Involutive and SAGBI Bases for Skew PBW Extensions

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TITLE

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TÍTULO

BASES INVOLUTIVAS Y SAGBI PARA EXTENSIONES PBW TORCIDAS

ABSTRACT: In this thesis, we study homological properties and SAGBI and Involutive bases of the noncommutative rings known as skew PBW extensions. First, we present some ring-theoretical notions of these extensions that are necessary throughout the thesis. With the aim of showing the generality of these objects in areas such as ring theory and noncommutative geometry, we include a non-exhaustive list of noncommutative algebras that are particular examples of these rings. Second, we characterize several homological properties of these extensions. We provide a new and more general filtration to these extensions, and introduce the notion of σ-filtered skew PBW extension with the aim of studying its homological properties. We show that the homogenization of a σ-filtered skew PBW extension over a coefficient ring is a graded skew PBW extension over the homogenization of such a ring. By using this fact, we prove that if the homogenization of the coefficient ring is Auslander-regular, then the homogenization of the extension is a domain Noetherian, Artin-Schelter regular, Zariski and (ungraded) skew Calabi-Yau. Third, we present our proposal of SAGBI bases theory for skew PBW extensions over algebras. We consider the notion of reduction which is necessary in the characterization of these bases, and then establish an algorithm to find the normal form of an element. Then, we define what a SAGBI basis is, and formulate a criterion to determine when a subset of a skew PBW extension over a field is a SAGBI basis. In addition, we establish an algorithm to find a SAGBI basis from a subset contained in a subalgebra of a skew PBW extension. We illustrate our results with different examples of noncommutative algebras. We also investigate the problem of polynomial composition for SAGBI bases of subalgebras of skew PBW extensions. Finally, we present a theory of Involutive bases for skew PBW extensions over fields. We consider the notions of weak and strong Involutive bases, and then we define the *involutive reduction process* and *involutive remainder* that are necessary for the characterization of weak (strong) Involutive bases. Next, we introduce the notion of standard Involutive representation for elements of a subset of a skew PBW extension. Also, we give the definition of minimal Involutive basis and show the existence of a monic, involutively autoreduced, minimal Involutive basis. Finally, we establish different algorithms that compute involutive standard representations, principal involutive autoreduction, and an Involutive basis of a left ideal of a skew PBW extension. In this way, the existence of a finite Involutive basis for these ideals is proved by assuming that the involutive division is constructive Noetherian.

RESUMEN: En esta tesis, estudiamos propiedades homológicas y bases SAGBI e Involutivas de los anillos no conmutativos conocidos como extensiones PBW torcidas. Primero, presentamos algunas nociones teóricas de la teoría de anillos de estas extensiones que son necesarias a lo largo de la tesis. Con el propósito de mostrar la generalidad de estos objetos en áreas como la teoría de anillos y la geometría no conmutativa, incluimos una lista no exhaustiva de álgebras no conmutativas que son ejemplos particulares de estos anillos. Segundo, caracterizamos variadas propiedades homológicas de estas extensiones. Brindamos una nueva y más general filtración para estas extensiones, e introducimos la noción de extensión PBW torcida σ-filtrada con el propósito de estudiar sus propiedades homológicas. Mostramos que la homogenización de una extensión PBW torcida σ-filtrada sobre un anillo de coeficientes es una extensión PBW torcida graduada sobre la homogenización de dicho anillo. Utilizando este hecho, probamos que si la homogenización del anillo de coeficientes es Auslander-regular, entonces la homogenización de la extensión es un dominio noetheriano, Artin-Schelter regular, Zariski y Calabi-Yau torcida. Tercero, presentamos nuestra propuesta de teoría de bases SAGBI para extensiones PBW torcidas sobre álgebras. Consideramos la noción de reducción la cual es necesaria en la caracterización de estas bases, y luego establecemos un algoritmo para encontrar la forma normal de un elemento. Después, definimos lo que es una base SAGBI, y formulamos un criterio para determinar cuándo un subconjunto de una extensión PBW sobre un campo es una base SAGBI. De hecho, establecemos un algoritmo para encontrar una base SAGBI a partir de un subconjunto contenido en una subálgebra de una extensión PBW torcida. Ilustramos nuestros resultados con diferentes ejemplos de álgebras no conmutativas. También investigamos el problema de la composición polinomial para bases SAGBI de subálgebras de extensiones PBW torcidas. Finalmente, presentamos una teoría de bases Involutivas para extensiones PBW torcidas sobre campos. Consideramos las nociones de base Involutiva débil y fuerte, y luego definimos el proceso de reducción involutiva y el residuo involutivo que son necesarios para la caracterización de bases Involutivas débiles y fuertes. A continuación, presentamos la noción de representación involutiva estándar para elementos de un subconjunto de una extensión PBW torcida. Además, damos la definición de base Involutiva minimal y mostramos la existencia de una base Involutiva minimal, mónica, e involutivamente autorreducida. Finalmente, establecemos diferentes algoritmos que calculan representaciones estándar involutivas, autorreducción involutiva principal, y una base Involutiva de un ideal izquierdo de una extensión PBW torcida. De esta manera, la existencia de una base Involutiva finita para estos ideales se demuestra asumiendo que la división involutiva es noetheriana constructiva.

