UNIVERSIDAD NACIONAL DE COLOMBIA

Categorification of Chern-Weil theory and equivariant cohomology

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Sistemas locales, álgebra homotópica, homomorfismo de Chern-Weil, cohomología equivariante

Resumen

Esta tesis contempla la generalización de resultados de geomtría diferencial clásica en el contexto de los sistemas locales homotópicos. En particular, se realiza la construcción del homomorfismo de Chern-Weil y el teorema equivariante de de Rham en el contexto de las categorias diferenciales graduadas conformadas por los sistemas locales homotópicos.

Contents

Abstract

Let G be a compact connected Lie group acting on a smooth manifold M. We show that the DG categories $\mathbf{Loc}_\infty(BG)$ and $\mathbf{Loc}_\infty(M_G)$ of ∞-local systems on the classifying space of G and the homotopy quotient of M, respectively, can be described infinitesimally as the categories $\mathbf{InfLoc}_\infty(\mathfrak{g})$ of basic $\mathfrak{g}$-L_∞ spaces and $\mathbf{InfLoc}_\infty(\mathfrak{g},M)$ of $\mathfrak{g}$ graded G-equivariant vector bundles, respectively. Moreover, we show that, given a principal bundle $\pi\colon P \to X$ with structure group G and any connection θ on P, there are DG functors

$$\mathscr{CW}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \longrightarrow \mathbf{Loc}_\infty(X),$$

and

$$\mathscr{C}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g},M) \longrightarrow \mathbf{Loc}_\infty((P\times M)/G),$$

that corresponds to the pullback functor by the classifying map of P. An A_∞-natural isomorphism relates the functors associated with different connections. This construction categorizes the Chern-Weil homomorphism, which is recovered by applying the functor $\mathscr{CW}_\theta$ to the endomorphisms of the constant local system.

Finally, we obtain a categorification of the equivariant de Rham theorem for infinity local systems, namely, the A_∞-fuctor

$$\mathscr{DR}:\mathbf{InfLoc}_\infty(\mathfrak{g},M)\to \mathbf{Loc}_\infty(M_G).$$

Chapter 1

Introduction

The history of A_∞-structures begins in 1957 with the work of Masahiro Sugawara [1]. An H-space is a topological space X with a continuous multiplication $m : X \times X \to X$ with unit element e in X. The H-space is homotopy associative if the two maps $X \times X \times X \to X$ given by the two ways of associating are homotopic. Given a topological space X with a base point x_0 in X, the based loop space denoted by ΩX is the space of paths in X beginning and ending in x_0. The concatenation of paths $* : \Omega X \times \Omega X \to \Omega X$ is a binary operation on ΩX. Sugawara defines a space F to be group-like if it is a homotopy associative H-space satisfying some technical assumptions, which are satisfied by loop spaces. He obtains a criterion for space F to be a group-like space.

Following the ideas of Sugawara, J. Stasheff in the 1960's introduced the Stasheff polytopes or associahedra $\{K_n\}_{n\geq 2}$, where for any fixed n, the polytope K_n has dimension $n-2$, each vertex of the polytope represents a different way of composing n paths in ΩX, and the edges are given by the different homotopies between the ways of composing n paths. For example, for $n = 3$, K_3 has dimension 1 and consists of two vertices given by the two different ways of composing 3 paths and one edge representing the unique homotopy between that compositions. Notice that for any $n \geq 2$, the polytope K_n is a connected, orientable closed manifold; hence the top homology group is $\mathrm{H}_{n-2}(K_n, \mathbb{Z}) \cong \mathbb{Z}$. A choice of a generator of that homology is called the fundamental class denoted by κ_n. Gluing together all homotopies for each n, we obtain the maps $M_n : K_n \times (\Omega X)^n \to \Omega X$ for each $n \geq 2$. Stasheff introduced A_∞-spaces as a refinement of a homotopy associative H-space taking into account higher coherences [2, 3]. An A_∞-space is a topological space Y with operators $M_n : K_n \times Y^n \to Y$ for $n \geq 2$ such that coherence conditions are satisfied encoding certain topological homotopies. Given an A_∞-space Y with operators $M_n : K_n \times Y^n \to Y$, let $A = C_\bullet(Y)$ be the singular chain complex of Y with boundary operator $\partial : A \to A$, which we denote m_1. Taking push-forward on the operators M_n, we obtain maps $(M_n)_\# : C_\bullet(K_n) \otimes A^{\otimes n} \to A$, adn therefore we can define operators $m_n : A^{\otimes n} \to A$ for $n \geq 2$, by the formula

$$m_n(\sigma_1 \otimes \cdots \otimes \sigma_n) := (M_n)_\#(\kappa_n \otimes \sigma_1 \otimes \cdots \otimes \sigma_n),$$

where $\kappa_n \in C_\bullet(K_n)$ is the fundamental class of K_n. The topological homotopies encoded by $\{M_n\}$ turn into algebraic homotopies encoded by $\{m_n\}$. So, the concept of A_∞-algebras come as linearized versions of A_∞-spaces, and this is the origin of the theory of A_∞-algebras. These structures with their corresponding morphisms, namely A_∞-morphisms, A_∞-functors, and A_∞-natural transformations, are of special interest in the study of the Riemann-Hilbert correspondence and other results that we will study in this thesis.

Jim Stasheff and Tom Lada introduced L_∞-algebras in [4,5] based on the work of Barton Zwiebach in [6]. A differential graded Lie algebra is a graded Lie algebra $L = \bigoplus_i L_i$ equipped with a differential that acts as a graded derivation with respect to the Lie bracket. The Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a differential graded algebra of elements dual to $\mathfrak{g}$ whose differential encodes the Lie bracket on $\mathfrak{g}$. As in the case of the A_∞-algebras where associativity is given up to homotopy, the L_∞-algebras are a higher generalization of differential graded Lie algebras, since in an L_∞-algebra the Jacobi identity is allowed to hold up to homotopy. The morphisms between L_∞-algebras are called L_∞-morphisms. An element x of a differential graded Lie algebra with differential d is said to be a Maurer-Cartan element if $\mathrm{d}x + \frac{1}{2}[x,x] = 0$. The relationship between the set of L_∞-morphisms of two L_∞-algebras L and L' and the set of Maurer-Cartan elements of the differential graded Lie algebra $\mathrm{CE}(L) \otimes L'$ will play an essential role in developing our theory.

A local system on a topological space X is a representation of the fundamental groupoid $\pi_1 X$ of X. The ∞-groupoid $\pi_\infty(X_\bullet)$ of X is the simplicial set $X_\bullet$, where X_k is the set of smooth k-simplices in X. A representation up to homotopy of the groupoid is a $\mathbb{Z}$-graded vector bundle over X together with a way to assign holonomies to simplices of all dimensions in a coherent manner. Replacing the fundamental groupoid with the ∞-groupoid obtains the theory of ∞-local systems. An ∞-local system on X is a representation up to homotopy of $\pi_\infty(X_\bullet)$, which form a DG category denoted by $\mathbf{Rep}_\infty(\pi_\infty(X))$. Just like ordinary local systems can be described in several ways, such as flat vector bundles, the DG category of ∞-local systems also admits various descriptions.

Point of view	**∞-local system**
Infinitesimal	Flat superconnection
Simplicial	Representation of $\pi_\infty X_\bullet$
Topological	Representation of $\mathbb{C}_\bullet(\Omega^{\mathrm{M}} X)$

Here $\mathbb{C}_\bullet(\Omega^{\mathrm{M}} X)$ denotes the algebra of singular chains on the based Moore loop space of X. Each of these notions of ∞-local system can be organized into a DG category, and these categories are quasi-equivalent. The proofs of these results can be found in [7–10]. The equivalence between the "infinitesimal" and the "simplicial" points of view is known as the higher Riemann-Hilbert correspondence.

From the infinitesimal point of view, an ∞-local system over X is a $\mathbb{Z}$-graded vector bundle over X of finite rank and a flat $\mathbb{Z}$-graded connection. The DG category obtained in this way is denoted by $\mathbf{Loc}_\infty(X)$. In this setting, the higher Riemann-Hilbert correspondence states that there is an A_∞-equivalence $\mathbf{Loc}_\infty(X) \to \mathbf{Rep}_\infty(\pi_\infty(X_\bullet))$. In other words, a flat $\mathbb{Z}$-graded connection over X provides enough data to construct holonomies along simplices of all dimensions. The higher Riemann-Hilbert correspondence can be thought of as a categorical generalization of the de Rham theorem, where the last one states that the de Rham complex $\Omega^\bullet(X)$ and the cochain complex $C^\bullet(X)$ are equivalent in the sense that both have isomorphic cohomologies. The equivalence is proven via an integration map $\int : \Omega^\bullet(X) \to C^\bullet(X)$ that descends to an isomorphism of algebras in cohomology.

Following the analogy between the de Rham complex $\Omega^\bullet(X)$ and the category of ∞-local systems $\mathbf{Loc}_\infty(X)$, in [11] one of the fundamental properties of de Rham's cohomology is studied in the context of ∞-local systems. Namely, two homotopical maps $f,g : X \to Y$ induce the same morphism in cohomology, i.e. $f^* = g^* : \mathrm{H}(Y) \to \mathrm{H}(X)$. The ∞ version given in [11], states that if X and Y are smooth manifolds and let $f,g : X \to Y$ be smooth maps, if $h : [0,1] \times X \to Y$ is a homotopy with $h \circ i_0 = f$ and $h \circ i_1 = g$, then there is an A_∞-natural isomorphism $\mathrm{hol} : f \Rightarrow g$ which depends on h. As a corollary of the previous result, in [11] is obtained an A_∞ version of the Poincaré lemma. This states that if X is contractible, then $\mathbf{Loc}_\infty(X)$ is equivalent to the category of ∞-local systems over a point, which is the category DGVectR of differential graded vector spaces. These results are essential for obtaining the categorification of the Chern-Weil homomorphism.

The Chern-Weil theory was developed in the late 1940s by Shiing-Shen Chern [12] and André Weil [13] and gives us a bridge between the areas of algebraic topology and differential geometry. The main result in Chern-Weil theory is the Chern-Weil homomorphism, which allows us to compute topological invariants of vector bundles and principal bundles on a smooth manifold M in terms of connections and curvature representing classes in the de Rham cohomology ring of M. These classes are called the characteristic classes, and these have a large number of applications in both mathematics and physics.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, there exists a classifying space BG, such that isomorphism classes of principal G-bundles over X are in natural bijective correspondence with the set of homotopy classes $[X,BG]$. The correspondence is given by pulling back a universal principal G-bundle over BG. When G is discrete, BG is an Eilenberg-Maclane space of type $(G,1)$. When G is the unitary group $U(n)$, BG is homotopy equivalent to the infinite Grassmanian $G_n(\mathbb{C}^\infty)$. Given a function $f : X \to BG$ in $[X,BG]$, taking cohomology we get a homomorphism of rings $f^\bullet : \mathrm{H}^\bullet(BG) \to \mathrm{H}^\bullet(X)$. The cohomology of the classifying space BG is isomorphic to the algebra of invariant polynomials $(\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}}$ on $\mathfrak{g}$. When we take $G = U(n)$, we obtain the homomorphism $f^\bullet : \mathrm{H}^\bullet(G_n(\mathbb{C}^\infty)) \to \mathrm{H}^\bullet(X)$, where the cohomology of the infinite Grassmanian $\mathrm{H}^\bullet(G_n(\mathbb{C}^\infty))$ is the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$, with the variables $x_1,\ldots,x_n$ the Chern classes.

Let $\pi\colon P \to X$ be a principal G-bundle with Lie algebra $\mathfrak{g}$ and given any connection θ on P; the Chern-Weil homomorphism is an algebra map

$$c_{\theta*}\colon (\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}} \to \mathrm{H}^\bullet_{\mathrm{DR}}(X),$$

where $(\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}}$ is the algebra of invariant polynomials on the symmetric algebra of $\mathfrak{g}^*$, which can be described geometrically in terms of the curvature and corresponds to the map induced in cohomology by the classifying map $f\colon X \to BG$ of P, previously discussed. Our first goal is to categorify this construction by replacing the cohomology of BG with a more abstract invariant, the DG category $\mathbf{Loc}_\infty(BG)$ of ∞-local systems on BG.

Given a topological group G, a G-manifold M is a manifold M with a left G-action. Equivariant cohomology aims to compute the cohomology of a G-manifold M taking into account the symmetries of that space given by the action of the group G. In 1950, Cartan [14, 15] constructed a differential complex $((\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\mathrm{inv}}, D)$ out of the differential forms on a G-manifold M, with

G a compact, connected Lie group. Cartan's complex computes the real equivariant cohomology of a G-manifold M, in the same way that the de Rham complex of smooth differential forms computes the real cohomology of a manifold M. In 1960, Borel [16] defined equivariant cohomology of a smooth manifold M using the Borel construction. Let EG be the total space of the classifying space BG; since EG is a contractible space on which the group G acts freely, the space $EG\times M$ will have the same homotopy type as M and the diagonal action of G on $EG\times M$ will always be free. Hence the Borel construction or homotopy quotient M_G of the manifold M is defined to be the quotient of $EG\times M$ by the diagonal action of G. The equivariant cohomology $\mathrm{H}^\bullet_G(M)$ of the G-manifold M is defined to be the cohomology $\mathrm{H}^\bullet(M_G)$ of the homotopy quotient M_G. The equivariant de Rham theorem state that if G is a compact connected Lie group and M is a G-manifold, there is a graded-algebra isomorphism between the equivariant cohomology of M and the cohomology of the Cartan model $(\mathrm{S}^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M))_{\mathrm{inv}}$,

$$\mathrm{H}^\bullet_G(M)\cong\mathrm{H}^\bullet((\mathrm{S}^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M))_{\mathrm{inv}}).$$

The second goal of this thesis is to show the equivariant de Rham theorem in the context of ∞-local systems by replacing equivariant cohomology $\mathrm{H}^\bullet_G(M)$ of a G-manifold M by the analog in the context of ∞-local systems, the equivariant local systems $\mathbf{Loc}_\infty(M_G)$, where M_G is the homotopic quotient of M.

In analogy with the Chern-Weil construction, we introduce a DG category $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$, which is the infinitesimal counterpart of $\mathbf{Loc}_\infty(BG)$. This DG category is defined as follows. Given a Lie algebra $\mathfrak{g}$, we consider the DG Lie algebra $\mathbb{T}\mathfrak{g}$ universal for the Cartan relations. The DG category $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ is a certain subcategory of the DG category of L_∞-representations of $\mathbb{T}\mathfrak{g}$.

Our main first result is the construction of a DG functor that extends the Chern-Weil homomorphism [17].

Theorem A. *Let G be a Lie group and let $\pi\colon P\to X$ be a principal bundle with structure group G. Then, for any connection θ on P, there is a natural DG functor*

$$\mathscr{CW}_\theta\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g})\longrightarrow\mathbf{Loc}_\infty(X).$$

Moreover, for any two connections θ and θ' on P, there is a A_∞-natural isomorphism between $\mathscr{CW}_\theta$ and $\mathscr{CW}_{\theta'}$.

Our second result states that, as expected, the functor $\mathscr{CW}_\theta$ corresponds to the pullback functor by the classifying map [17].

Theorem B. *Given a compact connected Lie group G, there is a natural A_∞-functor*

$$\mathscr{W}\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g})\to\mathbf{Loc}_\infty(BG),$$

which is an A_∞-quasi-equivalence. Moreover, if we let $\pi\colon P\to X$ be a principal bundle with structure group G, θ be any connection on P, and $f\colon X\to BG_n$ be the classifying map of P, then there exists an A_∞-natural isomorphism between the A_∞-functors $\mathscr{I}\circ\mathscr{CW}_\theta$ and $(\varphi_n\circ f)^*_\bullet\circ\mathscr{W}$

from $\mathbf{InfLoc}_\infty(\mathfrak{g})$ *to* $\mathbf{Rep}_\infty(\pi_\infty X_\bullet)$. *Here* $\mathcal{I}$ *is the integration* A_∞*-functor provided by the higher Riemann-Hilbert correspondence, and* φ_n *is the canonical map from* BG_n *to* BG.

In analogy with the equivariant de Rham theorem, we introduce a DG category $\mathbf{InfLoc}_\infty(\mathfrak{g}, M)$, which is the infinitesimal counterpart of $\mathbf{Loc}_\infty(M_G)$.

Our third result is the generalization of Theorem A [18].

Theorem C. *Let* G *be a Lie group and let* $\pi : P \to X$ *be a principal bundle with structure group* G. *Then, for any connection* θ *on* P, *there is a natural DG functor*

$$\mathscr{C}_\theta : \mathbf{InfLoc}_\infty(\mathfrak{g}, M) \to \mathbf{Loc}_\infty((P \times M)/G).$$

Moreover, for any two connections θ *and* θ' *on* P, *there is an* A_∞*-natural isomorphism between* $\mathscr{C}_\theta$ *and* $\mathscr{C}_{\theta'}$.

With this in hand, the last result is the construction of a DG functor that extends the equivariant de Rham theorem in the context of ∞-local systems [18].

Theorem D. *Given a compact connected Lie group* G, *the* A_∞*-fuctor*

$$\mathscr{DR} : \mathbf{InfLoc}_\infty(\mathfrak{g}, M) \to \mathbf{Loc}_\infty(M_G)$$

is an A_∞*-quasi-equivalence. In addition, for any principal* G*-bundle* $\pi : P \to X$ *with connection* θ *and classifying map* $\overline{f} : P \to EG_n$, *given a smooth* G*-manifold* M *simply connected, there exists an* A_∞*-natural isomorphism between the* A_∞*-functors* $\mathcal{I} \circ \mathscr{C}_\theta$ *and* $(\varphi_n \circ \overline{f})^*_\bullet \circ \mathscr{DR}$ *from* $\mathbf{InfLoc}_\infty(\mathfrak{g}, M)$ *to* $\mathbf{Rep}_\infty(\pi_\infty((P \times M)/G_\bullet)$. *Here* $\mathcal{I}$ *is the integration* A_∞*-functor provided by the higher Riemann-Hilbert correspondence, and* φ_n *is the canonical map from* EG_n *to* EG.

Let us explain the relationship between our results and the DG category of modules over the algebra $\mathbb{C}_\bullet(G)$ of singular chains on G. The correspondence between representations of $\mathbb{T}\mathfrak{g}$ and modules over $\mathbb{C}_\bullet(G)$ was studied in [19, 20]. In these works, it is proved that for a simply-connected group G, the category of representations of $\mathbb{T}\mathfrak{g}$ is equivalent to the category of "sufficiently smooth" modules over $\mathbb{C}_\bullet(G)$. Moreover, if G is also compact, the DG enhancements of these categories are also A_∞-quasi-equivalent. Precisely, the following is proved in [19].

Theorem 1.1. Suppose that G is a compact and simply connected group. The DG categories $\mathbf{DGRep}(\mathbb{T}\mathfrak{g})$ of representations of $\mathbb{T}\mathfrak{g}$ and $\mathbf{DGMod}(\mathbb{C}_\bullet(G))$ of modules over $\mathbb{C}_\bullet(G)$ are A_∞-quasi-equivalent.

Given that the DG category $\mathbf{DGRep}(\mathbb{T}\mathfrak{g})$ is a subcategory of $\mathbf{InfLoc}_\infty(\mathfrak{g})$, one concludes that the category of modules over $\mathbb{C}_\bullet(G)$ consists of ∞-local systems on BG. This should be expected on topological grounds because G is A_∞ equivalent to $\Omega^{\mathrm{M}}BG$, and therefore, $\mathbb{C}_\bullet(G)$ and $\mathbb{C}_\bullet(\Omega^{\mathrm{M}}BG)$ are homotopy equivalent. However, the argument is infinitesimal in terms of the Lie algebra $\mathfrak{g}$.

The structure of the thesis

In chapter 2, we will give the preliminary concepts for this thesis. In section 1, we have the basics in A_∞ and L_∞-structures, namely, we review the basic concepts of L_∞-algebras and the idea of DG categories and their A_∞-natural transformations. Section 2 is devoted to studying ∞-local systems, and it is the principal object of the thesis. In section 2, we have the basics of iterated integrals and a posterior discussion of the A_∞ version of Poincare Lemma. Finally, this section reviews the notions of representations up to homotopy and the higher Riemann-Hilbert correspondence.

Chapter 3 is devoted to defining the DG category $\mathbf{Loc}_\infty(\mathfrak{g})$, which is the analog of the algebra of invariant polynomials $(\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}}$. Subsequently, the Chern-Weil isomorphism for ∞-local systems is proved [17]. The last section of this chapter proves that the DG category $\mathbf{Loc}_\infty(\mathfrak{g})$ is A_∞-quasi-equivalent to the DG category $\mathbf{Loc}_\infty(BG)$ of ∞-local systems on the classifying space of G.

Finally, in chapter 4, we generalize the equivariant de Rham theorem for ∞-local systems [18], which allows us to calculate ∞-local systems on the homotopy space M_G of M. To show this theorem, we define the DG category $\mathbf{Loc}_\infty(\mathfrak{g}, M)$ given an analog for the cohomology of the Weil model of the Lie algebra $\mathfrak{g}$.

Chapter 2

Preliminaries

This chapter recalls some terminology and results concerning DG Lie algebras, DG categories, and ∞-local systems. The books by Guillemin-Sternberg and Meinrenken [21, 22] discuss the construction of the Weil algebra of a Lie algebra. For a detailed exposition of L_∞-algebras and morphisms, we recommend [23]. The paper by Keller [24] provides an excellent introduction to DG categories. Our conventions on ∞-local systems and the higher Riemann-Hilbert correspondence are taken from [8, 9, 25, 26]. For A_∞ structures see [27].

2.1 A_∞ and L_∞-Structures

2.1.1 Graded Vector Spaces

This section will review the basic definitions of graded vector spaces and their respective morphisms. The following section will use these definitions to abstract the algebraic idea behind the geometric construction discussed. All vector spaces and algebras are defined over the field of real numbers $\mathbb{R}$. All cochain complexes are cohomologically graded; the differential increases the degree by 1.

A $\mathbb{Z}$-*graded vector space* V is a vector space which decomposes into a direct sum of vector spaces over $\mathbb{Z}$ of the form

$$V = \bigoplus_{n\in\mathbb{Z}} V^n,$$

where for every $n \in \mathbb{Z}$, the elements of V^n are called *homogeneous* elements of *degree* n, in this case if $v \in V^n$, we write $|v| = n$.

The *suspension* of the $\mathbb{Z}$-graded vector space V is denoted by $\mathsf{s}V$, that is the same vector space with a shift in degrees given by

$$(\mathsf{s}V)^n := V^{n-1},$$

and by $\mathsf{u}V$ its *unsuspension*, which is the same graded vector space with grading defined by

$$(\mathsf{u}V)^n := V^{n+1}.$$

A familiar example of graded vector space is the vector space of all polynomials in one or several variables where the homogeneous elements of degree n are linear combinations of monomials of degree n.

Let V and W be a $\mathbb{Z}$-graded vector spaces. A *homogeneous morphism* of degree m is a linear map $f : V \to W$ such that for any homogeneous element $v \in V$ with $|v| = n$, we have that $|f(v)| = n+m$, i.e., the linear map f shifts the degree of the elements in V by m. In this case we write $|f| = m$ and $f : V^{\bullet} \to W^{\bullet+m}$. For example the *identity morphism* denoted by $1_V : V \to V$ of a $\mathbb{Z}$-graded vector space V, is a homogeneous morphism of degree zero.

Let V_i be $\mathbb{Z}$-graded vector spaces, with $i = 1,\dots,n$, then the *tensor product* $V_1 \otimes \cdots \otimes V_n$ is also a $\mathbb{Z}$-graded vector spaces with degree

$$(V_1 \otimes \cdots \otimes V_n)^m = \bigoplus_{m_1+\cdots+m_n=m} V_1^{m_1} \otimes \cdots \otimes V_n^{m_n}.$$

If we have homogeneous elements $v_i \in V_i$, with $i = 1,\dots,n$, then the tensor product element $v_1 \otimes \cdots \otimes v_n$ is homogeneous with degree

$$|v_1 \otimes \cdots \otimes v_n| = |v_1| + \cdots + |v_n|.$$

Let V and W be $\mathbb{Z}$-graded vector spaces. For homogeneous elements $v \in V$ and $w \in W$, the *Koszul sign convention* states that when v moves past w, then a sign change of $(-1)^{|v||w|}$ is required.

Let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ be homogeneous maps. Using the previous Koszul sign convention we define $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ by

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|v_1||f_2|} f_1(v_1) \otimes f_2(v_2),$$

for any homogeneous elements $v_1 \in V_1$ and $v_2 \in V_2$. More generally, if $f_i : V_i \to W_i$ are homogeneous, with $i = 1,\dots,n$, then

$$(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = (-1)^{\sigma} f_1(v_1) \otimes \cdots \otimes f_n(v_n),$$

where $\sigma = |v_1||f_2| + \cdots + (|v_1 \otimes \cdots \otimes v_{n-1}|)|f_n|$, for any homogeneous elements $v_i \in V_i$, with $i = 1,\dots,n$.

For homogeneous maps of the form $f : V^{\otimes m} \to W$, taking suspensions on all factors we get a map $\hat{f} : (\mathsf{s}V)^{\otimes m} \to \mathsf{s}W$. The following commutative diagram determines the relation between the degrees $|f|$ and $|\hat{f}|$

$$\begin{array}{ccc} V^{\otimes m} & \xrightarrow{\;f\;} & W \\ {\scriptstyle \mathsf{s}^{\otimes m}}\big\downarrow & & \big\uparrow{\scriptstyle \mathsf{u}} \\ (\mathsf{s}V)^{\otimes m} & \xrightarrow[\hat{f}]{} & \mathsf{s}W, \end{array}$$

therefore we get

$$|f| = 1 + |\hat{f}| - m.$$

A *differential graded vector space* (DG vector space) or *cochain complex* is a $\mathbb{Z}$-graded vector space V with a homogeneous map $\mathrm{d}_V : V^\bullet \to V^{\bullet+1}$ called a *differential*, such that $\mathrm{d}_V^2 = 0$. The subscript in d_V will be dropped when V is clear from the context. Let (V, d_V) and (W, d_W) be DG vector spaces, then the morphisms between V and W are the homogeneous maps for all degrees. Therefore the DG vector spaces are the objects of the category denoted by DGVect, with morphisms given by

$$\mathrm{Hom}_{\mathsf{DGVect}}(V,W) = \bigoplus_n \mathrm{Hom}^n_{\mathsf{DGVect}}(V,W),$$

where $\mathrm{Hom}^n_{\mathsf{DGVect}}(V,W)$ is the set of homogeneous maps of degree n.

Notice that the category DGVect is a monoidal category. Let (V, d_V) and (W, d_W) be DG vector spaces, then the tensor product $V \otimes W$ has a differential $\mathrm{d}_{V\otimes W} = \mathrm{d}_V \otimes 1_W + 1_V \otimes \mathrm{d}_W$. Using the Koszul sign convention we can check that $\mathrm{d}^2_{V\otimes W} = 0$. More in general, let (V_i, d_{V_i}) be a DG vector spaces, with $i = 1, \ldots, n$, then the tensor product $V_1 \otimes \cdots \otimes V_n$ has a differential

$$\sum_{i=1}^n 1_{V_1} \otimes \cdots \otimes 1_{V_{i-1}} \otimes \mathrm{d}_{V_i} \otimes 1_{V_{i+1}} \otimes \cdots \otimes 1_{V_n}.$$

The condition $\mathrm{d}^2 = 0$ implies that

$$\mathrm{im}(\mathrm{d}\colon V^{n-1} \to V_n) \subset \ker(\mathrm{d}\colon V^n \to V^{n+1}),$$

for all n. Therefore we can define the *cohomology* of the DG vector space (V, d) by

$$H^\bullet(V) = \bigoplus_n H^n(V),$$

where

$$H^n(V) := \frac{\ker(\mathrm{d}\colon V^n \to V^{n+1})}{\mathrm{im}(\mathrm{d}\colon V^{n-1} \to V^n)}.$$

Given $v \in \ker(\mathrm{d})$, the class that it defines in cohomology is denoted by $\overline{v}$.

A homogeneous morphism $f : V \to W$ is called a *morphism of complexes* if

$$\mathrm{d}_W \circ f = (-1)^{|f|} f \circ \mathrm{d}_V.$$

Hence, we have the category of *cochain complexes* denote by cCh, it is a subcategory of DGVect. A morphism of complexes $f : V \to W$ induces a morphism of the same degree $\overline{f} : H^\bullet(V) \to H^\bullet(W)$ by the formula $\overline{f}(\overline{v}) = \overline{f(v)}$, for all $\overline{v} \in H^\bullet(V)$. If the morphism of complexes f induces an isomorphism in cohommology, then f is called a *quasi-isomorphism*. Two homogeneous morphisms of complexes $f, g : V \to W$ of degrees $|f| = |g| = n$ are *homotopic* if there is a morphism h (not necessarily of complexes) of degree $|h| = n-1$, such that

$$f - g = \mathrm{d}_W \circ h - (-1)^{|h|} h \circ \mathrm{d}_V.$$

If f and g are homotopic, then they define the same morphism in cohomology $\overline{f} = \overline{g}$.

$\mathrm{Hom}_{\mathsf{DGVect}}(V,W)$ is a DG vector space with differential denote by $D_{V,W}$, given by

$$D_{V,W}(f) = \mathrm{d}_W \circ f - (-1)^{|f|} f \circ \mathrm{d}_V,$$

for all homogeneous morphism f. Notice that f is a morphism of complexes if and only if $D_{V,W}(f) = 0$ and $|f| = 0$. Two morphisms of complexes f and g are homotopic if and only if there is h such that $D_{V,W}(h) = f - g$, and therefore they define the same class in cohomology $H(\mathrm{Hom}_{\mathsf{DGVect}}(V,W))$.

2.1.2 Differential Graded Lie Algebras and L_∞-Algebras

We discuss in this section the concept of L_∞-algebras. First, we introduce the more fundamental idea of differential graded Lie algebra (DG Lie algebra in short) to motivate the definition of L_∞-algebra.

A *differential graded Lie algebra* (DG Lie algebra) is a $\mathbb{Z}$-graded vector space $L = \bigoplus_i L_i$, with a linear map $\mathrm{d} : L \to L$ and a bilinear map $[-,-] : L \otimes L \to L$ called the *bracket*, such that:

- d is a differential of degree -1 that makes (L, d) into a chain complex, i.e., $\mathrm{d}^2 = 0$,
- d is a graded derivation of the bilinear pairing,

$$\mathrm{d}[x_1,x_2] = [\mathrm{d}x_1,x_2] + (-1)^{|x_1|}[x_1,\mathrm{d}x_2],$$

- the bilinear pairing is graded skew-symmetric,

$$[x_1,x_2] = -(-1)^{|x_1||x_2|}[x_2,x_1],$$

- the bilinear pairing satisfies the graded Jacobi identity,

$$[x_1,[x_2,x_3]] = [[x_1,x_2],x_3] + (-1)^{|x_1||x_2|}[x_2,[x_1,x_3]],$$

  for all homogeneous elements x_1, x_2 and x_3 of L.

A *strict morphism* of DG Lie algebras $\phi : L \to L'$ is a homomorphism of graded Lie algebras with $\phi \circ \mathrm{d} = \mathrm{d}' \circ \phi$.

For the first example of DGLA, any DG algebra with the graded commutator is a DG Lie algebra. Let $V = \bigoplus_{k\in\mathbb{Z}} V^k$ be a graded vector space and (V,∂) a chain complex with differential $\partial : V^k \to V^{k+1}$. Taking $\mathrm{End}(V) = \bigoplus_{k\in\mathbb{Z}} \mathrm{End}^k(V)$, where we denote by $\mathrm{End}^k(V)$ the endomorphisms of V of degree k. On $\mathrm{End}(V)$ we define the bilinear map and the derivation respectively by

$$[f,g] = fg - (-1)^{|f||g|} gf,$$

$$\mathrm{d}f = \partial f - (-1)^{|f|} f \partial.$$

With these operations, $\mathrm{End}(V)$ is a DGLA.

If $\mathfrak{g}$ is a finite dimensional Lie algebra, the bracket $[-,-] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ is associate a dual map $\delta = [-,-]^* : \mathfrak{g}^* \to (\Lambda^2\mathfrak{g})^* = \Lambda^2(\mathfrak{g}^*)$. Since $\Lambda^2\mathfrak{g}^*$ is a free algebra generated by $\mathfrak{g}^*$, we can extend the map δ to a map $\widetilde{\delta} : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$. From the Jacobi equation for the bracket $[-,-]$ we have that $\widetilde{\delta}^2 = 0$, and therefore we get the bijection

$$\left\{ \widetilde{\delta} \in \mathrm{Der}^1\left(\Lambda\mathfrak{g}^*\right) | \widetilde{\delta}^2 = 0 \right\} \cong \{\text{Lie brackets in } \mathfrak{g}\}.$$

The previous bijection gives us an alternative form to define a finite-dimensional Lie algebra, as a finite-dimensional vector space $\mathfrak{g}$ with a derivation $\delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ of degree 1 such that $\delta^2 = 0$. But in the case of infinite-dimensional $\mathfrak{g}$ we have some problems with the dual space of $\mathfrak{g}$, so to avoid these problems, we will formulate the definition in terms of coalgebras.

Let K be a field. A *graded K-coalgebra* is defined to be a triple (C, Δ, ε), where $C = \bigoplus_k C^k$ is a graded K-vector space and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to K$ are K-linear maps, called the *comultiplication* and *counit* of C, respectively, such that the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \Delta\otimes \mathrm{id}_C} \\ C \otimes C & \xrightarrow[\mathrm{id}_C \otimes \Delta]{} & C \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{\beta_C} & \downarrow{\scriptstyle\Delta} & {}^{\sigma_C}\searrow & \\ K \otimes C & & & & C \otimes K, \\ & \nwarrow_{\varepsilon\otimes \mathrm{id}_C} & & {}_{\mathrm{id}_C \otimes \varepsilon}\nearrow & \\ & & C \otimes C & & \end{array}$$

where $\beta_C : C \to K \otimes C$ and $\sigma_C : C \to C \otimes K$ are given by $\beta_C(x) = 1_K \otimes x$ and $\sigma_C(x) = x \otimes 1_K$ respectively.

A *morphism of K-coalgebras* $(C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D)$ is a K-linear map $g : C \to D$ such that the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g\otimes g} \\ D & \xrightarrow[\Delta_D]{} & D \otimes D, \end{array}$$

and

$$\begin{array}{ccc} C & & \\ & \searrow^{\varepsilon_C} & \\ {\scriptstyle g}\downarrow & & K. \\ & \nearrow_{\varepsilon_D} & \\ D & & \end{array}$$

A *coaugmentation* of a K-coalgebra (C,Δ,ε) is a morphism $\eta : K \to C$. A K-coalgebra (C,Δ,ε) is said to be *cocommutative* if the diagram

$$\begin{array}{ccc} & C & \\ {}^{\Delta}\swarrow & & \searrow^{\Delta} \\ C\otimes C & \xrightarrow[T_{C,C}]{} & C\otimes C \end{array}$$

is commutative, where $T_{C,C}$ is the twist isomorphism on $C\otimes C$ given by $T_{C,C}(x\otimes y) = (-1)^{|x||y|} y\otimes x$.

We identify a K-coalgebra (C,Δ,ε) with its underlying graded vector space C. A *coderivation* of degree $p \in \mathbb{Z}$ on a coalgebra C, is a linear map $\delta : C \to C$ of degree p such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta\otimes 1+1\otimes\delta} \\ C & \xrightarrow[\Delta]{} & C\otimes C. \end{array}$$

In addition, a coderivation δ of a coaugmented coalgebra (C,η) is a coderivation of C such that $\delta\circ\eta = 0$. A *differential* δ on a coalgebra C, is a coderivation of degree -1 such that $\delta\circ\delta = 0$. The pair (C,δ) is called a *differential graded coalgebra*.

Given a commutative DG algebra A and a DG Lie algebra L, the tensor product $A\otimes L$ has the structure of a DG Lie algebra. The bracket operation and the differential on $A\otimes L$ are

$$\begin{aligned} [a\otimes x, b\otimes y] &= (-1)^{|x||b|} ab\otimes [x,y], \\ \mathrm{d}(a\otimes x) &= \mathrm{d}a\otimes x + (-1)^{|a|} a\otimes \mathrm{d}x, \end{aligned}$$

for homogeneous $a,b\in A$ and $x,y\in L$.

The symmetric group on n letters is denoted by $\mathfrak{S}_n$. If $i_1+\cdots+i_d = n$, then $\mathfrak{S}_{i_1,\ldots,i_d}$ is the set of $(i_1,\ldots,i_d)$-shuffle permutations in $\mathfrak{S}_n$. That is, the set of permutations that preserve the order on each block $\{i_1+\cdots+i_l+1,\ldots,i_1+\cdots+i_{l+1}\}$.

