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Resumen

Ecuación de dínamo cosmológica bajo teoría de perturbaciones cosmológicas a primer orden

En este trabajo se pretende dar una introducción a las perturbaciones cosmológicas y aplicaciones desde el punto de vista de la Relatividad numérica, en particular se muestra como se pueden aplicar estas perturbaciones al formalismo 3+1. Las perturbaciones cosmológicas se dan a primer orden sobre la solución espacialmente plana de Friedman-Lemaitre-Robertson-Walker (FLRW), esto con miras a obtener la ecuación de dínamo cosmológico, bajo la aproximación de dínamo cinemático, para poder estudiar la evolución de los campos magnéticos primordiales y su amplificación. También se mostrará el estudio computacional de perturbaciones cosmológicas a partir de la Relatividad Numérica haciendo uso del software `Einstein Toolkit`, se hace énfasis en `FLRWSolver` para la solución numérica de las ecuaciones de campo de Einstein desde el punto de vista cosmológico.

Palabras clave: Cosmología, Relatividad numérica, Campos magnéticos cosmológicos, Einstein Toolkit, FLRWSolver.

Abstract

Cosmic dynamo equation under cosmological perturbation theory at first order

This thesis aims to give an introduction to cosmological perturbations and their applications from the point of view of numerical relativity, in particular it shows how these perturbations can be applied to the 3+1 formalism. The cosmological perturbations are given up to first order on the spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) solution, this looking to obtaining the cosmological dynamo equation, under the kinematic-dynamo approximation, in order to study the evolution of primordial magnetic fields and their amplification. The computational study of cosmological perturbations from Numerical Relativity will also be shown using the `Einstein Toolkit` software, emphasizing `FLRWSolver` for the numerical solution of the Einstein field equations from the cosmological point of view.

Keywords: Cosmology, Numerical relativity, Cosmological magnetic fields, Einstein Toolkit, FLRWSolver.

Contents

Introduction

The study of magnetic fields has a broad spectrum, has been playing a role in astronomy and cosmology. Magnetic field can be found on our solar system, planets, stars, galaxies, galaxy clusters and in the voids of the Large-Scale Structure (LSS). In each case there is a broad study of the magnetic fields, one example is the case of how the Earth's magnetic field has sustained for billion years [53], the study of magnetic cycles of the Sun [29], the micro-Gauss magnetic fields host in spiral galaxies [20, 19], the stochastic magnetic field in galaxy clusters and the origin and evolution of the weak magnetic field in the Intergalactic medium (IGM) voids [116]. The relevance of the study of the magnetic field is increase if we also look into the Hubble tension [117, 63], briefly speaking, the CMB and the standard candles measurements of the Hubble factor do not match, but if we include primordial magnetic fields before recombination, then recombination occurs much faster because magnetic pressure induced by the magnetic field and there would be less time to Big Bag Nucleosynthesis (BBN) to occur affecting the amount of primordial elements.

The case that we will work on this document are the magnetic fields present in the early universe, and because is about the early universe that we are talking about, these fields must be studied from the cosmology point of view. A first question that arise is how to determine if the field is primordial or not? It is possible to find primordial magnetic fields at the voids of the LSS as a relic from the early universe, and the primordial nature is conserved in these places because the fields were present before structure formation and did not suffer to much amplification [99]. After knowing if a field is primordial or not another question arises: How magnetic fields were created in the early universe? The origin of magnetic fields can be given by several causes and determining which each of cause is, or are, true, is extremely difficult. Here we will mention a few mechanisms of generation of primordial magnetic fields, we are clear that in this thesis we will not cover in full detail this topic, we will just mention it. First, we must consider different stages in the evolution of the universe because in each could be different phenomena that could induce the origin of the primordial magnetic fields. Let us start with inflation, during this stage to generate magnetic fields one has either to couple the electromagnetic field to the inflaton [82] or introduce another coupling which breaks conformal invariance, for example couple the electromagnetic field to curvature or helical-inflaton coupling [37]. Another possibility for creation of magnetic fields in the early universe is given by charge separation or generation of vortical currents at the moment of cosmological phase transitions, these transitions can be given by electroweak and QCD phase transitions [115, 65, 109]. Another thing that could be consider for generation during recombination is that electron scattering is more efficient and feel a greater drag than do the protons prior to recombination, this produces a net electric current that sources a magnetic field if there is turbulence during cosmological recombination [51, 22, 42]. To determine if these origins are possibly right, it is necessary to check if these effects leave a signature in the CMB to be detected[59, 35, 58]

Next question about these fields is: How these fields evolve along the history the universe? In this case the field is sustained by the dynamo action, these equations can be obtained from the Maxwell equations and will tell how the field will evolve in time [33, 98, 64, 68]. In this document we will focus on the dynamo equation obtained using cosmological perturbation setting a spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) solution for the Einstein's fields equations (EFE) and perturbing a background magnetic field up to first order [55]. The problem is that these equations need a magnetic seed, otherwise there could not be magnetic field, that why first we needed to know first about the origin of the magnetic field, origins that we mention earlier. The evolution of the field will be obtained, in this case, from numerical solution of the dynamo equation, then we also have to evolve Einstein's field equations along with the full relativistic dynamo equation. The evolution of cosmological perturbed EFE has been done before, for example codes like `CAMB` [73, 1] or `CLASS` [72, 2], evolves background and first order perturbations in the Fourier space using Boltzmann equations, there are several codes that also take this approach and even in modified theories of gravity [8, 94, 60, 61, 123]. Another evolution point of view is the evolution of cosmological equations, in particular non-local inhomogeneity, using Numerical Relativity [78, 62, 28, 21, 44, 45, 85, 119], there are several codes which already implement this approach like `CosmoGraph` [3], `GRChombo` [31], `Einstein Toolkit` [74, 46, 122, 15] and several others, here reference of some of these codes [67, 7, 10, 50]. Here we will be using `Einstein Toolkit` together with `FLRWSolver` [77] to set initial conditions for cosmological evolution in `Einstein Toolkit`.

As mentioned above, in this document we will not worry about the origin of the magnetic fields in the intergalactic medium (IGM) but keep in mind that it should have one, the real target is to study the evolution of magnetic fields given a seed field, therefore our main goal is to study the dynamo equation. To be able to achieve this, the present document presents five chapters: chapter 1 gives a brief introduction to numerical relativity presenting the 3+1 formalism and obtaining the Maxwell equations and the perfect fluid equations in the context of 3+1 formalism; chapter 2 gives an introduction to the 1+3 formalism used in cosmology, its most representative quantities and the Maxwell equations in this formalism; chapter 3 introduces the cosmological perturbations, here the background solution is set, spatially flat FLRW universe, together with the cosmological perturbations, then the perturbation are introduced for 3+1 and 1+3 formalism and the perturbations relations between both formalisms, some of the result here match with some previous works done on the matter [38, 119]; in chapter 4 we present the dynamo equation for the presented formalisms and also compare the equations with result already obtain other publications, we also see

how the magnetic field decay when there are no perturbations, here is important to remark that the dynamo equation must be set using Lagrangian observers (1+3 formalism), then an equivalence between electromagnetic fields for both formalisms must be achieve to obtain a dynamo equation in the 3+1 formalism [23, 26, 32]; in chapter 5 computational implementation is shown, `Einstein Toolkit` is used to evolve equations in the computational domain setting initial conditions with `FLRWSolver`, the simulations implemented in this work start in a redshift $z \approx 1100$ and ends at $z \approx 534$, this to keep the linearity in the perturbations [77], in this chapter also evolve the dynamo equation obtained in chapter 4, we do it here because it uses the velocity field obtained using `Einstein Toolkit`.
Along the text we will use natural units used in [18, 5] unless otherwise stated, then $G = c = 1$, for the Maxwell equation $\varepsilon_0 = 1$, to be able to obtain the measurements of the magnetic fields in Gauss units (G), a factor of $2{,}35537 \times 10^{15}$ G must be multiplied to the magnetic fields. The tensor indices are given by Greek letters (α, β, γ, ...) and will take the values from zero (0) to three (3), sometimes Latin indices will be used (i, j, k, ...) and will take the values from one (1) to three (3).

1. Brief introduction to 3+1 Numerical Relativity

This is the chapter where we introduce the 3+1 formalism of Numerical Relativity (NR), we will foliate the spacetime and rewrite the Einstein equations in the main quantities of this formalism. We also study the perfect fluid and the Maxwell's equations in 3+1 formalism. Here we follow mainly [103, 18, 47]

1.1. Einstein equations in 3+1 formalism

Here we will work under the General theory of Relativity (GR) given by E. Einstein, we will take the Einsteins field equations as in (A-58) to be able to split them in the 3+1 formalism.

1.1.1. Foliation of spacetime

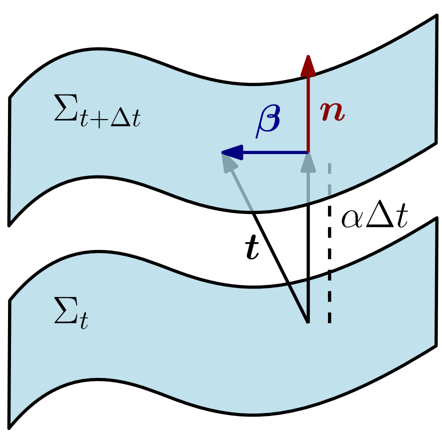

Figure 1-1.: This is an scheme to be able to understand how the lapse function α and the shift vector $\boldsymbol{\beta}$ are defined. The lapse function determines the physically proper time between two points on two neighboring spatial hypersurfaces Σ_t and $\Sigma_{t+\triangle t}$. The shift vector is the relative velocity between eulerian observers and the lines of constant spatial coordinates.

To write the Einstein field equations in the 3+1 formalism, we need first to take our spacetime $\mathcal{M}_4$ and make a foliation from a family of hypersurfaces, see appendix B, to be able to study the Einstein's equations as a Cauchy problem, we will denote this hypersurfaces as Σ_t, where the parameter of the foliation t corresponds to the coordinate time. From this we have a normal vector to the hypersurfaces $\boldsymbol{n}$, which is futurelike, and a projector to the hypersurfaces $\boldsymbol{\gamma}$ inducing a spatial metric on each hypersurface[1]. For a timelike 4-vector $\boldsymbol{t}$ on the spacetime tangent to the time axis, $t^\alpha = (\partial/\partial t)^\alpha$ and $t^\alpha \nabla_\alpha t = 1$, then we project $\boldsymbol{t}$ along $\boldsymbol{n}$ and $\boldsymbol{\gamma}$ in the following way

$$\alpha = -t^\alpha n_\alpha, \qquad \beta^\alpha = t^\beta \gamma^\alpha_\beta. \tag{1-1}$$

The functions α and β^α are called the lapse function and the shift vector, respectively, and the observers which his 4-velocity is $\boldsymbol{n}$ are called eulerian observers. Let us set up an induced coordinate system, taking a basis of spatial 3-vectors $\left\{\boldsymbol{E}^\alpha_{(i)}\right\}$ that reside on a particular time slice Σ_t[2], then $\boldsymbol{E}^\alpha_{(i)} \nabla_\alpha t = 0$. We Lie dragg the spatial vectors along $\boldsymbol{t}$,

$$\mathcal{L}_{\boldsymbol{t}} \boldsymbol{E}^\alpha_{(i)} = 0, \tag{1-2}$$

as a consequence, these basis vectors connect points with the same spatial coordinates on neighboring slices and as a temporal basis 4-vector we take $\boldsymbol{E}^\alpha_{(i)} = \boldsymbol{t}$ [18]. Let us consider two adjacent hypersurfaces Σ_t and $\Sigma_{t+\Delta t}$, given the induced coordinates we can write the metric tensor components as follow

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\alpha\beta} \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^j\beta^i/\alpha^2 \end{pmatrix}, \tag{1-3}$$

where $\beta^i = \gamma^{ij}\beta_j$,we should keep in mind that latin indices goes from 1 to 3, with the metric tensor components is possible to write the line element

$$ds^2 = \alpha^2 dt^2 + \gamma^{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right), \tag{1-4}$$

where $(x^i) = (x^1, x^2, x^3)$ represents the induced coordinate system over the hypersurfaces. The lapse function α determines the physical proper time between two points on two neighboring spatial hypersurfaces along $\alpha\boldsymbol{n}$ and the shift vector specifies the difference between $\alpha\boldsymbol{n}$ and $\boldsymbol{t}$ which determines the direction of the time axis for each spatial point [103], a scheme of the foliation is presented in figure **1-1**. The normal vector can be written in terms of α and β^i as follows

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), \qquad n_\mu = (-\alpha, 0, 0, 0). \tag{1-5}$$

The role of extrinsic curvature $\boldsymbol{K}$ defined in (B-6) in the foliation of spacetime is denoting the degree of difference of the normal vector field and its parallel transported version along a spatial geodesic [103], the components of $\boldsymbol{K}$ can be written as

$$K_{\alpha\beta} = -\gamma^\sigma_\alpha \nabla_\sigma n_\beta = -\left(\nabla_\alpha n_\beta + n_\alpha n^\sigma \nabla_\sigma n_\beta\right). \tag{1-6}$$

[1]Take into account that the projector is the diffeomorphism of the induced metric $\boldsymbol{\gamma}$. For details see appendix B

[2]The subscript $i = 1, 2, 3$ distinguishes the vectors

From above's expression we can see that $\boldsymbol{K}$ is purely spatial, this tensor is also symmetric, $K_{\alpha\beta} = K_{\beta\alpha}$.

1.1.2. Evolution and constrain equations

To write evolution equations we need first to decompose the energy-momentum tensor $\boldsymbol{T}$ projecting it along the normal vector and along the hypersurfaces Σ_t. For an Eulerian observer with 4-velocity $\boldsymbol{n}$, unit normal 4-vector to Σ_t, the full projection of $\boldsymbol{T}$along $\boldsymbol{n}$,

$$E = \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n}) = T_{\mu\nu} n^\mu n^\nu \tag{1-7}$$

is the matter energy density. Similarly, the mixed projection,

$$\boldsymbol{p} := \boldsymbol{T}(\boldsymbol{\gamma}(\_), \boldsymbol{n}) \Longrightarrow p_\alpha = -T_{\mu\nu} \gamma^\mu_\alpha n^\nu, \tag{1-8}$$

is the matter momentum density measured by the Eulerian observer[3]. The full projection along the hypersurface of $\boldsymbol{T}$[4],

$$\boldsymbol{S} := \bar{\gamma}^*_{\mathcal{M}} \boldsymbol{T} \Longrightarrow S_{\alpha\beta} = T_{\mu\nu} \gamma^\mu_\alpha \gamma^\nu_\beta, \tag{1-9}$$

is the matter-stress tensor. We can rewrite $\boldsymbol{T}$ as

$$\boldsymbol{T} = E\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} + \underline{\boldsymbol{n}} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \underline{\boldsymbol{n}} + \boldsymbol{S}, \tag{1-10}$$

in terms of its components

$$T_{\alpha\beta} = E n_\alpha n_\beta + n_\alpha p_\beta + p_\alpha n_\beta + S_{\alpha\beta}. \tag{1-11}$$

From this, it is possible to write $\nabla_\mu T^\mu_\alpha = 0$ as follows [47]

$$\nabla_\mu S^\mu_\alpha - K p_\alpha + n^\mu \nabla_\mu p_\alpha + n_\alpha \nabla_\mu p^\mu - p^\mu K_{\mu\alpha} - K E n_\alpha + E D_\alpha \ln \alpha + n^\mu n_\alpha \nabla_\mu E = 0. \tag{1-12}$$

If we project (1-12) onto Σ_t, using the induced coordinates, is obtained the expression

$$(\partial_t - \mathcal{L}_\beta) p_i + \alpha D_j S^j_i + S_{ij} D^j \alpha - \alpha K p_i + E D_i \alpha = 0, \tag{1-13}$$

which is known as the momentum conservation. Now projecting along the normal vector to Σ_t is obtained the expression

$$(\partial_t - \mathcal{L}_\beta) E + \alpha \left(D_i p^i - K E - K_{ij} S^{ij}\right) + 2 p^i D_i N = 0, \tag{1-14}$$

which is known as the local energy conservation.

[3]Here we can also take $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{\gamma}(\_))$, this represents the energy flux 1-form measured by the Eulerian observer, given the symmetry of $\boldsymbol{T}$ then $\boldsymbol{\varphi} = c^2 \boldsymbol{p}$

[4]The projector $\bar{\gamma}^*_{\mathcal{M}}$ is defined in appendix B

Now we can obtain the Einstein fields equations in its $3+1$ form. Let us star applying the operator $\bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}$ to the expression (A-58), then we fully project the equations over the hypersurface Σ_t, then we obtain

$$\mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} = -D_\alpha D_\beta \alpha + \alpha \left\{ R_{\alpha\beta} + K K_{\alpha\beta} - 2 K_{\alpha\sigma} K^\sigma_\beta + 4\pi \left[(S - E)\, \gamma_{\alpha\beta} - 2 S_{\alpha\beta} \right] \right\}, \quad (1\text{-}15)$$

where D is the covariant derivative given by $\boldsymbol{\gamma}$, see appendix B. which can be obtain also from the evolution equation (B-13), because (B-13) is the projection of the Riemann tensor along the normal 4-vector. The fully projection perpendicular to Σ_t is

$$R + K^2 - K_{\alpha\beta} K^{\alpha\beta} = 16\pi E, \quad (1\text{-}16)$$

known as the Hamiltonian constrain[5], we have to take into account that this equation can be obtain from the Gauss equation, because the Gauss equation is the full projection of the Riemann tensor over the hypersurface. The mixed projection of the Einstein equations is

$$D_\beta K^\beta_\alpha - D_\alpha K = 8\pi p_\alpha, \quad (1\text{-}17)$$

known as the momentum constrain. Similar to the Hamiltonian constrain, it can be obtained from the Codazzi equation which is the mixed projection of the Riemann tensor.
Now we can to obtain the evolution equations. In appendix B we obtain the evolution equations (B-19) for the induced metric, then using the induced coordinate system we rewrite equations (B-19), (1-15), (1-16) and (1-17) respectively in the following way [47, 18, 13]

$$\partial_t \gamma_{ij} = \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij} - 2\alpha K_{ij}, \quad (1\text{-}18)$$
$$\partial_t K_{ij} = \mathcal{L}_{\boldsymbol{\beta}} K_{ij} - D_i D_j \alpha + \alpha \left\{ R_{ij} + K K_{ij} - 2 K_{ik} K^k_j + 4\pi \left[(S - E)\, \gamma_{ij} - 2 S_{ij} \right] \right\}, \quad (1\text{-}19)$$
$$R + K^2 - K_{ij} K^{ij} = 16\pi E, \quad (1\text{-}20)$$
$$D_j K^j_i - D_i K = 8\pi p_i. \quad (1\text{-}21)$$

This system of equations is known as the 3+1 Einstein system. This system of equations is fulfilled in each of the hypersurfaces Σ_t for a time t, therefore it is also fulfilled in the hypersurface $\Sigma_{t+\Delta t}$ for a time $t + \Delta t$. This because the Lie derivative guarantees that for a vector field $\boldsymbol{v}$ evaluated at a time t lies into the tanget space of a point in Σ_t, then $\boldsymbol{v}$ evaluated at a time $t + \Delta t$ will lie into the tanget space of a point in $\Sigma_{t+\Delta t}$ [47], this can also be seen in expression (1-2). Therefore we can evolve the equations (1-18) and (1-19), but expressions (1-20) and (1-21) only have spatial derivatives but these still are fulfilled on each Σ_t and should evolve with expressions (1-18) and (1-19), if we take the derivatives of the constrains (1-20) and (1-21) we find that if these are fulfilled for an initial hypersurface, then they remain constant, therefore it not should be evolve with (1-18) and (1-19) [103].

[5]The name for this equations is because can be obtain from the ADM Hamiltonian formulation, its the result of the variation of the Hamiltonian with respect the lapse function α.

1.1.3. Choice of lapse and shift

Here we will impose coordinate conditions specifying the lapse function α and the shift vector $\boldsymbol{\beta}$. The lapse α determines how the shape of the hypersurfaces Σ_t changes in time, since it relates the advance of proper time to coordinate time along the normal vector $\boldsymbol{n}$. The shift $\boldsymbol{\beta}$ determines how change spatial points at rest with respect to an Eulerian observer, then the spatial gauge is determine by $\boldsymbol{\beta}$. The idea here is to mention a few choices. a full description of these choices are given in [18, 103, 23].

Geodesic slicing

The geodesic slicing is given by

$$\alpha = 1, \qquad \boldsymbol{\beta} = \mathbf{0}. \tag{1-22}$$

This means that the worldlines of the Eulerian observer are geodesics. The evolution equations in this case are given by

$$\partial_t \ln \gamma^{1/2} = -K, \tag{1-23}$$

$$\partial_t K = K_{ij} K^{ij} + 4\pi \left(E + S\right). \tag{1-24}$$

Maximal slicing

The maximal slicing correspond to vanishing the mean curvature of the hypersurfaces

$$K = 0, \tag{1-25}$$

in this case the volume of spatial surfaces is extremized [47]. With this choice the evolution equation for K becomes an elliptic equation for α

$$D^2\alpha = \alpha \left(K_{ij} K^{ij} + 4\pi \left(E + S\right)\right), \tag{1-26}$$

therefore it is possible to solve the lapse α independently of the shift $\boldsymbol{\beta}$.

Harmonic slicing

Here the DeDonder gauge is used $\Box x^\alpha = 0$, requiring that this condition holds for $x^0 = t$

$$\Box t = 0, \tag{1-27}$$

this is rewritten in the following way [47]

$$\left(\partial_t - \mathcal{L}_\beta\right) \alpha = -K\alpha^2. \tag{1-28}$$

Taking $\boldsymbol{\beta} = \mathbf{0}$

$$\partial_t \alpha = -K\alpha^2, \tag{1-29}$$

then

$$\alpha = C\left(x^i\right)\sqrt{\gamma}, \tag{1-30}$$

where $C\left(x^i\right)$ is an arbitrary function of the spatial coordinates. Ir is possible to generalize this slicing as follows

$$\left(\partial_t - \mathcal{L}_\beta\right)\alpha = -Kf\left(\alpha\right), \tag{1-31}$$

taking $f\left(\alpha\right) = 2/\alpha$ it is possible to write the shift as

$$\alpha = 1 + \ln\gamma. \tag{1-32}$$

1.2. Perfect fluid in numerical relativity

Here we will take into account the energy momentum-tensor for a perfect fluid given by

$$T^\mu_\nu = \left(\rho + p\right)u^\mu u_\nu + p\delta^\mu_\nu, \tag{1-33}$$

where ρ and p represent the energy matter density and the pressure, respectively, measure by the fluid frame, and $\boldsymbol{u}$ represents the 4-velocity of the fluid which is timelike and unitary, $\boldsymbol{u}\cdot\boldsymbol{u} = -1$. Let us define the Lorentz factor as

$$W = -\boldsymbol{n}\cdot\boldsymbol{u} \tag{1-34}$$

which represents the proportionality of the proper time of the Eulerian observer and the proper time of the observer. It is possible to make a 3+1 decomposition of the 4-velocity $\boldsymbol{u}$ as follows

$$\boldsymbol{u} = W\left(\boldsymbol{n} + \boldsymbol{U}\right), \tag{1-35}$$

where $\boldsymbol{n}\cdot\boldsymbol{U} = 0$, from above expression and using the normalization of $\boldsymbol{u}$

$$W = \frac{1}{\sqrt{1 - \boldsymbol{U}\cdot\boldsymbol{U}}}. \tag{1-36}$$

Another type of velocity is the fluid coordinate velocity $\boldsymbol{v}$, which gives information about the variation of displacement of the fluid worldline respect to the constant spatial coordinates. This velocity fulfills two relations

$$v^i = \frac{u^i}{u^0}, \tag{1-37}$$

and

$$\boldsymbol{U} = \alpha^{-1}\left(\boldsymbol{v} + \boldsymbol{\beta}\right). \tag{1-38}$$

From the decomposition of the energy-momentum tensor $E = T_{\alpha\beta}n^\alpha n^\beta$, then

$$E = W^2\left(\rho + p\right) - p, \tag{1-39}$$

for the case of $p_\alpha = -T_{\mu\nu}\gamma^\mu_\alpha n^\nu$ and using coordinates

$$p_i = (E + p)U_i, \tag{1-40}$$

and for $S_{\alpha\beta} = T_{\mu\nu}\gamma^\mu_\alpha \gamma^\nu_\beta$ also using coordinates

$$S_{ij} = p\gamma_{ij} + (E + p)\, U_i U_j. \tag{1-41}$$

From (1-13), using (1-14) and, because we already calculate p_i and S_{ij} for the perfect fluid case, it is possible to obtain the relativistic Euler equation [47]

$$\partial_t U_i + v^j D_j U_i = -\frac{1}{E+p}\left[\alpha D_i p + U_i \left(\partial_t P - \beta^j \partial_j P\right)\right] + U_j D_i \beta^i - D_i \alpha + U_i U^j \left(D_j \alpha - \alpha K_{jk} U^k\right). \tag{1-42}$$

1.3. Maxwell's equations in 3+1 formalism

In this section we will follow mostly [47] for the deduction of Maxwell's equations, but there are also other references to follow this procedure like [18, 103]. The electromagnetic field is represented by a 2-form $\boldsymbol{F}$ which is antisymmetric, this tensor is called the Faraday tensor. The electric field ($\boldsymbol{E}$) and the magnetic field ($\boldsymbol{B}$) measured by Eulerian observers defined in terms of $\boldsymbol{F}$ and the normal vector $\boldsymbol{n}$ is

$$\underline{\boldsymbol{E}} = \boldsymbol{F}\left(\_, \boldsymbol{n}\right), \tag{1-43}$$

$$\underline{\boldsymbol{B}} =^* \boldsymbol{F}\left(\boldsymbol{n}, \_\right) \tag{1-44}$$

where ${}^*\boldsymbol{F}$ is the Hodge dual of $\boldsymbol{F}$ and is given by

$${}^*F_{\alpha\beta} = \frac{1}{2}{}^4\varepsilon^{\mu\nu}_{\alpha\beta} F_{\mu\nu}, \tag{1-45}$$

where ${}^4\boldsymbol{\varepsilon}$ is the space-time Levi-Civita tensor. The fields are tangent to the hypersurface, then

$$\boldsymbol{n} \cdot \boldsymbol{E} = 0, \tag{1-46}$$

$$\boldsymbol{n} \cdot \boldsymbol{B} = 0, \tag{1-47}$$

this allow us to write the Faraday tensor as follows

$$\boldsymbol{F} = \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{E}} - \underline{\boldsymbol{E}} \otimes \underline{\boldsymbol{n}} +^4 \boldsymbol{\varepsilon}\left(\boldsymbol{n}, \boldsymbol{B}, \_, \_\right), \tag{1-48}$$

writing the Faraday tensor in terms of its components

$$F_{\alpha\beta} = n_\alpha E_\beta - E_\alpha n_\beta +^4 \varepsilon_{\mu\nu\alpha\beta} n^\mu B^\nu \tag{1-49}$$

and for the Hodge dual

$$\boldsymbol{F} = -\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{B}} + \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{n}} +^4 \boldsymbol{\varepsilon}\left(\boldsymbol{n}, \boldsymbol{E}, \_, \_\right). \tag{1-50}$$

The Maxwell equations in GR are given by

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0, \tag{1-51}$$

$$\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha, \tag{1-52}$$

with $\boldsymbol{j}$ the 4-curren, using ${}^*\boldsymbol{F}$, the equation (1-51) can be written as

$$\nabla_\beta {}^*F^{\alpha\beta} = 0. \tag{1-53}$$

We must now perform the 3+1 split of the expressions (1-53) and (1-52). Let us start with the decomposition of the 4-current

$$\boldsymbol{j} = \rho \boldsymbol{n} + \boldsymbol{J}, \tag{1-54}$$

where ρ is the electric charge density and $\boldsymbol{J}$ is the electric current, from (1-54)

$$\rho = -\boldsymbol{n} \cdot \boldsymbol{j}, \tag{1-55}$$

$$\boldsymbol{J} = \boldsymbol{\gamma}\left(\boldsymbol{j}\right), \tag{1-56}$$

$$\boldsymbol{n} \cdot \boldsymbol{J} = 0. \tag{1-57}$$

From these expressions we can see that the electric charge ρ is the projection of the 4-current along the normal vector, the electric current $\boldsymbol{J}$ is the projection of the 4-currrent on the hypersurface, therefore the projection of $\boldsymbol{J}$ along $\boldsymbol{n}$ is zero.
We are going to start performing the 3+1 split of (1-53), first we will write this expression in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$

$$\nabla_\mu \left(-n^\alpha B^\mu + B^\alpha n^\mu + {}^4\varepsilon^{\beta\sigma\alpha\mu} n_\beta E_\sigma\right) = 0. \tag{1-58}$$

Let us focus only on the magnetic field contribution, computing this contribution

$$\nabla_\mu \left(-n^\alpha B^\mu + B^\alpha n^\mu\right) = \mathcal{L}_{\boldsymbol{n}} B^\alpha - n^\alpha \nabla_\mu B^\mu - K, \tag{1-59}$$

where

$$\mathcal{L}_{\boldsymbol{n}} B^\alpha = n^\mu \nabla_\mu B^\alpha - B^\mu \nabla_\mu n^\alpha \text{ and } K = -B^\alpha \nabla_\mu n^\mu. \tag{1-60}$$