Keywords: SAGBI basis, involutive basis, skew PBW extension, quantum algebra, Auslander-regular, Artin-Schelter regular, skew Calabi-Yau.

Palabras clave: Base SAGBI, base involutiva, extensión PBW torcida, álgebra cuántica, regularidad de Auslander, regularidad de Artin-Schelter, Calabi-Yau torcida.

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To my husband Wilfredo, and my sons Károl and Johann

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CONTENTS

Introduction

The objects of our interest in this thesis are the *skew Poincaré-Birkhoff-Witt extensions* (skew PBW for short) introduced by Gallego and Lezama [GL11]. Following Lezama et al. [FGL+20, p. vii], “Skew PBW extensions represent a generalization of PBW extensions defined by Bell and Goodearl [BG88], and include the usual polynomials rings and a lot of other important classes of rings such as Weyl algebras, enveloping algebras of Lie algebras, and many examples of quantum algebras such as the Manin algebra of quantum matrices, q-Heisenberg algebra, Hayashi algebra, Witten’s deformation of the enveloping algebra of $\mathfrak{sl}(2)$”. Over the years, several authors have shown that skew PBW extensions also generalize families of noncommutative algebras such as Ore extensions of injective type [Ore31, Ore33], 3-dimensional skew polynomial algebras introduced by Bell and Smith [BS90], diffusion algebras defined by Isaev et al. [IPR01], ambiskew polynomial rings introduced by Jordan [Jor95, Jor00, Jor01, JW96], solvable polynomial rings introduced by Kandri-Rody and Weispfenning [KRW90], almost normalizing extensions defined by McConnell and Robson [MR01], skew bi-quadratic algebras recently introduced by Bavula [Bav23], and others (see [FGL+20, Chapter 2] for more details). Relations between skew PBW extensions and other noncommutative algebras having PBW bases can be found in [BG02, BGTV03, GT14, GJ04, Li02, NR23, RS17a, RS17c, Sei10]. Since its introduction, several mathematicians have studied ring-theoretic, homological and geometric properties of skew PBW extensions.

Precisely, our first topic of interest in this thesis is the research on homological properties of these extensions. We provide a new and more general filtration to skew PBW extensions that generalize the corresponding defined by Gallego and Lezama [GL11, Section 2]. With this, we prove different results on homogenizations[1], and properties such as Auslander-regular, Artin-Schelter regular, and skew Calabi-Yau. Our results contribute to the characterization of these properties for skew PBW extensions (e.g., [FGL+20, RS17b, SAR21, SCR21, SLR15, SLR17, SR17]).

Next, motivated by the treatments developed by different authors for Gröbner bases in the commutative case (e.g., Buchberger [Buc65], Adams and Loustaunau [AL94], Becker and Weispfenning [BW93], and Cox et al., [CLO15]) and noncommutative setting (e.g., Kandri-Rody and Weispfenning [KRW90] for algebras of solvable type, Bueso et al. [BGTV03] for PBW rings and PBW algebras, Levandovskyy [Lev05] for G-algebras, Li [Li02] for quadratic algebras), and the Gröbner bases theory developed by Gallego in her PhD thesis [Gal15] (and related papers with Lezama [GL11, Gal16a, GL17, Gal16b]) for skew PBW extensions, we present our proposal

[1] We have to say that the interest in the homogenizations of algebras is due to that, for example, Gaddis in his PhD Thesis [Gad13] used the technique of homogenization to study non $\mathbb{N}$-graded Artin-Schelter algebras.

for a theory of SAGBI[2] bases of skew PBW extensions over $\Bbbk$-algebras (note that in the literature we can find these bases for several commutative and noncommutative algebras, e.g., Kapur and Madlener [KM89], Robbiano et al., [RS90, KR05], Nordbeck [Nor98, Nor01a, Nor01b, Nor02], and Khan et al. [KKB19]). We also investigate the problem of polynomial composition for SAGBI bases of subalgebras of skew PBW extensions.