The *symmetric algebra* $\mathrm{S}^\bullet V$ of a graded vector space V is the quotient of the tensor algebra $\mathrm{T}^\bullet V$ by the graded ideal generated by elements of the form $u\otimes v - (-1)^{|u||v|} v\otimes u$, for homogeneous elements $u,v\in V$. We write $v_1\odot\cdots\odot v_k$ for the element represented by $v_1\otimes\cdots\otimes v_k$ in the quotient space $\mathrm{S}^k V$. Given a permutation $\sigma\in\mathfrak{S}_n$, we denote by $\varepsilon(\sigma;v_1,\ldots,v_1)$ the graded Koszul sign, which is defined via

$$v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma; v_1, \ldots, v_n) v_1 \odot \cdots \odot v_n.$$

We denote by $\bigodot^\bullet V$ the cocommutative coalgebra whose underlying vector space is $\mathrm{S}^\bullet V$ and has coproduct given by

$$\Delta(v_1 \odot \cdots \odot v_n) = \sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}_{i,j}} \varepsilon(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)} \otimes v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}.$$

The coalgebra $\bigodot^\bullet V$ is cocommutative, counital and coaugmented, with coaugmentation $c \colon \mathbb{R} \to \bigodot^\bullet V$ given by $1 \mapsto 1 \in \bigodot^0 V = \mathbb{R}$, and counit $\varepsilon \colon \bigodot^\bullet V \to \mathbb{R}$ given by the natural projection. We will refer to the coalgebra $\bigodot^\bullet V$ as the cocommutative coalgebra cogenerated by V. Given another graded vector space W and a linear map of degree zero $\varphi \colon \bigodot^\bullet W \to V$ there is unique coaugmented coalgebra map $\bar{\varphi} \colon \bigodot^\bullet W \to \bigodot^\bullet V$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigodot^\bullet W & \xrightarrow{\bar{\varphi}} & \bigodot^\bullet V \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \pi} \\ & & V. \end{array}$$

Here $\pi \colon \bigodot^\bullet V \to V$ is the natural projection map. The map $\overline{\varphi}$ is given explicitly by the formula

$$\begin{aligned} &\bar{\varphi}(v_1 \odot \cdots \odot v_n) \\ &\quad = \sum_{i_1+\cdots+i_p=n} \sum_{\sigma \in \mathfrak{S}_{i_1,\ldots,i_p}} \varepsilon(\sigma; v_1, \ldots, v_n) \frac{1}{p!} \prod_{k=1}^{p} \varphi(v_{\sigma(i_1+\cdots+i_{k-1}+1)} \odot \cdots \odot v_{\sigma(i_1+\cdots+i_k)}). \end{aligned}$$

Let L be a DG Lie algebra and consider the cocommutative coalgebra $\bigodot^\bullet(\mathsf{u}L)$. The differential and the bracket of L can be encoded in a single coderivation D on $\bigodot^\bullet(\mathsf{u}L)$. Explicitly, this coderivation is defined by setting for $x_1, \ldots, x_n \in L$,

$$\begin{aligned} D(\mathsf{u}x_1 \odot \cdots \odot \mathsf{u}x_n) = \sum_{i=1}^{n} \varepsilon(\sigma_i, \mathsf{u}x_1, \ldots, \mathsf{u}x_n) \mathsf{u}(dx_i) \odot \mathsf{u}x_1 \odot \cdots \odot \mathsf{u}x_{i-1} \odot \mathsf{u}x_{i+1} \odot \cdots \odot \mathsf{u}x_n + \\ \sum_{i<j} \varepsilon(\sigma_{ij}, \mathsf{u}x_1, \ldots, \mathsf{u}x_n)(-1)^{|v_i|} \mathsf{u}[x_i, x_j] \odot \mathsf{u}x_1 \cdots \odot \mathsf{u}x_{i-1} \odot \mathsf{u}x_{i+1} \odot \cdots \odot \mathsf{u}x_{j-1} \odot \mathsf{u}x_{j+1} \odot \cdots \odot \mathsf{u}x_n, \end{aligned}$$

where $\sigma_i \in \mathfrak{S}_n$ is the permutation which sends 1 to i, subtracts one from the integers $2, \ldots, i$ and fixes the other integers, and $\sigma_{ij} \in \mathfrak{S}_n$ is the permutation which sends 1 to i, sends 2 to j, removes two from the integers $2, \ldots, i+1$, removes one from the integers $i+2, \ldots, j$ and fixes the other integers. This coderivation satisfies $D^2 = 0$, so that the coalgebra $\bigodot^\bullet(\mathsf{u}L)$ has the structure of a differential graded coalgebra. Moreover, for any two DG Lie algebras L and L', a linear map $\phi : L \to L'$ of degree 0 is a morphism of DG Lie algebras if and only if $\overline{\mathsf{u}\phi} \circ D = D' \circ \overline{\mathsf{u}\phi}$, where D and D' are the codifferentials on $\bigodot^\bullet(\mathsf{u}L)$ and $\bigodot^\bullet(\mathsf{u}L')$, respectively, and $\overline{\mathsf{u}\phi} : \bigodot^\bullet(\mathsf{u}L) \to \bigodot^\bullet(\mathsf{u}L')$ is the coalgebra map associated to the linear map $\mathsf{u}\phi : \mathsf{u}L \to \mathsf{u}L'$.

We will consider a more general notion of morphism between DG Lie algebras, that of L_∞-morphism. Let L and L' be DG Lie algebras with corresponding codifferentials D and D'. A linear map $\Phi\colon \bigodot^\bullet(\mathrm{u}L) \to \mathrm{u}L'$ of degree 0 is called an *L_∞-morphism* between L and L' if

$$\bar{\Phi} \circ D = D' \circ \bar{\Phi}.$$

Such an L_∞-morphism can be written as the sum

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \dots,$$

where Φ_k is the restriction of Φ to the vector space $\bigodot^k(\mathrm{u}L)$. Strict morphisms of DG Lie algebras are a particular instance of L_∞-morphisms that correspond the case where $\Phi_k = 0$ for $k > 1$.

Let L be a DG Lie algebra. An element $x \in L$ of degree 1 is said to be a *Maurer-Cartan element* if

$$\mathrm{d}x + \frac{1}{2}[x,x] = 0.$$

This equation, which describes an abstract form of "flatness", is known as the *Maurer-Cartan equation*. Given a differential graded Lie algebra L, the dual vector space of the differential graded coalgebra $\bigodot^\bullet(\mathrm{u}L)$ is a differential graded Lie algebra known as the *Chevalley-Eilenberg DG algebra* of L. We shall denote it by $\mathrm{CE}(L)$ and write $\delta_{\mathrm{CE}} = D^*$ for the corresponding differential. In the special case where L is a Lie algebra, the definition reduces to that of the usual Chevalley-Eilenberg complex, which computes the Lie algebra cohomology of L.

The following result, which is a consequence of Proposition 3.3 of [28], will be used throughout the text.

Proposition 2.1. Let L and L' be DG Lie algebras, and suppose that L is finite-dimensional. Then, there is a natural identification between the set of L_∞-morphisms from L to L' and the set of Maurer-Cartan elements of $\mathrm{CE}(L) \otimes L'$.

The identification goes through the following sequence of isomorphisms of vector spaces:

$$\begin{aligned}
\left[\mathrm{CE}(L) \otimes L'\right]^1 &\cong \left[\left(\bigoplus_{k\geq 0} \bigodot^k(\mathrm{u}L)\right)^* \otimes L'\right]^1 \\
&\cong \prod_k \left[\left(\bigodot^k(\mathrm{u}L)\right)^* \otimes L'\right]^1 \\
&\cong \mathrm{Hom}^1\left(\bigodot^\bullet(\mathrm{u}L), L'\right) \\
&\cong \mathrm{Hom}^0\left(\bigodot^\bullet(\mathrm{u}L), \mathrm{u}L'\right).
\end{aligned}$$

Maurer-Cartan elements live on the space on the first vector space, L_∞-morphisms live on the last vector space, and the corresponding conditions map to one another.

The first example of L_∞-algebra is the DGLA's. Given a DGLA $\mathfrak{g}$ with bracket $[-,-] : \bigvee^2(\mathfrak{g}[1]) \to \bigvee^1(\mathfrak{g}[1])$ and derivation $\mathrm{d} : \bigvee^1(\mathfrak{g}[1]) \to \bigvee^1(\mathfrak{g}[1])$, operations of degree 1. Taking the linear map

$\mathrm{d}+[-,-]:\bigvee^1(\mathfrak{g}[1])\to\mathfrak{g}[1]$, we extend this map to the unique coderivation $\delta:\bigvee^1(\mathfrak{g}[1])\to\bigvee^1(\mathfrak{g}[1])$ such that $\delta^2=0$. The previous extension of $\mathrm{d}+[-,-]$ to δ, mean that if $\pi_1:\bigvee^\bullet(\mathfrak{g}[1])\to\mathfrak{g}[1]$ is the projection, then $\pi_1\delta=\mathrm{d}+[-,-]$. Therefore the category of DGLA's is a subcagory of the category of L_∞-algebras. This subcategory is not a full subcategory, so if $\mathfrak{g}$ and $\mathfrak{h}$ are DGLA's, then

$$\mathrm{Hom}_{\mathrm{DGLA}}(\mathfrak{g},\mathfrak{h})\subsetneq\mathrm{Hom}_{L_\infty}(\mathfrak{g},\mathfrak{h}).$$

2.1.3 DG Categories and A_∞-Natural Transformations

In this section, we will review the basic concepts that will allow us to study one of the main structures of the thesis, namely graded differential categories.

A *differential graded algebra* (DG algebra) is a DG vector space $(A,\mathrm{d}\,)$ together with a degree zero operator $m:A\otimes A\to A$ such that

- The Leibniz rule: for all $a_1\otimes a_2\in A\otimes A$ we have

$$\mathrm{d}\,(m(a_1\otimes a_2))=m(\mathrm{d}\,(a_1)\otimes a_2)+(-1)^{|a_1|}m(a_1\otimes\mathrm{d}\,(a_2)).$$

- Associativity: For all $a_1\otimes\ a_2\otimes a_3\in A\otimes A\otimes A$,

$$m(a_1\otimes m(a_2\otimes a_3))-m(m(a_1\otimes a_2)\otimes a_3)=0.$$

To keep an example in mind throughout the section, consider the de Rham complex $\Omega^\bullet(M)$ of a manifold M.

Notice that the product $m:A\otimes A\to A$ induces an associative multiplication $\overline{m}:H^\bullet(A)\otimes H^\bullet(A)\to H^\bullet(A)$ given by $\overline{m}(\overline{a_1}\otimes\overline{a_2})=\overline{m(a_1\otimes a_2)}$, for all $\overline{a_1},\overline{a_2}\in H^\bullet(A)$.

A *homogeneous morphism* of DG algebras $f:(A,\mathrm{d}_A,m_A)\to(B,\mathrm{d}_B,m_B)$ is a homogeneous linear map $f:A\to B$ of degree zero that is compatible with the DG algebra structures:

- The linear map f is a morphism of complexes

$$\mathrm{d}_B\circ f=(-1)^{|f|}f\circ\mathrm{d}_A.$$

- $f(m)=m(f\otimes f)$.

Hence we have the category DGAlg with objects the DG algebras and morphisms given by

$$\mathrm{Hom}_{\mathsf{DGAlg}}(A,B)=\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}^n_{\mathsf{DGAlg}}(A,B),$$

for all DG algebras A and B, where $\mathrm{Hom}^n_{\mathsf{DGAlg}}(A,B)$ is the set of all homogeneous DG algebra morphisms of degree n. Given a morphism $f:A\to B$ of DG algebras, we have the morphism of associative algebras $\overline{f}:H^\bullet(A)\to H^\bullet(B)$ given by $\overline{f}(\overline{a})=\overline{f(a)}$, with degree $|\overline{f}|=|f|$, for all homogeneous morphism f. Notice that the category DGAlg is monoidal. Given DG algebras A and B, the tensor product $A\otimes B$ has a differential

$$\mathrm{d}_{A\otimes B} = \mathrm{d}_A \otimes 1_B + 1_A \otimes \mathrm{d}_B,$$

and a product

$$m_{A\otimes B}((a_1\otimes b_1)\otimes(a_2\otimes b_2)) = (-1)^{|a_2||b_1|} m_A(a_1\otimes a_2)\otimes m_B(b_1\otimes b_2),$$

for all $a_1,a_2\in A$ and $b_1,b_2\in B$. Therefore $(A\otimes B,\mathrm{d}_{A\otimes B},m_{A\otimes B})$ is a DG algebra.

A *differential graded category* (DG category) is a linear category $\mathcal{C}$ such that for every two objects A and B the space of arrows $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is equipped with a structure of a cochain complex of vector spaces, and for every three objects A, B and C the composition map $\mathrm{Hom}_{\mathcal{C}}(B,C)\otimes\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(A,C)$ is a morphism of cochain complexes. Thus, by definition,

$$\mathrm{Hom}_{\mathcal{C}}(A,B) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{\mathcal{C}}(A,B)$$

is a graded vector space with a differential $\mathrm{d}\colon \mathrm{Hom}^n_{\mathcal{C}}(A,B)\to\mathrm{Hom}^{n+1}_{\mathcal{C}}(A,B)$. The elements $f\in\mathrm{Hom}^n_{\mathcal{C}}(A,B)$ are called *homogeneous of degree n*, and we write $|f|=n$. We denote the set of objects of $\mathcal{C}$ by $\mathrm{Ob}\,\mathcal{C}$.

Notice that a DG category with a single object is the same as a DG algebra. The real example of a DG category is the category of cochain complexes of vector spaces, which we denote by DGVect. Its objects are cochain complexes of vector spaces, and the morphism spaces $\mathrm{Hom}_{\mathsf{DGVect}}(V,W)$ are endowed with the differential defined as

$$\mathrm{d}(f) = \delta_W\circ f - (-1)^n f\circ\delta_V,$$

for any homogeneous element f of degree n.

Let $\mathcal{C}$ be a DG category and let $A\in\mathrm{Ob}\,\mathcal{C}$. Given a closed morphism $f\in\mathrm{Hom}^0_{\mathcal{C}}(B,C)$ we define $f_*\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(A,C)$ by $f_*(g)=f\circ g$ for $g\in\mathrm{Hom}_{\mathcal{C}}(A,B)$. It is not difficult to see that f_* is a morphism of cochain complexes. Similarly, if we define $f^*\colon \mathrm{Hom}_{\mathcal{C}}(C,A)\to\mathrm{Hom}_{\mathcal{C}}(B,A)$ by $f^*(h)=h\circ f$ for $h\in\mathrm{Hom}_{\mathcal{C}}(C,A)$, then f^* es a morphism of cochain complexes.

Given a DG category $\mathcal{C}$, one defines an ordinary category $\mathbf{Ho}(\mathcal{C})$ by keeping the same set of objects and replacing each Hom complex by its 0th cohomology. We call $\mathbf{Ho}(\mathcal{C})$ the *homotopy category* of $\mathcal{C}$.

If $\mathcal{C}$ and $\mathcal{D}$ are DG categories, a *DG functor* $F\colon\mathcal{C}\to\mathcal{D}$ is a linear functor whose associated map for $A,B\in\mathrm{Ob}\,\mathcal{C}$,

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{D}}(F(A),F(B)),$$

is a morphism of cochain complexes. Notice that any DG functor $F\colon\mathcal{C}\to\mathcal{D}$ induces an ordinary functor

$$\mathbf{Ho}(F)\colon \mathbf{Ho}(\mathcal{C})\to\mathbf{Ho}(\mathcal{D})$$

between the corresponding homotopy categories. A DG functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *quasi fully faithful* if for every pair of objects $A, B \in \operatorname{Ob}\mathcal{C}$ the morphism $F_{A,B}$ is a quasi-isomorphism. Moreover, the DG functor F is said to be *quasi essentially surjective* if $\mathbf{Ho}(F)$ is essentially surjective. A DG functor that is both quasi fully faithful and quasi essentially surjective is called a *quasi-equivalence*.

A morphism $f \in \operatorname{Hom}^0_{\mathcal{C}}(B,C)$ is said to be a *quasi-isomorphism* if it is closed and its equivalence class in $\mathbf{Ho}(\mathcal{C})$ is an isomorphism. The following lemma is an immediate consequence of the definition.

Lemma 2.1. *Let $f \in \operatorname{Hom}^0_{\mathcal{C}}(B,C)$ be a quasi-isomorphism. Then, for any object $A \in \operatorname{Ob}\mathcal{C}$, both $f_*\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{C}}(A,C)$ and $f^*\colon \operatorname{Hom}_{\mathcal{C}}(C,A) \to \operatorname{Hom}_{\mathcal{C}}(B,A)$ are quasi-isomorphisms.*

Proof. Let us write $[f]$ for the equivalence class of f in $\mathbf{Ho}(\mathcal{C})$. Then, by assumption, $[f]$ has an inverse $[g] \in \operatorname{Hom}_{\mathbf{Ho}(\mathcal{C})}(C,B)$. Let $g \in \operatorname{Hom}^0_{\mathcal{C}}(C,B)$ be a representative of $[g]$. Then, there exists $x \in \operatorname{Hom}^{-1}_{\mathcal{C}}(B,B)$ and $y \in \operatorname{Hom}^{-1}_{\mathcal{C}}(C,C)$ such that

$$\begin{aligned} g \circ f &= \mathrm{id}_A + \mathrm{d}x, \\ f \circ g &= \mathrm{id}_B + \mathrm{d}y. \end{aligned}$$

This implies that

$$\begin{aligned} g_* \circ f_* &= (g \circ f)_* = (\mathrm{id}_A + \mathrm{d}x)_*, \\ f_* \circ g_* &= (f \circ g)_* = (\mathrm{id}_B + \mathrm{d}y)_*. \end{aligned}$$

Since $(\mathrm{id}_A + \mathrm{d}x)_*$ and $(\mathrm{id}_B + \mathrm{d}y)_*$ induce the identity on cohomology, one concludes that f_* is a quasi-isomorphism. Similarly,

$$\begin{aligned} f^* \circ g^* &= (g \circ f)^* = (\mathrm{id}_A + \mathrm{d}x)^*, \\ g^* \circ f^* &= (f \circ g)^* = (\mathrm{id}_B + \mathrm{d}y)^*, \end{aligned}$$

and since $(\mathrm{id}_A + \mathrm{d}x)^*$ and $(\mathrm{id}_B + \mathrm{d}y)^*$ induce the identity on cohomology, one gets that f^* is a quasi-isomorphism. □

There is a more general notion of a functor between DG categories, that of A_∞-functor, where the composition is preserved only up to an infinite sequence of coherence conditions. It will be helpful to introduce first the Hochschild chain complex of a DG category.

Let $\mathcal{C}$ be a small DG category. The *Hochschild cochain complex* of $\mathcal{C}$ denoted $\mathrm{HC}(\mathcal{C})$ is the cochain complex defined as follows. As a vector space

$$\mathrm{HC}(\mathcal{C}) = \bigoplus_{A_0,\dots,A_n} \mathsf{u}\operatorname{Hom}_{\mathcal{C}}(A_{n-1},A_n) \otimes \cdots \otimes \mathsf{u}\operatorname{Hom}_{\mathcal{C}}(A_0,A_1),$$

where $A_0,\dots,A_n$ range through the objects of $\mathcal{C}$. The differential b is the sum of two components

b_1 and b_2, which are given by the formulas

$$b_1(f_{n-1}\otimes\cdots\otimes f_0)=\sum_{i=0}^{n-1}(-1)^{\sum_{j=i+1}^{n-1}|f_j|+n-i-1}f_{n-1}\otimes\cdots\otimes df_i\otimes\cdots\otimes f_0$$

and

$$b_2(f_{n-1}\otimes\cdots\otimes f_0)=\sum_{i=0}^{n-2}(-1)^{\sum_{j=i+2}^{n-1}|f_j|+n-i}f_{n-1}\otimes\cdots\otimes(f_{i+1}\circ f_i)\otimes\cdots\otimes f_0$$

for homogeneous elements $f_0\in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1),\dots,f_{n-1}\in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_{n-1},A_n)$. Here d denotes indistinctly the differential in any of the spaces $\mathrm{Hom}_{\mathcal{C}}(A_i,A_{i+1})$. It is easy to check that indeed $b^2=0$.

Let $\mathcal{C}$ and $\mathcal{D}$ be DG categories. An A_∞*-functor* $F\colon \mathcal{C}\to\mathcal{D}$ is the datum of a map of sets $F_0\colon \mathrm{Ob}\,\mathcal{C}\to \mathrm{Ob}\,\mathcal{D}$ and a collection of K-linear maps of degree 0

$$F_n\colon \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_{n-1},A_n)\otimes\cdots\otimes \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1)\to \mathrm{Hom}_{\mathcal{D}}(F_0(A_0),F_0(A_n))$$

for every collection $A_0,\dots,A_n\in\mathrm{Ob}\,\mathcal{C}$, such that the relation

$$b_1\circ F_n+\sum_{i+j=n}b_2\circ(F_i\otimes F_j)=\sum_{i+j+1=n}F_n\circ(\mathrm{id}^{\otimes i}\otimes b_1\otimes\mathrm{id}^{\otimes j})+\sum_{i+j+2=n}F_{n-1}\circ(\mathrm{id}^{\otimes i}\otimes b_2\otimes\mathrm{id}^{\otimes j})$$

is satisfied for any $n\geq 1$. We also require that $F_1(\mathrm{id}_A)=\mathrm{id}_{F_0(A)}$ for all objects A in $\mathcal{C}$, as well as $F_n(f_{n-2}\otimes\cdots\otimes f_i\otimes\mathrm{id}_{A_i}\otimes f_{i-1}\otimes\cdots\otimes f_0)=0$ for any $n\geq 1$, any $0\leq i\leq n-2$, and any chain of morphisms $f_0\in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1),\dots,f_{n-2}\in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_{n-2},A_{n-1})$.

The above relation when $n=1$ implies that F_1 is a morphism of cochain complexes. On the other hand, for $n=2$ we find that F_1 preserves the compositions on $\mathcal{C}$ and $\mathcal{D}$, up to a homotopy defined by F_2. In particular, a DG functor between $\mathcal{C}$ and $\mathcal{D}$ is identified with and A_∞-functor having $F_n=0$ for $n\geq 2$. It also follows that F_1 induces an ordinary functor

$$\mathbf{Ho}(F_1)\colon \mathbf{Ho}(\mathcal{C})\to\mathbf{Ho}(\mathcal{D}).$$

An A_∞-functor $F\colon\mathcal{C}\to\mathcal{D}$ is called A_∞*-quasi fully faithfull* if F_1 is a quasi-isomorphism of each pair of objects, F is called A_∞*-quasi essentially surjective* if $\mathbf{Ho}(F_1)$ is essentially surjective. Moreover, the A_∞-functor F is called a A_∞*-quasi-equivalence* if it is both quasi fully faithfull and quasi essentially surjective.

We need one more notion. Let $\mathcal{C}$ and $\mathcal{D}$ be DG categories and let $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{C}\to\mathcal{D}$ be DG functors. An A_∞*-natural transformation* $\lambda\colon F\Rightarrow G$ is the datum of a closed morphism $\lambda_0(X)\in\mathrm{Hom}^0_{\mathcal{D}}(F(A),G(A))$ for each $A\in\mathrm{Ob}\,\mathcal{C}$ and a collection of K-linear maps of degree 0

$$\lambda_n\colon \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_{n-1},A_n)\otimes\cdots\otimes \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1)\to\mathrm{Hom}_{\mathcal{D}}(F(A_0),G(A_n))$$

for every collection $A_0,\dots,A_n\in\mathrm{Ob}\,\mathcal{C}$, such that for all composable chains of homogeneous morphisms $f_0 \in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1),\dots,f_{n-1} \in \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_{n-1},A_n)$ the relation

$$\begin{aligned} G(f_{n-1}) \circ \lambda_{n-1}(f_{n-2} \otimes \cdots \otimes f_0) - (-1)^{\sum_{i=1}^{n-1} |f_i| - n+1} \lambda_{n-1}(f_{n-1} \otimes \cdots \otimes f_1) \circ F(f_0) \\ = \lambda\left(b(f_{n-1} \otimes \cdots \otimes f_0)\right) - d\left(\lambda_n(f_{n-1} \otimes \cdots \otimes f_0)\right) \end{aligned}$$

is satisfied for any $n \geq 1$. The λ on the right denotes the direct sum of the various λ_n. For $n = 1$ this yields the condition

$$G(f_0) \circ \lambda_0(A_0) - \lambda_0(A_1) \circ F(f_0) = \lambda_1\left(\mathrm{d}(f_0)\right) - \mathrm{d}\left(\lambda_1(f_0)\right).$$

Since the map $\lambda_1 \colon \mathsf{u}\mathrm{Hom}_{\mathcal{C}}(A_0,A_1) \to \mathrm{Hom}_{\mathcal{D}}(F(A_0),G(A_1))$ has degree -1 when considered as a map defined over $\mathrm{Hom}_{\mathcal{C}}(A_0,A_1)$, this implies that the diagram

$$\begin{array}{ccc} F(A_0) & \xrightarrow{\lambda_0(A_0)} & G(A_0) \\ {\scriptstyle F(f_0)}\downarrow & & \downarrow{\scriptstyle G(f_0)} \\ F(A_1) & \xrightarrow[\lambda_0(A_1)]{} & G(A_1) \end{array}$$

commutes up to a homotopy given by λ_1.

As usual, A_∞-natural transformations can be composed: if $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{C} \to \mathcal{D}$ and $H \colon \mathcal{C} \to \mathcal{D}$ are three DG functors from the DG category $\mathcal{C}$ to the DG category $\mathcal{D}$, and $\lambda \colon F \Rightarrow G$ and $\mu \colon G \Rightarrow H$ are two A_∞-natural transformations, then the formula

$$(\mu \circ \lambda)_n = \sum_{i=0}^{n} \mu_i \circ \lambda_{n-i}$$

defines a new A_∞-natural transformation $\mu \circ \lambda \colon F \Rightarrow H$. An *$A_\infty$-natural isomorphism* between functors from $\mathcal{C}$ to $\mathcal{D}$ is an A_∞-natural transformation λ such that $\lambda_0(A)$ is an isomorphism for all $A \in \mathrm{Ob}\,\mathcal{C}$.

We close this subsection with the following observation.

Lemma 2.2. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be DG categories and let $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{C} \to \mathcal{D}$ and $H \colon \mathcal{D} \to \mathcal{E}$ be DG functors. Then for each A_∞-natural transformation $\lambda \colon F \Rightarrow G$ there is an induced A_∞-natural transformation $H \circ \lambda \colon H \circ F \Rightarrow H \circ G$. Moreover, if λ is an A_∞-natural isomorphism, then $H \circ \lambda$ is an A_∞-natural isomorphism.*

Proof. The formula for $H \circ \lambda$ reads

$$(H \circ \lambda)_n = \sum_{k=2}^{n+2} \sum_{i+j=k} H_{k-1} \circ (G^{\otimes(i-1)} \otimes \lambda_{n+2-k} \otimes F^{\otimes(j-1)}),$$

for any $n \geq 1$. Let us check that this indeed defines an A_∞-natural transformation between $H \circ F$ and $H \circ G$. For this purpose, let m indistinctly denote the composition operation in $\mathcal{C}$, $\mathcal{D}$ or $\mathcal{E}$. In this notation, what we need to show is that

$$m\circ((H\circ G)\otimes(H\circ\lambda)_{n-1})-(-1)^n m\circ((H\circ\lambda)_{n-1}\otimes(H\circ F))=(H\circ\lambda)\circ b-\mathrm{d}\circ(H\circ\lambda)_n,$$

wherein

$$b=\sum_{i=1}^{n-1}(-1)^{n-i-1}\mathrm{id}^{\otimes(n-i-1)}\otimes\mathrm{d}\otimes\mathrm{id}^{\otimes i}+\sum_{i=0}^{n-2}(-1)^{n-i}\mathrm{id}^{\otimes(n-i-2)}\otimes m\otimes\mathrm{id}^{\otimes i}.$$

Let us start with the right-hand side. Using the definition of $H\circ\lambda$ and the above expression for b, this becomes

$$\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^{n-l-1}\sum_{k=2}^{n+2}\sum_{i+j=k}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes\mathrm{d}\otimes\mathrm{id}^{\otimes l})\\
&\qquad+\sum_{l=0}^{n-2}(-1)^{n-l}\sum_{k=2}^{n+1}\sum_{i+j=k}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-2)}\otimes m\otimes\mathrm{id}^{\otimes l})\\
&\qquad-\mathrm{d}\circ\left[\sum_{k=2}^{n+2}\sum_{i+j=k}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\right]\\
&\quad=\sum_{k=2}^{n+2}\sum_{i+j=k}\left\{\sum_{l=0}^{n-1}(-1)^{n-l-1}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes\mathrm{d}\otimes\mathrm{id}^{\otimes l})\right.\\
&\qquad\qquad+\sum_{l=0}^{n-2}(-1)^{n-l}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-2)}\otimes m\otimes\mathrm{id}^{\otimes l})\\
&\qquad\qquad\left.-\mathrm{d}\circ H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\right\}.
\end{aligned}$$

If in the right-hand side of the last equality we separate out the term $k=2$ and, in the first sum over

l, the terms in which l varies over the set $\{i-1,\dots,n-j\}$, we get

$$\sum_{l=0}^{n-1}(-1)^{n-l-1}H_1\circ\lambda_n\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes l})$$
$$+\sum_{l=0}^{n-2}H_1\circ\lambda_{n-1}\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes m\otimes \mathrm{id}^{\otimes l})+\mathrm{d}\circ H_1\circ\lambda_n$$
$$+\sum_{k=3}^{n+2}\sum_{i+j=k}\Bigg\{\sum_{\substack{l=0\\ l\notin\{i-1,\dots,n-j\}}}^{n-2}(-1)^{n-l-1}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes l})$$
$$+\sum_{l=i-1}^{n-j}(-1)^{n-l-1}H_{k-1}\circ\Big[G^{\otimes(i-1)}\otimes(\lambda_{n+2-k}\circ(\mathrm{id}^{\otimes(n-l-i)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes l-j+1}))\otimes F^{\otimes(j-1)}\Big]$$
$$+\sum_{l=0}^{n-2}(-1)^{n-l}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-2)}\otimes m\otimes \mathrm{id}^{\otimes l})$$
$$-\mathrm{d}\circ H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\Bigg\}.$$

On the other hand, keeping in mind the previous writing, the condition that $\lambda: F\Rightarrow G$ be an A_∞-natural transformation yields

$$\lambda_n\circ\left(\sum_{i=1}^{n-1}(-1)^{n-i-1}\mathrm{id}^{\otimes(n-i-1)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes i}\right)$$
$$=m\circ(G\otimes\lambda_{n-1})-(-1)^n m\circ(\lambda_{n-1}\otimes F)+\lambda_{n-1}\circ\left(\sum_{i=0}^{n-2}(-1)^{n-i}\mathrm{id}^{\otimes(n-i-2)}\otimes m\otimes \mathrm{id}^{\otimes i}\right)-\mathrm{d}\circ\lambda_n.$$

Using this in the terms of the second sum within the curly bracket gives,

$$H_1\circ m\circ(G\otimes\lambda_{n-1})-(-1)^n H_1\circ m\circ(\lambda_{n-1}\otimes F)$$
$$+\sum_{k=3}^{n+2}\sum_{i+j=k}\Bigg\{\sum_{\substack{l=0\\ l\notin\{i-1,\dots,n-j\}}}^{n-2}(-1)^{n-l-1}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\circ(\mathrm{id}^{\otimes(n-l-1)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes l})$$
$$+H_{k-1}\circ\Big[G^{\otimes(i-1)}\otimes(m\circ(G\otimes\lambda_{n+1-k})-(-1)^n m\circ(\lambda_{n+1-k}\otimes F)-\mathrm{d}\circ\lambda_{n+2-k})\otimes F^{(j-1)}\Big]$$
$$+\mathrm{d}\circ H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\Bigg\},$$

and, after some reordering,

$$\sum_{k=2}^{n+2}\sum_{i+j=k}\left\{\sum_{l=0}^{k-2}(-1)^l H_{k-1}\circ(\mathrm{id}^{\otimes(n-l-2)}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes l})\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\right.$$
$$+\sum_{l=0}^{k-3}(-1)^l H_{k-2}\circ(\mathrm{id}^{\otimes(n-l-3)}\otimes m\otimes \mathrm{id}^{\otimes l})\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})$$
$$\left.+\,\mathrm{d}\circ H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+2-k}\otimes F^{\otimes(j-1)})\right\}.$$

Now, we use the fact that H is a DG functor. This implies, in particular.

$$\sum_{p+q=k}H_p\circ H_q=\sum_{p+q+1=k}H_k\circ(\mathrm{id}^{\otimes p}\otimes \mathrm{d}\otimes \mathrm{id}^{\otimes q})+\sum_{p+q+2=k}H_k\circ(\mathrm{id}^{\otimes p}\otimes m\otimes \mathrm{id}^{\otimes q})+\mathrm{d}\circ H_k.$$

Plugging this back into the last expression above, we obtain

$$m\circ\left\{(m\circ(H\otimes G))\otimes\left[\sum_{k=2}^{n+1}\sum_{i+j=k}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})\right]\right\}$$
$$-(-1)^n m\circ\left\{\left[\sum_{k=2}^{n+1}\sum_{i+j=k}H_{k-1}\circ(G^{\otimes(i-1)}\otimes\lambda_{n+1-k}\otimes F^{\otimes(j-1)})\right]\otimes(m\circ(H\otimes F))\right\},$$

which, attending to the definitions, gives the desired.

For the second part, we observe that $(H\circ\lambda)_0=H_1\circ\lambda_0$ and, by our hypothesis on λ, we know that $\lambda_0(A)$ is an isomorphism for all $A\in\mathrm{Ob}\,\mathcal{C}$. Thus, for all $A\in\mathrm{Ob}\,\mathcal{C}$,

$$H_1(\lambda_0(A))\circ H_1(\lambda_0(A)^{-1})=H_1(\lambda_0(A)\circ\lambda_0(A)^{-1})=H_1(\mathrm{id}_A)=\mathrm{id}_{H_0(A)}.$$

This implies that $(H\circ\lambda)_0(A)$ is an isomorphism for all $A\in\mathrm{Ob}\,\mathcal{C}$, as shown. □

2.2 Local Systems

2.2.1 ∞-Local Systems

Let $E=\bigoplus_{k\in\mathbb{Z}}E^k$ be a $\mathbb{Z}$-graded vector bundle over a manifold M. The space of E-valued differential forms on M,

$$\Omega^\bullet(M,E)=\Gamma(\Lambda^\bullet T^*M\otimes E),$$

is graded respect to the total degree, where a form $\omega\in\Omega^n(M,E^k)$ has form-degree n and inner-degree k and therefore the total degree of ω is $|\omega|=n+k$. A *superconnection* (or *$\mathbb{Z}$-graded connection*) on E is an operator

$$D:\Omega^\bullet(M,E)\to\Omega^\bullet(M,E),$$

of degree 1 which satisfies the Leibniz rule

$$D(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge D\beta,$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\bullet(M,E)$. The operator D^2 is the *curvature* of D, which is an $\Omega^\bullet(M)$-linear operator on $\Omega^\bullet(M,E)$ of degree 2 which is given by multiplication by an element of

$$\Omega^\bullet(M, \mathrm{End}(E)).$$

If $D^2 = 0$, then the operator D is called *flat superconnection*. An ∞-*local system* on M is a graded vector bundle E equipped with a flat superconnection D. We will denote such an ∞-local system by (E,D).

Notice that the flatness condition makes $(\Omega^\bullet(M,E),D)$ into a DG vector space. Let (E,D) and (E',D') be ∞-local systems, we denote by $\mathrm{Hom}(E,E')$ the vector bundle morphism from E to E'. Taking homogeneous elements $\alpha \otimes T \in \Omega^\bullet(M) \otimes_{C^\infty(M)} \Gamma(\mathrm{Hom}(E,E'))$ and $\omega \otimes s \in \Omega^\bullet(M) \otimes_{C^\infty(M)} \Gamma(E)$, we define

$$(\alpha \otimes T)(\omega \otimes s) := (-1)^{|T||\omega|} \alpha\omega \otimes T(s) \in \Omega^\bullet(M) \otimes_{C^\infty(M)} \Gamma(E'),$$

and therefore, forms in $\Omega^\bullet(M, \mathrm{Hom}(E,E'))$ define maps $\Omega^\bullet(M,E) \to \Omega^\bullet(M,E')$.

Given two ∞-local systems (E,D) and (E',D'), the space of morphisms is the graded vector space $\Omega^\bullet(M, \mathrm{Hom}(E,E'))$ with the differential $\partial_{D,D'}$ given by

$$\partial_{D,D'}\varphi = D'\varphi - (-1)^k \varphi D,$$

for any homogeneous element $\varphi \in \Omega^\bullet(M, \mathrm{Hom}(E,E'))$ of degree k. The category of ∞-*local systems* is denoted by $\mathbf{Loc}_\infty(M)$.

