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Constructible sets in lattice-valued models.

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Conjuntos construibles en modelos valuados en retículos.

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Why? Because it's there.

George Mallory.

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Abstract

We investigate different set-theoretic constructions in Residuated Logic based on Fitting's work on Intuitionistic Kripke models of Set Theory.

Firstly, we consider constructable sets within valued models of Set Theory. We present two distinct constructions of the constructable universe: $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$, and prove that the they are isomorphic to V (von Neumann universe) and L (Gödel's constructible universe), respectively.

Secondly, we generalize Fitting's work on Intuitionistic Kripke models of Set Theory using Ono and Komori's Residuated Kripke models. Based on these models, we provide a generalization of the von Neumann hierarchy in the context of Modal Residuated Logic and prove a translation of formulas between it and a suited Heyting valued model. We also propose a notion of universe of constructable sets in Modal Residuated Logic and discuss some aspects of it.

Keywords: Valued models, abstract logics, residuated lattices, Kripke models, constructible sets

Resumen

Investigamos diferentes construcciones de la teoría de conjuntos en Lógica Residual basados en el trabajo de Fitting sobre los modelos intuicionistas de Kripke de la Teoría de Conjuntos. En primer lugar, consideramos conjuntos construibles dentro de modelos valuados de la Teoría de Conjuntos. Presentamos dos construcciones distintas del universo construible: $\mathfrak{L}^{\mathbb{B}}$ y $\mathbb{L}^{\mathbb{B}}$, y demostramos que son isomorfos a V (universo von Neumann) y L (universo construible de Gödel), respectivamente.

En segundo lugar, generalizamos el trabajo de Fitting sobre los modelos intuicionistas de Kripke de la teoría de conjuntos utilizando los modelos residuados de Kripke de Ono y Komori. Con base en estos modelos, proporcionamos una generalización de la jerarquía de von Neumann en el contexto de la Lógica Modal Residuada y demostramos una traducción de fórmulas entre ella y un modelo Heyting valuado adecuado. También proponemos una noción de universo de conjuntos construibles en Lógica Modal Residuada y discutimos algunos aspectos de la misma.

Palabras clave: Modelos valuados, lógicas abstractas, retículos residuales, modelos de Kripke, conjuntos construibles

Contents

Symbol list

Symbols

Symbol	Meaning	Definition
$\mathcal{P}(A)$	the set of all the subsets of A,	1
V	von Neumann's Universe,	1.1.2
L	Gödel's Universe,	1.1.8
$\mathcal{L}_\in$	the language of Set Theory,	1.1.1
$\Vdash_p$	Kripke forcing relation,	3.1.2
$\&$	strong conjunction,	1.3.11
$\mathbb{P}^+$	hereditary subsets of $\mathbb{P}$,	3.1.4
$D(\mathbb{P})$	$\cap$-closed subsets of $\mathbb{P}$,	3.1.23
$\mathbb{P}^*$	strongly hereditary subsets of $\mathbb{P}$,	3.1.24

Symbol	**Meaning**	**Definition**
$\mathbb{B}$	Boolean algebra,	1.2.2
$\mathbb{H}$	Heyting algebra,	1.2.6
$\mathbb{Q}$	quantale,	1.2.23
$\hat{\cdot}$	canonical immersion into $V^{\mathbb{B}}$,	2.2.18
$<$	well-founded relation on $V^{\mathbb{B}}$,	2.2.9
$\Diamond$	possibility symbol,	1.3.2
$\Box$	necessity symbol,	1.3.2
φ^G	Gödel-Gentzen translation of φ,	1.3.5
φ^{GK}	Gödel-Kolmogorov translation of φ,	1.3.6
α^{GMT}	Gödel–McKinsey–Tarski translation of α,	1.3.8
j	nucleus on $\mathbb{H}$,	1.2.16
γ	quantic nucleus on $\mathbb{Q}$,	1.2.43
δ	conucleus on $\mathbb{P}$,	3.1.29
$\vdash_i$	Intuitionistic deductions,	1.3.1
$\vdash_r$	Residuated deductions,	1.3.17
γ_δ	quantic nucleus on $\mathbb{P}^*$ determined by δ,	3.1.37

Symbol	Meaning	Definition
$\mathcal{F}_\gamma$	filter determined by γ,	1.2.58
$\mathcal{F}_{\neg\neg}$	filter of dense element of $\mathbb{H}$,	1.2.20
$\mathbb{H}/\mathcal{F}$	quotient of $\mathbb{H}$ modulo $\mathcal{F}$,	1.2.14
$\mathbb{Q}/\mathcal{F}$	quotient of $\mathbb{Q}$ modulo $\mathcal{F}$,	1.2.42
$\mathbb{H}_j$	set of fixed points of a nucleus j on $\mathbb{H}$,	1.2.16
$\mathbb{Q}_\gamma$	set of fixed points of a quantic nucleus γ on $\mathbb{Q}$,	1.2.46
$(\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$	complete SO-commutative monoid,	3.1.14
$(\mathbb{P}, \leqslant, \Vdash, \mathcal{P})$	Intuitionistic Kripke $\mathcal{L}$-model,	3.1.1
$(\mathbb{P}, \leqslant, \Vdash, D)$	Residuated Kripke $\mathcal{L}$-model,	3.1.17
$(\mathbb{P}, \leqslant, \delta, \Vdash, D)$	Modal Residuated Kripke $\mathcal{L}$-model,	3.1.34
$\mathcal{V}^{\mathbb{P}*}$	MR-Kripke model of Set Theory,	3.2.20
$R^{\mathbb{B}}$	Fitting's Boolean-valued universe,	3.2.9
$R^{\mathbb{H}}$	Fitting's Heyting-valued universe,	3.2.23
$V^{\mathbb{B}}$	the Boolean-valued universe,	2.2.1
$V^{\mathbb{H}}$	the Heyting-valued universe,	2.2.20
$V^{\mathbb{Q}}$	the quantale-valued universe,	2.2.23

Symbol	**Meaning**	**Definition**
$\mathfrak{L}^{\mathbb{B}}$	the class of $\mathbb{B}^*$-definable sets,	2.3.2
$\mathbb{L}^{\mathbb{B}}$	the class of $\mathbb{B}$-definable sets,	2.3.11

Introduction

The notion of constructibility (in Set Theory) started with Gödel's work [Göd38] on the consistency of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH). Gödel considers the class of **definable** sets in **Classical first-order Logic** in the language of Set Theory, now called Gödel's constructable universe.

At the beginning, Gödel's idea of considering definable sets in some logic was not widely used in the construction of new inner models, but rather, using different set-theoretical techniques, new inner models, such as HOD or $L[\mathcal{U}]$, were defined that allowed the advancement of Set Theory, especially in the realm of independence results. Nonetheless, a couple of attempts were made to generalize Gödel's idea of a class of definable sets: Scott and Myhill [SM71] showed that the well-known model HOD can be obtained as the definable sets in **second-order logic** and Chang [Cha71] showed that if one considers the definable sets in the **infinitary logic** L_{ω_1,ω_1}, an inner model is obtained that is characterized by being the smallest inner model that is closed under countable sequences. Although these results are interesting, no meaningful study of inner models arising from different logics was considered for a very long time.

It was not until the work of Kennedy, Magidor and Väänänen [KMV20] that inner models of Set Theory that arise when considering definable sets in generalized logics were systematically studied. The logics considered were **strengthenings** of first-order logic constructed by using generalized quantifiers or by allowing infinite disjunctions, conjunctions, or quantification. Some notable examples include the Stationary Set Theory, logics with cofinality quantifiers, the Härtig quantifier and the Magidor-Malitz quantifier. Such models made it possible to study new independence results in Set Theory.

Therefore, one could ask if such constructions could be done in logics that are **weakenings** (rather than strengthenings, as we just saw) of Classical first-order Logic. We would like to study logics general enough to capture the most important logical examples, such as **Intuitionistic** and **Fuzzy Logic**, but not so general that we lose too many structural rules, such as the commutativity of the premises in a deduction. Therefore, we are interested in studying constructibility in the context of the so called **substructural logics** without contraction (but with the exchange rule). Essentially, we consider a weakening of Intuitionistic Logic in which we consider two types of conjunctions: & and $\wedge$. The strong conjunction (denoted &) is no longer idempotent, that is,

$$\alpha \to (\alpha \& \alpha)$$

no longer holds for all formulas α. The defining feature of this connective is that is the left adjoint to the implication (just as $\wedge$ is for the classical and intuitionistic case),

$$\alpha \& \beta \Rightarrow \gamma \text{ if and only if } \alpha \Rightarrow \beta \to \gamma.$$

We also consider a weak conjunction (denoted $\wedge$) closer to the intuitionistic one, but that is not necessarily the adjoint to the implication.

Therefore, in this logic, called **Residuated Logic** (also **Monoidal Logic** in [Höh94]), it is the case that the strength of the premises changes depending on how many of the same hypothesis we have (due to the lack of idempotency), such as it can be seen in the deduction theorem for Residuated Propositional Logic (RPC):

Theorem 0.0.1 (Deduction theorem for RPC, [Mac96])**.** If there is a deduction in RPC of θ from the set of formulas $\alpha, \beta, ..., \gamma, \delta$, and the deduction used δ n-times, then there is a deduction in RPC of $(\delta^n \to \theta)$ from $\alpha, \beta, ..., \gamma$ (where $\delta^n = \delta \& \delta \& ... \& \delta$ is the n-fold conjunction of δ with itself).

Also in regards to the equality, we have that the usual substitution of equal elements

$$(x = y) \& \theta(x) \Rightarrow \theta(y)$$

is not going to hold in general, but rather, we have that

$$(x = y)^n \& \theta(x) \Rightarrow \theta(y)$$

where n occurrences of x in $\theta(x)$ that have been replaced by y to form $\theta(y)$.

With these key features in mind, one could ask what kind of models are we going to use to study these logics, and more specifically, how do we can find natural models of Set Theory for these logics. We do this in two ways: first by considering **lattice-valued models** and then using **Kripke-like models**.

Lattice-valued models were first introduced by Scott and Solovay in [SS67]. They considered **Boolean-valued models of Set Theory** in order to provide a more intuitive presentation of **Cohen's forcing**. To achieve this, they took a complete Boolean algebra $\mathbb{B}$ and built a "model" of Set Theory $V^{\mathbb{B}}$ in which the truth values of formulas take values in $\mathbb{B}$ instead of the trivial Boolean algebra $\{0, 1\}$.

Based on the construction of Scott and Solovay, several generalizations of the previous construction have been considered by taking other lattices instead of Boolean algebras. For example, Heyting lattices give rise to **Intuitionist models of Set Theory** [Gra79], $BL\Delta$-algebras give rise to **models of Fuzzy Set Theory** in the Fuzzy Logic $BL\forall\Delta$ [HH01, HH03] and topological complete residuated lattices (i.e. topological commutative integral quantale) give rise to **Modal models of Residuated Set Theory** [Lan92a]. These kind of valued models serve as natural models of logics weaker than first-order.

Until now, as far as the author is aware, there has been no in-depth study of what would be a "class of definable sets" in the context of valued models. The closest attempt to this was done by Fitting [Fit69], where, as motivation for his definition of class of constructable sets using Kripke models, it was shown how to construct L (or more precisely a model "isomorphic" to L) using two-valued characteristic functions that are definable by some formula.

Following Fitting's idea, we propose new definitions of the notion of **definable subset** within a Boolean-valued model of Set Theory and with them, we propose two new constructions of the constructable universe: $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$. Moreover, we prove that these models are, in fact, **two-valued**, since our definition of definable is too restrictive and forces the models to only take these values. Furthermore, we prove that $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$ are "isomorphic" to V (von Neumann universe) and L (Gödel's constructible universe), respectively.

When trying to generalize these notions of definability to the context of quantale-valued models, we found that the resulting classes of constructable sets are also **two valued**, and therefore are not suitable to study Residuated Logic. Hence, we instead focus on developing the notion of constructable sets in the realm of **Kripke models**, where these kind of problems are avoided.

As it turns out, there is a precedent to the idea of considering constructable sets over Kripke models: Fitting [Fit69] constructed several models of **Intuitionistic Set Theory** generalizing both the universes of von Neumann and of Gödel using **Intuitionistic Kripke models** and then went on to show how these models can be used to obtain **classical** proofs of independence in Set Theory.

Thus, since we would like to generalize Fitting's Intuitionistic Kripke models of Set Theory, we would need first a notion of Kripke models for Residuated Set Theory. Ono and Komori [OK85] introduced the notion of semantics for substructural logics without contraction and exchange. These models, that we call **Residuated Kripke models** (shortly, *R-Kripke models*), generalize the notion of Intuitionistic Kripke models and then, following the ideas of Lano [Lan92a], we further generalize these models to **Modal Residuated Kripke models** (shortly, *MR-Kripke models*). The definition for the interpretation of the modality in our definition is original, and allows for a suitable translation between Kripke models and lattice-valued models. Moreover, using these MR-Kripke models together with Fitting's ideas on the intuitionistic case, we construct new **Modal Residuated models of Set Theory**.

We define the model $\mathcal{V}^{\mathbb{P}^*}$ (see Definition 3.2.18), that generalizes the von Neumann hierarchy for Modal Residuated Logic, and we prove, in Corollary 3.2.30, that there is a **Gödel–McKinsey–Tarski-like translation** between this model and a suited Heyting valued model $R^{\mathbb{H}}$ (see Definition 3.2.23). This translation is obtained by first constructing an "isomorphism" (see Theorem 3.2.29) between $\mathcal{V}^{\mathbb{P}^*}$ and $R^{\mathbb{H}}$ and then proving how this result implies that if φ is an $\mathcal{L}_{\in}$-sentence that is valid in $R^{\mathbb{H}}$, then $\Diamond\varphi$ is valid in $\mathcal{V}^{\mathbb{P}^*}$.

Then, we study superficially the notion of constructible set in the context of Modal Residuated Kripke models of Set Theory. We only outline a propose for a construction in this context.

In Chapter 1, we begin by providing some basic definitions and facts about models of Set Theory. In particular, we introduce the **von Neumann hierarchy of sets** and **Gödel's hierarchy of constructable sets**, since these are the key notions that we are interested in studying in the context of Residuated Logic. In Section 1.2, we present some preliminary concepts of lattice theory. We are especially interested in the study of Boolean and Heyting algebras, commutative integral quantale and modal operators (i.e., quantic nucleus) on these lattices. In Section 1.3, we discuss some basic aspects of the logics that we encounter throughout this work. Two concepts of note are the **Gödel-Kolmogorov** translation between Classical and Intuitionistic Logic and the **Gödel-McKinsey-Tarski** translation between Intuitionistic (Classical) Logic and Modal Logic S_4 (S_5), since our results in Chapter 3 are inspired by this kind of translation theorems between logics.

In Chapter 2, we start by introducing the notion of lattice-valued models for different lattices (Boolean and Heyting algebras, and commutative integral quantale) and discuss their relationship with several logics (Classical, Intuitionistic and Residuated Logic). In Section 2.2 we discuss the construction of valued models of Set Theory, and prove some basic results about them. In Section, 2.3, we propose two definitions for the class of constructable sets in the context of Boolean-valued models of Set Theory, $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$. We show that both $\mathbb{B}$-valued models are in fact two-valued. Furthermore, we prove that $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$ are "isomorphic" to V and L, respectively. At the end of the chapter, we remark that if we use our notion of definable class for quantale-valued models, we obtain a model that is also two-valued and thus not suitable for Residuated Logic.

In Chapter 3, we start by introducing different notions of **Kripke model** for the different logics that we consider in this chapter. In Subsection 3.1.1 we discuss some basic aspect of the Intuitionistic Kripke models and then, in Subsection 3.1.2, we showcase some connections between these models and Cohen's forcing. In Subsection 3.1.3, we discuss Ono and Komori's R-Kripke models and adapt some of their results in our context. Lastly, in Subsection 3.1.4, we consider MR-Kripke models, whose definition is due to the author. These models introduce a modal operator in the definition of Residuated Kripke models.

In Section 3.2, we start by discussing Fitting's results [Fit69] and emphasizing some aspects of his argument. This is done to make it clear how to generalize his results for Residuated Logic. In Subsection 3.2.2, we propose a new definition for the von Neumann universe in the context of Residuated Logic using the MR-Kripke models that we introduced in Subsection 3.1.4. We provide a translation between our MR-Kripke models and a suited Heyting valued model via the possibility operator.

Finally, in Section 3.3 we begin by discussing Fitting's notion of class of constructable sets in Intuitionistic Kripke models. In Subsection 3.3.2, we propose a notion for the class of constructable sets in Modal Residuated Logic, and consider some properties that might be necessary for proving results similar to the ones from [Fit69]. We finish by discussing some open problems and conjectures.

1 Preliminaries

Throughout this document, we will work on ZFC, the Zermelo-Fraenkel axioms of Set Theory including the Axiom of Choice.

The notation that we use in this work is standard: Given a set A, $\mathcal{P}(A)$ denotes the *set of subsets* of A. If B is also a set, $X \subseteq A$ and $f : A \to B$ is a function, $f\restriction_X$ denotes the function that is obtain when the domain of f is *restricted* to X, $dom(f)$ denotes the *domain* of f and $ran(f)$ denotes its *range*. ON denotes the *class of all ordinals*. We use the symbol $\approx$ to denote an equivalence relation on A, $|a|_\approx$ to denote the equivalence class of $a \in A$ modulo $\approx$ and $A/_\approx$ to denote the set of $\approx$-equivalence classes.

1.1 Models of Set Theory

In this work we are (for the most part) only interested in structures which carry one binary relation $\in$ and that validate the axioms of ZFC (or some variation of them).

Definition 1.1.1. $\mathcal{L}_\in = \{\in\}$ denotes the *language of Set Theory*, which consists of a binary relation symbol $\in$.

Definition 1.1.2. We define the *universe of sets* as $V = \{x : x = x\}$. From Russell's paradox, we know that V is a proper class.

1.1.1 von Neumann Universe

Although this hierarchy carries his name, von Neumann was not the first mathematician to investigate this hierarchy, but Ernst Zermelo in 1930 [Zer30]. This class can be understood as the class of hereditary well-founded sets, which under the axiom of regularity, happens to coincide with the class of all sets and therefore allows us to divide the set universe into "levels" indexed by ordinals.

Definition 1.1.3 (von Neumann hierarchy of sets)**.** We define by transfinite recursion over the ordinals:

1. $V_0 := \varnothing$.
2. $V_{\alpha+1} := \mathcal{P}(V_\alpha)$.
3. $V_\alpha := \bigcup\limits_{\beta<\alpha} V_\beta$, for $\alpha \neq 0$ limit ordinal.

The von Neumann hierarchy allows us to see how every set can be obtained from the void and the power set operation, as long as the axiom of regularity holds.

Proposition 1.1.4 ([Jec03], Lemma 6.3)**.** For every set x there is an ordinal α such that $x \in V_\alpha$. Therefore, we can write $V = \bigcup\limits_{\alpha \in ON} V_\alpha$ and we call V *the von Neumman universe.*

Definition 1.1.5. Given a set x, we define the *rank* of x, $rank(x)$, as the smallest ordinal α such that $x \in V_{\alpha+1}$. Thus, we notice that from this definition $V_\alpha = \{x : rank(x) < \alpha\}$

One of the goals of this work is to study some generalizations of the von Neumann hierarchy for logics that are more general than first-order logic. That is, we want to generalize the notion of **subset**, and we will do that by associating this concept to the notion of **characteristic function** and then generalizing this notion to wider contexts.

1.1.2 Gödel's Universe

In [Göd38] Gödel constructed a class model consisting of $\mathcal{L}_\in$-definable sets in Classical first-order Logic and it was used to prove the consistency of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH). We present briefly those results on this subsection, but for anyone interested in a more in depth discussion of this model and the proofs for these theorems see [Kun11] or [Jec03].

Definition 1.1.6. Let us consider an $\mathcal{L}_\in$-structure $(M, \in)$ and $X \subseteq M$. We say that X is *definable* in $(M, \in)$ if there exists a (classical) first-order $\mathcal{L}_\in$-formula $\varphi(x_1, ..., x_n)$ and $a_1, \ldots, a_n \in M$ such that

$$X = \{x \in M \mid (M, \in) \vDash \varphi(x, a_1, \ldots, a_n)\}$$

Based on the above, we define $Def(M) = \{X \subseteq M \mid X \text{ is definable in } (M, \in)\} \subseteq \mathcal{P}(M)$.

Definition 1.1.7 (Gödel's hierarchy of constructible sets)**.** We now define the hierarchy L_α recursively over the ordinals:

1. $L_0 := \varnothing$.

2. $L_{\alpha+1} := Def(L_\alpha)$.

3. $L_\alpha := \bigcup_{\beta<\alpha} L_\beta$ for $\alpha \neq 0$ limit ordinal.

Definition 1.1.8 (Gödel's constructible universe)**.** Abusing the notation, we define the class of *Gödel's constructible sets* as $L := \bigcup_{\alpha \in ON} L_\alpha$.

The above is just shorthand for the $\mathcal{L}_\in$-formula $\varphi(x) = \exists \alpha \in ON(x \in L_\alpha)$. Thus, we say that $a \in L$ if and only if $\varphi(a)$ holds. Notice that, strictly speaking, L is not a set, but rather a proper class of sets that satisfy the property given by the formula $\varphi(x)$. We will use shorthands like these throughout this work, so we will not mention this convention again.

Definition 1.1.9. We say that a set x is *constructible* if $x \in L$.

We can view L as a more rigid and controlled version of the universe, where instead of considering all subsets of a set in the successor step, we only consider *definable* subsets of that set. This distinction allows us to control the cardinality of the power set in the L model, as we see in the next theorem.

Proposition 1.1.10. ([Jec03], Theorems 13.3, 13.16, 13.18, 13.20) Gödel's universe L is a model of $ZFC + GCH$.

One goal of this work is to propose generalizations of Gödel's universe L in more general contexts than Classical Logic. Therefore, we must first find natural models for logics weaker than Classical Logic. This will be done in Chapter 2 by considering models valued on different lattices, and in Chapter 3 with Kripke models and their generalizations.

1.2 Lattices

Throughout this section, we present some preliminary concepts of lattice theory, as they will be of use for the construction of models for different types of logics. For instance, lattices are widely used in algebraic logic as they can be understood as models of most logics via a construction similar to the Lindenbaum–Tarski algebra for Classical Logic.

Lattices are used for several reasons:

1. The notions of meet ($\wedge$) and join ($\vee$) serve as natural interpretations of the conjunction and disjunction, respectively.

2. The order in the lattice can be understood as formalizing the notion of "stronger than" in the logical sense.

3. The operation of implication $\rightarrow$ in the lattice is usually defined in terms of an **adjunction**, usually to the operation of meet or to some product in the lattice. This is done so that we can algebraically capture the logical rule of **Modus Ponens**.

1.2.1 Boolean and Heyting algebras.

Definition 1.2.1. Let $(\mathbb{P}, \leqslant)$ be a partially ordered set (or poset). We say that $\mathbb{P}$ is a *lattice* if for every pair of elements $x, y \in \mathbb{P}$ there exists the supremum and infimum of the set $\{x, y\}$ and we denote

$$x \vee y := \sup\{x, y\} \text{ and } x \wedge y := \inf\{x, y\}.$$

We say that a lattice $\mathbb{P}$ is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice $\mathbb{P}$ is said to be *bounded* if there are elements $0, 1 \in \mathbb{P}$ such that for all $x \in \mathbb{P}$

$$0 \leqslant x \leqslant 1.$$

We say that a bounded lattice $\mathbb{P}$ is *complemented* if for all $x \in \mathbb{P}$, there exists $y_x \in \mathbb{P}$ such that $x \vee y_x = 1$ and $x \wedge y_x = 0$ and we say $\mathbb{P}$ is *pseudocomplemented* if there exists a greatest element $y_x \in \mathbb{P}$ with the property $x \wedge y_x = 0$. In distributive lattices, complements are unique and we denote them as $\neg x := y_x$.

We say that $\mathbb{P}$ is a *complete* lattice if for all $X \subseteq \mathbb{P}$, there exists $\sup X$ in $\mathbb{P}$.

Definition 1.2.2. We say that $\mathbb{B} = (\mathbb{B}, \leqslant, \vee, \wedge, 0, 1, \neg)$ is a *Boolean algebra* if $\mathbb{B}$ is a distributive complemented lattice.

Example 1.2.3. The following are Boolean algebras:

1. The set $\{0, 1\}$ with the order $0 \leqslant 1$ is a Boolean algebra, usually called *the trivial Boolean algebra*. This algebra is used as the truth values for first-order Classical Logic.

2. If we take a set X and consider $\mathcal{P}(X)$ with the order $\subseteq$ we get a complete Boolean algebra.

3. Consider a topological space $(X, \Omega(X))$. We say that a subset $A \subseteq X$ is *clopen* if it is both open and closed in $(X, \Omega(X))$ and we say that an open set U if *regular* if it is equal to the interior of its closure, that is, $\overset{\circ}{\overline{U}} = U$. The set of clopen sets $C(X)$ and of regular open sets $RO(X)$ form a Boolean algebra with the order $\subseteq$.

Boolean algebras have an important role in Classical Logic given that the Lindenbaum-Tarski algebra of a (classical) theory is a Boolean algebra and therefore is possible to form a link between Classical Logic and Boolean algebras, in the sense that Boolean algebras can be used as a semantical counterpart (i.e. models) to the syntactical axioms of Classical Logic. We will study this link further in Section 2.2.1.

The next notion is crucial for the rest of the work, since it allows to characterize the notion of implication as an adjoint to the conjunction, and so we can capture its fundamental important properties.

Definition 1.2.4 (left and right adjoints)**.** Let $(A, \leqslant)$ and $(B, \leqslant)$ be two partially ordered sets and $F : A \to B$ and $G : B \to A$ be two monotone functions. We say that F is a *left adjoint* of G and G is a *right adjoint* of F, if for all $a \in A$ and $b \in B$, we have

$$F(a) \leqslant b \text{ if and only if } a \leqslant G(b),$$

we denote this by $F \dashv G$ or $G \vdash F$.

We say that F is a is *a left adjoint* if there exists a monotone function $G : B \to A$ such that $F \dashv G$. Similarly, we say that G is *a right adjoint* if there exists a monotone function $F : A \to B$ such that $F \dashv G$.

Theorem 1.2.5 (Adjoint Functor Theorem for preorders, [DP02], Proposition 7.34)**.** Let $(A, \leqslant)$ and $(B, \leqslant)$ be two partially ordered sets and $F : A \to B$ and $G : B \to A$ two monotone functions.

1. Suppose that $(A, \leqslant)$ has all joins. Then, $\mathcal{F}$ preserves all joins if and only if $\mathcal{F}$ is a left adjoint.

2. Suppose that $(B, \leqslant)$ has all meets. Then, G preserves all meets if and only if G is a right adjoint.

Just as Boolean algebras semantically capture Classical Logic, Heyting algebras capture Intuitionistic Logics. They were introduced by Arend Heyting in [Are30] to formalize Intuitionistic Logic.

Definition 1.2.6. We say that $\mathbb{H} = (\mathbb{H}, \wedge, \vee, \to, 0, 1)$ is a *Heyting algebra* if:

1. $(\mathbb{H}, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.

2. For all $x, y, z \in \mathbb{H}$, we have

$$z \wedge x \leqslant y \text{ if and only if } z \leqslant x \to y.$$

That is, $\_ \wedge x \dashv x \to \_$ ($\_ \wedge x$ is a left adjoint of $x \to \_$) for all $x \in \mathbb{H}$.

Remark 1.2.7. The condition 2. is key since it allows us to prove the inference rule **Modus Ponens** when we translate this algebraic property into a logical one.

Fact 1.2.8. If $\mathbb{H}$ is a complete lattice, by Theorem 1.2.5, we could remove the operator $\to$ together with the condition 2. from the Definition 1.2.6 and replaced them by the equivalent condition

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \text{ for all } x, y_i \in \mathbb{H} \text{ with } i \in I$$

and $\to$ can be defined as $x \to y := \bigvee\{z \in \mathbb{H} : x \wedge z \leqslant y\}$ so that $\mathbb{H}$ with this implication forms a Heyting algebra.

Definition 1.2.9. We define $\neg : \mathbb{H} \to \mathbb{H}$ as $\neg x := x \to 0$ for every $x \in \mathbb{H}$. It is straightforward to show that $\neg$ is a pseudocomplement on $\mathbb{H}$.

Example 1.2.10. From the definition, it is straightforward to prove that every Boolean algebra $\mathbb{B} = (\mathbb{B}, \leqslant, \vee, \wedge, 0, 1, \neg)$ is a Heyting algebra by defining $a \to b := \neg a \vee b$.

Remark 1.2.11 ([Bel05], Proposition 0.2)**.** Let $\mathbb{H}$ be a Heyting algebra that satisfies one of the following equivalent conditions.

1. The pseudocomplementation is of order 2: For every $x \in \mathbb{H}$, $\neg\neg x = x$.

2. Pseudocomplements are complements: For every $x \in \mathbb{H}$, $\neg x \vee x = 1$.

Then $\mathbb{H}$ is actually a Boolean algebra.

Example 1.2.12 ([Ros90], Chapter 1, Example 1.)**.** Let X be a topological space and let $\Omega(X)$ denote the lattice of open subsets of X. Then, the distributive law for $\cap$ and $\bigcup$ shows that $\Omega(X)$ is a complete Heyting algebra. If U and V are open in X, then

1. $U \to V = \bigcup\{W \in \Omega(X) : U \cap W \subseteq V\}$

2. $\neg U = (X \backslash U)^\circ$

3. $\neg\neg U = \overset{\circ}{\overline{U}}$

Notice that an open set U is **regular** if and only if $\neg\neg U = U$.

1.2.2 Nuclei and filters on Heyting algebras

Throughout this subsection, $\mathbb{H}$ denotes a **complete Heyting algebra**. We will use filters on Heyting algebras to construct new Heyting algebras (or even Boolean algebras) related to the original in some crucial ways. These applications will be studied in subsection 3.2.1

Definition 1.2.13 ([RS63], Chapter 1, Section 8)**.** We say that a nonempty subset $\mathcal{F}$ of $\mathbb{H}$ is called a *filter* if for every $x, y \in \mathbb{H}$

1. If $x \leqslant y$ and $x \in \mathcal{F}$, then $y \in \mathcal{F}$.

2. If $x, y \in \mathcal{F}$, then $x \wedge y \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *proper* if $F \neq \mathbb{H}$, that is, if $0 \notin \mathcal{F}$.

Consider a filter $\mathcal{F}$ on $\mathbb{H}$, then, the relation $\approx_{\mathcal{F}}$ defined on $\mathbb{H}$ by

$$x \approx_{\mathcal{F}} y \text{ if and only if } x \rightarrow y \text{ and } y \rightarrow x \in \mathcal{F}$$

is an equivalence relation on $\mathbb{H}$.

Definition 1.2.14 ([RS63], Chapter 1, Section 13)**.** The quotient algebra $\mathbb{H}/_{\approx_{\mathcal{F}}}$, denoted by $\mathbb{H}/\mathcal{F}$, becomes a Heyting algebra in a natural way, with the operations induced from those of $\mathbb{H}$. For $x \in \mathbb{H}$ we denote $|x|_{\mathcal{F}}$ the congruence class of x modulo $\approx_{\mathcal{F}}$. If there is no confusion, we denote $|x| := |x|_{\mathcal{F}}$. The order relation on $\mathbb{H}/\mathcal{F}$ is given by

$$|x| \leqslant |y|, \text{ if and only if, } x \rightarrow y \in \mathcal{F}$$

Notice that $|x| = |1|$, if and only if, $x \in \mathcal{F}$. We call all subsets of the form $\mathbb{H}/\mathcal{F}$ *quotients* of $\mathbb{H}$.

Nuclei on Heyting algebras, also known as **modal operators**, are closure operators that respect the meet operation. Just as with filters, they allow us to create new Heyting algebras.

Definition 1.2.15 ([Ros90], Definition 1.1.2)**.** Let $(\mathbb{P}, \leqslant)$ be a poset. We say that a function $j : \mathbb{P} \rightarrow \mathbb{P}$ is a *closure operator* if for every $x, y \in \mathbb{P}$, we have the following:

1. Expansivity: $x \leqslant j(x)$.

2. Idempotency with respect to compositions: $j(j(x)) = j(x)$.

3. Monotonicity: If $x \leqslant y$, then $j(x) \leqslant j(y)$.

Definition 1.2.16 ([Ros90], Definition 3.1.1.)**.** We say that a closure operator $j : \mathbb{H} \to \mathbb{H}$ is a *nucleus* (or a *modal operator*) if for every $x, y \in \mathbb{H}$

$$j(x \wedge y) = j(x) \wedge j(y)$$

and we denote

$$\mathbb{H}_j := \{x \in \mathbb{H} : j(x) = x\}$$

Proposition 1.2.17 ([Ros90], Proposition 1.2.4.)**.** If $\mathbb{H}$ is a complete Heyting algebra and j is a nucleus on $\mathbb{H}$, then $\mathbb{H}_j$ is a complete Heyting algebra with the order inherited from $\mathbb{H}$. Furthermore, if $\overset{j}{\bigvee}$ and $\overset{j}{\bigwedge}$ denote the supremums and infimums calculated in $\mathbb{H}_j$, respectively, then for every $x_i \in \mathbb{H}_j$ with $i \in I$

$$\overset{j}{\bigvee_{i \in I}} x_i = j\left(\bigvee_{i \in I} x_i\right)$$
$$\overset{j}{\bigwedge_{i \in I}} x_i = \bigwedge_{i \in I} x_i$$

Example 1.2.18 ([Ros90], Chapter 1, Examples (1))**.** Take $a \in \mathbb{H}$. Then, the following functions are nucleus on $\mathbb{H}$:

1. $c_a := a \vee \_$

2. $u_a := a \to \_$

3. $w_a := (\_ \to a) \to a$. An important example occurs when $a = 0$, since we obtain the *double negation operation* $\neg\neg$ on $\mathbb{H}$. $\mathbb{H}_{w_0} = \mathbb{H}_{\neg\neg}$ is the largest Boolean algebra quotient of $\mathbb{H}$.

Let us examine the last point more closely. Recall than in Example 1.2.12, we said that an open set U was regular if and only if $U = \overset{\circ}{\overline{U}} = \neg\neg U$. Notice also that in the topological context, a set is **dense** if and only if $(X \overset{\circ}{\setminus} U) = \varnothing$, that is, $\neg U = \varnothing$. This motivates the following definition.

Definition 1.2.19. We say that an element $x \in \mathbb{H}$ is

1. *regular* if $\neg\neg x = x$.

2. *dense* if $\neg x = 0$, or equivalently, if $\neg\neg x = 1$.

Remark 1.2.20. Notice that the set

$$\mathcal{F}_{\neg\neg} := \{x \in \mathbb{H} : \neg\neg x = 1\}$$

is a filter on $\mathbb{H}$.

Now, with the following two theorems, it should be clear what we meant when we claimed that $\mathbb{H}_{w_0}$ is the largest Boolean algebra quotient of $\mathbb{H}$.

Theorem 1.2.21 ([RS63], 5.8)**.** If a filter G in a Heyting algebra $\mathbb{H}$ contains all the dense elements, then $\mathbb{H}/G$ is a Boolean algebra.

Theorem 1.2.22 ([RS63], 6.7)**.** Let $\mathcal{F}_{\neg\neg}$ be the filter of all dense elements in $\mathbb{H}$. The mapping

$$h(a) := |a|_{\mathcal{F}_{\neg\neg}} \text{ for } a \in \mathbb{H}_{w_0} = \mathbb{H}_{\neg\neg}.$$

is a Boolean isomorphism from the Boolean algebra $\mathbb{H}_{\neg\neg}$ onto the Boolean algebra $\mathbb{H}/\mathcal{F}_{\neg\neg}$.

1.2.3 Quantales

Structures like quantales (i.e. ordered monoids with a product that distributes over arbitrary supremums) have been studied at least since Ward and Dilworth's work on residuated lattices [WD38, Dil39, War38], where their motivations were more algebraic, since they were studying the lattice of ideals in a ring: Given a ring R, the set of ideals of R, denoted as $Id(R)$, forms a complete lattice defining infimum and supremum as the intersection and sum of ideals, respectively. The monoid operation $\cdot$ on this lattice would be given by multiplication of ideals, and the element R in $Id(R)$ would be the identity of this operation.

But it was not until the work of Mulvey [Mul86], where the term **quantale** was coined as a combination of **"quantum"** and **"locale"** and proposed their use for studying **Quantum Logic** and non-commutative C^*-algebras.

Our motivation for the study of quantales is somewhat different. We are not interested in quantales that are non-commutative - as was the case for Mulvey - but rather quantales that are **not necessary idempotent**. We are interested in studying quantales since they semantically capture both intuitionistic and fuzzy logic, so we will focus on the study of **commutative integral quantales**. This kind of structures are widely use in the field of **substructural logics** as semantical counterparts for those logics.

Definition 1.2.23 (Quantale)**.** We say that $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \rightarrow_r, \rightarrow_l, \top, \bot)$ is a *quantale* if:

1. $(\mathbb{Q}, \wedge, \vee, \top, \bot)$ is a complete bounded lattice with $\top$ as top element and $\bot$ as bottom

element.

2. $(\mathbb{Q}, \cdot)$ is a semigroup (that is, $\cdot$ is associative).

3. For all $x, y, z \in \mathbb{Q}$, we have the two conditions

$$x \cdot z \leqslant y \text{ if and only if } z \leqslant x \to_r y$$

$$z \cdot x \leqslant y \text{ if and only if } z \leqslant x \to_l y$$

That is, $x \cdot \_ \dashv x \to_r \_$ and $\_ \cdot x \dashv x \to_l \_$ for all $x \in \mathbb{Q}$.

Fact 1.2.24. By Theorem 1.2.5, we could remove the operators $\to_r$ and $\to_l$ together with the condition 3. and replaced them by the equivalent conditions

$$x \cdot \left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I}(x \cdot y_i) \text{ and } \left(\bigvee_{i \in I} y_i\right) \cdot x = \bigvee_{i \in I}(y_i \cdot x) \text{ for all } x, y_i \in \mathbb{Q} \text{ with } i \in I$$

and $\to_r$ and $\to_l$ could be defined as

$$x \to_r y := \bigvee\{z \in \mathbb{Q} : x \cdot z \leqslant y\} \text{ and } x \to_l y := \bigvee\{z \in \mathbb{Q} : z \cdot x \leqslant y\}$$

Definition 1.2.25. Let $\mathbb{Q}$ be a quantale. We say that

1. $\mathbb{Q}$ has an *unity* if there exists $1 \in \mathbb{Q}$ such that $(\mathbb{Q}, \cdot, 1)$ is a monoid.

2. $\mathbb{Q}$ is *commutative* if $\cdot$ is commutative.

