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Internal and external aspects of continuous logic and categorical logic for sheaves over quantales

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Quisiera dedicar este tiempo, amor y trabajo a mi madre que siempre ha sido mi gran maestra... también quisiera dedicarmelo a mi mismo por la fortaleza de traerme hasta acá y por lo que viene...

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Aspectos internos y externos de lógica continua y lógica categórica para haces sobre cuantales

Resumen

En este texto exploramos y proponemos nociones de haces sobre cuantales conmutativos e integrales, basadas en extensiones de resultados de la teoría de haces sobre locales: la interacción de los haces como conjuntos valuados y la analogía de los haces como categorías enriquecidas. Sobre estas propuestas, definimos lógicas que encuentran su semántica en estos objetos tipo haz; por un lado, una lógica categórica que caracteriza la noción de haces asociada a conjuntos valuados completos como un modelo de cierta construcción interna, y en contraste, una lógica definida externamente cuya naturaleza se basa en la lógica continua para espacios métricos, la cual encuentra en la propuesta de haces como categorías enriquecidas una estructura para interpretar su semántica.

Palabras clave: Haces, cuantales, categorías enriquecidas, espacios métricos, lógica cuantal-valuada.

Internal and external aspects of continuous logic and categorical logic for sheaves over quantales

Abstract

In this text we explore and propose notions of sheaves over commutative, integral quantales, which are based on extensions of results of the theory of sheaves over locales: the interplay of sheaves as valued-sets and the analogy of sheaves as enriched categories. Over these proposals, we define logics that find semantics in these sheaf-like objects, on the one hand, a categorical logic that characterize the notion of sheaves associated to complete valued sets as a model of certain internal construction, and in contrast an externally defined logic whose nature is based on continuous logic for metric spaces which finds in the proposal of sheaves as enriched categories an structure for interpret the semantic.

Keywords: Sheaves, quantales, enriched categories, metric spaces, quantale valued logic.

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Introduction

In this thesis, we study and propose a notion of sheaves over quantales in terms of enriched categories. Under this representation, we define a "continuous" logic (in the spirit of (co)quantale-valued logic ([DP21]) which in turn is based on the proposed continuous logic for metric spaces ([BYBHU08]) and, in contrast, a categorical semantics based on the notion of hyper-doctrines ([Hyl80]). This work resides within mathematical logic, based on the interplay between metric spaces and sheaves mediated by enriched category theory. We utilize this interplay to propose the continuous (external) - categorical (internal) contrast in our definitions of syntax and semantics. We present the aforementioned proposals, demonstrate their extension beyond the classical Cartesian case (the case of locales), and perform specific calculations.

The notion of sheaf on topological spaces and, in general, on locales, is rich in presentations and approaches, showcasing equivalences between geometric versions (in the original intention of the theory as an object of algebraic/differential topology and geometry), functorial version (in purely categorical terms), in terms of valued-sets (a viewpoint stemming from mathematical logic particularly from set-theoretical perspectives), and prominently, in this text, in terms of enriched categories that serve as a bridge to the metric-theoretical perspective.

Based on these multiple equivalences we explore in the context of quantales, categories of what we call, sheaf-like objects, mean that they extend to quantale based case objects that result equivalent to sheaves in the case of locales. Specifically, in this work there is two main versions of what a sheaf over a commutative integral (in general semi-cartesian) quantale is, the first version of

Alvim, Mendez, Mariano proposed and developed in [AdAMM23b], [AdAMM23a] of sheaves over semicartesian commutative quantales as valued sets with a suitable notion of completeness, we revisit the general definitions of this notion and propose it as a result of a generic construction based on categorical semantics, however, we do not delve deeply into its theory directly because our interest lies in focusing on the perspective we propose, grounded in enriched categories. The second version is based in a construction (presented in section 2.3) which extends for a several classes of quantales the proposal of [Wal81] given by Walters, that aloud present the category of sheaves over a locale as a category of certain enriched categories over a locally ordered bicategory constructed from the locale, in this perspective sheaves over quantales are certain (Cauchy complete) enriched categories over suitable ordered bi-categories, this perspective naturally extend the theory of continuity spaces (see [Fla97]) as categories enriched over quantales and the logical framework for this introduced in [DP21][1] is also extended in section 3.3 where this kind of sheaf-like object arise as the semantic structure for the proposed logic.

So in one hand we have a in nature categorical logic in which the notion of sheaf over a quantale proposed by Alvim, Mendez, Mariano constitutes a semantical example, and a proposal of notion of sheaf over quantales which extends the presentation of Walters in [Wal81] with a semantic externally defined which enables this sheaf-like object as a structure. These proposed notions of sheaves on quantales are based on extensions of the multiple (of course equivalent) versions that exist in the case over locales, we briefly present them in the section 1.3.

In addition to the aforementioned, an important part of the work is focused on establishing a relationship between both notions of sheaf over a quantale proposals that are studied. We do this through the work of Isar Stubbe who, interested in the concept of internal order in categories of sheaves over quantaloids ([Stu05a]), introduces the notion of enriched totally regular semicategory on a quantaloid and its Cauchy completions [Stu05b], [Stu05c] notion that is surprisingly parallel to the one developed at IME-USP by Mariano's students, not just by the basic axioms

[1] This work constitutes the undergraduate thesis of the author of this text.

they find for construct the theory but also for the way the theory goes, both of them connect the completion notions with relational morphism and has pretty much the same crucial main theorems, so related their works can be consider as a bibliographic accomplished, and is because a note in Stubbe work that we find that our construction in section 2.3 is an equivalent of a canonical construction called Karoubi envelope or split-idempotent completion (Cauchy completion for categories, see [nLa23]) which enable us to recognize the similarity between the mentioned approaches in the section 2.4.

The main technical concept is that of Cauchy complete enriched (totally regular semi-) categories over quantal(oids), for our perspective and proposed is crucial the relationship between enriched categories and (generalized) metric spaces, the foundational article of Lawvere [Law73] where he already points out the importance of this analogy as a methodological object to export notions and techniques between the theory of categories and that of metric spaces, in addition to his pioneering approach in using logical translation subject to change the base quantale as a mechanism for that. This is why the short article [Wal81] in its simplicity it is one of our greatest inspirations, since in a quite neat way it not only poses sheaves in terms of enriched categories but also reveals Cauchy's completeness as the adequate translation of the gluing condition for pre-sheaves.
In the following, we provide a brief summary of what is covered in each chapter, aiming to offer a general perspective of the work before delving into the main body.
The thesis is divided into 3 chapters:

Chapter 1: Preliminaries

- Enriched categories: monoidal categories, enriched categories over a monoidal category. We focus on enriched categories over quantales, emphasizing their similarity to metric spaces. The primary example in this context lies within the monoidal category $([0,\infty],\geq,+,0)$, where enriched categories result in generalized metric spaces termed Lawvere spaces. Finally, we recapitulate multiple equivalences in the context of topological spaces (and locales in general) concerning the notion of sheaf: geometric, functorial, based on valued sets (Omega-Sets), and notably based on enriched categories (work by Walters [Wal81],[Wal82] foundational for our perspective). Through this last representation, we introduce the notion of quantaloid and discuss some generalities. Everything presented in this chapter is part of known literature, and proper citations are made.

Chapter 2: Sheaf-like categories over quantales

This chapter comprises 4 sections, the first two being monographic work, while the latter two mainly consist of original work:

Section 2.1: The proposal of sheaves over quantales extending the representation in terms of Ω-Sets for sheaves over a local Ω. These are termed Q-sets ([AdAMM23b],[AdAMM23a]), where Q is a commutative, semicartesian quantale. This proposal, developed by students under the guidance of the director of this thesis, is compared with our proposal towards the end of the chapter.

Section 2.2: Following Isar Stubbe's work ([Stu05a],[Stu05c],[Stu05b]) on sheaves over quantaloids Q (he works directly in this more general context) as totally regular, symmetric, and Cauchy-complete semicategories over Q. This is important for us due to its work in enriched structures and its clean reflection of the relationship with the proposal in the previous section.

Section 2.3: Original work. Based on a commutative and integral quantale Q, we construct a quantaloid Rel(Q) resulting from "dividing Q". The main achievements of this construction are:

1. To define it in a way that extends the construction enabling the connection with sheaves in the case of locales.

2. To provide the definition that also integrates the "Lawvere quantale" $([0,\infty],\geq,+,0)$, allowing us to draw an analogy with continuous logic.

We specify sufficient conditions on the quantale Q for Rel(Q) to indeed be a quantaloid. For this class of quantales, we propose our notion of sheaf: A sheaf over Q is an enriched category over Rel(Q) that is symmetric, skeletal (a technically transparent condition in its meaning), and

Cauchy complete.

Section 2.4: Building on the clarity of Section 2 and considering our definition in Section 3, we independently find that Rel(Q)-categories correspond to Q-semicategories that are totally regular. Therefore, the proposals from Section 2 (Stubbe) and Section 3 (ours) are equivalent, and it is also transparently evident that these are included in the proposal of Section 1. This comparison is crucial in our contexts, where for us, the gluing condition is Cauchy completeness, while for them, it is a notion (indeed, they work with several) of completeness called Scott-completeness, slightly weaker, making their context more general. This sharp comparison contributes significantly to the ongoing research and stands out as a monographic achievement in this work.

Chapter 3: Except for some definitions in the initial part, the content of this chapter is original: As mentioned, the aim is to contrast the continuous (external)/categorical (internal) semantics.

Sections 3.1 and 3.2 are dedicated to introducing and developing a notion we propose in this thesis: the notion of first-order monoidal hyper-doctrine, formulated for quantales as an extension in the monoidal context of the Cartesian notion of first-order hyper-doctrine. We use this notion to characterize the category of Q-sets (with morphisms as relations) as a category of sets and relations internal to the hyper-doctrine of Q-families. We also calculate the hyper-doctrine associated with the sub-object functor in the category of Q-sets with functional morphisms. All these computations, as well as the proposal, constitute original work.

Section 3.3: We define a logic interpreted over Rel(Q)-categories, extending the quantale-valued logic in the context of quantaloids, which further extends the continuous logic for metric spaces. Hence, we term it continuous logic for enriched categories over quantaloids. An interesting initial finding is that this logic specializes into: - Classical logic ($0,1$-valued) for pre-orders. - Intuitionistic logic (local-valued) for sheaves over locales. - Continuous logic $[0,\infty]^{op}$-valued for Lawvere spaces (generalized metric spaces).

As future work, using these logics to understand our sheaves over quantales is intended, along with the development of theoretical model notions within these logics.

Original proposes and achivements

- Showing sufficient conditions to construct a quantaloid from a quantale, to extend the construction of the subsection 1.3.5 for which the relation with sheaf categories is based. See section 2.3. In particular, Theorem 2.3.6 seems to be a novelty.

- Propose a notion of sheaf over commutative, integral, divisible and strict linear quantales, that extent Walter´s characterization of the localic case.

- Using this presentation to naturally give an interpretation of continuous (external) first-order logic in analogy with [BYBHU08] and extending [DP21]. See section 3.3.

- Proposing a generalized notion of first-order hyperdoctrines based in quantales giving a framework to study categorical (internal) logic of this sheaf like categories. See section 3.2.1.

- Use the aforementioned notion to present $Q-Set_{rel}$ as a category of partial internal equivalence relation of the monoidal hyperdoctrine of Q—families. See section 3.2.3.

- Calculate the hyperdoctrine (characterize equality, quantifiers and connectives) associated to the sub-object functor over $Q-Set_{fun}$. See sub-section 3.2.2

Monograph achievements

- Applying the theory of [Stu05c], [Stu05a] to compare the set-valued version of sheaves with the enriched-category version.

- Exposing the similitude of the work of Alvim, Mariano, Mendez in the theory of $Q-sets$ and the theory of totally regular semi-categories enriched over quantaloids introduced by Stubbe.

1 Preliminaries

In this chapter we introduce the elements and results that will be generalized and used as a basis in subsequent chapters.

We begin by exposing the general framework of the theory of enriched categories since many of the structures at stake in the thesis are of this type, and it is through this concept that one of the most important analogies for us is established, that of metric spaces as categories. We quickly focus on the type of basic category that we want to enrich, the quantales. Since one of the intentions of this text is to connect at the level of quantales the theory of enriched categories with that of sheaves, we make a section dedicated to studying multiple representations under equivalence of the category of sheaves on a local, one of which (that we put at the end) is the bridge with enriched categories, it reveals the general type of structure that interests us in the base, the so-called quantaloids, we dedicate a final section to some generalities of these.

1.1 Enriched categories

The basic structure in categorical thinking is the monoid, beyond the fact that a single object category is just a rephrasing of the structure of a set-theoretical monoid, the existence of an associative binary operation with identity element is at the core of the principal concepts in category theory, often through the structural amalgamation of various monoids it is possible to uniformly re-capture notions and concepts coming from almost any field of mathematics. This is the case of monoidal categories which take over the multi-monoidal structure of category, an additional external structure of monoid, the first examples come from real analysis, commutative algebra, topology, and other areas that at first seem independent of categorical approach. The main references we use for these preliminaries is founded in the encyclopedic book of Kelly [Kel05] for the basic theory of enriched categories over monoidal categories, for us is central to the step to enriched categories over bicategories, a very general theory that is introduced by Benabou in [J.67] and explored to in [Ben73].

Definition 1.1.1. *A **monoidal category** $\mathcal{V} = (V, \otimes, I)$ consist in a category V, a functor*[1]

$$\otimes : V \times V \to V$$

an object I in V and for every $X, Y, Z \in Obj(V)$, natural isomorphism

$$a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$
$$l_X : I \otimes X \to X$$
$$r_X : X \otimes I \to X$$

Axioms of coherence are imposed over a requiring associativity of $(V, \otimes)$ and the role of module of (I, l, r) respect to $\otimes$.

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle a \otimes 1_Z} & & & & \uparrow{\scriptstyle 1_W \otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\qquad\qquad a \qquad\qquad} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a} & X \otimes (I \otimes Y) \\ \downarrow{\scriptstyle r_X \otimes 1_Y} & \swarrow{\scriptstyle 1 \otimes l_Y} & \\ X \otimes Y & & \end{array}$$

*The category is said to be **strict** if a, r and l are equality's rather than natural isomorphism ´s.*

[1] Consider in the domain the product category structure.

Definition 1.1.2. *A monoidal category* $(V, \otimes, I)$ *is said to be* ***symmetric*** *if for every* A, B *objects of* V *there are is a natural ismorphism* $A \otimes B \to B \otimes A$

Again we say that $(V, \otimes, I)$ if is symmetric and the cited isomorphism becomes an identity.

Definition 1.1.3. *A monoidal category* $(V, \otimes, I)$ *is said to be* ***closed*** *when for every* Y *object in* V *the functor* $- \otimes Y : V \to V$ *has a right adjoint usually note as* $[Y, -] : V \to V$*. If the monoidal product is not naturally commutative then is necessarily a right adjoint for* $Y \otimes -$*, noted by* $[-, Y]$*.*

Examples 1.1.4. *A structural source of examples of monoidal categories is the cartesian categories: Those whose monoidal product is given by the cartesian product, i.e. categories with product. Particular interest examples are:*

- Set *the category of sets and functions with the usual product of sets performed by sets of order pairs. The exponentiation of sets* $A^B = \{f : B \to A : f$ *is a function*$\}$ *determines a right adjoint of the product, so this a monoidal (cartesian) closed category.*

- *For any category* C*, its associated presheaf category* $\mathsf{Set}^{C^{op}}$ *is a cartesian category with product calculated pointwise so as exponentiation, then is closed to.*

- Top *the category of topological spaces and continuous functions with the usual product topology for the underline product of sets, the space based in the singleton* $\{\star\}$ *is a module. This category have some interesting subcategories for which the product finds a right adjoint, but* $(\mathsf{Top}, \times, \{\star\})$ *is not closed.*

- Ord *the category of order sets and isotone functions determines a cartesian category with products based in the underlying product of sets and component by component order.*

- $(L, \leq)$ *a lattice as a category which are precisely the complete pre-orders as categories with finite products (meets) and coproducts (joins). The requirement of being closed is exactly the definition of Heyting algebra, so the cartesian closed pre-orders as categories are the Heyting algebras.*

Some interesting non cartesian examples are

Examples 1.1.5. *For a field* k *the category of vector spaces over* k*, determines a monoidal category with tensorial product of vector spaces*

$$V_1 \otimes_k V_2 = (V_1 \times V_2)/\sim$$

Which is given by the quotient of the equivalence relation generated by the necessarily identifications that allows turn bi-linear maps $V_1 \times V_2 \to V_3$ *to linear maps* $V_1 \otimes_k V_2 \to V_3$ *through the the projection to the quotient* $(v_1, v_2) \mapsto [(v_1, v_2)]$*. The module object is* k *itself as a* $k-$*vector space, this naturally arise the associativity of this monoidal product. Precisely the cited property of turn the bi-linear to linear, makes that* $\mathrm{Hom}(V_1, V_2)$ *the vector space of linear maps* $V_1 \to V_2$ *between vector spaces, constitutes a right adjoint* $- \otimes_k V \dashv \mathrm{Hom}(V, -)$*. So this an example of monoidal closed category.*

Examples 1.1.6. *The collection of endofunctors of a small category* C, $\{F : C \to C : F$ *is a functor*$\}$, *like composition is associative and the identical functor is a module for composition, this collection with this monoidal structure define a monoidal category.*

Examples 1.1.7. *Consider the natural numbers and co-functions as a subcategory of* Set^{op} *then the products in this category are the coproducts of sets and for natural numbers* $n, m \in \mathbb{N}$ *this coproduct corresponds to the sum* $n + m$ *off course the module object is the* 0*.*

Thinking this is cartesian structure the importance of this example finds in it is the initial pointed cartesian category, indeed define a functor that respects products with this domain is just choosing an image of 1.

Examples 1.1.8. *The closed interval* $[0, 1]$ *with the usual order and usual product of reals determines a non-cartesian monoidal category, indeed like the category structure is given by the order the cartesian product is the wedge that in this linear order is the minimum, so the product constitutes a monoidal extra structure.* $([0, 1], \leq, \cdot, 1)$

In general a monoidal structure over the unit interval is called a $t-$norm.

Examples 1.1.9. $([0,\infty],\geq_{usual},+,0)$ *again we are taking the pre-order* $([0,\infty],\geq_{usual})$ *as a category and the sum* $a+b$ *as the monoidal product of the objects* a,b *then* 0 *is a module, off course this operation is associative. Note that truncated subs-traction* $a\hat{-}b = max\{0, a-b\}$ *defines the adjoint relation* $a\hat{-}b \leq c$ *if and only if* $a \leq b+c$ *so this is a monoidal closed category.*

The function $-\ln : [0,1] \to [0,\infty]$, $0 \mapsto \infty$, $x \mapsto -\ln(x)$ determines an isomorphism of monoidal categories, so $([0,1],\leq,\cdot,1)$ is closed to.

The main example for monoidal closed us are **quantales**

Definition 1.1.10. *A* ***quantale*** *is triple* $\mathcal{Q} = (Q,\leq,\otimes)$ *such that* $(Q,\leq)$ *is a complete lattice,* $(Q,\otimes)$ *is a semigroup and the followings distributions always holds:*

$$a \otimes \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \otimes a_i),$$

$$(\bigvee_{i\in I} a_i) \otimes a = \bigvee_{i\in I}(a_i \otimes a)$$

Fact 1.1.11. *A quantale with a unit is a monoidal closed category.*

Indeed in the framework of pre-orders as categories for a functor have a right adjoint is enough to respect arbitrary colimits that is to say preserve joins, so an equivalent definition of a quantale with module could be: a pre-order that as a category is a complete monoidal closed one.

Definition 1.1.12. *Given a quantale* Q *and an element* $a \in Q$*, the residuum of*

$$-\otimes a : Q \to Q$$

and

$$a \otimes - : Q \to Q$$

are denoted by $a \implies_r - : Q \to Q$*, and* $a \implies_l - : Q \to Q$ *respectively.*

This means that for all $b,c \in Q$, $a \otimes b \leq c$ iff $b \leq a \implies_l c$ and $b \otimes a \leq c$ iff $b \leq a \implies_r c$.

A locale is a quantale where $\otimes = \wedge$, as we mentioned before, this corresponds to the version for categories that are orders of the conception of a monoidal product like a generalization of the cartesian product.

The poset of all open sets of a topological space X constitute a locale. Locales coincide with complete Heyting algebras[2].

We give the previous definition, because of the generality of the frame-work that bring the theory of enriched-categories over a monoidal closed category, but in the practice this text use only monoidal closed categories that are quantales or their inmediate 2-structural[3] generalization that are called quantaloids.

Definition 1.1.13. *For a monoidal category $\mathcal{V} = (V, \otimes, I)$, a* ***enriched category*** *over $\mathcal{V}$ or a $\mathcal{V}$-category A consist on: a set A of objects, a function*

$$A(-,-) : A \times A \to \mathrm{Obj}(V)$$

Such that every $a, b, c \in A$, there are morphisms in V

- *(Identities)* $id_a : I \to A(a,a)$
- *(Composition)* $\circ_{abc} : A(a,b) \otimes A(b,c) \to A(a,c)$

that make the later diagrams commute

$$\begin{array}{ccccc}
(A(c,d)\otimes A(b,c))\otimes A(a,b) & & \xrightarrow{\quad a \quad} & & A(c,d)\otimes(A(b,c)\otimes A(a,b)) \\
\downarrow{\scriptstyle \circ_{bcd}\otimes 1_{A(a,b)}} & & & & \downarrow{\scriptstyle 1_{A(c,d)}\otimes\circ_{abc}} \\
A(b,d)\otimes A(a,b) & & & & A(c,d)\otimes A(a,c) \\
& \searrow{\scriptstyle \circ_{abd}} & & \swarrow{\scriptstyle \circ_{acd}} & \\
& & A(a,d) & &
\end{array}$$

$$\begin{array}{ccccc}
A(b,b)\otimes A(a,b) & \xrightarrow{\circ_{abb}} & A(a,b) & \xleftarrow{\circ_{aab}} & A(a,b)\otimes A(a,a) \\
\uparrow{\scriptstyle id_b\otimes 1_{A(a,b)}} & \nearrow{\scriptstyle l} & & \nwarrow{\scriptstyle r} & \uparrow{\scriptstyle 1_{A(a,b)}\otimes id_a} \\
I\otimes A(a,b) & & & & A(a,b)\otimes I
\end{array}$$

[2]Recall that the class of all Heyting algebras provides the natural algebraic semantics for the intuitionistic propositional logic, that is the “constructive fragment" of the classical propositional logic.

[3]in the sense of the n−categorical framework

The former diagrams are a good example of what it means to export in terms of morphism and commutative diagrams the version of a concept that is clear in set-theoretical diagrams, with concrete objects indeed the first diagram is the commutativity of the procedures of associating and composing, that is, the associativity of composition and the second one in concrete terms (well-pointed among other) means that the element that "choose" the $\mathtt{id}_a$ morphism is a local identity of this locally defined composition, let us make this our first example:

Examples 1.1.14. *For the monoidal (cartesian) closed category* $(\mathrm{Set}, \times, \{\star\})$ *a* $\mathrm{Set}-$*enriched category is exactly the definition of small category, in that sense enriched category theory extent category theory as sets and functions constitute a category.*

Examples 1.1.15. *Our example of vector spaces over a field* k *as a monoidal category in which the monoidal structure is not the cartesian product can be generalized to modules over a commutative ring* R*, in particular the category* ***Ab*** *of abelian groups (modules over* $\mathbb{Z}$*) is a monoidal closed category, an* ***Ab*** *enriched category is an election of not just a set of morphism between two objects but an abelian group of it, all the involved morphism are group homomorphism, this structures are called additive categories.*

The following example constitute the kind of object which in this thesis extensively about, is the first example of an enriched category over a quantale, and from this example is extracted huge part of the intuition in at least three level, at a logical level (is the grounded classical example) and at a categorical level in which the permanent translation of the concepts of pre-sheaf theory for enriched categories let us an spacial notion, and as is already mentioned as a kind of metric space, intuition that comes from Hausdorff himself [F.14].

Examples 1.1.16. *Take the boolean algebra* $2 = (\{\bot, \top\}, \bot \leq \top, \wedge)$*, like in particular is a Heyting algebra, constitute (as a pre-order) a cartesian category and then monoidal category, for a set objects* X *under the translation* $X(a, b) = 1$ *if and only if* aRb*, for* $R \subseteq X$ *the composition and identity arrows are respectively transitivity and reflexivity of* R*. Then the* $2-$*categories are the pre-orders.*

As mentioned, we will focus on the case where the monoidal category on which it is enriched is a quantale, the previous is the first example but this matter deserves a separate hole section.

Before this we make some remark about concepts which are going to be studied in a slightly more general context in chapter 2 but which are of importance in representing the category theory-metric space theory connection, which is found in categories enriched over quantales. Again [Kel05] is a good reference for the results, but [Law73] it is central because it was in this article that the axiomatic analogy that leads to the formulation of metric spaces as categories is evidenced, in addition to capturing the notion of Cauchy completeness through the bi-categorical notion of distributor introduced by Benabou in [Ben73].

For it To formulate this we make a few brief notes on the theory of V—functors and V-natural transformations, for a monoidal closed (and usually symmetric) category $(V, \otimes, I)$. A V—functor $X \xrightarrow{F} Y$ (or an enriched functor over V) between V—categories X and Y has two parts: one is given by a function

$$\begin{gathered} Obj(X) \xrightarrow{F_0} Obj(Y) \\ a \mapsto F(a) \end{gathered}$$

that maps objects of X into objects of Y and the other is a coherent assignation of morphism in V indexed with couples of objects of X

$$\begin{gathered} Obj(X) \times Obj(X) \xrightarrow{F_1} Morp(V) \\ (a, b) \mapsto X(a, b) \xrightarrow{F_{ab}} Y(F(a), F(b)) \end{gathered}$$

subject to axioms (commutative diagrams in V) who assert that this assignment respects the composition maps on X and Y as well as carrying identities in identities.

$$\begin{array}{ccc} X(a,b)\otimes X(b,c) & \xrightarrow{\circ^X_{abc}} & X(a,c) \\ \downarrow{\scriptstyle F_{ab}\otimes F_{bc}} & & \downarrow{\scriptstyle F_{ac}} \\ Y(F(a),F(b))\otimes Y(F(b),F(c)) & \xrightarrow{\circ^Y_{FaFbFc}} & Y(F(a),F(c)) \end{array}$$

$$\begin{array}{ccc} & & X(a,a) \\ & \nearrow^{id_a} & \downarrow{\scriptstyle F_{aa}} \\ I & \xrightarrow[id_{F(a)}]{} & Y(F(a),F(a)) \end{array}$$

With this in mind can be formulated the notion of V—natural transformation between parallel V—functors as an object of V, then perform the V—category of functors. For this consider $F,G:X\to Y$ V—functors then the V—objects of natural transformations is noted by $Y^X(F,G)$ and calculated as the V—object that makes the following an equalizer diagram in V.

$$Y^X(F,G)\longrightarrow\prod_{a\in X}Y(F(a),G(a))\ \overset{r}{\underset{s}{\rightrightarrows}}\ \prod_{a,b\in X}V(X(a,b),Y(Fa,Gb))$$

This proposes the V—category of V—functors between two V—categories $Y^X=\{X\xrightarrow{F}Y\}$, with homs as above. In the usual category a natural question is the role of this construction as an exponentiation, i.e. the question if it is the right adjoint of a certain product. The answer is affirmative and the mentioned product work by component perfectly well, this means that if X and Y are V—categories then taken the set-product of the collections of objects, making $X\otimes Y((x_1,y_1)(x_2,y_2)):=X(x_1,x_2)\otimes Y(y_1,y_2)$ and multiplying composition morphism and identities determine a V—category $X\otimes Y$, the thing is that there is a natural and bijective correspondence of V—functors

$$(X\otimes A\to Y)\leftrightarrows(A\to Y^X)$$

.

The context of monoidal symmetric closed categories is robust enough that the basic theory of categories, functors, and natural transformations can be fully translated into the enriched case. We will limit ourselves to citing the Yoneda lemma in a schematic version that allows us to stress and introduces[4] the central notion of chapter 2, the concept of a complete Cauchy enriched category.

