Representaciones de Racks y Quandles y conexiones con los g-digrupos

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Rack and Quandle Representations and Connections to the g-digroup Structure

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Resumen

En este trabajo estudiamos algunas propiedades algebraicas de las estructuras de rack y quandle así como la teoría de representaciones de estos objetos. Concretamente, demostramos que existe una correspondencia entre las representaciones fuertes e irreducibles de un rack finito y conexo con las representaciones irreducibles de su grupo finito envolvente, lo cual implica que podemos estudiar las representaciones fuertes de un rack finito y conexo a través de la teoría de representaciones de grupos finitos.
Por último, estudiamos la estructura de digrupo generalizado y su relación con la estructura de rack.

Palabras clave: Racks, Quandles, Representaciones, g-digrupos, Grupo asociado.

Abstract

In this research we study some algebraic properties of the rack and quandle structure as well as the representation theory of these objects. We establish a correspondence between the irreducible strong representations of a finite, connected rack with the irreducible representation of its finite enveloping group, which implies that the study of strong representations of a finite, connected rack can be approached through the representation theory of finite groups.
Finally, we study the g-digroup structure and its connection to the rack structure.

Key words: Racks, Quandles, Representations, g-digroups, Enveloping group.

Contents

Introduction

The algebraic structures have played a fundamental role in the solution of some problems in different branches of mathematics. Since the 19th century, mathematicians have been studying those algebraic structures that have emerged a long of mathematical research. Some of these structures are relatively new, such as the *racks* and *quandles*, which are the focus of this work.
Racks and quandles theory has its origins in the middle of the twenty century, when Mituhisa Takasaki, in [17], introduced a new structure that he called *kei* motivated by the abstract notion of symmetry in the context of finite geometries. In 1959 John Conway and Gavin Wraith studied the object given by a group G with the binary operation of conjugation. They called it a *wrack*. In the early 1980s, Joyce [8] and Mateev [11] introduced, independently, the same structure in their study of knot theory. Joyce named it *quandle*. A quandle is a set X with a binary operation

$$\begin{aligned} \triangleright : X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \triangleright y, \end{aligned}$$

such that it satisfies the following axioms:

(Q1) For all $x \in X$, we have $x \triangleright x = x$.

(Q2) For all $x \in X$, the left multiplication map $L_x : X \longrightarrow X$ defined by $L_x(y) := x \triangleright y$ is a bijective function.

(Q3) For all $x, y, z \in X$, we have $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.

Joyce assigned to each knot K a *knot quandle* $Q(K)$. He proved that if the knot quandles $Q(K)$ and $Q(K')$ of two knots K and K' are isomorphic, then the knots are equivalent, that is, the knot quandle is a complete invariant and it can be used in the knot classification problem. In fact, the three axioms in the quandle definition correspond whit the three Reidemeister movements, see the Figure 1.

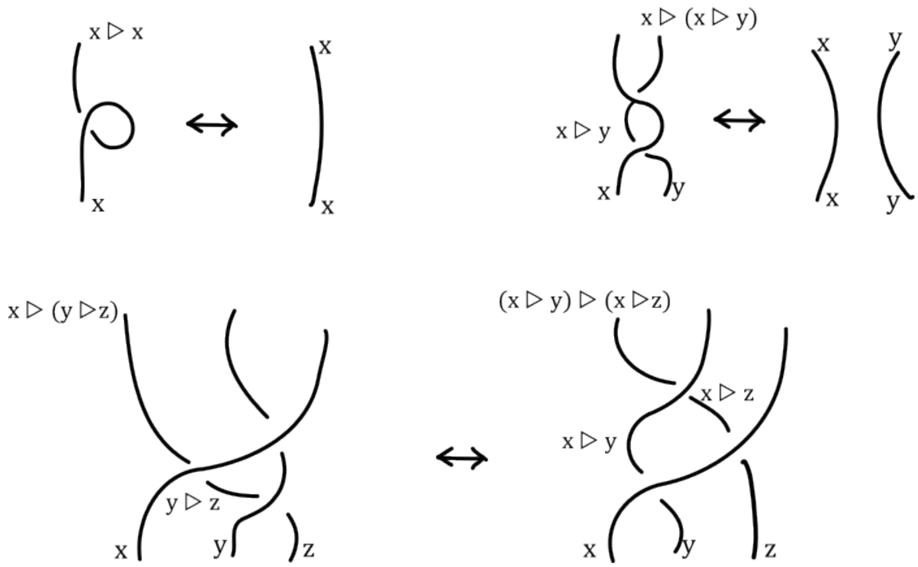

Figure 1: Quandle axioms and Reidemeister movements

In 1993, Fenn and Rourke introduced in [5] the concept of *rack*, as a generalization of quandle. A rack is a set X with a binary operation $\triangleright$ that satisfies the axioms $(Q2)$ and $(Q3)$. The rack and quandle structures emerged naturally in various mathematical contexts and they become objects of especial interest in algebra and low dimension topology, for example, they have important applications in knot theory [8], in the study of Yang-Baxter equation [10] and in the study of another algebraic structures such as rings [2], Hopf algebras [1] and Lie groups [9].

Due to the diverse range of applications that both racks and quandles have, it is important to study these objects in a purely way as algebraic entities on their own right, rather than solely based on their connections with other branches of mathematics. Several researchers have adopted this approach, and even some recent investigations have begun with the study of the *rack representation theory* [4]. Representation theory dates back to the late 19th century, when finite groups were first represented in this way. Since then, it has become a fundamental tool with applications in many areas of mathematics, both pure and applied.

In this work, we study rack and quandle structures from a purely algebraic perspective, with a special focus on rack representation theory. Elhamdadi and Moutuou, introduced rack representation theory in [4] and established general properties. Here, we study their results and present new findings. Specifically, we show a relation between strong irreducible representations of racks and irreducible representations of finite groups. We prove that (see chapter 2),

Given a finite connected rack X then, every strong irreducible representation $\rho : X \longrightarrow Conj(GL(V))$ of X induces an irreducible representation $\bar{\rho} : G_X \longrightarrow GL(V)$ of its finite enveloping group G_X, defined by $\bar{\rho}_{g_x\langle g_{x_0}^n\rangle} := \rho_x$ for all $x \in X$. Furthermore, if $\rho : X \longrightarrow Conj(GL(V))$ and $\phi : X \longrightarrow Conj(GL(V'))$ are two irreducible strong representations of X such that ρ is equivalent to ϕ, then the induced representations $\bar{\rho}$ and $\bar{\phi}$ of the group G_X are also equivalent.

The previous result allows us to study strong representations of a finite connected rack through a representation of a finite group, which is a more familiar and extensively studied subject. Furthermore, we also prove that, *Given a finite connected rack X then, every irreducible representation of its finite enveloping group $\bar{\rho}: G_X \longrightarrow GL(V)$ induces an irreducible representation $\rho: X \longrightarrow Conj(GL(V))$ of the rack X, defined by $\rho_x := \bar{\rho}_{g_x \langle g_{x_0}^n \rangle}$ for all $x \in X$. Furthermore, if $\bar{\rho}: G_X \longrightarrow GL(V)$ and $\bar{\phi}: G_X \longrightarrow GL(V')$ are two irreducible representations of G_X such that $\bar{\rho}$ is equivalent to $\bar{\phi}$, then the induced representations ρ and ϕ of the rack X are also equivalent.*

Observe that, in the previous result, the induced rack representation is not necessarily strong, we prove that if the finite enveloping group has trivial center, then it is strong. Therefore, under these conditions we have a bijective correspondence between strong irreducible rack representations and irreducible group representations.

We also study the relation between racks and an object that generalize the group structure, it is called *g-digroup* and it was defined by Salazar et al in [15]. Since quandles and groups are closely related, then one can think that this relation can be extended to g-digroups. In fact, we can get a rack from a g-digroup and we can associate a g-digroup to a rack. We study this relation in the last chapter.

This document is organized as follows. In Chapter 1, we review some basic concepts about rack and quandle structure and we study the relation between quandle structure and group structure. We define a new quandle that we have called *permutation quandle*, denoted by $\mathbb{P}_n$, and prove that its finite enveloping group $G_{\mathbb{P}_n}$, is exactly the symmetric group $\mathbb{S}_n$. Further, we study a new type of racks called *finitely stable racks*, which is a concept introduced in [4]. Finally, analogous to group theory, we study the notion of the *rack ring*. In Chapter 2 we study rack actions and rack representation theory, with a focus in strong representations. In Chapter 3 we review some general and basic properties about g-digroups and study the relation between rack structure and g-digroup structure.

Chapter 1

Racks and Quandles

In this chapter we give a short review of some general concepts about rack and quandle structures. We begin with the definition of quandle and we give some important examples, all of them introduced by Joyce in [7]. In section 1.2 we study a relation between racks and groups. In section 1.3 we define a new quandle, that we called *permutation quandle* and we establish some properties of it, specifically, we give a complete description of its finite enveloping group. In sections 1.4 and 1.5, we study finitely stable racks, a new concept introduced by Elhamdadi and Moutou in [4]. Finally, in section 1.7, analogous to the case of groups, we study the notion of rack ring.

1.1 Preliminaries

We start this section with the following definition.

Definition 1.1. A ***quandle*** is a set X with a binary operation

$$\begin{aligned} \rhd : X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \rhd y, \end{aligned}$$

such that it satisfies the following axioms:

(Q1) For all $x \in X$, we have $x \rhd x = x$.

(Q2) For all $x \in X$, the left multiplication map $L_x : X \longrightarrow X$ defined by $L_x(y) := x \rhd y$ is a bijective function.

(Q3) For all $x, y, z \in X$, we have $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$.

If $(X, \rhd)$ satisfies axioms $(Q2)$ and $(Q3)$, but does not satisfy axiom $(Q1)$, then X is called a ***rack***.

Note that, according to the previous definition, every quandle is a rack. Therefore, we can consider the rack structure as a generalization of the quandle structure.

Notation 1.1. For every element x, y of a rack X, we write $x \rhd^{-1} y$ for the inverse function $L_x^{-1}(y)$.

If $(X, \triangleright)$ satisfies that $L_x \circ L_x = id$, for all $x \in X$, that means, the function L_x is its own inverse, then we call X an ***involutive quandle*** or an ***involutive rack***, depending on whether it satisfies the axiom (Q1) or not. Mituhisa Takasaki [17] used the term *kei* to refer to the structure of an involutive quandle. Observe that, if X is a involutive quandle then we have

$$x \triangleright (x \triangleright y) = y; \quad \text{for all } x, y \in X.$$

Rack and quandle structure emerge in various context, let us take a look at some examples:

Example 1.1. *Let X be a set with the operation $x \triangleright y = y$ for all $x, y \in X$, then X is a quandle. This quandle is called* ***trivial quandle****.*

Proof. Let us check that the quandle axioms are satisfied

(Q1) $x \triangleright x = x$ for all $x \in X$.

$(Q2)$ Let $x, y, z \in X$ such that $L_x(y) = L_x(z)$. Then $x \triangleright y = x \triangleright z$. Thus, from the definition of $\triangleright$, $y = z$. Thereby L_x is injective. Now, for all $z \in X$ we have $L_x(z) = x \triangleright z = z$. Then L_x is bijective for all $x \in X$.

$(Q3)$ Let $x, y, z \in X$ then $x \triangleright (y \triangleright z) = x \triangleright z = z$. On the other hand we have $(x \triangleright y) \triangleright (x \triangleright z) = y \triangleright z = z$. Therefore, axiom $(Q3)$ holds.

▲

Example 1.2. *Let G be a group. We define the binary operation*

$$\begin{aligned} \triangleright : G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \triangleright h := ghg^{-1}. \end{aligned}$$

The set G with the operation $\triangleright$ is a quandle, which is known as the ***conjugacy quandle****. We denote this quandle by* ***Conj(G)****.*

Proof. Let us check that the quandle axioms are satisfied.

$(Q1)$ Note that $x \triangleright x = xxx^{-1} = x$ for all $x \in G$.

$(Q2)$ Let $x, y, z \in G$ such that $L_x(y) = L_x(z)$. Then $x \triangleright y = x \triangleright z$. Therefore, $xyx^{-1} = xzx^{-1}$, and so $y = x$. Hence L_x is injective. Besides, for all $z \in G$, we have $L_x(x^{-1}zx) = x \triangleright (x^{-1}zx) = xx^{-1}zxx^{-1} = z$. Then L_x is surjective for all $x \in G$.

$(Q3)$ Let $x, y, z \in G$ then $x \triangleright (y \triangleright z) = x \triangleright (yzy^{-1}) = xyzy^{-1}x^{-1}$. On the other hand we have $(x \triangleright y) \triangleright (x \triangleright z) = (xyx^{-1}) \triangleright (xzx^{-1}) = (xyx^{-1})(xzx^{-1})(xyx^{-1})^{-1} = xyzy^{-1}x^{-1}$.

Therefore axiom $(Q3)$ holds.

▲

Example 1.3. *Let G be a group. We define the binary operation*

$$\begin{aligned} \triangleright : G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \triangleright h := gh^{-1}g. \end{aligned}$$

The set G with the operation $\triangleright$ is a quandle. We denote this quandle by ***Core(G)****.*

Proof. Let us check that the quandle axioms are satisfied.

$(Q1)$ Note that $x \triangleright x = xx^{-1}x = x$ for all $x \in G$.

$(Q2)$ Let $x, y, z \in G$ such that $L_x(y) = L_x(z)$. Then $x \triangleright y = x \triangleright z$. Therefore, $xy^{-1}x = xz^{-1}x$, and so $y = x$. Hence L_x is injective. Now, for all $z \in G$ we have $L_x(xz^{-1}x) = x \triangleright (xz^{-1}x) = xx^{-1}zx^{-1}x = z$. Then L_x is surjective for all $x \in G$.

$(Q3)$ Let $x, y, z \in G$ then $x \triangleright (y \triangleright z) = x \triangleright (yz^{-1}y) = xy^{-1}zy^{-1}x$. On the other hand, we have
$$(x \triangleright y) \triangleright (x \triangleright z) = (xy^{-1}x) \triangleright (xz^{-1}x) = (xy^{-1}x)(xz^{-1}x)^{-1}(xy^{-1}x) = xy^{-1}zy^{-1}x.$$
Therefore axiom $(Q3)$ holds.

▲

The following example introduces what is called the Takasaki quandle.

Example 1.4. *Let G be an abelian group. We define the binary operation*

$$\begin{aligned} \triangleright : G \times G &\longrightarrow G \\ (x, y) &\longmapsto x \triangleright y := 2x - y. \end{aligned}$$

The set G with the operation $\triangleright$ is an involutive quandle. This quandle is known as Takasaki quandle.

Proof. Remark that the Takasaki quandle is the same Core(G) since, $x \triangleright y = 2x - y = x + (-y) + x$. Thus, from the previous example, we have that it is a quandle. The involution property is easy to check. Let $x, y \in G$ then
$$x \triangleright (x \triangleright y) = x \triangleright (2x - y) = 2x - (2x - y) = y.$$

▲

The following example introduces what is called the Alexander quandle.

Example 1.5. *Let Γ be a group and γ a non-trivial element in Γ. Let $\mathcal{A}$ be a left $\mathbb{Z}[\gamma^{\pm 1}]$ -module. Define the operation $\triangleright_\gamma$: $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ by*

$$x \triangleright_\gamma y = \gamma \cdot (y - x) + x.$$

Then $(\mathcal{A}, \triangleright_\gamma)$ is a quandle, known as the Alexander quandle.

Proof. Let us verify that the quandle axioms are satisfied. In fact,

$(Q1)$ Note that $x \triangleright_\gamma x = \gamma \cdot (x - x) + x = x$ for all $x \in \mathcal{A}$.

$(Q2)$ For the second axiom we have to prove that L_x is invertible, for all $x \in \mathcal{A}$. We claim that $L_x^{-1}(y) = \gamma^{-1} \cdot (y - x) + x$, for all $y \in A$, is the desired inverse of L_x. In fact, we have,

$$\begin{aligned} (L_x \circ L_x^{-1})(y) &= L_x(L_x^{-1}(y)) \\ &= L_x(\gamma^{-1} \cdot (y - x) + x) \\ &= \gamma \cdot [\gamma^{-1} \cdot (y - x) + x - x] + x \\ &= (\gamma\gamma^{-1}) \cdot (y - x) + x = y - x + x = y. \end{aligned}$$

and,

$$\begin{aligned}(L_x^{-1} \circ L_x)(y) &= L_x^{-1}(L_x(y)) \\ &= L_x(\gamma \cdot (y-x) + x) \\ &= \gamma^{-1} \cdot [\gamma \cdot (y-x) + x - x] + x \\ &= (\gamma^{-1}\gamma) \cdot (y-x) + x = y - x + x = y.\end{aligned}$$

Then L_x is bijective.

$(Q3)$ Let $x, y, z \in \mathcal{A}$, then

$$\begin{aligned}x \rhd (y \rhd z) &= x \rhd (\gamma \cdot (z-y) + y) \\ &= \gamma \cdot [\gamma \cdot (z-y) + y - x] + x \\ &= \gamma^2(z-y) + \gamma y - \gamma x + x \\ &= \gamma^2 z + \gamma(1-\gamma)y + (1-\gamma)x.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(x \rhd y) \rhd (x \rhd z) &= (\gamma \cdot (y-x) + x) \rhd (\gamma \cdot (z-x) + x) \\ &= \gamma[(\gamma \cdot (z-x) + x) - (\gamma \cdot (y-x) + x)] + (\gamma \cdot (y-x) + x) \\ &= \gamma(\gamma z - \gamma y) + \gamma y - \gamma x + x \\ &= \gamma^2 z + \gamma(1-\gamma)y + (1-\gamma)x.\end{aligned}$$

Therefore axiom $(Q3)$ holds.

▲

Definition 1.2. Let $(X, \rhd)$ and $(Y, *)$ be racks.

(I) A map $\phi : X \longrightarrow Y$ is called a rack *homomorphism* if it satisfies

$$\phi(x \rhd h) = \phi(x) * \phi(h), \text{ for all } x, h \in X.$$

If ϕ is bijective then it is called an isomorphism. If $X = Y$ and ϕ is bijective then it is called an automorphism.

(II) We write $Hom_{rack}(X, Y)$ for the set of rack homomorphisms from X to Y:

$$Hom_{rack}(X,Y) := \{\phi : X \longrightarrow Y \mid \phi \text{ is a function and } \phi(x \rhd h) = \phi(x) * \phi(h), \text{ for all } x, h \in X\}$$

(III) We write $Sym(X)$ or $\mathbb{S}_X$ for the group of all permutations of the rack X and $Aut(X)$ for the subgroup of $Sym(X)$, of all automorphisms of the rack X. Thus,

$$\begin{aligned}Sym(X) &:= \{\sigma : X \longrightarrow X \mid \ \sigma \ is \ bijective\}, and \\ Aut(X) &:= \{\phi \in Sym(X) \mid \ \phi(x \rhd y) = \phi(x) \rhd \phi(y) \ \ for \ all \ x, y \in X\}.\end{aligned}$$

(IV) A subset Q of a rack $(X, \triangleright)$ is called *subrack* if Q is closed under the operation $\triangleright$. That is, for every $q, p \in Q$, we have that $q \triangleright p \in Q$.

Example 1.6. *Let X be a rack. For all $x \in X$, the map $L_x : X \longrightarrow X$ defined by $L_x(y) := x \triangleright y$ is an automorphism of X. Indeed, from axiom (Q3) we have*

$$L_x(y \triangleright z) = x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) = L_x(y) \triangleright L_x(z), \quad \forall z, y \in X.$$

Example 1.7. *Let $(X, \triangleright)$ and $(Y, *)$ be racks and $\phi : X \longrightarrow Y$ a rack homomorphism. Then the image of ϕ, $Im(\phi)$ is a subrack of Y. In fact, let $y_1, y_2 \in Im(\phi)$ then, there exists $x_1, x_2 \in X$ such that $y_1 = \phi(x_1)$ and $y_2 = \phi(x_2)$. Since ϕ is a rack homomorphism,*

$$y_1 * y_2 = \phi(x_1) * \phi(x_2) = \phi(x_1 \triangleright x_2).$$

Therefore, $y_1 * y_2 \in Im(\phi)$. The result follows for all $y_1, y_2 \in Im(\phi)$, thus, $Im(\phi)$ is a subrack of Y.

Lemma 1.1. *Let G be a group. Consider the quandle $Conj(G)$, then every conjugacy class of G is a subquandle of $Conj(G)$.*

Proof. Let $g \in G$ and let $C_g = \{hgh^{-1} \mid h \in G\}$ be the conjugacy class represented by g. Let $q, p \in C_g$, we have to prove that $q \triangleright p \in C_g$. First, suppose that $q = p$, then $q \triangleright p = p \triangleright p = ppp^{-1} = p \in C_g$.
Now, suppose that $q \neq p$. Since $q, p \in C_g$ then there exists $h_1, h_2 \in G$ such that $p = h_1 g h_1^{-1} = h_1 \triangleright g$ and $q = h_2 g h_2^{-1} = h_2 \triangleright g$. We have,

$$\begin{aligned} q \triangleright p &= qpq^{-1} \\ &= (h_2 g h_2^{-1})(h_1 g h_1^{-1})(h_2 g h_2^{-1})^{-1} \\ &= (h_2 g h_2^{-1} h_1) g (h_1^{-1} h_2 g^{-1} h_2^{-1}) \\ &= (h_2 g h_2^{-1} h_1) g (h_2 g h_2^{-1} h_1)^{-1}. \end{aligned}$$

Therefore, $q \triangleright p \in C_g$. Thus, C_g is a subquandle of $Conj(G)$.

■

1.2 Related groups of a rack

There are three important groups associated to racks and quandles. These groups play an important role in the understanding of those concepts themselves, because a group is a more familiar and more studied structure. This relation allows us to translate some questions about quandles into questions about groups and vice versa. Let us begin this section with the following definition.

Definition 1.3. Let X be a rack. Let $F(X)$ the free group on the set X and let N be the normal subgroup generated by the words of the form $(x \triangleright y)xy^{-1}x^{-1}$ where $x, y \in X$. We define the ***associated group***, denoted by $As(X)$, to be the quotient group $F(X)/N$.

$$As(X) = F(X)/\langle x \triangleright y = xyx^{-1}, \quad x, y \in X\rangle$$

This group is also called the ***enveloping group***.

Observe that, we have two onto maps, the inclusion map $\iota : X \hookrightarrow F(X)$ defined by $\iota(x) := x$ for all $x \in X$ and the canonical homomorphism $\pi : F(X) \longrightarrow As(X)$ defined by $\pi(x) := \bar{x}$ for all $x \in X$. So we have a natural onto map, $\eta : X \longrightarrow As(X)$ defined by $\eta(x) := (\pi \circ i)(x) = \bar{x}$. This map is not necessarily injective, as we will see in the Example 1.11.

Notation 1.2. We will write g_x to represent the imagine $\eta(x)$. That is, $\eta(x) = g_x$.

Note that, from definition of the associated group $As(X)$, it has the presentation

$$As(X) = \langle g_x \ , \ x \in X \mid g_{x \triangleright y} = g_x g_y g_x^{-1}, \ x, y \in X \rangle.$$

Therefore, in $As(X)$, we have the relation $g_{x \triangleright y} = g_x g_y g_x^{-1}$, for all $x, y \in X$. Now, let $z = x \triangleright^{-1} y \in X$, then $x \triangleright z = y$, therefore,

$$\begin{aligned} g_y &= g_{x \triangleright z} \\ &= g_x g_z g_x^{-1}. \end{aligned}$$

hence $g_z = g_x^{-1} g_y g_x$.

Theorem 1.1. *Let $(X, \triangleright)$ and $(Y, *)$ be racks and $\phi : X \longrightarrow Y$ be a rack homomorphism. Then ϕ induces a group homomorphism $\hat{\phi} : As(X) \longrightarrow As(Y)$ such that $\hat{\phi}(g_x) = g_{\phi(x)}$ for all $x \in X$.*

Proof. Consider the map $\eta_Y \circ \phi : X \longrightarrow As(Y)$. From the universal property of free groups, there exists a unique surjective group homomorphism $f_\phi : F(X) \longrightarrow As(Y)$ such that $f_\phi(x) = g_{\phi(x)}$ for all $x \in X$. Notice that, in $As(Y)$ it holds that $g_{\phi(x) * \phi(z)} = g_{\phi(x)} g_{\phi(z)} g_{\phi(x)}^{-1}$, for all $x, z \in X$. So, we have

$$\begin{aligned} f_\phi((x \triangleright z) x z^{-1} x^{-1}) &= f_\phi(x \triangleright z) f_\phi(x) f_\phi(z)^{-1} f_\phi(x)^{-1} \\ &= g_{\phi(x \triangleright z)} g_{\phi(x)} g_{\phi(z)}^{-1} g_{\phi(x)}^{-1} \\ &= g_{\phi(x) * \phi(z)} g_{\phi(x)} g_{\phi(z)}^{-1} g_{\phi(x)}^{-1} \\ &= g_{\phi(x)} g_{\phi(z)} g_{\phi(x)}^{-1} g_{\phi(x)} g_{\phi(z)}^{-1} g_{\phi(x)}^{-1} \\ &= 1. \end{aligned}$$

This implies that $(x \triangleright z) x z^{-1} x^{-1} \subset Ker f_\phi$, for all $x, z \in X$. It follows that f_ϕ define a group homomorphism $\hat{\phi} : As(X) \longrightarrow As(Y)$ such that $\hat{\phi}(g_x) = f_\phi(x) = g_{\phi(x)}$ for all $x \in X$. That is, $\hat{\phi}$ makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ As(X) \arrow[r, "\hat{\phi}"'] & As(Y). \end{tikzcd}$$

∎

As is a functor from the category of racks into the category of groups and $Conj$ is a functor from the category of groups into the category of racks. Moreover, the functor As is the left adjoint to the functor $Conj$ (see [13] [8]). Namely, for any rack X and any group G we have

$$Hom_{gr}(As(X), G) \cong Hom_{rack}(X, Conj(G)).$$

This claim is proved in the following theorem.

