3. Innovation dynamics in the standard-special coffee market: the emergence of diversity through branching

Once we have a stable singular strategy $\bar{q}_1$, closed forms for the coexistence (4-14) and divergence (4-16) conditions were obtained and have been handled numerically to illustrate results. Fig. **4-6**A shows the coexistence condition in the $(h, \beta)-$plane when consumers favor low quality coffee ($f_1 = 1/1.1$, panel A), observe that coexistence condition is satisfied in the shaded red region corresponding to the stability region (compare with

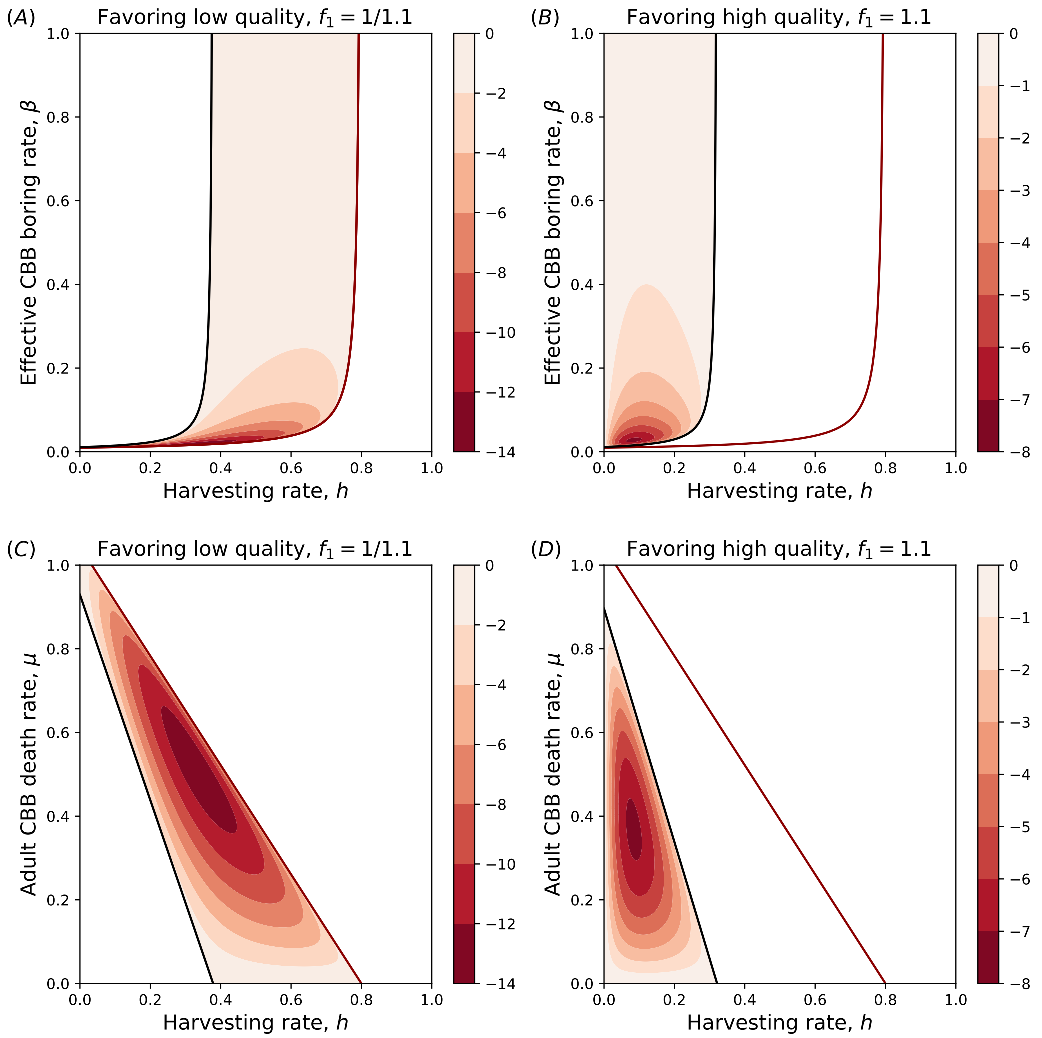

Figure 4-6.: The value corresponding to coexistence condition (4-14) at $\bar{q}_1$ is shown (red shading when negative), considering the parameter values as in Table **4-1**. The black curves correspond to the fold bifurcation and red curves to $B_0 = 1$. The $(h, \beta)-$plane is considered with $\mu = 0.2$ for $f_1 = 1/1.1$ (panel A) and $f_1 = 1.1$ (panel B). The $(h, \mu)-$plane is considered in lower panels with $\beta = 0.05$, for $f_1 = 1/1.1$ (panel C) and $f_1 = 1.1$ (panel D). Notice that the coexistence condition holds (red shaded region) in the whole definition and stability region of $\bar{q}_1$ (compare with Fig. **4-5**).

Fig. **4-5**A), i.e., stability implies coexistence; the same situation can be observed in the Fig. **4-6**B where the coexistence condition is illustrated when consumers favor high quality coffee ($f_1 = 1.1$). Similarly, Figures **4-6**C and **4-6**D illustrate the coexistence condition in the $(h, \mu)-$plane at $\bar{q}_1$; the results are equivalent to those obtained for h and β, i.e., the coexistence condition is satisfied in the same region of stability of $\bar{q}_1$.

To illustrate when the selection is disruptive, i.e., the divergence condition (4-16).

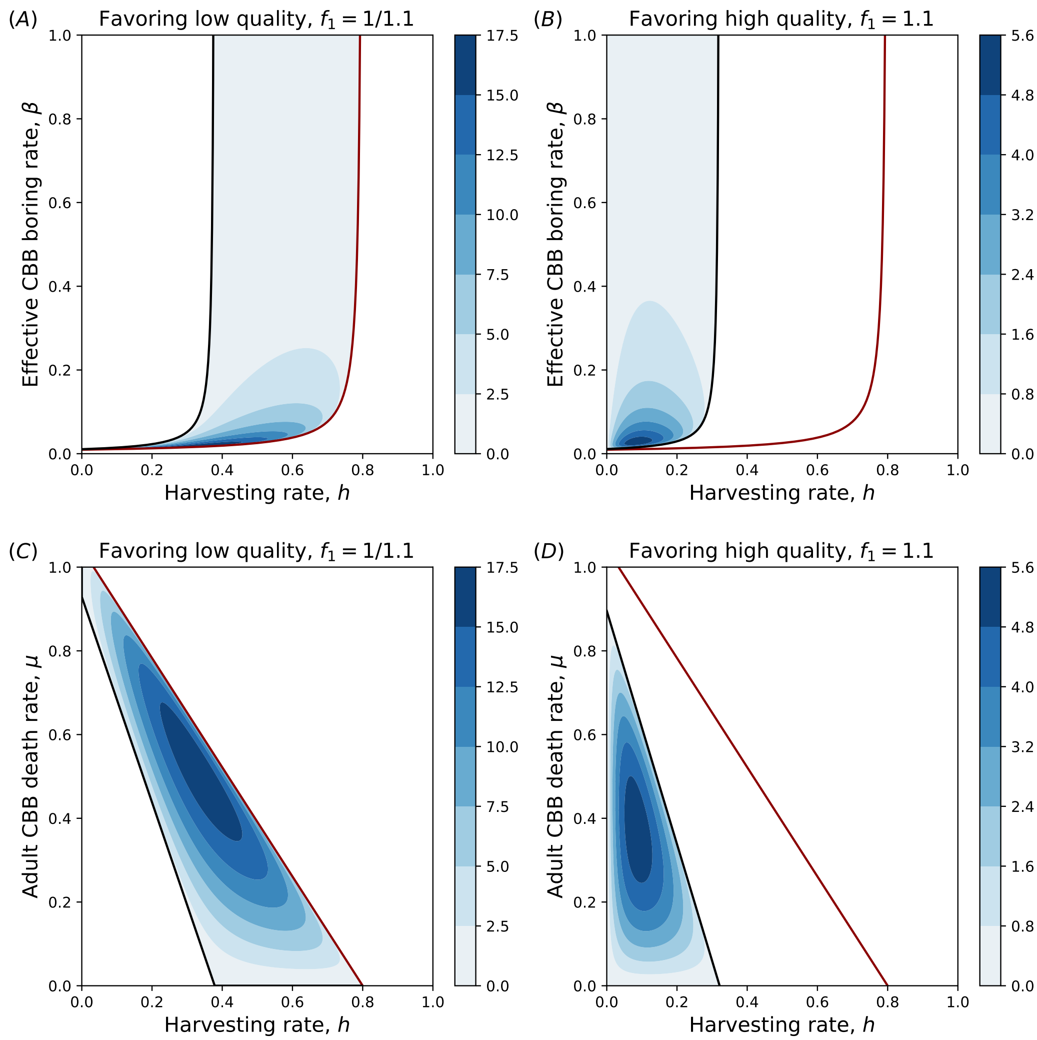

Figure 4-7.: The contour map illustrates in blue shading the positive values of the divergence condition (4-16) at $\bar{q}_1$. The parameters are considered as in Table **4-1**. The black curves correspond to the fold bifurcation and red curves to $B_0 = 1$. (A) It is considered the $(h, \beta)-$plane with $f_1 = 1/1.1$, and $\mu = 0.2$. (B) The $(h, \beta)-$plane but $f_1 = 1.1$, with $\mu = 0.2$. (C) Considers the $(h, \mu)-$plane with $\beta = 0.05$ and $f_1 = 1/1.1$ and (D) considers the $(h, \mu)-$plane with $\beta = 0.05$ and $f_1 = 1.1$.

Shaded blue regions in Fig. **4-7** show the divergence condition at $\bar{q}_1$ in the $(h, \beta)-$plane (panels A and B), respectively in the $(h, \mu)-$plane (panels C and D), when consumers favor low quality coffee ($f_1 = 1/1.1$) in panel A [resp. C] and when consumers favor high quality coffees ($f_1 = 1.1$) in panel B [resp. D]. It is worth noting that the divergence regions are the same obtained for stability and coexistence. Then, the shaded regions correspond to points where evolutionary branching takes place, allowing for the emergence of market diversity. The major result here is that strategy $\bar{q}_1$ satisfies the divergence condition both for $f_1 > 1$ and $f_1 < 1$, in the same regions of convergence stability and coexistence, defining the evolutionary branching regions.

The importance of identifying the region of evolutionary branching lies in the following. If the parameters are set as in Table **4-2** (for instance $h = 0.2$ and $\mu = 0.6$ in Fig. **4-7**C), the AD canonical equation (5-18) will reach a convergence stable strategy (a branching point) for the coffee quality type 1 ($\bar{q}_1$) allowing for the market diversification; i.e., an innovative coffee of quality q_2 can invade and diversify the market, in deed, both sold amounts of standard (N_1) and special (N_2) coffees will share the market. From there on, the evolutionary dynamics of both coffee qualities is driven by the 2D canonical equation (4-21) and will reach a two dimensional equilibrium (q_1^* and q_2^*), where the star superscript is used to differentiate this new equilibruim for coffee type 1 from the former one $\bar{q}_1$. In the next two subsections, we analyze the case in which branching develops, creating diversity, and then we perturb μ to illustrate the impact of varied control polices on diversification.

Results equivalent to those obtained in the $(h, \beta)-$plane were also attained (but not reported here) in the $(h, \epsilon)-$ and $(h, \omega)-$planes, which indicates a very close relationship between the effective CBB boring rate β, CBB reproductive efficiency ϵ and CBB maturation rate ω; it means that a control strategy affecting any of said rates seems to have direct or indirect impacts on the others. Similarly, results equivalent to those in the $(h, \mu)-$plane are also obtained in the $(h, \delta)-$plane, i.e., for the purposes of the model considered herein, control strategies for adult and immature CBB are equally effective. If this study is repeated focussed on the other parameters, CBB control policies that guarantee diversification in the market may be obtained, such as those found for β, μ and h in Table **4-2**.

4.3.4. Innovation dynamics under consumers' preference for high-quality

Dynamics in the market timescale before innovation is governed by the standard coffee model (4-1, 4-2, 4-6). Fig. **4-8**, in solid black, shows the dynamics for $q_1 = 1$, $h = 0.2$, $\beta = 0.05$, μ as indicated in the panel titles, initial conditions $C(0) = 1$, $C_b(0) = 0$, $M(0) = 1$, $I(0) = 0$, and $N_1(0) = 0.1$ and considering preference for high quality coffee ($f_1 = 1.1$), such that we meet the policies described in Table **4-2**. In those cases, for $\mu = 0.2$ we get $B_0 = 2.2645$ and for $\mu = 0.5$ is $B_0 = 1.3718$, then an equilibrium E is reached setting the market as dominated by the single standard coffee (type 1). To illustrate the evolutionary dynamics, Fig. **4-9**A shows the simulation of the canonical equation (5-18) in dashed black, with initial condition $q_1(0) = 1$; from which the evolutionary trajectory describes the standard coffee quality growing up to the branching point $\bar{q}_1 = 4.8831$; simultaneously, Fig. **4-9**B tracks the sold amount of standard coffee at equilibrium $\bar{N}(q_1)$ (dashed black), where q_1 is the numeric solution of (5-18) shown in Fig. **4-9**A.

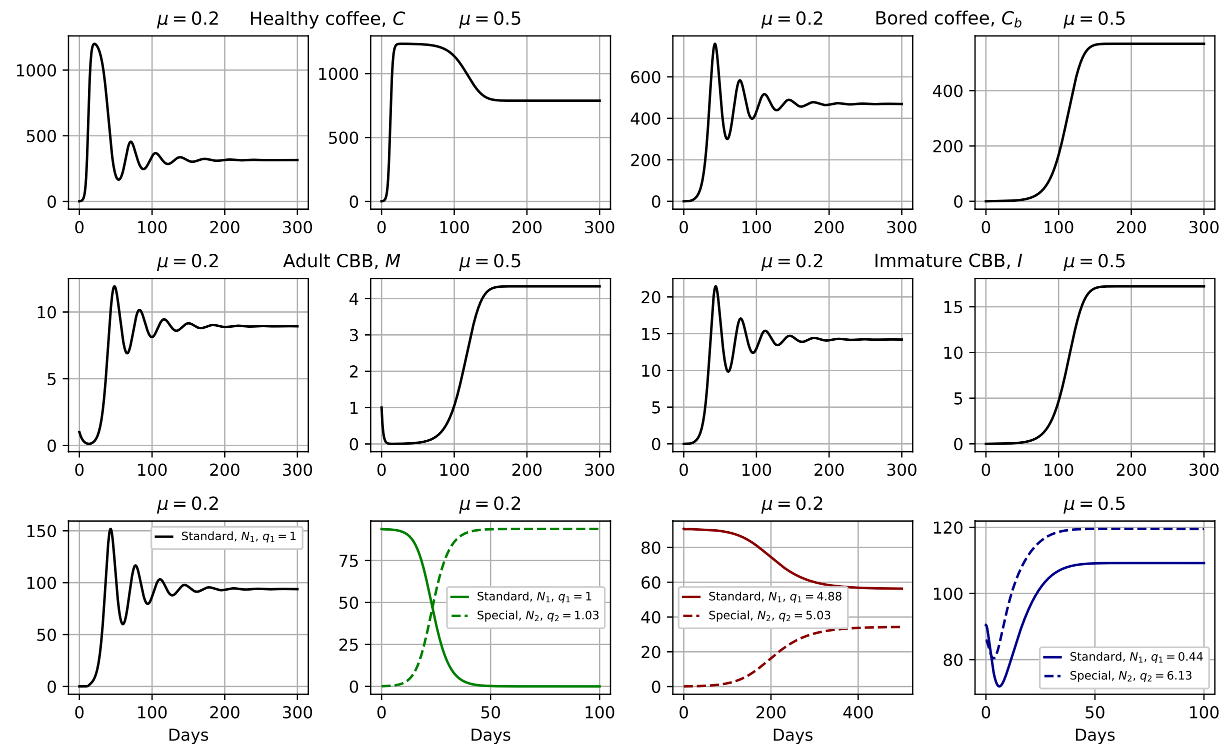

Figure 4-8.: Dynamics before and after innovation, considering the parameters in Table **4-1**, with μ as indicated in the panel title, $\beta = 0.05$ and $h = 0.2$. To illustrate the scenario when consumers' preference favors high quality coffee we set $f_1 = 1.1$ (see the right panels in Fig. **4-7**). Before the innovation occurs (black), the simulation corresponds to the standard coffee model (4-1, 4-2 4-6). After the innovation, the simulation corresponds to the standard-special coffee model (4-1, 4-2, 4-4) and illustrate the escenarios of substitution (green) and diversification (red). Fanally the impact of increasing control practices after diversification is shown (blue).

The last three panels in the bottom of Fig. **4-8** (red, green and blue) show the dynamics after innovation of the standard and special coffee governed by the eqs. (4-1, 4-2, 4-4). To illustrate a substitution scenario (green) we use $q_1 = 1$, $q_2 = 1.03q_1$ and the initial conditions at E. Notice the sales of special coffee N_2 increases from a very scarce average until it dominates the market, while the standard coffee is eliminated.

In of Fig. **4-8** (red), the coexistence scenario is shown, in this case, $q_1 = \bar{q}_1$, $q_2 = 1.03\bar{q}_1$ and $N_1(0) = 0.1$ are used, with the other initial conditions for the agro-ecological compartment at E. Note both sold amounts of special and standard coffees manages to share the market, creating diversity; indeed, the evolutionary dynamics of standard q_1 and special q_2 qualities are governed by the 2D canonical equation (4-21), illustrated in Fig. **4-9**A (red) with initial conditions at $q_1(1) = \bar{q}_1$ and $q_2(1) = 1.03\bar{q}_1$. Notice that special quality (dashed red) coexists and diverges from the standard quality (solid red), particu-

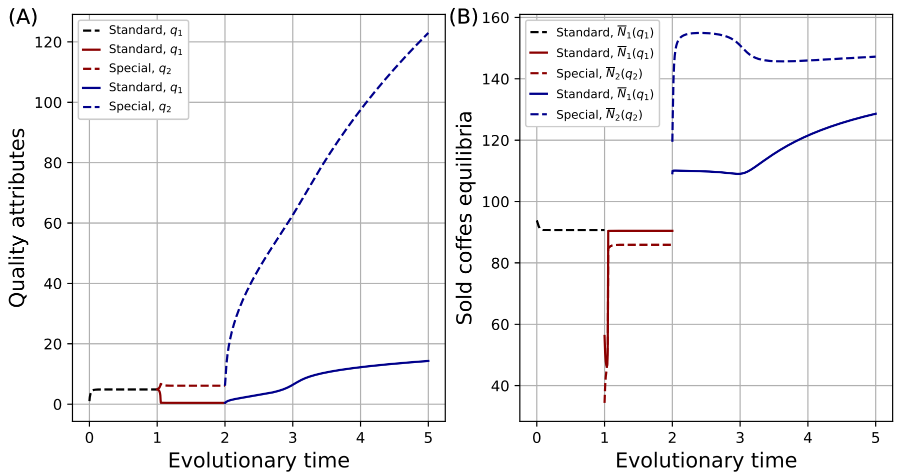

Figure 4-9.: (A) Shows the canonical eq. (5-18) before innovation (black) and the 2-dimensional canonical eqs. (4-21) after innovation, considering the parameters in Table **4-1**, $\beta = 0.05$, with $\mu = 0.2$ (red) and $\mu = 0.5$ (blue) to illustrate the impact that adult CBB death rate has on diversification. This scenario corresponds to consumers' preference favoring high quality coffee ($f_1 = 1.1$). (B) Shows the plots of standard coffee at equilibrium $\bar{N}(q_1)$ before innovation (dashed black). After innovation, shows both standard $\bar{N}_1(q_1, q_2)$ and special $\bar{N}_2(q_1, q_2)$ equilibria with $\mu = 0.2$ (solid red and dashed red resp.) and $\mu = 0.5$ (solid blue and dashed blue resp.)

larly, the special quality reach the equilibrium $q_2^* = 6.1326$ while the standard reach the equilibrium $q_1^* = 0.4375$. Fig. **4-9**B (red) shows the corresponding standard and special coffees' sold amounts at the equilibria $\bar{N}_1(q_1, q_2)$ and $\bar{N}_2(q_1, q_2)$, particularly we have at the end of these curves $\bar{N}_1(q_1^*, q_2^*) = 90.4880$ (dashed red) and $\bar{N}_2(q_1^*, q_2^*) = 85.9428$ (solid red).

Once diversification arises, and the sold amounts of special N_1 and standard N_2 coffees are stablished into the market, the value of the adult CBB death rate has been increased to $\mu = 0.5$, in order to illustrate the effect that a more intense control policy would have (see Fig. **4-8** in blue). Here the standard-special coffee model (4-1, 4-2, 4-4) is simulated with $q_1 = q_1^*$, $q_2 = q_2^*$ and initial conditions at the standard and special coffee sales at equilibrium $\bar{N}_1(0) = \bar{N}_1(q_1^*, q_2^*)$ and $N_2(0) = \bar{N}_2(q_1^*, q_2^*)$. It can be seen both standard (solid blue) and special (dashed blue) coffees manages to share the market. Regarding the evolutionary dynamics governed by the 2D canonical equation (4-21), in Fig. **4-9**A, once the qualities q_1 and q_2 reach their equilibrium values (final point in the red curves) the adult CBB death rate has been increased to $\mu = 0.5$ (blue curves); not only both qualities persist, but they begin to grow unbounded, that is to say, that the quality of the coffee improves progressively as time passes, particularly with the quality of special coffee q_2 growing much faster than the quality of standard coffee q_1. A similar analysis can be done from Figs. **5-7**B and **5-7**D, where increasing the CBB death rate from $\mu = 0.2$ to $\mu = 0.5$ cause the disappearance of the singular strategy $\bar{q}_1$, and permits the coffee quality to increase unboundedly. Fig. **4-9**A (in blue) tracks the quantities sold of special and standard coffees at equilibrium $\bar{N}_1(q_1, q_2)$ and $\bar{N}_2(q_1, q_2)$, where q_1 and q_2 are the solutions of the 2D canonical equation (in blue) shown in Fig. **4-9**A.

Mathematically speaking, here we could note that a fold scenario, similar to the one found in the canonocal equation (5-18), is most likely present also in the 2D canonical equation, i.e., the equilibrium present for $\mu = 0.2$ seems not to be anymore present for $\mu = 0.5$. Nevertheless, the detailed analysis of the 2D canonical equation is outside the scope of the present paper.