The importance of a category V to being monoidal closed is that of the existence of an internal hom for every pair of object $V(u,v)$, this is just the fact that V it itself a $V-$category if we make $V(u,v) = u \implies v$ (recall that the monoidal structure is symmetric the left and right implications are isomorphic). Then we can perform V^A for every $V-$category A, and this behaves like a usual functorial category. Remark that the central notion is that of pre-sheaf then take better $V^{A^{op}}$, this is just the remark that if A is $V-$category, define A^op with the same objects of A and $A^{op}(a,b) = A(b,a)$ for every couple of objects. With this, we can cite the enriched Yoneda-lemma.

Lemma 1.1.17. *(**enriched Yoneda lemma**, see[Kel05], [Law73]) For every $V-$category A, object a of A and $V-$functor $A^{op} \xrightarrow{F} V$ there is a natural correspondence between natural transformations $h_A(a) \to F$ and morphism $I \to F(a)$.*

Where $h_A(a) : A^{op} \to V$, $b \mapsto A(b,a)$ is the (in A) pre-sheaf of the object a, and a morphism $I \to F(a)$ is the version of an "**element**" of $F(a)$. As in usual category theory, the above implies the existence of an enriched Yoneda immersion that behaves like an (enriched) immersion, in the sense that meets the enriched versions of being full and faithfully.

$$A \to V^{A^{op}}$$
$$a \mapsto h_A(a)$$

In this sense an enriched functor $X \to V^{Y^{op}}$ is a generalized enriched functor from X to Y, because of the adjunction cited earlier, this generalized functor corresponds to an actually enriched arrow $Y^{op} \otimes X \to V$ which can be thinking as an “valued relation" on V, which measures “truth value of the relatedness of an element of Y to an element of X" ([Law73]).

This information arises as the concept of bimodule where the category of enriched in **Ab**, the

[4] and justify its name

monoidal category of abelian groups, or as the bicategorical concept of distributors (also called profunctors) (see [Ben73]). In this general setting, the name of distributors is inherited.

A **distributor** between V—categories

$$\phi : X \multimap\!\!\!\rightarrow Y$$

is given by a function $Obj(Y) \times Obj(X) \to Obj(V)$ such that for every x, x' objects of X and y, y' objects of Y there is arrows: $Y(y', y) \otimes \phi(y, x) \to \phi(y', x)$ and $\phi(y, x) \otimes X(x, x') \to \phi(y, x')$, that are coherent for the associativity and unities in both X and Y, and commutes with the composition in both V-categories.

As is natural to think every "normal" arrow (V—functor) induce a “generalized" arrow (a distributor). In fact induces two an this has an spacial behavior which allows the formulation of the Cauchy completeness.

Given a V—functor $X \xrightarrow{f} Y$ there is a couple of distributors $f_* : X \multimap\!\!\!\rightarrow Y$ and $f^* : Y \multimap\!\!\!\rightarrow X$ with $f_*(y, x) = Y(y, f(x))$ and $f^*(x, y) = Y(f(x), y)$. There is an adjunction property (in the 2—Category V—**Cat** of V—categories, V—functors and V—natural transformations) between them, and a category is said to be **Cauchy complete** if every couple of distributors with this adjoint relation arises from a V—functor in the form of f_* and f^*, in a nutshell: *Every adjunction between distributors (relations/generalized maps) is induced by a functor (map).*

We give this notion a bit vague because the concept will be well treated in chapter 2 (in the midst of a structural elevation as will be seen), however we are left with the intuition of what happens in pre-orders and Lawvere’s central result.

Theorem 1.1.18. *(see [Law73]) For a metric space seen as* $([0, \infty], \geq, +, 0)$*—category to being Cauchy complete means that every Cauchy succession (as in real analysis) converge in it (as in real analysis).*

1.2 On Quantales

Quantales are an algebraic-theoretic generalization of Locales, proposed in [C.86] as a non-commutative, non-idempotent version of the locale of open sets of a topological space which is the prototype example of locale, then open a framework for non-commutative topology, that aloud interpretation in diverse fields inside and outside of strictly pure mathematics. Our intention is sheaf theory and its related logic(s) over quantales and this is a partially very well field of study, however, there are many efforts and victories in the part of the theory related with commutative (or sometimes no) but importantly *idempotent* quantales, see for example [BF86],[FU98],[U98],[P.11], our work has the focus on an "orthogonal"[5] kind of quantales. The following definitions show the general framework and the usual extra properties, but in general, we focus on commutative integral quantales.

Definition 1.2.1. *We said that a quantale* $\mathcal{Q}$ *is*

- *Commutative If* $(Q, \otimes)$ *is commutative.*
- *Unital If* $(Q, \otimes)$ *is a monoid. In this case, the unity will be denoted by* 1.
- *Semi-cartesian If for all* $a, b \in Q$ *we have* $a \otimes b \leq a, b$.
- *Integral If it is unital and the module element coincides with the top element of the lattice structure:* $1 = \top$.
- *Right-sided If for all* $a \in Q$ *we have* $a \otimes \top = a$.
- *Idempotent If all the elements of* Q *are idempotent in the semigroup structure* $(Q, \otimes)$.
- *Right divisible If for all* $a, b \in Q$ *such that* $a \leq b$ *then exist* c *such that* $a = b \otimes c$
- *Left strict monotone If for all* $a, b, c \in Q$ *such that* $a < b$ *and* $c \neq 0$*, then* $c \otimes a < c \otimes b$

[5] first propositions makes clear the use of this expression

In the section 2.1 in [AdAMM23a], we can find the following results:

Proposition 1.2.2. 1. *locale $\Longrightarrow$ divisible $\Longrightarrow$ integral $\Longrightarrow$ unital, semicartesian*

2. *locale $\Longrightarrow$ divisible $\Longrightarrow$ T·T= T*

3. *locale $\Longrightarrow$ idempotent $\Longrightarrow$ strong $\Longrightarrow$ T·T= T*

4. *idempotent+semicartesian $\Longrightarrow$ locale*

5. *A commutative quantale $(Q, \leq, \otimes)$ is divisible if and only if for all $a, b \in Q$, if $a \leq b$ then $a = b \otimes (b \implies a)$*

Examples 1.2.3. 1. $([0, \infty], \geq, +)$*: is a commutative, integral and divisible quantale. Moreover, it is a left/right strict monotone quantale.*

2. *$(\omega, \leq, +)$: is a commutative, integral, and divisible quantale. Moreover, it is a left/right strict monotone quantale.*

3. *The inclusion poset of the ideals of commutative unital rings endowed with the product of ideals: is a commutative and integral quantale; moreover it is a divisible quantale whenever the ring is a PID.*

4. *The poset of closed right ideals of a $C^*-\mathtt{algebra}$ endowed with the operation of the topological closure of product of the right ideals is an idempotent and right-sided quantale.*

5. *Denote by Δ the set of left continuous functions from $[0, \infty]$ to $[0, 1]$ with the pointwise ordering, a structure of integral commutative quantale over it $\otimes : \Delta \times \Delta \to \Delta$ is called a triangle function, in [SS] are presented many kinds of triangle functions and then many kinds of (actually continuous) commutative quantales.*

6. *For any set R the collection, the collection of $\subseteq$-down closed collection of finite subsets of R, constitutes a quantale with the order given by inclusion and the operation given by intersection.*

This constitutes a locale and then a commutative idempotent integral quantale. We refer to this example as the ***free locale*** *over* $\mathbb{R}$.

Notation 1.2.4. *For* $\mathcal{Q}$ *an integral quantale we write* (X, d_X) *denoting a enriched category over* $\mathcal{Q}$ *as a monoidal closed category.*

Explicitly $\mathcal{Q} = (Q, \leq, \otimes, 1)$, a **enriched category** over Q consist in a set X of objects, and a function

$$d : X \times X \to Q$$

such that for every $x, y, z \in X$ we have

$$1 \leq d(x, x)$$
$$d(a, b) \otimes d(b, c) \leq d(a, c)$$

As is already mentioned in example 1.1.16 the 2-categories are the pre-orders, and in certain way a pre-order is a kind of space (a hierarchy of positions), the following examples aims to stress the intuition that enriched categories over **commutative, integral** quantales are in a own-way certain notion of space.

Examples 1.2.5.

1. *A* $([0, \infty], \geq, +, 0)-$*category is a generalized pseudo-metrical space, a kind of metric space where the distant assignation is not necessarily symmetric, the existence of different elements at distance 0 is admitted, as well as elements at infinite distance.*

2. *Probabilistic distributions spaces, this notion was defined in [SS], this theoretical approximation allows us to consider a kind of parameterized metric space where for each non-negative real* r *there is an associated metric structure* δ_r *that calculates the probability between points in the space of being at distance* r.

3. *In [Fla97] is show that for any topological space* (X, τ) *the free locale (as in the last example of 1.2.3) over* τ*, called* $\Omega(\tau)$*, the* $\Omega(\tau)-$*categories carries a metric topology that coincides*

with τ*. This is, any topology arises a "metric" topology where, "metric" means enriched over a quantale.*

As mentioned, the previous examples try to justify the analogy of enriched categories over commutative integral quantales as a kind of (perhaps quite general) metric spaces. One of the main points of this preliminaries chapter is to expose how this notion of enriched category also connect with the notion of sheaf for the case of locales, this connection can be made more clear if we note the axiomatic similarity between a enriched categories and a valued set, and how valued sets connects with sheaf theory, for all of this lets an overview of the case for locales that is the ground for the constructions presented in chapter 2.

1.3 Sheaves-like categories: the localic case

The notion of sheaf of structures on a topological space depends only on the structure of the lattice of the open subsets of the space: this lattice constitutes a complete Heyting algebra that coincides with the notion of locale.

There are many equivalent descriptions of the notion of the sheaf of sets over a locale. We reserve this section to briefly present these notions since some of these are generalized to the quantalic setting.

1.3.1 Geometric and functorial sheaves over topological spaces

The following results can be taken as folklore of sheaf theory, [LM92] is a good reference for the proofs and details.

Let X be a topological space. A **geometric sheaf** of sets over X is just a local homeomorphism from a topological space Y into X, $f : Y \to X$. Note that a local homeomorphism is automatically a continuous open map and that the for each $x \in X$, the topological subspace $p^{-1}[\{x\}] \subset Y$ is a (possibly empty) discrete subspace of Y. A morphism between geometric sheaves $(Y, f) \to (Y', f')$

is a continuous function $h : Y \to Y'$ such that $f' \circ h = f$. Such continuous function is automatically a local homeomorphism, thus it is an open map too. With the obvious definition of composition and identities, this data provides a category, $GSh(X)$, the category of geometric sheaves (of sets) over the space X.

Now denote by $\mathcal{O}(X)$ the category associated to the poset of all open sets of X. A **presheaf of sets** is a functor $F : \mathcal{O}(X)^{op} \to Set$, and a morphism of presheaves is just a natural transformation between functors. Vertical composition of natural transformations (with the obvious identities) provides a category, $pSh(X)$, the category of presheaves (of sets) over the space X.

Consider a presheaf $F : \mathcal{O}(X)^{op} \to Set$. Given inclusions $U \subseteq V$, we use $s_{|^V_U}$ (or just $s_{|_U}$) to denote the “restriction map" from $F(V)$ to $F(U)$. If $U \subseteq X$ is open and $U = \bigcup_{i \in I} U_i$ is an open cover, a presheaf F is a **(functorial) sheaf** (of sets) when we have the following diagram

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \cap U_j)$$

is an equalizer in the category Set, where:

1. $e(t) = \{t_{|_{U_i}} \mid i \in I\}, \quad t \in F(U)$

2. $p((t_k)_{k \in I}) = (t_{i_{|_{U_i \cap U_j}}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|_{U_i \cap U_j}}})_{(i,j) \in I \times I}, \ (t_k)_{k \in I} \in \prod_{k \in I} F(U_k)$

We denote $Sh(X)$ the full subcategory of $pSh(X)$ determined by the class of all functorial sheaves of sets over X.

The traditional development of sheaf theory provides the construction of a pair of functors that establishes an equivalence of categories between the category of geometric sheaves on X and the category of functorial sheaves over X, $GSh(X) \overset{\simeq}{\leftrightarrows} Sh(X)$.

1.3.2 (Functorial) sheaves on a locale

Note that in the definition of functorial sheaf over a topological space we did not use the points of the space, that is, only their **locale structure** was necessary. In fact, we can define sheaves for a presheaf $F : \mathcal{H}^{op} \to Set$, where $\mathcal{L}$ is the category associated to a locale H, since it is a poset. Consider a presheaf $F : \mathcal{H}^{op} \to Set$. Given $u, v \in \mathcal{H}$, if $u \leq v$, then $s_{|_u^v}$ denote the "restriction map" from $F(v)$ to $F(u)$. If $u, u_i \in \mathcal{H}$ and $u = \bigvee_{i \in I} u_i$ is a "cover", a presheaf F is a *(functorial) sheaf* (of sets) when we have the following diagram

$$F(u) \xrightarrow{e} \prod_{i \in I} F(u_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j \in I} F(u_i \wedge u_j)$$

is an equalizer in the category Set, where:

1. $e(t) = \{t_{|_{u_i}} \mid i \in I\}, \quad t \in F(U)$

2. $p((t_k)_{k \in I}) = (t_{i_{|_{u_i \wedge u_j}}})_{(i,j) \in I \times I}$

   $q((t_k)_{k \in I}) = (t_{j_{|_{u_i \wedge u_j}}})_{(i,j) \in I \times I}, \quad (t_k)_{k \in I} \in \prod_{k \in I} F(u_k)$

Another generalization of the notion of sheaf over sets is available for small categories by introducing an abstract idea of coverings of an object: this is the theory of Grothendieck topos.

1.3.3 Locale valued sets: sheaves in terms of sets with valued sameness

Let $\mathcal{H}$ be a locale. In this subsection, our presentation is close to [Bor94].

• A $\mathcal{H}$-**set** is a pair (X, δ) where X is a set and $\delta : X \times X \to \mathcal{H}$ is a function satisfying

$$\delta(x, y) = \delta(y, x)$$

$$\delta(x, y) \wedge \delta(y, z) \leq \delta(x, z)$$

As immediate consequences, we have that: (i) $\delta(x, y) \leq \delta(x, x) \wedge \delta(y, y)$; (ii) $\delta(x, y) \wedge \delta(y, y) = \delta(x, y)$.

• If $F : (\mathcal{H}, \leq)^{op} \to Set$ is a sheaf over the locale $\mathcal{H}$, then assignment

$$F \mapsto (X_F, \delta_F), \text{ where}$$

$$X_F := \coprod_{b \in \mathcal{H}} F(b)$$

$$\delta_F : X_F \times X_F \to \mathcal{H} \text{ is such that}$$

$$\delta_F((s,b),(t,c)) \mapsto \bigvee \{d \leq b \wedge c : s \restriction^b_d = t \restriction^c_d\}$$

determines a $\mathcal{H}$-set. In fact, it is a *complete* $\mathcal{H}$-set, see [Bor94].

• A *functional* morphism of $\mathcal{H}$-sets, $f : (X, \delta) \to (X', \delta')$, is a function $f : X \to X'$ such that:

$$\delta'(f(x), f(x)) = \delta(x, x)$$

$$\delta'(f(x), f(y)) \geq \delta(x, y)$$

The usual identity function and composition of functions induces a structure of category on the classes of $\mathcal{H}$-sets and functional morphisms: $\mathcal{H} - sets^{func}$. We denote $comp\mathcal{H} - sets^{func}$ the full subcategory of $\mathcal{H} - sets^{func}$ of all complete $\mathcal{H}$-sets.

• A *relational* morphism of $\mathcal{H}$-sets, $\phi : (X, \delta) \to (X', \delta')$, is a function $\phi : X \times X' \to \mathcal{H}$ such that "ϕ is a functional relation":

$$\delta'(x', y') \wedge \phi(x, y') \leq \phi(x, x')$$

$$\delta(x, y) \wedge \phi(x, x') \leq \phi(y, x')$$

$$\phi(x, x') \wedge \phi(x, y') \leq \delta'(x', y')$$

$$\bigvee_{x' \in X'} \phi(x, x') = \delta(x, x)$$

$\delta : (X, \delta) \to (X, \delta)$ is a relational morphism denoted $id_{(X,\delta)}$. There is natural way to compose relational morphisms: if $\phi' : (X', \delta') \to (X'', \delta'')$ is a relational morphism then $\phi' \circ \phi : X \times X'' \to \mathcal{H}$ is given by $\phi' \circ \phi(x, x'') = \bigvee_{y' \in X'} \phi'(y', x'') \wedge \phi(x, y')$. In this way we obtain a category, $\mathcal{H}-sets^{rel}$ or simply $\mathcal{H}-sets$; $comp\mathcal{H}-set$ denotes the full subcategory of $\mathcal{H}-sets$ of all complete $\mathcal{H}$-sets.

• The following categorial relations hold:

$$Sh(\mathcal{H}) \simeq comp\mathcal{H}-sets^{func} \cong comp\mathcal{H}-sets^{rel} \simeq \mathcal{H}-set^{rel};$$

see sections 2.8 and 2.9 in [Bor94] for the details.

1.3.4 The category of Heyting valued models of sets

Let $\mathcal{H}$ be a complete Heyting algebra, i.e., a locale. In this subsection, our presentation is close to [Bel05] and [ACM22].

- Define for a successor ordinal and for a limit ordinal λ:

$$V_{\mathcal{H}_0} = \emptyset$$

$$V_{\mathcal{H}_{\alpha+1}} = \left\{ f : dom(f) \subseteq V_\alpha^{(\mathcal{H})} \wedge im(f) \subseteq \mathcal{H} \right\}$$

$$V_{\mathcal{H}_\lambda} = \bigcup_{\beta<\lambda} V_\beta^{(\mathcal{H})}$$

• From this recursive construction we obtain the proper class

$$V_{\mathcal{H}} = \bigcup_{\alpha \in Ord} V_\alpha^{(\mathcal{H})}$$

and also a rank map $\rho : V_{\mathcal{H}} \to Ord$:

$$\rho(x) = \min\{\alpha \in Ord : dom(x) \subseteq V_\alpha^{(\mathcal{H})}\}$$

- *Heyting-valued semantics*:

There is a "natural interpretation" of set theory in $V^{(\mathcal{H})}$, at least a Heyting one. More precisely:

There are valuations functions:

$$[\![\cdot \in \cdot]\!]_{\mathcal{H}}, [\![\cdot = \cdot]\!]_{\mathcal{H}} : V^{(\mathcal{H})} \times V^{(\mathcal{H})} \longrightarrow \mathcal{H}$$

They are defined by simultaneous recursion on the well-founded relation:

$$(f,g) \prec (x,y) \iff f = x \wedge g \in \mathrm{dom}(y) \vee f \in \mathrm{dom}(x) \wedge g = y$$

Defined by:

$$[\![y \in x]\!]_{\mathcal{H}} = \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [\![y = t]\!]_{\mathcal{H}}$$

$$[\![y = x]\!]_{\mathcal{H}} = \bigwedge_{\substack{u \in \mathrm{dom}(x) \\ v \in \mathrm{dom}(y)}} (x(u) \to [\![u \in y]\!]_{\mathcal{H}}) \wedge (y(v) \to [\![v \in x]\!]_{\mathcal{H}})$$

We can extend these valuations to a valuations of formulae in general. The connectives are defined naturally with the functions of the $\mathcal{H}$.

For quantifiers we put:

$$[\![\forall x : \varphi(x)]\!]_{\mathcal{H}} = \bigwedge_{\hat{x} \in V^{(\mathcal{H})}} [\![\varphi(\hat{x})]\!]_{\mathcal{H}}$$

$$[\![\exists x : \varphi(x)]\!]_{\mathcal{H}} = \bigvee_{\hat{x} \in V^{(\mathcal{H})}} [\![\varphi(\hat{x})]\!]_{\mathcal{H}}$$

• The resulting semantic identifies a lot of sets as $[\![\cdot = \cdot]\!]_{\mathcal{H}}$.

• If $\mathcal{H} = \mathcal{B}$ is a complete boolean algebra, and we put that $V^{(\mathcal{B})} \vDash \varphi \iff [\![\varphi]\!]_{\mathcal{B}} = 1_{\mathcal{B}}$ then the structure $V^{(\mathcal{B})}$ satisfies all axioms of ZF set theory, preserves inference and doesn't validate falsehoods.

• When we are simply in the context of locales (=cHA) $\mathcal{H}$, we have that the corresponding Heyting Universe models intuitionistic ZF (IZF), in an intuitionistic sense.

• $V^{\mathcal{H}}$ and $\mathcal{H}-sets$

In [ACM22] is presented an extension of a result sketched in [Bel05] to the setting of complete Heyting algebras, we present, for the reader's convenience, many equivalent description of category of sheaves of a cHA $\mathcal{H}$, $\mathbf{Sh}(\mathcal{H}) \simeq \mathcal{H}$-$\mathbf{Set} \simeq \mathbf{Set}^{(\mathcal{H})}$, where the later is obtained by the cumulative hierarchy $V^{(\mathcal{H})}$ by taking quotients as below:

Consider the equivalence relation in $V^{(\mathcal{H})}$ given by $f \equiv g$ if, and only if, $[\![f = g]\!] = 1$. The category $\mathbf{Set}^{(\mathcal{H})}$ is defined as:

$$\mathrm{Obj}\,(\mathcal{S}^{(\mathcal{H})}) := V^{(\mathcal{H})}/\equiv$$

$$\mathbf{Set}^{(\mathcal{H})}\,([x],[y]) := \left\{ [\phi] \in \mathbf{Set}^{(\mathcal{H})} \mid [\![\mathrm{fun}(\phi : x \to y)]\!] = 1 \right\}$$

The arrows do not depend on the choice of representative of the equivalence classes $[x]$ and $[y]$. The composition and identity are defined as in **Set**.

Finally, to show the equivalence between **H-Set** and $\mathbf{Set}^{(\mathcal{H})}$, two constructions in $V^{(\mathcal{H})}$ (here we follow closely [Bel05]). Firstly, given $x \in V$, we define its "natural representative" $\hat{x}$ in $V^{(\mathcal{H})}$ using recursion over the (well-founded) membership relation: $\hat{x} := \{\langle y, 1\rangle \mid y \in x\}$. This allows us to define an ordered pair in $V^{(\mathcal{H})}$: given $u, v \in V^{(\mathcal{H})}$,

$$\{u\}^{(H)} := \{\langle u, 1\rangle\} \qquad\qquad \{u, v\}^{(H)} := \{u\}^{(H)} \cup \{v\}^{(H)}$$

$$\langle u, v\rangle^{(\mathcal{H})} := \left\{ \{u\}^{(H)}, \{u, v\}^{(H)} \right\}^{(H)}$$

Now let $\langle X, \delta\rangle$ be an $\mathcal{H}$-set. For each $x \in X$, define $\dot{x} \in V^{(\mathcal{H})}$ as:

$$\mathrm{dom}\dot{x} := \{\hat{z} \mid z \in X\} \quad \text{and} \quad \dot{x}(\hat{z}) := \delta(x, z), \text{ for all } z \in X$$

Then, define $X^{\dagger} \in V^{(\mathcal{H})}$ as

$$\mathrm{dom}X^{\dagger} := \{\dot{x} \mid x \in X\} \quad \text{and} \quad X^{\dagger}(\dot{x}) := \delta(x, x), \text{ for all } x \in X$$

Similarly, given a morphism $\phi : \langle X, \delta\rangle \to \langle X', \delta'\rangle$ of $\mathcal{H}$-sets, we may consider $\varphi^{\dagger} \in V^{(\mathcal{H})}$ given by:

$$\mathrm{dom}\phi^{\dagger} := \left\{ \langle \dot{x}, \dot{x}'\rangle^{(\mathcal{H})} \mid x \in X, x' \in X' \right\}$$

$$\phi^{\dagger}\,(\langle \dot{x}, \dot{x}'\rangle^{(\mathcal{H})}) := \phi(x, x'), \text{ for all } x \in X, x' \in X'$$

Since $V^{(\mathcal{H})} \models \mathrm{fun}(\phi^{\dagger})$, we may define a functor $\Phi : \mathbf{H\text{-}Set} \to \mathbf{Set}^{(\mathcal{H})}$ by taking $\Phi(X, \delta) = [X^{\dagger}]$, for every $\mathcal{H}$-set $\langle X, \delta\rangle$, and $\Phi(\phi) = \phi^{\dagger}$, for every arrow ϕ **H-Set**.

On the other hand, given $u \in V^{(\mathcal{H})}$, define $X_u := \mathrm{dom}\, u$ and $\delta_u : X_u \times X_u \to \mathcal{H}$ as

$$\delta_u(x,y) := [\![x \in u]\!] \wedge [\![x = y]\!] \wedge [\![y \in u]\!]\,,\ \text{for all } x, y \in X_u$$

Notice, however, that $[\![u = u']\!] = 1$ does not imply $X_u = \mathrm{dom}\, u = \mathrm{dom}\, u' = X_{u'}$, and that we may not define an $\mathcal{H}$-set using $[\mathrm{dom}\, u]$ since this class is not a set (later we will show that $\left\{u' \in V^{(H)} \mid [\![u = u']\!] = 1\right\}$ is a proper class). In that case, we will use Scott's trick to define a functor $\Psi : \mathbf{Set}^{(\mathcal{H})} \to \mathcal{H}\text{-}\mathbf{Set}$.

Firstly, if $[\![u = u']\!] = 1$, then $\langle X_u, \delta_u\rangle \cong \langle X_{u'}, \delta_{u'}\rangle$. Indeed, define $\lambda_{u,u'} : \langle X_u, \delta_u\rangle \to \langle X_{u'}, \delta_{u'}\rangle$ such that

$$\lambda_{u,u'}(x,x') := [\![x \in u]\!] \wedge [\![x = x']\!] \wedge [\![x' \in u']\!]\,,\ \text{for all } x \in \mathrm{dom}\, u, x' \in \mathrm{dom}\, u'$$

Now, for each $[u] \in \mathbf{Set}^{(\mathcal{H})}$, let $I^{[u]}$ be the category given by:

$$\mathrm{Obj}\,(I^{[u]}) := [u]_m \qquad\qquad \mathrm{Arr}\,(I^{[u]}) := [u]_m \times [u]_m$$

where $[u]_m$ is the equivalence class of the elements with minimum rank. Consider the functor $F^{[u]} : I^{[u]} \to \mathbf{H}\text{-}\mathbf{Set}$ such that

$$F^{[u]}(u') := \langle X_u, \delta_u\rangle\,,\ \text{for all } u' \in [u]_m$$

$$F^{[u]}(u', u'') := \lambda_{u',u''} : \langle X_{u'}, \delta_{u'}\rangle \to \langle X_{u''}, \delta_{u''}\rangle\,,\ \text{for all } u', u'' \in [u]_m$$

At last, we may define the functor $\Psi : \mathbf{Set}^{(\mathcal{H})} \to \mathcal{H}\text{-}\mathbf{Set}$ as $\Psi([u]) = \lim\limits_{u' \in [u]_m} F^{[u]}(u')$.

1.3.5 Sheaves in terms of enriched categories

Remark that in the definition of enriched category over a monoidal category (see 1.1.13) the existence of "the identity of an object a" is given by a morphism in the monoidal category from the module object to every "reflexive hom"

$$I \xrightarrow{id_a} A(a,a)$$

In the setting of enriched categories over *integral* quantales this axioms becomes

$$\top = 1 \leq d(a, a)$$

So if we are based ourselves in the analogy of sheaves over a locale like sets with a locale-valued sameness as is put in the subsection 1.3.3 , the reflexivity axiom says that:

$$\top = 1 \leq \delta(a, a)$$

In the connection with sheaves $\delta(a, a)$ calculates the domain of the section a, so this axioms becomes

$$a \ \textit{is a global section}$$

So this collapse sheaf theory over locales to set theory: sheaves with only global sections or equivalently sheaves over a space with the trivial topology, can be taken as sets that vary about a point i.e. just sets. In this situation for make a connection with sheave theory in terms of enriched categories this axioms must be dropped, and the theory decide between two natural choices

1. Assume the lack of identities morphism and search for *weaker versions* of it that still allows us to make certain nodal constructions, importantly *completions*.

