Existencia de soluciones positivas para un problema semipositón con p-Laplaciano fraccionario

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To my parents.

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Resumen

En esta tesis haremos un breve estudio de los espacios Fraccionarios de Sobolev. Daremos dos definiciones equivalentes de estos espacios usando espacios de interpolación y la transformada de Fourier en el caso $p = 2$. Finalmente, probaremos la existencia de al menos una solución positiva para el problema semipositón no local

$$\begin{cases} (-\Delta)_p^s(u) &= \lambda f(u) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

cuando $\lambda > 0$ es un parámetro suficientemente pequeño, $\Omega \subseteq \mathbb{R}^N$ es un dominio con frontera $C^{1,1}$, $2 \leqslant p < N$, $s \in (0,1)$ y f es superlineal y subcrítica. Si $\lambda > 0$ es escogido suficientemente pequeño, el funcional de energía del problema tendrá una estructura de paso de montaña y por lo tanto un punto crítico u_λ, que es una solución débil. Después de esto lograremos probar que esta solución es positiva usando nuevos resultados de regularidad hasta la frontera y un lema de Hopf.

Palabras Clave: Teorema de Paso de Montaña, problema Semipositón, soluciones positivas, p-Laplaciano Fraccionario, principios de comparación.

Abstract

In this thesis we will make a brief study of Fractional Sobolev spaces. We will give two equivalent definitions of these spaces using interpolation spaces and the Fourier transform in the case $p = 2$. Finally, we prove the existence of at least one positive solution for the nonlocal semipositone problem

$$\begin{cases} (-\Delta)_p^s(u) &= \lambda f(u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a sufficiently small parameter. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $2 \leqslant p < N$, $s \in (0,1)$ and f is superlineal and subcritical. We prove that if $\lambda > 0$ is chosen sufficiently small the associated energy functional to the problem has a mountain pass structure and, therefore, it has a critical point u_λ, which is a weak solution. After that we manage to prove that this solution is positive by using new regularity results up to the boundary and a Hopf's Lemma.

Keywords: Mountain Pass Theorem, Semipositone problem, positive solutions, fractional p-Laplacian, comparison principles.

Introduction

In the area of partial differential equations, a boundary value problem (B.V.P) consists of a differential equation defined in a domain $\Omega \subseteq \mathbb{R}^N$ and a condition that must be satisfied by the unknown function, u, in the boundary of Ω (or in our case in $\mathbb{R}^N \setminus \Omega$). There are many questions that result in relation to this type of problems. For example: does the problem have a solution? if so, is this solution unique? or otherwise, at least how many solutions does the problem have? What is the behavior of these solutions? that is, they are bounded or explode, they are oscillatory, have only one sign, etc. One of the main question that has been studied is about the regularity or smoothness of the solutions. In our context, the solutions are not understood in a classical sense but in a weak sense (concept that will be explained in this work). In order to respond this last question it is necessary to make *a priori estimates*.
We will consider the problem

$$\begin{cases} (-\Delta)_p^s(u) & = & \lambda f(u) & \text{in } \Omega \\ u & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{0.1}$$

where $\lambda > 0$ is a sufficiently small parameter. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $2 \leqslant p < N$, $s \in (0,1)$ and f is superlineal and subcritical and $(-\Delta)_p^s$ is the s-fractional p-Laplacian operator defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y.$$

In order to deal with this type of problems, it is then necessary to have sufficient knowledge about the fractional Sobolev spaces $W^{s,p}$. There is an extensive theory in the literature about these spaces which are very similar to the theory of the Sobolev spaces $W^{k,p}$, where k is a positive integer. Another extremely important theoretical tool for the study of our problem is the theory of critical points: *minimizing functionals*, *saddle points*, the Mountain Pass Theorem, etc. To achieve *a priori estimates* that allow us to know some qualitative properties of the possible solutions of our problem, the regularity study is required.
Recently in the literature a deep interest for Fractional Sobolev spaces, and the corresponding partial differential equations involving nonlocal operators has arisen due to its applications in a wide range of contexts, such as optimization, image processing, quantum mechanics, conservation laws, finance, among others see [18], [21], [6], [33]. The main objective of this thesis will be to prove the existence of at least one positive solution for the problem 0.1. A well known fact is that when the function f (non-linearity) has subcritical growth (this is closely related to the numbers p and s) we can define the functional E in the Fractional Sobolev space $W_0^{s,p}$. This, together with other hypotheses, allows us to give to the problem a variational approach, which is relatively simple and successful with a large classes of problems. It basically says that the solution of a differential equation coupled with a boundary condition can be obtained as a

critical point of an appropriate functional. In this work we will consider the functional $E : W_0^{s,p}(\Omega) \to \mathbb{R}$ defined as

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\,\mathrm{d}y - \lambda\int_\Omega F(u)\mathrm{d}x, \tag{0.2}$$

where F is an appropriate anti derivative of f. We will prove that if $\lambda > 0$ is chosen sufficiently small the associated energy functional E to the problem has a mountain pass structure and, therefore, it has a critical point u_λ, which is a weak solution. After that we manage to prove that this solution is positive by using new regularity results up to the boundary and a Hopf's Lemma.
For the sake of clarity in this thesis, we will cover the following topics.

In section 1 we define the Fractional Sobolev spaces $W^{s,p}$ and we investigate some of their basic properties. In section 2 we will give two different definitions of Fractional Sobolev Spaces. The first one is that, when $p = 2$ we will show that the space $W^{s,2}$ coincides with the Hilbert space H^s. The second one involves a family of intermediate spaces between L^p and $W^{1,p}$. More precisely, we will show that the Fractional Sobolev Space $W^{s,p}$ is an interpolation space between L^p and $W^{1,p}$. Section 3 contains the main contribution of this work which extends the result in [11] where the authors considered the problem for the p-Laplacian operator, $(2 \leqslant p < N)$. Finally in section 4 we have included an appendix that contains some technical results and we mention some classical definitions and theorems that we will use throughout this work, we will not prove the majority of statements there, so we will leave the references for the reader. In this thesis N will be a fixed natural number, C will denote a positive constant, not the same at each occurrence, and Ω will be an open set in $\mathbb{R}^N$.

Basic Notation

Ω^c or $\mathbb{R}^N \setminus \Omega$	complement of the set Ω in $\mathbb{R}^N$
Ω^2	cartesian product $\Omega \times \Omega$
X^*	dual space
$B(0,1)$	unit ball in $\mathbb{R}^N$
B_R	open ball of radious R
p'	conjugate exponent of p i.e. $p' = \frac{p}{p-1}$
$\lvert\Omega\rvert$	the Lebesgue measure of the set Ω
$\hat{u}$	Fourier transform
$\check{u}$	inverse Fourier transform
τ_z	translation operator $\tau_z u(x) \coloneqq u(x+z)$
$\mathrm{d}_\Omega(x)$	distance from x to Ω^c
supp u	support of the function u
a.e.	almost every where
$\rightharpoonup$	weak convergence
p^*	Sobolev critical exponent, $p^* = \frac{Np}{N-p}$
p_s^*	fractional Sobolev critical exponent, $p^* = \frac{Np}{N-sp}$

Contents

1 Fractional Sobolev Spaces

1.1 Motivation

Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $p \in [1,\infty)$ and $s \in (0,1)$. After the introduction of Fractional Sobolev spaces in the 1950's by Aronszajn [3], Gagliardo [23] and Slobodeckij [32], this spaces have found applications in a vast number of questions involving differential equations. These spaces arose in an attempt to fill the gaps between $L^p(\Omega), W^{1,p}(\Omega), W^{2,p}(\Omega), \dots$. The condition $s \in (0,1)$ is essential to avoid trivialities when $s \geq 1$ (see Remark 1.1). Another attempt to fill the gaps between the classical spaces is provided when $\Omega = \mathbb{R}^N$ by the spaces $H_p^s(\mathbb{R}^N)$ (see [19] for the definition of this space) defined via the Fourier transform. That is

$$H_p^s(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N) \text{ if } p \in (1,\infty) \text{ and } s \in \mathbb{N}$$

see [[19], Page 19]. But if $s \in (0,1)$, then

$$H^s(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N) \text{ if and only if } p = 2,$$

see Theorem 2.3. These facts suggest that the most natural extension of the classical Sobolev spaces involving an arbitrary smoothness parameter s is not $W^{s,p}(\mathbb{R}^N)$ but $H^s(\mathbb{R}^N)$. However, the most explicit definition of the norm on $W^{s,p}(\mathbb{R}^N)$ has advantages, notably in connection with the description of trace spaces see [[16], Page 184].

1.2 Definitions and Basic Properties

Definition 1.1. For $s \in (0,1)$ and $p \in [1,\infty)$ define

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

to be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (\|u\|_p^p + [u]_{s,p}^p)^{1/p}$$

where the term

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}$$

is called the Gagliardo seminorm of u.

Remark 1.1. If s is an integer then the Fractional space $W^{s,p}(\mathbb{R}^N)$ coincides with the Sobolev space $W^{k,p}(\mathbb{R}^N)$ up to equivalence of norms see [[1], page 253]. On the other hand, if s is not an integer and $s > 1$, then the space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) = \{u \in W^{[s],p}(\mathbb{R}^N) : D^\alpha u \in W^{s-[s],p}(\mathbb{R}^N) \ \forall \alpha \text{ with } |\alpha| = [s]\},$$

where $[s]$ is the integer part of s. This space endowed with the norm

$$\left(\|u\|_{[s],p}^p + \sum_{|\alpha|=[s]} \|D^\alpha u\|_{s-[s],p}^p \right)^{1/p}$$

is a Banach space.

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $p \in [1, \infty)$ and $s \in (0,1)$. Define

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega\}$$

which is a closed linear subspace of $W^{s,p}(\mathbb{R}^N)$ see [[28], page 2], and can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$.

Remark 1.2. The norms $\|\cdot\|_{s,p}$ and $[\cdot]_{s,p}$ are equivalent. Indeed, let C, C', B be positive constants. It is clear that $[u]_{s,p}^p \leq \|u\|_{s,p}^p$. On the other hand, from Fractional Sobolev embedding Theorem 1.3 and since $p < q+1$ for all $q \in [p, p_s^*]$, we have that

$$\begin{aligned}
\|u\|_{s,p} &= ([u]_{s,p}^p + \|u\|_p^p)^{1/p} \\
&\leq C([u]_{s,p} + \|u\|_p) \\
&\leq C[u]_{s,p} + C\|u\|_{q+1} \\
&\leq C[u]_{s,p} + C'[u]_{s,p} \\
&= B[u]_{s,p}.
\end{aligned}$$

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $p \in [1, \infty)$ and $s \in (0,1)$. The space $W^{s,p}(\Omega)$ with the norm $\|\cdot\|_{s,p}$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^{s,p}(\Omega)$, in particular $\{u_n\}$ is a Cauchy sequence in $L^p(\Omega)$. Since L^p is a Banach space, then $\{u_n\}$ converges to a function $u \in L^p(\Omega)$. Then, from Proposition 4.2 it admits a subsequence $\{u_{n_j}\}$ such that $u_{n_j}(x) \to u(x)$ a.e. in Ω. Moreover, the sequence of functions $v_{n_j}(x,y) := \frac{u_{n_j}(x) - u_{n_j}(y)}{|x-y|^{N/p+s}}$ is Cauchy in $L^p(\Omega^{2N})$, hence, it converges to certain $v(x,y)$ in $L^p(\Omega^{2N})$. Therefore, there exists a subsequence $\{v_{n_j}\}$ and a function $h \in L^p(\Omega)$ such that $v_{n_j}(x) \to v(x)$ a.e. in Ω and $|v_{n_j}(x)| \leq h(x)$ a.e. in Ω. By the Lebesgue's Dominated Convergence Theorem 4.5 we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_{n_j}(x) - u_{n_j}(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

This means that

$$\lim_{j\to\infty} [u_{n_j} - u]_{s,p}^p = 0.$$

Therefore,

$$\lim_{j\to\infty} ||u_{n_j} - u||_{s,p}^p = \lim_{j\to\infty} (||u_{n_j} - u||_p^p + [u_{n_j} - u]_{s,p}^p) = 0$$

Then we have shown that $u_n \to u$ in $W^{s,p}(\Omega)$. Finally, the fact that $u \in W^{s,p}(\Omega)$ is straightforward since

$$[u]_{s,p}^p = \int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}} dxdy = \int_{\Omega^2} |v(x,y)|^p dxdy < \infty.$$

□

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $p \in [1,\infty)$ and $s \in (0,1)$. Then*

1. *$W^{s,p}(\Omega)$ is reflexive.*
2. *$W^{s,p}(\Omega)$ is separable.*

Proof.

1. Let $J : W^{s,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega^2)$ be the function defined as

$$J(u) = \left(u, \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}} \right)$$

where $W^{s,p}(\Omega)$ is endowed with the norm $[\cdot]_{s,p}$ and $L^p(\Omega) \times L^p(\Omega^2)$ is endowed with the norm

$$\|(u,v)\| = \left(\int_\Omega |u|^p dx + \int_{\Omega^2} |v(x,y)|^p \mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

Then we obtain that

$$\begin{aligned} \|J(u)\| &= \left(\int_\Omega |u|^p dx + \int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{1/p}, \\ &= \|u\|_{s,p} \end{aligned}$$

which proves that J is an isometry. Since $W^{s,p}(\Omega)$ is a Banach space, $J(W^{s,p}(\Omega))$ is a closed subspace of $L^p(\Omega) \times L^p(\Omega^2)$. It follows that $J(W^{s,p}(\Omega))$ is reflexive. Consequently $W^{s,p}(\Omega)$ is also reflexive.

2. Similarly we consider J as in the previous case. We have that J is an isometry and $J(W^{s,p}(\Omega))$ is a closed subspace of $L^p(\Omega) \times L^p(\Omega^2)$, then the separability of $W^{s,p}(\Omega)$ follows from the separability of $L^p(\Omega) \times L^p(\Omega^2)$.

□

Proposition 1.2. *Let $p \in [1,\infty)$ and $s \in (0,1)$. If $\zeta \in C_0^\infty(\Omega)$ and $u \in W^{s,p}(\mathbb{R}^N)$ then $\zeta u \in W^{s,p}(\mathbb{R}^N)$.*

Proof. Since $u \in W^{s,p}(\mathbb{R}^N)$ and $\zeta \in C_0^\infty(\Omega)$ we have that $\zeta u \in L^p(\mathbb{R}^N)$. On the other hand, in order to prove that $[\zeta u]_{s,p}$ is finite, note that

$$\begin{aligned} [\zeta u]_{s,p} &= \int_{\mathbb{R}^{2N}} \frac{|\zeta(x)u(x)-\zeta(y)u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^{2N}} \frac{|\zeta(x)u(x)-\zeta(x)u(y)+\zeta(x)u(y)-\zeta(y)u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\leq 2^{p-1} \int_{\mathbb{R}^{2N}} \left(\frac{|\zeta(x)u(x)-\zeta(x)u(y)|^p}{|x-y|^{N+sp}} + \frac{|\zeta(x)u(y)-\zeta(y)u(y)|^p}{|x-y|^{N+sp}} \right) \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Let us consider the integrals of the above term separately. In the first integral, as the function ζ is bounded then,

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|\zeta(x)(u(x)-u(y))|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y &\leq C \int_{\mathbb{R}^{2N}} \frac{|(u(x)-u(y))|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &= C[u]^p_{s,p} < \infty.\end{aligned}$$

In the second integral, the Mean Value Theorem 4.1 and the Change of Variables Theorem 4.6 imply that for some $c \in [0,1]$,

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|\zeta(x)-\zeta(y)|^p|u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y &\leq \int_\Omega \int_{\Omega\cap|x-y|<1} \frac{|\nabla\zeta((1-c)x+cy)|^p|u(y)|^p}{|x-y|^{N+sp-p}} \mathrm{d}x\mathrm{d}y \\ &+ \int_\Omega \int_{\Omega\cap|x-y|\geq 1} \frac{|\nabla\zeta((1-c)x+cy)|^p|u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\leq \|\nabla\zeta\|^p_\infty \int_\Omega |u(y)|^p \left(\int_{|x-y|<1} \frac{1}{|x-y|^{N+sp-p}}\right. \\ &\left. + \int_{|x-y|\geq 1} \frac{1}{|x-y|^{N+sp}}\right) \mathrm{d}x\mathrm{d}y \\ &\leq C\|u\|^p_p \left(\int_{|z|<1} \frac{1}{|z|^{N+sp-p}} \mathrm{d}z + \int_{|z|\geq 1} \frac{1}{|z|^{N+sp}} \mathrm{d}z\right),\end{aligned}$$

where the last term is finite because $N+sp-p<N$ and $N+sp>N$, see Proposition 4.3. □

1.3 Fractional Sobolev Embeddings

This subsection is devoted to the embeddings of Fractional Sobolev spaces that will be useful later on. We give the relation between extensions and smooth domains and we will see the importance of the smoothness of Ω in some cases. For example in the forthcoming Proposition we have that in the limit case i.e., $s=1$, the space $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$ whenever the domain Ω is of class $C^{0,1}$.

Definition 1.3. Define the following sets

$$\begin{aligned}Q &= \{x=(x',x_N) \in \mathbb{R}^{N-1}\times\mathbb{R} : |x'|<1 \text{ and } |x_N|<1\} \\ Q_+ &= \{x=(x',x_N) \in \mathbb{R}^{N-1}\times\mathbb{R} : |x'|<1 \text{ and } 0<x_N<1\}, \\ Q_0 &= \{x\in Q : x_N=0\}.\end{aligned}$$

We say that Ω is of class $C^{k,\alpha}$ if for all $x\in\partial\Omega$ there is $r>0$ and a bijection $T : Q \to B(x,r)$ such that

$$T\in C^{k,\alpha}(\overline{Q}), \quad T^{-1}\in C^{k,\alpha}(\overline{B(x,r)}), \quad T(Q_+) = B(x,r)\cap\Omega, \quad \text{and}$$
$$T(Q_0) = B(x,r)\cap\partial\Omega.$$

Roughly speaking, a smooth domain is characterized by being locally the graph of a function of class $C^{k,\alpha}$.

Definition 1.4. We say that an open subset $\Omega \subseteq \mathbb{R}^N$ is an extension domain for $W^{s,p}$, if there exists a constant $C > 0$ which depends on N, s, p and Ω, such that for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $||\tilde{u}||_{s,p} \leq C||u||_{s,p}$.

The proof of the following Theorem can be found in [[16], page 92].