A morphism from (E,D) to (E',D') is called an *isomorphism* if the underlying map

$$\Phi \colon \Omega^\bullet(M,E) \to \Omega^\bullet(M,E')$$

is an isomorphism. Two ∞-local systems (E,D) and (E',D') are said to be *isomorphic* if there is an isomorphism from (E,D) to (E',D').

If (E,D) and (E',D') are trivialized over M as in the previous paragraph, then $\partial_{D,D'}$ may be expressed by

$$\partial_{D,D'}\omega = \mathrm{d}\omega + \alpha' \wedge \omega - (-1)^k \omega \wedge \alpha.$$

Notice that what we call a morphism from (E,D) to (E',D') is simply a closed element of $\Omega^\bullet(M, \mathrm{Hom}(E,E'))$ of degree 0.

Proposition 2.2. The map $\partial_{D,D'}\varphi : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E')$ satisfies $\partial_{D,D'}\varphi \in \Omega^\bullet(M, \mathrm{Hom}(E,E'))$. Therefore the category $\mathbf{Loc}_\infty(M)$ is a DG category.

Proof. Given $f \in C^\infty(M)$, by simple computation we have that

$$\begin{aligned}\partial_{D,D'}(\varphi)(f\omega) &= D'(\varphi(f\omega)) - (-1)^{|\varphi|}\varphi(D(f\omega))\\ &= D'(f\varphi(\omega)) - (-1)^{|\varphi|}\varphi(\mathrm{d}(f\omega) + f\mathrm{d}(\omega))\\ &= \mathrm{d}f\varphi(\omega) + fD'(\varphi(\omega)) - \mathrm{d}f\varphi(\omega) - (-1)^{|\varphi|}f\varphi(D(\omega))\\ &= f(D'(\varphi(\omega)) - (-1)^{|\varphi|}\varphi(D(\omega)))\\ &= f\partial_{D,D'}(\varphi)(\omega),\end{aligned}$$

□

As a simple example, consider a trivial vector bundle $M \times V$ where V is a complex of finite dimensional vector spaces $(\bigoplus_{k\in\mathbb{Z}} V^k, \partial)$. The differential $D = \mathrm{d} + \partial$ gives the vector bundle $M \times V$ the structure of a flat superconnection on M. We will refer to it as a *constant* ∞*-local system* on M.

Suppose that (E,D) is an ∞-local system on M. The Leibniz rule implies that D is completely determined by its restriction to $\Omega^0(M,E)$. Then we may decompose.

$$D = \sum_{k\geq 0} D_k,$$

where D_k is of partial degree k with respect to the $\mathbb{Z}$-grading on $\Omega^\bullet(M)$. It is clear that each D_k for $k \neq 1$ is $\Omega^\bullet(M)$-linear and therefore it is given by multiplication by an element $\alpha_k \in \Omega^k(M,\mathrm{End}(E)^{1-k})$. On the contrary, D_1 satisfies the Leibniz rule on each of the vector bundles E^k, so it must be of the form d_∇, where ∇ is an ordinary connection on E which preserves the $\mathbb{Z}$-grading. We can thus write

$$D = \mathrm{d}_\nabla + \alpha_0 + \alpha_2 + \alpha_3 + \cdots .$$

From this formula, it is straightforward to check that the flatness condition becomes equivalent to

$$\begin{aligned}\alpha_0^2 &= 0,\\ \mathrm{d}_\nabla\alpha_0 &= 0,\\ [\alpha_0,\alpha_2] + F_\nabla &= 0,\\ [\alpha_0,\alpha_{n+1}] + \mathrm{d}_\nabla\alpha_n + \sum_{k=2}^{n-1}\alpha_k\wedge\alpha_{n+1-k} &= 0,\ n\geq 2,\end{aligned}$$

where F_∇ is the curvature of the connection ∇. The first identity implies that we have a cochain complex of vector bundles with differential α_0. The second equation expresses that α_0 is covariantly constant concerning the connection ∇. The third equation indicates that the connection ∇ fails to be flat up to terms involving the homotopy α_2 and the differential α_0.

The following lemma gives a formula for the differential of a morphism between trivial ∞-local systems:

Lemma 2.3. *Let* $(E,D) = (M\times V, \mathrm{d}+\alpha)$ *and* $(E',D') = (M\times W, \mathrm{d}+\beta)$ *be trivial* ∞*-local systems.*

If $\varphi : (E,D) \to (E',D')$ is a homogenous morphism, then its differential is

$$D_{E,E'}(\varphi) = \mathrm{d}\varphi + \alpha'\varphi - (-1)^{|\varphi|}\varphi\alpha \tag{2.2.1}$$

Proof. The proof is straight forward

$$\begin{aligned} D_{E,E'}(\varphi) &= \mathrm{d}_{E'}\varphi - (-1)^{|\varphi|}\varphi \mathrm{d}_E \\ &= (\mathrm{d}+\alpha')\varphi - (-1)^{|\varphi|}\varphi(\mathrm{d}+\alpha) \\ &= \mathrm{d}\circ\varphi + \alpha'\varphi - (-1)^{|\varphi|}\varphi\circ \mathrm{d} - (-1)^{|\varphi|}\varphi\alpha \\ &= \mathrm{d}\varphi + (-1)^{|\varphi|}\varphi\circ\mathrm{d} + \alpha'\varphi - (-1)^{|\varphi|}\varphi\circ\mathrm{d} - (-1)^{|\varphi|}\varphi\alpha \\ &= \mathrm{d}\varphi + \alpha'\varphi - (-1)^{|\varphi|}\varphi\alpha. \end{aligned}$$ □

Now let us assume that E is trivialized over M. This means that $E = M \times V$ for some $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^k$. In this case, we have $\alpha_k \in \Omega^k(M, \mathrm{End}(V)^{1-k})$ for $k \neq 1$. Moreover, we can write $\mathrm{d}_\nabla = \mathrm{d} + \alpha_1$ for some $\alpha_1 \in \Omega^1(M, \mathrm{End}(V)^0)$. Thus, the $\mathbb{Z}$-graded superconnection D may be expressed as $D = \mathrm{d} + \alpha$, where $\alpha \in \Omega^\bullet(M, \mathrm{End}(V))$ is the homogeneous element of total degree 1 defined by $\alpha = \sum_{k\geq 0} \alpha_k$. In addition, a straightforward calculation gives

$$D^2 = \mathrm{d}\alpha + \alpha\wedge\alpha.$$

Consequently, the totality of equations of the flatness condition is equivalent to the single statement that α satisfies

$$\mathrm{d}\alpha + \alpha\wedge\alpha = 0.$$

This is known as the *Maurer-Cartan equation*.

Suppose we have another trivialization of E over M such that $E = M \times W$ for some $\mathbb{Z}$-graded vector space $W = \bigoplus_{k\in\mathbb{Z}} W^k$ and $D = \mathrm{d} + \beta$ for some homogeneous element $\beta \in \Omega^\bullet(M, \mathrm{End}(W))$ of total degree 1 satisfying the Maurer-Cartan equation. Then, we have a transition isomorphism between the two trivializations, which is realised by a linear isomorphism $g\colon \Omega^0(M,V) \to \Omega^0(M,W)$ that commutes with the operators $\mathrm{d}+\alpha$ and $\mathrm{d}+\beta$. If we think of g as an element of $\Omega^0(M, \mathrm{Hom}(V,W))$, the latter condition is equivalent to the requirement that

$$\alpha = g^{-1}\beta g + g^{-1}\mathrm{d}g.$$

The change from β to α given in this equation goes by the name of a *gauge transformation*.

For a pair of ∞-local systems (E,D) and (E',D') there is a natural notion of a strict morphism from (E,D) to (E',D'). Namely, such a morphism is a degree 0 linear map $\Phi\colon \Omega^\bullet(M,E) \to \Omega^\bullet(M,E')$ which is $\Omega^\bullet(M)$-linear and commutes with the $\mathbb{Z}$-graded superconnections D and D'. If both (E,D) and (E',D') are trivialized over M in such a way that $E = M \times V$ and $E' = M \times V'$ for some $\mathbb{Z}$-graded vector spaces V and V', and $D = \mathrm{d}+\alpha$ and $D' = \mathrm{d}+\alpha'$ for some homogeneous elements $\alpha \in \Omega^\bullet(M, \mathrm{End}(V))$ and $\alpha' \in \Omega^\bullet(M, \mathrm{End}(V'))$ of total degree 1 satisfying the Maurer-Cartan equation, then this condition is

$$(\mathrm{d}+\alpha')\circ\Phi = \Phi\circ(\mathrm{d}+\alpha),$$

or, interpreting Φ as an element of $\Omega^\bullet(M,\mathrm{Hom}(V,V'))$,

$$\mathrm{d}\Phi = \alpha\wedge\Phi - \Phi\wedge\alpha'.$$

With the gauge equivalence relation at hand, we are ready to construct pullbacks of ∞-local systems. For a smooth map $f\colon M\to N$ between two manifolds M and N, there is a DG functor $f^*\colon \mathbf{Loc}_\infty(N)\to\mathbf{Loc}_\infty(M)$ which sends E with structure superconnection

$$D = \mathrm{d}_\nabla + \alpha_0 + \alpha_2 + \alpha_3 + \cdots,$$

to f^*E endowed with

$$f^*D = \mathrm{d}_{f^*\nabla} + f^*\alpha_0 + f^*\alpha_2 + f^*\alpha_3 + \cdots,$$

Let $f: M\to N$ be a smooth map between manifolds. Then f induces a DG functor $f^*: \mathbf{Loc}_\infty(N)\to\mathbf{Loc}_\infty(M)$. Therefore the DG functor f^* is well defined. We refer to it as the *pullback functor* induced by f.

2.2.2 Iterated integrals

For the present section we will adopt a slightly different notation for simplices. We will denote by $\Delta_k(t)$ the *k-simplex* of width t for $t\in[0,1]$, this is

$$\Delta_k(t) = \{(s_1,\cdots,s_k)\in\mathbb{R} \quad | \quad t\geq s_1\geq\cdots\geq s_k\geq 0\}.$$

The components of the boundary of $\Delta_k(t)$ will be denoted $\partial_i\Delta_k(t)$ where

$$\begin{aligned}
\partial_0\Delta_k(t) &= \{(s_1,\cdots,s_k)\in\Delta_k(t) \quad | \quad s_1 = t\},\\
\partial_i\Delta_k(t) &= \{(s_1,\cdots,s_k)\in\Delta_k(t) \quad | \quad s_i = s_{i+1}\}, \quad i=1,\cdots,k-1,\\
\partial_k\Delta_k(t) &= \{(s_1,\cdots,s_k)\in\Delta_k(t) \quad | \quad s_k = 0\}.
\end{aligned}$$

For any smooth manifold M we denote by $\pi_{k,i,t}:\Delta_k(t)\times M\to[0,t]\times M$ the projection onto the ith component of the simplex. We will omit both k and t from the notation when the dimension and width of the simplex are clear from the context.

Let V be a vector space and consider the trivial vector bundles $\Delta_k\times M\times V$ and $M\times V$. The push forward $\int_{\Delta_k}:\Omega^\bullet(\Delta_k\times M)\to\Omega^\bullet(M)$ defined locally by

$$f(t_1,\cdots,t_n,x)dt_1\cdots dt_n dX \mapsto \left(\int_{\Delta_n} f(t_1,\cdots,t_n,x)dt_1\cdots dt_n\right)dX,$$

is extended to forms with values in V by

$$\int_{\Delta_k} \otimes \mathrm{Id} : \Omega^\bullet(\Delta_k \times M) \otimes V \to \Omega^\bullet(M) \otimes V.$$

Lemma 2.4. *The push forward $\int_{\Delta_k} : \Omega^\bullet(\Delta_n \times X) \to \Omega^\bullet(X)$ is a morphism of left $\Omega^\bullet(X)$-modules of degree $-n$. This is, fore every $\omega \in \Omega^\bullet(X)$ and $\alpha \in \Omega^\bullet(\Delta_n \times X)$ we have*

$$\int_{\Delta_k} (\pi^*(\omega) \wedge \alpha) = (-1)^{|\omega| n} \omega \wedge \int_{\Delta_k} (\alpha).$$

Proof. First we check that the push forward is a morphism of $\Omega^\bullet(X)$-modules, notice that it is enough to check this locally. Suppose $(x_1, \cdots, x_m)$ are coordinates for X and write

$$\begin{aligned} \omega &= f dx_{i_1} \cdots dx_{i_k} \in \Omega^k(X) \\ \alpha &= g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} \in \Omega^{n+l}(\Delta_n \times X). \end{aligned}$$

Then we have

$$\begin{aligned} \int_{\Delta_n} \pi^*(\omega) \wedge \alpha &= \int_{\Delta_n} f dx_{i_1} \cdots dx_{i_k} g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} f \left(\int_{\Delta_n} g dt_1 \cdots dt_n \right) dx_{i_1} \cdots dx_{i_k} dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} f dx_{i_1} \cdots dx_{i_k} \left(\int_{\Delta_n} g dt_1 \cdots dt_n \right) dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} \omega \wedge \int_{\Delta_n} \alpha. \end{aligned}$$

The fact that π_* has degree $-n$ is also clear from the previous computation. □

We will abuse the notation and just write $\int_{\Delta_k}$ for the map $\int_{\Delta_k} \otimes \mathrm{Id}$. Now consider the case where V is a graded vector space, and we take forms with values in $\mathrm{End}(V)$. The vector space $\Omega^\bullet(M, \mathrm{End}(V))$ may be provided with a DG algebra structure with the product

$$(\alpha \otimes T) \wedge (\beta \otimes S) = (-1)^{|\beta||T|} (\alpha \wedge \beta) \otimes (T \circ S) \quad \text{for} \quad \alpha, \beta \in \Omega^\bullet(M) \text{ and } T, S \in \mathrm{End}(V).$$

In a similar way, the action $\Omega^\bullet(M) \circlearrowright \Omega^\bullet(\Delta_k \times M)$ may be extended to an action

$$\Omega^\bullet(M, \mathrm{End}(V)) \circlearrowright \Omega^\bullet(\Delta_k \times M, \mathrm{End}(V)).$$

The Lemma 2.4 still holds in this context:

Lemma 2.5. *The map $\int_{\Delta_k} : \Omega^\bullet(\Delta_k \times M, \mathrm{End}(V)) \to \Omega^\bullet(M, \mathrm{End}(V))$ is a morphism of $\Omega^\bullet(M, \mathrm{End}(V))$-modules of degree $-k$.*

Proof. Take $\alpha \in \Omega^\bullet(M)$, $\beta \in \Omega^\bullet(\Delta_k \times M)$ and $T, S \in \mathrm{End}(V)$. Then

$$\begin{aligned}\int_{\Delta_k} (\alpha \otimes T) \wedge (\beta \otimes S) =& (-1)^{|\beta||T|} \left(\int_{\Delta_k} \alpha \wedge \beta \right) \otimes (T \circ S) \\ =& (-1)^{|\beta||T| - |\alpha|k} \left(\alpha \wedge \int_{\Delta_k} \beta \right) \otimes (T \circ S) \\ =& (-1)^{|\beta||T| - |\alpha|k + |T|(|\beta| - k)} (\alpha \otimes T) \wedge \left(\left(\int_{\Delta_k} \beta \right) \otimes S \right) \\ =& (-1)^{-(|\alpha| + |T|)k} (\alpha \otimes T) \wedge \left(\left(\int_{\Delta_k} \beta \right) \otimes S \right).\end{aligned}$$

Notice that the sign obtained is given by the Kozsul convention when considering the total degree of $\alpha \otimes T$, not just the form degree. □

For each $t \in [0,1]$ and a sequence of possibly non-homogeneous forms $\{\theta_i\}_{i \geq 1} \in \Omega^\bullet([0,1] \times M, \mathrm{End}(V))$ consider the series

$$\varphi(t) = \mathrm{id} + \sum_{k \geq 1} \int_0^t \pi_1^*(\theta_1) \int_0^{s_1} \pi_2^*(\theta_2) \cdots \int_0^{s_{k-1}} \pi_k^*(\theta_k). \tag{2.2.2}$$

If the forms are homogeneous, then the previous iterated integrals can be written as integrals over the simplices as follows:

$$\varphi(t) = \mathrm{id} + \sum_{k \geq 1} (-1)^{(k-1)|\theta_1| + (k-2)|\theta_2| + \cdots + |\theta_{k-1}|} \int_{\Delta_k(t)} \pi_1^*(\theta_1) \cdots \pi_k^*(\theta_k). \tag{2.2.3}$$

Notice that the sign $(-1)^{(k-1)|\theta_1| + (k-2)|\theta_2| + \cdots + |\theta_{k-1}|}$ is precisely the sign obtained when commuting forms with integrals. The sum above is not finite and is not necessarily convergent, however it will converge whenever the sequence of forms is uniformly bounded. Suppose each form can be written locally as $\theta_i = f_i(t,x) dt dX_i$. We say that the sequence $\{\theta_i\}_{i \geq 1}$ is uniformly bounded if there is a function $B : M \to \mathbb{R}$ such that $|f_i(t,x)| \leq B(x)$ for every $i \geq 1$ and every $t \in [0,1]$. Notice in particular that constant sequences and sequences with finitely many different forms are uniformly bounded.

Lemma 2.6. *Suppose the sequence $\{\theta_i\}_{i \geq 1} \in \Omega^\bullet([0,1] \times M, \mathrm{End}(V))$ is uniformly bounded. Then the series (2.2.3) is convergent.*

Proof. The volume of the k-simplex of width t is $t^k/k!$. If $B(x)$ is a bound for the sequence, then we have

$$\left| \int_{\Delta_k(t)} f_1(s_1, x) \cdots f_k(s_k, x) ds_k \cdots ds_1 \right| \leq \frac{t^k}{k!} B^k(x).$$

The convergence of the series follows. □

We are particularly interested in the case of constant sequences.

Definition 2.1. *Suppose $\theta_i = \alpha$ for every i. Then (2.2.3) can be written as*

$$\varphi(t) = \mathrm{id} + \sum_{k\geq 1} (-1)^{\sigma(k)|\alpha|} \int_{\Delta_k(t)} \pi_1^*(\alpha)\cdots\pi_k^*(\alpha),$$

where $\sigma(k) = 0+1+\cdots+(k-1)$. Since the sequence is uniformly bounded, the series is convergent and defines a form in $\Omega^\bullet(M,\mathrm{End}(V))$ which we will denote $\varphi^\alpha(t)$. In fact we have a smooth map $\varphi^\alpha : [0,1] \to \Omega^\bullet(M,\mathrm{End}(V))$.

Next we prove some facts about the forms φ^α. We denote by $\iota_t : M \to [0,1]\times M$ the inclusion at level t, this is, $\iota_t(x) = (t,x)$. Also, let $i_{\frac{\partial}{\partial t}}$ denote the contraction with the vector field $\frac{\partial}{\partial t}$. The first result we prove is that $\varphi^\alpha(t)$ defines a solution for the parallel transport differential equation.

Proposition 2.3. $\varphi^\alpha(t)$ satisfies the following differential equation.

$$\begin{cases} \frac{d\varphi^\alpha}{dt} = \iota_t^* i_{\frac{\partial}{\partial t}} \alpha \wedge \varphi^\alpha(t) \\ \varphi^\alpha(0) = \mathrm{id} \end{cases} \tag{2.2.4}$$

Proof. Clearly $\varphi^\alpha(0) = \mathrm{id}$. Let us compute the derivative.

$$\begin{aligned} \frac{d\varphi^\alpha}{dt} &= \frac{d}{dt}\int_0^t \pi_1^*(\alpha) + \sum_{k\geq 2} \frac{d}{dt}\int_0^t \pi_1^*(\alpha)\int_0^{s_1}\pi_2^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha) \\ &= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha + \iota_t^* i_{\frac{\partial}{\partial t}}\alpha \sum_{k\geq 2}\int_0^t \pi_2^*(\alpha)\int_0^{s_2}\pi_3^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha) \\ &= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\left(\mathrm{id} + \sum_{k\geq 2}\int_0^t \pi_2^*(\alpha)\int_0^{s_2}\pi_3^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha)\right) \\ &= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\varphi^\alpha(t). \end{aligned}$$

□

Next we prove the gauge invariance property of iterated integrals.

Proposition 2.4. Let $\alpha \in \Omega^\bullet([0,1]\times M,\mathrm{End}(V))$ and $\beta \in \Omega^\bullet([0,1]\times M,\mathrm{End}(W))$. If $\psi \in \Omega^0([0,1]\times M,\mathrm{Hom}(V,W))$ is invertible such that $\alpha = \psi^{-1}\beta\psi - \psi^{-1}d\psi$, then the following equation holds:

$$\varphi^\alpha(t) = (\iota_t^*\psi)^{-1}\varphi^\beta(t)(\iota_0^*\psi). \tag{2.2.5}$$

Proof. The strategy of the proof is to show that the right side of equation (2.2.5) satisfies the differential equation (2.2.4). By Proposition 2.3, φ^α also satisfies the equation making both sides of (2.2.5) equal.

Let us compute the derivative of $(\iota_t^*\psi)^{-1}\varphi^\beta(t)(\iota_0^*\psi)$ with respect to t:

$$\begin{aligned}\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) &= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\frac{d\varphi^\beta}{dt}\iota_0^*\psi \\ &= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\varphi^\beta(t)\iota_0^*\psi \\ &= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + (\iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\iota_t^*\psi)(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi). \end{aligned} \tag{2.2.6}$$

Since $\alpha = \psi^{-1}\beta\psi - \psi^{-1}d\psi$, we also have

$$\iota_t^* i_{\frac{\partial}{\partial t}}\alpha = \iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\iota_t^*\psi - \iota_t^*\psi^{-1}\frac{d}{dt}(\iota_t^*\psi).$$

Replacing in (2.2.6) we get

$$\begin{aligned}&\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) \\ &\quad = \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\frac{d}{dt}(\iota_t^*\psi)\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi.\end{aligned}$$

By the product rule we know that

$$d(\iota_t^*\psi^{-1})/dt = -\iota_t^*\psi^{-1}\big(d(\iota_t^*\psi)/dt\big)\iota_t^*\psi^{-1},$$

hence the first and last terms on the right cancel out, leaving us with

$$\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) = i_{\frac{\partial}{\partial t}}\alpha(t)\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi,$$

completing the proof. □

2.2.3 A_∞ Version of Poincare Lemma

Having developed our primary tool for the section, we are ready to prove some theorems regarding ∞-local systems.

Lemma 2.7. *Let $E = [0,1]\times M\times V$ be a trivial graded vector bundle and $\alpha \in \Omega^\bullet([0,1]\times M, \mathrm{End}(V))$ a Maurer-Cartan form so that $D = d+\alpha$ is a flat graded connection. The form $\varphi^\alpha(t)\in\Omega^\bullet(M,\mathrm{End}(V))$ defines a zero degree, closed isomorphism of ∞-local systems $\varphi^\alpha(t) : \iota_0^*(E,D)\to\iota_t^*(E,D)$.*

Proof. $\varphi^\alpha(t)$ defines a morphism of ∞-local systems. Let us check that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^\bullet(M,V) & \xrightarrow{d+\iota_0^*\alpha} & \Omega^\bullet(M,V) \\ {\scriptstyle\varphi^\alpha(t)}\downarrow & & \downarrow{\scriptstyle\varphi^\alpha(t)} \\ \Omega^\bullet(M,V) & \xrightarrow[d+\iota_t^*\alpha]{} & \Omega^\bullet(M,V) \end{array}$$

We need to prove that $\varphi^{\alpha}(t)(d+\iota_0^*\alpha) = (d+\iota_t^*\alpha)\varphi^{\alpha}(t)$. The previous equation is equivalent to $d\varphi^{\alpha}(t) = \varphi^{\alpha}(t)\iota_0^*\alpha - \iota_t^*\alpha\varphi^{\alpha}(t)$, which is demonstrated in [11].

To see that $\varphi^{\alpha}(t)$ is an isomorphism we write $\alpha = \sum_i \alpha_i$ where $\alpha_i \in \Omega^i([0,1]\times M, \mathrm{End}(V))$. Recall that α_1 is an ordinary connection over E and, according to the classical case, the iterated integral $\varphi^{\alpha_1}(t)$ defines a parallel transport which is invertible. Now write $\varphi^{\alpha}(t) = \varphi_0^{\alpha}(t) + \eta$ where $\varphi_0^{\alpha}(t)$ is the component of form degree zero and η is the sum of the rest of the components. Notice that $\varphi_0^{\alpha}(t) = \varphi^{\alpha_1}(t)$ and η is a nilpotent form, thus $\varphi^{\alpha}(t)$ is the sum of an invertible element and a nilpotent element, which means it is invertible. □

With the interpretation of the iterated integral as an isomorphism of ∞-local systems, the gauge invariance property given in the proposition 2.4 may be viewed as the commutativity of a diagram:

Lemma 2.8. *Suppose* $(E_1,D_1) = ([0,1]\times M\times V, d+\alpha)$ *and* $(E_2,D_2) = ([0,1]\times M\times W, d+\beta)$ *are* ∞*-local systems and* $\psi \in \Omega^\bullet([0,1]\times M, \mathrm{Hom}(V,W))$ *is a closed isomorphism. Then the following diagram is commutative*

$$\begin{array}{ccc} \iota_0^*(E_1,D_1) & \xrightarrow{\iota_0^*\psi} & \iota_0^*(E_2,D_2) \\ {\scriptstyle \varphi^{\alpha}(t)}\downarrow & & \downarrow{\scriptstyle \varphi^{\beta}(t)} \\ \iota_t^*(E_1,D_1) & \xrightarrow{\iota_t^*\psi} & \iota_t^*(E_2,D_2). \end{array}$$

Now we aim to generalize lemma 2.7 to non-trivial ∞-local systems. As expected, we rely on the gauge invariance property.

Theorem 2.1. Let (E,D) be an ∞-local system over $[0,1]\times M$. Then for each $t\in[0,1]$ there is a closed isomorphism $\varphi : \iota_0^*(E,D) \to \iota_t^*(E,D)$.

Proof. Take an open covering $\{U_i\}$ of M where each U_i is contractible so that

$$(E_i,D_i) := (E,D)|_{[0,1]\times U_i}$$

is a trivial vector bundle. Let $\varphi_i : \iota_0^*(E_i,D_i) \to \iota_t^*(E_i,D_i)$ be the closed isomorphism from lemma 2.7. If $\psi_{ij} : (E_i,D_i)|_{[0,1]\times(U_i\cap U_j)} \to (E_j,D_j)|_{[0,1]\times(U_i\cap U_j)}$ is the transition function, then by lemma 2.8 we have the commutativity of the following diagram:

$$\begin{array}{ccc} \iota_0^*(E_i,D_i)|_{[0,1]\times(U_i\cap U_j)} & \xrightarrow{\iota_0^*\psi_{ij}} & \iota_0^*(E_j,D_j)|_{[0,1]\times(U_i\cap U_j)} \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ \iota_t^*(E_i,D_i)|_{[0,1]\times(U_i\cap U_j)} & \xrightarrow{\iota_t^*\psi_{ij}} & \iota_t^*(E_j,D_j)|_{[0,1]\times(U_i\cap U_j)}. \end{array}$$

Thus the morphisms φ_i can be glued together into a closed isomorphism $\varphi : \iota_0^*(E,D) \to \iota_t^*(E,D)$. □

The correct statement is that the isomorphism φ of theorem 2.1 is the first component of an A_∞-natural isomorphism between the DG functors ι_0^* and ι_t^*.

Proposition 2.5. Let M be a smooth manifold, we denoted by $\iota_s : M \to M \times [0,1]$ the inclusion at height s given by $\iota_s(x) = (x,s)$. Let $\iota_0^*, \iota_1^* : \mathbf{Loc}_\infty(M \times [0,1]) \to \mathbf{Loc}_\infty(M)$ be the pullback functors of the inclusions ι_0 and ι_1 respectively. Then there exists an A_∞-natural isomorphism $\rho : \iota_0^* \Rightarrow \iota_1^*$.

Proof. Let $\Delta_k(t)$ be the k-simplex of width t for $t \in [0,1]$,

$$\Delta_k(t) := \{(t_1, ..., t_k) \in \mathbb{R}^k \mid t \geq t_1 \geq \cdots \geq t_k \geq 0\}.$$

We denoted by $\pi_{k,i,t} : \Delta_k(t) \times M \to [0,1] \times M$ the projection on the i-th component of the simplex. If the dimension and width of the simplex are clear from the context, then we will omit both k and t from the notation.

For each $t \in [0,1]$ and a sequence of possibly non-homogeneous forms $\{\theta_i\}_{i\geq 1}$ of $\Omega^\bullet([0,1] \times M, \mathrm{End}(V))$, we consider the series

$$\varphi(t) = \mathrm{id} + \sum_{k\geq 1} \int_0^t \pi_1^*(\theta_1) \int_0^{t_1} \pi_2^*(\theta_2) \cdots \int_0^{t_{k-1}} \pi_k^*(\theta_k).$$

In the case of homogeneous forms $\{\theta_i\}_{i\geq 1}$, the previous iterated integrals can be written as

$$\varphi(t) = \mathrm{id} + \sum_{k\geq 1} (-1)^{\maltese} \int_{\Delta_k(t)} \pi_1^*(\theta_1) \cdots \pi_k^*(\theta_k),$$

where $\maltese = (k-1)|\theta_1| + (k-2)|\theta_2| + \cdots + |\theta_{k-1}|$ is the sign obtained when commuting forms with integrals. Now suppose that we have a constant sequences $\{\alpha\}$, then the iterated integrals can be written as

$$\varphi^\alpha(t) = \mathrm{id} + \sum_{k\geq 1} (-1)^{\sigma(k)|\alpha|} \int_{\Delta_k(t)} \pi_1^*(\alpha) \cdots \pi_k^*(\alpha),$$

where $\sigma(k) = 0 + 1 + \cdots + (k-1)$. The previous series is convergent (see [11]) and defines a form in $\Omega^\bullet(M, \mathrm{End}(V))$, in fact a smooth map $\varphi^\alpha : [0,1] \to \Omega^\bullet(M, \mathrm{End}(V))$.

Let (E_i, D_i) be a trivialized local system on $M \times I$, for $i = 0, 1, ..., n$, then $E_i = M \times I \times V_i$ and $D_i = \mathrm{d} + \alpha_i$ with $\alpha_i \in \Omega^\bullet(M \times I, \mathrm{End}(V_i))$ of total degree 1 satisfying $\mathrm{d}\alpha_i + \alpha_i^2 = 0$. We define

$$\rho_n^\alpha(t) : \mathrm{Hom}(E_{n-1}, E_n) \otimes \cdots \otimes \mathrm{Hom}(E_0, E_1) \to \mathrm{Hom}(\iota_0^*(E_0), \iota_t^*(E_n))$$

as follows. For homogeneous elements $\Omega_i \in \mathrm{Hom}(E_i, E_{i+1}) = \Omega^\bullet(M \times I, \mathrm{Hom}(V_i, V_{i+1}))$, with $i = 0, 1, ..., n-1$. Taking $V = \bigoplus_i V_i$, then the forms $\{\alpha_i\}$ and $\{\omega_i\}$ can be seen as elements of $\Omega^\bullet(M \times I, \mathrm{End}(V))$. Let $\eta := \Sigma_i \alpha_i + \Sigma_i \omega_i$ and define

$$\overline{\rho}_n^\alpha(t)(\omega_0, ..., \omega_{n-1}) := \varphi^\eta(t) \in \Omega^\bullet(M, \mathrm{End}(V)).$$

For the $(0,n)$ component of this form which goes from V_0 to V_n, we define

$$\rho_n^\alpha(\omega_{n-1}, ..., \omega_0) := (\overline{\rho}_n^\alpha(t)(\omega_{n-1}, ..., \omega_0))_{0,n} \in \Omega^\bullet(M, \mathrm{Hom}(V_0, V_n)),$$

where a general term of ρ_n^α is an integral

$$\int_{\Delta_k(t)} \alpha_n \pi_{i_{n-1}}^*(\omega_{n-1})\alpha_{n-1}\cdots\alpha_1\pi_{i_0}^*(\omega_0)\lambda_0,$$

where $\alpha_j = \pi_{i_{j-1}+p_j}^*(\alpha_j)\cdots\pi_{i_{j-1}+1}^*(\alpha_j)$. In words, the integrands contain pullbacks of all the $\{\omega_i\}_{i=0}^{n-1}$ as factors, ordered with descending indices from left to right. Furthermore, there may be products of pullbacks of α_i between ω_i and ω_{i-1}. Notice that since the amount of factors in the integrand must equal k and all of the ω_i appear in the product, the integrals relevant to us are over $\Delta_k(t)$ with $k \geq n$. Also notice that since the $\{\alpha_i\}$ have total degree 1, the total degree of the forms yielded by the integrals is $\sum_{i=0}^{n-1}|\omega_i| - n$, which makes ρ_n^α a map of degree $-n$. Notice that ρ_0^α is just the isomorphism φ given in theorem 2.1, which we know is a closed isomorphism.

Next we prove that this construction does not depend on the trivializations chosen for each E_i. Suppose that (E_i, D_i) is trivialized both as $([0,1]\times M\times V_i, d+\alpha_i)$ and $([0,1]\times M\times W_i, d+\beta_i)$, and for each i let $\psi_i \in \Omega^0([0,1]\times M, \mathrm{Hom}(V_i, W_i))$ be an isomorphism between trivializations. Take $\omega_i \in \Omega^\bullet([0,1]\times M, \mathrm{Hom}(V_i, V_{i+1}))$ and define $\mu_i \in \Omega^\bullet([0,1]\times M, \mathrm{Hom}(W_i, W_{i+1}))$ by the formula $\omega_i = \psi_{i+1}^{-1}\mu_i\psi_i$. We want to prove the following equation

$$\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) = (i_t^*\psi_n)^{-1}\rho_n^\beta(\mu_{n-1},\cdots,\mu_0)(i_0^*\psi_0). \tag{2.2.7}$$

Consider the direct sums $V = \bigoplus V_i$ and $W = \bigoplus W_i$. Then we have an isomorphism $\psi = \bigoplus \psi_i : V \to W$. Furthermore, if we call $\eta = \sum\alpha_i + \sum\omega_i$ and $\theta = \sum\beta_i + \sum\mu_i$, then the forms are gauge equivalent, i.e., $\eta = \psi^{-1}\theta\psi - \psi^{-1}d\psi$. By Proposition 2.4, for every $t\in[0,1]$ the following equation holds

$$\varphi^\eta(t) = (i_t^*\psi)^{-1}\varphi^\theta(t)(i_0^*\psi).$$

In particular this implies that the $(0,n)$ components are equal. That is precisely equation (2.2.7).

Finally we check that the A_∞ relations are satisfied. It is enough to check the relations locally, hence we need to verify that

$$\begin{aligned}\iota_t^*\omega_{n-1}\wedge\rho_{n-1}^\alpha(\omega_{n-2}\otimes\cdots\otimes\omega_0)-(-1)^{\sum_{j=1}^{n-1}|\omega_j|-n+1}\rho_{n-1}^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_1)\wedge\iota_0^*\omega_0 \\ = \rho^\alpha(b(\omega_{n-1}\otimes\cdots\otimes\omega_0)) + D\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0),\end{aligned}$$

Using equation (2.2.1) we can write the differential D in the previous relation as follows:

$$\begin{aligned}&D\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)\\ &= d\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) - i_t^*(\alpha_n)\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) + (-1)^{|\omega_{n-1}|+\cdots+|\omega_0|-n}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)i_0^*(\alpha_0).\end{aligned}$$

Therefore the equation we need to verify is

$$\begin{aligned}
&d\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0)\\
&\quad=\iota_t^*\omega_{n-1}\wedge\rho_{n-1}^\alpha(\omega_{n-2}\otimes\cdots\otimes\omega_0)-(-1)^{\sum_{j=1}^{n-1}|\omega_j|-n+1}\rho_{n-1}^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_1)\wedge\iota_0^*\omega_0\\
&\qquad+\iota_t^*\alpha_n\wedge\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0)-(-1)^{\sum_{j=0}^{n-1}|\omega_j|-n}\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0)\wedge\iota_0^*\alpha_0\\
&\qquad-\rho^\alpha(b(\omega_{n-1}\otimes\cdots\otimes\omega_0)),
\end{aligned}$$

Therefore this construction does not depend on the trivializations chosen for each E_i and in addition ρ_n^α satisfies the A_∞ relations. See [11] for more details. □

2.2.4 The higher Riemann-Hilbert correspondence

This subsection reviews the higher Riemann-Hilbert correspondence, which relates ∞-local systems on X to representations up to homotopy of the smooth ∞-groupoid of X. Intuitively, the higher Riemann-Hilbert correspondence is the statement that, just as a flat connection can be integrated into a representation of the fundamental groupoid, a flat superconnection can be integrated into a representation of the ∞-groupoid. The details of the proof can be found in [8, 9].