3. $\mathbb{Q}$ is *idempotent* if $x \cdot x = x$ for all $x \in \mathbb{Q}$.

4. $\mathbb{Q}$ is *integral* if it has unit and $1 = \top$.

Remark 1.2.26. If $\mathbb{Q}$ is a commutative quantale, then $\to_r$ and $\to_l$ are the same, and we denote them simply as $\to$. If $\mathbb{Q}$ is a integral quantale , we denote $\bot$ as 0.

We now proceed to define the notion of residuated lattice, that precedes the notion of quantale by a few years. Although initially it was studied for its algebraic utility, nowadays residuated lattices are widely used in the field of substructural logics for the construction of semantical models.

These lattices have been known under many names: BCK-latices in [BP89], full BCK-algebras in [Kru24], FL_{ew}-algebras in [OT99], and integral, residuated, commutative l-monoids in [Höh95] and, as we will see, commutative integral quantale (when the residuated lattice is complete).

Definition 1.2.27 (Residuated lattice)**.** We say that $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ is a *residuated lattice* if:

1. $(\mathbb{Q}, \wedge, \vee, 1, 0)$ is a bounded lattice.
2. $(\mathbb{Q}, \cdot, 1)$ is a commutative[1] monoid.
3. For all $x, y, z \in \mathbb{Q}$, we have that

$$x \cdot z \leqslant y \text{ if and only if } z \leqslant x \rightarrow y$$

   That is, $x \cdot \_ \dashv x \rightarrow \_$ for all $x \in \mathbb{Q}$.

Remark 1.2.28. Notice that a **complete residuated lattice** is just a **commutative integral quantale**. We will use these notions interchangeably throughout this document.

We now introduce the notion of t-norms, which are a key example, since they are a fundamental operation in the context of fuzzy logics. Here $[0, 1]$ denotes the subset of real numbers between 0 and 1.

[1] Some authors do not include the commutativity in the definition of residuated lattice

Definition 1.2.29. A function $\cdot : [0,1]^2 \to [0,1]$ is called *t-norm* if for all $x, y, a, b \in [0,1]$:

1. Commutativity: $x \cdot y = y \cdot x$.
2. Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. Monotonicity: If $x \leqslant a$ and $y \leqslant b$, then $x \cdot y \leqslant a \cdot b$.
4. Identity: $x \cdot 1 = x$.

Definition 1.2.30. Let $\cdot : [0,1]^2 \to [0,1]$ be a t-norm. Then, $\cdot$ is said to be

1. *left continuous*, if it is left continuous as a function from $[0,1]^2$ to $[0,1]$ with the usual metric.
2. *continuous*, if it is continuous as a function from $[0,1]^2$ to $[0,1]$ with the usual metric.

Example 1.2.31. The following operations are left continuous t-norms:

1. The Łukasiewicz t-norm: $x \cdot_L y = max\{x + y - 1, 0\}$.
2. The product t-norm: $x \cdot_p y = x \cdot y$, where $\cdot$ denotes the usual product on $\mathbb{R}$.
3. The Gödel-Dummett t-norm: $x \cdot_{GD} y = min\{x, y\}$.

Example 1.2.32. The following structures are commutative integral quantale:

1. Boolean algebras.
2. Heyting algebras.
3. The order $([0,1], \leqslant)$ endowed with the t-norm of Łukasiewicz, Gödel-Dummett or the product t-norm.

4. More generally, every structure $([0,1], \leqslant \wedge, \vee, \cdot, 0, 1)$, where $\leqslant$ is the usual order and $\cdot$ is any left continuous t-norm.

As we mentioned before, we focus on integral commutative quantales, since these structures naturally generalize both to Heyting algebra (intuitionistic logic) and $[0,1]$ endowed with some left continuous t-norm (fuzzy logics).

Theorem 1.2.33 ([BP14], p. 2)**.** Let $\mathbb{Q}$ be a commutative integral quantale and $x, y, z \in \mathbb{Q}$ with $i \in I$. Then:

1. $x \leqslant y$ if and only if $(x \to y) = 1$.
2. $x \cdot (x \to y) \leqslant y$.
3. $(1 \to y) = y$
4. $0 = x \cdot 0 = 0 \cdot x$.
5. $(0 \to y) = 1$
6. If $x \leqslant y$, then $x \cdot z \leqslant y \cdot z$.
7. $x \cdot y \leqslant x \wedge y$.
8. If $x \leqslant y$, then $y \to z \leqslant x \to z$.
9. If $x \leqslant y$, then $z \to x \leqslant z \to y$.
10. $(x \cdot y) \to z = x \to (y \to z)$.

Definition 1.2.34. Let $\mathbb{Q}$ be a commutative integral quantale and $x, y \in \mathbb{Q}$. We define

1. $\sim x := x \to 0$ (negation),

2. $x \equiv y := (x \to y) \cdot (y \to x)$ (equivalence),

3. $x^0 = 1$ and $x^{n+1} = x \cdot x^n$ for $x \in \mathbb{N}$ (exponentiation).

Remark 1.2.35. Throughout this document, we make a distinction between the negation (equivalence) in a quantale, denoted by $\sim$ ($\equiv$), and the negation (equivalence) in a Heyting algebra, denoted by $\neg$ ($\leftrightarrow$).

Theorem 1.2.36 (cf. [RS63], Chapter IV., 7.2)**.** Let $\mathbb{Q}$ be a commutative integral quantale and $x, y, y_i, x_i \in \mathbb{Q}$ with $i \in I$. Then:

1. $x \cdot \left(\bigwedge_{i\in I} y_i\right) \leqslant \bigwedge_{i\in I}(x \cdot y_i)$

2. $x \to \left(\bigwedge_{i\in I} y_i\right) = \bigwedge_{i\in I}(x \to y_i)$.

3. $\left(\bigvee_{i\in I} x_i\right) \to y = \bigwedge_{i\in I}(x_i \to y)$.

4. $\sim \left(\bigvee_{i\in I} x_i\right) = \bigwedge_{i\in I}(\sim x_i)$.

Proof. 1. Notice that $\bigwedge_{i\in I} y_i \leqslant y_i$ for all $i \in I$. Then, by Theorem 1.2.33 item 6., we can deduce that

$$x \cdot \bigwedge_{i\in I} y_i \leqslant x \cdot y_i.$$

Therefore, by definition of infimum, $x \cdot \left(\bigwedge_{i\in I} y_i\right) \leqslant \bigwedge_{i\in I}(x \cdot y_i)$.

2. Similarly as in the last item, $\bigwedge_{i\in I} y_i \leqslant y_i$ for all $i \in I$. Then, by Theorem 1.2.33 item 9., we have

$$x \to \bigwedge_{i \in I} y_i \leqslant x \to y_i.$$

That is, $x \to \left(\bigwedge_{i \in I} y_i\right)$ is an upper bound of $\{x \to y_i : i \in I\}$. To see that it is the biggest upper bound, take $z \in \mathbb{Q}$ such that

$$z \leqslant x \to y_i \text{ for every } i \in I.$$

Then, since $x \to \_$ is a right adjoint of $x \cdot \_$,

$$x \cdot z \leqslant y_i \text{ for every } i \in I.$$

Therefore, by definition of infimum $x \cdot z \leqslant \bigwedge_{i \in I} y_i$ and since $x \to \_$ is a right adjoint of $x \cdot \_$, we can conclude that

$$z \leqslant x \to \left(\bigwedge_{i \in I} y_i\right)$$

3. It is proved similarly as the previous item.

4. It is a consequence of the previous item, when $y = 0$.

□

Theorem 1.2.37 ([BP14], p. 2)**.** Let $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \to, 1, 0)$ be a commutative integral quantale and let $x, y, y_i, x_i \in \mathbb{Q}$ for $i \in I$. Then:

1. $x \cdot (\sim x) = 0$, but in general it is not true that $x \vee \sim x = 1$.

2. $x \leqslant (\sim\sim x)$, but in general it is not true that $\sim\sim x \leqslant x$.

3. $\sim (x \vee y) = (\sim x) \cdot (\sim y)$ (De Morgan's Law), but it is not generally true that $\sim (x \cdot y) = (\sim x) \vee (\sim y)$.

4. If $x \leqslant y$, then $(\sim y) \leqslant (\sim x)$ and $(\sim\sim x) \leqslant (\sim\sim y)$.

5. $\sim 0 = 1$ and $\sim 1 = 0$.

6. $x = y$ if and only if $(x \equiv y) = 1$.

7. $(\sim\sim x) \cdot (\sim\sim y) \leqslant (\sim\sim (x \cdot y))$.

8. $\sim\sim\sim x = \sim x$.

If we consider a commutative integral quantale that is also **idempotent**, then the structure collapses to a Heyting algebra. For this reason, we are interested in studying commutative integral quantale that are not necessarily idempotent.

Theorem 1.2.38 (Folklore)**.** If $\mathbb{Q}$ is a commutative idempotent integral quantale, then $\mathbb{Q}$ is a Heyting algebra, where $x \cdot y = x \wedge y$ for all $x, y \in \mathbb{Q}$.

Proof. By Theorem 1.2.33 item 7., we have that $x \cdot y \leqslant x \wedge y$. To see the other inequality, notice that $x \wedge y \leqslant x$ and $x \wedge y \leqslant y$. Therefore, since $\cdot$ is idempotent,

$$x \wedge y = (x \wedge y) \cdot (x \wedge y) \leqslant x \cdot y$$

and we can conclude that $x \cdot y = x \wedge y$. □

Theorem 1.2.39. Let $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ be a commutative integral. Then, if $x, y \in \{0, 1\}$,

1. $x \rightarrow y \in \{0, 1\}$

2. $x \wedge y \in \{0, 1\}$

3. $x \vee y \in \{0, 1\}$

4. $x \cdot y \in \{0,1\}$

5. $\sim x \in \{0,1\}$.

Proof. 1. By Theorem 1.2.33 items 3. and 4., $(0 \to y) = 1$ and $(1 \to y) = y$ for all $y \in \{0,1\}$, and this implies that $x \to y \in \{0,1\}$ for all $x, y \in \{0,1\}$.

2. Since 0 is the minimum and 1 the maximum of $\mathbb{Q}$, we have $x \wedge y = 1$ if $x = y = 1$ and $x \wedge y = 0$ if any of them is equal to 0.

3. It is proved in a similar way as the previous item.

4. Since 1 is the module of the monoid $(\mathbb{Q}, \cdot, 1)$, we have $x \cdot 1 = 1 \cdot x = x$ for all $x \in \{0,1\}$, but on the other hand, by Theorem 1.2.33 6., we have $x \cdot 0 = 0 \cdot x = 0$ for all $x \in \{0,1\}$. The above implies that $x \cdot y \in \{0,1\}$ for all $x, y \in \{0,1\}$.

5. By Theorem 1.2.37 5., we have $\sim 0 = 1$ and $\sim 1 = 0$, which implies that $\sim x \in \{0,1\}$ for all $x, y \in \{0,1\}$.

□

Corollary 1.2.40. If $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \to, 1, 0)$ is a commutative integral quantale, then $\{0,1\} \subseteq \mathbb{Q}$ is a Boolean algebra with the operations inherited from $\mathbb{Q}$ and $x \cdot y = x \wedge y$ for all $x, y \in \{0,1\}$.

1.2.4 Filters and nuclei on quantales

Throughout this subsection, $\mathbb{Q}$ denotes a **commutative integral quantale**. The goal of this subsection is to generalize the results of Subsection 1.2.13 in the context of commutative integral quantale. That is, we use filters on quantales to construct new quantales (or even Heyting algebras or Boolean algebras) related to the original in some crucial ways. These applications will be studied in subsection 3.2.2

Definition 1.2.41 ([BP14], Definition 2)**.** A nonempty subset $\mathcal{F}$ of $\mathbb{Q}$ is said to be a *filter* if for every $x, y \in \mathbb{Q}$

1. If $x \leqslant y$ and $x \in \mathcal{F}$, then $y \in \mathcal{F}$.

2. If $x, y \in \mathcal{F}$, then $x \cdot y \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *proper* if $\mathcal{F} \neq \mathbb{Q}$, that is, if $0 \notin \mathcal{F}$.

Notice that if $\mathbb{Q}$ is a Heyting algebra, this definition coincides with Definition 1.2.13. Consider a filter $\mathcal{F}$ on $\mathbb{Q}$. The relation $\approx_{\mathcal{F}}$ defined on $\mathbb{Q}$ by

$$x \approx_{\mathcal{F}} y \text{ if and only if } x \to y \text{ and } y \to x \in \mathcal{F}$$

is an equivalence relation on $\mathbb{Q}$.

Definition 1.2.42 ([BP14], pp. 2 and 3)**.** The quotient algebra $\mathbb{Q}/_{\approx_{\mathcal{F}}}$, denoted by $\mathbb{Q}/\mathcal{F}$, becomes a complete residuated lattice in a natural way, with the operations induced from those of $\mathbb{Q}$. For $x \in \mathbb{Q}$, we denote by $|x| := |x|_{\mathcal{F}}$ the congruence class of x modulo $\approx_{\mathcal{F}}$. The order relation on $\mathbb{Q}/\mathcal{F}$ is defined by

$$|x| \leqslant |y|, \text{ if and only if, } x \to y \in \mathcal{F}$$

and the following equalities hold:

$$|A| \wedge |B| = |A \wedge B|,$$

$$|A| \vee |B| = |A \vee B|,$$

$$|A| \rightarrow |B| = |A \rightarrow B|,$$

$$| \sim A| = \sim |A|.$$

But with respect to quantic nucleus, our goal is to capture, as best as we can, all the properties of the double negation $\neg\neg$ so that we can replicate Fitting's results [Fit69] in the context of residuated logics.

Definition 1.2.43 ([Ros90], Definition 3.1.1)**.** We say that a closure operator $\gamma : \mathbb{Q} \rightarrow \mathbb{Q}$ is a *quantic nucleus* if for every $x \in \mathbb{Q}$

$$\gamma(x) \cdot \gamma(y) \leqslant \gamma(x \cdot y)$$

Lemma 1.2.44 ([Ros90], p. 29)**.** If $\gamma : \mathbb{Q} \rightarrow \mathbb{Q}$ is a quantic nucleus, then for every $x, y \in \mathbb{Q}$

$$\gamma(x \cdot y) = \gamma(\gamma(x) \cdot \gamma(y)) = \gamma(\gamma(x) \cdot y) = \gamma(x \cdot \gamma(y))$$

Lemma 1.2.45. [[Ros90], Proposition 3.1.1] A function $\gamma : \mathbb{Q} \rightarrow \mathbb{Q}$ is a quantic nucleus, if and only if, for every $x, y \in \mathbb{Q}$, $\gamma(x) \rightarrow \gamma(y) = x \rightarrow \gamma(y)$.

Theorem 1.2.46 ([Ros90], Theorem 3.1.1 + Lemma 3.2.1 + Lemma 3.2.2)**.** Let $\gamma : \mathbb{Q} \rightarrow \mathbb{Q}$ be a quantic nucleus on $\mathbb{Q}$. The set of fixed points of γ

$$\mathbb{Q}_\gamma := \{x \in \mathbb{Q} : \gamma(x) = x\}$$

is a commutative integral quantale with the order inherited from $\mathbb{Q}$ and the product

$$x \cdot_\gamma y = \gamma(x \cdot y).$$

Furthermore, if $\overset{\gamma}{\bigvee}$ and $\overset{\gamma}{\bigwedge}$ represent the supremums and infimums calculated in $\mathbb{Q}_\gamma$, respectively, then for every $x_i \in \mathbb{Q}_\gamma$ with $i \in I$

$$\overset{\gamma}{\bigvee_{i\in I}} x_i = \gamma\left(\bigvee_{i\in I} x_i\right)$$
$$\overset{\gamma}{\bigwedge_{i\in I}} x_i = \bigwedge_{i\in I} x_i$$

Corollary 1.2.47 ([Ros90], Chapter 3, Section 1, Corollary 1)**.** If $\gamma : \mathbb{Q} \to \mathbb{Q}$ is a quantic nucleus, then for every $x, y \in \mathbb{Q}$

$$\gamma(x \to y) \leqslant x \to \gamma(y) = \gamma(x) \to \gamma(y)$$

Theorem 1.2.48 ([Ros90], Proposition 3.1.2)**.** If $A \subseteq \mathbb{Q}$, then $A = \mathbb{Q}_\gamma$ for some quantic nucleus γ if and only if A is closed under infimums and if $x \in \mathbb{Q}$ and $y \in A$, then $x \to y \in A$.

Example 1.2.49. By Theorem 1.2.37 items 2., 4., 7. and 8., it is clear that the operator $\sim\sim$ is a quantic nucleus on a $\mathbb{Q}$. Furthermore, by Lemma 1.2.45 and Theorem 1.2.48, we have that

$$x \to\sim\sim y =\sim\sim x \to\sim\sim y \text{ and } \sim\sim(\sim\sim x \to\sim\sim y) =\sim\sim x \to\sim\sim y$$

Definition 1.2.50. We say that a quantic nucleus $\gamma : \mathbb{Q} \to \mathbb{Q}$ *respects implications if*

$$\gamma(x \to y) = 1 \text{ if and only if } \gamma(x) \to \gamma(y) = 1$$

Notice that by Lemma 1.2.45, this condition is equivalent to

$$\gamma(x \to y) = 1 \text{ if and only if } x \to \gamma(y) = 1$$

Lemma 1.2.51 (cf. [Fit69] Lemma 5.3)**.** Let $\gamma : \mathbb{Q} \to \mathbb{Q}$ be a quantic nucleus that respects implications. For every $x_i, y \in \mathbb{Q}$, with $i \in I$,

$$\gamma(\bigwedge_{i\in I}(x_i \to y)) = 1 \text{ if and only if } \bigwedge_{i\in I}\gamma(x_i \to y) = 1$$

Proof. Notice that, since γ respects implications,

$$1 = \gamma(\bigwedge_{i\in I}(x_i \to y)) = \gamma((\bigvee_{i\in I} x_i) \to y)$$

holds, if and only if

$$1 = (\bigvee_{i\in I} x_i) \to \gamma(y) \text{ holds.}$$

But then, since γ respects implications, the line given above is equivalent to

$$1 = \gamma((\bigvee_{i\in I} x_i) \to y) = \bigwedge_{i\in I}(x_i \to \gamma(y)).$$

Thus, by definition of infimum, the line given above holds, if and only if,

$$x_i \to \gamma(y) = 1 \text{ for every } i \in I$$

and since γ respects implications, that is equivalent to

$$\gamma(x_i \to y) = 1 \text{ for every } i \in I.$$

which is just

$$\bigwedge_{i\in I}\gamma(x_i \to y) = 1$$

□

Definition 1.2.52 ([Ros90], Definition 3.2.4.)**.** We say that a quantic nucleus $\gamma : \mathbb{Q} \to \mathbb{Q}$ is *idempotent with respect to products* if for every $x \in \mathbb{Q}$

$$\gamma(x^2) = \gamma(x)$$

Theorem 1.2.53 (cf. [Ros90], Lemma 3.2.4.)**.** If γ is a quantic nucleus that is idempotent with respect to products, then for every $x, y \in \mathbb{Q}$

$$\gamma(x \cdot y) = \gamma(x) \wedge \gamma(y).$$

We call a quantic nucleus that satisfies the property above *localic*.

Proof. Since $x \cdot y \leqslant x \cdot 1 = x$, we have $\gamma(x \cdot y) \leqslant \gamma(x)$. In a similar way we prove that $\gamma(x \cdot y) \leqslant \gamma(y)$ and therefore $\gamma(x \cdot y)$ is a lower bound of $\{\gamma(x), \gamma(y)\}$.

To see that it is the greatest lower bound, take $c \in \mathbb{Q}$ lower bound of $\{\gamma(x), \gamma(y)\}$ that is

$$c \leqslant \gamma(x) \text{ and } c \leqslant \gamma(y)$$

Therefore

$$c^2 \leqslant \gamma(x) \cdot \gamma(y)$$

and so

$$\begin{aligned}
c &\leqslant \gamma(c) && (\gamma \text{ is expansive}) \\
&= \gamma(c^2) && (\gamma \text{ is idempotent with respect to products}) \\
&\leqslant \gamma(\gamma(x) \cdot \gamma(y)) && (\gamma \text{ is monotone}) \\
&= \gamma(x \cdot y) && (\text{by Theorem 1.2.44})
\end{aligned}$$

□

Theorem 1.2.54 ([Ros90], Lemma 3.2.3)**.** Let $\gamma : \mathbb{Q} \to \mathbb{Q}$ be a quantic nucleus idempotent with respect to products on $\mathbb{Q}$. Then, $\mathbb{Q}_\gamma = \{x \in \mathbb{Q} : \gamma(x) = x\}$ is a idempotent commutative integral quantale, that is, by Theorem 1.2.38, $\mathbb{Q}_\gamma$ is a Heyting algebra.

Definition 1.2.55. We say that a quantic nucleus $\gamma : \mathbb{Q} \to \mathbb{Q}$ *respects the bottom element* if

$$\gamma(0) = 0.$$

Lemma 1.2.56. Let $\gamma : \mathbb{Q} \to \mathbb{Q}$ be a quantic nucleus that respects the bottom element. Then,

$$\sim\sim \gamma(x) = \sim\sim (\gamma(\sim\sim \gamma(x)),$$

that is, $\sim\sim \gamma$ is idempotent.

Proof. Notice that $\gamma(0) = 0 \in \mathbb{Q}_\gamma$ and by Theorem 1.2.48, we can deduce that for every $x \in \mathbb{Q}$, $x \to 0 = \sim x \in \mathbb{Q}_\gamma$ and thus $\sim\sim x \in \mathbb{Q}_\gamma$, that is,

$$\gamma(\sim\sim x) = \sim\sim x.$$

Therefore,

$$\sim\sim (\gamma(\sim\sim \gamma(x)) = \sim\sim (\sim\sim \gamma(x)) = \sim\sim \gamma(x).$$

□

Definition 1.2.57. We say that a quantic nucleus $\gamma : \mathbb{Q} \to \mathbb{Q}$ is *standard* if γ is idempotent with respect to products, respects implications and the bottom element.

We introduce the new notion of **standard quantic nucleus** since it captures all the necessary properties from the double negation operator that are used in Chapter 3 to generalize Fitting's results.

Theorem 1.2.58. Let $\gamma : \mathbb{Q} \to \mathbb{Q}$ be a quantic nucleus on $\mathbb{Q}$. The set

$$\mathcal{F}_\gamma := \{x \in \mathbb{Q} : \gamma(x) = 1\}$$

is a filter on $\mathbb{Q}$.

Proof. 1. Since γ is expansive, $1 \leqslant \gamma(1)$, and then $1 \in \mathcal{F}_\gamma$

2. Take $x \in \mathcal{F}_\gamma$ and $y \in \mathbb{Q}$ such that $x \leqslant y$. Since γ is monotone function, $\gamma(x) \leqslant \gamma(y)$ and since $x \in \mathcal{F}_\gamma$, $\gamma(x) = 1$. Then, $\gamma(y) = 1$ and $y \in \mathcal{F}_\gamma$.

3. Take $x, y \in \mathcal{F}_\gamma$, that is, $\gamma(x) = \gamma(y) = 1$. Then, since γ is a nucleus, $\gamma(x \cdot y) \geqslant \gamma(x) \cdot \gamma(y) = 1 \cdot 1 = 1$, that is, $x \cdot y \in \mathcal{F}_\gamma$.

□

For the rest of the section, $|x|$ denotes the class of $x \in \mathbb{Q}$ modulo $\approx_{\mathcal{F}_\gamma}$.

Theorem 1.2.59. Let γ be a quantic nucleus that respects implications and take $x, y \in \mathbb{Q}$. Then, $x \approx_{\mathcal{F}_\gamma} y$ if and only if $\gamma(x) = \gamma(y)$.

Proof.

$|x| = |y|$ iff $x \to y \in \mathcal{F}_\gamma$ and $y \to x \in \mathcal{F}_\gamma$	(by definition of $\approx_{\mathcal{F}_\gamma}$)
iff $\gamma(x \to y) = 1 = \gamma(y \to x)$	(by definition of $\mathcal{F}_\gamma$)
iff $\gamma(x) \to \gamma(y) = 1 = \gamma(y) \to \gamma(x)$	(γ respects implications)
iff $\gamma(x) \leqslant \gamma(y)$ and $\gamma(y) \leqslant \gamma(x)$	(by Theorem 1.2.33 item 1.)
iff $\gamma(x) = \gamma(y)$	

□

Corollary 1.2.60. If γ is a quantic nucleus that respects implications and $x \in \mathbb{Q}$,

$$|\gamma(x)| = |x|.$$

Theorem 1.2.61. Let γ be a quantic nucleus idempotent with respects to products that respects implications. Then, $\mathbb{Q}/\mathcal{F}_\gamma$ is a Heyting algebra.

Proof. By Theorems 1.2.42 and 1.2.58, we know that $\mathbb{Q}/\mathcal{F}_\gamma$ is a complete residuated lattice. By Theorem 1.2.38, is enough to prove that the product in $\mathbb{Q}/\mathcal{F}_\gamma$ is idempotent. If $|x| \in \mathbb{Q}/\mathcal{F}_\gamma$, then

$$\begin{aligned}
|x| \cdot |x| &= |x \cdot x| && \text{(by definition of $\cdot$ on $\mathbb{Q}/\mathcal{F}_\gamma$)} \\
&= |\gamma(x \cdot x)| && \text{(by Corollary 1.2.60)} \\
&= |\gamma(x)| && \text{(since γ is idempotent with respect to products)} \\
&= |x| && \text{(by Corollary 1.2.60)}
\end{aligned}$$

□

Theorem 1.2.62 (cf. [Fit69], Theorem 5.4)**.** Let γ be a quantic nucleus on $\mathbb{Q}$ that respects implications. For every $x_i \in \mathbb{Q}$ with $i \in I$, we have that

$$|\bigvee_{i \in I} x_i| = \bigvee_{i \in I} |x_i|$$

Proof. For every $i \in I$, $x_i \leqslant \bigvee_{i \in I} x_i$. Thus,

$$x_i \to \bigvee_{i \in I} x_i = 1,$$

therefore,

$$\gamma(x_i \to \bigvee_{i \in I} x_i) = 1,$$

and by definition of $\leqslant$ we deduce that $|x_i| \leqslant |\bigvee_{i \in I} x_i|$. Thus, $|\bigvee_{i \in I} x_i|$ is an upper bound of $\{|x_i| : i \in I\}$.

To see that it is the smallest upper bound, take $|b| \in \mathbb{Q}/\mathcal{F}_\gamma$ an upper bound of $\{|x_i| : i \in I\}$, that is,

$$|x_i| \leqslant |b| \text{ for every } i \in I,$$

that implies, by definition of $\leqslant$ on $\mathbb{Q}/\mathcal{F}_\gamma$,

$$\gamma(x_i \to b) = 1 \text{ for every } i \in I.$$

Therefore,

$$\bigwedge_{i \in I}(\gamma(x_i \to b)) = 1,$$

then, by Lemma 1.2.51

$$\gamma(\bigwedge_{i \in I}(x_i \to b)) = 1,$$

hence, by Theorem 1.2.36 item 3,

$$\gamma((\bigvee_{i \in I} x_i) \to b) = 1,$$

thus, by definition of $\leqslant$,

$$|\bigvee_{i \in I} x_i| \leqslant |b|$$

□

1.3 Substructural logics

Substructural logics are **non-classical** logics **weaker** than Classical Logic, notable for the absence of **structural rules** present in Intuitionistic Logic when formulated as Gentzen-style systems (we will not go over this kind of systems in this work, for an introduction to this topic see [Ono03] or [GJKO07]), such as weakening, contraction, exchange, commutativity or associativity. Substructural logics include, among other logics, Fuzzy Logics and Intuitionistic Logics.

1.3.1 Intuitionistic Logic

Intuitionistic Logic is a product of Brouwer's project of intuitionistic mathematics, whose systematic formalization was started by Brouwer's student, Arend Heyting, in 1928. One can understand this logic as an attempt to formalize the notion of constructive proofs, so the axioms of Classical Logic such as the principle of excluded middle, double negation elimination and the Axiom of Choice are not valid.

Intuitionistic Logic is a weakening of Classical Logic where one of the following logical axioms is not valid:

1. LEM (law of excluded middle) $\varphi \vee \neg\varphi$
2. LDN (law of double negation) $\neg\neg\varphi \to \varphi$

These axioms are intuitionistically equivalent and if we added one of these axioms to the axioms of propositional Intuitionistic Logic, we would obtain an axiomatization of Propositional Classical Logic.

We will not provide an axiomatization of this logic, but one can be found in [vD04] Section 1.4 by eliminating the rule **reductio ad absurdum** (RAA) from the rules of Classical Logic (see also [vD04], section 5.2 "Intuitionistic Propositional and Predicate Logic").

Remark 1.3.1. To avoid any confusion, we denote by $\vdash$ deductions made by using the axioms and inference rules of Classical Logic and $\vdash_i$ for Intuitionistic Logic.

In terms of lattices, we have that there is a fundamental relation with Heyting algebras and Intuitionistic Logic, as we can see in the following example:

Example 1.3.2 (Lindenbaum-Tarski algebra)**.** Let us consider a consistent theory T in an Intuitionistic first-order language $\mathcal{L}$ and consider the equivalence relation $\approx$ on the set of $\mathcal{L}$-formulas by $\varphi \approx \psi$ if and only if $T \vdash_i \varphi \leftrightarrow \psi$. Let $H(T)$ be the set of equivalence classes of $\mathcal{L}$-formulas and consider the relation $\leqslant$ on $H(T)$ given by $[\varphi] \leqslant [\psi]$ if and only if $T \vdash_i \varphi \to \psi$. Then, $(H(T), \leqslant)$ is a Heyting algebra. In this sense, Heyting algebras are the natural algebraic models for Intuitionistic Logic.

Since Intuitionistic Logic is a weakening of Classical Logic, we have that every formula that can be deduced in Intuitionistic Logic is also deducible in Classical Logic, that is

$$\text{if } \vdash_i \varphi, \text{ then } \vdash \varphi.$$

The opposite implication is not true in general. But we do have a crucial connection from Classical Logic into Intuitionistic Logic via the **double negation operator**.

Theorem 1.3.3 (Glivenko's theorem, [vD04], Theorem 5.2.10)**.** If φ is a propositional formula, then

$$\vdash \varphi \text{ if and only if } \vdash_i \neg\neg\varphi$$

Theorem 1.3.4 ([vD04], Theorem 5.2.6)**.** If φ does not contain $\vee$ or $\exists$ and all atoms but $\perp$ in φ are negated, then

$$\vdash_i \varphi \leftrightarrow \neg\neg\varphi$$

Furthermore, Gödel and Gentzen proved that by reinterpreting the classical disjunction and existence quantifier, we can embed Classical Logic into Intuitionistic Logic.

Definition 1.3.5 (Gödel-Gentzen translation)**.** Given a formula φ, we define its *Gödel-Gentzen translation* φ^G as follows:

1. $\varphi^G := \neg\neg\varphi$ if φ is an atomic formula.
2. $(\varphi \wedge \psi)^G := \varphi^G \wedge \psi^G$
3. $(\varphi \vee \psi)^G := \neg(\neg\varphi^G \wedge \neg\psi^G)$
4. $(\varphi \rightarrow \psi)^G := \varphi^G \rightarrow \psi^G$
5. $(\neg\varphi)^G := \neg(\varphi^G)$
6. $(\forall x\varphi(x))^G := \forall x(\varphi^G(x))$
7. $(\exists x\varphi(x))^G := \neg\forall x(\neg\varphi^G(x))$

There exists another translation that makes the use of the double negation operator even more explicit.

Definition 1.3.6 (Gödel-Kolmogorov translation)**.** Given a formula φ, we define its *Gödel-Kolmogorov translation* φ^{GK} as follows:

1. $\varphi^{GK} := \neg\neg\varphi$ if φ is an atomic formula.
2. $(\varphi \wedge \psi)^{GK} := \neg\neg(\varphi^{GK} \wedge \psi^{GK})$
3. $(\varphi \vee \psi)^{GK} := \neg\neg(\varphi^{GK} \vee \psi^{GK})$

4. $(\varphi \to \psi)^{GK} := \neg\neg(\varphi^{GK} \to \psi^{GK})$

5. $(\neg\varphi)^{GK} := \neg\neg(\neg(\varphi^{GK}))$

6. $(\forall x \varphi(x))^{GK} := \neg\neg\forall x(\varphi^{GK}(x))$

7. $(\exists x \varphi(x))^{GK} := \neg\neg\exists x(\varphi^{GK}(x))$

It is clear from these definitions that, classically speaking, the formulas φ, φ^G and φ^{GK} are all logically equivalent. Intuitionistically though, only φ^G and φ^{GK} are logically equivalent, and only in some cases we have that φ is equivalent to φ^G and φ^{GK}, as we see in the following theorem.

Lemma 1.3.7 ([BN04], Lemma 0.9 and 0.10.). 1. $\vdash_i \varphi^G$ if and only if $\vdash \varphi$.

2. $\vdash_i \varphi^G \leftrightarrow \varphi^{GK}$.

3. If every $\forall$ in φ is followed by an $\neg$, then

$$\vdash_i \varphi^{GK} \leftrightarrow \neg\neg\varphi$$

4. If φ' is the formula obtained from φ by replacing every occurrence of $\forall$ by $\neg\exists\neg$, then

$$\vdash_i \varphi^{GK} \leftrightarrow \varphi'$$

The last item is key, since in Subsection 3.2.1 we see how Fitting (see Corollary 3.2.12) essentially uses this fact to find a translation between intuitionistic Kripke models and Boolean valued models.

1.3.2 Modal Logics

The language of Classical Modal Logic has the same symbols as Classical Logic together with a unary connective of *necessity* $\Box$. The formulas are constructed by recursion in the same way as formulas in Classical Logic, but now we have that if φ is a formula, $\Box\varphi$ is also a formula and we call them *Modal* formulas. We define the symbol of *possibility* as $\Diamond\varphi := \neg\Box\neg\varphi$. In most modern presentations of Modal Logic, one takes the axioms as Classical Logic and augments them with some axioms that involve the operator of necessity and possibility. Some of those axioms are:

1. N: If φ is a theorem, then $\Box\varphi$ is likewise a theorem, that is, if $\vdash \varphi$, then $\vdash \Box\varphi$
2. T: $\Box\varphi \to \varphi$,
3. K: $\Box(\varphi \to \psi) \Rightarrow (\Box\varphi \to \Box\psi)$
4. 4: $\Box\varphi \Rightarrow \Box\Box\varphi$
5. 5: $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$.

If we add the axioms $N, T, K, 4$ to Classical Logic we obtain the (Classical) Modal Logic S_4 and if we add $N, T, K, 5$ we obtain the (Classical) Modal Logic S_5. We use the symbols $\vdash_{S_4}$ and $\vdash_{S_5}$ to denote deductions made in the systems S_4 and S_5, respectively.

Similarly as the translation between Classical and Intuitionistic Logic via the double negation operator, one can find a suitable translation between Intuitionistic (Classical) Logic and the Modal Logic S_4 (S_5).

Definition 1.3.8 (Gödel–McKinsey–Tarski translation)**.** Given a propositional Intuitionistic formula α, we can define its *Gödel–McKinsey–Tarski translation* α^{GMT} as follows:

1. $p^{GMT} = \Box p$ if p is a propositional variable.

2. $\bot^{GMT} = \bot$.

3. $(\alpha \wedge \beta)^{GMT} = \alpha^{GMT} \wedge \beta^{GMT}$

4. $(\alpha \vee \beta)^{GMT} = \alpha^{GMT} \wedge \beta^{GMT}$

5. $(\alpha \rightarrow \beta)^{GMT} = \Box(\alpha^{GMT} \rightarrow \beta^{GMT})$

6. $(\neg\alpha)^{GMT} = \Box(\neg(\alpha^{GMT}))$

Theorem 1.3.9 ([WZ14], p. 3)**.** An Intuitionistic formula α is derivable in Intuitionistic Propositional Logic if and only if α^{GMT} is derivable in the (Classical) Modal Logic S_4, that is,

$$\vdash_i \alpha \text{ if and only if } \vdash_{S_4} \alpha^{GMT}.$$

Theorem 1.3.10 ([WZ14], p. 3)**.** A classical formula α is derivable in Classical Propositional Logic if and only if α^{GMT} is derivable in the (Classical) Modal Logic S_5, that is,

$$\vdash \alpha \text{ if and only if } \vdash_{S_5} \alpha^{GMT}.$$

1.3.3 Fuzzy Logic

Fuzzy Logics seek to capture *imprecise*, *vague* concepts, or situations where we have *non-numeric* or *partial* information. We have that the propositions in these logics are true to *a* *certain degree*, varying from totally false to totally true, with a continuum of intermediate values.

The first examples of Fuzzy Logics were studied by Łukasiewicz and Tarski in the 1920s, in their study of multivalued logics, but it was not until Lotfi Zadeh's work on Fuzzy Set Theory [Zad65] that Fuzzy Logics were really born as we know them today. In Zadeh's work, the set $\{0,1\}$ in the definition of the characteristic function was replaced by the interval of real numbers $[0,1]$ and the properties of the resulting sets, called *fuzzy sets*, that is, sets in which the membership takes as truth values real numbers in the interval $[0,1]$. Similarly, in later work on Fuzzy Logics, the idea of taking the additionally structured lattice $[0,1]$ as a set of truth values for propositions in logic was continued. For a detailed historical treatment of Fuzzy Logics see [BDK17].

Basic Fuzzy Logic (BL) is a multivalued logic introduced by Petr Hájek in [Háj98a] and developed in [Háj98b]. This system seeks to capture the logic of continuous t-norms and their adjoints in order to capture the most important examples in the field of fuzzy logics up to that time: the logic of Łukasiewicz, Gödel, and product.

On the other hand, **Monoidal t-norm Based Logic** (MTL) was introduced in [EG01] in order to capture the logic of left-continuous t-norms. The distinction between left continuous and continuous is crucial, since left continuity is the necessary and sufficient condition for the existence of an adjunction for the t-norm, and it is this adjunction that allows us to define the implication in the logic. MTL is weaker than the BL and stronger than the **Residuated Logic** (or **Höhle's Monoidal Logic**) and seeks to connect to the previous logics to create a more compressible map of the Fuzzy Logics.

1.3.4 Residuated Logic (Monoidal Logic)

This logic was introduced by Ulrich Höhle [Höh94] under the name *Monoidal Logic* in order to present a general framework for the study of Fuzzy Logics based on t-norms, Intuitionistic Logic and Girard's Linear Logic. In his article, Höhle considers residuated integral commutative 1-monoids (i.e. complete residuated lattices in our terms) as a set of truth values of his logic, presents a completeness and soundness theorem, and shows some interactions of it with the other logics mentioned.