2. Interpret the strength of the reflexivity axiom in terms of a locality that can be discarded in a structural elevation of the base object, that is, if the problem is that the identities are "too big", just *split them*.

The first choice lets us in a weaker framework that the enriched category theory but the role of the quantale of "truth values" remains, in the second choice we stay in enriched category theory but we must to change the base quantale for a higher structure.

In the next chapter we will briefly present the construction that expose sheaf theory in terms of enriched categories, framework developed by R. Walters in [Wal81] for sheaves over locales and pretty soon widely extended to sheaves over sites in [Wal82]. This construction exposes what we mean in the second way of assume the necessarily absence of identities for categories enriched in a locale if we search an analogy with sheaves over it.

Consider $\mathcal{H}$ a complete Heyting algebra from it defines the following category $\mathrm{Rel}(\mathcal{H})$:

Objects: elements of $|\mathcal{H}|$

Morphism: $u \xrightarrow{R} v$ is given by $R \leq u \wedge v$.

Composition is given by $S \circ R = S \wedge R$ of $\mathcal{H}$. Note that for every $u, v \in |\mathcal{H}|$ if we denote by $\mathrm{Rel}(H)(u,v) := \{R \mid R \leq u \wedge v\}$ this result in a complete Heyting algebra restricting the structure of $\mathcal{H}$. Then we have a category in which every hom-set has an extra structure of Heyting Algebra and the composition preserves that structure.

The idea is consider a category enriched over this “2-monoidal" (actually “2-cartesian") object, this is just as in the quantale enriched case but with an extra assignment which specify coherently the Heyting algebra. More precisely a $\mathrm{Rel}(H)-$**enriched category** consist in, a set of objects X together with a domain assignment

$$e : X \to \mathrm{Obj}(\mathrm{Rel}(\mathcal{H})) = |\mathcal{H}|$$

and a hom-assignment

$$d : X \times X \to \mathrm{Morp}(\mathrm{Rel}(H))$$

such that for every $x, y, z \in X$ the following holds:

1. $d(x,y) : e(y) \to e(x)$ (i.e. $d(x,y) \leq e(x) \wedge e(y)$)

2. $1_{e(x)} \leq d(x,x)$

3. $d(z,y) \wedge d(y,x) \leq d(x,z)$

Note that the only additional axiom is the first one that speaks about the domains, then locally this object is an enriched category over a Heyting algebra.

Now let's note how this consideration of **split** the original Heyting algebra $\mathcal{H}$ solves the problem of formalizing pre-sheafs with not only global section as an enriched categories, even more with certain main enriched-category notion can be characterized sheaf (functorial)condition, i.e. gluing property.

Consider a pre-sheaf of sets over $\mathcal{H}$ as a category (like a pre-order) $F : \mathcal{H}^{op} \to Set$, the following define a enriched category over $Rel(H)$.

$$X_F = \coprod_{u \in |\mathcal{H}|} F(u)$$
$$e_F : X_F \to \mathcal{H}$$
$$(s, u) \mapsto u$$
$$d_F : X_F \times X_F \to Morph(Rel(Q))$$
$$((s,u),(t,v)) \mapsto \bigvee\{w \in |\mathcal{H}| \ : w \leq u \wedge v, s|_w = t|_w \}$$

Is a direct calculation matter prove that this actually defines an enriched category, even more note that for every sheaf the following extra properties always holds:

symmetry $d_F(((s,u),(t,v))) = d_F((t,v),(s,u))$

Note also that the condition of being a **separated** [6] pre-sheaf can be formalized in this language very directly, by the following property:

skeletal $d_F((s,u)) = e_F((s,u)) = e_F((t,v))$ implies $s = t$.

The following notion is central in general for the whole text because being the main concept of the real analysis, captures the sheaf theoretic essence to, the existence of lifting for local properties to a global ones.

We say that an enriched category over $Rel(H)$, $(X.d, e)$ is **Cauchy complete** if for every $u \in |\mathcal{H}|$, and every couple of functions $\phi, \psi : X \to morph(Rel(Q))$ such that

1. $\phi(x) : u \to e(x)$, $\psi(x) : e(x) \to u$

[6]means that sections that has the same restrictions in an open cover of their domain are equals

2. $d(x,x') \wedge \phi(x) \leq \phi(x')$, $\psi(x) \wedge d(x',x) \leq \psi(x')$

3. $u \leq \bigvee_{x \in X} (\psi(x) \wedge \phi(x))$

4. $\phi(x') \wedge \psi(x) \leq d(x,x')$

exist an element $a \in X$, such that $\phi(x) = d(a,x)$ and $\psi(x) = d(x,a)$ for every $x \in X$.

Note the analogy of this definition and the mentioned in the end of first section of this chapter, here is unfolded what it means and internal adjunction, plus the (always necessary) first domain (or type) axiom, natural consequence of this 2—categorical stance.

Now the strong results presented in [Wal81] and brings much of the inspiration of the original work of this thesis.

Theorem 1.3.1. *([Wal81]) The category of sheaves over a Heyting algebra $\mathcal{H}$ is bi-equivalent to the full subcategory of* $\mathrm{Rel}(H) - \mathrm{Cat}_{\mathrm{fun}}$ *given by the Cauchy complete, symmetric, skeletal enriched categories over* $\mathrm{Rel}(H)$.

More impressive is the (rather elegant) generalization of the result to any Grothendieck topos.

Theorem 1.3.2. *([Wal82]) Given any small site* (C,J) *there is local monoidal (cartesian) category* $\mathrm{Rel}(C,J)$ *such that the category* $\mathrm{Sh}(C,J)$ *of sheaves over that site is bi-equivalent to the full subcategory of* $\mathrm{Rel}(C,J) - \mathrm{Cat}$ *given by the Cauchy complete, symmetric, skeletal enriched categories over* $\mathrm{Rel}(C,J)$.

This local monoidal categorically structure, has a proper name and is the topic of the next section. In resume, this kind of object can be constructed starting from a local or more generally from a site, an enables to capture the existence of gluing sections for compatible families of section in pre-sheafs over the site as the concept of being Cauchy complete for enriched categories over it.

1.4 Quantaloids

A bit of the moral of Walters' $\mathsf{Rel}(H)$ construction is to divide a single base on which you are working and allow for multiple bases. So if we conceptualize a quantal as an object that only has one underlying point and allow this situation to be multiplied, we get the following definition. This section is based in the monographic work of Isar Stubbe in the study of quantaloids and their relation with sheaf theory and other areas. Much of the monographic work that we do in the second chapter is also in reference to the work of Stube. For this section see [Stu05a].

Definition 1.4.1. *A **quantaloid** Q is a category enriched over the category Sup of complete lattices and join-preserving lattices morphisms.*

This means that for every $a, b \in \mathsf{Obj}(Q)$, the Hom-set has an extra structure of complete lattice and the composition of the category distributes arbitrary joins.
Concretely for every $a, b \in \mathsf{Obj}(Q)$, $Q(a,b)$ is a complete lattice and for every $a, b, c \in \mathsf{Obj}(Q)$, if $\{f_i\}_{i\in I} \subseteq Q(a,b)$ and $g \in Q(b,c)$ then

$$g \circ \bigvee_{i\in I} f_i = \bigvee_{i\in I} (g \circ f_i)$$

also if $\{g_i : i \in I\} \subseteq Q(b,c)$ and $f \in Q(a,b)$ then

$$\left(\bigvee_{i\in I} g_i\right) \circ f = \bigvee_{i\in I} (g_i \circ f)$$

Examples 1.4.2. *1. A quantaloid with only one object is just a quantal.*

2. An example of quantaloid that is not a quantal is the category Set_{rel} whose class of objects is the class of all sets and $\mathsf{Hom}_{rel}(X,Y) = \{(X,R,Y) : R \subseteq X \times Y\}$. Note that $\mathsf{Hom}_{rel}(X,Y)$ is a complete lattice where sup = reunion and, since the composition of relations distributes over suprema, then Set_{rel} is a quantaloid.

Like in a pre-order view as a category the colimits are joins, that composition distributes over arbitrary local joins for arrows in quantaloid means that for every $f : a \to b$ in a quantaloid $\mathcal{Q}$ the induced Yoneda morphism

$$\mathcal{Q}(-,a) \xrightarrow{f\circ -} \mathcal{Q}(-,b)$$

and also

$$\mathcal{Q}(b,-) \xrightarrow{-\circ f} \mathcal{Q}(a,-)$$

are no just natural transformation between hom sets, but locally a morphism between complete categories (lattices because theay are categories as pre-ordes) with right adjoints. Indeed for every c object in the category $\mathcal{Q}$, the property of being a quantaloids means that

$$\mathcal{Q}(b,c) \xrightarrow{-\circ f} \mathcal{Q}(a,c)$$
$$g \mapsto g \circ f$$

and also

$$\mathcal{Q}(-,a) \xrightarrow{f\circ -} \mathcal{Q}(-,b)$$
$$h \mapsto f \circ h$$

preserves colimits, so in the context of pre-orders as categories this is equivalent to say that this morphism have right adjoints (calculation can be made by hand but if one prefers can use the adjoint functor theorem) respectively noted by

$$\mathcal{Q}(a,-) \xrightarrow{\{f,-\}} \mathcal{Q}(b,-), -\circ f \dashv \{f,-\}$$
$$\mathcal{Q}(-,b) \xrightarrow{[f,-]} \mathcal{Q}(-,a)\ f\circ - \dashv [f,-]$$

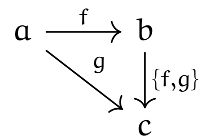
Explicitly

$$\{f,g\} := \{f,-\}(g) = \bigvee\{h \in Morp(Q) : h\circ f \leq g\}$$

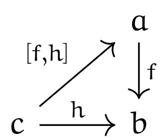
and

$$[f,h] := [f,-](h) = \bigvee\{p \in Morp(Q) : f\circ p \leq h\}$$

The adjuntion equation lets us think in $\{f,h\}$ as the **extension of** h **through** f, for every possible composable morphism in $\mathcal{Q}$ we have: $h\circ f \leq g$ if and only if $h \leq \{f,g\}$



For $[f, h]$ there is an intuition of it as: the **lifting of** h **through** f: $f \circ p \leq h$ if and only if $p \leq [f, h]$



A quantaloid is in particular a kind of bicategory, so the notion of adjunction that is a sentence that talks about categories, functors and natural transformations, say objects, arrows and 2-cells can be interpreted in any bicategory, so in particular in a quantaloid comes with the facility that every diagram of 2-cells commute (because in quantaloids this is a pre-order structure) so just with the existence of unities and co-unities is enough for the adjunction, because the usual triangular factorization speak about diagrams of natural transformations i.e. 2-cells.

Definition 1.4.3. *Given a quantaloid* $\mathcal{Q}$ *and two morphisms* $F : a \to b$, $G : b \to a$ *we say that* F ***is a left adjoint of*** G *if* $1_a \leq G \circ F$ *and* $F \circ G \leq 1_b$

The definitions that we have just put in this section are what is necessary to state and contextualize the subsequent chapters. However there is a fair amount of theory that we are omitting and it is quite interesting, a monographic presentation of this can be found in the first pages of [Stu05a].

2 Sheaf-like categories over quantales

In this chapter, are presented generalizations of the wide gamma of (equivalent) categorical presentations of sheaves over locales to the quantalic setting. The first section is limited to give the principal definition of the work of Tenorio [TdAMM22] in functorial versions of sheaf over quantales and Mendez, Alvim, Mariano in [AdAMM23b],[AdAMM23a] defining the extension of the theory of valued-sets over Locales to the case of commutative semi-cartesian quantales, they have calculated a large part of the categorical structure of these objects and have studied in depth several notions of completion that, in the case of Locales (as it was exposed in the preliminaries chapter), allow us to connect by equivalence with the theory of sheaves. It is in this sense that is in its own right a theory of sheaves over quantales.

Their results are used in chapter 3 for some original calculations, and at the end of this chapter to show how it connects with an independent work done by Stubbe ([Stu05c]) on the notion of totally regular semicategory enriched over a quantaloid, and with another equivalent presentation that we give in the section 2.3.

2.1 $\mathcal{Q}$-sets and sheaves over $\mathcal{Q}$

In this short section we describe some recent (in fact, ongoing) generalizations of the notions of sheaves over a locale and locale-valued sets, described in the previous chapter, to the setting of commutative, semicartesian quantales Q.

Definition 2.1.1. *(see [TdAMM22]) Let Q be a commutative, semicartesian quantale.*

A presheaf over Q is just a contravariant functor, $F : Q^{op} \to \mathrm{Set}$. A morphism from presheaf F into a presheaf G, both over Q, is just a natural transformation $\eta : F \to G$. $\mathrm{pSh}(Q)$ denotes the category of all presheaves over Q with the composition of natural transformation as composition.

A presheaf $F : Q^{op} \to \mathrm{Set}$ is a ***sheaf*** *if for any cover $u = \bigvee_{i\in I} u_i$ of any element $u \in Q$, the following diagram is an equalizer*

$$F(u) \xrightarrow{e} \prod_{i\in I} F(u_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(u_i \otimes u_j)$$

where

$$e(t) = \{t_{\restriction u_i} \ : \ i \in I\}, \quad p((t_k)_{k\in I}) = (t_{i_{\restriction u_i \otimes u_j}})_{(i,j)\in I\times I}$$
$$q((t_k)_{k\in I}) = (t_{j_{\restriction u_i \otimes u_j}})_{(i,j)\in I\times I}$$

Note that, these maps make sense since $u_i \otimes u_j \leq u_i, u_j$, because we have assumed that Q is a semicartesian quantale.

Denote by $\mathrm{Sh}(Q)$ the full subcategory of $\mathrm{pSh}(Q)$ determined by the subclass of all sheaves over Q.

In [TdAMM22] are established many interesting properties of the category $\mathrm{Sh}(Q)$, in particular it: is a complete and cocomplete category; is a monoidal closed reflexive subcategory of $\mathrm{pSH}(Q)$; it is not a topos, in general.

Definition 2.1.2. *(see [AdAMM23b],[AdAMM23a]) For a commutative, semicartesian quantale Q, a $Q-\mathrm{set}$ (X, δ) is given by a set X together with a $Q-$valued binary predicate over X.*

$$\delta : X \times X \to Q$$

such that

1. $\delta(x, y) = \delta(y, x)$

2. $\delta(x,y) \otimes \delta(y,z) \leq \delta(x,z)$

3. $\delta(x,y) \otimes \delta(y,y) = \delta(x,y)$

Since Q is a semicartesian quantale, it follows that $\delta(x,y) = \delta(x,x) \otimes \delta(x,y) \otimes \delta(y,y) \leq \delta(x,x) \otimes \delta(y,y)$.

Notation 2.1.3. *For* $x \in X$ *a* $Q-\text{set}$ *lets write* $\delta(x,x)$ *as* $E(x)$*, note that the axiom* 3. *in the definition of* $Q-\text{set}$ *implies that* $E(x)$ *is an idempotent of* Q.

Definition 2.1.4. *If* (X,δ_X) *and* (Y,δ_Y) *are* $Q-\text{sets}$ *a* ***functional morphism*** *from* (X,δ_X) *to* (Y,δ_Y) *is given by a function* $f : X \to Y$ *such that:*

$$\delta_X(x,y) \leq \delta_Y(f(x),f(y))$$
$$\delta_X(x,x) = \delta_Y(f(x),f(x))$$

If the context is clear we use indistinct the notation $f : X \to Y$ for a functional morphism and for the function that brings with it.

Definition 2.1.5. *If* (X,δ_X) *and* (Y,δ_Y) *are* $Q-\text{sets}$ *a* ***relational morphism*** *from* (X,δ_X) *to* (Y,δ_Y) *is given by a function* $\varphi : Y \times X \to Q$ *such that*

$$\varphi(y,x) \otimes \delta_X(x,x') \leq \varphi(y,x')$$
$$\delta_Y(y,y') \otimes \varphi(y',x) \leq \varphi(y,x)$$
$$\varphi(y,x) \otimes E(x) = \varphi(y,x)$$
$$E(y) \otimes \varphi(y,x) = \varphi(y,x)$$
$$\varphi(y,x) \otimes \varphi(y',x) \leq \delta_Y(y',y)$$
$$\bigvee_{y \in Y} \varphi(y,x) = E(x)$$

In [AdAMM23a] is explored the categorical properties of the category of Q-sets and functional morphisms and, in [AdAMM23b], are developed connections between diverse (sub)categories of Q-sets endowed with functional morphisms and relational morphisms.

2.2 Enriched semicategories and categories

As is said in the subsection 1.3.5 for formalizing sheaves as enriched categories in view of the existence of identities axiom there are (at least) two alternatives, one is to **split** the base object and make a kind of $\mathrm{Rel}(H)$ construction (this alternative is explored in a posterior section and constitutes original work, section 2.3) and the other alternative is to just not work with this axioms, work in categories with out identities, this makes the problem “disappear", now the problem is: What kind of (enriched)category theory can you do without identities?. First of all, the observations of [Stu05c] tells that for beginning the Yoneda lemma can not hold, and for pre-sheaf theory which is the one that finally leads to the Cauchy completeness (a key concept that gives us the connection with sheaves), this could be a serious structural problem.

In the following sections we will follow (outlining) a part of the path that Isar Stubbe built in great detail in [Stu05a],[Stu05c],[Stu05b] to minutely scale in structure from the categories without identities to where it is strictly necessary and sufficient to get a good notion of Cauchy completeness.

Definition 2.2.1. *For a quantaloid* $\mathcal{Q}$ *an* ***enriched semi-category*** *over* $\mathcal{Q}$ *is given by a triple* (X, d_x, e_x) *where* X *is a set,* $d : X \times X \to \mathrm{Morp}(\mathcal{Q})$ *and* $e : X \to \mathrm{Obj}(\mathcal{Q})$ *are functions such that for all* $x, y, z \in X$

1. $d(x, y) : e(y) \to e(x)$
2. $d(x, y) \circ d(y, z) \leq d(x, z)$

Definition 2.2.2. *Given* $\mathcal{Q}$ *a quantaloid, an* ***enriched category*** *over* $\mathcal{Q}$ *is a semi-category* (X, d_X, e_X) *with the extra property that for every* $x \in X$

$$1_e(x) \leq d_X(x, x)$$

Definition 2.2.3. *Given* $\mathcal{Q}$ *a quantaloid and* $\mathcal{Q}$*-enriched semi-categories* (X, d_X, e_X) *and* (Y, d_Y, e_Y) *an* ***enriched functor***

$$f : (X, d_X, e_X) \rightarrow (Y, d_Y, e_Y)$$

is given by a function $f : X \rightarrow Y$ *such that for all* $x, y \in X$

$$e_X(x) = e_Y(f(x))$$
$$d_X(x, y) \leq d_Y(f(x), f(y))$$

For measuring the gap between semi-categories and categories think that the most simple example of enriched category over quantale(oids) is the enrichment over $2 = \{\top, \bot\}$ for which as say in example 1.1.16, the $2-$Categories are pre-orders the existence of identities corresponds to reflexivity, then the $2-$Semicategories are sets with transitive binary relations, for example, $(\mathbb{R}, <)$ is a $2-$semicategory but isn´t a $2-$category.
The following can be proven very directly, and are presented in [Stu05a] for the theory over enriched categories and in [Stu05c] for the theory over semicategories.

Proposition 2.2.4. *For every quantaloid* $\mathcal{Q}$*, enriched semi-categories over* $\mathcal{Q}$ *and enriched functors determine a category with the composition given by composing the underlying functions between the sets of objects, we note this category by:*

$$Q - \mathsf{SCat}_{fun}$$

Proposition 2.2.5. *For every quantaloid* $\mathcal{Q}$ *enriched categories over* $\mathcal{Q}$ *and enriched functors determine a full sub-category of* $Q - \mathsf{SCat}_{fun}$*, we note this category by:*

$$Q - \mathsf{Cat}_{fun}$$

Definition 2.2.6. *Given* $\mathcal{Q}$ *a quantaloid and* $\mathcal{Q}$*-enriched semi-categories* (X, d_X, e_X) *and* (Y, d_Y, e_Y) *a* ***distributor****:*

$$\Psi : (X, d_X, e_X) \multimap (Y, d_Y, e_Y)$$

is given by a function $\Psi : Y \times X \rightarrow \mathsf{Morp}(\mathcal{Q})$*, such that for all* $x, x' \in X$ *and* $y, y' \in Y$ *we have*

$$\Psi(y,x) : e_X(x) \to e_Y(y)$$
$$d_Y(y',y) \circ \Psi(y,x) \leq \Psi(y',x)$$
$$\Psi(y,x) \circ d_X(x,x') \leq \Psi(y,x')$$

Lemma 2.2.7. *For* (X, d_x, e_X), (Y, d_Y, e_Y) *and* (Z, d_Z, e_Z) *enriched semi-categories over a quantaloid* $\mathcal{Q}$, *and distributors,* $\phi : (X, d_X, e_X) \longrightarrow\!\!\!\!\!\!\!\!\ominus\;\; (Y, d_Y, e_Y)$, $\psi : (Y, d_Y, e_Y) \longrightarrow\!\!\!\!\!\!\!\!\ominus\;\; (Z, d_Z, e_Z)$, *the following assignment for every* $x \in X$ *and* $z \in Z$ *determines a distributor between* (X, d_X, e_X) *and* (Z, d_Z, e_Z)

$$\psi \otimes_Y \phi : (X, d_x, e_X) \longrightarrow\!\!\!\!\!\!\!\!\ominus\;\; (Z, d_Z, e_Z)$$
$$(x,z) \mapsto \bigvee_{y \in Y} (\psi(z,y) \circ \phi(y,x))$$

Proof. We are taking joins over sets of morphisms, like $\mathcal{Q}$ is a quantaloid this join is actually an arrow, the two action inequalities are direct consequences of that compose with an arrow in an quantaloid is an join-preserving and in particular monotone application between the hom lattices, together with the hypothesis that ψ and ϕ are distributors.

$$\begin{aligned} d_Z(z',z) \circ \psi(z,y) &\leq \psi(z',y) \\ d_Z(z',z) \circ \psi(z,y) \circ \phi(y,x) &\leq \psi(z',y) \circ \phi(y,x) \\ \bigvee_{y \in Y} d_Z(z',z) \circ \psi(z,y) \circ \phi(y,x) &\leq \bigvee_{y \in Y} \psi(z',y) \circ \phi(y,x) \end{aligned}$$

so

$$\begin{aligned} d_Z(z',z) \circ (\psi \otimes_Y \phi)(z,x) &= d_Z(z',z) \circ \bigvee_{y \in Y} \psi(z,y) \circ \phi(y,x) \\ &= \bigvee_{y \in Y} d_Z(z',z) \circ \psi(z,y) \circ \phi(y,x) \\ &\leq \bigvee_{y \in Y} \psi(z',y) \circ \phi(y,x) \\ &= (\psi \otimes_Y \phi)(z',x) \end{aligned}$$

The other axiom is verified analogously. □

Although the previous lemma asserts that in the context of semi-categories and distributors between them the above defined operation determines a well defined composition, unfortunately *there is not a category of enriched semi-categories over a quantaloid and distributors* (at least with this composition), the main reason is that **there is not identities for that composition**, for this is quite essential at least a weaker version of the existence of identity morphisms axiom ($1_{e(x)} \leq d(x,x)$) in the semi-enriched categories. So by the moment we can perform a category of enriched *categories* and distributors between them.

Proposition 2.2.8. *Given a quantaloid $\mathcal{Q}$ the collection of enriched categories over $\mathcal{Q}$ and distributors between them with composition as in lemma 2.2.7, determines a category which we note as:*

$$Q - Cat_{rel}$$

We put the proof for stress the role of the existence of identities axiom in this structure.

Proof. The previous lemma asserts that the composition is well defined, clearly is associative, then only rest to see that for every (X, d_X, e_X) there is an identity distributor over it, for this we take

$$1_X = d_X : X \times X \to Mor(\mathcal{Q})$$

That (X, d_X, e_X) is (in particular) an enriched semi-category is exactly that d_X is a distributor from X to X

$$d_X : X \multimap X$$

Now for every distributor $\phi : X \multimap Y$, $x, x' \in X$ and $y \in Y$ we have

$$\begin{aligned} \phi(y,x') \circ d_X(x',x) &\leq \phi(y,x) \\ \bigvee_{x' \in X} \phi(y,x') \circ d_X(x',x) &\leq \phi(y,x) \\ (\phi \otimes_X d_X)(y,x) &\leq \phi(y,x) \end{aligned}$$

But also we have the following (note the use of the extra axiom $1_{e_X(x)} \leq d_X(x,x)$ which is the difference between categories and semi-categories).

$$
\begin{aligned}
\phi(y,x) &= \phi(y,x) \circ 1_{e_X(x)} \\
&\leq \phi(y,x) \circ d_X(x,x) \\
&\leq \bigvee_{x' \in X} \phi(y,x') \circ d_X(x',x) \\
&= (\phi \otimes_X d_X)(y,x)
\end{aligned}
$$

Proof that d_X is also neutral for this composition on the left is analogous. □

There is a main fact about quantaloid enriched categories and relational morphisms (distributors); they form quantaloids again. Indeed, given any enriched categories (X, d_X, e_X) and (Y, d_Y, e_Y), the set of distributors with domain X and codomain Y, $Q - \mathcal{C}at_{rel}(X,Y)$ can be ordered "pointwise", $\phi \leq \psi$ if $\phi(y,x) \leq \psi(y,x)$ for every $x \in X$ and $y \in Y$, then is straightforward that it is a complete lattice, with joins calculated pointwise, moreover like composition is also a join, and of course joins commute with joins, composition distributes over arbitrary joins taken in this local hom lattices, that is to say, the category $Q - \mathcal{C}at_{rel}$ is a (large) quantaloid.
Explicitly for a family of distributors with the same domain and codomain $\{\phi_i : X \mathrel{-\!\!\ominus\!\!\rightarrow} Y\}_{i \in I}$, the join of the family is given by $(\bigvee_{i\in I} \phi_i)(y,x) = \bigvee_{i \in I}(\phi_i(y,x))$, then for every $\psi : Z \mathrel{-\!\!\ominus\!\!\rightarrow} X$ and $\varphi : Y \mathrel{-\!\!\ominus\!\!\rightarrow} W$

$$
(\bigvee \phi_i) \otimes_X \psi = \bigvee(\phi_i \otimes_X \psi)
$$

$$
\varphi \otimes_Y (\bigvee \phi_i) = \bigvee(\varphi \otimes_Y \phi_i)
$$

The importance of this specific structure over quantaloid enriched categories and relation morphisms lies on the following functor, that assigns to every enriched functor their "graph" as a relation, more over this assignation always comes with an extra structure over the graph; it is an internal adjoint (in the sense of 1.4.3) in the quantaloid $Q - \mathcal{C}at_{rel}$

$$
Q - \mathcal{C}at_{fun} \to Q - \mathcal{C}at_{rel}
$$

$$(X, d_X, e_X) \mapsto (X, d_X, e_X)$$
$$X \xrightarrow{F} Y \mapsto d_Y(-, F-) : X \multimap Y$$

There is also a contravariant form of this graph assignation

$$X \xrightarrow{F} Y \mapsto d_Y(F-, -) : Y \multimap X$$

For clarity, the distributor that we call the graph of $F : X \to Y$ is given by:

$$d_Y(-, F-) : X \multimap Y$$
$$d_Y(-, F-) : Y \times X \to \mathrm{Morp}(\Omega)$$
$$d_Y(y, F(x)) : e_X(x) = e_X(F(x)) \to e_Y(y)$$

where $e_X(x) = e_X(F(x))$ is because the functoriality of F.

The theory of enriched semi-categories stars with the lack of unities, but searching for exactly enough structure to make Cauchy-completion theory work gives the extra axioms for enriched structures that we are going to use.

The following notation is actually pretty usual in the context of enriched categories and category theory in general. If we have an (small) enriched category **C** with a set C of objects, is usual to denote by

$$C(a, b)$$

the set (or object in a monoidal category) of morphisms between two objects a and b in C, so in our context of enriched categories over a quantaloid (a kind of bicategory) seems natural to denote by $X(a, b)$ for the arrow of morphisms between to objects $a, b \in X$, off course we already give the notation $d_X(a, b)$ for this, but we allow ourselves to change the notation when we are talking about categories (or semi-categories) and relational morphism between them.