For every $k \geq 0$ we denote $[k] = \{0,\cdots,k\}$. The *simplex category*, denoted Δ, is the category whose objects are the sets $[k]$ for $k \geq 0$. A morphism $f : [k] \to [l]$ is a function that preserves the order, i.e. if $m,n \in [k]$ with $m \leq n$, then $f(m) \leq f(n)$. Among the morphisms there are some particularly important ones:

- The *co-face* morphisms are maps $\tilde{p}_i^k : [k] \to [k+1]$, with $i \in [k+1]$. The map $\tilde{p}_i^k$ is the only injective, order preserving map whose image does not contain the element i.
- The *co-degeneracy* morphisms are maps $\tilde{s}_i^k : [k] \to [k-1]$, with $i \in [k-1]$. The map $\tilde{s}_i^k$ is the only surjective, order preserving map such that $\tilde{s}_i^k(i) = \tilde{s}_i^k(i+1)$.
- The $k+1$ possible ways to include $[0]$ in $[k]$ which we denote by $\tilde{v}_i^k : [0] \to [k]$ with $i \in [k]$.

The simplex category has a geometrical realization that will be convenient for our purposes. However, it is to be noted that the relevance of the simplex category lies in its combinatorial properties, not the geometric properties that may arise from the geometric description.

Let k be a non-negative integer, the *k-dimensional simplex*, denoted Δ_k, is the set

$$\Delta_k := \{(t_1,\cdots,t_k) \in \mathbb{R}^k \mid 1 \geq t_1 \geq \cdots \geq t_k \geq 0\}$$

The 0-simplex Δ_0 is merely the single point space $\{0\}$. The k-simplex may be included in the $k+1$-simplex as a face in $k+2$ different ways. This *inclusions* are denoted $\hat{p}_i^k : \Delta_k \to \Delta_{k+1}$ for $i \in [k+1]$ and are defined by the formula

$$\hat{p}_i^k(t_1,\cdots,t_k) = \begin{cases} (1,t_1,\cdots,t_k) & \text{if} \quad i=0 \\ (t_1,\cdots,t_{i-1},t_i,t_i,t_{i+1},\cdots,t_k) & \text{if} \quad 1 \leq i \leq k \\ (t_1,\cdots,t_k,0) & \text{if} \quad i=k+1 \end{cases}$$

The components of the *boundary* of Δ_{k+1} will be denoted $\partial_i \Delta_{k+1}$ where

$$\begin{aligned}
\partial_0 \Delta_{k+1} &= \hat{p}_0^k(\Delta_k) = \{(1, t_1, \cdots, t_k) \quad | \quad (t_1, \cdots, t_k) \in \Delta_k\} \\
\partial_i \Delta_{k+1} &= \hat{p}_i^k(\Delta_k) = \{(t_1, \cdots, t_{i-1}, t_i, t_i, t_{i+1}, \cdots, t_k) \quad | \quad (t_1, \cdots, t_k) \in \Delta_k\} \\
\partial_{k+1} \Delta_{k+1} &= \hat{p}_{k+1}^k(\Delta_k) = \{(t_1, \cdots, t_k, 0) \quad | \quad (t_1, \cdots, t_k) \in \Delta_k\}
\end{aligned}$$

On the other hand we have k *projection maps* $\hat{s}_i^k : \Delta_k \to \Delta_{k-1}$ for $i \in [k-1]$ that collapse the k-simplex onto one of its faces. The projections are defined by

$$\hat{s}_i^k(t_1, \cdots, t_k) = (t_1, \cdots, t_i, t_{i+2}, \cdots, t_k)$$

The $k+1$ different ways of including the 0-simplex as a vertex in the k simplex will be denoted $\hat{v}_i^k : \Delta_0 \to \Delta_k$ for $i \in [k]$. These inclusions may be obtained as compositions of the $\hat{p}_i^k$ as follows

$$\hat{v}_i^k = p_k^{k-1} \cdots p_{i+1}^i p_0^{i-1} \cdots p_0^0.$$

Put simply, $\hat{v}_i^k(0) = (1, \cdots, 1, 0, \cdots, 0)$ where the tuple has i ones and $k-i$ zeroes.

The correspondence between the combinatorial and geometric descriptions of the simplex category is straightforward: the set $[k]$ maps to the simplex Δ_k. If $\tilde{f} : [k] \to [n]$ is an order-preserving map, then we have a corresponding function $\hat{f} : \Delta_k \to \Delta_n$ mapping the i-th vertex of Δ_k into the $\tilde{f}(i)$-th vertex of Δ_n. This assignment maps $\tilde{p}_i^k$, $\tilde{s}_i^k$ and $\tilde{v}_i^k$ into $\hat{p}_i^k$, $\hat{s}_i^k$ and $\hat{v}_i^k$ respectively.

A *simplicial set* is a contravariant functor $X : \Delta \to \text{Set}$ from the simplex category to the category of sets. We usually denote the set $X(\Delta_k)$ by X_k and the whole simplicial set by $X_\bullet$. The co-face and co-degeneracy maps turn into face and degeneracy maps, respectively

$$X(\hat{p}_i^k) = p_i^k : X_{k+1} \to X_k, \quad X(\hat{s}_i^k) = s_i^k : X_{k-1} \to X_k.$$

The vertex maps turn into constant functions $v_i^k : X_k \to X_0$. When the simplex dimension is clear from the context we will simply write p_i, s_i and v_i for the face, degeneracy and vertex maps.

The face and degeneracy maps satisfy the following relations called the *simplicial identities*:

1. $p_i^{k-1} p_j^k = p_{j-1}^{k-1} p_i^k$ if $i < j$.
2. $s_i^{k+1} s_j^k = s_{j+1}^{k+1} s_i^k$ if $i \leq j$.
3. $p_i^{k-1} s_j^k = s_{j-1}^{k-1} p_i^{k-2}$ if $i < j$.
4. $p_i^{k-1} s_j^k = \text{Id}$ if $i = j$ or $i = j+1$.
5. $p_i^{k-1} s_j^k = s_j^{k-1} p_{i-1}^{k-2}$ if $i > j+1$.

Let M be a smooth manifold. We define a simplicial set $X_\bullet$ by $X_k = \text{Hom}_{\text{Smooth}}(\Delta_k, M)$, the set of smooth maps from Δ_k to M. The whole simplicial set is called the infinity groupoid of M and is denoted $\pi_\infty(M)$.

We aim to define representations up to homotopy of simplicial sets; for this purpose, we need the following: Let A be an algebra and $X_\bullet$ a simplicial set. A *cochain of degree* k with values in A is a map $F : X_k \to A$.

The algebra structure in A allows us to provide the set of cochains with some extra structure. First, we need some notation. There is a map $P^r_k : X_k \to X_r$ called the *r-dimensional frontal face map*. For an element $x \in X_k$, we define

$$P^r_k(x) = p^r_{r-1} \cdots p^{k-2}_{k-1} p^{k-1}_k(x).$$

Similarly, the *r-dimensional back face* is $Q^r_k : X_k \to X_r$ is defined by

$$Q^r_k(x) = p^r_0 \cdots p^{k-1}_0(x).$$

We use the front and back maps to define the cup product of cochains with values in A. Let F and G be cochains with degrees k and l respectively. The *cup product*, denoted $F \cup G$, is a cochain of degree $k+l$ defined by the formula

$$F \cup G(x) = F(P^k_{k+l}(x))G(Q^l_{k+l}(x)), \quad x \in X_{k+l}.$$

In the case of the infinity groupoid, the back and frontal face maps are the pullbacks of the following maps, respectively:

$$U_i : \Delta_i \to \Delta_n, \quad (t_1 \cdots, t_i) \mapsto (1, \cdots, 1, t_1 \cdots, t_i)$$
$$V_i : \Delta_i \to \Delta_n, \quad (t_1 \cdots, t_i) \mapsto (t_1 \cdots, t_i, 0, \cdots, 0).$$

Let $X_\bullet$ be a simplicial set. A *representation up to homotopy* of $X_\bullet$ is comprised of

- A $\mathbb{Z}$-graded vector space $E_x = \bigoplus_k E^k_x$ for each 0-simplex $x \in X_0$.
- A sequence of cochains $\{F_k\}_{k \geq 0}$ such that F_k is a cochain of degree k with

  $$F_k(x) \in \mathrm{Hom}^{1-k}(E_{v_k(x)}, E_{v_0(x)}),$$

  for $x \in X_k$.

For each $k \geq 0$ we require the following relation to be satisfied:

$$\sum_{j=1}^{k-1} (-1)^j F_{k-1}(p_j(x)) - \sum_{j=0}^{k} (-1)^j (F_j \cup F_{k-j})(x) = 0. \tag{2.2.8}$$

The cup product used in relation (2.2.8) is defined with compositions. Explicitly we have

$$(F_j \cup F_{k-j})(x) = F_j(P^j_k(x)) \circ F_{k-j}(Q^{k-j}_k(x)),$$

$$E_{v_k(x)} \xrightarrow{F_{k-j}(Q^{k-j}_k(x))} E_{v_j x} \xrightarrow{F_j(P^j_k(x))} E_{v_0(x)}.$$

As usual we will explore the meaning of the relations for low values of k.

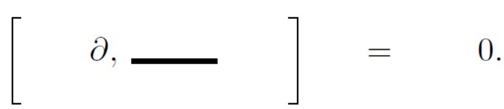
- For $k = 0$ we have a point $x \in X_0$ and a linear map $F_0(x) : E_x \to E_x$ of degree 1 with the property $F_0(x) \circ F_0(x) = 0$. Therefore $(E_x, F_0(x))$ is a cochain complex.

- For $k = 1$ we have that a 1-simplex γ is a path in M from x_0 to x_1. $F_1(\gamma)$ is a morphism $\mathrm{Hom}^0(E_{x_1}, E_{x_0})$ such that
$$F_0(x_0) \circ F_1(\gamma) = F_1(\gamma) \circ F_0(x_1).$$
In other words, $F_1(\gamma)$ is a morphism of cochain complexes.

- For $k = 2$ consider a 2-simplex σ with vertices x_0, x_1, x_2 and edges $\gamma_{i,j}$ connecting vertex x_i to x_j. The relation reads
$$F_1(\gamma_{0,1}) \circ F_1(\gamma_{1,2}) - F_1(\gamma_{0,2}) = F_0(x_0) \circ F_2(\sigma) + F_2(\sigma) \circ F_1(x_2)$$
This means that the cochain complex morphisms going from E_{x_2} to E_{x_0} defined by applying F_1 to the edges of σ are homotopic via a homotopy given by $F_2(\sigma)$.

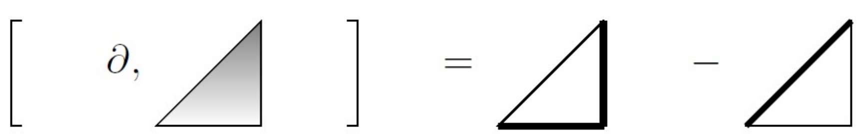
When the simplicial set is the infinity groupoid of a manifold, a representation up to homotopy is the assignment of holonomies for simplices of all dimensions in a coherent manner. In this case we have the following pictorial representation of the first three relations. A shaded face of a simplex represents the holonomy assigned to it:

- For a path we have .



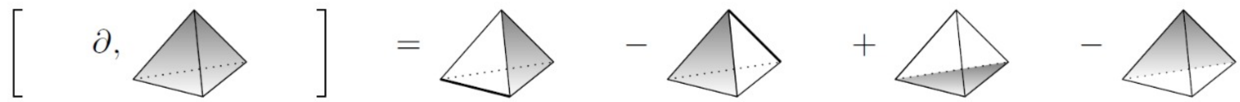
This relation states that the holonomy assigned to a path is a morphism between the cochain complexes lying over the path's endpoints.

- For the triangle we get ,



which means that the holonomy assigned to a 2-simplex is an (algebraic) homotopy between the cochain maps assigned to its edges.

- For the tetrahedron the relation is



Let $X_\bullet$ be a simplicial set and $(E,F_\bullet),(E',F'_\bullet)$ representations up to homotopy of $X_\bullet$. A *morphism* of representations of degree n is a sequence $\varphi = \{\varphi_k\}_{k\geq 0}$ with φ_k a k-cochain such that for a k-simplex $\sigma \in X_k$, we have

$$\varphi_k(\sigma) \in \mathrm{Hom}^{n-k}(E_{v_k(\sigma)}, E'_{v_0(\sigma)}).$$

The identity morphism is the sequence $\varphi_0 = \mathrm{Id}$ and $\varphi_k = 0$ for $k \geq 1$.

If $\varphi : (E,F) \to (E',F')$ and $\varphi' : (E',F') \to (E'',F'')$ are morphisms, then the composition is the sequence $\{(\varphi' \circ \varphi)_k\}_{k\geq 0}$ where

$$(\varphi' \circ \varphi)_k := \sum_{i+j=k} (-1)^{jn} (\varphi'_j \cup \varphi_i)$$

The differential of φ is the sequence $\{D(\varphi)_k\}_{k\geq 0}$ where

$$D(\varphi)_k(\sigma) := \sum_{i+j=k} (-1)^{jn} F'_j \cup \varphi_i(\sigma) + \sum_{i+j=k} (-1)^{n+j+1} \varphi_j F_i(\sigma) + \sum_{i=1}^{k-1} (-1)^{j+n} \varphi_{k-1}(p_i(\sigma)).$$

Proposition 2.6. Representations up to homotopy of $X_\bullet$ form a DG category.

The category of representations up to homotopy of $X_\bullet$ will be denoted $\mathbf{Rep}_\infty(X_\bullet)$. We should remark that the DG category $\mathbf{Rep}_\infty(X_\bullet)$ is functorial with respect to simplicial maps. More precisely, for a simplicial map $f_\bullet \colon K_\bullet \to L_\bullet$ between two simplicial sets $K_\bullet$ and $L_\bullet$, there is a DG functor $f_\bullet^* \colon \mathbf{Rep}_\infty(L_\bullet) \to \mathbf{Rep}_\infty(K_\bullet)$ which sends $(E,F_\bullet)$ to $(f_0^* E, F_\bullet \circ f_\bullet)$. It is straightforward to check that the latter is indeed a representation up to homotopy of $K_\bullet$, so that the DG functor $f_\bullet^*$ is well defined. We call it the *pullback functor* induced by $f_\bullet$.

Let X be a smooth manifold. The simplicial set $\pi_\infty X_\bullet$, called the *smooth fundamental ∞-groupoid* of X, is defined by setting $\pi_\infty X_p$ to be the set of smooth maps from the standard p-simplex Δ_p to X. The simplicial maps are defined by pulling back along with the cosimplicial maps between the simplices.

It turns out that the DG category $\mathbf{Rep}_\infty(\pi_\infty X_\bullet)$ is a global version of the DG category $\mathbf{Loc}_\infty(X)$ of ∞-local systems on X. This is the content of the *higher Riemann-Hilbert correspondence*, which is the following result, proved in [8].

Theorem 2.2. There exists an integration A_∞-functor

$$\mathscr{I} \colon \mathbf{Loc}_\infty(X) \longrightarrow \mathbf{Rep}_\infty(\pi_\infty X_\bullet),$$

which is an A_∞-quasi-equivalence of DG categories.

2.3 The equivariant de Rham Theorem

2.3.1 Equivariant Cohomology

Equivariant cohomology is a cohomology theory that considers the symmetries of space. From now on, we will denote the identity element of a group by e.

A *left action* (or G-*action*) of a topological group G on a topological space X is a continuous map $\varphi : G \times X \to X$, written $\varphi(g,x) = g \cdot x$, such that

(i) $e \cdot x = x$,

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$ and $x \in X$.

A *right action* is defined similarly but with $(x \cdot h) \cdot g = x \cdot (hg)$ instead of (ii). Any left action can be turned into a right action and vice versa via $g \cdot x = x \cdot g^{-1}$. For an action of a group G on a set X, the *stabilizer* of a point $x \in X$ is

$$\mathrm{Stab}(x) = \{g \in G | g \cdot x = x\}.$$

The action is said to be *free* if $\mathrm{Stab}(x) = \{e\}$, for every $x \in X$.

As an example, given the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$, rotation of $\mathbb{S}^2$ about the z-axis is an action of the circle $\mathbb{S}^1$ on the sphere $\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$, given explicitly by

$$e^{it} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, t \in [0, 2\pi]. \tag{2.3.1}$$

A G-*space* X, is a topological space X with a continuous action of a topological group G. A G-*equivariant map* is a continuous map $f : X \to Y$ between two G-spaces X and Y such that $f(g \cdot x) = g \cdot f(x)$, for all $x \in X$ and $g \in G$. For a given group G, the collection of G-spaces and G-maps forms a category. Let's take the smooth manifolds category instead of the topological spaces category. A smooth manifold M with a smooth action of a Lie group G is called a G-*manifold*. For a given Lie group G, the collection of smooth G-manifolds and smooth G-maps also forms a category.

Let G be a compact Lie group acting on a smooth manifold M. If the G-action is free, then the quotient M/G is a smooth manifold, in which case we can define the equivariant cohomology of M as the singular cohomology of M/G by

$$\mathrm{H}^\bullet_G(M) := \mathrm{H}^\bullet(M/G).$$

But if the action is not free, then the quotient M/G can be non-Hausdorff in general, so M/G is not a manifold again. In that case, if we take the same definition for equivariant cohomology for the manifold M, namely $\mathrm{H}^\bullet_G(M) := \mathrm{H}^\bullet(M/G)$, then this construction will give us less information about the action of the group G on the manifold M, in particular, the information about the stabilizer is lost. For example, if G is the circle group $\mathbb{S}^1$ acting on the sphere $M = \mathbb{S}^2$ by rotation about the z-axis as before, then the poles of $\mathbb{S}^2$ remain fixed by the $\mathbb{S}^1$ action. Therefore this action is not free with the two poles of the sphere having non-trivial stabilizers. The quotient $\mathbb{S}^2/\mathbb{S}^1$ can be identified with a closed interval $[-1,1]$, which is contractible and hence their cohomology is trivial, $\mathrm{H}^\bullet_{\mathbb{S}^1}(\mathbb{S}^2) = \{0\}$. The problem then is that we lose the information about the stabilizers of the G-action on M. Equivariant cohomology aims to keep some trace of the non-trivial stabilizers of the G-action.

Before presenting the definition of equivariant cohomology, we will see some essential preliminaries.

A principal G-bundle $\pi : EG \to BG$ is called a *universal G-bundle* if the following conditions are satisfied:

1. For any principal G-bundle P over a CW complex X, a continuous map $f : X \to BG$ exists such that P is isomorphic to the pullback f^*EG over X.

2. If $f, g : X \to BG$ pull EG back to isomorphic bundles $f^*EG \cong g^*EG$ over a CW complex X, then f and g are homotopic.

By a construction due to Milnor, for any topological space G, we can construct a universal G-bundle $EG \to BG$, where the base space BG is called a *classifying space* for G, and EG is a weakly contractible space. If the topological group G is a CW complex and the group operations are cellular, the areas EG and BG coming from the Milnor construction are CW complexes. By Whitehead's theorem, in the case of CW complexes, weakly contractible is the same as contractible.

If a group G acts on a space E freely, then no matter how G acts on a space M, we know that the diagonal action of G on $E \times M$ given by $g \cdot (x,y) = (g \cdot x, g \cdot y)$, is free. Since $EG \to BG$ is a principal G-bundle, the group G acts freely on the right on EG, and given a smooth G-manifold M (left-action as before), the diagonal action on $EG \times M$ given by $g \cdot (x,y) = (xg^{-1}, gx)$, will be free. In addition, if EG is a contractible space, then $EG \times M$ will have the same homotopy type as M. The *homotopy quotient* of M by G, denoted by M_G, is defined to be the orbit space of $EG \times M$ by the diagonal action of G.

Given a smooth G-manifold M, the *equivariant cohomology* of M by G is defined to be the singular cohomology of the homotopy quotient M_G,

$$\mathrm{H}^\bullet_G(M,R) := \mathrm{H}^\bullet(M_G,R),$$

where R is any coefficient ring. This definition is independent of the choice of the contractible space EG on which G acts freely. For more details see [29]. For a given topological group G, equivariant cohomology will be a contravariant functor

$$\mathrm{H}^\bullet_G(-) : \{G - \text{spaces}\} \to \{\text{rings}\}.$$

For example, the equivariant cohomology of a compact Lie group G with action given by left multiplication is $\mathrm{H}^\bullet_G(G) := \mathrm{H}^\bullet(G/G) = \mathrm{H}^\bullet(*)$, the singular cohomology of a point. If $M = \{*\}$, then we have that

$$\mathrm{H}_G(*) = \mathrm{H}^\bullet((* \times EG)/G) = \mathrm{H}^\bullet(BG).$$

So the equivariant cohomology of a point is the singular cohomology of the classifying space BG. Let G a compact group, if M is a G-space given by the trivial action $g \cdot m = m$, for all $g \in G$ and $m \in M$, then the homotopy quotient M_G is the cartesian product $BG \times M$ and therefore the equivariant cohomology of M with real coefficients is $\mathrm{H}^\bullet_G(M,\mathbb{R}) = \mathrm{H}^\bullet(BG,\mathbb{R}) \otimes \mathrm{H}^\bullet(M,\mathbb{R})$.

For example, if G is the infinite cyclic group $\mathbb{Z}$, then the classifying space and total space are $BG = \mathbb{S}^1$ and $EG = \mathbb{R}$ respectively. Therefore $\mathrm{H}^k_{\mathbb{Z}}(*,\mathbb{R}) = \mathrm{H}^k(\mathbb{S}^1,\mathbb{R}) = \mathbb{R}$, for $k = 1,2$.

Consider again the example of $\mathbb{S}^1$ acting on $\mathbb{S}^2$ by rotating about the z-axis. The homotopy quotient $\mathbb{S}^2_{\mathbb{S}^1}$ is a fiber bundle over $B\mathbb{S}^1 = \mathbb{CP}^\infty$ with fiber $\mathbb{S}^2$. The spectral sequence of this fiber bundle degenerates at the second page E_2, so that the additive structure of $\mathrm{H}^\bullet_{\mathbb{S}^1}(\mathbb{S}^2)$ is

$$\begin{aligned}
\mathrm{H}^\bullet_{\mathbb{S}^1}(\mathbb{S}^2,\mathbb{R}) &= \mathrm{H}^\bullet(\mathbb{S}^2_{\mathbb{S}^1},\mathbb{R}) = E_\infty = E_2 \\
&\cong \mathrm{H}^\bullet(B\mathbb{S}^1,\mathbb{R}) \otimes \mathrm{H}^\bullet(\mathbb{S}^2,\mathbb{R}) \\
&\cong \mathbb{R}[u] \otimes (\mathbb{R}[\omega]/(\omega^2)) \\
&= \mathbb{R}[u] \otimes (\mathbb{R} \oplus \mathbb{R}\omega) \\
&\cong \mathbb{R}[u] \oplus \mathbb{R}[u]\omega,
\end{aligned}$$

where ω is the volume form on $\mathbb{S}^2$ and $\deg u = 2$. For more details on this computation see [29]. Therefore, the equivariant cohomology $\mathrm{H}^\bullet_{\mathbb{S}^1}(\mathbb{S}^2,\mathbb{R})$ is a free $\mathbb{R}[u]$-module of rank 2, with one generator in degree 0 and one generator in degree 2.

We end the section with a result on G-principal bundles used in future sections.

Proposition 2.7. If $P \to X$ is a principal G-bundle and M is a left G-manifold, then under the diagonal action $(p,m) \cdot g = (pg, g^{-1}m)$, the projection $P \times M \to (P \times M)/G$ is a principal G-bundle.

2.3.2 The equivariant de Rham Theorem

This section outlines the more significant results in equivariant cohomology, which will give us a route to obtain the respective theorems in the case of ∞-local systems.

Let $\mathfrak{g}$ be the Lie algebra of the compact Lie group G be a Lie group. The most explicit realization of the Weil algebra $\mathrm{W}\mathfrak{g}$ is as follows. The underlying graded commutative algebra of $\mathrm{W}\mathfrak{g}$ is the tensor product

$$\mathrm{W}\mathfrak{g} = \Lambda^\bullet \mathfrak{g}^* \otimes \mathrm{S}^\bullet \mathfrak{g}^*,$$

where $\mathrm{S}^\bullet \mathfrak{g}^*$ is the symmetric algebra of $\mathfrak{g}^*$ and where we associate to each $\xi \in \mathfrak{g}^*$ the degree 1 generators $t(\xi) \in \Lambda^1 \mathfrak{g}^*$ and the degree 2 generators $w(\xi) \in \mathrm{S}^1 \mathfrak{g}^*$. The differential on $\mathrm{W}\mathfrak{g}$ is characterised by the formulas

$$\begin{aligned}
d_{\mathrm{W}}(t(\xi)) &= w(\xi) + \delta_{\mathrm{CE}}(t(\xi)), \\
d_{\mathrm{W}}(w(\xi)) &= \delta_{\mathrm{CE}}(w(\xi)),
\end{aligned}$$

where δ_{CE} is the differential of the Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$. The operators i_x and L_x are given on generators by

$$\begin{aligned} i_x(t(\xi)) &= \langle t(\xi), x\rangle, \\ i_x(w(\xi)) &= 0, \\ L_x(t(\xi)) &= \mathrm{ad}_x^*(t(\xi)), \\ L_x(w(\xi)) &= \mathrm{ad}_x^*(w(\xi)), \end{aligned}$$

for $\xi \in \mathfrak{g}^*$, and extended uniquely as derivations.

It will be useful to express the differential d_{W} and the operator i_x and L_x in terms of a dual basis e^a of $\mathfrak{g}^*$ and the structure constants $f^a_{\ bc}$ of $\mathfrak{g}$. If we write $t^a = t(e^a)$ and $w^a = w(e^a)$, they are as follows:

$$\begin{aligned} d_{\mathrm{W}} t^a &= w^a - \frac{1}{2} f^a_{\ bc} t^b t^c, \\ d_{\mathrm{W}} w^a &= f^a_{\ bc} w^b t^c, \\ i_b t^a &= \delta_b^{\ a}, \\ i_b w^a &= 0, \\ L_b t^a &= -f^a_{\ bc} t^c, \\ L_b w^a &= -f^a_{\ bc} w^c. \end{aligned}$$

It is clear that the Weil algebra is freely generated by t^a and $d_{\mathrm{W}} t^a$. This implies that $\mathrm{W}\mathfrak{g}$ is acyclic concerning d_{W}.

Let M be a G-manifold and let $\mathfrak{g}$ be the Lie algebra of G. An element $\alpha \in \mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M)$ is *basic* if and only if for all $x \in \mathfrak{g}$ we have that

$$i_x \alpha = 0,$$

$$L_x \alpha = 0,$$

where $i_x = i_x \otimes 1 + 1 \otimes i_x$, $L_x = L_x \otimes 1 + 1 \otimes L_x$. The *Weil model* is given by the DG algebra $(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\mathrm{bas}}$ of basic elements of $\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M)$, with differential given by $\delta = \mathrm{d}_W \otimes 1 + 1 \otimes \mathrm{d}_M$. We are now ready to state the main theorems that we will obtain for the case of ∞-local systems.

Theorem 2.3. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : M \to X$ be a smooth G-principal bundle. Then there is an algebra isomorphism

$$\mathrm{H}^\bullet(X) \cong \mathrm{H}^\bullet((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\mathrm{bas}}, \delta).$$

Proof. More than a complete demonstration, we will give an outline of it (For a complete demonstration see [29]).

1. Since the principal G-bundle is surjective, then $\pi^* : \Omega^\bullet(X) \to \Omega^\bullet(M)$ is injective.

2. The image $\pi^*(\Omega^\bullet(X))$ is the basic subcomplex of $\Omega^\bullet(M)$, hence

$$\Omega^\bullet(M)_{\text{bas}} = \pi^*(\Omega^\bullet(X)) \cong \Omega^\bullet(X).$$

3. By the Rham theorem, there are algebra isomorphisms

$$\mathrm{H}^\bullet(X) \cong \mathrm{H}^\bullet(\Omega^\bullet(X)) \cong \mathrm{H}^\bullet(\Omega^\bullet(M)_{\text{bas}}).$$

4. Let $i : \Omega^\bullet(M) \to \mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M)$ be the inclusion map given by $\omega \mapsto 1 \otimes \omega$. Notice that i commutes with δ, i_x and L_x for all $x \in \mathfrak{g}$. Therefore i is a morphism of $\mathfrak{g}$-differential graded algebras. As such, it takes basic elements to basic elements

$$i : \Omega^\bullet(M)_{\text{bas}} \to (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}}.$$

5. The morphism of $\mathfrak{g}$-differential graded algebras i is a quasi-isomorphism. Taking $B = \Omega^\bullet(M)_{\text{bas}}$ and $\widetilde{B} = (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}}$, and looking at the short exact sequence of complexes

$$0 \to B \xrightarrow{i} \widetilde{B} \to \widetilde{B}/B \to 0,$$

it is shown that $\mathrm{H}^\bullet(\widetilde{B}/B) = 0$, and then the long exact sequence in cohomology will give an isomorphism

$$i^* : \mathrm{H}^\bullet(B) \to \mathrm{H}^\bullet(\widetilde{B}).$$

6. Taking the two isomorphisms of 3 and 5, we obtain the desired isomorphism

$$\mathrm{H}^\bullet(X) \cong \mathrm{H}^\bullet(\Omega^\bullet(M)_{\text{bas}}) \cong \mathrm{H}^\bullet((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}}, \delta).$$ □

The following lemma marks the transition from the Weil model $(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}}$ and the Cartan model.

Lemma 2.9. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let M be a G-manifold. There is a graded-algebra isomorphism $F : (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{hor} \to \mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M)$, which induces an isomorphism on the basic sub algebras $F : (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}} \to (\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\text{inv}}$.*

The complex $(\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\text{inv}}$ is called the *Cartan model*, and the map F the *Weil-Cartan isomorphism*. The previous isomorphism F carries the Weil differential δ to a differential D on the Cartan model, defined by the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}} & \xleftarrow{H} & (\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\text{inv}}, \\ \delta \downarrow & & \downarrow D \\ (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\text{bas}} & \xrightarrow[F]{} & (\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\text{inv}} \end{array}$$

where H is the inverse map of F.

Finally, we have the equivariant de Rham theorem, which will be demonstrated for the case of ∞-local systems.

Theorem 2.4 (Equivariant de Rham Theorem)**.** Suppose a compact connected Lie group G with Lie algebra $\mathfrak{g}$ acts on a smooth manifold M, there is a graded-algebra isomorphism between equivariant cohomology and the cohomology of the Cartan model,

$$\mathrm{H}^\bullet_G(M) \cong \mathrm{H}^\bullet((\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\mathrm{inv}}, D).$$

Proof. We will give a sketch of the proof, for more details see [29].

1. By Theorem 2.3, if $P \to N$ is a principal G-bundle, then

$$\mathrm{H}^\bullet(N) \cong \mathrm{H}^\bullet((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(P))_{\mathrm{bas}}, \delta).$$

2. For a connected Lie group G, by Weil-Cartan isomorphism there is an algebra isomorphism

$$(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(P))_{\mathrm{bas}} \cong (\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(P))_{\mathrm{inv}}.$$

3. For a compact connected Lie group G, the total space is the infinite Stiefel variety $\mathrm{V}_k(\mathbb{C}^\infty)$ for some k. It can be approximated by

$$EG_n = \mathrm{V}_k(\mathbb{C}^{n+k+1}), \quad \text{for } n \gg 0$$

in the sense that

$$\mathrm{H}^q(\mathrm{V}_k(\mathbb{C}^{n+k+1})) = \mathrm{H}^q(\mathrm{V}_k(\mathbb{C}^\infty)),$$

for all $q \leq n$. Similarly, the homotopy quotient $M_G = (EG \times M)/G$ can be approximated by

$$M_{G_n} = (EG_n \times M)/G,$$

so that

$$\mathrm{H}^q(M_{G_n}) = \mathrm{H}^q(M_G),$$

for all $q \leq n$.

4. Let N and M be G-manifolds and let $f : N \to M$ be a G-equivariant map. If $f : N \to M$ induces an isomorphism in cohomology up to a certain dimension $m := n + \frac{1}{2}(n+1)n$, then it induces an isomorphism in

$$\mathrm{H}^\bullet((\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(-))_{\mathrm{inv}}) \tag{2.3.2}$$

up to dimension n.

Since EG_n is a manifold, so is $EG_n \times M$ and therefore the de Rham complex $\Omega^\bullet(EG_n \times M)$ makes sense. So we have that for $q \leq n$ and by Theorem 2.3, for $EG_n \times M \to M_{G_n}$,

$$\mathrm{H}^q_G(M) = \mathrm{H}^q(M_G) \cong \mathrm{H}^q(M_{G_n}) \cong \mathrm{H}^q((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(EG_n \times M))_{\mathrm{bas}}).$$

By Weil-Cartan isomorphism and 2.3.2 we have that

$$\begin{aligned}\mathrm{H}^q_G(M) \cong \mathrm{H}^q((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(EG_n \times M))_{\mathrm{bas}}) &\cong \mathrm{H}^q((\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(EG_n \times M))_{\mathrm{inv}}) \\ &= \mathrm{H}^q((\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M))_{\mathrm{inv}}). \qquad \square\end{aligned}$$

From now on, this will be what we refer to as the *equivariant de Rham Theorem*. The sketch of the previous proofs will be used fundamentally in the proofs of our results for the case of ∞-local systems.

As an important consequence is the cohomology of a classifying space.

Corollary 2.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then the cohomology of its classifying space BG is*

$$\mathrm{H}^\bullet(BG, \mathbb{R}) = (\mathrm{S}^\bullet\mathfrak{g}^*)_{inv}.$$

Given a complex finite dimensional vector space V with basis $\{e_1, \ldots, e_n\}$, and let $\mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$, we know that there is an isomorphism of rings $\mathrm{S}^\bullet V^* \to \mathbb{C}[x_1, \ldots, x_n]$ given by $P \mapsto \tilde{P}(x_1, \ldots, x_n) = P(v, \ldots, v)$, where $v = \sum_{i=1}^n x_i e_i$.

Next, we will give one example of classifying space. If G is the unitary group $\mathrm{U}(n)$ with Lie algebra the skew-Hermitian matrices $\mathfrak{u}(n)$, the classifying space is the Grassmannian of n-planes in $\mathbb{C}^\infty$, $BG = \mathrm{G}_n(\mathbb{C}^\infty)$, and the total space is $EG = \mathrm{V}_n(\mathbb{C}^\infty)$, the Stiefel manifold of n-dimensional orthonormal frames in $\mathbb{C}^\infty$. Taking a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{u}(n)$, by Corollary 2.1 and the previous isomorphism, we know that

$$\begin{aligned}\mathrm{H}^\bullet_{\mathrm{U}(n)}(*) &= \mathrm{H}^\bullet(\mathrm{G}_n(\mathbb{C}^\infty)) \\ &= (\mathrm{S}^\bullet(\mathfrak{u}(n))^*)_{\mathrm{inv}} \\ &= \mathbb{C}[x_1, \ldots, x_n],\end{aligned}$$

where $\mathbb{C}[x_1, \ldots, x_n]$ is the ring of invariant polynomials under the action of $\mathrm{U}(n)$ in the variables $x_1, \ldots, x_n$. Finally, notice that for the Chern-Weil homomorphism, the variables x_i are the Chern classes of the tautological bundle.

Chapter 3

Chern-Weil theory for ∞-local systems and Weil A_∞-functor

3.1 Chern-Weil theory for ∞-local systems

In this section we show that, given a principal bundle $\pi : P \to X$ with structure group G and any connection θ on P, there is DG functor $\mathscr{CW}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(X)$, where $\mathbf{InfLoc}_\infty(\mathfrak{g})$ is a the DG category of basic $\mathfrak{g}$-L_∞ spaces. Moreover, we show that, given a different connection θ', the functors $\mathscr{CW}_\theta$ and $\mathscr{CW}_{\theta'}$ are related by an A_∞-natural isomorphism. This construction provides a categorification of the Chern-Weil homomorphism.