Theorem 1.2. (Universal property)
Let G be a group, X be a rack and $\phi : X \longrightarrow Conj(G)$ be a rack homomorphism. Then ϕ induces a unique group homomorphism $\hat{\phi} : As(X) \longrightarrow G$ such that $\hat{\phi}(g_x) = \phi(x)$ for all $x \in X$. Conversely, let $\psi : As(X) \longrightarrow G$ a group homomorphism then ψ induces a rack homomorphism $\hat{\psi} : X \longrightarrow Conj(G)$ such that $\hat{\psi}(x) = \psi(g_x)$ for all $x \in X$.

Proof. Let $\phi : X \longrightarrow Conj(G)$ be a rack homomorphism. Remark that the quandle $Conj(G)$ is the set G with the conjugation operation $g * h = ghg^{-1}$. Since $G = Conj(G)$ as sets, then the function $\phi : X \longrightarrow G$ makes sense. From the universal property of free groups, there exists an unique group homomorphism $f_\phi : F(X) \longrightarrow G$ such that $f_\phi(x) = \phi(x)$ for all $x \in X$. Now, in G, we have that $\phi(x) * \phi(z) = \phi(x)\phi(z)\phi(x)^{-1}$, for all $x, z \in X$. So,

$$\begin{aligned} f_\phi((x \triangleright z)xz^{-1}x^{-1}) &= f_\phi(x \triangleright z)f_\phi(x)[f_\phi(z)]^{-1}[f_\phi(x)]^{-1} \\ &= \phi(x \triangleright y)\phi(x)[\phi(z)]^{-1}[\phi(x)]^{-1} \\ &= [\phi(x) * \phi(z)]\phi(x)[\phi(z)]^{-1}[\phi(x)]^{-1} \\ &= 1. \end{aligned}$$

This implies that $(x \triangleright z)xz^{-1}x^{-1} \subset Kerf_\phi$, for all $x, z \in X$. It follows that f_ϕ defines an unique group homomorphism $\hat{\phi} : As(X) \longrightarrow G$ such that $\hat{\phi}(g_x) = f_\phi(x) = \phi(x)$ for all $x \in X$. That is, $\hat{\phi}$ makes the following diagram commute,

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\phi"] & As(X) \arrow[d, "\hat{\phi}"] \\ & G. \end{tikzcd}$$

Conversely, let $\psi : As(X) \longrightarrow G$ be a group homomorphism. Since in $As(X)$ we have $g_{x \triangleright z}g_x g_y^{-1}g_x^{-1} = 1$ then for every $x, z \in X$,

$$\begin{aligned} \psi(1) &= \psi(g_{x\triangleright z}g_x g_y^{-1}g_x^{-1}) \\ &= \psi(g_{x\triangleright z})\psi(g_x)\psi(g_z)^{-1}\psi(g_x)^{-1} \\ &= 1 \\ \therefore\ \psi(g_{x\triangleright z}) &= \psi(g_x)\psi(g_z)\psi(g_x)^{-1}. \end{aligned}$$

Consider the natural map $\eta : X \longrightarrow As(X)$, then the map $\hat{\psi} : X \longrightarrow Conj(G)$ defined by $\hat{\psi}(x) = (\psi \circ \eta)(x) = \psi(g_x)$ is a rack homomorphism. Indeed, $\hat{\psi}(x \triangleright z) = \psi(g_{x\triangleright z}) = \psi(g_x)\psi(g_z)\psi(g_x)^{-1} = \hat{\psi}(x)\hat{\psi}(y)\hat{\psi}(x)^{-1}$. Moreover, $\hat{\psi}$ makes the following diagram commute,

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\hat{\psi}"] & As(X) \arrow[d, "\psi"] \\ & G. \end{tikzcd}$$

■

Proposition 1.1. *The associated group $As(X)$ of a rack X is an infinite set.*

Proof. Let the map $\Phi : X \longrightarrow Conj(\mathbb{Z})$ defined by $\Phi(x) = 1$ for all $x \in X$. Denote by $*$ the operation that makes $Conj(\mathbb{Z})$ into a rack. Note that for every $x, y \in X$ we have

$$\begin{aligned}\Phi(x \triangleright y) &= 1 \\ &= 1 + 1 - 1 \\ &= \Phi(x) + \Phi(y) - \Phi(x) \\ &= \phi(x) * \phi(y).\end{aligned}$$

Therefore, the map Φ is a rack homomorphism. By Theorem 1.2, it induces a group homomorphism $\hat{\Phi} : As(X) \longrightarrow \mathbb{Z}$ defined by $\hat{\phi}(g_x) = 1$ for all $x \in X$. Observe that $\hat{\Phi}$ is surjective. In fact, let $n \in \mathbb{Z}$ and $x \in X$, then $\hat{\Phi}(g_x^n) = \hat{\Phi}(g_x) + \hat{\Phi}(g_x) + \cdots + \hat{\Phi}(g_x) = n\hat{\Phi}(g_x) = n$. Hence, it is surjective. Therefore, there exists an injective function $f : \mathbb{Z} \longrightarrow As(X)$. Thus, $As(X)$ is an infinite set.

■

Now, we introduce another group associated to a rack, called the *inner automorphism group.*

Definition 1.4. The ***inner automorphism group***, denoted by $Inn(X)$, is defined as the subgroup of $Aut(X)$, generated by the permutation functions L_x. Concisely, the group is

$$Inn(X) := \langle L_x \mid x \in X \rangle.$$

Notation 1.3. From now on we write fg for the composition function $f \circ g$ and id for the identity function.

Proposition 1.2. *Let X be a rack, the map $L : X \longrightarrow Conj(Inn(X))$ defined by $L(x) := L_x$, is a rack homomorphism. That means, $L_{x \triangleright y} = L_x L_y L_x^{-1}$ for all $x, y \in X$.*

Proof. Let $x, y, z \in X$, by axiom $(Q3)$ we have

$$\begin{aligned}L(x \triangleright y)(z) &= L_{x \triangleright y}(z) \\ &= L_{x \triangleright y}(L_x(L_x^{-1}(z)) \\ &= (x \triangleright y) \triangleright (x \triangleright (x \triangleright^{-1} z)) \\ &= x \triangleright (y \triangleright (x \triangleright^{-1} z)) \\ &= L_x(L_y(L_x^{-1}(z))).\end{aligned}$$

Therefore, $L_{x \triangleright y} = L_x L_y L_x^{-1}$.

■

Corollary 1.1. *Let X be a rack then we have $L_{x \triangleright^{-1} y} = L_x^{-1} L_y L_x$.*

Proof. Let $x, y \in X$, suppose that $z = x \triangleright^{-1} y$, thus $x \triangleright z = y$. It follows that

$$\begin{aligned} L_y &= L_{x \triangleright z} \\ &= L_x L_z L_x^{-1} \end{aligned}$$

Therefore, $L_{x \triangleright^{-1} y} = L_x^{-1} L_y L_x$.

■

If G is a group then the inner automorphism group of the conjugacy quandle $Inn(Conj(G))$, is exactly the usual one that we have in group theory, as we show in the next example.

Example 1.8. *Let G be a group. Consider the map $L : G \longrightarrow Inn(Conj(G))$, defined by $L(g)(x) := L_g(x) = gxg^{-1}$ for all $g, x \in G$. Let $x, g, h \in G$ then,*

$$L(xg)(h) = L_{xg}(h) = (xg)h(xg)^{-1} = x(ghg^{-1})x^{-1} = x[L_g(h)]x^{-1} = L_x(L_g(h)) = (L(x)L(g))(h).$$

Hence, L is a group homomorphism. Furthermore, it is surjective. Indeed, given $f \in Inn(Conj(G))$, it has the form $f = L_{x_n}^{\epsilon_n} L_{x_{n-1}}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}$, where $x_i \in G$ and $\epsilon_i \in \{1, -1\}$ for all $i \in \{1, ..., n\}$. Besides, note that $f = L_{x_n}^{\epsilon_n} L_{x_{n-1}}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1} = L(x_n^{\epsilon_n} x_{n-1}^{\epsilon_{n-1}} \cdots x_1^{\epsilon_1})$. So it is surjective.
Let g be an element of the center $Z(G)$ of G. For every $x \in G$ we have

$$L_g(x) = gxg^{-1} = x.$$

Thus, $L(g) = id$, it implies that $Z(G) \subset KerL$. Conversely, let $g \in Ker(L)$, then for all $x \in X$, we have $L_g(x) = x$. From the definition of L_g, $gxg^{-1} = x$, hence $gx = xg$. Therefore, $g \in Z(G)$. From the previous calculation, $Z(G) = KerL$. From the first isomorphism theorem, $Inn(Conj(G)) \cong G/Z(G)$.

▲

We can think in a natural action of the inner automorphism group $Inn(X)$ of a rack X, over the underlying set X.

Lemma 1.2. *Let X be a rack, then the function*

$$\begin{aligned} \bullet : Inn(X) \times X &\longrightarrow X \\ (L_x, y) &\longmapsto L_x \bullet y := L_x(y) = x \triangleright y, \end{aligned}$$

is a left action of the group $Inn(X)$ over the set X.

Proof. Let us see that the function is well defined. Let $(L_x, y) = (L_{x'}, y') \in Inn(X) \times X$, then we have that $L_x = L_{x'}$ and $y = y'$, thus, $L_x(y) = L_{x'}(y')$, therefore $L_x \bullet y = L_{x'} \bullet y'$. So, it is well defined.
For the identity map $id \in Inn(X)$ we have $id \bullet x = id(x) = x$ for all $x \in X$. Further, given $x, y, z \in X$ then

$$L_x \bullet (L_y \bullet z) = L_x(L_y(z)) = (L_x L_y)(z) = (L_x L_y) \bullet z.$$

So, it is a group action.

■

The orbits of this action are known as ***the connected components*** of X. The case when the rack has just one orbit is very interesting.

Definition 1.5. A rack X is said to be ***connected*** (or *indecomposable*) if the action of $Inn(X)$ on X is transitive, that means, if it has only one orbit.

Let X be a connected rack and let $x \in X$ be an arbitrary element. Since X is connected then the orbit of x, denoted $\mathscr{O}_x$, is equal to X.

$$\mathscr{O}_x = \{y \in X \mid \exists \phi \in Inn(X) : \phi(x) = y\} = X.$$

Thus, given $y \in X$ there exists $\phi \in Inn(X)$ such that $\phi(x) = y$. Since $\phi \in Inn(X)$ then it has the form $\phi = L_{x_n}^{\epsilon_n} L_{x_{n-1}}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}$, where $x_i \in X$ and $\epsilon_i \in \{1, -1\}$, for all $i \in \{1, ..., n\}$. So, we have

$$\begin{aligned} y &= \phi(x) \\ &= L_{x_n}^{\epsilon_n} L_{x_{n-1}}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}(x) \\ &= x_n \rhd^{\epsilon_n} (x_{n-1} \rhd^{\epsilon_{n-1}} (\cdots \rhd^{\epsilon_2} (x_1 \rhd^{\epsilon_1} x) \cdots). \end{aligned}$$

In other words, a rack X is connected if and only if for all $x, y \in X$ there exists $x_1, ..., x_n \in X$ such that $L_{x_n}^{\epsilon_n} L_{x_{n-1}}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}(x) = y$, where $\epsilon_i \in \{1, -1\}$, for all $i \in \{1, .., n\}$.

In the following examples we describe the associated group and the inner automorphism group of some quandles.

Example 1.9. *Let us consider the cyclic group of order two $\mathbb{Z}_2 = \{0, 1\}$. Consider the Takasaki quandle $(\mathbb{Z}_2, \rhd)$ with the operation $x \rhd y = 2x - y$. The table of this quandle is:*

$\rhd$	***0***	***1***
0	*0*	*1*
1	*0*	*1*

From the table we have $L_0 = L_1 = id$. Then, the inner automorphism group is the trivial one. That is, $Inn(\mathbb{Z}_2) = \{id\}$. Note that, the quandle is not connected, it has two connected components $\mathscr{O}_0 = \{0\}$ and $\mathscr{O}_1 = \{1\}$.
Let us find the group $As(\mathbb{Z}_2)$. Consider the function

$$\begin{aligned} \phi : \mathbb{Z}_2 &\longrightarrow Conj(\mathbb{Z} \times \mathbb{Z}) \\ 0 &\longmapsto (1, 0) \\ 1 &\longmapsto (0, 1). \end{aligned}$$

Note that, $\phi(0 \rhd 1) = \phi(1) = \phi(1) + \phi(0) - \phi(0)$. Similarly, we have $\phi(1 \rhd 0) = \phi(0) + \phi(1) - \phi(1)$. It follows that ϕ is a rack homomorphism, then by Theorem 1.2, ϕ induces a group homomorphism

$$\begin{aligned} \hat{\phi} : As(\mathbb{Z}_2) &\longrightarrow \mathbb{Z} \times \mathbb{Z} \\ g_0 &\longmapsto \phi(0) \\ g_1 &\longmapsto \phi(1). \end{aligned}$$

Observe that, in $As(\mathbb{Z}_2)$, we have $g_{0\triangleright 1} = g_1 = g_0 g_1 g_0^{-1}$, therefore $g_0 g_1 = g_1 g_0$. It follows that, every $w \in As(\mathbb{Z}_2)$ is of the form $w = g_0^i g_1^j$, where $i, j \in \mathbb{Z}$. In particular, for $w \in Ker\hat{\phi}$,

$$\begin{aligned}(0,0) &= \hat{\phi}(w)\\ &= \hat{\phi}(g_0^i g_1^j)\\ &= \hat{\phi}(g_0)^i + \hat{\phi}(g_1)^j\\ &= \phi(0)^i + \phi(1)^j\\ &= (i,0) + (0,j).\end{aligned}$$

Therefore, $i = j = 0$ and so $w = g_0^i g_1^j = 1$. It follows that $ker(\hat{\phi}) = \{1\}$, and thus $As(\mathbb{Z}_2) \cong \mathbb{Z} \times \mathbb{Z}$.

▲

Example 1.10. *Let us denote the cyclic group of order three by $\mathbb{Z}_3 = \{0, 1, 2\}$. Consider the Takasaki quandle $(\mathbb{Z}_3, \triangleright)$, with the operation $x \triangleright y = 2x - y$. The table of such quandle is:*

$\triangleright$	***0***	***1***	***2***
0	*0*	*2*	*1*
1	*2*	*1*	*0*
2	*1*	*0*	*2*

Note that, this quandle is connected. Let us find the group $Inn(\mathbb{Z}_3)$, since Takasaki quandle is involutive we have $L_0^2 = L_1^2 = L_2^2 = id$. From Proposition 1.2 we have $L_0 = L_{1\triangleright 2} = L_1 L_2 L_1^{-1} = L_1 L_2 L_1$ and $L_0 = L_{2\triangleright 1} = L_2 L_1 L_2$. It follows that $L_1 L_2 L_1 = L_2 L_1 L_2$, so $(L_1 L_2)^3 = id$, and,

$$\begin{aligned}L_1 &= L_2(L_1L_2)^2\\ &= L_{1\triangleright 0}(L_1L_2)^2\\ &= L_1L_0L_1L_1L_2L_1L_2\\ &= L_{0\triangleright 2}L_0L_2L_1L_2\\ &= L_0L_2L_0L_0L_2L_1L_2\\ &= L_0(L_1L_2).\end{aligned}$$

Further,

$$\begin{aligned}L_2 &= L_{0\triangleright 1} = L_0L_1L_0\\ &= L_0(L_0L_1L_2)L_0\\ &= (L_1L_2)L_0.\end{aligned}$$

Then, we can take the automorphism L_0 and $L_1 L_2$ to be the generators of the group $Inn(\mathbb{Z}_3)$. Note that, $L_1^2 = (L_0(L_1L_2))^2 = id$. Hence, we have $L_0(L_1L_2) = (L_2L_1)L_0$. Since $L_1 = L_0(L_1L_2)$ and $L_2 = (L_1L_2)L_0$, then $L_0(L_1L_2) = (L_2L_1)L_0 = [((L_1L_2)L_0)(L_0(L_1L_2))]L_0 = (L_1L_2)^2 L_0$. Therefore, every $\phi \in Inn(\mathbb{Z}_3)$ is of the form $\phi = L_0^i(L_1L_2)^j$ where $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$. This implies that the group $Inn(\mathbb{Z}_3)$ has six elements

$$Inn(\mathbb{Z}_3) = \{id, L_0, L_0(L_1L_2), L_1L_2, (L_1L_2)^2, L_0(L_1L_2)^2\}.$$

Since $L_0(L_1L_2) = (L_1L_2)^2L_0$, then it is not an abelian group , so $Inn(\mathbb{Z}_3) \cong \mathbb{S}_3$.

▲

Example 1.11. *Let $X = \{1,2,3\}$ and $\sigma = (1\ 2\ 3) \in \mathbb{S}_3$. The set X is a rack with the binary operation $\triangleright$: $X \times X \longrightarrow X$, defined by $i \triangleright j = \sigma(j)$, for all $i,j \in X$. Indeed, note that $L_i = \sigma$ for all $i \in X$, since σ is bijective then L_i is bijective, for every $i \in X$. Furthermore, $i \triangleright (j \triangleright k) = i \triangleright \sigma(k) = \sigma^2(k)$. On the other hand, $(i \triangleright j) \triangleright (i \triangleright k) = \sigma(j) \triangleright \sigma(k) = \sigma^2(k)$. Therefore, $(X, \triangleright)$ is a rack. The table of this rack is*

$\triangleright$	**1**	**2**	**3**
1	2	3	1
2	2	3	1
3	2	3	1

Since $L_1 = L_2 = L_3$ then, the inner automorphism group $Inn(X) = \langle L_1 \rangle \cong \mathbb{Z}_3$.
Note that, $g_{1\triangleright 1} = g_2 = g_1g_1g_1^{-1}$, thereby, we have $g_2 = g_1$. Similarly, $g_{1\triangleright 2} = g_3 = g_1g_2g_1^{-1} = g_1$. Then $As(X) = \langle g_1 \rangle \cong \mathbb{Z}$.

▲

In general, for a rack X, it is not always easy to give a complete description of the group $Inn(X)$. In [3], they compute the inner automorphism group of some quandles.
Graña et al, in [6], introduce another group associated to finite connected racks. They named it, the *finite enveloping group*, and as the name suggests, this group is finite. We follow the construction given by them, but for the finiteness we give a slightly different proof.

Notation 1.4. Let X be a rack, $x, y \in X$ and $k \in \mathbb{N}$; we write

$$x \triangleright^k y = L_x^k(y) = x \triangleright (x \triangleright (\cdots \triangleright (x \triangleright y)) \cdots)\ , \qquad x \text{ multiplying } k\text{-times.}$$

Lemma 1.3. *Let X be a rack and $n \in \mathbb{N}$. Let $x_1, ..., x_n, y, z \in X$ such that*

$$L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}(z) = y.$$

where $\epsilon_i = \pm 1$, for all $i \in \{1, ..., n\}$. Then, $L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1} L_z = L_y L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}$.

Proof. Since $L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}(z) = y$, then $x_n \triangleright^{\epsilon_n} (x_{n-1} \triangleright^{\epsilon_{n-1}} (\cdots \triangleright^{\epsilon_2} (x_1 \triangleright^{\epsilon_1} z) \cdots)) = y$. From Proposition 1.2, it follows that

$$\begin{aligned}
L_y &= L_{x_n \triangleright^{\epsilon_n} (x_{n-1} \triangleright^{\epsilon_{n-1}} (\cdots \triangleright^{\epsilon_2} (x_1 \triangleright^{\epsilon_1} z) \cdots))} \\
&= L_{x_n}^{\epsilon_n} [L_{x_{n-1} \triangleright^{\epsilon_{n-1}} (\cdots \triangleright^{\epsilon_2} (x_1 \triangleright^{\epsilon_1} z) \cdots)}] L_{x_n}^{-\epsilon_n} \\
&= L_{x_n}^{\epsilon_n} [L_{x_{n-1}}^{\epsilon_{n-1}} [L_{x_{n-2} \triangleright^{\epsilon_{n-2}} (\cdots \triangleright^{\epsilon_2} (x_1 \triangleright^{\epsilon_1} z) \cdots)}] L_{x_{n-1}}^{-\epsilon_{n-1}}] L_{x_n}^{-\epsilon_n} \\
&\ \vdots \\
&= L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1} L_z L_{x_1}^{-\epsilon_1} \cdots L_{n-1}^{-\epsilon_{n-1}} L_{x_n}^{-\epsilon_n}.
\end{aligned}$$

Hence, $L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1} L_z = L_y L_{x_n}^{\epsilon_n} L_{n-1}^{\epsilon_{n-1}} \cdots L_{x_1}^{\epsilon_1}$.

■

Lemma 1.4. *Let X be a rack and $n \in \mathbb{N}$, then we have the following relations in the associated group $As(X)$,*

(1) $g_x^n g_y = g_{x \rhd^n y} g_x^n$, $\forall x, y \in X$.

(2) $g_x g_y^n = g_{x \rhd y}^n g_x$, $\forall x, y \in X$.

Proof. We use induction on n at the same time for the numerals (1) and (2). Let $n = 1$, since in the associated group, we have the relation $g_{x \rhd y} = g_x g_y g_x^{-1}$, for all $x, y \in X$, then $g_x g_y = g_{x \rhd y} g_x$. Therefore, the equalities (1) and (2) are satisfied for $n = 1$. Now,

(1) Suppose that $g_x^n y = g_{x \rhd^n y} g_x^n$, we have to prove that $g_x^{n+1} y = g_{x \rhd^{n+1} y} g_x^{n+1}$. In fact, from the definition of $As(X)$, we have

$$\begin{aligned} g_{x \rhd^{n+1} y} g_x^{n+1} &= g_{x \rhd (x \rhd^n y)} g_x^{n+1} \\ &= g_x g_{x \rhd^n y} g_x^{-1} g_x^{n+1} \\ &= g_x (g_x^n g_y g_x^{-n}) g_x^{-1} g_x^{n+1} \\ &= g_x^{n+1} g_y. \end{aligned}$$

Getting in this way what we want to prove.

(2) Suppose that $g_x g_y^n = g_{x \rhd y}^n g_x$, we have to prove that $g_x g_y^{n+1} = g_{x \rhd y}^{n+1} g_x$. From definition of $As(X)$ we have

$$\begin{aligned} g_{x \rhd y}^{n+1} x &= g_{x \rhd y} g_{x \rhd y}^n g_x \\ &= g_{x \rhd y} (g_x g_y^n g_x^{-1}) g_x \\ &= (g_x g_y g_x^{-1})(g_x g_y^n g_x^{-1}) g_x \\ &= g_x g_y^{n+1}. \end{aligned}$$

Thus, we get the proof.

■

Proposition 1.3. *Let X be a rack and $x \in X$ such that L_x has finite order n, then in the associated group $As(X)$, we have that $g_x^n \in Z(As(X))$, where $Z(As(X))$ is the center of the group $As(X)$.*

Proof. Suppose that $L_x^n = id$. Note that, for every $y \in X$ we have $x \rhd^n y = L_x^n(y) = y$. Therefore, from Lemma 1.4 (1),

$$g_x^n g_y = g_{x \rhd^n y} g_x^n = g_y g_x^n.$$

Hence, the element g_x^n commutes with all the generators of $As(X)$, it follows that $g_x^n \in Z(As(X))$.