Notation 2.2.9. *For now on we are going to write the distributor* $d_X : X \multimap X$ *as*

$$X : X \multimap X$$

The proposition 2.2.8 says that for an enriched category (X, d_X, e_X) and any distributors $\phi : X \multimap Y$ and $\psi : Y \multimap X$, we have

$$\phi \otimes_X X = \phi$$
$$X \otimes_X \psi = \psi$$

2.2.1 Regular semicategories

In this section we explore the very necessarily axioms which enables the quantaloid structure in $Q - Cat_{rel}$, how we can explore this theory based in the study of pre-sheaf (enriched) categories are as well as the following result and definition, is all work of [Stu05c].

Definition 2.2.10. *For* (X, d_X, e_X) *and* (Y, d_Y, e_Y) *semicategories enriched over a quantaloid* $\mathcal{Q}$*, a distributor*

$$\phi : X \multimap Y$$

is said to be a ***regular distributor*** *if*

$$\phi \otimes_X X = \phi$$
$$Y \otimes_Y \phi = \phi$$

This means that for every $x \in X$ and $y \in Y$ the following identities holds

$$\bigvee_{x' \in X} (\phi(y, x') \circ d_X(x', x)) = \phi(y, x)$$
$$\bigvee_{y' \in Y} (d_Y(y, y') \circ \phi(y', x)) = \phi(y, x)$$

Definition 2.2.11. *For a semicategory* (X, d_X, e_X) *enriched over a quantaloid* $\mathcal{Q}$ *we say that is a* ***regular semicategory*** *if the distributor*

$$X : X \multimap X$$
$$(x, y) \mapsto d_X(x, y)$$

is regular

That is to say $X \otimes_X X = X$ which in terms of factorization of morphism means that

$$\bigvee_{y \in X}(d_X(x,y) \circ d_X(y,z)) = d_X(x,z)$$

Definition 2.2.12. *Regular semicategories and regular distributors with is defined composition determines a category that we note as:*

$$Q - \mathsf{RSCat}_{rel}$$

Note 2.2.13. *Thinking regular semi-categories are in particular semi-categories we can put enriched functors as morphims of a obviously category of enriched regular semi-categories and enriched functors, but it happens that the induced graphs of certain functors are no regular distributors and then there will no be a canonical induced functor from the functional side to the relational (quantaloid) side, so the functional morphism must be reduced to only those whose induced graph are regular distributor*

Definition 2.2.14. *An enriched functor between regular semi-categories* $F : (X, d_X, e_X) \to (Y, d_Y, e_Y)$ *is said to be a* ***regular*** *if*

$$d_Y(-, F-) = d_Y(-, F-) \otimes_X X$$

and

$$d_Y(F-, -) = X \otimes_X d_Y(FX, Y)$$

.

Lemma 2.2.15. $F : X \to Y$ *is a enriched regular functor between enriched regular semi-categories if and only if* $d_Y(-, F-)$ *and* $d_Y(F-, -)$ *are regular distributors*

Proof. The definition of being regular for an enriched functor is the half of the regularity of the graphs, the point is because of the functoriality of F is given the other half.

$$\begin{aligned}
d_Y(-,F-)(y,x) &= d_Y(,F-)\otimes_X X(y,x)\\
&= \bigvee_{x'\in X}(d_Y(y,F(x'))\circ d_X(x',x))\\
&\leq \bigvee_{x'\in X}(d_Y(y,F(x'))\circ d_X(F(x'),F(x)))\\
&\leq \bigvee_{y'\in Y}(d_Y(y,y')\circ d_X(y',F(x)))\\
&= Y\otimes_Y d_X(-,F-)(y,x)
\end{aligned}$$

Then $d_Y(-,F-) = d_Y(,F-)\otimes_X X$ and the functoriality of F implies $d_Y(-,F-)\leq Y\otimes_Y d_X(-,F-)$ and thinking $d_Y(-,F-)\geq Y\otimes_Y d_X(-,F-)$ is just the structure of semicategory in (Y,d_Y,e_Y), we can say that $d_Y(-,F-)$ is a regular distributor. The prove of the regularity of $d_Y(F-,-)$ is completely analogous. □

With this definition there is a clear category of regular semi-categories and regular functors, and a functor that assigns (in a covariant way) graphs.

$$Q-RSCat_{fun}\to Q-RSCat_{rel}$$

2.2.2 Cauchy completions

This central idea of how work the following construction is the key of the extra axiom for regular semicategories that conduce to the precisely structure that is taken as a sheaf. The point is that the enriched categories are "well-pointed". The original work is fund in [Stu05b], there are all the proofs and a more extensive discussion about it.

Remark 2.2.16. *For every enriched functor* $F:(X,d_X,e_X)\to(Y,d_Y,e_Y)$ *the following is an adjunction in* $Q-Cat_{rel}$

$$d_Y(-,F-)\dashv d_Y(F-,-).$$

This observation are the framework for the notion of Cauchy-complete enriched structure that in the localic case is the key to connect sheaves with enriched categories. Note that the proposition 2.2.8 tells us that every distributor between $Q-$categories is a regular distributor, so lets give first the discussion over $Q-$categories where we can use all the distributors.

Definition 2.2.17. *For (X, d_X, e_X) and (Y, d_Y, e_Y) enriched categories over a quantaloid $\mathcal{Q}$, we say that a distributor $\phi : X \multimap Y$* ***converge*** *if exist an enriched functor $F : X \to Y$ such that $d_Y(-, F-) = \phi$.*

Definition 2.2.18. *For (X, d_X, e_X) and (Y, d_Y, e_Y) enriched categories over a quantaloid $\mathcal{Q}$, we say that a distributor $\phi : X \multimap Y$ is a* ***Cauchy distributor*** *if exist a distributor $\psi : Y \multimap X$ such that $\phi \dashv \psi$ in $Q - Cat_{rel}$*

Notation 2.2.19. *Wherever $\phi : X \multimap Y$ is a Cauchy distributor, we write $\phi^\star$ for the right adjoint of ϕ*

Remark 2.2.16 tells us that every convergent distributor is a Cauchy one, Cauchy completeness means that the reverse implication also holds.

Definition 2.2.20. *For a $Q-$enriched category (Y, d_Y, e_Y) we say that is* ***Cauchy complete*** *if every Cauchy distributor $\phi : X \multimap Y$ converge*

As will be mentioned later, to verify Cauchy completeness, it is only necessary to review certain types of distributors. We give examples based on quantales and in the $Rel(H)$ quantaloid induced by a Locale H.

Examples 2.2.21. *A $2-$Category i.e. a pre-order is Cauchy complete if every ideal-filter pair in the pre-order that determines a gap (exactly what it means a Dedekind cut) is determined by a unique element a in the pre-ordr in the sense that the filter corresponds to the upper bounds and the ideal corresponds to the lower bounds. This is $2-$category is Cauchy-complete if and only if as a pre-order is Dedekind complete.*

Examples 2.2.22. *The Cauchy-completeness for a metric space as a* $([0,\infty],\geq,+,0)-$*category means that every Cauchy succession in the space converges in the space.*

Examples 2.2.23. *In [Wal81] is shown that for a locale* H*, there is an associated quantaloid* $\mathrm{Rel}(H)$ *(as in section 1.3.5) such that given a pre-sheaf* F *over* H*, this induces a* $\mathrm{Rel}(Q)-$*category for which the Cauchy-completeness means that every compatible family of sections has glue.*

To prove that an enriched category is Cauchy complete it is enough to see certain simple kind of Cauchy distributors converge. Consider a fixed object u in a quantaloid $\mathcal{Q}$, then we can form a canonical enriched $Q-$category associating to u

Definition 2.2.24. *Given* $\mathcal{Q}$ *a quantaloid and* $u \in \mathrm{Obj}(\mathcal{Q})$ *consider the following category*

$$\hat{u} = (\{\star\}, d_u, e_u)$$

Where $e_u(\star) = u$ *and* $d_u(\star,\star) = 1_u$

Is straightforward that this define a $\mathcal{Q}-$enriched category, so in base of this we can in certain way decompose distributors through the notion of enriched presheaf.

Definition 2.2.25. *Given a quantaloid* $\mathcal{Q}$ *and a object* $u \in \mathrm{Obj}(\mathcal{Q})$*, for a* $\mathcal{Q}-$*enriched category* (X, d_X, e_X) *a* ***enriched presheaf of type*** u *over* (X, d_X, e_X) *is a distributor with domain* $\hat{u}$ *and codomain* (X, d_X, e_X)

Proposition 2.2.26. *Consider a quantaloid* $\mathcal{Q}$ *and a* $Q-$*enriched category* (Y, d_Y, e_Y)*, the collection of all presheaves (of any type) over* Y *determines a* $Q-$*enriched category* PY *where for presheaves* $\phi : \hat{u} \multimap Y$*,* $\psi : \hat{v} \multimap Y$*, define* $e_{PY}(\phi) = u$ *and*

$$d_{PY}(\phi,\psi) := [\phi,\psi](\star,\star)$$

where $[\phi,\psi] : \hat{v} \multimap \hat{u}$ *is the lifting of* ψ *through* ϕ *in* $Q - \mathrm{Cat}_{rel}$ *(see section 1.4)*

For an enriched category it is enough to see that the Cauchy pre-sheaf converge to know that all the Cauchy distributors converge. We put this as a result whose proof is simply based on indexing pre-sheaves to form distributors.

Theorem 2.2.27. *An enriched category* (X, d, e) *is Cauchy-complete if and only if every Cauchy presheaf*[1] $\phi : \hat{u} \multimap X$ *converge.*

Another basic result is that "pre-sheaves classifies distributors"

Theorem 2.2.28. *For every* $Q-$*enriched categories* (X, d_X, e_X), (Y, d_Y, e_Y) *the following assignation determines a isomorphism of pre-orders*

$$Q - Cat_{rel}(X, Y) \cong Q - Cat_{fun}(X, PY)$$
$$\phi : X \multimap Y \mapsto \overline{\phi} : X \to PY$$

Where $\overline{\phi}(x) : \widehat{e_X(x)} \multimap Y$ *makes* $\overline{\phi}(x)(y, \star) := \phi(y, x)$

Then is clear that

$$Q - Cat_{cc,fun} \cong Map(Q - Cat_{cc,rel})$$

The construction of the Cauchy-completion for an enriched category lets the equivalence of categories $Q - Cat_{cc,rel} \cong Q - Cat_{rel}$, we briefly expose this construction

Cauchy completion of an enriched category

Compare this idea of what is the Cauchy completion of a enriched category with the for example: the Cauchy completion of a metric space, or a uniform space, a sheafification functor or any completion of the style "put what is missing" and recall the role of the immersion of the original category in the completion as the collection of "constant objects" (constant successions, diagonals, representable functors). The main note that conduces to the totally regular axiom is that

[1] i.e. with a right adjoint in $Q - Cat_{rel}$

the representable enriched presheaf of any object in an enriched category is a Cauchy distributor and this is because we can identify any object in the category by a point (in a categorical sense) of the category.

Again all this theory is constructed and explained in the original work of [Stu05b], we explore this through some remarks and propositions that are usually left without proof. The work on enriched categories is quite standard and can be considered practically folklore but the work on semi-categories is part of Stubbe's main contribution in [Stu05b].

For (X, d_X, e_X) and enriched Q-category consider X_{cc} the full sub(enriched)category of the enriched category PX of presheaves over X, whose objects are the Cauchy presheaves, the Yoneda immersion factorize through the inclusion of X_{cc} into PX, because every representable is a Cauchy distributor, so let us examine how this is concluded in the case of categories and from this the axiom on semi-categories is extracted. First a note about well-pointness of an enriched category.

Proposition 2.2.29. *If* (X, d_X, e_X) *is an* $Q-$*enriched category then there is a correspondence between elements* $x \in X$*, and functors with domain a singleton enriched-categories* $\hat{u}$*, the assignation*

$$\star \longmapsto x$$

determines a functor

$$\Delta_x : e\widehat{(x)} \longrightarrow X$$

and every functor $F : \hat{u} \longrightarrow X$ *determines a object* $F(\star) \in X$

The important note is that the existence of identities axiom for the enriched **category** $e\widehat{(x)}$ is exactly the functoriality of Δ_x

$$d_{e\widehat{(x)}}(\star, \star) = 1_{e(x)} \leq d_X(x, x) = d_X(\Delta_x(\star), \Delta_x(\star))$$

like the the graph of any functor is a Cauchy distributor (see 2.2.16) then

$$d_X(-, \Delta_x(-)) = d_X(-, x) : e_X \multimap X$$

the representable enriched presheaf of the object x is left adjoint of the covariant version $d_X(x,-)$, this way the Yoneda embedding factorize thought the enriched functor

$$k_X : X \to X_{cc}$$
$$x \mapsto d_X(-,x)$$

Proposition 2.2.30. *For every* $Q-$*enriched category* (X, d_X, e_X) *the graph of the enriched functor* k_X *is an isomorphism in* $Q - Cat_{rel}$

$$X \multimap X_{cc}$$
$$(\phi, x) \mapsto d_{X_{cc}}(\phi, k_X(x)) = [\phi, d_X(-,x)]$$

For $\phi : \hat{u} \multimap X$, $[\phi, d_X(-,x)]$ is the lifting of the representable presehaf of x throughout ϕ, explicitly this calculate $[\phi, d_X(-,x)] = \bigwedge_{y \in X}[\phi(y,\star), d_X(y,x)]$ where the lifting´s and wedge are calculated in $\mathcal{Q}$.

Now lets us examine what means in the context of **regular semi-categories** that an object can be pointed-out by a functor from a singleton **regular semi-category**.

Note 2.2.31. *Consider* u *an object in a quantaloid* $\mathcal{Q}$*, define a one-object enriched regular semicategory with domain* u*, is choose an* ***idempotent*** *morphism* $i : u \to u$ *in* $\mathcal{Q}$

Indeed given a enriched semi-category $(\star, e, d)$ with only one object $\star$, and domain $e : \star \mapsto u$, the hom assignation must comply for the structure of semi-category

$$d(\star,\star) : u \to u$$
$$d(\star,\star) \circ d(\star,\star) \leq d(\star,\star)$$

up here any endomorphism of u it works to define as $d(\star,\star)$, but the regular axioms is exactly the idempotency of this morphism

$$d(\star,\star) \circ d(\star,\star) = d(\star,\star)$$

so there is a correspondence between idempotents $i : u \to u$ and structures of enriched regular one object semi-category with domain u. This note makes the following definition and proposition very natural.

Definition 2.2.32. *For an object $x \in X$ in an enriched regular semi-category (X, d_X, e_X) we say that x is **stable** when exist a functor from a one object regular semi-category to X such that the image of the single object in the domain is x.*

Lemma 2.2.33. *For an object x in a regular semi-category (X, d_X, e_X) the following are equivalent*

1. *x is stable*
2. *There is an idempotent arrow $i : e_X(x) \to e_X(x)$ with $i \le d_X(x,x)$ such that for all $y \in X$, $d_X(y,x) \circ i = d_X(y,x)$ and $i \circ d_X(x,y) = d_X(x,y)$.*
3. *For all $y \in X$, $d_X(y,x) \circ d_X(x,x) = d_X(y,x)$ and $d_X(x,x) \circ d_X(x,y) = d_X(x,y)$*

Proof. The previous note tells us that the structure of regular semicategory over a singleton is an idempotent of the base quantaloid, so if we write u_i for the regular semicategory induced by an idempotent $i : u \to u$, the existence of an enriched regular functor $F : u_i \to X$ such that $\star \mapsto x$, tells us that necessarily the type of x is u, $u = e(\star) = e_X(F(\star)) = e_X(x)$, so the idempotent is an endomorphims of the type of x, $i = d(\star,\star) \le d_X(F(\star), F(\star)) = d_X(x,x)$ and the regularity of F is $d_X(y,x) = d_X(y, F(\star)) = d_X(y, F(\star)) \circ d_X(F(\star), F(\star)) = d_X(y,x) \circ d_X(x,x)$, similarly $d_X(x,y) = d_X(x,x) \circ d_X(x,y)$, so 1 and 2 are equivalent, and for 2) implies 3) the observation is that

$$\begin{aligned}
d_X(y,x) \circ d_X(x,x) &\le \bigvee_{x' \in X} (d_X(y,x') \circ d_X(x',x)) \\
&= d_X(y,x) \\
&= d_X(y,x) \circ i \\
&\le d_X(y,x) \circ d_X(x,x)
\end{aligned}$$

Then $d_X(y,x) = d_X(y,x) \circ d_X(x,x)$, that $d_X(x,y) = d_X(x,x) \circ d_X(x,y)$ follows a similar argument. If 3) happens is directly that $d_X(x,x) : e(x) \to e(x)$ is an idempotent of $\mathcal{Q}$, so we have 2) with $i = d_X(x,x)$ □

So the kind of enriched semi-category where Cauchy completion can be raised and is strictly weaker than enriched categories where the analogy with sheaf theory disappears, is brought by a regular semicategories (the quantaloid structure with the relational morphism is completely basic for the notion of Cauchy-completeness) in which all of its objects can be pointed-out by a regular functor. They are named as totally regular enriched semi-categories.

Definition 2.2.34. *For a semicategory* (X, d_X, e_X) *we said that is* ***totally regular*** *is for every* $x, y \in X$ $d_X(x,x) \circ d_X(x,y) = d_X(x,y)$ *and* $d_X(y,x) \circ d_X(x,x) = d_X(y,x)$

Note that this is stronger than the regular axiom for a semicategory, because $d(x,y) = d_X(x,x) \circ d_X(x,y) \leq \bigvee_{x' \in X} (d_X(x,x') \circ d_X(x',y) \leq d_X(x,y)$, so every every totally regular semicategory is a regular semicategory, with this objects verifying the regularity of the functional and relational morphism is more direct.

Lemma 2.2.35. *If* (X, d_X, e_X) *and* (Y, d_Y, e_Y) *are totally regular enriched semicategory then for an enriched functor* $F : X \to Y$ *to be regular is enough that for all* $x, y \in X$*,*

$$d_Y(y, F(x)) \circ d_X(x,x) = d_Y(y, F(x))$$

and

$$d_X(x,x) \circ d_Y(F(x), y) = d_Y(F(x), y)$$

also if $\phi : X \mathrel{\longrightarrow\!\!\!\!\!\!\!\ominus} Y$ *is a distributor for being regular is enough*

$$\phi(y,x) \circ d_X(x,x) = \phi(y,x)$$

and

$$d_Y(y,y) \circ \phi(y,x) = \phi(y,x)$$

Definition 2.2.36. *totally regular* $\mathcal{Q}$*-semicategories as objects and regular enriched functors determines a category with the usual composition of enriched functors. This category is denoted by* $\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{fun}}$

Definition 2.2.37. *totally regular* $\mathcal{Q}$*-semicategories as objects and regular distributors determines a quantaloid with the usual composition and local point-wise order of distributors. This category is denoted by* $\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{rel}}$

there is also a canonical covariant graph immersion with this respective adjunction in $\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{rel}}$

$$\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{fun}} \to \mathsf{Q}-\mathsf{TRSCat}_{\mathsf{rel}}$$
$$F : X \to Y \mapsto d_Y(-, F-)$$
$$d_Y(-, F-) \dashv d_Y(F-, -)$$

Given the well-pointness of a totally regular semi-category every enriched representable functor is the induced graph of an enriched functor, then comes with a right adjoint in $\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{rel}}$ this is the framework of the Cauchy-completeness for enriched categories

Cauchy complete totally regular semi-categories

For enriched totally regular semicategories the definition of **cauchy regular distributor**, **convergence** and **cauchy completeness** are the same that in the context of enriched categories, that is to say a regular distributor between totally regular semicategories is one with a right adjoint in $\mathsf{Q}-\mathsf{TRSCat}_{\mathsf{rel}}$, we say that a regular distributor converge if exist a enriched (necessarily regular) functor such that the distributor is the graph of the functor, and finally we say that a totally regular semicategory is cauchy complete if all the cauchy regular distributors with this codomain converge, the following lemma characterize cauchy completeness of a enriched totally regular semicategory and is an extension of an enriched categories result.

Lemma 2.2.38. *A enriched totally regular* $\mathcal{Q}$ *semicategory* (X, d_X, e_X) *is Cauchy complete if only* *if for every* $i : u \to u$ *idempotent arrow in* $\mathcal{Q}$*, the induced one object totally regular semicategory* $\hat{u}_i$ *is such that every Cauchy regular distributor* $\phi : \hat{u}_i \longrightarrow X$ *converge*

Lets note as ϕ^* the right adjoint of a distributor ϕ in the quantaloid $Q - \mathrm{TRSCat}_{rel}$, for the Cauchy completeness of (X, d_X, e_X) is enough to check the convergence of the left adjoint distributors over X defined in a single object totally regular semicategory, so considering over the following collection

$$X_{cc} = \{\phi : \hat{u}_i \longrightarrow X : \phi \dashv \phi^*\}$$

the natural structure $e_{cc}(\phi : \hat{u}_i \longrightarrow X) = u$ and $d_{cc}(\psi, \phi) = [\psi, \phi](\star, \star) = \bigwedge_{x \in X} [\psi(x, \star), \phi(x, \star)]$ where $[\psi(x, \star), \phi(x, \star)]$ is a lifting in $\mathcal{Q}$ determines again a totally regular semicategory, in what follows we cite the results that characterize $X_c c$ as the Cauchy completion of (X, d_x, e_x) and how this implies the equivalence between taken cauchy complete object and take relational morphism.

Proposition 2.2.39. *For every totally regular* $\mathcal{Q}$ *semicategory* (X, d_X, e_X)*, the structure* (X_{cc}, e_{cc}, d_{cc}) *define a totally regular* $\mathcal{Q}$ *semicategory*

So is in this part where the totally regular axioms plays a role, because, for every object x in X, there is regular functor

$$\Delta_x : \widehat{e(x)}_{d(x,x)} \to X$$

, $\star \mapsto x$ so the induced graph of Δ_x which is the representable distributor of x, is a Cauchy distributor $d_X(-, \Delta_x -) = d_X(-, x) \dashv d_X(x, -) = d_X(\Delta_x -, -)$. Then there is induced enriched regular functor

$$K_X : X \to X_{cc}$$

$$x \mapsto d_X(-, x)$$

which reflects enriched Yoneda lemma

Proposition 2.2.40. *For every totally regular* $\mathcal{Q}$ *semicategory* (X, d_X, e_X)*, every Cauchy distributor* $\phi \in X_{cc}$ *and every* $x \in X$*,*

$$d_{cc}(d_X(-, x), \phi) = \phi(x)$$

we are writing $\phi(x)$ as an abbreviation of $\phi(x,\star)$.

The following propositioon lets us identify up to equivalence the category of totally regular semicategories with relational morphism with its full subcategory of Cauchy complete objects, first the main isomorphism

Theorem 2.2.41. *For every totally regular semicategory* (X, d_X, e_X) *the induced graph of the partial Yoneda immersion* $K_X : X \to X_{cc}$ *is an* ***isomorphism*** *in* $Q-\mathsf{TRSCat}_{rel}$

$$d_{cc}(-,K_x-) : X \mathrel{-\!\!\!\circ\!\!\!\!\to} X_{cc}$$

So the full reflexive subcategory of $Q-\mathsf{TRSCat}_{rel}$ given by the Cauchy completions determines a equivalent subcategory, naturally this is formed by the Cauchy complete objects.

Lemma 2.2.42. *If* (X, d_X, e_X) *totally regular semicategory, its cauchy completion is a Cauchy complete totally regular semicategory*

So the indicated equivalence is

$$Q-\mathsf{TRSCat}_{rel} \cong Q-\mathsf{TRSCat}_{rel,cc}$$

which says that with relational morphism the category of totally regular semicategories is equivalent to its full subcategory of cauchy complete objects, but in the later this is equivalent to keep cauchy complete objects but changing morphism with functional ones, says enriched regular functors, the conclusion of this is that we can take equivalently relational morphism and all the totally regular semicategories *or* take functional morphism but keeping only the cauchy complete totally regular semicategories. For this result consider the following reflection

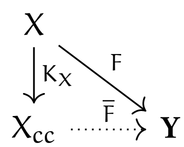
Proposition 2.2.43. *The inclusion* $i : Q-\mathsf{TRSCat}_{fun,cc} \to Q-\mathsf{TRSCat}_{fun}$ *of the category of enriched Cauchy complete totally regular semicategories and regular distributors in* TRSCat_{fun}*, find in the functor based in the construction* $(X, d_X, e_X) \mapsto (X_{cc}, d_{cc}, e_{cc})$

$$(-)_{cc} : \mathsf{TRSCat}_{fun} \to Q-\mathsf{TRSCat}_{fun,cc}$$

a left adjoint and the unity of this adjunction is given by the partial Yoneda embedding in the Cauchy completion $K_X : X \to X_{cc}$ *for* (X, d_X, e_X) *in* $Q-\mathrm{TRSCat}_{fun}$

$$(-)_{cc} \dashv i$$

this says that for every totally regular semicategory X if $\mathbf{Y}$ is a Cauchy complete totally regular semicategory then every regular functor $F : X \to \mathbf{Y}$ corresponds to a canonical extension to X_{cc} through K_X.



So the construction $K_{(-)} : (-) \to (-)_{cc}$ induces an isomorphism with relational morphism and is a reflection with functional morphism, lets examine the consequence of this:
Because of the very definition of Cauchy completeness if (X, d_X, e_x) and (Y, d_X, e_X) are Cauchy complete totally regular semicategories then $Q-\mathrm{TRSCat}_{fun}(X,Y) \cong Q-\mathrm{TRSCat}_{rel}(X,Y)$, so the graph assignation become locally an equivalence between ordered categories and is the identity in objects, given a biequivalence

$$Q-\mathrm{TRSCat}_{fun,cc} \sim \mathrm{Map}(Q-\mathrm{TRSCat}_{rel,cc}) \sim \mathrm{Map}(Q-\mathrm{TRSCat}_{rel})$$

where the later equivalence is because even as quantaloids $Q-\mathrm{TRSCat}_{rel,cc} \sim Q-\mathrm{TRSCat}_{rel}$ and $\mathrm{Map}(-)$ makes reference to the category structure (even a fragment of the quantaloid structure with the relational morphism can be translated to the functional context but the resulting pre-orders are no necessarily co-complete or antisymmetrical).
A final but important note.

Note 2.2.44. *Take a semicartesian commutative quantale* Q*, seeing as a quantaloid with only one object, a* $Q-\mathrm{set}$ *corresponds with a symmetric totally regular semi-category, the* $Q-\mathrm{Set}_{rel}$ *and* $Q-\mathrm{TRSCat}_{rel}$ *has the same objects. Now note that every adjunction of distributors* $Q-$

$\mathsf{TRSCat}_{rel,sym,cc} \cong Q-\mathsf{TRSCat}_{rel,sym}$ *induces a relational morphism of* $Q-$*sets, indeed if* $\phi \dashv \phi^*$ *is an adjunction of presheaves, if we take only* ***Cauchy complete and symmetric*** *totally regular semicategories then* ϕ *converges, so there is an element* a *that makes* $\phi(b) = d(b,a) = d(a,b) = \phi^*(b)$ *then* $\phi = \phi^*$*. With this in mind is immediate that the axioms which verifies the adjunction implies the relational nature of* ϕ*.*

Then we have the following

Theorem 2.2.45. *If we take a semicartesian commutative quantale* Q *as a one object quantaloid, the identity assignation in objects and morphism determines a full and faithfully functor.*

$$Q-\mathsf{TRSCat}_{rel,sym} \to Q-\mathsf{Set}_{rel}$$

We explore this situation but we the other notion of sheave that we are interested to, for this is important the following construction, that recover "the other way to get to sheaves".