3.1.1 $\mathfrak{g}$-DG spaces and $\mathfrak{g}$-L_∞ spaces

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider the DG Lie algebra $\mathbb{T}\mathfrak{g}$ defined as follows. As a vector space, $\mathbb{T}\mathfrak{g} = u\mathfrak{g} \oplus \mathfrak{g}$. For $x \in \mathfrak{g}$, we denote by $i(x) \in \mathbb{T}\mathfrak{g}^{-1}$ and $L(x) \in \mathbb{T}\mathfrak{g}^0$ the corresponding generators. The Lie bracket of $\mathbb{T}\mathfrak{g}$ is given by the Cartan relations

$$
\begin{aligned}
[i(x), i(y)] &= 0,\\
[L(x), L(y)] &= L([x,y]),\\
[L(x), i(y)] &= i([x,y]).
\end{aligned}
$$

The differential is defined by

$$
\begin{aligned}
d(i(x)) &= L(x),\\
d(L(x)) &= 0.
\end{aligned}
$$

By a $\mathfrak{g}$-*DG space* we mean a cochain complex V together with a DG Lie algebra homomorphism $\rho : \mathbb{T}\mathfrak{g} \to \mathrm{End}(V)$. That is, it consists of a representation of $\mathbb{T}\mathfrak{g}$ on V, where the operators $i_x \in \mathrm{End}(V)^{-1}$ and $L_x \in \mathrm{End}(V)^0$ correspond to $i(x)$ and $L(x)$, respectively. The operators i_x are called *contractions* and the operators L_x are called *Lie derivatives*. Given a $\mathfrak{g}$-DG space V, one defines the *basic subspace* V_{bas} to be the cochain subcomplex consisting of all $v \in V$ with $i_x v = 0$ and $L_x v = 0$ for all $x \in \mathfrak{g}$. Equivalently, V_{bas} is the subspace fixed by the action of $\mathbb{T}\mathfrak{g}$.

If V and W are $\mathfrak{g}$-DG spaces, a *homomorphism* $f\colon V \to W$ is just a morphism of cochain complexes commuting with the operators i_x and L_x. Such a homomorphism induces a morphism between the corresponding basic subspaces V_{bas} and W_{bas}.

We also need to consider $\mathfrak{g}$-DG algebras. A $\mathfrak{g}$-DG algebra is a $\mathfrak{g}$-DG space A endowed with the structure of DG algebra such that the action of $\mathbb{T}\mathfrak{g}$ is by derivations. Homomorphisms of $\mathfrak{g}$-DG algebras are homomorphisms of $\mathfrak{g}$-DG spaces, which are also homomorphisms of graded algebras.

The canonical example of a $\mathfrak{g}$-DG algebra is the De Rham complex $\Omega^\bullet(P)$ of a principal bundle P over a smooth manifold X with structure group G. Here the differential is the exterior derivative of forms d_P, and, if we let ρ denote the infinitesimal action of the Lie algebra $\mathfrak{g}$ on P, i_x is the inner product of a form with $\rho(x)$, and L_x is the Lie derivative of the form along with $\rho(x)$.

Another example of a $\mathfrak{g}$-DG algebra is the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Even though we already gave a general description of the Chevalley-Eilenberg algebra of a DG Lie algebra, we will also use the following, more explicit description in the case of a Lie algebra. As a graded algebra it is the exterior algebra $\Lambda^\bullet\mathfrak{g}^*$, where $\mathfrak{g}^*$ has degree 1. For $\xi \in \Lambda^1\mathfrak{g}^*$, $\delta_{\text{CE}}\xi$ is the element in $\Lambda^2\mathfrak{g}^*$ defined by

$$(\delta_{\text{CE}}\xi)(x,y) = -\xi([x,y]),$$

for all $x,y \in \mathfrak{g}$; δ_{CE} is then canonically extended to a derivation on $\Lambda^\bullet\mathfrak{g}^*$. It follows from the Jacobi identity that δ_{CE} defined in this manner squares to zero. The derivations i_x and L_x are defined on generators $\xi \in \Lambda^1\mathfrak{g}^*$, by

$$\begin{aligned} i_x\xi &= \langle \xi, x\rangle, \\ L_x\xi &= \mathrm{ad}^*_x\,\xi, \end{aligned}$$

where ad^*_x denotes the infinitesimal coadjoint action of the element x. Both are then canonically extended as derivations of degree -1 and 0, respectively, to all of $\Lambda^\bullet\mathfrak{g}^*$.

Explicit formulas for these various maps, which will be useful later on, are obtained by introducing a basis for $\mathfrak{g}^*$. Let e_a be a basis for $\mathfrak{g}$ with dual basis e^a and structure constants $f^a{}_{bc} = \langle e^a, [e_b, e_c]\rangle$, and write i_a and L_a for the contraction i_{e_a} and the Lie derivative L_{e_a} acting on $\mathrm{CE}(\mathfrak{g})$. Then the explicit formulas for δ_{CE}, i_a and L_a are the following:

$$\begin{aligned} \delta_{\text{CE}}e^a &= -\frac{1}{2}f^a{}_{bc}e^b \wedge e^c, \\ i_b e^a &= \delta_b{}^a, \\ L_b e^a &= -f^a{}_{bc}e^c, \end{aligned}$$

where $\delta_b{}^a$ is the Dirac function. Here and throughout the text, the convention that repeated indices are summed over is in place.

Given a commutative $\mathfrak{g}$-DG algebra A, an *algebraic connection* is a map $\theta\colon \mathfrak{g}^* \to A^1$, which satisfies the relations

$$
\begin{aligned}
i_x(\theta(\xi)) &= \langle \xi, x\rangle,\\
L_x(\theta(\xi)) &= \theta(\mathrm{ad}_x^*\xi),
\end{aligned}
$$

for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Given a principal bundle P over a smooth manifold X with structure group G and setting $A = \Omega^\bullet(P)$, this is equivalent to the usual definition of a connection on P.

The Weil algebra $\mathrm{W}\mathfrak{g}$, associated to the Lie algebra $\mathfrak{g}$, is the universal commutative $\mathfrak{g}$-DG algebra, with a connection $\iota\colon \mathfrak{g}^* \to \mathrm{W}^1\mathfrak{g}$. Thus, given a commutative $\mathfrak{g}$-DG algebra A, with connection θ, there exists a unique $\mathfrak{g}$-DG algebra homomorphism $c_\theta\colon \mathrm{W}\mathfrak{g} \to A$ such that $c_\theta \circ \iota = \theta$. We will refer to c_θ as the *characteristic homomorphism* for the connection θ.

Finally, given a commutative $\mathfrak{g}$-DG algebra A and a connection $\theta\colon \mathfrak{g}^* \to A^1$, the characteristic homomorphism $c_\theta\colon \mathrm{W}\mathfrak{g} \to A$ is defined on the generators of $\mathrm{W}\mathfrak{g}$, as follows:

$$
\begin{aligned}
c_\theta(t(\xi)) &= \theta(\xi),\\
c_\theta(w(\xi)) &= \delta_A(\theta(\xi)) - \theta(\delta_{\mathrm{CE}}(t(\xi))).
\end{aligned}
$$

Checking the definitions shows that c_θ is a chain map concerning the differential d_{W} of $\mathrm{W}\mathfrak{g}$.

For example, if we let θ be any connection on a principal bundle $P \to X$ with structure group G, then, by assigning to each $\xi \in \mathfrak{g}^*$ the form $\xi \circ \theta \in \Omega^1(P)$, we obtain a linear map $\mathfrak{g}^* \to \Omega^1(P)$; in view of the above, this map can be canonically extended to a $\mathfrak{g}$-DG homomorphism $c_\theta\colon \mathrm{W}\mathfrak{g} \to \Omega^\bullet(P)$, which in turn induces a morphism of cochain complexes on the basic subspaces $c_\theta\colon (\mathrm{W}\mathfrak{g})_{\mathrm{bas}} \to \Omega^\bullet_{\mathrm{bas}}(P)$. As $\mathrm{S}^\bullet\mathfrak{g}^*$ is precisely the set of elements in $\mathrm{W}\mathfrak{g}$ killed by i_x for $x \in \mathfrak{g}$, it follows that $(\mathrm{W}\mathfrak{g})_{\mathrm{bas}}$ coincides with the algebra of invariant polynomials $(\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}}$ on $\mathfrak{g}$. On the target complex we have on the other hand that $\Omega^\bullet_{\mathrm{bas}}(P)$ is canonically isomorphic to $\Omega^\bullet(X)$, so that in fact $c_\theta\colon (\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}} \to \Omega^\bullet(X)$. Since the differential d_{W} vanishes on $(\mathrm{W}\mathfrak{g})_{\mathrm{bas}} = (\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}}$, it follows that c_θ induces a cohomology map $c_{\theta*}\colon (\mathrm{S}^\bullet\mathfrak{g}^*)_{\mathrm{inv}} \to \mathrm{H}^\bullet_{\mathrm{DR}}(X)$. This is the Chern-Weil homomorphism for the principal bundle $P \to X$.

We conclude this subsection by introducing a generalization of the notion of $\mathfrak{g}$-DG space which will play a key role in the sequel. By a $\mathfrak{g}$-L_∞ *space* we mean a cochain complex V together with an L_∞-morphism $\Phi\colon \mathbb{T}\mathfrak{g} \to \mathrm{End}(V)$. To contrast this notion with that $\mathfrak{g}$-DG space, recall from § 2.1.2 that such a morphism corresponds to a collection of linear maps $\Phi_n\colon \bigodot^n(\mathrm{u}\mathbb{T}\mathfrak{g}) \to \mathrm{u}\,\mathrm{End}(V)$ of degree zero which satisfy the constraints coming from the condition that $\bar{\Phi}$ commutes with the codifferentials of $\bigodot^n(\mathrm{u}\mathbb{T}\mathfrak{g})$ and $\bigodot^n(\mathrm{u}\,\mathrm{End}(V))$. Upon setting $i_x = \Phi_1(i(x)) \in \mathrm{End}(V)^{-1}$ and

$L_x = \Phi_1(L(x)) \in \mathrm{End}(V)^0$, for low values of n, the constraints read

$$\begin{aligned}
[i_x, \delta_V] &= L_x,\\
[L_x, \delta_V] &= 0,\\
[i_x, i_y] &= \Phi_2(i(x), L(y)) - \Phi_2(L(x), i(y)) - \delta_V(\Phi_2(i(x), i(y))),\\
L_{[x,y]} - [L_x, L_y] &= \delta_V(\Phi_2(L(x), L(y))),\\
i_{[x,y]} - [L_x, i_y] &= \Phi_2(L(x), L(y)) + \delta_V(\Phi_2(L(x), i(y))).
\end{aligned}$$

From there, we gather that $\mathfrak{g}$-L_∞ spaces are generalizations of $\mathfrak{g}$-DG spaces where the higher maps Φ_n are homotopical corrections to the failure of the Cartan relations.

It is clear from the definitions that the Weil DG algebra $\mathrm{W}\mathfrak{g}$ of $\mathfrak{g}$ is isomorphic to the Chevalley-Eilenberg DG algebra of $\mathbb{T}\mathfrak{g}$. In view of Proposition 2.1, one concludes that a $\mathfrak{g}$-L_∞ space can be equivalently specified by a cochain complex V together with a Maurer-Cartan element of $\mathrm{W}\mathfrak{g} \otimes \mathrm{End}(V)$. This fact will be used throughout the text.

3.1.2 Basic ∞-local systems

Assume that $\pi\colon P \to X$ is a principal bundle with structure group G and $E = \bigoplus_{k\in\mathbb{Z}} E^k$ is a graded G-equivariant vector bundle on P. By the latter we mean a graded vector bundle $E = \bigoplus_{k\in\mathbb{Z}} E^k$ on P together with a right action of G on E that preserves the decomposition for which the projection from E to P is G-equivariant and G acts linearly on the fibers. Such an action induces a right action of G on $\Lambda^\bullet T^*P \otimes E$ turning it into graded G-equivariant vector bundle on P. Thus we get a natural left action of G on the space of E-valued differential forms $\Omega^\bullet(P,E)$: if $\varphi \in \Omega^r(P,E)$ and $g \in G$, then $\varphi \cdot g$ is the element of $\Omega^r(P,E)$ whose value at any $p \in P$ and any $v_1, \ldots, v_r \in T_pP$ is

$$(g\cdot\varphi)_p(v_1,\ldots,v_r) = \varphi_{p\cdot g}\big((d\sigma_g)_p(v_1),\ldots,(d\sigma_g)_p(v_r)\big)\cdot g^{-1}.$$

Let $\mathfrak{g}$ be the Lie algebra of G. For $x \in \mathfrak{g}$, we write i_x^E for the contraction operator on $\Omega^\bullet(P,E)$. We also denote by L_x^E the corresponding infinitesimal action on $\Omega^\bullet(P,E)$. An element $\varphi \in \Omega^\bullet(P,E)$ is called *basic* if it satisfies

$$\begin{aligned}
i_x^E\varphi &= 0,\\
L_x^E\varphi &= 0,
\end{aligned}$$

for all $x \in \mathfrak{g}$. Since each i_x^E and L_x^E are derivations, the basic elements are a graded subspace of $\Omega^\bullet(P,E)$. This subspace will be denoted by $\Omega^\bullet_{\mathrm{bas}}(P,E)$.

A special case which will be important for us occurs when E is trivialised over P in such a way that $E = P \times V$ for some graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^k$ together with a representation ρ of G on V that preserves the decomposition. In this case, $\Omega^\bullet(P,E)$ coincides with the space of V-valued differential forms $\Omega^\bullet(P,V) = \Omega^\bullet(P)\otimes V$, and if for each $x \in \mathfrak{g}$, we write i_x for the contraction operator on $\Omega^\bullet(P)$ and L_x for both the contraction operator acting on $\Omega^\bullet(P)$ and that acting on V, we have that $i_x^E = i_x \otimes 1$ and $L_x^E = L_x \otimes 1 + 1 \otimes L_x$.

Next we consider the homomorphism $\pi^* \colon \Omega^\bullet(X, E/G) \to \Omega^\bullet(P, \pi^*(E/G))$ and the isomorphism $\Phi \colon \Omega^\bullet(P,E) \to \Omega^\bullet(P, \pi^*(E/G))$ induced by the natural isomorphism $E \to \pi^*(E/G)$. We define

$$\pi^\# \colon \Omega^\bullet(X, E/G) \to \Omega^\bullet(P, E)$$

to be the composition $\pi^\# = \Phi^{-1} \circ \pi^*$. The following result is standard.

Proposition 3.1. The homomorphism $\pi^\#$ is injective. The image of $\pi^\#$ consists precisely of the basic elements.

This proposition shows that $\pi^\#$ can be considered as an isomorphism

$$\pi^\# \colon \Omega^\bullet(X, E/G) \longrightarrow \cong \Omega^\bullet_{\text{bas}}(P, E).$$

If E is trivialized over P as in the previous paragraph, then

$$\pi^\# \colon \Omega^\bullet(X, P \times_\rho V) \longrightarrow \cong \Omega^\bullet_{\text{bas}}(P, V),$$

where $P \times_\rho V$ is the associated vector bundle determined by ρ.

Now we come to the definition of basic superconnection. Let $\pi \colon P \to X$ be a principal bundle with structure group G and let $E = \bigoplus_{k \in \mathbb{Z}} E^k$ be a graded G-equivariant vector bundle on P. A *basic superconnection* on P is a superconnection D on E which is G-equivariant and satisfies the property that

$$[D, i_x^E] = L_x^E,$$

for all $x \in \mathfrak{g}$. The reason for this definition is made clear by the following result.

Lemma 3.1. *If D is a basic superconnection on E, then D preserves the graded subspace $\Omega^\bullet_{\text{bas}}(P, E)$.*

Proof. Since D is a superconnection on E which commutes with the action of G on $\Omega^\bullet(P, E)$, we see that

$$[D, L_x^E] = 0,$$

for all $x \in \mathfrak{g}$. This, combined with the defining relation, implies that if $\varphi \in \Omega^\bullet_{\text{bas}}(P, E)$ then $D\varphi \in \Omega^\bullet_{\text{bas}}(P, E)$. □

If D is a basic superconnection on E, we will also denote its restriction to the graded subspace $\Omega^\bullet_{\text{bas}}(P, E)$ by D. By a *basic ∞-local system* on P we mean a graded G-equivariant vector bundle E on P endowed with a flat basic superconnection D. As usual, we will denote such a basic ∞-local system as a pair (E, D).

Just as with ordinary ∞-local systems, all basic ∞-local system on a principal bundle $\pi \colon P \to X$ can be naturally organised into a DG category, we denote by $[\mathbf{Loc}_\infty(P)]_{\text{bas}}$. Its objects are, of course, basic ∞-local systems (E, D) on P. Given two basic ∞-local systems (E, D) and (E', D') we define the space of morphism to be the graded vector space $\Omega^\bullet_{\mathrm{bas}}(P,\mathrm{Hom}(E,E'))$ with the differential $\partial_{D,D'}$ acting as

$$\partial_{D,D'}\varphi = D'\wedge\varphi-(-1)^k\varphi\wedge D,$$

for any homogeneous basic element φ of degree k. It is important to note that $[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ is a non full DG subcategory of $\mathbf{Loc}_\infty(P)$.

We discuss next the geometric significance of $[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$. For this purpose, consider the pullback DG functor $\pi^*\colon\ \mathbf{Loc}_\infty(X)\to\mathbf{Loc}_\infty(P)$. We have the following fundamental result.

Proposition 3.2. For every object (E,D) in $[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ there is an isomorphism between (E,D) and an object of the form $\pi^*(\bar{E},\bar{D})$ with $(\bar{E},\bar{D})$ in $\mathbf{Loc}_\infty(X)$.

Proof. Let (E,D) be an object in $[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$. We know from Lemma 3.1 that D preserves the graded subspace $\Omega^\bullet_{\mathrm{bas}}(P,E)$. Set $\bar{E}=E/G$ and define an operator $\bar{D}\colon\ \Omega^\bullet(X,\bar{E})\to\Omega^\bullet(X,\bar{E})$ by means of the diagram

$$\begin{array}{ccc}\Omega^\bullet(X,\bar{E}) & \xrightarrow{\ \bar{D}\ } & \Omega^\bullet(X,\bar{E})\\ {\scriptstyle\pi^\#}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\pi^\#}\\ \Omega^\bullet_{\mathrm{bas}}(P,E) & \xrightarrow{\ D\ } & \Omega^\bullet_{\mathrm{bas}}(P,E).\end{array}$$

Then, it is immediate to verify that $\bar{D}$ is a flat superconnection on $\bar{E}$. Thus, the pair $(\bar{E},\bar{D})$ defines an object in $\mathbf{Loc}_\infty(X)$. Now consider the isomorphism $\Phi\colon\ \Omega^\bullet(P,E)\to\Omega^\bullet(P,\pi^*\bar{E})$ defined as above. Since the isomorphism $E\to\pi^*\bar{E}$ is G-equivariant and the contraction operator i^E_x only acts on $\Omega^\bullet(P)$, it follows that Φ restricts to an isomorphism from $\Omega^\bullet_{\mathrm{bas}}(P,E)$ to $\Omega^\bullet_{\mathrm{bas}}(P,\pi^*\bar{E})$, which we also denote by Φ. Notice that this isomorphism is of degree 0. Moreover, bearing in mind the definition of $\bar{D}$, we obtain the commutative diagram

$$\begin{array}{ccc}\Omega^\bullet_{\mathrm{bas}}(P,E) & \xrightarrow{\ D\ } & \Omega^\bullet_{\mathrm{bas}}(P,E)\\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi}\\ \Omega^\bullet_{\mathrm{bas}}(P,\pi^*\bar{E}) & \xrightarrow{\ \pi^*\bar{D}\ } & \Omega^\bullet_{\mathrm{bas}}(P,\pi^*\bar{E}).\end{array}$$

This shows that Φ is an isomorphism from (E,D) to $\pi^*(\bar{E},\bar{D})$. □

The previous proposition shows that π^* can be considered as a DG functor

$$\pi^*\colon\ \mathbf{Loc}_\infty(X)\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}.$$

We obtain a lot more.

Proposition 3.3. The DG functor $\pi^*\colon\ \mathbf{Loc}_\infty(X)\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ is a quasi-equivalence.

Proof. By virtue of Proposition 3.2, we only have to show that $\pi^*\colon\ \mathbf{Loc}_\infty(X)\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ is quasi fully faithful. So for any pair of objects $(\bar{E},\bar{D})$ and $(\bar{E}',\bar{D}')$ in $\mathbf{Loc}_\infty(X)$ consider the associated map

$$\pi^*\colon\ \Omega^\bullet(X,\mathrm{Hom}(\bar{E},\bar{E}'))\longrightarrow\Omega^\bullet_{\mathrm{bas}}(P,\mathrm{Hom}(\pi^*\bar{E},\pi^*\bar{E}')).$$

Then, if $\Psi\colon \Omega^\bullet(X,\mathrm{Hom}((\pi^*\bar{E})/G,(\pi^*\bar{E}')/G)) \to \Omega^\bullet(X,\mathrm{Hom}(\bar{E},\bar{E}'))$ denotes the isomorphism induced by the natural isomorphisms $(\pi^*\bar{E})/G \to \bar{E}$ and $(\pi^*\bar{E}')/G \to \bar{E}'$, is not hard to check that $\pi^*\circ\Psi = \pi^\#$. This shows that π^* is an isomorphism. On the other hand, if $\omega \in \Omega^\bullet(X,\mathrm{Hom}(\bar{E},\bar{E}'))$ is a homogeneous element of degree k, we have

$$\pi^*(\partial_{\bar{D},\bar{D}'}\omega) = \pi^*(\bar{D}'\wedge\omega-(-1)^k\omega\wedge\bar{D}) = \pi^*\bar{D}'\wedge\pi^*\omega-(-1)^k\pi^*\omega\wedge\pi^*\bar{D} = \partial_{\pi^*\bar{D},\pi^*\bar{D}'}(\pi^*\omega).$$

It follows that π^* is, in fact, an isomorphism of cochain complexes and, thus, in particular, a quasi-isomorphism. □

We shall see the importance of Proposition 3.3 in the following section.

3.1.3 The Chern-Weil DG functor

This part describes the construction of a characteristic DG functor that extends the Chern-Weil homomorphism for principal bundles to the realm of ∞-local systems. We begin by recording several preliminary observations.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let V be a $\mathfrak{g}$-L_∞ space. For $x\in\mathfrak{g}$, by a slight abuse of notation, we will indistinctly write i_x and L_x for the contraction and Lie derivative operators acting on $\mathrm{W}\mathfrak{g}$ or V. With this caveat, it is a fact that $\mathrm{W}\mathfrak{g}\otimes V$ acquires the structure of a $\mathfrak{g}$-L_∞ space, where the differential, contraction and Lie derivative operators are $d_{\mathrm{W}}\otimes 1+1\otimes\delta_V$, $i_x\otimes 1$ and $L_x\otimes 1+1\otimes L_x$, respectively. Thus, we may consider the basic subspace $(\mathrm{W}\mathfrak{g}\otimes V)_{\mathrm{bas}}$.

On the other hand, recall from the remark made at the end of §3.1.1 that the $\mathfrak{g}$-L_∞ space V determines and is determined by a Maurer-Cartan element of $\mathrm{W}\mathfrak{g}\otimes\mathrm{End}(V)$, which we write as α_V. We shall say that V is *basic* if the following identities are satisfied

$$[\alpha_V, i_x\otimes 1] = 1\otimes L_x,$$
$$[\alpha_V, L_x\otimes 1+1\otimes L_x] = 0,$$

for all $x\in\mathfrak{g}$. This definition is justified by the following construction.

We wish to construct a derivation $D\colon \mathrm{W}\mathfrak{g}\otimes V \to \mathrm{W}\mathfrak{g}\otimes V$ of homogeneous degree 1, such that $D^2=0$. To do so, we may simply define

$$D = d_{\mathrm{W}}\otimes 1+1\otimes\delta_V+\alpha_V.$$

That D^2 is zero, followed by a straightforward calculation. Also, the following property holds.

Lemma 3.2. *If V is a basic $\mathfrak{g}$-L_∞ space, then D preserves the graded subspace* $(\mathrm{W}\mathfrak{g}\otimes V)_{\mathrm{bas}}$.

Proof. We wish to show that if $\varphi\in(\mathrm{W}\mathfrak{g}\otimes V)_{\mathrm{bas}}$ then $D\varphi\in(\mathrm{W}\mathfrak{g}\otimes V)_{\mathrm{bas}}$. For this, it will be enough to verify that

$$[D, L_c\otimes 1+1\otimes L_c] = 0,$$

and

$$[D, i_c \otimes 1] = L_c \otimes 1 + 1 \otimes L_c.$$

Fix an element of $\mathrm{W}\mathfrak{g} \otimes V$ of the form $\varphi \otimes v$. Then a straightforward computation gives

$$\begin{aligned} D\left((L_c \otimes 1 + 1 \otimes L_c)(\varphi \otimes v)\right) &= d_{\mathrm{W}}(L_c\varphi) \otimes v + (-1)^{|\varphi|} L_c\varphi \otimes \delta_V v \\ &\quad + d_{\mathrm{W}}\varphi \otimes L_c v + (-1)^{|\varphi|}\varphi \otimes \delta_V(L_c v) \\ &\quad + \alpha_V\left((L_c \otimes 1 + 1 \otimes L_c)(\varphi \otimes v)\right), \end{aligned}$$

and

$$\begin{aligned} (L_c \otimes 1 + 1 \otimes L_c)\left(D(\varphi \otimes v)\right) &= L_c(d_{\mathrm{W}}\varphi) \otimes v + (-1)^{|\varphi|} L_c\varphi \otimes \delta_V v \\ &\quad + d_{\mathrm{W}}\varphi \otimes L_c v + (-1)^{|\varphi|}\varphi \otimes L_c(\delta_V v) \\ &\quad + (L_c \otimes 1 + 1 \otimes L_c)\left(\alpha_V(\varphi \otimes v)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} [D, L_c \otimes 1 + 1 \otimes L_c](\varphi \otimes v) &= [d_{\mathrm{W}}, L_c]\varphi \otimes v + (-1)^{|\varphi|}\varphi \otimes [\delta_V, L_c]v \\ &\quad + [\alpha_V, L_c \otimes 1 + 1 \otimes L_c](\varphi \otimes v) \\ &= 0. \end{aligned}$$

Thus the first identity is established. On the other hand, again, by direct computation,

$$\begin{aligned} D\left((i_c \otimes 1)(\varphi \otimes v)\right) &= d_{\mathrm{W}}(i_c\varphi) \otimes v - (-1)^{|\varphi|} i_c\varphi \otimes \delta_V v \\ &\quad + \alpha_V\left((i_c \otimes 1)(\varphi \otimes v)\right), \end{aligned}$$

and

$$\begin{aligned} (i_c \otimes 1)\left(D(\varphi \otimes v)\right) &= i_c(d_{\mathrm{W}}\varphi) \otimes v + (-1)^{|\varphi|} i_c\varphi \otimes \delta_V v \\ &\quad + (i_c \otimes 1)\left(\alpha_V(\varphi \otimes v)\right). \end{aligned}$$

Hence,

$$\begin{aligned} [D, i_c \otimes 1](\varphi \otimes v) &= [d_{\mathrm{W}}, i_c]\varphi \otimes v + [\alpha_V, i_c \otimes 1](\varphi \otimes v) \\ &= L_c\varphi \otimes v + (1 \otimes L_c)(\varphi \otimes v) \\ &= (L_c \otimes 1 + 1 \otimes L_c)(\varphi \otimes v), \end{aligned}$$

and, consequently, the second identity also holds. □

The content of the previous discussion is that, provided V is a basic $\mathfrak{g}$-L_∞ space, the differential D can be regarded as a flat basic superconnection on the graded vector bundle $EG \times V$. As a result, ignoring the technical problem with infinite dimensionality, the pair $(EG \times V, D)$ defines a basic ∞-local system on EG in the sense of §3.1.2.

The preceding discussion allows us to define a DG category, which we call the *DG category of infinitesimal ∞-local systems* on $\mathfrak{g}$, by the following data. The objects of this DG category are all basic $\mathfrak{g}$-L_∞ spaces. For any two $\mathfrak{g}$-L_∞ spaces V and V', with corresponding differentials D and D', the space of morphisms is the graded vector space $(\mathrm{W}\mathfrak{g}\otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}}$ with the differential $\partial_{D,D'}$ acting according to the formula

$$\partial_{D,D'}\varphi = D'\circ\varphi - (-1)^k\varphi\circ D,$$

for any homogeneous element φ of degree k. The DG category given by this data will be denoted by $\mathbf{InfLoc}_\infty(\mathfrak{g})$.

We are now in a position to state and prove the main result of this section.

Theorem A. *Let G be a Lie group and let $\pi\colon P\to X$ be a principal bundle with structure group G. Then, for any connection θ on P, there is a natural DG functor*

$$\mathscr{CW}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \longrightarrow \mathbf{Loc}_\infty(X).$$

Moreover, for any two connections θ and θ' on P, there is an A_∞-natural isomorphism between $\mathscr{CW}_\theta$ and $\mathscr{CW}_{\theta'}$.

Proof. In view of Proposition 3.3, the pullback DG functor $\pi^*\colon \mathbf{Loc}_\infty(X)\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ is a quasi-equivalence. Hence, it will suffice to show that for any connection θ on P, there is a natural DG functor

$$\overline{\mathscr{CW}}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \longrightarrow [\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}.$$

Let us thus fix a connection θ on P and keep the notation as above. We define the DG functor $\overline{\mathscr{CW}}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g})\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ as follows. For each object V in $\mathbf{InfLoc}_\infty(\mathfrak{g})$ consider the operator $D_\theta\colon \Omega^\bullet(P,V)\to\Omega^\bullet(P,V)$ determined by the formula

$$D_\theta = d_P\otimes 1 + 1\otimes\delta_V + (c_\theta\otimes 1)\alpha_V,$$

where as usual $c_\theta\colon \mathrm{W}\mathfrak{g}\to\Omega^\bullet(P)$ is the characteristic homomorphism for the connection θ. Here we note that the fact that c_θ is a $\mathfrak{g}$-DG algebra homomorphism implies that $(c_\theta\otimes 1)\alpha_V$ is a Maurer-Cartan element of $\Omega^\bullet_{\mathrm{bas}}(P,\mathrm{End}(V))$. Then the argument given in the proof of Lemma3.2 can be repeated to show that D_θ defines a flat basic superconnection on $P\times V$. That being the case, we define $\overline{\mathscr{CW}}_\theta(V)$ to be the basic ∞-local system $(P\times V, D_\theta)$. On the other hand, suppose $\varphi\in(\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V,V'))_{\mathrm{bas}}$ is an arbitrary morphism between two objects V and V' in $\mathbf{InfLoc}_\infty(\mathfrak{g})$. Then, by the foregoing remark, we again have that $(c_\theta\otimes 1)\varphi\in\Omega^\bullet_{\mathrm{bas}}(P,\mathrm{Hom}(V,V'))$. In this way we get a morphism of graded vector spaces

$$\overline{\mathscr{CW}}_\theta\colon (\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V,V'))_{\mathrm{bas}} \longrightarrow \Omega^\bullet_{\mathrm{bas}}(P,\mathrm{Hom}(V,V')).$$

We may note further that, since c_θ commutes with the differentials d_{W} and d_P, this morphism is a cochain map with respect to both differentials $\partial_{D,D'}$ and $\partial_{D_\theta,D'_\theta}$. It is a straightforward exercise to check that this data defines a DG functor $\overline{\mathscr{CW}}_\theta\colon \mathbf{InfLoc}_\infty(\mathfrak{g})\to[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$.

To show the second part, suppose that θ and θ' are two different connections on P. Define an interpolating connection θ_t by

$$\theta_t = \theta + t(\theta - \theta')$$

so that $\theta_0 = \theta$ and $\theta_1 = \theta'$. Then, if we let $\mathrm{pr}_1 \colon P \times [0,1] \to P$ be the projection onto the first factor, we have that $\hat{\theta} = \mathrm{pr}_1^* \theta_t$ defines a connection on $P \times [0,1]$. Hence, by the foregoing, we get a DG functor $\overline{\mathscr{CW}}_{\hat{\theta}} \colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to [\mathbf{Loc}_\infty(P \times [0,1])]_{\mathrm{bas}}$. If, on the other hand, $\iota_t \colon P \to P \times [0,1]$ denotes the inclusion of height t, then the pullback DG functor $\iota_t^* \colon \mathbf{Loc}_\infty(P \times [0,1]) \to \mathbf{Loc}_\infty(P)$ induces DG functors between the DG subcategories $[\mathbf{Loc}_\infty(P \times [0,1])]_{\mathrm{bas}}$ and $[\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$, which we denote by the same symbol, and further we have that $\overline{\mathscr{CW}}_{\theta} = \iota_0^* \circ \overline{\mathscr{CW}}_{\hat{\theta}}$ and $\overline{\mathscr{CW}}_{\theta'} = \iota_1^* \circ \overline{\mathscr{CW}}_{\hat{\theta}}$. In addition, by virtue of Proposition 2.5, there exists an A_∞-natural isomorphism between ι_0^* and ι_1^*. By restricting the latter to the full DG subcategory of $[\mathbf{Loc}_\infty(P \times [0,1])]_{\mathrm{bas}}$ consisting of objects of the form $\overline{\mathscr{CW}}_{\hat{\theta}}(V)$ with V an object of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$, we obtain an A_∞-natural isomorphism between $\overline{\mathscr{CW}}_{\theta}, \overline{\mathscr{CW}}_{\theta'} \colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to [\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$, as wished. □

We shall henceforth refer to the DG functor $\mathscr{CW}_{\theta} \colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(X)$ induced by a connection θ on P as the *Chern-Weil DG functor of P*.

3.1.4 An example: the Gauss-Manin ∞-local system

In the following we conserve the notations of §3.1.1. Let $\mathfrak{g}$ be a Lie algebra and consider the DG Lie algebra $\mathrm{W}\mathfrak{g} \otimes \mathrm{End}(\mathrm{CE}(\mathfrak{g}))$. Fix a basis e_a of $\mathfrak{g}$ with structure constants $f^a_{\ bc}$ and recall that t^a stands for the degree 1 generators of $\Lambda^1\mathfrak{g}^*$ and w^a stands for the degree 2 generators of $\mathrm{S}^1\mathfrak{g}^*$.

Our starting point is the following observation.

Lemma 3.3. *The element $\alpha_{\mathrm{CE}} = t^a \otimes L_a - w^a \otimes i_a$ is a Maurer-Cartan element of* $\mathrm{W}\mathfrak{g} \otimes \mathrm{End}(\mathrm{CE}(\mathfrak{g}))$.

Proof. On the one hand,

$$\begin{aligned}(d_{\mathrm{W}} \otimes 1)\alpha_{\mathrm{CE}} &= (d_{\mathrm{W}} \otimes 1)(t^a \otimes L_a - w^a \otimes i_a) \\ &= d_{\mathrm{W}} t^a \otimes L_a - d_{\mathrm{W}} w^a \otimes i_a \\ &= w^a \otimes L_a - \frac{1}{2} f^a_{\ bc} t^b t^c \otimes L_a - f^a_{\ bc} w^b t^c \otimes i_a,\end{aligned}$$

and

$$\begin{aligned}(1 \otimes \delta_{\mathrm{CE}})\alpha_{\mathrm{CE}} &= (1 \otimes \delta_{\mathrm{CE}})(t^a \otimes L_a - w^a \otimes i_a) \\ &= -t^a \otimes [\delta_{\mathrm{CE}}, L_a] - w^a \otimes [\delta_{\mathrm{CE}}, i_a] \\ &= -w^a \otimes L_a.\end{aligned}$$

Hence,

$$(d_{\mathrm{W}} \otimes 1 + 1 \otimes \delta_{\mathrm{CE}})\alpha_{\mathrm{CE}} = -f^a_{\ bc} w^b t^c \otimes i_a - \frac{1}{2} f^a_{\ bc} t^b t^c \otimes L_a$$

On the other hand,

$$\begin{aligned}[\alpha_{\mathrm{CE}},\alpha_{\mathrm{CE}}] &= [t^b\otimes L_b - w^b\otimes i_b, t^c\otimes L_c - w^c\otimes i_c]\\ &= t^bt^c\otimes[L_b,L_c] - t^bw^c\otimes[L_b,i_c] + w^bt^c\otimes[i_b,L_c] + w^bw^c\otimes[i_b,i_c]\\ &= f^a_{\ bc}t^bt^c\otimes L_a - f^a_{\ bc}t^bw^c\otimes i_a - f^a_{\ cb}w^bt^c\otimes i_a\\ &= f^a_{\ bc}t^bt^c\otimes L_a + 2f^a_{\ bc}w^bt^c\otimes i_a.\end{aligned}$$

In conclusion, we obtain

$$(d_{\mathrm{W}}\otimes 1 + 1\otimes\delta_{\mathrm{CE}})\alpha_{\mathrm{CE}} + \frac{1}{2}[\alpha_{\mathrm{CE}},\alpha_{\mathrm{CE}}] = 0,$$

as required. □

By the discussion of the preceding section, the Maurer-Cartan element α_{CE} endows the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ with the structure of a $\mathfrak{g}$-L_∞ space. The following lemma deals with the basicness of such $\mathfrak{g}$-L_∞ space.