■

Theorem 1.3. *Let X be a finite connected rack. Then for every $x \in X$, the permutations $L_x \in Inn(x)$ have the same order. Furthermore, if n is the order of all the permutations L_x then, in the associated group $As(X)$, we have the relation $g_x^n = g_y^n$ for all $x, y \in X$.*

Proof. Since X is finite, then the symmetric group $Sym(X)$ of X is finite. From the definition of rack, $L_x \in Sym(X)$, thus they have finite order.
Since X is connected, then for every $x, y \in X$ there exits $\phi \in Inn(X)$ such that $\phi(x) = y$. From the definition of $Inn(X)$, the function ϕ is the form $\phi = L_{x_r}^{\epsilon_r} \cdots L_{x_1}^{\epsilon_1}$, where $x_1, .., x_r \in X$ and $\epsilon_i = \pm 1$. Then, $L_{x_r}^{\epsilon_r} \cdots L_{x_1}^{\epsilon_1}(x) = y$. From Lemma 1.3 , it follows that $L_{x_r}^{\epsilon_r} L_{r-1}^{\epsilon_{r-1}} \cdots L_{x_1}^{\epsilon_1} L_x = L_y L_{x_r}^{\epsilon_r} L_{r-1}^{\epsilon_{r-1}} \cdots L_{x_1}^{\epsilon_1}$, and so, $L_x = \phi^{-1} L_y \phi$. Thereby, all the permutations are conjugate elements. Thus, they have the same order.
Suppose that n is the order of all permutations L_x. Then for $x, y \in X$ we have

$$
\begin{aligned}
g_y &= g_{L_{x_r}^{\epsilon_r} \cdots L_{x_1}^{\epsilon_1}(x)} \\
&= g_{x_r \rhd^{\epsilon_r} (x_{r-1} \rhd^{\epsilon_{r-1}} (\cdots \rhd^{\epsilon_2} (x_1 \rhd^{\epsilon_1} x) \cdots))} \\
&= g_{x_r}^{\epsilon_r} g_{x_{r-1} \rhd^{\epsilon_{r-1}} (\cdots \rhd^{\epsilon_2} (x_1 \rhd^{\epsilon_1} x) \cdots)} g_{x_r}^{-\epsilon_r} \\
&= g_{x_r}^{\epsilon_r} g_{x_{r-1}}^{\epsilon_{r-1}} g_{x_{r-2} \rhd^{\epsilon_{r-2}} (\cdots \rhd^{\epsilon_2} (x_1 \rhd^{\epsilon_1} x) \cdots)} g_{x_{r-1}}^{-\epsilon_{r-1}} g_{x_r}^{-\epsilon_r} \\
&\vdots \\
&= (g_{x_r}^{\epsilon_r} \cdots g_{x_1}^{\epsilon_1}) g_x (g_{x_1}^{-\epsilon_1} \cdots g_{x_r}^{-\epsilon_r}).
\end{aligned}
$$

Let $\omega := g_{x_r}^{\epsilon_r} \cdots g_{x_1}^{\epsilon_1}$ then we have $g_y = \omega g_x \omega^{-1}$. From Lemma 1.4 (2),

$$
\begin{aligned}
g_{x \rhd y}^n g_x &= g_x g_y^n \\
&= g_x (\omega g_x \omega^{-1})^n \\
&= g_x (\omega g_x \omega^{-1})(\omega g_x \omega^{-1})(\omega g_x \omega^{-1})^{n-2} \\
&= g_x (\omega g_x^2 \omega^{-1})(\omega g_x \omega^{-1})^{n-2} \\
&\vdots \\
&= g_x (\omega g_x^n \omega^{-1}).
\end{aligned}
$$

From Proposition 1.3, the elements g_x^n, g_y^n and $g_{x \rhd y}^n$ belongs to $Z(As(X))$, then $g_{x \rhd y}^n g_x = g_x g_{x \rhd y}^n$. On the other hand, $g_x(\omega g_x^n \omega^{-1}) = g_x g_x^n$, therefore, $g_x^n = g_{x \rhd y}^n$. Again, from Lemma 1.4 (2),

$$
g_x^n = g_{x \rhd y}^n = g_x g_y^n g_x^{-1}.
$$

So, $g_x^n = g_y^n$.

∎

Let X be a finite and connected rack, let $x_0 \in X$ and n be the order of the function L_{x_0}. Observe that, from the previous theorem and Proposition 1.3, the group $\langle g_{x_0}^n \rangle \subset As(X)$ generated by $g_{x_0}^n$, is a normal subgroup. Therefore, we can consider the quotient group $As(X)/\langle g_{x_0}^n \rangle$.

Definition 1.6. Let X be a finite and connected rack. The quotient group $As(X)/\langle g_{x_0}^n \rangle$, denoted by G_X, is called the ***finite enveloping group***.

Theorem 1.4. *Let X be finite and connected rack, then the group G_X is finite.*

Proof. Let $X=\{x_0,...,x_{k-1}\}$ be a connected rack. Suppose that $n\in\mathbb{N}$ is the order of L_{x_i} for all $i\in\{0,...,k-1\}$. From Theorem 1.3, for every $i\in\{0,...,k-1\}$, we have that $g_{x_i}^n\langle g_{x_0}^n\rangle=\langle g_{x_0}^n\rangle$, thus, $g_{x_i}^{-1}\langle g_{x_0}^n\rangle=g_{x_i}^{n-1}\langle g_{x_0}^n\rangle$.
We claim that every word $\omega\in G_X$ is of the form $\omega=g_{x_{i_1}}^{u_1}g_{x_{i_2}}^{u_2}\cdots g_{x_{i_m}}^{u_m}\langle g_{x_0}^n\rangle$ where $m\leq k$, $u_t\in\{0,1,...,n-1\}$, $x_{i_t}\in X$ for all $t\in\{1,...,m\}$ and $g_{x_{i_r}}\neq g_{x_{i_s}}$, for every $r\neq s$. In fact, let $\omega\in G_X$ thus, from the definition of G_X, $\omega=g_{x_{j_1}}^{e_1}g_{x_{j_2}}^{e_2}\cdots g_{x_{j_{\hat{m}}}}^{e_{\hat{m}}}\langle g_{x_0}^n\rangle$, where $e_t\in\{0,1,...,n-1\}$. Suppose that there exists $r<s$ such that $g_{x_{j_r}}=g_{x_{j_s}}$ and for every $t\in\{r+1,...,s-1\}$ it satisfies that $g_{x_{j_t}}\neq g_{x_{j_s}}$. Therefore, we have

$$\omega=(g_{x_{j_1}}^{e_1}g_{x_{j_2}}^{e_2}\cdots g_{x_{j_{r-1}}}^{e_{r-1}}\langle g_{x_0}^n\rangle)(g_{x_{j_r}}^{e_r}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s}\langle g_{x_0}^n\rangle)(g_{x_{j_{s+1}}}^{e_{s+1}}g_{x_{j_{s+2}}}^{e_{s+2}}\cdots g_{x_{j_{\bar{m}}}}^{e_{\bar{m}}}\langle g_{x_0}^n\rangle).$$

First, we prove that in the associated group $As(X)$ we have

$$g_{x_{j_r}}^{e_r}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{s}=g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r}.$$

We use induction on e_r. Let $e_r=1$ then, by Lemma 1.4 (2), we have that $g_{x_{j_r}}g_{x_{j_t}}^{e_t}=g_{x_{j_r}\triangleright x_{j_t}}^{e_t}g_{x_{j_r}}$ for every $t\in\{r+1,...,s-1\}$, it follows that,

$$\begin{aligned}
g_{x_{j_r}}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s} &= g_{x_{j_r}\triangleright x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}}g_{x_{j_{r+2}}}^{e_{r+2}}\cdots g_{x_{j_s}}^{e_s}\\
&= g_{x_{j_r}\triangleright x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}\triangleright x_{j_{r+2}}}^{e_{r+2}}g_{x_{j_r}}\cdots g_{x_{j_s}}^{e_s}\\
&\vdots\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}}g_{x_r}g_{x_{j_s}}^{e_s}\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+1}.
\end{aligned}$$

So, the result follows for $e_r=1$. Now, suppose that

$$g_{x_{j_r}}^{e_r-1}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{s}=g_{x_{j_r}\triangleright^{e_r-1}x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_r}\triangleright^{e_r-1}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r-1}.$$

Then, from Lemma 1.4 (2),

$$\begin{aligned}
g_{x_{j_r}}^{e_r}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{s} &= g_{x_{j_r}}[g_{x_{j_r}}^{e_r-1}g_{x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{s}]\\
&= g_{x_{j_r}}[g_{x_{j_r}\triangleright^{e_r-1}x_{j_{r+1}}}^{e_{r+1}}\cdots g_{x_{j_r}\triangleright^{e_r-1}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r-1}]\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}}g_{x_{j_r}\triangleright^{e_r-1}x_{j_{r+2}}}^{e_{r+2}}\cdots g_{x_{j_r}\triangleright^{e_r-1}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r-1}\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}\triangleright^{e_r}x_{j_{r+2}}}^{e_{r+2}}g_{x_{j_r}}\cdots g_{x_{j_r}\triangleright^{e_r-1}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r-1}\\
&\vdots\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}\triangleright^{e_r}x_{j_{r+2}}}^{e_{r+2}}\cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_r}}g_{x_{j_s}}^{e_s+e_r-1}\\
&= g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}}g_{x_{j_r}\triangleright^{e_r}x_{j_{r+2}}}^{e_{r+2}}\cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}}g_{x_{j_s}}^{e_s+e_r}.
\end{aligned}$$

And the result follows. Now, observe that $g_{x_{j_r}\triangleright^{e_r}x_{j_t}}\neq g_{x_{j_r}}$, for every $t\in\{r+1,...,s-1\}$. In effect, suppose that $g_{x_{j_r}\triangleright^{e_r}x_{j_t}} = g_{x_{j_r}}$ for some $t \in \{r+1, ..., s-1\}$. From the definition of $As(X)$,

$$\begin{aligned}
g_{x_{j_r}} &= g_{x_{j_r}\triangleright^{e_r}x_{j_t}} \\
&= g_{x_{j_r}\triangleright(x_{j_r}\triangleright^{e_r-1}x_{j_t})} \\
&= g_{x_{j_r}} g_{x_{j_r}\triangleright^{e_r-1}x_{j_t}} g_{x_{j_r}}^{-1} \\
&= g_{x_{j_r}} g_{x_{j_r}\triangleright(x_{j_r}\triangleright^{e_r-2}x_{j_t})} g_{x_{j_r}}^{-1} \\
&= g_{x_{j_r}}^2 g_{x_{j_r}\triangleright^{e_r-2}x_{j_t}} g_{x_{j_r}}^{-2} \\
&\vdots \\
&= g_{x_{j_r}}^{e_r-1} g_{x_{j_r}\triangleright x_{j_t}} g_{x_{j_r}}^{-e_r+1} \\
&= g_{x_{j_r}}^{e_r-1} (g_{x_{j_r}} g_{x_{j_t}} g_{x_{j_r}}^{-1}) g_{x_{j_r}}^{-e_r+1} \\
&= g_{x_{j_r}}^{e_r} g_{x_{j_t}} g_{x_{j_r}}^{-e_r}.
\end{aligned}$$

Therefore, we have $g_{x_{j_t}} = g_{x_{j_r}}^{e_r} g_{x_{j_r}} g_{x_{j_r}}^{-e_r} = g_{x_{j_r}}^{e_r-e_r+1} = g_{x_{j_r}} = g_{x_{j_s}}$, which contradicts the assumption that for every $t \in \{r+1, ..., s-1\}$, $g_{x_{j_t}} \neq g_{x_{j_s}}$. Thus, $g_{x_{j_r}\triangleright^{e_r}x_{j_t}} \neq g_{x_{j_r}}$ for every $t \in \{r+1, ..., s-1\}$. From the above, we can write the word ω as

$$\begin{aligned}
\omega &= (g_{x_{j_1}}^{e_1} g_{x_{j_2}}^{e_2} \cdots g_{x_{j_{r-1}}}^{e_{r-1}} \langle g_{x_0}^n \rangle)(g_{x_{j_r}}^{e_r} g_{x_{j_{r+1}}}^{e_{r+1}} \cdots g_{x_{j_{s-1}}}^{e_{s-1}} g_{x_{j_s}}^{e_s} \langle g_{x_0}^n \rangle)(g_{x_{j_{s+1}}}^{e_{s+1}} g_{x_{j_{s+2}}}^{e_{s+2}} \cdots g_{x_{j_{\bar{m}}}}^{e_{\bar{m}}} \langle g_{x_0}^n \rangle) \\
&= (g_{x_{j_1}}^{e_1} g_{x_{j_2}}^{e_2} \cdots g_{x_{j_{r-1}}}^{e_{r-1}} \langle g_{x_0}^n \rangle)(g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}} \cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}} g_{x_{j_s}}^{e_s+e_r} \langle g_{x_0}^n \rangle)(g_{x_{j_{s+1}}}^{e_{s+1}} g_{x_{j_{s+2}}}^{e_{s+2}} \cdots g_{x_{j_{\bar{m}}}}^{e_{\bar{m}}} \langle g_{x_0}^n \rangle) \\
&= g_{x_{j_1}}^{e_1} g_{x_{j_2}}^{e_2} \cdots g_{x_{j_{r-1}}}^{e_{r-1}} g_{x_{j_r}\triangleright^{e_r}x_{j_{r+1}}}^{e_{r+1}} \cdots g_{x_{j_r}\triangleright^{e_r}x_{j_{s-1}}}^{e_{s-1}} g_{x_{j_s}}^{e_s+e_r} g_{x_{j_{s+1}}}^{e_{s+1}} g_{x_{j_{s+2}}}^{e_{s+2}} \cdots g_{x_{j_{\bar{m}}}}^{e_{\bar{m}}} \langle g_{x_0}^n \rangle \\
&= g_{x_{i_1}}^{u_1} \cdots g_{x_{i_m}}^{u_m} \langle g_{x_0}^n \rangle.
\end{aligned}$$

From where $g_{x_{i_r}} \neq g_{x_{i_s}}$, for every $r \neq s$. If $m > k$. Then, we would have repeated occurrences and we can apply the same process shown above to reduce the word. Therefore, every word $\omega \in G_X$ is of the form $\omega = g_{x_{i_1}}^{u_1} \cdots g_{x_{i_m}}^{u_m} \langle g_{x_0}^n \rangle$, where $m \leq k$, $u_t \in \{0, 1, ..., n-1\}$, $x_{i_t} \in X$ for all $t \in \{1, ..., m\}$ and $g_{x_{i_r}} \neq g_{x_{i_s}}$ for every $r \neq s$. It follows that G_X is finite.

■

Example 1.12. *Let $X = \{1, 2, 3\}$ be the rack of Example 1.11. The table of this rack is*

$\triangleright$	***1***	***2***	***3***
1	*2*	*3*	*1*
2	*2*	*3*	*1*
3	*2*	*3*	*1*

Note that this rack is connected. Indeed, we have that $L_1(1) = 2$, $L_1^2(1) = 3$ and $L_1^3(1) = 1$, thus $\mathscr{O}_1 = X$. From the table we have that $L_i^3 = id$ for all $i \in X$. Since $g_1 = g_2 = g_3$ (see Example 1.11) then $g_1\langle g_1^3 \rangle = g_2\langle g_1^3 \rangle = g_3\langle g_1^3 \rangle$. Therefore $G_X = span\{g_1\langle g_1^3 \rangle\} \cong \mathbb{Z}_3$.

1.3 The Permutation Quandle

In general, it is not easy to describe the finite enveloping group of a rack. Vendramin, in [18], computed (with a GAP Package) the finite enveloping group of connected quandles of order less

than 36. Based on his results, we have identified the finite enveloping group of a quandle that we refer to as the *permutation quandle*.
It is well known that the set of all transpositions of the symmetric group $\mathbb{S}_n$, forms a conjugacy class of $\mathbb{S}_n$. Thus, from Lemma 1.1, that set is a subquandle of the quandle $Conj(\mathbb{S}_n)$.

Definition 1.7. Consider the conjugacy quandle $Conj(\mathbb{S}_n)$ of the symmetric group $\mathbb{S}_n$ with $n \geq 3$. We define the ***permutation quandle***, denoted by $\mathbb{P}_n$, as the set of all transpositions of $\mathbb{S}_n$ with conjugation operation. That is, the permutation quandle $\mathbb{P}_n$ is the set

$$\mathbb{P}_n := \{(i\ j) \mid (i\ j) \in \mathbb{S}_n\},$$

with the operation $(i\ j) \rhd (k\ r) = (i\ j)(k\ r)(i\ j)^{-1} = (i\ j)(k\ r)(i\ j)$, where $(i\ j)$ denotes the transposition that interchanges i and j.

As we mentioned in the paragraph previous to Definition 1.7, $\mathbb{P}_n$ is closed under the conjugation operation $(i\ j) \rhd (k\ r) = (i\ j)(k\ r)(i\ j)$. Thus, $(\mathbb{P}_n, \rhd)$ is a subquandle of $Conj(\mathbb{S}_n)$. The name of the permutation quandle will be justified later on. For now, note that for every $(i\ j), (k\ r) \in \mathbb{P}_n$, we have

$$\begin{aligned} (i\ j) \rhd [(i\ j) \rhd (k\ r)] &= (i\ j) \rhd [(i\ j)(k\ r)(i\ j)] \\ &= (i\ j)[(i\ j)(k\ r)(i\ j)](i\ j) \\ &= (k\ r). \end{aligned}$$

So, it is an *involutive* quandle.

Example 1.13. *Consider the permutation quandle* $\mathbb{P}_3 = \{(i\ j) \mid (i\ j) \in \mathbb{S}_3\}$*. That is,* $\mathbb{P}_3 = \{(1\ 2), (1\ 3), (2\ 3)\}$*. Let us do the table of this quandle.*

$$\begin{aligned} (1\ 2) \rhd (2\ 3) = (1\ 2)(2\ 3)(1\ 2) = (1\ 3) \\ (1\ 2) \rhd (1\ 3) = (1\ 2)(1\ 3)(1\ 2) = (2\ 3) \\ (1\ 2) \rhd (1\ 2) = (1\ 2)(1\ 2)(1\ 2) = (1\ 2) \end{aligned}$$

$$\begin{aligned} (1\ 3) \rhd (2\ 3) = (1\ 3)(2\ 3)(1\ 3) = (1\ 2) \\ (1\ 3) \rhd (1\ 3) = (1\ 3)(1\ 3)(1\ 3) = (1\ 3) \\ (1\ 3) \rhd (1\ 2) = (1\ 3)(1\ 2)(1\ 3) = (2\ 3) \end{aligned}$$

$$\begin{aligned} (2\ 3) \rhd (2\ 3) = (2\ 3)(2\ 3)(2\ 3) = (2\ 3) \\ (2\ 3) \rhd (1\ 3) = (2\ 3)(1\ 3)(2\ 3) = (1\ 2) \\ (2\ 3) \rhd (1\ 2) = (2\ 3)(1\ 2)(2\ 3) = (1\ 3) \end{aligned}$$

Then,

$\rhd$	***(2 3)***	***(1 3)***	***(1 2)***
(2 3)	*(2 3)*	*(1 2)*	*(1 3)*
(1 3)	*(1 2)*	*(1 3)*	*(2 3)*
(1 2)	*(1 3)*	*(2 3)*	*(1 2)*

$\blacktriangle$

Lemma 1.5. *The permutation quandle $\mathbb{P}_n$ is connected.*

Proof. Let $(i\ j), (l\ t) \in \mathbb{P}_n$. Since $(i\ j) = (j\ i)$, for all $(i\ j) \in \mathbb{P}_n$, then without loss of generality, we have three cases

(I) If $(i\ j) = (l\ t)$ then $L_{(i\ j)}[(l\ t)] = (i\ j) \triangleright (l\ t) = (i\ j)(l\ t)(i\ j) = (i\ j)$.

(II) Suppose that $i \neq l$ and $j = t$, then $L_{(i\ l)}[(l\ t)] = (i\ l) \triangleright (l\ t) = (i\ l)(l\ t)(i\ l) = (i\ t) = (i\ j)$.

(III) If $i \neq l$ and $j \neq t$ then $L_{(j\ t)}L_{(i\ l)}[(l\ t)] = (j\ t) \triangleright [(i\ l) \triangleright (l\ t)] = (j\ t) \triangleright [(i\ l)(l\ t)(i\ l)] = (j\ t) \triangleright (i\ t) = (j\ t)(i\ t)(j\ t) = (i\ j)$.

Therefore the permutation quandle $\mathbb{P}_n$ is connected.

$\blacksquare$

The next proposition provides us a complete description of the finite enveloping group $G_{\mathbb{P}_n}$ of the permutation quandle $\mathbb{P}_n$. Specifically, we prove that $G_{\mathbb{P}_n} \cong \mathbb{S}_n$, which justifies the name "*permutation quandle*".

Proposition 1.4. *Let $\mathbb{P}_n$ be the permutation quandle then its finite enveloping group $G_{\mathbb{P}_n}$ is isomorphic to the symmetric group $\mathbb{S}_n$.*

Proof. Consider the function $\psi : \mathbb{P}_n \longrightarrow Conj(\mathbb{S}_n)$, defined by $\psi[(i\ j)] := (i\ j)$ for all $(i\ j) \in \mathbb{P}_n$. Since $\mathbb{P}_n$ is a subquandle of $Conj(\mathbb{S}_n)$, then for every $(i\ j), (k\ r) \in \mathbb{P}_n$ we have that $(i\ j) \triangleright (k\ r) \in \mathbb{P}_n$. That is, there exists $(l\ t) \in \mathbb{P}_n \subset \mathbb{S}_n$, such that $(i\ j) \triangleright (k\ r) = (l\ t)$, then

$$\begin{aligned}\psi[(i\ j) \triangleright (k\ r)] &= \psi[(l\ t)]\\ &= (l\ t)\\ &= (i\ j) \triangleright (k\ r)\\ &= \psi[(i\ j)] \triangleright \psi[(k\ r)].\end{aligned}$$

Which implies that ψ is a quandle homomorphism. Now, from Theorem 1.2 there exists a group homomorphism $\hat{\psi} : As(\mathbb{P}_n) \longrightarrow \mathbb{S}_n$ such that $\hat{\psi}(g_{(ij)}) = \psi[(i\ j)]$. Since the permutation quandle is involutive and connected, then $L^2_{(i\ j)} = id$, for all $(i\ j) \in \mathbb{P}_n$. Which implies that $g^2_{(ij)} = g^2_{(kr)}$, for all $(i\ j), (k\ r) \in \mathbb{P}_n$. Observe that,

$$\hat{\psi}(g^2_{(12)}) = \hat{\psi}^2(g_{(12)}) = (1\ 2)^2 = 1_{\mathbb{S}_n}.$$

Therefore, $\langle g^2_{(12)} \rangle \subset ker(\hat{\psi})$. Thus, $\hat{\psi}$ induces a group homomorphism $\bar{\psi} : G_{\mathbb{P}_n} \longrightarrow \mathbb{S}_n$ such that $\bar{\psi}(g_{(ij)}\langle g^2_{(12)}\rangle) = \hat{\psi}(g_{(ij)}) = \psi[(i\ j)] = (i\ j)$. Let us see that $\bar{\psi}$ is bijective. In fact, let $\sigma \in \mathbb{S}_n$, since the set of all transpositions is a generating set of $\mathbb{S}_n$ then there exist $(i_1\ j_1), .., (i_k\ j_k) \in \mathbb{S}_n$ such that $\sigma = (i_1\ j_1)(i_2\ j_2) \cdots (i_k\ j_k)$. Therefore,

$$\begin{aligned}\sigma &= (i_1\ j_1)(i_2\ j_2) \cdots (i_k\ j_k)\\ &= \bar{\psi}(g_{(i_1 j_1)}\langle g^2_{(12)}\rangle)\bar{\psi}(g_{(i_2 j_2)}\langle g^2_{(12)}\rangle) \cdots \bar{\psi}(g_{(i_k j_k)}\langle g^2_{(12)}\rangle)\\ &= \bar{\psi}(g_{(i_1 j_1)}g_{(i_2 j_2)} \cdots g_{(i_k j_k)}\langle g^2_{(12)}\rangle).\end{aligned}$$

Hence, $\bar{\psi}$ is surjective.
Now, we use the presentation of the symmetric group $\mathbb{S}_n = \langle \sigma_i, ..., \sigma_{n-1} \mid \sigma_i^2 = 1, (\sigma_i\sigma_{i+1})^3 = 1, (\sigma_i\sigma_j)^2 = 1, |j-1| > 1 \rangle$, where $\sigma_i = (i \;\; i+1)$ for all $i \in \{1, .., n-1\}$ to prove the injectivity.
Let $(k\ r) \in \mathbb{P}_n$, since $(k\ r) = (r\ k)$ without loss of generality we suppose that $k < r$. Therefore, $r - k > 0$. If $r - k = 1$, then $r = k+1$ and we have that $(k\ r) = (k \;\; k+1)$. If $r - k > 1$, note that,

$$\begin{aligned}(k\ r) &= (k \;\; k+1)(k+1 \;\; r)(k \;\; k+1)\\ &= (k \;\; k+1) \triangleright (k+1 \;\; r),\end{aligned}$$

$$\begin{aligned}(k+1 \;\; r) &= (k+1 \;\; k+2)(k+2 \;\; r)(k+1 \;\; k+2)\\ &= (k+1 \;\; k+2) \triangleright (k+2 \;\; r),\\ &\vdots\\ (r-2 \;\; r) &= (r-2 \;\; r-1)(r-1 \;\; r)(r-2 \;\; r-1)\\ &= (r-2 \;\; r-1) \triangleright (r-1 \;\; r).\end{aligned}$$

Therefore,

$$(k\ r) = (k \;\; k+1) \triangleright [(k+1 \;\; k+2) \triangleright [\cdots \triangleright [(r-2 \;\; r-1) \triangleright (r-1 \;\; r)] \cdots].$$