Throughout this work, we will call Höhle's Monoidal Logic as *Residuated Logic*, following Lano's notation [Lan92a] in his study of Residuated Logic and fuzzy sets, where this logic is studied in its modal variant and is applied in the context of set-theoretic models valued on residuated lattices.

Definition 1.3.11 (Logical symbols for the propositional case, [Lan92a])**.** The fundamental difference between Classical (or Intuitionistic) Logic and Fuzzy (or Residuated) Logic is that we consider different basic logical symbols, namely, in Residuated Logic, we consider two types of **conjunction**, a **weak conjunction** ($\wedge$) and a **strong conjunction** ($\&$). With that in mind, the basic symbols for the propositional case are the following:

1. A countable set of propositional variables $Var = \{p_i : i \in \omega\}$,

2. Strong conjunction & (binary),

3. Implication $\rightarrow$ (binary),

4. Weak conjunction $\wedge$ (binary),

5. Contradiction $\perp$ (constant),

6. Disjunction $\vee$ (binary),

and the following symbols that are definable, using the previous ones:

1. Equivalence $\varphi \equiv \psi := (\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$,

2. Negation $\sim \varphi := \varphi \rightarrow \perp$,

3. Tautology $\top :=\sim \perp$.

Definition 1.3.12 (Propositional formulas, [Lan92a]). The construction of propositional formulas is done by recursion on a manner analogous to how it is done in Classical Propositional Logic. To differentiate these formulas from formulas in Classical (or Intuitionistic) Logic, we call them *Residuated (Propositional) formulas*, or R-formulas for short.

Definition 1.3.13 (Axioms of Residuated Propositional Logic, [Lan92a], pp. 203 and 204). Let $\alpha, \beta, \gamma, \delta$ be propositional R-formulas. The axioms of Propositional Residuated Logic are:

1. $(\alpha\&\beta) \rightarrow \alpha$.

2. $(\alpha \& \beta) \to (\beta \& \alpha)$, $((\alpha \& \beta) \& \gamma) \to (\alpha \& (\beta \& \alpha))$.

3. $((\alpha \& \beta) \to (\alpha \& \gamma)) \to (\alpha \& (\beta \& \gamma))$.

4. $((\alpha \to \beta) \& (\beta \to \gamma)) \to (\alpha \to \gamma)$.

5. $\beta \to (\alpha \to \beta)$.

6. $(\alpha \& (\alpha \to \beta)) \to \beta$.

7. $\alpha \to (\alpha \vee \beta)$

8. $(\alpha \vee \beta) \to (\beta \vee \alpha)$, $((\alpha \vee \beta) \vee \gamma) \to (\alpha \vee (\beta \vee \gamma))$.

9. $((\alpha \to \gamma) \& (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma)$.

10. $\neg\alpha \to (\alpha \to \beta)$.

11. $\alpha \to (\beta \to (\alpha \& \beta))$.

12. $(\alpha \& (\beta \vee \gamma)) \to ((\alpha \& \gamma) \vee (\alpha \& \gamma))$.

13. $((\alpha \& \beta) \to \gamma) \equiv (\alpha \to (\beta \to \gamma))$.

14. $(\alpha \to \beta) \& (\gamma \to \delta) \to ((\alpha \& \gamma) \to (\beta \& \delta))$.

15. $(\beta \to (\beta \& \neg\beta)) \to \neg\beta$.

The inference rule used is Modus Ponens.

Definition 1.3.14 (Axioms for the weak conjunction $\wedge$, [Lan92a], p. 205)**.** The following are the axioms for the weak conjunction $\wedge$:

1. $(\alpha \& \beta) \to (\alpha \wedge \beta)$.

2. $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma))$.

3. $(\alpha \to \beta) \wedge (\gamma \to \beta) \to ((\alpha \vee \gamma) \to \delta)$.

4. $\wedge$ is commutative and associative.

5. $(\alpha \to \beta) \to ((\alpha \wedge \gamma) \to (\beta \wedge \gamma))$.

6. $(\alpha \wedge \beta) \to \alpha$.

Remark 1.3.15 ([Lan92a], p. 203)**.** If we add the axiom

$$\alpha \to (\alpha \& \alpha) \text{ for all } \alpha \text{ } R\text{-formulas},$$

to the axioms of Propositional Residuated Logic, we would obtain an axiomatization for Propositional Intuitionistic Logic.

Definition 1.3.16 ([Lan92a], p. 204)**.** Just as in Classical Logic, we can add the quantifiers $\forall$ and $\exists$ to construct the set of predicate formulas in a given (Classical first-order) language $\mathcal{L}$ and we call them *Residuated* $\mathcal{L}$*-formulas* or $R - \mathcal{L}$*-formulas*, for short. Let φ and ψ be $R - \mathcal{L}$-formulas. The axioms of *Residuated Predicate Logic* are as follows:

1. $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$.

2. $(\sigma = \tau \& \varphi(\sigma)) \to \varphi(\tau)$, where φ is atomic and there is at most one occurrence of x in φ.

3. $\exists x(\varphi \to \psi) \to (\forall x\varphi \to \exists x\psi)$.

4. $\forall x(\varphi \to \psi) \to (\exists x\varphi \to \exists x\psi)$.

5. $(\exists x\varphi \to \forall x\psi) \to \forall x(\varphi \to \psi)$.

6. $\varphi \to \forall x\varphi$, with non-free x in φ.

7. $\exists x(\sigma = x)$, with x not occurring in σ.

8. $\forall x\varphi \to \varphi(\sigma/x)$, with σ free for x in φ.

9. $\exists x\varphi \to \varphi$, with non-free x in φ.

10. $=$ is an equivalence relation.

11. $\varphi(\sigma/x) \to \exists x\varphi$, with σ free for x in φ.

12. $\exists x(\varphi \& \alpha) \to ((\exists x\varphi)\&\alpha)$, with non-free x in α.

We consider the usual (classical) rules of inference for the introduction of $\forall$ and $\exists$.

Definition 1.3.17. We say that T is an $R-\mathcal{L}$-*theory* is T is a set of $R-\mathcal{L}$-sentences. We say that φ is an *R-logical consequence of* T, denoted by $T \vdash_r \varphi$, if there exists a deduction from T of φ that uses the axioms and deduction rules given above.

1.3.5 t-norm Logics

By t-norm Logics we mean any logic whose semantic counterpart is based on some t-norm over the interval $[0,1]$. Since we want to ensure the existence of an adjoint $\to$ of the product $\cdot$, we require different forms of distributivity of the product with respect to arbitrary joins (see Theorem 1.2.5). The most important examples of this kind of logics are:

1. Monoidal t-norm based Logic (MTL) : This is the logic associated with **left-continuous t-norms** (see Definition 1.2.30), that is, the logic where the truth take values in $[0,1]$, endowed of lattice structure with its usual order, a left-continuous t-norm. An axiomatization of MTL can be obtained by taking the axioms of Residuated Logic and adding the axiom scheme of *prelinearity* (Lin):

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

2. Basic Fuzzy Logic (BL): This is the logic associated with the **continuous t-norms** (see Definition 1.2.30). An axiomatization of the logic BL can be obtained by taking the axioms of MTL and adding the axiom scheme of *divisibility* (Div):

$$(\varphi \wedge \psi) \to (\varphi \& (\varphi \to \psi)).$$

3. Łukasiewicz Logic: This is a continuum-many valued generalization of the n-valued Łukasiewicz Logic (see Example 1.2.31). An axiomatization of the Łukasiewicz Logic can be obtained by taking the axioms of BL and adding the axiom of *double negation* (Inv)

$$(\neg\neg\varphi) \to \varphi.$$

4. Product Logic: This is the logic associated to the usual product of $\mathbb{R}$. An axiomatization of the Product Logic can be obtained by taking the axioms of BL and adding the axiom scheme of *weak contraction* (Weak-Con)

$$(\varphi \wedge \neg\varphi) \to \bot.$$

and the axiom scheme Π_1

$$(\neg\neg\varphi)\&(((\varphi\&\psi) \to (\varphi\&\chi)) \to (\psi \to \chi)).$$

5. Gödel-Dummett Logic: In 1932, Gödel [Göd32] introduces a family of n-valued logics in his study of Intuitionistic Logic. Gödel shows that there is no finite structure of truth values suitable for Heyting's axiomatization of Intuitionistic Logic such that one has a completeness-soundness theorem. That is, there might be n-valued models of intuitionistic logic, but these models will not capture all the tautologies of this logic.

   Continuing Gödel's work, Dummett [Dum59] proposes a generalization of Gödel's Logic that considers an infinite set of truth values. Dummett presents two logics, one that has a countable set of truth values and one that has a continuum of truth values, and then shows that the tautologies of both logics coincide. We focus on studying Gödel's Logic for the case where the set of truth values is the interval $[0, 1]$ (see Example 1.2.31).

   An axiomatization of the Gödel-Dummett Logic can be obtained by taking the axioms of BL and adding the axiom scheme of *contraction* (Con)

$$\varphi \to (\varphi\&\varphi).$$

   Or equivalently, taking the axioms of Intuitionistic Logic and adding to them the axiom of prelinearity (Lin)

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

1.3.6 Modal Residuated Logic.

The language of Modal Residuated Logic has the same symbols as Residuated Logic together with an unary connective of *possibility* $\Diamond$. The idea is to interpret some **quantic nucleus** as the **possibility** operator when we deal with the semantics. The formulas in a given first-order language $\mathcal{L}$ are constructed by recursion in the same way as for the residuated case and we call them formulas *Modal Residuated $\mathcal{L}$-formulas*, or $MR - \mathcal{L}$*-formulas*, for short. These formulas satisfy the same axioms as Residuated Logic does, but we need to add new axiom schemes for the possibility operator. We do not have an axiomatization for the logic of "complete residuated lattices with a quantic nucleus" (i.e. *Modal Residuated Logic*). Some basic axioms schemes that are valid in the logic that we would like to axiomatize are

1. $\varphi \to \Diamond\varphi$
2. $\Diamond\Diamond\varphi \to \Diamond\varphi$
3. $\Diamond(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$
4. $\Diamond\varphi \& \Diamond\psi \to \Diamond(\varphi \& \psi)$.

We will focus only on the semantical aspect of this logic; hence we will not prove a Soundness and Completeness Theorem for this logic.

2 Constructions over valued models

Valued models substitute the standard set of logical values $\{0, 1\}$ of Classical Logic for different lattices that carry with them logical properties via their algebraic properties.

It is known that it is enough to consider the trivial Boolean algebra $\{0, 1\}$ to represent the truth values of Classical Logic, so it is not required to study $\mathbb{B}$-valued models for a every Boolean algebra $\mathbb{B}$ to understand Classical Logic, we just need to focus on the two-valued case.

On the other hand, Gödel proved that Intuitionistic Propositional Logic cannot be represented as a finite many-valued logic [Göd32]. Since it is not so easy to represent Intuitionistic Logic via an infinite many-valued logic, it has been a common practice to study Intuitionistic Logic via Heyting valued models or Kripke models.

In this chapter, we focus on the study of valued models in different lattices and in constructions that generalize von Neumann's and Gödel's hierarchy in these models.

First, in Section 2.1, we expose an overview of lattice-valued models. As set of truth values, we consider Boolean algebras, Heyting algebras and commutative integral quantales with and without a quantic nucleus, and then we show how sentences in their respective logics are interpreted.

Then, in Section 2.2 we construct set-theoretic lattice-valued models and discuss some essential facts about them. We focus on the proofs for some of the properties of quantale-valued models of Set Theory for the following reason: Even though it is well known that these models are quantale-valued models in the sense of Definition 2.1.8, there is no, as far as we are aware, proof in the literature that justifies this fact, thus we will prove this in Theorem 2.2.31.

Lastly, in the context of valued models, we adapt Fitting's idea [Fit69], where, as motivation for his definition of class of constructable sets using Kripke models, he showed how to construct L (or more precisely a model "isomorphic" to L) using two-valued characteristic functions that are definable by some formula.

We propose new definitions of the notion of **definable subset** within a Boolean-valued model of Set Theory, and with these notions, we propose two new constructions of the constructable universe: $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$. Moreover, we prove that these models are, in fact, **two-valued**, since our definition of definable is too restrictive and forces the models to only take these values. Furthermore, we prove that $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$ are "isomorphic" to V (von Neumann universe) and L (Gödel's constructible universe), respectively.

Then, we discuss the problem of trying to generalize these notions of definability to the context of quantale-valued models. We found that the resulting classes of constructable models are also **two valued**, and therefore are not suitable to study Residuated Logic. Hence, in the next chapter, we instead focus on developing the notion of constructable sets in the realm of **Kripke models**, where these kind of problems are avoided and the constructions seem more promising.

2.1 Preliminaries of valued models

Valued models are a natural and simple way to find a semantical interpretation for various logics, as long as one can find a lattice (or some other algebraic or ordered structure) associated to them.

2.1.1 Boolean models

As mentioned before, by the completeness theorem of first-order logic, one only really needs two truth values to understand Classical Logic semantically. Regardless, we will study Boolean models, since it was here where von Neumann's hierarchy was first generalized [SS67] and also, since we are interested in generalizing Gödel's constructions for these lattice-valued models.

For the rest of the chapter, let $\mathcal{L}$ denote a first-order language.

Definition 2.1.1. Given a complete Boolean algebra $\mathbb{B} = (\mathbb{B}, \leqslant, \vee, \wedge, 0, 1, \neg)$, we say that a $\mathbb{B}$-*valued* $\mathcal{L}$-*structure* $\mathcal{M}$ consists of:

1. A non-empty class M.
2. A function $[\![\cdot = \cdot]\!]^{\mathcal{M}}_{\mathbb{B}} : M^2 \to \mathbb{B}$, which we call *equality*, such that for all $f, g, h \in M$:
    a. Reflexivity: $[\![f = f]\!]^{\mathcal{M}}_{\mathbb{B}} = 1$.
    b. Symmetry: $[\![f = g]\!]^{\mathcal{M}}_{\mathbb{B}} = [\![g = f]\!]^{\mathcal{M}}_{\mathbb{B}}$.
    c. Transitivity $[\![f = g]\!]^{\mathcal{M}}_{\mathbb{B}} \wedge [\![g = h]\!]^{\mathcal{M}}_{\mathbb{B}} \leqslant [\![f = h]\!]^{\mathcal{M}}_{\mathbb{B}}$.

3. For each $n-$ary relational symbol $R \in \mathcal{L}$, a function $R^{\mathcal{M}} : M^n \to \mathbb{B}$ such that for every $g_i, f_i \in M$ with $i = 1, ..., n$

$$\bigwedge_{i=1}^{n} [\![f_i = g_i]\!]_{\mathbb{B}}^{\mathcal{M}} \wedge R^{\mathcal{M}}(f_1, ..., f_n) \leqslant R^{\mathcal{M}}(g_1, ..., g_n).$$

4. For each m-ary function symbol $f \in \mathcal{L}$, a function $f^{\mathcal{M}} : M^m \to M$ such that for every $g_i, f_i \in M$ with $i = 1, ..., m$,

$$\bigwedge_{i=1}^{m} [\![f_i = g_i]\!]_{\mathbb{B}}^{\mathcal{M}} \leqslant [\![f^{\mathcal{M}}(f_1, ..., f_m) = f^{\mathcal{M}}(g_1, ..., g_m)]\!]_{\mathbb{B}}^{\mathcal{M}}.$$

5. For each constant symbol $c \in \mathcal{L}$, an element in the universe $c^{\mathcal{M}} \in M$.

Definition 2.1.2. Let $\mathcal{M}$ be a $\mathbb{B}$-valued $\mathcal{L}$-structure. We define the language

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L} \cup \{c_a : a \in M\} \text{ where } c_a \text{ is a constant symbol for each } a \in M.$$

and let

$$Sent(\mathcal{L}_{\mathcal{M}})$$

denote the first-order (classical) sentences constructed in the language $\mathcal{L}_{\mathcal{M}}$.

Definition 2.1.3. Given a $\mathbb{B}$-valued $\mathcal{L}$-structure $\mathcal{M}$, we can define a valuation $[\![\cdot]\!]_{\mathbb{B}}^{\mathcal{M}} : Sent(\mathcal{L}_{\mathcal{M}}) \to \mathbb{B}$ by recursion over the complexity of the $\mathcal{L}_{\mathcal{M}}$-formula:

1. For atomic formulas we consider

    a. $[\![f = g]\!]_{\mathbb{B}}^{\mathcal{M}} := [\![f^{\mathcal{M}} = g^{\mathcal{M}}]\!]_{\mathbb{B}}^{\mathcal{M}}$, where f, g are closed $\mathcal{L}_{\mathcal{M}}$-terms.

    b. $[\![Rf_1, ..., f_n]\!]_{\mathbb{B}}^{\mathcal{M}} := R^{\mathcal{M}}(f_1^{\mathcal{M}}, ..., f_n^{\mathcal{M}})$, where R is an n-ary relation, f_i and g_i are closed $\mathcal{L}_{\mathcal{M}}$-terms and $i = 1, 2, ..., n$.

2. Let ψ and φ be $\mathcal{L}_{\mathcal{M}}$-sentences and $\theta(x)$ a $\mathcal{L}_{\mathcal{M}}$-formula with free variable x.

a. $[\![\neg\psi]\!]^{\mathcal{M}}_{\mathbb{B}} := \neg[\![\psi]\!]^{\mathcal{M}}_{\mathbb{B}}$

b. $[\![\psi \to \varphi]\!]^{\mathcal{M}}_{\mathbb{B}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{B}} \to [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{B}}$

c. $[\![\psi \wedge \varphi]\!]^{\mathcal{M}}_{\mathbb{B}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{B}} \wedge [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{B}}$

d. $[\![\psi \vee \varphi]\!]^{\mathcal{M}}_{\mathbb{B}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{B}} \vee [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{B}}$

e. $[\![\exists x\theta(x)]\!]^{\mathcal{M}}_{\mathbb{B}} := \bigvee_{a\in M} [\![\theta(c_a)]\!]^{\mathcal{M}}_{\mathbb{B}}$

f. $[\![\forall x\theta(x)]\!]^{\mathcal{M}}_{\mathbb{B}} := \bigwedge_{a\in M} [\![\theta(c_a)]\!]^{\mathcal{M}}_{\mathbb{B}}$

Definition 2.1.4. If φ is a $\mathcal{L}_{\mathcal{M}}$-sentence, we say that $\mathcal{M} \models \varphi$ if $[\![\varphi]\!]^{\mathcal{M}}_{\mathbb{B}} = 1$.

Definition 2.1.5. Let T be a $\mathcal{L}$-theory and φ be a $\mathcal{L}$-sentence. we say that $T \models \varphi$ if for every complete Boolean algebra $\mathbb{B}$ and every $\mathbb{B}$-valued $\mathcal{L}$-structure $\mathcal{M}$, if $\mathcal{M} \models T$, then $\mathcal{M} \models \varphi$.

We say that $T \vdash \varphi$, if φ is syntactically deducible from T and from the axioms of Classical first-order Logic using the classical deduction rules.

Theorem 2.1.6 (Completeness and Soundness Theorem, [Bel05])**.** Given a $\mathcal{L}$-theory T and φ a $\mathcal{L}$-sentence,

$$T \models \varphi \text{ if and only if } T \vdash \varphi.$$

2.1.2 Heyting-valued models

Definition 2.1.7. Given a complete Heyting algebra $\mathbb{H} = (\mathbb{H}, \leqslant, \vee, \wedge, 0, 1, \neg)$, we can define the notion of a $\mathbb{H}$-valued $\mathcal{L}$-structure just as in the previous subsection, and we have a Completeness and Soundness Theorem for these models and Intuitionistic Logic (see [Bel05], Chapter 8 for more on these models).

2.1.3 Quantale-valued models

The development of these models can be found in [Lan92a], where Lano mentions the Completeness and Soundness of these models (where he calls them **residuated algebra valued models**) and then goes on to prove the Completeness Theorem for **topological residuated algebra valued models** and **Residuated Modal Logic** RS_5. In this subsection, we focus on the constructions of these models without the use of modal operators.

Definition 2.1.8 (ℚ-valued models, [Lan92a])**.** Let $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \neg, \cdot, \rightarrow, 0, 1)$ be a commutative integral quantale. A $\mathbb{Q}$-*valued* $\mathcal{L}$-*structure* $\mathcal{M}$ consists of:

1. A non-empty class M.
2. An equality function $[\![\cdot = \cdot]\!]^{\mathcal{M}}_{\mathbb{Q}} : M^2 \to \mathbb{Q}$ such that for all $f, g, h \in M$
   a. Reflexivity: $[\![f = f]\!]^{\mathcal{M}}_{\mathbb{Q}} = 1$.
   b. Symmetry: $[\![f = g]\!]^{\mathcal{M}}_{\mathbb{Q}} = [\![g = f]\!]^{\mathcal{M}}_{\mathbb{Q}}$.
   c. Transitivity: $[\![f = g]\!]^{\mathcal{M}}_{\mathbb{Q}} \cdot [\![g = h]\!]^{\mathcal{M}}_{\mathbb{Q}} \leqslant [\![f = h]\!]^{\mathcal{M}}_{\mathbb{Q}}$.
3. For each $n-$ary relational symbol $R \in \mathcal{L}$, a function $R^{\mathcal{M}} : M^n \to \mathbb{Q}$ such that for every $g_i, f_i \in M$ with $i = 1, ..., n$
$$\prod_{i=1}^{n} [\![f_i = g_i]\!]^{\mathcal{M}}_{\mathbb{Q}} \cdot R^{\mathcal{M}}(f_1, ..., f_n) \leqslant R^{\mathcal{M}}(g_1, ..., g_n).$$
4. For each m-ary function symbol $f \in \mathcal{L}$, a function $f^{\mathcal{M}} : M^m \to M$ such that for every $g_i, f_i \in M$ with $i = 1, ..., m$,
$$\prod_{i=1}^{m} [\![f_i = g_i]\!]^{\mathcal{M}}_{\mathbb{Q}} \leqslant [\![f^{\mathcal{M}}(f_1, ..., f_m) = f^{\mathcal{M}}(g_1, ..., g_m)]\!]^{\mathcal{M}}_{\mathbb{Q}}.$$

5. For each constant symbol $c \in \mathcal{L}$, an element in the universe $c^{\mathcal{M}} \in M$.

Remark 2.1.9. Notice that the only difference between Boolean models and quantale models is that the operation of $\wedge$ is replaced by $\cdot$.

Definition 2.1.10. We define the language

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L} \cup \{c_a : a \in M\} \text{ where } c_a \text{ is a constant symbol for each } a \in M.$$

and let

$$Sent^R(\mathcal{L}_{\mathcal{M}})$$

denote the first-order **Residuated** sentences (R-sentences) constructed in the language $\mathcal{L}_{\mathcal{M}}$.

Definition 2.1.11. Given a $\mathbb{Q}$-valued $\mathcal{L}$-structure $\mathcal{M}$, we define the valuation $[\![\cdot]\!]^{\mathcal{M}}_{\mathbb{Q}} : Sent^R(\mathcal{L}_{\mathcal{M}}) \to \mathbb{Q}$ by recursion on the complexity of the sentences.

1. For atomic sentences we consider

   a. $[\![f = g]\!]^{\mathcal{M}}_{\mathbb{Q}} := [\![f^{\mathcal{M}} = g^{\mathcal{M}}]\!]^{\mathcal{M}}_{\mathbb{Q}}$, where f, g are closed $\mathcal{L}_{\mathcal{M}}$-terms.

   b. $[\![Rf_1, ..., f_n]\!]^{\mathcal{M}}_{\mathbb{Q}} := R^{\mathcal{M}}(f_1^{\mathcal{M}}, ..., f_n^{\mathcal{M}})$, where f_i, g_i are closed $\mathcal{L}_{\mathcal{M}}$-terms for all $i \in \{1, 2, ..., n\}$.

2. Let ψ and φ be $R - \mathcal{L}_{\mathcal{M}}$-sentences and $\theta(x)$ an $R - \mathcal{L}_{\mathcal{M}}$-formula with free variable x.

   a. $[\![\psi \& \varphi]\!]^{\mathcal{M}}_{\mathbb{Q}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{Q}} \cdot [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{Q}}$

   b. $[\![\psi \to \varphi]\!]^{\mathcal{M}}_{\mathbb{Q}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{Q}} \to [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{Q}}$

   c. $[\![\psi \wedge \varphi]\!]^{\mathcal{M}}_{\mathbb{Q}} := [\![\psi]\!]^{\mathcal{M}}_{\mathbb{Q}} \wedge [\![\varphi]\!]^{\mathcal{M}}_{\mathbb{Q}}$

d. $[\![\psi \vee \varphi]\!]_{\mathbb{Q}}^{\mathcal{M}} := [\![\psi]\!]_{\mathbb{Q}}^{\mathcal{M}} \vee [\![\varphi]\!]_{\mathbb{Q}}^{\mathcal{M}}$

e. $[\![\exists x\theta(x)]\!]_{\mathbb{Q}}^{\mathcal{M}} := \bigvee_{a\in M} [\![\theta(c_a)]\!]_{\mathbb{Q}}^{\mathcal{M}}$

f. $[\![\forall x\theta(x)]\!]_{\mathbb{Q}}^{\mathcal{M}} := \bigwedge_{a\in M} [\![\theta(c_a)]\!]_{\mathbb{Q}}^{\mathcal{M}}$

Definition 2.1.12. If $\mathcal{M}$ is a $\mathbb{Q}$-valued $\mathcal{L}$-structure and φ is an $R-\mathcal{L}_{\mathcal{M}}$-sentence, we say that $\mathcal{M} \models_r \varphi$ if $[\![\varphi]\!]_{\mathbb{Q}}^{\mathcal{M}} = 1$.

Definition 2.1.13. Let T be an $R-\mathcal{L}$-theory and φ be an $R-\mathcal{L}$-sentence.

1. We say that $T \models_r \varphi$ if for every complete residuated lattice $\mathbb{Q}$ and every $\mathbb{Q}$-valued $\mathcal{L}$-structure $\mathcal{M}$ we have that $\mathcal{M} \models_r T$, implies $\mathcal{M} \models_r \varphi$.

2. Recall that $T \vdash_r \varphi$ means that there exists a deduction from T of φ by using the axioms and deduction rules given in Section 1.3.4.

Definition 2.1.14 (Completeness and Soundness Theorem, [Lan92b], p. 204)**.** Given an $R-\mathcal{L}$-theory T and an $R-\mathcal{L}$-sentence φ,

$$T \models_r \varphi \text{ if and only if } T \vdash_r \varphi.$$

2.1.4 Quantale-valued modal models

The following subsection is inspired in the ideas of Lano [Lan92a]. These models are constructed just as the quantale-valued models of the previous subsection, but now we consider a commutative integral quantale $\mathbb{Q}$ together with a **quantic nucleus** γ so that we can extend the valuation $[\![\cdot]\!]$ of a $\mathbb{Q}$-valued $\mathcal{L}$-model $\mathcal{M}$ to all $RM-\mathcal{L}$-sentences by interpreting the symbol of possibility as follows:

$$[\![\Diamond\varphi(x)]\!] := \gamma\left([\![\varphi(x)]\!]\right)$$

We call these structures $\mathbb{Q}$-*valued Modal models*, or $\mathbb{Q}$-*valued M-models*, for short.

Theorem 2.1.15. Every quantale-valued M-model $\mathcal{M}$ satisfies the axioms of Modal Residuated Logic stated in Subsection 1.3.6. That is, for every $MR-\mathcal{L}_{\mathcal{M}}$-sentence φ

1. $[\![\varphi]\!] \leqslant [\![\Diamond\varphi]\!]$.
2. $[\![\Diamond\Diamond\varphi]\!] \leqslant [\![\Diamond\varphi]\!]$.
3. $[\![\Diamond(\varphi \to \psi)]\!] \leqslant [\![\Diamond\varphi]\!] \to [\![\Diamond\psi]\!]$.
4. $[\![\Diamond\varphi \& \Diamond\psi]\!] \leqslant [\![\Diamond(\varphi\&\psi)]\!]$.

Proof. Items .1, 2. and 4. follow immediately from the definition of the interpretation of $\Diamond$ and the properties of the quantic nucleus γ. Item 3. follows from Corollary 1.2.47. □

2.2 Valued models of Set Theory

To understand the motivation behind the definition of valued models, it is convenient to study their relation to characteristic functions: Let X be a set and let us take $A \subseteq X$. We know that every subset A can be represented via its characteristic function χ_A

$$\chi_A : X \to \{0,1\}, \text{ where } \chi_A(x) = 1, \text{ if and only if, } x \in A.$$

Therefore, if one wanted to generalize the notion of a subset, a natural way would be to change the notion of a characteristic function and use any Boolean algebra $\mathbb{B}$ instead of the trivial Boolean algebra $\{0,1\}$. This is essentially what is done in the model proposed by Scoot and Solovay [SS67].

Although these models were first studied to find a simpler more intuitive way of understanding Cohen's forcing, now they are widely used as natural models for a variety of logics and set theories within them, such as Heyting-valued models for Intuitionistic Logic [Gra79], $BL\Delta$-algebra valued models for the Fuzzy Logic $BL\forall\Delta$ [HH01, HH03], and Topological Residuated algebra valued models for Modal Residuated Logic [Lan92a], among many others.

2.2.1 Boolean models of Set Theory

Throughout this section, $\mathbb{B}$ denotes be a complete Boolean algebra

Definition 2.2.1 ([Bel05], p. 21, (1.4))**.** We define V_α by transfinite recursion over the ordinals

1. $V_0^{\mathbb{B}} := \varnothing$
2. $V_{\alpha+1}^{\mathbb{B}} := \{f \in V : f$ is a function with $dom(f) \subseteq V_\alpha^{\mathbb{B}}$ and $ran(f) \subseteq \mathbb{B}\}$
3. $V_\alpha^{\mathbb{B}} := \bigcup_{\beta<\alpha} V_\beta^{\mathbb{B}}$ with $\alpha \neq 0$ limit ordinal.
4. $V^{\mathbb{B}} := \bigcup_{\alpha \in ON} V_\alpha^{\mathbb{B}}$.

Fact 2.2.2 ([Bel05], p. 21, (1.6))**.** From the definition, we have that $f \in V^{\mathbb{B}}$ if and only if f is a function with $dom(f) \subseteq V^{\mathbb{B}}$ and $ran(f) \subseteq \mathbb{B}$.

Definition 2.2.3 ([Bel05])**.** Given $f \in V^{\mathbb{B}}$, we define $rank_{V^{\mathbb{B}}}(x)$ as the smallest ordinal α such that $f \in V_{\alpha+1}^{\mathbb{B}}$, that is, the only ordinal such that $f \in V_{\alpha+1}^{\mathbb{B}} \backslash V_\alpha^{\mathbb{B}}$.

Theorem 2.2.4 (Principle of induction for $V^{\mathbb{B}}$, [Bel05] p. 21, 1.7)**.** For every Classical first-order $\mathcal{L}_\in$-formula $\varphi(x)$, we have

$$\forall x \in V^{\mathbb{B}}((\forall y \in dom(x)\varphi(y)) \to \varphi(x)) \to (\forall x \in V^{\mathbb{B}}\varphi(x)).$$

Fact 2.2.5. If $\alpha \in ON$, then $V^{\mathbb{B}}_{\alpha} \subseteq V^{\mathbb{B}}_{\alpha+1}$.

Corollary 2.2.6. If $\alpha < \beta \in ON$, then $V^{\mathbb{B}}_{\alpha} \subseteq V^{\mathbb{B}}_{\beta}$.

Definition 2.2.7. We define the (class) language $\mathcal{L}_{\mathbb{B}}$ as $\mathcal{L}_{\mathbb{B}} := \mathcal{L}_{V^{\mathbb{B}}} = \mathcal{L}_{\in} \cup \{c_a : a \in V^{\mathbb{B}}\}$, where each c_a is a constant symbol. From now on, we call $\mathcal{L}_{\mathbb{B}}$-sentences (formulas) to the classical first-order sentences (formulas) constructed in the language $\mathcal{L}_{\mathbb{B}}$.

Definition 2.2.8 ([Bel05], p. 23, (1.15) and (1.16))**.** We define interpretations of $\in$ and $=$ in $V^{\mathbb{B}}$ as follows:

1. $[\![f \subseteq g]\!]_{\mathbb{B}} = \bigwedge_{x \in dom(f)} (f(x) \to [\![x \in g]\!]_{\mathbb{B}})$
2. $[\![f = g]\!]_{\mathbb{B}} = [\![f \subseteq g]\!]_{\mathbb{B}} \wedge [\![g \subseteq f]\!]_{\mathbb{B}} = \bigwedge_{x \in dom(f)} (f(x) \to [\![x \in g]\!]_{\mathbb{B}}) \wedge \bigwedge_{y \in dom(g)} (g(y) \to [\![y \in f]\!]_{\mathbb{B}})$
3. $[\![f \in g]\!]_{\mathbb{B}} = \bigvee_{y \in dom(g)} (g(y) \wedge [\![y = f]\!]_{\mathbb{B}})$

Remark 2.2.9 ([Bel05], p. 23)**.** Notice that in order to define $[\![\cdot = \cdot]\!]_{\mathbb{B}}$ we are using $[\![\cdot \in \cdot]\!]_{\mathbb{B}}$ and vice versa. This is possible because we are defining both relations simultaneously by recursion on a well-founded relation $<$: Given $(x, y), (u, v) \in V^{\mathbb{B}} \times V^{\mathbb{B}}$, let

$$(x, y) < (u, v) \text{ if and only if either } (x \in dom(u) \text{ and } y = v) \text{ or } (x = u \text{ and } y \in dom(v))$$

Remark 2.2.10. We have that $(V^{\mathbb{B}}, [\![\cdot \in \cdot]\!]_{\mathbb{B}}, [\![\cdot = \cdot]\!]_{\mathbb{B}})$ is a $\mathbb{B}$-valued $\mathcal{L}_{\in}$-structure (see Theorem 2.2.14) and thus we can define a valuation $[\![\cdot]\!]_{\mathbb{B}} : Sent(\mathcal{L}_{\mathbb{B}}) \to \mathbb{B}$ as in Definition 2.1.3.

Remark 2.2.11 ([Bel05], Chapter 1, Remarks 1)**.** Observe that there is a considerable "duplication" of elements in the Boolean universe $V^{\mathbb{B}}$, that is, for every $f \in V^{\mathbb{B}}$ there exist a proper class of elements $g \in V^{\mathbb{B}}$ such that $[\![f = g]\!]_{\mathbb{B}} = 1$. For example, let us take $\alpha \in ON$ and define $Z^{\alpha} = \{(x, 0_{\mathbb{B}}) : x \in V^{\mathbb{B}}_{\alpha}\}$. It is easy to see that $[\![\varnothing = Z^{\alpha}]\!]_{\mathbb{B}} = 1$ for every $\alpha \in ON$ so each member of the proper class $\{Z^{\alpha} : \alpha \in ON\}$ represents the empty set in $V^{\mathbb{B}}$. Furthermore, given $f \in V^{\mathbb{B}}$, let us consider $\beta = rank_{V^{\mathbb{B}}}(f)$, i.e. the ordinal β such that $f \in V^{\mathbb{B}}_{\beta+1} \backslash V^{\mathbb{B}}_{\beta}$. Then, for every $\alpha > \beta$ consider the function f^{α} with $dom(f^{\alpha}) = V^{\mathbb{B}}_{\alpha}$, $f^{\alpha}(x) = f(x)$ for $x \in dom(f)$ and $f^{\alpha}(x) = 0_{\mathbb{B}}$ for $x \in dom(f^{\alpha}) \backslash dom(f)$. Then, for every $\alpha > \beta$, we have that $[\![f = f^{\alpha}]\!]_{\mathbb{B}} = 1$. Therefore, it is helpful to think of the members of $V^{\mathbb{B}}$ as 'representatives or labels' for sets (or even 'potential' sets), on which (Boolean-valued) equality is defined as an equivalence relation with very big equivalence classes.

Definition 2.2.12. If φ is a $\mathcal{L}_{\mathbb{B}}$-sentence, we say that $V^{\mathbb{B}} \models \varphi$, if and only if, $[\![\varphi]\!]_{\mathbb{B}} = 1$.

Remark 2.2.13. In case it is clear from the context, we write $[\![\psi]\!]$ instead of $[\![\psi]\!]_{\mathbb{B}}$.

Theorem 2.2.14 ([Bel05], Theorem 1.17)**.** All the axioms of first-order calculus with equality and first-order inference rules are valid on $V^{\mathbb{B}}$. In particular, if $f, g, h \in V^{\mathbb{B}}$ and $\varphi(x)$ is a $\mathcal{L}_{\mathbb{B}}$-formula:

1. Reflexivity: $[\![f = f]\!] = 1$.

2. If $x \in dom(f)$, then $f(x) \leqslant [\![x \in f]\!]$.

3. Symmetry: $[\![f = g]\!] = [\![g = f]\!]$.

4. Transitivity: $[\![f = g]\!] \wedge [\![g = h]\!] \leqslant [\![f = h]\!]$.

5. $[\![f = g]\!] \wedge [\![f \in h]\!] \leqslant [\![g \in h]\!]$.

6. $[\![g = h]\!] \wedge [\![f \in g]\!] \leqslant [\![f \in h]\!]$.

7. $[\![f = g]\!] \wedge [\![\varphi(f)]\!] \leqslant [\![\varphi(g)]\!]$.

Theorem 2.2.15 ([Bel05], Corollary 1.18)**.** If $f \in V^{\mathbb{B}}$ and $\varphi(x)$ are $\mathcal{L}_{\mathbb{B}}$-formulas:

1. $[\![\exists x \in f \varphi(x)]\!] = \bigvee_{x \in dom(f)} (f(x) \wedge [\![\varphi(x)]\!])$.

2. $[\![\forall x \in f \varphi(x)]\!] = \bigwedge_{x \in dom(f)} (f(x) \to [\![\varphi(x)]\!])$.

Definition 2.2.16. Given any subclass $M \subseteq V^{\mathbb{B}}$, we can endow M with a Boolean model structure by

1. $[\![\cdot \in \cdot]\!]^M_{\mathbb{B}} = [\![\cdot \in \cdot]\!]_{\mathbb{B}} \restriction_{M \times M}$

2. $[\![\cdot = \cdot]\!]^M_{\mathbb{B}} = [\![\cdot = \cdot]\!]_{\mathbb{B}} \restriction_{M \times M}$

And we call all Boolean models of the form $(M, [\![\cdot \in \cdot]\!]_{\mathbb{B}} \restriction_{M \times M}, [\![\cdot = \cdot]\!]_{\mathbb{B}} \restriction_{M \times M})$ *submodels* of $V^{\mathbb{B}}$.

Remark 2.2.17. Recall that V denotes the von Neumann universe (see Definition 1.1.2 and Proposition 1.1.4).