Introducing the normal evolution vector $\boldsymbol{m} = \alpha \boldsymbol{n}$

$$\mathcal{L}_{\boldsymbol{n}} B^\alpha = \frac{1}{\alpha}\left[\left(\partial_t - \mathcal{L}_{\boldsymbol{\beta}}\right) B^\alpha + B^\mu \alpha \nabla_\mu n^\alpha\right], \tag{1-61}$$

where we used that $\boldsymbol{B}$ is tangent to the hypersurface and that $\nabla \cdot \boldsymbol{B} = D \cdot \boldsymbol{B} + \boldsymbol{B} \cdot D \ln \alpha$, using

$$\nabla_\mu \left(-n^\alpha B^\mu + B^\alpha n^\mu\right) = \frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} B^\alpha - K B^\alpha - n^\alpha D_\mu B^\mu. \tag{1-62}$$

On the other hand

$$\nabla_\mu \left({}^4\varepsilon^{\rho\sigma\alpha\mu} n_\rho E_\sigma \right) = - {}^4\varepsilon^{\rho\sigma\alpha\mu} n_\mu E_\sigma D_\rho \ln \alpha + {}^4\varepsilon^{\rho\sigma\alpha\mu} n_\rho \nabla_\mu E_\sigma, \tag{1-63}$$

where $K_{\mu\rho}$ appears, but because ${}^4\varepsilon^{\rho\sigma\alpha\mu}$ is antisymmetric and $K_{\mu\rho}$ is symmetric ${}^4\varepsilon^{\rho\sigma\alpha\mu} K_{\mu\rho} = 0$. Now, ${}^4\varepsilon^{\rho\sigma\alpha\mu}$ has a temporal and a spatial orientation, the temporal induced orientation is given by $\boldsymbol{n}$, then

$${}^4\varepsilon^{\rho\sigma\alpha\mu} n_\rho \nabla_\mu E_\sigma = {}^4\varepsilon^{\mu\alpha\rho\sigma} n_\mu D_\rho E_\sigma, \tag{1-64}$$

on the other hand $D_\rho E_\sigma = \alpha^{-1} \left[D_\rho \left(\alpha E_\sigma \right) - E_\sigma D_\rho \alpha \right]$, replacing this expression in (1-64) and the resulting equation in (1-63)

$$\nabla_\mu \left({}^4\varepsilon^{\rho\sigma\alpha\mu} n_\rho E_\sigma \right) = \frac{1}{\alpha} n_\mu {}^4\varepsilon^{\mu\alpha\rho\sigma} D_\rho \left(\alpha E_\sigma \right). \tag{1-65}$$

Defining $\boldsymbol{\varepsilon} = {}^4\boldsymbol{\varepsilon} \left(\boldsymbol{n}, \_, \_, \_ \right)$ and replacing (1-62) and (1-65) in (1-58)

$$\boldsymbol{\mathcal{L}_m} B^\alpha - \alpha K B^\alpha - \alpha D_\mu B^\mu n^\alpha + \epsilon^{\alpha\beta\sigma} D_\beta \left(\alpha E_\sigma \right) = 0. \tag{1-66}$$

Taking the induced coordinates to the hypersurface, if we project the equation (1-66) along the normal vector we obtain the divergence of the magnetic field for an Eulerian observer

$$D_i B^i = 0 \tag{1-67}$$

and projecting over the hypersurface, because we are using the induced coordinates to the hypersurface $\boldsymbol{m} = \partial_t - \boldsymbol{\beta}$, we obtain the 3+1 Faraday equation

$$\left(\partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) B^i - \alpha K B^i + \epsilon^{ijk} D_j \left(\alpha E_k \right) = 0. \tag{1-68}$$

Following the same procedure for (1-52), replacing the Faraday tensor in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$

$$\nabla_\mu \left(n^\alpha E^\mu - E^\alpha n^\mu +^4 \varepsilon_{\beta\sigma\alpha\mu} n_\beta B_\sigma \right) = \mu_0 j^\alpha. \tag{1-69}$$

Similar to the case of the homogeneous equations, the last expression can be rewritten as

$$- \boldsymbol{\mathcal{L}_m} E^\alpha + \alpha K E^\alpha + \epsilon^{\alpha\beta\sigma} D_\beta \left(\alpha B_\sigma \right) + \alpha n^\alpha D_\mu E^\mu = \mu_0 \alpha \left(\rho_e n^\alpha + J^\alpha \right), \tag{1-70}$$

under the induced coordinate system to the hypersurface, projecting (1-70) along $\boldsymbol{n}$ we obtain the 3+1 Gauss equation

$$D_i E^i = \mu_0 \rho_e, \tag{1-71}$$

and projecting (1-70) over the hypersurface we obtain the 3+1 Ampère equation

$$\left(\partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) E^i - \alpha K E^i - \epsilon^{ijk} D_j \left(\alpha B_k \right) = -\mu_0 \alpha J^i. \tag{1-72}$$

1.4. Chapter conclusions

A brief introduction to the 3+1 formalism of general relativity was given where the spacetime was foliated using a set of spatial hypersurfaces, with this set of quantities appear and just like the metric tensor describe the spacetime, this new quantities will describe the spacetime as well, these are: the lapse function α, the shift vector $\boldsymbol{\beta}$, the spatial metric $\boldsymbol{\gamma}$ and the extrinsic curvature $\boldsymbol{K}$. Apart from these, the energy-momentum tensor was decomposed in the 3+1 formalism together with $\nabla_\mu T^\mu_\alpha = 0$, obtaining new quantities and equations in the 3+1 formalism. The Einstein field equations were written in a set partial differential equations involving the 3+1 quantities mention above: one set of this equations evolves respect to a parameter foliation t, evolving the quantities γ_{ij} and K_{ij}, the another two equations are constrain equations that must be fulfilled in every hypersurface. Some choices for lapse and shift were mention to evolve α and $\boldsymbol{\beta}$ together with the field equations. The perfect fluid case was given for the energy-momentum tensor, and also the Maxwell equations where written in the 3+1 formalism.

2. 1+3 Formalism

Before introducing cosmological perturbations, we will describe the spacetime geometry using Lagrangian observers, which corresponds to the 1+3 formalism. As in the last chapter where the 3+1 formalism was introduced, this is a general splitting of the spacetime, sometimes it can match with the 3+1 but not always. Here we will mainly follow [41, 40].

2.1. Coordinates and 4-velocity

The coordinates taken are such that for a three dimensional hypersurface, wich will be denoted as S, each the world line intersect the hypersurface only once, the values of the spatial coordinates are maintained along each world line and the time coordinate increases along each flow line, we label the spatial coordinates as y^i. Let t be the time coordinate along the fluid, then the adapted coordinates to the flow lines (t, y^i) are the comoving coordinates. It can be taken a normalized time $s = \tau + s_0$, where τ is the time proper time measure by the world lines from the taken hyprersurface and s_0 is an arbitrary constant. With the world lines in terms of local coordinates x^μ such that $x^\mu = x^\mu(\tau)$ where τ is the proper time along the world lines, the preferred 4-velocity is the unit timelike vector

$$u^\alpha = \frac{dx^\alpha}{d\tau} \text{ where } u^\mu u_\mu = -1. \tag{2-1}$$

In normalized comoving coordinates

$$u^\mu = \delta^\mu_0 \text{ if and only if } \frac{ds}{d\tau} = 1, \; \frac{dy^i}{d\tau} = 0. \tag{2-2}$$

This implies that the vector $\boldsymbol{u}$ is tangent to the direction where all the y^i are constant. Let $\boldsymbol{T}$ be a type $\binom{p}{q}$ tensor, the time derivative of $\boldsymbol{T}$ along the fluid lines is

$$\dot{T}^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} = u^\sigma \nabla_\sigma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}. \tag{2-3}$$

This determines the accelerations vector

$$\dot{u}^\alpha = u^\sigma \nabla_\sigma u^\alpha, \tag{2-4}$$

and as a consequence $\dot{u}^\alpha u_\alpha = 0$, giving in that way a projection contribution along the 4-velocity.

2.2. Spatial projection

Similar to the 3+1 formalism, we have an induced metric tensor for the hypersurfaces S defined as

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \tag{2-5}$$

this induced metric tensor is also a projector over S. Just as in 3+1 formalism, it is posible to project along $\boldsymbol{h}$ and $\boldsymbol{u}$, let $\boldsymbol{X}$ be a 4-vector, then

$$X_\perp^\alpha = h^\alpha_\beta X^\beta \text{ and } X_\parallel^\alpha = -u^\alpha u_\beta X^\beta, \tag{2-6}$$

where $\boldsymbol{X}_\perp$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{X}_\parallel$ is parallel to $\boldsymbol{u}$. From expression (2-5) we can write the line elements as follows

$$ds^2 = -\left(\delta t\right)^2 + \left(\delta l\right)^2, \tag{2-7}$$

where

$$\delta t = u_\mu dx^\mu, \tag{2-8}$$

$$\delta l = h_{\mu\nu} dx^\mu dx^\nu. \tag{2-9}$$

This is a decomposition of an arbitrary displacement into a time difference δt and a spatial distance δl measured by an observer moving with 4-velocity.

In the case of S the normal vector to the hypersurface is $\boldsymbol{u}$ and the observers with 4-velocity $\boldsymbol{u}$, observer along the fluid, are Lagrangian observers. Here we have to take into account that the hypersurfaces S do not always match with the hypersurfaces Σ from the 3+1 formalism, then the normal vector $\boldsymbol{n}$ to Σ neither match with $\boldsymbol{u}$. From the mathematical point of view, the induced metric $\boldsymbol{\gamma}$ from 3+1 formalism and $\boldsymbol{h}$ share several properties and the geometry given in appendix B also applies for $\boldsymbol{h}$.

Projections orthogonal to $\boldsymbol{u}$ which are also symmetric tracefree, for a two rank tensor $\boldsymbol{T}$ are given by

$$T_{\langle\alpha\beta\rangle} = \left\{ h^\gamma_{(\alpha} h^\sigma_{\beta)} - \frac{1}{3} h_{\alpha\beta} h^{\gamma\sigma} \right\} T_{\gamma\sigma}, \tag{2-10}$$

and we can use this to write any two rank tensor as follow

$$\begin{aligned} T_{\alpha\beta} &= \left(h^\gamma_\alpha + u_\alpha u^\gamma\right)\left(h^\delta_\beta + u_\beta u^\delta\right) T_{\gamma\delta} \\ &= \frac{1}{3} h_{\alpha\beta} h^{\gamma\delta} T_{\gamma\delta} + T_{\langle\alpha\beta\rangle} + h^\gamma_\alpha h^\delta_\beta T_{[\gamma\delta]} - h^\gamma_\alpha T_{\gamma\delta} u^\delta u_\beta - u_\alpha u^\gamma T_{\gamma\delta} h^\delta_\beta + u_\alpha u_\beta u^\gamma u^\delta T_{\gamma\delta}. \end{aligned} \tag{2-11}$$

This expression will be useful in the next section. Similar to the derivative $\boldsymbol{D}$ in 3+1 formalism, the projector $\boldsymbol{h}$ induced an operator $\bar{\nabla}$, this operator it is also the projection of the covariant derivative $\boldsymbol{\nabla}$ along the hypersurfaces S, see appendix B.

2.3. Kinematic quantities

Let us define $\boldsymbol{V}$ as follows

$$V_{\alpha\beta} = h^{\gamma}_{\alpha} h^{\delta}_{\beta} \nabla_{\delta} u_{\gamma} = \bar{\nabla}_{\beta} u_{\alpha}, \tag{2-12}$$

we can split $\boldsymbol{V}$ into its symmetric and skew-symmetric part

$$V_{\alpha\beta} = V_{(\alpha\beta)} + V_{[\alpha\beta]}, \tag{2-13}$$

let $\Theta_{\alpha\beta} = V_{(\alpha\beta)} = \bar{\nabla}_{(\beta} u_{\alpha)}$ and $\omega_{\alpha\beta} = V_{[\alpha\beta]} = \bar{\nabla}_{[\beta} u_{\alpha]}$, the tensor $\Theta_{\alpha\beta}$ is known as the expansion tensor and $\omega_{\alpha\beta}$ is known as the vorticity tensor. From (2-10)

$$\Theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}, \tag{2-14}$$

where $\sigma_{\alpha\beta} = \Theta_{\langle\alpha\beta\rangle} = \bar{\nabla}_{\langle\beta} u_{\alpha\rangle}$ is known as the shear tensor and Θ is the expansion rate. Given the spatial projection and the projection along $\boldsymbol{u}$, it is possible to write the covariant derivative of the 4-velocity in the following way [41]

$$\begin{aligned} \nabla_{\beta} u_{\alpha} &= h^{\gamma}_{\alpha} h^{\delta}_{\beta} \nabla_{\delta} u_{\gamma} - \dot{u}_{\alpha} u_{\beta} \\ &= \bar{\nabla}_{\beta} u_{\alpha} - \dot{u}_{\alpha} u_{\beta}. \end{aligned} \tag{2-15}$$

From (2-13) and the definitions of $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$ and Θ the covariant derivative of $\boldsymbol{u}$ is written as

$$\nabla_{\beta} u_{\alpha} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_{\alpha} u_{\beta}. \tag{2-16}$$

Let us see how the terms Θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ behaves. Lets us consider how a sphere of fluid particles changes during the elapse of s small increment in proper time, let us set the zero coordinates in the center of the sphere, the figure **2-1** shows the action of each one of this terms separately. The tensor $\Theta_{\alpha\beta}$ determines the rate of change of distance of neighboring particles in the fluid and the volume expansion of the fluid is given by Θ, the Hubble parameter is defined therefore as

$$\mathcal{H} = \frac{1}{3}\Theta \tag{2-17}$$

for a pure expansion case. The shear tensor $\sigma_{\alpha\beta}$ leaves the volume invariant but determines the distortion arising in the fluid flow, the directions that remains unchanged (principal directions) are eigenvectors of $\sigma_{\alpha\beta}$, other directions are changed. The vorticity tensor $\omega_{\alpha\beta}$ determines a rigid rotation preserving the relative distances, the magnitude of vorticity is $\sqrt{\omega^{\alpha\beta}\omega_{\alpha\beta}}$. To determine the rotation axis its define the vorticity vector

$$\omega^{\delta} = \frac{1}{2}\omega_{\alpha\beta} u_{\gamma} \epsilon^{\alpha\beta\gamma\delta}, \tag{2-18}$$

then the vorticity is also given by $\sqrt{\frac{1}{2}\omega^{\alpha}\omega_{\alpha}}$.

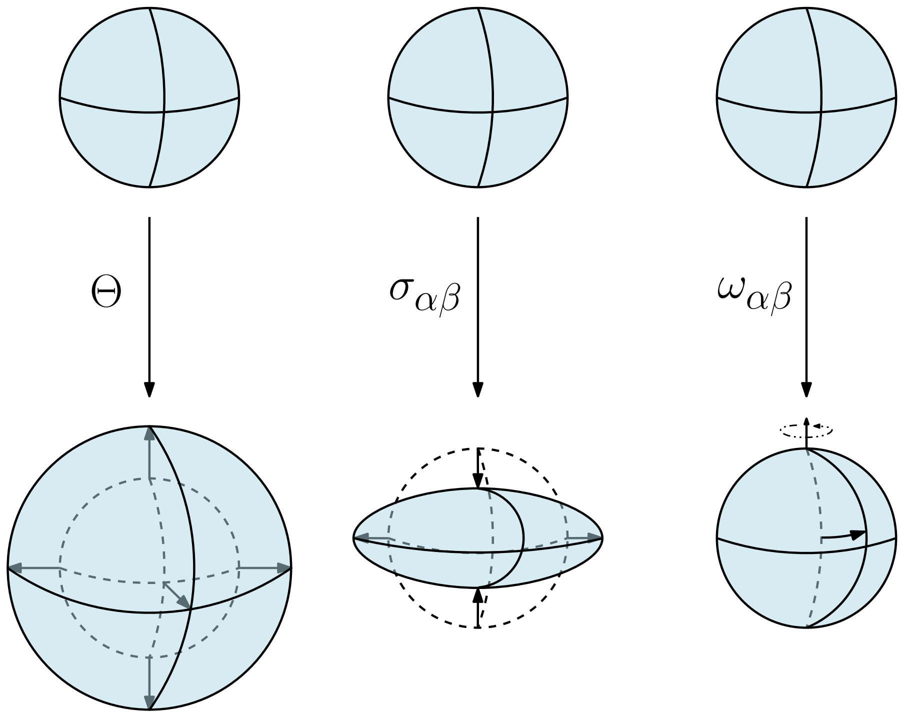

Figure 2-1.: Here we consider how a sphere of fluid particles changes during the elapse of a small increment in proper time. On the left is possible to see the action of Θ, transforming the original sphere into a sphere of bigger volume but the orientation does not change. In the center of the image can be seen how $\sigma_{\alpha\beta}$ distorts the sphere, leaving the volume constant and the direction of the axes remain unchained. On the right, the action of $\omega_{\alpha\beta}$ alone is the rigid rotation leaving one direction fixed. Reference image, figure 1 from [40].

2.4. Energy-momentum tensor and propagation equations

In the case of 1+3 formalism it is possible to decompose the energy-momentum tensor as follows [41]

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + q_\alpha u_\beta + u_\alpha q_\beta + p h_{\alpha\beta} + \Pi_{\alpha\beta}, \tag{2-19}$$

where

$$\rho = T_{\alpha\beta} u^\alpha u^\beta, \tag{2-20}$$

is the energy density relative to $\boldsymbol{u}$,

$$q^\alpha = -T_{\beta\gamma} u^\beta h^{\gamma\alpha}, \tag{2-21}$$

is the relativistic momentum density, which is also the energy flux relative to $\boldsymbol{u}$,

$$p = \frac{1}{3} T_{\alpha\beta} h^{\alpha\beta}, \tag{2-22}$$

is the isotropic preasure and

$$\Pi_{\alpha\beta} = T_{\gamma\delta} h^{\gamma}_{\langle\alpha} h^{\delta}_{\beta\rangle}, \tag{2-23}$$

is the anisotropic preasure. Taking the case of the perfect fluid $q_\alpha = 0$ and $\Pi_{\alpha\beta} = 0$, then

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p h_{\alpha\beta}, \tag{2-24}$$

taking into account (2-5) we have the same expression as in (1-33).
The energy-momentum tensor written in 1+3 formalism allow us to write the Einstein Field equations projections in this formalism, taking (A-58) and assuming $\Lambda = 0$ then [41]

$$h^{\alpha\gamma} h^{\beta\delta (4)} R_{\gamma\delta} = 8\pi \Pi^{\alpha\beta} + 4\pi \left(\rho - p\right) h^{\alpha\beta}, \tag{2-25}$$

$$h^{\alpha\gamma} u^{\beta (4)} R_{\gamma\beta} = -8\pi q^\alpha, \tag{2-26}$$

$$u^\alpha u^{\beta (4)} R_{\alpha\beta} = 4\pi \left[\rho + 3p\right]. \tag{2-27}$$

Now we are going to obtain the propagation equations, but before that we will obtain the electric and magnetic Weyl parts, for this let us decompose the Riemann tensor in the following way

$${}^{(4)}R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(g_{\alpha\gamma}{}^{(4)}R_{\beta\delta} + g_{\beta\delta}{}^{(4)}R_{\alpha\gamma} - g_{\beta\gamma}{}^{(4)}R_{\alpha\delta} - g_{\alpha\delta}{}^{(4)}R_{\beta\gamma}\right) + \frac{1}{6}{}^{(4)}R\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right), \tag{2-28}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. It is possible to decompose the Weyl tensor into its irreducible parts

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \qquad H_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\gamma\delta} C^{\gamma\delta}_{\ \ \beta\varepsilon} u^\varepsilon, \tag{2-29}$$

this allows to write the Weyl tensor as

$$C_{\alpha\beta}^{\ \ \gamma\delta} = 4\left(u_{[\alpha} u^{[\gamma} + h_{[\alpha}^{[\gamma}\right) E_{\beta]}^{\ \delta]} + 2\epsilon_{\alpha\beta\varepsilon} u^{[\gamma} H^{\delta]\varepsilon} + 2u_{[\alpha} H_{\beta]\varepsilon} \epsilon^{\gamma\delta\varepsilon}. \tag{2-30}$$

It is possible now to obtain the propagation equations, first let us use the commutation relation

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) u_\gamma = {}^{(4)}R_{\alpha\beta\gamma\delta} u^\delta, \tag{2-31}$$

The expression (2-31) can be written in the following ways [86]

$$\left(\nabla_\alpha u_\beta\right)^{\cdot} - \nabla_\alpha \dot{u}_\beta + \left(\nabla^\alpha u_\gamma\right)\left(\nabla^\gamma u_\beta\right) = -{}^{(4)}R_{\gamma\alpha\beta\delta} u^\gamma u^\delta, \tag{2-32}$$

$$\left(\nabla_\alpha u^\alpha\right)^{\cdot} - \nabla_\alpha \dot{u}^\alpha + \left(\nabla^\alpha u^\beta\right)\left(\nabla_\beta u_\alpha\right) = -{}^{(4)}R_{\alpha\beta} u^\alpha u^\beta, \tag{2-33}$$

from (2-33), using (2-27) and the kinematic decomposition of $\boldsymbol{u}$ we obtain

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}\left(\rho + 3p\right) - 2\left(\sigma^2 - \omega^2\right) + D^\alpha \dot{u}_\alpha + \dot{u}_\alpha \dot{u}^\alpha \tag{2-34}$$

which is the Raychaudhuri equation. If we obtain the symmetric trace-free part of (2-32) and use expressions (2-28) and (2-30) we obtain the shear propagation equation [41, 40]

$$\dot{\sigma}_{\langle\alpha\beta\rangle} = -\frac{2}{3}\Theta\sigma_{\alpha\beta} - \sigma_{\gamma\langle\alpha}\sigma^{\gamma}{}_{\beta\rangle} - \omega_{\langle\alpha}\omega_{\beta\rangle} + D_{\langle\alpha}\dot{u}_{\beta\rangle} + \dot{u}_{\langle\alpha}\dot{u}_{\beta\rangle} - E_{\alpha\beta} + \frac{1}{2}\Pi_{\alpha\beta}, \tag{2-35}$$

similarly, projecting (2-32) with $\epsilon^{\alpha\beta\gamma}$, we obtain the vorticity propagation equation

$$\dot{\omega}_{\langle\alpha\rangle} = -\frac{2}{3}\Theta\omega_{\alpha} - \frac{1}{2}\epsilon_{\alpha\beta\gamma}D^{\beta}\dot{u}^{\gamma} + \sigma_{\alpha\beta}\omega^{\beta}. \tag{2-36}$$

Just like the propagation equations it is also possible to obtain constrain equations, from the $(0,\alpha)$ component of the projection of (2-31)

$$D^{\beta}\sigma_{\alpha\beta} = \frac{2}{3}D_{\alpha}\Theta + \epsilon_{\alpha\beta\gamma}D^{\beta}\omega^{\gamma} + 2\epsilon_{\alpha\beta\gamma}\dot{u}^{\beta}\omega^{\gamma} - q_{\alpha}, \tag{2-37}$$

the contraction of (2-31) with $\epsilon^{\alpha\beta\gamma}$ give us

$$D_{\alpha}\omega^{\alpha} = \omega^{\alpha}\dot{u}_{\alpha} \tag{2-38}$$

and the contraction of the symmetric trace-free part of (2-32) with $\epsilon^{\gamma\delta\epsilon}$

$$H_{\alpha\beta} = \mathrm{curl}\sigma_{\alpha\beta} + D_{\langle\alpha}\omega_{\beta\rangle} + 2\dot{u}_{\langle\alpha}\omega_{\beta\rangle}. \tag{2-39}$$

2.5. Maxwell equations in 1+3 Formalism

Here we are going to consider the electromagnetic field from the point of view of Lagrangian observers, this means that if we consider a perfect fluid with 4-velocitu $\boldsymbol{u}$, the observer is moving along the fluid with such velocity. The electric and magnetic fields measure by this observer are going to be denoted by $\boldsymbol{e}$ and $\boldsymbol{b}$ respectively. The components of the Faraday tensor for this case are given by

$$F_{\mu\nu} = u_{\mu}e_{\nu} - e_{\mu}u_{\nu} + \epsilon_{\mu\nu\delta\gamma}b^{\delta}u^{\gamma} \tag{2-40}$$

and the Hodge dual is given by

$$^{*}F_{\mu\nu} = b_{\mu}u_{\nu} - u_{\mu}b_{\nu} + \epsilon_{\mu\nu\delta\gamma}e^{\delta}u^{\gamma}. \tag{2-41}$$

We are going to make the 1+3 splitting of the Maxwell equations, from $\nabla_{\beta}{}^{*}F^{\alpha\beta} = 0$, projecting along $\boldsymbol{u}$

$$u_{\alpha}\nabla_{\beta}\left(^{*}F^{\alpha\beta}\right) = \nabla_{\beta}\left(u_{\alpha}{}^{*}F^{\alpha\beta}\right) - u_{\alpha}\nabla_{\beta}\left(^{*}F^{\alpha\beta}\right) = 0, \tag{2-42}$$

replacing (2-41) in (2-42) we obtain

$$-\nabla_{\beta}b^{\beta} + \left(b^{\alpha}u^{\beta} + \epsilon^{\gamma\delta\alpha\beta}e_{\delta}u_{\mu}\right)\nabla_{\beta}u_{\alpha} = 0. \tag{2-43}$$

Using the general decomposition (2-16) and (2-18)

$$e_\delta u_\gamma \epsilon^{\gamma\delta\alpha\beta}\nabla_\beta u_\alpha = 2\omega^\alpha e_\alpha. \tag{2-44}$$

On the other side, because $\bar{\nabla}_\beta b^\alpha = h^\mu_\beta h^\alpha_\nu \nabla_\mu b^\nu$ then

$$\bar{\nabla}_\alpha b^\alpha = \nabla_\beta b^\beta - u^\beta b^\nu \nabla_\beta u_\nu, \tag{2-45}$$

from equations (2-43),(2-44) and (2-45)

$$\bar{\nabla}_\alpha b^\alpha = 2\omega^\alpha e_\alpha, \tag{2-46}$$

this is the divergence equation for the magnetic field in 1+3 formalism. Now we make a spatial projection

$$h^\gamma_\alpha \nabla_\beta \left(-b^\alpha u^\beta + u^\alpha b^\beta + \epsilon^{\mu\nu\alpha\beta} u_\mu e_\nu\right) = 0, \tag{2-47}$$

then

$$-h^\gamma_\alpha \dot{b}^\alpha - b^\gamma \nabla_\beta u^\beta + h^\gamma_\alpha b^\beta \nabla_\beta u^\alpha + h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} e_\nu \nabla_\beta u_\mu + h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} u_\mu \nabla_\beta e_\nu = 0. \tag{2-48}$$

We have that $h^\gamma_\alpha b^\beta \nabla_\beta u^\alpha = b^\beta \nabla_\beta u^\gamma$, then

$$b^\gamma \nabla_\beta u^\beta - b^\beta \nabla_\beta u^\gamma = -\left(\sigma^\gamma_\beta + \omega^\gamma_\beta - \frac{2}{3}\Theta\delta^\gamma_\beta\right) b^\beta. \tag{2-49}$$

On the other hand

$$h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} e_\nu \nabla_\beta u_\mu = h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} e_\nu \left(\omega_{\mu\beta} - \dot{u}_\mu u_\beta\right), \tag{2-50}$$

because $\omega_{\mu\beta}$ and e_ν contribute only spatially $h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} e_\nu \omega_{\mu\beta} = 0$. Therefore we can rewrite equation (2-48) as

$$h^\gamma_\alpha \dot{b}^\alpha = \left(\sigma^\gamma_\beta + \omega^\gamma_\beta - \frac{2}{3}\Theta\delta^\gamma_\beta\right) b^\beta - \epsilon^{\gamma\mu\nu\beta} u_\mu \nabla_\beta e_\nu - h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} \dot{u}_\mu u_\beta e_\nu, \tag{2-51}$$

the last equation is knows as the Faraday equation in 1+3 formalism.
Now we split the equation $\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha$, therefore we need to decompose the 4-current, this decomposition is given by

$$\boldsymbol{j} = \rho_u \boldsymbol{u} + \boldsymbol{J}_u, \tag{2-52}$$

where ρ_u is the charge density and $\boldsymbol{J}$ is the electric current measure by the observer. These quantities fulfill the following properties

$$\rho_u = -\boldsymbol{u}\cdot\boldsymbol{j}, \tag{2-53}$$

$$\boldsymbol{J}_u = \boldsymbol{h}\left(\boldsymbol{j}\right), \tag{2-54}$$

$$\boldsymbol{u}\cdot\boldsymbol{J}_u = 0. \tag{2-55}$$

Projecting over $\boldsymbol{u}$

$$u_\alpha \nabla_\beta F^{\alpha\beta} = \nabla_\beta \left(u_\alpha F^{\alpha\beta} \right) - F^{\alpha\beta} \nabla_\beta u_\alpha = 4\pi u_\beta j^\beta, \tag{2-56}$$

replacing the Faraday expression

$$\nabla_\beta e^\beta + \left(-e^\alpha u^\beta + \epsilon^{\gamma\delta\alpha\beta} b_\delta u_\gamma \right) \nabla_\beta u_\alpha = 4\pi \rho_u \tag{2-57}$$

and following the procedure that was made when we obtained the expression for the divergence of the magnetic field, we have

$$\bar{\nabla}_\alpha e^\alpha = 4\pi\rho_u - 2\omega^\alpha b_\alpha, \tag{2-58}$$

which is the Gauss equation in 1+3 formalism. Now projecting spatially, following the procedure to obtain the Faraday equation but adding the electric current we have

$$h^\gamma_\alpha \dot{e}^\alpha = \left(\sigma^\gamma_\beta + \omega^\gamma_\beta - \frac{2}{3}\Theta \delta^\gamma_\beta \right) e^\beta + \epsilon^{\gamma\mu\nu\beta} u_\mu \nabla_\beta b_\nu + h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} \dot{u}_\mu u_\beta b_\nu - 4\pi J^\gamma_u, \tag{2-59}$$

which is the Ampère equation in 1+3 formalism. Finally, the Ohm's law is given by

$$j_\nu = \sigma e_\mu, \tag{2-60}$$

where we are considering only the isotropic part of the Ohm's law, there are more terms representing anisotropies due to the presence of the magnetic field [23].