2.2.3 Skeletal categories

One of the conditions for characterizing sheaves through enriched categories is that the category associated with a sheaf must reflect the locality condition of the sheaf. That is, if two sections are locally equal (their restrictions to a covering are equal), they must be the same section. In terms of enriched categories, if s, s' are objects of an enriched category (X, d_X, e_X), then if $d_X(s,s') = d_X(s',s) = e_X(s) = e_X(s')$, then $s = s'$. This condition for a category is called to be **skeletal**.
This condition, if not fulfilled by a category, can always be assumed modulo equivalence, as demonstrated by the following claim found in [Stu05a] (**Prop 4.7**).

Fact 2.2.46. *For any enriched category over a quantaloid* $\mathcal{Q}$, $(X, d_X.e_X)$ *there is a associated equivalent skeletal category* X_{ske}*:*
Objects: $\{[x] : x \in X\}$ *where* $[x]$ *is the class of* $x \in X$ *under the equivalence relation* $x \sim x'$ *iff* $d_X(x,x') = d_X(x',x) = e_X(x) = e_X(x')$.

Morphism: $d_{X_{ske}}([x],[y]) = d_X(x,y)$

Type: $e_{X_{ske}}([x]) = e_X(x)$. *Specifically*

$$X \longrightarrow X_{ske}$$

$$x \longmapsto [x]$$

determines an equivalence of $\mathcal{Q}$*—enriched categories.*

For this reason, even though our proposal concerning sheaves on quantales is grounded in enriched categories that should be particularly skeletal, it's not a hypothesis we're emphasizing in our theory, as the aforementioned fact allows us to assume this modulo equivalence of enriched categories.

2.3 Rel(Q) construction

The previous sections shows a kind of structural climbing from the theory of semi-categories (categories with out identities) to the totally regular semi-categories that a little loose but without lying, constitute the necessary and sufficient structure to raise the Cauchy completeness that allows the connection with sheaves.
In this section we explore the other alternative put in and developed for the case of locales in subsection 1.3.5, this alternative lead us in enriched category setting, in where we have Yoneda lemma and other wonderfully structural result, the exchange for stay in this paradise is that we must divide carefully our base object. The goal of this section is make this construction in a way that extent Walters construction for locales and admit some other interesting examples of quantales.
The work and calculations in this section are originals.

Consider a commutative, integral quantal Q, if Q has certain properties, we can construct a quantaloid $\mathrm{Rel}(Q)$.

Definition 2.3.1. *Let* Q *be a commutative and integral quantale, then define the following category*

Objects	u	$u \in Q$
Morphisms	$u \xrightarrow{R} v$	$R \in Q, R \leq u \otimes v$
Composition	$\begin{array}{ccc} u & \xrightarrow{R} & v \\ & \searrow^{S \circ R} & \downarrow S \\ & & w \end{array}$	$R \leq u \otimes v$ $S \leq v \otimes w$

$$S \circ R = (\hat{v} \implies R) \otimes (\hat{v} \implies S)$$

$$\hat{v} = v \implies (v \otimes v)$$

The lattice structure in $\mathrm{Rel}(Q)(u, v)$ *is just the restriction of the lattice structure* $(Q, \leq)$

Lemma 2.3.2. *If* v *is an idempotent of* Q*, then for all* $S, R \in Q$ *we have*

$$(\hat{v} \implies S) \otimes (\hat{v} \implies R) = S \otimes R$$

Proof. $\hat{v} = v \implies (v \otimes v)$, if v is an idempotent, $v \otimes v = v$, then $\hat{v} = v \implies v = 1$, leaving:

$$\begin{aligned} (\hat{v} \implies S) \otimes (\hat{v} \implies R) &= (1 \implies S) \otimes (1 \implies R) \\ &= \bigvee \{r \in Q : r \otimes 1 \leq S\} \otimes \bigvee \{r \in Q : r \otimes 1 \leq R\} \\ &= \bigvee \{r \in Q : r \leq S\} \otimes \bigvee \{r \in Q : r \leq R\} \\ &= S \otimes R \end{aligned}$$

□

Remark 2.3.3. *If the quantal* Q *is a locale, then this construction corresponds to Walter´s construction.*

Lemma 2.3.4. *If* $(Q, \leq, \otimes, 1)$ *is a commutative, integral, divisible and strict monotone quantale, then for all* $a, b, c \in Q$ *if* $b \leq c$ *and* $c \neq 0 = \min Q$ *then*

$$c \implies (a \otimes b) = a \otimes (c \implies b)$$

Proof. First note that $c \implies b \leq c \implies b$ then $(c \implies b) \otimes c \leq b$, this way $a \otimes (c \implies b) \otimes c \leq a \otimes b$, using the adjunction find $a \otimes (c \implies b) \leq c \implies (a \otimes b)$.

Next note that like $a \otimes b \leq b \leq c$ and the quantale is divisible we obtain

1. $a \otimes b = c \otimes (c \implies (a \otimes b))$
2. $b = c \otimes (c \implies b)$

With this in mind and that we already have $a \otimes (c \implies b) \leq c \implies (a \otimes b)$, if $a \otimes (c \implies b) < c \implies (a \otimes b)$, like $c \neq 0$ and the quantale is strict linear, gets

$$\begin{aligned} a \otimes b &= a \otimes c \otimes (c \implies b) \\ &< c \otimes (c \implies (a \otimes b)) \\ &= a \otimes b \end{aligned}$$

A contradiction that comes from the strict inequality, then what really happens is $a \otimes (c \implies b) = c \implies (a \otimes b)$

□

Lemma 2.3.5. *If* $(Q, \leq, \otimes)$ *is an integral, strict monotone quantale then the only idempotent are the top and the bottom elements.*

Proof. Consider $v \in Q$, such that $0 < v < 1$, then like $0 \neq v$ and 1 is the unity, if v is an idempotent for the strict monotony we have

$$\begin{aligned} v &= v \otimes v \\ &< v \otimes 1 \\ &= v \end{aligned}$$

leaving $v < v$, a contradiction that comes from suppose that v is an idempotent, then the only idempotent are 0 and 1. □

Theorem 2.3.6. *Consider* Q *a commutative and integral quantale, if* Q *is divisible and strict monotone, then the construction* $\mathsf{Rel}(Q)$ *produces a quantaloid.*

Proof. First of all, we prove the composition is well defined, considering $u \xrightarrow{R} v \xrightarrow{S} w$, that means $R \le u \otimes v$ and $S \le v \otimes w$, we divide this part of the prove in two cases:

v is idempotent:

If v is idempotent, then the lemma 2.3.2, tell us that

$$\begin{aligned} S \circ R &:= (\hat{v} \Longrightarrow S) \otimes (\hat{v} \Longrightarrow R) \\ &= S \otimes R \\ &\le (u \otimes v) \otimes (v \otimes w) \\ &\le u \otimes w \end{aligned}$$

from where we say that $u \xrightarrow{S \circ R} w$.

v is not an idempotent:

First note that since v is not an idempotent $v \neq 0$, then we apply lemma 2.3.4 and obtain

$$\hat{v} = v \Longrightarrow (v \otimes v) = v \otimes (v \Longrightarrow v) = v \otimes 1 = v$$

from where we say that

$$S \circ R = (\hat{v} \Longrightarrow R) \otimes (\hat{v} \Longrightarrow S) = (v \Longrightarrow R) \otimes (v \Longrightarrow S)$$

Since Q is divisible and $R \le u \otimes v \le v$ for the lemma 5 we obtain

1. $R = u \otimes v \otimes ((u \otimes v) \Longrightarrow R)$

2. $R = v \otimes (v \Longrightarrow R)$

in particular from 1

$$R \ge (u \otimes v) \otimes ((u \otimes v) \Longrightarrow R)$$

then using the adjunction we conclude that

$$v \implies R \geq u \otimes ((u \otimes v) \implies R)$$

if $v \implies R > u \otimes ((u \otimes v) \implies R)$ then like Q is linear and $v \neq 0$ (because is not an idempotent), we have

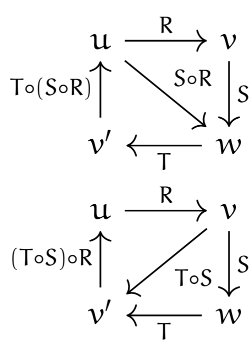
$$\begin{aligned} R &= v \otimes (v \implies R) \\ &> v \otimes (u \otimes ((u \otimes v) \implies R)) \\ &= (v \otimes u) \otimes ((u \otimes v) \implies R) \\ &= R \end{aligned}$$

that is to say $R > R$ a contradiction. Therefore $v \implies R = u \otimes ((u \otimes v) \implies R) \leq u$, analogously we obtain $v \implies S \leq w$, therefore

$$S \circ R = (v \implies R) \otimes (v \implies S) \leq u \otimes w$$

from where we say that $u \xrightarrow{S \circ R} w$.

Next, we focus in the associativity of this well-defined composition



$T \circ (S \circ R) = (\hat{w} \implies T) \otimes (\hat{w} \implies ((\hat{v} \implies S) \otimes (\hat{v} \implies R)))$

$(T \circ S) \circ R = (\hat{v} \implies R) \otimes (\hat{v} \implies ((\hat{w} \implies S) \otimes (\hat{w} \implies T)))$ Divide the proof of the associativity in the idempotency of v and w

v is idempotent

Then $\hat{v} = v \implies (v \otimes v) = v \implies v = 1$, then $\hat{v} \implies X = 1 \implies X = X$ for all $X \in Q$, then:

$T \circ (S \circ R) = (\hat{w} \implies T) \otimes (\hat{w} \implies (S \otimes R))$

$(T \circ S) \circ R = R \otimes (\hat{w} \implies S) \otimes (\hat{w} \implies T)$

If w is also an idempotent then $\hat{w} = 1$ and $\hat{w} \implies X = X$ for all $X \in Q$, from where we say that

$$\begin{aligned} T \circ (S \circ R) &= T \otimes (S \otimes R) \\ &= (T \otimes S) \otimes R \\ &= (T \circ S) \circ R \end{aligned}$$

If w is not an idempotent, then $w \neq 0$ and for lemma 2.3.4 we can conclude:

$\hat{w} = w \implies (w \otimes w) = w \otimes (w \implies w) = w \otimes 1 = w$, so

$T \circ (S \circ R) = (w \implies T) \otimes (w \implies (S \otimes R))$

$(T \circ S) \circ R = R \otimes (w \implies S) \otimes (w \implies T)$

Now note that $S \leq v \otimes w \leq w$ because S is an arrow from v to w, like w is not an idempotent then $w \neq 0$, so we apply again lemma 2.3.4 and obtain

$w \implies (S \otimes R) = R \otimes (w \implies S)$, then

$$\begin{aligned} T \circ (S \circ R) = (w \implies T) \otimes (w \implies (S \otimes R)) &= (w \implies T) \otimes (R \otimes (w \implies S)) \\ &= (w \implies T) \otimes R \otimes (w \implies S) \\ &= R \otimes (w \implies S) \otimes (w \implies T) \\ &= (T \circ S) \circ R \end{aligned}$$

Then we can say that if v is an idempotent the composition we study is associative.

If v is not an idempotent

Then we study in two cases, when w is an idempotent which is completely analogous to the case where v is an idempotent and w it is not, and the case of interest where neither are idempotent.

w is not and idempotent In this case like neither are idempotent we have: $\hat{v} = v$ and $\hat{w} = w$ so

$T \circ (S \circ R) = (w \implies T) \otimes (w \implies ((v \implies S) \otimes (v \implies R)))$

$(T \circ S) \circ R = (v \implies R) \otimes (v \implies ((w \implies S) \otimes (w \implies T)))$ in this case we use the study we did when we wanted to see the composition was well defined. Remember that when v is not and idempotent and $S : u \to v$ that is to say $S \le u \otimes v$, we find $v \implies S \le u$, use this for say that like $S : v \to w$ and v, w are not idempotent, we have: $v \implies S \le w$ and $w \implies S \le v$, using lemma 2.3.4 we find:

$(w \implies ((v \implies S) \otimes (v \implies R))) = (v \implies R) \otimes (w \implies (v \implies S))$ and

$(v \implies ((w \implies S) \otimes (w \implies T))) = (w \implies T) \otimes (v \implies (w \implies S))$.

Now is a general fact about quantales that: $a \implies (b \implies c) = (a \otimes b) \implies c$, so combining this, we obtain

$$\begin{aligned}
T \circ (S \circ R) &= (w \implies T) \otimes (w \implies ((v \implies S) \otimes (v \implies R))) \\
&= (w \implies T) \otimes ((v \implies R) \otimes (w \implies (v \implies S))) \\
&= (w \implies T) \otimes (v \implies R) \otimes ((v \otimes w) \implies S) \\
&= (v \implies R) \otimes ((w \implies T) \otimes ((v \otimes w) \implies S)) \\
&= (v \implies R) \otimes ((w \implies T) \otimes (v \implies (w \implies S))) \\
&= (v \implies R) \otimes (v \implies ((w \implies T) \otimes (w \implies S))) \\
&= (T \circ S) \circ R
\end{aligned}$$

Finally, we are going to prove that for every $u \in Obj(Rel(Q))$ exist a morphism $1_u \in Rel(Q)(u, u)$ which is an identity for the well defined and associative composition.

So for every $u \in Obj(Rel(Q)) = Q$ take

$$1_u = u \otimes u$$

lets prove that $R \circ 1_u = R$ if $R : u \to v$, and $1_u \circ S = S$ if $S : v \to u$, for every $v \in Obj(Rel(Q)) = Q$. Again we divide the proof based in the idempotency.

If u is an idempotent

Using lemma 2.3.5, we have that $u = 0$ or $u = 1$. If u is the bottom element of the quantale, then $1_u = u \otimes u = 0 \otimes 0 = 0$, so $1_u = u = 0$ and then if $R : u \to v$, we have $R \le u \otimes v = 0 \otimes v = 0$, so necessarily $R = 0$, so we have

$$\begin{aligned} R \circ 1_u &= (\hat{u} \Longrightarrow 1_u) \otimes (\hat{u} \Longrightarrow R) \\ (u \text{ is an idempotent}) \quad &= 1_u \otimes R \\ &= 0 \otimes 0 \\ &= 0 \\ &= R \end{aligned}$$

Is analogous show that $1_u \circ S = S$.

If $u = 1$ then $1_u = u \otimes u = 1 \otimes 1 = 1$, and then $R \circ 1_u = 1_u \otimes R = 1 \otimes R = R$, equally $1_u \circ S = S$

If u is not an idempotent. Then $\hat{u} = u$, and like $u \neq 0$, we can use lemma 2.3.4 to say that $u \Longrightarrow (u \otimes u) = u \otimes (u \Longrightarrow u)$, additionally like $R \leq u$ for lemma 5, $u \otimes (u \Longrightarrow R) = R$ so we have:

$$\begin{aligned} R \circ 1_u &= (u \Longrightarrow 1_u) \otimes (u \Longrightarrow R) \\ &= (u \Longrightarrow (u \otimes u)) \otimes (u \Longrightarrow R) \\ &= (u \otimes (u \Longrightarrow u)) \otimes (u \Longrightarrow R) \\ &= (u \otimes 1) \otimes (u \Longrightarrow R) \\ &= u \otimes (u \Longrightarrow R) \\ &= R \end{aligned}$$

□

Remark 2.3.7. *If the quantale* Q *of the previous construction is locale, we obtain the Walter´s construction.*

Indeed if every element is idempotent then the composition is just the quantale product, see lemma 2.3.2, based in this and the result in the final section of chapter 1, we consider Cauchy complete, symmetric, and skeletal $\mathrm{Rel}(Q)$ categories as sheaves over Q.

$\mathsf{Rel}(Q)$ is a closed bicategory, so we can enrich over the monoidal structure given by the composition of arrows, then there is focus in the category

$$\mathsf{Rel}(Q) - \mathsf{Cat}_{fun,cc,sym} \cong \mathsf{Rel}(Q) - \mathsf{Cat}_{rel,sym}$$

The following lemmas let us characterize the $\mathsf{Rel}(Q)$ construction as a canonical construction, proper of the enriched category theory.

For the following definition we take as a reference the nLab article [nLa23].

Definition 2.3.8. *In a category* C *a* $C-$*arrow* $a \xrightarrow{e} a$ *is said has a* ***retract*** *if there is a* $C-$*arrows* $r : b \to a$, $s : a \to b$ *such that* $r \circ s = e$ *and* $s \circ r = 1_b$, *we also said that* b ***is a retract*** *of* a

Note that if an endomorphism $a \xrightarrow{e} a$ has a retract then is an idempotent arrow. Indeed $e \circ e = (r \circ s) \circ (r \circ s) = r \circ (s \circ r) \circ s = r \circ 1_b \circ s = r \circ s = e$. The property that every idempotent in a category arises in this way, is called Cauchy-completeness for categories (because is the Cauchy-completeness over the monoidal category, Set.[2])

It happens that for every category C it exist a category D and an immersion $C \xrightarrow{i} D$ such that D is idempotent-complete in the sense that every idempotent of D has a retraction and is the smaller one in the sense that every object of D is a retract of the image of some object of C, because of this D is considered a completion of C, for which it is a call **split-idempotent completion** of C. Always exists because is defined in terms of pre-sheaves, but as we will see admits several different constructions. To detect when we are in the presence of this completion there is a test lemma that comes from the very definition. See [nLa23]

Lemma 2.3.9. *For any full and faithful embedding* $i : C \to D$, *if*

1. $i(e)$ *splits for any idempotent* $e : x \to x \in C$.

2. *For every object* d *in* D *exist a object* c_d *in* C *such that* d *is a retract of* $i(c_d)$ *in* D.

[2] Remark that $\mathsf{Set}-$categories are (small) usual categories

then (D, i) *represent a split-idempotent completion of* C.

Proposition 2.3.10. $\mathrm{Rel}(Q)$ *is the split-idempotent completion of* Q *like a one-object quantaloid.*

Proof. Let´s see that

$$
\begin{aligned}
i : Q &\to \mathrm{Rel}(Q) \\
\star &\mapsto 1 \\
\star \xrightarrow{u} \star &\mapsto 1 \xrightarrow{u} 1
\end{aligned}
$$

is in the hypothesis of lemma 2.3.9. Because 1 is an idempotent i is a functor, clearly is full and faithful. The rest is see that

$$i(\star \xrightarrow{e} \star) := 1 \xrightarrow{e} 1$$

splits in $\mathrm{Rel}(Q)$ if e is an idempotent of Q, besides of showing that every $u \in \mathrm{Obj}(\mathrm{Rel}(Q)) = Q$ is a retract of $i(\star) = 1$. Both facts are very clear, in the light of the commutativity of the following diagrams in $\mathrm{Rel}(Q)$.

$$
\begin{array}{ccc}
1 & \xrightarrow{e} & e \\
 & \searrow^{e\otimes e = e} & \downarrow e \\
 & & 1
\end{array}
\qquad
\begin{array}{ccc}
e & \xrightarrow{e} & 1 \\
 & \searrow^{e\otimes e = 1_e} & \downarrow e \\
 & & e
\end{array}
$$

$$
\begin{array}{ccc}
u & \xrightarrow{u} & 1 \\
 & \searrow^{1_u = u\otimes u} & \downarrow u \\
 & & u
\end{array}
$$

□

With this, we can give a comparison between the both notions of quantales in study.

2.4 $Q-\mathrm{sets}$ vs $\mathrm{Rel}(Q)-\mathrm{Cat}_{sym}$

Fact 2.4.1. *Taking a quantale as one object quantaloid the axioms of* $Q-\mathrm{set}$ *(see 2.1.2) and the axioms of* $Q-\mathrm{TRSCat}$ *(see 2.2.36) are the same*

For functional morphisms is pretty literal the correspondence, for relational morphism we need to take note, since:

$$Q-\mathbb{TRSCAT}_{rel,cc} \sim Q-\mathbb{TRSCAT}_{rel}$$

for comparing $Q-\mathbb{TRSC}at_{rel}$ with $Q-Set_{rel}$ we can suppose that our objects are Cauchy complete totally regular semi-categories and consider that the morphism are Cauchy distributors this plus the symmetry implies that if $\phi : X \multimap Y$ is a regular Cauchy distributor then the right adjoint $\phi^* : Y \multimap X$ makes $\phi(y,x) = \phi^*(x,y)$, indeed like the semicategories are Cauchy complete then exist a functor $F : X \to Y$ such that $\phi(y,x) = d_Y(y, F(x)) = d_Y(F(x), y) = \phi^*(x,y)$ then the unity and co-unity of the adjunction becomes

$$d_X(x,x') \leq \bigvee_{y \in Y} \phi^*(x,y) \otimes \phi(y,x') = \bigvee_{y \in Y} \phi(y,x) \otimes \phi(y,x')$$

and

$$d_Y(y,y') \geq \bigvee_{x \in X} \phi(y,x) \otimes \phi^*(x,y') = \bigvee_{y \in Y} \phi(y,x) \otimes \phi(y',x)$$

The co-unity is explicitly present in the definition of relational morphism between $Q-sets$ but there is only a trace of the unity that is only required over the diagonal $\delta(x,x)$ and in a weaker version. Explicitly the extra axiom for relational morphism in $Q-Set$ and the only who is not explicitly translated to regular Cauchy distributors is:

$$\delta(x,x) \leq \bigvee_{y \in Y} \phi(y,x)$$

which is clearly implied by the unity of the adjunction $\phi \dashv \phi$ in $Q-\mathbb{TRSC}at_{rel}$.

So the identical assignations naturally give a functor

$$Q-\mathbb{TRSC}at_{sym,cc,rel} \to Q-Set_{rel}$$

$$(X,d,e) \mapsto (X,d)$$

$$\phi \mapsto \phi$$

Proposition 2.4.2. *The following define a functor* $Q-Set_{fun} \to Rel(Q)-Cat_{fun,sym}$

$$(X,\delta) \mapsto (X,d,e)$$
$$d(x,y) := \delta(x,y)$$
$$e(x) := \delta(x,x)$$

and is identical in morphism.

$$F : X \to Y \mapsto F : X \to Y$$

Proof. This is pretty direct, since $d(x,y) = \delta(x,y) = \delta(x,x) \otimes \delta(x,y) \otimes \delta(y,y) \leq \delta(x,x) \otimes \delta(y,y) = e(x) \otimes e(y) = e(y) \otimes e(x)$ then $d(x,y) : e(y) \to e(x)$ is an arrow in $\mathrm{Rel}(Q)$, the composition inequality and the symmetry are explicit in the axioms of $Q-set$ and the existence of identities axiom comes from the idempotency of $\delta(x,x)$, $1_{e(x)} = e(x) \otimes e(x) = \delta(x,x) \otimes \delta(x,x) = \delta(x,x) = d(x,x)$ so this map objects to objects, and given that the axioms for functional morphism between $Q-Sets$ and enriched functors between enriched categories are the same, the identical assignation in morphism determines a functor. □

Remark 2.4.3. *Note that for the assignation* $(X,d,e) \mapsto (X,d)$ *from* $\mathrm{Obj}(\mathrm{Rel}(Q)-\mathrm{Cat}_{sym,rel,cc})$ *to* $\mathrm{Obj}(Q-\mathrm{Set}_{rel})$ *happens that in the definition of enriched category over quantaloids of the form* $\mathrm{Rel}(Q)$ *the hom arrows are always composed over idempotent elements of* Q*, so as the distributors. Indeed note that the diagrams presented in the definitions are*

$$\begin{array}{ccc} e(y) & \xrightarrow{d(x,y)} & e(x) \\ {\scriptstyle d(y,z)}\uparrow & \nearrow_{d(x,y)\circ d(y,z)} & \\ e(z) & & \end{array} \qquad \begin{array}{ccc} e(x) & \xrightarrow{\phi(y,x)} & e(y) \\ {\scriptstyle d(x,x')}\uparrow & \nearrow_{\phi(y,x)\circ d(x,x')} & \\ e(x') & & \end{array}$$

and because of the composition and identities in $\mathrm{Rel}(Q)$ *happens that* $1_{e(x)} = e(x) \otimes e(x)$

With all this just remaind a theorem of [Stu05b], that relates enriched categories over splitt idempotent completions with totally regular semicategories.

Theorem 2.4.4. *For any quantaloid* $\mathcal{Q}$ *if* $\overline{\mathcal{Q}}$ *note is the (a) splitt-idempotent completion of* $\mathcal{Q}$*, then there is an equivalence*

$$Q-\mathrm{TRSCat}_{rel} \cong \overline{Q}-\mathrm{Cat}_{rel}$$

With this in mind and the result of lemma 2.3.10, if Q is a quantale and we take as a one object quantaloid, then:

$$Rel(Q)-Cat_{sym,cc,fun} \cong Rel(Q)-Cat_{sym,rel} \cong Q-TRSCat_{rel,sym} \cong Q-TRSCat_{fun,cc,sym}$$

Then with the result of the theorem 2.2.45, gives

Theorem 2.4.5. *If Q is a semicartesian commutative quantale the identity assignation determines the following arrow, which is an immersion*

$$Rel(Q)-Cat_{rel,sym} \cong Q-TRSCat_{rel,sym} \to Q-Set_{rel}$$

3 Internal and external logic over sheaf-like categories

The intention of this chapter is to define logics for which the sheaf-like structures of the previous chapter become semantics, it is the interest of this text that in nature we present two different perspectives, one extending the categorical approach of hyperdoctrines (see [Hyl80], [Pit99]) and other extending the perspective presented in [DP21] which in turn extends continuous logic for metric spaces. We give a section to each of these purposes, in both original work is presently related to basic definitions that extend the classical theory based on locales to the quantale-based context and the formulation and resolution of test questions regarding the operation and interpretation of these logics.

3.1 Categorical logic

3.1.1 Hyperdoctrines and (Heyting) tripos

The notion of tripos (Hyland, Johnstone, and Pitts 1980; Pitts 1981) provides a unified approach of two very different classes of toposes: the realizability toposes and the localic toposes (in the sense Higg's description of sheaf toposes as H-valued sets, see Section 4 of Chapter 1)

A tripos is a first-order hyperdoctrine with equality satisfying an additional property that allows it to interpret impredicative higher-order logic as well. In particular, every tripos gives rise to a corresponding topos and every topos gives rise to a tripos. The tripos construction can be seen as the universal solution to the problem of realizing the predicates of a first-order hyperdoctrine as subobjects in a **l**ogos (a.k.a. Heyting category) with effective equivalence relations.