Definition 2.2.18. Take $\mathbb{B} = 2$. We define a class function $\hat{\cdot} : V \to V^2$ as follows: Given $x \in V$, take

$$\hat{x} = \{(\hat{y}, 1) : y \in x\}.$$

This is a definition by recursion on the well-founded relation $y \in x$. Notice that, for all $x \in V$,

$$\hat{x} \in V^2 \subseteq V^{\mathbb{B}}.$$

Theorem 2.2.19 ([Bel05], Theorem 1.23)**.** Let $x, y, a_1, ..., a_n \in V$ and $\varphi(x_1, ..., x_n)$ be a $\mathcal{L}_{\in}$-formula. Then

1. $[\![\hat{x} = \hat{y}]\!]_{\mathbb{B}} = [\![\hat{x} = \hat{y}]\!]_2 \in 2$.

2. $[\![\hat{x} \in \hat{y}]\!]_{\mathbb{B}} = [\![\hat{x} \in \hat{y}]\!]_2 \in 2$.

3. $[\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{\mathbb{B}}, [\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_2 \in 2$

4. $\hat{\cdot}$ is injective.

5. $\hat{\cdot}$ is surjective in the following way: For every $u \in V^2$ there exists a **unique** $a \in V$ such that $V^{\mathbb{B}} \models u = \hat{a}$.

6. $\hat{\cdot}$ is an "isomorphism" in the following way

$$\varphi(a_1, ..., a_n) \text{ holds in } V\text{, if and only if, } [\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{V^2} = 1$$

7. If $\varphi(x_1, ..., x_n)$ is an $\mathcal{L}_{\in}$-formula with bounded quantifiers (i.e. if each of its quantifiers occurs in the form $\forall x \in a$ or $\exists x \in b$)

$$\varphi(a_1, ..., a_n) \text{ holds in } V\text{, if and only if, } [\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{V^{\mathbb{B}}} = 1$$

2.2.2 Heyting models of Set Theory

Heyting models of Set Theory were introduced by Grayson in [Gra79], where he introduces an *Intuitionistic Zermelo-Fraenkel Set Theory* (IZF) and proves the validity of this theory in his Heyting-valued models.

Definition 2.2.20. Given a complete Heyting lattice $\mathbb{H}$, we can construct $V^{\mathbb{H}}$ and $[\![\cdot]\!]_{\mathbb{H}}$ analogously as in the complete Boolean algebras case (see Definition 2.2.1).

Definition 2.2.21 (Axioms of IZF, [Bel05], pp. 158 and 163)**.** The axioms of IZF are just the axioms of ZF, with the caveat that we write the Axiom of Regularity in the following way

$$\forall x((\forall y \in x\varphi(y)) \to \varphi(x)) \to \forall x\varphi(x) \text{ ($\in$-induction scheme)}$$

This is done since the Axiom of Regularity in its usual form (existence of $\in$-minimal elements) implies the law of excluded middle (LEM).

Theorem 2.2.22 ([Bel05], Chapter 8, pp. 165 and 166)**.** $V^{\mathbb{H}} \models \varphi$ for every axiom of first-order Intuitionistic Logic, for every rule of inference, and for every axiom of IZF. Furthermore, $V^{\mathbb{H}}$ validates Zorn's lemma, even though it does not generally validate the Axiom of Choice.

Many interesting independence results from IZF can be proved by using these Heyting-valued models. For instance, Fourman and Hyland [FH79] construct models such that

1. Every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous.
2. The sets of Cauchy and Dedekind real numbers do not coincide.
3. The field of complex numbers is not algebraically closed.

2.2.3 Quantale-valued models of Set Theory

In [Lan92a], Lano studied Topological Residuated Algebra valued models for **Modal** Residuated Logic. In this section, we aim to study these models without the modality (necessity) that Lano used. In this subsection, we focus on the proofs for some of the properties of quantale-valued models of Set Theory for the following reason: Even though it is well known that these models are quantale-valued models in the sense of Definition 2.1.8, there is not, as far as we are aware, a prove in the literature that shows this fact, so we prove this in Theorem 2.2.31.

Definition 2.2.23. We define $V_\alpha^{\mathbb{Q}}$ by recursion on ordinals:

1. $V_0^{\mathbb{Q}} := \varnothing$
2. $V_{\alpha+1}^{\mathbb{Q}} := \{f : f \text{ is a function with } dom(f) \subseteq V_\alpha^{\mathbb{Q}} \text{ and } ran(f) \subseteq \mathbb{Q}\}$
3. $V_\alpha^{\mathbb{Q}} := \bigcup_{\beta<\alpha} V_\beta^{\mathbb{Q}}$ with $\alpha \neq 0$ limit ordinal.
4. $V^{\mathbb{Q}} := \bigcup_{\alpha \in ON} V_\alpha^{\mathbb{Q}}$.

Definition 2.2.24. We define the language $\mathcal{L}_{\mathbb{Q}}$ as $\mathcal{L}_{\mathbb{Q}} := \mathcal{L}_{V^{\mathbb{Q}}} = \mathcal{L}_\in \cup \{c_a : a \in V^{\mathbb{Q}}\}$, where each c_a is a constant symbol.

Definition 2.2.25. Given $x \in V^{\mathbb{Q}}$, we define $rank_{V^{\mathbb{Q}}}(x)$ as the smallest ordinal α such that $x \in V_{\alpha+1}^{\mathbb{Q}}$.

Theorem 2.2.26 (Principle of induction for $V^{\mathbb{Q}}$, cf. [Bel05], p. 21, 1.7)**.** For every Classical first-order $\mathcal{L}_\in$-formula $\varphi(x)$, we have

$$(\forall x \in V^{\mathbb{Q}}((\forall y \in dom(x)\varphi(y)) \to \varphi(x))) \to (\forall x \in V^{\mathbb{Q}}\varphi(x))$$

Definition 2.2.27. We define interpretations of $\in$ and $=$ in $V^{\mathbb{Q}}$ as

1. $[\![f \subseteq g]\!]_{\mathbb{Q}} := \bigwedge_{x \in dom(f)} (f(x) \to [\![x \in g]\!]_{\mathbb{Q}})$
2. $[\![f = g]\!]_{\mathbb{Q}} := [\![f \subseteq g]\!]_{\mathbb{Q}} \cdot [\![g \subseteq f]\!]_{\mathbb{Q}}$

3. $[\![f \in g]\!]_{\mathbb{Q}} := \bigvee_{x \in dom(g)} (g(x) \cdot [\![x = f]\!]_{\mathbb{Q}})$

Remark 2.2.28. We have that $(V^{\mathbb{Q}}, [\![\cdot \in \cdot]\!]_{\mathbb{Q}}, [\![\cdot = \cdot]\!]_{\mathbb{Q}})$ is a $\mathbb{Q}$-valued $\mathcal{L}_{\in}$-structure and thus we can define a valuation $[\![\cdot]\!]_{\mathbb{Q}} : Sent^R(\mathcal{L}_{\mathbb{Q}}) \to \mathbb{Q}$ as in the Definition 2.1.11.

Definition 2.2.29. If φ is an $R - \mathcal{L}_{\mathbb{Q}}$-sentence, we say that $V^{\mathbb{Q}} \models \varphi$ if $[\![\varphi]\!]_{\mathbb{Q}} = 1$.

Remark 2.2.30. As long as it is clear from the context, we denote $[\![\cdot]\!]_{\mathbb{Q}}$ as $[\![\cdot]\!]$

Theorem 2.2.31 (cf. [Bel05], Theorem 1.17)**.** Let $f, g, h, h, f_1, f_2, g_1, g_2 \in V^{\mathbb{Q}}$ and $\varphi(x)$ be an $R - \mathcal{L}_{\mathbb{Q}}$-formula that does not contain the symbols & and $\to$. Then:

1. Reflexivity: $[\![f = f]\!] = 1$.

2. If $x \in dom(f)$, then $f(x) \leqslant [\![x \in f]\!]$.

3. Symmetry: $[\![f = g]\!] = [\![g = f]\!]$.

4. Transitivity: $[\![f = g]\!] \cdot [\![g = h]\!] \leqslant [\![f = h]\!]$.

5. $[\![f = g]\!] \cdot [\![f \in h]\!] \leqslant [\![g \in h]\!]$.

6. $[\![g = h]\!] \cdot [\![f \in g]\!] \leqslant [\![f \in h]\!]$.

7. $[\![f_1 = g_1]\!] \cdot [\![f_2 = g_2]\!] \cdot [\![f_1 \in f_2]\!] \leqslant [\![g_1 \in g_2]\!]$.

8. $[\![f = g]\!] \cdot [\![\varphi(f)]\!] \leqslant [\![\varphi(g)]\!]$.

9. $[\![\exists x \in f \varphi(x)]\!] = \bigvee_{x \in dom(f)} (f(x) \cdot [\![\varphi(x)]\!])$.

10. $[\![\forall x \in f \varphi(x)]\!] = \bigwedge_{x \in dom(f)} (f(x) \to [\![\varphi(x)]\!])$.

Proof. 1. We prove it by induction on $V^{\mathbb{Q}}$ (see Theorem 2.2.26).

Induction hypothesis: Suppose that $\llbracket x = x \rrbracket = 1$ for all $x \in dom(f)$.

Now, if $x \in dom(f)$, then

$$f(x) = f(x) \cdot 1 = f(x) \cdot \llbracket x = x \rrbracket \leqslant \bigvee_{y \in dom(f)} f(y) \cdot \llbracket x = y \rrbracket = \llbracket x \in f \rrbracket.$$

i.e., $f(x) \leqslant \llbracket x \in f \rrbracket$ and then, by Theorem 1.2.33 item 1., we have that $(f(x) \rightarrow \llbracket x \in f \rrbracket) = 1$. Then,

$$\llbracket f = f \rrbracket = \bigwedge_{x \in dom(f)} (f(x) \rightarrow \llbracket x \in f \rrbracket) \cdot \bigwedge_{y \in dom(f)} (f(y) \rightarrow \rrbracket y \in f \rrbracket) = \bigwedge_{x \in dom(f)} 1 \cdot \bigwedge_{y \in dom(f)} 1 = 1 \cdot 1 = 1.$$

2. It is obtained by using Theorem 2.2.31 item 1. and an argument similar to the previous item.

3. It is immediately obtained by the commutativity of the product $\cdot$:

$$\begin{aligned} \llbracket f = g \rrbracket &= \bigwedge_{x \in dom(f)} (f(x) \rightarrow \llbracket x \in g \rrbracket) \cdot \bigwedge_{y \in dom(g)} (g(y) \rightarrow \llbracket y \in f \rrbracket) \\ &= \bigwedge_{y \in dom(g)} (g(y) \rightarrow \llbracket y \in f \rrbracket) \cdot \bigwedge_{x \in dom(f)} (f(x) \rightarrow \llbracket x \in g \rrbracket) = \llbracket g = f \rrbracket \end{aligned}$$

4. We prove it by induction on $V^{\mathbb{Q}}$.

Induction hypothesis: Suppose that $\llbracket x = g \rrbracket \cdot \llbracket g = h \rrbracket \leqslant \llbracket x = h \rrbracket$ for all $x \in dom(f)$ and $g, h \in V^{\mathbb{Q}}$.

Then, in particular, for all $y \in dom(g)$ and $z \in dom(h)$ we have to $\llbracket x = y \rrbracket \cdot \llbracket y = z \rrbracket \leqslant \llbracket x = z \rrbracket$. Then, $\llbracket x = y \rrbracket \cdot \llbracket y = z \rrbracket \cdot h(z) \leqslant \llbracket x = z \rrbracket \cdot h(z)$. Taking supremums over

$z \in dom(h)$ we get that

$$\bigvee_{z \in dom(h)} [\![x = y]\!] \cdot [\![y = z]\!] \cdot h(z) \leqslant \bigvee_{z \in dom(h)} [\![x = z]\!] \cdot h(z) = [\![x \in h]\!].$$

But we have

$$\begin{aligned}\bigvee_{z \in dom(h)} [\![x = y]\!] \cdot [\![y = z]\!] \cdot h(z) = [\![x = y]\!] \cdot \bigvee_{z \in dom(h)} [\![y = z]\!] \cdot h(z) \\ = [\![x = y]\!] \cdot [\![y \in h]\!].\end{aligned}$$

And putting all the previous results together, we get that

$$[\![x = y]\!] \cdot [\![y \in h]\!] \leqslant [\![x \in h]\!]$$

On the other hand, we have to

$$\begin{aligned}g(y) \cdot [\![g \subseteq h]\!] &= g(y) \cdot \bigwedge_{y' \in dom(g)} (g(y') \rightarrow [\![y' \in h]\!]) \\ &\leqslant g(y) \cdot (g(y) \rightarrow [\![y \in h]\!]) \\ &\leqslant [\![y \in h]\!] \qquad \text{(by Theorem 1.2.33 item 2.)}\end{aligned}$$

i.e., $g(y) \cdot [\![g \subseteq h]\!] \leqslant [\![y \in h]\!]$.

And then,

$$\begin{aligned}[\![g \subseteq h]\!] \cdot [\![x = y]\!] \cdot g(y) &\leqslant [\![x = y]\!] \cdot [\![y \in h]\!] \\ &\leqslant [\![x \in h]\!].\end{aligned}$$

i.e., $[\![g \subseteq h]\!] \cdot [\![x = y]\!] \cdot g(y) \leqslant [\![x \in h]\!]$.

Taking supremum over $y \in dom(g)$,

$$\begin{aligned}
[\![g \subseteq h]\!] \cdot [\![x \in g]\!] &= [\![g \subseteq h]\!] \cdot \bigvee_{y \in dom(g)} ([\![x = y]\!] \cdot g(y)) \\
&= \bigvee_{y \in dom(g)} [\![g \subseteq h]\!] \cdot [\![x = y]\!] \cdot g(y) \\
&\leqslant [\![x \in h]\!].
\end{aligned}$$

i.e., $[\![g \subseteq h]\!] \cdot [\![x \in g]\!] \leqslant [\![x \in h]\!]$

Given that $[\![f \subseteq g]\!] \cdot f(x) \leqslant [\![x \in g]\!]$, using the result given above, we get

$$[\![f \subseteq g]\!] \cdot [\![g \subseteq h]\!] \cdot f(x) \leqslant [\![x \in h]\!].$$

And since $\cdot$ is left adjoint to $\rightarrow$, we have for all $x \in dom(f)$

$$[\![f \subseteq g]\!] \cdot [\![g \subseteq h]\!] \leqslant f(x) \rightarrow [\![x \in h]\!]$$

Hence, by definition of infimum

$$[\![f \subseteq g]\!] \cdot [\![g \subseteq h]\!] \leqslant \bigwedge_{x \in dom(f)} (f(x) \rightarrow [\![x \in h]\!]) = [\![f \subseteq h]\!] \ (*)$$

On the other hand, using the symmetry of $[\![\cdot = \cdot]\!]$, the induction hypothesis implies that for all $g, h \in V^{\mathbb{Q}}$ and all $x \in dom(f)$

$$[\![h = g]\!] \cdot [\![g = x]\!] \leqslant [\![h = x]\!]$$

Thus, using an argument similar to get $(*)$, we arrive at

$$[\![h \subseteq g]\!] \cdot [\![g \subseteq f]\!] \leqslant \bigwedge_{z \in dom(h)} (h(z) \rightarrow [\![z = f]\!]) = [\![h \subseteq f]\!] \ (**)$$

Thus, using $(*)$, $(**)$ and since $\cdot$ is monotone (Theorem 1.2.33 item 5.), we may say that

$[\![f = g]\!] \cdot [\![g = h]\!] \leqslant [\![f = h]\!]$.

5. If $z \in dom(h)$, then by the transitivity of $[\![\cdot = \cdot]\!]$ and the monotonicity of $\cdot$, we have

$$[\![f = g]\!] \cdot [\![f = z]\!] \cdot h(z) \leqslant [\![g = z]\!] \cdot h(z)$$

Taking supremums on both sides over $z \in dom(h)$, we have

$$\begin{aligned}
[\![f = g]\!] \cdot [\![f \in h]\!] &= [\![f = g]\!] \cdot \bigvee_{z \in dom(h)} ([\![f = z]\!] \cdot h(z)) && (\text{ definition of } [\![f \in h]\!]) \\
&= \bigvee_{z \in dom(h)} ([\![f = g]\!] \cdot [\![f = z]\!] \cdot h(z)) && (\cdot \text{ distributes over } \textstyle\bigvee) \\
&\leqslant \bigvee_{z \in dom(h)} [\![g = z]\!] \cdot h(z) = [\![g \in h]\!]
\end{aligned}$$

6. If $y \in dom(g)$, then

$$\begin{aligned}
[\![g = h]\!] \cdot g(y) &= [\![g \subseteq h]\!] \cdot [\![h \subseteq g]\!] \cdot g(y) && (\text{definition of } [\![g = h]\!]) \\
&\leqslant [\![g \subseteq h]\!] \cdot g(y) \\
&= \bigwedge_{y' \in dom(g)} (g(y')) \to [\![y' \in h]\!]) \cdot g(y) \\
&\leqslant (g(y)) \to [\![y \in h]\!]) \cdot g(y) \\
&\leqslant [\![y \in h]\!] && (\text{by Theorem 1.2.33 item 2.})
\end{aligned}$$

Therefore, by using the monotonicity of $\cdot$ and the item 5. on this theorem, we get

$$\begin{aligned}\llbracket g = h \rrbracket \cdot \llbracket f = y \rrbracket \cdot g(y) &\leqslant \llbracket y \in h \rrbracket \cdot \llbracket f = y \rrbracket \\ &\leqslant \llbracket f \in h \rrbracket\end{aligned}$$

Taking the supremum over $y \in dom(g)$, we get

$$\begin{aligned}\llbracket g = h \rrbracket \cdot \llbracket f \in g \rrbracket = \llbracket g = h \rrbracket \cdot \bigvee_{y \in dom(g)} \llbracket f = y \rrbracket \cdot g(y) \qquad &(\cdot \text{ distributes over } \textstyle\bigvee) \\ = \bigvee_{y \in dom(g)} \llbracket g = h \rrbracket \cdot \llbracket f = y \rrbracket \cdot g(y) & \\ \leqslant \llbracket f \in h \rrbracket &\end{aligned}$$

7. It follows from the previous items:

$$\begin{aligned}\llbracket f_1 = g_1 \rrbracket \cdot \llbracket f_2 = g_2 \rrbracket \cdot \llbracket f_1 \in f_2 \rrbracket &\leqslant \llbracket f_1 = g_1 \rrbracket \cdot \llbracket f_1 \in g_2 \rrbracket \quad &\text{(by Theorem 2.2.31 item 6.)} \\ &\leqslant \llbracket g_1 \in g_2 \rrbracket &\text{(by Theorem 2.2.31 item 5.)}\end{aligned}$$

8. We prove it by induction on the complexity of $R - \mathcal{L}_{\mathbb{Q}}$-formulas

   a. For the atomic case, it follows from items 4., 5. and 6 of this theorem.

   b. Induction hypothesis 1: Let $\varphi(x)$ and $\psi(x)$ be $R - \mathcal{L}_{\mathbb{Q}}$-formulas such that

$$\llbracket f = g \rrbracket \cdot \llbracket \varphi(f) \rrbracket \leqslant \llbracket \varphi(g) \rrbracket$$

$$\llbracket f = g \rrbracket \cdot \llbracket \psi(f) \rrbracket \leqslant \llbracket \psi(g) \rrbracket$$

i. Disjunction:

$$\begin{aligned}
[\![f = g]\!] \cdot [\![(\varphi \vee \psi)(f)]\!] &= [\![f = g]\!] \cdot ([\![\varphi(f)]\!] \vee [\![\psi(f)]\!]) \\
&= ([\![f = g]\!] \cdot [\![\varphi(f)]\!]) \vee ([\![f = g]\!] \cdot [\![\psi(f)]\!]) \quad (\cdot \text{ distributes over } \bigvee) \\
&\leqslant [\![\varphi(g)]\!] \vee [\![\psi(g)]\!] \qquad \text{(by the induction hypothesis 1)} \\
&= [\![(\varphi \vee \psi)(g)]\!]
\end{aligned}$$

ii. Conjunction:

$$\begin{aligned}
&[\![f = g]\!] \cdot [\![(\varphi \wedge \psi)(f)]\!] = [\![f = g]\!] \cdot ([\![\varphi(f)]\!] \wedge [\![\psi(f)]\!]) \\
&\leqslant ([\![f = g]\!] \cdot [\![\varphi(f)]\!]) \wedge ([\![f = g]\!] \cdot [\![\psi(f)]\!]) \qquad \text{(by Theorem 1.2.36 item 1.)} \\
&\leqslant [\![\varphi(g)]\!] \wedge [\![\psi(g)]\!] \qquad \text{(by the induction hypothesis 1)} \\
&= [\![(\varphi \wedge \psi)(g)]\!]
\end{aligned}$$

c. Induction hypothesis 2: Suppose that for all $h \in V^{\mathbb{Q}}$

$$[\![f = g]\!] \cdot [\![\varphi(h, f)]\!] \leqslant [\![\varphi(h, g)]\!]$$

i. Universal quantifier:

$$
\begin{aligned}
[\![f = g]\!] \cdot [\![\forall x \varphi(x, f)]\!] &= [\![f = g]\!] \cdot \bigvee_{h \in V^{\mathbb{Q}}} [\![\varphi(h, f)]\!] && \text{(by definition of } [\![\forall x \varphi(x, f)]\!] \text{)} \\
&= \bigvee_{h \in V^{\mathbb{Q}}} [\![f = g]\!] \cdot [\![\varphi(h, f)]\!] && (\cdot \text{ distributes over } \bigvee) \\
&\leqslant \bigvee_{h \in V^{\mathbb{Q}}} [\![\varphi(h, g)]\!] && \text{(by induction hypothesis 2)} \\
&= [\![\forall x \varphi(x, g)]\!]
\end{aligned}
$$

ii. Existential quantifier:

$$
\begin{aligned}
[\![f = g]\!] \cdot [\![\exists x \varphi(x, f)]\!] &= [\![f = g]\!] \cdot \bigwedge_{h \in V^{\mathbb{Q}}} [\![\varphi(h, f)]\!] && \text{(by definition of } [\![\exists x \varphi(x, f)]\!]) \\
&\leqslant \bigwedge_{h \in V^{\mathbb{Q}}} [\![f = g]\!] \cdot [\![\varphi(h, f)]\!] && \text{(by Theorem 1.2.36 item 1.)} \\
&\leqslant \bigwedge_{h \in V^{\mathbb{Q}}} [\![\varphi(h, g)]\!] && \text{(by induction hypothesis)} \\
&= [\![\exists x \varphi(x, g)]\!]
\end{aligned}
$$

9. Let us first see that for all $x \in V^{\mathbb{Q}}$

$$\bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \cdot [\![\varphi(y)]\!] = [\![\varphi(x)]\!].$$

By item 8. of this theorem, we have that for all $y \in V^{\mathbb{Q}}$,

$$[\![x = y]\!] \cdot [\![\varphi(y)]\!] \leqslant [\![\varphi(x)]\!].$$

Therefore,

$$\bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \cdot [\![\varphi(y)]\!] \leqslant [\![\varphi(x)]\!]$$

But on the other hand,

$$\begin{aligned} [\![\varphi(x)]\!] &= [\![\varphi(x)]\!] \cdot 1 = [\![\varphi(x)]\!] \cdot [\![x = x]\!] \\ &\leqslant \bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \cdot [\![\varphi(y)]\!] \qquad \text{since } x \in V^{\mathbb{Q}} \end{aligned}$$

and thus $[\![\varphi(x)]\!] = \bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \cdot [\![\varphi(y)]\!]$. Using the equality given above, we get

$$\begin{aligned} [\![\exists x \in f \varphi(x)]\!] = [\![\exists x (x \in f \& \varphi(x))]\!] &= \bigvee_{y \in V^{\mathbb{Q}}} [\![y \in f \& \varphi(y)]\!] \quad \text{(by definition of } \exists) \\ &= \bigvee_{y \in V^{\mathbb{Q}}} [\![y \in f]\!] \cdot [\![\varphi(y)]\!] \quad \text{(by definition of \&)} \\ &= \bigvee_{y \in V^{\mathbb{Q}}} \bigvee_{x \in dom(f)} f(x) \cdot [\![x = y]\!] \cdot [\![\varphi(y)]\!] \quad \text{(by definition of } [\![y \in f]\!]) \\ &= \bigvee_{x \in dom(f)} \bigvee_{y \in V^{\mathbb{Q}}} f(x) \cdot [\![x = y]\!] \cdot [\![\varphi(y)]\!] \quad \text{(by exchanging the supremums)} \\ &= \bigvee_{x \in dom(f)} f(x) \cdot \bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \cdot [\![\varphi(y)]\!] \quad (\cdot \text{ distributes over } \textstyle\bigvee) \\ &= \bigvee_{x \in dom(f)} f(x) \cdot [\![\varphi(x)]\!] \end{aligned}$$

10. In a similar fashion as the previous item, we can prove that

$$[\![\varphi(x)]\!] = \bigvee_{y \in V^{\mathbb{Q}}} [\![x = y]\!] \rightarrow [\![\varphi(y)]\!]$$

$$
\begin{aligned}
[\![\forall x \in f \varphi(x)]\!] = [\![\forall x(x \in f \to \varphi(x))]\!] &= \bigwedge_{y \in V^{\mathbb{Q}}} [\![y \in f \to \varphi(y)]\!] = \bigwedge_{y \in V^{\mathbb{Q}}} [\![y \in f]\!] \to [\![\varphi(y)]\!] \\
&= \bigwedge_{y \in V^{\mathbb{Q}}} \left(\bigvee_{x \in dom(f)} (f(x)) \cdot [\![x = y]\!]) \right) \to [\![\varphi(y)]\!] \quad \text{(by definition of } [\![y \in f]\!]) \\
&= \bigwedge_{x \in dom(f)} \bigwedge_{y \in V^{\mathbb{Q}}} ((f(x)) \cdot [\![x = y]\!]) \to [\![\varphi(y)]\!]) \quad \text{(by Theorem 1.2.36 item 3.)} \\
&= \bigwedge_{x \in dom(f)} \bigwedge_{y \in V^{\mathbb{Q}}} (f(x) \to ([\![x = y]\!] \to [\![\varphi(y)]\!])) \quad \text{(by Theorem 1.2.33 item 10.)} \\
&= \bigwedge_{x \in dom(f)} f(x) \to \bigwedge_{y \in V^{\mathbb{Q}}} ([\![x = y]\!] \to [\![\varphi(y)]\!]) \quad \text{(by Theorem 1.2.36 item 2.)} \\
&= \bigwedge_{x \in dom(f)} f(x) \to [\![\varphi(x)]\!]
\end{aligned}
$$

□

Corollary 2.2.32. $V^{\mathbb{Q}}$ is a $\mathbb{Q}$-valued model and therefore $V^{\mathbb{Q}}$ is a model of all of the axioms of Residuated Logic.

2.3 Constructible sets on valued models

There has not been a significant attempt at trying to generalize Gödel's universe within the framework of valued models of Set Theory. The closest we found was Fitting's work [Fit69] in which he re-states the definition of L by using standard (i.e. two valued) characteristic functions.

More precisely, Fitting takes a set M and v a truth function on the set of first-order formulas with constants from M (here Fitting considers only two possible truth values) and then says that a (characteristic) function f is *definable* over (M, v) if

1. $dom(f) = M$

2. $ran(f) \subseteq \{T, F\}$

3. There exists some formula $X(x)$ with one free variable and all constants from M such that for all $a \in M$

$$f(a) = v(X(a))$$

And then, with this notion of definable set, Fitting re-states the definition of Gödel's universe in these terms. We adapt Fitting's notion of definable sets in the context of Boolean valued models and quantale valued models to propose some new versions of L. Although reasonable at first, this attempt was not successful since the proposals for new models that we considered ended up collapsing to **two-valued** universes, and therefore they are essentially classical models.

2.3.1 Constructible sets in Boolean models.

We start by motivating our approach to constructible sets in Boolean models by stating Gödel's construction in a very precise way, so that it is clear how we translate this definition to the Boolean valued case.

1. $L_0 := \varnothing$,

2. $L_{\alpha+1} := \{X \subseteq L_\alpha :$ there is a classical first-order $\mathcal{L}_\in$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in L_\alpha^{|\bar{y}|}$ such that $X = \{a \in L_\alpha : (L_\alpha, \in \restriction_{L_\alpha}) \models \varphi(a, \bar{b})\}$,

3. $L_\alpha := \bigcup_{\beta<\alpha} L_\beta$ for $\alpha \neq 0$ limit ordinal.

Let $L := \bigcup_{\alpha \in ON} L_\alpha$.

Taking this into account, we start from a Boolean set-theoretic model

$$(V^{\mathbb{B}}, [\![\cdot \in \cdot]\!]_{\mathbb{B}}, [\![\cdot = \cdot]\!]_{\mathbb{B}})$$

and we take subsets (in the sense of V) $L^{\mathbb{B}}_{\alpha} \subseteq V^{\mathbb{B}}$ in order to take the restrictions of the interpretations of $\in$ and $=$ on the universe $V^{\mathbb{B}}$ and then using them for the set $L^{\mathbb{B}}_{\alpha}$ and considering the "definable sets" in the Boolean submodel

$$(L^{\mathbb{B}}_{\alpha}, [\![\cdot \in \cdot]\!]_{L^{\mathbb{B}}_{\alpha}}, [\![\cdot = \cdot]\!]_{L^{\mathbb{B}}_{\alpha}}) := (L^{\mathbb{B}}_{\alpha}, [\![\cdot \in \cdot]\!]_{\mathbb{B}}\restriction_{L^{\mathbb{B}}_{\alpha}}, [\![\cdot = \cdot]\!]_{\mathbb{B}}\restriction_{L^{\mathbb{B}}_{\alpha}}).$$

We propose two definitions of definable sets in this context: $\mathbb{B}^*$-definable and $\mathbb{B}$-definable, both inspired in [Fit70], and with those definitions we construct the models $\mathfrak{L}^{\mathbb{B}}$ and $\mathbb{L}^{\mathbb{B}}$, respectively.

Throughout this subsection, $\mathbb{B}$ is going to denote a complete Boolean algebra.

Definition 2.3.1. Let $M \subseteq V^{\mathbb{B}}$ be a subclass. Recall that we can view M as a $\mathbb{B}$-valued $\mathcal{L}_{\in}$-structure by taking the restrictions on M from the interpretations of $\in$ and $=$ on $V^{\mathbb{B}}$. We say that f is a $\mathbb{B}^*-$*definable subset* of M if:

1. $f \in V^{\mathbb{B}}$.

2. $dom(f) \subseteq M$.

3. There is a first-order $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M^{|\bar{y}|}$ such that for every $a \in dom(f)$

$$f(a) = [\![\varphi(a, \bar{b})]\!]_M.$$

And we define the class of $\mathbb{B}^*$-definable subsets of M as

$$Def^{\mathbb{B}^*}(M) := \{f \in V^{\mathbb{B}} : f \text{ is a } \mathbb{B}^*\text{-definable subset of } M\}.$$

Definition 2.3.2. We define $\mathfrak{L}^{\mathbb{B}}_{\alpha}$, by transfinite recursion over the ordinals, as follows

1. $\mathfrak{L}^{\mathbb{B}}_{0} := \varnothing$.

2. $\mathfrak{L}^{\mathbb{B}}_{\alpha+1} := Def^{\mathbb{B}*}(\mathfrak{L}^{\mathbb{B}}_{\alpha}) = \{f \in V^{\mathbb{B}} : dom(f) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha}$ and there exist a $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in (\mathfrak{L}^{\mathbb{B}}_{\alpha})^{|\bar{y}|}$ such that for every $a \in dom(f),\ f(a) = [\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}}\}$.

3. $\mathfrak{L}^{\mathbb{B}}_{\alpha} := \bigcup_{\beta<\alpha} \mathfrak{L}^{\mathbb{B}}_{\beta}$ for $\alpha \neq 0$ limit ordinal.

4. $\mathfrak{L}^{\mathbb{B}} := \bigcup_{\alpha\in ON} \mathfrak{L}^{\mathbb{B}}_{\alpha}$.

Proposition 2.3.3. For all $\alpha \in ON$, $\mathfrak{L}^{\mathbb{B}}_{\alpha} \subseteq V^{\mathbb{B}}_{\alpha}$.

Proof. A straightforward induction on α proves this claim. □

Theorem 2.3.4. For all $\alpha \in ON$, $\mathfrak{L}^{\mathbb{B}}_{\alpha} \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$.

Proof. We prove this by induction on α. The case $\alpha = 0$ and $\alpha \neq 0$ limit ordinal are simple, so we only focus on the successor step. Let us assume that

$$\mathfrak{L}^{\mathbb{B}}_{\alpha} \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha+1} \text{ (induction hypothesis)}$$

We have to prove that $\mathfrak{L}^{\mathbb{B}}_{\alpha+1} \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha+2}$. Hence, take $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, that is,

1. $f \in V^{\mathbb{B}}$,

2. $dom(f) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha}$,

3. there exist a classical first-order $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in (\mathfrak{L}^{\mathbb{B}}_{\alpha})^{|\bar{y}|}$ such that for all $a \in dom(f)$

$$f(a) = [\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}}.$$

By induction hypothesis, we have that $dom(f) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha} \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, so all we need to do is to find a $\mathcal{L}_{\in}$-formula **with parameters from** $\mathfrak{L}^{\mathbb{B}}_{\alpha+1}$ defines f . Consider the $\mathcal{L}_{\in}$-formula

$$\psi(x, y) := x \in y,$$

since $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, we want to prove that $\psi(x, f)$ defines f, that is, we want to prove that for every $a \in dom(f)$,

$$[\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} = [\![a \in f]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha+1}}.$$

Notice that, for every $a \in dom(f)$,

$$\begin{aligned} [\![a \in f]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} &= \bigvee_{c \in dom(f)} f(c) \wedge [\![a = c]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha+1}} && \text{(by definition of } [\![a \in f]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}}) \\ &= \bigvee_{c \in dom(f)} [\![\varphi(c, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}} \wedge [\![a = c]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} && \text{(by induction hypothesis).} \end{aligned}$$

Hence,

$$\begin{aligned} [\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} \cdot 1 &= [\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} \cdot [\![a = a]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha+1}} \\ &\leqslant \bigvee_{c \in dom(f)} [\![\varphi(c, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}} \wedge [\![a = c]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} && \text{(since } a \in dom(f)) \end{aligned}$$

On the other hand, by Theorem 2.2.14 item 7., for every $c \in dom(f)$,

$$[\![\varphi(c, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}} \wedge [\![a = c]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} \leqslant [\![\varphi(a, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}}$$

Then, by definition of join,

$$\bigvee_{c \in dom(f)} [\![\varphi(c, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}} \wedge [\![a = c]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} \leqslant [\![\varphi(a, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha}}$$

Therefore,

$$[\![a \in f]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} = \bigvee_{c \in dom(f)} [\![\varphi(c, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_\alpha} \wedge [\![a = c]\!]^{\mathbb{B}}_{\mathfrak{L}_{\alpha+1}} = [\![\varphi(a, \bar{b})]\!]^{\mathbb{B}}_{\mathfrak{L}_\alpha} = f(a)$$

and thus, $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+2}$. □

Corollary 2.3.5. For all $\alpha, \beta \in ON$, if $\alpha < \beta$, then $\mathfrak{L}^{\mathbb{B}}_{\alpha} \subseteq \mathfrak{L}^{\mathbb{B}}_{\beta}$.

Theorem 2.3.6. If $f \in \mathfrak{L}^{\mathbb{B}}$, then $ran(f) \subseteq \{0, 1\} = 2$.

Proof. We prove it by induction on ordinals, showing that for all $\alpha \in ON$, if $f \in \mathfrak{L}_\alpha$, then $ran(f) \subseteq 2$.

Induction hypothesis 1: Take $\alpha \in ON$ such that for all $a \in \mathfrak{L}^{\mathbb{B}}_{\alpha}$, $ran(a) \subseteq 2$.

Let us see that if $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, then $ran(f) \subseteq \{0, 1\} = 2$. Since $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, there is a $\mathcal{L}_\in$-formula $\varphi(x, \bar{y})$ with $|\bar{y}| = n$ and $\bar{b} \in (\mathfrak{L}^{\mathbb{B}}_{\alpha})^n$ such that for all $a \in dom(f) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha}$

$$f(a) = [\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}}.$$

Notice that $a, b_1, b_2, ..., b_n \in \mathfrak{L}^{\mathbb{B}}_{\alpha}$, so we can use the induction hypothesis 1 on them, so that $ran(a), ran(b_i) \subseteq 2$ for all $1 \leqslant i \leqslant n$.

We will prove that if $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, then $ran(f) \subseteq \{0, 1\} = 2$, using induction on formulas, where the formulas can take parameters from $\mathfrak{L}^{\mathbb{B}}_{\alpha}$.

We start with the atomic case. We want to prove that $[\![a \in b]\!], [\![b \in a]\!], [\![a = b]\!] \in 2$ for all $a, b \in \mathfrak{L}^{\mathbb{B}}_{\alpha}$.

We prove the statement given above by induction on the well-founded relation $<$ on $V^{\mathbb{B}}$ (see Definition 2.2.9), where

$$(v, w) < (a, b) \text{ if and only if } (v = a \text{ and } w \in dom(b)) \text{ or } (v \in dom(a) \text{ and } w = b), \text{ where}$$
$$a, b, f, g \in V^{\mathbb{B}}.$$

Induction hypothesis 2: for all <-predecessors (v, w) of (f, g), we have that

$$[\![v \in w]\!], [\![w \in v]\!], [\![v = w]\!] \in 2.$$

Notice that the <-predecessors (v, w) of (a, b) have the form

$$(a, w) \text{ and } (v, b), \text{ where } v \in dom(a) \text{ and } w \in dom(b).$$

Then, the induction hypothesis 2 means that for all $v \in dom(a)$ and $w \in dom(b)$ we have that

$$[\![a \in w]\!], [\![w \in a]\!], [\![v \in b]\!], [\![b \in v]\!], [\![a = w]\!], [\![v = b]\!] \in 2.$$

Let us see then that

$$[\![a \in b]\!], [\![b \in a]\!], [\![a = b]\!] \in 2$$

By definition of $[\![\cdot \in \cdot]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}}$, we have that

$$[\![a \in b]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}} = [\![a \in b]\!]_V = \bigvee_{w \in dom(b)} b(w) \wedge [\![a = w]\!]$$

Given that $b \in \mathfrak{L}_\alpha^{\mathbb{B}}$, by induction hypothesis 1, we have that $b(w) \in 2$. On the other hand, the induction hypothesis 2 implies that $[\![a = w]\!] \in 2$, so $b(w) \wedge [\![a = w]\!] \in 2$ and $\bigvee_{w \in dom(b)} b(w) \wedge [\![a = w]\!] = [\![a \in b]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}} \in 2$.