4.3.5. Innovation dynamics under consumers' preference for low quality

Fig. **4-10** shows in solid black the standard model (4-1, 4-2 4-6) before innovation under consumers' preference for low quality ($f_1 = 1/1.1$), with $q_1 = 30$, $\mu = 0.6$ and $\mu = 0.3$ as indicated in the panels title. To meet the policies in Table **4-2** we have set $h = 0.2$ and $\beta = 0.05$, the other parameters are as described in Table **4-1**. In this case $B_0 = 1.2124$ for $\mu = 0.6$ and $B_0 = 1.8608$ for $\mu = 0.3$. In both scenarios an equilibrium E is reached such that, as expected, the standard coffee N_1 is the only product available and dominates the market (as illustrated for $\mu = 0.6$ in the third row-first column panel). Evolutionary dynamics for the standard quality q_1 is illustrated in Fig. **4-11**A. It corresponds to the numerical simulations of the canonical eq. (5-18) (dashed black), with initial condition $q_1(0) = 30$; notice q_1 decreases until reaching the branching point $\bar{q}_1 = 24.8104$. Fig. **4-11**B, the corresponding sold amount of standard coffee at equilibrium $\bar{N}(q_1)$ is shown, where q_1 is the numerical solution to the AD canonical equation.

After innovation, the dynamics is governed by the standard-special coffee model (4-1, 4-2, 4-4). Firstly, the escenario where substitution takes place is illustrated for $\mu = 0.6$, $q_1 = 30$ and $q_2 = 0.97q_1$ (green in Fig. **4-10**) with initial conditions at E and $N_2(0) = 0.1$. In this scenario, coexistence is not posible, in deed, the small initial sold amount of special coffee (dashed red) invades the market and grows until manages to eliminate the standard coffee N_1 (solid red). There is no place for diversification since at the end only type 2 coffee will be available in the market.

The escenario of market diversification is illustrated Fig. **4-10** (red), where the only change w.r.t. the previous simulation is $q_1 = \bar{q}_1$ and $q_2 = 0.97q_1$. Here the sold amount of special coffee N_2 (dashed red) can invade the market but do not eliminate the standard coffee N_1 (solid red); both types of coffee share the market, giving origin to diversity. The evolutionary dynamics of the standard and special qualities q_1 and q_2 is governed by the 2D canonical equation (4-21), with initial condition at $q_1(20) = \bar{q}_1$ and $q_2(20) = 0.97q_1$,

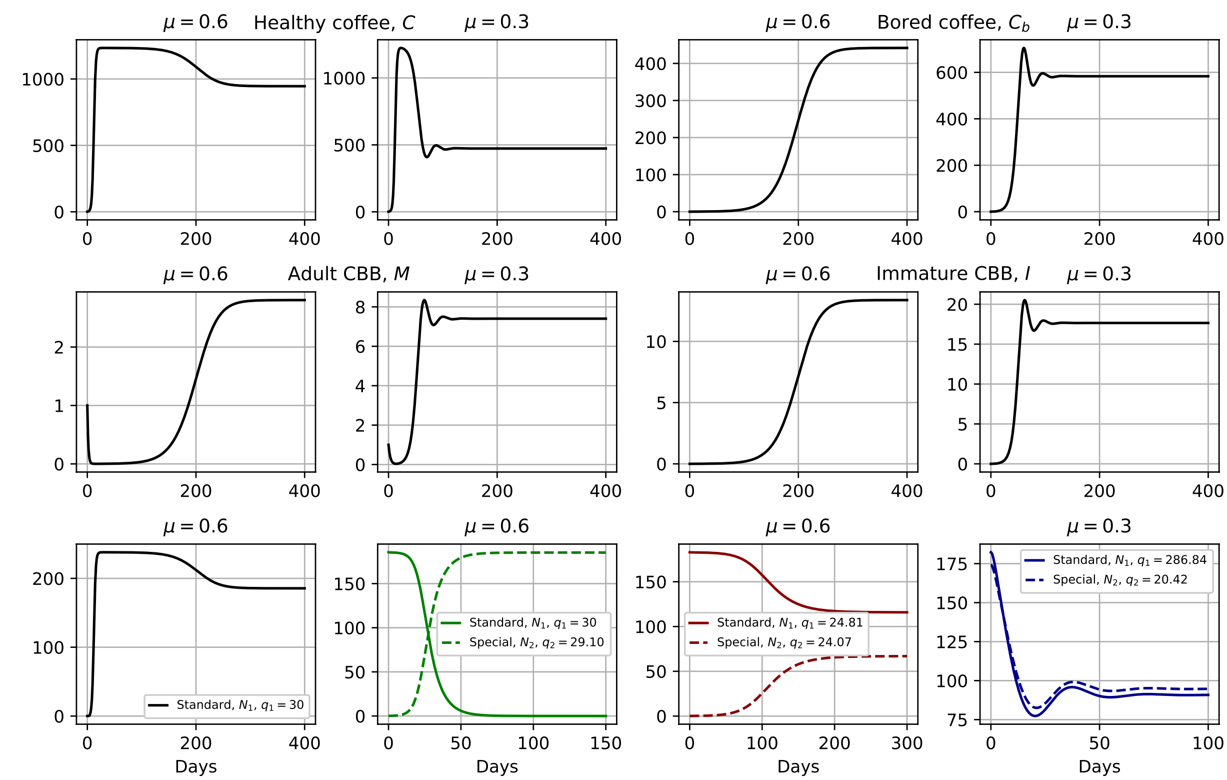

Figure 4-10.: agro-industrial dynamics before and after innovation, considering the parameters in Table **4-1**, with $\mu = 0.6$, $\beta = 0.05$ and $f_1 = 1/1.1$ (consumers' preference favors low quality coffee). Before the innovation (black), the simulation corresponds to the standard coffee model (4-1, 4-2 4-6). After the innovation, the simulation corresponds to the standard-special coffee model (4-1, 4-2, 4-4) with $\mu = 0.6$ (red) and $\mu = 0.3$ (blue).

and illustrated in Fig. **4-11**A (red). It can be seen that both qualities stabilize at different equilibrium levels, $q_1^* = 286.8386$ (solid red) and $q_2^* = 20.4216$ (dashed red) respectively. As for Fig. **4-11**B, it tracks the equilibrium sold amounts of standard $\bar{N}_1(q_1, q_2)$ (in solid red) and special $\bar{N}_2(q_1, q_2)$ (dashed red) coffees, where q_1 and q_2 are the solutions of the 2D canonical equation.

Once the market is diversifies, i.e., N_1 and N_2 are stablished coffees in the market, in the log term they reach equilibrium levels $\bar{N}_1(q_1^*, q_2^*) = 182.2967$ and $\bar{N}_2(q_1^*, q_2^*) = 174.2238$ for quality attributes also at equilibrium q_1^* and q_2^*. Then adult CBB death rate is decreased to $\mu = 0.3$ (blue in Fig. **4-10**) to illustrate the effect of a weaker pest control policy. In this case, initial conditions are at E, but $N_1(0) = \bar{N}_1(q_1^*, q_2^*)$ and $N_2(0) = \bar{N}_2(q_1^*, q_2^*)$. Notice both the standard coffee N_1 (solid blue) and special coffee N_2 (dashed blue) share the market at very similar levels. Nevertheless, as illustrated in Fig. **4-11**A, both standard quality q_1 (solid blue) and special quality q_2 (dashed blue), corresponding to the 2Dcano-

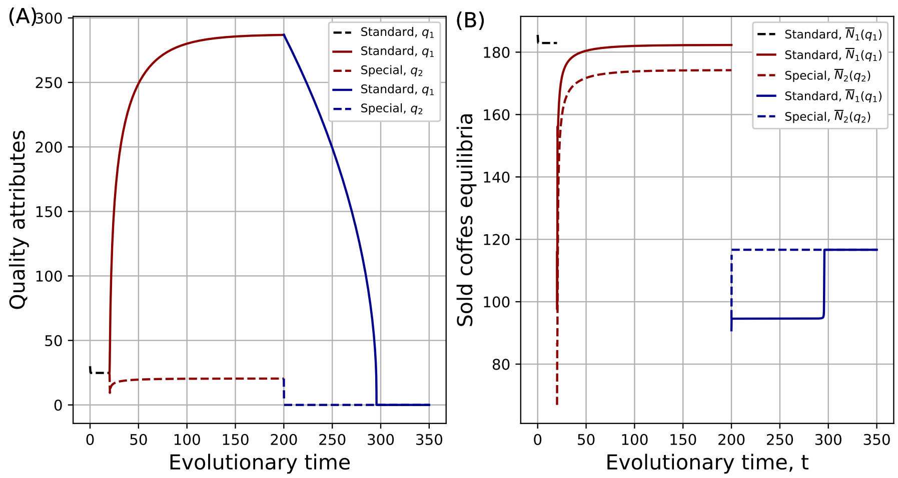

Figure 4-11.: (A) Shows the canonical eq. (5-18) before innovation (black) and the 2-dimensional canonical eqs. (4-21) after innovation, considering the parameters in Table **4-1**, $\beta = 0.05$, with $\mu = 0.6$ (red) and $\mu = 0.3$ (blue and green) when consumers' preference favors low quality coffee ($f_1 = 1/1.1$). (B) Shows the plots of standard coffee at equilibrium $\bar{N}(q_1)$ before innovation (dashed black). After innovation, shows both standard $\bar{N}_1(q_1, q_2)$ and special $\bar{N}_2(q_1, q_2)$ equilibria with $\mu = 0.6$ (solid red and dashed red resp.) and $\mu = 0.5$ (solid blue and dashed blue resp.)

nical equation (4-21), with initial conditions at $q_1(200) = q_1^*$ and $q_2(200) = 0.97q_1^*$, goes to zero, i.e., in the long term, both coffees will evolve to the worst quiality (coffees made up from large proportion of bored coffee). On the other hand, Fig. **4-11**B shows the sold amounts of standard coffee $\bar{N}_1(\bar{q}_1, \bar{q}_2)$ (solid blue) and special coffee $\bar{N}_2(\bar{q}_1, \bar{q}_2)$ (dashed blue), where q_1 and q_2 are the solutions of the canonical eqs. (4-21) illustrated in Fig. **4-11**A (blue). Note, both equilibrium densities increases and stabilizes at the same equilibrium value. In this case, with $\mu = 0.3$, we fall into a region where the singular strategy does not exist, that is, the reduced CBB control policy operates against the diversification in the market. In deed, in the long term, only one type of coffee of quality $q_1 = q_2 = 0$ (virtually splited in the sold amounts N_1 and N_2) will remain in the market.

This situation can be analyzed from the Fig. **4-7**C, if we start in a colored point with $\mu = 0.6$ (recall $h = 0.2$ in this scenario), and then decrease μ, the singular solution $\bar{q}_1$ is a branching point until the black boundary, then $\bar{q}_1$ disappears (note in Fig. **5-7**C with $\mu = 0.3$, there is no intersection between green and black lines), then the quality q_1 start to decrease to zero. This interesting result implies that once a branching point is reached and the market is diversified, reducing adult CBB control will favor the commercialization of lower quality coffees, which makes perfect sense, since a low adult CBB death rates means more damage to coffee production.

4.4. Discussion and Conclusions

A deterministic model describing the agro-industrial dynamics of coffee was formulated. Adult and immature CBB populations were considered to reflect the damage caused by their reproduction and feeding habits, as well as their impact on coffee quality. Harvested coffee was divided into two categories depending on quality, determined by the proportion of bored coffee beans used in production, here we have called low-quality coffees as standard and high-quality coffees as special, although this decision is arbitrary from the mathematical point of view. Quality is considered to be a quantitative differentiating attribute between competing coffee types. The role of consumer preference favoring high or low quality is considered in competition as a way to describe their budget limitations, therefore, indirectly the model considers the role of coffee price.

The analysis showed that coffee production strongly depends upon the harvesting rate that should be lower than the net coffee growth rate, so as to guarantee a non-vanishing coffee production. Although a complete stability analysis was left for a future paper, it is shown that CBB presence in the crop is determined by a ”net reproduction rate“ grater than 1 as sufficient and necessary condition. Thresholds to guarantee CBB presence relating the harvesting rate to the effective boring rate, and to the adult CBB death rate were found. These thresholds can be used by the farmer as means to reach pest elimination, mainly through parasitism [52, 77] and predation [42, 68], but also chemical and cultural control strategies are available; indeed, integrated control, as combination of several of control strategies, is considered to be the best [42, 52, 68, 77].

Under the presence of CBB, there is at least one stable equilibrium corresponding to the presence of every state variable considered in the model: the invasion equilibrium, whose stability helped to answer the question of under which conditions an innovative coffee type can penetrate, spread and compete with a stablished one into the market. The long-term dynamics of quality traits were studied from the perspective of adaptive dynamics, in order to establish the conditions under which evolutionary competition between both types of coffees results in market diversification. In the present context, the net reproduction rate, the consumers' preference regarding coffee quality, and the bifurcation thresholds play important roles in diversification through innovation, and permitted the formulation of policies on the adult CBB death rate, the effective CBB boring rate and harvesting effort which, in turn, determine the possibility of market diversification.

The main insight/recommendations for practitioners and policy makers that follows from the analysis of our model, is that the effective CBB boring rate plays an important role not only in the agro-ecological dynamics, but the evolutionary dynamics of quality, and therefore CBB control practices must be carefully analyzed. Indeed, the decision to increase or decrease pest control after diversification, is closely related to consumers' preference for high or low quality coffees, and has an impact on the evolutionary dynamics of quality in the long term. Under consumer's preference for higher quality, when reducing the CBB control the market remains diversified, but increasing the action of control strategies to reduce its population, may work against diversification, in this case, coffees are made more and more with healthy beans, preventing, in the long-term, for sustained coffee types in the market of differentiable quality. On the other hand, under consumer's preference for lower quality, increasing the CBB control gives only the top quality when the CBB goes extinct, while reducing the control polices permits an increase in the CBB population and the production of low quality coffees, but direct evolution to a vanishing quality, then the market will be composed by a single coffee quality virtually separated into two types. In this last case, the intermediate control strategies, allowing for coffees of different qualities to be produced, that must be considered in order to maintain diversification.

To briefly discuss pest control strategies that are be used to affect boring or death rates, we focus on biological control by parasitoids and predators, as one of the most widely used; in the first case, wasps are highlighted, mainly *Cephalonomia stephanoderis* and *Prorops nasuta*, but also *Phymastichus coffea* and *Cephalonomia hyalinipennis*, in smaller proportions [77]. In the second case, predation with coleoptera *Leptophloeus* and *Cathartus quadricollis* [42], or with lizards of the *Anolis* genus [66]. Effective predators for CBB control include ants that, depending on the species, can consume CBB in either adult or immature stages [68]. An additional type of biological control is the use of entomopathogens, in particular *Beauveria bassiana*, a fungus capable to attain mortality levels up to 84 % in field conditions; however, it has a slow infection process, allowing the adult CBB to live long enough to bore into coffee beans [51]. Chemical control methods are a variety of insecticides used to kill adult insects (affecting the boring rates and the adult CBB death rate); has been demonstrated, through fieldwork, that it have little or no effect on immature stages, as they live inside the bean and the chemicals thus cannot reach them [72]. Our analysis allows to determine the maximum harvesting rate tolerated by CBB population given the adult CBB death rate or the effective CBB boring rate; CBB control through harvesting is a cultural control defined by the FNC as *Re-Re* (*Recoger* and *Repasar*, in Spanish), and makes reference to collecting ripe and overripe grains from the coffee plantation and a few days later to check and collect again in order to prevent adult CBB insects from finding refuge and therefore preventing their reproduction [3,72].

It is important to highlight that our model considers the proportion of healthy and bored coffee used in production to be the differentiating factor in coffee quality. Further studies should be performed, in order to consider alternate forms of quality differentiation, such as the introduction of innovative agro-industrial processes that affect coffee washing, drying, roasting, or other crucial processes in coffee production, transformation, or commercialization.

Finally, although the National Federation of Coffee Growers of Colombia periodically publishes statistical information on coffee production and marketing, this is limited to green coffee and some with industrial treatment such as decaffeinated green, roasted in beans, roasted and ground, and extract and soluble. These statistics do not include specialty coffees in any of their main denominations (origin, preparation, sustainability, etc.), thus, we cannot compare our model with real data from the Colombian coffee market, indeed, as a final recommendation, it is essential that in Colombia a system allowing to collect, analyze and disseminate accurate information on the production, processing and sale of specialty coffees be defined, even more considering that these products become the main source of income for small producers that can access better economic benefits by producing high quality coffees valued by consumers for their attributes, for which they are willing to pay higher prices, which results in higher producer income and welfare.

5. Model for the competition among public transport systems

5.1. Introduction

Public transport is important to ensure population's mobility, however, congestion in cities continues to be one of the main problems, and government policies have failed to promote the use of public transport in an effective and sustainable way [71]. Paulley et. al., [74] address the problem of demand, presenting a synthesis of published and unpublished information, considering the effect of variables such as travel cost, quality of service and vehicle ownership as the most significant affecting the dynamics of demand for public transportation. However, these factors should not be studied in isolation; therefore, should be considered the impact of many other variables, among which include developments in transportation and technology such as pricing, changes in the size of the vehicle, control of emissions, etc.

The main barrier to the growth of public transport use is related to the limited budget, which has promoted the scheme of service provision through private operators competing in the market, while governments are mainly responsible for the aspects related to the regulation of the competition conditions, and the infrastructure for operation of the service [71]. This public-private scheme, which applies in most urban areas, is based on the principle that greater investment by the private sector promotes competition and should promote efficiency; moreover, it should also promote public transport operators to be more innovative in order to maintain their position in the market. However, although there are many studies that can be found focused on studying the economic effect of regulations, there are few studies focused on studying the concept of innovation in the public transport sector [60].

One of the reasons to consider, is pointed out from the work of Shummpeter [82], who establishes the need to differentiate between invention, innovation and diffusion. In its definition, innovation refers to the economic application of an invention (the development of a new "product"), while diffusion refers to the generalization of its use by buyers and production by different firms [71]. While this concept of associating innovation with the emergence of new products is valid, and in fact it has been fundamental in the growth of global industries, it must be considered that economic development is not limited to the industrial sector, but the service sector as a source of great business opportunities.

However, the services sector is usually considered as "intangible" and "interactive", so the concept of innovation has been more difficult to define. In that sense, Gallouj [43] considers the fact that innovations in services are influenced by a set of forces (driving forces) that he identifies as incentives or obstacles to the innovation process, divided into what he calls "trajectories", and corresponding to the professional, technological and social management and that are composed by different agents in each level, such as clients, competitors, the government, etc. [60].

One of the aspects that exerts more pressure on innovation in transport systems is the growth of cities and the subsequent need for efficient public transport services. In 1960, only seven megacities existed, understood as large cities, or metropolitan areas, with more than ten million inhabitants and large population densities; by 2010, this number had increased to 27, and by 2020, it is projected that this number will grow to 37 [7,54]. In this growth process, cities cannot ignore fundamental aspects of their own economic and demographic development, or the complex network of interactions generated thereby. One fundamental question is the relationship between population growth, demographic development, and public transport infrastructure. Bogotá - Colombia, in particular, is going through a key decision-making moment regarding the possibility of incorporating a metro system as one of its leading forms of transport. In contrast, the current mass-transit system, Transmilenio, operates using articulated buses. Unquestionably, the project involves innovations at all the levels mentioned above and the obvious need to address the question: under what conditions could a mass-transport system invade, expand in the market, and coexist with current, established city transport systems, in the long term.

This type of question is closely related to other studies from the standpoint of evolutionary biology, and which have allowed the development of adaptive dynamics as a useful mathematical framework for the study of these questions. However, beyond biology, this theoretical framework has recently been used to model a broad spectrum of non-genetic innovation phenomena; for example in the technological context with works like [23, 29] where the origin of technological diversity is explored from the interaction in the market; e.g., existing products compete in the market with innovative products, resulting in a continuous and slow evolution of the characteristic attributes of successful products. A similar work has addressed the problem of determining conditions on the energy market diversification from adaptive dynamics perspective [85], considering technological innovation in the energy generation processes. Analogies between the ecological processes of competition and collaboration with the dynamics of markets are powerful conceptual tools when used in the appropriate contexts; indeed, Nair et. al., [69] establishes that the complexity of technological change, the ecological and institutional dynamics can allow regimes of coexistence of competing technologies. In [44] is presented a model devoted to the study of an evolutionary system where similar individuals are competing for the same resources, and [70] studied model designed for a user-resource scenario in the context of technological innovation.

Here, a generalized model has been formulated for the competition between transport systems in a city, considering that interaction occurs under the same market platform and competition is determined by the proportion of users adopting each transport system. To measure the budget that the investor makes available, in order to promote the transportation system expansion among users is considered. The model is a generalization of the one presented in [84], as a product of a research project on urban metabolism for Bogotá Colombia, where just a local stability analysis of the $1-$dimensional case is performed, to understand the dynamics of the city with only one transport system, then, some discussion on the invasion conditions on the resident/innovative model was shown. The generalization consists in considering that there are N established transport systems in the city, and when they are theoretically in "equilibrium" an innovative competitor arises increasing the model to $N+1$ transport systems, therefore, the generalization consists in formulating the $N-$resident/$1-$innovative model and describing some general theory (as invasion conditions) related to that model.

Using the theoretical framework of adaptive dynamics it is possible to determine general conditions that must be met to guarantee or not the success of the innovation as the one managing to penetrate and expand into the market; this information is obtained from study of the sign of the fitness function for specific model coefficient expressions. Later, the canonical equation, corresponding to ordinary differential equation, is presented to describe the behavior over time of the characteristic attribute as a result of innovation processes. The approach is used to establish the long term dynamics of the quantitative attribute and allows the classification of the equilibria as terminal points (those in which the evolution definitively halts), like the points where substitution takes place.

5.2. Generalized model for competition of public transport systems

In order to formulate a generalized model, consider a city with N interacting transport systems (TS), quantity assumed to occur in a finite number. Each individual on the population uses a specific TS, which is characterized through a particular continuous attribute u as a real number and can be assumed bounded or not, depending on its meaning, in general it can be any physical measure in a suitable scale, associated with aspects such as comfort, security, availability, travel cost or travel duration among others; in our examples below, we consider u to be the average number of passengers the TS can transport. Particularly, $u_1, u_2, \ldots, u_N$ are the respective attributes that characterize (differentiate) the N TSs. For simplicity, we will refer as $i-$th TS when it is characterized by the attribute u_i.