Definition 3.1.1. *Let* C *be a category with finite products. A* ***doctrine*** P *over* C *is specified by a contravariant functor* $P : C^{op} \longrightarrow$ ***Poset****, the category of partially ordered sets and monotone functions, we call it a* ***hyperdoctrine*** *if meets the following conditions*

- *For each object* X *in* C*, the poset* $P(X)$ *is a Heyting algebra, i.e. has top element (*$\top$*) a bottom element (*$\bot$*), binary meets (*$\bigwedge$*), finite joins (*$\bigvee$*) and pseudo complements (*$\Longrightarrow$ *).*
- *For each morphism* $X \xrightarrow{f} Y$ *in* C*, the monotone function* $P(Y) \xrightarrow{P(f)} P(X)$ *is a morphism of Heyting algebras.*

we say that is a ***first-order hyperdoctrine*** *if additionally have the following properties*

1. *For every diagonal morphism in* C, $X \xrightarrow{\Delta_X} X \times X$ *there is a left adjoint for* $P(X \times X) \xrightarrow{P(\Delta_X)} P(X)$ *at the level of the top element of* $P(X)$, $\top_X = \top_{P(X)} \in P(X)$*. That is to say, exist an element* $=_X \in P(X \times X)$ *such that for every* $A \in P(X)$

$$\top_X \leq P(\Delta)(A) \text{ if and only if } =_x \leq A$$

2. *For every projection in* C $Y \times X \xrightarrow{\pi_Y} Y$ *its image* $P(Y) \xrightarrow{P(\pi_Y)} P(Y \times X)$ *has a left and a right adjoint, that are noted respectively by* $(\exists_X)_Y$ *and* $(\forall_X)_Y$*. This means that for all* $A \in P(Y)$ *and* $B \in P(Y \times X)$ *we have*

$$(\exists_X)_Y(B) \leq A \text{ if and only if } B \leq P(\pi_Y)(A)$$
$$P(\pi_Y)(A) \leq B \text{ if and only if } A \leq (\forall_X)_Y(B)$$

Besides this we must ask for the naturality of this construction, this is that for every morphism $Y \xrightarrow{f} Y'$ *in* C *the following diagrams commute*

$$\begin{array}{ccc} P(Y' \times X) & \xrightarrow{P(f \times 1_X)} & P(Y \times X) \\ {\scriptstyle (\exists_X)_{Y'}}\downarrow & & \downarrow{\scriptstyle (\exists_X)_Y} \\ P(Y') & \xrightarrow[P(f)]{} & P(Y) \end{array} \qquad \begin{array}{ccc} P(Y' \times X) & \xrightarrow{P(f \times 1_X)} & P(Y \times X) \\ {\scriptstyle (\forall_X)_{Y'}}\downarrow & & \downarrow{\scriptstyle (\forall_X)_Y} \\ P(Y') & \xrightarrow[P(f)]{} & P(Y) \end{array}$$

We call the elements of the Heyting algebras $P(X)$ P—**predicates** and the Heyting algebras morphism $P(f) : P(Y) \to P(X)$ **re-indexing** morphism (through f if we want to specify the morphisms where is come from), this terminology comes from the following main example:

Examples 3.1.2. *For* $(H, \leq, \wedge, 1)$ *a Heyting algebra consider the functor*

$$\mathsf{Set}^{op} \to \textbf{\textit{Poset}}$$
$$X \xrightarrow{f} Y \mapsto H^Y \xrightarrow{-\circ f} H^X$$

which assigns to every set X*, the Heyting algebra* $H^X := \{f : X \to H : f$ *is a function*$\}$ *with operation and order defined pointwise, and to every function* $f : X \to Y$ *the Heyting algebra morphism given by precomposition with* f

$$H^Y \xrightarrow{-\circ f} H^X$$
$$Y \xrightarrow{\alpha} H \mapsto X \xrightarrow{\alpha\circ f} H$$

Let's examine the structure of this specific hyperdoctrine: The fact that for every set X, defined pointwise the structure of Heyting algebra over H^X is completely straightforward (the particular structure in H can be translated for the fold product H^X). The equality $=_X \in H^X$ is a delta Kronecker function $=_X : X \times X \to H$ defined by:

$$=_X (x, y) = \begin{cases} \top & \text{if} \quad x = y \\ \\ \bot & \text{if} \quad x \neq y \end{cases}$$

From this is clear that for $\top_X : X \to H$, $x \mapsto \top$ and any $\alpha : X \times X \to H$ we have for all $x, y \in X$, $=_X (x, y) \leq \alpha(x, y)$ if and only if $\top \leq \alpha(x, x)$ which is $\top_X(x) \leq \alpha(\Delta_X(x)) = \alpha \circ \Delta_X(x)$, so $=_X \leq \alpha$ if and only if $\top_X \leq - \circ \Delta_X(\alpha)$. Note that for the only property is that $(H, \leq, \wedge, \top)$ have top and bottom elements.

Lets see how are the quantifiers in this hyperdoctrine. Consider $\beta : Y \times X \to H, \gamma : Y \to H$ arbitrary functions, so $\beta \leq -\circ\pi_Y(\gamma)$ means that for every $(y,x) \in Y \times X$ we have $\beta(y,x) \leq -\circ\pi_Y(\gamma)(y,x) = \gamma\circ\pi_Y(y,x) = \gamma(y)$, so this is equivalent to ask that $\bigvee_{x\in X} \beta(y,x) \leq \gamma(y)$, so defining $(\exists_X)_Y(\beta)(y) := \bigvee_{x\in X} \beta(y,x)$, find out that

$$(\exists_X)_Y \dashv - \circ \pi_Y$$

Dual argument shows that if we define for all $y \in Y$, $(\forall_X)_Y(\beta)(y) := \bigwedge_{x\in X} \beta(y,x)$ then

$$- \circ \pi_Y \dashv (\forall_X)_Y$$

. The naturality is simply because of the commutativity of the process of pre-compose and taking joins: for every functions $\beta : Y \times X \to H$, $g : Y \to Y'$ and $y \in Y$ we have

$$\begin{aligned} (\bigvee_{x\in X} \beta(-,x) \circ g)(y) &= \bigvee_{x\in X} \beta(-,x)(g(y)) \\ &= \bigvee_{x\in X} (\beta(g(y),x) \\ &= (\bigvee_{x\in X} (\beta(g(-),x))(y) \end{aligned}$$

Similar happens for meets.

A way of thinking in a function $X \xrightarrow{\alpha} H$ is as its image, that is to say as a family of elements of H indexed by elements of X as $\{\alpha(x)\}_{x\in X}$ so we can refer to the elements of the Heyting algebra H^X as H—families over X, lists of elements of H indexed by the set X, and the role of the image of a function $f : Y \to X$ is just to **re-index** this H—families, $\{\alpha(x)\}_{x\in X} \mapsto \{\alpha(f(y))\}_{y\in Y}$.

The specific distribution property of Heyting algebras is not used here, so lets make the following note

Note 3.1.3. *The argument for showing the existence of the equality predicate and quantifiers can be perform in any complete lattice.*

So this justify in first instance the name of re-indexing morphism for $P(f) : P(Y) \to P(X)$, the name of predicate over X for an element $a \in P(X)$ comes clearly by taking in the previous example the boolean algebra $2 = \{\bot, \top\}$, and the correspondence between subsets, characteristic functions and first-order properties over a set X. So there is this intuition of thinking that the elements of $P(X)$ are generalized first-order properties (more accurately local properties) over X.

Another interesting example of first-order hyperdoctrine came from the notion of **Heyting category (a.k.a. logos)** C: This is a first-order hyperdoctrine where the contravariant functor is just the subobject poset functor $\mathrm{Sub} : C^{op} \to \mathrm{HeytAlg}$ and, for each morphism $f : a \to a'$ in C, the Heyting algebra morphism $f^* : \mathrm{Sub}(a') \to \mathrm{Sub}(a)$ has a left adjoint, $\exists_f : \mathrm{Sub}(a) \to \mathrm{Sub}(a')$, and a right adjoint $\forall_f : \mathrm{Sub}(a) \to \mathrm{Sub}(a')$.

3.1.2 C[P] construction

The main first construction in the theory of hyperdoctrines concerns is based in the fact that we can interpret and extract first-order signatures in arbitrary first-order hyperdoctrines, for this let's remark first on the interpretation part:

Consider $\Sigma = (\mathfrak{X}, \mathfrak{F}, \mathfrak{R})$ a multi-sorted first-order signature where $\mathfrak{X} = \{X_i\}_{i \in I}$ is a set of sorts, $\mathfrak{F} = \{f_j(x_1 : X_1, \ldots, x_n : X_n) : Y\}_{j \in J}$ a set of function symbols and $\mathfrak{R} = \{R_k : X_1, \ldots, X_m\}_{k \in K}$ is a set of predicate symbols. In any quantalic tripos (C, P) can be interpreted this signature by giving an object of C for every sort (type) symbol, a morphism for every function symbol, and a P—predicate for every predicate symbol, so lets put between double lines the interpretation of a symbol $\| - \|$, then we just said that

$$X \in \mathfrak{X} \xrightarrow{\|\|} \|X\| \in \mathrm{Obj}(C)$$

$$f(x_1 : X_1, \ldots, x_n : X_n) : Y \in \mathfrak{F} \xrightarrow{\|\|} \|f\| : \|X_1\| \times \cdots \times \|X_n\| \to \|Y\|$$

$$R(X_1, \ldots X_n) \in \mathfrak{R} \xrightarrow{\|\|} \|R\| \in P(\|X_1| \times \cdots \times \|X_n\|)$$

Proceeding by induction there is an interpretation for every first-order term over the signature Σ as a morphism in C. So consider a term $t(x_1 : X_1, \ldots, x_n : X_n) : Y$, define $\|t\| : \|X_1\| \times \cdots \times \|X_n\| \to \|Y\|$ recursively in the complexity of t by:

- If t is $x_i : X_1 \times \ldots \times X_n$ then $\|t\| := \|X_1\| \times \ldots \times \|X_n\| \xrightarrow{\pi_i} \|X_i\|$

- If t is $f(t_1(x_1,\ldots x_n) : Y_1,\ldots,t_m(x_1,\ldots,x_n) : Y_m) : Y$ where for every $1 \leq j \leq m$, $t_j(x_1,\ldots,x_n) : Y_j$ is a term such that there is already defined an interpretation $\|t_j\| : \|X_1\| \times \cdots \times \|X_n\| \to \|Y_j\|$ and $f(y_1 : Y_1,\ldots,y_m : Y_m) : Y$ is a function symbol, define $\|t\| = \|f\| \circ (\|t_1\|,\ldots,\|t_m\|)$ where $(\|t_1\|,\ldots,\|t_m\|)$ is a product of morphism in C given by the universal property of cartesian product.

Off course there is also a recursive canonical interpretation of first-order formulas over the signature Σ, predicates are already interpreted, and with the structure of Heyting Algebra in each $P(X)$ interpret equality of terms, disjunctions and conjunctions, the morphism $(\exists_X)_Y$ and $(\forall_X)_Y$ are used to interpret existential and universal quantifiers.

$$\|t_1 : Y = t_2 : Y\| : =_Y (\|t_1\|,\|t_2\|)$$
$$\|\phi \wedge \psi\| = \|\phi\| \wedge \|\psi\|$$
$$\|\phi \vee \psi\| = \|\phi\| \wedge \|\psi\|$$
$$\|\phi \implies \psi\| = \|\phi\| \implies \|\psi\|$$
$$\|\phi \implies \psi\| = \|\phi\| \implies \|\psi\|$$
$$\|\forall x \phi(x,y)\| = (\forall_X)_Y(\|\phi\|)$$
$$\|\exists x \phi(x,y)\| = (\exists_X)_Y(\|\phi\|)$$

Given a first-order hyperdoctrine $P : C^{op} \to$ **Poset** there is a **internal language** of it, given by the first-order signature conformed by a symbol of type (a sort) for every object X of C, a symbol of function for every morphisms $X \xrightarrow{f} Y$ in C and a symbol of predicate $R(x)$ for every element $R \in P(X)$, there is a canonical interpretation for this signature in $C^{op} \to$ **Poset**, formulas based in this signature and interpreted in this obvious canonical way express properties that speak about the category C and more precisely the functor P, and can be used to make constructions using the internal logic. More precisely if the validity of a sentence in this language is required by ask for the interpretation of the sentence be the top element of the respective Heyting algebra in which is interpreted, we perform several sentences that codify for example the behavior of categories such as $Q - Sets_{rel}$, for this consider the following construction.

Given $P : C^{op} \to$ **Poset** a first-order hyperdoctrine, consider the category C[P] whose objects are couples (X, E) where X is an object of C and E is binary predicate over X, i.e. $E \in P(X \times X)$ such that is a partial equivalence relation in the internal logic, that is to say the following sentences are true:

$$\forall x, y : X \; E(x,y) \implies E(y,x)$$

$$\forall x, y, z : X \; E(x,y) \wedge E(y,z) \implies E(x,z)$$

The morphisms of this category are binary predicates that in the internal logic are well-defined functions between the "co-sets" given by the partially equivalence relations in its domain and co-domain. Concretely a morphism $(X, E) \xrightarrow{F} (Y, E')$ is given by a predicate $F \in P(X \times Y)$ such that the following sentences are true in the internal logic:

$$\textbf{1)} \forall x : X \; \forall y : Y \; F(x,y) \implies E(x,x) \wedge E(y,y)$$

$$\textbf{2)} \; \forall x, x' : X \; \forall y, y' : Y \; E(x,x') \wedge E'(y,y') \wedge F(x,y) \implies F(x',y')$$

$$\textbf{3)} \; \forall x : X \; \forall y, y' : Y \; F(x,y) \wedge F(x,y') \implies E'(y,y')$$

$$\textbf{4)} \; \forall x : X \; E(x,x) \implies \exists y : Y \; F(x,y)$$

Proposition 3.1.4. *Given a first-order hyperdoctrine* $P : C^{op} \to$ ***Poset*** *the internal partial equivalence relations* (X, E) *as objects and internal well defined functions* $(X, E) \xrightarrow{F} (Y, E')$ *as morphisms determines a category with the following composition and identities:for* $(X, E) \xrightarrow{F} (Y, E')$ *and* $(Y, E') \xrightarrow{G} (Y, E'')$, $G \circ F(x,z) := \exists y : Y (F(x,y) \wedge G(y,z))$ *and* $\mathrm{id}_{(X,E)} = E$

Theorem 3.1.5. *For every first-order hyperdoctrine* $P : C^{op} \to$ ***Poset****, the category* $C[P]$ *is a logos with effective equivalence relations, and classifies in the category of logos with effective equivalence relations the interpretation of* P*-predicates.*

Recall what means for the hyperdoctrine of H—families for a fixed complete Heyting algebra H as in example 3.1.2, the construction C[P]: A binary predicate $E \in P(X \times X)$ is given by a function $X \times X \xrightarrow{E} H$ and to being an internal partial equivalence relation is given by ask that the following equality's holds in H, $\bigwedge_{x,y \in X} (E(x,y) \implies E(y,x)) = 1$, $\bigwedge_{x,y,z \in X} ((E(x,y) \wedge E(y,z)) \implies E(x,z)) = 1$, that is to say that for every $x,y,z \in X$ it happens that $E(x,y) \implies E(y,x) = 1$ and $((E(x,y) \wedge E(y,z)) \implies E(x,z)) = 1$ the later is equivalent to the inequalities $E(x,y) \leq E(y,x)$ and $E(x,y) \wedge E(y,z) \leq E(x,z)$, note that they are exactly the axioms 1. and 2. of the definition 2.1.2 taking H as a commutative semicartesian quantale.

Now lets note that the first axiom for a morphism in C[P] for this hyperdoctrine of H—families; a binary predicate $F \in P(X \times Y)$ is given by a function $F : X \times Y \to H$, that the axiom

$$\forall x : X\ \forall y \in Y\ F(x,y) \implies E(x,x) \wedge E(y,y)$$

comes true in this specific hyperdoctrine means that $\bigwedge_{x \in X,\ y \in Y} (F(x,y) \implies (E(x,x) \wedge E(y,y))) = 1$ so for every $x \in X$ and $y \in Y$, it happens that $F(x,y) \implies (E(x,x) \wedge E(y,y)) = 1$ which is equivalent to $F(x,y) \leq E(x,x) \wedge E(y,y)$ but this in Heyting algebras only means that $F(x,y) \leq E(x,x)$ and $F(x,y) \leq E(y,y)$ or in a more useful presentation for us that $F(x,y) \wedge E(x,x) = F(x,y)$ and $F(x,y) \wedge E(y,y) = F(x,y)$, so this axiom can replaced by the axioms:

$$\forall x : X \forall y : Y\ F(x,y) \wedge E(x,x) = F(x,y)$$

$$\forall x : X \forall y : Y\ F(x,y) \wedge E(y,y) = F(x,y)$$

Now lets examine this applied to the second axiom for morphism in C[P], $\forall x,x' : X\ \forall y,y' : Y\ E(x,x') \wedge E'(y,y') \wedge F(x,y) \implies F(x',y')$ which means that for all $x,x' \in X$ and $y,y' \in Y$, $E(x,x') \wedge E'(y,y') \wedge F(x,y) \leq E'(y,y')$, so taking $x = x'$ and using the equivalent version of the axiom **1)** obtain $E(x,x) \wedge E'(y,y') \wedge F(x,y) = E'(y,y') \wedge F(x,y)$ the axioms says that

$$E'(y,y') \wedge F(x,y) \leq F(x,y')$$

and making $y = y'$ we obtain

$$E(x,x') \wedge F(x,y) \leq F(x',y)$$

Reciprocally assuming the later two inequalities obtain the axiom **2)**: $E(x,x') \wedge E'(y,y') \wedge F(x,y) \leq E(x,x') \wedge F(x,y') \leq F(x',y')$. From this is clear that for the example of $H-$families if we change the first two axioms for morphisms C[P] by the following axioms (in the first two we prefer fall in redundant writing for make an explicit translation of the axiom to the quantale case.)

$$\textbf{I)}\ \forall x : X\ \forall y : Y\ F(x,y) \implies (F(x,y) \wedge E(x,x))$$
$$\textbf{II)}\ \forall x : X\ \forall y : Y\ F(x,y) \implies (F(x,y) \wedge E(y,y))$$
$$\textbf{III}\ \forall x : X\ \forall y,y' : Y\ E'(y,y') \wedge F(x,y) \implies F(x,y')$$
$$\textbf{IV)}\ \forall x,x' : X\ \forall y : Y\ E(x,x') \wedge F(x,y) \implies F(x',y)$$

This notes are for make explicit the very direct relation of the construction C[P] for the hyperdoctrine of $H-$families $H^{(-)} : \mathrm{Set}^{op} \to \textbf{Poset}$ with the definition of the category $Q - \mathrm{Set}_{rel}$ applied to a Heyting algebra. Indeed for objects, define an $H - set$ is equivalent that define an internal partial equivalence relation with the extra property $\forall x,y : X\ E(x,y) \implies E(x,x)$ and the morphism are exactly the same, compare what means the axioms **3)** and **4)** in this hyperdoctrine and note that is exactly the last two axioms of the definition 2.1.5 of relational morphism between Q-sets applied to the Heyting algebra $(H, \leq, \wedge, \top)$ as a quantale. The axioms **1)** and **2)** are equivalent to the axioms $\textbf{I}, \textbf{II}, \textbf{III}, \textbf{IV}$ and this are exactly the first four axioms of the definition 2.1.5. So we put this as a fact that we use in the next section for give a version of the C[P] construction in the context of first-order doctrines valued over quantales.

Fact 3.1.6. *Given a complete Heyting algebra* H*, for the hyperdoctrine of* $H-$*families*

$$H^{(-)} : \mathrm{Set}^{op} \to \textbf{\textit{Poset}}$$
$$X \mapsto H^X$$

the logos $\mathrm{Set}[H^{(-)}]$ *is isomorphic to the topos* $H - \mathrm{Set}_{rel} \sim \mathrm{Sh}(H)$

3.1.3 Generalized quantifiers and Beck-Chevalley condition

The interpretation of the adjoints to the induced maps of projections are interpreted as the usual quantifiers but, we can ask to for adjoints of the induced arrow of every morphism, and then get to the concept of generalized quantifiers. The existence of it actually only depends on the existence of the usual ones, that is to say if re-index along projections finds adjoint, then re-indexing along any morphism finds adjoints to. We won't focus on these generalized quantifiers then, but the extra logical property of replacing terms in formulas with quantifiers is a property that is categorically caught using the so-called Beck-Chevalley property. Something important to note is that having it for reindexing morphisms through projections, that is, having it on classical quantifiers does not imply having it on all morphisms, which is why it is usually requested in general.

Note 3.1.7. *Ask for left and right adjoints of the images of projections implies right and left adjoint for the image of every morphism*

The Beck-Chevalley condition for a hyperdoctrine $P : C^{op} \to \mathsf{Poset}$ says that for every morphism $Y \xrightarrow{f} Y'$ in C, the horizontal composition specified in the following diagram

$$\begin{array}{ccccccc}
P(X\times Y) & \xleftarrow{id_{P(X\times Y)}} & P(X\times Y) & \xleftarrow{P(id_X\times f)} & P(X\times Y') & \xleftarrow{P(\pi_{Y'})} & P(Y') \\
{\scriptstyle id_{P(X\times Y)}}\downarrow & \overset{\epsilon_{X,Y}}{\Leftarrow} & \downarrow{\scriptstyle (\exists_X)_Y} & \overset{id_\alpha}{\Leftarrow} & \downarrow{\scriptstyle (\exists_X)'_Y} & \overset{\eta_{X,Y'}}{\Leftarrow} & \downarrow{\scriptstyle id_{P(Y')}} \\
P(X\times Y) & \xleftarrow[P(\pi_Y)]{} & P(Y) & \xleftarrow[P(f)]{} & P(Y') & \xleftarrow[id_{P(Y')}]{} & P(Y')
\end{array}$$

is an isomorphism.

As it said this categorical property corresponds to the logical property that replaces terms in formulas commute with the process of taking quantifiers. To make explicit this relation, is necessary to consider a **syntactic hyperdoctrine** over a category formed by context (order list of variables) and arrows as terms, the functor that gives the hyperdoctrine takes a context and leaves the set of formulas in this variables, re-indexing along a term is given by replacement. Interpret the previous diagram over this syntactic hyperdoctrine makes appear the mentioned logical property,

moreover we can extend this to every hyperdoctrine because as is said at the beginning of section 3.1.2, every hyperdoctrine arises a language.
The theory that we want to extend to quantale based hyperdoctrines don't use this logical/categorical property, we omit get deeper into it because of this, but if one want to make some serious applications (for example model-theoretic questions) then it should be taken into account as a hypothesis.

3.2 Monoidal hyperdoctrines

In this section we propose an extension of the theory of first-order hyperdoctrines with values in quantales rather than in Heyting algebra, and explore how the category $Q-\mathsf{Set}_{rel}$ appears as an example of the $C[P]$ construction in this context.

Definition 3.2.1. *Given* C *a category with finite products, a doctrine* $C^{op} \to$ ***Poset*** *is said to be a* ***monoidal hyperdoctrine*** *if:*

1. *For every object* X *of* C *the poset* $P(X)$ *has structure of commutative integral quantale natural in the variable* X*, that is:*
2. *For every morphism* $X \xrightarrow{f} Y$ *in* C *its image* $P(Y) \xrightarrow{P(f)} P(X)$ *is a morphism of quantales (a morphism of complete lattices that preserve the monoidal product).*

we say that is a ***first-order monoidal hyperdoctrine*** *if as a functor meets exactly the two extra properties explicit in the definition 3.1.1 that makes an hyperdoctrine first-order (i.e. has a equality predicate and quantifiers)*

As in the cartesian case (with Heyting algebras) we call **predicates** to the elements of the quantales $P(X)$ for X in C, and **re-indexing along** f to the image of an arrow $X \xrightarrow{f} Y$. This corresponds to the very direct fact that the hyperdoctrine of set indexed families of a fixed quantale, makes an example of first-order monoidal hyperdoctrine.

Examples 3.2.2. *Consider a commutative integral quantale* $(Q, \leq, \otimes, 1)$*, the functor*

$$\mathrm{Set}^{op} \xrightarrow{Q^{(-)}} \textbf{\textit{Poset}}$$
$$X \xrightarrow{f} Y \mapsto Q^X \xrightarrow{-\circ f} Q^Y$$

is a monoidal first-order hyperdoctrine.

The quantale structure in Q^X is defined by components and is direct calculation matter check this. The first-order structure can be justified for the note 3.1.3, so as a quantale is in particular a complete lattice has arbitrary meets and joins, given the first-order quantifiers $\forall$ and $\exists$. Note that for this example the equality predicate has a discrete nature.

3.2.1 $C[P]$ construction for monoidal first-order hyperdoctrines

The extra monoidal structure naturally requires additional syntax that allows us to talk about it, so for this consider the usual syntax of first-order logic, i.e. logical symbols for equality (=), conjunction ($\wedge$), disjunction ($\vee$), a symbol for implication ($\implies$), symbols for quantifiers, existential ($\exists$) and universal ($\forall$), and no logical symbols given by symbols of function, relation and constant (this one can be put as certain class of function symbol), this comes with arities and types (sorts) for each as implicitly we perform them in the hyperdoctrines section, the new syntaxes comes with a new symbol & which plays the logic role of a strong conjunction, because its interpretation $\otimes$ in semi-cartesian quantales meets $a \otimes b \leq a \wedge b$ which a posteriori says that $\phi \& \psi \leq \phi \wedge \psi$, there is another change which no comes with a new symbol or with withdraw a old one, but with re-interpreted the meaning of the implication symbol ($\implies$), this is because the lattices structure is not necessarily a Heyting algebra, so calculate infima with a fixed element has no necessarily a right adjoint which becomes the interpretation of the (cartesian) implication, instead of this we have a right adjoint with the process of make monoidal products with fixed elements, then a monoidal implication, for which in the general theory use the same logical symbol.

So in resume, the logical framework is that of a weak first-order logic in which the conjunction symbol is & and weak means that the contraction rule for first-order deductive calculus is dropped, remark that this says that from a formula ϕ we can deduce $\phi\&\phi$ which force the idempotency axiom on $\otimes$ and makes or quantale a locale, so explicitly avoid this deductive rule, additionally have a weak conjunction symbol $\wedge$ that responds to the complete lattice structure in our quantales that are the algebraic models of this logical gadget that we are describing (in a somehow parallel way) this means that contraction, weakening and exchange are valid for this connective.

Definition 3.2.3. *For a first-order monoidal hyperdoctrine* $\mathsf{C}^{op} \xrightarrow{P}$ ***Poset*** *consider the category* $\mathsf{C}[\mathsf{P}]$ *whose objects are couples* (X,E) *where* X *is an object of* C *and* $\mathsf{E} \in \mathsf{P}(\mathsf{X}\times\mathsf{X})$ *is a binary predicate over* X *such that the following sentences of the internal language are true in the canonical interpretation of it in* $\mathsf{C}^{op} \xrightarrow{P}$ ***Poset***

$$\forall x,y:X\ E(x,y) \implies E(y,x)$$
$$\forall x,y,z:X\ E(x,y)\&E(y,z) \implies E(x,z)$$
$$\forall x,y\in X\ E(x,y) \implies (E(x,y)\&E(x,x))$$

A morphism between two objects $(X,E_1) \xrightarrow{F} (Y,E_2)$ *is given by a binary predicate* $F\in P(Y\times X)$ *such that the following sentences are true (again in their canonical interpretation)*

$$\forall x,x':X\ \forall y:Y\ F(y,x)\&E_1(x,x') \implies F(y,x')$$
$$\forall y,y':Y\forall x:X\ E_2(y,y')\&F(y',x) \implies F(y,x)$$
$$\forall y:Y\forall x:X\ F(y,x)\&E(x,x) \iff F(y,x)$$
$$\forall y:Y\forall x:X\ E(y,y)\&F(y,x) \iff F(y,x)$$
$$\forall y,y':Y\forall x:X\ F(y,x)\&F(y',x) \implies E_2(y,y')$$
$$\forall x:X \bigvee_{y\in Y} F(y,x) \iff E(x,x)$$

The composition of two (composable) morphism $G\circ F := \exists y:Y(G(z,y)\&F(y,x))$ *and the identities are given by* $1_{(X,E)} = E(x,y)$

3.2.2 Internal equivalence relations in the sub-object doctrine

see [AdAMM23a]

Fact 3.2.4. *The monomorphism of* $Q-\mathrm{sets}_{\mathrm{fun}}$ *are given by underling injective functions, the epimorphism by underling surjective functions and the isomorphism by bijective* δ*-preserving functions.*

Note that in particular $Q-\mathrm{set}_{\mathrm{fun}}$ is not a balanced category, that is can be mono+epi morphism that are not isomorphism, take for example the Lawvere´s quantale $([0,\infty],\geq,+,0)$, then a bijective contraction determines a mono and an epi but no an iso.

Given a $Q-\mathrm{set}$ (X,δ) the slice category $Q-\mathrm{Set}_{\mathrm{fun}}/X$ of functional maps into X induce a poset if we consider the quotient of it by the isomorphism relation, i.e. if $f:Y\to X$, $g:Z\to X$ are functional maps such that exist an isomorphism $Y\xrightarrow{h}Z$ with $g\circ h=f$ then we take two objects as the same, because of this is immediate that the relation $(A\xrightarrow{p}X)\leq(B\xrightarrow{q}X)$ if exist an arrow in $Q-\mathrm{Set}_{\mathrm{fun}}/X$ from $A\xrightarrow{p}X$ to $B\xrightarrow{q}X$ determines a poset that will be noted by $\mathrm{Sub}(X)$, lets put this into a doctrine.

Definition 3.2.5. *Consider the doctrine*

$$\mathrm{Sub}(-):Q-\mathrm{set}_{\mathrm{fun}}\to \textbf{\textit{Poset}}$$
$$(X,\delta)\mapsto \mathrm{Sub}(X)$$

For give a more precisely structure of this doctrine we use the following results of [AdAMM23a].