In a similar way as before, it is shown that $[\![b \in a]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}} \in 2$

By definition of $[\![\cdot = \cdot]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}}$, we have that

$$[\![a = b]\!]_{\mathfrak{L}_\alpha^{\mathbb{B}}} = [\![a = b]\!]_V = \left(\bigwedge_{v \in dom(a)} a(v) \to [\![v \in b]\!]\right) \wedge \left(\bigwedge_{w \in dom(b)} b(w) \to [\![w \in a]\!]\right)$$

Since $a, b \in \mathfrak{L}_\alpha^{\mathbb{B}}$, by induction hypothesis 1, we have that $a(v), b(w) \in 2$. On the other hand, the induction hypothesis 2 implies that $[\![v \in b]\!], [\![w \in a]\!] \in 2$, thus

$$(a(v) \to [\![v \in b]\!]), (b(w) \to [\![w \in a]\!]) \in 2.$$

Then,

$$\bigwedge_{v \in dom(a)} a(v) \to [\![v \in b]\!], \bigwedge_{w \in dom(b)} b(w) \to [\![w \in a]\!] \in 2.$$

Therefore,

$$\left(\bigwedge_{v \in dom(a)} a(v) \to [\![v \in b]\!] \wedge \bigwedge_{w \in dom(b)} b(w) \to [\![w \in a]\!] \right) = [\![a = b]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} \in 2.$$

Thus, by induction on the well-founded relation $<$, we have that $[\![a \in b]\!], [\![b \in a]\!], [\![a = b]\!] \in 2$ for all $a, b \in \mathfrak{L}^{\mathbb{B}}_{\alpha}$. this proves the case for atomic $\mathcal{L}_{\in}$-formulas.

By induction on $\mathcal{L}_{\in}$-formulas, it is straightforward to show that for every $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ with $|y| = n$, $\bar{b} \in (\mathfrak{L}^{\mathbb{B}}_{\alpha})^n$y $a \in dom(f) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha}$ we have that

$$[\![\varphi(a, \bar{b})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} = f(a) \in 2,$$

since if Boolean combinations of formulas that only take values in $\{0, 1\}$ are made, the result of evaluating these formulas is 0 or 1.

Thus, we have that, for all $f \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$, $ran(f) \subseteq 2$.

Checking the limit ordinal case is straightforward. Then, by induction on the ordinals, we have that for all $f \in \mathfrak{L}^{\mathbb{B}}$, $ran(f) \subseteq 2$. □

Remark 2.3.7. The previous theorem tells us that $\mathfrak{L}^{\mathbb{B}} \subseteq V^2$.

Theorem 2.3.8. For all $x \in V$, $\hat{x} \in \mathfrak{L}^{\mathbb{B}}$.

Proof. We prove it by induction on the well-founded relation $\in$.

Induction hypothesis: Suppose that for all $y \in x$, $\hat{y} \in \mathfrak{L}^{\mathbb{B}}$. Take $\alpha \in ON$ such that for all $y \in x$, $\hat{y} \in \mathfrak{L}^{\mathbb{B}}_{\alpha}$. By definition of $\hat{\cdot}$ (see Theorem 2.2.18),

$$\hat{x} = \{(\hat{y}, 1) : y \in x\}$$

Notice that $dom(\hat{x}) \subseteq \mathfrak{L}^{\mathbb{B}}_{\alpha}$ and that if we take the formula $\varphi(v) : v = v$, we have that for all $\hat{y} \in dom(\hat{x})$,

$$[\![\varphi(\hat{y})]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} = [\![\hat{y} = \hat{y}]\!]_{\mathfrak{L}^{\mathbb{B}}_{\alpha}} = 1 = \hat{x}(\hat{y}).$$

Therefore $\hat{x} \in \mathfrak{L}^{\mathbb{B}}_{\alpha+1}$. □

Remark 2.3.9. Notice that the previous theorem tells us that the function $\hat{\cdot}$ is such that $\hat{\cdot} : V \to \mathfrak{L}^{\mathbb{B}} \subseteq V^2$. Recall that $\hat{\cdot}$ is an "isomorphism" between V and V^2 (see Theorem 2.2.19). We want to see that it is also an "isomorphism" between V and $\mathfrak{L}^{\mathbb{B}}$.

We know that $\hat{\cdot}$ is injective by item 4. of Theorem 2.2.19. Since $\mathfrak{L}^{\mathbb{B}} \subseteq V^2$, and since for all $f \in \mathfrak{L}^{\mathbb{B}}$, there exists $a \in V$ such that $[\![\hat{a} = f]\!]$ we have that $\hat{\cdot} : V \to \mathfrak{L}^{\mathbb{B}}$ is also "surjective". On the other hand, by Theorem 2.2.19 item 6., we get that for all $a_1, ..., a_n \in V$ and $\mathcal{L}_{\in}$-formula $\varphi(x_1, ..., x_n)$,

$$\varphi(a_1, ..., a_n) \text{ holds in } V\text{, if and only if, } [\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{V^2} = 1.$$

But we can prove that

$$[\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{\mathfrak{L}^{\mathbb{B}}} = [\![\varphi(\hat{a_1}, ..., \hat{a_n})]\!]_{V^2}$$

holds (we will not prove it here, since it is a standard, but long and tedious proof), so $\mathfrak{L}^{\mathbb{B}}$ is "isomorphic" to V.

Taking these results into account, we need to change our definition of definable set. We start by changing the definition of definability as follows:

Definition 2.3.10. Let $M \subseteq V^{\mathbb{B}}$. We say that $f \in V^{\mathbb{B}}$ is a $\mathbb{B}-$*definable* subset of M if f satisfies the following

1. $dom(f) = M$ (before it was $dom(f) \subseteq M$).

2. There is a classical first-order $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M^{|\bar{y}|}$ such that for all $a \in dom(f)$

$$f(a) = [\![\varphi(a, \bar{b})]\!]_M$$

And we define the set of $\mathbb{B}$-definable subsets of M as

$$Def^{\mathbb{B}}(M) := \{f \in V^{\mathbb{B}} : f \text{ is a } \mathbb{B}\text{-definable subset of } M\}.$$

Definition 2.3.11. Given a complete Boolean algebra $\mathbb{B}$, we define by transfinite recursion over the ordinals

1. $\mathbb{L}_0^{\mathbb{B}} := \varnothing$

2. $\mathbb{L}_{\alpha+1}^{\mathbb{B}} := Def^{\mathbb{B}}(\mathbb{L}_\alpha^{\mathbb{B}}) \cup \mathbb{L}_\alpha^{\mathbb{B}} = \{f \in V^{\mathbb{B}} : dom(f) = \mathbb{L}_\alpha^{\mathbb{B}}$ and there exist a $\mathcal{L}_{\in}$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in (\mathbb{L}_\alpha^{\mathbb{B}})^{|\bar{y}|}$ such that for all $a \in dom(f)$, $f(a) = [\![\varphi(a, \bar{b})]\!]_{\mathbb{L}_\alpha^{\mathbb{B}}}\} \cup \mathbb{L}_\alpha^{\mathbb{B}}$

3. $\mathbb{L}_\alpha^{\mathbb{B}} := \bigcup_{\beta<\alpha} \mathbb{L}_\beta^{\mathbb{B}}$ for $\alpha \neq 0$ limit ordinal.

4. $\mathbb{L}^{\mathbb{B}} := \bigcup_{\alpha \in ON} \mathbb{L}_\alpha^{\mathbb{B}}$.

Notice that two changes were made to the original definition. First, the domain of functions was changed, we went from considering functions with $dom(f) \subseteq \mathbb{L}_\alpha^{\mathbb{B}}$ to $dom(f) = \mathbb{L}_\alpha^{\mathbb{B}}$. In this way, it is not always the case that $\hat{x} \in \mathbb{L}^{\mathbb{B}}$ for all $x \in V$. Second, the successor step was defined so that

$$\mathbb{L}^{\mathbb{B}}_{\alpha} \subseteq \mathbb{L}^{\mathbb{B}}_{\alpha+1} \text{ holds for all } \alpha \in ON.$$

These changes are very important on a technical level, since they make many proofs easier later on.

Remark 2.3.12. We can prove that $\mathbb{L}^{\mathbb{B}} \subseteq V^2$ mimicking the argument that we used for the model $\mathfrak{L}^{\mathbb{B}}$ in Theorem 2.3.6.

Theorem 2.3.13 (cf. Theorem 2.3.6)**.** If $f \in \mathbb{L}^{\mathbb{B}}$, then $ran(f) \subseteq \{0,1\} = 2$.

Lemma 2.3.14. If M is a $\mathbb{B}$-valued $\mathcal{L}_\in$-structure, then for all $f, g \in M$, if $[\![f = g]\!]_M = 1$, then, for all $\mathcal{L}_\in$-formula $\varphi(x, \bar{y})$ and $\bar{a} \in M^{|\bar{y}|}$,

$$[\![\varphi(f, \bar{a})]\!]_M = [\![\varphi(g, \bar{a})]\!]_M.$$

Proof. By Theorem 2.2.14 item 7., we have that

$$[\![f = g]\!]_M \wedge [\![\varphi(f, \bar{a})]\!]_M \leqslant [\![\varphi(g, \bar{a})]\!]_M \text{ and } [\![f = g]\!]_M \wedge [\![\varphi(g, \bar{a})]\!]_M \leqslant [\![\varphi(f, \bar{a})]\!]_M$$

And since $[\![f = g]\!]_M = 1$, we have that

$$[\![\varphi(f, \bar{a})]\!]_M \leqslant [\![\varphi(g, \bar{a})]\!]_M \text{ and } [\![\varphi(g, \bar{a})]\!]_M \leqslant [\![\varphi(f, \bar{a})]\!]_M, \text{ i.e., } [\![\varphi(f, \bar{a})]\!]_M = [\![\varphi(g, \bar{a})]\!]_M$$

□

Lemma 2.3.15. Let $f, g \in V^{\mathbb{B}}$. Suppose f is an extension of g, i.e. $dom(g) \subseteq dom(f)$ and $f\restriction_{dom(g)} = g$. If $f(a) = 0$ for all $a \in dom(f) \backslash dom(g)$, then $V^{\mathbb{B}} \models f = g$.

Proof. We want to see that

$$[\![f := g]\!] = [\![f \subseteq g]\!] \wedge [\![g \subseteq f]\!] := \left(\bigwedge_{a \in dom(f)} f(a) \rightarrow [\![a \in g]\!] \right) \wedge \left(\bigwedge_{b \in dom(g)} g(b) \rightarrow [\![b \in f]\!] \right) = 1.$$

Notice that

$$
\begin{aligned}
[\![f \subseteq g]\!] :&= \bigwedge_{a \in dom(f)} f(a) \to [\![a \in g]\!] && \text{(by definition of } [\![f \subseteq g]\!]) \\
&= \bigwedge_{b \in dom(g)} f(b) \to [\![b \in g]\!] \wedge \bigwedge_{a \in dom(f)\backslash dom(g)} f(a) \to [\![a \in g]\!] && (dom(f) = dom(f) \cup (dom(g)\backslash dom(g))) \\
&= \bigwedge_{b \in dom(g)} g(b) \to [\![b \in g]\!] \wedge \bigwedge_{a \in dom(f)\backslash dom(g)} 0 \to [\![a \in g]\!] && (f(a) = 0 \text{ for all } a \in dom(f)\backslash dom(g)) \\
&= \bigwedge_{b \in dom(g)} g(b) \to [\![b \in g]\!] \wedge 1 && \text{(since } 0 \to x = 1 \text{ for } x \in \mathbb{B}) \\
&= \bigwedge_{b \in dom(g)} g(b) \to [\![b \in g]\!] \\
&= [\![g \subseteq g]\!] = 1
\end{aligned}
$$

On the other hand, since $dom(g) \subseteq dom(f)$ and $f\restriction_{dom(g)} = g$, we have for all $b \in dom(g)$

$$
\begin{aligned}
[\![b \in f]\!] :&= \bigvee_{a \in dom(f)} f(a) \wedge [\![a = b]\!] \\
&\geqslant f(b) \wedge [\![b = b]\!] && \text{(since } b \in dom(g)) \\
&= g(b) \wedge 1 && \text{(since } f\restriction_{dom(g)} = g) \\
&= g(b).
\end{aligned}
$$

Therefore, $g(b) \leqslant [\![b \in f]\!]$ for all $b \in dom(g)$, i.e., $(g(b) \to [\![b \in f]\!]) = 1$ and

$$[\![g \subseteq f]\!] = \bigwedge_{b \in dom(g)} g(b) \to [\![b \in f]\!] = \bigwedge_{b \in dom(g)} 1 = 1$$

From this, we conclude that

$$[\![f = g]\!] = [\![f \subseteq g]\!] \wedge [\![g \subseteq f]\!] = 1 \wedge 1 = 1.$$

□

The following theorem shows us that, for all Boolean algebras $\mathbb{B}$, L is "isomorphic" to $\mathbb{L}^{\mathbb{B}}$ in the following way:

Theorem 2.3.16. There exists a class function $j : L \to \mathbb{L}^{\mathbb{B}}$ such that for all $\alpha \in ON$, the restriction $j{\restriction}_{L_\alpha} : L_\alpha \to \mathbb{L}^{\mathbb{B}}$ satisfies

1. $ran(j{\restriction}_{L_\alpha}) \subseteq \mathbb{L}_\alpha$.
2. $j{\restriction}_{L_\alpha}$ is injective.
3. $j{\restriction}_{L_\alpha}$ is surjective in the following sense: for all $Y \in \mathbb{L}_\alpha$, there exists $X \in L_\alpha$ such that
$$\mathbb{L}_\alpha \models j(X) = Y.$$
4. $j{\restriction}_{L_\alpha}$ is an elementary embedding in the following sense: For every $\mathcal{L}_\in$-formula $\varphi(\bar{x})$ and $\bar{a} \in L_\alpha^{|\bar{x}|}$,
$$L_\alpha \models \varphi(\bar{a}) \text{ if and only if } [\![\varphi(j(\bar{a}))]\!]_{\mathbb{L}_\alpha} = 1.$$
5. $rank_L(X) = rank_{\mathbb{L}^{\mathbb{B}}}(j(X))$.

Proof. We prove this by induction on ordinals: The case $\alpha = 0$ is trivial.

Suppose that we have already defined $j{\restriction}_{L_\beta}$ and that it satisfies the conditions of the theorem for all $\beta \leqslant \alpha$, where $\alpha \in ON$. We define j for $L_{\alpha+1} \backslash L_\alpha$: Given $X \in L_{\alpha+1} \backslash L_\alpha$, we have that $X \subseteq L_\alpha$ and that there exists a first-order $\mathcal{L}_\in$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in L_\alpha^{|\bar{y}|}$ such that

$$X = \{a \in L_\alpha : L_\alpha \models \varphi(a, \bar{b})\}.$$

We define

$$j(X) : \mathbb{L}_\alpha \to \mathbb{B} \text{ as } j(X)(c) = [\![\varphi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} \text{ for all } c \in \mathbb{L}_\alpha$$

Notice that, from the definition, $j(X) \in \mathbb{L}_{\alpha+1} \backslash \mathbb{L}_\alpha$, and thus $ran(j \restriction_{L_{\alpha+1}}) \subseteq \mathbb{L}_{\alpha+1}$ and $rank_L(X) = rank_{\mathbb{L}^\mathbb{B}}(j(X))$.

Let us see that the function $j(X)$ is indeed well-defined, that is, that the function does not depend on the choice of formulas and parameters.

Consider some $\mathcal{L}_\in$-formulas $\varphi(x, \bar{y})$, $\psi(z, \bar{w})$ and parameters $\bar{b} \in L_\alpha^{|\bar{y}|}$ and $\bar{d} \in L_\alpha^{|\bar{w}|}$ such that

$$X = \{a \in L_\alpha : L_\alpha \models \varphi(a, \bar{b})\} = \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{d})\}.$$

From the previous equality, we have that for all $a \in L_\alpha$,

$$L_\alpha \models \varphi(a, \bar{b}) \text{ if and only if } L_\alpha \models \psi(a, \bar{d}).$$

But using the induction hypothesis item 4., we have that

$$L_\alpha \models \varphi(a, \bar{b}) \text{ if and only if } [\![\varphi(j(a), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ and}$$
$$L_\alpha \models \psi(a, \bar{d}) \text{ if and only if } [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 1$$

Combining these results, we get

$$[\![\varphi(j(a), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ if and only if } [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 1, \text{ namely,}$$
$$[\![\varphi(j(a), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} \text{ for all } a \in L_\alpha$$

And thus the function $j(X)$ is well-defined for every $c \in j(\mathbb{L}_\alpha) \subseteq \mathbb{L}_\alpha$.

To see that this is also true for all $c \in \mathbb{L}_\alpha$, and not only for all $j(a) \in j(L_\alpha)$, we use Lemma 2.3.14 and the "surjectivity" of j. Given $c \in \mathbb{L}_\alpha$, there exists $a_c \in L_\alpha$ such that $[\![j(a_c) = c]\!]_{\mathbb{L}_\alpha} = 1$. Then,

$$[\![\varphi(j(a_c), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ if and only if } [\![\varphi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ and } [\![\psi(j(a_c), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ if and only if } [\![\psi(c, j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 1$$

we conclude that

$$[\![\varphi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ if and only if } [\![\psi(c, j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 1, \text{ i.e., } [\![\varphi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = [\![\psi(c, j(\bar{d}))]\!]_{\mathbb{L}_\alpha} \text{ for all } c \in \mathbb{L}_\alpha.$$

And therefore the function $j(X)$ is well-defined for $X \in \mathbb{L}_{\alpha+1}$.

Let us see that $j\restriction_{\mathbb{L}_\alpha}$ is injective.

If $X, Y \in L_\alpha$, then, by induction hypothesis item 2., $j(X) \neq j(Y)$ provided that $X \neq Y$.

If $X \in L_{\alpha+1}$ and $Y \in L_\alpha$, then we have that there exists $\beta < \alpha$ such that $Y \in L_{\beta+1}$ and therefore $dom(X) = \mathbb{L}_\alpha$ and $dom(Y) = \mathbb{L}_\beta$, so we have $j(X) \neq j(Y)$, since $dom(j(X)) \neq dom(j(Y))$.

Let $X, Y \in L_{\alpha+1} \backslash L_\alpha$ be such that $X \neq Y$. Thus, there are $\mathcal{L}_\in$-formulas $\varphi(x, \bar{y})$, $\psi(z, \bar{w})$ and parameters $\bar{b} \in L_\alpha^{|\bar{x}|}$ and $\bar{d} \in L_\alpha^{|\bar{w}|}$ such that

$$X = \{a \in L_\alpha : L_\alpha \models \varphi(a, \bar{b})\} \text{ and } Y = \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{d})\}.$$

Since $X \neq Y$, we may assume, without loss of generality, that there exists $a \in L_\alpha$ such that $a \in X$ and $a \notin Y$, i.e.

$$L_\alpha \models \varphi(a, \bar{b}) \text{ and } L_\alpha \not\models \psi(a, \bar{d}).$$

Then,

$$j(X)(j(a)) = [\![\varphi(j(a), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ and } j(Y)(j(a)) = [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 0$$

i.e., $j(X) \neq j(Y)$ and $j\upharpoonright_{\mathbb{L}_\alpha}$ is injective.

Let us now show that $j\upharpoonright_{\mathbb{L}_\alpha}$ is surjective in the sense of item 3. of this theorem. Let $Y \in \mathbb{L}_{\alpha+1} \backslash \mathbb{L}_\alpha = Def^{\mathbb{B}}(\mathbb{L}_\alpha)$. Then, there is a $\mathcal{L}_\in$-formula $\psi(x, \bar{y})$ and parameters $\bar{d} \in L_\alpha^{|\bar{x}|}$ such that for all $c \in \mathbb{L}_\alpha = dom(Y)$

$$Y(c) = [\![\psi(c, \bar{d})]\!]_{\mathbb{L}_\alpha}.$$

By the induction hypothesis item 3., there exists $\bar{b} \in L_\alpha^{|\bar{y}|}$ such that $\mathbb{L}_\alpha \models j(\bar{b}) = \bar{d}$. Let us define

$$X := \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{b})\} \in L_{\alpha+1}.$$

We then have two cases:

1. Suppose that $X \in L_\alpha$, then, we have that there is an ordinal $\beta < \alpha$ such that $X \in L_{\beta+1}$. Thus, we have that $X \subseteq L_\beta$ and that there is a first-order $\mathcal{L}_\in$-formula $\varphi(w, \bar{z})$ and parameters $\bar{f} \in L_\alpha^{|\bar{z}|}$ such that

$$X = \{e \in L_\beta : L_\beta \models \varphi(e, \bar{f})\} = \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{b})\}$$

Thus, since $L_\beta \subseteq L_\alpha$, we have that for all $e \in L_\beta$,

$$L_\beta \models \varphi(e, \bar{f}) \text{ if and only if } L_\alpha \models \psi(e, \bar{b})$$

by the induction hypothesis item 4., we get

$$[\![\varphi(j(e), j(\bar{f}))]\!]_{\mathbb{L}_\beta} = [\![\psi(j(e), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = [\![\psi(j(e), \bar{d})]\!]_{\mathbb{L}_\alpha} \text{ for all } e \in L_\beta$$

By using the "surjectivity" of j, we can generalize this to all $g \in dom(j(X)) = \mathbb{L}_\beta$, obtaining

$$j(X)(g) = [\![\varphi(g, j(\bar{f}))]\!]_{\mathbb{L}_\beta} = [\![\psi(g, \bar{d})]\!]_{\mathbb{L}_\alpha} = Y(g) \text{ for all } g \in dom(j(X))$$

In this way, we conclude that Y is an extension of $j(X)$. Let us see how Y behaves in $\mathbb{L}_\alpha \backslash \mathbb{L}_\beta$. Given $a \in L_\alpha \backslash L_\beta$, since $a \notin L_\beta$ and $X \subseteq L_\beta$, we have $a \notin X$ and therefore

$$L_\alpha \not\models \psi(e, \bar{b}), \text{ i.e., } [\![\psi(j(e), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = [\![\psi(j(e), \bar{d})]\!]_{\mathbb{L}_\alpha} = 0.$$

By the "surjectivity" of j, we can generalize the equality given above to

$$j(Y) = [\![\psi(g, \bar{d})]\!]_{\mathbb{L}_\alpha} = 0 \text{ for all } g \in \mathbb{L}_\alpha \backslash \mathbb{L}_\beta.$$

In this way, since Y is an extension of $j(X)$ such that for all $g \in dom(Y) \backslash dom(j(X))$, $Y(g) = 0$, we conclude that the functions $j(X)$ and Y are equal in the sense of $[\![\cdot = \cdot]\!]$ by Lemma 2.3.15, i.e.,

$$\mathbb{L}_\alpha \models j(X) = Y$$

as we wanted.

2. Suppose $X \in L_{\alpha+1} \backslash L_\alpha$. In this case, we have $j(X) = Y$ since $dom(j(X)) = \mathbb{L}_\alpha = dom(Y)$ and for all $c \in \mathbb{L}_\alpha$,

$$j(X)(c) = [\![\psi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = [\![\psi(c, \bar{d})]\!]_{\mathbb{L}_\alpha} = Y(c), \text{ i.e., } j(X) = Y$$

Let us see that we have the property 4 by induction on formulas:

1. $\in$: Let $X, Y \in L_{\alpha+1}$, we see that

$$L_{\alpha+1} \models X \in Y \text{ if and only if } \mathbb{L}_{\alpha+1} \models j(X) \in j(Y).$$

($\Rightarrow$) Suppose $L_{\alpha+1} \models X \in Y$. Thus, since $X \in Y \subseteq L_\alpha$, we have $X \in L_\alpha$. Consider an $\mathcal{L}_\in$-formula $\psi(x, \bar{y})$ and parameters $\bar{b} \in L_\alpha^{|\bar{x}|}$ such that

$$Y = \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{b})\}$$

Then, since $X \in Y$, we have $L_\alpha \models \psi(X, \bar{b})$ and therefore, by the induction hypothesis, $\mathbb{L}_\alpha \models \psi(j(X), j(\bar{b}))$, i.e.,

$$j(Y)(j(X)) = [\![\psi(j(X, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1$$

And since $[\![j(X) = j(X)]\!]_{\mathbb{L}_\alpha} = 1$, we have $j(Y)(j(X)) \wedge [\![j(X) = j(X)]\!]_{\mathbb{L}_\alpha} = 1$ and since $j(X) \in dom(Y) \subseteq \mathbb{L}_\alpha$, we have

$$[\![j(X) \in j(Y)]\!]_{\mathbb{L}_\alpha} = \bigvee_{c \in dom(j(Y))} j(Y)(c) \wedge [\![c = j(X)]\!]_{\mathbb{L}_\alpha} = 1$$

as desired.

($\Leftarrow$) Let us suppose that

$$[\![j(X) \in j(Y)]\!]_{\mathbb{L}_\alpha} = \bigvee_{c \in dom(j(Y))} j(Y)(c) \wedge [\![c = j(X)]\!]_{\mathbb{L}_\alpha} = 1$$

Thus, since $j(Y)(c) \wedge [\![c = j(X)]\!]_{\mathbb{L}_\alpha}$ can only be 0 or 1, there exists $c \in dom(j(Y)) \subseteq \mathbb{L}_\alpha$ such that

$$j(Y)(c) \wedge [\![c = j(X)]\!]_{\mathbb{L}_\alpha} = 1,$$

then,

$$j(Y)(c) = 1 \text{ and } [\![c = j(X)]\!]_{\mathbb{L}_\alpha} = 1$$

Since $j(Y)(c) = [\![\psi(c, j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1$ and $[\![c = j(X)]\!]_{\mathbb{L}_\alpha} = 1$, we have that

$$j(Y)(j(X)) = [\![\psi(j(X), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1.$$

By induction hypothesis, we have that

$$L_\alpha \models \psi(X, \bar{b})$$

and then $X \in Y$.

2. Equality: Let $X, Y \in L_{\alpha+1} \backslash L_\alpha$. Let us see that

$$L_{\alpha+1} \models X = Y \text{ if and only if } \mathbb{L}_{\alpha+1} \models j(X) = j(Y).$$

By definition, we have that there exist first-order $\mathcal{L}_\in$-formulas $\varphi(x, \bar{y})$, $\psi(z, \bar{w})$ and parameters $\bar{b} \in L_\alpha^{|\bar{x}|}$ and $\bar{d} \in L_\alpha^{|\bar{w}|}$ such that

$$X = \{a \in L_\alpha : L_\alpha \models \varphi(a, \bar{b})\} \text{ and } Y = \{a \in L_\alpha : L_\alpha \models \psi(a, \bar{d})\}.$$

If $X = Y$, then, since j is a function, $j(X) = j(Y)$ and therefore $\mathbb{L}_{\alpha+1} \models j(X) = j(Y)$.

Suppose now that $X \neq Y$. Without loss of generality, suppose there exists $a \in X$ with $a \notin Y$, i.e.,

$$L_\alpha \models \varphi(a, \bar{b}) \text{ and } L_\alpha \not\models \psi(a, \bar{d})$$

By the induction hypothesis item 4., this means that

$$j(X)(j(a)) = [\![\varphi(j(a), j(\bar{b}))]\!]_{\mathbb{L}_\alpha} = 1 \text{ and } j(Y)(j(a)) = [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} \neq 1.$$

but since $[\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha}$ can only take values in $\{0, 1\}$, we get

$$j(Y)(j(a)) = [\![\psi(j(a), j(\bar{d}))]\!]_{\mathbb{L}_\alpha} = 0.$$

Let us see that $[\![j(a) \in j(Y)]\!]_{\mathbb{L}_\alpha} = 0$.

Notice that

$$j(X)(j(a)) = 1 \text{ and } [\![j(a) \in j(Y)]\!]_{\mathbb{L}_\alpha} = 0, \text{ i.e., } j(X)(j(a)) \to [\![j(a) \in j(Y)]\!]_{\mathbb{L}_\alpha} = 0$$

and therefore

$$[\![j(X) \subseteq j(Y)]\!]_{\mathbb{L}_\alpha} = \bigwedge_{c \in dom(j(X))} j(X)(c) \to [\![c \in j(Y)]\!]_{\mathbb{L}_\alpha} = 0$$

and we have $[\![j(X) = j(Y)]\!]_{\mathbb{L}_\alpha} = 0$, as desired.

The rest of the induction is straightforward, and therefore, by induction on formulas, we have the theorem.

□

2.3.2 Constructible sets in quantale-valued models

Definition 2.3.17. Let $M \subseteq V^{\mathbb{Q}}$. Notice that we can view M as a $\mathbb{Q}$-valued model by taking the restrictions on M from the interpretations of $\in$ and $=$ on $V^{\mathbb{Q}}$. We say that $f \in V^{\mathbb{Q}}$ is a $\mathbb{Q}-$*definable* ($\mathbb{Q}^*-$*definable*) subset of M if f satisfies the following:

1. $dom(f) = M$ $(dom(f) \subseteq M)$

2. There is an $R - \mathcal{L}_\in$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M^{|\bar{y}|}$ such that for all $a \in dom(f)$

$$f(a) = [\![\varphi(a, \bar{b})]\!]_M.$$

Given a commutative integral quantale $\mathbb{Q}$, we can define, by transfinite recursion on the ordinals, the class models $\mathfrak{L}^{\mathbb{Q}}$ and $\mathbb{L}^{\mathbb{Q}}$ in the same fashion as we did for the Boolean case (see Definitions 2.3.2 and 2.3.11). Furthermore, by using Theorem 1.2.39, we can prove that $\mathfrak{L}^{\mathbb{Q}} \subseteq V^2$ and $\mathbb{L}^{\mathbb{Q}} \subseteq V^2$ by using a similar argument as in Theorem 2.3.6.

Since the logic for these models is two-valued, there is no reason to continue exploring this definition of definability on Residuated Logic.

3 Constructions over Kripke models

As it was done in the previous chapter, we would like to generalize both the von Neumann's and Gödel's hierarchy, only that now out aim is to do it in the context of Kripke models. Since in [Fit69] Fitting studied exactly these hierarchies for Intuitionistic Kripke models, we generalize his results for a more general kind of Kripke models.

First, in Section 3.1, we start by overviewing the topic of Intuitionistic Kripke models and showing some well-known relations between it and Cohen's forcing. Then, we study Ono and Komori's notion of semantics for substructural logics without contraction and exchange (see [OK85]). These models generalize the notion of Intuitionistic Kripke models and are a suitable semantical counterpart for **Residuated Logic**. Then, following the ideas of Lano [Lan92a], we further generalize Ono and Komori's models to the context of **Modal Residuated Logic**. The definition for the interpretation of the modality in our definition is original, and allows for a smooth transition between Kripke models and lattice-valued models.

Then, in Section 3.2, we start by exposing Fitting's results from [Fit69], where he constructs a generalization of von Neumann's hierarchy using Intuitionistic Kripke models. Then, using the notion of MR-Kripke models that we defined in the previous section, together with Fitting's ideas on the intuitionistic case, we construct new **Modal Residuated model of Set Theory**.

We consider the model $\mathcal{V}^{\mathbb{P}^*}$ (see Definition 3.2.18), that generalizes the von Neumann hierarchy for Modal Residuated Logic, and we prove, in Corollary 3.2.30, that there is a **Gödel–McKinsey–Tarski-like translation** between this model and a suited Heyting valued model $R^{\mathbb{H}}$ (see Definition 3.2.23). This translation is obtained by first constructing an "isomorphism" (see Theorem 3.2.29) between $\mathcal{V}^{\mathbb{P}^*}$ and $R^{\mathbb{H}}$ and then proving how this result implies that if φ is an $\mathcal{L}_\in$-sentence that is valid in $R^{\mathbb{H}}$, then $\Diamond\varphi$ is valid in $\mathcal{V}^{\mathbb{P}^*}$.

Finally, in Section 3.3, we start by stating Fitting's main result for constructable sets in Intuitionistic Kripke models, and then we proceed to superficially study the notion of constructible set in the context of Modal Residuated Kripke models of Set Theory. We only outline a propose for a construction in this context and indicate some possible conditions that we believe are necessary for the generalization of Fitting's results.

3.1 Kripke models

Kripke models allow intuitive interpretations of different kinds of non-classical logics such as Modal (see [Kri59, Kri63b]) or Intuitionistic Logic (see [Kri63a]). Intuitively, this is done via a collection of *possible universes* or *states of knowledge* connected by an *accessibility relation* between them and a notion of *local truth* on each world represented by the forcing relation.

Although these models were originally used for Modal or Intuitionistic Logic, there are a lot generalizations into different kind of logics such as Fuzzy Logics [SS18], the Logic of Gelfand

Quantales [AM01] or, as we see in subsection 3.1.3, Residuated Logic [Ono85, OK85, Mac96] (Ono and Komori call this logic L_{BCK}, which is essentially a substructural logic without the contraction rule, but with the exchange rule).

3.1.1 Intuitionistic Kripke models

We do not follow Kripke's original notation nor his definition for his models for Intuitionistic Logic [Kri63a], but rather Fitting's definition [Fit69], but written in modern terms (such as in [Cai95]).

Throughout this chapter, $\mathcal{L}$ denotes a first-order language.

Definition 3.1.1 (Intuitionistic Kripke Model, [Fit69], Chapter 4, Section 2)**.** We say that $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ is an *Intuitionistic Kripke $\mathcal{L}$-model* if

1. $\mathbb{P} \neq \varnothing$ and $(\mathbb{P}, \leqslant)$ is a partial order.

2. $\mathcal{D}$ is a function with domain $\mathbb{P}$ to non-empty sets of parameters

3. $\Vdash$ is a relation between elements of $\mathbb{P}$ and atomic sentences in the language

$$\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup \bigcup_{p \in \mathbb{P}} \mathcal{D}(p).$$

   where each element of $\bigcup_{p \in \mathbb{P}} \mathcal{D}(p)$ is considered as a constant symbol. We denote $(p, \varphi) \in \Vdash$ by $\mathcal{A} \Vdash_p \varphi$ and we say that φ *is forced in $\mathcal{A}$ at p.*

4. Given $p, q \in \mathbb{P}$ and φ an atomic $\mathcal{L}_{\mathcal{A}}$-sentence, we require that $\mathcal{D}$ and $\Vdash$ satisfy the following conditions:

   a. If $p \leqslant q$, then $\mathcal{D}(p) \subseteq \mathcal{D}(q)$.

b. If $p \leqslant q$ and $\mathcal{A} \Vdash_p \varphi$, then $\mathcal{A} \Vdash_q \varphi$.

c. If $\mathcal{A} \Vdash_p \varphi$, then $\varphi \in Sent(\mathcal{L} \cup \mathcal{D}(p))$.

Definition 3.1.2 (Intuitionistic Kripke forcing, [Fit69], Chapter 4, Section 2)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ a Kripke $\mathcal{L}$-model, we can extend the forcing relation $\mathcal{A} \Vdash_p \varphi$ to all $\mathcal{L}_{\mathcal{A}}$-sentences by recursion on the complexity of φ

1. $\mathcal{A} \Vdash_p (\varphi \vee \psi)$, if and only if, $\mathcal{A} \Vdash_p \varphi$ or $\mathcal{A} \Vdash_p \psi$.

2. $\mathcal{A} \Vdash_p (\varphi \wedge \psi)$, if and only if, $\mathcal{A} \Vdash_p \varphi$ and $\mathcal{A} \Vdash_p \psi$.

3. $\mathcal{A} \Vdash_p (\varphi \rightarrow \psi)$, if and only if, for all $q \in \mathbb{P}$, if $q \geqslant p$ and $\mathcal{A} \Vdash_q \varphi$, then $\mathcal{A} \Vdash_q \psi$.

4. $\mathcal{A} \Vdash_p \neg\varphi$, if and only if, for all $q \in \mathbb{P}$, if $q \geqslant p$, then $\mathcal{A} \nVdash_q \varphi$.

5. $\mathcal{A} \Vdash_p \exists x\varphi(x)$, if and only if, there exists $a \in \mathcal{D}(p)$ such that $\mathcal{A} \Vdash_p \varphi(a)$.

6. $\mathcal{A} \Vdash_p \forall\varphi(x)$, if and only if, for all $q \in \mathbb{P}$, if $q \geqslant p$ and $\forall b \in \mathcal{D}(q)$, then $\mathcal{A} \Vdash_q \varphi(b)$.

Definition 3.1.3 ([Fit69], Chapter 4, Section 2)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ a Kripke $\mathcal{L}$-model and an $\mathcal{L}$-sentence φ, we say that φ is *true* in the structure $\mathcal{A}$ (and we denote it by $\mathcal{A} \models \varphi$) if for all $p \in \mathbb{P}$, $\mathcal{A} \Vdash_p \varphi$.

Definition 3.1.4. Given $(\mathbb{P}, \leqslant)$ a partial order and $A \subseteq \mathbb{P}$ a non-empty set, we say that A is *hereditary*, if and only if,

$$\text{whenever } p \in A \text{ and } q \geqslant p,\ q \in A$$

and we denote the collection of hereditary subsets of $\mathbb{P}$ by $\mathbb{P}^+$.

Theorem 3.1.5 (Folklore)**.** The set $\mathbb{P}^+$ with the order $\subseteq$ is a complete Heyting algebra.

Proof. Since the order considered is the set the subset relation, it is enough to prove that the intersection and union of a collection of hereditary sets is hereditary. We only prove the case for intersection, since the case for the union is proved in a similar way. Let us consider $A_i \in \mathbb{P}^+$ for $i \in I$ and take $x \in \bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$ and $y \in \mathbb{P}$ such that $x \leqslant y$. Then, for every $i \in I$, $x \in A_i$ and since A_i is hereditary and $x \leqslant y$, $y \in A_i$. Therefore, $y \in \bigcap_{i \in I} A_i$, that is, $\bigcap_{i \in I} A_i$ is hereditary. □

The set $\mathbb{P}^+$ is used as a set of **truth values** once we define the notion of generalized subsets in a Kripke model.

Theorem 3.1.6 ([Fit69], Chapter 4, Theorem 4.2)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ a Kripke $\mathcal{L}$-model, $p \in \mathbb{P}$ and an $\mathcal{L}_{\mathcal{A}}$-sentence φ, we have that if $\mathcal{A} \Vdash_p \varphi$ and $q \in \mathbb{P}$ is such that $p \leqslant q$, then $\mathcal{A} \Vdash_q \varphi$. That is, the set

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$$

is hereditary.

Definition 3.1.7. Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ a Kripke $\mathcal{L}$-model, we denote *the universe* of the Kripke $\mathcal{L}$-model $\mathcal{A}$ as $|\mathcal{A}| := \bigcup_{p \in \mathbb{P}} \mathcal{D}(p)$.