From where, it follows that

$$\begin{aligned}g_{(k\ r)} &= g_{(k\ k+1)\triangleright[(k+1\ k+2)\triangleright[\cdots\triangleright[(r-2\ r-1)\triangleright(r-1\ r)]\cdots]}\\ &= g_{(k\ k+1)}g_{(k+1\ k+2)\triangleright[\cdots\triangleright[(r-2\ r-1)\triangleright(r-1\ r)]\cdots]}g^{-1}_{(k\ k+1)}\\ &= g_{(k\ k+1)}g_{(k+1\ k+2)}g_{(k+2\ k+3)\triangleright[\cdots\triangleright[(r-2\ r-1)\triangleright(r-1\ r)]\cdots]}g^{-1}_{(k+1\ k+2)}g^{-1}_{(k\ k+1)}\\ &\vdots\\ &= g_{(k\ k+1)}g_{(k+1\ k+2)}\cdots g_{(r-2\ r-1)}g_{(r-1\ r)}g^{-1}_{(r-2\ r-1)}\cdots g^{-1}_{(k+1\ k+2)}g^{-1}_{(k\ k+1)}.\end{aligned}$$

Therefore, the set of elements $\{g_{(i\ i+1)} \in As(\mathbb{P}_n) \mid i = 1, ..., n-1\}$ is a generating set of $As(\mathbb{P}_n)$ and thus, the set $\{g_{(i\ i+1)}\langle g^2_{(12)}\rangle \in G_{\mathbb{P}_n} \mid i = 1, ..., n-1\}$ is a generating set of $G_{\mathbb{P}_n}$. Now, note that for every $i = 1, ..., n-1$, we have that $(i\ i+1) \triangleright (i+1\ i+2) = (i\ i+1)(i+1\ i+2)(i\ i+1) = (i\ i+2)$ and $(i+1\ i+2) \triangleright (i\ i+1) = (i+1\ i+2)(i\ i+1)(i+1\ i+2) = (i\ i+2)$. Therefore,

$$\begin{aligned}(g_{(i\ i+1)}g_{(i+1\ i+2)})^3 &= g_{(i\ i+1)}g_{(i+1\ i+2)}g_{(i\ i+1)}g_{(i+1\ i+2)}g_{(i\ i+1)}g_{(i+1\ i+2)}\\ &= g_{(i\ i+1)\triangleright(i+1\ i+2)}g_{(i+1\ i+2)\triangleright(i\ i+1)}\\ &= g_{(i\ i+2)}g_{(i\ i+2)} = g^2_{(i\ i+2)}.\end{aligned}$$

It follows that $(g_{(i\ i+1)}g_{(i+1\ i+2)}\langle g^2_{(12)}\rangle)^3 = g^2_{(i\ i+2)}\langle g^2_{(12)}\rangle = \langle g^2_{(12)}\rangle = 1_{G_{\mathbb{P}_n}}$.
Let $i, j \in \{1, ..., n-1\}$ such that $|j-i| > 1$. We want to prove that $(g_{(i\ i+1)}g_{(j\ j+1)}\langle g^2_{(12)}\rangle)^2 = 1_{G_{\mathbb{P}_n}}$. Since $|j-i| > 1$ we have two cases:

- If $j - i > 1$, then $j > 1 + i$ then $(i \;\; i+1) \triangleright (j \;\; j+1) = (i \;\; i+1)(j \;\; j+1)(i \;\; i+1) = (j \;\; j+1)$, therefore

$$\begin{aligned}
(g_{(i\ \ i+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle)^2 &= (g_{(i\ \ i+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle)(g_{(i\ \ i+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle)\\
&= (g_{(i\ \ i+1)}g_{(j\ \ j+1)}g_{(i\ \ i+1)})g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle\\
&= g_{(i\ \ i+1)\rhd(j\ \ j+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle\\
&= g_{(j\ \ j+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle\\
&= \langle g_{(12)}^2\rangle = 1_{G_{\mathbb{P}_n}}.
\end{aligned}$$

- If $j-i<-1$, then $j<i-1$ then $(i\ \ i+1)\rhd(j\ \ j+1)=(i\ \ i+1)(j\ \ j+1)(i\ \ i+1)=(j\ \ j+1)$, therefore

$$(g_{(i\ \ i+1)}g_{(j\ \ j+1)}\langle g_{(12)}^2\rangle)^2 = \langle g_{(12)}^2\rangle = 1_{G_{\mathbb{P}_n}}.$$

Then, the group $G_{\mathbb{P}_n}$ satisfies all the relations of the presentation of $\mathbb{S}_n$. It follows that, there exists a group epimorphism $\phi:\mathbb{S}_n\longrightarrow G_{\mathbb{P}_n}$, which implies that $|G_{\mathbb{P}_n}|\leq n!$. Besides, we have that the homomorphism $\bar{\psi}:G_{\mathbb{P}_n}\longrightarrow\mathbb{S}_n$ is surjective, then $|\mathbb{S}_n|\leq|G_{\mathbb{P}_n}|$. Because $|G_{\mathbb{P}_n}|\leq n!$, then $|G_{\mathbb{P}_n}|=n!$. Thus, $\bar{\psi}$ must be bijective, i.e, the map

$$\begin{aligned}
\bar{\psi}: G_{\mathbb{P}_n} &\longrightarrow \mathbb{S}_n\\
g_{(i\ j)}\langle g_{(12)}^2\rangle &\longmapsto (i\ j),
\end{aligned}$$

is a group isomorphism.

■

1.4 Finitely stable racks

Elhamdadi and Moutuou in [4] defined a new class of racks, called finitely stable, in an attempt to capture the notion of identity and center in the category of racks and quandles. We study this racks and complete the details of some examples and proofs.

Notation 1.5. If $u_1,u_2,...,u_n$ are elements in a rack X, then for a $x\in X$ we will write

$$(u_i)_{i=1}^n\rhd x = u_n\rhd(...(u_3\rhd(u_2\rhd(u_1\rhd x)))...).$$

Definition 1.8. A *stabilizer* u in a rack X is an element such that

$$u\rhd x = x \text{ for all } x\in X.$$

Note that, a stabilizer u in a rack X satisfies that $L_u(x)=x$ for all $x\in X$, then L_u is the identity function in the group $Inn(X)$.

Example 1.14. *Let G be a group and u a stabilizer in $Conj(G)$, we have*

$$\begin{aligned}
g &= u\rhd g\\
&= ugu^{-1}.
\end{aligned}$$

Hence, $ug=gu$. The result follows for all $g\in G$. Thus u belongs to the center $Z(G)$ of G . Then, the stabilizers of $Conj(G)$ are the elements of $Z(G)$.

The previous example shows us that the definition of stabilizer allows us to capture the notion of the center of a rack. Elhamdadi and Moutuou in [4] take the property that a rack has a stabilizer and they weaken it by the following definition:

Definition 1.9. Let X be a rack

1. A **stabilizing family of order** n for X is a finite set $\{u_1, u_2, ..., u_n\} \subset X$ such that

$$(u_i)_{i=1}^n \rhd x = x, \ \forall x \in X.$$

   in other words, $L_{u_n} L_{u_{n-1}} ... L_{u_1} = id$.

2. The n**-center** X, denoted by $S^n(X)$, is the collection of all stabilizing families of order n for X.

3. The collection

$$S(X) := \underset{n\in\mathbb{N}}{\cup} S^n(X),$$

   of all stabilizing families for X is called the **center of** X.

Lemma 1.6. *Let X be a rack, $\{u_i\}_{i=1}^n \in S^n(X)$ and σ an element of the symmetric group $\mathbb{S}_n$. If $\sigma \in \langle (2\ 3\ 4\ ...\ n\ 1) \rangle$, then $\{u_{\sigma(i)}\}_{i=1}^n \in S^n(X)$.*

Proof. Let $\sigma \in \langle (2\ 3\ 4...\ n\ 1) \rangle$, then $\sigma = (2\ 3\ 4...\ n\ 1)^k$, where $1 \leq k \leq n$. We proceed by induction on k. Suppose that $k = 1$, then $\sigma = (2\,3\,4...\,n\,1)$. Let $n \in \mathbb{N}$ and $\{u_i\}_{i=1}^n$ be a stabilizing family of order n for X. From the definition of stabilizing family, $L_{u_n} L_{u_{n-1}} ... L_{u_1} = id$. Since L_{u_1} is invertible, it follows that $L_{u_n} L_{u_{n-1}} ... L_{u_2} = L_{u_1}^{-1}$. Thus, we have

$$\begin{aligned} id &= L_{u_1} L_{u_1}^{-1} \\ &= L_{u_1}(L_{u_n} ... L_{u_3} L_{u_2}) \\ &= L_{u_{\sigma(n)}} L_{u_{\sigma(n-1)}} ... L_{u_{\sigma(2)}} L_{u_{\sigma(1)}}. \end{aligned}$$

Therefore, $\{u_{\sigma(i)}\}_{i=1}^n \in S^n(X)$. Suppose that the result holds for some $k < n$. Let $\sigma = (2\,3\,4...\,n\,1)^{k+1}$ and let $\tau = (2\,3\,4...\,n\,1)^k$. Note that, $\sigma = (2\,3\,4...\,n\,1)^k (2\,3\,4...\,n\,1) = \tau(2\,3\,4...\,n\,1)$. Thereby, $\sigma(n) = \tau(1)$ and $\sigma(i) = \tau(i+1)$, for every $i \in \{1, 2, ..., n-1\}$. Let $\{u_i\}_{i=1}^n$ be a stabilizing family of order n for X. Since the result holds for k, then $L_{u_{\tau(n)}} L_{u_{\tau(n-1)}} ... L_{u_{\tau(2)}} L_{u_{\tau(1)}} = id$. Applying the same process shown for the case $k = 1$, we have $L_{u_{\tau(1)}} L_{u_{\tau(n)}} L_{u_{\tau(n-1)}} ... L_{u_{\tau(2)}} = id$ which implies that $L_{u_{\sigma(n)}} L_{u_{\sigma(n-1)}} ... L_{u_{\sigma(2)}} L_{u_{\sigma(1)}} = id$. So, the result follows for $k + 1$.

■

Definition 1.10. A rack X is said to be ***finitely stable*** if $S(X) \neq \emptyset$. Further, if $S^n(X) \neq \emptyset$ for some $n \in \mathbb{N}$, then the rack is said to be ***n-stable***.

The following examples illustrate the previous definitions.

Example 1.15. *Every finite rack is finitely stable. In fact, let X be a finite rack of order n. Then, the symmetric group of X, $Sym(X)$, is finite with order $n!$. Let $x \in X$, since $L_x \in Sym(X)$ then*

$$L_x^{n!} = id.$$

Hence, the family $\{x\}_{i=1}^{n!}$ *is a stabilizing family for* X*. So,* $S(X) \neq \emptyset$.

▲

Example 1.16. *Let* $\mathbb{R}$ *the set of real numbers with the Takasaki quandle structure*

$$x \rhd y = 2x - y;\ \ x, y \in \mathbb{R}.$$

Let $t \in \mathbb{R}$ *and* $\{x_1, x_2, ..., x_n\}$ *an arbitrary subset of real numbers. Notice that*

$$\begin{aligned}
(x_i)_{i=1}^n \rhd t &= x_n \rhd (.. \rhd (x_3 \rhd (x_2 \rhd (x_1 \rhd t)))...) \\
&= x_n \rhd (... \rhd (x_3 \rhd (x_2 \rhd (2x_1 - t)))...) \\
&= x_n \rhd (... \rhd (x_3 \rhd (2x_2 - 2x_1 + t))...) \\
&= x_n \rhd (... \rhd (2x_3 - 2x_2 + 2x_1 - t)...) \\
&= 2[\sum_{i=1}^{n} (-1)^{i+1} x_{n-i+1}] + (-1)^n t.
\end{aligned}$$

Then, we can form a stabilizing family of order $2n$ *by setting* $\{x_1, x_1, x_2, x_2, x_3, x_3, ..., x_n, x_n\}$*. Thus, the Takasaki quandle* $(\mathbb{R}, \rhd)$*, admits infinite stabilizing families of even order.*

▲

Example 1.17. *Let* G *be a non-trivial group with the* $Core(G)$ *quandle structure*

$$g \rhd h := gh^{-1}g,\ \ g, h \in G.$$

Let $h \in G$ *and* $\{g_1, g_2, ..., g_n\}$ *an arbitrary subset of* G*. Then*

$$\begin{aligned}
(g_i)_{i=1}^n \rhd h &= g_n \rhd (.. \rhd (g_3 \rhd (g_2 \rhd (g_1 \rhd h)))...) \\
&= g_n \rhd (... \rhd (g_3 \rhd (g_2 \rhd (g_1 h^{-1} g_1)))..) \\
&= g_n \rhd (... \rhd (g_3 \rhd (g_2 g_1^{-1} h g_1^{-1} g_2))...) \\
&= g_n \rhd (... \rhd (g_3 g_2^{-1} g_1 h^{-1} g_1 g_2^{-1} g_3)...) \\
&= [\prod_{i=0}^{n-1} g_{n-i}^{(-1)^i}] \cdot h^{(-1)^n} \cdot [\prod_{j=1}^{n} g_j^{(-1)^{n-j}}].
\end{aligned}$$

Then, we can form a stabilizing family of order $2n$ *by setting* $\{g_1, g_1, g_2, g_2, g_3, g_3, ..., g_n, g_n\}$.

▲

Example 1.18. *Let* V *a complex vector space equipped with the Alexander quandle structure given by*

$$x \rhd y = iy + (1 - i)x,\ \ \forall x, y \in V.$$

Let $u \in V$ be an arbitrary vector, then the family $\{u,u,u,u\}$ is a stabilizing family or order 4 for V. Indeed, let $w \in V$ then

$$\begin{aligned}(u)_{i=1}^4 \rhd w &= u \rhd (u \rhd (u \rhd (u \rhd w)))) \\ &= u \rhd (u \rhd (u \rhd [(iw + (1-i)u]))) \\ &= u \rhd (u \rhd ([iw + u - iu]i + (1-i)u)) \\ &= u \rhd (u \rhd (-w + 2u)) \\ &= u \rhd ((-w + 2u)i + (1-i)u) \\ &= u \rhd (-iw + iu + u) \\ &= (-iw + iu + u)i + (1-i)u \\ &= w - u + iu + u - iu = w.\end{aligned}$$

▲

The next theorem characterizes the stabilizing families of the conjugacy quandle $Conj(G)$ of a group G.

Theorem 1.5. *Let G be a group, then $\{u_i\}_{i=1}^n \in S^n(Conj(G))$ if and only if $u_n u_{n-1} ... u_1 \in Z(G)$.*

Proof. Let $\{u_i\}_{i=1}^n \in S^n(Conj(G))$ and $g \in G$, then

$$\begin{aligned}(u_i)_{i=1}^n \rhd g &= u_n \rhd (\cdots \rhd (u_3 \rhd (u_2 \rhd (u_1 \rhd g))) \\ &= u_n \rhd (\cdots \rhd (u_3 \rhd (u_2 \rhd (u_1 g u_1^{-1})))) \\ &= u_n \rhd (\cdots \rhd (u_3 \rhd (u_2 u_1 g u_1^{-1} u_2^{-1}) \\ &= (u_n \cdots u_2 u_1) g (u_1^{-1} u_2^{-1} \cdots u_n^{-1}) \\ &= (u_n \cdots u_2 u_1) g (u_n \cdots u_2 u_1)^{-1}.\end{aligned}$$

Since $\{u_i\}_{i=1}^n$ is a stabilizing family, $(u_n \cdots u_2 u_1) g (u_n \cdots u_2 u_1)^{-1} = g$. Then, $(u_n ... u_2 u_1) g = g(u_n ... u_2 u_1)$. Since g is arbitrary, the result follows for all $g \in G$. Then $u_n ... u_2 u_1 \in Z(G)$. Conversely, if $u_n ... u_2 u_1 \in Z(G)$, then $(u_n ... u_2 u_1) g = g(u_n ... u_2 u_1)$, for all $g \in G$, which implies that $g = (u_n \cdots u_2 u_1) g (u_n \cdots u_2 u_1)^{-1} = (u_i)_{i=1}^n \rhd g$.

■

In Example 1.17, we see that in the quandle $Core(G)$ of a group G, we can form stabilizing families of even order. The next theorem characterizes the groups whose $Core(G)$ quandles have stabilizing families of odd order.

Theorem 1.6. *Let G be a group and $k \in \mathbb{N}$. The quandle $Core(G)$ is $(2k+1)$-stable if and only if all the elements of the group G has order two. In other words, if $G \cong \bigoplus_{i \in I} \mathbb{Z}_2$, for some set I.*

Proof. Let $u_1, ..., u_{2k+1}$ a stabilizing family of order $2k+1$ of the quandle $Core(G)$. From Example 1.17, for all $g \in G$

$$(u_i)_{i=1}^{2k+1} \triangleright g = [\prod_{i=0}^{2k} u_{2k+1-i}^{(-1)^i}] \cdot g^{-1} \cdot [\prod_{j=1}^{2k+1} u_j^{(-1)^{2k+1-j}}].$$

Then, for all $g \in G$

$$g = [\prod_{i=0}^{2k} u_{2k+1-i}^{(-1)^i}] \cdot g^{-1} \cdot [\prod_{j=1}^{2k+1} u_j^{(-1)^{2k+1-j}}].$$

We define $\alpha := \prod_{i=0}^{2k} u_{2k+1-i}^{(-1)^i}$ and $\beta := \prod_{j=1}^{2k+1} u_j^{(-1)^{2k+1-j}}$. Then $g = \alpha g^{-1}\beta$, $\forall g \in G$. In particular, we have $\alpha\beta = \alpha(\alpha\beta)^{-1}\beta$, then $(\alpha\beta)^{-1} = 1$. Furthermore; $g^{-1} = \alpha(g^{-1})^{-1}\beta = \alpha g\beta$, $\forall g \in G$. Now, we define

$$\begin{aligned} \Phi : G &\longleftarrow G \\ g &\longmapsto g^{-1}. \end{aligned}$$

Note that, for all $g, h \in G$ we have

$$\Phi(gh) = (gh)^{-1} = \alpha(gh)\beta = (\alpha g\beta)(\alpha\beta)^{-1}(\alpha h\beta) = g^{-1}h^{-1} = \Phi(g)\Phi(h).$$

Thus, Φ is a group homomorphism, since all $g \in G$ has inverse and it is unique, it follows that Φ is an automorphism of G. Note that, for all $g, h \in G$, $(gh)^{-1} = g^{-1}h^{-1}$. On the other hand, $(gh)^{-1} = h^{-1}g^{-1}$. Thereby, $gh = hg$. So, the group is abelian. Thus, for all $g \in G$, $g = \alpha g^{-1}\beta = \alpha\beta g^{-1}$, which implies that, $g^2 = \alpha\beta$. Since $(\alpha\beta)^{-1} = 1$ then $\alpha\beta = 1$; therefore $g^2 = 1$, for all $g \in G$. Conversely, if $g^2 = 1$, for all $g \in G$, then $g = g^{-1}$ and G is an abelian group. Let $h \in G$ an arbitrary element, for all $g \in G$ we have $h \triangleright g = hg^{-1}h = h^2g = g$. It implies that G has a stabilizing family of order 1, that means, G is $(2k+1)$-stable. Note that, under this conditions, $Core(G)$ is a trivial rack.

▲

1.5 Stable Alexander quandles

Alexander quandles (see Example 1.5) have been studied by many authors due to their applications in knot theory [8], algebra [1], and topology [12]. Elhamdadi et al. in [4] provided necessary and sufficient conditions for Alexander quandles to be finitely stable. Furthermore, they also developed a general algorithm for calculating a stabilizing family in these quandles.

Notation 1.6. Let G be a group. We write $Aut_\circ(G)$ for the set of all group automorphism of G.

$$Aut_\circ(G) = \{\psi : G \longrightarrow G \mid \ \psi \textit{ is bijective and } \psi(gh) = \psi(g)\psi(h) \ \ \forall g, h \in G\}.$$

Definition 1.11. Let G be a group and $\phi \in Aut_\circ(G)$. The ϕ-conjugate of G, denoted by $Conj_\phi(G)$, is the set G with the operation

$$g \triangleright h = g\phi(h)\phi(g^{-1}), \;\; g, h \in G.$$

Lemma 1.7. *Let G be a group and $\phi \in Aut_\circ(G)$, then $Conj_\phi(G)$ is a quandle.*

Proof. First, we have that for every $g \in G$,

$$g \triangleright g = g\phi(g)\phi(g^{-1}) = g\phi(g)\phi^{-1}(g) = g.$$

Let $g, h_1, h_2 \in G$, such that $L_g(h_1) = L_g(h_2)$, then $g \triangleright h_1 = g \triangleright h_2$, thus, $g\phi(h_1)\phi(g^{-1}) = g\phi(h_2)\phi(g^{-1})$, thereby, $\phi(h_1) = \phi(h_2)$. Since ϕ is an automorphism, then $h_1 = h_2$, so L_g is injective.
Let $z \in G$, then

$$\begin{aligned} L_g[\phi(g)\phi(z^{-1})g] &= g \triangleright [\phi(g)\phi(z^{-1})g] \\ &= g\phi[\phi(g)\phi(z^{-1})g]\phi(g^{-1}) \\ &= g\phi[\phi^{-1}(g^{-1})]\phi[\phi^{-1}(z)]\phi(g)\phi^{-1}(g) \\ &= z. \end{aligned}$$

Therefore, L_g is surjective. Finally, let $g, h, z \in G$ then

$$\begin{aligned} g \triangleright (h \triangleright z) &= g \triangleright (h\phi(z)\phi(h^{-1})) \\ &= g\phi[h\phi(z)\phi(h^{-1})]\phi(g^{-1}) \\ &= g\phi(h)\phi^2(z)h\phi(g^{-1}) \\ &= [g\phi(h)\phi(g^{-1})][\phi(g)\phi(z)\phi^{-2}(g)][\phi^2(g)h\phi(g^{-1})] \\ &= [g\phi(h)\phi(g^{-1})]\phi[g\phi(z)\phi(g^{-1})]\phi([g\phi(h)\phi(g^{-1})]^{-1}) \\ &= [g\phi(h)\phi(g^{-1})] \triangleright [g\phi(z)\phi(g^{-1})] \\ &= (g \triangleright h) \triangleright (g \triangleright z). \end{aligned}$$

■

Note that if $\phi = id$, then $Conj_\phi(G) = Conj(G)$.

Proposition 1.5. *Let G be a group and $\phi \in Aut_\circ(G)$, then, $Conj_\phi(G)$ is finitely stable if and only if there exists an integer n and $z \in G$, such that for all $g \in G$, $\phi^n(g) = z^{-1}g\phi(z)$. Furthermore, $\{u_i\}_{i=1}^n \in S^n(Conj_\phi(G))$ if and only if $\phi^n(g) = [\prod_{i=1}^n \phi^{n-i}(u_i)^{-1}]g[\prod_{i=0}^{n-1} \phi^{i+1}(u_{n-i})]$, for all $g \in G$.*

Proof. Let us suppose that $Conj_\phi(G)$ is finitely stable. Then there exits $\{u_i\}_{i=1}^n \in S^n(Conj_\phi(G))$. Note that, for all $g \in G$ we have

$$\begin{aligned} (u_i)_{i=1}^n \triangleright g &= u_n \triangleright (u_{n-1} \triangleright (...(u_2 \triangleright (u_1 \triangleright g))...) \\ &= u_n \triangleright (u_{n-1} \triangleright (...(u_2 \triangleright (u_1\phi(g)\phi(u_1^{-1})...) \\ &= u_n \triangleright (u_{n-1} \triangleright (... \triangleright (u_2\phi(u_1)\phi^2(g)\phi^2(u_1^{-1})\phi(u_2^{-1}))...) \\ &= [\prod_{i=0}^{n-1} \phi^i(u_{n-i})]\phi^n(g)[\prod_{i=0}^{n-1} \phi^{n-i}(u_{i+1})^{-1}]. \end{aligned}$$

Therefore, $g = [\prod_{i=0}^{n-1} \phi^i(u_{n-i})]\phi^n(g)[\prod_{i=0}^{n-1} \phi^{n-i}(u_{i+1})^{-1}]$. If we define $z := \prod_{i=0}^{n-1} \phi^i(u_{n-i})$, then $\phi(z^{-1}) = \prod_{i=0}^{n-1} \phi^{n-i}(u_{i+1})^{-1}$, thus

$$z\phi^n(g)\phi(z^{-1}) = g,$$

where the result follows: $\phi^n(g) = z^{-1}g\phi(z) = [\prod_{i=1}^{n} \phi^{n-i}(u_i)^{-1}]g[\prod_{i=0}^{n-1} \phi^{i+1}(u_{n-i})]$.
Conversely, if there exists $z \in G$ such that $\phi^n = z^{-1}g\phi(z)$, then we get the family $\{u_i\}_{i=1}^{n} \subset G$, where $u_1 = u_2 = ... = u_{n-1} = 1$ and $u_n = z$. Note that, for all $g \in G$,

$$\begin{aligned}(u_i)_{i=1}^{n} \rhd g &= u_n \rhd (... \rhd (u_2 \rhd (u_1 \rhd g))...)\\ &= z \rhd (... \rhd (1 \rhd (1 \rhd g))...)\\ &= z \rhd \phi^{n-1}(g) = z\phi^n(g)\phi(z^{-1})\\ &= z(z^{-1}g\phi(z))\phi(z^{-1})\\ &= g.\end{aligned}$$

So, the subset $\{1, 1, ..., 1, z\}$ is a stabilizing family of order n.