Theorem 1.2. *Any domain Ω of class $C^{k,\alpha}$ is an extension domain.*

Proposition 1.3. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset of class $C^{0,1}$, $p \in [1, \infty)$ and $s \in (0, 1)$. Then $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$ and consider

$$[u]^p_{s,p} = \left(\int_\Omega \int_{\Omega \cap |x-y|<1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} + \int_\Omega \int_{\Omega \cap |x-y|\geq 1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \right) \mathrm{d}x\mathrm{d}y. \tag{1.1}$$

In the first integral, taking into account the smoothness assumptions of the domain Ω, we have that for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$. This fact, together with the Change of Variables Theorem 4.6 and Jensen's inequality 4.2 imply that

$$\begin{aligned}
\int_\Omega \int_{\Omega \cap |x-y|<1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y &\leq \int_\Omega \int_{|z|<1} \frac{|u(z+y) - u(y)|^p}{|z|^{N+sp}} \mathrm{d}z\mathrm{d}y \\
&\leq \int_\Omega \int_{|z|<1} \frac{|\int_0^1 \nabla u(y+tz) \cdot z|^p \mathrm{d}t}{|z|^{N+sp}} \mathrm{d}z\mathrm{d}y \\
&= \int_{\mathbb{R}^N} \int_{|z|<1} \frac{|\int_0^1 \nabla \tilde{u}(y+tz)|^p}{|z|^{N+sp-p}} \mathrm{d}t\mathrm{d}z\mathrm{d}y \\
&= \int_{|z|<1} \int_0^1 \frac{\|\nabla \tilde{u}\|^p_p}{|z|^{N+sp-p}} \mathrm{d}t\mathrm{d}z \\
&= C\|\nabla \tilde{u}\|^p_p \int_{|z|<1} \frac{1}{|z|^{N+sp-p}} \mathrm{d}z \\
&\leq C'\|\nabla \tilde{u}\|^p_p = C'\|,
\end{aligned}$$

where in the last inequality we have used Propostion 4.3 since $N + sp - p < N$, and the fact that $\|\tilde{u}\|_{1,p} \leq C'\|u\|_{1,p}$.
On the other hand, notice that, using again the Change of Variables Theorem 4.6 we get:

$$\begin{aligned}
\int_\Omega \int_{|x-y|\geq 1} \frac{|u(x)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y &= \int_\Omega |u(x)|^p \int_{|x-y|\geq 1} \frac{1}{|x - y|^{N+sp}} \mathrm{d}y\mathrm{d}x \\
&= \|u\|^p_p \int_{|z|\geq 1} \frac{1}{|z|^{N+sp}} \mathrm{d}z < \infty,
\end{aligned}$$

where the last inequality holds since $N+sp>N$. Therefore using the previous estimation in the second integral of (1.1) we have that:

$$\int_\Omega\int_{\Omega\cap|x-y|\geq 1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \leq 2^{p-1}\left(\int_\Omega\int_{|x-y|\geq 1}\frac{|u(x)|^p+|u(y)|^p}{|x-y|^{N+sp}}\right)\mathrm{d}x\mathrm{d}y.$$
$$\leq C''\|u\|_p^p,$$

for some constant $C''>0$.
Finally we obtain that:

$$\|u\|_{s,p}^p=\|u\|_p^p+[u]_{s,p}^p\leq \tilde{C}\|u\|_{1,p}^p$$

for some constant $\tilde{C}>0$. □

Proposition 1.4. *Let $\Omega\subseteq\mathbb{R}^N$ be an open subset, $p\in[1,\infty)$ and $0<s'\leq s<1$. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s',p}(\Omega)$.*

Proof.
Let $u\in W^{s,p}(\Omega)$ and consider:

$$[u]_{s',p}^p=\left(\int_\Omega\int_{\Omega\cap|x-y|<1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}+\int_\Omega\int_{\Omega\cap|x-y|\geq 1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\right)\mathrm{d}x\mathrm{d}y.$$

Suppose that $|x-y|<1$. Since $s'\leq s$ then $\frac{1}{|x-y|^{N+s'p}}\leq\frac{1}{|x-y|^{N+sp}}$, thus the first integral is bounded by

$$\int_\Omega\int_{\Omega\cap|x-y|<1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y.$$

On the other hand, arguing as in Proposition 1.3 we have that:

$$\int_\Omega\int_{\Omega\cap|x-y|\geq 1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\mathrm{d}x\mathrm{d}y\leq C\|u\|_p^p.$$

Therefore,

$$\begin{aligned}[u]_{s',p}^p &\leq \int_\Omega\int_{\Omega\cap|x-y|<1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y+C\|u\|_p^p\\ &\leq \int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y+C\|u\|_p^p\\ &=[u]_{s,p}^p+C\|u\|_p^p,\end{aligned}$$

hence,

$$\|u\|_{s',p}^p=\|u\|_p^p+[u]_{s',p}^p\leq C'\|u\|_{s,p}^p$$

for some constant $C'>0$. □

Proposition 1.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded subset and $0 < \tilde{s} < s < 1$, $1 \leq \tilde{p} < p < \infty$. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{\tilde{s},\tilde{p}}(\Omega)$.*

Proof. For $u \in W^{s,p}(\Omega)$ we have

$$[u]^{\tilde{p}}_{\tilde{s},\tilde{p}} = \left(\int_\Omega \int_{\Omega \cap |x-y|<1} \frac{|u(x)-u(y)|^{\tilde{p}}}{|x-y|^{N+\tilde{s}\tilde{p}}} + \int_\Omega \int_{\Omega \cap |x-y|\geq 1} \frac{|u(x)-u(y)|^{\tilde{p}}}{|x-y|^{N+\tilde{s}\tilde{p}}} \right) \mathrm{d}x\mathrm{d}y.$$

In the above expression, the first term can be written as

$$\int_\Omega \int_{\Omega \cap |x-y|<1} \frac{|u(x)-u(y)|^{\tilde{p}}}{|x-y|^{N\tilde{p}/p+s\tilde{p}}} \frac{1}{|x-y|^q} \mathrm{d}x\mathrm{d}y,$$

where $q = N(p-\tilde{p})/p - \tilde{p}(s-\tilde{s})$. Thus by Hölder inequality with the exponents $p/\tilde{p}$ and $p/p-\tilde{p}$, it is bounded by

$$[u]^{\tilde{p}}_{s,p} \left(\int_{|x-y|<1} \left(\frac{1}{|x-y|^{qp/(p-\tilde{p})}} \right) \mathrm{d}x\mathrm{d}y \right)^{\frac{p-\tilde{p}}{p}}.$$

Applying the Change of Variables Theorem 4.6 and Proposition 4.3 since $N - \frac{\tilde{p}p(s-\tilde{s})}{p-\tilde{p}} < N$, we found that the last expression is bounded by $C[u]^{p'}_{s,p}$ for some constant $C > 0$. On the order hand the term

$$\int_\Omega \int_{\Omega \cap |x-y|\geq 1} \frac{|u(x)-u(y)|^{p'}}{|x-y|^{N+s'p'}} \mathrm{d}x\mathrm{d}y$$

is bounded by $C'\|u\|^{p'}_{p'}$ as in Proposition 1.3. Therefore we have:

$$\|u\|^{p'}_{s',p'} = \|u\|^{p'}_{p'} + [u]^{p'}_{s',p'} \leq \tilde{C}\|u\|^p_{s,p}$$

for some constant $\tilde{C} > 0$. □

Proposition 1.6. *Let $p \in [1,\infty)$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. For any $k \in \mathbb{N}$ and $x \in \mathbb{R}^N$ we define $u_M(x) = \max\{\min\{u(x), M\}, -M\}$, then*

$$\lim_{M\to\infty} \|u_M\|_p = \|u\|_p.$$

Proof. Notice that the function u_M can be interpreted as the function u cut on the lines M and $-M$. According to the definition of u_M we have that $\liminf_{M\to\infty} |u_M(x)| = |u(x)|$, using this fact and the Fatou's Lemma we obtain:

$$\liminf_{M\to\infty} \|u_M\|^p_p = \liminf_{M\to\infty} \int_{\mathbb{R}^N} |u_M(x)|^p \mathrm{d}x \geq \int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x = \|u\|^p_p.$$

□

We will state two lemmas that will be useful to prove Proposition 1.7, the proofs of these lemmas can be found in [[14], pages 551 and 552].

Lemma 1.1. *Let $p \in [1, \infty)$ and $s \in (0,1)$ such that $sp < N$. Fix $T > 1$; let $k \in \mathbb{Z}$ and let a_n be a bounded, nonnegative, decreasing sequence with $a_n = 0$ for any $n \geq k$. Then,*

$$\sum_{n \in \mathbb{Z}} a_n^{(N-sp)/N} T^n \leq C \sum_{n \in \mathbb{Z}} a_{n+1} a_n^{-sp/N} T^n,$$

for some constant $C > 0$, depending on N, p, s and T.

Lemma 1.2. *Let $p \in [1, \infty)$ and $s \in (0,1)$ such that $sp < N$. Let $u \in L^\infty(\mathbb{R}^N)$ be compactly supported. For any $n \in \mathbb{Z}$ let $a_n := |\{|u| > 2^n\}|$. Then,*

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \geq C \sum_{n \in \mathbb{Z}} a_{n+1} a_n^{-sp/N} 2^{pn},$$

for some constant $C > 0$ depending on N, p and s.

Proposition 1.7. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable and compactly supported function such that $u \in L^\infty(\mathbb{R}^N)$, then*

$$\|u\|_{p^*} \leq C[u]_{s,p}$$

for some constant $C > 0$ depending on N, s, p and Ω. Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$.

Proof. For any $n \in \mathbb{Z}$, define $A_n = \{|u| > 2^n\}$. Thus

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(x)|^{p^*} \mathrm{d}x &= \sum_{n \in \mathbb{Z}} \int_{A_n \setminus A_{n+1}} |u(x)|^{p^*} \mathrm{d}x \\
&\leq \sum_{n \in \mathbb{Z}} \int_{A_n \setminus A_{n+1}} |2^{n+1}|^{p^*} \mathrm{d}x \\
&\leq \sum_{n \in \mathbb{Z}} 2^{(n+1)p^*} |A_n|.
\end{aligned}$$

The last estimation and Lemma 1.1 imply that:

$$\|u\|_{p^*}^p \leq 2^p \sum_{n \in \mathbb{Z}} 2^{np} |A_n|^{(N-sp)/N} \leq C \sum_{n \in \mathbb{Z}} 2^{np} |A_{n+1}||A_n|^{-sp/N}.$$

for some constant $C > 0$.
Finally the inequality

$$C \sum_{n \in \mathbb{Z}} 2^{np} |A_{n+1}||A_n|^{-sp/N} \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y$$

that follows from Lemma 1.2, gives us the desired result. □

Theorem 1.3. *Let $p \in [1, \infty)$ and $s \in (0,1)$. Then $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^{p_s^*}(\mathbb{R}^N)$.*

Proof. Suppose that

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y < \infty, \tag{1.2}$$

otherwise we have nothing to prove. In particular (1.2) also holds for the functions u_M defined in Proposition 1.6. We claim that for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\frac{|u_M(x)-u_M(y)|}{|x-y|^{N/p+s}} \leq \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}}.$$

Indeed, let us define $A := \{x \in \mathbb{R}^N : u(x) \geq M\}$ and suppose that $u(x) \geq 0$ for all $x \in \mathbb{R}^N$.

- If $(x,y) \in A \times A$, then $u_M(x) = u_M(y) = M$ and consequently

$$\frac{|u_M(x)-u_M(y)|}{|x-y|^{N/p+s}} = 0.$$

- If $(x,y) \notin A \times A$, we have the following three cases:

  - $x \in A$ and $y \notin A$ imply that $u_M(x) = M \leq u(x)$ and $u_M(y) = u(y) \leq M$, therefore,

$$\frac{|u_M(x)-u_M(y)|}{|x-y|^{N/p+s}} = \frac{|M-u(y)|}{|x-y|^{N/p+s}} = \frac{M-u(y)}{|x-y|^{N/p+s}} \leq \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}}.$$

  - $x \notin A$ and $y \in A$ is similar to the previous case.
  - $x \notin A$ and $y \notin A$ imply that $u_M(x) = u(x)$ and $u_M(y) = u(y)$, therefore,

$$\frac{|u_M(x)-u_M(y)|}{|x-y|^{N/p+s}} = \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}}.$$

Arguing similarly we obtain the same result for $u(x) \leq 0$ for all $x \in \mathbb{R}^N$. Then by the Lebesgue's Dominated Convergence Theorem 4.5 we have

$$\lim_{M\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_M(x)-u_M(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y,$$

which means that

$$\lim_{M\to\infty} [u_M]^p_{s,p} = [u]^p_{s,p}. \tag{1.3}$$

Since $u_M \in L^\infty(\mathbb{R}^N)$, the Proposition 1.7 implies that:

$$||u_M||^p_{p_s^*} \leq C[u_M]^p_{s,p}.$$

Taking the limit in the above expression when $M \to \infty$, we get from Proposition 1.6 and equation (1.3) that

$$||u||^p_{p_s^*} \leq C[u]^p_{s,p}.$$

□

Remark 1.3. As a consequence of the previous Theorem we have that $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p_s^*]$. Indeed, using the reverse Hölder inequality see [[10], Page 137], we get:

$$\|u\|_{p_s^*}^{p_s^*} = \int_{\mathbb{R}^N} |u(x)|^{p_s^*} \mathrm{d}x \geq \left(\int_{\mathbb{R}^N} |u(x)|^{\frac{p_s^* q}{p_s^*}} \mathrm{d}x \right)^{\frac{p_s^*}{q}} \left(\int_{\mathbb{R}^N} |1|^{\frac{q}{q-p_s^*}} \mathrm{d}x \right)^{\frac{q-p_s^*}{q}} = C\|u\|_q^{p_s^*}$$

for some constant $C > 0$. This implies that $C\|u\|_q \leq \|u\|_{p_s^*}$ and therefore $\|u\|_q \leq C[u]_{s,p}$.

In order to study deeper properties of Fractional Sobolev spaces we need procedures for approximating a function in a Fractional Sobolev space by smooth functions. These approximation procedures allow us to consider smooth functions and then extend the statements to functions in the Fractional Sobolev spaces by density arguments. We are going to prove that smooth functions are in fact dense in $W^{s,p}(\mathbb{R}^N)$. The method of mollifiers provides the tool.

Definition 1.5. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be given by

$$\eta(x) = \begin{cases} \frac{Ce}{|x|^2-1} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with the constant $C > 0$ chosen such that $\int_{\mathbb{R}^N} \eta(x)\mathrm{d}x = 1$. For each $\delta > 0$ we define

$$\eta_\delta(x) = \frac{1}{\delta^N} \eta\left(\frac{x}{\delta}\right).$$

We call η the standard mollifier and η_δ the rescaling function.

Remark 1.4. Notice that:

- $\eta \in C_c^\infty(\mathbb{R}^N)$.
- $\eta_\delta \in C^\infty(\mathbb{R}^N)$ and satisfies:

$$\int_{\mathbb{R}^N} \eta_\delta(x)\mathrm{d}x = 1 \quad \text{and} \quad \operatorname{supp}(\eta_\delta) \subseteq B(0,\delta).$$

Definition 1.6. If $\Omega \subseteq \mathbb{R}^N$ is an open subset with $\partial\Omega \neq \emptyset$, we define

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

For each $u \in L^1_{\mathrm{loc}}(\Omega)$, its standard convolution mollification $u_\delta : \Omega_\delta \to \mathbb{R}$ is given by

$$u_\delta(x) = (u * \eta_\delta)(x) = \int_{\Omega_\delta} u(y)\eta_\delta(x-y)\mathrm{d}y.$$

Proposition 1.8. *Let $u \in W^{s,p}(\mathbb{R}^N)$. Given a sequence of rescaling functions $\{\eta_\delta\}_{\delta\in\mathbb{N}}$ we have $\|\eta_\delta u - u\|_{s,p} \to 0$ as $\delta \to \infty$.*

Proof. Let $u \in W^{s,p}(\mathbb{R}^N)$. Notice that the fact that

$$\|\eta_\delta u - u\|_p \to 0 \tag{1.4}$$

as $\delta \to \infty$ is straightforward, since $\eta_\delta u(x) \to u(x)$ pointwise for all $x \in \mathbb{R}^N$ and $|\eta_\delta u - u| \leq |u|$. Applying the Lebesgue's Dominated Convergence (Theorem 4.5) we obtain the desired result.
On the other hand, let us consider:

$$\begin{aligned}
[\eta_\delta u - u]_{s,p} = & \int_\Omega \int_{\Omega \cap |x-y|<1} \frac{|(\eta_\delta u - u)(x) - (\eta_\delta u - u)(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
& + \int_\Omega \int_{\Omega \cap |x-y|\geq 1} \frac{|(\eta_\delta u - u)(x) - (\eta_\delta u - u)(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{1.5}$$

The first term of (1.5) can be written as

$$\int_\Omega \int_{\Omega \cap |x-y|<1} \left(\frac{|1-\eta_\delta(y)|^p |u(x)-u(y)|^p}{|x-y|^{N+sp}} + \frac{|u(x)|^p |\eta_\delta(x) - \eta_\delta(y)|^p}{|x-y|^{N+sp}} \right) \mathrm{d}x\mathrm{d}y$$

where both expressions are bounded by

$$\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^N)$$

and

$$\frac{|u(x)|^p}{|x-y|^{N+p(s-1)}} \in L^1(\mathbb{R}^N)$$

respectively. Thus by the Lebesgue's Dominated Convergence we get the convergence of the first term of (1.5) when $\delta \to \infty$. Finally, using the idea of Proposition 1.3 we can estimate the second term of (1.5) by

$$\|\eta_\delta u - u\|_p^p$$

which goes to zero when $\delta \to \infty$.