Theorem 3.1.8 ([Fit69], Chapter 6, Theorem 6.1)**.** If φ is a $\mathcal{L}$-sentence with no universal quantifiers and $\nVdash_i \varphi$, then there is a model $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ in which φ is not true and $\mathcal{D}$ is a constant function.

Definition 3.1.9. If $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ is a Kripke $\mathcal{L}$-model where $\mathcal{D}$ is a constant function (i.e. $ran(\mathcal{D}) = \{D\}$), we denote $(\mathbb{P}, \leqslant, \Vdash, D) := (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ and we call them *Kripke models with constant universe.*

3.1.2 Connections with Cohen's forcing and valued models.

Now we present a well-known connection between Intuitionistic Kripke models and valued-models that is going to be useful later. We present it first in the context of **Cohen's forcing** and then show how it can be altered into the context of Kripke models. For a thorough treatment of set theoretical forcing, see [Kun11].

Given a partial order $\mathbb{P}$, we can construct a complete Boolean algebra $\mathbb{B}$ and a dense embedding $i : \mathbb{P} \to \mathbb{B}$. The Boolean algebra $\mathbb{B}$ is called the *completion of* $\mathbb{P}$ and it is unique up to isomorphism. Thus, if we need to force with a poset $\mathbb{P}$, we could take its Boolean completion $\mathbb{B}$ and force with it instead.

Now, let us consider a complete Boolean algebra $\mathbb{B}$ and consider Cohen's forcing relation $\Vdash$ on $\mathbb{B}$, where $\mathbb{B}$ is viewed as a forcing poset. Then, if φ is any $\mathcal{L}_\in$-sentence, the **truth value of** φ can be defined as

$$[\![\varphi]\!]_{\mathbb{B}} = \bigvee\{p \in \mathbb{B} : p \Vdash \varphi\}$$

and we have that the following holds:

Theorem 3.1.10 ([Kun11], Lemma IV.4.19)**.** For every $p \in \mathbb{B}$

$$p \Vdash \varphi, \text{ if and only if, } p \leqslant [\![\varphi]\!]_{\mathbb{B}}.$$

Furthermore, by [Kun11] Exercise IV.4.20, IV.4.21 and IV.4.24, it is clear that the valuation that we defined satisfies the conditions of Definition 2.1.1 and 2.1.3.

The only problem with this definition is that in Cohen's forcing the order relation works in the opposite way as with Kripke models, that is, $p \leqslant q$ is understood as "p has more information than q" rather than the other way around. Or in a more formal way, in the context of Cohen's forcing we have that if $q \Vdash \varphi$ and $p \leqslant q$, then $p \Vdash \varphi$. Therefore, since Cohen's forcing preserves truth **downwards** and Kripke's forcing preserves truth **upwards**, if we want to formulate Theorem 3.1.10 in the context of Kripke's forcing, we have to consider the **opposite order** on the Kripke model. Therefore, we would expect to have the following:

Remark 3.1.11. Consider a Kripke model $\mathcal{A} = (\mathbb{B}, \leqslant^{op}, \Vdash, M)$ and the Boolean valued model $\mathcal{M} = (M, [\![\cdot]\!])$ determined by $\mathcal{A}$. Then, for every $p \in \mathbb{B}$,

$$\mathcal{A} \Vdash_p \varphi \text{ iff } p \leqslant [\![\varphi]\!]$$

In this way, we could construct valued models from a given Kripke model using

$$[\![\varphi]\!]_{\mathbb{B}} := \bigvee \{p \in \mathbb{B} : p \Vdash \varphi\}$$

as a definition for the valuation.

A similar process can be considered and we can take a valued model and from it construct a Kripke model. Given that constructing valued models is simpler than defining Kripke models, we may use valued models together with the relation given in Theorem 3.1.11 to generate new definitions for Kripke models.

We will see an example of that in the Subsection 3.1.4, where we propose a definition of the modal operator of *possibility* on Kripke models using this relation.

3.1.3 Residuated Kripke models

In [OK85], Ono and Komori generalize the notion of Intuitionistic Kripke models for (propositional) logics without contraction, that is, substructural logics without the idempotency of the conjunction. Then, in [Ono85], Ono defines the notion of Kripke models for the predicate case. For the most part, we follow their notation and conventions, but with small variations, since we are not interested in working with Gentzen-style sequences. Even though these models were initially used for the study of substructural logics, MacCaull [Mac96] showed how the models in [OK85] can be seen as models for Residuated Logic.

Since we are working in the context of Residuated Logic, we need a more robust structure than just a poset $(\mathbb{P}, \leqslant)$ in order to properly interpret the operation of strong conjunction & and to capture the subtleties of the Residuated Logic, so it would be natural to work with some kind of **ordered monoid**, just as quantales are defined. But there is a small issue: in the Kripke semantics convention, we would expect the forcing relation to preserve truth **upwards**, that is:

if a sentence φ is forced at p and $p \leqslant q$, then the sentence is forced at q, where $p, q \in \mathbb{P}$.

So if we were to consider the order as we have been doing it with quantales, this property would hold **backwards** (i.e. there would be truth preservation **downwards**). That is why some of the properties that we require for our orders in this section are the dual ones of the properties of quantales.

Definition 3.1.12 ([OK85], Section 3)**.** We say that $(\mathbb{P}, \leqslant, \cdot, 1)$ is a *partially ordered commutative monoid* if

1. $(\mathbb{P}, \leqslant, 1)$ is a partial order with 1 as the bottom element.

2. $(\mathbb{P}, \cdot, 1)$ is a commutative monoid.

3. For all $a, b, c \in \mathbb{P}$, if $a \leqslant b$, then $a \cdot c \leqslant b \cdot c$.

Definition 3.1.13 ([OK85], Section 3)**.** We say that $(\mathbb{P}, \leqslant, \wedge, \cdot, 1)$ is an *SO-commutative monoid* if

1. $(\mathbb{P}, \wedge, \leqslant)$ is a meet-semilattice.

2. $(\mathbb{P}, \leqslant, \cdot, 1)$ is a partially ordered commutative monoid.

Definition 3.1.14 ([Ono85], p. 189)**.** We say that an SO-commutative monoid $(\mathbb{P}, \leqslant, \wedge, \cdot, 1)$ is *complete* if

1. $(\mathbb{P}, \wedge)$ is a complete meet-semilattice (thus, $\mathbb{P}$ has a top element, denoted by ∞)

2. For every $a, b_i \in M$ with $i \in I$, $a \cdot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \cdot b_i)$

Remark 3.1.15. Notice that since $a\_\cdot$ preserves arbitrary meets, by the Adjoint Functor Theorem for preorders (Theorem 1.2.5), $a \cdot \_$ is a right adjoint, that is, there exists a function $a \to \_ : \mathbb{P} \to \mathbb{P}$ such that for all $b, c \in \mathbb{P}$

$$a \to c \leqslant b, \text{ if and only if, } c \leqslant a \cdot b$$

Furthermore, this implication allow us to define a negation on $\mathbb{P}$ by taking

$$\sim a := a \to \infty$$

Notice that we take $a \to \infty$ as the definition of $\sim a$ since ∞ is the top element of $\mathbb{P}$, and the definitions and properties on this order are the duals of those in the context of quantales.

Theorem 3.1.16. Take an SO-commutative complete monoid $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$. Then, $\mathbb{P}$ satisfies (for the most part) the dual properties of $\mathbb{Q}$ (see Theorems 1.2.33, 1.2.36 and 1.2.37). More specifically, if $a, b, c \in \mathbb{P}$ with $i \in I$, then:

1. $a \leqslant b$, if and only if, $(b \to a) = 1$.

2. $b \leqslant a \cdot (a \to b)$.

3. If $a \leqslant b$, then $b \to c \leqslant a \to c$.

4. If $a \leqslant b$, then $c \to a \leqslant c \to b$.

5. $(a \cdot b) \to c = a \to (b \to c)$.

6. $a \cdot (\sim a) = \infty$.

We focus on Residuated Kripke models with constant universe (i.e. $\mathcal{D}(p) = D$ for all $p \in \mathbb{P}$). We follow the presentation of [Ono85], where he calls this kind of models *total strong frame with constant domain* or simply *total CD-frame*. Since all the Residuated Kripke models that we consider have constant universe, we will not call attention on this fact from now on.

Definition 3.1.17 (Residuated Kripke Model, [Ono85], p. 189)**.** We say that $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ is a *Residuated Kripke $\mathcal{L}$-model* (or *R-Kripke $\mathcal{L}$-model* for short) if

1. $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ is a complete SO-commutative monoid.

2. $\Vdash$ is a relation between elements of $\mathbb{P}$ and atomic sentences in the language

$$\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup D,$$

where each element of D is considered as a constant symbol. We denote $(p, \varphi) \in \Vdash$ by $\mathcal{A} \Vdash_p \varphi$ and we say that φ *is forced in* $\mathcal{A}$ *at* p.

3. Given $p_i, q \in \mathbb{P}$, with $i \in I$ and φ an atomic $\mathcal{L}_{\mathcal{A}}$-sentence, we require that $\Vdash$ satisfies the following conditions:

   a. If $\bigwedge_{i \in I} p_i \leqslant q$ and for each $i \in I$ $\mathcal{A} \Vdash_{p_i} \varphi$, then $\mathcal{A} \Vdash_q \varphi$.

   b. $\mathcal{A} \Vdash_\infty \varphi$ for every atomic $R - \mathcal{L}_{\mathcal{A}}$-sentence φ.

   c. $\mathcal{A} \Vdash_p \bot$ if and only if $p = \infty$. Recall that $\bot$ is the symbol of contradiction.

Definition 3.1.18 (Residuated Kripke forcing [Ono85], p. 189)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ a Residuated Kripke $\mathcal{L}$-model, $p \in \mathbb{P}$ and an $R - \mathcal{L}_{\mathcal{A}}$-sentence φ, we can extend the forcing relation $\mathcal{A} \Vdash_p \varphi$ to all $R - \mathcal{L}_{\mathcal{A}}$-sentences by recursion on the complexity of φ

1. $\mathcal{A} \Vdash_p (\varphi \& \psi)$, if and only if, there are $q, r \in \mathbb{P}$ such that $p \geqslant q \cdot r$, $\mathcal{A} \Vdash_q \varphi$ and $\mathcal{A} \Vdash_r \psi$.

2. $\mathcal{A} \Vdash_p (\varphi \vee \psi)$, if and only if, there are $q, r \in \mathbb{P}$ such that $p \geqslant q \wedge r$, and both ($\mathcal{A} \Vdash_q \varphi$ or $\mathcal{A} \Vdash_q \psi$) and ($\mathcal{A} \Vdash_r \varphi$ or $\mathcal{A} \Vdash_r \psi$) hold.

3. $\mathcal{A} \Vdash_p (\varphi \wedge \psi)$, if and only if, $\mathcal{A} \Vdash_p \varphi$ and $\mathcal{A} \Vdash_p \psi$.

4. $\mathcal{A} \Vdash_p (\varphi \to \psi)$, if and only if, for all $q, r \in \mathbb{P}$ if $\mathcal{A} \Vdash_q \varphi$ and $p \cdot q \leqslant r$, then $\mathcal{A} \Vdash_r \psi$.

5. $\mathcal{A} \Vdash_p \exists x \varphi(x)$ if and only there exist an index set I such that for every $i \in I$, there exists $d_i \in D$ and $q_i \in \mathbb{P}$ such that $\bigwedge_{i \in I} q_i \leqslant p$ and $\mathcal{A} \Vdash_{q_i} \varphi(d_i)$

6. $\mathcal{A} \Vdash_p \forall \varphi(x)$, if and only if, for all $b \in D$, $\mathcal{A} \Vdash_p \varphi(b)$.

Remark 3.1.19. By definition of $\sim \varphi$, it is straightforward to prove that

$$\mathcal{A} \Vdash_p \sim \varphi, \text{ if and only if, for all } q, r \in \mathbb{P} \text{ if } \mathcal{A} \Vdash_q \varphi \text{ and } p \cdot q \leqslant r, \text{ then } r = \infty.$$

Remark 3.1.20. Ono and Komori (see [OK85] Section 6) proved that the propositional version of these models generalizes the usual Intuitionistic Propositional Kripke Models.

Definition 3.1.21 ([Ono85], p. 189)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ an R-Kripke $\mathcal{L}$-model and an $R - \mathcal{L}_{\mathcal{A}}$-sentence φ, we say that φ is *true* in the structure $\mathcal{A}$ (denoted by $\mathcal{A} \models \varphi$) if for all $p \in \mathbb{P}$, $\mathcal{A} \Vdash_p \varphi$.

Theorem 3.1.22 (Completeness and Soundness Theorem for the Kripke semantics, [Ono85] Lemma 2.2 and Theorem 2.3)**.** Given an $R - \mathcal{L}$-theory T and φ an $R - \mathcal{L}$-sentence, we have that $T \vdash_r \varphi$, if and only if, for every R-Kripke $\mathcal{L}$-model $\mathcal{A}$, if $\mathcal{A} \models T$, then $\mathcal{A} \models \varphi$.

Now we proceed to analyze the behavior of the sets of conditions that force a given formula in the context of these models.

Definition 3.1.23 ([OK85], p. 194)**.** We say that $A \subseteq \mathbb{P}$ is $\cap$-*closed* A if A is hereditary and closed under (finite) meets, that is:

1. For all $a \in A$ and $b \in \mathbb{P}$, if $a \leqslant b$, then $b \in A$.

2. For all $a, b \in A$, $a \wedge b \in A$.

We denote

$$D(\mathbb{P}) := \{A \subseteq \mathbb{P} : A \text{ is } \cap\text{-closed}\}$$

Ono and Komori work with the notion of $\cap$-closed in [OK85] for the propositional case. Since we work in the predicate version of this kind of logics, we have to change the notion of $\cap$-closed to a new notion that we called **strongly hereditary** set. This had to be done, since in the case of the Predicate Logic the set of elements of $\mathbb{P}$ that force a formula are characterized by a stronger assumption.

Definition 3.1.24. We say that a **non-empty** set $A \subseteq \mathbb{P}$ is *strongly hereditary* if for all $c_i \in A$ and $d \in \mathbb{P}$ for $i \in I$, if $\bigwedge_{i \in I} c_i \leqslant d$, then $d \in A$. Notice that from the definition is clear that is A is strongly hereditary, then A is hereditary. We denote

$$\mathbb{P}^* := \{A \subseteq \mathbb{P} : A \text{ is strongly hereditary}\}$$

As we mentioned before, we need to work with the notion of strongly hereditary and not with Ono's notion of $\cap$-closed since the set

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$$

is strongly hereditary and not just $\cap$-closed for a given $R - \mathcal{L}$-sentence φ.

Definition 3.1.25 (cf. [OK85], p. 194)**.** Let $(\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ be a complete SO-commutative monoid. Take $A, B, A_i \subseteq \mathbb{P}$, with $i \in I$ and $x \in \mathbb{P}$. We define

$$A \cdot B := \{c \in \mathbb{P} : \text{ there exist } a \in A, b \in B \text{ such that } c \geqslant a \cdot b\}$$

$$x \cdot A := \{x\} \cdot A = \{c \in \mathbb{P} : \text{ there exists } a \in A \text{ such that } c \geqslant a \cdot x\}$$

$$A \to B := \{c \in \mathbb{P} : c \cdot A \subseteq B\}$$

$$0_{\mathbb{P}*} := \{\infty\}.$$

$$1_{\mathbb{P}*} := \mathbb{P}.$$

$$A \vee B := \{c \in \mathbb{P} : \text{ there exist } a, b \in A \cup B \text{ such that } c \geqslant a \wedge b\}$$

$$A \wedge B := A \cap B.$$

$$\bigvee_{i\in I} A_i := \{c \in \mathbb{P} : \text{ there exists an index set } J \text{ such that for all } j \in J, \text{ there are } a_j \in \bigcup_{i\in I} A_i \text{ such that } \bigwedge_{j\in J} a_j \leqslant c\}$$

$$\bigwedge_{i\in I} A_i := \bigcap_{i\in I} A_i$$

Our definition of the arbitrary join differs from the one given in [OK85]. Ono and Komori defined the arbitrary join operation as follows:

$$\bigvee_{i\in I} A_i := \{c \in \mathbb{P} : \text{ there exists a } \textbf{finite} \text{ index set } J \text{ such that for all } j \in J, \text{ there are } a_j \in \bigcup_{i\in I} A_i \text{ such that } \bigwedge_{j\in J} a_j \leqslant c\}$$

We had to change this, since we are working on the set $\mathbb{P}^*$ and not on the set $D(\mathbb{P})$, which happens to be larger, and therefore the definition of joins might not coincide.

On the other hand, Ono and Komori proved that $D(\mathbb{P})$ is a complete full BCK-algebra (see [OK85] for a definition of this algebra) with the operations they defined. Since a complete full BCK-algebra happens to be a complete Residuated Lattice, we use their ideas to prove that the set $\mathbb{P}^*$ is a complete Residuated Lattice with the operations that we defined above.

Theorem 3.1.26 (cf. [OK85], Lemma 8.3)**.** Let $(\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ be a complete SO-commutative monoid. Then, $\mathbb{P}^*$ endowed with the operations of Definition 3.1.25 and the order $\subseteq$ forms a complete Residuated Lattice.

Proof. We start by showing that $\mathbb{P}^*$ is closed under the operations we defined in Definition 3.1.25. Consider $A, B, A_i \in \mathbb{P}^*$, with $i \in I$.

1. Take $c_i \in A \cdot B$, $d \in \mathbb{P}$ for $i \in I$ and assume that $\bigwedge_{i\in I} c_i \leqslant d$. Since $c_i \in A \cdot B$, there are $a_i \in A$ and $b_i \in B$ such that $a_i \cdot b_i \leqslant c_i$. Then, $\bigwedge_{i\in I}(a_i \cdot b_i) \leqslant \bigwedge_{i\in I} c_i \leqslant d$. Notice that

$\left(\bigwedge_{i\in I} a_i\right)\cdot\left(\bigwedge_{i\in I} b_i\right) \leqslant \bigwedge_{i\in I}(a_i\cdot b_i)$, since, by monotonicity, $\left(\bigwedge_{i\in I} a_i\right)\cdot\left(\bigwedge_{i\in I} b_i\right) \leqslant (a_i\cdot b_i)$ for all $i\in I$. Now, since A and B are strongly hereditary, we have that $\bigwedge_{i\in I} a_i \in A$ and $\bigwedge_{i\in I} b_i \in B$ and we can conclude that $d\in A\cdot B$.

2. Take $c_i \in A\to B$ for $i\in I$, $d\in\mathbb{P}$ and assume that $\bigwedge_{i\in I} c_i \leqslant d$. Since $c_i\in A\to B$, $c_i\cdot A\subseteq B$. Take $c\in d\cdot A$, and $a\in A$ such that $d\cdot a\leqslant c$. Then, $\bigwedge_{i\in I}(c_i\cdot a) = \left(\bigwedge_{i\in I} c_i\right)\cdot a =\leqslant d\cdot a\leqslant c$, that is, $\bigwedge_{i\in I}(c_i\cdot a)\leqslant c$. Notice that $c_i\cdot a\in c_i\cdot A\subseteq B$ and since B is strongly hereditary and $\bigwedge_{i\in I}(c_i\cdot a)\leqslant c$, we can conclude that $c\in B$. This shows that $c\in d\cdot A\subseteq B$ and therefore $d\in A\to B$.

3. Take $c_j\in \bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$, $d\in\mathbb{P}$ for $j\in J$ and assume that $\bigwedge_{j\in J} c_j\leqslant d$. Since $c_j\in A_i$ and A_i is strongly hereditary for each $i\in I$, we have that $d\in A_i$ for every $i\in I$, that is, $d\in\bigcap_{i\in I} A_i$.

4. Take $c_j\in\bigvee_{i\in I} A_i$, $d\in\mathbb{P}$ for $j\in J$ and assume that $\bigwedge_{j\in J} c_j\leqslant d$. Since $c_j\in\bigvee_{i\in I} A_i$, there exists K_j such that for all $k\in K_j$, there is $a_k\in\bigcup_{i\in I} A_i$ such that $\bigwedge_{k\in K_i} a_k\leqslant c_j$. Consider $K:=\bigcup_{i\in I} K_i$, then, $\bigwedge_{k\in K} a_k\leqslant\bigwedge_{j\in J} c_j\leqslant d$ and we conclude that $d\in\bigvee_{i\in I} A_i$.

By the definition of $\mathbb{P}^*$ and since the order is $\subseteq$, is clear that $0_{\mathbb{P}*} := \{\infty\}$ and $1_{\mathbb{P}*} := \mathbb{P}$. Let us see that $(\mathbb{P}^*,\cdot,\mathbb{P})$ is a commutative monoid. The commutativity of $(\mathbb{P}^*,\cdot)$ follows from the commutativity of $(\mathbb{P},\cdot)$. The identity of $(\mathbb{P}^*,\cdot)$ is $\mathbb{P}$, since $1\in\mathbb{P}$. For the associativity of the product, take $z\in A\cdot(B\cdot C)$, therefore, there exist $a\in A$ and $y\in(B\cdot C)$ such that $z\geqslant a\cdot y$. Then, by definition of $B\cdot C$, there exist $b\in B, c\in C$ such that $y\geqslant b\cdot c$, by monotonicity, we get $z\geqslant a\cdot y\geqslant a\cdot b\cdot c$, then, if we take $x=a\cdot b$, we have that $x\in A\cdot B$ and then, $z\in(A\cdot B)\cdot C$, i.e. $A\cdot(B\cdot C)\subseteq(A\cdot B)\cdot C$. The converse is proved similarly.

To see the adjunction condition, we have to prove that

$$A \cdot B \subseteq C, \text{ if and only if, } A \subseteq B \to C$$

1. Assume that $A \cdot B \subseteq C$ and take $x \in A$. Notice that the hypothesis implies that $x \cdot B \subseteq C$ and therefore $x \in B \to C$

2. Assume that $A \subseteq B \to C$ and take $x \in A \cdot B$. Then, there are $a \in A$ and $b \in B$ such that $a \cdot b \leqslant x$. Since $a \in A$ and $A \subseteq B \to C$, we deduce that $a \in B \to C$, that is, $a \cdot B \subseteq C$. Therefore, since $x \in a \cdot B$, be conclude that $x \in C$.

All we have left to see is that

$$\bigvee_{i \in I} A_i = \{c \in \mathbb{P} : \text{ there exists an index set } J \text{ such that for all } j \in J, \text{ there are } a_j \in \bigcup_{i \in I} A_i \text{ such that } \bigwedge_{j \in J} a_j \leqslant c\}$$

is in fact the $\subseteq$-supremum of the set $\{A_i : i \in I\}$. Clearly, $A_i \subseteq \bigvee_{i \in I} A_i$ for every $i \in I$. Now, take $B \in \mathbb{P}^*$ an $\subseteq$-upper bound for $\{A_i : i \in I\}$. Let $x \in \bigvee_{i \in I} A_i$, then, there exists J such that for each $j \in J$ there is $a_j \in \bigcup_{i \in I} A_i$ such that $\bigwedge_{j \in J} a_j \leqslant x$. Since $a_j \in \bigcup_{i \in I} A_i \subseteq B$, if follows that $a_j \in B$ and since B is strongly hereditary, we conclude that $x \in B$. Thus, $\bigvee_{i \in I} A_i \subseteq B$.

□

Lemma 3.1.27 ([Ono85], Lemma 2.1)**.** For every $R - \mathcal{L}_{\mathcal{A}}$-sentence φ and $a_i, b \in \mathbb{P}$, with $i \in I$. If $\mathcal{A} \Vdash_{a_i} \varphi$ for every $i \in I$ and $\bigwedge_{i \in I} a_i \leqslant b$, then $\mathcal{A} \Vdash_b \varphi$. Notice that this lemma implies that the set $\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$ is strongly hereditary.

Theorem 3.1.28 (cf. [Fit69])**.** Take a R-Kripke $\mathcal{L}$-model $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$. If φ and ψ are $R - \mathcal{L}_{\mathcal{A}}$-sentences and $\theta(x)$ is an $R - \mathcal{L}_{\mathcal{A}}$-formula, then

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi \& \psi\}$$

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \vee \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi \vee \psi\}$$

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cap \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi \wedge \psi\}$$

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \to \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi \to \psi\}$$

$$\sim \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \sim \varphi\}$$

$$\bigcap_{d \in D} \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \theta(d)\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \forall x \theta(x)\}$$

$$\bigvee_{d \in D} \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \theta(d)\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \exists x \theta(x)\}$$

Proof. Take $p \in \mathbb{P}$.

1. Suppose that $\mathcal{A} \Vdash_p \varphi \& \psi$, that is, there are $q, r \in \mathbb{P}$ such that $p \geqslant q \cdot r$, $\mathcal{A} \Vdash_q \varphi$ and $\mathcal{A} \Vdash_r \psi$, but this is equivalent to $p \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$.

2. Suppose that $\mathcal{A} \Vdash_p \varphi \vee \psi$, that is, there are $q, r \in \mathbb{P}$ such that $p \geqslant q \wedge r$ and both ($\mathcal{A} \Vdash_q \varphi$ or $\mathcal{A} \Vdash_q \psi$) and ($\mathcal{A} \Vdash_r \varphi$ or $\mathcal{A} \Vdash_r \psi$), that is, both ($q \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cup \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$) and ($r \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cup \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$), that is, $r, q \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \cup \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$ and this the definition of $p \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \vee \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$.

3. The conjunction and universal quantifier cases are straightforward.

4. Suppose that $\mathcal{A} \Vdash_p (\varphi \to \psi)$, that is, for all $q, r \in \mathbb{P}$ if $\mathcal{A} \Vdash_q \varphi$ and $p \cdot q \leqslant r$, then $\mathcal{A} \Vdash_r \psi$. This means that for all $q, r \in \mathbb{P}$ if $q \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$ and $p \cdot q \leqslant r$, then $r \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$, that is, for all $r \in \mathbb{P}$, if $r \in p \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$, then $r \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$, but this is equivalent to $p \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \subseteq \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$, but this is equivalent to $p \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \to \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \psi\}$.

5. By the previous item, if we take $\psi = \bot$, we get

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \to \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \bot\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi \to \bot\}.$$

Then, by Definition 3.1.17 item 3. a., and by definition of $\sim \varphi$, we have

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \to \{\infty\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \sim \varphi\},$$

but $\{\infty\} = 0_{\mathbb{P}*}$. Thus

$$\sim \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \sim \varphi\}$$

6. Suppose that $\mathcal{A} \Vdash_p \exists x\theta(x)$, that is, there exists an index set I such that for for every $i \in I$, there exists $d_i \in D$ and $q_i \in \mathbb{P}$ such that $\bigwedge\limits_{i \in I} q_i \leqslant p$ and $\mathcal{A} \Vdash_{q_i} \theta(d_i)$, that is, for every $i \in I$, there are $q_i \in \bigcup\limits_{d \in D} \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \theta(d)\}$ such that $\bigwedge\limits_{i \in I} q_i \leqslant p$ but this the definition of $p \in \bigvee\limits_{d \in D} \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \theta(d)\}$.

□

3.1.4 Residuated Kripke models with modal operators

Throughout this subsection, $\mathbb{Q}$ denotes a **complete Residuated Lattice**.

This section is inspired in the work of Lano [Lan92a], where he considers a special kind of **interior operator** I on a residuated lattice (structure that he calls **Topological Residuated Lattice**) and constructs models of *Residuated Modal Set Theory*. He introduces an operator of **necessity** (denoted $\square$) to be interpreted by an interior operator I and then provides an axiomatization of this logic and proves some results for the model of Set Theory that he constructs.

Unlike Lano, we consider a special kind of **closure operator** (a **standard quantic nucleus**) on our residuated lattice and develop a similar kind of model where we augment the logic with a **possibility** operator (denoted $\Diamond$). This is done since the order that Lano considers for his models preserves truth downwards (following Cohen's convention) and not upwards (following Kripke's). See Subsection 3.1.2 for a more in-depth discussion.

Thus, we would like to extend the definition of Kripke model so that we can interpret formulas of the form $\Diamond\varphi$. To get an idea of how we should define them, we start from the more natural definition of $\Diamond\varphi$ in a valued model. So consider a $\mathbb{Q}$-valued modal model $\mathcal{M}$ and a quantic nucleus γ on $\mathbb{Q}$ (see Definition 1.2.43).

Then, by definition (see Subsection 2.1.4), we have that

$$[\![\Diamond\varphi]\!] := \gamma([\![\varphi]\!]).$$

Let us assume that the formula φ satisfies the relation

$$\mathcal{A} \Vdash_p \varphi \text{ iff } p \leqslant [\![\varphi]\!] \text{ (induction hypothesis)}$$

and let us try to define $\mathcal{A} \Vdash_p \Diamond\varphi$ using only the forcing relation $\Vdash$ and the quantic nucleus on γ by using the relation given by Theorem 3.1.11. Therefore, we want to prove the following equivalences:

$$
\begin{aligned}
\mathcal{A} \Vdash_p \Diamond\varphi \text{ iff } & p \leqslant [\![\Diamond\varphi]\!] && \text{(we want Thm. 3.1.10 to hold)}\\
\text{iff } & p \leqslant \gamma([\![\varphi]\!]) && \text{(by definition of } [\![\Diamond\varphi]\!])\\
\text{iff } & \text{there exists } q \in \mathbb{P} \text{ such that } p \leqslant \gamma(q) \text{ and } q \leqslant [\![\varphi]\!] && \text{(since } \gamma \text{ is monotone)}\\
\text{iff } & \text{there exists } q \in \mathbb{P} \text{ such that } p \leqslant \gamma(q) \text{ and } \mathcal{A} \Vdash_q \varphi && \text{(by induction hypothesis)}
\end{aligned}
$$

Therefore, we can define $\mathcal{A} \Vdash_p \Diamond\varphi$ only in terms of forcing and some nucleus γ on $\mathbb{P}$. But since the order of the underlining order of the Kripke model is reversed in Theorem 3.1.11, we need to reverse the order in last line of equivalences above. Also, we have to reverse the order in the conditions defining a quantic nucleus. That lead us to the following definition.

Definition 3.1.29. We say that a function $\delta : \mathbb{P} \to \mathbb{P}$ is a *conucleus* on a complete SO-monoid $(\mathbb{P}, \leqslant, \cdot)$ if for all $p, q, p_i \in \mathbb{P}$ with $i \in I$

1. $\delta(p) \leqslant p$.
2. If $p \leqslant q$, then $\delta(p) \leqslant \delta(q)$.
3. $\delta(\delta(p)) = \delta(p)$.
4. $\delta(p \cdot q) \leqslant \delta(p) \cdot \delta(q)$.
5. $\delta(\bigwedge_{i\in I} p_i) = \bigwedge_{i\in I} \delta(p_i)$

Remark 3.1.30. Notice that the conditions $1 - 4$ are the dual ones of the conditions in the definition of a quantic nucleus (see Definitions 1.2.15 and 1.2.43) and therefore δ is a **interior operator** rather than a closure one. The condition 5 was added to been able to prove that:

1. The set $\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$ is strongly hereditary for every MR-formula φ,
2. the operation γ_δ (we will define this operation in Theorem 3.1.37) is well defined on the set $\mathbb{P}^*$.

Remark 3.1.31. Notice that since δ distributes over arbitrary meets, by the Adjoint Functor Theorem for preorders (Theorem 1.2.5), there exists an operator $\rho : \mathbb{P} \to \mathbb{P}$ that is the left adjoint of δ. As it is usual in Modal Logic (for example in S_5), the necessity and the possibility form an adjoint pair. Therefore, we may use the operator ρ to define a notion of **necessity** in our Kripke models. We will not do so in our work, since there is no real need for this operator for our results.

Remark 3.1.32. This notion of conucleus is different the notion defined by Rosenthal in [Ros90]. A conucleus in [Ros90] is an interior operator that satisfies the condition

$$\delta(p) \cdot \delta(q) \leqslant \delta(p \cdot q).$$

which is the opposite of what we require.

Definition 3.1.33. We say that $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty, \delta)$ is a *complete modal SO-commutative monoid* if

1. $(\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ is a complete SO-commutative monoid.

2. δ is a conucleus on $(\mathbb{P}, \leqslant, \cdot)$.

Definition 3.1.34 (Modal Residuated Kripke model, cf. [Ono85] p. 189)**.** We say that $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ is a *Modal Residuated Kripke $\mathcal{L}$-model* (or MR-Kripke $\mathcal{L}$-model, for short) if

1. $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty, \delta)$ is a complete **modal** SO-commutative monoid.

2. $\Vdash$ is a relation between elements of $\mathbb{P}$ and atomic sentences in the language

$$\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup D,$$

   where each element of D is considered as a constant symbol. We denote $(p, \varphi) \in \Vdash$ as $\mathcal{A} \Vdash_p \varphi$.

3. Given $p_i, q \in \mathbb{P}$, with $i \in I$ and φ an atomic $\mathcal{L}_{\mathcal{A}}$-sentence, we require that $\Vdash$ satisfy the following conditions:

   a. If $\bigwedge_{i \in I} p_i \leqslant q$ and for each $i \in I$ $\mathcal{A} \Vdash_{p_i} \varphi$, then $\mathcal{A} \Vdash_q \varphi$.

   b. $\mathcal{A} \Vdash_\infty \varphi$ for every atomic $\mathcal{L}_{\mathcal{A}}$-sentence.

   c. $\mathcal{A} \Vdash_p \bot$, if and only if, $p = \infty$. Recall that $\bot$ is the symbol of contradiction.

We propose now an extension of the forcing relation based on the ideas we exposed in the introduction of this subsection.

Definition 3.1.35 (Modal Residuated Kripke forcing)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ a MR-Kripke $\mathcal{L}$-model, $p \in \mathbb{P}$ and an $MR - \mathcal{L}_{\mathcal{A}}$-sentence φ, we can extend the forcing relation $\mathcal{A} \Vdash_p \varphi$ to all $MR - \mathcal{L}_{\mathcal{A}}$-sentences by recursion on the complexity of φ. The definition of $\Vdash$ for the usual symbols is the same as in Definition 3.1.18, so we only define $\Diamond$.

1. $\mathcal{A} \Vdash_p \Diamond\varphi$, if and only if, there exists $q \in \mathbb{P}$ such that $\mathcal{A} \Vdash_q \varphi$ and $\delta(q) \leqslant p$.

Notice that the definition of $\Diamond$ depends on δ, so if we change the conucleus δ, we would obtain different notions of possibility $\Diamond$.

Lemma 3.1.36 (cf. [Ono85] Lemma 2.1)**.** Let φ be an $MR - \mathcal{L}_{\mathcal{A}}$-sentence and $a_i, b \in \mathbb{P}$, with $i \in I$. If $\mathcal{A} \Vdash_{a_i} \varphi$ for every $i \in I$ and $\bigwedge_{i \in I} a_i \leqslant b$, then $\mathcal{A} \Vdash_b \varphi$. Notice that this lemma implies that the set

$$\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}$$

is strongly hereditary.

Proof. By Lemma 3.1.27, we just need to see what happens with $\Diamond\varphi$ for a given φ such that

$$\text{for all } p, p_i \in \mathbb{P} \text{ with } i \in I, \text{ if } \mathcal{A} \Vdash_{p_i} \varphi \text{ and } \bigwedge_{i\in I} p_i \leqslant p, \text{ then } \mathcal{A} \Vdash_p \varphi \text{ (induction hypothesis).}$$

Take $p, p_i \in \mathbb{P}$ with $i \in I$ such that $\mathcal{A} \Vdash_{p_i} \Diamond\varphi$ and $\bigwedge_{i\in I} p_i \leqslant p$. By definition of $\Vdash$, there exists $q_i \in \mathbb{P}$ such that

$$\delta(q_i) \leqslant p_i \text{ and } \mathcal{A} \Vdash_{q_i} \varphi$$

By the induction hypothesis, we have $\mathcal{A} \Vdash_q \varphi$, where $q = \bigwedge_{i\in I} q_i$. Notice that

$$\begin{aligned}
\delta(q) = \delta\left(\bigwedge_{i\in I} q_i\right) &= \bigwedge_{i\in I} \delta(q_i) \quad && \text{(by Definition 3.1.29 item 5.)} \\
&\leqslant \bigwedge_{i\in I} p_i && \text{(since } \delta(q_i) \leqslant p_i \text{ for all } i \in I) \\
&\leqslant p
\end{aligned}$$

Then, by definition of $\Vdash$, we conclude that $\mathcal{A} \Vdash_p \Diamond\varphi$ □

Theorem 3.1.37. The operation $\gamma_\delta : \mathbb{P}^* \to \mathbb{P}^*$ defined as

$$\gamma_\delta(A) := \{p \in \mathbb{P} : \text{there is } q \in A \text{ such that } \delta(q) \leqslant p\}$$

is a quantic nucleus on $(\mathbb{P}^*, \subseteq, \cdot)$. If there is no ambiguity, we denote $\gamma := \gamma_\delta$.

Proof. Take $A \in \mathbb{P}^*$. We want to see that $\gamma(A) \in \mathbb{P}^*$, so let us take $b_i \in \gamma(A)$ with $i \in I$ and $p \in \mathbb{P}$ such that $\bigwedge_{i\in I} b_i \leqslant p$. Then, by definition of γ, there exists $a_i \in A$ such that $\delta(a_i) \leqslant b_i$. Then, by monotonicity and idempotence of δ, $\delta(a_i) = \delta(\delta(a_i)) \leqslant \delta(b_i)$. Notice now that

$$\begin{aligned}
\delta\left(\bigwedge_{i\in I} a_i\right) &= \bigwedge_{i\in I} \delta(a_i) && \text{(by Definition 3.1.29 item 5.)}\\
&\leqslant \bigwedge_{i\in I} \delta(b_i) && \text{(since } \delta(a_i) \leqslant \delta(b_i) \text{ for all } i \in I\text{)}\\
&= \delta(\bigwedge_{i\in I} b_i) && \text{(by Definition 3.1.29 item 5.)}\\
&\leqslant \delta(p) && \text{(since } \bigwedge_{i\in I} b_i \leqslant p \text{ and } \delta \text{ is a monotone function)}\\
&\leqslant p && \text{(by Definition 3.1.29 item 1.)}
\end{aligned}$$

Since A is strongly hereditary, $\bigwedge_{i\in I} a_i \in A$. Therefore, it follows that $p \in \gamma(A)$ and we conclude that $\gamma(A) \in \mathbb{P}^*$. We proceed to prove that γ is a quantic nucleus:

1. We want to see that $A \subseteq \gamma(A)$. Take $p \in A$. Since $\delta(p) \leqslant p$ and by definition of γ, $p \in \gamma(A)$.

2. We want to see that if $A \subseteq B$, then $\gamma(A) \subseteq \gamma(B)$. Assume that $A \subseteq B$ and take $p \in \gamma(A)$. Then, there exists $q \in A$ such that $\delta(q) \leqslant p$, but since $A \subseteq B$ we have that $q \in B$ and this implies that $p \in \gamma(B)$.