■

The next result shows that when the group G is abelian, the stabilizing families of $Conj_\phi(G)$ are related to the torsion of ϕ in the automorphism group of G, $Aut(G)$.

Proposition 1.6. *Let G be a non-trivial abelian group and $\phi \in Aut(G)$. Take the $Conj_\phi(G)$ quandle with the structure*

$$g \rhd h = g + \phi(h) - \phi(g) = \phi(h) + (id - \phi)(g), \quad g, h \in G.$$

Then, the following statements are equivalents:

i) $Conj_\phi(G)$ is finitely stable.

ii) ϕ is a torsion element of Aut(G).

Proof. Let $u_1, u_2, ..., u_n$ a stabilizing family of $Conj_\phi(G)$. Then we have that

$$\begin{aligned}(u_i)_{i=1}^{n} \rhd g &= u_n \rhd (u_{n-1} \rhd (...(u_2 \rhd (u_1 \rhd g))...)\\ &= u_n \rhd (u_{n-1} \rhd (...(u_2 \rhd (\phi(g) + (id - \phi)(u_1))...)\\ &= u_n \rhd (u_{n-1} \rhd (...(\phi^2(g) + (id - \phi)(\phi(u_1) + u_2))...).\end{aligned}$$

Therefore, $g = \phi^n(g) + (id - \phi)(\sum_{i=1}^{n} \phi^{n-i}(u_i))$ for all $g \in G$.
Then $\{u_1, u_2, ..., u_n\}$ is a stabilizing family if and only if $\phi^n = id$ and $\sum_{i=1}^{n} \phi^{-i}(u_i) = 0$.

■

Lemma 1.8. *Let Γ be a group and γ a non- trivial element in Γ. Let $\mathcal{A}$ be a left $\mathbb{Z}[\gamma^{\pm 1}]$- module. Take the Alexander quandle $(\mathcal{A}, \rhd_\gamma)$ with the operation*

$$x \rhd_\gamma y = \gamma \cdot (y - x) + x, \ \forall x, y \in \mathcal{A}.$$

Then, the Alexander quandle $(\mathcal{A}, \rhd_\gamma)$ is the same $conj_{\phi_\gamma}(\mathcal{A})$, where $\phi_\gamma : \mathcal{A} \longrightarrow \mathcal{A}$ is the automorphism defined by $\phi_\gamma(a) = \gamma \cdot a$, for all $a \in \mathcal{A}$.

Proof. Since $\mathcal{A}$ is a left $\mathbb{Z}[\gamma^{\pm 1}]$-module, it is easy to prove that ϕ_γ is an automorphism of $\mathcal{A}$. Let $x, y \in \mathcal{A}$, we have that

$$\phi_\gamma(x+y) = \gamma \cdot (x+y) = \gamma \cdot x + \gamma \cdot y = \phi_\gamma(x) + \phi_\gamma(y).$$

Then, ϕ_γ is homomorphism. Now, let $a, x \in \mathcal{A}$ such that $\phi_\gamma(a) = \phi_\gamma(x)$, then $\gamma \cdot a = \gamma \cdot x$. It follows that $a = x$, which implies that ϕ_γ is injective. Further, if we take $\gamma^{-1} \cdot a \in \mathcal{A}$, $\phi_\gamma(\gamma^{-1} \cdot a) = \gamma \cdot (\gamma^{-1} \cdot a) = (\gamma\gamma^{-1}) \cdot a = a$. Thus, ϕ_γ is bijective.
Consider the quandle $Conj_{\phi_\gamma}(\mathcal{A})$, then for all $x, y \in \mathcal{A}$ we have

$$x \rhd y = x + \phi_\gamma(y) - \phi_\gamma(x) = x + \gamma \cdot y - \gamma \cdot x = \gamma \cdot (y - x) + x = x \rhd_\gamma y.$$

Therefore, $Conj_{\phi_\gamma}(\mathcal{A})$ is the same as the Alexander quandle $(\mathcal{A}, \rhd_\gamma)$.

■

Theorem 1.7. *Let Γ be a group and γ a non- trivial element in Γ. Let $\mathcal{A}$ be a left $\mathbb{Z}[\gamma^{\pm 1}]$-module with the operation*

$$x \rhd^\gamma y = \gamma \cdot (y - x) + x, \quad \forall x, y \in \mathcal{A}.$$

For $n \in \mathbb{N}$, let $F_\gamma : \mathcal{A}^n \longrightarrow \mathcal{A}$ be a function defined by

$$F_\gamma(x_1, ..., x_n) := \textstyle\sum_{i=1}^{n} \gamma^{n-i} \cdot x_i.$$

Then $(\mathcal{A}, \rhd^\gamma)$ has a stabilizing family of order n if and only if γ is of order n. Furthermore, $S^n(M)$ is the vector space of all solutions of the equation $F_\gamma(x_1, ..., x_n) = 0$.

Proof. Define

$$\begin{aligned} \phi_\gamma : \mathcal{A} &\longrightarrow \mathcal{A} \\ a &\longmapsto \gamma \cdot a. \end{aligned}$$

From Lemma 1.8, the Alexander quandle is the same quandle $Conj_{\phi_\gamma}(\mathcal{A})$. From Proposition 1.6 the Alexander quandle is $n-$stable if and only if $\phi_\gamma^n = id$, that means

$$\phi_\gamma^n(a) = \gamma^n \cdot a = a, \ \forall a \in \mathcal{A}.$$

It follows that $\gamma^n = 1$. Furthermore, $u_1, ..., u_n \in \mathcal{A}$ is a stabilizing family if and only if

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \phi_\gamma^{n-i}(u_i) \\ &= \sum_{i=1}^{n} \gamma^{n-i} \cdot u_i \\ &= F_\gamma(u_1, ..., u_n). \end{aligned}$$

■

Corollary 1.2. *Let V a complex vector space. Let $\alpha \in \mathbb{C}$ such that $\alpha \neq 0$ and $\alpha \neq 1$. Consider the Alexander quandle structure on the vector space V defined by $v \triangleright_\alpha w = \alpha(w - v) + v$, for all $v, w \in V$. Then, V has a stabilizing family of order n if and only if*

i) α is an n^{th} root of unity.

ii) There exists $r \in \mathbb{Z}$ such that $0 < r < n$ and $\sum_{k=1}^{n} e^{-\frac{2i\pi}{n}kr} u_k = 0$.

Proof. Consider the multiplicative group $\mathbb{C}^*$ and consider the vector space V as a $\mathbb{Z}[\mathbb{C}^*]$-module. Let $v_1, ..., v_n \in V$; from Theorem 1.7, $v_1, ..., v_n$ is a stabilizing family of order n if and only if $\alpha^n = 1$, then α is an n^{th} root of unity. Further, we have that $\sum_{k=1}^{n} \alpha^{n-k} v_k = \sum_{k=1}^{n} \alpha^{-k} v_k = 0$. Since α is an n^{th} root of unity then it is the form $\alpha = e^{\frac{2\pi r}{n}}$ for some integer $0 < r < n$. Therefore, $\sum_{k=1}^{n} e^{\frac{-2\pi}{n} rk} v_k = 0$.

■

1.6 Quandle associated to a rack

We can go from quandles to racks with the identity map because every quandle is a rack and, therefore, every morphism between quandles is also a morphism between racks. Now, we consider the reciprocal, i.e, we endow any rack X of quandle structure through an automorphism of the rack. In order to get that, we follow the construction given in [16].

Lemma 1.9. *Let X be a rack. For every $x \in X$ there exists an unique element $h \in X$ such that $x = h \triangleright h$.*

Proof. Let $x \in X$, since the function $L_x : X \longrightarrow X$ defined by $L_x(y) = x \triangleright y$ is bijective then there exists an unique $h \in X$ such that $x = L_x(h) = x \triangleright h$. Note that

$$x \triangleright (h \triangleright h) = (x \triangleright h) \triangleright (x \triangleright h) = x \triangleright x.$$

Therefore, $L_x(h \triangleright h) = L_x(x)$. Since L_x is injective, it follows that $h \triangleright h = x$.

■

Lemma 1.10. *Let X be a rack. For every $x, y \in X$, we have the relation*

$$(x \triangleright x) \triangleright y = x \triangleright y.$$

Proof. Let $x, y \in X$. Since the function L_x is onto, there exists $z \in X$ such that $L_x(z) = x \triangleright z = y$. Note that,

$$\begin{aligned}(x \triangleright x) \triangleright y &= (x \triangleright x) \triangleright (x \triangleright z) \\ &= x \triangleright (x \triangleright z) \\ &= x \triangleright y.\end{aligned}$$

■

Proposition 1.7. *Let X be a rack. The function $F : X \longrightarrow X$, defined by $F(x) = x \triangleright x$, is an automorphism of X.*

Proof. Let us see that F is an homomorphism of racks. Let $x, y \in X$, we have

$$F(x \triangleright y) = (x \triangleright y) \triangleright (x \triangleright y) = x \triangleright (y \triangleright y).$$

From Lemma 1.10, $x \triangleright (y \triangleright y) = (x \triangleright x) \triangleright (y \triangleright y) = F(x) \triangleright F(y)$. Thus, for every $x, y \in X$ it follows that $F(x \triangleright y) = F(x) \triangleright F(y)$.
Now, from Lemma 1.9, for every $x \in X$, there exists a unique $h \in X$ such that $F(h) = x$, therefore F is a bijective function.

■

Definition 1.12. Let $(X, \triangleright)$ be a rack, the automorphism F defined by $F(x) = x \triangleright x$ is called the ***canonical automorphism of X***.

Note that, since F is an automorphism of X, then its inverse F^{-1} is also an automorphism of X. Now, from Lemma 1.10, for every $x, y \in X$ we have $F(x) \triangleright y = x \triangleright y$. Therefore,

$$\begin{aligned} F^{-1}(x \triangleright y) &= F^{-1}(F(x) \triangleright y) \\ &= F^{-1}(F(x)) \triangleright F^{-1}(y) \\ &= x \triangleright F^{-1}(y). \end{aligned}$$

Theorem 1.8. *Let X be a rack. Define the operation*

$$\begin{aligned} \triangleright_F \colon X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \triangleright_F y := F^{-1}(x \triangleright y) = x \triangleright F^{-1}(y). \end{aligned}$$

Then $(X, \triangleright_F)$ is a quandle.

Proof. Let us check that the quandle axioms are satisfied. In fact,

(Q1) $x \triangleright_F x = F^{-1}(x \triangleright x) = F^{-1}(F(x)) = x$, for all $x \in X$.

(Q2) To prove the second axiom, we need to show that the function $\ell_x : X \longrightarrow X$ defined by $\ell_x(y) := x \triangleright_F y$, is bijective. We have $\ell_x(y) = x \triangleright_F y = x \triangleright F^{-1}(y) = L_x(F^{-1}(y))$. Thus, for every $x \in X$, the function ℓ_x is the composition of two bijective functions, L_x and F^{-1}, thereby, ℓ_x is bijective as well.

(Q3) Let $x, y, z \in X$,

$$\begin{aligned} x \triangleright_F (y \triangleright_F z) &= x \triangleright_F (y \triangleright F^{-1}(z)) \\ &= x \triangleright F^{-1}(y \triangleright F^{-1}(z) \\ &= x \triangleright (F^{-1}(y) \triangleright F^{-2}(z)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (x \triangleright_F y) \triangleright_F (x \triangleright_F z) &= (x \triangleright F^{-1}(y)) \triangleright F^{-1}(x \triangleright F^{-1}(z)) \\ &= (x \triangleright F^{-1}(y)) \triangleright (x \triangleright F^{-2}(z)) \\ &= x \triangleright (F^{-1}(y) \triangleright F^{-2}(z)). \end{aligned}$$

Therefore, $x \triangleright_F (y \triangleright_F z) = (x \triangleright_F y) \triangleright_F (x \triangleright_F z)$.

So, $\triangleright_F$ defines a quandle structure on X.

$\blacksquare$

Definition 1.13. Let X be a rack. The ***quandle associated to X***, denoted by $\mathcal{Q}_X$, is the set X with the binary operation $\triangleright_F$; where F is the canonical automorphism of X and $\triangleright_F$ is defined by $x \triangleright_F y := F^{-1}(y \triangleright x)$.

Note that if X is already a quandle, then $F = id$ and thus $\triangleright = \triangleright_F$. This implies that, in this case, the quandle associated to X is the same as X itself.
As we see, we can naturally obtain a quandle from any rack X. Additionally, any morphism between racks induces a morphism between their associated quandles, as it is shown in the next theorem.

Theorem 1.9. *Let $(X, \triangleright)$ and $(Y, *)$ be racks, and $\phi : X \longrightarrow Y$ be a rack homomorphism. Then $\phi : \mathcal{Q}_X \longrightarrow \mathcal{Q}_Y$ is also a quandle homomorphism.*

Proof. Since $\mathcal{Q}_X = X$ and $\mathcal{Q}_Y = Y$ as sets then, the map $\phi : \mathcal{Q}_X \longrightarrow \mathcal{Q}_Y$ makes sense. Let F_X and F_Y the canonical automorphism of X and Y, respectively. Note that for all $x \in X$ we have

$$\phi(F_X(x)) = \phi(x \triangleright x) = \phi(x) * \phi(x) = F_Y(\phi(x)).$$

Therefore, $\phi F_x = F_Y \phi$. This implies that $F_Y^{-1}\phi = \phi F_X^{-1}$. So, for every $x, h \in \mathcal{Q}_X$ we have

$$\phi(x \triangleright_{F_X} h) = \phi(F_X^{-1}(x \triangleright h)) = F_Y^{-1}(\phi(x \triangleright h)) = F_Y^{-1}(\phi(x) * \phi(h)) = \phi(x) *_{F_Y} \phi(h).$$

Hence, ϕ is a morphism between $\mathcal{Q}_X$ and $\mathcal{Q}_Y$.

$\blacksquare$

Corollary 1.3. *Let X be a rack and Q be a quandle and $\phi : X \longrightarrow Q$ be a rack homomorphism. Then, $\phi : \mathcal{Q}_X \longrightarrow Q$ is a quandle homomorphism.*

Proof. Since Q is a quandle then its associated quandle is simply Q itself. From Theorem 1.9, $\phi : \mathcal{Q}_X \longrightarrow Q$ is a quandle homomorphism.

$\blacksquare$

Corollary 1.4. *Let X be a rack, then the canonical automorphism F belongs to the center $Z(Aut(X))$ of $Aut(X)$.*

Proof. Let $\phi \in Aut(X)$, then from Theorem 1.9, ϕ is also an automorphism of the associated quandle $\mathcal{Q}_X$, that means, $\phi F^{-1} = F^{-1}\phi$. Therefore $F\phi = \phi F$, and the result follows.

$\blacksquare$

Proposition 1.8. *Let X be an involutive rack, then $\mathcal{Q}_X$ is an involutive quandle.*

Proof. Since X is involutive, then $L_x^2 = id$ for all $x \in X$. Thus, for every $x \in X$, we have

$$\begin{aligned} x &= L_x^2(x) \\ &= x \rhd (x \rhd x) \\ &= (x \rhd x) \rhd (x \rhd x) \\ &= F(x) \rhd F(x) \\ &= F(x \rhd x) \\ &= F^2(x). \end{aligned}$$

Therefore, $F^2 = id$. In other words we have $F = F^{-1}$. Now, in the quandle $\mathcal{Q}_X$ we have $\ell_x = L_x F^{-1} = L_x F$. Since $F \in Z(Aut(X))$, then for every $x \in X$, we have $\ell_x^2 = (L_x F)^2 = L_x^2 F^2 = id$. Thus, $\mathcal{Q}_X$ is involutive.

■

1.7 The rack ring

Some authors have studied the notion of rack ring, for example in [2], an analogous theory of group rings was proposed for quandles and racks.
In this work, we give two different constructions for the rack ring and prove that those constructions are isomorphic. In particular, we are interested in the rack ring $\mathbb{C}X$, for a finite rack X.

Definition 1.14. Let R be a commutative ring with unity and let X be a rack. Define the set

$$RX := \{f : X \longrightarrow R \mid f(x) = 0 \text{ except for finite many x's}\}.$$

Then, RX is a ring with addition and multiplication in the usual way , $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $(f_1 f_2)(x) = f_1(x) f_2(x)$ for all $f_1, f_2 \in RX$ and $x \in X$. The set RX is called the ***rack ring***. If X is a quandle, then we calle it the ***quandle ring***.

As in group rings, we can see the rack ring in terms of formal sums.

Proposition 1.9. *Let R be a commutative ring with unity and X be a rack. Define the set of all formal finite R-linear combinations of elements of X,*

$$R[X] := \left\{ \sum_{x \in X} r_x x \mid r_x \in R,\ \forall x \in X \ \ \textit{and} \ \ r_x = 0 \ \textit{except for finite many x's} \right\}.$$

Then R[X] is a non-associative ring with addition in the usual way, $\sum\limits_{x \in X} r_x x + \sum\limits_{x \in X} s_x x = \sum\limits_{x \in X} (r_x + s_x) x$ and multiplication $(\sum\limits_{x \in X} r_x x) \cdot (\sum\limits_{y \in X} s_y y) := \sum\limits_{x, y \in X} r_x s_y (x \rhd y)$.

Proof. The set $R[X]$ with the usual addition is an abelian group. Now observe that

$$\begin{aligned}
(\sum_{x\in X}\alpha_x x)\cdot(\sum_{z\in X}r_z z+\sum_{z\in X}s_z z) &= (\sum_{x\in X}\alpha_x x)\cdot(\sum_{z\in X}(r_z+s_z)z)\\
&= \sum_{x,z\in X}\alpha_x(r_z+s_z)(x\triangleright z)\\
&= \sum_{x,z\in X}(\alpha_x r_z+\alpha_x s_z)(x\triangleright z)\\
&= \sum_{x,z\in X}\alpha_x r_z(x\triangleright z)+\sum_{x,z\in X}\alpha_x s_z(x\triangleright z)\\
&= (\sum_{x\in X}\alpha_x x)\cdot(\sum_{z\in X}r_z z)+(\sum_{x\in X}\alpha_x)\cdot(\sum_{z\in X}s_z z).
\end{aligned}$$

In a similar way, we prove the distribution by left multiplication. Since the the operation on the rack $\triangleright$ is not necessarily associative , then neither is the multiplication defined on $R[X]$.

■

Let X be a rack and R be a commutative ring with unity. The set RX is a R-module with the usual scalar multiplication $(rf)(x) = rf(x)$. Consider the set $\delta := \{\delta_x \mid x \in X\}$, where the function $\delta_x : X \longrightarrow R$ is defined by

$$\delta_x(y) := \begin{cases} 0, & \text{si } y \neq x \\ 1, & \text{si } y = x. \end{cases}$$

The set δ is a base of RX seen as R-module. Indeed, for every $f \in RX$ and $x \in X$, we have

$$f(x) = \sum_{x\in X} f(x)\delta_x(x).$$

Since $f(x) \neq 0$ for a finite set of x's, then

$$f = \sum_{x\in X} f(x)\delta_x \in \langle\delta\rangle.$$

Furthermore, observe that $\sum_{x\in X} r_x\delta_x = 0$, implies that for every $x \in X$,

$$\sum_{x\in X} r_x\delta_x(x) = r_x = 0.$$

Therefore, δ is R- linearly independent.
Now, $R[X]$ is also an R- module with scalar multiplication defined by $r\sum_{x\in X} s_x x = \sum_{x\in X}(rs_x)x$. From definition, the set X form a basis for $R[X]$ as an R-module.

Proposition 1.10. *Let X be a rack and R be a commutative ring with unity. Then the R-modules RX and $R[X]$ are isomorphic.*

Proof. Let the map $\phi : RX \longrightarrow R[X]$ be defined by $\phi(\sum_{x\in X} f(x)\delta_x) := \sum_{x\in X} f_x x$, where $f_x = f(x)$ for all $x \in X$. Let $r \in R$ and $f, g \in RX$, then

$$\begin{aligned}\phi(r(\sum_{x\in X} f(x)\delta_x) + \sum_{x\in X} g(x)\delta_x) &= \phi(\sum_{x\in X} r(f(x)) + g(x))\delta_x \\ &= \phi(\sum_{x\in X}[(rf)(x) + g(x)]\delta_x \\ &= \sum_{x\in X}(rf_x + g_x)x \\ &= r\sum_{x\in X} f_x x + \sum_{x\in X} g_x x \\ &= r\phi(\sum_{x\in X} f(x)\delta_x) + \phi(\sum_{x\in X} g(x)\delta_x).\end{aligned}$$

Therefore, ϕ is a R-module homomorphism. Now, let $\sum_i f(x_i)\delta_{x_i} \in Ker\phi$. Then,

$$\begin{aligned}\phi(\sum_{x\in X} f(x)\delta_x) = \sum_{x\in X} f_x x = 0 \\ \therefore f(x) = f_x = 0, \ \forall x \in X.\end{aligned}$$

Thus, ϕ is an injective function.
Let $\sum_{x\in X} r_x x \in R[X]$, define the function $f : X \longrightarrow R$ by $f(x) := r_x$. Since $r_x \neq 0$ for a finite many $x's$, then $f(x) \neq 0$ for a finite set of $x's$, it follows that $f \in RX$. Observe that $\phi(f) = \phi(\sum_{x\in X} f(x)\delta_x) = \sum_{x\in X} f_x x = \sum_{x\in X} r_x x$, which implies that ϕ is surjective. Therefore, ϕ is an isomorphism.

■

Chapter 2

Rack Actions and Rack Representations

Representation theory studies different algebraic structures by representing their elements as linear transformations over a vector space. This perspective allows us to use several tools of linear algebra and obtain significant results. Representation theory is closely related to the notion of action, which is another important tool in mathematics. Thus, if we have the concept of a rack action, then we get the concept of a rack representation. In this chapter we study rack actions and, also rack representation theory. We follow the approach of Elhamdadi and Moutuou in [4], providing additional details for some proofs and presenting new results.

2.1 Rack actions and approximate units

Let X be a rack, then there is a natural action of the group $Inn(X)$ on X, given by $L_x \cdot y = L_x(y) = x \triangleright y$, for all $x, y \in X$. This group action can be seen as a "quandle action" of $Conj(Inn(X))$ on the set X, where $Conj(Inn(X))$ is the set $Inn(X)$ with the conjugation operation $L_x \triangleleft L_y = L_x L_y L_x^{-1}$. Note that

$$L_x \cdot (L_y \cdot z) = (L_x L_y) \cdot z = (L_x L_y L_x^{-1}) \cdot (L_x \cdot z) = (L_x \triangleleft L_y) \cdot (L_x \cdot z).$$

Further, suppose that the rack X is finitely stable. Let $\{x_1, ..., x_n\}$ be a stabilizing family of X, then we have $L_{x_n} \cdots L_{x_1} = id$. From Lemma 1.6, for all $\sigma \in \langle (2\ 3\ 4\ ...\ n\ 1) \rangle$, satisfies that $L_{x_{\sigma(n)}} \cdots L_{x_{\sigma(1)}} = id$, thus $L_{x_{\sigma(n)}} \cdots L_{x_{\sigma(1)}} = L_{x_n} \cdots L_{x_1}$. Therefore,

$$\begin{aligned} L_{x_n} \cdot (L_{x_{n-1}} \cdot (\cdot ... (L_{x_1} \cdot z))...) &= (L_{x_n} L_{x_{n-1}} \cdots L_{x_1}) \cdot z \\ &= (L_{x_{\sigma(n)}} \cdots L_{x_{\sigma(1)}}) \cdot z \\ &= L_{x_{\sigma(n)}} \cdot (L_{x_{\sigma(n-1)}} \cdot (\cdot ... (L_{x_{\sigma(1)}} \cdot z))...). \end{aligned}$$

These observations motivate the following definition which was introduced in [4].

Definition 2.1. A left ***rack action*** of a rack X on a set M is a map

$$\begin{aligned} \bullet : X \times M &\longrightarrow M \\ (x, m) &\longmapsto x \cdot m, \end{aligned}$$

which satisfies the following conditions,

(i) For all $m \in M$ and $x, y \in X$,

$$x \cdot (y \cdot m) = (x \triangleright y) \cdot (x \cdot m).$$

(ii) Let $\{u_1, u_2, ..., u_n\} \in S(X)$, then

$$(u_n \cdot (u_{n-1} \cdot (...(u_1 \cdot m)...) = (u_{\sigma(n)} \cdot (u_{\sigma(n-1)} \cdot (...(u_{\sigma(1)} \cdot m)...),$$

for all $m \in M$ and all $\sigma \in \langle (2\ 3\ ...\ n\ 1) \rangle$.

Notation 2.1. Let X be a rack acting over a set M. Let $u_1, u_2, ..., u_n$ be elements in X and m an element of M, we write: $(u_i)_{i=1}^n \cdot m = u_n \cdot (...(u_3 \cdot (u_2 \cdot (u_1 \cdot m)))...)$.

The following example is a generalization of what we describe in the paragraph previous to the Definition 2.1.