Denote $x_i = x_i(t)$; $0 \leq x_i \leq 1$, for $i = 1, \ldots, N$ the proportion of people who adopt the $i-$th TS, thus, the city's population is subdivided into N subpopulations such that

$\sum_{i=1}^{N} x_i = 1$. To formulate a model describing "competition" of the N TS through the adoption that population makes of each of them, consider a reformulation and a generalization of the model studied by [70], designed for a user-resource scenario. Consider that the N populations x_i interact through the generalized system:

$$\dot{x}_i = r(y_i; u_i)x_i \left(1 - x_i - \sum_{k=1,k\neq i}^{N} c(u_i, u_k)x_k\right), \quad i = 1, \ldots, N, \tag{5-1}$$

Note that it has been assumed that competition between TSs of the same class is symmetrical, therefore $c(u_i, u_i) = 1$, a fact that is reflected in the term $-x_i$ appearing in the parentheses and also in the restriction $k \neq i$ on the sum. We assume that financial resources are exclusively allocated for the enhancement of the innovation adoption, the size of this resource budget is represented by $y_i = y_i(t)$ which can take values, after rescaling, on $0 \leq y_i \leq 1$, for $i = 1, \ldots, N$; then $r(y_i; u_i)$ is some expression describing the intrinsic growth rate of the $i-$th TS by new users, depending on the state variable y_i and the $i-$th TS attribute.

On the other hand, $c(u_i, u_k)$ is the interaction rate between $i-$th and $k-$th TS. A number of situations are then obtained; first, if $c(u_i, u_k) > 1$, inter-system competition prevails over intra-system competition. As a simple example of this, if $i-$th TS corresponds to a city taxi system, while system k corresponds to a public bus system, then $c(u_i, u_k) > 1$ implies that taxi competition with buses is stronger than the competition between the taxis themselves. If $0 \leq c(u_i, u_k) \leq 1$, then intra-system competition prevails over inter-system competition. Returning to the public taxi and bus example, in this scenario, competition between the taxis themselves is stronger than competition between taxis and buses. Particularly, $c(u_i, u_k) = 0$ indicates that there is no interaction between the two TS, and $c(u_i, u_k) = 1$ indicates that the interaction between the two TS is symmetrical, i.e., affects both systems equally. Finally, if $c(u_i, u_k) < 0$, the interaction between transport systems does not correspond to competition, but rather cooperation, a situation which can describe integrated TSs, and is not going to be considered in this model.

The simplest way to define the adoption rate r is through the linear dependency:

$$r(y_i; u_i) = \alpha(u_i)y_i - \delta(u_i), \quad i = 1, \ldots, N, \tag{5-2}$$

where $\alpha(u_i)$ is the instantaneous adoption rate of the $i-$th TS based on the characteristic attribute u_i and $\delta(u_i)$ is the rate at which the $i-$th TS is abandoned (multiple reasons may justify this fact, as dissatisfaction or users getting their own car), for a fixed value u_i; this definition implies that the adoption rate $r(y_i)$ increases in direct proportion to the size of the allocated present budget $\alpha(u_i)y_i$, and decreases according to adoption failure $\delta(u_i)$, and decreases at a rate δ. The dimensions of $\alpha(u_i)$ and $\delta(u_i)$ are $1/\text{time}$ (y_i is adimensional after rescaling) which is consistent with the units of the intrinsic growth rate. In general it is assumed $\alpha(u_i) > \delta(u_i)$, for all $i = 1, \ldots, N$, however, even for small values of y_i, it is possible to have $r(y_i) < 0$, for this reason the equation (5-1) is modified to have,

$$\dot{x}_i = r(y_i; u_i)x_i - |r(y_i; u_i)|\, x_i^2 - |r(y_i; u_i)|\, x_i \sum_{k=1, k\neq i}^{N} c(u_i, u_k)x_k, \quad i = 1, \ldots, N,$$

then, by factoring in the intrinsic rate of growth r and x_i, we can write in the form

$$\dot{x}_i = r(y_i; u_i)\left(1 - \text{sign}\left(r(y_i; u_i)\right) x_i - \text{sign}\left(r(y_i; u_i)\right) \sum_{k=1, k\neq i}^{N} c(u_i, u_k)x_k\right) x_i, \quad i = 1, \ldots, N. \tag{5-3}$$

To establish budget dynamics y_i, it must be considered that investment can not be unlimited, then it is assumed that the investor places resources in direct proportion to the budget not yet placed, i.e., $l(u_i)(1 - y_i)$, where $l(u_i)$ is the rate with which the unallocated budget is allocated. Finally, the budget for the expansion of the $i-$th TS is used or "consumed" in a magnitude $\epsilon(u_i)\alpha(u_i)y_i x_i$, where $\epsilon(u_i)\alpha(u_i)$ denotes the efficiency with which the resources y_i are "converted" into new users. Therefore, the term assumes that budget use is proportional to the product of instantaneous rate of technology adoption $\epsilon(u_i)\alpha(u_i)$ and $x_i y_i$ representing the interaction between users and TSs (a mass action law). The equations for the budget is as follows:

$$\dot{y}_i = l(u_i)(1 - y_i) - \epsilon_i(u_i)\alpha(u_i)x_i y_i, \quad i = 1, \ldots, N. \tag{5-4}$$

In conclusion, the equations (5-3) and (5-4) define the $2N-$differential equations, given by:

$$\begin{cases} \dot{x}_i = r(y_i; u_i)\left[1 - \text{sign}\left(r(y_i; u_i)\right)\left(x_i + \displaystyle\sum_{k=1, k\neq i}^{N} c(u_i, u_k)x_k\right)\right] x_i \\ \dot{y}_i = l(u_i)(1 - y_i) - \epsilon(u_i)\alpha(u_i)x_i y_i, \end{cases} \tag{5-5}$$

for $i = 1, \ldots, N$. Model (5-5) will be referred to as the "resident" model from now on. System (5-5) is subject to non negative initial conditions $x_i(0)$ and $y_i(0)$, for $i = 1, \ldots, N$ and defined in the region:

$$\Omega_N = \{x_i, y_i \in \mathbb{R} : 0 \leq x_i \leq 1, 0 \leq y_i \leq 1, i = 1, \ldots, N\} \tag{5-6}$$

5.2.1. Innovation in the generalized model for public transport systems

Model (5-5) for N transport systems should have at least one locally asymptotically stable equilibrium inside Ω_N in which they can "coexist". Denote $\mathbf{u} = (u_1, \ldots, u_N)$ to the vector of all the characteristic attributes, then, the coexistence equilibrium can be written as,

$$E_N = (\overline{x}_1(\mathbf{u}), \overline{y}_1(\mathbf{u}), \ldots, \overline{x}_N(\mathbf{u}), \overline{y}_N(\mathbf{u})). \tag{5-7}$$

Suppose a technological innovation occur in the $j-$th attribute u_j, for some $j = 1, \ldots, N$; changing the value of u_j to $\tilde{u}_j$. The innovation introduced corresponds to some change that physically modifies the $j-$th TS or the organization processes thereof; in general, it is assumed an small innovation with small effect, which allows the interaction between transportation systems to be carried out under the same market platform. The innovative attribute gives rise to a small proportion of users $\tilde{x}_j(t)$ of the innovative TS, entering to compete with the N established TS, and to the corresponding initial budget fully disponible for allocation ($\tilde{y}_i = 1$). The innovations success or failure can be studied by extending the system (5-5) in two equations: one for $\tilde{x}_j$ and another for the corresponding budget $\tilde{y}_j$. Explicitly it will be:

$$\begin{cases} \dot{x}_i = r(y_i; u_i) \left[1 - \text{sign}\left(r(y_i; u_i)\right) \left(x_i + c(u_i, \tilde{u}_j)\tilde{x}_j + \displaystyle\sum_{k=1, k\neq i}^{N} c(u_i, u_k) x_k \right) \right] x_i \\ \dot{y}_i = l(u_i)(1 - y_i) - \epsilon(u_i)\alpha(u_i) x_i y_i \\ \dot{\tilde{x}}_j = r(\tilde{y}_j; \tilde{u}_j) \left[1 - \text{sign}\left(r(\tilde{y}_j; \tilde{u}_j)\right) \left(\tilde{x}_j + c(\tilde{u}_j, u_i) x_i + \displaystyle\sum_{k=1, k\neq i}^{N} c(\tilde{u}_j, u_k) x_k \right) \right] \tilde{x}_j \\ \dot{\tilde{y}}_j = l(\tilde{u}_j)(1 - \tilde{y}_j) - \epsilon(\tilde{u}_j)\alpha(\tilde{u}_j)\tilde{x}_j \tilde{y}_j, \end{cases} \tag{5-8}$$

for $i = 1, \ldots, N$. Model (5-8) is going to be called "resident-innovative" model from now on. Note that the model (5-5) corresponds to the model (5-8) when $\tilde{x}_j = 0$ and $\tilde{y}_j = 1$. Even though, in the equation of the innovative TS, is a bit strange to keep the competition between the innovative and the $i-$th TS out of the sum, our decision is based on symmetry reasons. In Table **5-1**, find a description of state variables and of the coefficients used in the resident system.

In the following section a local stability analysis is performed to describe the model behavior for $N = 1$ and latter the evolutionary behavior of the characteristic attribute is studied for a situation with two TS, one stablished and one innovative. Such analysis are made considering the parameters as described here, but particularly for its numerical study, certain considerations have been made on the coefficients, it has been assumed

Table 5-1.: Description of state variables and parameters

	Description of the state variables	Value
x_i	Proportion of people using the $i-$th TS	-
y_i	Budget disponible to allocate for the expansion of the $i-$th TS	-
	Description of parameters	Value
u_i	Characteristic attribute describing the $i-$th TS	Varies
$\alpha(u_i)$	Instant rate of adoption of the $i-$th TS	-
$\delta(u_i) = d$	Rate at which the $i-$th TS is abandoned by users	0.2
$l(u_i) = l$	Rate at which new resources are allocated for expansion of the $i-$th TS	Varies
$\epsilon(u_i) = \varepsilon$	Efficiency of the $i - th$ TS in converting the investment in new users	0.9
$c(u_i, u_k)$	Interaction rate between $i-$th and $k-$th TS's	-
a	Maximum adoption rate of the $i-$th TS	1
a_1	Users' sophistication sensitivity	0.8
a_2	Most absorbable number of passengers	$\sqrt{250}$
f_1	Users' preference (high (>1), low (<1))	Varies
f_2	Strength of similar coffees' competition	1

that, the proportion at which new resources $l(u) = l$ are allocated, TS efficiency to "convert" the investment into new users $\epsilon(u) = \varepsilon$, and the rate at which the TS is abandoned by users $\delta(u) = d$, are constants. It has been assumed that the rate of instant adoption depends on the characteristic attribute u, through the function:

$$\alpha(u) = a \exp\left(-\frac{1}{2a_1^2}\ln^2\left(\frac{u}{a_2^2}\right)\right). \tag{5-9}$$

The rate $\alpha(u)$ makes perfect sense when u is small, and has no competition from other transport systems [23]. A maximum a occurs when $u = a_2^2$, in order to indicate the value of the attribute which is easiest to absorb. On the other hand, for a transport system with a very low or very high number of users, $\alpha(u)$ tends to vanish out with sensitivity controlled by a_1. It is assumed that $0 \leq a \leq 1$ and a_1, a_2 are non negative (see Figure **5-1**).

The values of the parameters used in simulations are stated *ad hoc* for illustration purposes; however, the value of attribute u_i, is based partially on the capacity of the Transmilenio's articulated buses (massive TS in the city of Bogotá, Colombia) around 160-person capacity, then, we assume $u_i = 160$ (Figure **5-2**-left shows one of these buses). The value of $a_2^2 = 250$ correspond to the actual capacity of the Transmilenio's bi-articulated buses (see Figure **5-2**-right), and correspond to the current average number of users to reach the maximum adoption rate per users.

On the other hand, $c(u_i, u_k)$ is the interaction rate between TS's i and k, as described

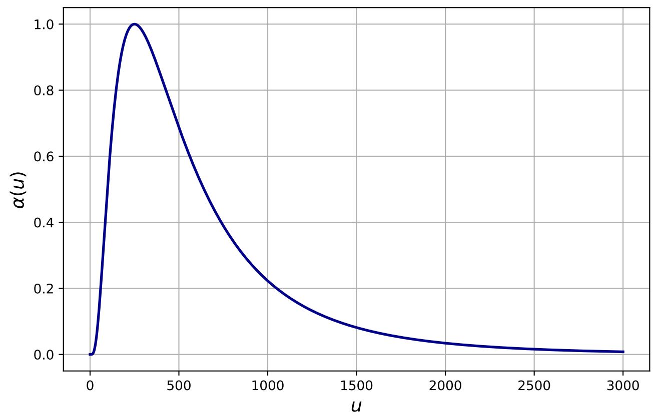

Figure 5-1.: Function $\alpha(u)$ chart for parameters $a = 1$, $a_1 = 0.8$, $a_2 = \sqrt{250}$.

Figure 5-2.: **Left**: Photography of an articulated Transmilenio bus, the mass-transport company in the city of Bogota, Colombia. They have a 160-passenger capacity. **Right**: Photograph of a biarticulated bus of Transmilenio: they have a 250-passenger capacity. This type of bus is a recent incorporation in the busses fleet. Reproduced from transmilenio.gov.co.

in the generalized model formulation and is assumed as in [23] to take the following expression:

$$c(u_i, u_j) = \exp\left(\frac{\ln^2 f_1}{2f_2^2}\right) \exp\left(-\frac{1}{2f_2^2} \ln^2\left(\frac{f_1 u_i}{u_j}\right)\right)$$

Observe that the interaction rate between TSs $c(u_i, u_j)$ depends on the u_i/u_j ratio, and tends toward zero when it tends toward zero, or when it tends toward infinity, which reflects that TSs which are very different compete weakly. A graphic representation of the function is shown in Figure **5-3**.

If $f_1 > 1$, the TSs that move the greatest average of passengers tend to have a competitive advantage. On the other hand, if $0 < f_1 < 1$, the TSs that move a lower average number of passengers will be those which have the advantage. A large f_2 value implies

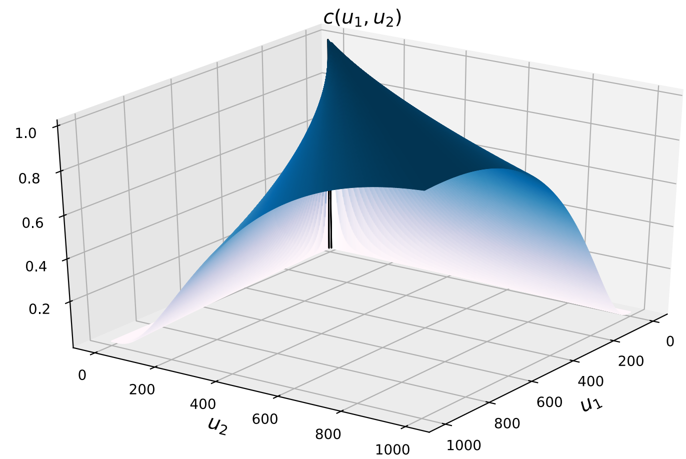

Figure 5-3.: Chart of function $c(u_1, u_2)$, for parameters $f_1 = 1.05$ and $f_2 = 1$, for illustration purposes.

that very different TSs also compete strongly. When $f_1 = 1$, competition between TSs is symmetrical.

Competition among the $N + 1$ TSs in the city is assumed under a common market platform, regulated by governmental entities in charge of establishing public policies in this regard; in this sense, the participation of TSs that intervene in the market outside the legal framework is not considered in this model. This assumption is reflected in the fact that the differential equation for the proportion of users and the budge are described by a pair of functions f and g depending on the proportion of users, the budget and the attributes values; in other words, the "relative diffusion rate" of the innovative TS with attribute $\tilde{u}_j$, for some j is given by the $f-$function defined as,

$$\begin{aligned} \frac{\dot{\tilde{x}}_j}{\tilde{x}_j} &= r(\tilde{y}_j;\tilde{u}_j)\left[1 - \text{sign}\left(r(\tilde{y}_j;\tilde{u}_j)\right)\left(\tilde{x}_j + c(\tilde{u}_j, u_i)x_i + \sum_{k=1,k\neq i}^{N} c(\tilde{u}_j, u_k)x_k\right)\right] \\ &= f(x_1, y_1, \ldots, x_N, y_N, \tilde{x}_j, \tilde{y}_j; u_1, \ldots, u_N, \tilde{u}_j); \end{aligned} \quad (5\text{-}10)$$

particularly, the relative diffusion rate $\dot{x}_1/x_1$ of the TS with attribute u_1 is obtained when $\tilde{u}_j = u_1$ and replacing $\tilde{x}_j$ with x_1 and $\tilde{y}_j$ with y_1. Similarly,

$$\begin{aligned} \dot{\tilde{y}}_j &= l(\tilde{u}_j)(1 - \tilde{y}_j) - \epsilon(\tilde{u}_j)\alpha(\tilde{u}_j)\tilde{x}_j\tilde{y}_j \\ &= g(x_1, y_1, \ldots, x_N, y_N, \tilde{x}_j, \tilde{y}_j; u_1, \ldots, u_N, \tilde{u}_j), \end{aligned}$$

is the corresponding function for the proportion of budget.

Due to the symmetry of model (5-8), it is not really important which of the two transport systems related by the innovation is considered the innovative TS and which is the resident, so by exchanging x_j for $\tilde{x}_j$ y y_j for $\tilde{y}_j$ it is verified,

$$f(x_1, y_1, \ldots, \tilde{x}_j, \tilde{y}_j, \ldots, x_N, y_N, x_j, y_j; \mathbf{u}, \tilde{u}_j) = f(x_1, y_1, \ldots, x_i, y_i, \ldots, x_N, y_N, \tilde{x}_j, \tilde{y}_j; \mathbf{u}, \tilde{u}_i) \tag{5-11}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$ anew for simplicity. This is an important property of f that will be exploited in the derivation of invasion conditions in the next subsection. Meanwhile, consider that

$$E_{N+1} = (\overline{x}_1(\mathbf{u}), \overline{y}_1(\mathbf{u}), \ldots, \overline{x}_N(\mathbf{u}), \overline{y}_N(\mathbf{u}), 0, 1), \tag{5-12}$$

is an equilibrium of (5-8) just after the innovation is introduced, referred as the "invasion equilibrium". In other words, at the time of innovation, by the assumption of local and asymptotically stability, the city's TSs were at the equilibrium E_N given in (5-7), which implies that the Jacobian matrix $J(E_N)$ of (5-5) has $2N$ eigenvalues with real negative part. By extending the resident model (5-5) to the model (5-8), we want to determine conditions under which the innovative TS can "invade" into the market. To do this, stability conditions must be studied. In deed, if equilibrium E_{N+1} is unstable, it implies that a scarce proportion of users of the innovative TS will grow and allow its expansion in the market; on the contrary, if E_{N+1} is LAS, then that initial proportion of users will tend to disappear in finite time.

To determine the local stability of equilibrium (5-12), a small perturbation is made around it and the behavior of the associated linear system is studied. The corresponding Jacobian matrix takes the form:

$$J(E_{N+1}) = \left[\begin{array}{c|cc} J(E_N)_{N\times N} & \multicolumn{2}{c}{\mathbf{A}_{N\times 2}} \\ \hline & [\alpha(\tilde{u}_j) - \delta(\tilde{u}_j)]\left(1 - c(\tilde{u}_j, u_i)\overline{x}_i - \sum_{k=1,k\neq i,j}^{N} c(\tilde{u}_j, u_k)\overline{x}_k\right) & 0 \\ \mathbf{0}_{2\times N} & & \\ & -\epsilon(\tilde{u}_j)\alpha(\tilde{u}_j) & -l(\tilde{u}_j) \end{array}\right] \tag{5-13}$$

Note that $J(E_{N+1})$ corresponds to a diagonal matrix by blocks and that the upper left block coincides with $J(E_N)$, which is the Jacobian matrix of the resident model, and therefore contributes $2N$ eigenvalues with negative real part. On the other hand, the lower right block, associated with the equations of the innovative TS, is a diagonal block with an eigenvalue given by $-l(\tilde{u}_j) < 0$ and therefore the local stability of E_{N+1} is determined by:

$$\lambda_j(u_1, \ldots, u_j, \ldots, u_N, \tilde{u}_j) = \left[\alpha(\tilde{u}_j) - \delta(\tilde{u}_j)\right] \left(1 - c(\tilde{u}_j, u_i)\overline{x}_i - \sum_{k=1, k \neq i,j}^{N} c(\tilde{u}_j, u_k)\overline{x}_k\right), \quad (5\text{-}14)$$

defined as the "fitness function" of the innovative TS and represents the relative diffusion rate $\dot{\tilde{x}}_j / \tilde{x}_j$ at the invasion equilibrium; therefore, it can be obtained directly from the f function defined above; particularly from (5-10) at E_{N+1}, to get:

$$\lambda_j(\mathbf{u}, \tilde{u}_j) = f(\overline{x}_1, \overline{y}_1, \ldots, \ldots, \overline{x}_N, \overline{y}_N, 0, 1; \mathbf{u}, \tilde{u}_j) \quad (5\text{-}15)$$

In the adaptive dynamics language, the f function is referred as the "fitness generating function" [31]. For simplicity, in the next section we write

$$\lambda_j(\mathbf{u}, u_j, \tilde{u}_j) = \lambda_j(u_1, \ldots, u_j, \ldots, u_N, \tilde{u}_j)$$

where $\mathbf{u} = (u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$ in that suitable order.

5.2.2. Invasion conditions and canonical equation of adaptive dynamics

The theory of adaptive dynamics allows to relate the fitness function obtained from the market model, with the long-term evolutionary dynamics of the innovative attribute, result achieved through the so called "canonical equation of the adaptive dynamics" [31], to be studied with more detail in the following sections. In order to relate both results, we will obtain the invasion conditions of the innovative attribute through the first order expansion in Taylor's series of (5-14). The following description is generic in this kind of models, it is included here for comprehension purposes. The first order expansion in Taylor series around $\tilde{u}_j = u_j$ neglecting high order terms, takes the form:

$$\lambda_j(\mathbf{u}, u_j, \tilde{u}_j) = \lambda_j(\mathbf{u}, u_j, u_j) + (\tilde{u}_j - u_j)\frac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j). \quad (5\text{-}16)$$

where $\mathbf{u} = (u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$. On the other hand, $\lambda(u_j, u_j) = 0$ by means of (5-15) and then using property (5-11). So, the first order expansion takes the form:

$$\lambda_j(\mathbf{u}, u_j, \tilde{u}_j) = (\tilde{u}_j - u_j)\frac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j). \quad (5\text{-}17)$$

The expression $\frac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j)$ is called "selection gradient" in the language of adaptive dynamics, it is responsible for "selecting " the attributes that will be favored in the competition; in fact, $\lambda_j(\mathbf{u}, u_j, \tilde{u}_j) > 0$ is obtained when one of the two invasion conditions stated below is satisfied:

C1. $\dfrac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j) < 0$ and $\tilde{u}_j < u_j$.