2.6. Chapter conclusions

An introduction to the $1+3$ formalism was given, this formalism is essential in the dynamo approach. Similar to the $3+1$ decomposition, here there is also a decomposition along an hypersurface S through a projector $\boldsymbol{h}$ and along a 4-vector $\boldsymbol{u}$ normal to S. The difference with the 3+1 formalism is that the observer with 4-velocity $\boldsymbol{u}$, called Lagrangian observer, go along the fluid, the decomposition of tensors here is a symmetric trace free projection and the temporal derivative is the derivative along $\boldsymbol{u}$. The covariant derivative of $\boldsymbol{u}$ is decomposed in kinematic quantities that describe the actions over the fluid. The energy-momentum tensor is also decompose in $1+3$ formalism, also the Einstein field equations and the Weyl tensor, together with the commutation relation of the covariant derivative applied to $\boldsymbol{u}$, evolution equations for the kinematic quantities and constrain equation are obtained. Finally the Maxwell equations are presented together with the isotropic part of the Ohm's law.

3. Cosmological perturbations

In this chapter the cosmological perturbations are presented. We start by perturbing the spatially flat Friedman-Lemaître-Robertson-Walker (FLRW) solution, the 3+1 quantities are obtain in the perturbed formalism, the Maxwell equations are also perturbed in the case of 3+1 and 1+3 formalism.

3.1. Perturbed FLRW equations

To obtain the perturbed equations first we must to fix a background, in this case we assume a spatially flat FLRW solution. After fixing the background solution it is possible to perturb the geometric and matter quantities to obtain the desire equations.

3.1.1. Background equations

The line element for the FLRW metric is [88]

$$ds^2 = -dt^2 + \left(\frac{a^2(t)}{1+\frac{1}{4}kr^2}\right)\delta_{ij}dx^i dx^j, \tag{3-1}$$

where $r^2 = x^2 + y^2 + z^2$, if $k > 1$ the solution represents a closed universe, if $k < 1$ the solution represents an open universe and if $k = 0$ the solutions represents a flat universe. In this case we are taking a spatially flat universe, therefore the line element is given by

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j. \tag{3-2}$$

The line element can also be written in terms of the conformal time η, which is define in such a way that $d\eta = a^{-1}dt$, this allow us to write the line element as

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right). \tag{3-3}$$

Given (3-3) we can calculate the geometric quantities associated to this solution, given in Appendix E. In the background evolution, the energy-momentum tensor is taken as a perfect fluid, the components of the tensor are given by

$$T^\alpha_\beta = (\rho + p)\,u^\alpha u_\beta + p\delta^\alpha_\beta, \tag{3-4}$$

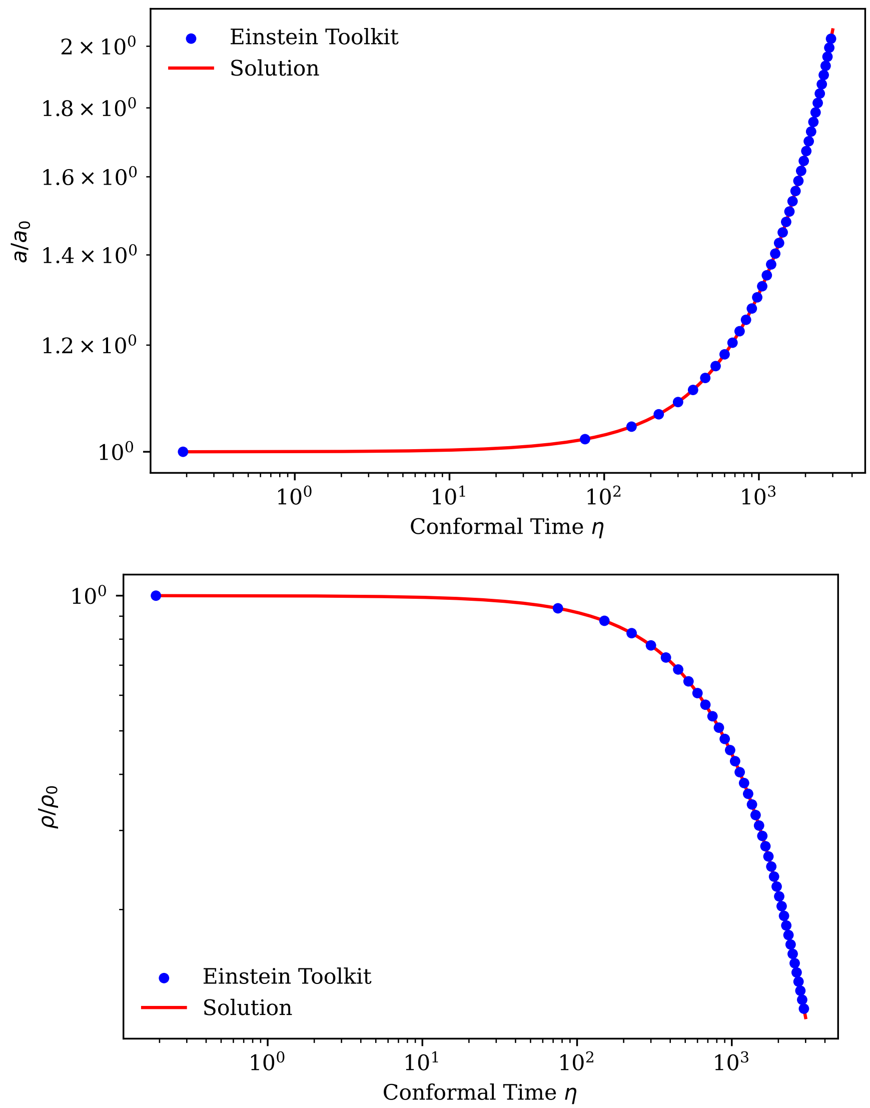

Figure 3-1.: These graphics shows the evolution of a/a_0 and ρ/ρ_0 respect to conformal time η. The solution in the graphic, red line, is given by equations (3-10) and (3-11). It is also shown, in blue dots, a numerical solution using `Einstein Toolkit`, matching with the results obtained in [77]. The discussion of these results will be one of the main topics of chapter 5.

where ρ is the energy density and p the pressure. In the rest of frame of the fluid

$$u^{\alpha} = (1, 0, 0, 0) \text{ and } u_{\beta} = (-1, 0, 0, 0), \tag{3-5}$$

therefore

$$T^0_0 = -\rho(\eta), \qquad T^i_0 = 0, \qquad T^i_j = p(\eta)\, \delta^i_j. \tag{3-6}$$

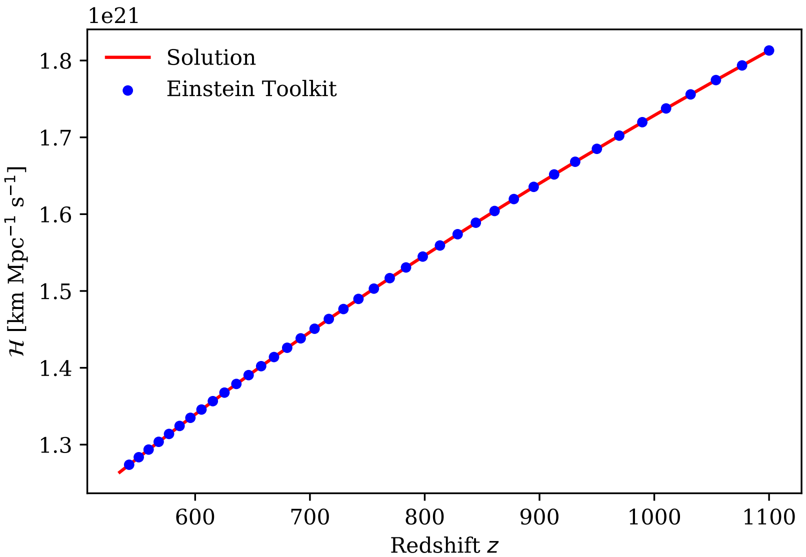

Figure 3-2.: Evolution for the Hubble parameter for matter domination era, the red line shown the evolution using expression (3-8). There is also a numerical evolution obtained from the results of the numerical simulations in `Einstein Toolkit`, using the fact that $\mathcal{H} = a'/a$ and that $z = -1 + (z_{\mathrm{CMB}} + 1)/a$ where $z_{\mathrm{CMB}} = 1100$. The ρ used in this case is the same one that appears in figure **3-1**. Computational implementation will be discussed in chapter 5.

From the energy and momentum conservation $\nabla_\mu T^\mu_\alpha = 0$, setting $\alpha = 0$ the conservation equation is obtained

$$\rho' + 3\mathcal{H}\left(\rho + p\right) = 0, \tag{3-7}$$

where $\mathcal{H} = a'/a$, which is known as the Hubble parameter. Using (A-57) setting $\alpha = 0$ and $\beta = 0$ the Friedman equation is obtained

$$\mathcal{H}^2 = \frac{8\pi}{3}a^2\rho, \tag{3-8}$$

its evolution is shown in figure **3-2**, and from the (i, j) components of (A-57) combined with Friedman equation

$$\frac{a''}{a} = \frac{4\pi}{3}\left(\rho a^2 + 3p\right). \tag{3-9}$$

The solution for (3-7), (3-8), (3-9) is given by [79]

$$a = a_0\xi^2, \tag{3-10}$$

$$\rho = \rho_0\xi^{-6}, \tag{3-11}$$

$$\xi = 1 + \eta\sqrt{\frac{2}{3}\pi\rho_0 a_0^2}, \tag{3-12}$$

where $a_0 = a(\eta_0)$ and $\rho_0 = \rho(\eta_0)$ for the initial time $\eta_0 = 0$, the evolution of a and ρ is shown in figure **3-1**. There is a redundancy given by the diffeomorphism invariance of GR, therefore the system of equations (3-7), (3-8) and (3-9) is closed by the equation of state

$$p(\eta) = w(\eta)\,\rho(\eta)\,. \tag{3-13}$$

Different values of w represents different epochs of the universe: $w = 1/3$ represents radiation dominance epoch and $w = 0$ represents matter dominance epoch. Replacing the equation of state in the Friedman equation

$$a(\eta) \propto \begin{cases} \eta^{2/(1+3w)} & w \text{ constant and } w \neq 1, \\ \eta^2 & w = 0\,(\text{Matter dominance})\,, \\ \eta & w = 1/3\,(\text{Radiation dominance})\,. \end{cases} \tag{3-14}$$

The total energy density is given by

$$\rho = \rho_R + \rho_M, \tag{3-15}$$

and the total pressure is

$$p = p_R + p_M, \tag{3-16}$$

where the index R and M represents the radiation and matter term respectively. The present value of the density, knows as the critical density, is given by

$$\rho_0 = \frac{3H_0^2}{8\pi}, \tag{3-17}$$

in terms of conformal time

$$\rho_c(\eta) = \frac{3\mathcal{H}^2(\eta)}{8\pi a(\eta)}, \tag{3-18}$$

then

$$\rho_R(\eta) = \rho_0\Omega_R a^{-2}(\eta)\,, \qquad \rho_M(\eta) = \rho_0\Omega_M a^{-1}(\eta)\,, \tag{3-19}$$

where

$$\Omega_R = \frac{\rho_R(\eta)}{\rho_c(\eta)}, \qquad \Omega_M = \frac{\rho_M(\eta)}{\rho_c(\eta)}, \tag{3-20}$$

are values of energy fraction. This defined quantities allow us to rewrite the Friedman equation as

$$\mathcal{H}(a) = \mathcal{H}_0\sqrt{\frac{\Omega_R}{a^2} + \frac{\Omega_M}{a}}. \tag{3-21}$$

All the backgroud quantities were obtained using `Sagemath` and `Sagemanifolds` [48, 49] and the codes are shown in appendix F.

3.1.2. Perturbed equations

Here we will apply the perturbation theory shown in Appendix C. The metric tensor can be split in a background plus a perturbed contribution

$$\boldsymbol{g} = \bar{\boldsymbol{g}} + \delta \boldsymbol{g}, \tag{3-22}$$

the term $\bar{\boldsymbol{g}}$ in this case represents the FLRW metric tensor and $\delta \boldsymbol{g}$ are the perturbations of the FLRW spacetime. The components of the metric tensor can be written in the following way [25, 90]

$$g_{00} = -a^2(\eta)\left(1 + 2\sum_{n=1}^{\infty}\frac{\psi^{(n)}}{n!}\right), \tag{3-23}$$

$$g_{0i} = a^2(\eta)\sum_{n=1}^{\infty}\frac{\omega_i^{(n)}}{n!}, \tag{3-24}$$

$$g_{ij} = a^2(\eta)\left[\left(1 - 2\sum_{n=1}^{\infty}\frac{\phi^{(n)}}{n!}\right)\delta_{ij} + \sum_{n=1}^{\infty}\frac{\chi_{ij}^{(n)}}{n!}\right], \tag{3-25}$$

where $\psi^{(n)}$ and $\phi^{(n)}$ are scalar perturbations, $\omega_i^{(n)}$ are vector perturbations and $\chi_{ij}^{(n)}$ are tensor perturbation all of order n. Writing the metric tensor components $g_{\alpha\beta}$ up to first order and the contravariant metric tensor components $g^{\alpha\beta}$ in its matrix representation

$$g_{\alpha\beta} = a^2\begin{pmatrix} -(1+2\psi) & \omega_i \\ \omega_j & (1-2\phi)\delta_{ij} + \chi_{ij} \end{pmatrix}, \quad g^{\alpha\beta} = a^{-2}\begin{pmatrix} -(1-2\psi) & \omega^i \\ \omega^j & (1+2\phi)\delta^{ij} - \chi^{ij} \end{pmatrix}. \tag{3-26}$$

where, for simplicity, we removed the index (1), then the perturbations from now on are ψ, ϕ, ω_i and χ_{ij}.We will also consider perturbed matter quantities, therefore for the density ρ, the pressure p, the 4-velocity $\boldsymbol{u}$, the electromagnetic field $\boldsymbol{e}$ and $\boldsymbol{b}$ and the 4-current $\boldsymbol{j}$

$$\rho = \rho_{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!}\rho_{(n)}, \tag{3-27}$$

$$p = p_{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!}p_{(n)}, \tag{3-28}$$

$$u^{\alpha} = \frac{1}{a(\eta)}\left(\delta^{\alpha}_{0} + \sum_{k=1}^{\infty} \frac{1}{k!}v^{\alpha}_{(k)}\right), \tag{3-29}$$

$$e^{i} = \frac{1}{a^2(\eta)}\left(\sum_{n=1}^{\infty} \frac{1}{n!}e^{i}_{(n)}\right), \tag{3-30}$$

$$b^{i} = \frac{1}{a^2(\eta)}\left(\sum_{n=1}^{\infty} \frac{1}{n!}b^{i}_{(n)}\right), \tag{3-31}$$

$$j^{\mu} = \frac{1}{a(\eta)}\left(\sum_{n=1}^{\infty} \frac{1}{n!}j^{i}_{(n)}\right). \tag{3-32}$$

The velocity v^{α} is the peculiar velocity, we can obtain an expression for the peculiar velocity using the fact that the norm of u^{α} is constant

$$u^{\mu}u_{\mu} = u^{\mu}u^{\nu}g_{\nu\mu} = -1, \tag{3-33}$$

using (3-29) and (3-26)

$$v^{0}_{(1)} = -\psi, \tag{3-34}$$

therefore

$$\left(u^{\alpha}\right) = \left(u^{0}, u^{i}\right) = a^{-1}\left(1-\psi, v^{i}_{(1)}\right), \tag{3-35}$$

$$\left(u_{\alpha}\right) = \left(u_{0}, u_{i}\right) = a\left(-1-\psi, \omega_i + v^{(1)}_{i}\right). \tag{3-36}$$

According to (1-37) and (1-38)

$$v^{i} = v^{i}_{(1)}, \tag{3-37}$$

$$U^{i} = a^{-1}\left(v^{i}_{(1)} + \omega^{i}\right), \tag{3-38}$$

Using this expressions, it is possible to write the relativistic Euler equation perturbed at first order

$$\left(U^{(1)}_{i}\right)' = \frac{1}{\rho_{(0)}+p_{(0)}}\left[\alpha_{(1)}\partial_i p_{(0)} + \alpha_{(0)}\partial_i p_{(1)} + U^{(1)}_{i}p'_{(0)} + \left(\rho_{(1)}+p_{(1)}\right)\alpha_{(0)}p_{(0)}\right], \tag{3-39}$$

where we used the fact that it is possible to split the lapse function into a background and a perturbed contribution, just like in (3-57). Keeping in mind that $\boldsymbol{U}$ is the spatial contribution of $\boldsymbol{u}$, the equation is rewritten as

$$\left(\rho_{(0)}+p_{(0)}\right)\left[a\left(\omega_i^{(1)}+v_i^{(1)}\right)\right]'=a\phi\partial_i p_{(0)}+a\partial_i p_{(1)}+a\left(\omega_i^{(1)}+v_i^{(1)}\right)p_{(0)}'+\left(\rho_{(1)}+p_{(1)}\right)ap_{(0)}. \tag{3-40}$$

From (3-26) it is possible to write the line element ds^2, considering the conformal Newtonian gauge [75]

$$ds^2=a^2\left(\eta\right)\left[-\left(1+2\Psi\right)d\eta^2+\left(1-2\Phi\right)\delta_{ij}dx^i dx^j\right], \tag{3-41}$$

considering perturbations at first order

$$G_{\mu\nu}^{(1)}=8\pi T_{\mu\nu}^{(1)},$$

where $G_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(1)}$ are the Einstein tensor and the energy-momentum tensor at first order respectively. It is possible to write this equations as follow[1] [77]

$$\nabla^2\Phi-3\mathcal{H}\left(\Phi'+\mathcal{H}\Psi\right)=4\pi\rho\delta a^2, \tag{3-42}$$

$$\mathcal{H}\partial_i\Psi+\partial_i\Phi'=-4\pi\rho a^2\delta_{ij}v_{(1)}^j, \tag{3-43}$$

$$\Phi''+\mathcal{H}\left(\Psi'+2\Phi'\right)=\frac{1}{2}\nabla^2\left(\Phi-\Psi\right), \tag{3-44}$$

$$\left[\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2\right]\left(\Phi-\Psi\right)=0, \tag{3-45}$$

where

$$\delta=-1+\rho^{(1)}/\rho, \tag{3-46}$$

for ρ the background density, (3-42) and (3-43) are the Hamiltonian and the momentum constrains respectively. From $\nabla_\alpha T_\mu^\alpha=0$

$$\delta'=3\Phi'-\partial_i v_{(1)}^i, \tag{3-47}$$

$$\left(v_{(1)}^i\right)'=-\partial^i\Psi-\mathcal{H}v_{(1)}^i. \tag{3-48}$$

Taking the linear case $\Psi=\Phi$, the potential takes the general form [36]

$$\Phi=f\left(x^i\right)-\frac{g\left(x^i\right)}{5\xi^5}, \tag{3-49}$$

where the functions f and g are spatially dependent, these represents the growing and decaying modes of the density perturbations [36, 88]. From the Hamiltonian constrain we obtain

$$\delta=C_1\xi^2\nabla^2 f\left(x^i\right)-2f\left(x^i\right)-C_2\xi^{-3}g\left(x^i\right)-\frac{3}{5}\xi^{-5}g\left(x^i\right), \tag{3-50}$$

[1] To write the equations in this way we followed [79] and used the Mathematica library `xPand` [95], the code is shown in appendix F.

where

$$C_1 = \frac{1}{4\pi\rho_0 a_0^2}, \qquad C_2 = \frac{1}{20\pi\rho_0 a_0^2}. \tag{3-51}$$

From the momentum constrain

$$v^i_{(1)} = C_3 \xi \partial^i f\left(x^i\right) + \frac{3}{10} C_3 \xi^{-4} \partial^i g\left(x^i\right) \tag{3-52}$$

where

$$C_3 = -\sqrt{\frac{1}{6\pi\rho_0 a_0}}. \tag{3-53}$$

Taking only the growing modes

$$\Phi = f\left(x^i\right), \tag{3-54}$$
$$\delta = C_1 \xi^2 \nabla^2 f\left(x^i\right) - 2f\left(x^i\right), \tag{3-55}$$
$$v^i_{(1)} = C_3 \xi \partial^i f\left(x^i\right), \tag{3-56}$$

last three equations will be useful in the computational results chapter.

3.2. Cosmological perturbations and Numerical Relativity

Given the cosmological perturbation theory, here we will apply it into NR, then we will perturb $3+1$ quantities at first order. For this we will make a direct comparison between (1-3) and (3-26), but before this we need to take into account that it is possible to make such an equality because it is possible to set up a set of basis in such a way that $\boldsymbol{u}$ and $\boldsymbol{n}$ are perpendicular to the same hypersurface, it is possible to see that such bases exist because for the FLRW background solution, four-vectors $\boldsymbol{u}$ and $\boldsymbol{n}$ match. We can decompose the lapse function α, the shift vector $\boldsymbol{\beta}$ and the induced metric $\boldsymbol{\gamma}$ in the background and in the first order perturbation contribution [38]

$$\alpha = \alpha^{(0)} + \alpha^{(1)}, \tag{3-57}$$
$$\beta_i = \beta_i^{(0)} + \beta_i^{(1)}, \tag{3-58}$$
$$\gamma_{ij} = \gamma_{ij}^{(0)} + \gamma_{ij}^{(1)}. \tag{3-59}$$

Comparing directly (1-3) and (3-26) the background and the perturbed quantities for the metric are given by

$$\alpha^{(0)} = a\left(\eta\right), \qquad \alpha^{(1)} = a\left(\eta\right)\psi, \tag{3-60}$$
$$\beta_i^{(0)} = 0, \qquad \beta_i^{(1)} = a^2\left(\eta\right)\omega_i, \tag{3-61}$$
$$\gamma_{ij}^{(0)} = a^2\left(\eta\right)\delta_{ij}, \qquad \gamma_{ij}^{(1)} = a^2\left(\eta\right)\left(-2\phi\delta_{ij} + \chi_{ij}\right). \tag{3-62}$$

Doing the same but with quantities with the covariant index

$$\beta^i_{(0)} = 0, \qquad \beta^i_{(1)} = \omega^i, \tag{3-63}$$

$$\gamma^{ij}_{(0)} = a^{-2}(\eta)\,\delta^{ij}, \qquad \gamma^{ij}_{(1)} = a^{-2}(\eta)\left(2\phi\delta^{ij} - \chi^{ij}\right). \tag{3-64}$$

This allow us to write the normal vector to the hypersurface perturbed at first order

$$(n^\mu)^{(0)} = a^{-1}(1,0,0,0), \qquad (n^\mu)^{(1)} = a^{-1}\left(-\psi, -\omega^i\right), \tag{3-65}$$

$$(n_\mu)^{(0)} = -a(1,0,0,0), \qquad (n_\mu)^{(1)} = -a(\psi,0,0,0). \tag{3-66}$$

Let us calculate now the extrinsic curvature, from the evolution equation (1-19) we can calculate K_{ij}, but first we must calculate $\partial_\eta \gamma_{ij}$ and $\mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij}$, from (3-63) and (3-62)

$$\partial_\eta \gamma_{ij} = 2aa'\left[(1-2\phi)\,\delta_{ij} + \chi_{ij}\right] + a^2\left(-2\phi'\delta_{ij} + \chi'_{ij}\right), \tag{3-67}$$

$$\mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = 2a^2\partial_{(i}\omega_{j)}, \tag{3-68}$$

therefore

$$K_{ij} = -a'\left[(1-2\phi+\psi)\,\delta_{ij} + \chi_{ij}\right] + a\left[\phi'\delta_{ij} - \frac{1}{2}\chi'_{ij} + \partial_{(i}\omega_{j)}\right]. \tag{3-69}$$

In a similar way we calculate K^{ij}, from (3-63) and (3-64)

$$\partial_\eta \gamma^{ij} = -2a^{-2}\mathcal{H}\left[(1+2\phi)\,\delta^{ij} - \chi^{ij}\right] + a^{-2}\left[2\phi' - \left(\chi^{ij}\right)'\right], \tag{3-70}$$

$$\mathcal{L}_{\boldsymbol{\beta}}\gamma^{ij} = -2a^{-2}\partial^{(i}\omega^{j)}, \tag{3-71}$$

therefore

$$K^{ij} = a^{-3}\left[\phi'\delta^{ij} + \mathcal{H}\psi\delta^{ij} - \mathcal{H}(1+2\phi)\,\delta_{ij} + \mathcal{H}\chi_{ij} - \frac{1}{2}\chi_{ij} + \frac{1}{2}\left(\delta^{kj}\partial_k\omega^i + \delta^{ki}\partial_k\omega^j\right)\right]. \tag{3-72}$$

Now we can compute the extrinsic curvature trace $K = \gamma^{ij}K_{ij}$, from (3-64) and (3-69)

$$K = a^{-1}\left[3\left(-\mathcal{H} + \phi' + \mathcal{H}\psi\right) + \delta^{ij}\partial_{(i}\omega_{j)}\right]. \tag{3-73}$$

We will compute now the Christoffel symbols for the Levi-Civita connection D associated with the metric $\boldsymbol{\gamma}$, we replace (3-64) and (3-62) in (A-46)

$$\Gamma^k_{ij} = -2\delta^k_{(i}\partial_{j)}\phi + \partial_{(i}\chi^k_{j)} + \delta_{ij}\delta^{kl}\partial_l\phi - \frac{1}{2}\delta^{kl}\partial_l\chi_{ij}. \tag{3-74}$$

To be able to write all the quantities necessary to write Einstein field equations, the perturbed quantities of the projected momentum energy tensor E, p_α and $S_{\alpha\beta}$ must be calculated

$$E = E^{(0)} + E^{(1)}, \tag{3-75}$$

$$p_\alpha = p^{(0)}_\alpha + p^{(1)}_\alpha, \tag{3-76}$$

$$S_{\alpha\beta} = S^{(0)}_{\alpha\beta} + S^{(1)}_{\alpha\beta}, \tag{3-77}$$

then let us write the momentum-energy tensor in the following form

$$T^{\alpha\beta} = T^{\alpha\beta}_{(0)} + T^{\alpha\beta}_{(1)}. \tag{3-78}$$

We already calculate the perturbed induced metric and normal vector over an induced coordinate system, then replacing (3-62) and (3-66) in (1-7), (1-8) and (1-9)

$$E^{(0)} = a^2 T^{00}_{(0)}, \qquad E^{(1)} = a^2\left(2\psi T^{00}_{(0)} + T^{00}_{(1)}\right), \tag{3-79}$$

$$p^{(0)}_i = a^3 T^{0j}_{(0)}\delta_{ij}, \qquad p^{(1)}_i = a^3\left\{T^{0j}_{(0)}\left[(\psi - 2\phi)\delta_{ij} + \chi_{ij}\right] + T^{00}_{(0)}\omega_i + T^{j0}_{(1)}\delta_{ij}\right\}, \tag{3-80}$$

$$S^{(0)}_{ij} = a^4 T^{kl}_{(0)}\delta_{ki}\delta_{lj}, \qquad S^{(1)}_{ij} = a^4\left\{T^{kl}_{(0)}\left[(\chi_{ki} - 2\phi\delta_{ki})\delta_{lj} + (\chi_{lj} - 2\phi\delta_{lj})\delta_{ki}\right] + T^{kl}_{(1)}\delta_{ki}\delta_{lj}\right\}. \tag{3-81}$$

Let us see what happens with the Lorentz factor (1-34), if we equate the metric tensor components written in terms of projectors $\boldsymbol{h}$ and $\boldsymbol{\gamma}$ and normal vectors $\boldsymbol{u}$ and $\boldsymbol{n}$ then

$$h_{\alpha\beta} - u_\alpha u_\beta = \gamma_{\alpha\beta} - n_\alpha n_\beta, \tag{3-82}$$

if we take a look at perturbed expressions for $\boldsymbol{u}$, (3-35) and (3-36), and perturbed equations for $\boldsymbol{n}$, (3-66) and (3-65)

$$u_\alpha u_\beta \text{ and } n_\alpha n_\beta \sim (\text{Background}) + (\text{Second order terms}), \tag{3-83}$$

as a consequence $h_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ are the same up to first order, then the foliation for both formalism perturbing up to first order over a flat FLRW solution is the same, this implies that $\boldsymbol{n}$ and $\boldsymbol{u}$ are co-lineal because both are perpendicular to the same foliation, but the are not the same, this implies that $W = 1$. Another way to check this is looking at (1-34) and using perturbed expressions for $\boldsymbol{u}$ and $\boldsymbol{n}$ directly in (1-34), then

$$n_\alpha u_\beta \sim (\text{Background}) + (\text{Second order terms}), \tag{3-84}$$

therefore $W = 1$ up to first order. Now let us calculate the energy-momentum tensor for the perfect fluid case

$$E^{(0)} = \rho_{(0)}, \qquad E^{(1)} = \rho_{(1)}, \tag{3-85}$$

$$p^{(0)}_i = 0, \qquad p^{(1)}_i = a\left(\rho_{(0)} + p_{(0)}\right)(v_i + \omega_i), \tag{3-86}$$

$$S^{(0)}_{ij} = a^2 p_{(0)}\delta_{ij}, \qquad S^{(1)}_{ij} = a^2\left\{p_{(0)}\left(\chi_{ij} - 2\phi\delta_{ij}\right) + p_{(1)}\delta_{ij}\right\}, \tag{3-87}$$

taking into account that in this case $W = 1$, then matches with the perfect fluid quantities obtained in expressions (1-39), (1-40) and (1-41).
Now that we have the perturbed 3+1 expressions, we are able to write the Einstein's field equations perturbed in the 3+1 formalism. Something that we can immediately write are

the background field equations using all the background quantities shown in this section and the field equations (1-18), (1-19), (1-20) and (1-21). In the case of (1-18) and (1-21) the background contribution is trivial[2], in the case of (1-20) we obtain the Friedman equation (3-8) and for (1-19) the expression (3-7). Let us see what we obtain at first order, in the case of (1-21) we obtain

$$\mathcal{H}\partial_i\psi + \partial_i\phi' + \frac{1}{4}\partial_j\left(\chi_i^j\right)' - \frac{1}{4}\partial^j\chi_{ij} - \partial^j\partial_{(j}\omega_{i)} + \delta^{kl}\partial_i\partial_{(k}\omega_{l)} = -4\pi\rho_{(0)}a^2\delta_{ij}v^j, \tag{3-88}$$

in the case of the Newtonian gauge the expression coincides with (3-43). In the case of the hamiltonian constrain (1-20) at first order

$$\nabla^2\phi + \frac{1}{4}\delta^{ij}\partial_k\partial_{(i}\chi_{j)}^k - 3\mathcal{H}\left[\phi' + \mathcal{H}\psi + \delta^{ij}\partial_{(i}\omega_{j)}\right] = 4\pi a^2\rho^{(1)}, \tag{3-89}$$

in the case of the Newtonian gauge the expression coincides with (3-42). The evolution equation (1-19) is given by

$$\phi'' + \mathcal{H}\left(3\phi' + \psi\right) + \mathcal{H}^2\left[\phi' + 2\left(\phi - \psi\right) + a\psi\right] + \nabla^2\left(\psi - \phi\right) = 4\pi\rho_{(0)}\left[a^3\left(\psi - \phi\right) + 2a\phi + a^2\psi\right]$$

and (1-18) at first order due to the geometrical terms is trivial [103].