□

Proposition 1.9. *Let $p \in [1, \infty)$, $s \in (0,1)$, $u \in W^{s,p}(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^N)$. Then $f * u \in W^{s,p}(\mathbb{R}^N)$ and $\|f * u\|_{s,p} \leq \|f\|_1 \|u\|_{s,p}$.*

Proof. The fact that $f * u \in L^p(\mathbb{R}^N)$ and $\|f * u\|_p \leq \|f\|_1 \|u\|_p$ follows from Theorem 4.15 in [[7], page 104]. On the other hand, using the Change of Variables

Theorem we get

$$
\begin{aligned}
[f*u]_{s,p}^p &= \int_{\mathbb{R}^{2N}} \frac{|f*u(x)-f*u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^{2N}} \frac{|f*u(z+y)-f*u(y)|^p}{|z|^{N+sp}}\mathrm{d}z\mathrm{d}y \\
&= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} |f*(u(z+y)-u(y))|^p \mathrm{d}y\mathrm{d}z \\
&= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} |f*(\tau_z u(y)-u(y))|^p \mathrm{d}y\mathrm{d}z \\
&= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \|f*(\tau_z u-u)\|_p^p \mathrm{d}z,
\end{aligned}
$$

where τ_z denotes the translation operator. Notice that the norm in the last term is bounded by $\|f\|_1\|u\|_p$, thus

$$
\begin{aligned}
[f*u]_{s,p}^p &\leq \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \|f\|_1^p \|\tau_z u-u\|_p^p \mathrm{d}z \\
&= \|f\|_1^p \int_{\mathbb{R}^{2N}} \frac{|u(y+z)-u(y)|^p}{|z|^{N+sp}}\mathrm{d}y\mathrm{d}z \\
&= \|f\|_1^p [u]_{s,p}^p.
\end{aligned}
$$

□

Proposition 1.10. *Let $p \in [1,\infty)$ and $s \in (0,1)$. Given a mollifier η and $u \in W^{s,p}(\mathbb{R}^N)$ we have that $\|\eta_\delta * u - u\|_{s,p} \to 0$ as $\delta \to \infty$.*

Proof. The convergence in $L^p(\mathbb{R}^N)$ holds, see Proposition 4.22 in [[7], page 109]. On the other hand, using the Change of Variables Theorem:

$$
\begin{aligned}
[\eta_\delta u - u]_{s,p}^p &= \int_{\mathbb{R}^{2N}} \frac{|\eta_\delta * u(x) - u(x) - \eta_\delta * u(y) + u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^{2N}} \frac{|\eta_\delta * u(z+y) - u(z+y) - \eta_\delta * u(y) + u(y)|^p}{|z|^{N+sp}}\mathrm{d}z\mathrm{d}y.
\end{aligned}
\tag{1.6}
$$

Notice that

$$
\begin{aligned}
\eta_\delta * u(z+y) - u(z+y) &= \int_{\mathbb{R}^N} \eta_\delta(r)[u(y+z-r)-u(z+y)]\mathrm{d}r \\
&= \int_{\mathbb{R}^N} \eta_\delta(r)[\tau_z u(y-r) - \tau_z u(y)]\mathrm{d}r
\end{aligned}
$$

and

$$
\eta_\delta * u(y) - u(y) = \int_{\mathbb{R}^N} \eta_\delta(r)[u(y-r)-u(y)]\mathrm{d}r,
$$

where τ_z is the translation operator. Therefore from Minkowski's integral inequality, (1.6) is bounded by:

$$\left(\int_{\mathbb{R}^N} \eta_\delta(r) \left(\int_{\mathbb{R}^{2N}} \frac{|\tau_z u(y-r) - u(y-r) - (\tau_z u(y) - u(y))|^p}{|z|^{N+sp}} \mathrm{d}y\mathrm{d}z\right)^{1/p} \mathrm{d}r\right)^p$$
$$= \left(\int_{\mathbb{R}^N} \eta_\delta(r) \left(\int_{\mathbb{R}^{2N}} \left|\frac{\tau_r u(x) - \tau_r u(y)}{|x-y|^{N/p+s}} - \frac{u(x)-u(y)}{|x-y|^{N/p+s}}\right|^p \mathrm{d}y\mathrm{d}x\right)^{1/p} \mathrm{d}r\right)^p$$
$$= \left(\int_{\mathbb{R}^N} \eta_\delta(r) \left\|\tau_r v - v\right\|_p \mathrm{d}r\right)^p$$

where $v : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is defined in Theorem 1.1. From Corollary in [[26], Page 245] we have that $\|\tau_r v - v\|_p \to 0$ when $r \to 0$, i.e. for all $\epsilon > 0$ there exists $\sigma > 0$ such that $\|\tau_r v - v\|_p < \epsilon$ for $r > \sigma$. Therefore all the previous inequalities imply that,

$$[\eta_\delta * u - u]_{s,p} < \int_{\mathbb{R}^N} \eta_\delta \epsilon = \epsilon.$$

□

Theorem 1.4. *Let $p \in [1, \infty)$ and $s \in (0,1)$. The space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$.*

Proof. Let $u \in W^{s,p}(\mathbb{R}^N)$, for each $\delta \in \mathbb{N}$ consider $K_\delta = \overline{B(0,\delta)}$ and define the characteristic function

$$\chi_{K_\delta}(x) = \begin{cases} 1 & \text{if } x \in K_\delta \\ 0 & \text{if } x \notin K_\delta \end{cases}.$$

Let η_δ be a mollification. Thus $\eta_\delta * \chi_{K_\delta} u \in C_0(\mathbb{R}^N)$, see [Brezis, Remark 10, Page 106]. Moreover $\eta_\delta * \chi_{K_\delta} u \in C_0^\infty(\mathbb{R}^N)$ see [Brezis, Proposition 4.20]. From Proposition 1.9 we obtain that

$$\begin{aligned} \|\eta_\delta * \chi_{K_\delta} u - u\|_{s,p} &= \|\eta_\delta * \chi_{K_\delta} u - \eta_\delta * u + \eta_\delta * u - u\|_{s,p} \\ &\le \|\eta_\delta * \chi_{K_\delta} u - \eta_\delta * u\|_{s,p} + \|\eta_\delta * u - u\|_{s,p} \\ &= \|\eta_\delta * (\chi_{K_\delta} u - u)\|_{s,p} + \|\eta_\delta * u - u\|_{s,p} \\ &\le \|\eta_\delta\|_1 \|\chi_{K_\delta} u - u\|_{s,p} + \|\eta_\delta * u - u\|_{s,p}. \end{aligned}$$

The result follows using the Propositions 1.8 and 1.10 when $\delta \to \infty$ in the last inequality. □

Remark 1.5. As we saw in the previous theorem any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ can be approximated by a sequence of smooth functions with compact support. However, if Ω is an open set of $\mathbb{R}^N$, the space $C_0^\infty(\Omega)$ is not necessarily dense in $W_0^{s,p}(\Omega)$. We can see this in the following example.

Example 1.1. Let $N = 1$, $p = 2$, $\Omega = (-1,0)\cup(0,1)$ and $s \in (1/2, 1)$. Consider the smooth fixed function $\psi : \mathbb{R} \to \mathbb{R}$ in $W^{s,2}(\Omega)$ with compact support in $(-1,1)$ such that $\psi(0) = 1$ and define

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

For any $s \in (0,1)$ we have that

$$\|\phi\|_{s,2} = \|\psi\|_{s,2},$$

which implies that $\|\phi\|_{s,2}$ is finite. By definition, ϕ also vanishes outside of Ω and therefore $\phi \in W_0^{s,2}(\Omega)$. Now consider a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ with compact support in Ω. Since $0 \notin \text{supp}(\Omega)$ we see that $\rho(0) = 0$. As in the proof of Theorem 8.2 in [[14], page 562] if a function $u \in L^2(\Omega)$ then

$$\|u\|_\infty \leq \|u\|_\infty + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^s} \leq C\|u\|_{s,2}.$$

Hence

$$1 = \lim_{\substack{x\to 0 \\ x\in\Omega}} (\phi - \rho)(x) \leq \|\phi - \rho\|_\infty \leq C\|\phi - \rho\|_{s,2}.$$

In order to establish a similar result to the one known, we have to deal with it when $\partial\Omega$ is a graph of a continuous function.

Definition 1.7. An open set $\Omega \subseteq \mathbb{R}^N$ is an hypograph if there exists a continuous function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, up to a rigid motion,

$$\Omega = \{(x', x_n) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_n < f(x')\}.$$

Thus the density result can be stated in this way:

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a hypograph. Then $C_0^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$.*

Thee proof of this Theorem can be found in [[22], page 4].

To end this section we have a brief summary of how the spaces that we studied are related. For $p \in [1,\infty)$ and $s \in (0,1)$ we have the following relations depending on the properties of the domain.

1. For $\mathbb{R}^N$:
   - $\overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{s,p}} = W^{s,p}(\mathbb{R}^N)$.
   - $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*]$.
2. For an open set Ω of $\mathbb{R}^N$:
   - $W^{s,p}(\Omega) \hookrightarrow W^{\tilde{s},\tilde{p}}(\Omega)$ for $\tilde{s} < s$ and $\tilde{p} < p$.
   - $W^{s,p}(\Omega) \hookrightarrow W^{\tilde{s},p}(\Omega)$ for $\tilde{s} < s$ and $p > 1$.
   - $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ for Ω being a domain of class $C^{0,1}$.
   - $\overline{C_0^\infty(\Omega)}^{\|\cdot\|} = W_0^{s,p}(\Omega)$ for Ω being a hypograph.

2 Equivalent Definitions of The Fractional Sobolev Spaces

We will talk about the relation between the Fracional Sobolev Spaces and the Fourier Transform. We will show an interesting result involving the Hilbert space H^s. More precisely we will have that when $p = 2$, $W^{s,2}$ is a Hilbert space. This fact is a consequence of the equality of norms between the spaces H^s and $W^{s,2}$

2.1 An Approach Via the Fourier Transform

From now on we will use the multi-index notation: If $\alpha = (\alpha, \dots, \alpha_N)$ and $\beta = (\beta_1, \dots, \beta_N)$, where for al $i = 1, \dots N$, α_i and β_i are integers, then

$$x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N},$$

if $x = (x_1, \dots, x_N)$ and (for derivatives)

$$D^\beta u := \partial_1^{\beta_1} \cdots \partial_N^{\beta_N} u.$$

Also, $\alpha! := \alpha_1! \cdots \alpha_N!$, $|\alpha| := \alpha_1 + \cdots + \alpha_N$,

$$\binom{\beta}{\alpha} := \frac{\beta!}{\alpha!(\beta - \alpha)!}$$

and $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$

Definition 2.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is a topological vector space of functions $u : \mathbb{R}^N \to \mathbb{C}$ such that $u \in C^\infty(\mathbb{R}^N)$ and for all multi-indices α and β and, there is a constant $C > 0$ depending on N, α and β that satisfies $|x^\alpha D^\beta u(x)| \leq C$ for all $x \in \mathbb{R}^N$. The space $\mathcal{S}(\mathbb{R}^N)$ has a natural topology generated by the following countable family of seminorms:

$$\|u\|_{\alpha,\beta} := \|x^\alpha D^\beta u\|_\infty.$$

Indeed, we say that a sequence (u_n) converges to u in $\mathcal{S}(\mathbb{R}^N)$ provided that $\|u_n - u\| \to 0$ as $n \to \infty$, for all α, β. This definition gives us the notion of closed set and thus we have a topology on $\mathcal{S}(\mathbb{R}^N)$.

Remark 2.1. Notice that the condition $|x^\alpha D^\beta u(x)| \leq C$ is equivalent to the fact that $x^\alpha D^\beta u \in L^\infty(\mathbb{R}^N)$. Moreover this implies that for all α, β

$$\lim_{|x| \to \infty} x^\alpha D^\beta u(x) = 0. \tag{2.1}$$

Theorem 2.1. *The space $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $\psi \in C_c^\infty(\mathbb{R}^N)$ with $0 \le \psi \le 1$ and $\psi = 1$ in the ball $B(0,1)$. If we define the sequence $\varphi_n(x) = \psi\left(\frac{x}{n}\right)\varphi(x)$ then $\varphi_n \in C_c^\infty(\mathbb{R}^N)$. Using the Leibniz's rule we have

$$\begin{aligned} D^\beta \varphi_n(x) &= D^\beta\left(\psi\left(\frac{x}{n}\right)\varphi(x)\right) \\ &= \psi\left(\frac{x}{n}\right)D^\beta\varphi(x) + \sum_{0\neq\gamma\le\beta}\binom{\beta}{\gamma}\frac{1}{n^{|\gamma|}}D^\gamma\psi\left(\frac{x}{n}\right)D^{\beta-\gamma}\varphi(x). \end{aligned}$$

Observe that there exists a constant C such that for all multi-index γ and all $x \in \mathbb{R}^N$, $D^\gamma\varphi(\frac{x}{n}) \le C$. It follows that

$$\begin{aligned} &\sup_{x\in\mathbb{R}^N}|x^\alpha D^\beta\varphi_n(x) - x^\alpha D^\beta\varphi(x)| \\ &= \sup_{x\in\mathbb{R}^N}|x^\alpha|\left|\psi\left(\frac{x}{n}\right)D^\beta\varphi(x) + \sum_{0\neq\gamma\le\beta}\binom{\beta}{\gamma}\frac{1}{n^{|\gamma|}}D^\gamma\psi\left(\frac{x}{n}\right)D^{\beta-\gamma}\varphi(x) - D^\beta\varphi(x)\right| \\ &= \sup_{x\in\mathbb{R}^N}|x^\alpha|\left|\left(\psi\left(\frac{x}{n}\right)-1\right)D^\beta\varphi(x) + \sum_{0\neq\gamma\le\beta}\binom{\beta}{\gamma}\frac{1}{n^{|\gamma|}}D^\gamma\psi\left(\frac{x}{n}\right)D^{\beta-\gamma}\varphi(x)\right| \\ &\le \sup_{x\in\mathbb{R}^N} C\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{1}{n^{|\gamma|}}\left|x^\alpha D^{\beta-\gamma}\varphi(x)\right|, \end{aligned}$$

which gives the desired result, since each term $\left|x^\alpha D^{\beta-\gamma}\varphi(x)\right|$ is bounded by a constant independent of n and therefore it goes to zero as $n \to \infty$. □

Definition 2.2. Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz space. Then for any $u \in \mathcal{S}(\mathbb{R}^N)$ we define its Fourier transform $\mathcal{F}u = \hat{u}$ by

$$\hat{u}(y) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-iy\cdot x}u(x)\mathrm{d}x,$$

and its inverse Fourier transform $\mathcal{F}^{-1}u = \check{u}$ by

$$\check{u}(y) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{iy\cdot x}u(x)\mathrm{d}x.$$

The Theorems 2.1 and 4.4 and the inclusion of $\mathcal{S}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ imply the density of $S(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$. This allow us to define the Fourier transform of $L^p(\mathbb{R}^N)$ functions.

The proof of the following Proposition can be found in [[20], page 189].

Proposition 2.1. *Let $u \in L^2(\mathbb{R}^N)$.*

1. *Let $v \in L^2(\mathbb{R}^N)$. Then $\int_{\mathbb{R}^N} u\overline{v}\mathrm{d}x = \int_{\mathbb{R}^N}\hat{u}\overline{\hat{v}}\mathrm{d}y$.*
2. *Let $D^\alpha u \in L^2(\mathbb{R}^N)$ for some multi index α. Then $\widehat{D^\alpha u}(y) = (iy)^{|\alpha|}\hat{u}(y)$.*

3. *$u = \check{\hat{u}}$.*

4. *For fixed $z \in \mathbb{R}^N$, $\hat{u}(x+z) = e^{iz\cdot x}\hat{u}$.*

Definition 2.3. Let $s \geq 1$. We define the space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (1+|y|^{2s})\hat{u}(y) \in L^2(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|(1+|\cdot|^{2s})\hat{u}\|_2.$$

Theorem 2.2. *The space $H^s(\mathbb{R}^N)$ endowed with the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $H^s(\mathbb{R}^N)$. Then for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that for all $m, n > n_\varepsilon$,

$$\|u_m - u_n\|_{H^s(\mathbb{R}^N)} = \|(1+|y|^{2s})\hat{u}_n - (1+|y|^{2s})\hat{u}_m\|_2 < \varepsilon.$$

This implies that $\{(1+|y|^{2s})\hat{u}_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, thus there exists $u \in L^2(\mathbb{R}^N)$ such that

$$\|(1+|y|^{2s})\hat{u}_n - u\|_2 \to 0 \text{ as } n \to \infty. \tag{2.2}$$

We point out that, since $s \geq 1$ the function defined by $f = \frac{1}{1+|y|^{2s}}$ is bounded in $\mathbb{R}^N$ then $g = fu \in L^2(\mathbb{R}^N)$. Hence

$$\begin{aligned}
\|u_n - \check{g}\|_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |(1+|y|^{2s})\hat{u}_n(y) - (1+|y|^{2s})fu(y)|^2 \mathrm{d}y \\
&= \int_{\mathbb{R}^N} \left|(1+|y|^{2s})\hat{u}_n(y) - u(y)\right|^2 \mathrm{d}y \\
&= \|(1+|y|^{2s})\hat{u}_n - u\|_2.
\end{aligned}$$

From (2.2) the last term goes to zero when $n \to \infty$. □

Proposition 2.2. *The space $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$.*

Proof. Let $v \in H^s(\mathbb{R}^N)$. From the density of $C_0^\infty(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ there is $\psi_n \in C_0^\infty(\mathbb{R}^N)$ such that $\psi_n \to (1+|y|^{2s})\hat{v}$ in $L^2(\mathbb{R}^N)$ when $n \to \infty$. Since $s \geq 1$ the function $f = \frac{1}{1+|y|^{2s}}$ is bounded in $\mathbb{R}^N$, therefore $\varphi_n := \frac{1}{1+|y|^{2s}}\psi_n \in C_0^\infty(\mathbb{R}^N)$. Hence,

$$\begin{aligned}
\|\check{\varphi}_n - v\|_{H^s(\mathbb{R}^N)} &= \|(1+|y|^s)\hat{\check{\varphi}}_n - (1+|y|^{2s})\hat{v}\|_2 \\
&= \left(\int_{\mathbb{R}^N} \left|(1+|y|^{2s})\frac{\psi_n}{1+|y|^{2s}} - (1+|y|^{2s})\hat{v}\right|^2 \mathrm{d}y\right)^{1/2} \\
&= \|\psi_n - (1+|y|^{2s})\hat{v}\|_2.
\end{aligned}$$

Since the last term goes to 0 when $n \to \infty$, we obtain the result. □

The following proposition will allow us to show that the space $H^s(\mathbb{R}^N)$ coincides with the space $W^{s,2}(\mathbb{R}^N)$ in the fractional sense; i.e. when $s \in (0,1)$.