The other topic of interest in this thesis are the Involutive bases which represent a special kind of Gröbner bases with additional combinatorial properties. As one can appreciate in the literature, these bases have been developed for commutative and noncommutative algebras (e.g., Apel [Ape95, Ape98], Blinkov et al. [Ger99, GB98a, GB98b, ZB93], Evans [Eva05], Hausdorff et al. [HSS02], Saito et al. [SST00], and Seiler [Sei09, Sei10]). In particular, of interest for us is the theory of Involutive bases presented by Seiler [Sei10, Chapter 3] for his *polynomial algebras of solvable type* (c.f. [KRW90]), since as he recognized, his introduction of these bases is closely modelled on a classical approach to Gröbner bases [Sei10, p. 64]. Due to the similarities between the algebra of these polynomials and skew PBW extensions, the approach to Involutive bases of these extensions that we propose takes into account the Gröbner bases theory developed by Gallego and Lezama, and the ideas formulated by Gerd, Blinkov and Seiler.

On the structure of the thesis, this is based on a collection of papers. Chapter 1 presents some definitions and properties of skew PBW extensions that are necessary in the next chapters. With the aim of showing the generality of these objects in areas such as ring theory and noncommutative geometry, we include a non-exhaustive list of remarkable noncommutative algebras. Next, Chapter 2 presents the first original results on skew PBW extensions obtained in the thesis. We provide a new and more general filtration to the class of skew PBW extensions. We introduce the notion of *σ-filtered skew PBW extension* and study some of its homological properties. We show that the homogenization of a σ-filtered skew PBW extension A over a ring R is a graded skew PBW extension over the homogenization of R. Using this fact, we prove that if the homogenization of R is Auslander-regular, then the homogenization of A is a domain Noetherian, Artin-Schelter regular, and A is Noetherian, Zariski and (ungraded) skew Calabi-Yau. In Chapter 3, we present a first approach toward a theory of SAGBI bases for skew PBW extensions over $\Bbbk$-algebras. We formulate the notion of *reduction* which is necessary in the characterization of SAGBI bases, and then establish an algorithm to find the SAGBI normal form of an element. Then, we define what a SAGBI basis is, and formulate a criterion to determine when a subset of a skew PBW extension over a field is a SAGBI basis. In addition, we establish an algorithm to find a SAGBI basis from a subset contained in a subalgebra of a skew PBW extension. We illustrate our results with examples concerning algebras appearing in Lie theory, and noncommutative algebraic geometry. We also investigate the problem of polynomial composition for SAGBI bases of subalgebras of skew PBW extensions. Finally, in Chapter 4, we present a theory of Involutive bases for skew PBW extensions over fields. We consider the notions of *weak involutive basis* and (*strong*) *involutive basis*, and then we define the *involutive reduction process* and *involutive remainder* that are necessary for the characterization of weak (strong) Involutive bases. Next, we introduce the notion of *standard involutive representation* for elements of a subset of a skew PBW extension. Also, we give the definition of *minimal* Involutive *basis* and show the existence of at most one monic, involutively autoreduced, minimal Involutive basis. Finally,

[2]It is an acronym for Subalgebras Analogue to Gröbner Basis for Ideals. This term was used for the first time by Robbiano and Sweedler [RS90].

we establish different algorithms that compute involutive standard representations, principal involutive autoreduction, and an `Involutive` basis of a left ideal of a skew PBW extension. In this way, the existence of a finite `Involutive` basis for these ideals is proved by assuming that the involutive division is constructive Noetherian.

Notation and some terminology

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers including the zero element
$\mathbb{Z}$	The ring of integer numbers
$\mathbb{Z}_{>0}$	The set of positive integer numbers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{C}$	The field of complex numbers
R	Associative ring (not necessarily commutative) with identity
R^*	The set of non-zero elements of the ring R
K	Commutative ring with identity
$\Bbbk$	Field
$\Bbbk[x_1,\dots,x_n]$	Commutative ring of polynomials in n indeterminates over $\Bbbk$
$\Bbbk\{X\}=\Bbbk\{x_1,\dots,x_n\}$	Free associative algebra over $\Bbbk$ in the set of noncommutative indeterminates $X=\{x_1,\dots,x_n\}$
$\dim(V)$	dimension of a $\Bbbk$-vector space V

Throughout the thesis, the word ring means an associative ring (not necessarily commutative) with identity. All algebras are $\Bbbk$-algebras, all modules are left modules, and the tensor product $\otimes$ means $\otimes_{\Bbbk}$.

Statement of contributions

The chapters two, three and four in this thesis correspond to the following publications and preprints containing original results.