3.3. 3+1 Maxwell equations perturbed at first order

Here we will apply apply the perturbations of the electromagnetic field and obtain the 3+1 Maxwell equations at first order

$$B^i = \frac{1}{a^2}\left(B^i_{(0)} + B^i_{(1)}\right), \qquad E^i = \frac{1}{a^2}\left(E^i_{(0)} + E^i_{(1)}\right), \tag{3-90}$$

$$B_i = a^2\left(B_i^{(0)} + B_i^{(1)}\right), \qquad E_i = a^2\left(E_i^{(0)} + E_i^{(1)}\right). \tag{3-91}$$

For the 4-current

$$j^\mu = \frac{1}{a}\left(j^\mu_{(0)} + j^\mu_{(1)}\right) \text{ where } j^\mu = \left(\rho_e, J^i\right). \tag{3-92}$$

We will calculate the background and first order contribution from (1-67), then we need to calculate the covariant derivative, which is function of Γ^i_{ik} appears, using (3-74)

$$\Gamma^i_{ik} = -3\partial_k\phi. \tag{3-93}$$

For (1-67) the background contribution is

$$\partial_i B^i_{(0)} = 0, \tag{3-94}$$

[2] Trivial here means that the obtain expression is $0 = 0$

and the first order contribution is given by

$$\partial_i B^i_{(1)} - 3B^i_{(0)}\partial_i\phi = 0. \tag{3-95}$$

Before writing (1-68) up to first order, we have to calculate first the following quantities

$$\mathcal{L}_{\boldsymbol{\beta}} B^i = \omega^k a^{-2}\partial_k B^i_{(0)} - a^{-2}B^k_{(0)}\partial_k\omega^i, \tag{3-96}$$

$$\partial_\eta B^i = -2a^{-2}\mathcal{H}\left(B^i_{(0)} + B^i_{(1)}\right) + a^{-2}\partial_\eta\left(B^i_{(0)} + B^i_{(1)}\right), \tag{3-97}$$

$$\alpha K B^i = a^{-1}B^i_{(0)}\left[3\mathcal{H}a^{-1}\left(\psi - 1\right) + a^{-1}\left(3\phi' + \delta\partial_{(k}\omega_{j)}\right)\right] - 3\mathcal{H}a^{-2}\left(B^i_{(1)} + \psi B^i_{(0)}\right), \tag{3-98}$$

$$\alpha E_k = a^3\left(E_i^{(0)} + E_i^{(1)} + \psi E_i^{(0)}\right), \tag{3-99}$$

$$\epsilon^{\ ij}_{0k}\Gamma^k_{jl} = \epsilon^{\ ij}_{0k}\left(\delta_{jl}\delta^{km}\partial_m\phi - \frac{1}{2}\delta^{km}\partial_m\chi_{jl}\right). \tag{3-100}$$

For (1-68) the background contribution is

$$\left(B^i_{(0)}\right)' + \mathcal{H}B^i_{(0)} + a^5\epsilon^{ijk}\partial_j E^{(0)}_k = 0, \tag{3-101}$$

and the first order contribution is given by

$$\begin{aligned}\left(B^i_{(1)}\right)' + \mathcal{H}B^i_{(1)} - B^i_{(0)}\left(3\phi' + \delta^{kj}\partial_{(j}\omega_{k)}\right) + B^k_{(0)}\partial_k\omega^i - \omega^k\partial_k B^i_{(0)} \\ + a^5\epsilon^{ijk}\left[\partial_j E^k_{(1)} + E^k_{(0)}\partial_j\psi + E^l_{(0)}\left(\delta_{jl}\delta^{km}\partial_m\phi - \frac{1}{2}\delta^{km}\partial_m\chi_{jl}\right)\right] = 0.\end{aligned} \tag{3-102}$$

For (1-71), following a similar procedure to obtain (3-94) and (3-95), the background contribution is given by

$$\partial_i E^i_{(0)} = a4\pi\rho_e^{(0)}, \tag{3-103}$$

and the first order contribution is given by

$$\partial_i E^i_{(1)} - 3E^j_{(0)}\partial_i\phi = 0. \tag{3-104}$$

Finally, following a similar procedure to obtain (3-101) and (3-102) considering also the 4-current, the background contribution is given by

$$\left(E^i_{(0)}\right)' + \mathcal{H}E^i_{(0)} + a^5\epsilon^{ijk}\partial_j B^{(0)}_k = -4\pi a^2 J^i_{(0)}, \tag{3-105}$$

and the first order contribution is given by

$$\begin{aligned}\left(E^i_{(1)}\right)' + \mathcal{H}E^i_{(1)} - E^i_{(0)}\left(3\phi' + \delta^{kj}\partial_{(j}\omega_{k)}\right) + E^k_{(0)}\partial_k\omega^i - \omega^k\partial_k E^i_{(0)} \\ - a^5\epsilon^{ijk}\left[\partial_j B^k_{(1)} + \psi\partial_j B^k_{(0)} - B^l_{(0)}\left(\delta_{jl}\delta^{km}\partial_m\phi - \frac{1}{2}\delta^{km}\partial_m\chi_{jl}\right)\right] = -4\pi a^2\left(J^i_{(1)} + \psi J^i_{(0)}\right).\end{aligned} \tag{3-106}$$

3.4. 1+3 Maxwell equations perturbed at first order

Here we will apply the perturbations of the electromagnetic field and obtain the 1+3 Maxwell equations at first order, the field $\boldsymbol{e}$ and $\boldsymbol{b}$ perturbed at first order are given by

$$b^i = a^{-2}\left(b^i_{(0)} + b^i_{(1)}\right), \qquad e^i = a^{-2}\left(e^i_{(0)} + e^i_{(1)}\right), \tag{3-107}$$

$$b_i = a^2\left(b_i^{(0)} + b_i^{(1)}\right), \qquad e_i = a^2\left(e_i^{(0)} + e_i^{(1)}\right). \tag{3-108}$$

For the 4-current

$$j^\mu = a^{-1}\left(j^\mu_{(0)} + j^\mu_{(1)}\right) \text{ where } j^\mu = \left(\rho_u, J^i_u\right) \tag{3-109}$$

and for the 4-velocity of the fluid

$$u^\mu = a^{-1}\left(1-\psi, v^i_{(1)}\right), \qquad u_\mu = a\left(-1-\psi, \omega_i^{(1)} + v_i^{(1)}\right). \tag{3-110}$$

First we calculate the vorticity perturbed at first order, from the general decomposition of $\nabla_\beta u_\alpha$

$$\omega_{ij} = a\left(\partial_{[i}\omega^{(1)}_{j]} + \partial_{[i}v^{(1)}_{j]}\right), \tag{3-111}$$

$$\sigma_{ij} = a\left(\partial_{(i}\omega^{(1)}_{j)} + \partial_{(i}v^{(1)}_{j)}\right) - \frac{a^2}{3}\left[\left(-\mathcal{H}\psi^{(1)} - \phi^{(1)}\right)\delta_{ij} + \mathcal{H}\left(-2\phi^{(1)}\delta_{ij} + \chi^{(1)}_{ij}\right)\right], \tag{3-112}$$

$$\Theta = 3a^{-1}\mathcal{H} + a^{-1}\left(\partial_j v^j - 3\phi\right). \tag{3-113}$$

We must calculate also σ_i^j and ω_i^j

$$\omega_i^j = a^{-1}\delta^{jk}\left(\partial_{[k}\omega^{(1)}_{i]} + \partial_{[k}v^{(1)}_{i]}\right), \tag{3-114}$$

$$\sigma_i^j = a^{-1}\delta^{jk}\left(\partial_{(k}\omega^{(1)}_{i)} + \partial_{(k}v^{(1)}_{i)}\right) - \frac{1}{3}\left[\left(-\mathcal{H}\psi^{(1)} - \phi^{(1)}\right)\delta_i^j + \mathcal{H}\left(-2\phi^{(1)}\delta_i^j + \chi_i^{j(1)}\right)\right], \tag{3-115}$$

now we are able to obtain the Maxwell equations perturbed at first order.
For (2-46), the background contribution is

$$\partial_j b^j_{(0)} = 0, \tag{3-116}$$

and the first order contribution is

$$\partial_j b^j_{(1)} + b^k_{(0)}\left[-3\partial_k\phi - \partial_k\psi - 2\mathcal{H}\omega_k + \frac{1}{2}\partial_j\chi^j_k - \frac{1}{2}\delta^j_l\partial_l\chi_{jk}\right] + b^j_{(0)}\left[a'\left(\omega_j + v_j\right) + a\left(\omega_j + v_j\right)'\right] = 0. \tag{3-117}$$

For (2-51), the background contribution is

$$\left(b^i_{(0)}\right)' + \mathcal{H}b^i_{(0)} + a^5\epsilon^{ijk}\partial_j e_k^{(0)} = 0, \tag{3-118}$$

and the first order contribution is

$$\left(b^i_{(1)}\right)' + \mathcal{H}b^i_{(1)} + b^i_{(0)}\left[\frac{2}{3}\left(\partial_k v^k - 3\phi\right) - \psi\mathcal{H}\right] + v^j\partial_j b^i_{(0)}$$
$$= b^j_{(0)}\left\{a^{-1}\delta^{ik}\left(\partial_k\omega_j + \partial_k v_j\right) - \frac{1}{3}\left[\mathcal{H}\left(2\phi\delta^i_j - \chi^i_j\right) - \left(\mathcal{H}\psi + \phi\right)\delta^i_j\right] - \frac{1}{2}\left(\chi^i_j\right)' + \left(\phi'\right)\delta^i_j\right\}$$
$$- a^5\epsilon^{ijk}\left[\partial_k e^{(1)}_j + e^l_{(0)}\left(\partial_j\phi\delta_{kl} - \mathcal{H}\omega_j\delta_{kl} - \frac{1}{2}\partial_j\chi_{kl}\right)\right]. \quad (3\text{-}119)$$

For (2-58), the background contribution is

$$\partial_j e^j_{(0)} = a4\pi\rho^{(0)}_u, \quad (3\text{-}120)$$

and the first order contribution is

$$\partial_j e^j_{(1)} + e^k_{(0)}\left[-3\partial_k\phi - \partial_k\psi - 2\mathcal{H}\omega_k + \frac{1}{2}\partial_j\chi^j_k - \frac{1}{2}\delta^j_l\partial_l\chi_{jk}\right]$$
$$+ e^j_{(0)}\left[a'\left(\omega_j + v_j\right) + a\left(\omega_j + v_j\right)'\right] = a4\pi\rho^{(1)}_u. \quad (3\text{-}121)$$

For (2-59), the background contribution is

$$\left(e^i_{(0)}\right)' + \mathcal{H}e^i_{(0)} + a^5\epsilon^{ijk}\partial_j b^{(0)}_k = -a^2 4\pi J^i_{u(0)}, \quad (3\text{-}122)$$

and the first order contribution is

$$\left(e^i_{(1)}\right)' + \mathcal{H}e^i_{(1)} + e^i_{(0)}\left[\frac{2}{3}\left(\partial_k v^k - 3\phi\right) - \psi\mathcal{H}\right] - \psi a^5\epsilon^{ijk}\partial_k b^j_{(0)} + v^j\partial_j e^i_{(0)}$$
$$= e^j_{(0)}\left\{a^{-1}\delta^{ik}\left(\partial_k\omega_j + \partial_k v_j\right) - \frac{1}{3}\left[\mathcal{H}\left(2\phi\delta^i_j - \chi^i_j\right) - \left(\mathcal{H}\psi + \phi\right)\delta^i_j\right] - \frac{1}{2}\left(\chi^i_j\right)' + \left(\phi'\right)\delta^i_j\right\}$$
$$- a^5\epsilon^{ijk}\left[\partial_k b^{(1)}_j + b^l_{(0)}\left(\partial_j\phi\delta_{kl} - \mathcal{H}\omega_j\delta_{kl} - \frac{1}{2}\partial_j\chi_{kl}\right)\right] - a^2 4\pi\left(\psi J^i_{u(0)} + J^i_{u(1)}\right). \quad (3\text{-}123)$$

In the case of the Omh's law the background equation is given by

$$J^i_{u(0)} = \sigma e^i_{(0)}, \quad (3\text{-}124)$$

and the first order contribution

$$J^i_{u(1)} + \left(J^j_{u(1)}\chi^i_j - 2\phi J^i_{u(1)}\right) - \rho^{(0)}_u v_i = \sigma\left[-2e^i_{(0)}\left(\phi - \frac{\psi}{2}\right) + e^j_{(0)}\chi^i_j + e^i_{(1)} + \epsilon^{ijk}\left(\omega_j + v_j\right)b^{(0)}_k\right]. \quad (3\text{-}125)$$

3.5. Chapter conclusions

In this chapter the cosmological perturbations were introduced fixing the background solution with a spatially flat FLRW solution, where the background behavior is shown. Together with appendix C, it was shown how it is possible to perturb the background FLRW solution and obtain the Einstein field equations perturbed at first order, taking the Newtonian gauge are shown the complete expressions for the potential and perturbed quantities. Then it is shown how the $3+1$ quantities are expressed in terms of the background and perturbations quantities shown together with the $3+1$ Einstein field perturbed equations and the perturbed Maxwell equations in $3+1$ and $1+3$ formalism.

4. Cosmic dynamo equation

In this chapter we obtain the cosmic dynamo equation perturbed at first order, but first we give a review on basics of dynamo theory. Because the approach of the dynamo equation is given by the observer who goes along with the lines of the fluid, Lagrangian observer, then the dynamo equation is first obtain in the 1+3 formalism and then, with the electromagnetic field equivalence of the 3+1 and 1+3 formalism, it is possible to obtain the dynamo equation.

4.1. Dynamo theory and mean-field MHD

The dynamo theory takes care of the way that magnetic fields are generated and maintained in different systems of interest, this is equally valid for highly conducting fluids, metal liquids or ionized gas, all under rotation effects and convective movement. A system that can maintain its own magnetic field through self movements in electrically conducting fluids its called an hydromagnetic dynamo. In what follows we will review the basic aspects of dynamo theory under magnetohydrodynamics (MHD).

Let us consider a conductive fluid, in the fluid is possible to measure the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively. The relation of electric currents $\boldsymbol{J}$ and the electric field in the local reference frame is given by the Omh's law

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \tag{4-1}$$

where σ is the electric conductivity. The fluid can be accelerated , so in this local frame the fluid is not inertial, then it is necessary to reformulate the Omh's law in terms of fields measure in an inertial field. Let us consider a medium with velocity $\boldsymbol{u}$ such that this is a non-relativistic velocity, this means that $|\boldsymbol{u}| \ll 1$. Transforming the electric field

$$\boldsymbol{E} \rightarrow \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}, \tag{4-2}$$

where the right hand side represents the inertial fields. In the case of high conductivity there is no electric field, this can be seen taking the limit $\sigma \rightarrow \infty$ in the equation (4-1), then

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}. \tag{4-3}$$

In the case of the magnetic field, the transformation is given by

$$\boldsymbol{B} \rightarrow \boldsymbol{B} + \boldsymbol{u} \times \boldsymbol{E}. \tag{4-4}$$

As a consequence of (4-3) and the fact that $\boldsymbol{u}$ is non-relativistic, the transformation of the magnetic field is given by

$$\boldsymbol{B} \to \boldsymbol{B}. \tag{4-5}$$

From the Maxwell equations

$$\rho_e \sim \frac{u}{c^2} J, \tag{4-6}$$

where ρ_e is the charge density. Transforming the current $\boldsymbol{J}$

$$\boldsymbol{J} \to \boldsymbol{J} + \rho_e \boldsymbol{u}, \tag{4-7}$$

together with (4-6)

$$\boldsymbol{J} \to \boldsymbol{J}. \tag{4-8}$$

From this, the Ohm's law can be written as follows

$$\boldsymbol{J} = \sigma \left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \right). \tag{4-9}$$

Using the Maxwell equations it is possible to obtain the induction equations [33, 110].

$$\partial_t \boldsymbol{B} = \nabla \times \left(\boldsymbol{u} \times \boldsymbol{B} \right) + \left(4\pi\sigma \right)^{-1} \nabla^2 \boldsymbol{B}. \tag{4-10}$$

To obtain the induction equation it is necessary to take into account that. If we take into account the Lorentz force

$$\boldsymbol{F} = \frac{1}{4\pi} \left[\left(\boldsymbol{B} \cdot \nabla \right) \boldsymbol{B} - \frac{1}{2} \nabla \left| \boldsymbol{B} \right|^2 \right]. \tag{4-11}$$

The first part of the right hand side of Lorentz force equation is the magnetic tension, while the second part is the magnetic pressure.
Now we will describe in brief the mean field MHD applied to the large scale dynamo [98], here we must consider mean fields and also small fluctuations over the mean fields, then we can split the magnetic field $\boldsymbol{B}$ as a mean contribution $\langle \boldsymbol{B} \rangle$ and a deviation from this mean $\boldsymbol{B}_d$, this averaging $\langle \cdots \rangle$ must follow the Reynolds averaging rules. Let us see how this averaging process works, if F and G are fluctuating fields then

$$F = \langle F \rangle + F_d, \qquad G = \langle G \rangle + G_d. \tag{4-12}$$

The Reynolds averaging rules are given by [68, 97]

$$\langle \langle F \rangle \rangle = \langle F \rangle, \qquad \langle F_d \rangle = 0, \tag{4-13}$$

$$\langle F + G \rangle = \langle F \rangle + \langle G \rangle, \qquad \langle \langle F \rangle \langle G \rangle \rangle = \langle F \rangle \langle G \rangle, \tag{4-14}$$

$$\langle FG \rangle = \langle F \rangle \langle G \rangle + \langle F_d G_d \rangle, \qquad \langle \langle F \rangle G_d \rangle = 0 \tag{4-15}$$

$$\langle \partial_t F \rangle = \partial_t \langle F \rangle, \qquad \langle \partial_x F \rangle = \partial_x \langle F \rangle. \tag{4-16}$$

Then, the magnetic field $\boldsymbol{B}$ and the fluid velocity $\boldsymbol{u}$ are written as

$$\boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{B}_d, \tag{4-17}$$

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}_d. \tag{4-18}$$

Let us substitute expressions (4-17) and (4-18) into the induction equation (4-10), we can separate the induction equation into the following expressions

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = (\nabla \times \langle \boldsymbol{u} \rangle \times \langle \boldsymbol{B} \rangle) + (\nabla \times \boldsymbol{\mathcal{E}}) + (4\pi\sigma)^{-1} \langle \boldsymbol{B} \rangle , \tag{4-19}$$

$$\frac{\partial \boldsymbol{B}_d}{\partial t} = (\nabla \times \boldsymbol{u}_d \times \langle \boldsymbol{B} \rangle) + (\nabla \times \boldsymbol{u}_p \times \boldsymbol{B}_d) + (\nabla \times \langle \boldsymbol{u} \rangle \times \boldsymbol{B}_d) - (\nabla \times \boldsymbol{\mathcal{E}}) + (4\pi\sigma)^{-1} \boldsymbol{B}_d, \tag{4-20}$$

where $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u}_p \times \boldsymbol{B}_p \rangle$ is the electromotive force caused by the fluctuating motions. This leads to dynamos with turbulent motions and hence turbulent magnetic fields, therefore is known as the theory of the turbulent dynamo.

4.2. Electromagnetic field equivalence between 3+1 and 1+3 formalism

In the case of 3+1 formalism we have Eulerian observers with 4-velocity $\boldsymbol{n}$ perpendicular to the hypersurface, if we took the case of 1+3 formalism we have Lagrangian observers with 4-velocity $\boldsymbol{u}$, which is also the velocity of the fluid, then these observers move along with the fluid. The electromagnetic field measure for both observer is not necessarily the same, then we must know how to express the Lagrangian observed fields in terms of the Eulerian fields, we remark here that the physics describe in both cases is the same, even there is an equivalence. The Faraday tensor for an observer moving with the fluid is

$$F_{\mu\nu} = u_\mu e_\nu - e_\mu u_\nu + \epsilon_{\delta\gamma\mu\nu} u^\delta b^\gamma, \tag{4-21}$$

then the fields are given by

$$e^\mu = F^{\nu\mu} u_\nu \ , \quad b^\mu = {}^*F^{\mu\nu} u_\nu \tag{4-22}$$

where $e^\mu u_\mu = b^\mu u_\mu = 0$. On the other hand for the Eulerian observers

$$F_{\mu\nu} = n_\mu E_\nu - E_\mu n_\nu + \epsilon_{\delta\gamma\mu\nu} n^\delta B^\gamma \tag{4-23}$$

and

$$E^\mu = F^{\nu\mu} n_\nu \ , \quad B^\mu = {}^*F^{\mu\nu} n_\nu. \tag{4-24}$$

From (4-22) and (4-23)

$$e^\mu = W E^\nu - (E^\nu u_\nu)\, n^\mu + \epsilon^{\delta\gamma\mu\nu} B_\gamma n_\delta u_\nu, \tag{4-25}$$

$$b^\mu = W B^\mu + (B^\nu u_\nu)\, n^\mu + \epsilon^{\delta\gamma\mu\nu} E_\gamma n_\delta u_\nu. \tag{4-26}$$

Now we must obtain the 3+1 decomposition from these fields, projecting along the normal vector and the hypersurfaces

$$e^{\mu}n_{\mu} = E^{\nu}u_{\nu}, \qquad \gamma_{\mu\nu}e^{\nu} = WE_{\mu} + \epsilon_{\mu}^{\delta\gamma\alpha}B_{\gamma}n_{\delta}u_{\alpha}, \tag{4-27}$$

$$b^{\mu}n_{\mu} = -B^{\nu}u_{\nu}, \qquad \gamma_{\mu\nu}b^{\nu} = WB_{\mu} + \epsilon_{\mu}^{\delta\gamma\alpha}E_{\gamma}n_{\delta}u_{\alpha}. \tag{4-28}$$

Under the induced coordinate system over the hypersurfaces

$$e^{\mu}n_{\mu} = E^{j}u_{j}, \qquad e_{i} = WE_{i} + \epsilon_{i}^{jk}B_{j}u_{k}, \tag{4-29}$$

$$b^{\mu}n_{\mu} = -B^{j}u_{j}, \qquad b_{i} = WB_{i} + \epsilon_{i}^{jk}E_{j}u_{k}. \tag{4-30}$$

In chapter 2 we saw that the decomposition of the 4-current for a lagrangian observer is given by $\boldsymbol{j} = \rho_u \boldsymbol{u} + \boldsymbol{J}_u$, projecting along the normal vector and the hypersurface

$$\rho = -W\rho_u + J_u^{\mu}n_{\mu}, \qquad J_{\mu} = \rho_u\left(\gamma_{\mu\nu}u^{\nu}\right) + \gamma_{\mu\nu}J_u^{\nu}. \tag{4-31}$$

4.3. Dynamo equation at first order

Here we will obtain the dynamo equation at first order, the steps that are going to be followed here are given in [83] where the full dynamo equation for 1+3 formalism is shown, but in this reference is not clear enough how several expressions were obtained, the steps to obtain this expressions are shown in appendix D, therefore the dynamo equation obtained in this section can be obtained also with the general dynamo equation (D-26), here we apply the same steps shown in appendix D but taking the particular case where a FLRW flat solution is perturbed up to first order. Let us obtain the dynamo equations, let us start from the Maxwell equations (3-118) and (3-122), if we apply isotropy and homogeneity conditions into the fields, which means that $\partial_i b_j^{(0)} = \partial_i e_j^{(0)} = 0$ and $b_i^{(0)} = b^{(0)}$, the equation is given by

$$\left(b_{(0)}\right)' + \mathcal{H}b_{(0)} = 0, \tag{4-32}$$

therefore $b_{(0)} \propto a^{-1}$ in the background as is shown in figure **4-1**. Usually the equation (4-32) is written with a $2\mathcal{H}b_{(0)}$, this guaranties that the magnetic field decays as a^{-2} [55, 57], for this a frame choice is made and the field is written as $\boldsymbol{b} = (0, ab^i)$, then replacing this field in Faraday's equation (1-68) then $b^i \propto a^{-2}$ [37, 106, 107], therefore in this work the frame is choosen in such a way that the field decay as a^{-1}. Now let us obtain the first order dynamo equation, before obtaining the curl of the curl of $b^i_{(1)}$, following [83], let us write the curl of $b^i_{(1)}$, using the Maxwell equations, as follow

$$\begin{aligned} a^5\epsilon^{ijk}\partial_j b_k^{(1)} = {} & \left(e^i_{(1)}\right)' + \mathcal{H}e^i_{(1)} - e^i_{(0)}\left[\phi'\delta^i_j - \frac{1}{2}\left(\chi^i_j\right)' + \partial^i\omega_j + \partial^i v_j + \frac{a}{3}\left(\mathcal{H}\psi - \phi + 2\mathcal{H}\phi\right)\delta^i_j - \frac{a}{3}\chi^i_j\right] \\ & - a^5\epsilon^{ijk}\left[-b_l^{(0)}\left(-2\delta^l_{(j}\partial_{k)}\phi + \delta_{jk}\partial^l\phi - \mathcal{H}\omega^l\delta_{jk} + \partial_{(j}\chi^l_{k)} - \frac{1}{2}\partial^l\chi_{jk}\right) - \left(\omega_j + v_j - \partial_j\psi\right)\right] \\ & + 4\pi a^2\left(\psi J^i_{u(0)} + J^i_{u(1)}\right), \end{aligned} \tag{4-33}$$

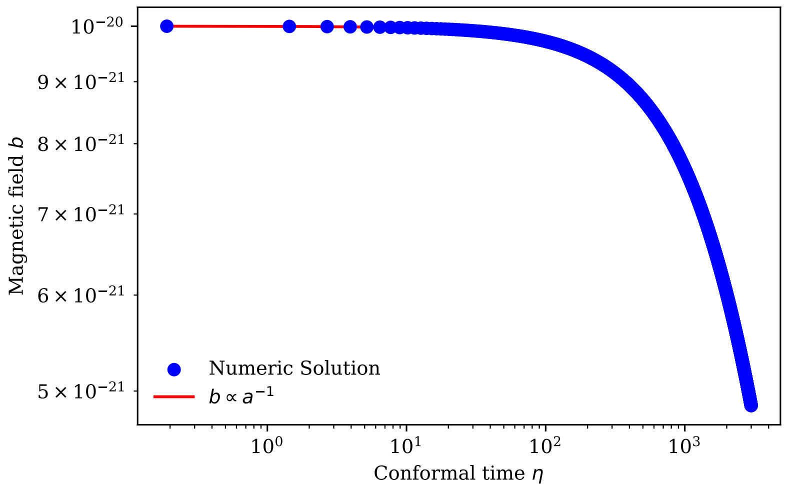

Figure 4-1.: This figure shows the decay of the magnetic field in the background, it shows the numerical evolution of (4-32) and the proportionality between the background magnetic field $b_{(0)}$ and the scale factor a.