Theorem 2.3. *Let $s \in (0,1)$. The space $H^s(\mathbb{R}^N)$ coincides with $W^{s,2}(\mathbb{R}^N)$.*

Proof. Let $u \in H^s(\mathbb{R}^N)$. Using the Change of Variables Theorem and the Plancherel's Theorem we have:

$$\begin{aligned}
[u]_{s,2}^2 &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \frac{u(z+y)-u(y)}{|z|^{N/2+s}} \right|^2 \mathrm{d}y \right) \mathrm{d}z \\
&= \int_{\mathbb{R}^N} \left\| \frac{u(z+\cdot)-u}{|z|^{N/2+s}} \right\|_2^2 \mathrm{d}z \\
&= \int_{\mathbb{R}^N} \left\| \frac{\hat{u}(z+\cdot)-\hat{u}}{|z|^{N/2+s}} \right\|_2^2 \mathrm{d}z.
\end{aligned}$$

From Proposition 2.1, for a fixed z we obtain that

$$\begin{aligned}
\hat{u}(z+y)-\hat{u}(y) &= e^{iz\cdot y}\hat{u}(y)-\hat{u}(y) \\
&= (e^{iz\cdot y}-1)\hat{u}(y).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^N} \left\| \frac{\hat{u}(z+\cdot)-\hat{u}(\cdot)}{|z|^{N/2+s}} \right\|_2^2 \mathrm{d}z &= \int_{\mathbb{R}^{2N}} \frac{|e^{iz\cdot y}-1|^2|\hat{u}(y)|^2}{|z|^{N+2s}} \mathrm{d}y\mathrm{d}z \\
&= \int_{\mathbb{R}^{2N}} \frac{|\cos(z\cdot y)+i\sin(z\cdot y)-1|^2|\hat{u}(y)|^2}{|z|^{N+2s}} \mathrm{d}y\mathrm{d}z \\
&= 2\int_{\mathbb{R}^{2N}} \frac{(1-\cos(z\cdot y))|\hat{u}(y)|^2}{|z|^{N+2s}} \mathrm{d}y\mathrm{d}z.
\end{aligned}$$

We claim that $\int \frac{1-\cos(z\cdot y)}{|z|^{N+2s}} \mathrm{d}z = C|y|^{2s}$ for some $C>0$. Indeed, if $y=(y_1,y_2,\dots,y_N) \in \mathbb{R}^N$ then

$$\frac{1-\cos(y_1)}{2} \le \frac{y_1^2}{4} < \frac{y_1^2}{2} < |y_1|^2.$$

Thus

$$\int_{\mathbb{R}^N} \frac{1-\cos(y_1)}{|y|^{N+2s}} \mathrm{d}z < \int_{\mathbb{R}^N} \frac{1}{|y|^{N+2s}} \mathrm{d}y < \infty.$$

Let us consider the function $\rho : \mathbb{R}^N \to \mathbb{R}$ defined as

$$\rho(y) = \int_{\mathbb{R}^N} \frac{1-\cos(z\cdot y)}{|z|^{N+2s}} \mathrm{d}z.$$

We will show that $\rho(y) = \rho(|y|e_1)$, where $e_1 = (1, 0, \cdots, 0)$. Consider the rotation R for which $R(|y|e_1) = y$. Thus

$$\rho(y) = \int_{\mathbb{R}^N} \frac{1 - \cos(R(|y|e_1) \cdot z)}{|z|^{N+2s}} \mathrm{d}z = \int_{\mathbb{R}^N} \frac{1 - \cos((|y|e_1) \cdot (R^T z))}{|z|^{N+2s}} \mathrm{d}z. \tag{2.3}$$

Here we used the fact that the dot product is invariant under rotations, i.e.

$$R(|y|e_1) \cdot z = R^T R(|y|e_1) \cdot R^T z = |y|e_1 \cdot R^T z$$

and $R^T R = I$. Making the substitution $\tilde{z} = R^T z$ in (2.3) we get that $\rho(y) = \rho(|y|e_1)$. Therefore the substitution $y = |\xi|z$ gives us:

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{1 - \cos(\xi \cdot z)}{|z|^{N+2s}} \mathrm{d}z = \rho(\xi) = \rho(|\xi|e_1) &= \int_{\mathbb{R}^N} \frac{1 - \cos(|\xi| \cdot z)}{|z|^{N+2s}} \mathrm{d}z \\
&= \int_{\mathbb{R}^N} \frac{1 - \cos(y)}{|y|^{N+2s}} |\xi|^{2s} \mathrm{d}y \\
&= C|\xi|^{2s}.
\end{aligned}$$

Summarizing, we have that:

$$\begin{aligned}
[u]_{s,2}^2 = 2 \int_{\mathbb{R}^{2N}} \frac{(1 - \cos(z \cdot y))|\hat{u}(y)|^2}{|z|^{N+2s}} \mathrm{d}y\mathrm{d}z &= C \int_{\mathbb{R}^N} |y|^{2s} |\hat{u}(y)|^2 \mathrm{d}y \\
&\leq C \int_{\mathbb{R}^N} (1 + |y|^{2s})^2 |\hat{u}(y)|^2 \mathrm{d}y < \infty.
\end{aligned}$$

On the other hand, suppose that $u \in W^{s,2}(\mathbb{R}^N)$. Then

$$\begin{aligned}
\|u\|_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} (1 + |y|^{2s})^2 |\hat{u}(y)|^2 \mathrm{d}y \\
&= \int_{\mathbb{R}^N} \left(|\hat{u}(y)|^2 + 2|y|^{2s} |\hat{u}(y)|^2 + |y|^{4s} |\hat{u}(y)|^2 \right) \mathrm{d}y.
\end{aligned}$$

Using the Plancherel's Theorem we have that

$$\int_{\mathbb{R}^N} |\hat{u}(y)|^2 \mathrm{d}y = \int_{\mathbb{R}^N} |u(y)|^2 \mathrm{d}y < \infty$$

and

$$\int_{\mathbb{R}^N} 2|y|^{2s} |\hat{u}(y)|^2 = C[u]_{s,2}^2 < \infty.$$

Thus $\|u\|_{H^s(\mathbb{R}^N)} < \infty$. □

2.2 An Approach Via the Interpolation Spaces

In this subsection we present an equivalent definition for the Fractional Sobolev Spaces that involves a family of intermediate spaces between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$. More precisely, the Fractional Sobolev Space $W^{s,p}(\mathbb{R}^N)$ is an interpolation space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$.

Definition 2.4. Let X and Y be Banach spaces. The couple (X, Y) is said to be an interpolation couple if both X and Y are continuously embedded in a Hausdorff topological vector space V.

Notice that $X \cap Y$ and $X + Y$ are vector subspaces of V and Banach spaces endowed with the norms defined by

$$\|u\|_{X\cap Y} = \max\{\|u\|_X + \|u\|_Y\}$$

and

$$\|u\|_{X+Y} = \inf_{x\in X, y\in Y, x+y=u} (\|x\|_X + \|y\|_Y)$$

respectively.

Definition 2.5. If (X, Y) is an interpolation couple, an intermediate space is any Banach space B such that

$$X \cap Y \subset B \subset X + Y.$$

An interpolation space between X and Y is any intermediate space B such that if $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ then $T \in \mathcal{L}(B)$.

Definition 2.6. Let I be any interval in $(0, \infty)$. Then $L^p_*(I)$ is the space L^p with respect to the measure dt/t in I. In particular, $L^\infty_*(I) = L^\infty(I)$.

Definition 2.7. Let (X, Y) be an interpolation couple. For every $x \in X + Y$ and $t > 0$, we define

$$K(t, u, X, Y) = \inf_{u=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y).$$

We shall often write $K(t, u)$ instead $K(t, u, X, Y)$.

Now we define a family of Banach spaces by means of the function K.

Definition 2.8. Let $p \in [1, \infty]$ and $s \in (0, 1)$. Then the space

$$(X, Y)_{s,p} = \{u \in X + Y : t \mapsto t^{-s}K(t, u) \in L^p_*(0, \infty)\}$$

endowed with the norm

$$\|u\|_{(X,Y)_{s,p}} = \|t^{-s}K(t, u)\|_{L^p_*(0,\infty)}$$

is called real interpolation space.

Theorem 2.4. *The space $(X, Y)_{s,p}$ endowed with the norm $\|\cdot\|_{(X,Y)_{s,p}}$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $(X, Y)_{s,p}$. Since $(X, Y)_{s,p}$ is continuously embedded in $X + Y$, $\{u_n\}$ is a Cauchy sequence in $X + Y$, thus $\{u_n\}$

converges to an element $u \in X+Y$. Therefore, for all $n, m \in \mathbb{N}$ and $t>0$ we have

$$t^{-s}K(t,u_n-u) \leq t^{-s}K(t,u_n-u_m)+t^{-s}K(t,u_m-u). \tag{2.4}$$

We claim that

$$K(t,u_m-u) \leq \max\{1,t\}\|u_m-u\|_{X+Y}.$$

Indeed, let $a \in X$ and $b \in Y$. Consider two cases: If $t<1$,

$$\begin{aligned} K(t,u_m-u) &\leq \|a\|_X + t\|b\|_Y \\ &\leq \|a\|_X + \|b\|_Y \\ &= \|u_m-u\|_{X+Y} \\ &= \max\{t,1\}\|u_m-u\|_{X+Y}. \end{aligned}$$

On the other hand, if $t \geq 1$,

$$\begin{aligned} K(t,u_m-u) &\leq \|a\|_X + t\|b\|_Y \\ &\leq t(\|a\|_X + \|b\|_Y) \\ &= t\|u_m-u\|_{X+Y} \\ &= \max\{1,t\}\|u_m-u\|_{X+Y}. \end{aligned}$$

Therefore (2.4) is bounded by

$$t^{-s}K(t,u_n-u_m)+t^{-s}\max\{1,t\}\|u_m-u\|_{X+Y},$$

which goes to zero when $n \to \infty$. □

Proposition 2.3. *Let $p \in [1,\infty]$ and $s \in (0,1)$. Then $(X,Y)_{s,p}$ is continuously embedded in $X+Y$.*

Proof. Let us show first that $(X,Y)_{s,p}$ is continuously embedded in $(X,Y)_{s,\infty}$. Indeed, let $t>t_0>0$, using the fact that $K(\cdot,u)$ is increasing we get:

$$\begin{aligned} \|u\|^p_{(X,Y)_{s,p}} &= \int_{t_0}^{\infty} t^{-sp}|K(t,u)|^p \frac{\mathrm{d}t}{t} \\ &\geq \int_{t_0}^{\infty} t^{-sp}|K(t_0,u)|^p \frac{\mathrm{d}t}{t} \\ &= \frac{t_0^{-sp}}{sp}|K(t_0,u)|^p. \end{aligned}$$

Hence, $t_0^{-s}|K(t_0,u)| \leq C\|u\|_{(X,Y)_{s,p}}$ for some constant $C>0$. i.e.,

$$\|u\|_{(X,Y)_{s,\infty}} = \|t^{-s}K(t,u)\|_{L^\infty_*(0,\infty)} \leq C\|u\|_{(X,Y)_{s,p}}.$$

Therefore

$$\|u\|_{X+Y} = K(1,u) \leq \|u\|_{(X,Y)_{s,\infty}} \leq C\|u\|_{(X,Y)_{s,p}}.$$

□

In the next Theorem we will show how $W^{s,p}(\mathbb{R}^N)$ can be constructed from $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ via an interpolation method.

Theorem 2.5. *Let $p \in [1,\infty)$ and $s \in (0,1)$. Then*

$$(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p} = W^{s,p}(\mathbb{R}^N).$$

Proof. Let $u \in (L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p}$ such that $u = a + b$ with $a \in L^p(\mathbb{R}^N)$ and $b \in W^{1,p}(\mathbb{R}^N)$. From Proposition 4.5 we have

$$\begin{aligned}
\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^{2N}} \frac{|a(x)+b(x)-a(y)-b(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&\le 2^{p-1}\int_{\mathbb{R}^{2N}} \left(\frac{|a(x)-a(y)|^p}{|x-y|^{N+sp}} + \frac{|b(x)-b(y)|^p}{|x-y|^{N+sp}}\right)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^N} |x-y|^{-N-sp}(2^{2p-2}\|a\|_p^p + 2^{p-1}|x-y|^p\|b\|_{1,p}^p)\mathrm{d}y \\
&\le C\int_{\mathbb{R}^N} |x-y|^{-N-sp}(\|a\|_p + |x-y|\|b\|_{1,p})^p\mathrm{d}y \\
&= C\int_{\mathbb{R}^N} |x-y|^{-N-sp}K(|x-y|,u)^p\mathrm{d}y,
\end{aligned}$$

for some constant $C > 0$. Applying polar coordinates to the last integral we have

$$\begin{aligned}
C\int_{\mathbb{R}^N} |x-y|^{-N-sp}K(|x-y|,u)^p\mathrm{d}y &= C\int_0^\infty \frac{K(r,u)^p}{r^{N+sp}}\mathrm{d}r\int_{\partial B(0,1)}\mathrm{d}s \\
&= C'\int_0^\infty (r^{-s}K(r,u))^p\frac{\mathrm{d}r}{r} \\
&= C'\|u\|^p_{(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}},
\end{aligned}$$

for some constant $C' > 0$. Thus we have shown that

$$[u]_{s,p} \le C'\|u\|_{(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}}.$$

We know that $(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p}$ is continuously embedded in $L^p(\mathbb{R}^N) + W^{1,p}(\mathbb{R}^N)$ and since $W^{1,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$, then $L^p(\mathbb{R}^N)+W^{1,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$. Therefore that $\|u\|_p \le C\|u\|_{(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}}$.
On the other hand let $u \in W^{s,p}(\mathbb{R}^N)$ and $\varphi \in C_0^\infty(B(0,1))$ such that $\int_{\mathbb{R}^N}\varphi(x)\mathrm{d}x = 1$. For every $t > 0$ define

$$a_t(x) = u(x) - b_t(x) \quad \text{where} \quad b_t(x) = \frac{1}{t^N}u(y)\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y, \quad x \in \mathbb{R}^N.$$

Notice that using the Change Variables Theorem we get:

$$\begin{aligned}
\frac{1}{t^N}u(x)\int_{\mathbb{R}^N}\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y &= \frac{1}{t^N}u(x)\int_{\mathbb{R}^N} t^N\varphi(u)\mathrm{d}u \\
&= u(x)\int_{\mathbb{R}^N}\varphi(u)\mathrm{d}u \\
&= u(x).
\end{aligned}$$

Hence we can rewrite $a_t(x)$ as

$$a_t(x) = \frac{1}{t^N}\int_{\mathbb{R}^N}(u(x)-u(y))\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y.$$

Also we have,

$$D_ib_t(x) = \frac{1}{t^{N+1}}\int_{\mathbb{R}^N}u(x)D_i\left(\frac{x-y}{t}\right)\mathrm{d}y.$$

Since $\int_{\mathbb{R}^N} D_i\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y = 0$, then

$$D_ib_t(x) = \frac{1}{t^{N+1}}\int_{\mathbb{R}^N}(u(y)-u(x))D_i\left(\frac{x-y}{t}\right)\mathrm{d}y.$$

Using the definition of a_t and Jensen's inequality with the measure $\mathrm{d}\mu = t^{-N}\left|\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|$ we obtain that

$$\begin{aligned}
\|a_t\|_p^p &= \int_{\mathbb{R}^N}\left|\frac{1}{t^N}\int_{\mathbb{R}^N}(u(x)-u(y))\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|^p\mathrm{d}x\\
&\le \int_{\mathbb{R}^{2N}}|u(x)-u(y)|^p\frac{1}{t^N}\left|\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|\mathrm{d}x.
\end{aligned}$$

Therefore using this estimate we have

$$\begin{aligned}
\int_0^\infty t^{-sp}\|a_t\|_p^p\frac{\mathrm{d}t}{t} &\le \int_0^\infty \frac{t^{-sp}}{t^N}\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^p\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\mathrm{d}x\frac{\mathrm{d}t}{t}\\
&\le \|\varphi\|_\infty\int_{|x-y|}^\infty\frac{t^{-sp}}{t^{N+1}}\mathrm{d}t\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^p\mathrm{d}y\mathrm{d}x\\
&= \|\varphi\|_\infty\frac{|x-y|^{-sp-N}}{sp+N}\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^p\mathrm{d}y\mathrm{d}x\\
&= \frac{\|\varphi\|_\infty}{sp+N}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}}\mathrm{d}x\mathrm{d}y\\
&= C[u]_{s,p}^p < \infty.
\end{aligned}$$

Similarly from the definition of D_ib_t and Jensen's inequality with the measure $\mathrm{d}\mu = t^{-N}|D_i\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y|$ we have that

$$\begin{aligned}
\|D_ib_t\|_p^p &= \int_{\mathbb{R}^N}\left|\frac{1}{t^{N+1}}\int_{\mathbb{R}^N}(u(y)-u(x))D_i\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|^p\mathrm{d}x\\
&= \frac{1}{t^p}\int_{\mathbb{R}^N}\left|\frac{1}{t^N}\int_{\mathbb{R}^N}(u(y)-u(x))D_i\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|^p\mathrm{d}x\\
&\le \frac{1}{t^p}\int_{\mathbb{R}^{2N}}|u(y)-u(x)|^p\frac{1}{t^N}\left|D_i\varphi\left(\frac{x-y}{t}\right)\mathrm{d}y\right|\mathrm{d}x.
\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^\infty t^{(1-s)p}\|D_i b_t\|_p^p\frac{\mathrm{d}t}{t} &\le \int_0^\infty \frac{t^{(1-s)p}}{t^{p+N}}\int_{\mathbb{R}^{2N}}|u(y)-u(x)|^p\left|D_i\varphi\left(\frac{x-y}{t}\right)\right|\mathrm{d}y\mathrm{d}x\frac{\mathrm{d}t}{t}\\ &\le \|D_i\varphi\|_\infty\int_{|y-x|}^\infty \frac{t^{(1-s)p}}{t^{p+N+1}}\mathrm{d}t\int_{\mathbb{R}^{2N}}|u(y)-u(x)|^p\mathrm{d}y\mathrm{d}y\\ &= \|D_i\varphi\|_\infty\frac{|y-x|^{-sp-N}}{sp+N}\int_{\mathbb{R}^{2N}}|u(y)-u(x)|^p\mathrm{d}y\mathrm{d}y\\ &= \frac{\|D_i\varphi\|_\infty}{sp+N}\int_{\mathbb{R}^{2N}}\frac{|u(y)-u(x)|^p}{|y-x|^{sp+N}}\mathrm{d}x\mathrm{d}y\\ &= C[u]_{s,p}^p<\infty.\end{aligned}$$

In addition the estimate:

$$\|b_t\|_p \le \|u\|_p\|\varphi\|_1 = \|u\|_p$$

follows from Hölder inequality and the fact that $\int_{\mathbb{R}^N}\varphi(x)\mathrm{d}x = 1$. Thus

$$\begin{aligned}t^{-s}K(t,u) &= t^{-s}\inf_{u=a_t+b_t}(\|a_t\|_p + t\|b_t\|_{1,p})\\ &\le t^{-s}\|a_t\|_p + t^{1-s}\|b_t\|_{1,p}\in L_*^p(0,1),\end{aligned}$$

which shows that $u\in(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}$. □

3 A Semipositone Type Problem

3.1 Description of the Problem

We are interested in the study of the existence of positive solutions to the problem

$$\begin{cases}(-\Delta)_p^s(u) &= \lambda f(u) \text{ in } \Omega\\ u &= 0 \text{ in } \mathbb{R}^N\setminus\Omega,\end{cases}\tag{3.1}$$

where $N>2$ is an integer, $\Omega\subseteq\mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $s\in(0,1)$, $1<p$, $sp<N$ and $\lambda>0$. Besides $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and $(-\Delta)_p^s$ is the s-fractional p-Laplacian operator defined as

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to0^+}\int_{B^c(x,\varepsilon)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\mathrm{d}y.$$

Let us set for all $s\in\mathbb{R}$

$$\Phi_p(s) = |s|^{p-2}s.$$

Define $F(s)=\int_0^s f(t)\mathrm{d}t$. We will consider a negative *non-linearity* at 0 ($f(0)<0$), so we will be facing a *semipositone* type problem. We assume that f satisfies an Ambrosetti-Rabinowitz type condition. More specifically, we will assume that there exist $\theta>p$ and $M\in\mathbb{R}$ such that for all $s\in\mathbb{R}$,

$$sf(s)\geqslant\theta F(s)+M.\tag{3.2}$$

Remark 3.1. The existence of at least one solution to our problem can be stated under the assumption $q \in (p-1,\, p_s^*-1)$. We will see that the restriction $p-1 < q < \min\{\frac{sp}{N}p_s^*, p_s^*-1\}$ is necessary to prove the positiveness of this.