- **Chapter 2.** Suárez, H., Reyes, A. and **Suárez, Y.** Homogenized skew PBW extensions. *Arabian Journal of Mathematics* (*Springer*) 12 (1) 247–263 (2023). Available online at https://link.springer.com/article/10.1007/s40065-022-00410-z
- **Chapter 3.** Reyes, A. and **Suárez, Y.** On SAGBI bases theory of skew Poincaré-Birkhoff-Witt extensions (2024) [Manuscript submitted for publication].
- **Chapter 4.** Reyes, A. and **Suárez, Y.** Some remarks about involutive bases of skew Poincaré-Birkhoff-Witt extensions (2024) [Manuscript submitted for publication].

CHAPTER 1

SKEW PBW EXTENSIONS

In this chapter, we recall some definitions and elementary properties of skew PBW extensions that are necessary in the next chapters (Section 1.1). Next, in Section 1.2 we present a list (not exhaustive) of noncommutative algebraic structures that are particular examples of skew PBW extensions. Our aim in this section is to show explicitly the generality of these rings in the setting of ring theory, noncommutative algebraic geometry and noncommutative differential geometry.

1.1 DEFINITIONS AND KEY PROPERTIES

As we said in the Introduction, skew PBW extensions were defined by Gallego and Lezama [GL11] with the aim of extending PBW extensions introduced by Bell and Goodearl [BG88] (these algebras generalize enveloping algebras of finite-dimensional Lie algebras, any differential operator ring formed from commuting derivations, differential operators introduced by Rinehart [Rin63], the twisted or smash product differential operator ring, and universal enveloping rings [BG88, Section 5]) and Ore extensions of injective type defined by Ore [Ore31, Ore33]. During the last years, it has been shown that skew PBW extensions also include other families of noncommutative rings such as almost normalizing extensions defined by McConnell and Robson [MR01], solvable polynomial rings introduced by Kandri-Rody and Weispfenning [KRW90], diffusion algebras defined by Isaev, Pyatov, and Rittenberg [IPR01], 3-dimensional skew polynomial algebras introduced by Bell and Smith [BS90], bi-quadratic algebras having PBW bases recently defined by Bavula [Bav23], and some examples of Auslander-Gorenstein rings, Calabi-Yau and skew Calabi-Yau algebras, Artin-Schelter regular algebras, Koszul algebras, and quantum universal enveloping algebras. For more details about the generality of skew PBW extensions in noncommutative ring theory, see [FGL$^+$20, Chapter 2] or [LR14, Section 3]. Several ring-theoretical properties of skew PBW extensions have been investigated by some authors (e.g., [AT24, AL15, Art15, Bav23, Cha22, HKA17, HKG19, LR20, NR23, Rey14, Rey19, RR21, RS16a, RS16b, RS17b, RS18b, RS20, SR17, TRS20, Ven20], and references therein).

DEFINITION 1.1 ([GL11, DEFINITION 1]). Let R and A be rings. We say that A is a *skew PBW extension* (also known as σ - *PBW extension*) over R if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$ there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n, \tag{1.1}$$

or equivalently,

$$x_j x_i = d_{i,j} x_i x_j + r_{0_{j,i}} + r_{1_{j,i}} x_1 + \cdots + r_{n_{j,i}} x_n, \tag{1.2}$$

where $d_{i,j}, r_{0_{j,i}}, r_{1_{j,i}}, \dots, r_{n_{j,i}} \in R$, for $1 \leq i, j \leq n$.

Under these conditions, we write $A = \sigma(R)\langle x_1, \dots, x_n\rangle$.

Remark 1 ([GL11, Remark 2]). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R.

(i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ in Definition 1.1 are unique.

(ii) If $r = 0$, then $c_{i,0} = 0$. In Definition 1.1 (iv), $d_{i,i} = 1$. This follows from $x_i^2 - d_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_j \in R$, which implies $1 - d_{i,i} = 0 = s_j$, for $0 \leq j \leq n$.

(iii) Let $i < j$. By (1.1) there exist elements $d_{j,i}, d_{i,j} \in R$ such that $x_i x_j - d_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, and hence $1 = d_{j,i} d_{i,j}$, that is, for each $1 \leq i < j \leq n$, $d_{i,j}$ has a left inverse and $d_{j,i}$ has a right inverse. In general, the elements $d_{i,j}$ are not two-sided invertible. For instance, $x_1 x_2 = d_{2,1} x_2 x_1 + p = d_{21}(d_{1,2} x_1 x_2 + q) + p$, where $p, q \in R + Rx_1 + \cdots + Rx_n$, so $1 = d_{2,1} d_{1,2}$, since $x_1 x_2$ is a basic element of $\mathrm{Mon}(A)$. Now, $x_2 x_1 = d_{1,2} x_1 x_2 + q = d_{1,2}(d_{2,1} x_2 x_1 + p) + q$, but we cannot conclude that $d_{1,2} d_{2,1} = 1$ because $x_2 x_1$ is not a basic element of $\mathrm{Mon}(A)$.