let us define the following tensors

$$P^i_j = \phi' \delta^i_j - \frac{1}{2}\left(\chi^i_j\right)' + \partial^i \omega_j + \partial^i v_j + \frac{a}{3}\left(\mathcal{H}\psi - \phi + 2\mathcal{H}\phi\right)\delta^i_j - \frac{a}{3}\chi^i_j,$$
$$F^l_{jk} = -2\delta^l_{(j}\partial_{k)}\phi + \delta_{jk}\partial^l\phi - \mathcal{H}\omega^l\delta_{jk} + \partial_{(j}\chi^l_{k)} - \frac{1}{2}\partial^l\chi_{jk}, \tag{4-34}$$

this will help to reduce our calculations. Now let us obtain the curl of the curl of $b^i_{(1)}$

$$a^5\epsilon^{im}_{\ \ l}\epsilon^{ljk}\partial_m\partial_j b^{(1)}_k = \left[\epsilon^{im}_{\ \ l}\partial_m e^l_{(1)}\right]' + \mathcal{H}\epsilon^{im}_{\ \ l}\partial_m e^l_{(1)} - \epsilon^{im}_{\ \ l}\partial_m\left(e^j_{(0)}P^l_j\right)$$
$$- a^5\epsilon^{im}_{\ \ l}\epsilon^{ljk}\partial_m\left[-b^{(0)}_l F^l_{jk} - \left(\omega_j + v_j - \partial_j\psi\right)e^{(0)}_k\right] + 4\pi a^2\epsilon^{im}_{\ \ l}\partial_m\left(\psi J^i_{u(0)} + J^i_{u(1)}\right), \tag{4-35}$$

after some calculations using Maxwell equations and the Ohm's law then the dynamo equation is given by

$$-a^5\partial^j\partial_j b^i_{(0)} = 4a^{-5}\mathcal{H}\left[\left(b^i_{(1)}\right)' + \mathcal{H}b^i_{(1)}\right] - a^{-5}\left[\left(b^i_{(1)}\right)'' + \mathcal{H}'b^i_{(1)} + \mathcal{H}\left(b^i_{(1)}\right)' + b^j_{(0)}P^i_j\right] - \epsilon^{ij}_{\ \ k}\partial_j\left(e^l_{(0)}P^k_j\right)$$
$$- a^5\left\{\partial^k e^{(0)}_k\left(\omega^i + v^i - \partial^i\psi\right) + e^{(0)}_k\partial^k\left(\omega^i + v^i - \partial^i\psi\right) - \partial^j\left[e^i\left(\omega^i + v^i - \partial^i\psi\right)\right]\right\}$$
$$+ \epsilon^{ijk}\mathcal{H}\left[e^{(0)}_l F^l_{jk} + e^{(0)}_k\left(\omega_j + v_j - \partial_j\psi\right) + F^l_{jk}\left(e^{(0)}_l\right)' + \left(\omega_j + v_j - \partial_j\psi\right)\left(e^{(0)}_k\right)'\right].$$
$$+ 4\pi a^2\left\{\rho^{(0)}v^i + \sigma\left[e^i_{(1)} + \epsilon^{ijk}\left(\omega_j + v_j\right)b^{(0)}_k\right]\right\} \tag{4-36}$$

Now let us take into account the average, here the values in the background must respect isotropy and homogeneity, in this case we supposed that the magnetic field $b^i_{(0)}$ is homogeneous and sufficiently random such that[1] $\left\langle b^i_{(0)}\right\rangle = 0$ and $\left\langle b_i^{(0)} b^i_{(0)}\right\rangle \neq 0$ [16], this implies that we have that the following terms are non-zero; $b^2_{(0)}$, $e^2_{(0)}$ and $e^i_{(0)}$. Because part of the cosmological history the universe is conductor, at big scales, because is composed of charged particles coupled by interactions, we must consider the Ohm's law. Under these assumptions

$$\lim_{\sigma\to\infty} \frac{J^i_{u(0)}}{\sigma} = \lim_{\sigma\to\infty} e^i_{(0)} = 0. \tag{4-37}$$

This does not implies that the current is zero, but if the current is not zero the charges separate breaking homogeneity in the background therefore $J^i_{u(0)} = 0$, also from the Gauss law $\rho_u^{(0)} = 0$. Finally the only non zero term in the background is $b^2_{(0)}$ and

$$b^i_{(0)} = e^2_{(0)} = e^i_{(0)} = J^i_{u(0)} = \rho_u^{(0)} = 0. \tag{4-38}$$

Under conditions (4-38) the Ohm law is written in the following way

$$J^i_{u(1)} = \sigma \left[e^i_{(1)} + \epsilon^{ijk} \left(\omega_j + v_j\right) b_k^{(0)}\right]. \tag{4-39}$$

The Faraday and the Ampère expressions at first order in 1+3 formalism, (3-119) and (3-123) respectively, rearranging indices take the form

$$\left(b^i_{(1)}\right)' + \mathcal{H} b^i_{(1)} + \epsilon^{ij}{}_k \partial_j e^j_{(1)} = 0, \tag{4-40}$$

$$\left(e^i_{(1)}\right)' + \mathcal{H} e^i_{(1)} + \epsilon^{ij}{}_k \partial_j b_k^{(1)} = -a^2 4\pi J^i_{u(1)}. \tag{4-41}$$

Now let us obtain the curl of the curl of $b^i_{(1)}$ using (4-41) and (4-39)

$$\epsilon^{im}{}_l \epsilon^{ljk} \partial_m \partial_j b_k^{(1)} = -\epsilon^{im}{}_l \partial_m \left(e^l_{(0)}\right)' - \epsilon^{im}{}_l \mathcal{H} \partial_m e^l_{(1)} - 4\pi a^2 \epsilon^{im}{}_l \partial_m J^l_{u(1)}, \tag{4-42}$$

using (4-40) to obtain $\epsilon^{ijk}\partial_k e_j^{(1)}$, the dynamo equation is given by [55]

$$\begin{aligned}\left(b^i_{(1)}\right)'' = \partial_j \partial^j b^i_{(1)} - \left[2\mathcal{H} + 4\pi a^2 \sigma\right] \left(b^i_{(1)}\right)' - \left[\mathcal{H}' + \mathcal{H}^2 + 4\pi a^2 \sigma\right] b^i_{(1)} \\ + 4\pi a^2 \sigma \partial_j \left[b^j_{(0)} \left(\omega^i + v^i\right) - b^i_{(0)} \left(\omega^j + v^j\right)\right],\end{aligned} \tag{4-43}$$

this equation will be studied from the numerical point of view in chapter 5. Now we want to obtain the dynamo equation in the 3+1 formalism, for this we will use the equivalence equations (4-29) and (4-30), because the electromagnetic fields are only on the hypersuface, they do not have normal contribution, therefore we restrict to the spatial contribution of the fields. The equivalence perturbed up to first order and under the conditions (4-38) are

$$e^i_{(1)} = E^i_{(1)} + a\epsilon^{ij}{}_k B^k_{(0)} \left(\omega_j + v_j\right), \tag{4-44}$$

$$b^i_{(1)} = B^i_{(1)} + a\epsilon^{ij}{}_k E^k_{(0)} \left(\omega_j + v_j\right). \tag{4-45}$$

[1] The $\langle \cdots \rangle$ is the expected value of the magnetic field

Replacing these expressions in (4-43) we obtain the dynamo equation at first order in 3+1 formalism

$$\left(B^i_{(1)}\right)'' + \epsilon^i{}_{jk}\left[aE^j_{(0)}\left(v^k+\omega^k\right)\right]'' = \partial_j\partial^j\left[B^i_{(1)} + \epsilon^{ij}{}_k aE^k_{(0)}\left(v_j+\omega_j\right)\right] - \left(2\mathcal{H}+4\pi a^2\sigma\right)\left[B^i_{(1)} + \epsilon^{ij}{}_k aE^k_{(0)}\left(v_j+\omega_j\right)\right]' - \left(\mathcal{H}'+\mathcal{H}^2+4\pi a^2\sigma\right)\left[B^i_{(1)} + \epsilon^{ij}{}_k aE^k_{(0)}\left(v_j+\omega_j\right)\right] + 4\pi a^2\sigma\partial_j\left[b^j_{(0)}\left(\omega^i+v^i\right) - b^i_{(0)}\left(\omega^i+v^j\right)\right]. \quad (4\text{-}46)$$

There are some things that should be point out, one of them is that these dynamo equations do not have a gauge fixed, the information about how the perturbations affect the equation are in v_i, then given velocity field v_i, it is possible to see how this velocity field affects the magnetic field. This is what is called the kinetic dynamo [98, 33], it is possible to apply this type of regime because Lorentz force terms, that can be seen in (4-11) are second order contributions to the field, then there is no need to solve the non-linear dynamo.

4.4. Chapter conclusions

The dynamo theory and mean-field approximation of MHD helped us to obtain the dynamo equation through the induction equation. Before obtaining the dynamo equation, the equivalence of $3+1$ and $1+3$ formalism was obtained, this equivalence plays the same role of the reference frames transformations given in section 4.1. Finally, using the background and perturbed Maxwell equations with the Reynolds averaging rules, it is possible to obtain the dynamo equation.

5. Computational implementation

Here we will show the computational results together with the implementation of the software `Einstein Toolkit`. Also, we will review some hyperbolic partial differential equations and BSSN formalism, this because BSSN formalism is the one used by `Einstein Toolkit`, and part of its numerical success is supported by the hyperbolic partial differential equations theory.

5.1. Hyperbolic partial differential equations

The 3+1 evolution equations is known in mathematics as a set of hyperbolic PDEs, but the problem of these equations is that do not behave well in the numerical simulations, this can be seen from the mathematical properties of PDEs that we will show here and after that we can see how this could be applied to the 3+1 evolution equations. A system of hyperbolic PDEs at first order can be written in general as follow

$$\partial_t \boldsymbol{u} + \sum_i \mathbb{M}^i \cdot \partial_i \boldsymbol{u} = \boldsymbol{S}\left(\boldsymbol{u}\right), \tag{5-1}$$

where $\boldsymbol{u}$ is an n dimensional column vector known as solution vector, $\mathbb{M}^i$ are $n \times n$ matrices known as the velocity matrices, in this case with constant components, and $\boldsymbol{S}$ is an n dimensional columns vector known as the source vector. Another way to write this system, applying the sum convention, is as follow

$$\partial_t \boldsymbol{u} + \partial_i F^i\left(\boldsymbol{u}\right) = \boldsymbol{S}\left(\boldsymbol{u}\right), \tag{5-2}$$

where

$$M^i_{ab} = \frac{\partial F^i_a}{\partial u_b}, \tag{5-3}$$

here we will stick to the first case. Let us consider $\boldsymbol{S} = 0$, then

$$\partial_t \boldsymbol{u} + \mathbb{M}^i \cdot \partial_i \boldsymbol{u} = 0. \tag{5-4}$$

The system of equations (5-4) is said to be well-posed if it is possible to define a norm $\|\cdot\|$ such that

$$\left\|\boldsymbol{u}\left(t, x^i\right)\right\| \leq k_1 e^{k_2 t} \left\|\boldsymbol{u}\left(0, x^i\right)\right\|, \tag{5-5}$$

where k_1 and k_2 are constants independent of the initial conditions $\boldsymbol{u}\left(0, x^i\right)$[1].
Let $\boldsymbol{n}$ and unitary arbitrary vector and let $\mathbb{P}$ be defined as follow

$$\mathbb{P} = \mathbb{M}^i n_i, \tag{5-6}$$

this matrix is known as the principal symbol. The hyperbolocity is defined as follows, the system is called

- Symmetric hyperbolic if $\mathbb{P}$ can be symmetrized in a way independent of $\boldsymbol{n}$,
- Strongly hyperbolic if, for all unit vectors $\boldsymbol{n}$, $\mathbb{P}$ has a set of real eigenvalues and a complete set of eigenvectors,
- and Weakly hyperbolic if $\mathbb{P}$ has real eigenvalues but not a complete set of eigenvectors.

Let us take the eigenvalues and eigenvectors for $\mathbb{P}$

$$\mathbb{P}\boldsymbol{e}_a = \lambda_a \boldsymbol{e}_a, \tag{5-7}$$

where $\{\boldsymbol{e}_a\}$ is the set of eigenvectors and λ_a are the eigenvalues. Let us define the matrix $\mathbb{E}$ such that the columns of the matrix are the eigenvectors $\boldsymbol{e}_a$

$$\mathbb{E} = \left(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\right). \tag{5-8}$$

From this a new matrix $\mathbb{H}$, called the symmetrizator, is defined as follow

$$\mathbb{H} = \left(\mathbb{E}^{-1}\right)^T \mathbb{E}^{-1}, \tag{5-9}$$

which is hermetic and defined positive. If $\mathbb{H}$ is independent of $\boldsymbol{n}$, then the system is symmetric hyperbolic. From $\mathbb{H}$ it is possible to define an inner product and a norm as follow

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \boldsymbol{u}^\dagger \mathbb{H} \boldsymbol{v}, \tag{5-10}$$

$$\|\boldsymbol{u}\|^2 := \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \boldsymbol{u}^\dagger \mathbb{H} \boldsymbol{u}. \tag{5-11}$$

This norm will be the norm that is used in the well-posed system, this because the system is well-posed if and only if the system of PDEs is strongly hyperbolic [54]. If we want to apply this to the evolution equations in the 3+1 formalism (1-18) and (1-19) there is a problem, these equations have second order terms, but its is possible to rewrite this system as a first order hyperbolic PDE. This allows to show that the 3+1 evolution system is weakly hyperbolic [66], therefore the idea is to rewrite the 3+1 formalism such that the evolution PDEs are strongly hyperbolic.

[1] If we consider $\boldsymbol{S} \neq 0$ we can also apply this definition of well-posedness, however it must be take into account that this is valid only if $\boldsymbol{S}$ is linear, then $\boldsymbol{S} = S\boldsymbol{u}$ where S is constant in space and time.

5.2. BSSN formalism

Here we will show in brief one of several formalisms used to evolve Einstein's equation, this formalism is called BSSN (Baumgarte-Shapiro-Shibata-Nakamura) developed by Baumgarte and Shapiro [17] and Shibata and Nakamura [104]. Let us take the spatial metric $\boldsymbol{\gamma}$ and decomposed into a conformal metric $\bar{\boldsymbol{\gamma}}$ and a conformal factor as follow

$$\gamma_{ij} = e^{4\phi}\bar{\gamma}_{ij}. \tag{5-12}$$

The extrinsic curvature tensor is decompose into its trace and traceless part

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \tag{5-13}$$

where A_{ij} is traceless. Because the conformal factor

$$\widetilde{A}_{ij} = e^{-4\phi}A_{ij}, \tag{5-14}$$

then

$$K_{ij} = e^{4\phi}\widetilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{5-15}$$

Under $\bar{\boldsymbol{\gamma}}$ we have a covariante derivative $\bar{\boldsymbol{D}}$ such that $\bar{\boldsymbol{D}}\bar{\boldsymbol{\gamma}} = 0$, this allow to write the Hamiltonian and momentum constrains as follow, respectively

$$\bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\widetilde{A}_{ij}\widetilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}E = 0, \tag{5-16}$$

$$\bar{D}_j\left(e^{6\phi}\widetilde{A}_{ij}\right) - \frac{2}{3}e^{6\phi}\bar{D}^i K - 8\pi e^{6\phi}p^i = 0. \tag{5-17}$$

The evolution equation for $\boldsymbol{\gamma}$ is splitted in two equations, an evolution equation for ϕ and another one for $\bar{\boldsymbol{\gamma}}$

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^l\partial_l\phi + \frac{1}{6}\partial_l\beta^l, \tag{5-18}$$

$$\partial_t\bar{\gamma}_{ij} = -2\alpha\widetilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k. \tag{5-19}$$

Similar to the case of $\boldsymbol{\gamma}$, in the case of $\boldsymbol{K}$the evolution equations are for K and $\widetilde{A}_{ij}$

$$\partial_t K = -\gamma^{ij}D_jD_i\alpha + \alpha\left(\widetilde{A}_{ij}\widetilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha\left(E + S\right) + \beta^k\partial_k K, \tag{5-20}$$

$$\partial_t\widetilde{A}_{ij} = e^{-4\phi}\left(-(D_iD_j\alpha)^{\mathrm{TF}} + \alpha(R_{ij}^{\mathrm{TF}} - 8\pi S_{ij}^{\mathrm{TF}})\right) + \alpha\left(K\widetilde{A}_{ij} - 2\widetilde{A}_{il}\widetilde{A}^l_j\right) + \beta^k\partial_k\widetilde{A}_{ij} + \widetilde{A}_{ik}\partial_j\beta^k + \widetilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\widetilde{A}_{ij}\partial_k\beta^k, \tag{5-21}$$

where the TF means Trace-Free. On the other hand, $\bar{\boldsymbol{\gamma}}$ allows to write the Ricci tensor as follow

$$R_{ij} = \bar{R}_{ij} + R^{\phi}_{ij}, \tag{5-22}$$

where $\bar{R}_{ij}$ is given by $\bar{\boldsymbol{\gamma}}$ for R_{ij} and R^{ϕ}_{ij} by ϕ, these terms are written explicitly as follow

$$R^{\phi}_{ij} = -2\left(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi\right) + 4\left((\bar{D}_i\phi)(\bar{D}_j\phi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l\phi)(\bar{D}_m\phi)\right), \tag{5-23}$$

$$\bar{R}_{ij} = -\frac{1}{2}\gamma^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}\left(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj}\right), \tag{5-24}$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols given by $\bar{\boldsymbol{\gamma}}$ and $\bar{\Gamma}^i = \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\gamma^{ij}$. The term $\bar{\Gamma}^i$ is taken as an independent function, then we have a new constrain equation

$$\bar{\Gamma}^i + \partial_j\gamma^{ij} = 0 \tag{5-25}$$

and an evolution equation for $\bar{\Gamma}^i$

$$\partial_t\bar{\Gamma}^i = -2\widetilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\widetilde{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}p_j + 6\widetilde{A}^{ij}\partial_j\phi\right) + \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j + \bar{\gamma}^{lj}\partial_j\partial_l\beta^i. \tag{5-26}$$

The constrains equations (5-16), (5-17) and (5-25) together with (5-18), (5-19), (5-20), (5-21) and (5-26) form the BSSN equations. The advantage of these set of evolution equations are strongly hyperbolic PDEs [100].

5.3. Einstein Toolkit

In chapter 1 we presented the Einstein's field equations written in a hyperbolic form, this allows that the equations can evolve in a computational way, which is known as the Einstein's equations in the 3+1 formalism. The main problem of this set of equations is that is weakly hyperbolic, therefore these equations are rewritten in what is known as the BSSN formalism. The main problem now is how to evolve the equations in a computational way, it must be taken into account that these equations are computationally demanding [14]. This leads to the development of code made to evolve efficiently the Eintein's field equations. There are several options available for free download used for the evolution of the equations, [71] mentions a few useful codes in section four, `GRChombo` [31] and `Dendro-GR` [92, 43] are other example of this kind of codes.
To evolve the Einstein equations we used the free open-source code `Einstein Toolkit` (`ET`) [74, 122, 15]. The `ET` perform high-level numerical-relativity simulations, even while operating within the constraints of desktop-level computational power, some examples of its success are simulations on binary black hole merger [96, 120], binary neutron stars merger [80, 81] among others. `ET` uses a set of core modules that provide the infrastructure to build complex simulations codes, this is done using the `Cactus` framework [46] with a central core known as the “flesh”, this provides the interaction between modules to be able to compile, the modules are called “thorns”. Several thorn are used in each simulation on the `ET`, here we

will mentions just a few used in this thesis given their importance. Here we will not focus on installation and use of `ET`, for this we recommend the `ET` tutorial for new users[2], the videos on ET youtube channel[3] and [30]. Let us mention some of the most relevant thorns used in this Thesis: `McLachlan` [6] are a set of thorns used to evolve the spacetime using the BSSN formalism, `GRHydro` [87] is used to evolve MHD quantities in flat and curve spaces, to implement the evolution `GRHydro` uses the thorn `MoL` [11] which implements the method of lines to evolve partial differential equations. For analyzing the output the typical thorns used are `CarpetIOASCII` and `CarpetIOScalar`, which writes `.asc` or `.xg` files of the values of specified variables, and also `CarpetIOHDF5` which writes outputs in `.h5` files, these torns belongs to the `Carpet` arrangement, `Carpet` [101, 102] is a mesh refinement driver for `Cactus` framework capable to run with multiple grid patches.

There are several works which applied numerical relativity to cosmology using cosmological perturbation and evolving the field equations using `ET` [21, 119]. In this case we will focus our attention in a particular thorn called `FLRWSolver` [77], this thorn is written in `fortran90` which sets initial conditions for cosmological perturbations at first order using a FLRW flat background using the Newtonian gauge[4]. The cosmological perturbation can be turn-off, then it could be used for the evolution of the background field equations for the FLRW solution.

Here we will briefly show how to run a simulation using `ET` assuming that is already installed, first we need to initialized it with `SimFactory` [12]. and for this it is necessary to run the following command in the Carpet directory, which is created during the installation

```
./simfactory/bin/sim setup-silent
```

After running this command it is possible to star running our simulations. To run it we need first a parameter file, these files end with `.par` and a few examples can be found in the par directory inside `/Cactus` directory, these files have all the necessary parameters to run a simulation in `ET`. We also need a name for the simulation, let us assume that the parameter file that we need to use is called `parameters.par` and it is located on the `/par` directory, the name that we want to give to our simulation is `MySimulation`, then to run the simulation the following command have to be written on the terminal

```
./simfactory/bin/sim create MySimulation --configuration sim
                                        --parfile=par/parameters.par
```

the command `create` creates the directory where the results of the simulation will be stored, this directory is created in the `/simulations` directory, `/simulations` is created in the same directory where `/Cactus` is, also after `--parfile` the location of the parameter file must be specified. To be able to finally execute our simulation we need to run the following command

[2]This tutorial can be found on https://einsteintoolkit.org/documentation/new-user-tutorial.html

[3]ET youtube channel https://www.youtube.com/channel/UC8IObWZ7_wEbWnbIKVIQRYQ/featured

[4]We will work only under this gauge, but this does not mean that other gauge can be taken, for example in [119] tensor perturbation are included to the study gravitational waves

```
./simfactory/bin/sim MySimulation --cores=2 --num-threads=1
                                              --walltime=0:20:00
```

here another options appears, `cores` is related to the amount of cores of the computer, `--num-threads` is the number of threads per process to use and `walltime` is the limit amount of time for the implementation of our simulation. These two lines can be written as one without problems in the following way

```
./simfactory/bin/sim create-sumit MySimulation --parfile=par/parameters.par
      --cores=2 --num-threads=1 --walltime=0:20:00
```

this is possible because the command `create-sumit`. This commands will not show the status of the simulation, to be able to see itwe need to rnu the command

```
./simfactory/bin/sim list-simulations MySimulation
```

if the simulation ended running we will see `[ACTIVE (FINISHED)`, or `[ACTIVE (RUNNING)` if it is still running. Using only `list-simulations` will show the status of all the simulations in the local machine, but does not follow the simulation, for this it is possible to use the following command

```
./simfactory/bin/sim show-output --follow MySimulation
```

and if we do not want to keep following the simulation we must interrupt the kernel, this will not interrupt our simulation[5].
To analyze the files obtained in the simulation several tools can be used, for `.dat` files `pyhton` can be used, the difficulties are in the `.hdf5` files, these can be read using the `h5py` library, but even with this library the handling of these kind of files could turn out very difficult. To be able to handle with all the different types of files that produces `ET` are tools made specially for this purpose, one of them is the library `SimulationTools` [9] made for `Mathematica`, the one that are going to use is called `kuibit` [24], a free `pyhton` library for `ET` post-processing.
Before analysing the simulations that will be presented, we will show how to include `FLRWSolver`, this because this thorn is not (yet) included with the default thorns that came with `ET`[6]. First it must be downloaded from https://github.com/hayleyjm/FLRWSolver_public, after cloning the repository it must be placed in `/Cactus/arragements/EinsteinInitialData` and change directory's name to `FLRWSolver`, then we need to include `FLRWSolver` to the list of thorns that are going to be compiled, this list is in the file `einsteintoolkit.th` used for the ET installation, what should be added is the line `EinsteinInitialData/FLRWSolver` at the end of the list coresponding to the `EinsteinInitialData` thorns. Finally we must rebuild `ET` with the following command

[5]The list of all `SimFactory` commands that can be used can be found on http://simfactory.org/info/documentation/userguide/commands.html

[6]Here we will show how to install a version of `FLRWSolver` which only includes the necessary files to solve the FLRW spacetime with no perturbations and with a sinlge mode linear perturbation, currently there is a version of `FLRWSolver` which includes gaussian random linear perturbations but this did not worked for us.

```
./simfactory/bin/sim build -j2 --thornlist ../einsteintoolkit.th
```

Sometimes `Meudon_BH` or `Lorene` thorns does not allow to build `FLRWSolver`, therefore these must be comented in the `einsteintoolkit.th` file if necessary. Inside the `FLRWSolver` is a directory called `/par`, here are some parameter files to check if everything went the right way.
Before showing some results obtained with `FLRWSolver` some comments, first we need to call `FLRWSolver` in order to use it, solve the FLRW background equations we must specify that we do not want perturbations in our simulation, this is done in the parameter file that we are going to use, this is done by witting

```
ActiveThorns = "FLRWSolver"
FLRWSolver::FLRW_perturb = "no"
```

In case that we want to perform a simulation including single-mode perturbations

```
ActiveThorns = "FLRWSolver"
FLRWSolver::FLRW_perturb                 = "single_mode"
FLRWSolver::FLRW_perturb_direction       = "all"
FLRWSolver::single_perturb_wavelength    = 1.0
FLRWSolver::phi_perturb_amplitude        = 1.e-6
```

As can be seen, the second line of the above code includes the single-mode perturbations, `FLRW_perturb_direction` indicates the spatial direction of the perturbation, could be `x`, `y`, `z` or `all`. The parameter `single_perturb_wavelength` is the wavelength of the scalar perturbation mode and `phi_perturb_amplitude` is the amplitude of the perturbation.

5.4. Results

First we will show the evolution of FLRW spacetime without perturbation and then with perturbations, it is possible to compare the numerical and the analytical results using the expressions obtained in section 3.1. In this case we ran a simulation over a 40^3 grid in a 1 Gpc size box, the simulation starts from a redshift $z = 1100$, which corresponds to CMB photon decoupling [88, 78], the final redshift is $z \approx 534$, which corresponds to $\eta \approx 3000$, we decide to evolve up to this point because beyond $\eta \approx 10^4$ the difference between solutions and numerical results start to differ significantly [77]. For both cases, background and perturbations, the equation of state is given by (3-13). According to the thorn documentation of `EOS_Omni` [4], thorn used to provided the equation of state for the `ET` simulations, it is possible to implement the polytropic equation

$$p = K_{\text{poly}}\rho^{\gamma} \tag{5-27}$$

where we fix values to K_{poly} and γ. Looking at (3-13) then $K_{\text{poly}} = 0$ and $\gamma = 1$, but in `ET` is not possible to assign zero to K_{poly}, therefore we assign a value close to zero, in this case $K_{\text{poly}} = 10^{-4}$, lowest values lead to `NaN` values in the simulation. Using this values for simulations in ET with FLRWSolver will result problemati, because assigning $\gamma = 1$ lead to NaN values, just like in Kpoly case, then the value assign is $\gamma = 2$, therefore the equation of state for the simulations is the polytropic equation

$$p = K_{\mathrm{poly}}\rho^2. \tag{5-28}$$

The form to include this values in the parameter file is as follows:

```
ActiveThorns = "EOS_Omni"
EOS_Omni::poly_k       = 1.e-4
EOS_Omni::poly_gamma   = 2.0
```

Now let us see the backgroung evolution results. In figure **3-1** it is shown the evolution for a/a_0 and ρ/ρ_0, as can be seen analytical results match with the numerical results, the maximum relative error for a was 0,005 % and for ρ was 0,016 %, the minimum relative error was 0,004 % and 0,011 % respectively [77, 79]. After the evolution of a and ρ is obtained, we proceed to calculate the Hubble parameter with the numerical values of the scale factor a obtained with ET, because $\mathcal{H} = a'/a$ then it is possible to use finite center-differences [70] and then compare with expression (3-8). This comparison is shown in figure **3-2**, just like in the case of a and ρ there is a good match between the numeric solution and the analytic solution, the maximum relative error in this case was 0,008 % and the minimum was 0,006 %. If we look now at the perturbed EFE, we must compare the quantities δ and v^i, the solutions for these quantities are given by (3-54), (3-55) and (3-56). The initial conditions are set up by FLRWSolver [77] with the following spatial function

$$\Phi = \Phi_0 \sum_{i=1}^{3} \sin\left(\frac{2\pi}{L}x_i\right), \tag{5-29}$$

which is a solution for the perturbed EFE, with the solutions (3-55) and (3-56) for δ and v^i, respectively, then

$$\delta = \left[C_1\left(\frac{2\pi}{L}\right)^2 - 2\right]\Phi_0 \sum_{i=1}^{3} \sin\left(\frac{2\pi}{L}x_i\right), \tag{5-30}$$

$$v^i = \frac{2\pi}{L}C_3\Phi_0\cos\left(\frac{2\pi}{L}x_i\right). \tag{5-31}$$

To compare this solutions with the numerical results given by ET we used the background solutions, using (3-46) it is possible to obtain δ. In the case of the velocity ET gives the projected four velocity, which is given by (3-38), taking $\omega^i = 0$, it is possible to obtain the velocity (3-37) in terms of (3-38). The comparison between the numerical and the analytical solutions are given by figures **5-1**, which corresponds to the δ case solution, and **5-2** for the velocity solution case, for the simulations it was always used a wavelength of value 1. In figures **5-1** and **5-2** we show two types of figures, the first one is fixing a point and then see how this point evolves in time, the other type of figure is fixing a space slice, in this case

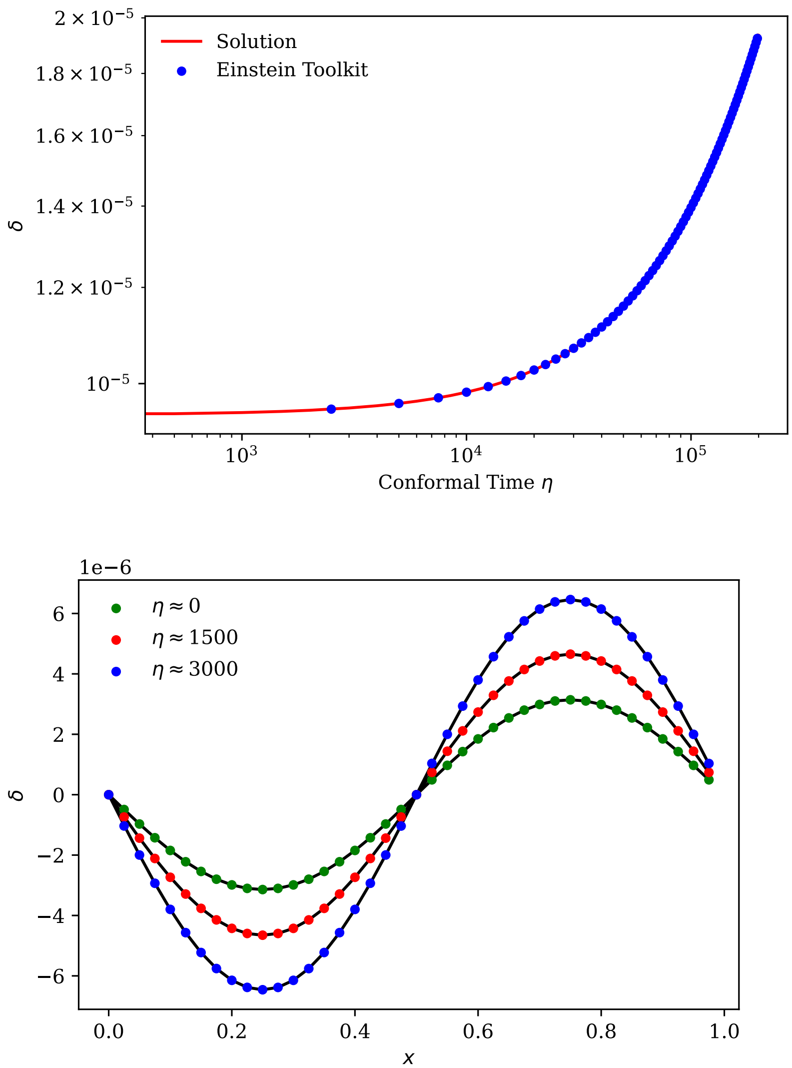

Figure 5-1.: Evolution for δ with respect the conformal time and how evolves along the x coordinate in different times, in both cases analytic and numeric solutions are presented, the points represent the numeric solution.

along x axis and see how numerical solutions evolve in time. For both cases the numerical solution matches with the analytic solutions (3-55) and (3-56), the maximum relative error for the velocity was 0,026 %, in the case of δ the maximum relative error is 0,007 %. As we can see in figure **5-1**, the modes of δ increase, beside the close numerical error, this is a expected behaviour for δ.