Lemma 3.4. *The* $\mathfrak{g}$-L_∞ *space* $\mathrm{CE}(\mathfrak{g})$ *is basic.*

Proof. It suffices to show that

$$[\alpha_{\mathrm{CE}}, i_c\otimes 1] = 1\otimes L_c,$$

and

$$[\alpha_{\mathrm{CE}}, L_c\otimes 1 + 1\otimes L_c] = 0.$$

So fix an element of $\mathrm{W}\mathfrak{g}\otimes\mathrm{CE}(\mathfrak{g})$ of the form $\varphi\otimes\xi$. A simple calculation shows that

$$\alpha_{\mathrm{CE}}((i_c\otimes 1)(\varphi\otimes\xi)) = (t^a i_c\varphi)\otimes L_a\xi + (-1)^{|\varphi|}(w^a i_c\varphi)\otimes i_a\xi,$$

and

$$(i_c\otimes 1)(\alpha_{\mathrm{CE}}(\varphi\otimes\xi)) = \delta^a_c\varphi\otimes L_a\xi - (t^a i_c\varphi)\otimes L_a\xi - (-1)^{|\varphi|}(w^a i_c\varphi)\otimes i_a\xi.$$

Thus,

$$[\alpha_{\mathrm{CE}}, i_c\otimes 1](\varphi\otimes\xi) = \delta^a_c\varphi\otimes L_a\xi = \varphi\otimes L_c\xi = (1\otimes L_c)(\varphi\otimes\xi).$$

Hence the first identity holds. Furthermore, another straightforward calculation shows that

$$\begin{aligned}\alpha_{\mathrm{CE}}((L_c\otimes 1 + 1\otimes L_c)(\varphi\otimes\xi)) &= (t^aL_c\varphi)\otimes L_a\xi - (-1)^{|\varphi|}(w^aL_c\varphi)\otimes i_a\xi\\ &\quad + (t^a\varphi)\otimes L_a(L_c\xi) - (-1)^{|\varphi|}(w^a\varphi)\otimes i_a(L_c\xi),\end{aligned}$$

and

$$
\begin{aligned}
(L_c \otimes 1 + 1 \otimes L_c)(\alpha_{\mathrm{CE}}(\varphi \otimes \xi)) = &-(f^a{}_{bc} t^b \varphi) \otimes L_a \xi + (t^a L_c \varphi) \otimes L_a \xi \\
&+ (-1)^{|\varphi|}(f^a{}_{cb} w^b \varphi) \otimes i_a \xi - (-1)^{|\varphi|}(w^a L_c \varphi) \otimes i_a \xi \\
&+ (t^a \varphi) \otimes L_c(L_a \xi) - (-1)^{|\varphi|}(w^a \varphi) \otimes L_c(i_a \xi).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
[\alpha_{\mathrm{CE}}, L_c \otimes 1 + 1 \otimes L_c] &= (f^a{}_{cb} t^b \varphi) \otimes L_a \xi - (-1)^{|\varphi|}(f^a{}_{cb} w^b \varphi) \otimes i_a \xi \\
&\quad + (t^a \varphi) \otimes [L_a, L_c]\xi + (-1)^{|\varphi|}(w^a \varphi) \otimes [L_c, i_a]\xi \\
&= (f^a{}_{cb} t^b \varphi) \otimes L_a \xi - (-1)^{|\varphi|}(f^a{}_{cb} w^b \varphi) \otimes i_a \xi \\
&\quad + (f^b{}_{ac} t^a \varphi) \otimes L_b \xi + (-1)^{|\varphi|}(f^b{}_{ca} w^a \varphi) \otimes i_b \xi \\
&= 0,
\end{aligned}
$$

implying that the second identity also holds. □

In view of this result, the $\mathfrak{g}$-L_∞ space $\mathrm{CE}(\mathfrak{g})$ defines an object of the DG category $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$. The corresponding differential takes the form

$$
D = d_{\mathrm{W}} \otimes 1 + 1 \otimes \delta_{\mathrm{CE}} + \alpha_{\mathrm{CE}} = d_{\mathrm{W}} \otimes 1 + 1 \otimes \delta_{\mathrm{CE}} + t^a \otimes L_a - w^a \otimes i_a.
$$

As pointed out in the previous section, in the case in which $\mathfrak{g}$ is the Lie algebra of a connected Lie group, D can be thought of as a flat superconnection on the graded vector bundle $EG \times \mathrm{CE}(\mathfrak{g})$. Thus, ignoring the issue of infinite dimensionality, the pair $(EG \times \mathrm{CE}(\mathfrak{g}), D)$ defines an ∞-local system on EG, which we will call the *universal Weil ∞-local system*.

Let now $\pi\colon P \to X$ be a principal bundle with structure group G and consider the coadjoint action ad^* of G on $\mathfrak{g}^*$. This action extends uniquely to a representation of G on $\mathrm{CE}(\mathfrak{g})$, turning $P \times \mathrm{CE}(\mathfrak{g})$ into a graded G-equivariant vector bundle on P. Choose a connection θ on P with curvature Ω and write $\theta = \theta^a \otimes e_a$ and $\Omega = \Omega^a \otimes e_a$. By following the construction presented in the proof of Theorem A, the operator $D_\theta\colon \Omega^\bullet(P, \mathrm{CE}(\mathfrak{g})) \to \Omega^\bullet(P, \mathrm{CE}(\mathfrak{g}))$ given by

$$
D_\theta = d_P \otimes 1 + 1 \otimes \delta_{\mathrm{CE}} + (c_\theta \otimes 1)\alpha_{\mathrm{CE}} = d_P \otimes 1 + 1 \otimes \delta_{CE} + \theta^a \otimes L_a - \Omega^a \otimes i_a,
$$

defines a flat basic superconnection on $P \times \mathrm{CE}(\mathfrak{g})$. In other words, we obtain a basic ∞-local system $(P \times \mathrm{CE}(\mathfrak{g}), D_\theta)$, which is nothing but the image $\overline{\mathscr{CW}}_\theta(\mathrm{CE}(\mathfrak{g}))$ through the DG functor $\overline{\mathscr{CW}}_\theta\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to [\mathbf{Loc}_\infty(P)]_{\mathrm{bas}}$ of $\mathrm{CE}(\mathfrak{g})$.

Next we consider the graded vector bundle $\mathrm{CE}(P) = P \times_{\mathrm{ad}^*} \mathrm{CE}(\mathfrak{g})$ associated with P and the coadjoint action ad^* of G on $\mathrm{CE}(\mathfrak{g})$. On account of Proposition 3.2, the flat basic superconnection D_θ defined above induces a flat superconnection $\bar{D}_\theta$ on $\mathrm{CE}(P)$. Thus, the pair $(\mathrm{CE}(P), \bar{D}_\theta)$ constitutes an ∞-local system on the base manifold X, which again by the argument given in the proof of Theorem A, is exactly the image of the image $\mathscr{CW}_\theta(\mathrm{CE}(\mathfrak{g}))$ under the Chern-Weil DG functor

$\mathscr{C}\mathscr{W}_\theta \colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(X)$ of $\mathrm{CE}(\mathfrak{g})$. We call it the *Gauss-Manin ∞-local system for θ*.

We close with the following result.

Proposition 3.4. Let G be a compact connected Lie group and let $\pi\colon P \to X$ be a principal bundle with structure group G. Then, for any connection θ on P, the cohomology of the corresponding Gauss-Manin ∞-local system $(\mathrm{CE}(P), \bar{D}_\theta)$ is isomorphic to $\mathrm{H}^\bullet_{\mathrm{DR}}(P)$.

Proof. Let us denote by $\Omega^\bullet(P)^G$ the subspace of G-invariant elements of $\Omega^\bullet(P)$, and write $\underline{\mathbb{R}}$ for the constant ∞-local system $(X \times \mathbb{R}, d)$. We will prove a stronger result, namely that the space of morphisms $\mathrm{Hom}(\underline{\mathbb{R}}, (\mathrm{CE}(P), \bar{D}_\theta))$ is isomorphic as a cochain complex to $\Omega^\bullet(P)^G$. That this implies our desired result is a consequence of the fact that the cohomology of $(\mathrm{CE}(P), \bar{D}_\theta)$ isomorphic to that to that of $\mathrm{Hom}(\underline{\mathbb{R}}, (\mathrm{CE}(P), \bar{D}_\theta))$, and that, since G is compact and connected, the inclusion $\Omega^\bullet(P)^G \to \Omega^\bullet(P)$ is a quasi-isomorphism. By the preceding remark, we know that $(\mathrm{CE}(P), \bar{D}_\theta)$ coincides with the image of $\mathrm{CE}(\mathfrak{g})$ under the Chern-Weil DG functor $\mathscr{C}\mathscr{W}_\theta \colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(X)$. Thus, by the construction of the latter, $\mathrm{Hom}(\underline{\mathbb{R}}, (\mathrm{CE}(P), \bar{D}_\theta))$ is isomorphic to $(\Omega^\bullet(P) \otimes \mathrm{CE}(\mathfrak{g}))_{\mathrm{bas}}$. Thus, it will be enough to show that $(\Omega^\bullet(P) \otimes \mathrm{CE}(\mathfrak{g}))_{\mathrm{bas}}$ is isomorphic to $\Omega^\bullet(P)^G$. For this, let us consider the subspace $(\Omega^\bullet(P) \otimes \mathrm{CE}(\mathfrak{g}))_{\mathrm{hor}}$ consisting of all the elements of $\Omega^\bullet(P) \otimes \mathrm{CE}(\mathfrak{g})$ that are sent to zero by the contraction $i_x \otimes 1$ for all $x \in \mathfrak{g}$. Then the connection θ induces an isomorphism of cochain complexes between $(\Omega^\bullet(P) \otimes \mathrm{CE}(\mathfrak{g}))_{\mathrm{hor}}$ and $\Omega^\bullet(P)$. The result we are after follows immediately from the fact that this isomorphism is G-equivariant. $\square$

3.2 ∞-Local systems and classifying spaces

In this section we prove our second main result. This states that if G is a compact connected Lie group with Lie algebra $\mathfrak{g}$, then, the DG category $\mathbf{InfLoc}_\infty(\mathfrak{g})$ is A_∞-quasi-equivalent to the DG category $\mathbf{Loc}_\infty(BG)$ of ∞-local systems on the classifying space of G. Moreover, given a principal bundle $\pi\colon P \to X$ with structure group G and any connection θ on P, the Chern-Weil functor $\mathscr{C}\mathscr{W}_\theta \colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(X)$ corresponds to the pullback functor associated to the classifying map of P.

3.2.1 Canonical connections on Stiefel bundles

Here we recall the existence of canonical connections on Stiefel bundles which will be needed in the text below. Although the results are well-known, proofs will be given for completeness.

For nonnegative integers $k \le n$, let $\mathrm{V}_k(\mathbb{C}^n)$ denote the Stiefel manifold which parametrises orthonormal k-frames in $\mathbb{C}^k$. It can be thought of as a set of $n \times k$ matrices by writing a k-frame as a matrix of k column vectors in $\mathbb{C}^n$. The orthonormality condition is expressed by $A^*A = I_k$ where A^* denotes the conjugate transpose of A and I_k denotes the $k \times k$ identity matrix. The topology on $\mathrm{V}_k(\mathbb{C}^n)$ is the subspace topology inherited from $\mathbb{C}^{n\times k}$. With this topology $\mathrm{V}_k(\mathbb{C}^n)$ is a compact manifold whose dimension is $(2n-k)k$. There is a natural projection $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$ from the Stiefel manifold $\mathrm{V}_k(\mathbb{C}^n)$ to the Grassmannian of k-planes in $\mathbb{C}^n$ which sends a k-frame to the subspace spanned by that frame. The fiber over a given point W in $\mathrm{G}_k(\mathbb{C}^n)$ is the set of all orthonormal k-frames contained in the space W. There is also a natural right action of $\mathrm{U}(k)$ on $\mathrm{V}_k(\mathbb{C}^n)$ which rotates a k-frame in the space it spans. This action is free and the quotient is precisely $\mathrm{G}_k(\mathbb{C}^n)$. It follows that $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$ is indeed a principal bundle with structure group $\mathrm{U}(k)$.

There are natural inclusions $i_n\colon \mathrm{G}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^{n+1})$ and $j_n\colon \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{V}_k(\mathbb{C}^{n+1})$ induced by the standard inclusion $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. Moreover, the latter is equivariant, so that we have a commutative diagram of inclusions and principal bundles with structure group $\mathrm{U}(k)$,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{V}_k(\mathbb{C}^{n-1}) & \xrightarrow{j_{n-1}} & \mathrm{V}_k(\mathbb{C}^n) & \xrightarrow{j_n} & \mathrm{V}_k(\mathbb{C}^{n+1}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \\
\cdots & \longrightarrow & \mathrm{G}_k(\mathbb{C}^{n-1}) & \xrightarrow{i_{n-1}} & \mathrm{G}_k(\mathbb{C}^n) & \xrightarrow{i_n} & \mathrm{G}_k(\mathbb{C}^{n+1}) & \longrightarrow & \cdots.
\end{array}$$

In the following, we shall write simply j to denote any of the inclusions in the upper line of the diagram.

Proposition 3.5. The principal bundles $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$ admit a canonical connection ω_{c} such that $j^*\omega_{\mathrm{c}} = \omega_{\mathrm{c}}$.

Proof. We first identify $\mathrm{V}_k(\mathbb{C}^n)$ with $n\times k$ matrices A satisfying $A^*A = I_k$. Then the tangent space of $\mathrm{V}_k(\mathbb{C}^n)$ at A identified with the space of all $n \times k$ matrices M such that

$$M^*A + A^*M = 0.$$

For each point A in $\mathrm{V}_k(\mathbb{C}^n)$ and for each tangent vector M of $\mathrm{V}_k(\mathbb{C}^n)$ at A, we consider the $k\times k$ matrix-valued 1-form ω_{c} on $\mathrm{V}_k(\mathbb{C}^n)$ defined as

$$[\omega_{\mathrm{c}}(A)](M) = A^*M.$$

Because of the above condition, it is clear that ω_{c} takes actually values in the Lie algebra $\mathfrak{u}(k)$ of $\mathrm{U}(k)$. We claim that ω_{c} is a connection form. In fact, if $u \in \mathrm{U}(k)$ and r_u denotes the right translation, then for each point A in $\mathrm{V}_k(\mathbb{C}^n)$ and for each tangent vector M of $\mathrm{V}_k(\mathbb{C}^n)$ at A,

$$[r_u^*\omega_{\mathrm{c}}(A)](M) = [\omega_{\mathrm{c}}(Au)](Mu) = u^*A^*Mu = u^{-1}A^*Mu = \mathrm{Ad}_{u^{-1}}[\omega_{\mathrm{c}}(A)](M).$$

On the other hand, if $x \in \mathfrak{u}(k)$ and $x^\sharp$ denotes the fundamental vector field generated by x, then for each point A in $\mathrm{V}_k(\mathbb{C}^n)$,

$$[\omega_{\mathrm{c}}(A)](x^\sharp(A)) = [\omega_{\mathrm{c}}(A)]\left(\frac{d}{dt}\bigg|_{t=0} A\exp(tx)\right) = A^*Ax = x.$$

Finally, the condition $j^*\omega_{\mathrm{c}} = \omega_{\mathrm{c}}$ follows immediately from the fact that j is linear. □

Now recall that every compact Lie group is isomorphic to a compact subgroup of $\mathrm{U}(k)$. Therefore, given such a Lie group G, one can fix an embedding $G \subset \mathrm{U}(k)$ and restrict the action of $\mathrm{U}(k)$ on

$\mathrm{V}_k(\mathbb{C}^n)$ to G. The quotient of $\mathrm{V}_k(\mathbb{C}^n)$ by this action is a set BG_n, which can be given the structure of a smooth manifold. Thus we get a new principal bundle $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to BG_n$ with structure group G.

The inclusions $\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \cdots$ give inclusions $BG_n \to BG_{n+1} \to \cdots$ and taking the direct limit we arrive at the classifying space BG. Furthermore, as before, we have a commutative diagram of inclusions and principal bundles with structure group G,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{V}_k(\mathbb{C}^{n-1}) & \xrightarrow{j_{n-1}} & \mathrm{V}_k(\mathbb{C}^n) & \xrightarrow{j_n} & \mathrm{V}_k(\mathbb{C}^{n+1}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\ \cdots & \longrightarrow & BG_{n-1} & \xrightarrow{i_{n-1}} & BG_n & \xrightarrow{i_n} & BG_{n+1} & \longrightarrow & \cdots . \end{array}$$

Keeping the notation introduced above, we have the following result.

Proposition 3.6. The principal bundles $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to BG_n$ admit a canonical connection ω_G such that $j^*\omega_G = \omega_G$.

Proof. Pick an Ad-invariant inner product on the Lie algebra $\mathfrak{u}(k)$ of $\mathrm{U}(k)$. Then we have a direct sum decomposition

$$\mathfrak{u}(k) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^\perp$ is its orthogonal complement concerning this inner product. We put

$$\omega_G = \mathrm{pr}_{\mathfrak{g}} \circ \omega_{\mathrm{c}},$$

where $\mathrm{pr}_{\mathfrak{g}}$ is the canonical projection onto the first summand of the above decomposition and ω_{c} is the canonical connection of Proposition 3.5. We claim that ω_G satisfies the required conditions. Indeed, since the inner product on $\mathfrak{u}(k)$ is Ad-invariant, we have that $\mathrm{pr}_{\mathfrak{g}} \circ \mathrm{Ad}_u = \mathrm{Ad}_u \circ \mathrm{pr}_{\mathfrak{g}}$ for all $u \in \mathrm{U}(k)$. Thus, if $u \in \mathrm{U}(k)$, then for each point A in $\mathrm{V}_k(\mathbb{C}^n)$ and for each tangent vector M of $\mathrm{V}_k(\mathbb{C}^n)$ at A,

$$\begin{aligned}[r_u^*\omega_G(A)](M) = [\omega_G(Au)](Mu) = \mathrm{pr}_{\mathfrak{g}}\{[\omega_{\mathrm{c}}(Au)](Mu)\} &= \mathrm{pr}_{\mathfrak{g}}\{\mathrm{Ad}_{u^{-1}}[\omega_{\mathrm{c}}(A)](M)\} \\ = \mathrm{Ad}_{u^{-1}}\mathrm{pr}_{\mathfrak{g}}\{[\omega_{\mathrm{c}}(A)](M)\} &= \mathrm{Ad}_{u^{-1}}[\omega_G(A)](M).\end{aligned}$$

On the other hand, if $x \in \mathfrak{g}$, then for each point A in $\mathrm{V}_k(\mathbb{C}^n)$,

$$[\omega_G(A)](x^\sharp(A)) = \mathrm{pr}_{\mathfrak{g}}\{[\omega_{\mathrm{c}}(A)](x^\sharp(A))\} = \mathrm{pr}_{\mathfrak{g}} x = x.$$

Finally, the condition $j^*\omega_G = \omega_G$ follows by a straightforward calculation. □

3.2.2 Smooth approximation for classifying spaces

We prove a smooth approximation result for the classifying space of a compact Lie group. We start by summarizing some of the necessary definitions.

Let G be any compact Lie group, which we may suppose to be embedded in $\mathrm{U}(k)$ for some k, and let BG denote its classifying space. As discussed in the previous section, BG is the direct limit of finite-dimensional manifolds BG_n concerning natural inclusions. We may thus consider the smooth fundamental ∞-groupoid $\pi_\infty BG_{n\bullet}$ of each BG_n, and define the smooth fundamental ∞-groupoid $\pi_\infty BG_\bullet$ of BG to be the direct limit of these simplicial sets overall n. On the other hand, if BG is regarded as a topological space with the direct limit topology, we may also consider the continuous fundamental ∞-groupoid of BG, which we denote by $\pi_\infty^{\mathrm{c}} BG_\bullet$. This is defined by setting $\pi_\infty^{\mathrm{c}} BG_p$ to be the set of continuous maps from the standard p-simplex Δ_p to BG. By the universal property of direct limits, there is a map of simplicial sets $j_\bullet \colon \pi_\infty BG_\bullet \to \pi_\infty^{\mathrm{c}} BG_\bullet$. The proof of the following result uses an argument we learned from Neil Strickland's answer to a question on MathOverflow (see [30]).

Proposition 3.7. The map $j_\bullet \colon \pi_\infty BG_\bullet \to \pi_\infty^{\mathrm{c}} BG_\bullet$ is a weak equivalence.

Proof. Let $|\pi_\infty BG_\bullet|$ and $|\pi_\infty^{\mathrm{c}} BG_\bullet|$ be the geometric realisations of the simplicial sets $\pi_\infty BG_\bullet$ and $\pi_\infty^{\mathrm{c}} BG_\bullet$, and let $|j_\bullet| \colon |\pi_\infty BG_\bullet| \to |\pi_\infty^{\mathrm{c}} BG_\bullet|$ be the continuous map induced by $j_\bullet$. We must show that $|j_\bullet|$ is a weak homotopy equivalence. To this end, we consider the evaluation maps $\mathrm{ev} \colon |\pi_\infty BG_\bullet| \to BG$ and $\mathrm{ev}^{\mathrm{c}} \colon |\pi_\infty^{\mathrm{c}} BG_\bullet| \to BG$. Then clearly there is a commutative diagram

$$\begin{array}{ccc} |\pi_\infty BG_\bullet| & \xrightarrow{\;|j_\bullet|\;} & |\pi_\infty^{\mathrm{c}} BG_\bullet| \\ & {\scriptstyle \mathrm{ev}}\searrow \quad \swarrow{\scriptstyle \mathrm{ev}^{\mathrm{c}}} & \\ & BG. & \end{array}$$

We know that ev^{c} is a weak homotopy equivalence, therefore, it will be enough to show that ev is a weak homotopy equivalence. For this let $\eta \colon BG \to |\pi_\infty BG_\bullet|$ be the natural map which sends $x \in BG$ to the equivalence class $[\sigma_x, \mathbf{0}]$, where σ_x is the zero simplex mapping to the point x. It is immediate to check that $\mathrm{ev} \circ \eta = \mathrm{id}_{BG}$. On the other hand, consider the map $F \colon |\pi_\infty BG_\bullet| \times [0,1] \to |\pi_\infty BG_\bullet|$ defined for $\sigma \in \pi_\infty BG_p$, $\mathbf{t} \in \Delta_p$ and $\lambda \in [0,1]$ by

$$F([\sigma, \mathbf{t}], \lambda) = [\sigma_{\mathbf{t},\lambda}, \mathbf{t}],$$

where $\sigma_{\mathbf{t},\lambda} \in \pi_\infty BG_p$ here is given by $\sigma_{\mathbf{t},\lambda}(\mathbf{s}) = \sigma((1-\lambda)\mathbf{s} + \lambda \mathbf{t})$. Then F is continuous and is clearly a homotopy between id_{BG} and $\eta \circ \mathrm{ev}$. This shows the desired assertion. □

We also note the following general result by Holstein [7, Corollary 20].

Lemma 3.5. *Let $f_\bullet \colon K_\bullet \to L_\bullet$ be a weak equivalence of simplicial sets. Then the pull-back functor $f_\bullet^* \colon \mathbf{Rep}_\infty(L_\bullet) \to \mathbf{Rep}_\infty(K_\bullet)$ is a quasi-equivalence.*

Combining this lemma with Proposition 3.7, we have the following.

Proposition 3.8. The pullback functor $j_\bullet^* \colon \mathbf{Rep}_\infty(\pi_\infty^{\mathrm{c}} BG_\bullet) \to \mathbf{Rep}_\infty(\pi_\infty BG_\bullet)$ is a quasi-equivalence.

We conclude that the DG category $\mathbf{Rep}_\infty(\pi_\infty BG_\bullet)$ can be thought of as the category of ∞-local systems on BG. We will therefore abuse notation and write $\mathbf{Loc}_\infty(BG)$ instead of $\mathbf{Rep}_\infty(\pi_\infty BG_\bullet)$.

3.2.3 Auxiliary lemmas

In this part, we prove several technical results that will be crucial in our arguments later on, but the reader only interested in the main results might omit on the first reading. We use freely the notation, terminology and definitions of §2.2.1, §2.2.4 and §3.1.

Let X be a smooth manifold and let (E,D) be an ∞-local system on X. We know already that D can be expanded as

$$D = \mathrm{d}_\nabla + \alpha_0 + \alpha_2 + \alpha_3 + \cdots ,$$

where $\alpha_k \in \Omega^k(X, \mathrm{End}(E)^{1-k})$. Also, recall that $\alpha_0^2 = 0$. We will write $\mathcal{H}^\bullet(E)$ for the cohomology of the graded vector bundle E with respect to α_0. We also remark here that for a pair of objects (E,D) and (E',D') in $\mathbf{Loc}_\infty(X)$ a quasi-isomorphism from (E,D) to (E',D') is precisely a closed element of $\Omega^\bullet(X,\mathrm{Hom}(E,E'))$ of degree 0 for which its homogeneous component of partial degree 0 is a quasi-isomorphism of cochain complexes. The following result is a direct translation of Theorem 4.13 in [26].

Lemma 3.6. *The complex $\mathcal{H}^\bullet(E)$ can be given the structure of a flat superconnection $\overline{D}$ such that there is an isomorphism of ∞-local systems from $(\mathcal{H}^\bullet(E),\overline{D})$ onto (E,D). Moreover, there is a spectral sequence with second page $\mathcal{E}_2^{p,q} = \mathrm{H}^p(X,\mathcal{H}^q(E))$ and which converges to $\mathrm{H}^{p+q}(X,E)$.*

We consider the full DG subcategory $\mathbf{Loc}^0_\infty(X)$ of $\mathbf{Loc}_\infty(X)$ consisting of those ∞-local systems (E,D) for which $\alpha_0 = 0$. The next result is an immediate consequence of the preceding lemma.

Lemma 3.7. *The natural inclusion DG functor $\mathbf{Loc}^0_\infty(X) \to \mathbf{Loc}_\infty(X)$ is a quasi-equivalence.*

Next, let us make an observation that will be helpful later.

Lemma 3.8. *Let (E,D) and (E',D') be two ∞-local systems on X. Then the space of morphism $\Omega^\bullet(X,\mathrm{Hom}(E,E'))$ in the DG category $\mathbf{Loc}_\infty(X)$ carries a canonical decreasing filtration by partial degree. This filtration induces a spectral sequence with second page $\mathcal{E}_2^{p,q} = \mathrm{H}^p(X,\mathcal{H}^q(\mathrm{Hom}(E,E')))$ and which converges to $\mathrm{H}^{p+q}(\Omega^\bullet(X,\mathrm{Hom}(E,E')))$.*

Proof. Explicitly, the filtration is

$$F^p\Omega^\bullet(X,\mathrm{Hom}(E,E')) = \bigoplus_{k\geq p} \Omega^k(X,\mathrm{Hom}(E,E')).$$

The zeroth page of the spectral sequence is $\mathcal{E}_0^{p,q} = \Omega^p(X,\mathrm{Hom}(E,E')^q)$, where the differential is precisely the one induced by the differential on $\mathrm{Hom}(E,E')$. Therefore the first page of the spectral sequence is $\mathcal{E}_1^{p,q} = \Omega^p(X,\mathcal{H}^q(\mathrm{Hom}(E,E')))$, with first page differential given by the flat connection induced on $\mathcal{H}^\bullet(\mathrm{Hom}(E,E'))$. Thus, the second page of the spectral sequence becomes $\mathcal{E}_2^{p,q} = \mathrm{H}^p(X,\mathcal{H}^q(\mathrm{Hom}(E,E')))$. □

We also wish to consider a canonical filtration of the DG category $\mathbf{Loc}_\infty(X)$. If $m \in \mathbb{N}$, we let $F^m\mathbf{Loc}_\infty(X)$ be the full DG subcategory of $\mathbf{Loc}_\infty(X)$ consisting of ∞-local systems (E,D) for which $E^k = 0$ for all $k \in \mathbb{Z}$ such that $|k| > m$. Then it is obvious that $F^m\mathbf{Loc}_\infty(X) \subset F^n\mathbf{Loc}_\infty(X)$ whenever

$m \leq n$, and any object of $\mathbf{Loc}_\infty(X)$ is contained in some $F^m\mathbf{Loc}_\infty(X)$. This means that $\mathbf{Loc}_\infty(X)$ is an $\mathbb{N}$-filtered DG category.

Now let Y be another smooth manifold and let $f\colon X \to Y$ be a smooth map. Consider the pullback DG functor $f^*\colon \mathbf{Loc}_\infty(Y) \to \mathbf{Loc}_\infty(X)$. Then it can easily be seen that, for any $m \in \mathbb{N}$, $f^*(F^m\mathbf{Loc}_\infty(Y))$ is a full DG subcategory of $F^m\mathbf{Loc}_\infty(X)$. Thus f^* is an $\mathbb{N}$-filtered DG functor.

For the following discussion, let us write $\underline{\mathbb{R}}$ for both the trivial vector bundle $X \times \mathbb{R}$ of rank one and the constant ∞-local system $(X \times \mathbb{R}, \mathrm{d})$. Also, let (E,D) be an ∞-local system on X and let ξ be a closed element of $\Omega^k(X, \mathrm{Hom}(E, \underline{\mathbb{R}})^{l-k})$ for some $k \geq 0$ and some $l \geq 0$. The extension of (E,D) by ξ, which we denote by $(E,D) \rtimes_\xi \underline{\mathbb{R}}$, is the ∞-local system on X whose underlying graded vector bundle is the direct sum $E \oplus \underline{\mathbb{R}}[1-l]$ and whose flat graded superconnection is $D_\xi = D + \xi$.

Lemma 3.9. *Let (E,D) and (E',D') be two ∞-local systems on X and let Φ be a quasi-isomorphism from (E,D) onto (E',D'). If ξ' is a closed element of $\Omega^k(X, \mathrm{Hom}(E', \underline{\mathbb{R}})^{l-k})$ for some $k \geq 1$ and some $l \geq 0$, then $\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}$ defines a quasi-isomorphism from $(E,D) \rtimes_{\xi' \circ \Phi} \underline{\mathbb{R}}$ onto $(E',D') \rtimes_{\xi'} \underline{\mathbb{R}}$.*

Proof. Since, by definition, Φ is an element of $\Omega^\bullet(X, \mathrm{Hom}(E,E'))$ of degree 0, so is $\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}$ viewed as an element of $\Omega^\bullet(X, \mathrm{Hom}(E \oplus \underline{\mathbb{R}}[1-l], E' \oplus \underline{\mathbb{R}}[1-l]))$. We must show that $\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}$ commutes with the graded superconnections $D_{\xi' \circ \Phi}$ and $D'_{\xi'}$. For this we simply compute:

$$\begin{aligned} D'_{\xi'} \circ (\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}) &= (D' + \xi') \circ (\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}) \\ &= D' \circ \Phi + \xi' \circ \Phi \\ &= \Phi \circ D + \xi' \circ \Phi \\ &= (\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}) \circ (D + \xi' \circ \Phi) \\ &= (\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}) \circ D_{\xi' \circ \Phi}. \end{aligned}$$

Finally, $\Phi + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}$ is a quasi-isomorphism because its homogeneous component of partial degree 0 is $\Phi_0 + \mathrm{id}_{\underline{\mathbb{R}}[1-l]}$. □

Lemma 3.10. *Let (E,D) be an ∞-local system on X. If ξ is a closed element of $\Omega^k(X, \mathrm{Hom}(E, \underline{\mathbb{R}})^{l-k})$ and η is an element of $\Omega^k(X, \mathrm{Hom}(E, \underline{\mathbb{R}})^{l-1-k})$ for some $k \geq 0$ and some $l \geq 0$, then $\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta$ defines a quasi-isomorphism from $(E,D) \rtimes_\xi \underline{\mathbb{R}}$ onto $(E,D) \rtimes_{\xi + D\eta} \underline{\mathbb{R}}$.*

Proof. It is clear that $\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta$ viewed as an element of $\Omega^\bullet(X, \mathrm{Hom}(E \oplus \underline{\mathbb{R}}[1-l], E' \oplus \underline{\mathbb{R}}[1-l]))$ has degree 0. We need to check that $\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta$ commutes with the graded superconnections D_ξ and $D_{\xi + D\eta}$. Again, we simply compute:

$$\begin{aligned} D_{\xi + D\eta} \circ (\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta) &= (D + \xi + D\eta) \circ (\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta) \\ &= D + \xi + D\eta - D \circ \eta \\ &= D + \xi - \eta \circ D \\ &= (\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta) \circ (D + \xi) \\ &= (\mathrm{id}_{\underline{\mathbb{R}}[1-l]} - \eta) \circ D_\xi. \end{aligned}$$

Finally, notice that $\mathrm{id}_{\mathbb{R}[1-l]} - \eta$ is invertible with inverse $\mathrm{id}_{\mathbb{R}[1-l]} + \eta$. □

Let $K_\bullet$ be a simplicial set and let $(E, F_\bullet)$ be a representation up to homotopy of $K_\bullet$. In keeping with the notation used above, we will write $\mathcal{H}^\bullet(E)$ to represent the cohomology of the graded vector bundle E with respect to F_0. We will also say that $(E, F_\bullet)$ is *normalized* if each cochain F_p vanishes on degenerate simplices. The following result is analogous to Lemma 3.8.

Lemma 3.11. *Let $(E, F_\bullet)$ and $(E', F'_\bullet)$ be two normalized representations up to homotopy of $K_\bullet$. Then the space of morphism $\underline{\mathrm{Hom}}^\bullet((E, F_\bullet), (E', F'_\bullet))$ in the DG category $\mathbf{Rep}_\infty(K_\bullet)$ carries a decreasing filtration by cochain degree. This filtration induces a spectral sequence with second page $\mathcal{E}_2^{p,q} = \mathrm{H}^p(K_\bullet, \mathcal{H}^q(\mathrm{Hom}(E, E')))$ and which converges to $\mathrm{H}^{p+q}(\underline{\mathrm{Hom}}^\bullet((E, F_\bullet), (E', F'_\bullet)))$.*

Proof. The zeroth page of the spectral sequence $\mathcal{E}_0^{p,q}$ is given by the space of singular p-cochains on $K_\bullet$ with values in $\mathrm{Hom}(E, E')^q$ with the differential induced by that of $\mathrm{Hom}(E, E')$. Thus the first page of the spectral sequence $\mathcal{E}_1^{p,q}$ is the space p-cochains on $K_\bullet$ with values in $\mathcal{H}^q(\mathrm{Hom}(E, E'))$, with the first page differential given by the induced one from $\mathcal{H}^\bullet(\mathrm{Hom}(E, E'))$. Hence, the second page of the spectral sequence is indeed $\mathcal{E}_2^{p,q} = \mathrm{H}^p(K_\bullet, \mathcal{H}^q(\mathrm{Hom}(E, E')))$. □

Next we wish to consider a canonical filtration of $\mathbf{Rep}_\infty(K_\bullet)$, just as we did for the DG category of ∞-local systems. Thus, if $m \in \mathbb{N}$, we set $F^m\mathbf{Rep}_\infty(K_\bullet)$ to be the full DG subcategory of $\mathbf{Rep}_\infty(K_\bullet)$ consisting of all representations up to homotopy $(E, F_\bullet)$ for which $E^k = 0$ for all $k \in \mathbb{Z}$ such that $|k| > m$. This turns $\mathbf{Rep}_\infty(K_\bullet)$ into an $\mathbb{N}$-filtered DG category. Moreover, if $L_\bullet$ is another simplicial set and $f_\bullet \colon K_\bullet \to L_\bullet$ is a simplicial map, then, for any $m \in \mathbb{N}$, the pullback DG functor $f_\bullet^* \colon \mathbf{Rep}_\infty(L_\bullet) \to \mathbf{Rep}_\infty(K_\bullet)$ sends $F^m\mathbf{Rep}_\infty(L_\bullet)$ to a full DG subcategory of $F^m\mathbf{Rep}_\infty(K_\bullet)$. In other words, $f_\bullet^*$ is an $\mathbb{N}$-filtered DG functor.

Before moving on, we introduce some terminology and notation regarding simplicial sets. Denoting the usual category of finite ordinals by $\mathbf{\Delta}$, we obtain for each $n \geq 0$, a subcategory $\mathbf{\Delta}_{\leq n}$ determined by the objects $[k]$ of $\mathbf{\Delta}$ with $k \leq n$. A n-truncated simplicial set is a contravariant functor from $\mathbf{\Delta}_{\leq n}$ to the category of sets $\mathbf{Set}$. We denote the category of simplicial sets by $\mathbf{sSet}$ and the category of n-truncated simplicial sets by $\mathbf{sSet}_{\leq n}$. Restriction gives a truncation functor $\mathrm{tr}_n \colon \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$ which admits a right adjoint $\mathrm{cosk}_n \colon \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$ called the n-coskeleton functor. By abuse of language, one refers to the composite functor $\mathrm{Cosk}_n = \mathrm{cosk}_n \circ \mathrm{tr}_n$ also as the n-coskeleton functor. It can be shown that, given an object $K_\bullet$ in $\mathbf{sSet}$, the unit of the adjunction $\eta_K \colon K_\bullet \to \mathrm{Cosk}_n(K_\bullet)$ induces a bijection on simplices of dimension $k \leq n$. Furthermore, if $K_\bullet$ and $L_\bullet$ are two objects in $\mathbf{sSet}$ which in addition are Kan complexes, and $f_\bullet \colon K_\bullet \to L_\bullet$ is a morphism that induces an isomorphism in homotopy groups up to degree $k < n$, then the morphism $\mathrm{Cosk}_n(f_\bullet) \colon \mathrm{Cosk}_n(K_\bullet) \to \mathrm{Cosk}_n(L_\bullet)$ is a weak equivalence. For details of these constructions, we refer the reader to [31–33].