Theorem 3.2.6. *Consider a commutative semi-cartesian quantale* Q*, then the category of* $Q-\mathrm{sets}_{\mathrm{fun}}$ *is a complete and co-complete category*

Indeed the following are the (co)limits and (co)equalizers:

Consider (X,δ_X) and (Y,δ_Y) Q-sets and $f,g:(X,\delta_X)\to(Y,\delta_Y)$ functional morphism, and $((X_i,\delta_i))_{i\in I}$ a family of Q—sets indexed by a set I, if we denote by $E(x)=\delta(x,x)$ the extent of an element in a $Q-\mathrm{set}$ then:

$$\prod_{i\in I}(X_i,\delta_i)=(\{(x_i)_{i\in I}\in\prod_{i\in I}X_i \mid E(x_i)=E(x_j) \text{ for every } i,j\in I\},\delta_{\prod})$$
$$\delta_{\prod}((x_i)_{i\in I},(y_i)_{i\in I})=\bigwedge_{i\in I}\delta_i(x_i,y_i)$$
$$eq(f,g)=(\{x\in X\mid f(x)=g(x)\},\delta_X|_{\{x\in X|f(x)=g(x)\}})$$
$$\coprod_{i\in I}(X_i,\delta_i)=(\coprod_{i\in I}X_i,\delta_{\coprod})$$
$$\delta_{\coprod}((x,i),(y,j))=\begin{cases}\delta_i(x,y) & \text{if } i=j\\ \bot & \text{if } i\neq j\end{cases}$$
$$coeq(f,g)=(Y/\sim,d_{Y/\sim})\ {}^{1}$$
$$d_{Y/\sim}([y],[y'])=\bigvee_{a\sim y,a'\sim y'}\delta_Y(a,a')$$

Like in any category all the limits are formed by the equalizer over suitable arrows defined in a product, and the colimits arise as the co-equalizers of suitable morphism defined in a coproduct, in particular if $(A,\delta_A)\xrightarrow{f}(X,\delta_X)$ and $(B,\delta_B)\xrightarrow{g}(X,\delta_X)$ are monomorphism in $Q-Set_{fun}$ then the pullback is calculated based in the previous result as

$$A\wedge B=(\{(a,b)\in A\times B\mid E(a)=E(b),f(a)=g(b)\},\delta_{A\wedge B})$$

with $\delta_{A\wedge B}((a_1,b_1)(a_2,b_2))=\delta_A(a_1,a_2)\wedge\delta_B(b_1,b_2)$, the universal property of pullback is exactly the definition of wedge in the poset $Sub(X)$.

Joins are given by

$$A\vee B=((A\coprod B)/\sim_{f,g},\delta_{A\vee B})$$

where $(A\coprod B)/_{f,g}$ is the quotient of the coproduct $A\coprod B$ by the equivalence relation $\sim_{f,g}$ generated by $a\sim b$ if $f(a)=g(b)$, and $\delta_{A\vee B}([a],[b])$ is $\delta_A(a,a')$ if $b\in B$ and there is an $a'\in A$ such that $a'\sim b$, $\delta_A(a,b)$ if $a,b\in A$, is $\delta_B(a,b)$ if $a,b\in B$ and is $\bot$ if neither of the previous cases happens.

There is an important note made and developed in [AdAMM23a], is the fact that even if $Q-set_{fun}$ is a cartesian category this is not a closed one, for which they give multiples monoidal closed

[1]where $Y/\sim$ is the quotient in Y given by the equivalence relation $\sim\subseteq Y\times Y$ generated by the identification $f(x)\sim g(x)$

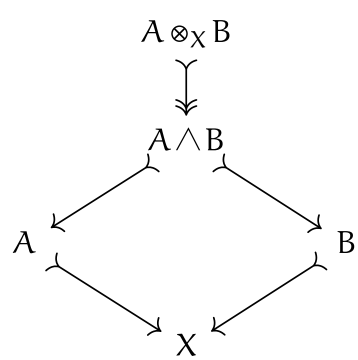
structures over this category, one for each suitable equivalence relation over the set of idempotent element of Q, for the quantale structure in the poset of subobjects of a given $Q-set$ we use the monoidal closed structure given by the equality relation, more precisely consider

$$A \otimes B = (\{(a,b) \in A \times B : E(a) = E(b)\}, \delta_{A\otimes B})\ \delta_{A\otimes B}((a,b),(a',b')) = \delta_A(a,a') \otimes \delta_B(b,b')$$

Now if $A \xrightarrow{f} X$, $B \xrightarrow{g} X$ are in $Sub(X)$ for calculated a monoidal product in $Sub(X)$ note that $A \otimes B$ has the same underling set as $A \times B$, given that $u \otimes v \leq u \wedge v$, the identity assignation $(a,b) \mapsto (a,b)$ is a bijective functional map (mono+epi), so following the projections and composing with f, g gives a couple of parallel arrows from $A \otimes B$ to X, the equalizer of this arrows works a the monoidal product operation of the quantale structure in $Sub(X)$, explicitly:

$$A \otimes_X B = (\{(a,b) \in A \times B \mid E(a) = E(b), f(a) = g(b)\}, \delta_{A \otimes_X B})$$
$$\delta_{A\otimes_X B}((a,b),(a',b')) = \delta_A(a,a') \otimes \delta_B(b,b')$$

Note that the underlying subset of $A \otimes_X B$ and $A \wedge B$ is the same and because of $\delta_A(a,a') \otimes \delta_B(b,b') \leq \delta_A(a,a') \wedge \delta_B(b,b')$ the identical function is a functional morphism which is a monomorphism and an epimorphism but isn't an isomorphism.



The square in the above diagram is a pullback, and like pulback preserves monos, all the arrows are monomorphism, then is clear that $A \otimes B \in Sub(X)$.

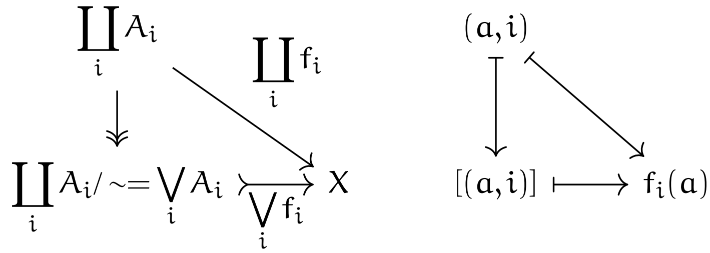
That this product is associative follows easily from the monoidal closed structure of $(Q{-}Set_{fun}, \otimes)$, but the module object for $\otimes_X$ is not the module object for $\otimes$. If we denote by $\mathcal{E}Q$ the subset of idempotents elements of Q is proven in [AdAMM23a] that for every (Y, δ_Y) there is a natural isomorphism $Y \otimes \mathcal{E}Q \cong Y$, the think is that $\mathcal{E}$ is not a subobject of every $Q-set$, then this unit must change with every product $\otimes_X$, for this we use the axiom of choice for pick a right inverse of the extent function $E_X : X \to \mathcal{E}Q$ $x \mapsto E_X(x) = \delta_X(x,x)$ this is not a necessarily surjective application so we restrict to $1_X = \{E(x) \in \mathcal{E}Q | x \in X\}$ with this codomain E_X is obviously a surjection so we ask for a right inverse $s : 1_X \to X$, $E \circ s = id_{1_X}$, from this s must be an injective map an then a subobject of X. We are taking the structure of $Q-set$ over 1_X as $\delta_{1_X}(e,e') = e \otimes e'$, so for a sub-object $A \xrightarrow{f} X$ the assignation

$$A \otimes_X 1_X \to A$$
$$(a, E(a)) \mapsto a$$

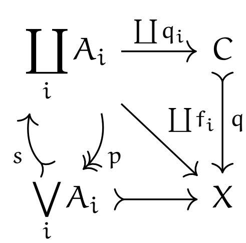
determines an isomorphism $A \otimes_X 1_X \cong A$, indeed like $s(E_X(x)) = x$, $\delta_{A \otimes_X 1_X}((a,E(a)),(b,E(b))) = \delta_A(a,b) \otimes \delta_{1_x}(E(a),E(b)) = \delta_A(a,b) \otimes E(a) \otimes E(b) = \delta_A(a,b)$, so the previous map preserves δ assignment.

This situation of given a definition dependingonf the choice of a section (a right inverse) of a certain map (typically a projection or an extent assignation), is present when we try for example to expose the universal property of a join that we define in the poset of the functional sub-objects of a given $Q-Set$.

The general picture will be: a $Q-Set$ (X, δ_X) and a collection of sub-objects of it indexed by a set $(A_i \xrightarrow{f_i} X)_{i \in I}$, the first intuitive candidate is the arrow given by the universal property of the co-product $\coprod_{i \in I} A_i \xrightarrow{\coprod f_i} X$, again the thing is that this is not, in general, a monomorphism even if each component is, the problem is that exist necessarily different $i, j \in I$ and $a_i \in A_i$, $a_j \in A_j$ such that $f_i(a_i) = f_j(a_j)$, so it necessarily identifies $a_i \sim a_j$ and actually with this is sufficient to for give the join of the sub-objects, this idea es the canonical epi-mono factorization which exists in the category of sets and in many concrete categories, but is, in general, a delicate question in sheaf-like categories over quantales (see [TdAMM22] for categories of functorial sheaves over semicartesian commutative quantales).



One role of this cited factorization is to expose the necessary (partial) inversion of the epi part of the factorization, i.e. the projection to the quotient. For this consider a sub-object of (X, δ_X) given by a monomorphism $C \xrightarrow{q} X$, if we assume that in $\mathrm{Sub}(X)$ this later sub-object is an upper bound of the family $(A_i \xrightarrow{f_i} X)_{i \in I}$, there is also exists (necessarily mono)morphism $A_i \xrightarrow{q_i} C$ such that $q \circ q_i = f_i$ for each $i \in I$. The joining property tells us that exists an arrow $\bigvee_{i \in I} A_i \xrightarrow{\varphi} C$ that let us $\bigvee_{i \in I} A_i \leq C$ in $\mathrm{Sub}(X)$ so followed by $C \xrightarrow{q} X$ factorize the injection $\bigvee_{i \in I} A_i \to X$, for find such an arrow is enough have a section to the quotient map to the join from the coproduct, indeed if s is a section (a right inverse) of the projection then $\varphi = \coprod q_i \circ s$ is a function that gives the desire factorization.



The quotient map $p : \coprod_i A_i \to \bigvee_i A_i$ possibly has many sections, each one deliver a factorization function $\varphi : \bigvee_i A_i \to C$ note that any of this determines a functional map between $Q-\mathrm{Set}$ i.e. such that $\delta_{\bigvee A_i}([(a,i)],[(b,j)]) \leq \delta_C(\varphi([(a,i)]), \varphi([(b,j)]))$, indeed first remember that for classes $c_1, c_2 \in \bigvee_{i \in I} A_i$, $\delta_{\bigvee A_i}(c_1, c_2)$ is $\delta_j(a,b)$ if exist $j \in I$ and $a, b \in A_j$ such that $[(a,j)] = c_1$ and $[(b,j)] = c_2$, and is $\bot$ in other case. So lets take classes c_1, c_2 in the previous situation where $\delta_{\bigvee A_i}(c_1, c_2) = \delta_j(a,b)$ if the section $s : \bigvee_i A_i \to \coprod_i A_i$ makes $s(c_1) = [(c,i)]$ and $s(c_2) = [(d,k)]$ then $[(a,j)] = [(c,i)]$ and $[(b,j)] = [(d,k)]$ which means $f_j(a) = f_i(c)$ and $f_j(b) = f_k(d)$ but like the q_i's factorize f_i through q, we have $q \circ q_j(a) = f_j(a) = f_i(c) = q \circ q_i(c)$ like q is injective we have $q_j(a) = q_i(c)$, also $q_j(b) = q_k(d)$ and like q_j is a functional map obtain

$\delta_{\bigvee\limits_i A_i}(c_1,c_2)=\delta_j(a,b)\leq\delta_C(q_j(a),q_j(b))=\delta_C(q_i(c),q_k(d))=\delta_C(\varphi(c_1),\varphi(c_2))$.

In resume $\bigvee\limits_i A_i \xrightarrow{\vee f_i} X$ is the join of the family $(A_i \xrightarrow{f_i} X)$ but the universal property depend of the existence of the section $s:\bigvee\limits_i A_i \to \coprod\limits_i A_i$ and is not a colimit in $Q-Set_{fun}$, this must affect the distributions over $\otimes_x$ which will be immediate if it is a colimit because the of monoidal closed structure in $(Q-set_{fun},\otimes,\mathcal{E}Q)$ however this property is easily obtained:

Take $(A_i \xrightarrow{X})_{i\in I}$ a family of monomorphism in $Q-set_{fun}$ and another sub-object of X, $B\xrightarrow{g} X$, then $\bigvee_{A\,i}\otimes_X B=(\{([(a,i)],b)\mid f_i(a)=g(b),E(a)=E(b)\},\delta)$ with $\delta(([(a_1,i)],b_1),([(a_2,j)],b_2))=\delta_{\vee A_i}([(a_1,i)],[(a_2,j)])\otimes\delta_B(b_1,b_2)$ and $\bigvee\limits_i(A_i\otimes_X B)=(\{[(a,b),i]\mid E(a)=E(b),f_i(a)=g(b)\},\delta')$ with $\delta'([(a_1,b_1),i],[(a_2,b_2),j])=\delta_{\vee(A_i\otimes_X B)}([(a_1,b_1),i],[(a_2,b_2),j])$, so the assignation $([(a,i)],b)\mapsto[(a,b),i]$ is a bijection and an isomorphism in $Q-Set_{fun}$, indeed, suppose in the previous presentation of δ and δ' that exist $k\in I$ and $c,d\in A_k$ such that $[(a_1,i)]=[(c,k)]$ and $[(a_2,j)]=[(d,k)]$ then $\delta_{\vee A_i}([(a_1,i)],[(a_2,j)])=\delta_k(c,d)$, under this hypothesis is true to that $[(a_1,b_1),i]=[(c,b_1),k]$ and $[(a_2,b_2),j]=[(d,b_2),k]$ because this only means that $f_i(a_1)=g(b_1)=f_k(c)$ and $f_j(a_2)=g(b_2)=f_k(d)$, because of this

$$\begin{aligned}\delta'([(a_1,b_1),i],[(a_2,b_2),j]) &= \delta_{\vee(A_i\otimes_X B)}([(a_1,b_1),i],[(a_2,b_2),j])\\ &= \delta_k(c,d)\otimes\delta_B(c,d)\\ &= \delta_{\vee A_i}([(a_1,i)],[(a_2,j)])\otimes\delta_B(c,d)\\ &= \delta(([(a_1,i)],b_1),([(a_2,j)],b_2))\end{aligned}$$

with justifies the mentioned isomorphism, then in the poset of subjects obtain $(\bigvee\limits_{i\in I}A_i)\otimes_X B=\bigvee\limits_{i\in I}(A_i\otimes_X B)$.

To this point we give a structure of quantale to $Sub(X)$ for each X in $Q-set_{fun}$, the usual pullback assignation which naturally works for morphism assignment of the doctrine of sub-objects must be a morphism of quantales, this is a morphism of complete lattices that preserves the monoidal product, is in general a delicate question (see [dCTCdAMJGAM2X] for this question over functorial sheaves over semicartesian quantales) and we restrict only to morphism like diagonal and projections for characterize equality's predicates and quantifiers.

So for the equality's predicate consider a diagonal map in $Q-set_{fun}$ which as in every cartesian category is given by the product of the identities, $(1_x, 1_x) = \Delta_X : X \xrightarrow{X} \times X$, pulling back through this arrow sub-object of $X \times X$ leads

$$\mathrm{Sub}(X \times X) \xrightarrow{\Delta_X^*} \mathrm{Sub}(X)$$
$$(Y \xrightarrow{g} X \times X) \mapsto (\Delta^*(Y) \xrightarrow{\pi_X} X)$$

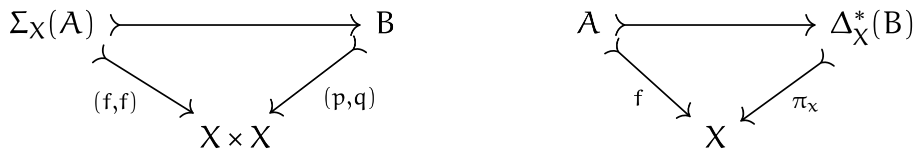
where $\Delta^*(Y) = (\{(y,x) : g(y) = (x,x)\}, \delta_{\Delta^*(Y)})$ with

$\delta_{\Delta^*(Y)}((y_1,x_1),(y_2,x_2)) = \delta_Y(y_1,y_2) \wedge \delta_X(x_1,x_2) \leq \delta_X(x_1,x_2)$, like g is injective the projection $(y,x) \mapsto x$ is a monomorphism, a routine count shows that this assignation preserves the monoidal product (the lattice structure is a folklore result). Let's give a left adjoint for this quantale map, consider

$$\Sigma_X : \mathrm{Sub}(x) \to \mathrm{Sub}(X \times X)$$
$$(A \xrightarrow{f} X) \mapsto (\Sigma_X(A) \xrightarrow{\Sigma(f)} X \times X)$$

where for a $Q-set$ (A, δ_A) and a monomorphism in $Q-set_{fun}$ $f : A \to X$ is assigned $\Sigma_X(A) = (\{(a,a) \mid a \in A\}, \delta_{\Sigma(A)})$ where $\delta_{\Sigma(A)}((a,a),(b,b)) = \delta_A(a,b) \wedge \delta_A(a,b) = \delta(a,b)$ and $\Sigma(f) = (f,f)$ i.e. $\Sigma(f)(a,a) = (f(a), f(a))$. Like f is injective, (f,f) is injective and is a functional morphism as the product of two (the same) functional morphisms. This determines a morphism of quantales, only proof that preserves the monoidal structure:

For what it follows, denote by x^2 a couple (x,x). Consider two subobjects of (X, δ_X), say $A \xrightarrow{f} X$ and $B \xrightarrow{g} X$, the elements in $\Sigma_X(A) \otimes_{X \times X} \Sigma_X(B)$ are of the form (a^2, b^2) with $E(a) = E(a^2) = E(b^2) = E(b)$ and the distance is given by $\delta((a_1^2, b_1^2),(a_2^2, b_2^2)) = \delta_{\Sigma(A)}(a_1^2, a_2^2) \otimes \delta_{\Sigma(B)}(b_1^2, b_2^2) = \delta_A(a_1,a_2) \otimes \delta_B(b_1,b_2)$. In the other hand the elements of $\Sigma_X(A \otimes_X B)$ are squares of couples $(a,b)^2$ and the structure is given by $\delta'((a_1,b_1)^2,(a_2,b_2)^2) = \delta_{A \otimes_X B}((a_1,b_1),(a_2,b_2)) = \delta_A(a_1,a_2) \otimes \delta_B(b_1,b_2)$, then the assignation $(a^2, b^2) \mapsto (a,b)^2$ determines an isomorphism that represent the equality of sub-objects $\Sigma(A) \otimes_{X \times X} \Sigma(B) = \Sigma(A \otimes_X B)$. With this morphism clear let express the adjunction $\Sigma_X(-) \dashv \Delta_X^*(-)$, that is: For every $(B \xrightarrow{(p,q)} X \times X) \in \mathrm{Sub}(X \times X)$ and every $(A \xrightarrow{f} X) \in \mathrm{Sub}(X)$ there is a natural bijective correspondence between the following commutative triangles of monomorphism in $Q-\mathrm{set}_{\mathrm{fun}}$



We give assignation for morphisms in $Q-\mathrm{set}_{\mathrm{fun}}/X \times X$ to morphism in $Q-\mathrm{set}_{\mathrm{fun}}/X$, so lets first go left to right, this is consider an arrow $\Sigma_X(A) \xrightarrow{h} B$ such that $(p,q) \circ h = (f,f)$ this means that $p \circ h(a^2) = f(a) = q \circ h(a^2)$ for every $a \in A$, this means precisely that $(f(a), h(a^2)) \in \Delta_X(B)$ so consider $A \xrightarrow{\hat{h}} \delta_X^*(B)$ given by $\hat{h}(a) = (f(a), h(a^2))$, then $\pi_x(\hat{h}(a)) = f(a)$ given the commutative of the right triangle. That h is an arrow in $Q-\mathrm{set}_{\mathrm{fun}}$ is given by $\delta_{\Sigma_X(A)}(a_1^2, a_2^2) = \delta_A(a_1, a_2) \leq \delta_B(h(a_1^2), h(a_2^2))$, for seeing that $\hat{h}$ is functional map to, lets first make a note about the structure in $\Delta_X^\star(B)$, for elements couples $(x_1, b_1), (x_2, b_2)$ being in $\Delta_X^*(B)$ means that $p(b_1) = x = q(b_1)$ and $p(b_2) = x_2 = q(b_2)$, so like $B \xrightarrow{(p,q)} X \times X$ is a functional map one obtains

$$
\begin{aligned}
\delta_B(b_1, b_2) \leq \delta_{X \times X}((p,q)(b_1), (p,q)(b_2)) &= \delta_X(p(b_1), p(b_2)) \wedge \delta_X(q(b_1), q(b_2)) \\
&\leq \delta_X(p(b_1), p(b_2))
\end{aligned}
$$

so $\delta_{\Delta_X^*(B)}((x_1, b_1), (x_2, b_2)) = \delta_X(x_1, x_1) \wedge \delta_B(b_1, b_2) = \delta_X(p(b_1), p(b_2)) \wedge \delta_B(b_1, b_2) = \delta_B(b_1, b_2)$. With this in mind the hypothesis of functional map for h is exactly

$$
\begin{aligned}
\delta_A(a_1, a_2) &= \delta_{\Sigma_X(A)}(a_1^2, a_2^2) \\
&\leq \delta_B(h(a_1^2), h(a_2^2)) \\
&= \delta_{\Delta_X^*(B)}((f(a_1), h(a_1^2)), (f(a_2), h(a_2^2))) \\
&= \delta_{\Delta_X^*(B)}(\hat{h}(a_1), \hat{h}(a_2))
\end{aligned}
$$

In the other direction a morphism $A \xrightarrow{t} \Delta_X^*(B)$ such that $\pi_x \circ t = f$ is given by an assignation $a \mapsto (f(a), b_a)$ with $p(b_a) = f(a) = q(f_a)$, so this induce a map $\check{t} : \Sigma_X(A) \to B$, $a^2 \mapsto b_a = \pi_B(t(a))$, that t is a morphism implies (an actually is) that

$$\begin{aligned}\delta_{\Sigma_X(A)}(a_1^2, a_2^2) &= \delta_A(a_1, a_2)\\ &\leq \delta_{\Delta_X^*(B)}(t(a_1), t(a_2))\\ &= \delta_{\Delta_X^*(B)}((f(a_1), b_{a1}), (f(a_2), b_{a2}))\\ &= \delta_B(b_{a1}, b_{a2})\\ &= \delta_B(\check{t}(a_1^2), \check{t}(a_2^2))\end{aligned}$$

and $(p,q)\circ\check{t}(a^2) = (p(\check{t}(a^2)), q(\check{t}(a^2))) = (p(b_a), q(a)) = (f(a), f(a)) = (f,f)(a)$ so $\check{t} : \Sigma_X(A) \to B$ is a morphism in $Q-set_{fun}/X\times X$ from $\Sigma_X(A) \xrightarrow{(f,f)} X\times X$ to $B \xrightarrow{(p,q)} X\times X$. Finally note that $\hat{\check{h}}(a^2) = h(a^2)$ and $\check{\hat{t}}(a) = t(a)$, so this is a bijective correspondence.

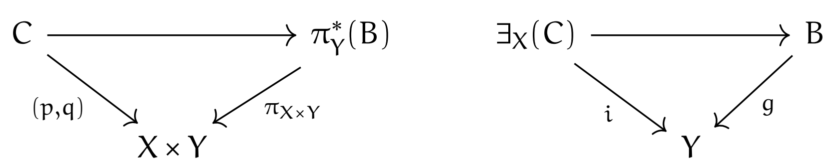
Now let's give attention to the quantifiers of this monoidal hyperdoctrine of sub-objects. For this, first remark how is pullback through projections monomorphism in $Q-set$, so for (X,δ_X) and (Y,δ_Y) take the projection to Y $X\times Y \xrightarrow{\pi_Y} Y$, pulling back a sub-object $(B \xrightarrow{g} Y)$ of (Y,δ_Y) through π_Y let $\Pi_Y^*(B) = (\{((x, g(b)), b) : x \in X, b \in B\}), \delta_{\Pi_Y^*(B)}$

$$\Pi_Y^*(B) \xrightarrow{\Pi_Y^*(g)=\pi_{X\times Y}} X\times Y$$
$$((x, g(b)), b) \mapsto (x, g(b))$$

with $\delta_{\Pi_Y^*(B)}(((x_1, g(b_1)), b_1), ((x_2, g(b_2)), b_2)) := \delta_X(x_1,x_2) \wedge \delta_B(b_1,b_2)$. The existential quantifier is given by a adjoint of $(\exists_X)_Y \dashv \pi_Y^*$ so defining

$$(\exists_X)_Y : Sub(X\times Y) \to Sub(Y)$$
$$(C \xrightarrow{(p,q)} X\times Y) \mapsto ((\exists_X)_Y(C) \xrightarrow{\exists(p,q)} Y)$$

given by $(\exists_X)_Y(C) = (\{q(c) \in Y \mid c \in C\}, \delta_{\exists_X})$ whit $\delta_{\exists_X}(q(c), q(c')) := \delta_C(c,c')$ and $\exists(p,q)(q(c)) = q(c)$, so $\exists(p,q) = i$ the inclusion of $(\exists_X)_Y(C)$ in Y, the adjunction then means that there is natural bijective correspondence between commutative triangles of sub-objects

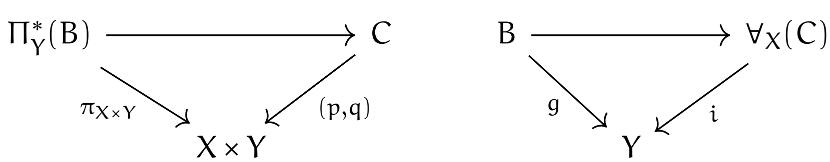


This is simple because give a functional map $C \xrightarrow{\alpha} \pi_Y^*(B)$ such that $\pi_{X\times Y} \circ \alpha = (p,q)$ is to choose for each $c \in C$ an element $b_c \in B$ such that $q(c) = g(b_c)$, indeed with this condition gives a well defined $\alpha(c) = ((p(c), q(c)), b_c) \in \Pi_Y^*(B)$ and such an election determines the assignation $q(c) \mapsto b_c = \pi_B(\alpha(c))$ which becomes a functional map $\exists_X(C) \xrightarrow{\hat{\alpha}} B$, like α is a functional map

$$\begin{aligned}
\delta_{\exists_X(C)}(q(c_1), q(c_2)) &= \delta_C(c_1, c_2) \\
&\leq \delta_{\pi_Y^*(B)}(\alpha(c_1), \alpha(c_2)) \\
&= \delta_{\pi_Y^*(B)}(((p(c_1), q(c_1)), b_{c_1}), ((p(c_2), q(c_2)), b_{c_2})) \\
&= \delta_X(p(c_1), p(c_2)) \otimes \delta_B(b_{c_1}, b_{c_2}) \\
&\leq \delta_B(b_{c_1}, b_{c_2})
\end{aligned}$$

Like $g(b_c) = q(c) = i(q(c))$ the right triangle commute. In the other direction a functional map $\exists_X(C) \xrightarrow{\beta} B$, is again chose for every $c \in C$, an element $b_c \in B$, such that $q(c) = g(b_c)$, for such an election $\beta(q(c)) = b_c$ is a well defined function with the required commutativity condition, the associated map $\check{\beta} : C \to \Pi_Y^*(B)$, $c \mapsto ((p(c), q(c)), \beta(q(c))) = ((p(c), q(c)), b_c)$, we already mentioned why this is well defined and makes commute the left triangle, the condition of β to being a functional map says that $\delta_{\exists_X(C)}(q(c_1), q(c_2)) = \delta_C(c_1, c_2) \leq \delta_B(b_{c_1}, b_{c_2})$ and like $C \xrightarrow{(p,q)} X \times Y$ is a morphism, $\delta_C(c_1, c_2) \leq \delta_X(p(c_1), p(c_2)) \wedge \delta_Y(q(c_1), q(c_2))$, so $\delta_C(c_1, c_2) \leq \delta_X(p(c_1), p(c_2)) \wedge \delta_B(b_{c_1}, b_{c_2}) = \delta_{\Pi_Y^*(B)}(\check{\beta}(c_1), \check{\beta}(c_2))$, then $\check{\beta}$ is a morphism. Note the assignation $\hat{-}$ and $\check{-}$ are inverses, because $\check{\hat{\alpha}}(c) = \alpha(c)$ and $\hat{\check{\beta}}(q(c)) = \beta(q(c))$, so this a bijective correspondence.