Example 2.1. *Every action of a group G on a set M extens to an action of the quandle $Conj(G)$ on M. In fact, let G be a group acting on a set M. Then:*
(i) Let $g, h \in Conj(G)$ and $m \in M$,

$$\begin{aligned} g \cdot (h \cdot m) &= (gh) \cdot m \\ &= (ghg^{-1}g) \cdot m \\ &= (ghg^{-1}) \cdot (g \cdot m) \\ &= (g \triangleright h) \cdot (g \cdot m). \end{aligned}$$

(ii) Let $\{u_i\}_{i=1}^n \in S^n(Conj(G))$, $m \in M$ and $\sigma = (2\ 3\ ...\ n\ 1)$,

$$\begin{aligned} (u_i)_{i=1}^n \cdot m &= u_n \cdot (...(u_3 \cdot (u_2 \cdot (u_1 \cdot m)))...) \\ &= (u_n ... u_2 u_1) \cdot m \\ &= (u_1^{-1} u_1 u_n ... u_2 u_1) \cdot m. \end{aligned}$$

Since $\{u_i\}_{i=1}^n \in S^n(Conj(G))$, from Lemma 1.6, $\{u_{\sigma(i)}\}_{i=1}^n \in S^n(Conj(G))$ then, from Theorem 1.5 $u_1 u_n ... u_2 \in Z(G)$. It follows that

$$(u_1^{-1} u_1 u_n ... u_2 u_1) \cdot m = (u_1 u_n ... u_2) \cdot m = u_1 \cdot (u_n \cdot (... \cdot (u_2 \cdot m))) = (u_{\sigma(i)})_{i=1}^n \cdot m.$$

Therefore, $(u_i)_{i=1}^n \cdot m = (u_{\sigma(i)})_{i=1}^n \cdot m$. From induction, the result follows for all $\sigma \in \langle (2\ 3\ ...\ n\ 1) \rangle$.

▲

The converse of the previous example is not always true, that is, a rack action of $Conj(G)$ does not necessarily define a group action of G. A counterexample is given in the next example.

Example 2.2. *Let $X = Conj(G)$, where G is an abelian group. Consider the rack ring $\mathbb{C}X$ seen as a complex vector space. Define the map*

$$\begin{aligned} \cdot : X \times \mathbb{C}X &\longrightarrow \mathbb{C}X \\ (x, f) &\longmapsto x \cdot f := f + \delta_x, \end{aligned}$$

where,

$$\delta_x(y) := \left\{ \begin{array}{ll} 0, & si\ y \neq x \\ 1, & si\ y = x \end{array} \right. \ for\ all\ x, y \in X.$$

Note that, $S^k(X) \neq \emptyset$ for every $k \in \mathbb{N}$. Indeed, let $x, y \in X$, then we have $x \triangleright y = x + y - x = y$. Therefore, $\{x, x, ..., x\}$ is a stabilizing family of order k. Let us see that $\cdot$ is an action of the quandle X over $\mathbb{C}X$.

(i) Let $f \in \mathbb{C}X$ and $x, y \in X$ then

$$\begin{aligned} x \cdot (y \cdot f) &= x \cdot (f + \delta_y) \\ &= f + \delta_y + \delta_x \\ &= f + \delta_x + \delta_y \\ &= y \cdot (f + \delta_x) \\ &= y \cdot (x \cdot f) \\ &= (x \triangleright y) \cdot (x \cdot f). \end{aligned}$$

(ii) Let $\{u_1, ..., u_k\} \in S^k(X)$, $f \in \mathbb{C}X$ and $\sigma = (2\ 3\ ...\ n\ 1)$ then

$$\begin{aligned} (u_i)_{i=1}^k \cdot f &= f + \delta_{u_1} + \delta_{u_2} + \cdots + \delta_{u_k} \\ &= f + \delta_{u_2} + \cdots + \delta_{u_k} + \delta_{u_1} \\ &= u_1 \cdot (u_k \cdot ...(u_2 \cdot f)...) \\ &= (u_{\sigma(i)})_{i=1}^k \cdot f. \end{aligned}$$

by induction as in the proof of Lemma 1.6 the result follows for all $\sigma \in \langle (2\ 3\ ...\ n\ 1) \rangle$. Then, $\cdot$ is an action of the quandle $Conj(G)$ but it is not an action of the group G on $\mathbb{C}X$. In fact, reasoning by contradiction, let us suppose that $\cdot$ is a group action. Let $x, y \in G$ be non trivial elements, then we have, $x \cdot (y \cdot f) = f + \delta_y + \delta_x$. On the other hand, $(x+y) \cdot f = f + \delta_{x+y}$, thus, if $x \cdot (y \cdot f) = (x+y) \cdot f$, then, $\delta_y + \delta_x = \delta_{x+y}$, but $(\delta_y + \delta_x)(x + y) = 0$ and $\delta_{x+y}(x + y) = 1$; therefore we can not get the equality $\delta_y + \delta_x = \delta_{x+y}$. It follows that $\cdot$ is not a group action.

▲

Definition 2.2. A rack action of a rack X on a set M is ***faithful*** if for each $m \in M$, the map $x \in X \longmapsto x \cdot m \in M$ is injective.

Example 2.3. *Let G be an abelian group. Consider the action of the Example 2.2 of the quandle $X = Conj(G)$ over the vector space $\mathbb{C}X$ given by $x \cdot f = f + \delta_x$. Then, this action is faithful. Indeed, let $x, y \in X$ and $f \in \mathbb{C}X$ such that $x \cdot f = y \cdot f$. Then, $f + \delta_x = f + \delta_y$, which implies that $x = y$.*

▲

Let X be a rack. Observe that we can define an action of X on its underlying set by $x \cdot y := x \triangleright y$ for all $x, y \in X$. Suppose that $\{t_1, ..., t_n\}$ is an stabilizing family of X, then for every $x \in X$ we have

$$\begin{aligned} x &= t_n \triangleright (... \triangleright (t_2 \triangleright (t_1 \triangleright x))..) \\ &= t_n \cdot (... \cdot (t_2 \cdot (t_1 \cdot x))...) \\ \therefore \quad (t_i)_{i=1}^n \cdot x &= x\ ,\ \forall x \in X. \end{aligned}$$

The following definition is motivated by the previous observation.

Definition 2.3. Let X be a rack acting on a set $M \neq \emptyset$.

(I) An ***approximate unit*** is a subset $\{t_i\}_{i=1}^n \subset X$ such that

$$(t_i)_{i=1}^n \cdot m = m, \quad \forall m \in M.$$

(II) A rack action is said to be ***strong*** if every stabilizing family of the rack is an approximate unit.

Lemma 2.1. *Let X be a rack acting on a set M and $\{x_i\}_{i=1}^n \subset X$, then we have the relation $y \cdot ((x_i)_{i=1}^n \cdot m) = (y \triangleright x_i)_{i=1}^n \cdot (y \cdot m)$.*

Proof. The proof is straightforward from the definition of a rack action. Let $\{x_i\}_{i=1}^n \subset X$ then

$$\begin{aligned} y \cdot ((x_i)_{i=1}^n \cdot m) &= y \cdot (x_n \cdot (...(x_1 \cdot m)...)) \\ &= (y \triangleright x_n) \cdot [y \cdot (x_{n-1} \cdot (...(x_1 \cdot m)...))] \\ &= (y \triangleright x_n) \cdot [(y \triangleright x_{n-1}) \cdot [y \cdot (x_{n-2} \cdot (...(x_1 \cdot m)))]] \\ &= (y \triangleright x_n) \cdot [(y \triangleright x_{n-1}) \cdot [... \cdot (y \triangleright x_1) \cdot (y \cdot m)]]. \end{aligned}$$

Therefore, we can write $\quad y \cdot ((x_i)_{i=1}^n \cdot m) = (y \triangleright x_i)_{i=1}^n \cdot (y \cdot m)$.

■

Notation 2.2. Let X be a rack acting on a set M. Let $x \in X$, $m \in M$ and $k \in \mathbb{N}$. We write

$$x \cdot^k m = x \cdot (x \cdot (...(x \cdot m)..)) \ , \quad x \text{ acting } k\text{-times.}$$

Theorem 2.1. *Let X be a rack acting strongly on a set M. Let $x \in X$ such that $\{x\}_{i=1}^k$ is a stabilizing family of order k. Then, every element $(x_i)_{i=1}^r \cdot m$, where x appears k-times in the sequence $x_1, x_2, ..., x_r$; can be written as $(y_n)_{n=1}^{r-k} \cdot m$.*

Proof. Let $x \in X$, such that $\{x\}_{i=1}^k$ is an stabilizing family of order k and

$$\{x_i\}_{i=1}^r = \{x_1, ..., x_{j_1}, x, x_{j_1+1}, ..., x_{j_1+j_2}, x, x_{j_1+j_2+1}, ..., x_{\sum_{p=1}^k j_p}, x, x_{\sum_{p=1}^k j_p+1}, ..., x_{\sum_{p=1}^k j_p+n}\},$$

be a sequence of r elements where x appears k times. Note that $r = k + n + \sum_{p=1}^k j_p$, then we have

$$\begin{aligned} (x_i)_{i=1}^r \cdot m &= (x_i)_{i=j_1+1}^r \cdot (x \cdot [(x_i)_{i=1}^{j_1} \cdot m]) \\ &= (x_i)_{i=j_1+1}^r \cdot [(x \triangleright x_i)_{i=1}^{j_1} \cdot (x \cdot m)] \\ &= (x_i)_{i=j_1+j_2+1}^r \cdot x \cdot ((x_i)_{i=j_1+1}^{j_1+j_2} \cdot [(x \triangleright x_i)_{i=1}^{j_1} \cdot (x \cdot m)]) \\ &= (x_i)_{i=j_1+j_2+1}^r \cdot [(x \triangleright x_i)_{i=j_1+1}^{j_1+j_2} \cdot [(x \triangleright^2 x_i)_{i=1}^{j_1} \cdot (x \cdot^2 m)]) \\ &= (x_i)_{i=\sum_{p=1}^k j_p+1}^r \cdot [(x \triangleright x_i)_{i=\sum_{p=1}^{k-1} j_p+1}^{\sum_{p=1}^k j_p} \cdot [\cdots [(x \triangleright^{k-1} x_i)_{i=j_1+1}^{j_1+j_2} \cdot [((x \triangleright^k x_i)_{i=1}^{j_1} \cdot (x \cdot^k m)]. \end{aligned}$$

Since the action is strong and $\{x\}_{i=1}^k$ is an stabilizing family, then $(x \cdot^k m) = m$. Therefore,

$$(x_i)_{i=1}^r \cdot m = (x_i)_{i=\sum_{p=1}^k j_p+1}^r \cdot [(x \triangleright x_i)_{i=\sum_{p=1}^{k-1} j_p+1}^{\sum_{p=1}^k j_p} \cdot [\cdots [(x \triangleright^{k-1} x_i)_{i=j_1+1}^{j_1+j_2} \cdot [(x_i)_{i=1}^{j_1} \cdot m)].$$

Note that, there is $n + \sum_{p=1}^k j_p = r - k$ elements, therefore $(x_i)_{i=1}^r \cdot m = (y_j)_{j=1}^{r-k} \cdot m$.

■

2.2 Rack representations

With the definition of a rack action we can define a rack representation.

Definition 2.4. A ***representation*** of a rack X is a vector space V equipped with an action of the rack X, such that for all $x \in X$ the function

$$\begin{aligned} \rho_x : V &\longrightarrow V \\ v &\longmapsto x \cdot v, \end{aligned}$$

is an automorphism of V. Equivalently, a representation of X consist of a vector space and a map

$$\begin{aligned} \rho : X &\longrightarrow Conj(GL(V)) \\ x &\longmapsto \rho_x, \end{aligned}$$

which is a rack homomorphism, i.e., $\rho_{x \triangleright y} = \rho_x \rho_y \rho_x^{-1}$ for all $x, y \in X$.

Example 2.4. *Let G be a group, then every representation of G defines a representation of the quandle $Conj(G)$. Indeed, let $\rho : G \longrightarrow GL(V)$ a group representation, that means, ρ is a group homomorphism. Let $g, h \in G$, note that*

$$\rho_{g \triangleright h} = \rho_{ghg^{-1}} = \rho_g \rho_h \rho_g^{-1}.$$

Therefore, the map $\rho : Conj(G) \longrightarrow Conj(GL(V))$ is also a rack homomorphism.

▲

As in representations of groups, we can define the regular representation of a rack X.

Lemma 2.2. *Let X be a finite rack and $\mathbb{C}X$ the complex vector space, seen as the formal sums*

$$\mathbb{C}X = \{f = \sum_{x \in X} f_x x \mid f_x \in \mathbb{C} \text{ and } f_x = 0 \text{ except for finite many } x\text{'s}\}.$$

Then, the map $\lambda : X \longrightarrow Conj(GL(\mathbb{C}X))$, defined by

$$\lambda_t(f) = \lambda_t(\sum_{x \in X} f_x x) := \sum_{x \in X} f_x (t \triangleright x) = \sum_{u \in X} f_{L_t^{-1} u} u = f \circ L_t^{-1},$$

is a representation of X.

Proof. Let $t \in X$ and $f \in \mathbb{C}X$, note that $\lambda_t^{-1}(f) = f(L_t)$, indeed, we have $\lambda_t(\lambda_t^{-1}(f)) = \lambda_t(f(L_t)) = f(L_t L_t^{-1}) = f$ and $\lambda_t^{-1}(\lambda_t(f)) = \lambda_t(f(L_t^{-1})) = f(L_t^{-1} L_t) = f$. Now, let $x \in X$, from Proposition 1.2, for every $z \in X$ we have,

$$\begin{aligned} \lambda_{t \triangleright x}(f)(z) &= f(L_{t \triangleright x}^{-1}(z)) \\ &= f((L_t L_x L_t^{-1})^{-1}(z)) \\ &= (f L_t L_x^{-1})(L_t^{-1}(z)) \\ &= \lambda_t((f L_t)(L_x^{-1}(z))) \\ &= \lambda_t \lambda_x (f(L_t(z))) \\ &= \lambda_t \lambda_x \lambda_t^{-1}(f)(z). \end{aligned}$$

Therefore, $\lambda_{t \triangleright x} = \lambda_t \lambda_x \lambda_t^{-1}$.

■

Definition 2.5. Let X be a rack, the representation of Lemma 2.2 is called the ***regular representation*** of X.

Example 2.5. *Take the cyclic group of order three $\mathbb{Z}_3 = \{0, 1, 2\}$. Consider the Takasaki quandle $Q = (\mathbb{Z}_3, \rhd)$ with the operation $x \rhd y = 2x - y$. The table of this quandle is:*

$\rhd$	***0***	***1***	***2***
0	*0*	*2*	*1*
1	*2*	*1*	*0*
2	*1*	*0*	*2*

As discussed in Section 1.7, the set of functions $\{\delta_0, \delta_1, \delta_2\}$ is a base for the quandle ring $\mathbb{C}Q$. The regular representation $\lambda : Q \longrightarrow Conj(GL(\mathbb{C}Q))$ is defined by the following equalities,

$$\begin{aligned}
\lambda_0(\delta_0(x)) &= \delta_0(L_0^{-1}(x)) = \delta_0(L_0(x)) = \delta_0(x),\\
\lambda_0(\delta_1(x)) &= \delta_1(L_0^{-1}(x)) = \delta_1(L_0(x)) = \delta_2(x),\\
\lambda_0(\delta_2(x)) &= \delta_2(L_0^{-1}(x)) = \delta_2(L_0(x)) = \delta_1(x),\\
\\
\lambda_1(\delta_0(x)) &= \delta_0(L_1^{-1}(x)) = \delta_0(L_1(x)) = \delta_2(x),\\
\lambda_1(\delta_1(x)) &= \delta_1(L_1^{-1}(x)) = \delta_1(L_1(x)) = \delta_1(x),\\
\lambda_1(\delta_2(x)) &= \delta_2(L_1^{-1}(x)) = \delta_2(L_1(x)) = \delta_0(x),\\
\\
\lambda_2(\delta_0(x)) &= \delta_0(L_2^{-1}(x)) = \delta_0(L_2(x)) = \delta_1(x),\\
\lambda_2(\delta_1(x)) &= \delta_1(L_2^{-1}(x)) = \delta_1(L_2(x)) = \delta_0(x),\\
\lambda_2(\delta_2(x)) &= \delta_2(L_2^{-1}(x)) = \delta_2(L_2(x)) = \delta_2(x).
\end{aligned}$$

Therefore, with the base $\{\delta_0, \delta_1, \delta_2\}$, we can describe the regular representation $\lambda : Q \longrightarrow Conj(GL(3, \mathbb{C}))$ in matrix form as follow:

$$\lambda_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \lambda_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that the regular representation of the quandle $(\mathbb{Z}_3, \rhd)$ is not a representation of the group $\mathbb{Z}_3$.

▲

Example 2.6. *Let the permutation quandle $\mathbb{P}_3$ (see Example 1.13). The table of this quandle is:*

$\rhd$	***(2 3)***	***(1 3)***	***(1 2)***
(2 3)	*(2 3)*	*(1 2)*	*(1 3)*
(1 3)	*(1 2)*	*(1 3)*	*(2 3)*
(1 2)	*(1 3)*	*(2 3)*	*(1 2)*

This quandle is isomorphic to the Takasaki quandle $\mathbb{Z}_3$, where the isomorphism is the function $\phi : \mathbb{Z}_3 \longrightarrow \mathbb{P}_3$ defined by $\phi(0) := (2\ 3)$, $\phi(1) := (1\ 3)$ and $\phi(2) := (1\ 2)$. So, the regular representation of this quandle is the same as $\mathbb{Z}_3$, that is $\lambda : \mathbb{P}_3 \longrightarrow Conj(GL(3, \mathbb{C}))$ defined by

$$\lambda_{(2\ 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \lambda_{(1\ 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_{(1\ 2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

▲

Example 2.7. *Let us consider $X = \{1, 2, 3\}$ with the operation $i \triangleright j = \sigma(j)$ for all $i, j \in X$, where $\sigma = (1\ 2\ 3) \in \mathbb{S}_3$. $(X, \triangleright)$ is a rack (see Example 1.11) with the table*

$\triangleright$	***1***	***2***	***3***
1	*2*	*3*	*1*
2	*2*	*3*	*1*
3	*2*	*3*	*1*

Note that $L_i^3 = id$ for all $i \in X$, then $L_i^2 = L_i^{-1}$ for all $i \in X$. The regular representation $\lambda : X \longrightarrow Conj(GL(\mathbb{C}X))$ is defined by

$$\begin{aligned} \lambda_1(\delta_1(x)) &= \delta_1(L_1^{-1}(x)) = \delta_1(L_1^2(x)) = \delta_2(x), \\ \lambda_1(\delta_2(x)) &= \delta_2(L_1^{-1}(x)) = \delta_2(L_1^2(x)) = \delta_3(x), \\ \lambda_1(\delta_3(x)) &= \delta_3(L_1^{-1}(x)) = \delta_3(L_1^2(x)) = \delta_1(x). \end{aligned}$$

Since $L_1 = L_2 = L_3$ then $\lambda_1 = \lambda_2 = \lambda_3$. Then we have that

$$\lambda_1 = \lambda_2 = \lambda_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

▲

Definition 2.6. Let $\rho : X \longrightarrow Conj(GL(V))$ and $\phi : X \longrightarrow Conj(GL(W))$ two representations of a rack X. A linear map $T : V \longrightarrow W$ is called ***X-linear*** if for all $x \in X$ the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\rho_x"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\phi_x"'] & W, \end{tikzcd}$$

that means, $\phi_x T = T\rho_x$, for all $x \in X$. The representations ϕ and ρ are said to be ***equivalent*** if T is an isomorphism. We use the notation $\phi \sim \rho$ for two equivalent representations.

Definition 2.7. Let $\rho : X \longrightarrow Conj(GL(V))$ a representation of a rack X and $W \subset V$ a subspace of V, such that $\rho_x(W) \subset W$ for all $x \in X$, then W is called a ***subrepresentation***.
A representation V of X is said to be ***irreducible*** if the only subrepresentations are $W = \{0\}$ and $W = V$.

Lemma 2.3. *Let $\rho : X \longrightarrow Conj(GL(V))$ and $\phi : X \longrightarrow Conj(GL(W))$ be representations of a rack X. Let $T : V \longrightarrow W$ be a linear map. If T is X-linear then $ker(T)$ and $Im(T)$ are subrepresentations of V and W, respectively.*

Proof. Let $x \in X$ and $T : V \longrightarrow W$ be a X-linear map. Then $T\rho_x = \phi_x T$ for all $x \in X$. Let $v \in Ker(T)$, note that $T(\rho_x(v)) = \phi_x(T(v)) = \phi_x(o) = o$, then $\rho_x(v) \in ker(T)$ for all $x \in X$. It follows that $\rho_x(Ker(T)) \subset ker(T)$ for all $x \in X$.
Now, let $w \in Im(T)$ then there is a $v \in V$ such that $w = Tv$. Observe that $\phi_x(w) = \phi_x(T(v)) = T(\rho_x(v))$, since $\rho_x(v) \in V$ then $\phi_x(w) \in Im(T)$ for all $x \in X$.It follows that $\phi_x(Im(T)) \subset Im(T)$ for all $x \in X$.

■

2.3 Strong representations

Elhamdadi and Moutuou in [4] introduced the *strong representations*. In this section, we study that concept and we present new and interesting results about this type of rack representations.

Definition 2.8. A representation $\rho : X \longrightarrow GL(V)$ of a rack X is said to be ***strong*** if the action of the rack over V, is strong. That is, for every $x_1, ..., x_n \in X$ such that $L_{x_n} \cdots L_{x_2} L_{x_1} = id$, we have that $\rho_{x_n} \cdots \rho_{x_2}\rho_{x_1} = id$.

Example 2.8. *Let X be a rack and V a vector space, the trivial representation $\rho : X \longrightarrow GL(V)$ defined by $\rho_x := id$ for all $x \in X$, is strong.*

Example 2.9. *Let G be an abelian group. Suppose that $\rho : Conj(G) \longrightarrow Conj(GL(V))$ is a strong representation of the quandle $Conj(G)$. Since G is abelian, then $Z(G) = G$, from Theorem 1.5 every element g of the group G is a stabilizer of $Conj(G)$, since the representation is strong then $\rho_g = id$ for all $g \in G$. Therefore, ρ is the trivial representation. That means, every strong representation of $Conj(G)$ where G is abelian, is the trivial representation.*

Proposition 2.1. *Let X be a rack. The regular representation $\lambda : X \longrightarrow Conj(GL(\mathbb{C}X))$ is strong.*

Proof. Let $\{u_1, ..., u_n\}$ a stabilizing family of the rack X and let $f \in \mathbb{C}X$. Note that

$$\begin{aligned}
(\rho_{u_n} \cdots \rho_{u_2}\rho_{u_1})(f) &= (\rho_{u_n} \cdots \rho_{u_2})(\rho_{u_1}(f)) \\
&= (\rho_{u_n} \cdots \rho_{u_2})(f L_{u_1}^{-1}) \\
&= (\rho_{u_n} \cdots \rho_{u_3})[\rho_{u_2}(f L_{u_1}^{-1}))] \\
&= (\rho_{u_n} \cdots \rho_{u_3})(f L_{u_1}^{-1} L_{u_2}^{-1}) \\
&= f L_{u_1}^{-1} L_{u_2}^{-1} \cdots L_{u_n}^{-1} \\
&= f(L_{u_n} \cdots L_{u_2} L_{u_1})^{-1}.
\end{aligned}$$

Since $\{u_1, ..., u_n\}$ is a stabilizing family, then $L_{u_n} \cdots L_{u_1} = id$, therefore

$$\begin{aligned}
(\rho_{u_n} \cdots \rho_{u_2}\rho_{u_1})(f) &= f(L_{u_n} \cdots L_{u_2} L_{u_1})^{-1} \\
&= f.
\end{aligned}$$

■

Regarding the dimension of the irreducible strong representations of a rack X, we have a result analogous to the irreducible representations of a finite group.