Definition 3.1. A weak solution of problem (3.1) is a function $u \in W_0^{s,p}(\Omega)$ such that

$$\int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = \lambda \int_\Omega f(u)\varphi \mathrm{d}x$$

for all $\varphi \in W_0^{s,p}(\Omega)$.

We shall give to this problem a variational approach. For each $\lambda > 0$ define the Energy Functional:

Definition 3.2. For each $\lambda > 0$, let us define the functional $E_\lambda : W_0^{s,p}(\Omega) \to \mathbb{R}$ as

$$E_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\,\mathrm{d}y - \lambda \int_\Omega F(u)\mathrm{d}x. \tag{3.3}$$

Observe that $E_\lambda(u) := \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(u)\mathrm{d}x$.

It is well known that critical points of E_λ are weak solutions of problem (3.1).

Proposition 3.1. *Let us assume that there exist $q \in (p-1, p_s^*-1)$ and $A, B > 0$ such that*

$$\begin{array}{cc} A(s^q-1) \le f(s) \le B(s^q+1) & \textit{for} \ \ s > 0 \\ f(s) = 0 & \textit{for} \ \ s \le -1 \end{array}. \tag{3.4}$$

Then there exist $A_1, B_1,\, C_1 > 0$ such that

$$\begin{array}{cc} F(u(x)) \le B_1(u(x)^{q+1}+1) & \textit{for} \ \ u \in \mathbb{R} \\ F(u(x)) \ge A_1(u(x)^{q+1}-C_1) & \textit{for} \ \ u \ge 0 \end{array}. \tag{3.5}$$

for all $x \in \mathbb{R}^N$.

Proof. Notice that $f(s) \le B(s^q+1)$ implies that $F(u) \le B\left(\frac{u^{q+1}}{q+1}+u\right)$. The proof is split into two different cases. First, if $u(x) > 1$ for all $x \in \mathbb{R}^N$, we have:

$$B\left(\frac{u(x)^{q+1}}{q+1}+u(x)\right) \le B\left(\frac{u(x)^{q+1}}{q+1}+u(x)^{q+1}\right).$$

Taking $\frac{B}{q+1}+B = B_1$ and since $\frac{u^{q+1}(x)}{u^{q+1}(x)+1} < 1$ we get:

$$\frac{u^{q+1}(x)}{u^{q+1}(x)+1}\left(\frac{B}{q+1}+B\right) \le \frac{B}{q+1}+B = B_1,$$

which implies that

$$F(u(x)) \leq B\left(\frac{u^{q+1}(x)}{q+1} + u^{q+1}(x)\right) \leq B_1(u^{q+1}(x)+1).$$

On the other hand, if $u(x) \leq 1$ for all $x \in \mathbb{R}^N$, we obtain that:

$$B\left(\frac{u^{q+1}(x)}{q+1} + u(x)\right) \leq B\left(\frac{u^{q+1}(x)}{q+1} + 1\right).$$

Taking $B_1 = B$ since $\frac{\frac{u^{q+1}(x)}{q+1}+1}{u^{q+1}(x)+1} \leq 1$, we have

$$B\left(\frac{\frac{u^{q+1}(x)}{q+1}+1}{u^{q+1}(x)+1}\right) \leq B = B_1$$

which implies that

$$F(u(x)) \leq B\left(\frac{u^{q+1}(x)}{q+1} + 1\right) \leq B_1(u^{q+1}(x)+1).$$

Finally if $A(s^q - 1) \leq f(s)$ then $A\left(\frac{u^{q+1}}{q+1} - u\right) \leq F(u)$. Let $A_1 = \frac{A}{2(q+1)}$ and define for all $u \geq 0$

$$H(u) = \frac{1}{2}\frac{A}{A_1(q+1)}u^{q+1} - \frac{A}{A_1}u.$$

Notice that $H(u) \to \infty$ as $u \to \infty$, therefore there exists $M > 0$ such that $H(u) \geq 1$ for all $u \in [M, \infty)$. Since H is a continuous function on $[0, M]$, we get $H(u) \geq b$ for some b. Taking $C_1 \geq -\min\{1, b\}$ we get that

$$-C_1 \leq \min\{1, b\} \leq H(u)$$

for all $u \geq 0$. With this choice of C_1 we have that

$$-C_1 \leq \frac{1}{A_1}\left(\frac{A}{q+1} - \frac{A}{2(q+1)}\right)u^{q+1} - \frac{A}{A_1}u,$$

and this implies that

$$A_1(u^{q+1} - C_1) \leq A\left(\frac{u^{q+1}}{q+1} - u\right).$$

□

Proposition 3.2. *The functional E_λ is Fréchet differentiable. Moreover its derivative is given by*

$$\langle E'_\lambda u, \varphi\rangle = \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\,\mathrm{d}y - \lambda\int_\Omega f(u)\varphi\mathrm{d}x. \quad (3.6)$$

Proof. In order to show that E_λ is Fréchet differentiable we will prove that E_λ is Gâteaux differentiable, and then we show that $(E_\lambda)_G$ is continuous. Thus, it is enough to prove that

$$\underbrace{\lim_{t\to 0}\frac{E_\lambda(u+t\varphi)-E_\lambda(u)}{t}}_{(1)} = \int_{\mathbb{R}^{2N}}\frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\,\mathrm{d}y$$
$$-\lambda\int_\Omega f(u)\varphi\mathrm{d}x.$$

Notice that, (1) is equal to

$$\underbrace{\lim_{t\to 0}\left[\frac{1}{pt}\int_{\mathbb{R}^{2N}}\left(\frac{|u(x)+t\varphi(x)-u(y)-t\varphi(y)|^p}{|x-y|^{N+sp}}-\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\right)\mathrm{d}x\mathrm{d}y\right.}_{(2)}$$
$$\left.-\frac{\lambda}{t}\int_\Omega[F(u(x)+t\varphi(x))-F(u(x))]\mathrm{d}x\right]. \tag{3.7}$$

Now by the Mean Value Theorem 4.1,

$$\begin{aligned}(2) &\le |u(x)+\theta\varphi(x)-u(y)-\theta\varphi(y)|^{p-1}|\varphi(x)-\varphi(y)|\\ &\le C(|u(x)-u(y)|^{p-1}+|\theta|^{p-1}|\varphi(x)-\varphi(y)|^{p-1})|\varphi(x)-\varphi(y)|\\ &= C|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|+C|\varphi(x)-\varphi(y)|^p\end{aligned}$$

for some $\theta<t<1$ and some constant $C>0$. Therefore

$$\frac{|u(x)+t\varphi(x)-u(y)-t\varphi(y)|^p}{|x-y|^{N+sp}}-\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}$$

$$\le\frac{C|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}+\frac{C|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}. \tag{3.8}$$

Taking $m=\frac{(p-1)(N+sp)}{p}$, $l=\frac{N+sp}{p}$ and applying Hölder inequality we have that

$$\begin{aligned}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^m|x-y|^l}\mathrm{d}x\mathrm{d}y\\ &\le\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{mp/p-1}}\mathrm{d}x\mathrm{d}y\right)^{p/p-1}\\ &\quad\left(\int_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{lp}}\mathrm{d}x\mathrm{d}y\right)^{1/p},\end{aligned}$$

where the last two terms are in $L^1(\Omega^2)$ because $u,\varphi\in W_0^{s,p}(\Omega)$. Thus (3.8) belongs to $L^1(\Omega^2)$ and by the Lebesgue's Dominated Convergence Theorem 4.5, we obtain:

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{pt}\int_{\mathbb{R}^{2N}}\left(\frac{|u(x)+t\varphi(x)-u(y)-t\varphi(y)|^p}{|x-y|^{N+sp}}-\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\right)\mathrm{d}x\mathrm{d}y\\
=&\int_{\mathbb{R}^{2N}}\lim_{t\to 0}\frac{1}{pt}\left(\frac{|u(x)+t\varphi(x)-u(y)-t\varphi(y)|^p}{|x-y|^{N+sp}}-\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\right)\mathrm{d}x\mathrm{d}y\\
&\qquad=\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where the last equality can be check considering the derivative of the function $f(t)=|u(x)-u(y)+t(\varphi(x)-\varphi(y))|^p$ at 0. On the other hand, from the Mean Value Theorem 4.1 and taking account the growth assumptions of f, see (3.4), we have

$$\begin{aligned}
\frac{1}{t}|F(u(x)+t\varphi(x))-F(u(x))|&\leq|f(u(x)+\theta\varphi(x))||\varphi(x)|\\
&\leq|\hat{B}((u(x)+\theta\varphi(x))^q+1)||\varphi(x)|\\
&\leq|\tilde{B}Cu^q(x)+\tilde{B}C\theta^q\varphi^q(x)+\tilde{B}||\varphi(x)|\\
&\leq\tilde{B}C|u^q(x)||\varphi(x)|+\tilde{B}C|\varphi^{q+1}(x)|+\tilde{B}|\varphi(x)|,
\end{aligned}$$

for some $\theta<t<1$ and some constants $C,\tilde{B}>0$. Applying the Hölder inequality we get that

$$\begin{aligned}
\int_\Omega|u^q(x)\varphi(x)|dx&\leq\left(\int_\Omega|u^q(x)|^{(q+1)/q}\mathrm{d}x\right)^{q/(q+1)}\left(\int_\Omega|\varphi(x)|^{q+1}dx\right)^{1/(q+1)}\\
&=\|u\|_{q+1}^q\|\varphi\|_{q+1},
\end{aligned}$$

where the last term is finite since $W_0^{s,p}(\Omega)\hookrightarrow L^{q+1}(\Omega)$ for $q+1\in(p,p^*)$. Similarly

$$\int_\Omega|\varphi^{q+1}(x)|\mathrm{d}x<\infty\quad\text{and}\quad\int_\Omega|\varphi(x)|\mathrm{d}x<\infty.$$

Therefore $\hat{B}C|u^q(x)||\varphi(x)|+\hat{B}C|\varphi^{q+1}(x)|+\hat{B}|\varphi(x)|\in L^1(\Omega)$.
Now if we fix $x\in\Omega$, the limit

$$\lim_{t\to 0}\frac{F(u(x)+t\varphi(x))-F(u(x))}{t}$$

is the directional derivative of the function F at the point $u(x)$ in the direction of $\varphi(x)$. Hence by the Lebesgue's Dominated Convergence Theorem 4.5 we get

$$\lim_{t\to 0}\lambda\int_\Omega\frac{F(u(x)+t\varphi(x))-F(u(x))}{t}dx=\lambda\int_\Omega f(u(x))\varphi(x)dx.$$

We have shown that the functional E_λ is Gâteaux differentiable. We complete the proof by checking that the function $E'_\lambda:W_0^{s,p}(\Omega)\to(W_0^{s,p}(\Omega))^*$ is continuous. Indeed, Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $W_0^{s,p}(\Omega)$. Note that

$$|(E'_\lambda(u_n) - E'_\lambda(u))\varphi| \leq \underbrace{\int_{\mathbb{R}^{2N}} \frac{\Phi_p(u_n(x)-u_n(y)) - \Phi_p(u(x)-u(y))}{|x-y|^{N+sp}} |\varphi(x)-\varphi(y)| \mathrm{d}x\mathrm{d}y}_{(3)}$$
$$+ \underbrace{\int_\lambda |f(u) - f(u_n)||\varphi| \mathrm{d}x}_{(4)}.$$

From Hölder inequality,

$$(3) \leq C \left(\int_{\mathbb{R}^{2N}} \left(\frac{|\varphi(x)-\varphi(y)|}{|x-y|^l} \right)^p \mathrm{d}x\mathrm{d}y \right)^{1/p},$$

where $C = \left(\int_{\mathbb{R}^{2N}} \left| \frac{\Phi(u_n(x)-u_n(y)) - \Phi(u(x)-u(y))}{|x-y|^m} \right|^{p'} \mathrm{d}x\mathrm{d}y \right)^{1/p'}$, $m = \frac{(p-1)(N+sp)}{p}$ and $l = \frac{N+sp}{p}$. Since $u_n \to u$ in $W_0^{s,p}(\Omega)$ then $\{u_n\}$ is a Cauchy sequence in $L^p(\Omega)$. It converges to a function $u \in L^p(\Omega)$. Hence $u_{n_j}(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Moreover, the sequence of functions $v_n(x,y) = \frac{u_n(x)-u_n(y)}{|x-y|^{N/p+s}}$ is Cauchy, hence it converges to certain $v(x,y)$ in $L^p(\mathbb{R}^{2N})$. Therefore, up to a sequence, there exists a function $h \in L^p(\mathbb{R}^N)$ such that $|u_n(x)| \leq h(x)$ and $|v_n(x,y)| \leq h(x,y)$. Using all this facts we obtain that

$$\begin{aligned} \left| \frac{\Phi(u_n(x)-u_n(y)) - \Phi(u(x)-u(y))}{|x-y|^m} \right|^{p'} &\leq C' \left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \right) \\ &\leq C' h^p(x,y) + C' \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \in L^1(\Omega^2). \end{aligned}$$

Also, since $\varphi \in W_0^{s,p}(\Omega)$ we have that,

$$\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}} \in L^1(\Omega^2).$$

Similarly, from Hölder inequality, (4) is bounded by

$$\begin{aligned} &\lambda \left(\int_\Omega |f(u) - f(u_n)|^{(q+1)/q} \mathrm{d}x \right)^{q/(q+1)} \left(\int_\Omega |\varphi|^{q+1} \mathrm{d}x \right)^{1/(q+1)} \\ &\leq \lambda \left(\int_\Omega |f(u)|^{(q+1)/q} + |f(u_n)|^{(q+1)/q} \mathrm{d}x \right)^{q/(q+1)} ||\varphi||_{q+1}. \end{aligned}$$

Notice that $\|\varphi\|_{q+1} < \infty$. Due to the growth assumptions of f, see (3.4), and the fact that $W_0^{s,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ for $q+1 \in (q, q^*)$ we have:

$$\begin{aligned} |f(u)|^{q/(q+1)} + |f(u_n)|^{q/(q+1)} &\leq \hat{B}(|u|^{q+1} + 1) + \hat{B}(|u_n|^{q+1} + 1) \\ &\leq \hat{C} + \hat{B}(h^{q+1} + 1) \in L^1(\Omega), \end{aligned}$$

where $\hat{B} = \max\{B, |f(0)|\}$. Therefore by the Lebesgue's Dominated Convergence Theorem 4.5

$$\lim_{n\to\infty} \|E'_\lambda(u_n) - E'_\lambda(u)\| \le \limsup_{n\to\infty} |E'_\lambda(u_n) - E'_\lambda(u)| = 0.$$

This shows that E'_λ is continuous. Hence E_λ is Fréchet differentiable, which concludes the proof.

□

3.2 Technical Results

In this subsection we shall establish some results that guarantee that E_λ has a critical point, u_λ, whenever $\lambda > 0$ is sufficiently small. After that, we present some technical important results relevant in the proof of our main theorem. The positive number

$$r := \frac{1}{q+1-p},$$

will be use repeatedly throughout this subsection. Let $\varphi \in W_0^{s,p}(\Omega)$ be a positive function with $\|\varphi\| = 1$ and let

$$c := \left(\frac{2}{pA_1\|\varphi\|_{q+1}^{q+1}}\right)^r > 0.$$

Lemma 3.1. *There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$ then $E_\lambda(c\lambda^{-r}\varphi) \le 0$.*

Proof. Let $l = c\lambda^{-r}$. From the growth assumptions of F see (3.5) and the facts that $\|\varphi\| = 1$ and $c^{q+1} = \frac{2c^p}{pA_1\|\varphi\|_{q+1}^{q+1}}$ we have

$$\begin{aligned}
E_\lambda(l\varphi) &= \frac{1}{p}\|l\varphi\|^p - \lambda \int_\Omega F(l\varphi)\mathrm{d}x \\
&\leqslant \frac{l^p}{p}\|\varphi\|^p - \lambda A_1 l^{q+1} \int_\Omega \varphi^{q+1}\mathrm{d}x + \lambda A_1 C_1|\Omega| \\
&\leqslant \frac{l^p}{p} - \lambda A_1 l^{q+1}\|\varphi\|_{q+1}^{q+1} + \lambda A_1 C_1|\Omega| \\
&= \frac{(c\lambda^{-r})^p}{p} - \lambda A_1 (c\lambda^{-r})^{q+1}\|\varphi\|_{q+1}^{q+1} + \lambda A_1 C_1|\Omega| \\
&= \lambda^{-rp}\left(\frac{c^p}{p} - 2\frac{c^p}{p} + \lambda^{1+rp}A_1C_1|\Omega|\right).
\end{aligned} \tag{3.9}$$

Thus, if $0 < \lambda < \left(\frac{c^p}{2pA_1C_1|\Omega|}\right)^{1/(1+rp)} =: \lambda_1$, then

$$E_\lambda(l\varphi) \leqslant -\frac{c^p}{2p}\lambda^{-rp} \leqslant 0.$$

□

Lemma 3.2. *There exist $\tau > 0$, $c_1 > 0$ and $\lambda_2 \in (0,1)$ such that if $\|u\| = \tau\lambda^{-r}$ then $E_\lambda(u) \geq c_1(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \lambda_2)$.*

Proof. Let $u \in W_0^{s,p}(\Omega)$ with $\|u\| = \lambda^{-r}\tau$, since $W_0^{s,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, there exists $K_1 > 0$ such that for all $\varphi \in W_0^{s,p}(\Omega)$, $\|\varphi\|_{q+1} \leq K_1\|\varphi\|$, define $\tau = \min\{(2pK_1^{q+1}B_1)^{-r}, c\}$ then,

$$\begin{aligned}
E_\lambda(u) &= \frac{1}{p}\|u\|^p - \lambda\int_\Omega F(u)\mathrm{d}x \\
&\geq \frac{1}{p}(\lambda^{-r}\tau)^p - \lambda B_1\|u\|_{q+1}^{q+1} - \lambda B_1|\Omega| \\
&\geq \frac{1}{p}(\lambda^{-r}\tau)^p - \lambda B_1(K_1\|u\|)^{q+1} - \lambda B_1|\Omega| \\
&= \frac{1}{p}(\lambda^{-r}\tau)^p - \lambda B_1 K_1^{q+1}(\lambda^{-r}\tau)^{q+1} - \lambda B_1|\Omega| \\
&\geq \lambda^{-rp}\left(\frac{\tau^p}{2p} - \lambda^{1+rp}|\Omega|B_1\right) \\
&\geq \lambda^{-rp}\frac{\tau^p}{4p}
\end{aligned}$$

taking $c_1 = \frac{1}{4p}$ and $\lambda_2 := \tau^{p/(1+rp)}(4pB_1|\Omega|)^{-1/(1+rp)}$ we obtain the result. $\square$

Lemma 3.3. *Let $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. Then, there exists a constant $c_2 > 0$ such that for all $\lambda \in (0, \lambda_3)$ the functional E_λ has a critical point u_λ which satisfies*

$$c_1\lambda^{-rp} \leqslant E_\lambda(u_\lambda) \leqslant c_2\lambda^{-rp},$$

where $c_1 > 0$ is the constant given in Lemma 3.2.