3. We want to see that $\gamma(\gamma(A)) = \gamma(A)$. By the item 1. on this list,

$$\gamma(A) \subseteq \gamma(\gamma(A)).$$

To see the other inequality, take $p \in \gamma(\gamma(A))$. Then, there exists $q \in \gamma(A)$ with $\delta(q) \leqslant p$, and thus there exists $r \in A$ with $\delta(r) \leqslant q$. Therefore,

$$\delta(r) = \delta(\delta(r)) \leqslant \delta(q) \leqslant p,$$

that is, $\delta(r) \leqslant p$ and we conclude that $p \in \gamma(A)$.

4. We want to see that $\gamma(A) \cdot \gamma(B) \subseteq \gamma(A \cdot B)$ so let us take $p \in \gamma(A) \cdot \gamma(B)$. Then, there exists $a \in \gamma(A)$ and $b \in \gamma(B)$ such that $a \cdot b \leqslant p$. Thus, there exists $q_a \in A$ and $q_b \in B$ such that $\delta(q_a) \leqslant a$ and $\delta(q_b) \leqslant b$. Notice that, by the monotonicity of $\cdot$,

$$\delta(q_a \cdot q_b) \leqslant \delta(q_a) \cdot \delta(q_b) \leqslant a \cdot b \leqslant p$$

and since $q_a \cdot q_b \in A \cdot B$, we conclude that $p \in \gamma(A \cdot B)$

□

Lemma 3.1.38. Let $A_i \in \mathbb{P}^*$ for $i \in I$. The quantic nucleus γ satisfies

$$\gamma(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} \gamma(A_i)$$

Proof. Take $p \in \gamma(\bigvee_{i \in I} A_i)$. Therefore,

$$\text{there exists } q \in \bigvee_{i \in I} A_i \text{ such that } \delta(q) \leqslant p.$$

And then, by definition of $\bigvee$ (see Definition 3.1.25), we have that

there exists an index set J such that for all $j \in J$ there exists $a_j \in \bigcup_{i \in I} A_i$ such that $\bigwedge_{j \in J} a_j \leqslant q$.

Now, since $\bigwedge_{j \in J} a_j \leqslant q$ and by Definition 3.1.29 item 5., we have that

$$\bigwedge_{j \in J} \delta(a_j) = \delta(\bigwedge_{j \in J} a_j) \leqslant \delta(q) \leqslant p$$

Therefore, since $\delta(a_j) \in \bigcup_{i \in I} \gamma(A_i)$, we conclude that $p \in \bigvee_{i \in I} \gamma(A_i)$.

On the other hand, if $p \in \bigvee_{i \in I} \gamma(A_i)$, we have that

there exists an index set J such that for all $j \in J$ there exists $a_j \in \bigcup_{i \in I} \gamma(A_i)$ such that

$$\bigwedge_{j \in J} a_j \leqslant p$$

Then, for every $j \in J$, there exist $i_j \in I$ such that $a_j \in \gamma(A_{i_j})$. Then, by definition of γ, there exist $q_j \in A_{i_j}$ such that $\delta(q_j) \leqslant a_j$ for every $j \in J$. This implies that

$$\begin{aligned}\delta(\bigwedge_{j\in j} q_j) &= \bigwedge_{j\in j} \delta(q_j) && \text{(by Definition 3.1.29 item 5.)}\\ &\leqslant \bigwedge_{j\in j} a_j && \text{(since } \delta(q_j) \leqslant a_j \text{ for every } j \in J)\\ &\leqslant p.\end{aligned}$$

Notice that $\bigwedge_{j\in j} q_j \in \bigvee_{i\in I} A_i$, and therefore $p \in \gamma(\bigvee_{i\in I} A_i)$.

□

Remark 3.1.39. Notice that this lemma implies, by the Adjoint Functor Theorem for preorders (Theorem 1.2.5), that γ has a right adjoint. This adjoint may be used to define a notion of necessity just as we mentioned before.

Definition 3.1.40. Let $\delta : \mathbb{P} \to \mathbb{P}$ be a conucleus.

1. δ is said to be *idempotent* if $\delta(p^2) := \delta(p \cdot p) = \delta(p)$, for every $p \in \mathbb{P}$.

2. δ is said to *respects the top element* if $\delta(\infty) = \infty$.

3. δ is said to *respect implications* if $\delta(p \to q) = 1$, if and only if, $p \to \delta(q) = 1$ for every $p, q \in \mathbb{P}$ (see Remark 3.1.15 for the definition of $\to$).

Theorem 3.1.41. Let $\delta : \mathbb{P} \to \mathbb{P}$ be a conucleus.

1. If δ is idempotent, then γ is idempotent.

2. If δ respects the top element, then γ respects the bottom element.

3. If δ respects implications, then γ respects implications.

Proof. Take $A, B \in \mathbb{P}^*$.

1. Assume that δ is idempotent. We want to see that $\gamma(A \cdot A) = \gamma(A)$. Since $(\mathbb{P}^*, \subseteq, \cdot)$ is a commutative integral quantale, by Theorem 1.2.33 item 6, $A \cdot A \subseteq A$ and then, since γ is monotone,

$$\gamma(A \cdot A) \subseteq \gamma(A).$$

Now take $p \in \gamma(A)$. Then, there exists $q \in A$ such that $\delta(q) \leqslant p$. Since δ is idempotent, $\delta(q \cdot q) = \delta(q) \leqslant p$. Therefore, since $q \cdot q \in A \cdot A$, we have that $p \in \gamma(A \cdot A)$, that is,

$$\gamma(A) \subseteq \gamma(A \cdot A).$$

2. Recall that $0_{\mathbb{P}^*} = \{\infty\}$, therefore

$$\begin{aligned}\gamma(\{\infty\}) &= \{p \in \mathbb{P} : \text{ there exists } q \in \{\infty\}(\delta(q) \leqslant p)\} = \{p \in \mathbb{P} : \delta(\infty) \leqslant p\} \\ &= \{p \in \mathbb{P} : \infty \leqslant p\} \qquad \text{(since } \delta \text{ respects the top element)} \\ &= \{\infty\}\end{aligned}$$

3. Assume now that δ respects implications. Since $1_{\mathbb{P}^*} = \mathbb{P}$, we want to prove that

$$\gamma(A \to B) = \mathbb{P}, \text{ if and only if, } A \to \gamma(B) = \mathbb{P}.$$

Since γ is a quantic nucleus on $\mathbb{P}^*$, by Corollary 1.2.47, we have that

$$\gamma(A \to B) \subseteq A \to \gamma(B).$$

And therefore $\gamma(A \to B) = \mathbb{P}$ implies that $A \to \gamma(B) = \mathbb{P}$.

Let us see converse. Let us assume that $A \to \gamma(B) = \mathbb{P}$.

Take $c \in \mathbb{P} = A \to \gamma(B)$, that is, $c \cdot A \subseteq \gamma(B)$. This means that for every $a \in A$, there exists $b_a \in B$ such that $\delta(b_a) \leqslant c \cdot a$. Notice now that

$$\begin{aligned}
\delta(b_a) \leqslant (c \cdot a) \cdot 1 \text{ iff } (c \cdot a) \to \delta(b_a) \leqslant 1 & \qquad \text{(by Theorem 3.1.16 item 1.)} \\
\text{iff } \delta((c \cdot a) \to b_a) = 1 & \qquad \text{(since } \delta \text{ respects implications)} \\
\text{iff } \delta(c \to (a \to b_a)) = 1 & \qquad \text{(by Theorem 3.1.16 item 5.)} \\
\text{iff } c \to \delta(a \to b_a) = 1 & \qquad \text{(since } \delta \text{ respects implications)} \\
\text{iff } \delta(a \to b_a) \leqslant c & \qquad \text{(by Theorem 3.1.16 item 1.)}
\end{aligned}$$

If we find some $a \in A$ such that $a \to b_a \in A \to B$, we get that $c \in \gamma(A \to B)$. Thus, take an enumeration $A = \{a_i : i \in I\}$ and define

$$a = \bigwedge_{i \in I} a_i.$$

Since A is strongly hereditary, $a \in A$. Let us see that $a \to b_a \in A \to B$, that is, $(a \to b_a) \cdot A \subseteq B$. So take any $j \in I$ and let us show that $(a \to b_a) \cdot a_j \in B$.

Since $a = \bigwedge_{i \in I} a_i \leqslant a_j$, by Theorem 3.1.16 item 3., we deduce that $a_j \to b_a \leqslant a \to b_a$. Therefore,

$$\begin{aligned}
b_a &\leqslant (a_j \to b_a) \cdot a_j & \quad \text{(by Theorem 3.1.16 item 2.)} \\
&\leqslant (a \to b_a) \cdot a_j & \quad (\cdot \text{ is a monotone function})
\end{aligned}$$

and since B is strongly hereditary and $b_a \in B$, we deduce $(a \to b_a) \cdot a_j \in B$. Then, since $a \to b_a \in A \to B$, we conclude that $c \in \gamma(A \to B)$.

□

Theorem 3.1.42. Take $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ a Modal Residuated Kripke $\mathcal{L}$-model, and an $MR - \mathcal{L}_{\mathcal{A}}$-sentence φ. Then,

$$\gamma(\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}) = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \Diamond\varphi\}$$

Proof. Take $p \in \mathbb{P}$, thus

$$\begin{aligned} p \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \Diamond\varphi\} &\text{ iff } \text{ there is } q \in \mathbb{P} \text{ such that } \delta(q) \leqslant p \text{ and } \mathcal{A} \Vdash_p \varphi && \text{(by definition of } \Vdash) \\ &\text{ iff } \text{ there is } q \in \mathbb{P} \text{ such that } \delta(q) \leqslant p \text{ and } q \in \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\} \\ &\text{ iff } p \in \gamma(\{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi\}) && \text{(by definition of } \gamma) \end{aligned}$$

□

3.2 Kripke models of Set Theory

Fitting [Fit69] constructed several models of Set Theory generalizing both the universes of von Neumann and of Gödel using Kripke models and then went on to show how these models can be used to obtain **classical** proofs of independence in Set Theory.

In this section, we aim to expose Fitting's results and then to propose some generalizations in the context of Modal Residuated Kripke models.

3.2.1 Intuitionistic Kripke models of set theory

This subsection focuses on Chapter 14 of [Fit69] "Additional classical model generalizations ". Our goal is to highlight some crucial points in Fitting's results so that we can generalize them in the next subsection in the context of Residuated Kripke models. One key aspect to notice on Fitting's results is the use of the **double negation** modal operator $\neg\neg$ to translate sentences from Classical Logic into Intuitionistic Logic (see Definition 1.3.5).

The notion of subset in these models is similar to the notion of generalized subsets we studied in Section 2.2. The key difference is that since we are working with a order $\mathbb{P}$, we need to find a suited Heyting algebra to take as a set of truth values. This is done via the Heyting algebra of hereditary subsets of $\mathbb{P}$.

Definition 3.2.1 ([Fit69], p. 166)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ an Intuitionistic Kripke $\mathcal{L}_\in$-model with constant universe. We say that a function f is a $\mathbb{P}^+$*-subset of* $\mathcal{A}$ if

1. $Dom(f) \subseteq D$

2. $Ran(f) \subseteq \mathbb{P}^+ = \{A \subseteq \mathbb{P} : A \text{ is hereditary}\}$

Recall that $\mathbb{P}^+$ is a Heyting algebra with the order $\subseteq$.

Definition 3.2.2 ([Fit69], p. 166)**.** Recall that an element $A \in \mathbb{P}^+$ is called regular if $\neg\neg A = A$ (see Definition 1.2.19), where $\neg$ is the operation of $\mathbb{P}^+$ as a Heyting algebra. We call a function with range $\mathbb{P}^+$ *regular* if every member of its range is regular.

Remark 3.2.3. Recall that if $\mathbb{H}$ is a Heyting algebra, then the set $\mathbb{B} = \{A \in \mathbb{H} : \neg\neg A = A\}$ is a Boolean algebra.

Definition 3.2.4 ([Fit69], p. 166)**.** Given $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ a Intuitionistic Kripke $\mathcal{L}_{\in}$-model, we say that a function from D to $\mathbb{P}^{+}$ is *extensional* if, for each $g, h \in D$

$$f(g) \cap \{p \in \mathbb{P} : \mathcal{A} \Vdash_p (\neg(\exists x)\neg(x \in g \rightarrow x \in h)) \wedge (\neg(\exists x)\neg(x \in h \rightarrow x \in g))\} \subseteq f(h)$$

and we denote

$$\mathcal{P}^{\mathbb{P}^{+}}(D) := \{f : f \text{ is a regular and extensional } \mathbb{P}^{+}\text{-subset of } \mathcal{A}\}$$

Definition 3.2.5 ([Fit69], p. 166)**.** We now define on induction over the ordinals a class of Intuitionistic Kripke models $\mathcal{V}_{\alpha}^{\mathbb{P}^{+}} := (\mathbb{P}, \leqslant, \Vdash, R_{\alpha}^{\mathbb{P}^{+}})$, all with the same underlying order $(\mathbb{P}, \leqslant)$, but changing the universe for each α.

1. $\mathcal{V}_{0}^{\mathbb{P}^{+}} := (\mathbb{P}, \leqslant, \Vdash, R_{0}^{\mathbb{P}^{+}})$ where $R_{0}^{\mathbb{P}^{+}} := \varnothing$.

2. $\mathcal{V}_{\alpha+1}^{\mathbb{P}^{+}} := (\mathbb{P}, \leqslant, \Vdash, R_{\alpha+1}^{\mathbb{P}^{+}})$ where $R_{\alpha+1}^{\mathbb{P}^{+}} := R_{\alpha}^{\mathbb{P}^{+}} \cup \mathcal{P}^{\mathbb{P}^{+}}(R_{\alpha}^{\mathbb{P}^{+}})$ and $\mathcal{V}_{\alpha+1}^{\mathbb{P}^{+}} \Vdash_p \varphi$ is defined as follows: If $p \in \mathbb{P}$ and $f, g \in R_{\alpha+1}^{\mathbb{P}^{+}}$ then we have the following cases:

   a. If $f, g \in R_{\alpha}^{\mathbb{P}^{+}}$, then

   $$\mathcal{V}_{\alpha+1}^{\mathbb{P}^{+}} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}_{\alpha}^{\mathbb{P}^{+}} \Vdash_p (f \in g)$$

   b. If $f \in R_{\alpha}^{\mathbb{P}^{+}}$ and $g \in R_{\alpha+1}^{\mathbb{P}^{+}} \backslash R_{\alpha}^{\mathbb{P}^{+}} = \mathcal{P}^{\mathbb{P}^{+}}(R_{\alpha}^{\mathbb{P}^{+}})$, then

   $$\mathcal{V}_{\alpha+1}^{\mathbb{P}^{+}} \Vdash_p (f \in g), \text{ if and only if, } p \in g(f).$$

   c. If $f \in R_{\alpha+1}^{\mathbb{P}^{+}} \backslash R_{\alpha}^{\mathbb{P}^{+}} = \mathcal{P}^{\mathbb{P}^{+}}(R_{\alpha}^{\mathbb{P}^{+}})$, then $\mathcal{V}_{\alpha+1}^{\mathbb{P}^{+}} \Vdash_p (f \in g)$, if and only if, there exist $h \in dom(g)$ such that

   $$p \in g(h) \text{ and } p \in (f(x) \leftrightarrow \{q \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}^{+}} \Vdash_q \neg\neg(x \in h)\}) \text{ for every } x \in R_{\alpha}^{\mathbb{P}^{+}}$$

3. If $\alpha \neq 0$ is a limit ordinal, then let $R_{\alpha}^{\mathbb{P}^{+}} := \bigcup_{\beta<\alpha} R_{\beta}^{\mathbb{P}^{+}}$ and given $f, g \in R_{\alpha}^{\mathbb{P}^{+}}$ take any $\eta < \alpha$ such that $f, g \in R_{\eta}^{\mathbb{P}}$ and let

$$\mathcal{V}_\alpha^{\mathbb{P}+} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}_\eta^{\mathbb{P}+} \Vdash_p (f \in g).$$

Remark 3.2.6 ([Fit69], Remark 4.2)**.** The expression

$$f(x) \leftrightarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}+} \Vdash_q \neg\neg(x \in h)\}$$

is an element in the Heyting algebra $\mathbb{P}^+$, where $\leftrightarrow$ is the operation in $\mathbb{P}^+$.

Definition 3.2.7 ([Fit69], p. 166)**.** Consider the Intuitionistic Kripke (class) $\mathcal{L}_\in$-model

$$\mathcal{V}^{\mathbb{P}+} := (\mathbb{P}, \leqslant, R^{\mathbb{P}+}), \text{ where } R^{\mathbb{P}+} := \bigcup_{\alpha \in ON} R_\alpha^{\mathbb{P}+},$$

and given $f, g \in R^{\mathbb{P}+}$ take any $\eta \in ON$ such that $f, g \in R_\eta^{\mathbb{P}}$ and let

$$\mathcal{V}^{\mathbb{P}+} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}_\eta^{\mathbb{P}+} \Vdash_p (f \in g).$$

We now construct a Boolean valued model $(R^{\mathbb{B}}, [\![\cdot]\!]^{\mathbb{B}})$ that is deeply related with $\mathcal{V}^{\mathbb{P}+} = (\mathbb{P}, \leqslant, \Vdash, R^{\mathbb{P}+})$.

Definition 3.2.8 ([Fit69], p. 164)**.** Given a Boolean valued model $(R, [\![\cdot]\!]^{\mathbb{B}})$, we say that a function $f : R \to \mathbb{B}$ is *extensional* if for all $g, h \in R$,

$$f(g) \wedge [\![(\forall x)(x \in g \leftrightarrow x \in h)]\!]^{\mathbb{B}} \leqslant f(h)$$

Definition 3.2.9 ([Fit69], p. 165)**.** We define on induction on ordinals a class of Boolean valued models $(R_\alpha^{\mathbb{B}}, [\![\cdot]\!]_\alpha^{\mathbb{B}})$ as follows:

1. $R_0^{\mathbb{B}} := \varnothing$ with $[\![\cdot]\!]_0^{\mathbb{B}} := \varnothing$.

2. $R_{\alpha+1}^{\mathbb{B}} := R_\alpha^{\mathbb{B}} \cup \{f : R_\alpha^{\mathbb{B}} \to \mathbb{B} : f \text{ is extensional}\}$ and given $f, g \in R_{\alpha+1}^{\mathbb{B}}$, we have, for the definition of $[\![f \in g]\!]_\alpha^{\mathbb{B}}$, the following cases:

a. If $f, g \in R^{\mathbb{B}}_{\alpha}$, then $[\![f \in g]\!]^{\mathbb{B}}_{\alpha+1} := [\![f \in g]\!]^{\mathbb{B}}_{\alpha}$.

b. If $f \in R^{\mathbb{B}}_{\alpha}$ and $g \in R^{\mathbb{B}}_{\alpha+1} \backslash R^{\mathbb{B}}_{\alpha}$, then $[\![f \in g]\!]^{\mathbb{B}}_{\alpha+1} := g(f)$.

c. If $f \in R^{\mathbb{B}}_{\alpha+1} \backslash R^{\mathbb{B}}_{\alpha}$, then

$$[\![f \in g]\!]^{\mathbb{B}}_{\alpha+1} := \bigvee_{h \in dom(g)} \{g(h) \wedge \bigwedge_{x \in R^{\mathbb{B}}_{\alpha}} (f(x) \leftrightarrow [\![x \in h]\!]^{\mathbb{B}}_{\alpha})\}$$

d. If $\alpha \neq 0$ is a limit ordinal, then let $R^{\mathbb{B}}_{\alpha} := \bigcup_{\beta < \alpha} R^{\mathbb{B}}_{\beta}$ and given $f, g \in R^{\mathbb{B}}_{\alpha}$ take any $\eta < \alpha$ such that $f, g \in R^{\mathbb{B}}_{\eta}$ and let $[\![f \in g]\!]^{\mathbb{B}}_{\alpha} := [\![f \in g]\!]^{\mathbb{B}}_{\eta}$.

Now let

$$R^{\mathbb{B}} := \bigcup_{\alpha \in ON} R^{\mathbb{B}}_{\alpha}$$

and given $f, g \in R^{\mathbb{B}}$ take any $\eta < \alpha$ such that $f, g \in R^{\mathbb{B}}_{\eta}$ and let $[\![f \in g]\!]^{\mathbb{B}} := [\![f \in g]\!]^{\mathbb{B}}_{\eta}$.

Remark 3.2.10. Recall that an element of $A \in \mathbb{P}^+$ is called **dense** if $\neg\neg A = 1 = \mathbb{P}$. Let $\mathcal{F}_{\neg\neg}$ be the collection of all dense elements of $\mathbb{P}^+$. We know that $\mathcal{F}_{\neg\neg}$ is a filter and that the relation $\approx_{\mathcal{F}_{\neg\neg}}$ given by

$$A \approx_{\mathcal{F}_{\neg\neg}} B, \text{ if and only if, } A \to B \in \mathcal{F}_{\neg\neg} \text{ and } B \to A \in \mathcal{F}_{\neg\neg}$$

is an equivalence relation. With that in mind, let us denote

$$\mathbb{B} := \mathbb{P}^+/\mathcal{F}_{\neg\neg} := \mathbb{P}^+/_{\approx_{\mathcal{F}_{\neg\neg}}} = \{|A| : A \in \mathbb{P}^+\},$$

where

$$|A| := |A|_{\approx_{\mathcal{F}_{\neg\neg}}} \text{ is the } \approx_{\mathcal{F}_{\neg\neg}} \text{ - equivalence class of } A.$$

Recall that by Theorem 1.2.21, $\mathbb{B} := \mathbb{P}^+/\mathcal{F}_{\neg\neg}$ is a complete Boolean algebra. This Boolean algebra determines a sequence of valued models $(R^{\mathbb{B}}_{\alpha}, [\![\cdot]\!]^{\mathbb{B}}_{\alpha})$ that is isomorphic to the sequence $\mathcal{V}^{\mathbb{P}^+}_{\alpha} = (\mathbb{P}, \leqslant, \Vdash, R^{\mathbb{P}^+}_{\alpha})$ in the following way:

Theorem 3.2.11 ([Fit69], Chapter 15, Theorem 5.5)**.** For every $\alpha \in ON$, there exist a bijection $j_\alpha : R_\alpha^{\mathbb{P}^+} \to R_\alpha^{\mathbb{B}}$ such that for every classical first-order $\mathcal{L}_\in$-formula with no universal quantifiers $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n \in R_\alpha^{\mathbb{P}^+}$,

$$[\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}^+} \Vdash_p \varphi(a_1, ..., a_n)\}] = [\![\varphi(j_\alpha(a_1), ..., j_\alpha(a_n))]\!]_\alpha^{\mathbb{B}}$$

Corollary 3.2.12 ([Fit69], Chapter 15, Corollary 5.6)**.** If φ is an $\mathcal{L}_\in$-sentence with no universal quantifiers, then φ is valid in the model $R_\alpha^{\mathbb{B}}$ (that is $[\![\varphi]\!]_\alpha^{\mathbb{B}} = 1$), if and only if, $\neg\neg\varphi$ is valid in $(\mathbb{P}, \leqslant, \Vdash, R_\alpha^{\mathbb{P}^+})$ (that is $R_\alpha^{\mathbb{P}^+} \Vdash_p \varphi$ for every $p \in \mathbb{P}$.)

Remark 3.2.13. Since in Classical Logic there is an equivalence between universal and existential quantifiers given by

$$\vdash \forall x \varphi(x), \text{ if and only if, } \vdash \neg\exists\neg\varphi(x)$$

we can always write a formula in such a way that it has no universal quantifiers. We use this rewriting in the following theorem.

Corollary 3.2.14 ([Fit69], Chapter 15, Corollary 5.7)**.** $(\mathbb{P}, \leqslant, \Vdash, R_\alpha^{\mathbb{P}^+})$ is an Intuitionistic ZFC model, that is, classical equivalents of all the axioms of Zermelo-Fraenkel with choice, expressed without the use of the universal quantifier, are valid in the model.

3.2.2 Modal Residuated Kripke models of Set Theory

The goal of this section is to generalize Fitting's results on Intuitionistic Kripke models that we presented in last subsection. Specifically, we want to prove analogs of Theorem 3.2.11 and Corollaries 3.2.12 and 3.2.14.

We were able to find a suitable generalization of Fitting model in the context of Residuated Logic (see Definition 3.2.18) such that there exists an isomorphism (see Theorem 3.2.29) between it and an appropriate Heyting valued model (see Definition 3.2.23). Furthermore, in Corollary 3.2.30, we find if φ is an $MR-\mathcal{L}_\in$-sentence that is valid in the Heyting model, then $\Diamond\varphi$ is valid in the Residuated Kripke model.

We show now the general structure of this subsection and how it relates to Fitting's original construction. We start by noticing that the double negation operator $\neg\neg$ is a modal (closure) operator on a Heyting algebra, and it is used to translate sentences (see Definition 1.3.6 or Corollary 3.2.12) from the Boolean valued model $R^{\mathbb{B}}$ (see Definition 3.2.9) into the Intuitionistic Kripke model $\mathcal{V}^{\mathbb{P}^+}$ (see Definition 3.2.5). The key points to notice are as follows: We start with an Intuitionistic Kripke model with constant universe $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$.

1. The set $\mathbb{P}^+ = \{A \subseteq \mathbb{P} : \text{A is hereditary}\}$ is a Heyting algebra (see Theorem 3.1.5).
2. The operator double negation $\neg\neg$ is a modal operator on a Heyting algebra (See Example 1.2.18).
3. $\neg\neg$ is used to translate sentences from the Boolean valued model into the intuitionistic Kripke models (see Corollary 3.2.12 and Lemma 1.3.7 items 1. and 4.).
4. The set $\mathcal{F}_{\neg\neg} := \{x \in \mathbb{P}^+ : \neg\neg x = 1\}$ is a filter on $\mathbb{P}^+$ such that $\mathbb{P}^+/\mathcal{F}_{\neg\neg}$ is a Boolean algebra (see Theorem 1.2.21).

Therefore, one could wonder if this kind of results can be obtain by starting with some MR-Kripke $\mathcal{L}$-model $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ and then finding a suitable valued model such that an analogous of Corollary 3.2.12 holds.

Throughout this subsection, fix a Residuated Kripke Modal $\mathcal{L}$-model $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ with $\delta : \mathbb{P} \to \mathbb{P}$ an idempotent conucleus that respects implications and the top element.

Definition 3.2.15 (cf. [Fit69], p. 166)**.** We say that a function f is a $\mathbb{P}^*$-*subset of* $\mathcal{A}$ if

1. $Dom(f) \subseteq D$
2. $Ran(f) \subseteq \mathbb{P}^* = \{A \subseteq \mathbb{P} : A \text{ is strongly hereditary}\}$

Recall Theorem 3.1.26 proved that $\mathbb{P}^*$ is a Residuated Lattice with the order $\subseteq$ and the product

$$A \cdot B = \{c \in \mathbb{P} : \text{ there exist } a \in A, b \in B \text{ such that } c \geqslant a \cdot b\}, \text{ where } A, B \in \mathbb{P}^*.$$

and that Theorems 3.1.37 and 3.1.41 state that $\gamma : \mathbb{P}^* \to \mathbb{P}^*$ defined by

$$\gamma(A) := \{p \in \mathbb{P} : \exists q \in A(\delta(q) \leqslant p)\}$$

is an idempotent quantic nucleus that respects implications and the bottom element, that is, γ is a standard quantic nucleus on $\mathbb{P}^*$.

Definition 3.2.16 (cf. [Fit69], p. 166)**.** Let $\mathbb{Q}$ be any complete Residuated Lattice and γ be any standard quantic nucleus on $\mathbb{Q}$. We call an element $x \in \mathbb{Q}$ γ-*regular* if $\sim\sim \gamma(x) = x$. This definition generalizes the notion of **regular** sets in a Heyting algebra (see Definition 1.2.19). We call a function with range $\mathbb{Q}$ γ-*regular*, if every member of its range is γ-regular.

Definition 3.2.17 (cf. [Fit69], p. 166)**.** We say that a function from D to $\mathbb{P}^*$ is *extensional* if, for each $g, h \in D$

$$f(g) \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p (g = h)\} \subseteq f(h)$$

where $(g = h)$ is an abbreviation defined by:

$$(g = h) := \Diamond \sim (\exists x) \sim (x \in g \rightarrow x \in h))\&(\Diamond \sim (\exists x) \sim (x \in h \leftarrow x \in g)).$$

And we denote

$$\mathcal{P}^{\mathbb{P}^*}(D) := \{f : f \text{ is a } \gamma\text{-regular and extensional } \mathbb{P}^*\text{-subset of } \mathcal{A}\}$$

Definition 3.2.18 (cf. [Fit69], p. 166)**.** We now define on induction on ordinals a class of MR-Kripke $\mathcal{L}_\in$-models $\mathcal{V}_\alpha^{\mathbb{P}^*} := (\mathbb{P}, \leqslant, \delta, \Vdash, R_\alpha^{\mathbb{P}^*})$ all with the same underlying order $(\mathbb{P}, \leqslant)$ but changing the universe for each ordinal α as follows:

1. $\mathcal{V}_0^{\mathbb{P}^*} := (\mathbb{P}, \leqslant, \delta, \Vdash, R_0^{\mathbb{P}^*})$ where $R_0^{\mathbb{P}^*} := \varnothing$.
2. $\mathcal{V}_{\alpha+1}^{\mathbb{P}^*} := (\mathbb{P}, \leqslant, \delta, \Vdash, R_{\alpha+1}^{\mathbb{P}^*})$ where $R_{\alpha+1}^{\mathbb{P}^*} := R_\alpha^{\mathbb{P}^*} \cup \mathcal{P}^{\mathbb{P}^*}(R_\alpha^{\mathbb{P}^*})$ and $\mathcal{V}_{\alpha+1}^{\mathbb{P}^*} \Vdash_p f \in g$ is defined as follows:

   If $p \in \mathbb{P}$ and $f, g \in R_{\alpha+1}^{\mathbb{P}^*}$ then we have the following cases:

   a. If $f, g \in R_\alpha^{\mathbb{P}^*}$, then

   $$\mathcal{V}_{\alpha+1}^{\mathbb{P}^*} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}_\alpha^{\mathbb{P}^*} \Vdash_p (f \in g).$$

   b. If $f \in R_\alpha^{\mathbb{P}^*}$ and $g \in R_{\alpha+1}^{\mathbb{P}^*} \backslash R_\alpha^{\mathbb{P}^*} = \mathcal{P}^{\mathbb{P}^*}(R_\alpha^{\mathbb{P}^*})$, then

   $$\mathcal{V}_{\alpha+1}^{\mathbb{P}^*} \Vdash_p (f \in g), \text{ if and only if, } p \in g(f).$$

   c. If $f \in R_{\alpha+1}^{\mathbb{P}^*} \backslash R_\alpha^{\mathbb{P}^*} = \mathcal{P}^{\mathbb{P}^*}(R_\alpha^{\mathbb{P}^*})$, then $\mathcal{V}_{\alpha+1}^{\mathbb{P}^*} \Vdash_p (f \in g)$, if and only if,

   $$p \in \bigvee_{h \in dom(g)} P_h$$

where

$$P_h := g(h) \cdot (P_{f \subseteq h} \cdot P_{h \subseteq f})$$

and

$$P_{f \subseteq h} := \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \rightarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\})$$

$$P_{h \subseteq f} := \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \leftarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\}).$$

3. If $\alpha \neq 0$ is a limit ordinal, then let $R_\alpha^{\mathbb{P}*} := \bigcup_{\beta < \alpha} R_\beta^{\mathbb{P}*}$ and given $f, g \in R_\alpha^{\mathbb{P}*}$ take any $\eta < \alpha$ such that $f, g \in R_\eta^{\mathbb{P}}$ and let

$$\mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}_\eta^{\mathbb{P}*} \Vdash_p (f \in g).$$

For every α we also define

$$\mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p \bot, \text{ if and only if, } p = \infty.$$

Remark 3.2.19 (cf. [Fit69], Remark 4.2)**.** The expression

$$\bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \rightarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\}) \cdot \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \leftarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\})$$

is an element in the Residuated Lattice $\mathbb{P}^*$, where $\cdot, \rightarrow, \leftarrow$ and $\bigcap$ are the operations on $\mathbb{P}^*$ as a Residuated Lattice (see Definition 3.1.25).

Definition 3.2.20 (cf. [Fit69], p. 166)**.** Consider the MR-Kripke (class) $\mathcal{L}_\in$-model

$$\mathcal{V}^{\mathbb{P}*} := (\mathbb{P}, \leqslant, \delta, \Vdash, R^{\mathbb{P}*}), \text{ where } R^{\mathbb{P}*} := \bigcup_{\alpha \in ON} R_\alpha^{\mathbb{P}*}$$

and given $f, g \in R^{\mathbb{P}*}$, take any $\eta \in ON$ such that $f, g \in R_\eta^{\mathbb{P}}$ and define

$$\mathcal{V}^{\mathbb{P}^*} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{V}^{\mathbb{P}^*}_\eta \Vdash_p (f \in g).$$

We need to see that this definition provides indeed a Modal Residuated Kripke model, that is, it satisfies the Definition 3.1.34 item 3. sub-items a., b. and c.. By definition of $\Vdash$, the model $\mathcal{V}^{\mathbb{P}^*}_\alpha$ satisfies condition c. and since ∞ is an element of every strongly hereditary set (see Definition 3.1.24) it is straightforward to see that condition b. also holds. Therefore, we just need to check that a. holds.

Theorem 3.2.21. For every $\alpha \in ON$, we have that if $p_i, q \in \mathbb{P}$ with $i \in I$,

$$\text{if } \mathcal{V}^{\mathbb{P}^*}_\alpha \Vdash_{p_i} (f \in g) \text{ for every } i \in I \text{ and } \bigwedge_{i \in I} p_i \leqslant q, \text{ then } \mathcal{V}^{\mathbb{P}^*}_\alpha \Vdash_q (f \in g).$$

Proof. We prove this by transfinite induction. Since the cases for $\alpha = 0$ and α a limit ordinal are trivial, we only consider what happens at the successor step, so let us suppose the following holds at α:

$$\text{if } \mathcal{V}^{\mathbb{P}^*}_\alpha \Vdash_{p_i} (f \in g) \text{ for } i \in I \text{ and } \bigwedge_{i \in I} p_i \leqslant q, \text{ then } \mathcal{V}^{\mathbb{P}^*}_\alpha \Vdash_q (f \in g) \text{ (induction hypothesis)},$$

and let us prove it at $\alpha + 1$. Assume that

$$\mathcal{V}^{\mathbb{P}^*}_{\alpha+1} \Vdash_{p_i} (f \in g) \text{ for } i \in I \text{ and } \bigwedge_{i \in I} p_i \leqslant q.$$

We have three cases:

1. If $f, g \in R^{\mathbb{P}^*}_\alpha$, then we have the result by the induction hypothesis.

2. If $f \in R^{\mathbb{P}^*}_\alpha$ and $g \in R^{\mathbb{P}^*}_{\alpha+1} \backslash R^{\mathbb{P}^*}_\alpha = \mathcal{P}^{\mathbb{P}^*}(R^{\mathbb{P}^*}_\alpha)$, then

$$\mathcal{V}^{\mathbb{P}^*}_{\alpha+1} \Vdash_{p_i} (f \in g) \text{ means that } p_i \in g(f).$$

but by definition of $g \in \mathcal{P}^{\mathbb{P}^*}(R^{\mathbb{P}^*}_\alpha)$ we know that the codomain of g is $\mathbb{P}^*$ and so $g(f)$ is strongly hereditary. Therefore, $q \in g(f)$ and thus

$$\mathcal{V}_{\alpha+1}^{\mathbb{P}*} \Vdash_q (f \in g)$$

3. If $f \in R_{\alpha+1}^{\mathbb{P}*} \backslash R_{\alpha}^{\mathbb{P}*} = \mathcal{P}^{\mathbb{P}*}(R_{\alpha}^{\mathbb{P}*})$, then

$$\mathcal{V}_{\alpha+1}^{\mathbb{P}*} \Vdash_{p_i} (f \in g) \text{ means that } p_i \in \bigvee_{h \in dom(g)} P_h = \bigvee_{h \in dom(g)} g(h) \cdot (P_{f \subseteq h} \cdot P_{h \subseteq f})$$

but

$$P_{f \subseteq h} := \bigcap_{x \in R_{\alpha}^{\mathbb{P}*}} (f(x) \to \{q \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\})$$

$$= \bigcap_{x \in R_{\alpha}^{\mathbb{P}*}} (f(x) \to \sim\sim \gamma\{q \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_q (x \in h)\}) \quad \text{(by Theorems 3.1.42 and 3.1.28).}$$

By the induction hypothesis, the set $\{q \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_q (x \in h)\}$ is strongly hereditary, and since $f \in \mathcal{P}^{\mathbb{P}*}(R_{\alpha}^{\mathbb{P}*})$, $f(x)$ is also strongly hereditary for every $x \in dom(g)$. Therefore, since the operations $\sim, \to, \cdot, \gamma$ and $\bigcap$ are all closed in $\mathbb{P}^*$ (see Theorems 3.1.26 and 3.1.37), we have that $P_{f \subseteq h} \in \mathbb{P}^*$. By using a similar argument, we can show that $P_{h \subseteq f} \in \mathbb{P}^*$ and since $g(h) \in \mathbb{P}^*$, we have that $\bigvee_{h \in dom(g)} P_h$ is strongly hereditary. Therefore, $q \in \bigvee_{h \in dom(g)} P_h$ and thus

$$\mathcal{V}_{\alpha+1}^{\mathbb{P}*} \Vdash_q (f \in g)$$

□

We now construct a Heyting valued model $(R^{\mathbb{H}}, [\![\cdot]\!]^{\mathbb{H}})$ that is related with $\mathcal{V}^{\mathbb{P}*} = (\mathbb{P}, \leqslant, R^{\mathbb{P}*})$, in a similar way as in Definition 3.2.9.

Definition 3.2.22 (cf. [Fit69], p. 164)**.** Given a Heyting valued model $(R, [\![\cdot]\!]^{\mathbb{H}})$, we say that a function $f : R \to \mathbb{H}$ is *extensional* if for all $g, h \in R$,

$$f(g) \wedge [\![\neg(\exists x)\neg(x \in g \to x \in h)]\!]^{\mathbb{H}} \wedge [\![\neg(\exists x)\neg(x \in g \leftarrow x \in h)]\!]^{\mathbb{H}} \leqslant f(h)$$

and we say that a function $f : R \to \mathbb{H}$ is *regular* if $\neg\neg f(x) = f(x)$ for every $x \in R$.