Let us now check how linear was the performed simulation, for this we have to check how the difference between the potential ϕ and ψ evolve along the simulation, to obtain this quantities in terms of $3+1$ quantities we used equations (3-60) and (3-62), the result of the difference between the potential is given in figure **5-3**. The linearity is given in the initial

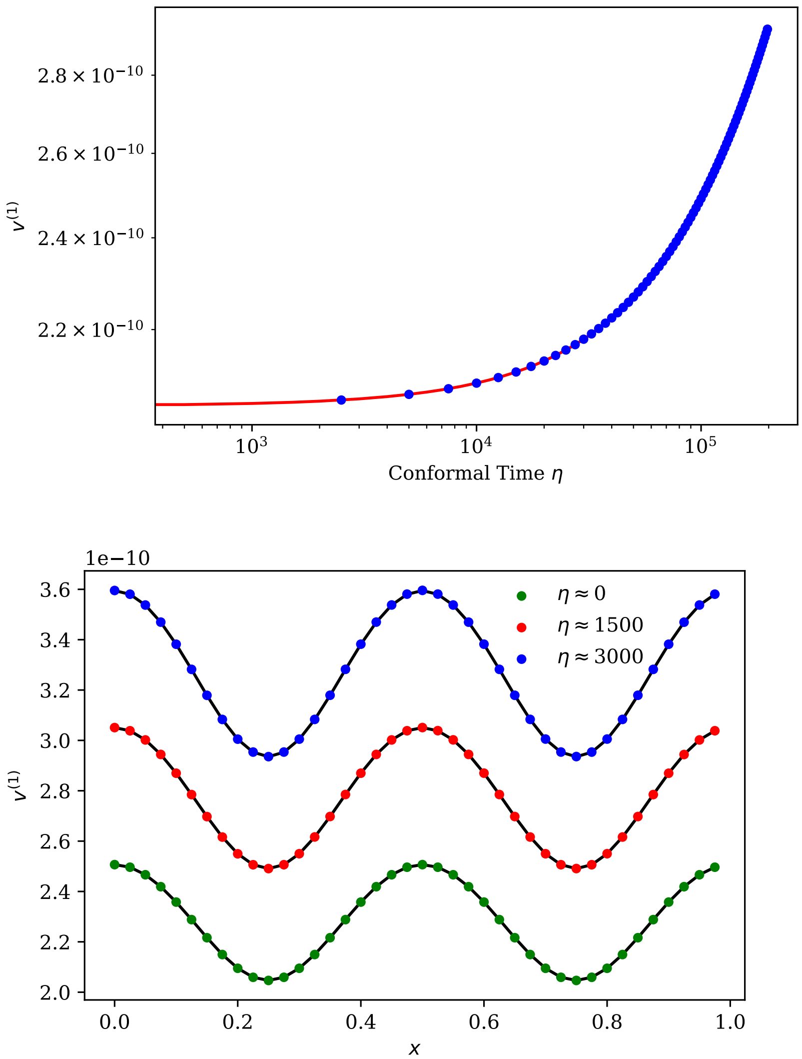

Figure 5-2.: Evolution for δ with respect the conformal time and how evolves along the x coordinate in different times, in both cases analytic and numeric solutions are presented, the points represent the numeric solution.

conditions, where the difference between the potentials is zero, along the simulation, due to numerical dissipation, the difference between both potential increases, according to [77] this difference will lead to a maximum of $6{,}5 \times 10^{-6}$, according to figure **5-3** this difference is much bigger than the one reported, in the final time reached in the simulation the difference was almost $|\phi - \psi| \approx 0{,}5$, which differs from the value reported in [77], this means that the deviation from linearity do not start in times around 10^4 like [77], in this case the deviation from linearity started at orders of 10^2 in time.

Now we will evolve equation (4-43) from the numerical point of view, let use the following notation to be brief $b^i \equiv b^i_{(1)}$, we will discretize this equation spatially and temporally, for the temporal evolution the discretization, using the index n for time discretizaton with a

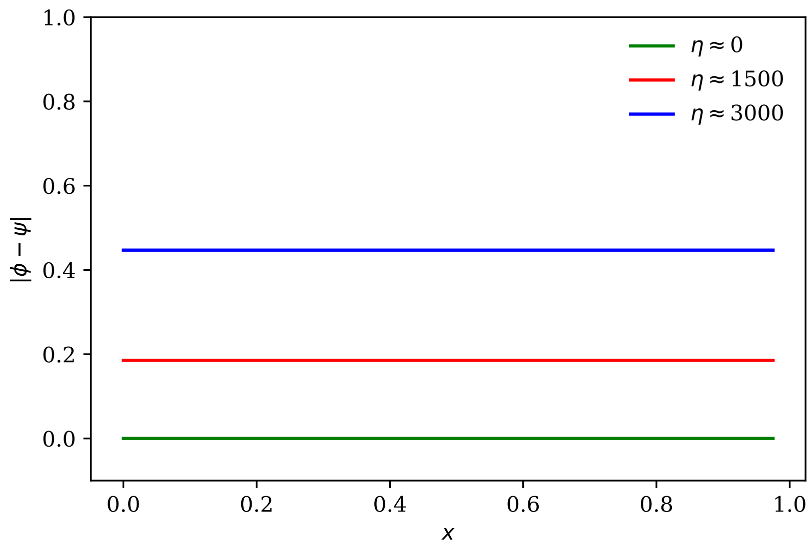

Figure 5-3.: Evolution of the difference between the potentials ϕ and ψ along the simulation, at the beginning of the simulation the linearity is maintained, but during the evolution of the simulation this difference increase almost to 0,5.

time step $\Delta\eta$, is given by [70]

$$\left(b^i\right)' \approx \left(b^i_{n+1} - b^i_n\right)/\Delta\eta + \mathcal{O}\left(\Delta\eta\right), \tag{5-32}$$

$$\left(b^i\right)'' \approx \left(b^i_{n+1} - 2b^i_n - b^i_{n-1}\right)/\left(\Delta\eta\right)^2 + \mathcal{O}\left(\Delta\eta\right)^2, \tag{5-33}$$

replacing this in (4-43), the time evolution discretization is given by

$$b^i_{n+1} \approx \left(1 + \Delta\eta D_1\right)^{-1} \times \left\{b^i_{n-1} + \left(1 - \Delta\eta^2 D_2\right) b^i_n + \Delta\eta^2 \partial^j \partial_j b^i_n + \Delta\eta^2 4\pi a^2 \sigma \left[\partial_j \left(b^j_{(0)} v^i\right)_n - \partial_j \left(b^i_{(0)} v^j\right)_n\right]\right\}, \tag{5-34}$$

where

$$D_1 = 2\mathcal{H}_n + 4\pi \left(a_n\right)^2 \sigma \tag{5-35}$$

$$D_2 = \left(\mathcal{H}'\right)_n + \left(\mathcal{H}_n\right)^2 + 4\pi \left(a_n\right)^2 \sigma \tag{5-36}$$

there is also an spatial derivative that must be solved, in this case finite center-differences is used for first and second derivatives. The discretization was done this way because, before using expression (5-34) the fourth order Runge-Kutta method was used [70], but the numerical evolution after a few iterations did not converge, then the numerical result were to much higger that usual.

To evolve (5-34) first we need to obtain the background quantities, $\mathcal{H}$ and a, these were obtain using `FLRWSolver`, then we add perturbations to the simulation, this give us the

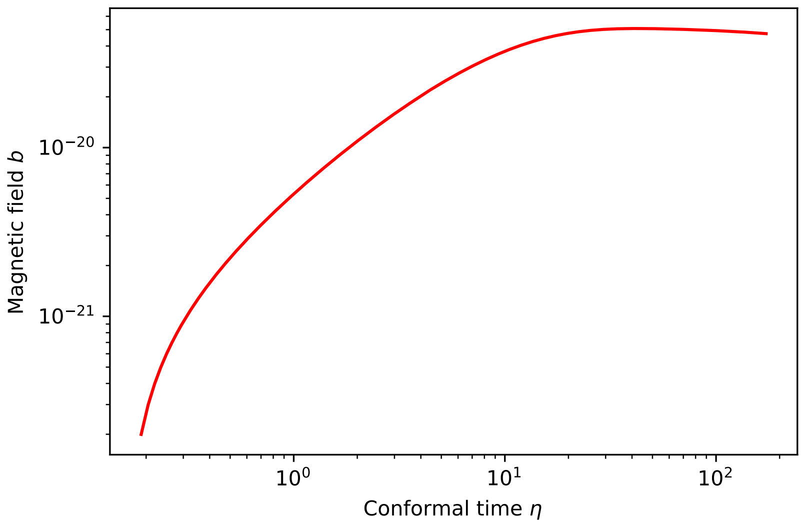

Figure 5-4.: Growth of magnetic field b_i given by the numerical solution of the equation (4-43), the conductivity for this case is $\sigma = 0{,}01$. What it is assumed here is that the average field is constant in the spatial domain.

velocity field that is shown in (4-43) and (5-34). Therefore, with the results obtained with `ET` and `FLRWSolver` it is possible to solve (4-43), then we notice how the magnetic field is affected given a velocity field, this is known as the kinematic dynamo [98, 33], it is possible to evolve the field in this way because Lorentz force terms, as can be seen in (4-11), are given by second order contributions. The results of the evolution using the discretization (5-34) are shown in figure **5-4**, in this case we are assuming that the initial average magnetic field is constant and that the conductivity is $\sigma = 0{,}01$, we set up this value because it was used in [83], the evolution shows is a growing magnetic field, as longer the time in the simulation, the growth of the field start to decrease, we need to take into account that this is only applied to the matter domination era after CMB photon decoupling, then cosmological constant effects and higher order effects are missing here.

6. Conclusions

In this work we presented two broad formalism of GR, each one of these can be really wide by their own. For the case of the $3+1$ formalism, presented in Chapter 1, the spacetime is foliated with spatial hypersurfaces, the observers which its four velocity is the normal vector to this surfaces are called Eulerian observers. The foliation of the spacetime allows to define a spatial projector, that also works as a metric for the hypersurfaces, in such a way that geometric quantities can be decompose in contributions along the normal vector and over the hypersurfaces. The decomposition let write the Einstein field equations into a set of hyperbolic partial differential equations to be evolve numerically. The particular case of the perfect fluid was presented in this formalism and also the Maxwell equations.
In chapter 2 the second mentioned formalism was presented, this is the $1+3$ formalism. From the mathematical point of view, it is just like $3+1$ formalism, because it also implements the geometry of foliations and hypersurfaces. The main difference is that the observer in the $1+3$ formalism, called Lagrangian observer, goes along the lines of fluid, and the four velocity of the fluid is the normal vector to the hypersurfaces in this particular case. This four velocity allows to obtain certain kinematic quantities that describe the cosmological fluid, how this quantities act over the fluid is shown in figure **2-1**. In this formalism it is also possible to decompose the Einstein's field equations, just like $3+1$ formalism, and together with the commutation of covariant derivative, the propagation equations of the mentioned kinematic quantities are obtained together with the Maxwell equations in this formalism.
Now that the two formalism were introduced, then it is possible to perturbed the main quantities for both formalism through cosmological perturbations, this was done in chapter 3. First, the background was fixed, it was used a flat FLRW solution, and then this solution was perturbed up to first order. This enables to show the perturbed $3+1$ Einstein field equations together with perturbed Maxwell equations in both formalism, this includes the Ohm's law.
Before setting up the cosmological dynamo equation, main goal of chapter 4, a review on classical dynamo theory is made starting from pre-Maxwell equations, the induction equation is obtained and then, with the averaging process the turbulent dynamo equation is obtain. Because the approach of the dynamo equation is along the Lagrangian observer, to know how the $3+1$ electromagnetic fields behaves in terms of the $1+3$ fields, then an equivalence between both electromagnetic fields is shown. Finally, the cosmological dynamo equation, with and with out average, is obtained, in the case of the averaged equation it should be keep in mind that this average process follows the average Reynolds rules. It was checked

that the field decays as a^{-1} in the absence of perturbations.
Finally, in chapter 5, it is shown how the software `Einstein Toolkit` (`ET`) was implemented, specially the thorn `FLRWSolver` which sets up initial conditions to the EFE, in such a way that these equations could evolve using the numerical methods already implemented in `ET`. The formalism used to evolve the EFE is the BSSN formalism, which is based on the $3+1$ formalism, what makes the BSSN formalism and optimal way to evolve the equations is that it is possible to write the EFE in a set of strongly hyperbolic partial differential equations, therefore a review on hyperbolic partial differential equations is made together with the forementioned BSSN formalism. After that, a quick introduction on `ET` is made, emphasizing on how to include `FLRWSolver`, this because till the date, `FLRWSolver` is freely available but not included on the `ET`. Then `FLRWSolver` was used to solve numerically the Friedmann equations, which correspond to the background solution, and also the perturbed Einstein field equations for a sinusoidal form of the perturbations in the Newtonian gauge. For the background case the maximum relative error found, comparing to the analytical solution, for the scale factor a of 0,004 % and ρ of 0,016 %, in the case of the perturbations some of the figures shown in [77] where reproduced using the functions δ and the velocity v^i, in this case the maximum relative error for the velocity was 0,026 %, in the case of δ the maximum relative error was 0,007 %, the difference between the potentials along the simulations was almost 0,5. This implementation using `ET` with the low values in the relative error shows that `ET` is capable of maintain the linearity of the perturbations δ and v^i up to times of order $\approx 10^3$. The linearity is measure by the gravitational slip $|\Phi - \Psi|$ and according to [77] the linearity is maintained up to $\approx 10^4$ order, where the evolution of the equations should show a deviation from the linear regime, but in this work that was not the case, at a time of order 10^2 the code start to show deviations, around a time of 1500 the deviation was almost 0,2 and at time order of 10^3 the deviation from linear regime was almost 0,5.
The dynamo equation (4-43) was evolved numerically, for this the background results were obtained with `FLRWSolver` together with the velocity field, where the perturbation information is. The value assigned to the conductivity was $\sigma = 0{,}01$, value used in [83], the discretization used is shown in (5-34) and the results of this discretization are shown in figure **5-4**, then the field is amplified but as long as the time increase, the increase of the field decay, then it tends to a certain value, after that start to decree but really slowly, still the field is higher than the initial field. This lead to future work of solving the dynamo equation in a non linear way, this mean to be able to set up initial conditions in `ET` to see how differ the kinematic dynamo from the non linear dynamo, also work with the dynamo equation in higher orders setting initial conditions for the magnetic field in a similar way than `SONG` does [8]. This is not limited to General Relativity, this also could be done in modified theories of gravity like $f(R)$ theories of gravity where a $3+1$ scheme is well stablish, see [27, 84], making possible a general evolution of the magnetic field.

A. Geometry

This appendix is a brief review of differential geometry needed to study General Relativity, along this appendix some references are given to a deeper study of the subject.

A.1. Differential Manifolds

The mathematical language of General Relativity (GR) is the differential geometry. In this section we are going to give some of the main geometric definitions and properties used in GR. For further details see [108, 89, 52].

Let $\mathcal{M}$ be a topological space, a coordinate chart $C_\alpha = (\varphi_\alpha, U_\alpha)$ over $\mathcal{M}$ is a homeomorphism

$$\varphi_\alpha : U_\alpha \subseteq \mathcal{M} \to \mathbb{R}^n, \tag{A-1}$$

where U_α is an open set over $\mathcal{M}$. We call a C^r-atlas over $\mathcal{M}$ to a chart collection $\{C_\alpha = (\varphi_\alpha, U_\alpha)\}_{\alpha\in I}$ such that

$$\mathcal{M} = \bigcup_{\alpha\in I} U_\alpha, \tag{A-2}$$

and if $U_\alpha \bigcap U_\beta \neq \emptyset$ then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \bigcap U_\beta\right) \subseteq \mathbb{R}^n \to \varphi_\beta \left(U_\alpha \bigcap U_\beta\right) \subseteq \mathbb{R}^n \tag{A-3}$$

is a C^r diffeomorfism. Two C^r atlas over a topological space $\mathcal{M}$ are compatible if the union of the atlas is a new atlas, then, the union of all compatible atlas over a topological space forms an equivalence class atlas, or a maximal altas. Then, a C^r differentiable manifold $\mathcal{M}$ is a Haurdorff topological space with a maximal atlas.

A manifold is orientable if there is an atlas $\{\phi_\alpha, U_\alpha\}_{\alpha\in I}$ such that, in every no-empty intersection $U_\alpha \bigcap U_\beta$ of open sets, the determinant of the matrix $\partial x^i/\partial x'^j$ is greater than zero, where x^i are U_α coordinates and x'^i are U_β coordinates. An atlas is locally finite, if every point in the manifold has an open neighborhood that intercepts only a finite number of neighborhood U_β. A manifold $\mathcal{M}$ is paracompact if for every atlas $\{\phi_\alpha, U_\alpha\}_{\alpha\in I}$, exist a a locally finite atlas $\{\psi_\beta, V_\beta\}_{\beta\in I}$ with each V_β contained in some U_α.

Tangent space , Dual space and Tensors

Tangent space

Let us define the set of all real value functions over a manifold $\mathcal{M}$ as

$$\mathcal{F}(\mathcal{M},\mathbb{R}) := \{f : \mathcal{M} \to \mathbb{R}\}, \tag{A-4}$$

where for $f, g \in \mathcal{F}(\mathcal{M},\mathbb{R})$ we have a vector space structure and if f is differentiable respect to a chart (ϕ_α, U_α), it is also differentiable respect to (ϕ_β, U_β). A tangent vector v_p to the manifold in a point p is a function

$$\begin{array}{rccc} v_p : & \mathcal{F}(\mathcal{M},\mathbb{R}) & \to & \mathbb{R} \\ & f & \mapsto & v_p(f) \end{array}, \tag{A-5}$$

such that v_p is lineal in $\mathbb{R}$ and meets the Leibniz product property. The space of all tangent vectors at the point p, denoted by $T_p\mathcal{M}$, is a vector real space. The partial derivatives, denoted as ∂_α where $i = 1, 2, ..., n$ and n is the manifolds dimension, are tanget vectors in $T_p\mathcal{M}$ and form a base for the tangent space. Then, for $v_p \in T_p\mathcal{M}$ and introducing the Einsten sum convention

$$v_p = \sum_\alpha v_p^\alpha \partial_\alpha \bigg|_p \equiv v_p^\alpha \partial_\alpha\big|_p . \tag{A-6}$$

Dual space

A one-form $\boldsymbol{\omega}$ in the point p is a real function over $T_p\mathcal{M}$

$$\begin{array}{rccc} \boldsymbol{\omega} : & T_p\mathcal{M} & \to & \mathbb{R} \\ & \boldsymbol{v} & \mapsto & \boldsymbol{\omega}(\boldsymbol{v}) \equiv \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \end{array}, \tag{A-7}$$

such that $\boldsymbol{\omega}$ is linear in $\mathbb{R}$. The space of all one-forms, denoted by $T_p^*\mathcal{M}$, is called the dual vector space. Every function $f \in \mathcal{F}(\mathcal{M},\mathbb{R})$ defines a one-form df in p, this one.form is called the differential of f in p. For the local coordinates $\phi_\alpha(p) = (x^1, x^2, ..., x^n)$ we have that the set of differentials

$$\left\{dx^1, dx^2, ..., dx^n\right\} \tag{A-8}$$

in p forms a base for the dual vector space, then

$$\boldsymbol{\omega} = \omega_\alpha dx^\alpha . \tag{A-9}$$

This space meets the condition

$$\langle dx^\alpha, \partial_\beta \rangle = \delta^\alpha_\beta, \tag{A-10}$$

where $\{\partial_\alpha\}$ is a base of $T_p\mathcal{M}$.

Tensors

Let Π_r^s be defined as

$$\Pi_r^s := \left\{ \left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s\right) — \boldsymbol{\eta}^i \in T_p^*\mathcal{M}, \boldsymbol{Y}_j \in T_p\mathcal{M} \right\}, \tag{A-11}$$

then a tensor $\boldsymbol{T}$ of type $\binom{r}{s}$ is a multilinear function over Π_r^s

$$\begin{array}{rccc} \boldsymbol{T}: & \Pi_r^s & \to & \mathbb{R} \\ & \left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s\right) & \mapsto & \boldsymbol{T}\left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s\right). \end{array} \tag{A-12}$$

The space of all tensors, denoted as T_s^r, is called the tensor product

$$T_s^r := \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{r\text{-times}} \otimes \underbrace{T_p^*\mathcal{M} \otimes \cdots \otimes T_p^*\mathcal{M}}_{s\text{-times}}, \tag{A-13}$$

this is a vector real space of dimension $r + s$. Given $\boldsymbol{T} \in T_s^r$ it can be written as

$$\boldsymbol{T} = T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \partial_{\alpha_1} \otimes \cdots \otimes \partial_{\alpha_r} \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_s}, \tag{A-14}$$

where $T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}$ are the components of $\boldsymbol{T}$, we call the index $\alpha_1 \cdots \alpha_r$ contravariant and $\beta_1 \cdots \beta_s$ covariant, then the components of the tensor $\boldsymbol{T}$ related to to the contravariant index are called contravariant components and the ones related to the covariant index are called covariant components. Under a change of coordinates the tensor components, the vector and the one-foms components as well, transform acording to

$$T'^{\gamma_1 \cdots \gamma_r}_{\sigma_1 \cdots \sigma_s} = \frac{\partial x'^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\gamma_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial x'^{\sigma_1}} \cdots \frac{\partial x^{\beta_s}}{\partial x'^{\sigma_s}} T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}. \tag{A-15}$$

Transformation between manifolds

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds. We can define a diffeomorfism between these manifolds

$$\phi : \mathcal{M} \to \mathcal{N}. \tag{A-16}$$

From this, we can induce a new function

$$\begin{array}{rccc} \tilde{\phi}: & \mathcal{F}(\mathcal{M}) & \to & \mathcal{F}(\mathcal{M}) \\ & f & \mapsto & \tilde{\phi}(f) \end{array} \tag{A-17}$$

defined as $\left[\tilde{\phi}(f)\right](p) = f(\phi(p))$, where $p \in \mathcal{M}$. We induce now the function

$$\begin{array}{rccc} \phi_*: & T_p\mathcal{M} & \to & T_{\phi(p)}\mathcal{N} \\ & \boldsymbol{X} & \mapsto & \phi_*(\boldsymbol{X}) \end{array} \tag{A-18}$$

such that

$$\phi_*\left(\boldsymbol{X}(f)\right)\big|_{\phi(p)} := \boldsymbol{X}\left(f \circ \phi\right)\big|_p, \tag{A-19}$$

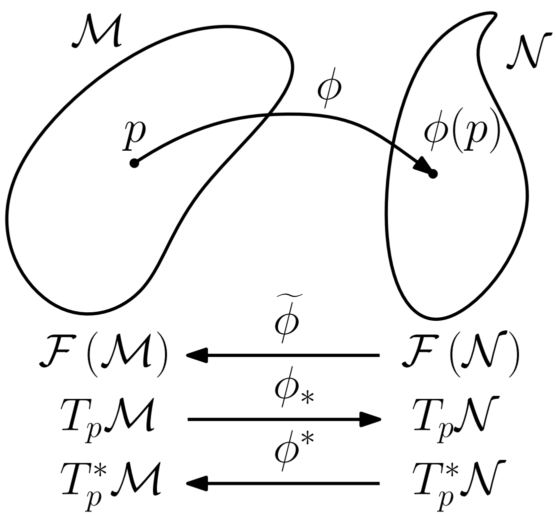

Figure **A-1**.: Diffeomorphism between the manifolds $\mathcal{M}$ and $\mathcal{N}$

the function ϕ_* is known as pushforward. Now, from the definition of ϕ_*, we define the function

$$\begin{array}{rccc} \phi^*: & T^*_{\phi(p)}\mathcal{N} & \to & T^*_p\mathcal{M} \\ & \boldsymbol{\omega} & \mapsto & \phi^*(\boldsymbol{\omega}) \end{array} \tag{A-20}$$

given by

$$\langle \phi^*(\boldsymbol{\omega}), \boldsymbol{X}\rangle = \langle \boldsymbol{\omega}, \phi_*(\boldsymbol{X})\rangle , \tag{A-21}$$

the function ϕ^* is known as pullback. These functions extends to tensors of type $(0, s)$ and $(r, 0)$, then extends to (r, s) tensors, a scheme of the diffeomorphism are in figure **A-1**.

A.2. Curvature

References for this section are [108, 93, 121]

Lie derivative, connection and covariant derivative

Let $\lambda(t)$ be a curve over a manifold $\mathcal{M}$, there exist only one maximal curve $\lambda(t)$ over $\mathcal{M}$ that goes through each $p \in \mathcal{M}$ such that $\lambda(0) = p$ and its tangent vector in the point $\lambda(t)$ is the vector $\boldsymbol{X}|_{\lambda(t)}$. The flux of a vector field $\boldsymbol{X}$ over $\mathcal{M}$ is a transformation

$$\begin{array}{rcl} \phi: \mathcal{M}\times\mathbb{R} & \longrightarrow & \mathcal{M} \\ (p,t) & \longrightarrow & \phi(p,t) := \lambda_t(p) \end{array} \tag{A-22}$$

if we fix t then we define a diffeomorfism that sends a point p in $\mathcal{M}$ to a point $\phi_t(p)$. With this we can define the Lie derivative of a tensor $\boldsymbol{T}$ with respect to a vector field $\boldsymbol{X}$ in the point p is

$$L_{\mathbf{X}}\, \boldsymbol{T}|_p = \operatorname*{Lim}_{t\to 0}\frac{1}{t}\left(\boldsymbol{T}|_p - (\phi_t)_*\, \boldsymbol{T}|_p\right). \tag{A-23}$$

Given the fact that under the change of coordinates de partial derivative is not invariant, we need to generalize this concept over a manifold. This generalization is given by the covariant derivative $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$, where $\boldsymbol{X}, \boldsymbol{Y} \in T_p\mathcal{M}$. Because $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ is a tensor, we can write it using the bases $\{\partial_\alpha\}$ and $\{dx^\alpha\}$, then the components of this tensor, denoted as $\nabla_\beta Y^\alpha$, are

$$\nabla_\beta Y^\alpha = \partial_\beta Y^\alpha + \Gamma^\alpha_{\beta\gamma} Y^\gamma. \tag{A-24}$$

The terms $\Gamma^\alpha_{\beta\gamma}$ are called Christoffel symbols of the second kind, these symbols are given by

$$\Gamma^\alpha_{\beta\gamma} = \left\langle dx^\alpha, \nabla_{\partial_\beta}\partial_\alpha\right\rangle. \tag{A-25}$$

Just like the partial derivative, it is linear and meets the Leibniz product property.
We extend the covariant derivative to arbitrary tensor. If $\boldsymbol{T} \in T^s_r$, then $\nabla\boldsymbol{T} \in T^s_{r+1}$, where the components of $\nabla\boldsymbol{T}$ are

$$\begin{aligned}\nabla_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = \partial_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} + \Gamma^{\alpha_1}_{\gamma\sigma} T^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s} + \cdots + \Gamma^{\alpha_r}_{\gamma\sigma} T^{\sigma\alpha_2\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s}\\ - \Gamma^\sigma_{\gamma\beta_1} T^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s} - \cdots - \Gamma^\sigma_{\gamma\beta_s} T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\sigma\beta_{s-1}},\end{aligned} \tag{A-26}$$

where it is still linear and meets the Leibniz product property. Given that is a tensor this have to transform like in (A-15), from this we can see that the transformation rule for the Christoffel symbols is given by

$$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\sigma}{\partial x'^\rho \partial x'^\gamma} + \frac{\partial x'^\alpha}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\tau}{\partial x'^\gamma}\Gamma^\sigma_{\rho\tau}. \tag{A-27}$$

With the covariant derivative and the Lie derivative defined, given $\boldsymbol{T} \in T^r_s(\mathcal{M})$, we can write the components of $L_{\mathbf{X}}\boldsymbol{T}$ in terms of partial derivatives

$$\begin{aligned}(L_{\mathbf{X}}\boldsymbol{T})^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = X^\sigma\partial_\sigma \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} - \boldsymbol{T}^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s}\partial_\sigma X^{\alpha_1} - \cdots \boldsymbol{T}^{\alpha_1\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s}\partial_\sigma X^{\alpha_r}\\ + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s}\partial_{\beta_1}X^\sigma + \cdots + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_{s-1}\sigma}\partial_{\beta_s}X^\sigma,\end{aligned} \tag{A-28}$$

and in terms of covariant derivatives

$$\begin{aligned}(L_{\mathbf{X}}\boldsymbol{T})^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = X^\sigma\nabla_\sigma \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} - \boldsymbol{T}^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s}\nabla_\sigma X^{\alpha_1} - \cdots \boldsymbol{T}^{\alpha_1\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s}\nabla_\sigma X^{\alpha_r}\\ + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s}\nabla_{\beta_1}X^\sigma + \cdots + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_{s-1}\sigma}\nabla_{\beta_s}X^\sigma.\end{aligned} \tag{A-29}$$

We need to keep in mind that in general relativity we work with a free torsion connection, a consequence of this is that the Christoffel symbols are symmetric in its index, i.e.