C2. $\dfrac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j) > 0$ and $\tilde{u}_j > u_j$.

If condition C1 or condition C2 holds, equilibrium E_{N+1} is LAS, therefore the innovative TS can invade into the market. On the other hand, if neither of such conditions holds, then E_{N+1} is unstable and the initial proportion of users introduced by the innovations vanishes, leading to the extinction of the innovative TS.

The dynamic of the attributes, henceforth called the evolutionary dynamic, will help to explain the characteristics of the innovation and competition process which acts on the market. Dercole et al., [31], describes the processes which should be considered for rigorous formulation of the canonical equation, which describes the evolutionary behavior (in the long term) of attribute u.

First, consider that the dynamics described by the resident-innovative model (5-8) occur in the "market timescale" measured by the variable t; while the dynamics of the attribute u, takes place on an "evolutionary timescale". Not completely separate timescales are considered, but a small scaling factor τ is introduced, which separates the two timescales considering that $\tau \to 0$. Thus, a large amount of time in the market scale dt/τ corresponds to a small amount of evolutionary time dt. F. Dercole and S. Rinaldi in [31], describe the stochastic processes that must be followed rigorously derive of the differential equation describing the evolutionary behavior of the attribute, arriving at the canonical equation of adaptive dynamics, in our context it looks like,

$$\dot{u}_j = \tau^2 \frac{1}{2} \mu(u_j) \sigma^2(u_j) \overline{x}_j(\mathbf{u}, u_j) \frac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j)$$

where $\mu(u_j)$ is the frequency of innovations and $\tau^2\sigma^2(u_j)$ is related to the variance (small innovations are been considered). The above equation is correct only when $\tau \to 0$, which completely separates the two timescales. Factor τ^2 is obtained due to the assumption (characteristic of the theory of adaptive dynamics) that innovations are rare and small events, which allows to consider one innovation at a time and to group the resident-innovative model in an infinitesimal amount of evolutionary time. On the other hand, considering small innovations, allows to "soften" the average evolutionary trajectory described by the equation (5-18) [31]. Rigorously speaking, the variable of evolutionary time must be defined as $\tau^2 t$, however, hoping that there are no confusions, the symbol t is used here for the time and the factor τ^2 is simply removed; then, the canonical equation takes the form,

$$\dot{u}_j = \frac{1}{2}\mu(u_j)\sigma^2(u_j)\overline{x}_j(\mathbf{u}, u_j)\frac{\partial \lambda_j}{\partial \tilde{u}_j}(\mathbf{u}, u_j, u_j) \tag{5-18}$$

For simplicity, we assume $\mu(u_j) = \mu$ and $\Omega^2(u_j) = \Omega^2$ as constant values in the model examples bellow. as a consequence, equilibria are constant solutions $(\overline{\mathbf{u}}, \overline{u}_j)$ in the equation where all derivatives in (5-19) are nullified, that is,

$$\frac{\partial \lambda_j}{\partial \tilde{u}_j}(\overline{\mathbf{u}}, \overline{u}_j, \overline{u}_j) = 0. \tag{5-19}$$

for each attribute u_j. At such points the canonical equation is meaningless, therefore, in a neighborhood of an evolutionary equilibrium, it is necessary to make a deeper analysis of the model, as will be illustrated in the next section.

The stability of the equilibria can be discussed through linearization of the canonical equation (5-18); however, it is important to keep in mind that $\mu(\overline{u}_j)$, $\Omega^2(\overline{u}_j)$ and $\overline{x}_j(\overline{\mathbf{u}}, \overline{u}_j)$ in general depend on j, therefore, small variations can affect the stability of the equilibria.

5.2.3. Sufficient conditions for market diversification

Evolutionary branching occur when the resident and innovative TSs can coexist and the selection is disruptive; that is, when the selection favors extreme quality attributes u instead of the intermediate ones. Evolutionary branching demand, in the first instance, that the resident attribute u_j is in a neighborhood of an asymptotically stable strategy $\overline{u}$ (a LAS constant solution of the canonical equation (5-18)) which vanishes the selection gradient and locally attracts evolutionary dynamics driven by rare and small innovations [22].

Second, coexistence has to be possible for $(\mathbf{u}, u_j, \tilde{u}_j)$ in a vicinity of the singularity; Geritz, Metz, et.al. in [45,46,64], show that coexistence in the special-standard model (5-8) is possible for u_j and $\tilde{u}_j$ close to $\overline{u}$, if

$$\frac{\partial^2 \lambda_j}{\partial u_j \partial \tilde{u}_j}(\overline{\mathbf{u}}, \overline{u}, \overline{u}) < 0. \tag{5-20}$$

Finally, for the selection to be disruptive; i.e., to have two products that coexist (one stablished and one innovative) and diverge in attributes, a sufficient condition is,

$$\frac{\partial^2 \lambda_j}{\partial \tilde{u}_j^2}(\overline{\mathbf{u}}, \overline{u}, \overline{u}) > 0. \tag{5-21}$$

A region where the equilibria exists, is locally asymptotically stable, and both coexistence and divergence conditions are satisfied or not, can be studied to classify the equilibria in three categories: (i) *Branching points (BP):* LAS equilibria in which quality attribute can branch, which occurs when both coexistence and divergence conditions are satisfied. (ii) *Terminal points (TP):* LAS equilibria, but they are not branching points. At these points the evolution stops completely. This case arises when any of the coexistence or divergence conditions is not satisfied, and (iii) *Bifurcation branching points (BBP):* Corresponds to border cases between branching points and terminal points.

After the innovation has occurred, conditions have been established for the occurrence of evolutionary ramifications, that is, when both the conditions of coexistence and divergence are satisfied. In that case, it is expected that the dynamics in the production timescale tend to some coexistence equilibrium

$$E_{N+1} = \left(\overline{x}_1(\mathbf{u}), \overline{y}_1(\mathbf{u}), \ldots, \overline{x}_N(\mathbf{u}), \overline{y}_N(\mathbf{u}), \overline{x}_{N+1}(\mathbf{u}), \overline{y}_{N+1}(\mathbf{u})\right)$$

of system (5-8). In that case, the innovative TS with $\tilde{x}_j$ and $\tilde{y}_j$ manages to penetrate and establish itself, becoming into an additional resident TS and reaching the equilibrium values $\overline{x}_{N+1}(\mathbf{u})$ and $\overline{y}_{N+1}(\mathbf{u})$. This phenomenon is called "origin of diversity".

At this point, a second innovation can occur in any of the $N+1$ resident TSs and all the process starts again. In general, a sequence of technological innovations can be studied through the methodology described, as it is going to be illustrated with some detail in the following section.

5.3. Model for one established and one innovative transport system

5.3.1. Resident model and local stability analysis

In this section of the paper, a detailed description of the case $N = 1$ is presented. To illustrate the model, we consider initially a city with a resident TS, characterized by a particular attribute, u_1, which is assumed to be positive and associated with the average number of passengers who are transported. Under the description of model (5-5), the equations for $\dot{x}_1$ and $\dot{y}_1$ conform a 2-dimensional resident system, given explicitly by the equations:

$$\begin{cases} \dot{x}_1 = (\alpha(u_1)y_1 - \delta(u_1))\,[1 - \text{sign}\,(\alpha(u_1)y_1 - \delta(u_1))\,x_1]\,x_1 \\ \dot{y}_1 = l(u_1)(1 - y_1) - \epsilon(u_1)\alpha(u_1)x_1y_1, \end{cases} \tag{5-22}$$

defined for non negative initial conditions $x_1(0)$ and $y_1(0)$ in the square,

$$\Omega_1 = \{x_1, y_1 \in \mathbb{R} : 0 \leq x_1 \leq 1, 0 \leq y_1 \leq 1\}.$$

Note that system (5-22) is non-smooth on the curve $C = \left\{(x_1, y_1) \in \Omega_1 : y_1 = \frac{\delta(u_1)}{\alpha(u_1)}\right\}$. Therefore the system must be studied independently in the regions

$$\Omega^+ = \left\{(x_1, y_1) \in \Omega_1 : \frac{\delta(u_1)}{\alpha(u_1)} < y_1 \leq 1\right\} \text{ and } \Omega^- = \left\{(x_1, y_1) \in \Omega_1 : 0 \leq y_1 < \frac{\delta(u_1)}{\alpha(u_1)}\right\}.$$

Some special techniques are necessary to understand the model behavior on the curve C.

Local stability analysis

First we consider the system (5-22) defined in the region Ω^-, that is, when $0 \leq y_1 < \frac{\delta(u_1)}{\alpha(u_1)}$, to have the differential equations,

$$\begin{cases} \dot{x}_1 = (\alpha(u_1)y_1 - \delta(u_1))\,[1 + x_1]\,x_1 \\ \dot{y}_1 = l(u_1)(1 - y_1) - \epsilon(u_1)\alpha(u_1)x_1y_1. \end{cases} \tag{5-23}$$

In this case one have the constant solutions:

- $E_1^a = (0, 1)$ for the absence of TS in the city, in practice have no interest beyond the purely mathematical, however, the Jacobian matrix of (5-23) at E_1^a is,

$$J\left(\overline{E}_1^a\right) = \begin{bmatrix} \alpha(u_1) - \delta(u_1) & 0 \\ -\epsilon(u_1)\alpha(u_1) & -l(u_1) \end{bmatrix},$$

  whose eigenvalues $L_1 = \alpha(u_1) - \delta(u_1)$ and $L_2 = -l(u_1)$, then if $\frac{\delta(u_1)}{\alpha(u_1)} > 1$ they are both negative and therefore the equilibrium is locally asymptotically stable (LAS), as illustrated in Figure **5-4**-left with the parameters as described in the caption.

- The second equilibrium is $E_1^p = \left(\frac{l(u_1)(\alpha(u_1)-\delta(u_1))}{\delta(u_1)\epsilon(u_1)\alpha(u_1)}, \frac{\delta(u_1)}{\alpha(u_1)}\right)$, corresponding to partial adoption by users. It only has sense if $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ and $\frac{l(u_1)(\alpha(u_1)-\delta(u_1))}{\delta(u_1)\epsilon(u_1)\alpha(u_1)} < 1$ to be in the square Ω_1. From the second condition, clearing the rate at which new resources are allocated for expansion of the TS we get the condition,

$$l(u_1) < l^*(u_1) = \frac{\delta(u_1)\epsilon(u_1)\alpha(u_1)}{\alpha(u_1) - \delta(u_1)}. \tag{5-24}$$

  Observe that $E_1^p \in C$, the non-smoothness curve, therefore it is not in Ω^- but on its boundary; however, the linearization matrix in E_1^p provides information on the local behavior of the system. The characteristic polynomial is given by:

$$p(L) = L^2 + \frac{l(u_1)\alpha(u_1)}{\delta(u_1)}L + \frac{l(u_1)(\alpha(u_1) - \delta(u_1))\left[\alpha(u_1)\delta(u_1)\epsilon(u_1) + l(u_1)(\alpha(u_1) - \delta(u_1))\right]}{\epsilon(u_1)\delta(u_1)\alpha(u_1)} \tag{5-25}$$

  Provided $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ the coefficients of $p(L)$ are all positive and by the two-dimensional case of Routh-Hurwitz criterion it can be concluded that E_1^p is LAS. Note that in this case, $\overline{E}_1^a = (0,1)$ is still an equilibrium but is not in Ω^- but in Ω^+, therefore its dynamics is governed by the system of equations (5-26) described below.

  In the Figure **5-4**-right it can be seen that all the orbits of the system cross from the region Ω^+ (above the switching curve) to the region Ω^- or vice versa, depending on whether the crossing point on C is respectively to the right or to the left of the partial adoption equilibrium point E_1^p.

- Finally, there is a third equilibrium is $E^* = \left(-1, -\frac{l}{\alpha(u_1)\epsilon(u_1)-l(u_1)}\right)$ which do not belongs to Ω_1 and will not be discussed here.

Now we consider the system (5-22) defined in the region Ω^+, that is, when $\frac{\delta(u_1)}{\alpha(u_1)} < y_1 \leq 1$, to have the differential equations,

$$\begin{cases} \dot{x}_1 = (\alpha(u_1)y_1 - \delta(u_1))\left[1 - x_1\right]x_1 \\ \dot{y}_1 = l(u_1)(1 - y_1) - \epsilon(u_1)\alpha(u_1)x_1y_1, \end{cases} \tag{5-26}$$

which has the constant solutions:

- $E_1^a = (0,1)$ for the absence of TS in the city, the Jacobian matrix of (5-26) at E_1^a is the same $J\left(E_1^a\right)$ showed above. Then, as $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ in Ω^+, then E_1^a is unstable (U) by the eigenvalues criterion.

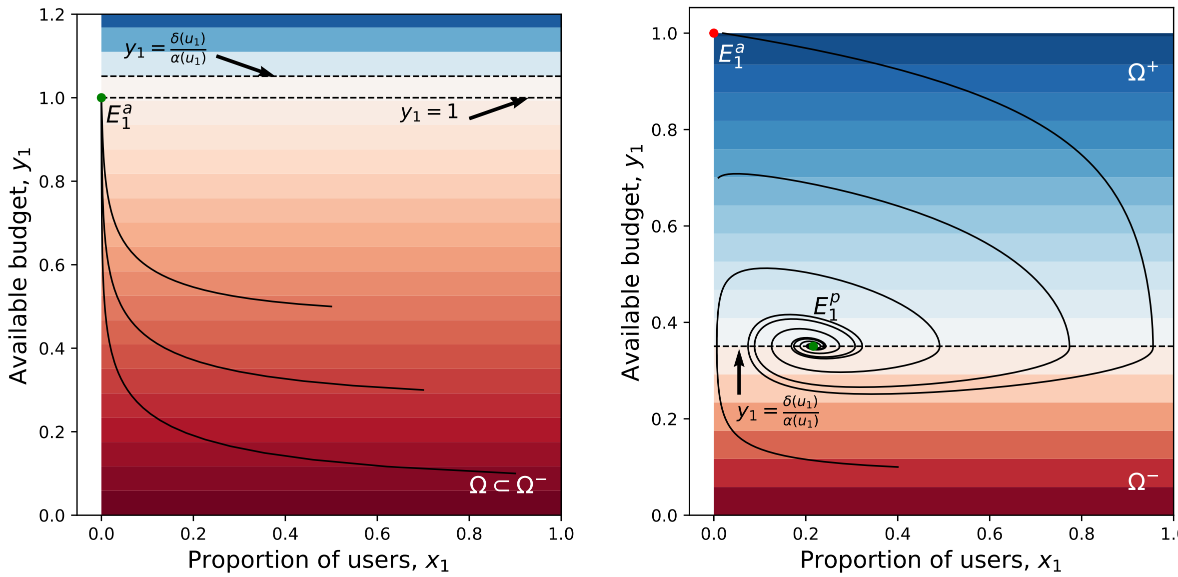

Figure 5-4.: Simulation scenarios for the resident model (5-22) in the fase plane with $a = 1$, $a_1 = 0.8$, $a_2 = \sqrt{250}$, $u_1 = 160$. **Left** $d = 0.9$ is used to have $\frac{\delta(u_1)}{\alpha(u_1)} = 1.0515 > 1$, then the absence equilibrium $\overline{E}^a = (0, 1)$ is the only equilibria in Ω and it is LAS. **Right**: fase portrait for $d = 0.3$ to have $\frac{\delta(u_1)}{\alpha(u_1)} = 0.3505 < 1$ and $l = 0.01 < 0.0462 = l^*(u_1)$, then the partial adoption equilibrium is $E_1^p = (0.3832, 0.2337)$ is LAS.

- Also, the equilibrium $E_1^p = \left(\frac{l(u_1)(\alpha(u_1)-\delta(u_1))}{\alpha(u_1)\delta(u_1)\epsilon(u_1)}, \frac{\delta(u_1)}{\alpha(u_1)}\right) \in C$, subject to the conditions $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ and $l(u_1) < l^*(u_1)$ as in (5-24). The local analysis in a vicinity of E_1^p, leads to the characteristic polynomial

$$p(L) = L^2 + \frac{l(u_1)\alpha(u_1)}{\delta(u_1)}L + \frac{l(u_1)(\alpha(u_1) - \delta(u_1))\left[\alpha(u_1)\delta(u_1)\epsilon(u_1) - l(u_1)(\alpha(u_1) - \delta(u_1))\right]}{\epsilon(u_1)\delta(u_1)\alpha(u_1)} \tag{5-27}$$

The condition $l(u_1) < l^*(u_1)$, as defined in (5-24) all the coefficients in (5-27) are positive, therefore it is LAS under this conditions, by the Routh-Hurwitz criterion.

- The third equilibria $E_1^t = \left(1, \frac{l(u_1)}{l(u_1)+\epsilon(u_1)\alpha(u_1)}\right)$ corresponds to total adoption of the TS by the users, or maximum possible adoption. Notice that $\frac{l(u_1)}{l(u_1)+\epsilon(u_1)\alpha(u_1)} < 1$ always, then to ensure that this constant solution is at Ω^+ it is only required that $\frac{\delta(u_1)}{\alpha(u_1)} < \frac{l(u_1)}{l(u_1)+\epsilon(u_1)\alpha(u_1)}$, then we get the condition:

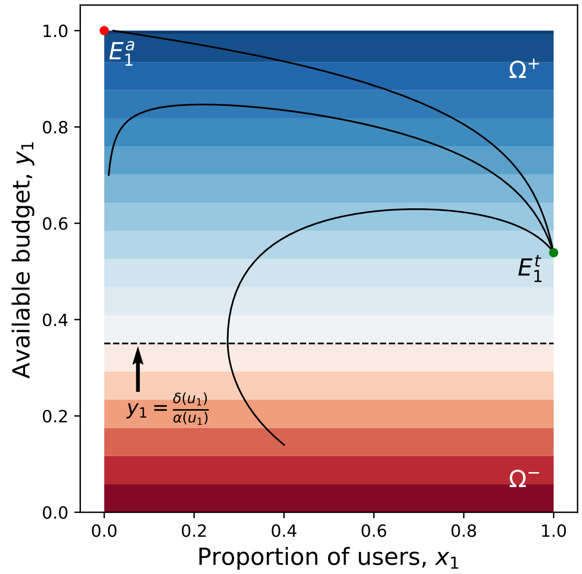

Figure 5-5.: A simulation scenario of the resident model (5-22) in the fase plane for a single TS, with $a = 1$, $a_1 = 0.8$, $a_2 = \sqrt{250}$, $d = 0.3$, $u_1 = 160$. In this case $\frac{\delta(u_1)}{\alpha(u_1)} = 0.3505 < 1$ and $l = 0.1 > 0.0462 = l^*(u_1)$ then the total adoption equilibrium $E_1^t = (1, 0.5388)$ is located above the non-smoothness curve C, i.e., it belongs to Ω^+, and is LAS, while $E_1^p \notin \Omega_1$.

$$l(u_1) > l^*(u_1) = \frac{\delta(u_1)\epsilon(u_1)\alpha(u_1)}{\alpha(u_1) - \delta(u_1)} \tag{5-28}$$

This implies that the equilibrium E_1^t is located in the region Ω^+ if and only if $l(u_1) > l^*(u_1)$, case in which $E^p \notin \Omega_1$. The eigenvalues of the linearization matrix in E_1^t are given by:

$$L_1 = \frac{\delta(u_1)\epsilon(u_1)\alpha(u_1) + l(u_1)\left(\delta(u_1) - \alpha(u_1)\right)}{\alpha(u_1)\epsilon(u_1) + l(u_1)} \quad \text{and} \quad L_2 = -\alpha\epsilon - l < 0$$

Observe that $L_1 < 0$, if and only if $\delta(u_1)\epsilon(u_1)\alpha(u_1) < l(u_1)\left(\alpha(u_1) - \delta(u_1)\right)$, then, since $\frac{\delta(u_1)}{\alpha(u_1)} < 1$, it is required $l(u_1) > l^*(u_2)$ as in (5-28) for E_1^t to exist in Ω^+ and to be LAS, see Figure **5-6** for illustration.

As shown in the analysis above, the condition $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ is sufficient to state that system (5-22) always have one stable equilibrium with either partial or total adoption of the TS. In Table **5-2**, the local stability results are summarized. Observe that, when system

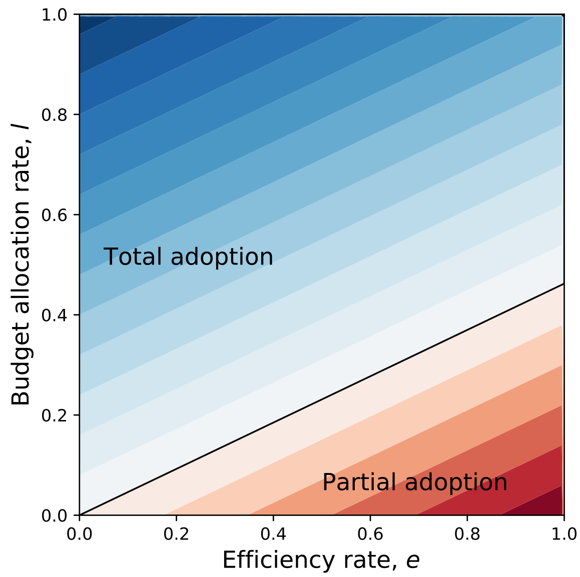

Figure 5-6.: Regions of partial adoption (red) and total adoption (blue) according to the local stability analysis summarized in the Table **5-2** with $a = 1$, $d = 0.3$, $a_1 = 0.8$, $a_2 = \sqrt{250}$ and $f_2 = 1$ and $f_1 = 0.9$.

adoption is total, the resources invested are never completely consumed as the equilibria value $0 < \frac{l(u_1)}{l(u_1)+\epsilon(u_1)\alpha(u_1)} < 1$ is achievedd

Table 5-2.: Resident model local stability results.