Theorem 2.2. *Let X be a finite rack , then every irreducible strong representation of X is either, trivial or finite dimensional.*

Proof. Let X be a rack with n elements and (V, ρ) be a nontrivial irreducible strong representation of X. Since X has order n, then the symmetric group of X, $Sym(X)$ has order $n!$. Since all the functions L_x belongs to $Sym(X)$, then we have $L_x^{n!} = id$ for all $x \in X$. Given that the representation is strong, it follows that $\rho_x^{n!} = id$ for every $x \in X$. Now, since the representation is non trivial then $V \neq \{0\}$. Fix a non-zero vector $v \in V$. We define the subspace

$$E_v := span\{(x_i)_{i=1}^s \cdot v \mid s = 0, 1, ..., (n+1)! \text{ and } x_1, ..., x_s \in X\}.$$

We use the convention $\emptyset \cdot v = v$. Note that, E_v is a finite dimensional subspace. Furthermore, it is a subrepresentation of V. Indeed, let $y \in X$ then we have
If $s = 0$ then

$$y \cdot ((x_i)_{i=1}^0 \cdot v) = y \cdot (\emptyset \cdot v) = y \cdot v \in E_v.$$

If $s = 1$ then

$$y \cdot ((x_i)_{i=1}^1 \cdot v) = y \cdot (x_1 \cdot v) \in E_v.$$

In a similar way, we prove that $y \cdot ((x_i)_{i=1}^s \cdot v) \in E_v$, for the cases $s = 2, 3, ..., (n+1)! - 1$. Now, suppose that $s = (n+1)!$. Since X has n elements then, any sequence $(x_i)_{i=1}^{(n+1)!}$ has at least one element repeated at least $n!$ times. From Theorem 2.1 the element $(x_i)_{i=1}^{(n+1)!} \cdot v$ can be written as $(z_j)_{j=1}^{(n+1)!-n!} \cdot v$, where $z_1, ..., z_{(n+1)!-n!} \in X$. It follows that

$$y \cdot ((x_i)_{i=1}^{(n+1)!} \cdot v) = y \cdot ((z_j)_{j=1}^{(n+1)!-n!} \cdot v) \in E_v.$$

Therefore, E_v is a subrepresentation of V. Since the representation is irreducible and $V \neq \{0\}$, then $V = E_v$, where the result follows.

■

With the previous properties and definitions given in [4] we have found new results. Precisely, for a finite and connected rack X there is an interesting relation between strong representations of the rack and representations of the finite enveloping group G_X. Specifically, every irreducible strong representation of a finite connected rack X induces a irreducible representation of the group G_X. Conversely, every irreducible representation of the group G_X induces an irreducible representation (not necessarily strong) of the rack X. We prove these claims in the next discussion.

Notation 2.3. From now on, for a finite connected rack X we write n for the order of all permutations L_x and x_0 for a fixed element of X.

Theorem 2.3. *Let X be a finite connected rack and $\rho : X \longrightarrow Conj(GL(V))$ a strong representation of X. Then ρ induces a representation $\bar{\rho} : G_X \longrightarrow GL(V)$ of the finite enveloping group G_X such that $\bar{\rho}_{g_x\langle g_{x_0}^n\rangle} := \rho_x$ for all $x \in X$. Furthermore, if ρ is an irreducible rack representation then $\bar{\rho}$ is an irreducible group representation.*

Proof. Since $\rho : X \longrightarrow Conj(GL(V))$ is a rack homomorphism then from Theorem 1.2, ρ induces a group homomorphism $\hat{\rho} : As(X) \longrightarrow GL(V)$ such that $\hat{\rho}_{g_x} = \rho_x$ for all $x \in X$. Now, fix $x_0 \in X$, note that $\hat{\rho}_{g_{x_0}^n} = \hat{\rho}_{g_{x_0}}^n = \rho_{x_0}^n$. Since n is the order of all permutations L_x, then $L_{x_0}^n = id$. Due to the representation ρ is strong, we have $\hat{\rho}_{g_{x_0}^n} = \rho_{x_0}^n = id$. Therefore, $\langle g_{x_0}^n\rangle \subset ker(\hat{\rho})$. This implies that there exists a group homomorphism $\bar{\rho} : As(X)/\langle g_{x_0}^n\rangle \longrightarrow GL(V)$ such that $\bar{\rho}_{g_x\langle g_{x_0}^n\rangle} = \rho_x$ for all $x \in X$.
Now, suppose that ρ is irreducible and suppose that W is a subspace of V such that $\bar{\rho}_h(W) \subset W$ for all $h \in G_X$. In particular, we have that $\bar{\rho}_{g_x\langle g_{x_0}^n\rangle}(W) = \rho_x(W) \subset W$ for all $x \in X$. Since ρ is irreducible then $W = \{0\}$ or $W = V$. Therefore, $\bar{\rho}$ is also irreducible.

■

Theorem 2.4. *Let X be a finite connected rack. Let $\rho : X \longrightarrow Conj(GL(V))$ and $\phi : X \longrightarrow Conj(GL(V'))$ be strong representations of X such that $\rho \sim \phi$. Then, the group representations $\bar{\rho} : G_X \longrightarrow GL(V)$ and $\bar{\phi} : G_X \longrightarrow GL(V')$ are also equivalents.*

Proof. Since $\rho \sim \phi$ then there exists an isomorphism $T : V' \longrightarrow V$ such that $\rho_x T = T\phi_x$ for all $x \in X$. Let $h \in G_X$, from the proof of Theorem 1.4 the element h is the form $h = g_{x_1}^{e_1} \cdots g_{x_m}^{e_m}\langle g_{x_0}^n\rangle$ where $m \leq |G_X|$, $x_i \in X$ for all $i \in \{1, ..., m\}$ and $e_i \in \{0, 1, ..., n-1\}$. Then,

$$\begin{aligned}
\bar{\rho}_h T &= \bar{\rho}_{g_{x_1}^{e_1}\cdots g_{x_m}^{e_m}\langle g_{x_0}^n\rangle} T \\
&= \bar{\rho}_{g_{x_1}\langle g_{x_0}^n\rangle}^{e_1} \cdots \bar{\rho}_{g_{x_m}\langle g_{x_0}^n\rangle}^{e_m} T \\
&= \rho_{x_1}^{e_1} \cdots \rho_{x_m}^{e_m} T.
\end{aligned}$$

Note that, for every $i \in \{1, ..., m\}$ we have that

$$\begin{aligned}
\rho_{x_i}^{e_i} T &= \rho_{x_i}^{e_i-1}\rho_{x_i} T \\
&= \rho_{x_i}^{e_i-1} T\phi_{x_i} \\
&= \rho_{x_i}^{e_i-2}\rho_{x_i} T\phi_{x_i} \\
&= \rho_{x_i}^{e_i-2} T\phi_{x_i}^2 \\
&\vdots \\
&= T\phi_{x_i}^{e_i}.
\end{aligned}$$

Therefore, $\rho_{x_1}^{e_1} \cdots \rho_{x_m}^{e_m} T = \rho_{x_1}^{e_1} \cdots T\phi_{x_m}^{e_m} = T\phi_{x_1}^{e_1} \cdots \phi_{x_m}^{e_m}$. Hence,

$$\begin{aligned}
\bar{\rho}_h T &= \rho_{x_1}^{e_1} \cdots \rho_{x_m}^{e_m} T \\
&= T\phi_{x_1}^{e_1} \cdots \phi_{x_m}^{e_m} \\
&= T\bar{\phi}_{g_{x_1}\langle g_{x_0}^n\rangle}^{e_1} \cdots \bar{\phi}_{g_{x_m}\langle g_{x_0}^n\rangle}^{e_m} \\
&= T\bar{\phi}_{g_{x_1}^{e_1}\langle g_{x_0}^n\rangle} \cdots \bar{\phi}_{g_{x_m}^{e_m}\langle g_{x_0}^n\rangle} \\
&= T\bar{\phi}_{g_{x_1}^{e_1}\cdots g_{x_m}^{e_m}\langle g_{x_0}^n\rangle} \\
&= T\bar{\phi}_h.
\end{aligned}$$

Hence, the result follows.

■

The theorems above allow us to translate questions about strong representations of a finite connected rack into questions about representations of a finite group. That means, we can apply the theory of representations of finite groups, such that Schur's lemma, orthogonal relations or character theory. For example, we have the following corollaries.

Corollary 2.1. *Let X be a finite connected rack, then the number of irreducible strong complex representations of X (up to equivalence) is less than or equal to the conjugacy classes of the finite enveloping group G_X.*

Proof. Suppose that the number of conjugacy classes of G_X is $k \in \mathbb{N}$. From representation theory of finite groups we have that the number of irreducible complex representations (up to equivalence) of the finite enveloping group G_X is equal to the number of conjugacy classes of G_X. Now, reasoning by contradiction, suppose that the number of irreducible strong representations of X is $m > k$. Let $\rho_1, ..., \rho_m$ be the distinct representatives of irreducible strong representations (up to equivalence) of X. Then, from Theorem 2.3 every representative ρ_i induces a irreducible representation $\bar{\rho}_i$ of the group G_X. Since $\rho_i \not\sim \rho_j$ for all $i \neq j \in \{1, 2..., m\}$, then from Theorem 2.4 $\bar{\rho}_i \not\sim \bar{\rho}_j$ for all $i \neq j \in \{1, 2..., m\}$. Therefore, G_X would have $m > k$ irreducible representations (up to equivalence), which is a contradiction. Thus $m \leq k$.

■

Corollary 2.2. *Let X be a finite connected rack whose finite enveloping group G_X is abelian. For every irreducible strong representation $\rho : X \longrightarrow Conj(GL(V))$ of the rack X, we have that the vector space V is one dimensional.*

Proof. Let $\rho : X \longrightarrow Conj(GL(V))$ be an irreducible strong representation of X. From Theorem 2.3, ρ induces a irreducible representation $\bar{\rho} : G_X \longrightarrow GL(V)$ such that $\bar{\rho}_{g_x\langle g_{x_0}^n\rangle} = \rho_x$ for all $x \in X$. Since G_X is an abelian group and $\bar{\rho}$ is irreducible, then V is one dimensional, that is, there exists $v \in V$ such that $\langle v \rangle = V$.

■

If we have a representation of the finite enveloping group G_X of a rack X then we can define a representation of the rack X as it is given in the next Theorem.

Theorem 2.5. *Let X be a finite connected rack. Let $\bar{\rho} : G_X \longrightarrow GL(V)$ a representation of the finite enveloping group of X. Consider the function $\rho : X \longrightarrow Conj(GL(V))$ defined by $\rho_x := \bar{\rho}_{g_x\langle g_{x_0}^n\rangle}$ for all $x \in X$. Then ρ is a representation of the rack X. Furthermore, if $\bar{\rho}$ is irreducible, ρ is too.*

Proof. Let us see that ρ is well defined. Let $x, y \in X$ such that $x = y$. Therefore, $g_x\langle g_{x_0}^n\rangle = g_y\langle g_{x_0}^n\rangle$. It follows that $\rho_x = \bar{\rho}_{g_x\langle g_{x_0}^n\rangle} = \bar{\rho}_{g_y\langle g_{x_0}^n\rangle} = \rho_y$, so ρ is well defined. From the definition of G_X, we have the relation $g_{x\rhd y}\langle g_{x_0}^n\rangle = g_x g_y g_x^{-1}\langle g_{x_0}^n\rangle$, for all $x, y \in X$. Then,

$$\rho_{x\rhd y} = \bar{\rho}_{g_{x\rhd y}\langle g_{x_0}^n\rangle} = \bar{\rho}_{g_x g_y g_x^{-1}\langle g_{x_0}^n\rangle} = \bar{\rho}_{g_x\langle g_{x_0}^n\rangle}\bar{\rho}_{g_y\langle g_{x_0}^n\rangle}\bar{\rho}^{-1}_{g_x\langle g_{x_0}^n\rangle} = \rho_x\rho_y\rho_x^{-1}.$$

Therefore, ρ is a representation of the rack X. Suppose that $\bar{\rho}$ is irreducible. Let W a subspace of V such that $\rho_x(W) \subset W$ for all $x \in X$. First, we claim that $\rho_x^k(W) \subset W$ for all $k \in \mathbb{N}$ and all $x \in X$. Indeed, suppose that for every $x \in X$ we have $\rho_x^t(W) \subset W$ for some $t \in \mathbb{N}$. Note that $\rho_x^{t+1}(W) = \rho_x \rho_x^t(W) = \rho_x[\rho_x^t(W)]$, since $\rho_x^t(W) \subset W$, then we have $\rho_x^{t+1}(W) = \rho_x[\rho_x^t(W)] \subset W$, so the result follows by induction.
Let $h \in G_X$, from the proof of Theorem 1.4, the element h is the form $h = g_{x_1}^{e_1} \cdots g_{x_m}^{e_m} \langle g_{x_0}^n \rangle$ where $m \leq |G_X|$, $x_i \in X$ for all $i \in \{1, ..., m\}$ and $e_i \in \{0, 1, ..., n-1\}$. Then,

$$\begin{aligned} \bar{\rho}_h(W) &= \bar{\rho}_{g_{x_1}^{e_1} \cdots g_{x_m}^{e_m} \langle g_{x_0}^n \rangle}(W) \\ &= \bar{\rho}^{e_1}_{g_{x_1} \langle g_{x_0}^n \rangle} \cdots \bar{\rho}^{e_m}_{g_{x_m} \langle g_{x_0}^n \rangle}(W) \\ &= \rho_{x_1}^{e_1} \cdots \rho_{x_m}^{e_m}(W). \end{aligned}$$

Since $\rho_{x_i}^{e_1}(W) \subset W$ and $\rho_{x_i}^{e_2}(W) \subset W$ then,

$$\rho_{x_1}^{e_1} \rho_{x_2}^{e_2}(W) = \rho_{x_1}^{e_1}[\rho_{x_2}^{e_2}(W)] \subset W.$$

Suppose that $\rho_{x_1}^{e_1} \cdots \rho_{x_i}^{e_i}(W) \subset W$ for some $i \in \{2, 3, .., m-1\}$. Since $\rho_{x_{i+1}}^{e_{i+1}}(W) \subset W$ it follows that $\rho_{x_1}^{e_1} \cdots \rho_{x_i}^{e_i} \rho_{x_{i+1}}^{e_{i+1}}(W) = (\rho_{x_1}^{e_1} \cdots \rho_{x_i}^{e_i})[\rho_{x_{i+1}}^{e_{i+1}}(W)] \subset W$. Thus, from induction over i we have $\rho_{x_1}^{e_1} \cdots \rho_{x_m}^{e_m}(W) \subset W$. Therefore, $\bar{\rho}_h(W) \subset W$ for all $h \in G_X$. Since $\bar{\rho}$ is irreducible then $W = \{0\}$ or $W = V$. Hence ρ is an irreducible representation of X.

■

Theorem 2.6. *Let X be a finite connected rack. Let $\bar{\rho} : G_X \longrightarrow GL(V)$ and $\bar{\phi} : G_X \longrightarrow GL(V')$ be representations of the group G_X such that $\bar{\rho} \sim \bar{\phi}$. Then, the rack representations $\rho : X \longrightarrow Conj(GL(V))$ and $\phi : X \longrightarrow Conj(GL(V'))$ defined as in the previous theorem, are also equivalent.*

Proof. Since $\bar{\rho} \sim \bar{\phi}$ then there exist an isomorphism $T : V' \longrightarrow V$ such that $\bar{\rho}_h T = T\bar{\phi}_h$ for all $h \in G_X$. In particular we have that, $\bar{\rho}_{g_x \langle g_{x_0}^n \rangle} T = T\bar{\phi}_{g_x \langle g_{x_0}^n \rangle}$ for all $x \in X$. Thus, $\rho_x T = \bar{\rho}_{g_x \langle g_{x_0}^n \rangle} T = T\bar{\phi}_{g_x \langle g_{x_0}^n \rangle} = T\phi_x$ for all $x \in X$.Therefore, $\rho \sim \phi$.

■

From Theorem 2.5, we can obtain a representation of a finite connected rack X from its finite enveloping group G_X. This representation may not necessarily be strong, but under certain conditions we can ensure this property.

Theorem 2.7. *Let X be a finite connected rack and $\bar{\rho} : G_X \longrightarrow GL(V)$ be a representation of the finite enveloping group G_X. If G_X has trivial center, that is, $Z(G_X) = \{1\}$ then the rack representation $\rho : X \longrightarrow Conj(GL(V))$ defined by $\rho_x := \rho_{g_x \langle g_{x_0}^n \rangle}$, is strong.*

Proof. Let $\{x_1, ..., x_k\}$ be a stabilizing family of the rack X. That means, $x_k \triangleright (x_{n-1} \triangleright (\cdots (x_1 \triangleright x) \cdots)) = x$ for all $x \in X$. Since in the associated group $As(X)$ we have the relation $g_{x \triangleright y} = g_x g_y g_x^{-1}$, for all $x, y \in X$, then for every $x \in X$ we have

$$\begin{aligned} g_x &= g_{x_k \triangleright (x_{k-1} \triangleright (\cdots (x_1 \triangleright x) \cdots))} \\ &= g_{x_k} g_{x_{k-1} \triangleright (\cdots (x_1 \triangleright x) \cdots)} g_{x_k}^{-1} \\ &= g_{x_k} g_{x_{k-1}} g_{x_{k-2} \triangleright (\cdots (x_1 \triangleright x) \cdots)} g_{x_{k-1}}^{-1} g_{x_k}^{-1} \\ &\vdots \\ &= (g_{x_k} \cdots g_{x_1}) g_x (g_{x_1}^{-1} \cdots g_{x_k}^{-1}) \\ \therefore \ (g_{x_k} \cdots g_{x_1}) g_x &= g_x (g_{x_k} \cdots g_{x_1}). \end{aligned}$$

Thus, the word $g_{x_k}g_{x_{k-1}}\cdots g_{x_1}$ belongs to the center of the group $As(X)$ and therefore $g_{x_k}g_{x_{k-1}}\cdots g_{x_1}\langle g_{x_0}^n\rangle \in Z(G_X)$. Since $Z(G_X)=\{1\}$, $g_{x_k}g_{x_{k-1}}\cdots g_{x_1}\langle g_{x_0}^n\rangle = 1$.
Note that,

$$\rho_{x_k}\cdots\rho_{x_1} = \bar{\rho}_{g_{x_k}\langle x_0^n\rangle}\cdots\bar{\rho}_{g_{x_1}\langle g_{x_0}^n\rangle} = \bar{\rho}_{g_{x_k}g_{x_{k-1}}\cdots g_{x_1}\langle g_{x_0}^n\rangle} = \bar{\rho}_1 = id.$$

It follows that the representation ρ is strong.

■

Corollary 2.3. *Let X be a finite connected rack. If $Z(G_X)=\{1\}$, then the number of irreducible strong complex representations of X (up to equivalence) is equal to the number of irreducible complex representations of the group G_X.*

Proof. By previous theorems there exists a bijective correspondence between irreducible strong representations of the rack X and the irreducible representations of the group G_X.

■

The next examples illustrate the previous results.

Example 2.10. *Let the permutation quandle $\mathbb{P}_3$. In Example 2.6 we found the regular representation of $\mathbb{P}_3$, which is $\lambda:\mathbb{P}_3\longrightarrow Conj(GL(3,\mathbb{C}))$ defined by*

$$\lambda_{(2\ 3)} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \lambda_{(1\ 3)} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\lambda_{(1\ 2)} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Remark that $G_{\mathbb{P}_3}\cong\mathbb{S}_3$ (see Proposition 1.4) where the isomorphism $\mu: G_{\mathbb{P}_3}\longrightarrow\mathbb{S}_3$ is given by $\mu_{g_{(ij)}\langle g_{(12)}^2\rangle} = (i\ j)$ for all $(i\ j)\in\mathbb{P}_3$. From Proposition 2.1 the regular representation λ is strong. Then it induces a representation $\bar{\lambda}:\mathbb{S}_3\longrightarrow GL(3,\mathbb{C})$ of the finite enveloping group $G_{\mathbb{P}_3}\cong\mathbb{S}_3$ defined by

$$\bar{\lambda}_{(2\ 3)} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \bar{\lambda}_{(1\ 3)} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\bar{\lambda}_{(1\ 2)} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Note that, the subspace $W = span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is invariant under the representation $\bar{\lambda}$. Thus, this representation is reducible and decomposable. By the representation theory of finite groups, we know that every representation can be written as an unique (up to equivalence) direct sum of irreducible representations. Specifically, we have that $\bar{\lambda}\sim\bar{\phi}\oplus\bar{\psi}$, where $\bar{\phi}:\mathbb{S}_3\longrightarrow\mathbb{C}^*$ and $\bar{\psi}:\mathbb{S}_3\longrightarrow GL(2,\mathbb{C})$ are irreducible representations defined by $\bar{\phi}(g)=1$ for all $g\in\mathbb{S}_3$ and

$$\bar{\psi}_{(1\ 2)} := \begin{bmatrix}-1&-1\\0&1\end{bmatrix},\ \bar{\psi}_{(1\ 3)} := \begin{bmatrix}1&0\\-1&-1\end{bmatrix},\ \bar{\psi}_{(2\ 3)} := \begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

From Theorem 2.5, the group representations $\bar{\phi}$ and $\bar{\psi}$ induce rack representations $\phi : \mathbb{P}_3 \longrightarrow Conj(GL(\mathbb{C}^*))$ and $\psi : \mathbb{P}_3 \longrightarrow Conj(GL(2,\mathbb{C}))$ defined by $\phi_x := 1$ for all $x \in \mathbb{P}_3$ and

$$\psi_{(1\ 2)} := \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},\ \psi_{(1\ 3)} := \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},\ \psi_{(2\ 3)} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From Theorem 2.6, we have that $\lambda \sim \phi \oplus \psi$. Indeed, let $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$. It can be checked that $(\phi \oplus \psi)_x T = T\lambda_x$ for all $x \in \mathbb{P}_3$.
Since the finite enveloping group $G_{\mathbb{P}_3} \cong \mathbb{S}_3$ has trivial center, then by Theorem 2.7 the representations ϕ and ψ are strong.
It is well known that the group $\mathbb{S}_3$ has three (up to equivalence) irreducible representations, then from Corollary 2.3, the number of irreducible strong representations of $\mathbb{P}_3$ is equal to 3. Previously, we found two irreducible strong representations of $\mathbb{P}_3$, from the knowledge of representations of finite groups we can find the last one. The other irreducible representation of $\mathbb{S}_3$ is $\bar{\tau} : \mathbb{S}_3 \longrightarrow GL(\mathbb{C}^*)$ defined by

$$\bar{\tau}_\sigma := \begin{cases} 1 & if \quad \sigma \text{ is even} \\ \\ -1 & if \quad \sigma \text{ is odd} \end{cases}$$

Thus, we have a rack representation $\tau : \mathbb{P}_3 \longrightarrow Conj(GL(\mathbb{C}^*))$ defined by

$$\begin{aligned} \tau_{(2\ 3)} &:= \bar{\tau}_{g_{(23)}\langle g^2_{(12)}\rangle} = \bar{\tau}_{(2\ 3)} = -1, \\ \tau_{(1\ 3)} &:= \bar{\tau}_{g_{(13)}\langle g^2_{(12)}\rangle} = \bar{\tau}_{(1\ 3)} = -1, \\ \tau_{(1\ 2)} &:= \bar{\tau}_{G_{(12)}\langle g^2_{(12)}\rangle} = \bar{\tau}_{(1\ 2)} = -1. \end{aligned}$$

Therefore, we have $\tau_{(i\ j)} = -1$ for all $(i\ j) \in \mathbb{P}_3$. From Theorem 2.7, the representation τ is strong. Hence, the permutation quandle $\mathbb{P}_3$ has three (up to equivalence) irreducible strong representations.

▲

Example 2.11. *Let $X = \{1,2,3\}$ be the rack given in the Example 2.7. The operation of this rack is defined as $i \triangleright j = \sigma(j)$, for all $i, j \in X$, where $\sigma = (1\ 2\ 3) \in \mathbb{S}_3$. The regular representation $\lambda : X \longrightarrow Conj(GL(\mathbb{C}X))$ is defined by*

$$\lambda_1 = \lambda_2 = \lambda_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Previously, we found the finite enveloping group of this rack, which is $G_X = sapn\{g_1\langle g_1^3\rangle\} \cong \mathbb{Z}_3$. Therefore, λ induces a group representation $\bar{\lambda} : G_X \longrightarrow GL(3,\mathbb{C})$ defined by:

$$\bar{\lambda}_{g_1\langle g_1^3\rangle} = \bar{\lambda}_{g_2\langle g_1^3\rangle} = \bar{\lambda}_{g_3\langle g_1^3\rangle} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The irreducible representations of $G_X \cong \mathbb{Z}_3$ are one dimensional and they are the cube roots of unity, that is $\bar{\rho}(g_1\langle g_1^3\rangle) = 1$, $\bar{\psi}(g_1\langle g_1^3\rangle) = w$, $\bar{\phi}(g_1\langle g_1^3\rangle) = w^2$ where $w = e^{2\pi i/3}$. The character table of this group representation is

	$\langle g_1^3 \rangle$	$g_1\langle g_1^3 \rangle$	$g_1^2\langle g_1^3 \rangle$
$\chi_{\bar{\rho}}$	*1*	*1*	*1*
$\chi_{\bar{\psi}}$	*1*	w	w^2
$\chi_{\bar{\phi}}$	*1*	w^2	w
$\chi_{\bar{\lambda}}$	*3*	*0*	*0*

From we have that $\bar{\lambda} \sim \bar{\rho} \oplus \bar{\psi} \oplus \bar{\phi}$. Now, the irreducible representations of G_X induces irreducible representations of the rack X defined by

$$\begin{aligned} \rho_i &= 1 \quad \forall i \in X, \\ \psi_i &= w \quad \forall i \in X, \\ \phi_i &= w^2 \quad \forall i \in X. \end{aligned}$$

Therefore, we have that $\lambda \sim \rho \oplus \psi \oplus \phi$. The representation ρ is the trivial one, so it is strong. We claim that the representations ψ and ϕ are also strong. Indeed, let $\{i_1, ..., i_k\}$ be a stabilizing family of the rack X. Note that for every $j \in X$ we have that

$$\begin{aligned} j &= i_k \triangleright (\cdots \triangleright (i_2 \triangleright (i_1 \triangleright j)) \cdots) \\ &= i_k \triangleright (\cdots \triangleright (i_2 \triangleright \sigma(j) \cdots) \\ &= i_k \triangleright (\cdots (i_3 \triangleright \sigma^2(j) \cdots) \\ &= \sigma^k(j) \end{aligned}$$

Therefore, $\sigma^k = id$. Since the order of σ is 3, then k is of the form $k = 3n$ for some $n \in \mathbb{N}$. Note that,

$$\begin{aligned} \psi_{i_k} \cdots \psi_{i_1} &= w^k = w^{3n} = 1 \\ \phi_{i_k} \cdots \phi_{i_1} &= w^{2k} = w^{6n} = 1 \end{aligned}$$

Therefore, the representations are strong. Hence, the rack X has 3 (up to equivalence) irreducible strong representations.