Definition 3.2.23 (cf. [Fit69], p. 165)**.** We now define on induction over the ordinals a class of Heyting valued models $(R^{\mathbb{H}}_{\alpha}, [\![\cdot]\!]^{\mathbb{H}}_{\alpha})$

1. $R^{\mathbb{H}}_0 := \varnothing$ with $[\![\cdot]\!]^{\mathbb{H}}_0 := \varnothing$.

2. $R^{\mathbb{H}}_{\alpha+1} := R^{\mathbb{H}}_{\alpha} \cup \{f : R^{\mathbb{H}}_{\alpha} \to \mathbb{H} : f \text{ is extensional and regular}\}$ and given $f, g \in R^{\mathbb{H}}_{\alpha+1}$, we have, for the definition of $[\![f \in g]\!]^{\mathbb{H}}_{\alpha}$, the following cases:

   a. If $f, g \in R^{\mathbb{H}}_{\alpha}$, define $[\![f \in g]\!]^{\mathbb{H}}_{\alpha+1} := [\![f \in g]\!]^{\mathbb{H}}_{\alpha}$.

   b. If $f \in R^{\mathbb{H}}_{\alpha}$ and $g \in R^{\mathbb{H}}_{\alpha+1} \backslash R^{\mathbb{H}}_{\alpha}$, define $[\![f \in g]\!]^{\mathbb{H}}_{\alpha+1} := g(f)$.

   c. If $f \in R^{\mathbb{H}}_{\alpha+1} \backslash R^{\mathbb{H}}_{\alpha}$, define

$$[\![f \in g]\!]^{\mathbb{H}}_{\alpha+1} := \bigvee_{h \in dom(g)} \{g(h) \wedge \bigwedge_{x \in R^{\mathbb{H}}_{\alpha}} (f(x) \leftrightarrow [\![\neg\neg(x \in h)]\!]^{\mathbb{H}}_{\alpha})\}$$

   d. If $\alpha \neq 0$ is a limit ordinal, then let $R^{\mathbb{H}}_{\alpha} := \bigcup_{\beta<\alpha} R^{\mathbb{H}}_{\beta}$ and given $f, g \in R^{\mathbb{H}}_{\alpha}$ take any $\eta < \alpha$ such that $f, g \in R^{\mathbb{H}}_{\eta}$ and let $[\![f \in g]\!]^{\mathbb{H}}_{\alpha} := [\![f \in g]\!]^{\mathbb{H}}_{\eta}$.

   Now let

$$R^{\mathbb{H}} := \bigcup_{\alpha \in ON} R^{\mathbb{H}}_{\alpha}$$

   and given $f, g \in R^{\mathbb{H}}$ take any $\eta < \alpha$ such that $f, g \in R^{\mathbb{H}}_{\eta}$ and let $[\![f \in g]\!]^{\mathbb{H}} := [\![f \in g]\!]^{\mathbb{H}}_{\eta}$.

Remark 3.2.24. The construction given above mimics Definition 3.2.9 but with two main differences:

1. Instead of using a Boolean algebra, we consider a Heyting algebra.

2. In condition 2. c., we consider the term $[\![\neg\neg(x \in h)]\!]^{\mathbb{H}}_{\alpha}$ rather than $[\![x \in h]\!]^{\mathbb{H}}_{\alpha}$. Clearly, in the Classical (Boolean) case, these expressions are equivalent, but in the Intuitionistic case they are not.

Definition 3.2.25. Let $\mathbb{Q}$ be any complete residuated lattice and γ be any quantic nucleus on $\mathbb{Q}$. An element of $x \in \mathbb{Q}$ is called γ-*dense* if $\gamma(x) = 1_{\mathbb{Q}}$.

Remark 3.2.26. The definition given above generalizes the notion of **dense** sets in a Heyting algebra (see Definition 1.2.19). We focus on the case where $\mathbb{Q} = \mathbb{P}^*$ and γ is the quantic nucleus determined by δ. Let $\mathcal{F}_\gamma$ be the collection of all γ-dense elements of $\mathbb{P}^*$. By Theorems 1.2.58 and 1.2.61 $\mathcal{F}_\gamma$ is a filter such that $\mathbb{P}^*/\mathcal{F}_\gamma$ is a Heyting algebra.

Remark 3.2.27. Recall (see Definition 1.2.42 and Theorem 1.2.61) that the relation $\approx_{\mathcal{F}_\gamma}$ given by

$$A \approx_{\mathcal{F}_\gamma} B, \text{ if and only if, } A \to B \in \mathcal{F}_\gamma \text{ and } B \to A \in \mathcal{F}_\gamma$$

is an equivalence relation. Also, we have that

$$\mathbb{H} := \mathbb{P}^*/\mathcal{F}_\gamma = \mathbb{P}^*/_{\approx_{\mathcal{F}_\gamma}} = \{|A| : A \in \mathbb{P}^*\}$$

is a complete Heyting algebra. Furthermore, if $|A|, |B| \in \mathbb{H}$

$$|A| \leqslant |B| \text{ iff } A \to B \in \mathcal{F}_\gamma$$

$$|A| \wedge |B| = |A \wedge B| = |A \cdot B| = |A| \cdot |B|$$

$$|A| \vee |B| = |A \vee B|$$

$$|A| \to |B| = |A \to B|$$

$$|\sim A| = \neg |A|$$

$$|A| = |\gamma(A)| \text{ (see Corollary 1.2.60)}$$

$$|\bigvee_{i\in I} A_i| = \bigvee_{i\in I} |A_i| \text{ (see Theorem 1.2.62)}$$

Remark 3.2.28 (cf. [Fit69], Remark 5.1)**.** The equality $|\bigwedge_{i\in I} A_i| = \bigwedge_{i\in I} |A_i|$ is not true in general, and thus MR-formulas with universal quantifiers behave poorly (since, in valued models, we usually interpret universal quantifiers as meets). This explains why we do not consider formulas with universal quantifiers in the following theorem.

The Heyting algebra $\mathbb{H} := \mathbb{P}^*/\mathcal{F}_\gamma$ determines a sequence of valued models $(R^{\mathbb{H}}_\alpha, [\![\cdot]\!]^{\mathbb{H}}_\alpha)$ that is isomorphic to the sequence $\mathcal{V}^{\mathbb{P}^*}_\alpha = (\mathbb{P}, \leqslant, \delta, \Vdash, R^{\mathbb{P}^*}_\alpha)$ in the following way:

Theorem 3.2.29 (cf. [Fit69], Chapter 15, Theorem 5.5)**.** For every $\alpha \in ON$, there exist a bijection between $R^{\mathbb{P}^*}_\alpha$ and $R^{\mathbb{H}}_\alpha$ (where if $f \in R^{\mathbb{P}^*}_\alpha$, f' denotes the image of f via this bijection) such that for every $\mathcal{L}_\in$-formula with no universal quantifiers $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n \in R^{\mathbb{P}^*}_\alpha$,

$$|\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}^*}_\alpha \Vdash_p \varphi(a_1, ..., a_n)\}| = [\![\varphi(a'_1, ..., a'_n)]\!]^{\mathbb{H}}_\alpha$$

Proof. We show this by induction on α. $R^{\mathbb{P}^*}_0$ and $R^{\mathbb{H}}_0$ are the same, so it holds for α.

Assume that there exists such a bijection $'$ between $R^{\mathbb{P}^*}_\alpha$ and $R^{\mathbb{H}}_\alpha$ (induction hypothesis 1).

Take $g \in R^{\mathbb{P}^*}_{\alpha+1} \backslash R^{\mathbb{P}^*}_\alpha = \mathcal{P}^{\mathbb{P}^*}(R^{\mathbb{P}^*}_\alpha)$ and define the function $g' : R^{\mathbb{H}}_\alpha \to \mathbb{H}$ in the following way: Since $' : R^{\mathbb{P}^*}_\alpha \to R^{\mathbb{H}}_\alpha$ is onto, every element $F \in R^{\mathbb{H}}_\alpha$ has the form $F = f'$ for some $f \in R^{\mathbb{P}^*}_\alpha$. Therefore, we can define g' by

$$g'(f') := |g(f)| \in \mathbb{H} = \mathbb{P}^*/\mathcal{F}_\gamma \text{ for every } f' \in R^{\mathbb{H}}_\alpha.$$

For now we assume that g is extensional if and only if g' is extensional. This will be proved at the end of this theorem.

Let us see that this map is injective. Take $g, h \in R^{\mathbb{P}^*}_{\alpha+1} \backslash R^{\mathbb{P}^*}_{\alpha} = \mathcal{P}^{\mathbb{P}^*}(R^{\mathbb{P}^*}_{\alpha})$ such that $|g(f)| = |h(f)|$ for every $f \in R^{\mathbb{P}^*}_{\alpha}$. Then, by definition of $=$ in $\mathbb{P}^*/\mathcal{F}_\gamma$, we have, in particular

$$g(f) \to h(f) \in \mathcal{F}_\gamma$$

that is, by definition of $\mathcal{F}_\gamma$,

$$\gamma(g(f) \to h(f)) = 1$$

and since by Corollary 1.2.47, $\gamma(g(f) \to h(f)) \leqslant g(f) \to \gamma(h(f))$, we have that

$$g(f) \to \gamma(h(f)) = 1$$

By Theorem 1.2.37 item 2., $\gamma(h(f)) \leqslant (\sim\sim \gamma(h(f)))$ and by Theorem 1.2.33 item 9.,

$$g(f) \to (\sim\sim \gamma(h(f))) = 1$$

But h is a γ-regular function, so

$$g(f) \to h(f) = 1$$

which implies, by Theorem 1.2.33 item 1., that $g(f) \subseteq h(f)$. In a similar fashion we can show that $h(f) \subseteq g(f)$ and therefore we conclude $g(f) = h(f)$ for every $f \in R^{\mathbb{P}^*}_{\alpha}$, that is, $g = h$.

To see that the map is surjective, let us take $h \in R^{\mathbb{H}}_{\alpha+1} \backslash R^{\mathbb{H}}_{\alpha} = \mathcal{P}^{\mathbb{H}}(R^{\mathbb{H}}_{\alpha})$, that is, $h : R^{\mathbb{H}}_{\alpha} \to \mathbb{H}$ is a regular and extensional function. We will construct a function $g \in \mathcal{P}^{\mathbb{P}^*}(R^{\mathbb{P}^*}_{\alpha})$ such that $g' = h$, that is, $g'(f') = h(f')$ for every $f' \in R^{\mathbb{H}}_{\alpha}$. Let s be any function from $R^{\mathbb{P}^*}_{\alpha}$ to $\mathbb{P}^*$ such that

for $f \in R^{\mathbb{P}^*}_{\alpha}$, $s(f)$ is a representative of the class $h(f') \in \mathbb{H} = \mathbb{P}^*/\mathcal{F}_\gamma$, that is, $h(f') = |s(f)|$. Let g be the function defined by

$$g(f) = \sim\sim \gamma(s(f)) \text{ for } f \in R^{\mathbb{P}*}_{\alpha}.$$

Then, by Theorem 1.2.56, g is γ-regular and since its domain is $R^{\mathbb{P}*}_{\alpha}$, we have that $g \in R^{\mathbb{P}*}_{\alpha+1} \backslash R^{\mathbb{P}*}_{\alpha} = \mathcal{P}^{\mathbb{P}*}(R^{\mathbb{P}*}_{\alpha})$. We want to see that $g' = h$, so let us take $f' \in R^{\mathbb{H}}_{\alpha}$.

$$\begin{aligned}
g'(f') &= |g(f)| && \text{(by definition of } g'\text{)}\\
&= |\sim\sim \gamma(s(f))| && \text{(by definition of } g\text{)}\\
&= \neg\neg|\gamma s(f)| && \text{(by definition of } \neg\text{)}\\
&= \neg\neg|s(f)| && \text{(by Corollary 1.2.60)}\\
&= \neg\neg h(f') && (s(f) \text{ is a representative of the class } h(f'))\\
&= h(f') && (h \text{ is regular function}).
\end{aligned}$$

Then, $g' = h$ and the function $'$ is surjective.

By the induction hypothesis 1, we may assume that for every $MR - \mathcal{L}_{\in}$-formula with no universal quantifiers $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n \in R^{\mathbb{P}*}_{\alpha}$,

$$|\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_p \varphi(a_1, ..., a_n)\}| = [\![\varphi(a'_1, ..., a'_n)]\!]^{\mathbb{H}}_{\alpha}$$

We will show that this result also holds for $R^{\mathbb{P}*}_{\alpha+1}$ by induction on formulas. We start with the atomic case. Let $f, g \in R^{\mathbb{P}*}_{\alpha+1}$. We have three cases:

1. If $f, g \in R^{\mathbb{P}*}_{\alpha}$, then we have the result by the induction hypothesis 1.

2. If $f \in R_\alpha^{\mathbb{P}*}$ and $g \in R_{\alpha+1}^{\mathbb{P}*} \backslash R_\alpha^{\mathbb{P}*}$, then

$$\llbracket f' \in g' \rrbracket_{\alpha+1}^{\mathbb{H}} = g'(f') \qquad \text{(by definition of } \llbracket \cdot \in \cdot \rrbracket_{\alpha+1}^{\mathbb{H}})$$

$$= |g(f)| \qquad \text{(by definition of } g')$$

$$= |\{p \in \mathbb{P} : p \in g(f)\}| \qquad \text{(by definition of } g(f))$$

$$= |\{p \in \mathbb{P} : \mathcal{V}_{\alpha+1}^{\mathbb{P}} \Vdash_p f \in g\}| \qquad \text{(by definition of } \mathcal{V}_{\alpha+1}^{\mathbb{P}*} \Vdash_p f \in g)$$

3. If $f \in R_{\alpha+1}^{\mathbb{P}*}$, recall that we denote

$$P_h := g(h) \cdot (P_{f \subseteq h} \cdot P_{h \subseteq f})$$

$$:= g(h) \cdot \left(\bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \to \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\}) \cdot \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \leftarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\}) \right)$$

and notice that

$$\{p \in \mathbb{P} : \mathcal{V}_{\alpha+1}^{\mathbb{P}*} \Vdash_p (f \in g)\} = \bigvee_{h \in dom(g)} P_h.$$

Furthermore,

$$P_{f \subseteq h} = \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \to \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q \sim\sim \Diamond(x \in h)\})$$

$$= \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (f(x) \to \sim\sim \gamma\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}) \qquad \text{(by Theorems 3.1.42 and 3.1.28)}$$

$$= \bigcap_{x \in R_\alpha^{\mathbb{P}*}} (\sim\sim \gamma(f(x)) \to \sim\sim \gamma\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}) \qquad \text{(since } f \text{ is } \gamma\text{-regular)}$$

$$= \bigcap_{x \in R_\alpha^{\mathbb{P}*}} \sim\sim (\sim\sim \gamma(f(x)) \to \sim\sim \gamma\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}) \qquad \text{(by Example 1.2.49)}$$

$$= \sim \bigvee_{x \in R_\alpha^{\mathbb{P}*}} \sim (\sim\sim \gamma(f(x)) \to \sim\sim \gamma\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}) \qquad \text{(by Theorem 1.2.36 item 4.)}$$

Thus,

$$|P_{f\subseteq h}| = \left|\bigcap_{x\in R_\alpha^{\mathbb{P}*}} (f(x) \to \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q {\sim\sim} \Diamond(x \in h)\})\right|$$

$$= \left|\sim \bigvee_{x\in R_\alpha^{\mathbb{P}*}} \sim ({\sim\sim}\, \gamma(f(x)) \to {\sim\sim}\, \gamma\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\})\right|$$

$$= \neg \bigcup_{x\in R_\alpha^{\mathbb{P}*}} \neg(\neg\neg|\gamma(f(x))| \to \neg\neg|\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}|) \qquad \text{(by Remark 3.2.27)}$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} \neg\neg(\neg\neg|\gamma(f(x))| \to \neg\neg|\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}|) \qquad \text{(by Theorem 1.2.36 item 4.)}$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} \neg\neg|\gamma(f(x))| \to \neg\neg|\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}| \qquad \text{(by Example 1.2.49)}$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} |{\sim\sim}\, \gamma(f(x))| \to \neg\neg|\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}| \qquad \text{(by definition of } \neg)$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} |f(x)| \to \neg\neg|\{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q (x \in h)\}| \qquad (f \text{ is } \gamma\text{-regular})$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} f'(x') \to \neg\neg[\![x' \in h']\!]_\alpha^{\mathbb{H}} \qquad \text{(by the induction hypothesis and the definition of } f')$$

$$= \bigcap_{x\in R_\alpha^{\mathbb{P}*}} f'(x') \to [\![\neg\neg(x' \in h')]\!]_\alpha^{\mathbb{H}} \qquad \text{(by definition of } \neg)$$

In a similar way we can prove that

$$|P_{h\subseteq f}| = \left|\bigcap_{x\in R_\alpha^{\mathbb{P}*}} (f(x) \leftarrow \{q \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_q {\sim\sim} \Diamond(x \in h)\})\right| = \bigcap_{x\in R_\alpha^{\mathbb{P}*}} f'(x') \leftarrow [\![\neg\neg(x' \in h')]\!]_\alpha^{\mathbb{H}}$$

Thus,

$$
\begin{aligned}
|P_h| &= |g(h)\cdot(P_{f\subseteq h}\cdot P_{h\subseteq f})| = |g(h)|\cdot(|P_{f\subseteq h}|\cdot|P_{h\subseteq f}|) && \text{(by Remark 3.2.27)}\\
&= |g(h)|\cdot\left(\bigcap_{x\in R_\alpha^{\mathbb{P}^*}} f'(x')\to[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}\cdot\bigcap_{x\in R_\alpha^{\mathbb{P}^*}} f'(x')\leftarrow[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}\right) && \text{(by the previous computations)}\\
&= |g(h)|\cap\left(\bigcap_{x\in R_\alpha^{\mathbb{P}^*}} f'(x')\to[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}\cap\bigcap_{x\in R_\alpha^{\mathbb{P}^*}} f'(x')\leftarrow[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}\right) && \text{(since } \cap = \cdot \text{ in } \mathbb{P}^*/\mathcal{F}_\gamma)\\
&= g'(h')\cap\bigcap_{x\in R_\alpha^{\mathbb{P}^*}}(f'(x')\leftrightarrow[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}) && \text{(properties of } \wedge \text{ and definition of } g')
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[\![f'\in g']\!]_{\alpha+1}^{\mathbb{H}} &= \bigcup_{h'\in dom(g')}(g'(h')\cap\bigcap_{x\in R_\alpha^{\mathbb{P}^*}}(f'(x')\leftrightarrow[\![\neg\neg(x'\in h')]\!]_\alpha^{\mathbb{H}}) && \text{(by definition of } [\![\cdot\in\cdot]\!]_{\alpha+1}^{\mathbb{H}})\\
&= \bigcup_{h'\in dom(g')}|P_h|\\
&= \left|\bigvee_{h\in dom(g)} P_h\right| && \text{(by Theorem 1.2.62)}\\
&= |\{p\in\mathbb{P} : \mathcal{V}_{\alpha+1}^{\mathbb{P}^*}\Vdash_p (f\in g)\}|
\end{aligned}
$$

Now we have the result for atomic MR-formulas. Hence, it is straightforward to prove the result for the rest of MR-formulas by induction on formulas and by using Theorems 3.1.28 and 3.1.42. For this reason, we show it only for the product and the existential quantifier: Assume that $\varphi(x_1,...,x_n)$ and $\psi(x_1,...,x_n)$ are $MR-\mathcal{L}_\in$-formulas such that for all $a_1,...,a_n\in R_\alpha^{\mathbb{P}^*}$

$$|\{p\in\mathbb{P} : \mathcal{V}_{\alpha+1}^{\mathbb{P}^*}\Vdash_p \varphi(a_1,...,a_n)\}| = [\![\varphi(a_1',...,a_n')]\!]_{\alpha+1}^{\mathbb{H}} \text{ and}$$

$$|\{p\in\mathbb{P} : \mathcal{V}_{\alpha+1}^{\mathbb{P}^*}\Vdash_p \psi(a_1,...,a_n)\}| = [\![\psi(a_1',...,a_n')]\!]_{\alpha+1}^{\mathbb{H}} \text{ (induction hypothesis 2).}$$

Then,

$$[\![(\varphi\&\psi)(a'_1,...,a'_n)]\!]^{\mathbb{H}}_{\alpha+1} = [\![\varphi(a'_1,...,a'_n)]\!]^{\mathbb{H}}_{\alpha+1} \cdot [\![\psi(a'_1,...,a'_n)]\!]^{\mathbb{H}}_{\alpha+1}$$

$$= |\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p \varphi(a_1,...,a_n)\}| \cdot |\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p \psi(a_1,...,a_n)\}| \qquad \text{(by induction hypothesis 2)}$$

$$= |\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p (\varphi\&\psi)(a_1,...,a_n)\}| \qquad \text{(by Theorem 3.1.28)}$$

Take an $MR{-}\mathcal{L}_{\in}$-formula $\varphi(x_1,...,x_n)$ and $a_1,...,a_n \in R^{\mathbb{P}*}_{\alpha+1}$. Assume that for every $a \in R^{\mathbb{P}*}_{\alpha+1}$, $\varphi(a_1,...,a_n,a)$ satisfies

$$|\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p \varphi(a_1,...,a_n,a)\}| = [\![(\varphi(a'_1,...,a'_n,a')]\!]^{\mathbb{H}}_{\alpha+1} \text{ (induction hypothesis 3).}$$

Then,

$$[\![\exists x\varphi(a'_1,...,a'_n,x)]\!]^{\mathbb{H}}_{\alpha+1} = \bigvee_{a' \in R^{\mathbb{H}}_{\alpha+1}} [\![(\varphi(a'_1,...,a'_n,a')]\!]^{\mathbb{H}}_{\alpha+1} \qquad \text{(by definition of } [\![\exists x\varphi(a'_1,...,a'_n,x)]\!]^{\mathbb{H}}_{\alpha+1})$$

$$= \bigvee_{a' \in R^{\mathbb{H}}_{\alpha+1}} |\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p \varphi(a_1,...,a_n,a)\}| \qquad \text{(by induction hypothesis 3)}$$

$$= |\{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p \exists x\varphi(a_1,...,a_n,x)\}| \qquad \text{(by Theorem 3.1.28)}$$

Let us see that g is extensional if and only if g' is extensional.

If g is extensional, then, for every $f, h \in R^{\mathbb{P}*}_{\alpha}$, we have that

$$g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_p (f = h)\} \subseteq g(h),$$

this implies, by Theorem 1.2.33 item 1., that

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_p (f = h)\}) \to g(h) = 1.$$

Thus,

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (f = h)\}) \to g(h) \in \mathcal{F}_{\gamma}.$$

By definition of $\leqslant$ on $\mathbb{P}^*/\mathcal{F}_{\gamma}$, we get

$$|g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (f = h)\}| \leqslant |g(h)|$$

by Remark 3.2.27, this implies that

$$|g(f)| \wedge |\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (f = h)\}| \leqslant |g(h)|.$$

Recall that

$$(g = h) := \Diamond \sim (\exists x) \sim (x \in g \to x \in h))\&(\Diamond \sim (\exists x) \sim (x \in h \leftarrow x \in g)).$$

Thus, by Theorems 3.1.28 and 3.1.42,

$$|\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (f = h)\}|$$

$$= |\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (\Diamond \sim (\exists x) \sim (x \in g \to x \in h))\&(\Diamond \sim (\exists x) \sim (x \in h \leftarrow x \in g)))\}|$$

$$= |\gamma\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (\sim (\exists x) \sim (x \in g \to x \in h))\}| \cdot |\gamma\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (\sim (\exists x) \sim (x \in g \leftarrow x \in h))\}|$$

Notice that

$$|\gamma\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (\sim (\exists x) \sim (x \in g \to x \in h))\}|$$

$$= |\{p \in \mathbb{P} : \mathcal{V}_{\alpha}^{\mathbb{P}*} \Vdash_p (\sim (\exists x) \sim (x \in g \to x \in h))\}| \qquad \text{(by Corollary 1.2.60)}$$

$$= [\![\sim (\exists x) \sim (x \in g \to x \in h))]\!]_{\alpha}^{\mathbb{H}} \qquad \text{(by induction hypothesis 1.)}$$

In a similar way, we prove that

$$|\gamma\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (\sim (\exists x) \sim (x \in g \leftarrow x \in h))\}| = [\![\sim (\exists x) \sim (x \in g \leftarrow x \in h))]\!]_\alpha^{\mathbb{H}}$$

Therefore,

$$|\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}| = [\![\sim (\exists x) \sim (x \in g \to x \in h))]\!]_\alpha^{\mathbb{H}} \wedge [\![\sim (\exists x) \sim (x \in h \leftarrow x \in g)]\!]_\alpha^{\mathbb{H}}$$

Thus, by definition of g'

$$g'(f') \wedge [\![\sim (\exists x) \sim (x \in g \to x \in h))]\!]_\alpha^{\mathbb{H}} \wedge [\![\sim (\exists x) \sim (x \in h \leftarrow x \in g)]\!]_\alpha^{\mathbb{H}} \leqslant g'(h')$$

which proves that g' is extensional.

On the other hand, if g' is extensional, by using (backwards) the argument given above, we get that

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \to g(h) \in \mathcal{F}_\gamma.$$

which means

$$\gamma((g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \to g(h)) = 1$$

but, since γ respects implications, we get

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \to \gamma(g(h)) = 1.$$

Hence, by Theorem 1.2.37 item 2., $\gamma(g(h)) \leqslant (\sim\sim \gamma(g(h)))$ and by Theorem 1.2.33 item 9., we have

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \to (\sim\sim \gamma(g(h))) = 1,$$

but g is a γ-regular function,

$$(g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \to g(h) = 1$$

which implies, by Theorem 1.2.33 item 1., that

$$g(f) \cdot \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p (f = h)\}) \subseteq g(h)$$

That is, g is extensional.

□

Corollary 3.2.30. If φ is an $\mathcal{L}_\in$-sentence with no universal quantifiers, then φ is valid in the model $R_\alpha^{\mathbb{H}}$ (that is $[\![\varphi]\!]_\alpha^{\mathbb{H}} = 1_{\mathbb{H}}$), if and only if, $\Diamond\varphi$ is valid in $(\mathbb{P}, \leqslant, \delta, \Vdash, R_\alpha^{\mathbb{P}*})$ (that is $R_\alpha^{\mathbb{P}*} \Vdash_p \Diamond\varphi$ for every $p \in \mathbb{P}$.)

Proof. We have that

$$\begin{aligned}
[\![\varphi]\!]_\alpha^{\mathbb{H}} = 1_{\mathbb{H}} \text{ iff } [\![\varphi]\!]_\alpha^{\mathbb{H}} = |\mathbb{P}| & \quad \text{(by definition of } 1_{\mathbb{H}} = 1_{\mathbb{P}*/\mathcal{F}_\gamma}\text{)} \\
\text{iff } |\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p \varphi\}| = |\mathbb{P}| & \quad \text{(by Theorem 3.2.29)} \\
\text{iff } \gamma\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p \varphi\} = \gamma(\mathbb{P}) & \quad \text{(by Theorem 1.2.59)} \\
\text{iff } \{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}*} \Vdash_p \Diamond\varphi\}] = \mathbb{P} & \quad \text{(by Theorem 3.1.42 and since } \gamma \text{ is expansive)}
\end{aligned}$$

□

3.3 Constructible sets over Kripke models

We now study the notion of constructibility in the context of Kripke models of Set Theory. We start with a review of Fitting's results [Fit69] and then proceed to outline a propose for a generalization of those results in the context of Residuated Models of Set Theory, using the tools that we develop in the prior section.

3.3.1 Intuitionistic Constructible sets.

Let us take an Intuitionistic Kripke model $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$. Recall that

$$\mathbb{P}^+ := \{A \subseteq \mathbb{P} : A \text{ is hereditary}\}$$

is a Heyting algebra. Thus, it is natural to consider $\mathbb{P}^+$ as a set of truth values for the notion of **definability**.

Definition 3.3.1 ([Fit69], p. 94)**.** We say that a function f is a $\mathbb{P}^+$*-definable subset* of $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ if

1. $Dom(f) = D$
2. $Ran(f) \subseteq \mathbb{P}^+$
3. There exists some (classical) first-order $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x)$ with no universal quantifiers such that for any $a \in D$

$$f(a) = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi(a)\}$$

and let us define

$$Def^{\mathbb{P}^+}(D) := \{f : f \text{ is a } \mathbb{P}^+\text{-definable subset of } (\mathbb{P}, \leqslant, D)\}$$

Definition 3.3.2 (The derived model [Fit69], pp. 94 and 95)**.** Let $D' = D \cup Def^{\mathbb{P}^+}(D)$ and let us extend the forcing relation to the model $\mathcal{A}' = (\mathbb{P}, \leqslant, \Vdash, D')$

1. If $f, g \in D$, then $\mathcal{A}' \Vdash_p (f \in g)$, if and only if, $\mathcal{A}_\alpha \Vdash_p (f \in g)$
2. If $f \in D$ and $g \in D' \backslash D = Def^{\mathbb{P}^+}(D)$, then $\mathcal{A}' \Vdash_p (f \in g)$, if and only if, $p \in g(f)$.

3. If $f \in D' \backslash D = Def^{\mathbb{P}^+}(D)$, and $\varphi(x)$ is one formula that defines f over $\mathcal{A}$, then we say that $\mathcal{A}' \Vdash_p (f \in g)$, if and only if, there exist $h \in D$ such that

$$\mathcal{A} \Vdash_p \neg(\exists x)\neg(x \in h \leftrightarrow \varphi(x)) \text{ and } \mathcal{A}' \Vdash_p (h \in g)$$

Let $\mathcal{A}_0 = (\mathbb{P}, \leqslant, \Vdash, D_0)$ be any Intuitionistic Kripke Model satisfying the following conditions:

1. $(\mathbb{P}, \leqslant, \Vdash, D_0)$ is a set.

2. D_0 is a collection of functions such that if $f \in D_0$, $Dom(f) \subseteq D_0$ and $Ran(f) \subseteq \mathbb{P}^+$

3. For $f, g \in D_0$, $\mathcal{A}_0 \Vdash_p (f \in g)$, if and only if, $p \in g(f)$

4. For $f, g, h \in D_0$, if $\mathcal{A}_0 \Vdash_p \neg(\exists x)\neg(x \in f \leftrightarrow x \in g)$ and $\mathcal{A}_0 \Vdash_p \neg(f \in h)$, then $\mathcal{A}_0 \Vdash_p \neg(g \in h)$

5. D_0 is well-founded with respect to the relation $x \in dom(y)$.

Next, let $\mathcal{A}_{\alpha+1} := (\mathbb{P}, \leqslant, \Vdash, D_{\alpha+1})$ be the derived model of $\mathcal{A}_\alpha = (\mathbb{P}, \leqslant, \Vdash, D_\alpha)$.

If $\alpha \neq 0$ is a limit ordinal, let

$$\mathcal{A}_\alpha := (\mathbb{P}, \leqslant, \Vdash, D_\alpha) \text{ be defined by } D_\alpha := \bigcup_{\beta<\alpha} D_\beta$$

and if $f, g \in D_\alpha$, take any $\eta < \alpha$ such that $f, g \in D_\eta$ and let

$$\mathcal{A}_\alpha \Vdash_p (f \in g), \text{ if and only if, } \mathcal{A}_\eta \Vdash_p (f \in g).$$

Let

$$\mathcal{A} := (\mathbb{P}, \leqslant, \Vdash, D) \text{ where } D := \bigcup_{\alpha \in ON} D_\alpha$$

and if $f, g \in D$, take any $\eta \in ON$ such that $f, g \in D_\eta$ and let

$$\mathcal{A} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{A}_\eta \Vdash_p (f \in g).$$

Theorem 3.3.3 ([Fit69], Chapter 7, Theorem 3.2)**.** $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, D)$ is an Intuitionistic ZFC model, that is, classical equivalents of all the axioms of Zermelo-Fraenkel with choice, expressed without the use of the universal quantifier, are valid in the model.

Lemma 3.3.4 ([vD04], Lemma 5.2.1 (19))**.** It is a theorem of Intuitionistic Logic that for every formula $\varphi(x)$

$$\vdash_i \neg(\exists x \varphi(x)) \leftrightarrow (\forall x \neg\varphi(x))$$

Thus, we could change the expressions

$$\mathcal{A} \Vdash_p \neg(\exists x)\neg(x \in h \leftrightarrow \varphi(x))$$

by

$$\mathcal{A} \Vdash_p (\forall x)\neg\neg(x \in h \leftrightarrow \varphi(x))$$

And still get the same results of this section. This will be of use in the next section, where we consider $\Diamond$ as a generalization of $\neg\neg$.

3.3.2 Residuated Constructible sets.

We propose a notion of class of constructable sets in the context of Modal Residuated Kripke models of Set Theory that generalizes Fitting's construction. We only outline a propose for this construction and indicate some possible conditions that we believe are necessary for the generalization of Fitting's results.

Take an $MR-$Kripke model $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$.

Definition 3.3.5 (cf. [Fit69], p. 94). We say that a function f is a $\mathbb{P}^*$-*definable subset* of $\mathcal{A} = (\mathbb{P}, \leqslant, \delta, \Vdash, D)$ if

1. $Dom(f) = D$
2. $Ran(f) \subseteq \mathbb{P}^* = \{A \subseteq \mathbb{P}^* : \ A \text{ is strongly hereditary}\}$
3. There exist some $MR - \mathcal{L}_{\mathcal{A}}$-formula $\varphi(x)$ with no universal quantifiers such that for all $a \in D$

$$f(a) = \{p \in \mathbb{P} : \mathcal{A} \Vdash_p \varphi(a)\},$$

and denote

$$Def^{\mathbb{P}^*}(D) := \{f : f \text{ is a } \mathbb{P}^*\text{-definable subset of } \mathcal{A}\}$$

Definition 3.3.6 (cf. [Fit69] pp. 94 and 95). Let $D' = D \cup Def^{\mathbb{P}^*}(D)$ and let us extend the forcing relation to the model $\mathcal{A}' = (\mathbb{P}, \leqslant, \delta, \Vdash, D')$ as follows:

1. If $f, g \in D$, then $\mathcal{A}' \Vdash_p (f \in g)$, if and only if, $\mathcal{A}_\alpha \Vdash_p (f \in g)$
2. If $f \in D$ and $g \in D' \backslash D = Def^{\mathbb{P}^*}(D)$, then

$$\mathcal{A}' \Vdash_p (f \in g), \text{ if and only if, } p \in g(f).$$

3. If $f \in D' \backslash D = Def^{\mathbb{P}^*}(D)$, and $\varphi(x)$ is an $MR - \mathcal{L}_\in$-formula that defines f over $\mathcal{A}$, then we say that $\mathcal{A}' \Vdash_p (f \in g)$, if and only if, there exist $h \in D$ such that

$$\mathcal{A} \Vdash_p ((\forall x) \sim\sim \Diamond(x \in h \to \varphi(x)))\&((\forall x) \sim\sim \Diamond(x \in h \leftarrow \varphi(x))) \text{ and } \mathcal{A}' \Vdash_p (h \in g)$$

Let $\mathcal{A}_0 = (\mathbb{P}, \leqslant, \delta, \Vdash, D_0)$ be any MR-Kripke $\mathcal{L}_\in$-Model satisfying the following conditions:

1. $(\mathbb{P}, \leqslant, \delta, \Vdash, D_0)$ is a set.

2. D_0 is a collection of functions such that if $f \in D_0$, $Dom(f) \subseteq D_0$ and $Ran(f) \subseteq \mathbb{P}^*$

3. For $f, g \in D_0$, $\mathcal{A}_0 \Vdash_p (f \in g)$, if and only if, $p \in g(f)$

4. For $f, g, h \in D_0$, if $\mathcal{A}_0 \Vdash_p \neg(\exists x)\neg(x \in f \leftrightarrow x \in g)$ and $\mathcal{A}_0 \Vdash_p \neg(f \in h)$, then $\mathcal{A}_0 \Vdash_p \neg(g \in h)$

5. D_0 is well-founded with respect to the relation $x \in dom(y)$.

Let $\mathcal{A}_{\alpha+1} := (\mathbb{P}, \leqslant, \Vdash, D_{\alpha+1})$ be the derived model of $\mathcal{A}_\alpha = (\mathbb{P}, \leqslant, \Vdash, D_\alpha)$.

If $\alpha \neq 0$ is a limit ordinal, let

$$\mathcal{A}_\alpha := (\mathbb{P}, \leqslant, \Vdash, D_\alpha) \text{ be defined by } D_\alpha := \bigcup_{\beta<\alpha} D_\beta$$

and if $f, g \in D_\alpha$, take any $\eta < \alpha$ such that $f, g \in D_\eta$ and let

$$\mathcal{A}_\alpha \Vdash_p (f \in g), \text{ if and only if, } \mathcal{A}_\eta \Vdash_p (f \in g).$$

Let

$$\mathcal{A} := (\mathbb{P}, \leqslant, \Vdash, D), \text{ where } D := \bigcup_{\alpha \in ON} D_\alpha$$

and if $f, g \in D$, take any $\eta \in ON$ such that $f, g \in D_\eta$ and let

$$\mathcal{A} \Vdash_p (f \in g), \text{ if and only if, } \mathcal{A}_\eta \Vdash_p (f \in g).$$

3.3.3 Further work

We would like to make some concluding remarks and talk about some open questions and conjectures.

1. We wonder what kind of axioms of Set Theory are valid in the model $R^{\mathbb{H}}$ (see Definition 3.2.23). We expect this structure to be a model of Intuitionistic Set Theory, or a transformed version of it, via a Gödel-Kolmogorov-like translation. We believe that there must be some kind of isomorphism between $R^{\mathbb{H}}$ and the usual valued model $V^{\mathbb{B}}$, where $\mathbb{B} = \mathbb{H}/\mathcal{F}_{\neg\neg}$.

2. We proposed a version of Gödel's universe in Subsection 3.3.2. We wonder if the conditions that we proposed for our Residuated model are enough to prove a result similar to Theorem 3.3.3. We believe that some translated version of the axioms of IZF is valid in this model, via some kind of Gödel–McKinsey–Tarski translation.

3. In Section 2.3, we proposed two definitions of class of constructible sets in the context of Boolean-valued models and quantale-valued models. These definitions ended up collapsing to two valued models, but we wonder if there exists a variation of these definitions of constructibility in which the resulting model is not two valued.

4. We wonder what kind of independence results in **Set Theory** can be achieved with our Modal Residuated models of Set Theory that are not achievable with Intuitionistic models.

5. We would like to find an axiomatization for the logic of "complete residuated lattices

with a quantic nucleus". Furthermore, we would like to find a proper axiomatization for these modal logics when we consider additional properties for our quantic nucleus, such as respecting implications or idempotency with respect to products.

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