Region	Condition	Absence of TS's	Partial adoption	Total adoption
$\Omega^- : 0 \leq y_1 < \frac{\delta(u_1)}{\alpha(u_1)}$	$1 < \frac{\delta(u_1)}{\alpha(u_1)}$	E_1^a is LAS	$E_1^p \notin \Omega^-$	-
	$\frac{\delta(u_1)}{\alpha(u_1)} < 1$ and $l(u_1) < l^*(u_1)$	$E_1^a \notin \Omega^-$	$E_1^p \in C$ is LAS	-
$\Omega^+ : \frac{\delta(u_1)}{\alpha(u_1)} < y_1 \leq 1$	$l(u_1) < l^*(u_1)$	E_1^a is U	$E_1^p \in C$ is LAS	$E_1^t \notin \Omega^+$
	$l(u_1) > l^*(u_1)$	E_1^a is U	$E_1^p \notin \Omega_1$	$E_1^t \in \Omega^+$ is LAS

When $\frac{\delta(u_1)}{\alpha(u_1)} = 1$, the jacobian matriz at E_1^a has a zero eigenvalue and the equilibria losses its hyperbolicity, in deed, in such scenario $\overline{E}_1^p = \overline{E}_1^a$ and a transcritical bifurcation occur, where both equilibria interchange stability. On the other hand, if $l(u_1) = l^*(u_1)$, then $E_1^t = E_1^p = \left(1, \frac{\delta(u_1)}{\alpha(u_1)}\right)$, i.e., both equilibria collide on the curve C as can be proved by simple algebraic procedures, again a transcritical bifurcation occur [33,48,56,62,75].

5.3.2. Resident innovative model and fitness function

As shown in the previous section, if $\frac{\delta(u_1)}{\alpha(u_1)} < 1$ the resident TS system (5-22) will stabilize at some non trivial equilibrium (partial or total adoption) denoted from now on by:

$$\overline{E}_1 = (\overline{x}_1(u_1), \overline{y}_1(u_1)).$$

Suppose that an innovation occur in the established TS, corresponding to some technological modification which physically affects the established TS, and leads to the appearance of an innovative TS characterized by the value of attribute u_2. In general, it is assumed that the innovation is small, and will have a minimal effect, which permits the interaction between transport systems to occur below the same conditions, and on the same market platform. In this case, the difference between the two TSs is in the average number of users which they can transport per mobile unit. The innovative TS gives rise to a small proportion of users $x_2 = x_2(t)$ who compete with the established TS. The success or failure of the innovation may be studied through the resident system (5-8) for $N = 2$. Explicitly, the 4-dimensional system will exist as follows:

$$\begin{cases} \dot{x}_1 = [\alpha(u_1)y_1 - \delta(u_1)]\,[1 - \text{sign}(\alpha(u_1)y_1 - \delta(u_1))\,(x_1 + c(u_1,u_2)x_2)]\,x_1 \\ \dot{y}_1 = l(u_1)(1-y_1) - \epsilon(u_1)\alpha(u_1)x_1y_1 \\ \dot{x}_2 = [\alpha(u_2)y_2 - \delta(u_2)]\,[1 - \text{sign}(\alpha(u_2)y_2 - \delta(u_2))\,(x_2 + c(u_2,u_1)x_1)]\,x_2 \\ \dot{y}_2 = l(u_2)(1-y_2) - \epsilon(u_2)\alpha(u_2)x_2y_2. \end{cases} \tag{5-29}$$

This model goes by the name *resident-innovative system*. Note that, for the model characteristics, must be satisfied that $0 \leq x_1 + x_2 \leq 1$.

At the time in which innovation occur, the established TS is at the equilibrium $\overline{E}_1$; in other words, it is assumed that this equilibrium is LAS, and as such, the Jacobian matrix denoted here $\mathbf{J}(\overline{E}_1)$ has two eigenvalues with a real negative part. When the resident-innovative system is studied, it may be of interest to determine the conditions for the innovative TS of attribute u_2 can “invade” the market. For this, stability conditions at the equilibrium:

$$\overline{E}_2 = (\overline{x}_1(u_1), \overline{y}_1(u_1), 0, 1),$$

of absence of innovative TS must be studied. In deed, the 0 and 1 values in the last two coordinates of $\overline{E}_2$ indicate that the innovative TS has not yet entered the market, and that the entirety of the budget is available for allocation. In order to determine local stability, a small disruption is created around it, and the behavior of the linear system associated is studied [75]. The corresponding Jacobian matrix takes diagonal form in blocks:

$$\mathbf{J}(\overline{E}_2) = \left[\begin{array}{c|cc} \mathbf{J}(\overline{E}_1) & \cdots & \\ \hline \mathbf{0} & \begin{array}{c}(\alpha(u_2)-\delta(u_2))\left[(1-\text{sign}(\alpha(u_2)-\delta(u_2))c(u_2,u_1)\overline{x}_1(u_1))\right] \\ -\epsilon(u_1)\alpha(u_1)\end{array} & \begin{array}{c}0 \\ -l(u_2)\end{array} \end{array}\right],$$

where $\mathbf{J}(\overline{E}_1)$ is a 2×2 matrix, with both of their eigenvalues with real negative parts. In deed, it corresponds to the Jacobian matrix of the resident system (5-22) at a LAS equilibrium. The $\mathbf{0}$ in the bottom left block is a null 2×2 matrix. Thus, $\mathbf{J}(\overline{E}_2)$'s eigenvalues are those contributed by $\mathbf{J}(\overline{E}_1)$, together with $\lambda_1 = -l(u_2)$, which evidently is negative, owing to the positivity of $l(u)$, and $\lambda_2 = (\alpha(u_2)-\delta(u_2))\left[(1-\text{sign}(\alpha(u_2)-\delta(u_2))c(u_2,u_1)\overline{x}_1(u_1))\right]$, whose sign will depend on the functions that it involves,

$$\lambda(u_1,u_2) = \lambda_2 = (\alpha(u_2)-\delta(u_2))\left[1-\text{sign}(\alpha(u_2)-\delta(u_2))c(u_2,u_1)\overline{x}_1(u_1)\right] \tag{5-30}$$

Through the study of the sign of this function for specific u_1 and u_2, the possibility of innovative TS invasion may be established. In order to numerically study the previous system, recall that the proportion in which new resources are allocated for transport system expansion is $l(u_i) = l$, that the TS efficiency to "convert" the investment into new users is given by $\epsilon(u_i) = \varepsilon$, and that the rate at which the TS is abandoned by users $\delta(u_i) = d$ are constants for $i = 1,2$.

In Figure **5-7** it is shown the contour map of the fitness function $\lambda(u_1,u_2)$ for $f_1 > 1$, which means that the TS capable of transporting a greater average of passenger will have the competitive advantage, the other parameters are as described in the caption. It can be seen that the fitness function is negative in the red regions and positive in the blue ones. The solid green line correspond to $u_2 = u_1$ and the black solid line is $l(u_1) = l^*(u_1)$ describing a switching curve since if $l < l^*$, then $\overline{x}_1(u_1) = \frac{l(\alpha(u_1)-d)}{d\varepsilon\alpha(u_1)}$ in (5-30), and if $l > l^*$, then $\overline{x}_1(u_1) = 1$. Finally, the red solid line correspond to the points where $\alpha(u_2) = d$ another switching curve for (5-30), in fact, for the points on this curve, $\lambda(u_1,u_2) = 0$.

5.3.3. AD canonical equation and its stability analysis

In order to study the evolutionary dynamics of an attribute u_1, it is necessary to determine the canonical equation of adaptive dynamics, given by the expression:

$$\dot{u}_1 = \frac{1}{2}\mu(u_1)\sigma^2(u_1)\overline{x}_1(u_1)\frac{\partial\lambda}{\partial u_2}(u_1,u_1) = f(u_1), \tag{5-31}$$

where $\mu(u_1) = \mu$ is innovation frequency, and $\sigma^2(u_1) = \sigma^2$ is variance. The canonical equation considers the frequency at which innovations are presented in the TS market,

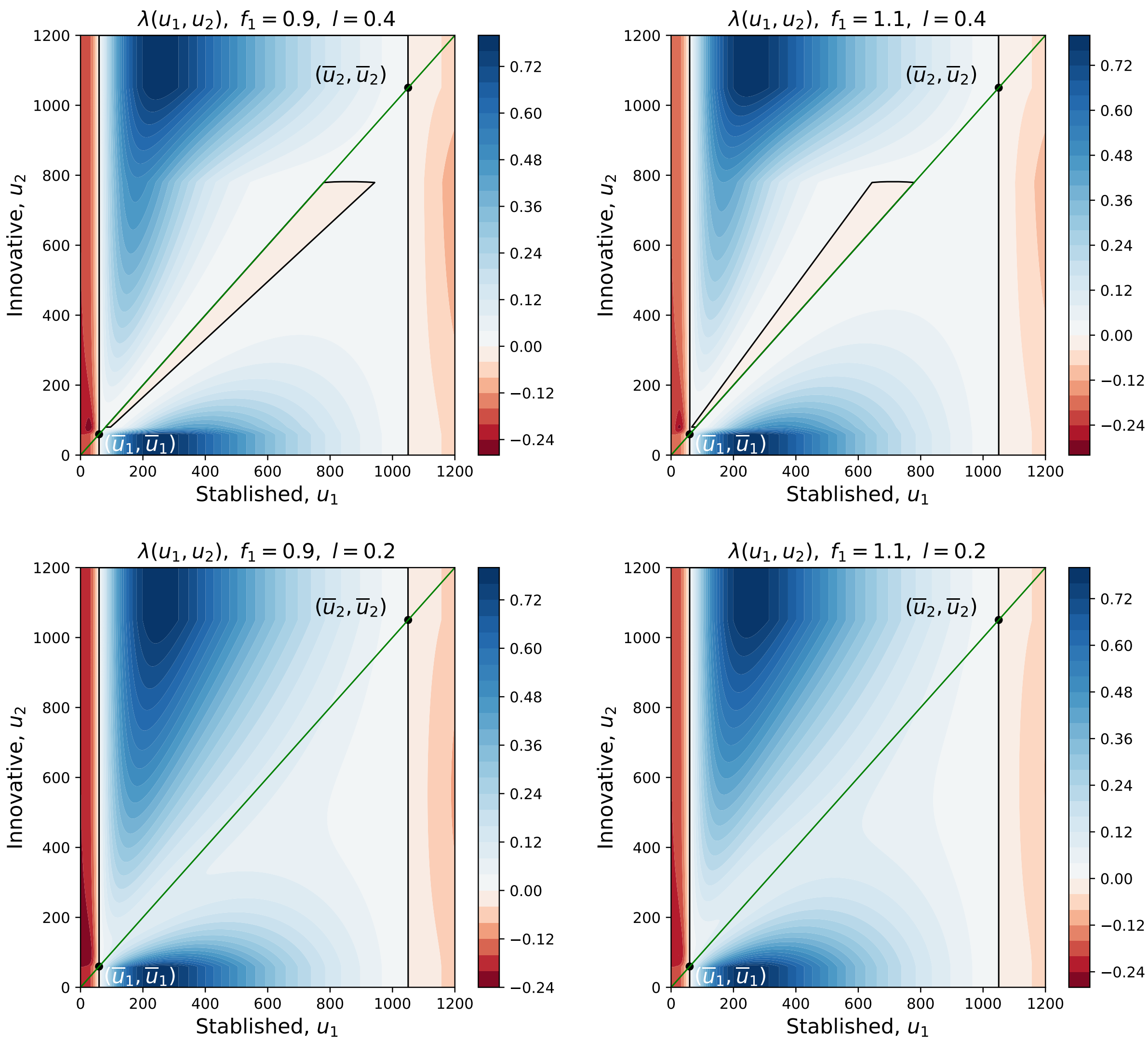

Figure 5-7.: Contour map of the fitness function (5-30) with the values of the parameters in Table **5-1**, and the panel title. The color range allows for establishing the regions in the $(u_1, u_2)-$plane where the fitness function is positive (blue regions), and therefore where the invasion of the innovative TS is possible. The solid black line correspond to $\lambda(u_1, u_2) = 0$, and the green solid line to points where $u_2 = u_1$, where also λ vanishes.

and the size of the variations obtained in each innovation. Consider the function f defined by the right hand side of (5-31), the $\overline{x}(u_1)$ value corresponds to the number of users of stablished TS at equilibrium before the innovation, therefore, based on the stability analysis of the resident model made in the previous section, provided $\alpha(u_1) > d$, we have two cases:

- If $l < l^*$, then $\overline{x}_1(u_1) = \frac{l(\alpha(u_1)-d)}{\alpha(u_1)d\varepsilon}$, corresponding to the partial adoption equilibrium in the resident model.

- If $l > l^*$, then $\overline{x}_1(u_1) = 1$, corresponding to the total adoption equilibrium in the resident model.

Therefore, $l = l^*$ is a switching curve for the differential equation. The case when $\alpha(u_1) \leq d$ should be discarded since it makes $\overline{x}(u_1) = 0$, and therefore we would have $\dot{u}_1 = 0$ by (5-31), which lacks practical interest. On the other hand, the partial derivative $\frac{\partial \lambda}{\partial u_2}(u_1, u_1)$, is called the *selection gradient*, and is associated with the forces of selection exerted by the TS users on the long-term dynamics of the characteristic attribute u_1. Specifically for the coefficients defined in our model (recall that $c(u_1, u_1) = 1$) and since $\alpha(u_1) > d$, the selection gradient is,

$$\frac{\partial \lambda}{\partial u_2}(u_1, u_1) = -\alpha'(u_1)\left(\overline{x}_1(u_1) - 1\right) - \left[\alpha(u_1) - d\right]\left[\overline{x}_1(u_1) + c_{u_2}(u_1, u_1)\overline{x}_1(u_1)\right],$$

where the subscript u_2 in $c_{u_2}(u_1, u_1)$ denotes partial differentiation. Equilibria of the AD canonical equation are constant solutions of equation (5-31), therefore, they are obtained by solving $f(u_1) = 0$. In either case, they correspond to the points where $\frac{\partial \lambda}{\partial u_2}(u_1, u_1) = 0$. Two explicit solutions are given by:

$$\overline{u}_1 = a_2^2 e^{-a_1\sqrt{-2\ln(d/a)}} \quad \text{and} \quad \overline{u}_2 = a_2^2 e^{a_1\sqrt{-2\ln(d/a)}}, \tag{5-32}$$

and additional equilibria occur only in the case of partial adoption, given by the expression,

$$\overline{u} = a_2^2 \exp\left(\frac{a_1^2 l(ae^{\overline{z}} - d)^2 \ln f_1}{a f_2^2 e^{\overline{z}}(a(d\varepsilon - l)e^{\overline{z}} + dl)}\right), \tag{5-33}$$

where $\overline{z}$ is a solution of the transcendent equation $g(z) = 0$, with,

$$\begin{aligned} g(z) \;=\; & a^4(2f_2^4(de-l)^2 z + a_1^2 l^2 \ln^2 f_1)e^{4z} - 4a^3 dl(-f_2^4(de-l)z + a_1^2 l \ln^2 f_1)e^{3z} \\ & + 2a^2 d^2 l^2 (f_2^4 z + 3a_1^2 \ln^2 f_1)e^{2z} - 4aa_1^2 d^3 l^2 \ln^2 f_1 e^z + a_1^2 d^4 l^2 \ln^2 f_1. \end{aligned} \tag{5-34}$$

Regarding the analytical equilibria in (5-32), note $\overline{u}_1$ and $\overline{u}_2$ are defined as positive real numbers if, and only if $a \geq d$, i.e., the maximum rate of instant adoption a must be greater or equal than the rate at which the technology is abandoned by users d, which makes perfect sense, since it ensures that the TS has the ability to stay on the market. Both equilibria satisfy $\overline{u}_1 \leq \overline{u}_2$, indeed, as $a \geq d$,

$$-a_1\sqrt{-2\ln(d/a)} \geq 0 \iff e^{-a_1\sqrt{-2\ln(d/a)}} \leq 1$$
$$\iff a_2^2 e^{-a_1\sqrt{-2\ln(d/a)}} \leq a_2^2$$
$$\iff \overline{u}_1 \leq a_2^2.$$

Similarly it can be shown that $a_2^2 \leq \overline{u}_2$. Therefore, $\overline{u}_1$ corresponds to the strategy of reducing in the long term the number of transported passengers per mobile unit, and $\overline{u}_2$ to the strategy of increasing in the long term that number (in deed, using the parameters described in Table **5-1**, particularly $a_2 = \sqrt{250}$ as usual here, it is obtained $\overline{u}_1 = 59.5111$ and $\overline{u}_2 = 1050.2242$). Particularly,

$$a = d \iff \overline{u}_1 = \overline{u}_2,$$

then, if $a = d$, both equilibria collide by means of a fold bifurcation.

The function g is an increasing for $z > 0$, but have some oscillations otherwise. A numeric method has been used to identify the solutions of $g(z) = 0$, varying the budget allocation rate for $l < l^*$ (partial adoption). The numerical method consists of evaluating the function $g(z)$ in the interval $[-10, 0.1]$ with a partition of 10000 nodes for a fixed l (wider intervals were used, but no additional roots were identified). For each pair of consecutive points in the partition ξ_j and ξ_{j+1} the possibility of changing the sign is found in order to establish an interval where possibly a root of the function is found; that is to say, if $\text{sign}(g(\xi_j)) \neq \text{sign}(g(\xi_{j+i}))$, then the interval $[a_j, b_j]$ must contain a root of $g(z)$, where $a_j = \text{mín}\{\xi_j, \xi_{j+1}\}$ and $b_j = \text{máx}\{\xi_j, \xi_{j+1}\}$ (one, two or four of these intervals were found depending on the value of a_2, but a further analysis of a_2 has not been developed yet, since our focus at this point is one parameter behavior: l). Each interval is used in a bisection method (10000 iterations) to identify a good aproximation of $\bar{z}_j$ (in every case, an error was obtained up the order at least of 10^{-18}. This procedure is repeated for every $l \in [0, l^*]$, with a partition of 500 vales for l. Once the solutions $\bar{z}$ are identified for each l, they are substituted into the expression (5-33) and scattered in a one parameter bifurcation diagram, with l in the horizontal axis and the equilibria $\overline{u}_i$ in the vertical axis. For $l > l^*$ (total adoption), only the analytical equilibria $\overline{u}_1$ and $\overline{u}_2$ (constants on l) were plot, recall l is defines as a proportion in $[0, 1]$.

Figure **5-8** illustrate the results and using $a_2 = \sqrt{250}$ (panels A and B) to illustrate a scenario with two numerical equilibria ($a_2 = \sqrt{500}$ is shown to illustrate a scenario with four numerical equilibria, see Figure **5-8**, panels C and D) and the other parameters are as described in Table **5-1**. The switching line $l = l^* = 0.2349$ (panels A and B) has been illustrated in dashed orange, therefore, at the left is the region of partial adoption, and at the right is the region of total adoption (for $a_2 = \sqrt{500}$, we get $l^* = 0.4013$, panels C and D). Numeric equilibria $\overline{u}_i$ are shown in Figure **5-8**A for $f_1 = 0.9$ to illustrate the case when users favor a TS capable of transporting a lower average number of passengers; the case when users favor a greater average number of passengers ($f_1 = 1.1$) is shown in Figure

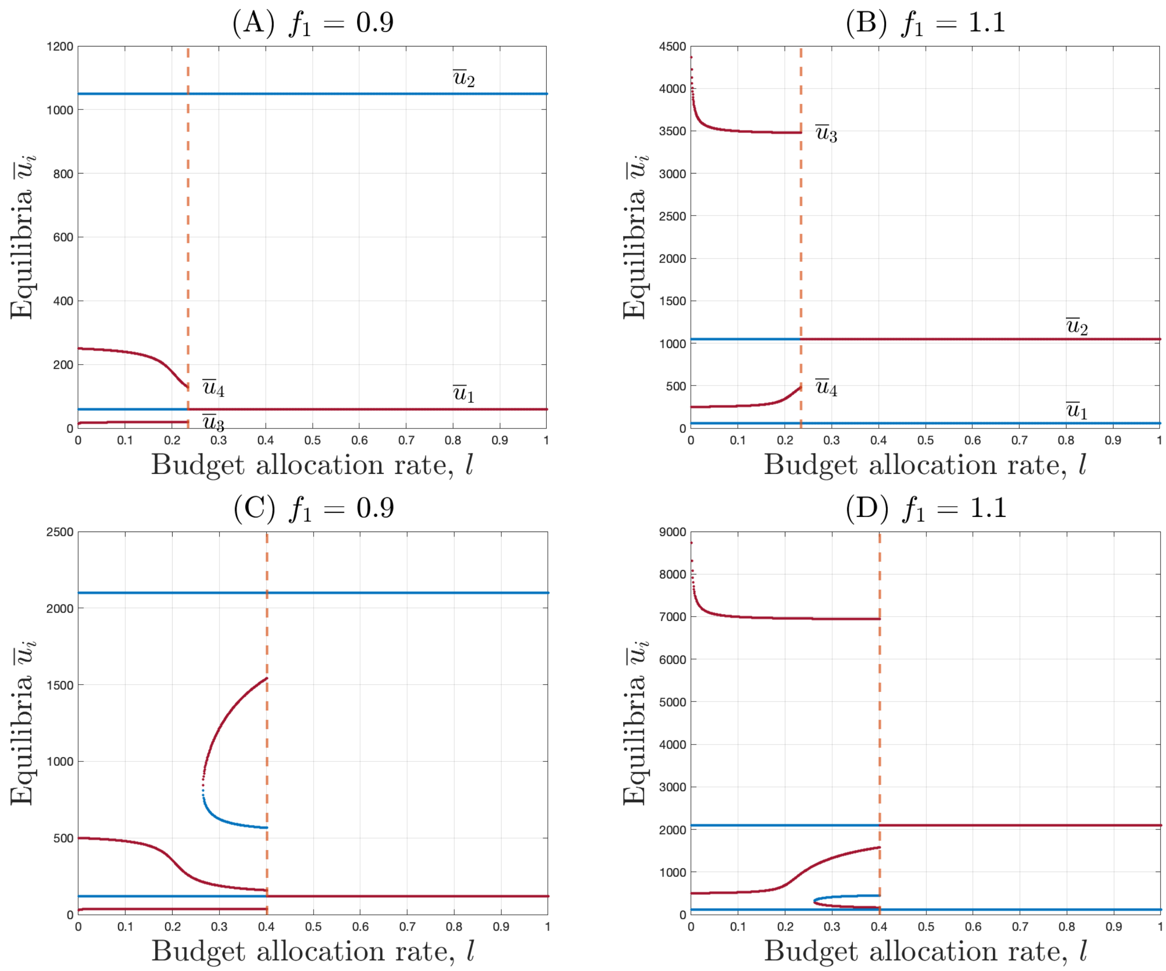

Figure 5-8.: AD canonical equation equilibria $\overline{u}_i$ for different values of l ($l = l^*$ in dashed orange), for $f_1 = 0.9$ (panel A) and for $f_1 = 1.1$ (panel B); in both cases have been used $a_2 = \sqrt{250}$. The line color has been used to illustrate the sign of the associated eigenvalue $f'(\overline{u}_i)$, blue when positive, then $\overline{u}_i$ is unstable, and red when $f'(\overline{u}_i) < 0$, so that $\overline{u}_i$ in those cases is LAS. Panels C and D illustrate the same situation considering $a_2 = \sqrt{500}$.