Proof. First of all, we show that, E_λ satisfies the Palais Smale condition. Let us assume that $\{u_n\} \subset W_0^{s,p}(\Omega)$ is a sequence such that $\{E_\lambda(u_n)\}$ is bounded and $E'_\lambda \to 0$, in $W_0^{s,p}(\Omega)$ as $n \to \infty$. Hence, for $\varepsilon = 1$ there exists $\nu > 0$ such that $\|E'_\lambda(u_n)\|_{(W_0^{s,p}(\Omega))^*} \leq 1$, for $n > \nu$. By definition of the norm we have that

$$\left|\langle E'_\lambda(u_n), \frac{u_n}{\|u_n\|}\rangle\right| \leq \sup_{\frac{u_n}{\|u_n\|} \leq 1}\left|\langle E'_\lambda(u_n), \frac{u_n}{\|u_n\|}\rangle\right| = \|E'_\lambda(u_n)\|_{(W_0^{s,p}(\Omega))^*} \leq 1$$

which implies that

$$|\langle E'_\lambda(u_n), u_n\rangle| \leq \|u_n\|.$$

Moreover, from (3.6) we have

$$\begin{aligned}
|\langle E'_\lambda(u_n), u_n\rangle| &= \left|\|u_n\|^p - \lambda\int_\Omega f(u_n)u_n\mathrm{d}x\right| \\
&= -\|u_n\|^p + \lambda\int_\Omega f(u_n)u_n\mathrm{d}x \leq \|u_n\| \qquad (3.10)
\end{aligned}$$

$$-\|u_n\|^p - \|u_n\| \leq -\lambda\int_\Omega f(u_n)u_n\mathrm{d}x, \quad \text{for}\, n > \nu.$$

Let $K > 0$ such that for all n, $|E_\lambda(u_n)| \leqslant K$. From the Ambrosetti-Rabinowitz condition equation (3.2) we see that

$$\begin{aligned}\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega f(u_n)u_n\mathrm{d}x + \frac{\lambda}{\theta}M|\Omega| &\leqslant \frac{1}{p}\|u_n\|^p - \lambda\int_\Omega F(u_n)\mathrm{d}x \\ &\leqslant K.\end{aligned} \tag{3.11}$$

Using (3.10) and (3.11) we obtain

$$\begin{aligned}\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p - \frac{1}{\theta}\|u_n\| &= \frac{1}{p}\|u_n\|^p + \frac{1}{\theta}\left(-\|u_n\|^p - \|u_n\|\right) \\ &\leq \frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega f(u_n)u_n\mathrm{d}x \\ &\leq K, -\frac{\lambda}{\theta}M|\Omega|.\end{aligned}$$

which proves that $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega)$. Therefore since $W_0^{s,p}(\Omega)$ is reflexive then up to a subsequence, $\{u_n\}$ converges weakly to the function $u \in W_0^{s,p}(\Omega)$. Furthermore, since $W_0^{s,p} \subset\subset L^{q+1}(\Omega)$ see [[16], page 216] for $p < q+1 < p_s^*$, then $u_n \to u$ (strongly) in $L^{q+1}(\Omega)$. Applying the Hölder inequality we get

$$\lambda\int_\Omega f(u_n)(u_n - u)\mathrm{d}x \leq \lambda\left(\int_\Omega |f(u_n)|^{\frac{q+1}{q}}\mathrm{d}x\right)^{\frac{q}{q+1}}\left(\int_\Omega |u_n - u|^{q+1}\mathrm{d}x\right)^{\frac{1}{q+1}}.$$

Thus,

$$\lim_{n\to\infty}\lambda\int_\Omega f(u_n)(u_n - u)\mathrm{d}x = 0.$$

Then, since $\lim_{n\to\infty} E'_\lambda(u_n) = 0$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2N}}\frac{\Phi_p(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x-y|^{N+sp}} = 0. \tag{3.12}$$

Using again that u is the weak limit of u_n we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2N}}\frac{\Phi_p(u(x) - u(y))((u_n - u)(x) - (u_n - u)(y))}{|x-y|^{N+sp}} = 0. \tag{3.13}$$

On the other hand, taking into account the Hölder inequality, we see that

$$\begin{aligned}&\int_\Omega \frac{\Phi_p(u_n(x) - u_n(y)) - \Phi_p(u(x) - u(y))}{|x-y|^{N+sp}}((u_n - u)(x) - (u_n - u)(y))\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega\left[\frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} - \frac{\Phi_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+sp}}\right. \\ &\left.-\frac{\Phi_p(u(x) - u(y))(u_n(x) - u_n(y))}{|x-y|^{N+sp}} + \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\right]\mathrm{d}x\mathrm{d}y \\ &\geqslant \|u_n\|^p - \|u_n\|^{p-1}\|u\| - \|u_n\|\|u\|^{p-1} + \|u\|^p \\ &= ([\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \geqslant 0.\end{aligned}$$

From (3.12), (3.13) we obtain

$$\lim_{n\to\infty}(\|u_n\|^{p-1}-\|u\|^{p-1})(\|u_n\|-\|u\|)=0,$$

which implies

$$\lim_{n\to\infty}\|u_n\|=\|u\|.$$

Since $u_n \rightharpoonup u$, then $u_n \to u$ strongly in $W_0^{s,p}(\Omega)$. This proves that E_λ satisfies the Palais-Smale condition.
Let us observe that, from (3.9), for all $0\leqslant l\leqslant c\lambda^{-r}$

$$E_\lambda(l\phi)\leqslant \frac{l^p}{p}+\lambda A_1C_1|\Omega|\leqslant \frac{c^p}{p}\lambda^{-rp}+A_1C_1|\Omega|\lambda^{-rp}=c_2\lambda^{-rp}.$$

where $c_2:=\frac{c^p}{p}+A_1C_1|\Omega|$. Therefore

$$\max_{0\leqslant l\leqslant c\lambda^{-r}}E_\lambda(l\phi)\leqslant c_2\lambda^{-rp}. \tag{3.14}$$

From Lemmas 3.1 and 3.2, and the Mountain Pass Theorem for each $\lambda\in(0,\lambda_3)$ there exist $u_\lambda\in W_0^{s,p}(\Omega)$ such that $E'_\lambda(u_\lambda)=0$. Furthermore, this critical point is characterized by

$$E_\lambda(u_\lambda)=\min_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1}E(\gamma(t)). \tag{3.15}$$

where Γ is the set of continuous functions $\gamma:[0,1]\to W_0^{s,p}(\Omega)$ with $\gamma(0)=0$, $\gamma(1)=c\lambda^{-r}\varphi$. Moreover, from (3.14), (3.15) and Lemma 3.2 we see that

$$c_1\tau^p\lambda^{-rp}\leqslant E_\lambda(u_\lambda)\leqslant c_2\lambda^{-rp}.$$

Note that c_1 c_2 are independent of λ. □

Remark 3.2. There exists a constant $C>0$ such that for all $0<\lambda<\lambda_3$

$$\|u_\lambda\|\leqslant C\lambda^{-r}. \tag{3.16}$$

In fact, since u_λ is a critical point of E_λ, then

$$\|u_\lambda\|^p=\lambda\int_\Omega f(u_\lambda)u_\lambda\mathrm{d}x.$$

From the Ambrosetti-Rabinowitz condition and Lemma 3.3 we see that

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_\lambda\|^p &\le \frac{1}{p}\|u_\lambda\|^p-\frac{\lambda}{\theta}\int_\Omega f(u_\lambda)u_\lambda\mathrm{d}x\\ &\le \frac{1}{p}\|u_\lambda\|^p-\frac{\lambda}{\theta}\int_\Omega f(u_\lambda)u_\lambda\mathrm{d}x+\frac{\lambda}{\theta}M|\Omega|\\ &\le \frac{1}{p}\|u_\lambda\|^p-\lambda\int_\Omega F(u_\lambda)\mathrm{d}x\\ &= E_\lambda(u_\lambda)\\ &\le c_2\lambda^{-rp}.\end{aligned}$$

The proof of the following lemma can be found in [[28], page 6]

Lemma. *Let $g \in L^q(\Omega)$ and let $u \in W_0^{s,p}(\Omega)$ be a weak solution of $(-\Delta)_p^s u = f(x)$ in Ω. Then*

$$\|u\|_r \leq C\|g\|_q^{1/(p-1)}$$

where

$$\begin{cases} \frac{N(p-1)q}{N-spq} & 1 < q < \frac{N}{sp} \\ \infty & \frac{N}{sp} < q \leq \infty \end{cases}$$

and C a positive constant depending on N, Ω, p, s and q. In particular if $g \in L^\infty(\Omega)$, then

$$\|u\|_\infty \leq C\|g\|_\infty^{1/(p-1)}.$$

Lemma 3.4. *There exist $\alpha \in (0, s]$ and a constant $C > 0$ such that for all $0 < \lambda < \lambda_3$, the solution u_λ of the problem (3.1) satisfies $u_\lambda/\mathrm{d}_\Omega^s \in C^\alpha(\overline{\Omega})$ and*

$$\left\| \frac{u_\lambda}{\mathrm{d}_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \leqslant C\lambda^{-r}.$$

Proof. Let t be such that $\frac{N}{sp} < t$ and $tq < p_s^*$ and $g := \lambda f \circ u_\lambda$. Since $W_0^{s,p}(\Omega) \hookrightarrow L^{tq}(\Omega)$ and $|g| \leqslant A_1\lambda(|u_\lambda|^q + 1)$ we have

$$\begin{aligned} \int_\Omega |\lambda f(u_\lambda)(x)|^t \mathrm{d}x &\leq \lambda^t \int_\Omega |A_1(u_\lambda^q + 1)|^t \mathrm{d}x \\ &\leq \lambda^t C \int_\Omega u_\lambda^{qt} + 1 \mathrm{d}x. \end{aligned}$$

Hence $g \in L^t(\Omega)$. According to the above Lemma,

$$\|u_\lambda\|_\infty \leqslant \|g\|_t^{\frac{1}{p-1}}. \tag{3.17}$$

But taking into account the Remark 3.2, we have

$$\|g\|_t \leqslant C\lambda\|u_\lambda\|_{tq}^q \leqslant C\lambda\|u_\lambda\|^q \leqslant C\lambda^{1-rq}.$$

Therefore, from (3.17) and setting $-r = (1-rq)/(p-1)$, we see that

$$\|u_\lambda\|_\infty \leq \|g\|_t^{1/(p-1)} \leq C\lambda^{-r}. \tag{3.18}$$

The inequality (3.18) together with the grow assumptions of f and that $1-rq = -r(p-1)$ imply that

$$\|\lambda f(u_\lambda)\|_\infty \leq C\lambda|u_\lambda^q + 1| \leq C\lambda\|u_\lambda^q\|_\infty + C\lambda \leq C\lambda^{-r(p-1)}.$$

Since $u_\lambda \in L^\infty(\Omega)$ then $g \in L^\infty(\Omega)$. From Theorem 1.1. in [[25], page 4], we see that there exists $\alpha \in (0, s]$ and $C > 0$, depending only on N, p, s and Ω, such that the solution u_λ satisfies $u_\lambda/\mathrm{d}_\Omega^s \in C^\alpha(\overline{\Omega})$ and

$$\left\| \frac{u_\lambda}{\mathrm{d}_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \leqslant C\|\lambda f(u_\lambda)\|_\infty^{\frac{1}{p-1}} \leqslant \lambda^{-r},$$

$\square$

Lemma 3.5. *Let u_λ be a weak solution of (3.1). Then there exists a constant C such that for all $0 < \lambda < \lambda_3$*

$$C\lambda^{-r} \leqslant \|u_\lambda\|_\infty.$$

Proof. From Lemma 3.3 there exists c_1 such that $c_1\lambda^{-rp} \leqslant E_\lambda(u_\lambda)$. Moreover, since $\min F > -\infty$ and $E'_\lambda(u_\lambda) = 0$ then

$$\begin{aligned}\lambda\int_\Omega f(u_\lambda)u_\lambda \mathrm{d}x = \|u_\lambda\|^p& \\ &= pE_\lambda(u_\lambda) + p\lambda\int_\Omega F(u_\lambda)\mathrm{d}x \\ &\geqslant pc_1\lambda^{-rp} + p|\Omega|\lambda\min F \\ &\geqslant C_1\lambda^{-rp},\end{aligned} \tag{3.19}$$

for some $C_1 > 0$. On the other hand, observe from (3.4) that there exists $B_2 > 0$ such that for all $s \in \mathbb{R}$, $f(s)s \leqslant B_2(|s|^{q+1} + |s|)$. Thus

$$\begin{aligned}\lambda\int_\Omega f(u_\lambda)u_\lambda\mathrm{d}x &\leqslant B_2\lambda\int_\Omega(|u_\lambda|^{q+1} + |u_\lambda|)\mathrm{d}x \\ &\leqslant B_2\lambda\int_\Omega(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty)\mathrm{d}x \\ &\leqslant B\lambda\|u_\lambda\|_\infty^{q+1},\end{aligned} \tag{3.20}$$

for some $B > 0$. From (3.19) and (3.20) we obtain the result. □

3.3 Proof of the Main Theorem

We will state some results that will be used in the proof of the main theorem. The proof of the following theorem can be found in [[25], page 4].

Theorem 3.1. *Let $p \geq 2$, Ω be a bounded domain with $C^{1,1}$ boundary and $d_\Omega(x) := dist(x, \partial\Omega)$. There exist $\alpha \in (0, s)$ and $C > 0$ depending on N, Ω, p and s, such that for all $f \in L^\infty(\Omega)$ the weak solution $u \in W_0^{s,p}(\Omega)$ to problem*

$$\begin{cases} (-\Delta)_p^s(u) &= \quad f(u) \quad in \quad \Omega \\ \qquad u &= \quad 0 \quad in \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

satisfies $u/d_\Omega^s \in C^\alpha(\overline{\Omega})$ and

$$\left\| \frac{u}{d_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C\|f\|_\infty^{1/(p-1)}.$$

Definition 3.3. Let $\Omega \subseteq \mathbb{R}^N$ be bounded, We set

$$\tilde{W}^{s,p}(\Omega) := \{u \in L^p_{\text{loc}} : \exists U \supset\supset \Omega \text{ s.t. } \|u\|_{s,p} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}}\mathrm{d}x < \infty\}.$$

Definition 3.4. Let $\Omega \subseteq \mathbb{R}^N$ be bounded. We say that $u \in \tilde{W}^{s,p}(\Omega)$ is a weak super(sub)-solution of $(-\Delta)_p^s u = f$ in Ω if

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \geq (\leq)\langle f, \varphi\rangle$$

for each $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \geq 0$.

Definition 3.5. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. We say that Ω satisfies the interior ball condition if for all $y \in \partial\Omega$ there exist $x \in \Omega$ and an open ball $B_r(x)$ such that $B_r(x) \subset \Omega$ and $y \in \partial B_r(x)$.

The proof of the following two theorems can be found in [[15], page 4].

Theorem 3.2. *Let $c \in C(\overline{\Omega})$ be a non-positive function and $u \in \tilde{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of*

$$(-\Delta)_p^s u = c(x)|u|^{p-2}u \quad in \quad \Omega. \tag{3.21}$$

If Ω is bounded, and $u \geq 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ then either $u > 0$ in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.

Theorem 3.3. *Let Ω satisfy the interior ball condition in $x_0 \in \partial\Omega$, $c \in C(\overline{\Omega})$, and $u \in \tilde{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of (3.21). If Ω is bounded, $c(x) \leq 0$ in Ω and $u \geq 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ then either $u = 0$ a.e. in $\mathbb{R}^N$ or*

$$\liminf_{x \to x_0} \frac{u(x)}{\delta_R(x)^s} > 0$$

where $B_R \subseteq \Omega, x \in B_R, x_0 \in \partial B_R$ and $\delta_R(x)$ is the distance from x to $\mathbb{R}^N \setminus B_R$.

The proof of the following proposition can be found in [[24], page 1364].

Proposition (Comparison principle)**.** *Let Ω be a bounded set, $u, v \in \tilde{W}^{s,p}(\Omega)$ satisfy $u \leq v$ in $\mathbb{R}^N \setminus \Omega$ and, for all $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \geq 0$ in Ω,*

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \leq \int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

Then $u \leq v$ in Ω.