▲

Elhamdadi and Moutou, in [4], stated the theorem: *"Theorem 9.11: Every strong irreducible representation of a finite connected involutive rack is one–dimensional"*. The reader can see that the Example 2.10 is a counterexample to this theorem. Indeed, the permutation quandle $\mathbb{P}_3$ is finite, connected and involutive, however it has one (up to equivalence) strong irreducible representation of dimension two. Furthermore, since for every $k \in \mathbb{N}$ we have $\mathbb{S}_k \cong G_{\mathbb{P}_k}$; then for all $k \in \mathbb{N}$, the permutation quandle $\mathbb{P}_k$, which is finite, involutive and connected, has at least one irreducible strong representation with dimension larger than one. Therefore, we can form an infinite family of strong representations that contradict such theorem.

Chapter 3

Connections Between Racks and g-digroups

The coquecigrue problem was proposed by J. L. Loday following the idea of a possible extension of Lie's third theorem for Leibniz algebras, that is, the problem consists in finding an appropriate structure that generalizes the concept of Lie group and whose algebra is the corresponding Leibniz algebra. One approximation to the solution of this problem was proposed independently by M. Kinyon, R. Felipe , and K. Liu , who defined the digroup structure which is generalization of the group structure with two products.
In this context, in [15] O. Salazar-Díaz, R. Velásquez and L. A. Wills-Toro proposed a structure called generalized digroups or simply g-digroups, as a more general extension of the group structure. Since g-digroups are a extension of groups, one might think that the relations that exist between quandles and groups can be extended to g -digroups. Actually, there is a interesting relation between the rack structure and the g-digroup structure. In this chapter we study this relation.

3.1 Generalized digroups

In this section we give a short review of some definitions and results about g-digroups, given in [15].

Definition 3.1. A set D is called a ***g-digroup (generalized digroup)*** if it has two binary operations $\vdash$ and $\dashv$ over D, which are associative (each separately), and satisfy the conditions:

(D1) $x \vdash (y \dashv z) = (x \vdash y) \dashv z$

(D2) $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
$(x \vdash y) \vdash z = (x \dashv y) \vdash z$

(D3) There exists (at least) an element e in D, such that for all $x \in D$, $x \dashv e = e \vdash x = x$.
The elements that satisfy this condition are called *bar - units* and the *set of bar - units* in D, denoted by E is called the *halo* of D.

(D4) For a fixed bar-unit e, we have that for each $x \in D$ there exist $x_{r_e}^{-1}$ and $x_{l_e}^{-1}$ in D (the right inverse of x and the left inverse of x, respectively) such that $x \vdash x_{r_e}^{-1} = e$ and $x_{l_e}^{-1} \dashv x = e$.

The following propositions summarizes some basic properties of g-digroups.

Proposition 3.1. *Let D be a g-digroup and e a fixed bar unit then*

(1) For all $x \in D$, $x_{l_e}^{-1} \vdash e$ is a right inverse of x and $e \dashv x_{r_e}^{-1}$ is a left inverse of x. Furthermore, $x_{l_e}^{-1} \vdash e = x_{r_e}^{-1}$ and $e \dashv x_{r_e}^{-1} = x_{l_e}^{-1}$

(2) For all $x \in D$, the elements $x_{l_e}^{-1} \vdash x$ and $x \dashv x_{r_e}^{-1}$ are bar-units.

Proof.

(1) Note that

$$\begin{aligned} x \vdash (x_{l_e}^{-1} \vdash e) &= x \vdash [x_{l_e}^{-1} \vdash (x \vdash x_{r_e}^{-1})] \\ &= x \vdash [(x_{l_e}^{-1} \vdash x) \vdash x_{r_e}^{-1}] \\ &= x \vdash [(x_{l_e}^{-1} \dashv x) \vdash x_{r_e}^{-1}] \\ &= x \vdash (e \vdash x_{r_e}^{-1} \\ &= x \vdash x_{r_e}^{-1} \\ &= e \end{aligned}$$

Then $x_{l_e}^{-1} \vdash e$ is a right inverse. Now, observe that $x_{l_e}^{-1} \vdash e = x_{l_e}^{-1} \vdash (x \vdash x_{r_e}^{-1}) = (x_{l_e}^{-1} \vdash x) \vdash x_{r_e}^{-1} = (x_{l_e}^{-1} \dashv x) \vdash x_{r_e}^{-1} = e \vdash x_{r_e}^{-1} = x_{r_e}^{-1}$. Similarly, we can prove that $e \dashv x_{r_e}^{-1} = x_{l_e}^{-1}$.

(2) We have

$$x \dashv (x_{l_e}^{-1} \vdash x) = x \dashv (x_{l_e}^{-1} \dashv x) = x \dashv e = x \text{ and } (x_{l_e}^{-1} \vdash x) \vdash x = (x_{l_e}^{-1} \dashv x) \vdash x = e \vdash x = x$$

Thus, $x_{l_e}^{-1} \vdash x$ is a bar- unit. Similarly we prove that $x \dashv x_{r_e}^{-1}$ is also a bar-unit.

■

The previous proposition suggest that for a fixed $e \in E$ the right inverse and the left inverse of $x \in D$ are unique. Indeed, suppose that there exists $y \in X$ such that $y \dashv x = e$ then

$$\begin{aligned} y &= y \dashv (x \vdash x_{r_e}^{-1}) \\ &= y \dashv (x \vdash (x_{l_e}^{-1} \vdash e)) \\ &= y \dashv (x \dashv (x_{l_e}^{-1} \vdash e)) \\ &= (y \dashv x) \dashv (x_{l_e}^{-1} \vdash e) \\ &= e \dashv (x_{l_e}^{-1} \vdash e) \\ &= \dashv x_{r_e}^{-1} \\ &= x_{l_e}^{-1} \end{aligned}$$

Therefore, the left inverse of x is unique. Similarly, we prove that $x_{r_e}^{-1}$ is unique.

Proposition 3.2. *Let D be a g-digroup and e a fixed bar unit. Then, for all $x, y \in D$,*

(1) Given $x \in D$ we have that $(x_{l_e}^{-1})_{l_e}^{-1} = e \dashv x$ and $(x_{r_e}^{-1})_{r_e}^{-1} = x \vdash e$.

(2) The inverse of the products are $(x \vdash y)_{l_e}^{-1} = (x \dashv y)_{l_e}^{-1} = y_{l_e}^{-1} \dashv x_{l_e}^{-1}$ and $(x \dashv y)_{r_e}^{-1} = (x \vdash y)_{r_e}^{-1} = y_{r_e}^{-1} \vdash x_{r_e}^{-1}$

Proof.

(1) Note that,

$$\begin{aligned}(e \dashv x) \dashv x_{l_e}^{-1} &= e \dashv (x \dashv x_{l_e}^{-1}) \\ &= e \dashv [x \dashv (x_{l_e}^{-1} \dashv e)] \\ &= e \dashv [x \dashv (x_{l_e}^{-1} \vdash e)] \\ &= e \dashv (x \dashv x_{r_e}^{-1}) \\ &= e \dashv e = e\end{aligned}$$

Thus, $(x_{l_e}^{-1})_{l_e}^{-1} = e \dashv x$. Similarly it can be proved that $(x_{r_e}^{-1})_{r_e}^{-1} = x \vdash e$.

(2) We have

$$\begin{aligned}(x \dashv y) \vdash (y_{r_e}^{-1} \vdash x_{r_e}^{-1}) &= (x \vdash y) \vdash (y_{r_e}^{-1} \vdash x_{r_e}^{-1}) \\ &= x \vdash [(y \vdash y_{r_e}^{-1}) \vdash x_{r_e}^{-1}] \\ &= x \vdash [e \vdash x_{r_e}^{-1}] \\ &= x \vdash x_{r_e}^{-1} \\ &= e\end{aligned}$$

Therefore, $(x \dashv y)_{r_e}^{-1} = (x \vdash y)_{r_e}^{-1} = y_{r_e}^{-1} \vdash x_{r_e}^{-1}$. In a similar way we can prove that $(x \vdash y)_{l_e}^{-1} = (x \dashv y)_{l_e}^{-1} = y_{l_e}^{-1} \dashv x_{l_e}^{-1}$.

■

Let $e \in E$ be a bar-unit. We define the sets of left and right inverses, denoted by G_l^e and G_r^e, respectively, as follows

$$G_l^e := \{x_{l_e}^{-1} \mid x \in D\} \text{ and } G_r^e := \{x_{r_e}^{-1} \mid x \in D\}$$

It is not hard to prove that $(G_l^e, \dashv)$ and $(G_r^e, \vdash)$ are isomorphic groups with identity e [15].
Observe that given $\xi, e \in E$ and $x \in D$ we have that

$$(\xi \dashv x_{l_e}^{-1}) \dashv x = \xi \dashv (x_{l_e}^{-1} \dashv x) = \xi \dashv e = \xi$$

Since the left inverse is unique we have that $x_{l_\xi}^{-1} = \xi \dashv x_{l_e}^{-1}$. Similarly, $x_{r_\xi}^{-1} = x_{r_e}^{-1} \vdash \xi$.
There is an interesting relation between g-digroups and pairs of the form (G, X) where G is a group and X is a $G-$set.

Theorem 3.1. *Let G be a group with unity e and X a $G-$set under the action $(g, x) \longmapsto g \bullet x$. Then $D := G \times X$ is a g-digroup with operations*

$$\begin{aligned}(a, \alpha) \vdash (b, \beta) &= (ab, a \bullet \beta) \\ (a, \alpha) \dashv (b, \beta)) &= (ab, \alpha)\end{aligned}$$

Proof. Let's see that the operations $\vdash$ and $\dashv$ satisfy the axioms of g-digroups

(D1) $(a, \alpha) \vdash [(b, \beta) \dashv (m, \mu)] = (a, \alpha) \vdash (bm, \beta) = (abm, a \bullet \beta)$, on the other hand $[(a, \alpha) \vdash (b, \beta)] \dashv (m, \mu) = [(ab, a \bullet \beta) \dashv (m, \mu) = (abm, a \bullet \beta)$.

(D2) $(a,\alpha) \dashv [(b,\beta) \dashv (m,\mu)] = (a,\alpha) \dashv (bm,\beta) = (abm,\alpha)$, on the other hand $(a,\alpha) \dashv [(b,\beta) \vdash (m,\mu)] = (a,\alpha) \dashv (bm, b \bullet \mu) = (abm,\alpha)$.
Further, note that $[(a,\alpha) \vdash (b,\beta)] \vdash (m,\mu) = (ab, a \bullet \beta) \vdash (m,\mu) = (abm, ab \bullet \mu)$, on the other hand $[(a,\alpha) \dashv (b,\beta)] \vdash (m,\mu) = (ab,\alpha) \dashv (m,\mu) = (abm, ab \bullet \mu)$.

(D3) Let $(e,\alpha) \in G \times X$. Note that, for every $(b,\beta) \in G \times X$ we have, $(b,\beta) \dashv (e,\alpha) = (be,\beta) = (b,\beta)$ and $(e,\alpha) \vdash (b,\beta) = (eb, e \bullet \beta) = (b,\beta)$. Therefore, (e,α) is a bar-unit for all $\alpha \in X$. Note that, if (g,α) a bar unit of D then $g = e$.

(D4) Let $\xi := (e,\alpha)$ a bar unit of D, for any $(b,\beta) \in G \times X$ we have $(b^{-1},\alpha) \dashv (b,\beta) = (b^{-1}b,\alpha) = (e,\alpha)$ and $(b,\alpha) \vdash (b^{-1}, b^{-1} \bullet \alpha) = (bb^{-1}, b \bullet (b^{-1} \bullet \alpha)) = (e,\alpha)$. Therefore, $(b,\beta)^{-1}_{l_\xi} = (b^{-1},\alpha)$ and $(b,\beta)^{-1}_{r_\xi} = (b^{-1}, b^{-1} \bullet \alpha)$.

■

Definition 3.2. Let D and D' are g-digroups, a map $\phi : D \longrightarrow D'$ is a ***g-digroup homomorphism*** if for any $x, y \in D$ we have

$$\phi(x \vdash y) = \phi(x) \vdash \phi(y) \text{ and } \phi(x \dashv y) = \phi(x) \vdash \phi(y)$$

If ϕ is bijective then it is a g-digroup *isomorphism*.

3.2 Racks and g-digroups

Let D be a g-digroup, e, ξ a pair of bar-units and $x, y \in D$. Note that

$$y \dashv x^{-1}_{l_\xi} = y \dashv (\xi \dashv x^{-1}_{l_e}) = (y \dashv \xi) \dashv x^{-1}_{l_e} = y \dashv x^{-1}_{l_e}$$

Thus, $y \dashv x^{-1}_{l_\xi} = y \dashv x^{-1}_{l_e}$ for any $e, \xi \in E$. Furthermore, by Proposition 3.1 (1),

$$y \dashv x^{-1}_{r_e} = y \dashv (x^{-1}_{l_e} \vdash e) = y \dashv (x^{-1}_{l_e} \dashv e) = y \dashv x^{-1}_{l_e}$$

By the previous observation we just write $y \dashv x^{-1}$, where x^{-1} can be a right or left inverse for any bar-unit. Similarly, we write $x^{-1} \vdash y$ where x^{-1} can be a right or left inverse for any bar-unit.

Theorem 3.2. *Let D be a g-digroup. We define the operation $x \triangleright y := x \vdash y \dashv x^{-1}$ for every $x, y \in D$. Then $(D, \triangleright)$ is a rack.*

Proof.

(Q2) Let $x, y, z \in D$ such that $L_x(y) = L_x(z)$ then

$$\begin{aligned} L_x(y) &= L_x(z) \\ x \triangleright y &= x \triangleright z \\ x \vdash y \dashv x^{-1} &= x \vdash z \dashv x^{-1} \\ x^{-1} \vdash (x \vdash y \dashv x^{-1}) &= x^{-1} \vdash (x \vdash z \dashv x^{-1}) \\ (x^{-1} \vdash x) \vdash (y \dashv x^{-1}) &= (x^{-1} \vdash x) \vdash (z \dashv x^{-1}) \end{aligned}$$

from Proposition 3.1, $x^{-1} \vdash x$ and $x \dashv x^{-1}$ are a bar-units, then

$$\begin{aligned}
(x^{-1} \vdash x) \vdash (y \dashv x^{-1}) &= (x^{-1} \vdash x) \vdash (z \dashv x^{-1}) \\
y \dashv x^{-1} &= z \dashv x^{-1} \\
(y \dashv x^{-1}) \dashv x &= (z \dashv x^{-1}) \dashv x \\
(y \dashv x^{-1}_{l_\xi}) \dashv x &= (z \dashv x^{-1}_{l_\xi}) \dashv x \\
y \dashv (x^{-1}_{l_\xi} \dashv x) &= z \dashv (x^{-1}_{l_\xi} \dashv x) \\
y \dashv \xi &= z \dashv \xi \\
y &= z
\end{aligned}$$

Therefore, the map L_x is injective. Now, note that

$$\begin{aligned}
x \triangleright (x^{-1} \vdash y \dashv x) &= x \vdash (x^{-1} \vdash y \dashv x) \dashv x^{-1} \\
&= [x \vdash (x^{-1} \vdash (y \dashv x))] \dashv x^{-1} \\
&= [x \vdash (x^{-1}_{r_\xi} \vdash (y \dashv x))] \dashv x^{-1} \\
&= [(x \vdash x^{-1}_{r_\xi}) \vdash (y \dashv x)] \dashv x^{-1} \\
&= [\xi \vdash (y \dashv x)] \dashv x^{-1} \\
&= (y \dashv x) \dashv x^{-1} \\
&= y \dashv (x \dashv x^{-1}) \\
&= y
\end{aligned}$$

Therefore the map L_x is surjective. It follows that L_x is bijective for every $x \in D$.

(Q3) Let $x, y, z \in D$, note that

$$\begin{aligned}
(x \triangleright y) \triangleright (x \triangleright z) &= (x \vdash y \dashv x^{-1}) \triangleright (x \vdash z \dashv x^{-1}) \\
&= (x \vdash y \dashv x^{-1}) \vdash (x \vdash z \dashv x^{-1}) \dashv (x \vdash y \dashv x^{-1})^{-1} \\
&= [x \vdash (y \dashv x^{-1}) \vdash (x \vdash (z \dashv x^{-1}))] \dashv (x \vdash y \dashv x^{-1})^{-1} \\
&= [x \vdash ((y \dashv x^{-1}) \vdash x) \vdash (z \dashv x^{-1})] \dashv (x \vdash y \dashv x^{-1})^{-1} \\
&= [x \vdash (y \dashv (x^{-1} \vdash x)) \vdash (z \dashv x^{-1})] \dashv (x \vdash y \dashv x^{-1})^{-1} \\
&= [x \vdash y \vdash (z \dashv x^{-1})] \dashv (x \vdash y \dashv x^{-1})^{-1} \\
&= x \vdash [y \vdash (z \dashv x^{-1}) \dashv (x \vdash y \dashv x^{-1})^{-1}]
\end{aligned}$$

Now, by Proposition 3.2, we have that

$$\begin{aligned}
(x \vdash y \dashv x^{-1})^{-1} &= ((x \vdash y) \dashv x^{-1}_{l_\xi})^{-1}_{l_\xi} \\
&= (x^{-1}_{l_\xi})^{-1}_{l_\xi} \dashv (x \vdash y)^{-1}_{l_\xi} \\
&= (\xi \dashv x) \dashv (x \vdash y)^{-1}_{l_\xi} \\
&= (\xi \dashv x) \dashv (y^{-1}_{l_\xi} \dashv x^{-1}_{l_\xi}) \\
&= [(\xi \dashv x) \dashv y^{-1}] \dashv x^{-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
x \vdash [y \vdash (z \dashv x^{-1}) \dashv (x \vdash y \dashv x^{-1})^{-1}] &= x \vdash [y \vdash (z \dashv x^{-1}) \dashv ([(\xi \dashv x) \dashv y^{-1}] \dashv x^{-1}) \\
&= x \vdash [y \vdash [(z \dashv ((x^{-1} \dashv \xi) \dashv x)] \dashv y^{-1})] \dashv x^{-1} \\
&= x \vdash [y \vdash [(z \dashv (x^{-1} \dashv x) \dashv y^{-1})] \dashv x^{-1} \\
&= x \vdash [y \vdash [(z \dashv \xi) \dashv y^{-1}] \dashv x^{-1} \\
&= x \vdash [y \vdash z \dashv y^{-1}] \dashv x^{-1} \\
&= x \rhd [y \vdash z \dashv y^{-1}] \\
&= x \rhd (y \rhd z)
\end{aligned}$$

Thus, $(x \rhd y) \rhd (x \rhd z) = x \rhd (y \rhd z)$.

■

Note that, the set D with operation $x \rhd y = x \vdash y \dashv x^{-1}$ is not a quandle. Indeed, $x \rhd x = x \vdash x \dashv x^{-1} = x \vdash (x \dashv x^{-1}) = x \vdash \xi \neq x$.
Now, if we have a rack X, we can associate a g-digroup to X. Before we see how this g-digroup is defined, we need the next observation and proposition. Given a rack X, by Proposition 1.2 the map $L : X \longrightarrow Conj(Inn(X))$ defined by $L(x) := L_x$ is a rack homomorphism. Then by universal property (Theorem 1.2) L induces a group homomorphism $\phi_L : As(x) \longrightarrow Inn(X)$ defined by $\phi_L(g_x) = L_x$. We can use this homomorphism to define an action of the group $As(X)$ on the set X.

Proposition 3.3. *Let X be a rack, then the function*

$$\begin{aligned}
\bullet : As(X) \times X &\longrightarrow X \\
(\omega, y) &\longmapsto \omega \bullet y := [\phi_L(\omega)](y)
\end{aligned}$$

is a left action of the group $As(X)$ over the set X.

Proof. Let's see that the map $\bullet$ is well defined. Let $(\omega, y) = (\omega', y') \in As(X) \times X$. Therefore we have that $\phi_L(\omega) = \phi_L(\omega')$ and $y = y'$, thus,

$$\begin{aligned}
[\phi_L(\omega)](y) &= [\phi_L(\omega')](y') \\
\omega \bullet y &= \omega' \bullet y'
\end{aligned}$$

So, it is well defined. Now, for the identity $1_{As(X)} \in As(X)$ we have $1_{As(X)} \bullet x = [\phi_L(1_{As(X)})](x) = id(x) = x$ for all $x \in X$. Further, given $\omega, h \in As(X)$ and $x \in X$ then

$$\omega \bullet (h \bullet x) = [\phi_L(\omega)]([\phi_L(h)](x)) = [\phi_L(\omega h)](x) = (\omega h) \bullet x$$

So, it is a group action.

■

Observe that, given a rack X, then it is a $As(X)-$set. Therefore, from Theorem 3.1 the set $As(X) \times X$ is a g-digroup.

Definition 3.3. Let X be a rack we define the **associated g-digroup** of X, denoted by ***g-As(X)***, as the set $As(X) \times X$ with operations,

$$\begin{aligned}(\omega, x) \vdash (h, y) &= (\omega h, \omega \bullet y) = (\omega h, [\phi_L(\omega)](y)) \\ (\omega, x) \dashv (h, y) &= (\omega, x)\end{aligned}$$

For all $\omega, h \in As(X)$ and $x, y \in X$.

Observe that by previous definition $g_x \bullet y = \phi_L(g_x)(y) = L_x(y) = x \triangleright y$, for all $x, y \in X$.

Theorem 3.3. *Let X be a rack and $\rho : X \longrightarrow Conj(GL(V))$ be a representation of X. Then, ρ induces a $g-$digroup homomorphism $\phi_\rho : As(X) \times X \longrightarrow As(Conj(Gl(V))) \times Conj(GL(V))$ defined by $\phi_\rho[(g_x, y)] := (\hat{\rho}(g_x), \rho_y)$ for all $x, y \in X$, where $\hat{\rho} : As(X) \longrightarrow As(Conj(GL(V)))$ is the group homomorphism of Theorem 1.1. Specifically, $\hat{\rho}$ makes commute the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Conj(GL(V)) \\ \big\downarrow{\scriptstyle \eta_X} & & \big\downarrow{\scriptstyle \eta} \\ As(X) & \xrightarrow{\hat{\rho}} & As(Conj(GL(V))) \end{array}$$

That means, $\hat{\rho}(g_x) = g_{\rho_x}$ for any $x \in X$.

Proof. Let's see that ϕ_ρ is a g-digroup homomorphism. Let $x, y, z, z' \in X$, Note that

$$\begin{aligned}\phi_\rho[(g_x, z) \vdash (g_y, z')] &= \phi_\rho[(g_x g_y, g_x \bullet z')] \\ &= (\hat{\rho}(g_x g_y), \rho_{x \triangleright z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \rho_x \rho_{z'} \rho_x^{-1})\end{aligned}$$

On the other hand

$$\begin{aligned}\phi_\rho[(g_x, z)] \vdash \phi_\rho[(g_y, z')] &= (\hat{\rho}(g_x), \rho_z) \vdash (\hat{\rho}(g_y), \rho_{z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \hat{\rho}(g_x) \bullet \rho_{z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), g_{\rho_x} \bullet \rho_{z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \rho_x \triangleright \rho_{z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \rho_x \rho_{z'} \rho_x^{-1})\end{aligned}$$

Therefore, $\phi_\rho[(g_x, z) \vdash (g_y, z')] = \phi_\rho[(g_x, z)] \vdash \phi_\rho[(g_y, z')]$. Also, we have that

$$\begin{aligned}\phi_\rho[(g_x, z) \dashv (g_y, z')] &= \phi_\rho[(g_x g_y, z)] \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \rho_z)\end{aligned}$$

on the other hand

$$\begin{aligned}\phi_\rho[(g_x, z)] \dashv \phi_\rho[(g_y, z')] &= (\hat{\rho}(g_x), \rho_z) \dashv (\hat{\rho}(g_y), \rho_{z'}) \\ &= (\hat{\rho}(g_x)\hat{\rho}(g_y), \rho_z).\end{aligned}$$

Thus, ϕ_ρ is a g-digroup homomorphism.

∎

Rodríguez Nieto et al, in [14] introduced some concepts in attempt to capture the notion of a g-digroup representation. We think that is possible to capture such notion trough the relation between racks and g-digroups, maybe Theorem 3.3 could be a first aproximation.

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