5-8B. Similarly, Figure **5-8**C corresponds to $f_1 = 0.9$, while Figure **5-8**D illustrates the case $f_1 = 1.1$.

The stability of any equilibria $\overline{u}_i$ is determined by the sign of the real part of the associated eigenvalue $f'(\overline{u}_i)$. The explicit expressions of that eigenvalue are determined for each equilibria $\overline{u}_1$ and $\overline{u}_2$ in the expression (5-32), however they are omitted here since they are very long. For the numerical equilibria, the explicit expression f' is handled numerically. In order to illustrate stability, it is shown in blue the point corresponding to an equilibrium $\overline{u}_i$ where $f'(\overline{u}_i) > 0$, and therefore a blue curve, is a curve of unstable equilibria. Conversely, the red curves correspond to curves of equilibria where $f'(\overline{u}_i) < 0$

and therefore LAS equilibria. Note that in both cases, the stable and unstable equilibria are intercalated, for example, in Figure **5-8**A, the equilibria $\overline{u}_3$ and $\overline{u}_4$ (for $l < l^*$) were obtained numerically, starting from the bottom up, we get that $\overline{u}_3$ is LAS (attractor), then $\overline{u}_1$ is unstable (repulsor), then $\overline{u}_4$ is LAS (attractor) and finally $\overline{u}_2$ unstable (repulsor). An interesting result is obtained for $a_2 = \sqrt{500}$ in Figure **5-8**C and D, at $l = 0.26$ approximately, that is the occurrence of a tangent bifurcation. The analysis of that results is beyond our objetives at this moment, but interesting dynamics arises when varying multiple parameter and a series of 2-parameter bifurcation analysis is left for a future work.

When an innovative TS transporting u_2 users per mobile unit in average, is made available, the established TS of attribute u_1 is at (or close to) its equilibrium $\overline{x}(u_1)$. Therefore the sign of the fitness function $\lambda(u_1, u_2)$ determines whether the innovation invades or quickly disappears. Moreover, the "invasion implies substitution" theorem [26, 31], says that if u_2 is sufficiently close to u_1, invasion under a nonzero "selection gradient" implies the substitution of the former attribute by the new one. After the substitution transient, the attribute u_1 is eliminated from the TS market and replaced by the attribute u_2, that can therefore be renamed u_1, i.e., the new established attribute. In other words, when an equilibria of the AD canonical equation is LAS, means that successive innovations which replace those previous, direct the attribute u_1 toward the value of equilibria $\overline{u}_i$, for i as appropriate. For instance, in the total adoption case $l > l^*$ and $f_1 = 1.1$ (see Figure **5-8**B), $\overline{u}_2$ is LAS, it implies that the successive innovations and substittutions increase in the long term the number of transported users per mobile unit from $\overline{u}_1 = 59.5111$ to a TS capable for transporting a greater average number of users per mobile unit $\overline{u}_2 = 1050.2242$.

5.3.4. Coexistence and divergence conditions

Close to an equilibria of the canonical equation, invasion does not necessarily imply substitution. Lets consider an attracting equilibrium of the AD canonical equation $\overline{u}$, towards where the innovation process directs the number of passengers transported by mobile unit, expanding the fitness function $\lambda(u_1, u_2)$ up to second-order w.r.t. both (u_1, u_2) at $(\overline{u}, \overline{u})$, both $\lambda(u_1, u_2)$ and $\lambda(u_2, u_1)$ can be positive close to $(\overline{u}, \overline{u})$, so that both attributes can invade a market established by the other. As stated in [45, 46, 64] and also included in [31]), this occurs when coexistence condition,

$$\frac{\partial^2 \lambda}{\partial u_1 \partial u_2}(\overline{u}, \overline{u}) < 0, \tag{5-35}$$

is satisfied. Considering that the coexistence condition only makes sense for a stable equilibrium of the canonical equation, in Figure **5-9**A, the coexistence condition is shown by the curve coloring for the equilibria $\overline{u}_3$, $\overline{u}_4$ (if $l < l^*$, partial adoption) and for $\overline{u}_1$ (if $l > l^*$ total adoption). As usual, red means that the coexistence condition is negative and blue indicates positive values.

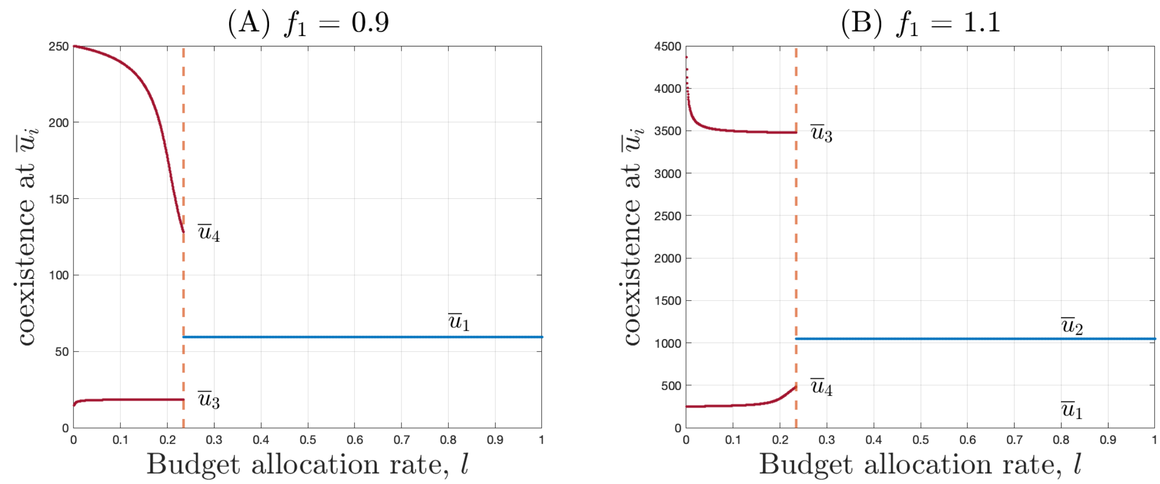

Figure 5-9.: (A) Illustration of the coexistence condition (5-35) for different values of l and $f_1 = 0.9$. (B) The same scenario but $f_1 = 1.1$. The line color indicates the sign if the coexistence condition in the labeled equilibrium, red when negative and blue when positive. The parameters are set as indicated in Table Parameters.

Note that for total adoption and $f_1 = 0.9$, although $\overline{u}_1$ is a LAS equilibria (the only one), the coexistence condition at $\overline{u}_1$ is not satisfied, then, under this conditions, is not possible that an innovative TS, which arises as a technological adaptation of the transport system characterized by the attribute u_1, can enter the market and coexist with it, this definitively eliminates the possibility of diversification. The same happens when $f_1 = 1.1$, in this case the LAS equilibrium is $\overline{u}_2$, but as illustrated in Figure **5-9** B, the coexistence condition is not satisfied either, then the invasion and subsequent permanence in the market of an innovative transport TS entering a market dominated by the TS characterized by the attribute u_1 is not possible either.

On the other hand, in the case of partial adoption (at the left of the dashed line $l = l^*$), both LAS equilibria $\overline{u}_3$ and $\overline{u}_4$ satisfy the coexistence condition, as is illustrated by the red equilibria curves. However, the coexistence of two different, although similar, TSs, is only the first step towards market diversification. In deed, what really generates two different TSs is that the innovation process is such that the successive innovations of the two coexisting TSs direct the attributes' evolution of u_1 and u_2 in opposite directions. Without going into the theoretical details, (see again [45, 46, 64] or [31]), we have that the fitness function in the market dominated by a single TS determines the possibility of diversification. Specifically, if

$$\frac{\partial^2 \lambda}{\partial u_2^2}(\overline{u}, \overline{u}) > 0, \tag{5-36}$$

then innovations u_2 in the attribute u_1, invade and replace the established TS of attribute

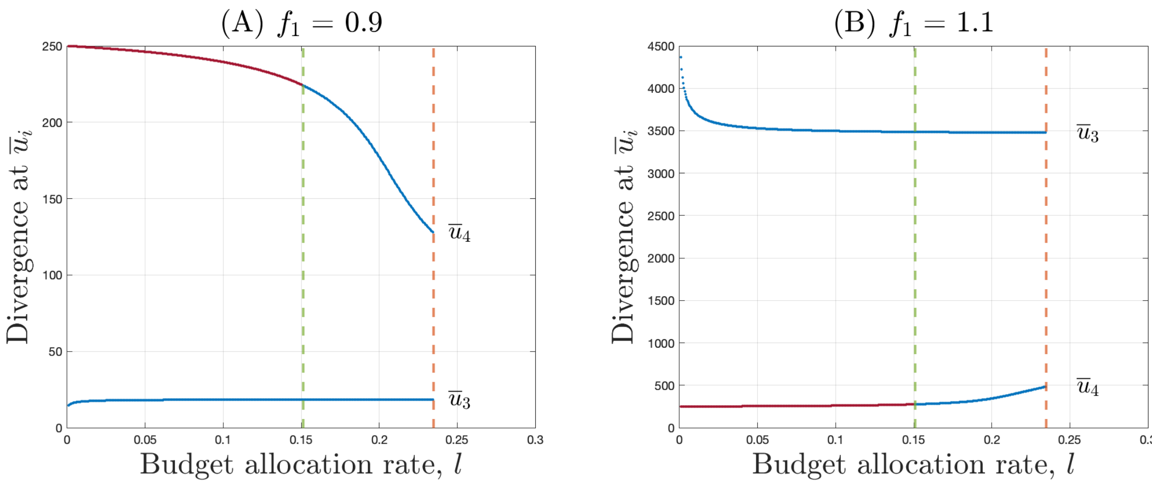

Figure 5-10.: (A) Illustration of the divergence condition (5-36) for different values of l and $f_1 = 0.9$ in the equilibria where coexistence is met. (B) The same scenario but $f_1 = 1.1$. The line color indicates the sign if the divergence condition in the labeled equilibrium, red when negative and blue when positive. The parameters are set as indicated in Table Parameters.

u_1 if $u_2 < u_1$, while the same would happen reciprocally if $u_2 > u_1$, as a result, the attributes u_1 and u_2 get further diversified and the selection in the market is said to be disruptive at the equilibrium.

According to the results for the coexistence condition, the analysis of the divergence condition (5-36) should be limited to the equilibria $\overline{u}_3$ and $\overline{u}_4$, obtained numerically for the case of partial adoption. Figure **5-10** illustrates the sign of this condition using the color assigned to the equilibrium curve, blue when positive and red when negative. Additionally, panel A illustrates the case $f_1 = 0.9$ while panel B illustrates for $f_1 = 1.1$, and the other parameters as in Table **5-1**. It can be seen that when users favor the TS transporting a lower average number of passengers ($f_1 = 0.9$) than the equilibrium $\overline{u}_3$ satisfies the divergence condition for any value $l < l^*$, while the equilibrium $\overline{u}_4$ only satisfies the condition for values between the green dashed line, let's say $\hat{l} = 0.1514$ (value that was determined approximately by identifying the change of sign of the condition between consecutive equilibria in the curve) and the threshold value $l^* = 0.2349$ (orange dashed). Something similar occurs when users favor a TS with the capacity to transport a large average number of passengers, in deed, in this case, it is the equilibrium $\overline{u}_4$ that only satisfies the divergence condition for values $\hat{l} < l < l^*$, while $\overline{u}_3$ satisfies this condition throughout the region of partial adoption $l < l^*$.

The most important of these results is that the region of divergence, which corresponds here to intervals on the parameter l, determines the conditions under which diversification in the TS market is possible. In fact, as previously indicated, when the divergence condition is satisfied, it means that a TS characterized by the attribute u_2 (although this choice is mathematically arbitrary) arisen from an innovation in the attribute u_1 of the established TS (at equilibrium) before innovation, can not only enter the market and coexist with the former (market share), but also the selection forces exerted by users direct the evolutionary dynamics of both attributes in different directions, allowing both TS differ from each other; that is, creating diversity into the market.

Also note the budget allocation rate l plays an important role. Consider for example a rate $l < \hat{l}$ in the scenario in which a lower number of passengers per mobile unit is favored ($f_1 = 0.9$) as shown in Figure **5-10**A (at the left of the green dashed line), diversification only will be possible in the smallest equilibrium $\overline{u}_3$, which increases from $\overline{u}_3 = 14.5256$ when $l = 0$ to $\overline{u}_3 = 18.3951$ when $l = \hat{l}$; on the other hand, when a TS with the capacity to transport a high average number of passengers ($f_1 = 1.1$) is favored (see Figure **5-10**B at the left of the green dashed line), the values of the budget allocation rate $l < \hat{l}$, will lead to diversification at the greater equilibrium, which in this case decreases from $\overline{u}_3 = 4365.7749$ when $l = 0$ to $\overline{u}_3 = 3485.8130$ when $l = \hat{l}$.

The situation is different when the budget allocation rate satisfies $\hat{l} < l < l^*$. Indeed, when the TS with a lower capacity ($f_1 = 0.9$) is favored (see **5-10**A at the right of the green dashed line), it is possible to obtain diversification in the smallest equilibrium $\overline{u}_3$ that varies between the values $\overline{u}_3 = 18.3951$ when $l = \hat{l}$ and $\overline{u}_3 = 18.4445$ when $l = l^*$; but diversification can also be obtained in the larger equilibrium, which decreases from $\overline{u}_4 = 225.1267$ when $l = \hat{l}$ to $\overline{u}_4 = 127.5442$ when $l = l^*$. An equivalent situation occurs when users favor the TS that transports a higher average number of passengers (see **5-10**B at the right of the green dashed line); in this case, diversification can be obtained in both equilibria $\overline{u}_3$ (the large one) and in $\overline{u}_4$ (the small one); the first decreases from $\overline{u}_3 = 3485.8130$ when $l = \hat{l}$ to $\overline{u}_3 = 3477.1164$ when $l = l^*$, while the second grows from $\overline{u}_4 = 274.8999$ when $l = \hat{l}$ up to $\overline{u}_4 = 484.1367$ when $l = l^*$.

5.3.5. Origin of diversity through branching

In Figure **5-11**, the scenarios before innovation corresponding to partial adoption ($l = 0.2$) and total adoption ($l = 0.4$) in the resident model (5-22) are illustrated in solid black with $u_1 = 250$, and considering user's preference in favor of a TS capable of transport a low number of passengers per mobile unit $f_1 = 1/1.1$. In the case of partial adoption the initial conditions are $x_1(0) = 0.01$ and $y_1(0) = 1$, indicating that the TS enters the market with all the budget available to invest and a scarce proportion of users, and reaches the partial adoption equilibrium $E_1^p = (\overline{x}_1, \overline{y}_1) = (0.8885, 0.2)$. Note that the TS is not being used to its maximum capacity and the available budget is not fully consumed either, however this TS no longer has the possibility of growing further. The same initial conditions were used for the total adoption scenario, but in this case, the equilibrium $E_1^t = (\overline{x}_1, \overline{y}_1) = (1, 0.3077)$ is reached, which clearly indicates that the TS is at its

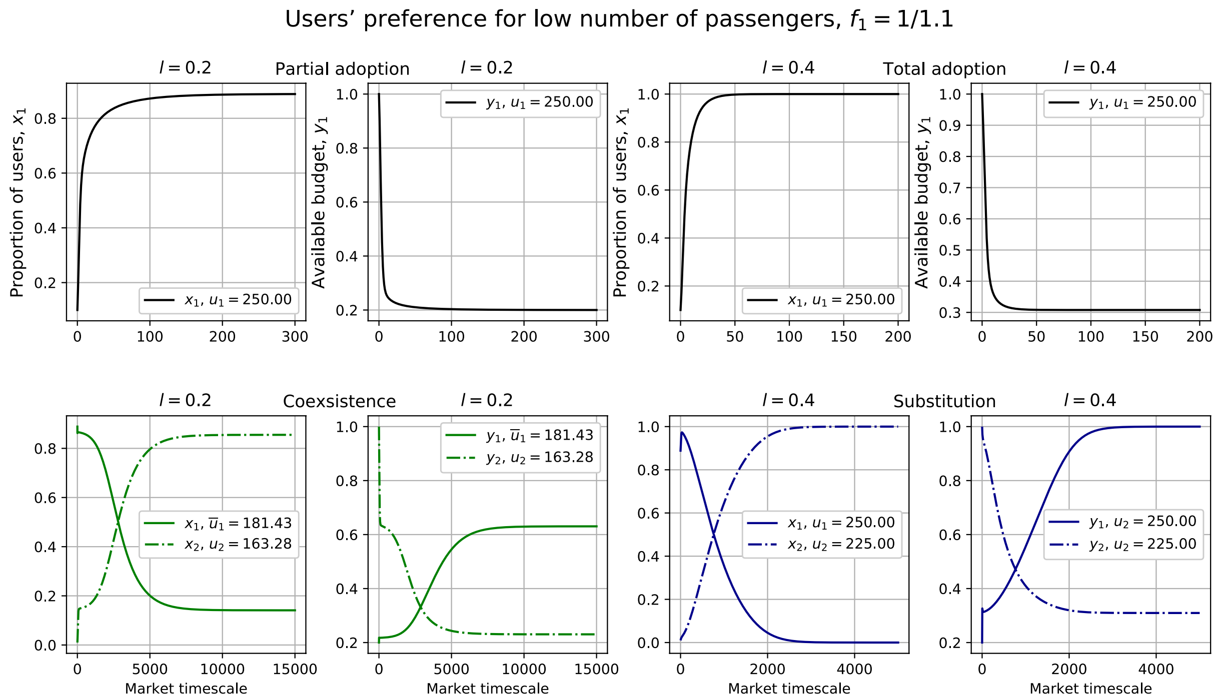

Figure 5-11.: Market dynamics before (solid black) and after diversificación (green and blue) for users' preference for low capacity TS ($f_1 = 1/1.1$). Before innovation the simulations correspond to the resident model (5-29) for partial adoption ($l = 0.2$) and total adoption ($l = 0.4$) and the other parameters as in Table **5-1**. After innovation the simulations correspond to the resident-innovative model (5-29). See the text for further details.

full capacity and is left an available budget to invest of approximately 30 %.

After the innovation, the dynamics is governed by the model (5-29); Figure **5-11** illustrates the substitution scenario in blue, while the green curves show the diversification scenario. As discussed in the previous section, in the total adoption region, neither of the two equilibria $\overline{u}_1$ nor $\overline{u}_2$ satisfies simultaneously the conditions of stability and coexistence, therefore, an innovation arising from a slight decrease (remember $f_1 < 1$) in the average number of transported passengers per mobile unit ($u_1 = 250$ while $u_2 = 225$ for the innovative TS), leads the number of users of the innovative TS, starting from a very small proportion $x_2(0) = 0.01$, grow to equilibrium $\overline{x}_2 = 1$, while the available budget, which starts at $y_2(0) = 1$, reaches the value equilibrium $\overline{y}_2 = 0.3076$ (dash-dot blue); that is, the innovative TS reaches the equilibrium of partial adoption; meanwhile, the previously established transport system (of attribute u_1) is removed from the market (solid blue), in deed, after innovation it reaches the equilibrium $E_1^a = (0, 1)$ which indicates that it does not has any market share.

Before describing the market diversification scenario, we first must consider the case

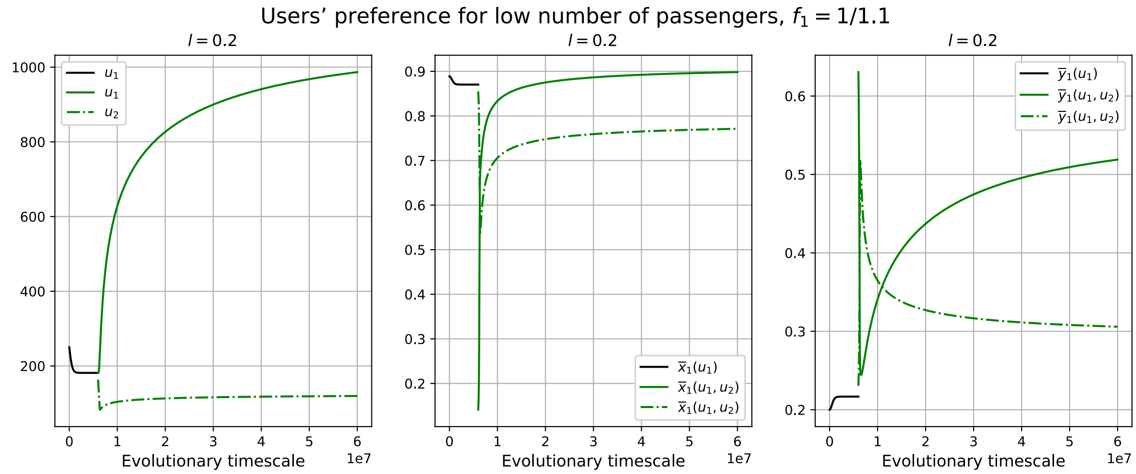

Figure 5-12.: Evolutionary dynamics of the number of transported passengers when consumer prefer a TS capable of transported a low number of passengers ($f_1 = 1/1.1$). In solid black the simulation of the AD canonical equation for u_1 is shown (left) and the corresponding market equilibrium values are shown at the center ($\overline{x}_1(u_1)$) and at the right ($\overline{y}_1(u1)$). Similarly, in green they are illustrated the solutions of the canonical equations after branching for u_1 and u_2 (right) and the corresponding equilibrium values in the market ($\overline{x}_{1,2}(u_1, u_2)$)-center and ($\overline{y}_{1,2}(u_1, u_2)$)-right.

of partial adoption to simulate the AD canonical equation (5-31), which allows us to observe the evolutionary dynamics of the attribute u_1, in the long term, as shown by the solid black curve in Figure **5-12**-left. The initial condition is $u_1(0) = 250$, and the other parameters as described in the caption. It can be seen that the number of passengers transported, after successive innovations that replace the previous ones, will decrease until reaching the equilibrium $\overline{u}_1 = 181.4252$, which is a branching point, in deed, the stability condition $f'(\overline{u}_1) = -0.000003$, the coexistence condition (5-35) in $\overline{u}_1$ is -0.000020 and the divergence condition (5-36) is 0.000012. In Figure **5-12**-center and -right, the value of $\overline{x}_1(u_1)$ and $\overline{y}_1(u_1)$ where u_1 is the solution for the AD canonical equation are shown, with the aim is to illustrate how the evolutionary dynamics of u_1 impacts the market equilibrium in the evolutionary timescale (solid black respectively).