Lemma 3.12. *Let $K_\bullet$ and $L_\bullet$ be simplicial sets and let $f_\bullet \colon K_\bullet \to L_\bullet$ be a simplicial map that induces an isomorphism in homotopy groups up to degree $2m$. Then the pullback functor $f_\bullet^* \colon \mathbf{Rep}_\infty(L_\bullet) \to \mathbf{Rep}_\infty(K_\bullet)$ induces a quasi-equivalence between $F^m\mathbf{Rep}_\infty(L_\bullet)$ and $F^m\mathbf{Rep}_\infty(K_\bullet)$.*

Proof. The proof makes use of Kan's fibrant replacement functor $\mathrm{Ex}^\infty \colon \mathbf{sSet} \to \mathbf{sSet}$ (see for instance §3 of [34]). There are natural maps $\nu_K \colon K_\bullet \to \mathrm{Ex}^\infty(K_\bullet)$ and $\nu_L \colon L_\bullet \to \mathrm{Ex}^\infty(L_\bullet)$ which are weak equivalences. Moreover, as the name suggests, $\mathrm{Ex}^\infty(K_\bullet)$ and $\mathrm{Ex}^\infty(L_\bullet)$ are Kan complexes. Consider the following commutative diagram

$$\begin{array}{ccc} K_\bullet & \xrightarrow{f_\bullet} & L_\bullet \\ \downarrow{\scriptstyle \nu_K} & & \downarrow{\scriptstyle \nu_L} \\ \mathrm{Ex}^\infty(K_\bullet) & & \mathrm{Ex}^\infty(L_\bullet) \\ \downarrow{\scriptstyle \eta_{\mathrm{Ex}^\infty(K)}} & & \downarrow{\scriptstyle \eta_{\mathrm{Ex}^\infty(L)}} \\ \mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(K_\bullet)) & \xrightarrow{\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(f_\bullet))} & \mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(L_\bullet)). \end{array}$$

In order to show that $f_\bullet^*$ induces an equivalence on homotopy categories, it suffices to show that the pullback functor associated to all the other maps in the diagram have the corresponding property. Using Lemma 3.5, we observe that, being weak equivalences, the pullback functors associated to ν_K and ν_L induce quasi-equivalences between $\mathbf{Rep}_\infty(\mathrm{Ex}^\infty(K_\bullet))$ and $\mathbf{Rep}_\infty(K_\bullet)$, and between $\mathbf{Rep}_\infty(\mathrm{Ex}^\infty(L_\bullet))$ and $\mathbf{Rep}_\infty(L_\bullet)$, respectively. On the other hand, in view of our previous remarks, we know that the maps $\eta_{\mathrm{Ex}^\infty(K)}$ and $\eta_{\mathrm{Ex}^\infty(L)}$ are bijections on simplices of dimension $k \leq 2m+1$ and therefore their associated pullback functors induce equivalences between the homotopy categories of $F^m\mathbf{Rep}_\infty(\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(K_\bullet)))$ and $F^m\mathbf{Rep}_\infty(\mathrm{Ex}^\infty(K_\bullet))$, and those of $F^m\mathbf{Rep}_\infty(\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(L_\bullet)))$ and $F^m\mathbf{Rep}_\infty(\mathrm{Ex}^\infty(L_\bullet))$, respectively. Besides, by our hypothesis on $f_\bullet$, the map $\mathrm{Ex}^\infty(f_\bullet)$ induces an isomorphism in homotopy groups up to degree $2m$. One concludes that $\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(f_\bullet))$ is a weak equivalence, using again Lemma 3.5, one concludes that its associated pullback functor induces an equivalence between the homotopy categories of $F^m\mathbf{Rep}_\infty(\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(K_\bullet)))$ and $F^m\mathbf{Rep}_\infty(\mathrm{Cosk}_{2m+1}(\mathrm{Ex}^\infty(L_\bullet)))$. This completes the proof of the lemma. □

Now let G be a compact and simply connected Lie group with Lie algebra $\mathfrak{g}$. If V is a $\mathfrak{g}$-L_∞ space, we shall write $\mathcal{H}^\bullet(V)$ for the cohomology of the graded vector space V concerning the zeroth component of its corresponding Maurer-Cartan element α_V. A result analogous to Lemmas 3.8 and 3.11 is as follows.

Lemma 3.13. *Let V and V' be two objects in* $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$. *Then the space of morphisms* $(\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}}$ *carries a canonical decreasing filtration by the degree in the Weil algebra* $\mathrm{W}\mathfrak{g}$. *This filtration induces a spectral sequence with second page* $\mathcal{E}_2^{p,q} = \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V'))$ *and which converges to* $\mathrm{H}^{p+q}((\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}})$.

Proof. The zeroth page of the spectral sequence is

$$\mathcal{E}_0^{p,q} = (\mathrm{W}^p\mathfrak{g} \otimes \mathrm{Hom}(V,V')^q)_{\mathrm{bas}} = (\mathrm{S}^{p/2}\mathfrak{g}^* \otimes \mathrm{Hom}(V,V')^q)_{\mathrm{inv}},$$

where the differential is induced by $\mathrm{Hom}(V,V')$, since G is compact, we know that taking cohomology commutes with taking invariants. Moreover, since V and V' are basic $\mathfrak{g}$-L_∞-spaces, the action of $\mathfrak{g}$ on $\mathcal{H}^\bullet(\mathrm{Hom}(V,V'))$ is trivial. Therefore,

$$\mathcal{E}_1^{p,q} = (\mathrm{S}^{p/2}\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{H}^q(\mathrm{Hom}(V,V')) = \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V')).$$

As for the first page differential, it vanishes for degree reasons. We therefore conclude that the second page of the spectral sequence is $\mathcal{E}_2^{p,q} = \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V'))$, as asserted. □

It should also be apparent that, just as for the DG categories of ∞-local systems and representations up to homotopy, there is a canonical filtration $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ consisting of all $\mathfrak{g}$-L_∞ spaces V for which $V^k = 0$ for all $k \in \mathbb{Z}$ such that $|k| > m$. This makes $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ into a filtered DG category.

There is one more result that will be useful as we go forward.

Lemma 3.14. *Let G be a compact connected Lie group, $\pi\colon P \to X$ be a principal G-bundle and θ a connection on P. Then, for any pair of objects V and V' of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$, the morphism of cochain complexes*

$$\mathscr{C}\mathscr{W}_\theta\colon (\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}} \longrightarrow \Omega^\bullet(X, \mathrm{Hom}(\mathscr{C}\mathscr{W}_\theta(V), \mathscr{C}\mathscr{W}_\theta(V'))),$$

determined by the Chern-Weil DG functor of P, is compatible with the filtrations described in Lemmas 3.11 and 3.13, and induces a morphism of spectral sequences. Moreover, if X is simply connected, this morphism is identified with the Chern-Weil homomorphism of P tensored with the identity on $\mathcal{H}^\bullet(\mathrm{Hom}(V,V'))$.

Proof. The first part follows immediately from the definition of the Chern-Weil DG functor of P. To prove the second part, we note that, since G acts trivially on cohomology, the graded vector bundle $\mathcal{H}^\bullet(\mathrm{Hom}(\mathscr{C}\mathscr{W}_\theta(V), \mathscr{C}\mathscr{W}_\theta(V')))$ is trivial, so if we look at the first pages of the associated spectral sequences, we get the morphism

$$\mathscr{C}\mathscr{W}_\theta\colon \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V')) \longrightarrow \Omega^p(X, \mathcal{H}^q(\mathrm{Hom}(V,V'))),$$

and this is simply the tensor product of the characteristic homomorphism of P with the identity on $\mathcal{H}^q(\mathrm{Hom}(V,V'))$. Furthermore, by the hypothesis that X is simply connected, we have that the graded vector bundle $\mathcal{H}^\bullet(\mathrm{Hom}(\mathscr{C}\mathscr{W}_\theta(V), \mathscr{C}\mathscr{W}_\theta(V')))$ is trivialized by the flat connection induced on it so that the induced morphism between the second pages of the spectral sequences is

$$\mathscr{C}\mathscr{W}_\theta\colon \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V')) \longrightarrow \mathrm{H}^p_{\mathrm{DR}}(X) \otimes \mathcal{H}^q(\mathrm{Hom}(V,V')),$$

and it is given by the tensor product of the Chern-Weil homomorphism of P with the identity on $\mathcal{H}^q(\mathrm{Hom}(V,V'))$, as claimed. □

We close by mentioning that a construction analogous to that of an extension of an ∞-local system can be carried out for basic $\mathfrak{g}$-L_∞ spaces. Let us write $\mathbb{R}$ for the trivial representation of $\mathbb{T}\mathfrak{g}$. We further let V be a basic $\mathfrak{g}$-L_∞ space and γ be a closed element of $(\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,\mathbb{R}))_{\mathrm{bas}}$ of degree l. Then the extension of V by γ, denoted by $V \rtimes_\gamma \mathbb{R}$, is the basic $\mathfrak{g}$-L_∞ space whose underlying graded vector space is the direct sum $V \oplus \mathbb{R}[1-l]$ and whose corresponding Maurer-Cartan element is $\alpha_V + \gamma$.

3.2.4 The Weil A_∞-functor for classifying spaces

This subsection proves an infinitesimal description of the ∞-local systems category on the classifying space BG of a compact connected Lie group G with Lie algebra $\mathfrak{g}$. First, let us recall our situation.

We consider a compact and simply connected Lie group G with Lie algebra $\mathfrak{g}$. For nonnegative integers $k \leq n$, we let $\pi\colon \mathrm{V}_k(\mathbb{C}^n) \to BG_n$ be the principal bundle with structure group G described in §3.2.1. The canonical connection on $\mathrm{V}_k(\mathbb{C}^n)$ constructed in Proposition 3.6 we denote simply by ω in this discussion. We recall that it satisfies $j^*\omega = \omega$, where $j\colon \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{V}_k(\mathbb{C}^{n+1})$ is the natural inclusion. We also write $\varphi_n\colon BG_n \to BG$ for the canonical map obtained from the direct limit construction.

We will now proceed to construct the A_∞-functor that connects the DG categories $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ and $\mathbf{Loc}_\infty(BG)$. By Theorem A, we know that for each $n \geq k$, there is a Chern-Weil DG functor $\mathscr{CW}^{(n)}_\omega\colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(BG_n)$. Because of the naturallity of the connection ω, these DG functors fit into a commutative diagram

$$\begin{array}{ccc} \mathbf{Loc}_\infty(BG_n) & \xleftarrow{\quad i^* \quad} & \mathbf{Loc}_\infty(BG_{n+1}) \\ & {\scriptstyle \mathscr{CW}^{(n)}_\omega} \nwarrow \quad \nearrow {\scriptstyle \mathscr{CW}^{(n+1)}_\omega} & \\ & \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) & \end{array}$$

where $i\colon BG_n \to BG_{n+1}$ is the natural inclusion. On the other hand, by the higher Riemann-Hilbert correspondence, for each n, there is an integration A_∞-functor $\mathscr{I}^{(n)}\colon \mathbf{Loc}_\infty(BG_n) \to \mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet})$, which is in addition an A_∞-quasi-equivalence. By the naturality of the construction, all these A_∞-functors fit into the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet}) & \xleftarrow{\; i^*_\bullet \;} & \mathbf{Rep}_\infty(\pi_\infty BG_{n+1\bullet}). \\ \uparrow {\scriptstyle \mathscr{I}^{(n)}} & & \uparrow {\scriptstyle \mathscr{I}^{(n+1)}} \\ \mathbf{Loc}_\infty(BG_n) & \xleftarrow{\; i^* \;} & \mathbf{Loc}_\infty(BG_{n+1}) \end{array}$$

For each $n \geq 0$, we let $\mathscr{W}^{(n)}\colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to \mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet})$ be the A_∞-functor defined as the composition

$$\mathscr{W}^{(n)} = \mathscr{I}^{(n)} \circ \mathscr{CW}^{(n)}_\omega.$$

It follows from the above discussion that these A_∞-functors fit into a commutative diagram

$$\begin{array}{ccc} \mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet}) & \xleftarrow{\quad i^*_\bullet \quad} & \mathbf{Rep}_\infty(\pi_\infty BG_{n+1\bullet}) \\ & {\scriptstyle \mathscr{W}^{(n)}} \nwarrow \quad \nearrow {\scriptstyle \mathscr{W}^{(n+1)}} & \\ & \mathbf{Inf\,Loc}_\infty(\mathfrak{g}). & \end{array}$$

Taking note of the definition of the DG category $\mathbf{Loc}_\infty(BG) := \mathbf{Rep}_\infty(\pi_\infty BG)$ given at the end of §3.2.2, and recalling that, for each $n \geq 0$, integrating an ∞-local system on BG_n amounts to assigning holonomies to smooth maps from the standard simplices to BG_n, we deduce the existence of an A_∞-functor

$$\mathscr{W}\colon \mathbf{InfLoc}_\infty(\mathfrak{g}) \longrightarrow \mathbf{Loc}_\infty(BG).$$

Our goal is to show that $\mathscr{W}$ is, in fact, an A_∞-quasi-equivalence. The following two preliminary results will clear our path.

Proposition 3.9. Let V and V' be two objects in $F^m\mathbf{InfLoc}_\infty(\mathfrak{g})$ and put $N = \frac{1}{2}(2k+l+6m)$. Then, for each $n \geq N$, the morphism of cochain complexes

$$\mathscr{C}\mathscr{W}^{(n)}_\omega\colon (\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}} \longrightarrow \Omega^\bullet(BG_n, \mathrm{Hom}(\mathscr{C}\mathscr{W}^{(n)}_\omega(V), \mathscr{C}\mathscr{W}^{(n)}_\omega(V')))$$

induces an isomorphism in cohomology up to degree l.

Proof. To start with, since $n \geq N$, it follows that $\mathrm{V}_k(\mathbb{C}^n)$ is $2(N-k)$-connected, see for instance [35] page 382. As a consequence, employing the long exact homotopy sequence, we deduce that the canonical map $\varphi_n\colon BG_n \to BG$ induces an isomorphism in homotopy groups up to degree $2(N-k)$. This in turn implies that the characteristic map $c_\omega\colon (\mathrm{W}\mathfrak{g})_{\mathrm{bas}} \to \Omega^\bullet(BG_n)$ induces an isomorphism in cohomology up to degree $2(N-k)$. On the other hand, by virtue of Lemma 3.14, the morphism of cochain complexes $\mathscr{C}\mathscr{W}^{(n)}_\omega$ is compatible with the canonical decreasing filtrations on $(\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V,V'))_{\mathrm{bas}}$ and $\Omega^\bullet(BG_n, \mathrm{Hom}(\mathscr{C}\mathscr{W}^{(n)}_\omega(V), \mathscr{C}\mathscr{W}^{(n)}_\omega(V')))$, and induces a morphism between the corresponding spectral sequences, whose rth pages we write respectively as $\mathcal{E}_r^{p,q}$ and $\mathcal{E}_r'^{p,q}$. Moreover, since BG_n is simply connected, this morphism is identified with the Chern-Weil homomorphism of $\mathrm{V}_k(\mathbb{C}^n)$ tensored with the identity of $\mathcal{H}^\bullet(\mathrm{Hom}(V,V'))$. These facts, together with Lemmas 3.8 and 3.13, imply that the induced morphism between the second pages of the spectral sequences

$$\mathscr{C}\mathscr{W}^{(n)}_\omega\colon \mathcal{E}_2^{p,q} \longrightarrow \mathcal{E}_2'^{p,q}$$

is an isomorphism for all $p \leq 2(N-k)$. But, by our hypothesis on V and V', it results that $\mathcal{H}^q(\mathrm{Hom}(V,V')) = 0$ if $q+2m < 0$. Thus, the above induced morphism is an isomorphism if $p+q \leq 2(N-k)-2m$. From this, by induction on r, we infer that the induced morphism between the rth pages of the spectral sequences

$$\mathscr{C}\mathscr{W}^{(n)}_\omega\colon \mathcal{E}_r^{p,q} \longrightarrow \mathcal{E}_r'^{p,q}$$

is an isomorphism if $p+q \leq 2(N-k)-2m-r+2$. In particular, if we take $r = 4m+2$, then the induced morphism

$$\mathscr{C}\mathscr{W}^{(n)}_\omega\colon \mathcal{E}_{4m+2}^{p,q} \longrightarrow \mathcal{E}_{4m+2}'^{p,q}$$

is an isomorphism if $p+q \leq 2(N-k)-6m = l$. At the same time, under this assumption on $p+q$, the rth page differentials vanish for all $r \geq 4m+2$. We therefore conclude that the induced morphism between the limiting spectral sequences

$$\mathscr{C}\mathscr{W}^{(n)}_\omega\colon \mathcal{E}_\infty^{p,q} \longrightarrow \mathcal{E}_\infty'^{p,q}$$

is an isomorphism if $p+q \leq l$. Appealing again to Lemmas 3.8 and 3.13, we get the desired conclusion. ☐

Proposition 3.10. Given $m \geq 0$, there exists a sufficiently large integer $n > 0$ such that the corresponding Chern-Weil DG funtor $\mathscr{CW}^{(n)}_{\omega}\colon \mathbf{Inf\,Loc}_\infty(\mathfrak{g}) \to \mathbf{Loc}_\infty(BG_n)$ induces a quasi essentially surjective DG functor between $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ and $F^m\mathbf{Loc}_\infty(BG_n)$.

Proof. Pick n sufficiently large so that $\mathrm{V}_k(\mathbb{C}^n)$ is $2m$-connected and BG_n is simply connected. We must show that for any object (E,D) of $F^m\mathbf{Loc}_\infty(BG_n)$, an object V of $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ can be found such that (E,D) is quasi-isomorphic to $\mathscr{CW}^{(n)}_{\omega}(V)$. Thus, let us fix such an object (E,D). Notice that, by Lemma 3.7, we may assume that (E,D) is in fact an object of $F^m\mathbf{Loc}^0_\infty(BG_n)$. Moreover, since BG_n is simply connected, we may further assume that (E,D) is trivialized over BG_n, that is to say, $E = BG_n \times W$ for some graded vector space $W = \bigoplus_{i=p}^{q} W^i$ with $-m \leq p \leq q \leq m$, and $D = d + \alpha$ for some Maurer-Cartan element $\alpha \in \Omega^\bullet(BG_n, \mathrm{End}(W))$. We will therefore argue by induction on the dimension of W. If this dimension is equal to 0, the result clearly holds. The result also holds if $p = q$, since in this case (E,D) is isomorphic to the constant ∞-local system $(BG_n \times \mathbb{R}, d)$. Suppose, then, that $p < q$ and that the result is true for all objects of $F^m\mathbf{Loc}^0_\infty(BG_n)$ arising from a graded vector space of dimension less than the dimension of W. Pick a nonzero vector in W^p and consider the one-dimensional subspace U spanned by this vector. We also fix an inner product on W^p and write $U^\perp$ for the orthogonal complement of U in W^p. Set $W' = U^\perp \oplus \left(\bigoplus_{i=p+1}^{q} W^i\right)$ so that $W = U \oplus W'$. Then the Maurer-Cartan element α can be decomposed as $\alpha = \alpha' + \xi$, where $\alpha' \in \Omega^\bullet(BG_n, \mathrm{End}(W'))$ and $\xi \in \Omega^\bullet(BG_n, \mathrm{Hom}(W', U))$ are homogeneous elements of total degree 1. In terms of this decomposition, the Maurer-Cartan equation for α becomes

$$0 = d\alpha + \alpha \wedge \alpha = d\alpha' + d\xi + \alpha' \wedge \alpha' + \xi \wedge \alpha',$$

which can be decoupled into two independent equations

$$\begin{aligned} d\alpha' + \alpha' \wedge \alpha' &= 0, \\ d\xi + \xi \wedge \alpha' &= 0. \end{aligned}$$

Hence, if we put $E' = BG_n \times W'$ and $D' = d + \alpha'$, the first of these equations implies that (E', D') defines an object of $F^m\mathbf{Loc}_\infty(BG_n)$. And since the dimension of W' is less than the dimension of W, our induction hypothesis ensures the existence of an object V' of $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ together with a quasi-isomorphism Φ from $\mathscr{CW}^{(n)}_{\theta}(V')$ onto (E', D'). In addition to this, it is clear from the construction above that there is an isomorphism of ∞-local systems from (E,D) onto the central extension $(E', D') \rtimes_\xi \mathbb{R}$. Therefore, by applying Lemma 3.10, we determine the existence of an isomorphism of ∞-local systems from (E,D) onto $\mathscr{CW}^{(n)}_{\omega}(V') \rtimes_{\xi\circ\Phi} \mathbb{R}$. On the other hand, because V' is an object of $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$, by virtue of Proposition 3.9, we know that, if $n \geq \frac{1}{2}(2k+1-p+6m)$, the morphism of cochain complexes

$$\mathscr{CW}^{(n)}_{\omega}\colon (\mathrm{W}\mathfrak{g} \otimes \mathrm{Hom}(V', \mathbb{R}))_{\mathrm{bas}} \longrightarrow \Omega^\bullet(BG_n, \mathrm{Hom}(\mathscr{CW}^{(n)}_{\omega}(V'), \mathbb{R}))$$

induces an isomorphism in cohomology up to degree $1-p$. This means, in particular, that we can find a closed element γ of $(\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V',\mathbb{R}))_{\mathrm{bas}}$ of degree 1 such that

$$\mathscr{CW}_\omega^{(n)}(\gamma)=\xi\circ\Phi+D\eta,$$

for some homogeneous element η of $\Omega^\bullet(BG_n,\mathrm{Hom}(\mathscr{CW}_\omega^{(n)}(V'),\underline{\mathbb{R}}))$ of degree 0. Therefore, combining the foregoing with Lemma 3.9, we obtain an isomorphism of ∞-local systems from (E,D) onto $\mathscr{CW}_\omega^{(n)}(V')\rtimes_{\mathscr{CW}_\omega^{(n)}(\gamma)}\underline{\mathbb{R}}$. But it is not hard to check that $\mathscr{CW}_\omega^{(n)}(V')\rtimes_{\mathscr{CW}_\omega^{(n)}(\gamma)}\underline{\mathbb{R}}$, as an object of $F^m\mathbf{Loc}_\infty(BG_n)$, is isomorphic to $\mathscr{CW}_\omega^{(n)}(V'\rtimes_\gamma\mathbb{R})$, where $V'\rtimes_\gamma\mathbb{R}$ is the extension of V' by γ. Hence the desired conclusion follows by taking $V=V'\rtimes_\gamma\mathbb{R}$. □

Now we can state and prove the general theorem we have been looking for.

Theorem B. *Given a compact connected Lie group G, the A_∞-functor*

$$\mathscr{W}:\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g})\longrightarrow\mathbf{Loc}_\infty(BG)$$

is a A_∞-quasi-equivalence. Moreover, for any principal G-bundle $\pi\colon P\to X$ with connection θ and classifying map $f\colon X\to BG_n$, there there exists an A_∞-natural isomorphism between the A_∞-functors $\mathscr{I}\circ\mathscr{CW}_\theta$ and $(\varphi_n\circ f)^*_\bullet\circ\mathscr{W}$ from $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ to $\mathbf{Rep}_\infty(\pi_\infty X_\bullet)$. Here $\mathscr{I}$ is the integration A_∞-functor provided by the higher Riemann-Hilbert correspondence, and φ_n is the canonical map from BG_n to BG.

Proof. We first prove that the functor A_∞-functor $\mathscr{W}:\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g})\to\mathbf{Loc}_\infty(BG)$ is A_∞-quasi fully faithful. Let V and V' two objects of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$. By definition, we have to show that

$$\mathscr{W}:\ (\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V,V'))_{\mathrm{bas}}\longrightarrow\underline{\mathrm{Hom}}^\bullet(\mathscr{W}(V),\mathscr{W}(V')),$$

is a quasi-isomorphism. Notice that this morphism is compatible with the canonical filtrations on $(\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V,V'))_{\mathrm{bas}}$ and $\underline{\mathrm{Hom}}^\bullet(\mathscr{W}(V),\mathscr{W}(V'))$, and induces a morphism between their associated spectral sequences. It, therefore, suffices to verify that the induced morphism between the second pages of the spectral sequences is an isomorphism. On the one hand, recall from Lemma 3.13 that the second page of the spectral sequence associated with the canonical filtration on $(\mathrm{W}\mathfrak{g}\otimes\mathrm{Hom}(V,V'))_{\mathrm{bas}}$ is

$$\mathcal{E}_2^{p,q}=\mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}})\otimes\mathcal{H}^q(\mathrm{Hom}(V,V')).$$

On the other hand, Lemma 3.11 says that the second page of the spectral sequence associated with the canonical filtration on $\underline{\mathrm{Hom}}^\bullet(\mathscr{W}(V),\mathscr{W}(V'))$ is

$$\mathcal{E}_2'^{p,q}=\mathrm{H}^p(\pi_\infty BG_\bullet,\mathcal{H}^q(\mathscr{W}(V),\mathscr{W}(V'))).$$

In addition, since BG is simply connected, the graded vector bundle $\mathcal{H}^\bullet(\mathscr{W}(V),\mathscr{W}(V'))$ is trivial with fiber $\mathcal{H}^\bullet(\mathrm{Hom}(V,V'))$, from which it follows that

$$\mathcal{E}_2'^{p,q}\cong\mathrm{H}^p(\pi_\infty BG_\bullet)\otimes\mathcal{H}^q(\mathrm{Hom}(V,V'))=\mathrm{H}^p(BG)\otimes\mathcal{H}^q(\mathrm{Hom}(V,V')).$$

Consequently, the induced morphism between the second pages of the spectral sequence is

$$\mathscr{W}\colon\ \mathrm{H}^p((\mathrm{W}\mathfrak{g})_{\mathrm{bas}})\otimes\mathcal{H}^q(\mathrm{Hom}(V,V'))\longrightarrow \mathrm{H}^p(BG)\otimes\mathcal{H}^q(\mathrm{Hom}(V,V')),$$

and it is given by the tensor product of the universal Weil homomorphism for BG with the identity on $\mathcal{H}^q(\mathrm{Hom}(V,V'))$. Since the former is an isomorphism, the desired assertion follows.

We next show that the A_∞-functor $\mathscr{W}\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g})\to\mathbf{Loc}_\infty(BG)$ is A_∞-quasi essentially surjective. Let us fix an object $(E,F_\bullet)$ of $\mathbf{Loc}_\infty(BG)$. Then there exists a sufficiently large integer $m>0$ such that $(E,F_\bullet)$ belongs to $F^m\mathbf{Loc}_\infty(BG)$. Since $\mathrm{V}_k(\mathbb{C}^n)$ is $2(n-k)$-connected, from the long exact homotopy sequence, we infer that, if $2m<2(n-k)$, the canonical map $\varphi_n\colon BG_n\to BG$ induces an isomorphism in homotopy group up to degree $2m$. Invoking Lemma 3.12, we conclude that the pullback functor $\varphi_{n\bullet}^*\colon\ \mathbf{Loc}_\infty(BG)\to\mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet})$ induces an equivalence between the homotopy categories of $F^m\mathbf{Loc}_\infty(BG)$ and $F^m\mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet})$. On the other hand, by construction, we have the following commutative diagram of A_∞-functors

$$\begin{array}{ccc}
F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g}) & \xrightarrow{\mathscr{W}} & F^m\mathbf{Loc}_\infty(BG)\\
\downarrow{\scriptstyle \mathscr{C}\mathscr{W}_\omega^{(n)}} & & \downarrow{\scriptstyle \varphi_{n\bullet}^*}\\
F^m\mathbf{Loc}_\infty(BG_n) & \xrightarrow{\mathscr{I}^{(n)}} & F^m\mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet})
\end{array}$$

where $\varphi_{n\bullet}^*$ is the pullback DG functor along the induced simplicial map $\varphi_{n\bullet}\colon\ \pi_\infty BG_{n\bullet}\to\pi_\infty BG_\bullet$. Thus, since $\mathscr{I}^{(n)}$ is an A_∞-quasi-equivalence, it will be enough to show that $\mathscr{C}\mathscr{W}_\omega^{(n)}$ is a quasi-essentially surjective. But, if n is sufficiently large, this is true by Proposition 3.10.

Finally, we prove the second assertion. To this end, we first notice that by the naturality of the integration A_∞-functors $\mathscr{I}^{(n)}$ and $\mathscr{I}$, the following diagram commutes

$$\begin{array}{ccc}
\mathbf{Loc}_\infty(BG_n) & \xrightarrow{f^*} & \mathbf{Loc}_\infty(X)\\
\downarrow{\scriptstyle \mathscr{I}^{(n)}} & & \downarrow{\scriptstyle \mathscr{I}}\\
\mathbf{Rep}_\infty(\pi_\infty BG_{n\bullet}) & \xrightarrow{f_\bullet^*} & \mathbf{Rep}_\infty(\pi_\infty X_\bullet),
\end{array}$$

where $f_\bullet^*$ is the pullback DG functor along the induced simplicial map $f_\bullet\colon\ \pi_\infty X_\bullet\to\pi_\infty BG_{n\bullet}$. Besides, by the naturality of the connection ω, we get a commutative diagram

$$\begin{array}{ccc}
\mathbf{Loc}_\infty(BG_n) & \xrightarrow{\qquad f^*\qquad} & \mathbf{Loc}_\infty(X)\\
{\scriptstyle \mathscr{C}\mathscr{W}_\omega^{(n)}}\nwarrow & & \nearrow{\scriptstyle \mathscr{C}\mathscr{W}_{f^*\omega}}\\
 & \mathbf{Inf\,Loc}_\infty(\mathfrak{g}). &
\end{array}$$

Combining these last two with the above commutative diagram gives

$$(\varphi_n\circ f)_\bullet^*\circ\mathscr{W}=f_\bullet^*\circ\varphi_{n\bullet}^*\circ\mathscr{W}=f_\bullet^*\circ\mathscr{I}^{(n)}\circ\mathscr{C}\mathscr{W}_\omega^{(n)}=\mathscr{I}\circ f^*\circ\mathscr{C}\mathscr{W}_\omega^{(n)}=\mathscr{I}\circ\mathscr{C}\mathscr{W}_{f^*\omega}.$$

On the other hand, according to Theorem A, there is an A_∞-natural isomorphism between $\mathscr{CW}_\theta$ and $\mathscr{CW}_{f^*\omega}$. Since $\mathscr{I}$ is an A_∞-quasi-equivalence, the desired conclusion now follows from Lemma 2.2.

□

Chapter 4

Equivariant de Rham Theorem for ∞-local systems

4.1 Infinitesimal ∞-local systems

Given a left G-manifold M and a principal bundle $\pi : P \to X$ with structure group G, we consider the orbit space $(P \times M)/G$ under the diagonal action given by

$$(p,m)\cdot g = (pg, g^{-1}m).$$

The main topic of this section is to show that, given any connection θ on P there is a DG functor

$$\mathscr{C}_\theta : \mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M) \to \mathbf{Loc}_\infty((P \times M)/G),$$

where $\mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M)$ is the DG category of basic $\mathfrak{g}$ graded G-equivariant vector bundles. Moreover, we show that, given a different connection θ', the functors $\mathscr{C}_\theta$ and $\mathscr{C}_{\theta'}$ are related by an A_∞-natural isomorphism.

We start by defining the analogous of the infinitesimal version of the equivariant ∞-local system. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. By a $\mathfrak{g}$ *graded G-equivariant vector bundle* we mean a graded G-equivariant vector bundle $E = \bigoplus_{k\in\mathbb{Z}} E^k$ over a manifold M, with differential $\delta^E : E^k \to E^{k+1}$ such that commutes with the Lie derivative $L^E_x : \Gamma(M,E) \to \Gamma(M,E)$,

$$[\delta^E, L^E_x] = 0.$$

For each $x \in \mathfrak{g}$, we will write i_x and L_x for the contraction and Lie derivative operators acting on the Weil algebra $\mathrm{W}\mathfrak{g}$ or the $\mathfrak{g}$-L_∞ space $\Omega^\bullet(M,E)$. Notice that $\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E)$ acquires the structure of a $\mathfrak{g}$-L_∞ space with differential, contraction and Lie derivative operators $d_W \otimes 1 + 1 \otimes \delta_E$, $i_x \otimes 1 + 1 \otimes i_x$ and $L_x \otimes 1 + 1 \otimes L_x$ respectively, where δ_E, i_x and L_x are the differential, contraction and Lie derivative operators of $\Omega^\bullet(M,E)$ given by $d_M \otimes 1 + 1 \otimes \delta^E$, $i_x \otimes 1$ and $L_x \otimes 1 + 1 \otimes L^E_x$, respectively. Thus, we may consider the basic subspace $(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E))_{\mathrm{bas}}$.

From §3.1.1, we know that a $\mathfrak{g}$-L_∞ space V determines and is determined by a Maurer-Cartan element of $\mathrm{W}\mathfrak{g}\otimes\mathrm{End}(V)$. Given a $\mathfrak{g}$ graded G-equivariant vector bundle E, the corresponding $\mathfrak{g}$-L_∞ space $\Omega^\bullet(M,E)$ give us a Maurer-Cartan element of $\mathrm{W}\mathfrak{g}\otimes\mathrm{End}(\Omega^\bullet(M,E))$, that we denoted by α_E. We shall say that E is *basic* if the following identities are satisfied

$$\begin{aligned}[]
[\alpha_E, i_x\otimes 1+1\otimes i_x] &= 1\otimes 1\otimes L_x^E,\\
[\alpha_E, L_x\otimes 1+1\otimes L_x] &= 0,
\end{aligned}$$

for all $x\in\mathfrak{g}$.

Lemma 4.1. *Let E be a basic $\mathfrak{g}$ graded G-equivariant vector bundle. The operator*

$$D:\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,E)\to\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,E)$$

defined by

$$D = d_{\mathrm{W}}\otimes 1\otimes 1+1\otimes d_M\otimes 1+1\otimes 1\otimes\delta^E+\alpha_E,$$

defines a flat basic super connection on $EG\times E$.