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}. \tag{A-30}$$

Parallel transport

Let $\boldsymbol{T}$ a tensor field and $\lambda(t)$ a curve over a manifold $\mathcal{M}$[1], let us define the covariant derivative along the curve as $\nabla_{\partial_t}\boldsymbol{T}$, then if $\boldsymbol{X}$ is the tangent vector to the curve $\lambda(t)$ then the components of the covariant derivative along the curve is $\nabla_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}X^\gamma$. We said that a vector is parallel transport along a curve $\lambda(t)$ if $\nabla_{\partial_t}\boldsymbol{T}=0$. In the particular case of a vector $\boldsymbol{Y}$ we choose a curve $\lambda(t)$ such that we have a coordinate system $x^\alpha(t)$ and $X^\alpha=\frac{dx^\alpha}{dt}$. We said that the curve is a geodesic curve if the tangent vector is parallel transported along the curve, this means that

$$\nabla_{\boldsymbol{X}}\boldsymbol{X}=0. \tag{A-31}$$

For the basis $\{\partial_\alpha\}$ and $\{dx^\alpha\}$ we can write this condition in the following way

$$\frac{d^2x^\alpha}{dt^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}=0. \tag{A-32}$$

Riemann tensor and metric tensor

In a euclidian space we have that we can commute the derivatives without problems, but this does not happens in a curve space. A measure of this non-commutativity is given by the Riemann tensor, the components of this tensor are given by

$$R^\alpha_{\beta\gamma\delta}=\partial_\gamma\Gamma^\alpha_{\delta\beta}-\partial_\delta\Gamma^\alpha_{\gamma\beta}+\Gamma^\alpha_{\gamma\sigma}\Gamma^\sigma_{\delta\beta}+\Gamma^\alpha_{\delta\sigma}\Gamma^\sigma_{\gamma\beta}, \tag{A-33}$$

these components have de following properties

$$R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}=R_{\alpha\beta\delta\gamma}=R_{\gamma\delta\alpha\beta}, \tag{A-34}$$
$$R_{\alpha\beta\gamma\delta}+R_{\alpha\beta\gamma\delta}+R_{\alpha\beta\gamma\delta}=0, \tag{A-35}$$
$$\nabla_\eta R^\alpha_{\beta\gamma\delta}+\nabla_\delta R^\alpha_{\beta\eta\gamma}+\nabla_\gamma R^\alpha_{\beta\eta\delta}=0. \tag{A-36}$$

From a contraction we define the Ricci tensor as

$$R_{\alpha\beta}=R^\sigma_{\alpha\sigma\beta}. \tag{A-37}$$

Over a manifold $\mathcal{M}$ we define the metric tensor as a symmetric tensor field of the type T^0_2. Given the basis $\{dx^\alpha\}$ we have that

$$\boldsymbol{g}=g_{\alpha\beta}dx^\alpha\otimes dx^\beta, \tag{A-38}$$

with the metric tensor we can define a norm, the cosine of an “angle” for two given vectors and the lenght of a path between two points, this allow us to write the distance along a curve of two infinitesimal close points as

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta. \tag{A-39}$$

[1] A curve λ is a function such that $\lambda: I\subset\mathbb{R}\longrightarrow\mathcal{M}$ where $\mathbb{R}$ are the real numbers

We said that the metric is not degenerated if the determinant of the metric is dinstict from zero, $\det|g_{\alpha\beta}| \neq 0$. This condition over the metric allows to define T_0^2 tensor type $g^{\alpha\beta}$ such that

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}. \tag{A-40}$$

From this we can built an isomorphism such that we can relate the covariant and contravariant index components of the tensors, if we take a vector with components X^α then

$$X_\alpha = g_{\alpha\beta}X^\beta \tag{A-41}$$

and using (A-40)

$$X^\alpha = g^{\alpha\beta}X_\beta. \tag{A-42}$$

We can generalize this to tensor components, for example, for a tensor $\boldsymbol{T}$ with components $T^{\alpha\beta\gamma}$

$$T^{\alpha\beta}_{\gamma} = g_{\gamma\sigma}T^{\alpha\beta\gamma} \tag{A-43}$$

$$T^{\alpha}_{\beta\gamma} = g_{\gamma\sigma_1}g_{\beta\sigma_2}T^{\alpha\sigma_2\sigma_1} \tag{A-44}$$

$$T_{\alpha\beta\gamma} = g_{\gamma\sigma_1}g_{\beta\sigma_2}g_{\alpha\sigma_3}T^{\sigma_3\sigma_2\sigma_1} \tag{A-45}$$

in a similar way we do this for $T_{\alpha\beta\gamma}$.
Let us define the signature of of the metric tensor as the number of positive eigenvalues less the number of negative eigenvalues. A particular case is the signature $n-2$, this is the case of a Minkowskian or Lorenzian metric, from here we are going to assume a lorenzian signature. A consequence of this is the values that can take the inner product of a vector defined by the metric tensor, we are going to divide this cases in three. For a vector $\boldsymbol{X}$ we said that this vector is

- Null if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = 0$
- Timelike if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) < 0$
- Spacelike if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) > 0$

To get a relation between the metric tensor components we use variational calculus[2], with this we get the geodesic equation but in terms of the components of the metric tensor, this gives the following relation between the Christoffel symbols and the metric tensor components

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\right). \tag{A-46}$$

With this in mind we can write the equation (A-26) in terms of the metric tensor components, a consequence of this is that

$$\nabla_\gamma g_{\alpha\beta} = 0 \text{ and } \nabla_\gamma g^{\alpha\beta} = 0. \tag{A-47}$$

[2]For details see [93]

Killing vector field

The metric, or the metric tensor components, $g_{\alpha\beta}$ is a form invariant under a transformation from x^α to x'^α if $g'_{\alpha\beta}(x')$ is the same fuction of x'^α as $g_{\alpha\beta}(x)$ is of x^α. We know that $g_{\alpha\beta}$ transform as a tensor, then if the metric is a form invariant

$$g_{\alpha\beta}(x) = \frac{\partial x'^\rho}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta} g_{\rho\sigma}(x'), \tag{A-48}$$

any transformation $x \to x'$ that satisfies (A-48) is called an isometry. Let us consider an infinitesimal coordinate transformation

$$x'^\alpha = x^\alpha + \epsilon\xi^\alpha \text{ with } |\epsilon| \ll 1, \tag{A-49}$$

to first order in ϵ we have that $g_{\alpha\beta}(x') \approx g_{\alpha\beta}(x) + \epsilon\xi^\lambda\partial_\lambda g_{\alpha\beta}(x)$, then we can write (A-48) as follows

$$g_{\alpha\sigma}\partial_\beta\xi^\sigma + g_{\rho\beta}\partial_\alpha\xi^\rho + \xi^\lambda\partial_\lambda g_{\alpha\beta} = 0. \tag{A-50}$$

This can be rewritten in terms of derivatives of the covariant components $\xi_\alpha = g_{\alpha\mu}\xi^\mu$, then

$$\begin{aligned}
0 &= g_{\alpha\sigma}\partial_\beta\xi^\sigma + g_{\rho\beta}\partial_\alpha\xi^\rho + \xi^\lambda\partial_\lambda g_{\alpha\beta} \\
&= \partial_\beta\xi_\alpha + \partial_\alpha\xi_\beta - \xi^\lambda\left(\partial_\beta g_{\alpha\lambda} + \partial_\alpha g_{\beta\lambda} - \partial_\lambda g_{\alpha\beta}\right) \\
&= \partial_\beta\xi_\alpha + \partial_\alpha\xi_\beta - 2\xi_\lambda\Gamma^\lambda_{\alpha\beta}
\end{aligned}$$

therefore

$$\nabla_\beta\xi_\alpha + \nabla_\alpha\xi_\beta = 0. \tag{A-51}$$

This is the Killing equation, every vector that satisfies this equation is called a Killing vector. The problem of determining all infinitesimal isometries of a given metric is now reduce to determining the Killing vectors. Any linear combination of Killing vectors, with constant coefficients, is a Killing vector. For more details see [121].

A.3. General relativity

Here we are going to name the postulates of general relativity and see which are the equations that rules the dynamics of the spacetime.

Postulates of General Relativity theory

This postulates are a motivation of the geometrization of the gravity force in classical mechanics.

Postulate 1 The spacetime is the collection of all events, it is described by the pair $(\mathcal{M}, \boldsymbol{g})$, where $\mathcal{M}$ is a Hausdorff smooth four-dimensional manifold and $\boldsymbol{g}$ is a lorenzian metric over $\mathcal{M}$.
Now let us introduce the postulates that involves the matter fields in the theory.

Postulate 2 The equations that satisfy the matter fields must fulfill that if, for $U \subset \mathcal{M}$ is convex and $p, q \in U$, then a signal can be send in U between p and q if and only if p and $q$can be join by a c^1-curve contained in U, which tangent vector everywhere is non-zero and timelike or null.

Postulate 3

There exist a symmetric tensor

$$T_{\alpha\beta} = T_{\beta\alpha} = T_{\alpha\beta}\left(\Psi_i, \nabla\Psi_i\right), \tag{A-52}$$

where Ψ_i are the matter fields and i index the different matter field, such that the depence of the matter fields is finite and

- $T_{\alpha\beta} = 0$ over $U \subset \mathcal{M}$ and open set, if and only if $\Psi_i = 0$ for every i over U.
- $\nabla_\beta T^{\alpha\beta} = 0$

For a further discussion of the postulates see [52].

Einstein Field Equations

The gravitational action is

$$S = S_E + S_M + S_\Lambda, \tag{A-53}$$

where

$$S_E = \frac{c^3}{16\pi G}\int {}^{(4)}R\sqrt{-g}d^4x \tag{A-54}$$

is the Einstein action, ${}^{(4)}R$ is the Ricci scalar, S_Λ is the contribution due to the cosmological constant

$$S_\Lambda = -\frac{c^3}{16\pi G}\int \frac{2\Lambda}{c^2}\sqrt{-g}d^4x, \tag{A-55}$$

and S_M the matter action. From the matter action we can define the energy-momentum tensor $T^{\mu\nu}$, and because the matter action is a function of the Lagrange density $\mathfrak{L}$, which is also function of matter fields Ψ_i, then [69, 34]

$$T^{\alpha\beta} = \sum_i \frac{\partial\mathfrak{L}}{\partial\left(\nabla_\alpha\Psi_i\right)}\nabla^\beta\Psi_i - g^{\alpha\beta}\mathfrak{L}. \tag{A-56}$$

Taking the variation of the total action with respect to $g_{\mu\nu}$ one finds the Einstein field equations with cosmological constant

$${}^{(4)}R_{\alpha\beta} - \frac{1}{2}{}^{(4)}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta}, \tag{A-57}$$

from now on we will use the natural unit system, therefore $G = c = 1$. The set of equations (A-57) can be written as

$$ {}^{(4)}R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) + \Lambda g_{\alpha\beta}. \quad \text{(A-58)} $$

We can deduce the Einstein field equations in vacuum from an action this action is called the Einstein-Hilbert action, taking $\Lambda = 0$ this action is given by

$$ S_{EH} = \frac{1}{16\pi G} \int R\sqrt{-g} d^4x, \quad \text{(A-59)} $$

making $\delta S_{EH} = 0$ leads to

$$ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0. \quad \text{(A-60)} $$

If we want to include the matter contribution to the field equations, we must add the following term to the Einstein-Hilbert action

$$ S_M = \int \mathcal{L}\left(\Psi_i\right) \sqrt{-g} d^4x, \quad \text{(A-61)} $$

where $\mathcal{L}$ is a lagrangian density. Making $\delta\left(S_{EH} + S_M\right) = 0$ we obtain

$$ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \quad \text{(A-62)} $$

these are ten non-linear coupled partial differential equations. For details see [89, 52, 108].

B. Hypersurfaces and Foliations

This appendix attempt to give some highlights in geometry of hypersurfaces and foliations. We start considering a spacetime $(\mathcal{M}, \boldsymbol{g})$ which is time orientable. For this appendix we will mainly follow [47, 103, 18].

B.1. Geometry of hypersurfaces

A hypersurface of $\mathcal{M}$ is the image Σ of a 3-dimansional manifold $\hat{\Sigma}$ by an embedding

$$\Phi : \hat{\Sigma} \longrightarrow \mathcal{M}, \tag{B-1}$$

then

$$\Sigma = \Phi\left(\hat{\Sigma}\right). \tag{B-2}$$

A hypersurface can be defined locally as the set of point for which a scalar field on $\mathcal{M}$ is constant. Let t be a scalar field over $\mathcal{M}$, setting the constant to zero, for all $p \in \mathcal{M}$, if $p \in \Sigma$ if and only if $t(p) = 0$. From Appendix A, the pullback Φ^* is defined in the following domains

$$\Phi^* : T_p^*\mathcal{M} \longrightarrow T_p^*\hat{\Sigma}, \tag{B-3}$$

this allows to define the induced metric on Σ as

$$\boldsymbol{\gamma} := \Phi^*\boldsymbol{g}, \tag{B-4}$$

which is also called the first fundamental form of Σ. The hypersurface is said to be

- Spacelike if and only if $\boldsymbol{\gamma}$ has signature $(+++)$;
- Timelike if and only if $\boldsymbol{\gamma}$ has signature $(-++)$;
- Null if and only if $\boldsymbol{\gamma}$ has signature $(0++)$.

If Σ is a spacelike or timelike hypersurface, then the induced metric $\boldsymbol{\gamma}$ is not degenerate. This implies that there is a unique connection (covariant derivative) $\boldsymbol{D}$ on the manifold Σ that is torsion-free and satisfies

$$\boldsymbol{D}\boldsymbol{\gamma} = 0. \tag{B-5}$$

Let $u, v \in T_p\mathcal{M}$, the extrinsic curvature tensor of Σ, or second fundamental form of the hypersurface Σ, is

$$\begin{aligned} \boldsymbol{K} : T_p\Sigma \times T_p\Sigma &\longrightarrow \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) &\longmapsto -\boldsymbol{u} \cdot (\nabla_{\boldsymbol{v}} \boldsymbol{n}) , \end{aligned} \tag{B-6}$$

where $\boldsymbol{n}$ is a vector normal to Σ.
Given a scalar field t on $\mathcal{M}$ such that the hypersurface Σ is defined as a level surface of t, the 1-form of the gradient of t is normal to Σ, the vector $\boldsymbol{t} = \nabla t$ satisfies the following properties

- $\boldsymbol{t}$ is timelike if and only if Σ is spacelike;
- $\boldsymbol{t}$ is spacelike if and only if Σ is timelike;
- $\boldsymbol{t}$ is null if and only if Σ is null.

In the case where Σ is not null, it can be define a unit vector by setting

$$\boldsymbol{n} := \frac{\boldsymbol{t}}{\pm \boldsymbol{t} \cdot \boldsymbol{t}}, \tag{B-7}$$

where

- $\boldsymbol{n} \cdot \boldsymbol{n} = -1$ if Σ is spacelike,
- $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ if Σ is timelike.

From now on we focus on hypersurfaces Σ such that the induced metric is defined positive, which will represents the spatial role of the physical system [13]. To be able to project any quantity over Σ, we define the orthogonal projector onto Σ as

$$\begin{aligned} \bar{\boldsymbol{\gamma}} : T_p\mathcal{M} &\longrightarrow T_p\Sigma \\ \boldsymbol{v} &\longmapsto \boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\, \boldsymbol{n}. \end{aligned} \tag{B-8}$$

As a direct consequence of $\boldsymbol{n} \cdot \boldsymbol{n} = -1$, $\bar{\boldsymbol{\gamma}}$ satisfies $\bar{\boldsymbol{\gamma}}(\boldsymbol{n}) = 0$. It is possible to generalize aboves expression to 1-forms, even to any multilinear form $\boldsymbol{A}$ in the following way

$$\begin{aligned} \bar{\boldsymbol{\gamma}}^*_{\mathcal{M}} : (T_p\mathcal{M})^n &\longrightarrow T_p\Sigma \\ (\boldsymbol{v}_1, ..., \boldsymbol{v}_n) &\longmapsto \boldsymbol{A}\left(\bar{\boldsymbol{\gamma}}\boldsymbol{v}_1, ..., \bar{\boldsymbol{\gamma}}\boldsymbol{v}_n\right). \end{aligned} \tag{B-9}$$

Given (B-9), if $\boldsymbol{v}_1, \boldsymbol{v}_2 \in T_p\Sigma$ then $\bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}\left[\boldsymbol{\gamma}\left(\boldsymbol{v}_1, \boldsymbol{v}_2\right)\right] = \boldsymbol{\gamma}\left(\boldsymbol{v}_1, \boldsymbol{v}_2\right)$, therefore from now on $\boldsymbol{\gamma} := \bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}(\boldsymbol{\gamma})$, similarly $\boldsymbol{K} := \bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}(\boldsymbol{K})$. For the case of a tensor $\boldsymbol{T}$ of type $\binom{p}{q}$ on $\mathcal{M}$, the projection is denoted as $\bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}(\boldsymbol{T})$, then for any basis $\{\boldsymbol{E}_\alpha\}$ of $T_p\mathcal{M}$ the projection of $\boldsymbol{T}$ onto $\mathcal{M}$ is

$$\left(\bar{\boldsymbol{\gamma}}^*_{\mathcal{M}}\boldsymbol{T}\right)^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \gamma^{\alpha_1}_{\mu_1} \cdots \gamma^{\alpha_p}_{\mu_p} \gamma^{\nu_1}_{\beta_1} \cdots \gamma^{\nu_q}_{\beta_q} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}. \tag{B-10}$$

Now that we are able to project onto Σ, it must be possible to relate somehow the curvature quantities of $\mathcal{M}$ with the ones of Σ, the relations that make this possible are the Gauss-Codazzi relations, which are the full projection of the Riemann tensor in the hypersuface and one index projection in to the normal vector, respectively [18]. The Gauss relations is given by

$$\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\gamma}_{\rho}\gamma^{\sigma}_{\delta}{}^{(4)}R^{\rho}_{\sigma\mu\nu} = R^{\gamma}_{\delta\alpha\beta} + K^{\gamma}_{\alpha}K_{\delta\beta} - K^{\gamma}_{\beta}K_{\alpha\delta}, \tag{B-11}$$

where ${}^{(4)}\boldsymbol{R}$ is the Riemann tensor in $\mathcal{M}$ and $\boldsymbol{R}$ is the Riemann tensor in Σ. The Codazzi relation is given by

$$\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\lambda}_{\rho}{}^{(4)}R^{\rho}_{\sigma\mu\nu}n^{\sigma} = D_{\beta}K^{\lambda}_{\alpha} - D_{\alpha}K^{\lambda}_{\beta}. \tag{B-12}$$

The Riemann tensor $\boldsymbol{R}$ is function of $\boldsymbol{\gamma}$, then what Gauss-Codazzi relations says is that the choice of $\boldsymbol{\gamma}$ and $\boldsymbol{K}$ cannot be arbitary, these must fullfill the relations (B-11) and (B-12). We can also project two indices of the Riemann tensor along the normal vector, from this we obtain the evolution equation

$$\mathcal{L}_{\boldsymbol{n}}K_{\alpha\beta} = n^{\delta}n^{\gamma}\gamma^{\sigma}_{\alpha}\gamma^{\rho}_{\beta}{}^{(4)}R_{\delta\rho\gamma\sigma} - \alpha^{-1}D_{\alpha}D_{\beta}\alpha - K^{\gamma}_{\beta}K_{\alpha\gamma}. \tag{B-13}$$

B.2. Geometry of foliations

The 3+1 formalism is based on a foliation on spacetime by a 1-parameter family of spacelike hypersurfaces, this is possible for the globally hyperbolic spaces [118]. A Cauchy surface is a spacelike hypersurface Σ in $\mathcal{M}$ such that each causal curve without end points intersects Σ once and only once. A spacetime $(\mathcal{M}, \boldsymbol{g})$ that admits a Cauchy surface Σ is said to be globally hyperbolic. Any hyperbolic spacetime $(\mathcal{M}, \boldsymbol{g})$ can be foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$, a foliation or slicing meant that there exist a smooth scalar field $\hat{t}$ on $\mathcal{M}$, which is regular, such that its gradient never vanishes and

$$\forall t \in \mathbb{R},\ \Sigma_t := \left\{p \in \mathcal{M} | \hat{t}(p) = t\right\}. \tag{B-14}$$

This hypersurfaces fulfill that

- $\Sigma_t \cap \Sigma_{t'} = \emptyset$ and $t \neq t'$,
- $\mathbb{M} = \cup_{t\in\mathbb{R}}\Sigma_t$.

Since we have a future directed unit vector normal to Σ_t, we can write $\boldsymbol{n} = \alpha\boldsymbol{t}$ where

$$\alpha = \left(-\boldsymbol{t}\cdot\boldsymbol{t}\right)^{-1/2}, \tag{B-15}$$

this function is known as the lapse function. With the lapse function we can define a normal vector to Σ_t, called the normal evolution vector, as

$$\boldsymbol{m} := \alpha\boldsymbol{n}. \tag{B-16}$$

Those observers which $\boldsymbol{n}$ is its 4-velocity are Eulerian observes. For a timelike 4-vector $\boldsymbol{t}$ on the spacetime tangent to the time axis, $t^\alpha = (\partial/\partial t)^\alpha$ and $t^\alpha \nabla_\alpha t = 1$, take into account that $\boldsymbol{t}$ is not always normal to Σ_t, then we project $\boldsymbol{t}$ along $\boldsymbol{n}$ and $\boldsymbol{\gamma}$ in the following way

$$\alpha = -t^\alpha n_\alpha, \qquad \beta^\alpha = t^\beta \gamma^\alpha_\beta. \tag{B-17}$$

The vector β^α is called the shift vector, respectively, then we can write $\boldsymbol{t}$ as

$$\boldsymbol{t} = \alpha \boldsymbol{n} + \boldsymbol{\beta}. \tag{B-18}$$

We are able to write the evolution equation for $\boldsymbol{\gamma}$ along $\boldsymbol{m}$, this expression is given by

$$\mathcal{L}_{\boldsymbol{m}} \boldsymbol{\gamma} = -2\alpha \boldsymbol{K} \Longrightarrow \boldsymbol{K} = \frac{1}{2} \mathcal{L}_{\boldsymbol{n}} \boldsymbol{\gamma}, \tag{B-19}$$

this allows to obtain the scalar curvature in terms of hypersurface quantities

$${}^{(4)}R = R + K^2 + K_{\mu\nu} K^{\mu\nu} - \frac{2}{\alpha} \mathcal{L}_{\boldsymbol{m}} K - \frac{2}{\alpha} D_\mu D^\mu \alpha. \tag{B-20}$$

C. Perturbation theory

The main idea of this appendix is to give a brief introduction into the mathematics of cosmological perturbation theory, main references for this appendix are [91, 90, 25, 56, 55]. Here we need two different spacetimes, the real spacetime, or physical spacetime $(\mathcal{M}_p, g_{\alpha\beta})$ and the background spacetime $(\mathcal{M}_0, \bar{g}_{\alpha\beta})$. The perturbation of any quantity $\boldsymbol{T}$ is the difference between the value that this quantity takes in real spacetime and the value in the background spacetime at a given point. To do this we need a diffeomorphism ϕ between $\mathcal{M}_0$ and $\mathcal{M}_p$, $\phi : \mathcal{M}_0 \to \mathcal{M}_p$ this is called a gauge choice, from this we can induce a pullback $\phi^* : T^*_{\phi(p)}\mathcal{M}_p \to T^*_p\mathcal{M}_0$ for $p \in \mathcal{M}$. Let $\boldsymbol{T}_0$ be a tensor defined on $\mathcal{M}_0$ and let $\boldsymbol{T}$ be a tensor defined on $\mathcal{M}_p$, then the perturbation $\Delta\boldsymbol{T}$ is defined as

$$\Delta\boldsymbol{T} = \phi^*\boldsymbol{T}|_{\mathcal{M}_0} - \boldsymbol{T}_0, \tag{C-1}$$

where it must be take in to account that this is given at each point of $\mathcal{M}_0$.

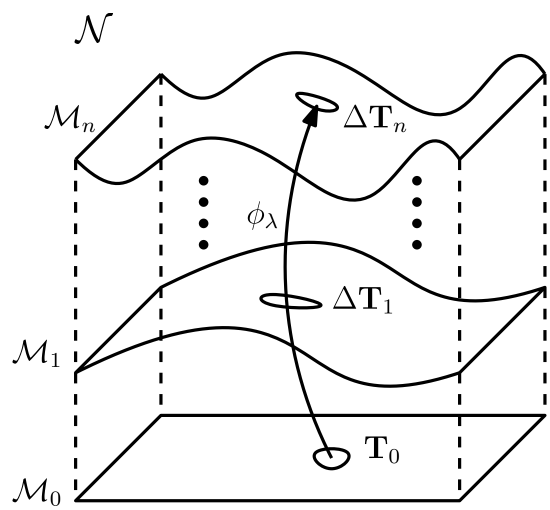

Figure **C-1**.: Scheme of sub-manifolds family $\mathcal{M}_\lambda$ in a five dimensional manifold $\mathcal{N}$. The comparison between manifolds is given by ϕ_λ.