Theorem 3.4. *Let us assume that Ω is a bounded domain with $C^{1,1}$ boundary. Then there is $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ problem (3.1) has at least one positive weak solution $u_\lambda \in C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$.*

Proof. Arguing by contradiction, let $\{\lambda_j\}$ a sequence of positive numbers such that $\lambda_j \to 0$, as $j \to \infty$ and such that $|\{x \in \Omega : \quad u_{\lambda_j}(x) \leq 0\}| > 0$. Let $w_j := \frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}$. Then

$$(-\Delta)_p^s(w_j) = \|u_{\lambda_j}\|_\infty^{1-p}(-\Delta)_p^s(u_{\lambda_j}) = \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}.$$

By previous Lemmas and Theorem 3.1, there exists $\alpha \in (0, s]$ such that

$$\begin{aligned}\left\|\frac{w_j}{\mathrm{d}_\Omega^s}\right\|_{C^\alpha(\overline{\Omega})} &\leq \|\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}\|_\infty^{\frac{1}{p-1}} \\ &\leq C\|\lambda_j f(u_{\lambda_j})\lambda^{-r(p-1)}\|_\infty^{\frac{1}{p-1}} \\ &= C\lambda_j^r\|\lambda_j f(u_{\lambda_j})\|_\infty^{\frac{1}{p-1}} \\ &\leq C.\end{aligned}$$

where C does not dependent on λ_j. Let us choose any $0 < \beta < \alpha$. Since $C^\alpha(\overline{\Omega}) \subset\subset C^\beta(\overline{\Omega})$ see Theorem 5.13, [[17], page 102] then, up to a subsequence, $\lim_{j\to\infty} \frac{w_j}{\mathrm{d}_\Omega^s} = \frac{w}{\mathrm{d}_\Omega^s}$ in $C^\beta(\overline{\Omega})$. Now, we will use comparison principle to prove that $w(x) \geqslant 0$. Let $v_0 \in W_0^{s,p}(\Omega)$ be the solution of

$$\begin{cases} (-\Delta)_p^s u = 1, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N - \Omega. \end{cases}$$

Let $K_j = \frac{\lambda_j}{\|u_{\lambda_j}\|_\infty^{p-1}} \min_{t\in\mathbb{R}} f(t)$. Observe that $K_j < 0$. Then, the solution $v_j \in W_0^{s,p}(\Omega)$ of

$$\begin{cases} (-\Delta)_p^s u = K_j, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N - \Omega, \end{cases}$$

is given by $v_j = -(-K_j)^{1/(p-1)}v_0$. Since

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{\Phi_p(w_j(x) - w_j(y))}{|x-y|^{N+sp}}(\varphi(x) - \varphi(y))\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}\varphi\mathrm{d}x \geq \int_\Omega K_j\varphi\mathrm{d}x \\ &= \int_{\mathbb{R}^{2N}} \frac{\Phi_p(v_j(x) - v_j(y))}{|x-y|^{N+sp}}(\varphi(x) - \varphi(y))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then $(-\Delta)_p^s(w_j) \geq (-\Delta)_p^s(v_j)$. By the comparison principle we have $w_j \geqslant v_j$. Since $v_j \to 0$, as $j \to \infty$, then $w(x) \geqslant 0$.
Let us observe that from grow assumptions of f and (3.18) we have

$$\begin{aligned}\lambda_j|f(u_{\lambda_j}(x))|\|u_{\lambda_j}\|_\infty^{1-p} &\leqslant C\lambda_j(|u_{\lambda_j}(x)|^q + 1)\|u_{\lambda_j}\|_\infty^{1-p} \\ &\leqslant C\lambda_j(\|u_{\lambda_j}\|_\infty^q + 1)\|u_{\lambda_j}\|_\infty^{1-p} \\ &\leqslant C\lambda_j(\lambda_j^{-rq} + 1)\lambda_j^{r(p-1)} \\ &\leqslant C\lambda_j\lambda_j^{-rq}\lambda_j^{r(p-1)} \\ &= C\lambda_j^{1-rq+r(p-1)} = C.\end{aligned}$$

Which implies that $\{\|\lambda_j f(u_{\lambda_j})\|u_{\lambda_j})\|_\infty^{1-p}\}$ is bounded by a constant independent of λ_j. Therefore there exists $t > 1$ such that $\{\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}\}_j$ is bounded in $L^t(\Omega)$. Thus, we may assume that it converges weakly in $L^t(\Omega)$. Let $z := \lim_{j\to 0} \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}$, its weak limit. Since f is bounded from below and $\lim_{j\to\infty} \lambda_j \|u_{\lambda_j}\|_\infty^{1-p} = 0$, then $z \geqslant 0$. We claim that $(-\Delta)_p^s(w) = z$. In fact, from remark 3.2 and Lemma 3.5, the sequence of functions

$$\psi_j(x,y) := \frac{|w_j(x) - w_j(y)|}{|x-y|^{\frac{N}{p}+s}},$$

is bounded in $L^p(\mathbb{R}^{2N})$. Therefore, following the same procedure made in Lemma 3.3 to prove the strong convergence of $\{u_n\}$ (see Lemma 4.2 in the appendix), we conclude that it converges to

$$\psi(x,y) := \frac{|w(x) - w(y)|}{|x-y|^{\frac{N}{p}+s}},$$

in $L^p(\mathbb{R}^{2N})$. Then there exists $h \in L^p(\mathbb{R}^{2N})$ such that $|\psi_j(x,y)| \leqslant h(x,y)$, a.e. (x,y). Hence, from the Young's inequality, for all $\varphi \in W_0^{s,p}(\Omega)$ we have

$$\frac{|w_j(x) - w_j(y)|^{p-1}|\varphi(x) - \varphi(y)|}{|x-y|^{N+sp}} = \frac{|w_j(x) - w_j(y)|^{p-1}|\varphi(x) - \varphi(y)|}{|x-y|^{\frac{N+sp}{p'}}|x-y|^{\frac{N+sp}{p}}} \tag{3.22}$$

$$\leqslant \frac{1}{p'}\frac{|w_j(x) - w_j(y)|^{(p-1)p'}}{|x-y|^{N+sp}} + \frac{1}{p}\frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+sp}} \tag{3.23}$$

$$\leqslant \frac{1}{p'}(h(x,y))^p + \frac{1}{p}\frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+sp}},$$

where p' stands for the conjugate Hölder exponent of p. Since the last function belongs to $L^1(\mathbb{R}^{2N})$, by the Lebesgue's Dominated Convergence Theorem 4.5 we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\,\mathrm{d}y \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{|w_j(x) - w_j(y)|^{p-2}(w_j(x) - w_j(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\,\mathrm{d}y \\
&= \lim_{j\to\infty} \int_\Omega \lambda_j f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}\varphi(x)\mathrm{d}x \\
&= \int_\Omega z(x)\varphi(x)\mathrm{d}x.
\end{aligned} \tag{3.24}$$

Observe that we also proved that $w_j \to w$ in $W_0^{s,p}(\Omega)$, and thus $w \in W_0^{s,p}(\Omega)$. This proves the claim. Thus w is a supersolution of the $(-\Delta)_p^s(w) = 0$ in Ω. Since Ω has $C^{1,1}$ boundary then it satisfies the interior ball condition (see Theorem 1.0.9 in [[5], page 7]). Therefore, by Theorems 3.2 and 3.3 we have $w > 0$ in Ω and for all $x_0 \in \partial\Omega$,

$$\liminf_{x\to x_0} \frac{w(x)}{\mathrm{d}^s_{B_R(x)}} > 0,$$

where $B_R \subseteq \Omega$ and $x_0 \in \partial B_R$. From Lemma 4.1 there exists j sufficiently large such that $w_j > 0$ in Ω. Absurd. $\square$

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4 Appendix

4.1 Two Technical Lemmas

In this subsection we shall prove some technical results. The first one is based on the Hopf's Lemma established in [15]. The second, follows the same lines in part of the proof of Lemma 3.3.

Lemma 4.1. *Let us assume that $\Omega \subseteq \mathbb{R}^N$ is bounded domain with $C^{1,1}$ boundary and $\frac{w_j}{\mathrm{d}_\Omega^s} \to \frac{w}{\mathrm{d}_\Omega^s}$ in $C^\beta\big(\overline{\Omega}\big)$ with $w(x) = w_j(x) = 0$, for all j and all $x \in \partial\Omega$. Let us assume that $w > 0$ in Ω and for all $x_0 \in \partial\Omega$*

$$m := \liminf_{x \to x_0} \frac{w(x)}{\mathrm{d}_{B_R}^s(x)} > 0. \tag{4.1}$$

Then there exists j such that $w_j(x) > 0$ for all $x \in \Omega$.

Proof. First of all, let us emphasize that, since $\frac{w}{\mathrm{d}_\Omega^s} \in C^\beta\big(\overline{\Omega}\big)$, then for all $x_0 \in \partial\Omega$, $\frac{w(x_0)}{\mathrm{d}_\Omega^s(x_0)}$ is well defined in terms of limits. Now, let $B_R \subseteq \Omega$ be an interior ball such that $x_0 \in \partial B_R$ and let be $\varepsilon_0 > 0$ such that for all $x \in B_R \cap B(x_0, \varepsilon_0)$,

$$\frac{w(x)}{\mathrm{d}_{B_R}^s(x)} > \frac{m}{2}.$$

Let us pick up a sequence $\{x_n\}$ in $B_R \cap B(x_0, \varepsilon_0)$ in the segment joining x_0 and the center of B_R and such that $x_n \to x_0$. So that for all n, $x_n - x_0$ is orthogonal to ∂B_R and $\partial\Omega$ and $\mathrm{d}_{B_R}(x_n) = \mathrm{d}_\Omega(x_n)$. Therefore

$$\frac{w(x_0)}{\mathrm{d}_\Omega^s(x_0)} = \lim_{n\to\infty} \frac{w(x_n)}{\mathrm{d}_\Omega^s(x_n)} = \lim_{n\to\infty} \frac{w(x_n)}{\mathrm{d}_{B_R}^s(x_n)} \geqslant \frac{m}{2} > 0.$$

And, obviously, $\frac{w(x)}{\mathrm{d}_\Omega^s(x)} > 0$ for all $x \in \Omega$. Thus $\frac{w}{\mathrm{d}_\Omega^s}$ is positive in the compact $\overline{\Omega}$. Let

$$\varepsilon := \min \frac{w}{\mathrm{d}_\Omega^s} > 0. \tag{4.2}$$

We claim that $w_j(x_0) > 0$ for some $x_0 \in \Omega$. Indeed by contradiction, suppose that $w_j(x_0) \leq 0$ for all $x_0 \in \Omega$, notice that $w(x_0) \geq w_j(x_0)$ and $\left\| \frac{w}{\mathrm{d}_\Omega^s} - \frac{w_j}{\mathrm{d}_\Omega^s} \right\| < \frac{\varepsilon}{2}$, therefore there exists j such that

$$\varepsilon \leq \frac{w(x_0)}{\mathrm{d}_\Omega^s(x_0)} \leq \frac{w(x_0)}{\mathrm{d}_\Omega^s(x_0)} - \frac{w_j(x_0)}{\mathrm{d}_\Omega^s(x_0)} < \frac{\varepsilon}{2}.$$

Finally let us prove that $w_j(x) > 0$ for all $x \in \Omega$. Let us argue by contradiction. If there exists $x_0 \in \Omega$ such that $w_j(x_0) \leq 0$ then, by the intermediate Value Theorem, there is $z_0 \in \Omega$ such that $w_j(z_0) = 0$. Thus, from (4.2) we have

$$\begin{aligned} \epsilon &\leqslant \left| \frac{w(z_0)}{\mathrm{d}_\Omega^s(z_0)} - \frac{w_j(z_0)}{\mathrm{d}_\Omega^s(z_0)} \right| \\ &\leqslant \left\| \frac{w}{\mathrm{d}_\Omega^s} - \frac{w_j}{\mathrm{d}_\Omega^s} \right\|_{C^\beta(\overline{\Omega})} < \frac{\epsilon}{2}. \end{aligned} \tag{4.3}$$

Absurd. $\square$

Lemma 4.2. *Let $\{w_j\}$ be a bounded sequence in $W_0^{s,p}(\Omega)$, such that*

$$\begin{cases} (-\Delta)_p^s(w_j) & = & \lambda_j g(w_j) \ \textit{in} & \Omega \\ w_j(x) & = & 0 \qquad \textit{in} & \mathbb{R}^N - \Omega, \end{cases}$$

with $\{\lambda_j g(w_j)\}$ bounded in $L^\infty(\Omega)$. Then w_j converges strongly in $W_0^{s,p}(\Omega)$.

Proof. Since $\{w_j\}$ is bounded in $W_0^{s,p}(\Omega)$, then, up to a subsequence, $\{w_j\}$ converges weakly to the function $v \in W_0^{s,p}(\Omega)$. Since $p < q+1 < p_s^*$, then $w_j \to v$ (strongly) in $L^{q+1}(\Omega)$. As $\{\lambda_j g(w_j)\}$ bounded in $L^\infty(\Omega)$, applying the Hölder inequality this implies that

$$\lim_{j\to\infty} \lambda_j \int_\Omega g(w_j)(w_j - v)\mathrm{d}x = 0.$$

Then, since $J'_{\lambda_j}(w_j) = 0$ (where J_λ is the associated Energy Functional to this problem), we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{\Phi_p(w_j(x) - w_j(y))((w_j - v)(x) - (w_j - v)(y))}{|x-y|^{N+sp}} = 0. \tag{4.4}$$

Using again that v is the weak limit of w_j we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^{2N}} \frac{\Phi_p(v(x) - v(y))((w_j - v)(x) - (w_j - v)(y))}{|x-y|^{N+sp}} = 0. \tag{4.5}$$

Thus, from the same argument that we use in the proof of Lemma 3.3 we obtain

$$\int_\Omega \frac{\Phi_p(w_j(x) - w_j(y)) - \Phi_p(v(x) - v(y))}{|x-y|^{N+sp}}((w_j - v)(x) - (w_j - v)(y))\mathrm{d}x\mathrm{d}y$$
$$\geqslant (\|w_j\|^{p-1} - \|v\|^{p-1})(\|w_j\| - \|v\|) \geqslant 0.$$

From (4.4), (4.5) we obtain

$$\lim_{j\to\infty} (\|w_j\|^{p-1} - \|v\|^{p-1})(\|w_j\| - \|v\|) = 0,$$

which implies

$$\lim_{j\to\infty} \|w_j\| = \|v\|.$$

Since $w_j \rightharpoonup v$, then $w_j \to v$ strongly in $W_0^{s,p}(\Omega)$. $\square$

4.2 Some known Results

We state some classical results that we have used through this thesis. We leave the references for the statements that we will not prove.

Definition 4.1. Let X be a vector space over $\mathbb{R}$, we recall that a functional is a function defined on X, or on some subspace of X, with values in $\mathbb{R}$. We denote by X^* the dual space of X that consists of all bounded linear functionals from X to $\mathbb{R}$. Furthermore X^* is a Banach space (whether or not X is) endowed with the norm

$$\|u\|_{X^*} = \sup_{\substack{\|x\|\leq 1\\ x\in X}} |u(x)| = \sup_{\substack{\|x\|\leq 1\\ x\in X}} u(x).$$

Given $u \in X^*$ and $x \in X$ we say that $\langle \ , \ \rangle$ is the duality product between X^* and X. We will often write $\langle u, x\rangle$ instead of $u(x)$.

Notation 4.1. Let X and Y be normed vector spaces. We denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators $A : X \to Y$ endowed with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x\in X, \|x\|\leq 1} \|Ax\|.$$

For the sake of simplicity of notation we will write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

Definition 4.2. Let X and Y be normed vector spaces with $X \subset Y$. We say that X is continuously embedded in Y and we will write $X \hookrightarrow Y$ if there exists a constant $C \geq 0$ such that

- $||u||_Y \leq C||u||_X$ for all $u \in X$,

and we say that X is compactly embedded in Y and we will write $X \subset\subset Y$ if $X \hookrightarrow Y$ and

- given $\{u_n\}$ is a bounded sequence in X, then there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ convergent in Y.

Definition 4.3. The weak topology on a normed vector space X, denoted by $\sigma(X, X^*)$ is the topology generated by the continuous linear functionals $A \in X^*$. If a sequence $\{u_n\}$ converges to u in the weak topology we write $u_n \rightharpoonup u$, and to emphasize strong convergence we write $u_n \to u$ meaning that $\|u_n - u\|_X \to 0$.

The proof of the following Theorem can be found in [[29], page 59].

Theorem 4.1 (Mean Value Theorem)**.** *Let Ω be an open set in $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be differentiable on Ω. If Ω contains the line segment between x and $x + h$, then there is a point $c = x + th$ with $0 < t < 1$ such that*

$$u(x + h) - u(x) = \nabla u(c) \cdot h. \tag{4.6}$$

Remark 4.1. The Cauchy-Schwarz inequality together with the equation (4.6) imply

$$|u(x + h) - u(x)| \leq |\nabla u(c)||h|.$$

Definition 4.4. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called convex if

$$u(tx + (1 - t)y) \leq tu(x) + (1 - t)u(y)$$

for every $x, y \in \mathbb{R}^N$ and every $t \in [0, 1]$.

The proof of the following Theorem can be found in [[20], Page 705].

Theorem 4.2 (Jensen's Inequality)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset such that $|\Omega| = 1$ and let $f : \mathbb{R}^M \to \mathbb{R}$ be a convex function. Assume that $u : \Omega \to \mathbb{R}^M$ is integrable. Then*

$$f\left(\int_\Omega u \mathrm{d}x\right) \leq \int_\Omega f(u)\mathrm{d}x.$$

Definition 4.5. Let k be a non-negative integer. The space $C^k(\Omega)$ consists of all functions with k continuous derivatives. The space $C_0^\infty(\Omega)$ consists of all functions with compact support in Ω and infinitely differentiable. The functions $\varphi \in C_0^\infty(\Omega)$ are called test functions.

4.3 L^p Spaces

The space of all measurable functions defined on a set X is denoted by $\mu(X)$.

Definition 4.6. We denote by $L^1(\Omega)$ the space of all integrable functions from Ω to $\mathbb{R}$. Let $p \in [1, \infty)$ and

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \ \text{ measurable and } |u|^p \in L^1(\Omega)\}$$

be the space of L^p-integrable functions over Ω with

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \mathrm{d}x\right)^{1/p}.$$

It is well known that $\| \cdot \|_p$ is a norm, see [[26], page 221].

The definition of L^∞ is not closely related to L^p for $p < \infty$ since integration is not involved. However, the only important aspect of the measure $|\Omega|$ is the definition of almost everywhere concept.

Definition 4.7. Let

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \ \text{ measurable and } |u(x)| \leq C \text{ a.e. in } \Omega \text{ for some } C\}$$

be the space of all essentially bounded measurable functions on Ω with

$$\|u\|_\infty = \inf\{C \ : \ |u(x)| \leq C \text{ a.e. on } \Omega\}$$

.

The following remark implies that $\| \cdot \|_\infty$ is a norm.

Remark 4.2. If $u \in L^\infty(\Omega)$ then $|u(x)| \leq \|u\|_\infty$ for a.e $x \in \Omega$. Indeed there exists a sequence C_n such that $C_n \to \|u\|_\infty$, and for all n $|u(x)| \leq C_n$ a.e. on Ω, i.e. $|u(x)| \leq C_n$ for all $x \in \Omega \setminus E_n$ where $|E_n| = 0$. Set $E = \cup_{n=1}^\infty E_n$, then $|E| = 0$, therefore $|u(x)| \leq C_n$ for all n and for all $x \in \Omega \setminus E$, taking the limit when $n \to \infty$ we get that $|u(x)| \leq \|u\|_\infty$ for all $x \in \Omega \setminus E$.