Finally for universal quantifiers, consider a monomorphism in $Q-set_{fun}$,$C \xrightarrow{(p,q)} X \times Y$ and consider $\forall_X(C) = (\{y \in Y \mid \text{for all } x \in X, \text{ exist } c \in C \text{ such that } x = p(c), y = q(c)\}, \delta_{\forall_X(C)})$ with $\delta_{\forall_X(C)})(y_1, y_2) = \delta_Y(y_1, y_2)$ and $\forall_X(C) \xrightarrow{\forall_X((p,q))=i} Y$, $y \mapsto y$. Then there is a correspondence between commutative triangles in $Q-Set_{fun}$



For the sub-object $B \xrightarrow{g} Y$ a functional map $\Pi_Y(B) \xrightarrow{\alpha} C$ is given by an assignation for each $x \in X$, $b \in B$ $((x, g(b)), b) \mapsto c_{xb}$ such that $x = p(c_{xb})$ and $g(b) = q(c_{xb})$ then the correspondence $b \mapsto g(b)$ is a well defined functional map $B \xrightarrow{\hat{\alpha}} \forall_X(C)$ because for every $x \in X$, c_{xb} is makes $p(c_{xb}) = x$ and $q(c_{xb}) = g(b)$, this correspondence is actually the only one who makes the right triangle commute. For the other correspondence the necessary assignation $b \mapsto g(b)$ says that for every $x \in X$ exist c_{xb} with $p(c_{xb}) = x$ and $q(c_{xb}) = g(b)$, then $((x, g(b)), b) \mapsto c_{xb}$ is a well defined function that makes the left triangle commute. Because of the dependence of the same condition this process are inverse each other.

All of this lets us understand that the structure of the doctrines of sub-objects over the category $Q - set_{fun}$ produces a very similar situation that the one obtained when we make the hyper-doctrine of sub-objects over the category of sets and functions Set, the main difference is that in general the sub-objects are given by subsets in which the structure of $Q - set$ seem more "tight", but the role of equality and quantifiers is exactly the interpretation in sub-objects of a set, then the axioms of the definition 3.2.3 lets the category of $Q - Set$ with partial equivalence relations and well defined functions between them.

3.3 Continuous logic

The intention is to define in an external way (in terms of structured sets and structure-preserving functions between them) a logic that use $Rel(Q)$-categories as a semantic and extent the quantale valued logic presented in [DP21] which in turn extent the continuous logic for metric spaces proposed in [BYBHU08]. Then the general structure of the following definitions comes inherited by the monograph [BYBHU08] of model theory for metric space, however some restrictions inherent of the fact that this is a more general framework, for example, our quantales are not necessarily continuous lattices, then there is not a clear version of continuous or uniform continuous function, the clear translation form the theory of metric spaces is that of 1-Lipschitz functions a stronger version of uniform continuity.

Even if the main intention is to use $\mathsf{Rel}(Q)$—categories as semantics is more general and clear work in the context of enriched categories over quantaloids (and not only over quantaloids of the form $\mathsf{Rel}(Q)$), because of this we will focus in the constructions and definitions over quantaloids, and in the later think in the very specific nature of this definitions over quantaloids of the form $\mathsf{Rel}(Q)$.

First of all, for the following definitions makes sense is important to make two clarities, one is related to the existence of products in the category $Q-\mathsf{Cat}_{fun}$ because the interpretation of formulas and terms use lists of variables which will be interpreted as an assignation over a product, the other issue is related with the understanding of the version of valued morphism in this context, because in continuous logic the interpretation of formulas comes through valued assignation over the unit interval $[0,1]$, in quantale valued logic this valuations are taken over the quantale Q that parameterize the logic, so in our context we must present a suitable notion of valued morphism.

For the discussion that follows take a fixed quantaloid $\mathcal{Q}$, consider (X, d_X, e_X) and (Y, d_Y, e_Y) two Q—categories, define the following structure

$$X \times Y = (X \hat{\times} Y, d_{X\times Y}, e_{X\times Y})$$

where $X \hat{\times} Y := \{(x,y) \in X \times Y \mid e_X(x) = e_Y(y)\}$, $d_{X\times Y} : X \hat{\times} Y \times X \hat{\times} Y \to \mathsf{Morp}(\mathcal{Q})$, is defined by $d_{X\times Y}((x_1,y_1),(x_2,y_2)) := d_X(x_1,x_2) \wedge d_Y(y_1,y_2)$ and $e_{X\times Y} : X \hat{\times} Y \to \mathsf{Obj}(\mathcal{Q})$ is defined by $e_{X\times Y}(x,y) := e_X(x)$ $(= e_Y(y))$. We claim that this define a Q—category and there is canonical projections to X and Y with enables to $X \times Y$ the universal property of products in the category $Q-\mathsf{Cat}_{fun}$. First lets check that the definitions we give actually determines a Q—category, for this note that like $(x,y) \in X \hat{\times} Y$ only if $e_X(x) = e_Y(y)$ and like (X, d_X, e_X) and (Y, d_Y, e_Y) already determines Q—categories then for $(x_1,y_1),(x_2,y_2) \in X \hat{\times} Y$ we have parallel arrows in $\mathcal{Q}$, $d_X(x_1,x_2), d_X(y_1,y_2) : e(x_1) \to e(x_2)$ so the meet $d_X(x_1,x_2) \wedge d_Y(y_1,y_2)$ is calculated in the lattice $Q(e_X(x_1), e_X(x_2))$ then is an arrow $d_{X\times Y}((x_1,y_1),(x_2,y_2)) := d_X(x_1,x_2) \wedge d_Y(y_1,y_2) :$

$e_{X\times Y}(x_1,y_1)\to e_{X\times Y}(x_2,y_2)$, the composition law (or transitivity axiom) is given because for every $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\hat{\times}Y$

$$\begin{aligned} & d_{X\times Y}((x_1,y_1),(x_2,y_2))\circ d_{X\times Y}((x_2,y_2),(x_3,y_3)) \\ = & (d_X(x_1,x_2)\wedge d_Y(y_1,y_2))\circ(d_X(x_2,x_3)\wedge d_Y(y_2,y_3)) \\ \leq & d_X(x_1,x_2)\circ(d_X(x_2,x_3) \\ \leq & d_X(x_1,x_3) \end{aligned}$$

and also

$$\begin{aligned} \leq & d_Y(y_1,y_2)\circ d_Y(y_2,y_3) \\ \leq & d_Y(y_1,y_3) \end{aligned}$$

then

$$\begin{aligned} & d_{X\times Y}((x_1,y_1),(x_2,y_2))\circ d_{X\times Y}((x_2,y_2),(x_3,y_3)) \\ \leq & d_X(x_1,x_3)\wedge d_Y(y_1,y_3) \\ = & d_{X\times Y}((x_1,y_1),(x_3,y_3)) \end{aligned}$$

Finally like $e_{X\times Y}(x,y)=e_X(x)=e_Y(y)$, the identity $1_{e_{X\times Y}(x,y)}=1_{e_X(x)}=1_{e_Y(y)}\leq d_X(x,x),d_Y(y,y)$ and then $1_{e_{X\times Y}(x,y)}\leq d(x,x)\wedge d_Y(y,y)=d_{X\times Y}((x,y),(x,y))$. With this its show that our definition actually gives a Q—category.

Rest to see the projections arrows which are based in the usual set-theoretical projections, indeed consider $\pi_X: X\hat{\times}Y\to X$, $(x,y)\mapsto x$, note that like $e_{X\times Y}(x,y)=e_X(x)=e_X(\pi_X(x,y))$ and $d_{X\times Y}((x_1,y_1),(x_2,y_2))=d_X(x_1,x_2)\wedge d_Y(y_1,y_2)\leq d_X(x_1,x_2)=d_X(\pi_X(x_1,y_1),\pi_x(x_2,y_2))$ then the map π_x determines a functional map $\pi_x: X\times Y\to X$ between Q—categories as well as $\pi_Y: X\times Y\to Y$. The universal property comes directly from the definition of this projections, for this if $Z\xrightarrow{f}X$, $Z\xrightarrow{g}Y$ are maps that determines arrows from (Z,d_Z,e_Z) to (X,d_X,e_X) and (Y,d_Y,e_Y), if $\varphi: Z\to X\hat{\times}Y$ is a map such that $\pi_X\circ\varphi=f$ and $\pi_X\circ\varphi=g$ then necessarily $\varphi=(f,g)$, which makes $z \mapsto (f(z), g(z))$ for every $z \in Z$, note that this application is well defined because like f and g are arrows in $Q-\mathcal{C}at_{fun}$, for every $z \in Z$ we have $e_Z(z) = e_X(f(z))$ and $e_Z(z) = e_Y(g(z))$ so $e_X(f(z)) = e_Y(g(z))$ then $(f(z), g(z)) \in X \hat{\times} Y$, also $d_Z(z_1, z_2) \leq d_X(f(z_1), f(z_2)), d_Y(g(z_1), g(z_2))$, which is equivalent to $d_Z(z_1, z_2) \leq d_X(f(z_1), f(z_2)) \wedge d_Y(g(z_1), g(z_2)) = d_{X\times Y}((f(z_1), g(z_1)), (f(z_2), g(z_2)))$, then $Z \xrightarrow{(f,g)} X \times Y$ is an arrow in $Q-\mathcal{C}at_{fun}$.

With this, it is clear the product structure in $Q-\mathcal{C}at_{fun}$, then we continue to interpret languages over Q—categories.

3.3.1 Interpret continuous languages in enriched categories over quantaloids

With the previous clarification on how it works the products in $Q-\mathcal{C}at_{fun}$, let's see how to define what will be the interpretations of the formulas of our logic. In the case of continuous logic for metric spaces the interpretation of formulas are continuously uniform functions $(M, d_M) \to [0, 1]$ taken the usual structure of metric space of the unit interval, the generalization for quantale valued logic (see [DP21]) are uniform continuous functions $(X, d_X) \to Q$, in that context the quantales are in additions continuous lattices and has filter of positive elements then the view that enables a clear generalization comes from the fact that the so-called continuous spaces are enriched categories over a quantale then a map $(X, d_X) \to Q$ is just and enriched presheaf, so the notion that naturally generalize the context of [DP21] are precisely presheaves over $\mathcal{Q}$ enriched categories.

Let's make a couple of important notes. For the following notes, we use the notation and remarks of section 1.4.

Note 3.3.1. *If $\mathcal{Q}$ is a quantaloid, for any u object of $\mathcal{Q}$, the following defines a $\mathcal{Q}$ enriched category structure over the collection of $\mathcal{Q}$-arrows over u.*

$$Q|_u := \{v \xrightarrow{f} u \in Morph(\mathcal{Q})\}$$

$$e_{Q|_u} : Q|_u \to Obj(\mathcal{Q}), v \xrightarrow{f} u \mapsto v$$

$$d_{Q|_u} : Q|_u \times Q|_u \to Morph(\mathcal{Q}), (f,g) \mapsto [f,g]$$

Where $[f,g]$ *is the lifting of* g *through* f*, recall that this is defined by the adjunction* $f \circ - \dashv [f,-]$.

$$\begin{array}{ccc} & \overset{[f,g]}{\nearrow} & v \\ & & \downarrow f \\ w & \xrightarrow[g]{} & u \end{array}$$

This is just because of the very definition $d_{Q|_u}(f,g) : w \to v$ and $w = e_{Q|_u}(g)$, $v = e_{Q|_u}(f)$ then $d_{Q|_u}(f,g) : e_{Q|_u}(g) \to e_{Q|_u}(f)$ and the co-unity of the adjunction tell us that $x \circ [x,y] \leq y$, then

$$\begin{aligned} f \circ ([f,q] \circ [q,r]) &= (f \circ [f,q]) \circ [q,r] \\ &\leq q \circ [q,r] \\ &\leq r \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_{Q|_u}(f,q) \circ d_{Q|_u}(q,r) &= [f,q] \circ [q,r] \\ &\leq [f,r] \\ &= d_{Q|_u}(f,r) \end{aligned}$$

additionally $1_{e_{Q|_u}(f)} = 1_v] = 1_{dom(f)} = [f,f] = d_{Q|_u}(f,f)$ in particular $1_v \leq d_{Q|_u}(f,f)$. We note as $Q|_u$ this enriched category.

The above structure codifies the notion of valued morphism as a functional map, lets make a note a bout this.

Proposition 3.3.2. *Enriched functors* $X \to Q|_u$ *and pre-sheaves of type* u *over* X, $\hat{u} \multimap X$ *are the same thing*

Proof. In fact, give an enriched functor $\varphi : X \to Q|_u$ is give an assignation $x \mapsto \cdot_x \xrightarrow{\phi(x)} u$, such that $e_X(x) = e_{Q|_u}(\phi(x)) = \cdot_x$, then is an $\mathcal{Q}$—arrow of the form $e_X(x) \xrightarrow{\phi(x)} u$ such that meet the functional axiom $d_X(x,y) \leq d_{Q|_u}(\phi(x),\phi(y)) = [\phi(x),\phi(y)]$ which is equivalent to $\phi(x) \circ$

$d_X(x,y) \leq \phi(y)$ which means that this assignation determines a distributor $\hat{u} \overset{\Theta}{\longrightarrow} X$, in chapter 2 this is the very definition 2.2.25 of what is call a pre-sheaf of type u over X. □

For a fixed a quantaloid Q, consider a set of formal symbols which we call logical symbols associated to it **LS** $:= \{C_Q, d, e, \bigvee, \bigwedge, \mathcal{X}\}$, d and e are symbols to refer the enriched category structure and play the role of equality predicate and an extent (or type) predicate respectively, $\bigvee, \bigwedge$ are symbols that we use as quantifiers, they are the version of the existential and the universal respectively, $\mathcal{X} = \{X_t \mid t \in \tau\}$ is set of sorts (or types) each one with a countable set of variables of that type $\{x_{t,n} : X_t\}_{n \in \mathbb{N}}$ and C_Q makes reference to the connectives.

In continuous logic as well as in (co)quantale valued logic a connective is given by a uniformly continuous function of a finite (Cartesian) power of the unit interval (in general of the valued quantale) on itself, as $Q^n \xrightarrow{c} Q$. Although it seems like an excessive amount of connectives, in continuous logic it happens that module uniform convergence is only necessary to add a connective for residuation and a connective for the process of "dividing by 2", in valued quantale logic it is an open question. In this context, we will define a theory that naturally extends the valued quantale case and we will propose as future work prospects that include a broader class of morphisms that approximate a notion of a uniformly continuous map. By C_Q, is noted the collection of connectives.

$$\begin{aligned} C_Q &:= \{Q|_{u_1} \times \cdots \times Q|_{u_n} \xrightarrow{c} Q|_v \ : u_1, \ldots, u_n, v \in Obj(\mathcal{Q}), c \text{ is a functional map}\} \\ &= \{c : \hat{v} \overset{\Theta}{\longrightarrow} Q|_{u_1} \times \cdots \times Q|_{u_n} \ : u_1, \ldots, u_n, v \in Obj(\mathcal{Q}), c \text{ is a pre-sheaf of type } v\} \end{aligned}$$

This corresponds to the fact that as we mentioned, the presheaves are our notion of valued morphism.

A **signature** $\Theta = (\mathcal{F}, \mathcal{R}, \mathcal{K})$ is given by a specifying a set of symbols of *function, relation* and *constant.*

$$\mathcal{F} = \{f_i(X_1 \cdots X_n) : Y\}_{i \in I}$$

$$\mathcal{R} = \{R_j : X_1 \cdots X_n\}_{j \in J}$$

$$\mathcal{K} = \{k_l : X\}_{l \in L}$$

Where the notation $f(X_1 \cdots X_n) : Y$ means that f is functional symbol, $X_1, \ldots, X_n, Y$ are sorts (not necessarily different) this notation also contains the information of the arity of f, (n). Also write $R : X_1 \cdots X_n$ to denote that R is a relation symbol of type $X_1 \cdots X_n$, and $k : X$ to denote that k is constant symbol in the sort X.

The **Terms** of the language is an inductive set of formal symbols, for which the basic elements are symbols of *constants* and *variables*, and the functions that generate the structure are given by concatenation with function symbols, this means that if $t_1 : X_1, \ldots, t_n : X_n$ are terms of the specified sorts and f is a function symbol such that $f(X_1, \ldots, X_n) : Y$ then the string $ft_1 \cdots t_n$ is a term. All the terms arise in this way or are basic terms.

The **Formulas** of the language again is a inductive set of formal symbols for which the basic elements are of the form dt_1t_2 and $Rt_1 \cdot\cdot t_n$ where $t_1, \ldots, t_n$ are terms, d is the logical symbols mentioned early and R is any relation symbol, the rest of the terms are given by concatenation of connectives and quantifiers, i.e. are of the form $c\phi_1 \cdots \phi_n$ and $(\bigvee)\bigwedge_{x_i} \phi(\overline{x})$ where c is a connective and $\phi_1, \ldots, \phi_n$ are formulas.

Consider Θ a continuous signature then an **interpretation** of Θ is given by an assignation

- For each type or sort X in Θ an enriched category over $\mathcal{Q}$, $[X] = (|X|, d_X, e_X)$, in such a way that:

- For a finite succession of sorts of the form $X_1 \cdots X_n$ the product $[X_1 \cdots X_n] = [X_1] \times \cdots \times [X_n]$ in $Q - Cat_{fun}$.

- The interpretation of a variable $x_i : X_1 \cdots X_n$ is given by the canonical projection to the i—th component.

- For each constant symbol $k : X$ an enriched functor from a one-object enriched category to X, $\hat{u} \xrightarrow{[k]} [X]$

- For each symbol of function $f : X_1 \cdots X_n \xrightarrow{f} X$ an arrow $[X_1] \times \cdots \times [X_n] \xrightarrow{f} [X]$ in $Q-\mathcal{C}at_{fun}$.

- For each relation symbol $R : X$ an enriched pre-sheaf of some type over $[X]$, $[R] : \hat{u} \multimap [X]$.

Associated with an interpretation there is a canonical extension to interpret the terms of the language and later to interpret the formulas of the language.

To **interpret the terms of the language** we make a recursive definition based in the inductive structure that this set of symbol has, for a symbol of constant $c : X$, we already have an interpretation, then like the terms are constructed by function symbols followed of previous terms and the basic elements are constants, only rest to say that this definition is given by composition, i.e. if $X_1 \cdots X_n \xrightarrow{f} Y$ and $\{X \xrightarrow{t_i} X_i\}_{1 \leq i \leq n}$ are terms with interpretations $\{[X] \xrightarrow{[t_i]} [X_i]\}_{1 \leq n}$, then the definition of $ft_1 \cdots t_n$ is given by: $[ft_1 \cdots t_n] := [f] \circ ([t_1], ..., [t_n])$.

The following remark is essential for the interpretation of formulas

For **interpret the formulas** we make to an recursive definition, for this consider $\{Y \xrightarrow{t_i} X_i\}_{1 \leq i \leq n}$ a finite set of terms, that have an interpretation $\{[Y] \xrightarrow{[t_i]} [X_i]\}_{1 \leq i \leq n}$, then:

- For interpret formulas of the type dt_1t_2 where t_1 and t_2 are terms is necessarily and sufficient that exist a $\Omega-$object u and a $\Omega-$arrow $\alpha_y : u \rightarrow e_{[Y]}(y)$ such that the compositions $d([t_1](y), [t_2](y)) \circ \alpha_y : u \rightarrow e_{[Y]}(y)$ for every $y \in [Y]$ determines a distributor that becomes the interpretation of $[dt_1t_2] : \hat{u} \multimap [Y]$

- if $[R] : \hat{u} \multimap [X_1] \times \cdots \times [X_n]$ is the interpretation of a relation symbol then the collection ($\Omega-$arrows $u \xrightarrow{[R]([t_1](y), ..., [t_n(y)])} e_{[Y]}(y))_{y \in [Y]}$ determines the enriched presheaf $[Rt_1 \cdots t_n] : \hat{u} \multimap Y$

- If $\{[\phi_i] : \hat{u_i} \multimap [X_i]\}_{1 \leq i \leq n}$ is the family of interpretations of a finite set of formulas $\{\phi_i\}_{1 \leq i \leq n}$ and $Q|_{u_1} \times \cdots \times Q|_{u_n} \xrightarrow{c} Q|_u$ is a connective then the arrow $[\phi_1] \times \cdots \times [\phi_n] := ([\phi_1] \circ \pi_1, ..., [\phi_n] \circ \pi_n)$ follow by c, as

$$[X_1] \times \cdots \times [X_n] \xrightarrow{[\phi_1] \times \cdots \times [\phi_n]} Q|_{u_1} \times \cdots \times Q|_{u_n} \xrightarrow{c} Q|_u$$

determines the distributor $[c\phi_1 \cdots \phi_n] : \hat{u} \multimap [X_1] \times \cdots \times [X_n]$

- If $\phi : X_1 \cdots X_n$ is a formula which is interpreted as $[\phi] : \hat{u} \multimap [X_1] \times \cdots \times [X_n]$, then for $i \in \{1, ..., n\}$ the interpretation of $\bigvee_{x_i} \phi$ is the presheaf of type u, determined by each $\overline{x} \in X_1 \times \cdots \times X_n$ (with out the i-th place) as the $\mathcal{Q}$—arrow given by $[\bigvee_{x_i} \phi](\overline{x}) = \bigvee\{[\phi](\overline{x} \cup x) : x \in [X_i]\}$.

- If $\phi : X_1 \cdots X_n$ is a formula which is interpreted as $[\phi] : \hat{u} \multimap [X_1] \times \cdots \times [X_n]$, then for $i \in \{1, ..., n\}$ the interpretation of $\bigwedge_{x_i} \phi$ is the presheaf of type u, determined by each $\overline{x} \in X_1 \times \cdots \times X_n$ (with out the i-th place) as the $\mathcal{Q}$—arrow given by $[\bigwedge_{x_i} \phi](\overline{x}) = \bigwedge\{[\phi](\overline{x} \cup x) : x \in [X_i]\}$.

The technical condition of the first literal, called for the existence of a $\mathcal{Q}$—object, u and a $\mathcal{Q}$—arrow $\alpha_y : u \to e_{[Y]}(y)$ such that for every couple of terms t_1, t_2 the family of compositions $(d_[Y]([t_1](y), [t_2](y)) \circ \alpha_y)_{y \in [Y]}$ determine a presheaf of type u over Y, is because for a homogeneous definition for the interpretation of formulas as enriched presheaves, it is required that the morphisms $d_{[Y]}([t_1](y), [t_2](y)) : e([t_1](y)) = e(y) \to e(y) = e([t_2](y))$ have a common domain.

Although this definition is given for any quantaloid (that meets the aforementioned technical condition) our main intention is based on quantaloids of the form $\mathrm{Rel}(Q)$ for a quantale Q. Precisely in this case the canonical morphism $\alpha_y : u \to e(y)$ is given by take $u = 1$ the top element of Q, and $\alpha_y = e(y) \otimes e(y) : 1 \to e(y)$ that in this kind of quantaloids corresponds to the identity of the object $e(y)$, then $1 \xrightarrow{d_Y([t_1](y),[t_2](y)) \circ (e(y) \otimes e(y))} e(y)$ is $1 \xrightarrow{d_Y([t_1](y),[t_2](y))} e(y)$ which determines a distributor $1 \multimap [Y]$.

With this, we already give an externally defined semantic which takes use $\mathrm{Rel}(Q)$—categories as a semantic, and that extends the aim of the quantale valued logic, presented in [DP21].

Finally, we give examples for some quantales Q, of what the semantics that we have just defined applied to the quantaloids $\mathrm{Rel}(Q)$ mean. We are going to consider that the categories are Cauchy complete and symmetric since naturally we can restrict our semantics so that it only admits interpretations in this class, thus making the connection with the theory of sheaves and metric spaces more present.

Examples 3.3.3. *If* Ω *is a complete Heyting algebra (a locale), the remark 2.3.7 tells us that* $\mathsf{Rel}(\Omega)$ *is Walter´s construction of section 1.3.5 and then*

$$\mathsf{Rel}(\Omega)-\mathsf{Cat}_{cc,sym,fun} \cong \mathsf{Sh}(\Omega)$$

So the associated logic admits sheaves over Ω *as structures, natural transformations to interpret functions and* $\Omega-$*valued assignations to interpret formulas. Future work lead to a deeper understanding and applications of this interplay.*

Examples 3.3.4. *Consider the quantale* $2=\{\top,\bot\}$*, in [Stu05b], pg 17 it is briefly shown that*

$$2-\mathsf{TRSCat}_{cc,fun} \cong 2-\mathsf{Cat}_{cc,fun}$$

then as $\mathsf{Rel}(2)-\mathsf{Cat}_{cc,fun} \cong 2-\mathsf{TRSCat}_{cc,fun}$*, we obtain that* $\mathsf{Rel}(2)-\mathsf{Cat}_{cc,fun} \cong 2-\mathsf{Cat}_{cc,fun}$*, then we define a logic to talk about Dedekind complete pre-orders (see 1.1.16) and isotone maps.*

Similarly what it happens to the 2 quantale, for the Lawvere quantale $([0,\infty],\geq,+,0)$ the Cauchy complete $[0,\infty]$ totally regular semicategories (and then the $\mathsf{Rel}([0,\infty])-$categories) correspond to $[0,\infty]-$ categories, then to generalized metric spaces (Lawvere spaces).

Examples 3.3.5. *The following equivalences happen to hold*

$$\mathsf{Rel}([0,\infty])-\mathsf{Cat}_{cc,sym,fun} \cong [0,\infty]-\mathsf{TRSCat}_{cc,sym,fun} \cong [0,\infty]-\mathsf{Cat}_{cc,sym,fun}$$

Then the defined logic finds in generalized (and symmetric) metric spaces their structures, the rest of the behavior imitates a fragment of the (co)quantale valued logic presented in [DP21]

Then the definition that we just give corresponds to a logic scheme that generalizes in the same framework, a Heyting valued logic for sheaves, a classical logic for pre-orders and a fragment of continuous logic for metric spaces. The interaction and use of these logics in the understanding of these mathematical objects is part of a future work.

Further work and conclusions

We give a list that contemplates future work and conclusions that this text leaves us.

- Go deeper in the theory of enriched categories on quantaloids of the type $\mathrm{Rel}(Q)$ for suitable quantales Q with particular attention to the interaction between metric/analytic/geometric interpretation and the categorical theoretical framework.

- Understand the role of the monoidal hyperdoctrines in a categorical (linear) logic setting, as a test question the categorical properties of the construction $C[P]$ presented in the section 3.2.3.

- Understand the role of completions respect to the logical definitions, specially the question that if we can reduce the to a smaller (maybe finite) collection of connectives for the "continuous logic" defined in the lat section, modulo some notion of convergence induced by the Cauchy completeness.

- In the recent dissertation of Moncayo [MVJR23] was developed a cumulative construction of von Neumann V^Q as a possibility of an approach to an "untyped" version of the universe with quantalic semantics , as it is the "continuous logic" in an adequate (co)quantale: this version would allow approaching (as in the case of locales, see 1.3) for example the categories $Q-sets$ with relational morphisms: it would be a typed logic, like in internal (categorical) logic of topos. This could provide a clue on the possibility of extending, in a future work, Schulman's stack semantics [Shu10] to the quantalic case.

- A full and complete description of categorical relationships between diverse categories of sheaf-like objects and the logical aspects that each one naturally supports is an interesting (and difficult) endeavor, reserved for future works in collaboration with the categorical community of IME-USP.

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