Let us consider a 5 dimensional manifold $\mathcal{N} = \mathcal{M}_p \times \mathbb{R}$, in $\mathcal{N}$ is embedded the family of 4 dimensional sub-manifolds $\mathcal{M}_\lambda$ where $\lambda \in \mathbb{R}$. A tensor $\boldsymbol{T}_\lambda$ living on $\mathcal{M}_\lambda$ can be extended to a tensor $\boldsymbol{T}$ on $\mathcal{N}$ evaluating in the point (p, λ), where $p \in \mathcal{M}$, then $\boldsymbol{T}(p, \lambda) = \boldsymbol{T}_\lambda(p)$. Each of these sub-manifolds represents a perturbed spacetime, for the background spacetime $\mathcal{M}_0$ corresponds $\lambda = 0$. To be able to compare a tensor in $\mathcal{M}_\lambda$ with a tensor in $\mathcal{M}_0$ it must be consider a flux ϕ_λ which is the integral curve of the vector field $\boldsymbol{X}$, because we have a five dimensional space, the components of the vector field are $\boldsymbol{X} = (X^0, X^1, X^2, X^3, X^4)$ where $X^4 = 1$, so the points lie on the same manifold. It is possible to make a Taylor expansion using ϕ_λ, the perturbation in this case is

$$\Delta\boldsymbol{T}_\lambda = \phi_\lambda^*\boldsymbol{T}|_{\mathcal{M}_0} - \boldsymbol{T}_0, \tag{C-2}$$

and the expansion is given by

$$\phi_\lambda^*\boldsymbol{T}|_{\mathcal{M}_0} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta_\phi^{(k)}\boldsymbol{T} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_{\boldsymbol{X}}^k, \tag{C-3}$$

where

$$\delta_\phi^{(k)}\boldsymbol{T} = \frac{d^k}{d\lambda^k}\left(\phi_\lambda^*\boldsymbol{T}\right)\bigg|_{\lambda=0,\mathcal{M}_0}. \tag{C-4}$$

Therefore the perturbation is given by

$$\Delta\boldsymbol{T}_\lambda = \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\delta_\phi^{(k)}\boldsymbol{T} \tag{C-5}$$

Due to the covariance of General Relativity, it is possible to choose another diffeomorphism ψ between $\mathcal{M}_0$ and $\mathcal{M}_p$, the change between this diffeomorfism is called gauge transformation. Let ψ_λ another gauge choice which is the integral curve of the vector $\boldsymbol{Y}$, where $\boldsymbol{X} \neq \boldsymbol{Y}$, then it is possible to have

$$\psi_\lambda^*\boldsymbol{T}|_{\mathcal{M}_0} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta_\psi^{(k)}\boldsymbol{T} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_{\boldsymbol{Y}}^k, \tag{C-6}$$

where

$$\delta_\psi^{(k)}\boldsymbol{T} = \frac{d^k}{d\lambda^k}\left(\psi_\lambda^*\boldsymbol{T}\right)\bigg|_{\lambda=0,\mathcal{M}_0}. \tag{C-7}$$

If $\phi_\lambda^*\boldsymbol{T} = \psi_\lambda^*\boldsymbol{T}$ for any ϕ and ψ, then $\boldsymbol{T}$ is gauge invariant. This allows to mention the Stewart-Walker lemma [105]: For every vector field $\boldsymbol{X}$ and $k \geq 1$

$$\mathcal{L}_{\boldsymbol{X}}\delta^k\boldsymbol{T} = 0, \tag{C-8}$$

if and only if $\boldsymbol{T}$ is gauge invariant at order k. It is not always possible to have invariant gauge quantities, in this case we must have a transformation between the gauge choices, this is called a gauge transformation, which is given by

$$\Phi_\lambda = \phi_{-\lambda}\circ\psi_\lambda. \tag{C-9}$$

This makes a difference between $\delta_\phi^{(k)}\boldsymbol{T}$ and $\delta_\psi^{(k)}\boldsymbol{T}$, for the case of first order perturbations the difference is given by

$$\delta_\psi^{(1)}\boldsymbol{T} - \delta_\phi^{(1)}\boldsymbol{T} = \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{T}_0, \tag{C-10}$$

where

$$\boldsymbol{\xi} = \boldsymbol{Y} - \boldsymbol{X}. \tag{C-11}$$

D. General Relativistic Dynamo Equation

In this appendix we will calculate the full dynamo equation in $1+3$ formalism given at [83] as equation (11), in [83] the steps of the calculations and geometrical properties used are dismissed, but here we will show the properties and most of the steps for the calculations. Here we will use the Maxwell equations obtained in chapter 2, but we want that the appendix be complete as possible, then we will write Maxwell equations assuming quasi-neutrality ($\rho_{\text{charge}} \approx 0$) and the Ohm's law using the same notation as in [83],

$$\dot{B}^{\langle a\rangle} = \left(\sigma^{ab} + \omega^{ab} - \frac{2}{3}\Theta h^{ab}\right) B_b - \text{curl}E^a - \epsilon^{abc}E_b\dot{u}_c, \tag{D-1}$$

$$\dot{E}^{\langle a\rangle} = \left(\sigma^{ab} + \omega^{ab} - \frac{2}{3}\Theta h^{ab}\right) E_b + \text{curl}B^a + \epsilon^{abc}B_b\dot{u}_c - 4\pi J^{\langle a\rangle}, \tag{D-2}$$

$$D_aE^a = -2\omega^a B_a, \qquad D_aB^a = 2\omega E_a, \tag{D-3}$$

$$J^{\langle a\rangle} = \frac{1}{4\pi\lambda}\left(E^a + \epsilon^{abc}v_bB_c\right) \tag{D-4}$$

where the $1+3$ electric and magnetic fields in this appendix are written as E and B respectively, $X^{\langle a\rangle} = h^{ab}X_b$, $\text{curl}X^a = \epsilon^{abc}D_bX_c$ and $\dot{X}^a = u^b\nabla_bX^a$, here D is the projection of the covariant derivative ∇ under the projector $\boldsymbol{h}$. The main idea is to calculate $\text{curl}\,(\text{curl}B)^a$ from two different ways, from a physical and from a geometrical point of view. In the case of the physical point of view is using Maxwell equations, form the geometrical point of view is using geometrical properties that can be obtained from the $1+3$ formalism. Therefore we will divide this appendix in two sections: Geometry and Physics. In Geometry we will obtain $\text{curl}\,(\text{curl}B)^a$ from the geometrical properties given by $1+3$ formalism and also $\text{curl}\dot{E}^{\langle a\rangle}$ which is given by equations (9) and (10) in [83]. In Physics we will obtain $\text{curl}\,(\text{curl}B)^a$ from Maxwell equations and then the full dynamo equation is obtained.

D.1. Geometry

As mention before, we will obtain $\text{curl}\,(\text{curl}B)^a$ from the geometrical point of view. From the definition of curl and using the fact that

$$\epsilon^{abc}\epsilon_{dec} = \delta^a_d\delta^b_e - \delta^b_d\delta^a_e, \tag{D-5}$$

then

$$\text{curl}\,(\text{curl}B)^a = \epsilon^{abc}\epsilon_{cde}D_b D^d B^e = -D^2 B^a + D_b D^a B^b. \tag{D-6}$$

Using the commutation of the D operator [111, 114]

$$2D_{[c}D_{d]}B_a = -2\epsilon_{cdb}\omega^d \dot{B}_{\langle a\rangle} + \mathcal{R}_{dabc}B^d, \tag{D-7}$$

then

$$\text{curl}\,(\text{curl}B)^a = -D^2 B^a + D^a\left(D_b B^b\right) + 2\epsilon_{cdb}\omega^d \dot{B}_{\langle a\rangle} + \mathcal{R}^{ab}B_b. \tag{D-8}$$

Now we need to find the expression for $\mathcal{R}_{ab}$, for this we will use the contracted Gauss equation obtained from (B-11), then we can decompose $\mathcal{R}_{ab}$ as follows [111, 114]

$$\mathcal{R}_{ab} = h^c_a h^d_b R_{cd} + R_{acbd}u^c u^d + (D_c u_a)(D_b u^c) - \Theta D_b u_a, \tag{D-9}$$

using the Einstein Field equations decomposition in 1+3 formalism, from (A-58), using the projector $\boldsymbol{h}$ is possible to obtain $h^c_a h^d_b R_{cd}$, which is given in (2-25), then using $R_{acbd}u^c u^d$, given in (2-32), together with the propagation equations and using the fact that

$$\nabla_b \dot{u}_a - (\nabla_b u_a)^{\cdot} = u_a\left(\frac{1}{3}\Theta\dot{u}_b - \dot{\sigma}_{bc}u^c + \epsilon_{bcd}\omega^c\dot{u}^d\right) + D_b\dot{u}_a - \dot{\sigma}_{ab} - \dot{\omega}_{ab} - \frac{1}{3}\left(\dot{\Theta}h_{ab} + \Theta\dot{h}_{ab}\right) + \dot{u}_a\dot{u}_b, \tag{D-10}$$

after some calculations [112, 39, 113]

$$\mathcal{R}_{ab} = \frac{2}{3}\left(\rho + \Lambda - \frac{1}{3}\Theta^2 - \sigma^2 + \omega^2\right)h_{ab} + \pi_{ab} - \dot{\sigma}_{\langle ab\rangle} + D_{\langle a}u_{b\rangle} - \frac{1}{3}\Theta\left(\sigma_{ab} - \omega_{ab}\right) - \dot{u}_{\langle a}\dot{u}_{b\rangle} - 2\left(\sigma^c_{\langle a}\sigma_{b\rangle c} + \omega_{\langle a}\omega_{b\rangle}\right) - 2\sigma_{c[a}\omega^c_{b]}. \tag{D-11}$$

Let us now obtain $\text{curl}\dot{E}^{\langle a\rangle}$, for this we will use the following property [76]

$$(D_a X_b)^{\cdot} = D_a\dot{X}_b - \frac{1}{3}\Theta D_a X_b - \sigma^c_a D_c X_b + H^d_a\epsilon_{dbc}X^c, \tag{D-12}$$

where $H_{ab} = \text{curl}\sigma_{ab} + D_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle}$, then

$$\text{curl}\dot{E}^{\langle a\rangle} = \epsilon^{abc}\left[\left(D_b E_{\langle c\rangle}\right)^{\cdot} + \frac{1}{3}\Theta D_b E_c + \sigma^d_b D_d E_c - \left(\text{curl}\sigma^{bd} + D^{\langle b}\omega^{d\rangle} + 2\dot{u}^{\langle b}\omega^{d\rangle}\right)\epsilon_{dce}E^e\right], \tag{D-13}$$

To compute this expression we used the following formulas together with the Maxwell equations

$$\epsilon^{abc}\left(D_bE_{\langle c\rangle}\right)^{\cdot} = \left(\epsilon^{abc}D_bE_{\langle c\rangle}\right)^{\cdot} - \dot{\epsilon}^{abc}D_bE_{\langle c\rangle}, \tag{D-14}$$

$$\left(\epsilon^{abc}D_bE_{\langle c\rangle}\right)^{\cdot} = -\ddot{B}^{\langle a\rangle} + \left[\left(\sigma^{ab}+\omega^{ab}-\frac{2}{3}\Theta h^{ab}\right)B_b\right]^{\cdot} - \dot{\epsilon}^{abc}E_b\dot{u}_c - \epsilon^{abc}\dot{E}_b\dot{u}_c - \epsilon^{abc}E_b\ddot{u}_c, \tag{D-15}$$

$$\dot{h}_{ab} = 2u_{(a}\,\dot{u}_{b)}, \tag{D-16}$$

$$\dot{\epsilon}_{abc} = 3u_{[a}\,\epsilon_{bc]d}\dot{u}^d, \tag{D-17}$$

$$\mathrm{curl}\sigma_{ab} = \epsilon_{cd(a}\,D^c\sigma^d_{\ b)}, \tag{D-18}$$

$$D^b\sigma_{ab} = \frac{2}{3}D_a\Theta + \mathrm{curl}\omega_a + 2\epsilon_{abc}\dot{u}^b\omega^c - q_a, \tag{D-19}$$

then

$$\mathrm{curl}\dot{E}^{\langle a\rangle} = -\ddot{B}^{\langle a\rangle} + \Xi^a \tag{D-20}$$

where

$$\begin{aligned}\Xi^a =& \left(\dot{\sigma}^{ab}+\dot{\omega}^{ab}-\frac{2}{3}\dot{\Theta}h^{ab}\right)B_b + \left(\sigma^{ab}+\omega^{ab}-\frac{2}{3}\Theta h^{ab}\right)\dot{B}_b + \frac{1}{3}\Theta\left(\sigma^{ab}+\omega^{ab}-\frac{2}{3}\Theta h^{ab}\right)B_b\\ &-\epsilon^{abc}\dot{u}_c\left[\left(\sigma_{bd}+\omega_{bd}-\frac{1}{3}\Theta h_{bd}\right)E^d + \epsilon_{bde}D^dB^e + \epsilon_{bde}\dot{u}^eB^d - 4\pi J_{\langle b\rangle}\right] + E_b\left[(\mathrm{curl}\sigma)^{ab} + D^{\langle a}\omega^{b\rangle} + 2\dot{u}^{\langle a}\omega^{b\rangle}\right]\\ &- E^a\epsilon_{deb}D^d\sigma^{eb} - \epsilon^{abc}q_bE_c + \epsilon^{abc}\left[D^d\left(E_c\sigma_{bd}\right) - \frac{2}{3}E_cD_b\Theta\right] - 2E_b\left(D^{[a}\omega^{b]} + \dot{u}^{[a}\omega^{b]}\right).\end{aligned} \tag{D-21}$$

D.2. Physics

Here we will calculate $\mathrm{curl}\,(\mathrm{curl}B)^a$ from Maxwell equations, let us start from the Ampère equation

$$\mathrm{curl}B^a = \dot{E}^{\langle a\rangle} - \left(\sigma^{ab}+\omega^{ab}-\frac{2}{3}\Theta h^{ab}\right)E_b - \epsilon^{abc}B_b\dot{u}_c + 4\pi J^{\langle a\rangle}, \tag{D-22}$$

taking the curl and using the Ohm's law

$$\begin{aligned}\mathrm{curl}\,(\mathrm{curl}B)^a =& \mathrm{curl}\dot{E}^{\langle a\rangle} - \epsilon^{abc}D_b\left[\epsilon_{cde}\dot{u}^eB^d\right] + \lambda^{-1}\epsilon^{abc}\epsilon_{cde}D_b\left(v^dB^e\right) - \epsilon^{abc}D_b\left[\sigma_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right]\\ &-\epsilon^{abc}D_b\left[\omega_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] + \lambda^{-1}\epsilon^{abc}D_bE_c + \frac{2}{3}\epsilon^{abc}D_b\left[4\pi\lambda\Theta h_{cd}J^{\langle d\rangle} - \Theta h_{cd}\epsilon^{def}v_eB_f\right],\end{aligned} \tag{D-23}$$

in the second line of above equation, distributing the derivative, using Faraday's equation and using Ohm's law we obtain

$$\begin{aligned}\mathrm{curl}\,(\mathrm{curl}B)^a =& \mathrm{curl}\dot{E}^{\langle a\rangle} - \epsilon^{abc}D_b\left[\epsilon_{cde}\dot{u}^eB^d\right] + \lambda^{-1}\epsilon^{abc}\epsilon_{cde}D_b\left(v^dB^e\right) - \epsilon^{abc}D_b\left[\sigma_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right]\\ &- \epsilon^{abc}D_b\left[\omega_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] + \frac{8\pi}{3}\lambda J_{\langle c\rangle}\epsilon^{abc}D_b\Theta - \frac{2}{3}\epsilon^{abc}\epsilon^{def}v_eB_fh_{cd}D_b\Theta\\ &+ \left(\frac{2}{3}\Theta + \lambda^{-1}\right)\left[\dot{B}^{\langle a\rangle} - \left(\sigma^{ab}+\omega^{ab}-\frac{2}{3}\Theta h^{ab}\right)B_b + \epsilon^{abc}\dot{u}_c\left(4\pi\lambda J_{\langle b\rangle} - \epsilon_{bde}v^dB^e\right)\right].\end{aligned} \tag{D-24}$$

Now that we have curl $(\mathrm{curl}B)^a$ obtained in a geometrical and in a physical way, let us equate both expressions

$$
\begin{aligned}
&- D^2B^a + D^a\left(D_bB^b\right) + \mathcal{R}^{ab}B_b = \\
&\mathrm{curl}\dot{E}^{\langle a\rangle} - \epsilon^{abc}D_b\left[\epsilon_{cde}\dot{u}^eB^d\right] + \lambda^{-1}\epsilon^{abc}\epsilon_{cde}D_b\left(v^dB^e\right) - \epsilon^{abc}D_b\left[\sigma_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] \\
&\quad - \epsilon^{abc}D_b\left[\omega_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] + \frac{8\pi}{3}\lambda J_{\langle c\rangle}\epsilon^{abc}D_b\Theta - \frac{2}{3}\epsilon^{abc}\epsilon^{def}v_eB_fh_{cd}D_b\Theta \\
&+ \left(\frac{2}{3}\Theta + \lambda^{-1}\right)\left[\dot{B}^{\langle a\rangle} - \left(\sigma^{ab} + \omega^{ab} - \frac{2}{3}\Theta h^{ab}\right)B_b + \epsilon^{abc}\dot{u}_c\left(4\pi\lambda J_{\langle b\rangle} - \epsilon_{bde}v^dB^e\right)\right],
\end{aligned}
\tag{D-25}
$$

multiplying by λ and rearranging the terms in the equation we finally obtain

$$
\begin{aligned}
&\dot{B}^{\langle a\rangle} + \lambda D^2B^a + \epsilon^{abc}\epsilon_{cde}D_b\left(v^dB^e\right) = \\
&-\frac{2}{3}\lambda\Theta\dot{B}^{\langle a\rangle} + 2\lambda D^a\left(\omega^bE_b\right) + \lambda\mathcal{R}^{ab}B_b + \lambda\epsilon^{abc}D_b\left[\sigma_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] + \lambda\epsilon^{abc}D_b\left[\epsilon_{cde}\dot{u}^eB^d\right] \\
&\quad + \left(\frac{2}{3}\lambda\Theta + 1\right)\left[\dot{B}^{\langle a\rangle} - \left(\sigma^{ab} + \omega^{ab} - \frac{2}{3}\Theta h^{ab}\right)B_b + \epsilon^{abc}\dot{u}_c\left(4\pi\lambda J_{\langle b\rangle} - \epsilon_{bde}v^dB^e\right)\right] \\
&+ \lambda\epsilon^{abc}D_b\left[\omega_{cd}\left(4\pi J^{\langle d\rangle} - \epsilon^{def}v_eB_f\right)\right] + \frac{8\pi}{3}\lambda^2J_{\langle c\rangle}\epsilon^{abc}D_b\Theta + \frac{2}{3}\lambda\epsilon^{abc}\epsilon^{def}v_eB_fh_{cd}D_b\Theta + \lambda\ddot{B}^{\langle a\rangle} - \Xi^a.
\end{aligned}
\tag{D-26}
$$

E. Geometric quantities

In this appendix it is shown some perturbed quantities at first order, in order to do this we used `Sagemanifolds`[48, 49] and `xPand` [95]. Let us remember that $\mathcal{H} = a'/a$ where the prime denotes the derivative respect to the conformal time.

E.1. Geometric quantities perturbed at first order

Christoffel symbols for the four dimensional spacetime can be written in the following way

$$\Gamma^{\alpha}_{\beta\gamma} = \delta^{(0)}\Gamma^{\alpha}_{\beta\gamma} + \delta^{(1)}\Gamma^{\alpha}_{\beta\gamma}, \tag{E-1}$$

where $\delta^{(0)}\Gamma^{\alpha}_{\beta\gamma}$ is the background contribution and $\delta^{(1)}\Gamma^{\alpha}_{\beta\gamma}$ is the contribution at first order. Therefore

$$\delta^{(0)}\Gamma^{0}_{00} = \mathcal{H}, \tag{E-2}$$
$$\delta^{(0)}\Gamma^{i}_{0j} = \mathcal{H}\delta^{i}_{j}, \tag{E-3}$$
$$\delta^{(0)}\Gamma^{0}_{ij} = \mathcal{H}\delta_{ij}, \tag{E-4}$$
$$\delta^{(0)}\Gamma^{i}_{00} = \delta^{(0)}\Gamma^{0}_{0i} = \delta^{(0)}\Gamma^{i}_{kj} = 0, \tag{E-5}$$

and

$$\delta^{(1)}\Gamma^{0}_{00} = (\psi)', \tag{E-6}$$
$$\delta^{(1)}\Gamma^{i}_{0j} = -(\phi)'\delta^{i}_{j} + \frac{1}{2}\left(\chi^{i}_{j}\right)', \tag{E-7}$$
$$\delta^{(1)}\Gamma^{0}_{ij} = -\partial_i\omega_j - 2\mathcal{H}\phi\delta_{ij} - (\phi)'\delta_{ij} - \mathcal{H}\chi_{ij} + \frac{1}{2}\left(\chi_{ij}\right)', \tag{E-8}$$
$$\delta^{(1)}\Gamma^{i}_{00} = \partial^i\psi + \mathcal{H}\omega^i + \left(\omega^i\right)', \tag{E-9}$$
$$\delta^{(1)}\Gamma^{0}_{0i} = \partial_i\psi + \mathcal{H}\omega_i, \tag{E-10}$$
$$\delta^{(1)}\Gamma^{i}_{kj} = -\partial_j\phi\delta^{i}_{k} - \partial_k\phi\delta^{i}_{j} + \partial^i\phi\delta_{jk} - \mathcal{H}\omega^i\delta_{jk} + \frac{1}{2}\partial_j\chi^{i}_{k} + \frac{1}{2}\partial_k\chi^{i}_{j} - \frac{1}{2}\partial^i\chi_{jk}. \tag{E-11}$$

The Ricci tensor in the background is given by

$$R_{00}^{(0)} = -3\frac{a''}{a} + 3\mathcal{H}, \tag{E-12}$$

$$R_{0i}^{(0)} = 0, \tag{E-13}$$

$$R_{ij}^{(0)} = \left(\frac{a''}{a} + \mathcal{H}\right)\delta_{ij}. \tag{E-14}$$

At first order

$$R_{00}^{(1)} = \mathcal{H}\partial_i\omega^i + \partial_i\left(\omega^i\right)' + \partial_i\partial^i\psi + 3\left(\phi\right)'' + 3\mathcal{H}\left(\phi\right)' + 3\mathcal{H}\left(\psi\right)', \tag{E-15}$$

$$R_{0i}^{(1)} = \frac{a''}{a}\omega_i + \mathcal{H}^2\omega_i + 2\partial_i\left(\phi\right)' + 2\mathcal{H}\partial_i\psi + \frac{1}{2}\partial_k\left(\chi_i^k\right)', \tag{E-16}$$

$$R_{ij}^{(1)} = \left[-\mathcal{H}\psi' - 5\mathcal{H}\phi' - 2\frac{a''}{a}\psi - 2\mathcal{H}^2\psi - 2\frac{a''}{a}\phi - 2\mathcal{H}^2\phi - \phi'' + \partial^k\partial_k\phi - \mathcal{H}\partial_l\omega^l\right]\delta_{ij}$$

$$- \partial_i\left(\omega_j\right)' - 2\mathcal{H}\partial_i\omega_j + \mathcal{H}\chi_{ij}' + \frac{a''}{a}\chi_{ij} + \mathcal{H}^2\chi_{ij} + \frac{1}{2}\chi_{ij}'' + \partial_i\partial_j\left(\phi - \psi\right) \tag{E-17}$$

$$+ \frac{1}{2}\partial_l\left(\partial_i\chi_j^l + \partial_j\chi_i^l\right) - \frac{1}{2}\partial_l\partial^l\chi_{ij}. \tag{E-18}$$

The curvature scalar in the background is given by

$$R^{(0)} = \frac{6a''}{a^3}. \tag{E-19}$$

At first order

$$R^{(1)} = \frac{1}{a^2}\left[-6\mathcal{H}\partial_i\omega^i - 2\partial_i\left(\omega^i\right)' - 2\partial_i\partial^i\psi - 6\phi'' - 6\mathcal{H}\psi' - 18\mathcal{H}\phi' - 12\frac{a''}{a}\psi + 4\partial_i\partial^i\phi + \partial_l\partial^i\chi_i^l\right]. \tag{E-20}$$

The Einstein tensor in the background is given by

$$\left(G_0^0\right)^{(0)} = -\frac{3}{a^2}, \tag{E-21}$$

$$\left(G_i^0\right)^{(0)} = \left(G_0^i\right)^{(0)} = 0, \tag{E-22}$$

$$\left(G_i^j\right)^{(0)} = -\frac{1}{a^2}\left(2\frac{a''}{a} - \mathcal{H}^2\right)\delta_i^j. \tag{E-23}$$

At first order

$$\left(G_0^0\right)^{(1)} = \frac{1}{a^2}\left(6\mathcal{H}^2\phi + 6\mathcal{H}\phi' + 2\mathcal{H}\partial_i\omega^i - 2\partial_i\partial^i\phi - \frac{1}{2}\partial_k\partial^i\chi_i^k\right), \tag{E-24}$$

$$\left(G_i^0\right)^{(1)} = \frac{1}{a^2}\left[-2\partial_i\left(2\mathcal{H}\psi + \phi'\right) - \frac{1}{2}\partial_k\chi_i^k + \frac{1}{2}\partial_k\partial^k\omega_i\right] \tag{E-25}$$

$$\left(G_j^i\right)^{(1)} = \frac{1}{a^2}\left\{\left[2\mathcal{H}\psi' + 4\frac{a''}{a}\psi - 2\mathcal{H}^2\psi + \partial_l\partial^l\psi + 4\mathcal{H}\phi' + 2\phi'' - \partial_l\partial^l\phi + 2\mathcal{H}\partial_l\omega^l + \partial_l\left(\omega^l\right)' + \frac{1}{2}\partial_k\partial^l\chi_l^k\right]\delta_j^i\right.$$

$$\left. +\partial^i\partial_j\left(\phi - \psi\right) - \partial^i\left(2\mathcal{H}\omega_j + \omega_j'\right) + \left(\mathcal{H}\left(\chi_j^i\right)' + \frac{1}{2}\left(\chi_j^i\right)''\right) + \frac{1}{2}\partial_l\partial^i\chi_j^l + \frac{1}{2}\partial_l\partial_j\chi^{il} - \frac{1}{2}\partial_l\partial^l\chi_j^i\right\} \tag{E-26}$$

F. Codes

The following Sagemath code calculates the Friedman equations, this also calculates the lapse function, the shift vector, the extrinsic curvature, the induced metric and the normal vector for the FLRW solution with conformal time

```
%display latex
Parallelism().set(nproc=8)

M = Manifold(4, 'M', structure="Lorentzian")
N = Manifold(3, 'N', ambient=M, structure="Riemannian")
print(M)
print(N)

C.<eta,x,y,z>=M.chart(r'eta:(-oo,oo):\eta x:(-oo,oo) y:(-oo,oo) z:(-oo,oo)')
r=sqrt(x^2+y^2+z^2)

var('k', domain='real'); k=0
a = M.scalar_field(function('a')(eta),name='a')
rho = M.scalar_field(function('rho')(eta), name='rho')
p = M.scalar_field(function('p')(eta), name='p')

g = M.metric()
g[0,0] = -a*a
g[1,1] = a*a/((1 + (k*r^2)/4)^2)
g[2,2] = a*a/((1 + (k*r^2)/4)^2)
g[3,3] = a*a/((1 + (k*r^2)/4)^2)
g.display()

nabla = g.connection()
g.christoffel_symbols_display()

Ricci = nabla.ricci()
Ricci.display_comp()

Ricci_scalar = g.ricci_scalar()
Ricci_scalar.display()

u = M.vector_field('u')
u[0] = 1/a u.display()

g(u,u).expr()
```

```
u_form = u.down(g)

T = (rho+p)*(u_form*u_form) + p*g
T.set_name('T')
print(T)
T.display()

Ttrace = g.inverse()['^ab']*T['_ab']
Ttrace.display()

E1 = Ricci - Ricci_scalar/2*g - (8*pi)*T

print("First Friedmann equation:\n")
E1[0,0].expr().expand() == 0

E2 = Ricci - (8*pi)*(T - Ttrace/2*g)
print("Second Friedmann equation:\n")
E2[0,0].expr().expand() == 0

tau = var(r'tau')

CN.<x0,y0,z0> = N.chart(r'x0:(-oo,oo) y0:(-oo,oo) z0:(-oo,oo)')

phi = N.diff_map(M, {(CN,C):[tau,x0,y0,z0]})
phi.display()

phi_inv = M.diff_map(N, {(C,CN):[x,y,z]})
phi_inv.display()

phi_inv_tau = M.scalar_field({C:eta})
phi_inv_tau.display()

N.set_embedding(phi, inverse=phi_inv, var=tau, t_inverse={tau: phi_inv_tau})
T = N.adapted_chart()
T

N.induced_metric().display()

N.induced_metric().inverse()[:]

N.normal().display()

g(N.normal(), N.normal()).display()

N.lapse().display()

N.shift().display()
```

```
N.second_fundamental_form().display()

K = N.induced_metric().inverse()['^ab']*N.second_fundamental_form()['_ab']
K.display()

N.induced_metric().connection().ricci().display_comp()
```

The following is a `Mathematica` code which calculates the Einstein field equations pertubed at first order on the Newtonian Gauge, this are the equations (7) on [77]

```
<< xAct`xPand`;

DefManifold[M, 4, {\[Alpha], \[Beta], \[Gamma], \[Mu], \[Nu], \[Lambda],
                                                            \[Sigma]}];
DefMetric[-1, g[-\[Alpha], -\[Beta]], CD, {";", "\[Del]"}];

DefMetricPerturbation[g, dg];
SetSlicing[g, n, h, cd, {"|", "D"}, "FLFlat"];

MyToxPand[expr_, gauge_, order_] := ToxPand[expr, dg, u, du, h, gauge,
                                                                order]

$FirstOrderVectorPerturbations = False;
$FirstOrderTensorPerturbations = False;

MyToxPand[EinsteinCD[-\[Mu], -\[Nu]], "NewtonGauge", 1]

ExtractComponents[%, h, {"Time", "Time"}]
ExtractComponents[%%, h, {"Time", "Space"}]

ExtractComponents[MyToxPand[EinsteinCD[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Space", "Space"}] // Simplify

DefTensor[Tmunu[-\[Mu], -\[Nu]], M] Tmunu[\[Mu]_, \[Nu]_] := (\[Rho][u][])
                                                         u[\[Mu]] u[\[Nu]]
MyToxPand[Tmunu[-\[Mu], -\[Nu]], "NewtonGauge", 1]

ExtractComponents[MyToxPand[Tmunu[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Time", "Time"}]
ExtractComponents[MyToxPand[Tmunu[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Time", "Space"}]
ExtractComponents[MyToxPand[Tmunu[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Space", "Space"}]

MyGR[\[Mu]_, \[Nu]_] := EinsteinCD[\[Mu], \[Nu]] - 8*Pi*Tmunu[\[Mu],
                                                               \[Nu]];
ExtractComponents[MyToxPand[MyGR[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Time", "Time"}] == 0 // Simplify
```

```
ExtractComponents[MyToxPand[MyGR[-\[Mu], -\[Nu]], "NewtonGauge", 1],
                  h, {"Time", "Space"}] == 0 // Simplify
ExtractComponents[STFPart[MyToxPand[EinsteinCD[-\[Mu], -\[Nu]],
                  "NewtonGauge", 1], h],
                  h, {"Space", "Space"}] // Simplify

MyToxPand[EinsteinCD[\[Mu], -\[Mu]], "NewtonGauge", 1]
MyToxPand[CD[-\[Mu]]@Tmunu[\[Mu], \[Nu]], "NewtonGauge", 1]

ExtractComponents[MyToxPand[CD[-\[Mu]]@Tmunu[\[Mu], \[Nu]], "NewtonGauge",
1], h, {"Time"}] == 0 // FullSimplify
ExtractComponents[MyToxPand[CD[-\[Mu]]@Tmunu[\[Mu], \[Nu]], "NewtonGauge",
1], h, {"Space"}] == 0 // FullSimplify
```

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