The proof of the following Theorem can be found in [[7], page 98].

Theorem 4.3. *Let $p \in [1, \infty)$. The space $L^p(\Omega)$ is separable.*

Proposition 4.1. *If $p \in (0, \infty)$ then $|a - b|^p \leq \max\{1, 2^{p-1}\}(|a|^p + |b|^p)$ see [[31], page 73].*

The proof of the following Theorem can be found in [[7], Page 97].

Theorem 4.4. *The space $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$.*

The proof of the following Theorem can be found in [[26], Page 133].

Theorem 4.5 (Lebesgue's Dominated Convergence Theorem)**.** *Let $\{u_n\} \subset L^1(\Omega)$ be a sequence such that*

- *$u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$;*
- *there exists $v \in L^1(\Omega)$ such that for all n, $|u_n(x)| \leq v(x)$ a.e. in Ω.*

Then $u \in L^1(\Omega)$ and $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

The proof of the following Proposition can be found in [[4], Page 10].

Proposition 4.2. *Let $p \in [1, \infty)$ and $\{u_n\} \in L^p(\Omega)$ be a sequence such that $u_n \to u$ in $L^p(\Omega)$. Then, there exist a subsequence $\{u_{n_j}\}$ and a function $h \in L^p(\Omega)$ such that*

- *$u_{n_j}(x) \to u(x)$ a.e. in Ω.*
- *For all j, $|u_{n_j}(x)| \leq h(x)$ a.e. in Ω.*

The proof of the following Lemma can be found in [[26], Page 129].

Lemma 4.3 (Fatou's Lemma)**.** *Let $\mathcal{M}$ will be a σ-algebra in a set X and μ be a positive measure on $\mathcal{M}$. Assume that for each n, u_n is nonnegative measurable function. Then*

$$\int_X \left(\liminf_{n\to\infty} u_n\right) \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X u_n \mathrm{d}\mu.$$

The proof of the following Theorem can be found in [[26], Page 129].

Theorem 4.6 (Change of Variables Theorem)**.** *Suppose that Ω and Ω' are open and bounded sets in $\mathbb{R}^N$ and assume that the function $\phi : \Omega \to \Omega'$ is a bijection of class C^1 whose inverse is also of class C^1. Let $u : \Omega' \to \mathbb{R}$ be a measurable function, then $u \circ \phi$ is measurable on Ω' and*

$$\int_{\Omega'} u(y)\mathrm{d}y = \int_\Omega u(\phi(x))|J(x)|\mathrm{d}x$$

where $J(x) = det\phi'(x)$ is the Jacobian determinant of ϕ at x.

The proof of the following Proposition can be found in [[26], Page 194].

Proposition 4.3. *The integrals on $\mathbb{R}^N$ satisfy:*

1. *$\int_{B(0,1)} \frac{1}{|x|^\alpha}\mathrm{d}x < \infty$ if and only if $\alpha < N$,*
2. *$\int_{B(0,1)^c} \frac{1}{|x|^\alpha}\mathrm{d}x < \infty$ if and only if $\alpha > N$.*

4.4 Sobolev Spaces

Before discussing the Classical Sobolev spaces we introduce the notion of Hölder spaces.

Definition 4.8. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset and $\gamma \in (0,1]$. The functions $u : \Omega \to \mathbb{R}$ that satisfy

$$|u(x) - u(y)| \leq C|x-y|^\gamma$$

for all $x, y \in \Omega$ and for some constant C, are called Hölder continuous with exponent γ.

The Hölder seminorm of $u : \Omega \to \mathbb{R}$ is defined as

$$[u]_{0,\gamma} \coloneqq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}$$

It is not difficult to see that this is not a norm, for example consider a constant nonzero function.
We define the Hölder norm as

$$\|u\|_{0,\gamma} \coloneqq \|u\|_{C(\overline{\Omega})} + [u]_{0,\gamma},$$

where $\|u\|_{C(\Omega)} \coloneqq \sup_{x\in\Omega} |u(x)|$. The proof of the fact that this is a norm is straightforward.

Definition 4.9. Let

$$C^{k,\gamma}(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} \ : \ u \text{ has } k \text{ continuous derivatives and } \|u\|_{k,\gamma} < \infty\}$$

be the Hölder space endowed with the norm

$$\|u\|_{k,\gamma} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{0,\gamma}.$$

Definition 4.10. A vector of the form $\alpha = (\alpha_1, \cdots, \alpha_N)$ is called a multi index of order $|\alpha| := \alpha_1 + \cdots + \alpha_N = k$. Each multi index defines a partial differential operator of order $|\alpha|$, given by

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u.$$

Notice that $\nabla u = (D^{e_1}u, D^{e_2}u, \cdots, D^{e_N}u)$ where $e_i = (0, \cdots, 1, \cdots, 0)$.

Definition 4.11. Let

$$L^1_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} \ \text{ measurable and } \int_K |u(x)|\mathrm{d}x < \infty \ \text{ for all } \ K \subseteq \Omega \text{ compact}\}.$$

be the space of locally integrable functions.

Definition 4.12. If $\partial\Omega$ is of class C^1, then along $\partial\Omega$ is defined the outward pointing unit normal vector

$$\nu = (\nu^1, \dots, \nu^N)$$

Theorem 4.7 (Integration by parts formula). *Let $u, v \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega u_{x_i} v \mathrm{d}x = -\int_\Omega u v_{x_i} \mathrm{d}x + \int_{\partial\Omega} u v \nu^i \mathrm{d}s \qquad i = 1, \dots, n.$$

Definition 4.13. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and α a multi index. If there exists $v \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u D^\alpha \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \varphi \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$, then v is called the $\alpha -$ th weak partial derivative of u denoted by

$$D^\alpha u = v.$$

If the function v does not exist we say that u has not α-th weak partial derivative

We now establish the definition of the function spaces that we call Sobolev Spaces whose functions have weak derivatives of various orders lying in L^p spaces.

Definition 4.14. Let k be a non-negative integer and $p \in [1, \infty)$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L^p(\Omega)$. It is endowed with the norm

$$||u||_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \mathrm{d}x \right)^{1/p}.$$

The proof of the following Proposition can be found in [[20], page 279].

Proposition 4.4. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded subset and $p \in [1, N)$. Then*

$$||u||_q \leq C ||u||_{W^{1,p}(\Omega)}$$

for all $q \in [1, p^*]$ and some constant $C > 0$ depending on N, p and Ω. Here $p^* = \frac{Np}{N-p}$ is called the Sobolev critical exponent of p.

The proof of the following Theorem can be found in [[7], page 265]

Theorem 4.8. *Let $u \in W^{1,p}(\Omega)$. Then there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that*

- *$u_n|_\Omega \to u$ in $L^p(\Omega)$ and*
- *$\nabla u_n|_{\Omega'} \to \nabla u|_{\Omega'}$ in $L^p(\Omega')^N$ for all $\Omega' \subset\subset \Omega$.*

In case $\Omega = \mathbb{R}^N$ *and* $u \in W^{1,p}(\mathbb{R}^N)$, *there exists a sequence* $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ *such that*

- $u_n \to u$ *in* $L^p(\mathbb{R}^N)$ *and*
- $\nabla u_n \to \nabla u$ *in* $L^p(\mathbb{R}^N)^N$.

Proposition 4.5. *Let* $u \in L^p(\Omega)$. *Then* $u \in W^{1,p}(\Omega)$ *if and only if there exists a constant* C *such that for all* $\Omega' \subset\subset \Omega$ *and* $h \in \mathbb{R}^N$ *with* $|h| < dist(\Omega', \mathbb{R}^N \setminus \Omega)$,

$$\int_{\Omega'} |u(x+h) - u(x)|^p \mathrm{d}x \leq C|h|^p.$$

Moreover, $C = (\int_\Omega |\nabla u(y)|^p \mathrm{d}y)^{1/p}$. *In particular taking* $\Omega = \mathbb{R}^N$ *we obtain*

$$\int_{\mathbb{R}^N} |u(x+h) - u(x)|^p \mathrm{d}x \leq |h|^p \int_{\mathbb{R}^N} |\nabla u(y)|^p \mathrm{d}y. \tag{4.7}$$

Proof. We consider two cases: $u \in C_0^\infty(\mathbb{R}^N)$ and $u \in W^{1,p}(\Omega)$. If $u \in C_0^\infty(\mathbb{R}^N)$, take $h \in \mathbb{R}^N$, $t \in \mathbb{R}$ and define $v_x(t) = u(x+th)$, then $v_x'(t) = \nabla u(x+th) \cdot h$. Using the Change of Variable Theorem and Fubini's Theorem we have,

$$\begin{aligned}
\int_{\Omega'} |u(x+h) - u(x)|^p \mathrm{d}x &= \int_{\Omega'} |v_x(1) - v_x(0)|^p \mathrm{d}x \\
&= \int_{\Omega'} \left| \int_0^1 v_x'(t) \mathrm{d}t \right|^p \mathrm{d}x \\
&= \int_{\Omega'} \left| \int_0^1 h \cdot \nabla u(x+th) \mathrm{d}t \right|^p \mathrm{d}x \\
&\leq |h|^p \int_0^1 \int_{\Omega'} |\nabla u(x+th)|^p \mathrm{d}x\mathrm{d}t \\
&\leq |h|^p \int_0^1 \int_{\Omega'+th} |\nabla u(y)|^p \mathrm{d}y\mathrm{d}t.
\end{aligned}$$

Consider $\delta = \mathrm{dist}(\Omega', \mathbb{R}^N \setminus \Omega) > 0$, $W = \{x \in \Omega : \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega) \geq \delta/2\} \supseteq \Omega'$ and $|h| < \frac{\delta}{2}$. We claim that given $t \in [0,1]$, $\Omega' + th \subseteq W \subset\subset \Omega$. To check this let $\tilde{x} \in \Omega' + th$ and $\tilde{x} \notin W$. Then $\tilde{x} = x + th$ with $x \in \Omega'$, $t \in [0,1]$ and $\mathrm{dist}(\tilde{x}, \mathbb{R}^N \setminus \Omega) < \frac{\delta}{2}$.
For all $y \in \mathbb{R}^N \setminus \Omega$ we have

$$\delta \leq |x - y| \leq |x - \tilde{x}| + |\tilde{x} - y| = t|h| + |\tilde{x} - y| < \frac{\delta}{2} + |\tilde{x} - y|,$$

Hence,

$$\frac{\delta}{2} < |\tilde{x} - y|. \tag{4.8}$$

Then

$$\frac{\delta}{2} \leq \mathrm{dist}(\tilde{x}, \mathbb{R}^N \setminus \Omega) < \frac{\delta}{2}. \tag{4.9}$$

which is absurd. Thus $\Omega' + th \subseteq W$.
On the other hand let us see that $W \subset\subset \Omega$. Indeed, suppose that $x \in \overline{W}$ and $x \in \mathbb{R}^N \setminus \Omega$, if we consider the open ball $B(x, \delta/4)$ we have that $B(x, \delta/4) \cap W \neq \emptyset$ which that there is $z \in B(x, \delta/4)$ and $z \in W$ such that $\operatorname{dist}(z, \mathbb{R}^N \setminus \Omega) \leq |z - x| < \delta/4$ and $\operatorname{dist}(z, \mathbb{R}^N \setminus \Omega) \geq \delta/2$, which is absurd, hence $\overline{W} \subseteq \Omega$. The above claim and the inequalities (4.8) and (4.9) allow us to conclude that

$$\int_{\Omega'} |u(x+h) - u(x)|^p \mathrm{d}x \leq |h|^p \int_W |\nabla u(y)|^p \mathrm{d}y. \tag{4.10}$$

On the other hand if $u \in W^{1,p}(\Omega)$ by Theorem 4.8 in there exists a sequence $\{u_n\}$ in $C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ in $L^p(\Omega')$ for all $\Omega' \subset\subset \Omega$. Taking the limit when $n \to \infty$ in (4.7) we obtain the desired result. □

4.5 Differentiability for Real Functionals

In this subsection X will be a Banach space. We start presenting two definitions about differentiability for real functionals: The Fréchet derivative which is a derivative defined on Banach spaces and the Gâteaux derivative which is a generalization of the well known concept of directional derivative.

Definition 4.15. Let $\Omega \subseteq X$ be an open subset and $E : \Omega \to \mathbb{R}$ be a functional. We say that E is Fréchet differentiable at $x \in \Omega$ if there exists $A \in X^*$ such that for all $h \in X$

$$\lim_{\|h\| \to 0} \frac{E(x+h) - E(x) - Ah}{\|h\|} = 0.$$

Definition 4.16. Let $\Omega \subseteq X$ be an open subset and $E : \Omega \to \mathbb{R}$ be a functional. We say that E is Gâteaux differentiable at $x \in \Omega$ if there exists $A \in X^*$ such that for all $h \in X$

$$\lim_{t \to 0} \frac{E(x+th) - E(x)}{t} = Ah.$$

Remark 4.3. If E is Gâteaux differentiable at x, then there exists a unique linear functional $A \in X^*$ which satisfies the previous equality. It is called the Gâteaux differential of E at x and is denoted by $E'_G(u)$.

The next Proposition points out an important result that technically says that it is easier to compute the Gâteaux derivative and then prove that it is continuous instead of proving Fréchet's differentiability directly. The proof can be found in [[4], page 14].

Proposition 4.6. *Let $\Omega \subseteq X$ be an open subset. If E is a Gâteaux differentiable functional in Ω and E'_G is continuous at $x \in \Omega$ then E is Fréchet differentiable at x and, $E'_G(x) = E'(x)$.*

Definition 4.17. We say $E \in C^1(\Omega, \mathbb{R})$ if $E'(u)$ exists for each $u \in \Omega$ and the mapping $E' : \Omega \to \Omega$ is continuous.

Notation 4.2.
1. We denote by $\mathcal{C}$ the collection of functions $E \in C^1(\Omega, \mathbb{R})$ such that $E' : \Omega \to \Omega$ is Lipschitz continuous on bounded subsets of Ω.
2. If $c \in \mathbb{R}$, we write $A_c = \{u \in \Omega | E(u) \leq c\}$ y $K_c = \{u \in \Omega | E(u) = c \ \text{ o } \ E'(u) = 0\}$.

Definition 4.18. Let $\Omega \subseteq X$ be an open subset and $E : \Omega \to \mathbb{R}$ be a differentiable functional. A critical point of E is a point $x \in \Omega$ such that $E'(x) = 0$. A real number c is a critical value of E if $E'(x) = 0$ and $E(x) = c$.

The next definition present a kind of compactness condition which is useful to guarantee the existence of certain critical points.

Definition 4.19. We say that a differentiable functional $E : X \to \mathbb{R}$ satisfies the Palais-Smale condition if: for every sequence $\{u_n\} \subset X$ such that

1. $\{E(u_n)\}$ is bounded in $\mathbb{R}$,
2. $E'(u_n) \to 0$ in X as $n \to \infty$,

there exists a convergent subsequence in X.

Definition 4.20. A deformation of Ω is a continuous function $\eta : [0,1] \times \Omega \to \Omega$ such $\eta(0, u) = u$ for all $u \in \Omega$.

The proof of the following theorem can be found in [[20], page 503].

Theorem 4.9 (Deformation Theorem)**.** *Assume $E \in \mathcal{C}$ satisfies the Palais-Smale condition. Let $K_c = \emptyset$ then for each sufficiently small $\epsilon > 0$, there exists a constant $0 < \delta < \epsilon$ and a deformation η such that for all $0 \leq t \leq 1$ and $u \in \Omega$ the functions $\eta_t(u) = \eta(t, u)$ satisfy*

1. *$\eta_0(u) = u$ with $u \in \Omega$.*
2. *$\eta_1(u) = u$ with $E(u) \notin (c - \epsilon, c + \epsilon)$.*
3. *$E(\eta_t(u)) \leq E(u)$ with $u \in \Omega$ and $0 \leq t \leq 1$.*
4. *$\eta_1(A_{c+\delta}) \subset A_{c-\delta}$.*

Theorem 4.10 (Mountain Pass Theorem)**.** *Let $E : X \to \mathbb{R}$ be a differentiable functional such that $E' : X \to \mathbb{R}$ is Lipschitz continuous on bounded subsets of X. Assume that E satisfies the Palais-Smale condition, suppose also that*

1. *$E(0) = 0$,*
2. *there exists positive constants r and a such that $E(u) \geq a$ if $\|u\| = r$,*
3. *there exists $v \in X$ such that $\|v\| > r$ and $E(v) \leq 0$.*
   If we define

$$\Gamma := \{\gamma([0,1]) \mid \gamma : [0,1] \to X \quad \textit{is continuous} \quad , \gamma(0) = 0, \gamma(1) = v\},$$

then

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} E(\gamma(t))$$

is a critical value of E*.*

Proof. By contradiction, suppose that c is not a critical value of E, i.e. $K_c = \emptyset$. Given a sufficiently small ϵ such that $0 < \epsilon < \frac{a}{2}$, from Theorem 4.10 there exist a constant $0 < \delta < \epsilon$, a deformation $\eta : \Omega \to \Omega$ with

$$\eta(A_{c+\delta}) \subset A_{c-\delta} \tag{4.11}$$

and

$$\eta(u) = u \ \text{ for each } \ E(u) \notin (c - \epsilon, c + \epsilon).$$

Let us take $\gamma \in \Gamma$ such that

$$\max_{0 \le t \le 1} E(\gamma(t)) \le c + \delta. \tag{4.12}$$

Consider $\phi : [0,1] \to \mathbb{R}$, by $\phi(t) = ||\gamma(t)||$, clearly $\phi(0) = 0$ and $\phi(1) = ||v|| > r$, now there is $t \in [0,1]$ such that $\phi(t) = ||\gamma(t)|| = r$, thus $E(\gamma(t)) \ge a$, and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} E(\gamma(t)) \ge a.$$

Define, $\hat{g} = \eta \circ \gamma$, since $\hat{g}(1) = (\eta \circ \gamma)(1) = \eta(v) = v$ and $\hat{g}(0) = (\eta \circ \gamma)(0) = 0$ for a sufficiently small ϵ, we get $\hat{g} \in \Gamma$. By (4.11) we have $E(\eta(u)) \le c - \delta$ and by (4.12) we obtain

$$\max_{0 \le t \le 1} I(\hat{g}(t)) \le c - \delta.$$

Hence

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)) \le c - \delta,$$

which is absurd. □