Proof. As α_E is a Maurer-Cartan element of $\mathrm{W}\mathfrak{g}\otimes\mathrm{End}(\Omega^\bullet(M,E))$, it follows that $D^2=0$. We just need to proof that D is basic. Fix an element $\eta\otimes\varphi\otimes\xi$ in $\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,E)$. Then we have

$$\begin{aligned}
&D((L_x\otimes 1\otimes 1+1\otimes L_x\otimes 1+1\otimes 1\otimes L_x^E)(\eta\otimes\varphi\otimes\xi))\\
&\quad=(d_{\mathrm{W}}\otimes 1\otimes 1+1\otimes d_M\otimes 1+1\otimes 1\otimes\delta^E+\alpha_E)\\
&\qquad(L_x\eta\otimes\varphi\otimes\xi+\eta\otimes L_x\varphi\otimes\xi+\eta\otimes\varphi\otimes L_x^E\xi)\\
&\quad=d_{\mathrm{W}}(L_x\eta)\otimes\varphi\otimes\xi+d_{\mathrm{W}}\eta\otimes L_x\varphi\otimes\xi+d_{\mathrm{W}}\eta\otimes\varphi\otimes L_x^E\xi\\
&\qquad+(-1)^{|\eta|}L_x\eta\otimes d_M\varphi\otimes\xi+(-1)^{|\eta|}\eta\otimes d_M(L_x\varphi)\otimes\xi+(-1)^{|\eta|}\eta\otimes d_M\varphi\otimes L_x^E\xi\\
&\qquad+(-1)^{|\eta|+|\varphi|}L_x\eta\otimes\varphi\otimes\delta^E\xi+(-1)^{|\eta|+|\varphi|}\eta\otimes L_x\varphi\otimes\delta^E\xi+(-1)^{|\eta|+|\varphi|}\eta\otimes\varphi\otimes\delta^E(L_x^E\xi)\\
&\qquad+\alpha_E((L_x\otimes 1\otimes 1+1\otimes L_x\otimes 1+1\otimes 1\otimes L_x^E)(\eta\otimes\varphi\otimes\xi)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(L_x\otimes 1\otimes 1+1\otimes L_x\otimes 1+1\otimes 1\otimes L_x^E)\\
&\quad((d_{\mathrm{W}}\otimes 1\otimes 1+1\otimes d_M\otimes 1+1\otimes 1\otimes\delta^E+\alpha_E)(\eta\otimes\varphi\otimes\xi))\\
&\quad=(L_x\otimes 1\otimes 1+1\otimes L_x\otimes 1+1\otimes 1\otimes L_x^E)\\
&\qquad(d_{\mathrm{W}}\eta\otimes\varphi\otimes\xi+(-1)^{|\eta|}\eta\otimes d_M\varphi\otimes\xi+(-1)^{|\eta|+|\varphi|}\eta\otimes\varphi\otimes\delta^E\xi+\alpha_E(\eta\otimes\varphi\otimes\xi)\\
&\quad=L_x(d_{\mathrm{W}}\eta)\otimes\varphi\otimes\xi+(-1)^{|\eta|}L_x\eta\otimes d_M\varphi\otimes\xi+(-1)^{|\eta|+|\varphi|}L_x\eta\otimes\varphi\otimes\delta^E\xi\\
&\qquad+d_{\mathrm{W}}\eta\otimes L_x\varphi\otimes\xi+(-1)^{|\eta|}\eta\otimes L_x(d_M\varphi)\otimes\xi+(-1)^{|\eta|+|\varphi|}\eta\otimes L_x\varphi\otimes\delta^E\xi\\
&\qquad+d_{\mathrm{W}}\eta\otimes\varphi\otimes L_x^E\xi+(-1)^{|\eta|}\eta\otimes d_M\varphi\otimes L_x^E\xi+(-1)^{|\eta|+|\varphi|}\eta\otimes\varphi\otimes L_x^E(\delta^E\xi)\\
&\qquad+(L_x\otimes 1\otimes 1+1\otimes L_x\otimes 1+1\otimes 1\otimes L_x^E)(\alpha_E(\eta\otimes\varphi\otimes\xi)).
\end{aligned}$$

Then

$$\begin{aligned}
&[D, L_x \otimes 1 \otimes 1 + 1 \otimes L_x \otimes 1 + 1 \otimes 1 \otimes L^E_x](\eta \otimes \varphi \otimes \xi)\\
&\quad = [d_{\mathrm{W}}, L_x]\eta \otimes \varphi \otimes \xi + (-1)^{|\eta|}\eta \otimes [d_M, L_x]\varphi \otimes \xi + (-1)^{|\eta|+|\varphi|}\eta \otimes \varphi \otimes [\delta^E, L^E_x]\xi\\
&\quad + [\alpha_E, L_x \otimes 1 \otimes 1 + 1 \otimes L_x \otimes 1 + 1 \otimes 1 \otimes L^E_x](\eta \otimes \varphi \otimes \xi)\\
&\quad = 0.
\end{aligned}$$

Also,

$$\begin{aligned}
&D((i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)(\eta \otimes \varphi \otimes \xi))\\
&\quad = (d_{\mathrm{W}} \otimes 1 \otimes 1 + 1 \otimes d_M \otimes 1 + 1 \otimes 1 \otimes \delta^E + \alpha_E)(i_x\eta \otimes \varphi \otimes \xi + (-1)^{|\eta|}\eta \otimes i_x\varphi \otimes \xi)\\
&\quad = d_{\mathrm{W}}(i_x\eta) \otimes \varphi \otimes \xi + (-1)^{|\eta|} d_{\mathrm{W}}\eta \otimes i_x\varphi \otimes \xi + (-1)^{|\eta|-1} i_x\eta \otimes d_M\varphi \otimes \xi + \eta \otimes d_M(i_x\varphi) \otimes \xi\\
&\qquad + (-1)^{|\eta|+|\varphi|-1} i_x\eta \otimes \varphi \otimes \delta^E\xi + (-1)^{|\varphi|-1}\eta \otimes i_x\varphi \otimes \delta^E\xi +\\
&\qquad + \alpha_E((i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)(\eta \otimes \varphi \otimes \xi)),
\end{aligned}$$

and

$$\begin{aligned}
&(i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)((d_{\mathrm{W}} \otimes 1 \otimes 1 + 1 \otimes d_M \otimes 1 + 1 \otimes 1 \otimes \delta^E + \alpha_E)(\eta \otimes \varphi \otimes \xi))\\
&\quad = (i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)(d_{\mathrm{W}}\eta \otimes \varphi \otimes \xi + (-1)^{|\eta|}\eta \otimes d_M\varphi \otimes \xi + (-1)^{|\eta|+|\varphi|}\eta \otimes \varphi \otimes \delta^E\xi\\
&\qquad + \alpha_E(\eta \otimes \varphi \otimes \xi))\\
&\quad = i_x(d_{\mathrm{W}}\eta) \otimes \varphi \otimes \xi + (-1)^{|\eta|} i_x\eta \otimes d_M\varphi \otimes \xi + (-1)^{|\eta|+|\varphi|} i_x\eta \otimes \varphi \otimes \delta^E\xi\\
&\qquad + (-1)^{|\eta|+1} d_{\mathrm{W}}\eta \otimes i_x\varphi \otimes \xi + \eta \otimes i_x(d_M\varphi) \otimes \xi + (-1)^{|\varphi|}\eta \otimes i_x\varphi \otimes \delta^E\xi\\
&\qquad + (i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)(\alpha_E(\eta \otimes \varphi \otimes \xi)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[D, i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1](\eta \otimes \varphi \otimes \xi)\\
&\quad = [d_{\mathrm{W}}, i_x]\eta \otimes \varphi \otimes \xi + \eta \otimes [d_M, i_x]\varphi \otimes \xi + \alpha_E((i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1)(\eta \otimes \varphi \otimes \xi))\\
&\quad = L_x\eta \otimes \varphi \otimes \xi + \eta \otimes L_x\varphi \otimes \xi + [\alpha_E, i_x \otimes 1 \otimes 1 + 1 \otimes i_x \otimes 1](\eta \otimes \varphi \otimes \xi)\\
&\quad = L_x\eta \otimes \varphi \otimes \xi + \eta \otimes L_x\varphi \otimes \xi + \eta \otimes \varphi \otimes L^E_x\xi\\
&\quad = (L_x \otimes 1 \otimes 1 + 1 \otimes L_x \otimes 1 + 1 \otimes 1 \otimes L^E_x)(\eta \otimes \varphi \otimes \xi),
\end{aligned}$$

as desired. □

By the previous result, the pair $(EG \times E, D)$ defines a basic ∞-local system on $EG \times M$.

Let M be a G-manifold and let $\mathfrak{g}$ be the Lie algebra of G. With the previous discussion, we can define the DG category of *infinitesimal ∞-local systems* on M, denoted by $\mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M)$, by the following data. The objects of this DG category are all basic $\mathfrak{g}$ graded G-equivariant vector bundles $E \to M$. For any two basic $\mathfrak{g}$ graded G-equivariant vector bundles E and E' with corresponding differentials D_E and $D_{E'}$, the space of morphisms is the graded vector space

$$(W\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}},$$

with the differential $\partial_{D_E,D_{E'}}$ given by the formula

$$\partial_{D_E,D_{E'}}\varphi = D_{E'}\circ\varphi-(-1)^k\varphi\circ D_E,$$

for any homogeneous element φ of degree k.

Next, we have the main result of this section.

Theorem C. *Let G be a Lie group and let $\pi : P \to X$ be a principal bundle with structure group G. Then, for any connection θ on P, there is a natural DG functor*

$$\mathscr{C}_\theta : \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to\mathbf{Loc}_\infty((P\times M)/G).$$

Moreover, for any two connections θ and θ' on P, there is an A_∞-natural isomorphism between $\mathscr{C}_\theta$ and $\mathscr{C}_{\theta'}$.

Proof. Given a connection θ on P, this connection can be extended to a connection θ on $P\times M$. By Proposition 3.3 it is suffice to show that there is a natural DG functor

$$\overline{\mathscr{C}}_\theta : \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to[\mathbf{Loc}_\infty(P\times M)]_{\mathrm{bas}}.$$

Given a fixed connection θ on $P\times M$, for each object E in $\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ consider the operator

$$D_\theta:\Omega^\bullet(P)\otimes\Omega^\bullet(M,E)\to\Omega^\bullet(P)\otimes\Omega^\bullet(M,E)$$

given by the formula

$$D_\theta = d_P\otimes 1\otimes 1+1\otimes d_M\otimes 1+1\otimes 1\otimes\delta_E+(c_\theta\otimes 1)\alpha_E,$$

where $(c_\theta\otimes 1)\alpha_E$ is a Maurer-Cartan element of $(\Omega^\bullet(P)\otimes\Omega^\bullet(M,E))_{\mathrm{bas}}$. Using the same argument given in the proof of Lemma 4.1, we can show that D_θ defines a flat basic superconnection on $P\times E$, defining $\overline{\mathscr{C}}_\theta(E)$ to be the basic ∞-local system $(P\times E,D_\theta)$. By the same arguments as Lemma 4.1, given a morphism $\varphi\in(W\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}}$, we get a cochain map

$$\overline{\mathscr{C}}_\theta:(W\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}}\to(\Omega^\bullet(P)\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}},$$

with respect to both differentials $\partial_{D,D'}$ and $\partial_{D_\theta,D'_\theta}$, concluding that $\overline{\mathscr{C}}_\theta$ is a DG functor.

Finally, given two connections θ and θ' on $P\times M$, we have the interpolating connection $\theta_t = \theta+t(\theta-\theta')$, where $\theta_0=\theta$ and $\theta_1=\theta'$. We define $\overline{\mathscr{C}}_\theta=\iota_0^*\circ\overline{\mathscr{C}}_{\hat\theta}$ and $\overline{\mathscr{C}}_{\theta'}=\iota_1^*\circ\overline{\mathscr{C}}_{\hat\theta}$, where

$$\overline{\mathscr{C}}_{\hat\theta}:\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to[\mathbf{Loc}_\infty(P\times M\times[0,1])]_{\mathrm{bas}},$$

is the DG functor given by the connection $\hat{\theta} = \mathrm{pr}_1^*\theta_t$ on $P \times M \times [0,1]$, with $\mathrm{pr}_1 : P \times M \times [0,1] \to P \times M$, and

$$\iota_t^* : \mathbf{Loc}_\infty(P \times M \times [0,1]) \to \mathbf{Loc}_\infty(P \times M),$$

is the pullback DG functor of the inclusion $\iota_t : P \times M \to P \times M \times [0,1]$ of height t. By virtue of proposition 2.5, there exists an A_∞-natural isomorphism between ι_0^* and ι_1^*. By restricting the latter to the full DG subcategory of $[\mathbf{Loc}_\infty(P \times M \times [0,1])]_{\mathrm{bas}}$ consisting of objects of the form $\overline{\mathscr{C}}_{\hat{\theta}}(E)$ with E an object of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M)$, we obtain an A_∞-natural isomorphism between

$$\overline{\mathscr{C}}_\theta, \overline{\mathscr{C}}_{\theta'} : \mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M) \to [\mathbf{Loc}_\infty(P \times M)]_{\mathrm{bas}}$$

as wished. □

Finally, notice that in the particular case of the previous result when M is a point, we get the Chern-Weil DG functor $\mathscr{CW}_\theta : \mathbf{Inf\,Loc}_\infty\,\mathfrak{g} \to \mathbf{Loc}_\infty(X)$.

Before going to the central result in the following section, we will summarize some results. The first of these is the Weil-Cartan isomorphism for the complex $\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E)$.

Lemma 4.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let M be a G-manifold. There is a graded-algebra isomorphism $F : (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E))_{hor} \to \mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M,E)$. This isomorphism induce an isomorphism on the basic sub algebras $F : (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E))_{\mathrm{bas}} \to (\mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M,E))_{\mathrm{inv}}$.*

Proof. Taking in account the same notation of §3.1.1, let e_a be a basis for $\mathfrak{g}$, t^a the dual basis for $\mathfrak{g}^*$ in the exterior algebra $\Lambda^\bullet\mathfrak{g}^*$, and w^a the dual basis for $\mathfrak{g}^*$ in the symmetric algebra $\mathrm{S}^\bullet\mathfrak{g}^*$. As shorthand, we write

$$\begin{aligned} x_A &= x_{a_1 \cdots a_r} \in \Omega^\bullet(M,E)[w^1, \dots, w^n], \\ i_a &= i_{e_a}, \quad L_a = L_{e_a}, \\ t^A &= t^{a_1} \cdots t^{a_r} = t^{a_1} \wedge \cdots \wedge t^{a_r}. \end{aligned}$$

Notice that

$$\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E) = \Lambda(t^1, \dots, t^n) \otimes \Omega^\bullet(M,E)[w^1, \dots, w^n].$$

Thus, an element α of $\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E)$ can be written as a linear combination of monomials $t^{a_1} \cdots t^{a_r}$, with $1 \leq a_1 < \cdots < a_r \leq n$, with coefficients in $\Omega^\bullet(M,E)[w^1, \dots, w^n]$,

$$\alpha = x + \sum_A t^A \otimes x_A, \quad x_A \in \Omega^\bullet(M,E)[w^1, \dots, w^n].$$

We define $F : (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M,E))_{\mathrm{hor}} \to \mathrm{S}^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M,E)$ as

$$\alpha = x + \sum_A t^A \otimes x_A \mapsto x.$$

First, we will verify that F commutes with the Lie derivative L_a. So we have that

$$\begin{aligned}(F\circ L_a)(\alpha) &= F\left(L_a\left(x+\sum_A t^A\otimes x_A\right)\right)\\ &= F\left(L_a x+\sum_A (L_a t^A)\otimes x_A+\sum_A t^A\otimes L_a x_A\right).\end{aligned}$$

Since $L_a t^c = -\sum_b c^c_{ab} t^b$, each term of $(L_a t^A)\otimes x_A$ is either 0 or has positive degree in the t^a's. Therefore

$$(F\circ L_a)(\alpha) = L_a x = (L_a\circ F)(\alpha).$$

This shows that if α is invariant, then $F(\alpha)$ is also invariant.

To prove that F is a graded-algebra isomorphism, let $K = S^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M,E)$ be the graded algebra with $i_a = i_a\otimes 1 + 1\otimes i_a$, where $i_a^2 = 0$ for all $a\in\mathfrak{g}$. We define

$$H_a := 1 - t^a\otimes i_a : \Lambda^\bullet\mathfrak{g}^*\otimes K\to\Lambda^\bullet\mathfrak{g}^*\otimes K$$

and

$$H = \prod H_a : \Lambda^\bullet\mathfrak{g}^*\otimes K\to\Lambda^\bullet\mathfrak{g}^*\otimes K.$$

Here the product $\prod$ is in the sense of composition of functions, so

$$H = H_1\circ H_2\circ\cdot\circ H_n$$

and i_a acts by the diagonal action on $\Lambda^\bullet\mathfrak{g}^*\otimes K$. Let

$$J := \bigcap_a \ker i_a = (\Lambda^\bullet\mathfrak{g}^*\otimes K)_{\mathrm{hor}}$$

We show that $H|_K : K\to J$ is the map inverse to F. On the generators of $\Lambda^\bullet\mathfrak{g}^*$, we have that

$$H_a(t^b) = t^b - t^a\otimes i_a(t^b) = t^b - t^a\delta_{ab}.$$

Therefore $H_a(t^b)$ is 0 if $b=a$ and t^b if $b\neq a$. In order to show that $H|_K$ is the map inverse to F, we need the following results:

1. First we show that H_a is a ring map, hence H is a ring map.

$$\begin{aligned}H_a(xy) &= xy - t^a\otimes i_a(xy)\\ &= xy - t^a\otimes((i_a x)y + (-1)^{|x|}x(i_a y))\\ &= xy - (t^a\otimes i_a x)y - x(t^a\otimes i_a y).\end{aligned}$$

$$\begin{aligned} H_a(x)H_b(y) &= (x - t^a \otimes i_a x)(y - t^a \otimes i_a y) \\ &= xy - (t^a \otimes i_a x)y - x(t^a \otimes i_a y) + (t^a \otimes i_a x)(t^a \otimes i_a y). \end{aligned}$$

These two expressions are equal since $t^a t^a = 0$.

2. On the other hand, let us see that $H_a H_b = H_b H_a$. Assume $a \neq b$. Then

$$\begin{aligned} H_a H_b &= (1 - t^a \otimes i_a)(1 - t^b \otimes i_b) \\ &= 1 - t^a \otimes i_a - t^b \otimes i_b + t^a \otimes i_a (t^b \otimes i_b) \\ &= 1 - t^a \otimes i_a - t^b \otimes i_b - t^a t^b \otimes i_a i_b. \end{aligned}$$

In the calculation above,

$$\begin{aligned} t^a \otimes i_a (t^b \otimes i_b) &= t^a \otimes (i_a t^b) i_b - t^a t^b \otimes i_a i_b \\ &= 0 - t^a t^b \otimes i_a i_b, \quad \text{since for } a \neq b, i_a t^b = \delta_{ab} = 0. \end{aligned}$$

Inverting a and b leaves the expression $1 - t^a \otimes i_a - t^b \otimes i_b - t^a t^b \otimes i_a i_b$ invariant, since

$$t^b t^a = -t^a t^b \text{ and } i_b i_a = -i_a i_b.$$

Thus, $H_a H_b = H_b H_a$.

3. Next, we show that $i_a H_a = 0$, therefore im $H_a \subset \ker i_a$.

$$i_a H_a = i_a(1 - t^a \otimes i_a) = i_a - (i_a t^a) \otimes i_a = i_a - i_a = 0,$$

because $i_a t^a = t^a(e_a) = 1$ and $i_a^2 = 0$. Thus, im $H_a \subset \ker i_a$. By (2), im $H \subset$ im $H_a \subset \ker i_a$. So im $H \subset J := \cap_a \ker i_a$.

4. Also, notice that $H|_J = 1_J$. $H|_J = \prod(1 - t^a \otimes i_a)|_J = 1_J$, since $i_a|_J = 0$. Thus, $H : \Lambda^\bullet \mathfrak{g}^* \otimes K \to J$ is surjective.

5. Finally, notice that $H_a(t^a) = 0$ and $H(K) = J$. Notice that H vanishes on t^a. Since H is a ring map, it vanishes on the ideal $(t^1, ..., t^n)$. By (4),

$$J = \text{im } H = H(\Lambda^\bullet \mathfrak{g}^* \otimes K) = H(K).$$

Therefore $H|_K : K \to J$ is surjective onto J.

Since $H(x)$ is of the form $x + \sum t^A \otimes x_A$, where $x, x_A \in K$ and $t^A \otimes x_A$ has positive degree in the t^a's, if $H(x) = H(y)$, then $x + \sum t^A \otimes x_A = y + \sum t^A \otimes y_A$. By comparing terms of degree 0 in the t^a's, we get $x = y$. Therefore $H|_K : K \to J$ is injective. By (5),

$$H|_K : \mathrm{S}^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M, E) \to (\Lambda^\bullet \mathfrak{g}^* \otimes \mathrm{S}^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M, E))_{\text{hor}} = (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M, E))_{\text{hor}}$$

is an isomorphism. □

Let G be a compact and connected Lie group with Lie algebra $\mathfrak{g}$. If E is a $\mathfrak{g}$ graded G-equivariant vector bundle, we shall write $\mathcal{H}^\bullet(E)$ for the cohomology of the graded vector space E concerning the zeroth component of its corresponding Maurer-Cartan element α_E. The following result is analogous to Lemma 3.13.

Lemma 4.3. *Let E and E' be two objects in $\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$. Then the space of morphisms $(\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}}$ carries a canonical decreasing filtration by the degree in the Weil algebra $\mathrm{W}\mathfrak{g}$ and in the differential forms $\Omega^\bullet(M)$. This filtration induces a spectral sequence with second page $\mathcal{E}_2^{p,q}=\mathrm{H}^p((\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M))_{\mathrm{bas}})\otimes\mathcal{H}^q(\mathrm{Hom}(E,E'))$ and which converges to $\mathrm{H}^{p+q}((\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}})$.*

Proof. We only show the second part. By Lemma 4.2, the zeroth page of the spectral sequence is

$$\mathcal{E}_0^{p,q}=(\mathrm{W}^r\mathfrak{g}\otimes\Omega^s(M,\mathrm{Hom}(E,E')^q))_{\mathrm{bas}}=(\mathrm{S}^{r/2}\mathfrak{g}^*\otimes\Omega^s(M)\otimes_{C^\infty(M)}\mathrm{Hom}(E,E')^q)_{\mathrm{inv}}$$

with $r+s=p$ and differential induced by that of $\mathrm{Hom}(E,E')$. We know that taking cohomology commutes with taking invariants by the compactness of G, and therefore

$$\mathcal{E}_1^{p,q}=\left(\mathrm{H}^\bullet\left(\mathrm{S}^{r/2}\mathfrak{g}^*\otimes\Omega^s(M)\otimes_{C^\infty(M)}\mathrm{Hom}(E,E')^q\right)\right)_{\mathrm{inv}}.$$

Since, the ring $C^\infty(M)$ is semisimple, then the $C^\infty(M)$-modules $\Omega^\bullet(M)$ and $\mathrm{Hom}(E,E')$ are projective, and hence that modules are flat. In that case, we have the Kunneth theorem, and therefore using also the fact that the differential acts only on the last component, the first page of the spectral sequence is

$$\mathcal{E}_1^{p,q}=(\mathrm{S}^{r/2}\mathfrak{g}^*\otimes\Omega^s(M))_{\mathrm{inv}}\otimes\mathcal{H}^q(\mathrm{Hom}(E,E'))=\mathrm{H}^p((\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M))_{\mathrm{bas}})\otimes\mathcal{H}^q(\mathrm{Hom}(E,E')).$$

For degree reasons the first page differential vanishes, so we conclude that the second page of the spectral sequence is $\mathcal{E}_2^{p,q}=\mathrm{H}^p((\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M))_{\mathrm{bas}})\otimes\mathcal{H}^q(\mathrm{Hom}(E,E'))$, as asserted. □

Notice that, as for the DG category $\mathbf{Inf\,Loc}_\infty(\mathfrak{g})$ there is a canonical filtration, the DG category $\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ has a canonical filtration $F^p\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ consisting of all $\mathfrak{g}$ graded G-equivariant vector bundles E for which $E^k=0$ for all $k\in\mathbb{Z}$ such that $|k|>p$.

The following result is similar to Lemma 3.14 and its proof runs along the same lines.

Lemma 4.4. *Let G be a compact connected Lie group, $\pi\colon P\to X$ be a principal G-bundle and θ a connection on P. Then, for any pair of objects E and E' of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$, the morphism of cochain complexes*

$$\mathscr{C}_\theta\colon(\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}}\longrightarrow\Omega^\bullet(X,\Omega^\bullet(M,\mathrm{Hom}(\mathscr{C}_\theta(E),\mathscr{C}_\theta(E'))))$$

determined by the Cartan's DG functor of P, is compatible with the filtrations described in Lemmas 3.11 and 4.3, and induces a morphism of spectral sequences. Moreover, if X and M are simply *connected, this morphism is identified with the Chern-Weil homomorphism of P tensored with the identity on* $\mathrm{H}^\bullet_{\mathrm{DR}}(M)\otimes\mathcal{H}^\bullet(\mathrm{Hom}(E,E'))$.

4.2 Equivariant ∞-Local systems

We will now proceed to construct the A_∞-functor that connects the DG categories $\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ and $\mathbf{Loc}_\infty(M_G)$, where the last DG category will be defined in the process. Let G be a Lie group, given the principal G-bundle $EG_n\to BG_n$ and a left G-manifold M, by Proposition 2.7, the projection $EG_n\times M\to(EG_n\times M)/G$ is a principal G-bundle. In addition, given a connection θ on EG_n, this connection can be extended naturally to $EG_n\times M$. Therefore, by Theorem C, we know that for each $n\geq 0$, there is a DG functor $\mathscr{C}^{(n)}_\theta:\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to\mathbf{Loc}_\infty((EG_n\times M)/G)$. By the higher Riemann-Hilbert correspondence 2.2.4, for each n, there is an A_∞-quasi-equivalence $\mathscr{I}^{(n)}\colon\mathbf{Loc}_\infty((EG_n\times M)/G)\to\mathbf{Rep}_\infty(\pi_\infty((EG_n\times M)/G_\bullet))$, and then by the naturality of the construction, all these A_∞-functors fit into the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{Rep}_\infty(\pi_\infty((EG_n\times M)/G_\bullet)) & \xleftarrow{\;i^*_\bullet\;} & \mathbf{Rep}_\infty(\pi_\infty((EG_{n+1}\times M)/G_\bullet)). \\
\uparrow{\scriptstyle \mathscr{I}^{(n)}} & & \uparrow{\scriptstyle \mathscr{I}^{(n+1)}} \\
\mathbf{Loc}_\infty((EG_n\times M)/G) & \xleftarrow{\;i^*\;} & \mathbf{Loc}_\infty((EG_{n+1}\times M)/G) \\
{\scriptstyle \mathscr{C}^{(n)}_\theta}\nwarrow & & \nearrow{\scriptstyle \mathscr{C}^{(n+1)}_\theta} \\
& \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M) &
\end{array}$$

where $i\colon EG_n\to EG_{n+1}$ is the natural inclusion. Therefore, for each $n\geq 0$, we define the A_∞-functor $\mathscr{DR}^{(n)}\colon\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to\mathbf{Rep}_\infty(\pi_\infty((EG_n\times M)/G_\bullet))$ as the composition

$$\mathscr{DR}^{(n)}=\mathscr{I}^{(n)}\circ\mathscr{C}^{(n)}_\theta.$$

Notice that these A_∞-functors fit into a commutative diagram

$$\begin{array}{ccc}
\mathbf{Rep}_\infty(\pi_\infty((EG_n\times M)/G_\bullet)) & \xleftarrow{\;i^*_\bullet\;} & \mathbf{Rep}_\infty(\pi_\infty((EG_{n+1}\times M)/G_\bullet)) \\
{\scriptstyle \mathscr{DR}^{(n)}}\nwarrow & & \nearrow{\scriptstyle \mathscr{DR}^{(n+1)}} \\
& \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M). &
\end{array}$$

Since for a compact connected Lie group G, the total space EG is the infinite Stiefel variety $\mathrm{V}_k(\mathbb{C}^\infty)$, for some integer k, then we can approximate EG_n by $\mathrm{V}_k(\mathbb{C}^{n+k+1})$, for n sufficiently large. Similarly, notice that the homotopy quotient $M_G=(EG\times M)/G$ can be approximated by $M_{G_n}=(\mathrm{V}_k(\mathbb{C}^{n+k+1})\times M)/G$. Therefore, taking note of the definition of the DG category $\mathbf{Loc}_\infty(BG)$ given at the end of §3.2.2, using the same arguments for the previous approximation of M_G we define the DG category $\mathbf{Loc}_\infty(M_G)$ by

$$\mathbf{Loc}_\infty(M_G):=\mathbf{Rep}_\infty(\pi_\infty((EG\times M)/G_\bullet))$$

called the *equivariant ∞-local systems* of M, and recalling that, for each $n \geq 0$, integrating an ∞-local system on $(EG_n \times M)/G_\bullet$ amounts to assigning holonomies to smooth maps from the standard simplices to $(EG_n \times M)/G_\bullet$, we deduce the existence of an A_∞-functor

$$\mathscr{DR}\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M) \longrightarrow \mathbf{Loc}_\infty(M_G),$$

that we will call the *equivariant De Rham A_∞-functor*, our goal is to show that $\mathscr{DR}$ is, in fact, an A_∞-quasi-equivalence, the following two preliminary results will clear our path. From now, we denote by M_{G_n} the space $(EG_n \times M)/G$.

Proposition 4.1. Let M be simply connected. Let E and E' be two objects in $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ and put $N = \frac{1}{2}(2k+l+6m)$. Then, for each $n \geq N$, the morphism of cochain complexes

$$\mathscr{C}_\theta^{(n)}\colon\ (\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}} \longrightarrow \Omega^\bullet(M_{G_n},\Omega^\bullet(M,\mathrm{Hom}(\mathscr{C}_\theta^{(n)}(E),\mathscr{C}_\theta^{(n)}(E'))))$$

induces an isomorphism in cohomology up to degree l.

Proof. Following the same lines of reasoning as Proposition 3.9, we know that the characteristic map $c_\omega\colon (\mathrm{W}\mathfrak{g})_{\mathrm{bas}} \to \Omega^\bullet(BG_n)$ induces an isomorphism in cohomology up to degree $2(N-k)$. On the other hand, by virtue of Lemma 4.4, the morphism of cochain complexes $\mathscr{C}_\theta^{(n)}$ is compatible with the canonical decreasing filtrations on $(\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,E')))_{\mathrm{bas}}$ and

$$\Omega^\bullet(M_{G_n},\Omega^\bullet(M,\mathrm{Hom}(\mathscr{C}_\theta^{(n)}(E),\mathscr{C}_\theta^{(n)}(E')))),$$

and induces a morphism between the corresponding spectral sequences, whose rth pages we write respectively as $\mathcal{E}_r^{p,q}$ and $\mathcal{E}_r'^{p,q}$. Moreover, since M_{G_n} is simply connected, this morphism is identified with the Chern-Weil homomorphism of $\mathrm{V}_k(\mathbb{C}^n)$ tensored with the identity of $H^\bullet_{\mathrm{DR}}(M)\otimes \mathcal{H}^\bullet(\mathrm{Hom}(E,E'))$. Following the same lines of reasoning as in Proposition 3.9 together with Lemmas 3.8 and 4.3, we get the desired conclusion. □

As in the case of ∞-local systems, the concept of central extension can be carried out for a $\mathfrak{g}$ graded G-equivariant vector bundles. Let us write $\mathbb{R}$ for the trivial representation of $\mathbb{T}\mathfrak{g}$. We further let E be a $\mathfrak{g}$ graded G-equivariant vector bundle and γ be a closed element of $(\mathrm{W}\mathfrak{g}\otimes\Omega^\bullet(M,\mathrm{Hom}(E,\mathbb{R})))_{\mathrm{bas}}$ of degree p. Then the central extension of E by γ, denoted by $E\rtimes_\gamma\mathbb{R}$, is the $\mathfrak{g}$ graded G-equivariant vector bundle whose underlying graded vector space is the direct sum $E\oplus\mathbb{R}[1-p]$ and whose corresponding Maurer-Cartan element is $\alpha_E+\gamma$.

Proposition 4.2. Let M be simply connected G-manifold. Given $m \geq 0$, there exists a sufficiently large integer $n > 0$ such that the corresponding Cartan's DG funtor

$$\mathscr{C}_\theta^{(n)}\colon\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M) \to \mathbf{Loc}_\infty(M_{G_n})$$

induces a quasi essentially surjective DG functor between $F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)$ and $F^m\mathbf{Loc}_\infty(M_{G_n})$.

Proof. Using the fact that M_{G_n} is simply connected since M is simply connected, together with Proposition 4.1, the result is obtained following the same arguments of Proposition 3.10. □

With all the previous results, we are prepared to state and prove the general theorem we have been looking for, which is a generalization of the classical equivariant de Rham theorem 2.4.

Theorem D. *Let M be simply connected G-manifold. Given a compact connected Lie group G, the A_∞-fuctor*

$$\mathcal{DR} : \mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M) \to \mathbf{Loc}_\infty(M_G)$$

is an A_∞-quasi-equivalence. In addition, for any principal G-bundle $\pi : P \to X$ with connection θ and classifying map $\overline{f} : P \to EG_n$, given a smooth G-manifold M simply connected, there exists an A_∞-natural isomorphism between the A_∞-functors $\mathcal{I} \circ \mathcal{C}_\theta$ and $(\varphi_n \circ \overline{f})^*_\bullet \circ \mathcal{DR}$ from $\mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M)$ to $\mathbf{Rep}_\infty(\pi_\infty((P \times M)/G_\bullet)$. Here $\mathcal{I}$ is the integration A_∞-functor provided by the higher Riemann-Hilbert correspondence, and φ_n is the canonical map from EG_n to EG.

Proof. Prove that the A_∞-functor $\mathcal{DR} :\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M) \to \mathbf{Loc}_\infty(M_G)$ is A_∞-quasi fully faithful, is equivalent to show that

$$\mathcal{DR} :\ (\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M, \mathrm{Hom}(E, E')))_{\mathrm{bas}} \longrightarrow \underline{\mathrm{Hom}}^\bullet(\mathcal{DR}(E), \mathcal{DR}(E'))$$

is a quasi-isomorphism, for E and E' two objects of $\mathbf{Inf\,Loc}_\infty(\mathfrak{g}, M)$. This morphism is compatible with the canonical filtrations on $(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M, \mathrm{Hom}(E, E')))_{\mathrm{bas}}$ and $\underline{\mathrm{Hom}}^\bullet(\mathcal{DR}(E), \mathcal{DR}(E'))$, and induces a morphism between their associated spectral sequences.

From Lemma 4.3 and Lemma 3.11, we know that the second page of the spectral sequence associated with the canonical filtrations on $(\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M, \mathrm{Hom}(E, E')))_{\mathrm{bas}}$ and on $\underline{\mathrm{Hom}}^\bullet(\mathcal{DR}(E), \mathcal{DR}(E'))$ are

$$\mathcal{E}_2^{p,q} = \mathrm{H}^p((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(E, E')),$$

and

$$\mathcal{E}'^{p,q}_2 = \mathrm{H}^p(\pi_\infty((EG \times M)/G_\bullet), \mathcal{H}^q(\mathcal{DR}(E), \mathcal{DR}(E'))),$$

respectively. Since M_G is simply connected, then the graded vector bundle $\mathcal{H}^\bullet(\mathcal{DR}(E), \mathcal{DR}(E'))$ is trivial with fiber $\mathcal{H}^\bullet(\mathrm{Hom}(E, E'))$, and then

$$\mathcal{E}'^{p,q}_2 \cong \mathrm{H}^p(\pi_\infty((EG \times M)/G)_\bullet) \otimes \mathcal{H}^q(\mathrm{Hom}(E, E')) = \mathrm{H}^p(M_G) \otimes \mathcal{H}^q(\mathrm{Hom}(E, E')).$$

From which, we get the induced morphism between the second pages of the spectral sequence

$$\mathcal{DR} :\ \mathrm{H}^p((\mathrm{W}\mathfrak{g} \otimes \Omega^\bullet(M))_{\mathrm{bas}}) \otimes \mathcal{H}^q(\mathrm{Hom}(E, E')) \longrightarrow \mathrm{H}^p(M_G) \otimes \mathcal{H}^q(\mathrm{Hom}(E, E')),$$

and it is given by the tensor product of the equivariant De Rham isomorphism with the identity on $\mathcal{H}^q(\mathrm{Hom}(E, E'))$.

Using the same arguments as Theorem B and invoking Lemma 3.12, we get an equivalence between the homotopy categories $\varphi_{n\bullet}^* : F^m\mathbf{Loc}_\infty(M_G) \to F^m\mathbf{Rep}_\infty(\pi_\infty((EG_n \times M)/G_\bullet))$, where $\varphi_{n\bullet}^*$ is the pullback DG functor along the induced simplicial map $\varphi_{n\bullet} :\ \pi_\infty((EG_n \times M)/G_\bullet) \to \pi_\infty((EG \times$

$M)/G_\bullet)$. In addition by construction, we have the following commutative diagram of A_∞-functors

$$\begin{array}{ccc} F^m\mathbf{Inf\,Loc}_\infty(\mathfrak{g},M) & \xrightarrow{\mathscr{D}\mathscr{R}} & F^m\mathbf{Loc}_\infty(M_G) \\ {\scriptstyle \mathscr{C}_\theta^{(n)}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{n\bullet}^*} \\ F^m\mathbf{Loc}_\infty((EG_n\times M)/G) & \xrightarrow{\mathscr{I}^{(n)}} & F^m\mathbf{Rep}_\infty(\pi_\infty((EG_n\times M)/G_\bullet)) \end{array}$$

Thus, in order to show that the A_∞-functor $\mathscr{D}\mathscr{R}:\ \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M)\to\mathbf{Loc}_\infty(M_G)$ is A_∞-quasi-essentially surjective, since $\mathscr{I}^{(n)}$ is an A_∞-quasi-equivalence, it will be enough to show that $\mathscr{C}_\theta^{(n)}$ is a quasi-essentially surjective. But, if n is sufficiently large, this is true by Proposition 4.2.

Finally, combining the commutative diagrams

$$\begin{array}{ccccc} \mathbf{Loc}_\infty(EG_n) & & \xrightarrow{\quad \overline{f}^* \quad} & & \mathbf{Loc}_\infty(P) \ , \\ & {\scriptstyle \mathscr{C}_\theta^{(n)}}\nwarrow & & \nearrow{\scriptstyle \mathscr{C}_{f^*\theta}} & \\ & & \mathbf{Inf\,Loc}_\infty(\mathfrak{g},M). & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Loc}_\infty(EG_n) & \xrightarrow{\overline{f}^*} & \mathbf{Loc}_\infty(P) \\ {\scriptstyle \mathscr{I}^{(n)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{I}} \\ \mathbf{Rep}_\infty(\pi_\infty EG_{n\bullet}) & \xrightarrow{(\overline{f}_\bullet)^*} & \mathbf{Rep}_\infty(\pi_\infty P_\bullet), \end{array}$$

where $(\overline{f}_\bullet)^*$ is the pullback DG functor along the induced simplicial map $\overline{f}_\bullet\colon\ \pi_\infty P_\bullet\to\pi_\infty EG_{n\bullet}$, and using the same calculation as Theorem B, we get the second assertion. □

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