Now the branching point $\overline{u}_1$ is used to simulate the coexistence scenario in the resident-innovative model (5-29) with $u_1 = \overline{u}_1 = 181.4252$ and $u_2 = 0.9u_1 = 163.2825$ (green in Figure **5-11**), $l = 0.2$ (partial adoption), with the initial conditions for x_1 and y_1 at the final values obtained in the simulation of the resident model (5-22) (partial adoption solid black) and $x_2(0) = 0.01$ and $y_2(0) = 1$. Solutions for the established TS are illustrated in solid green and solutions for the innovative TS are shown in dash-dot green. Note that the innovative TS manages to penetrate the market and coexist with the established TS and with an even greater market share $\overline{x}_2 = 0.8550$ (dash-dot green), while

$\overline{x}_1 = 0.1411$ (solid green), reflecting the preference of users for a lower average number of transported passengers ($f_1 < 1$); reciprocally, the TS with the greater market share is the one that finally has the least available budget. In order to illustrate what happens in the long term, the AD canonical equations for u_1 and u_2 are deduced (repeating the process described in the previous sections and using the fitness function (5-14)). The results are illustrated in Figure **5-12**-left, it can be seen that both attributes u_1 (solid green) and u_2 (dash-dot green) evolve independently, which makes perfect sense, since after branching, both are governed by different selection forces exerted from the market (the respective selection gradients). In this case, the market diversifies with two TS that, in the long term, correspond to one with a greater capacity u_1 and another with a smaller capacity u_2. The evolutionary behavior of the market equilibria $\overline{x}_1(u_1, u_2)$ and $\overline{x}_2(u_1, u_2)$ are shown in Figure **5-12**-center, and $\overline{y}_1(u_1, u_2)$ and $\overline{y}_2(u_1, u_2)$ are shown in Figure **5-12**-right, these last two panels allow us to estimate the equilibrium value of the proportion of passengers and available budget that there will be in the long term, as the number of passengers transported by each TS evolves.

Now we are going to show an example in which users prefer a TS with the capacity to transport a large number of passengers ($f_1 = 1.1$). Figure **5-13** shows the curves corresponding to the resident model (5-22) in solid black for the case of partial adoption ($l = 0.2$) and total adoption ($l = 0.4$), in both scenarios the initial conditions $x_1(0) = 0.01$ and $y_1(0) = 1$ and $u_1 = 160$ has been used. In the case of partial adoption, the TS reaches the equilibrium $E_1^p = (\overline{x}_1, \overline{y}_1) = (0.8514, 0.233)$. On the other hand, in the case of total adoption, the TS reaches the equilibrium $E_1^t = (\overline{x}_1, \overline{y}_1) = (1, 0.3418)$.

As shown previously, the total adoption scenario does not allows the possibility of obtaining evolutionary branching, for this reason, it has been used to illustrate the case of market substitution. Indeed, the blue curves in Figure **5-13** were obtained by simulating the model (5-29) with $u_1 = 160$ for the resident TS (solid blue), while $u_2 = 192$ for the innovative TS (dash-dot blue), note that $u_1 < u_2$ is set to reflect users' preference for higher capacity transport systems. As expected, the innovative TS enters the market and increases its participation to its maximum capacity $\overline{x}_2 = 1$ and ending with an available budget of $\overline{y}_2 = 0.3204$. Meanwhile, the TS that was established before the innovation (solid blue) is definitely eliminated; indeed, the established TS (which before innovation was in the total adoption equilibrium) reaches the equilibrium $E_1^a = (\overline{x}_1, \overline{y}_1) = (0, 1)$, in which there are no users and the entire budget is available to invest. It is important to note that even in the scenario of total adoption, which guarantees that the system is 100 % occupied, there is still a budget available (a little more than 32 % according to the parameters used here).

According to the analysis made in the previous sections, it is the partial adoption scenario that allows diversification. To illustrate this, we will first discuss Figure **5-14**-left that shows the evolutionary dynamics of the attribute u_1 in the long term, that is, the numerical solution of the AD canonical equation (5-31) (solid black) that corresponds

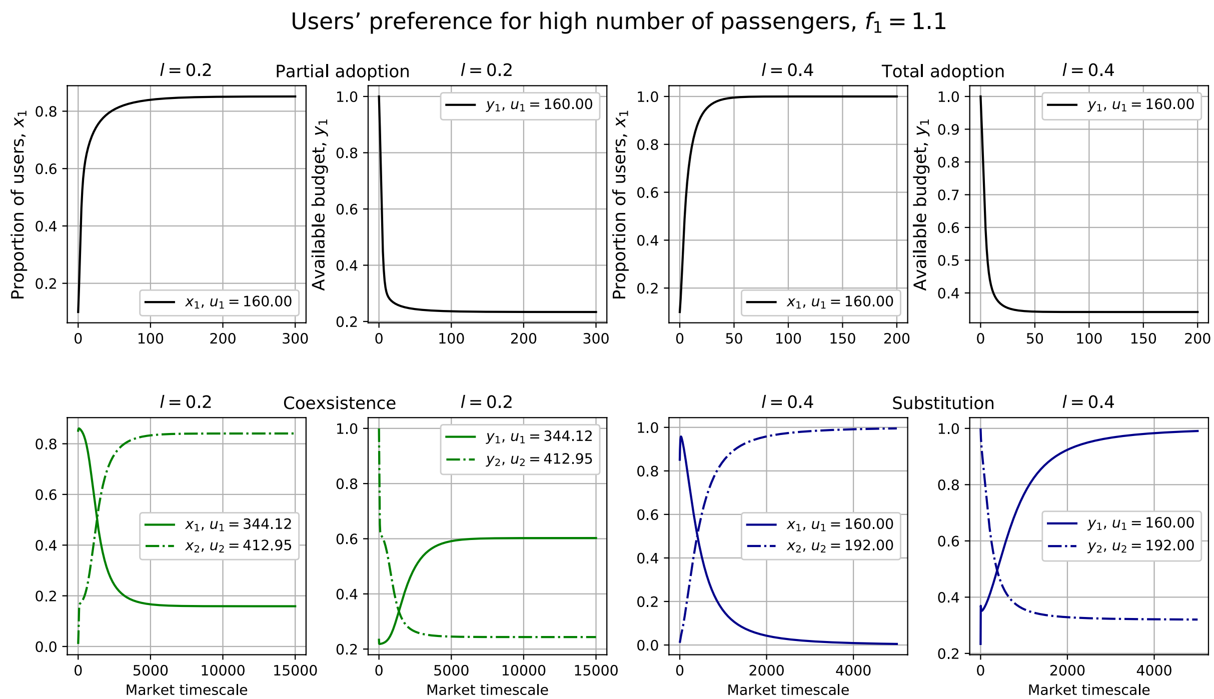

Figure 5-13.: Market dynamics before (solid black) and after diversificación (green and blue) when consumer prefer a TS capable of transported a high number of passengers ($f_1 = 1.1$). Before innovation the simulations correspond to the resident model (5-29) for partial adoption ($l = 0.2$) and total adoption ($l = 0.4$) and the other parameters as in Table **5-1**. After innovation the simulations correspond to the resident-innovative model (5-29). The other parameters as indicated in the panel, and Table **5-1**

to a branching point (the stability condition $f'(\overline{u}_1) = -0.000001$, the coexistence condition (5-35) in $\overline{u}_1$ is -0.000005 and the divergence condition (5-36) is 0.000003). Note u_1 grows from the initial condition $u_1(0) = 160$ to the evolutionary equilibrium value $\overline{u}_1 = 344.1248$. The behavior of u_1 has an impact on the values of the equilibria in the market $\overline{x}_1(u_1)$ and $\overline{y}_1(u_1)$ in the evolutionary timescale, as illustrated in the solid black curves in Figure **5-14**-center and -right.

Once the branching point $\overline{u}_1$ has been reached, an innovative TS, characterized by an attribute u_2 that arises from a small variation in the number of passengers transported, will be able to invade the market and coexist with the previously established TS. Figure **5-13** (green) shows this scenario, corresponding to the resident-innovative model (5-29) with $u_1 = \overline{u}_1 = 344.1248$ and $u_2 = 1.2u_1 = 412.9498$ and with the initial conditions for x_1 and y_1 at the final values obtained in the simulation of the resident model (5-22) (partial adoption solid black) and $x_2(0) = 0.01$ and $y_2(0) = 1$. In this case, both transport systems reach partial adoption equilibria with $\overline{x}_1 = 0.1588$ and $\overline{y}_1 = 0.6025$ for the TS with

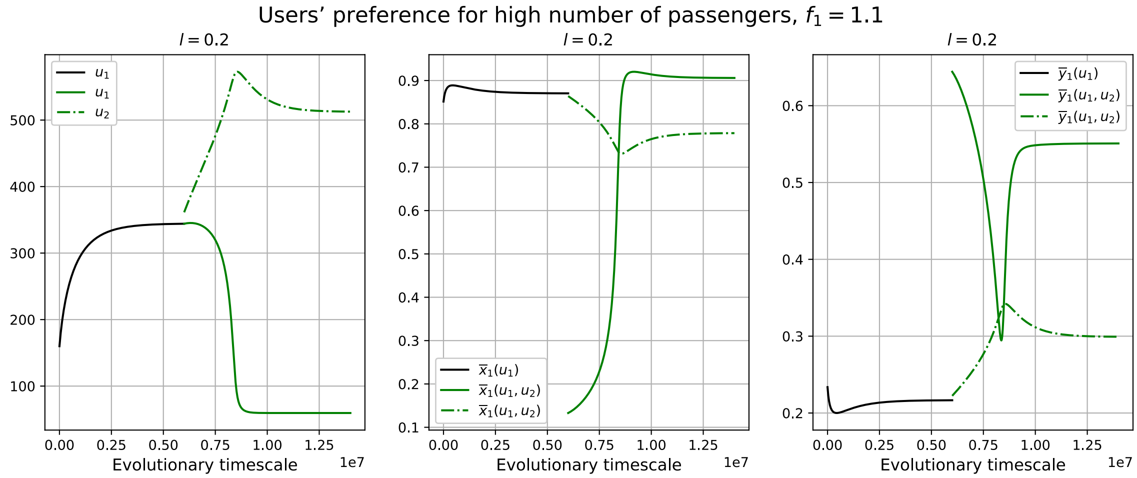

Figure 5-14.: Evolutionary dynamics of the number of transported passengers for users' preference for high capacity TS ($f_1 = 1.1$). In solid black the simulation of the AD canonical equation for u_1 is shown (left) and the corresponding market equilibrium values are shown at the center ($\overline{x}_1(u_1)$) and at the right ($\overline{y}_1(u1)$). Similarly, in green they are illustrated the solutions of the canonical equations after branching for u_1 and u_2 (right) and the corresponding equilibrium values in the market ($\overline{x}_{1,2}(u_1, u_2)$)-center and ($\overline{y}_{1,2}(u_1, u_2)$)-right. The other parameters as indicated in each panel, and Table **5-1**

attribute u_1, and $\overline{x}_2 = 0.8406$ and $\overline{y}_2 = 0.2435$ for the innovative TS with attribute u_2. The described situation effectively illustrates the origin of diversity through branching. Indeed, after diversification, each attribute evolves independently, attending to different selection forces exerted by users from the market, in fact, repeating the analysis described here, with the help of the fitness function (5-14), it is possible to deduce canonical equations for u_1 and u_2 that describe their long-term dynamics (see Figure **5-14**-left in green). As can be seen, both attributes evolve towards different equilibrium values, implying that in the long term the market will maintain the two transportation options, one with the capacity to transport a low number of passengers $\overline{u}_1 = 59.5111$ and the other with capacity to transport a greater number of passengers $\overline{u}_2 = 512.6789$. This can also be observed in Figure **5-14**-center and -right, where the equilibrium values in the market are tracked $\overline{x}_1(u_1, u_2)$ and $\overline{y}_1(u_1, y_2)$ (solid green) and $\overline{x}_2(u_1, u_2)$ y $\overline{y}_2(u_1, y_2)$ (dash-dot green), which allow to see how the attributes evolution affect market equilibria in the long term.

5.4. Results and Conclusions

A generalized model, called resident model, has been formulated for the competition between transport systems in a city, considering that the interaction occurs under the same market platform, that is, the intervention of irregular transport systems is not considered. Competition in the market is determined by the proportion of users adopting each transport system and, additionally, a state variable is included to measure the amount of budget that the investor makes available in order to promote the expansion of the transportation system among users. The model proposed here is an initial approach to the phenomenon, it allows for the study of the dynamics of a city's transport systems in various scenarios, and to learn under which conditions one transport system may spread and become stablished in the market as result of competition.

The example illustrated for one transport system, permits the explicit study of the basic dynamics and learn under which conditions it may be consistency in the market. For one transport system, it is possible to obtain explicit conditions under which an equilibrium of partial adoption or one of total adoption by users of the transport system is reached; in addition, according to this model, both cases are perfectly achievable without consuming all the available resources. In fact, the model makes possible to establish explicit conditions for the level of investment required in each case, information that may be useful in decision making.

Under the assumption that the resident model is at equilibrium (a consideration based on the fact that governmental entities in charge of formulating the public policies to govern the market, establish a priori the basic conditions for competition), a generalized model is formulated, called resident-innovative model, to describe competition in the market when that equilibrium is disturbed by the entry of an innovative transport system. The fitness function was determined and the local stability of equilibria of the AD canonical equation was numerically studied; when they are LAS, means that successive innovations which replace those previous, direct the number of passengers toward the value of equilibria, indicating the number of passenger the transport system will transport in the long term.

In the case of partial adoption, a much richer dynamic is obtained, it has been proven that the AD canonical equation can have multiple equilibria, in some of them branching is possible. The conditions under which these ramifications occur also depend on the users' preference for transport systems with the capacity to transport a high/low number of passengers. In the case of preference for transport systems with low capacity, it was shown that before the innovation, the transport system, initially considered to transport 250 passengers (the approximate capacity of a Transmilenio's bi-articulated in Bogotá), stabilizes at a capacity approximately of 180 passengers per mobile unit (a capacity very similar to that currently available for Transmilenio's articulated buses). After the innovation, a diversification scenario was shown in which a transportation system evolves towards approximately 1000 passengers per mobile unit (consider that the simulation time illustrated in Figure **5-12** was not sufficient to determine a numerical value of that equilibrium) that could be associated with a metro system like the one implemented in the city of Medellín, which has a capacity of approximately 1200 passengers for each three-wagons vehicle. Meanwhile, the other transportation system is evolving towards a capacity of approximately 100 passengers per mobile unit (could be associated with the bus system). Analyzing the levels of equilibrium in the market, but in the evolutionary time scale, it was observed that the transport system with the highest capacity is the one that reaches the highest levels of use by users while being the one with the most available budget to invest in new users.

On the other hand, in the scenario in which users prefer a transport system with greater capacity, it was observed that the number of transported passengers must increase from 160 (considered as the initial condition of AD canonical equation and corresponding to the current capacity from a Transmilenio's articulate) to 344 passengers per mobile unit, which far exceeds the capacity of the current bi-articulated vehicles, and suggests the need for a light metro or tram transport system. After the innovation, the transport market diversifies with two well-established transport systems, one of which reduces its capacity of transported passengers by mobile unit to 60 passengers (traditional bus system), while the other increases its capacity from the initial 344 passengers to 513 passengers per mobile unit approximately.

The main insight of this model is that in the presence of a single transportation system in the city, the users preference significantly influences the course of the evolution of the transportation system, with respect to the number of passengers; particularly, increasing the capacity if users prefer high-capacity transport systems, or reducing the number of passengers per mobile unit otherwise. Depending on the general preference of the users (interesting method to measure this preference should be implemented), that single transport system would have to transport from a minimum of 180 passengers to a maximum of 345 per movie unit. In a diversified transport system, the preference of the users is counterintuitive (in deed, the transport system with the highest capacity is obtained in the scenario of preference for low transported passengers and vice versa). This implies that, although preference influences, it is not the determining factor, since other factors (some of them considered in the selection gradients, the driving force of evolution) also influence long-term dynamics. In any case, it can be concluded that, regardless of the preference of the users, four possibilities arise (at least from the exercise here) regarding the capacity of the transport systems, which are, a transport system with low capacity ($\approx$ 60 passengers per mobile unit), two transport systems with intermediate capacity ($\approx$ 100 passengers one and $\approx$ 513 passengers the other); finally, the fourth diversification option involves a mass transportation system with the capacity to transport more than 1,000 passengers per mobile unit.

6. Conclusions and recommendations

In this thesis three mathematical models have been formulated from the perspective of the adaptive dynamics allowing to describe evolutionary branching, that is the coexistence between resident and similar innovative technologies and their further divergence in the market space.

The model in chapter three describe the dynamic interaction in the market of two types of energy, called standard and innovative. By analyzing the model, conditions on the possibility of invasion on an innovative generation technology can be established in a market dominated by a conventional generation technology. The adaptive dynamics canonical equation was studied to know the long-term behavior of the characteristic attributes and its impact on the market. Then we establish conditions under which evolutionary branching occur, that is to say, the requirements of coexistence and divergence at the singular strategies, whose occurrence leads to the origin of diversity in the energy market. Repeated process of innovation can give origin to a rich variety of different kinds of energy generation technologies. However, it is important to note that, this processes of emergence and disappearance of energy generation technologies is influenced by a wide range of external and internal factors, which may exert additional selection processes on innovations. Specific situations should be studied in greater depth and detail in order to achieve an informed decision making.

The conditions established in this study to classify evolutionary equilibria as branching points, terminal points or degenerate branching points, can be used as control strategies that allow to reach precise objectives in relation to the long-term behavior of the energy market, particularly on the stated decision to prevent/promote market diversification. It is inferred from the analysis of the model, under the assumptions considered here, that for the energy market to function in a "healthy" manner, it is necessary to exercise strict control over the taxes or subsidies that are decided to apply to the sources of energy generation, depending on the objectives that the regulatory agent wants to achieve. If the objective is to promote market diversification, it must establish interaction rules that locate the system in the branching region (favoring innovative generation technologies over the previously stablished), but if, on the contrary, the regulatory agent wants to avoid that new generation technologies have the possibility of entering the market, it is in the region of terminal points, or even the degenerate evolutionary branching region, where the system must be located. In this way, the tax/subsidy relationship together with the conditions of evolutionary branching, become control strategies that not only have a direct impact on the market in a short time scale, but also determine its structure in the long term. This way of analyzing control strategies (associated with diversification) is particularly interesting and differs significantly from the classical strategies of Mathematics and Engineering.

In chapter four, a deterministic model describing coffee production and harvesting of a coffee crop was formulated considering mature and immature coffee berry borer populations to reflect the damage caused by their reproduction and feeding habits, as well as their impact on coffee quality. Harvested coffee was divided into two categories in function of quality, low quality being when coffee is produced with a large proportion of bore-damaged grains and, conversely, high quality coffee being produced with a high proportion of healthy grains. Quality is considered to be a quantitative differentiating attribute between competing coffee types. Under coffee berry borer (CBB) persistence, there is at least one stable equilibrium that corresponded to the presence of every density considered in the model: the invasion equilibrium, whose local stability helps to show that the invasion equilibrium instability is related to the possibility of an initial density of special coffee to spread into the market. The study of the long-term dynamics of quality traits from the perspective of adaptive dynamics, allowed for the establishment of conditions under which evolutionary competition between standard and special coffees results in invasion, coexistence, and divergence, making clear that the net reproduction rate of coffee berry borer is a threshold of determining importance as does the consumers' preferences regarding coffee quality, and the bifurcation threshold. They play important roles in diversification through innovation, and permitted the formulation of policies for the control of mature CBB and the effective consumption rate of CBB, which, in turn, guarantee the possibility of market diversification. In deed, the decision to apply, or not, strict CBB population control strategies, directly impacts the possibility of diversification in the market, beyond the obvious scenarios in which eliminating CBB leads to top quality coffees, while strictly controlling the pest (looking for elimination) leads to coffees of the lowest quality. In general, the possibility of diversification in the market is closely linked to the existing relationship between the preference of users for high/low quality and the control strategies that are implemented in each case, before and after diversification.

Again, the close relationship that analysis through adaptive dynamics theory and control strategies leads us to believe that further studies should be performed, in order to consider alternate forms of quality differentiation, such as the introduction of innovative, agro-industrial processes that affect coffee washing, drying, roasting, or other crucial processes in coffee production, transformation, or commercialization. Finally, it is important that in Colombia a system that allows collecting, analyzing and disseminate accurate information on the production, processing and sale of specialty coffees be defined, even more considering that these products have become the main source of income for small producers that can access better economic benefits by producing high quality coffees valued by consumers for their consistent, verifiable, and sustainable attributes, for which they are willing to pay higher prices, which results in higher producer income and welfare.

In the fifth chapter a generalized model has been formulated for the competition between transport systems in a city, considering that the interaction occurs under the same market platform and competition is determined by the proportion of users adopting each transport system and, additionally, a measure of the amount of budget that the investor makes available in order to promote the expansion of the transportation system among users is considered. Later, under the assumptions of stability, a generalized model is formulated, to describe competition in the market when that stability is disturbed by the entry of an innovative transport system. From the perspective of the adaptive dynamics, it is possible to determine general conditions that must be met to guarantee or not the success of the innovation as the one managing to penetrate and expand into the market. Additionally, the approach through adaptive dynamics is used to establish the long term dynamics of the quantitative attribute and permits the classification of the evolutionary equilibra, particularly as evolutionary branching points; i.e., singular strategies in which diversification arises.

It was shown that in a single transport system scenario, the users' preference for high/low capacity systems directly influences the evolutionary dynamics of the number of transported passengers, bringing the system to a low value of approximately 180 passengers when low-capacity systems are favored, or at a value of about 344 passengers per mobile unit, when high-capacity transport systems are preferred. When the market diversifies, the user's preference becomes less influential. Indeed, under the diversification scenarios, four transport options were found that can be established in the market, two of them involving intermediate capacity systems (100 or 513 passengers per mobile unit) and two others that correspond to extreme values (one of low capacity with approximately 60 passengers and a massive system transporting over 1000 passengers per mobile unit). This result may indicate that the needs on the transport systems cannot be met using only buses; in deed, on the one hand, highlights the importance of diversifying the city with different systems of different capacities, on the other hand, really massive transport systems such as a metro or tram are required.

In the execution of this thesis, the necessary theoretical aspects surrounding the theory of adaptive dynamics were reviewed, which constitutes a referent that allows addressing the issue of technological innovations and how their evolution on the evolutionary time scale is influenced by the selection forces that users/consumers make in the market. Adaptive dynamics theory describes the evolution of attributes using an ordinary differential equation, which relates the dynamics on a market scale with evolutionary dynamics in the long term. Although this approach has been widely used in the study of phenomena in biology and ecology, are the applications to market contexts and technological innovation that makes it an attractive and certainly useful tool in different contexts of engineering, economics and the administration. In general, a judgmental analysis of the proposed models and robust simulation tools should be created, in order to make meaningful contributions and adequately support decision-making in those contexts. Particularly interesting is how the analysis of phenomena mediated by this tool, allows the discussion of control strategies to be carried out in an alternative way to the classic tools of dynamic systems and control theory, in deed, this tool allows us to predict the effect that control strategies implemented on a short time scale (let's say the market), can have on the long term on an evolutionary time scale. This is particularly important when contrasted with the stated intention of promoting or preventing diversification.

A. Glossary

Allele. Correspond to different subsequences of genetic letters A, C, G and T (see DNA) that are possible for a gene.

Attractor. Selection is an autonomous process (work in the absence of mutations and external influences) and, as such, drives the system toward a regime, which can be stationary as well as non stationary (periodic or wilder, so-called chaotic regime). Such regimes are called attractors of the dynamical process, since they attract nearby states.

Chromosomes. They are structures, the genetic material of all the species is organized in so called chromosomes, composed of a DNA molecule and a protein coat.

Demographic dynamics. The population dynamics driven by selection (ecological or short-term).

Demographic timescale. Characteristic timescale on which the process of selection drives the system toward one of its attractors.

Demography. Dynamical process that regulates population abundances. Is a non autonomous process; i.e. not solely determined by the current state.

Deterministic demographic models. Are justified only when the actual number of individuals in each population is sufficiently large to avoid accidental extinction risks (demographic stochasticity). We assume that even small abundances correspond to relatively large numbers of individuals.

State of population: determined by the genotypic distribution, which gives the abundance of all genotypes present in the population in a given time.

Dynamics of the population: are the changes in time of the genotypic distribution that result from birth, death, and migration of individuals.

Dimorphic. See polymorphism.

Disruptive selection. Also called diversifying selection, describes changes in population genetics in which extreme values for a trait are favored over intermediate values. In this case, the variance of the trait increases and the population is divided into two distinct groups.

DNA. A DNA molecule consist of two helices of alternating sugar and phosphate molecules, where each sugar binds to one of four possible molecules called DNA bases: adenine (A)-thymine (T) and cytosine (C)-guanine (G).

Evolutionary branching. Under the effect of disruptive selection a monomorphic population may turn dimorphic with respect to some relevant phenotypes, undergoing what is called an evolutionary branching.

Evolutionary dynamics. In the idealized case of extremely rare mutations, we can define evolutionary dynamics as the sequence of attractors visited by the demographic dynamics.

Evolutionary timescale. The timescale on which an evolutionary dynamics develops.

Fitness. Word used in the biological context to stand for overall ability to survive and reproduce. Quantitative the fitness function of an individual is defined as the abundance of its progeny in the next generation or, equivalently, as the per-capita growth rate of the group of individuals characterized by the same phenotypic values, i.e. *the abundance variation per unit of time relative to the total abundance of the group.*

This definitions say that the abundance of individuals characterized by a given set of phenotypic values is increasing, at a given time, if the associated fitness is, respectively, larger than one or positive.

Gene. The portion of the DNA molecule corresponding to a locus is a gene, and may take one of several forms, called alleles.

Genetic drift. At each generation, the genes that control selectively neutral phenotypes, or that are altered by mutations with no phenotypic effect, are a sample of the genes present in the parental population, whose offspring are not filtered by any selection pressure. The accumulation of differences in such genes (and related phenotypes) produces an evolutionary change that biologist call genetic drift.

Genome. The genome of a species is the set of all possible chromosomes characterizing an individual of the species and is therefore defined by all allelic forms of all genes of the species.

Genotype. The genotype of an individual is a particular genome relization, given by the chromosomes carried by the individual.

Loci. Plural of locus.

Locus. Is a particular subsequence of genetic letters (within the sequence of genetic letters representing a chromosome), whose number, lengths, and positions are specific of the type of chromosome.

Monomorphic. See polymorphism.

Mutation. Phenotypic changes that reflect heritable changes in the organism genetic material.

Phenotype. Any individual characteristic determined, to some extent, by the genotype is called a phenotype or phenotype trait, and is therefore a heritable characteristic from parents to the progeny.

Polymorphism. Is the phenotypic variability within populations. May take the form of discrete differences, or that of a continuous range of measurable values.

Population genetics. Focusses on the change of genotypic relative abundances, often called frequencies, but ignores, at least in its classical formulations, the change in genotypic absolute numbers.

Red Queen dynamics. Independently of sex, we refer the Red Queen dynamics as evolutionary dynamics that in the absence of external forcing (constant abiotic environment) lead to non stationary evolutionary regimes.

Species. Can be defined as a group of morphologically and genetically similar individuals that, when reproduction is sexual, are capable of interbreeding and reproductively isolated from other such groups.

Trait. Phenotypes with an effect on fitness are able to adapt to the environmental conditions experienced by individuals and are therefore say to be adaptive; they are often called adaptive traits and, most of the time simply traits.

Bibliografía

[1] Peter A Abrams. Prey evolution as a cause of predator-prey cycles. *Evolution*, 51:1740–1748, 1997.

[2] Peter A Abrams. Can adaptive evolution or behavior lead to diversification of traits determining a trade-off between foraging gain and predation risk? *Evolutionary Ecology Research*, 5(5):653–670, 2003.

[3] Alejandra Cruz Alarcón, Juan Francisco Barrera Gaytán, Jorge Jiménez Zili, Jorge Ernesto Valenzuela, Pablo Esau Cruz Domínguez, Carlos Roberto Cerdán Cabrera, and Gerardo Alvarado Castillo. Evaluación de tres tipos de trampas, efecto de altura y evaporación del atrayente para la broca del café hypothenemus hampei en la finca vegas, veracruz, méxico. *Fitosanidad*, 21(2):53–60, 2017.

[4] Benjamin Allen, Martin A Nowak, and Ulf Dieckmann. Adaptive dynamics with interaction structure. *The American Naturalist*, 181(6):E139–E163, 2013.

[5] J Arcila, FF Farfan, AM Moreno, LF Salazar, and E Hincapié. Sistemas de producción de café en colombia, 2007.

[6] Sergei P Atamas. Self-organization in computer simulated selective systems. *Biosystems*, 39(2):143–151, 1996.

[7] Peter Baccini and Paul H Brunner. *Metabolism of the anthroposphere: analysis, evaluation, design*. MIT Press, 2012.

[8] Roger K Butlin, Juan Galindo, and John W Grahame. Sympatric, parapatric or allopatric: the most important way to classify speciation? *Philosophical Transactions of the Royal Society B: Biological Sciences*, 363(1506):2997–3007, 2008.

[9] Raphael Calel and Antoine Dechezlepretre. Environmental policy and directed technological change: evidence from the european carbon market. *Review of economics and statistics*, 98(1):173–191, 2016.

[10] Nicolas Champagnat, R Ferrične, and G Ben Arous4. The canonical equation of adaptive dynamics: a mathematical view. *Selection*, 2(1-2):73–83, 2002.

[11] Nicolas Champagnat, Régis Ferrière, and Sylvie Méléard. Unifying evolutionary dynamics: from individual stochastic processes to macroscopic models. *Theoretical population biology*, 69(3):297–321, 2006.

[12] Pierre Collet, Sylvie Méléard, and Johan AJ Metz. A rigorous model study of the adaptive dynamics of mendelian diploids. *Journal of Mathematical Biology*, 67(3):569–607, 2013.

[13] Alessandro Colombo, Fabio Dercole, and Sergio Rinaldi. Remarks on metacommunity synchronization with application to prey-predator systems. *The American Naturalist*, 171(4):430–442, 2008.

[14] Luis Miguel Constantino, CE OLIVEROS, P BENAVIDES, CA SERNA, CA RAMIREZ, RD MEDINA, and A ARCILA. Dispositivo recolector de frutos de café del suelo para el manejo integrado de la broca. 2017.

[15] A Damon. A review of the biology and control of the coffee berry borer, hypothenemus hampei (coleoptera: Scolytidae). *Bulletin of entomological research*, 90(6):453–465, 2000.

[16] Fabio Della Rossa, Fabio Dercole, and Pietro Landi. The branching bifurcation of adaptive dynamics. *International Journal of Bifurcation and Chaos*, 25(07):1540001, 2015.

[17] F Dercole, R Ferriere, A Gragnani, and S Rinaldi. Coevolution of slow–fast populations: evolutionary sliding, evolutionary pseudo-equilibria and complex red queen dynamics. *Proceedings of the Royal Society of London B: Biological Sciences*, 273(1589):983–990, 2006.

[18] Fabio Dercole. Remarks on branching-extinction evolutionary cycles. *Journal of mathematical biology*, 47(6):569–580, 2003.

[19] Fabio Dercole. Border collision bifurcations in the evolution of mutualistic interactions. *International Journal of Bifurcation and Chaos*, 15(07):2179–2190, 2005.

[20] Fabio Dercole. Bpcont: An auto driver for the continuation of branch points of algebraic and boundary-value problems. *SIAM Journal on Scientific Computing*, 30(5):2405–2426, 2008.

[21] Fabio Dercole. The ecology of asexual pairwise interactions: The generalized law of mass action. *Theoretical ecology*, 9(3):299–321, 2016.

[22] Fabio Dercole, Fabio Della Rossa, and Pietro Landi. The transition from evolutionary stability to branching: A catastrophic evolutionary shift. *Scientific reports*, 6:26310, 2016.

[23] Fabio Dercole, Ulf Dieckmann, Michael Obersteiner, and Sergio Rinaldi. Adaptive dynamics and technological change. *Technovation*, 28(6):335–348, 2008.

[24] Fabio Dercole, Régis Ferrière, and Sergio Rinaldi. Ecological bistability and evolutionary reversals under asymmetrical competition. *Evolution*, 56(6):1081–1090, 2002.

[25] Fabio Dercole, Regis Ferriere, and Sergio Rinaldi. Chaotic red queen coevolution in three-species food chains. *Proceedings of the Royal Society of London B: Biological Sciences*, page rspb20100209, 2010.

[26] Fabio Dercole and Stefan AH Geritz. Unfolding the resident–invader dynamics of similar strategies. *Journal of theoretical biology*, 394:231–254, 2016.

[27] Fabio Dercole, Jean-Olivier Irisson, and Sergio Rinaldi. Bifurcation analysis of a prey-predator coevolution model. *SIAM Journal on Applied Mathematics*, 63(4):1378–1391, 2003.

[28] Fabio Dercole, Daniele Loiacono, and Sergio Rinaldi. Synchronization in ecological networks: a byproduct of darwinian evolution? *International Journal of Bifurcation and Chaos*, 17(07):2435–2446, 2007.

[29] Fabio Dercole, Charlotte Prieu, and Sergio Rinaldi. Technological change and fisheries sustainability: The point of view of adaptive dynamics. *Ecological Modelling*, 221(3):379–387, 2010.

[30] Fabio Dercole and Sergio Rinaldi. Evolution of cannibalistic traits: scenarios derived from adaptive dynamics. *Theoretical population biology*, 62(4):365–374, 2002.

[31] Fabio Dercole and Sergio Rinaldi. *Analysis of evolutionary processes: the adaptive dynamics approach and its applications: the adaptive dynamics approach and its applications*. Princeton University Press, 2008.

[32] Fabio Dercole and Sergio Rinaldi. Evolutionary dynamics can be chaotic: A first example. *International journal of bifurcation and chaos*, 20(11):3473–3485, 2010.

[33] Fabio Dercole and Sergio Rinaldi. Dynamical systems and their bifurcations. *Advanced Methods of Biomedical Signal Processing*, pages 291–325, 2011.

[34] Ulf Dieckmann and Richard Law. The dynamical theory of coevolution: a derivation from stochastic ecological processes. *Journal of mathematical biology*, 34(5-6):579–612, 1996.

[35] Ulf Dieckmann, Paul Marrow, and Richard Law. Evolutionary cycling in predator-prey interactions: population dynamics and the red queen. *Journal of theoretical biology*, 176(1):91–102, 1995.

[36] M Doebeli and U Dieckmann. Evolutionary branching and sympatric speciation caused by different types of ecological interactions [revised 19 april 2018]. 2000.

[37] Hong-Bo Duan, Lei Zhu, and Ying Fan. A cross-country study on the relationship between diffusion of wind and photovoltaic solar technology. *Technological Forecasting and Social Change*, 83:156–169, 2014.

[38] Fedecafé. Exportaciones de café colombiano. *Federación Nacional de Cafeteros, https://federaciondecafeteros.org/estadisticas-cafeteras/*, 2019.

[39] Silvestre Fernández and Julio Cordero. Biología de la broca del café hypothenemus hampei (ferrari)(coleoptera: Curculionidae: Scolytinae) en condiciones de laboratorio. *Bioagro*, 19(1):35–40, 2007.

[40] Régis Ferriere, Judith L Bronstein, Sergio Rinaldi, Richard Law, and Mathias Gauduchon. Cheating and the evolutionary stability of mutualisms. *Proceedings of the Royal Society of London B: Biological Sciences*, 269(1493):773–780, 2002.

[41] Regis Ferriere and Stéphane Legendre. Eco-evolutionary feedbacks, adaptive dynamics and evolutionary rescue theory. *Phil. Trans. R. Soc. B*, 368(1610):20120081, 2013.

[42] Peter A Follett, Andrea Kawabata, Robert Nelson, Glenn Asmus, Jen Burt, Kally Goschke, Curtis Ewing, Julie Gaertner, Eva Brill, and Scott Geib. Predation by flat bark beetles (coleoptera: Silvanidae and laemophloeidae) on coffee berry borer (coleoptera: Curculionidae) in hawaii coffee. *Biological Control*, 101:152–158, 2016.

[43] Faïz Gallouj and Olivier Weinstein. Innovation in services. *Research policy*, 26(4-5):537–556, 1997.

[44] S Genieys, Nikolai Bessonov, and Vitaly Volpert. Mathematical model of evolutionary branching. *Mathematical and computer modelling*, 49(11-12):2109–2115, 2009.

[45] Stefan AH Geritz, Géza Meszéna, Johan AJ Metz, et al. Evolutionarily singular strategies and the adaptive growth and branching of the evolutionary tree. *Evolutionary ecology*, 12(1):35–57, 1998.

[46] Stefan AH Geritz, Johan AJ Metz, Éva Kisdi, and Géza Meszéna. Dynamics of adaptation and evolutionary branching. *Physical Review Letters*, 78(10):2024, 1997.

[47] Arnulf Grübler. *Technology and global change*. Cambridge university press, 2003.

[48] Jack K Hale and Hüseyin Koçak. *Dynamics and bifurcations*, volume 3. Springer Science & Business Media, 2012.

[49] Josef Hofbauer and Karl Sigmund. Adaptive dynamics and evolutionary stability. *Applied Mathematics Letters*, 3(4):75–79, 1990.

[50] Jens Horbach, Christian Rammer, and Klaus Rennings. Determinants of eco-innovations by type of environmental impact—the role of regulatory push/pull, technology push and market pull. *Ecological economics*, 78:112–122, 2012.

[51] Francisco Infante. Pest management strategies against the coffee berry borer (coleoptera: Curculionidae: Scolytinae). *Journal of agricultural and food chemistry*, 66(21):5275–5280, 2018.

[52] Juliana Jaramillo, Adenirin Chabi-Olaye, Christian Borgemeister, Charles Kamonjo, Hans-Michael Poehling, and Fernando E Vega. Where to sample? ecological implications of sampling strata in determining abundance and impact of natural enemies of the coffee berry borer, hypothenemus hampei. *Biological Control*, 49(3):245–253, 2009.

[53] Jacob Johansson, Jörgen Ripa, and Nina Kuckländer. The risk of competitive exclusion during evolutionary branching: Effects of resource variability, correlation and autocorrelation. *Theoretical population biology*, 77(2):95–104, 2010.

[54] Chris Kennedy, Iain D Stewart, Nadine Ibrahim, Angelo Facchini, and Renata Mele. Developing a multi-layered indicator set for urban metabolism studies in megacities. *Ecological Indicators*, 47:7–15, 2014.

[55] Éva Kisdi and Stefan AH Geritz. Adaptive dynamics in allele space: evolution of genetic polymorphism by small mutations in a heterogeneous environment. *Evolution*, 53(4):993–1008, 1999.

[56] Yuri A Kuznetsov. *Elements of applied bifurcation theory*, volume 112. Springer Science & Business Media, 2013.

[57] Cecilia SL Lai, Simon E Fisher, Jane A Hurst, Faraneh Vargha-Khadem, and Anthony P Monaco. A forkhead-domain gene is mutated in a severe speech and language disorder. *Nature*, 413(6855):519, 2001.

[58] Pietro Landi and Fabio Dercole. The social diversification of fashion. *The Journal of Mathematical Sociology*, 40(3):185–205, 2016.

[59] Pietro Landi, Fabio Dercole, and Sergio Rinaldi. Branching scenarios in eco-evolutionary prey-predator models. *SIAM Journal on Applied Mathematics*, 73(4):1634–1658, 2013.

[60] Rafael Mendes Lübeck, Milton Luiz Wittmann, Luciana Flores Battistella, Angélica Skrebsky Richter, and Marcia Santos da Silva. Inovação em serviços de transporte público. *Revista Organizações em Contexto*, 8(15):65–86, 2012.

[61] Paul Marrow, Richard Law, and C Cannings. The coevolution of predator—prey interactions: Esss and red queen dynamics. *Proc. R. Soc. Lond. B*, 250(1328):133–141, 1992.

[62] HGE Meijer, F Dercole, and B Oldeman. Encyclopedia of complexity and systems science, chapter numerical bifurcation analysis, 2009.

[63] Geza Meszena, Éva Kisdi, Ulf Dieckmann, Stefan AH Geritz, and Johan AJ Metz. Evolutionary optimisation models and matrix games in the unified perspective of adaptive dynamics. *Selection*, 2(1-2):193–220, 2002.

[64] Johan AJ Metz, Stefan AH Geritz, Géza Meszéna, Frans JA Jacobs, and Joost S Van Heerwaarden. Adaptive dynamics: a geometrical study of the consequences of nearly faithful reproduction. 1995.

[65] Johan AJ Metz, Roger M Nisbet, and Stefan AH Geritz. How should we define 'fitness' for general ecological scenarios? *Trends in Ecology & Evolution*, 7(6):198–202, 1992.

[66] Ivan V Monagan Jr, Jonathan R Morris, Alison R Davis Rabosky, Ivette Perfecto, and John Vandermeer. Anolis lizards as biocontrol agents in mainland and island agroecosystems. *Ecology and evolution*, 7(7):2193–2203, 2017.

[67] Esther Cecilia Montoya-Restrepo. Caracterización de la infestación del café por la broca y efecto del daño en la calidad de la bebida. *Cenicafé*, 50(4):245–258, 1999.

[68] Jonathan R Morris and Ivette Perfecto. Testing the potential for ant predation of immature coffee berry borer (hypothenemus hampei) life stages. *Agriculture, Ecosystems & Environment*, 233:224–228, 2016.

[69] Anil Nair and David Ahlstrom. Delayed creative destruction and the coexistence of technologies. *Journal of Engineering and Technology Management*, 20(4):345–365, 2003.

[70] M Núñez-López. The dynamics of technological change under constraints: adopters and resources. 2014.

[71] Sumet Ongkittikul and Harry Geerlings. Opportunities for innovation in public transport: Effects of regulatory reforms on innovative capabilities. *Transport Policy*, 13(4):283–293, 2006.

[72] Alex Enrique Bustillo Pardey. Una revisión sobre la broca del café, hypothenemus hampei (coleoptera: Curculionidae: Scolytinae), en colombia. *Revista Colombiana de Entomología*, 32(2):101–117, 2006.

[73] Kalle Parvinen. Evolutionary branching of dispersal strategies in structured metapopulations. *Journal of mathematical biology*, 45(2):106–124, 2002.

[74] Neil Paulley, Richard Balcombe, Roger Mackett, Helena Titheridge, John Preston, Mark Wardman, Jeremy Shires, and Peter White. The demand for public transport: The effects of fares, quality of service, income and car ownership. *Transport Policy*, 13(4):295–306, 2006.

[75] Lawrence Perko. *Differential equations and dynamical systems*, volume 7. Springer Science & Business Media, 2013.

[76] Renato Ripa and Pilar Larral. *Manejo de plagas en paltos y cítricos*, volume Colección de libros INIA No 23. Instituto de Investigaciones Agropecuarias del Ministerio de Agricultura de Chile, 2008.

[77] Daniel Rodríguez, José Ricardo Cure, Andrew Paul Gutierrez, and José Miguel Cotes. A coffee agroecosystem model: Iii. parasitoids of the coffee berry borer (hypothenemus hampei). *Ecological Modelling*, 363:96–110, 2017.

[78] Juan Vicente Romero and Hernando Alfonso Cortina Guerrero. Tablas de vida de *Hypothenemus hampei* (coleoptera: Curculionidae: Scolytinae) sobre tres introducciones de café. *Revista colombiana de entomología*, 33(1):10–16, 2007.

[79] Ramiro Ruiz Cárdenas and Peter Baker. Life table of *Hypothenemus hampei* (ferrari) in relation to coffee berry phenology under colombian field conditions. *Scientia Agrícola*, 67(6):658–668, 2010.

[80] Pier P Saviotti et al. Technological evolution, variety and the economy. *Books*, 1996.

[81] Pier Paolo Saviotti. Variety, growth and demand. *New Dynamics of Innovation and Competition*, page 41, 2002.

[82] Joseph Alois Schumpeter et al. *Business cycles*, volume 1. McGraw-Hill New York, 1939.

[83] Andrew Stirling. On the economics and analysis of diversity. *Science Policy Research Unit (SPRU), Electronic Working Papers Series, Paper*, 28:1–156, 1998.

[84] Hernán Darío Toro-Zapata and Gerard Olivar-Tost. Mathematical model for the evolutionary dynamic of innovation in city public transport systems. *Copernican Journal of Finance & Accounting*, 7(2):77–98, 2018.

[85] Hernán Darío Toro-Zapata, Gerard Olivar-Tost, and Fabio Dercole. Conditions on the energy market diversification from adaptive dynamics. *Mathematical Problems in Engineering*, 2018, 2018.

[86] Juan David Velásquez Henao, Isaac Dyner Resonsew, and Reinaldo Castro Souza. ¿ por qué es tan difícil obtener buenos pronósticos de los precios de la electricidad en mercados competitivos? *Cuadernos de Administración*, 20(34), 2007.