(iv) Every element $f \in A \setminus \{0\}$ has a unique representation as a linear combination of monomials $f = \sum_{i=1}^{t} r_i X_i$, with $r_i \in R \setminus \{0\}$ and $X_i \in \mathrm{Mon}(A)$ for $1 \leq i \leq t$.

Proposition 1.1 evidences the relationship between skew PBW extensions and Ore extensions or skew polynomial rings (Section 1.2.1).

Proposition 1.1 ([GL11, Proposition 3]). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, $r \in R$.*

The notation $\sigma(R)\langle x_1, \dots, x_n\rangle$ and the name of the skew PBW extensions are due to Proposition 1.1.

DEFINITION 1.2 ([GL11, DEFINITION 4]). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R, $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, where σ_i and δ_i $(1 \le i \le n)$ are as in Proposition 1.1

(i) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 1.1 are replaced by

(iii') for each $1 \le i \le n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$;

(iv') for any $1 \le i, j \le n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

(ii) A is called *bijective* if σ_i is bijective for each $\sigma_i \in \Sigma$, and $d_{i,j}$ is invertible for any $1 \le i < j \le n$.

(iii) If $\delta_i = 0$ for every $\delta_i \in \Delta$, then we say that A is a skew PBW extension of *endomorphism type*.

PROPOSITION 1.2 ([GL11, THEOREM 7]). *Let A be a left polynomial ring over R with respect to $\{x_1, \dots, x_n\}$. A is a skew PBW extension over R if and only if the following conditions hold:*

(1) *For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \ne r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \ne 0$. Moreover, if r is left invertible, then r_α is left invertible.*

(2) *For every $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha+\beta|$ if $p_{\alpha,\beta} \ne 0$.*

Recall that a *filtered ring* is a ring R with a family $F(R) = \{F_n(R) \mid n \in \mathbb{Z}\}$ of subgroups of the additive group of R, where we have the ascending chain $\cdots \subset F_{n-1}(R) \subset F_n(R) \subset \cdots$ such that $R = \bigcup_{n\in\mathbb{Z}} F_n(R)$, $1 \in F_0(R)$, and $F_n(R)F_m(R) \subseteq F_{n+m}(R)$ for all $n, m \in \mathbb{Z}$. As is well-known, from a filtered ring R it is possible to construct its associated graded ring $G(R)$ by taking $G(R)_n := F_n(R)/F_{n-1}(R)$. A filtration $\{F_i(R)\}_{i\in\mathbb{Z}}$ of an algebra R is called *finite* if each $F_i(R)$ is a finite dimensional subspace.

The following proposition shows that skew PBW extensions are filtered rings and establishes its associated graded ring.

PROPOSITION 1.3 ([LR14, THEOREM 2.2]). *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over R, then A is a filtered ring with increasing filtration given by*

$$F_m(A) := \begin{cases} R, & \text{if } m = 0, \\ \{f \in A \mid \deg(f) \le m\} \cup \{0\}, & \text{if } m \ge 1, \end{cases} \tag{1.3}$$

and the corresponding graded ring $G(A)$ is a quasi-commutative skew PBW extension over R. If A is also bijective, then $G(A)$ is a quasi-commutative bijective skew PBW extension over R.

Next, we recall briefly some key facts about monomial orders in the setting of skew PBW extensions. In Section 3.1.1 we will present a detailed treatment of this topic in the commutative and noncommutative settings.

Definition 1.3 ([GL11, Definitions 6 and 11]). Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over R.

(i) For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $\sigma^{\alpha}:=\sigma_1^{\alpha_1}\circ\cdots\circ\sigma_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions, $|\alpha|:=\alpha_1+\cdots+\alpha_n$. If $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}^n$, then $\alpha+\beta:=(\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$.

(ii) For $X=x^{\alpha}\in \mathrm{Mon}(A)$, $\exp(X):=\alpha$ and $\deg(X):=|\alpha|$.

(iii) Since every element $f\in A$ can be written uniquely as $f=\sum_{i=1}^{t} r_iX_i$ (Remark 1 (iv)), let $\deg(f):=\max\{\deg(X_i)\}_{i=1}^{t}$.

(iv) Let $\preceq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^{\alpha}\preceq x^{\beta}$ but $x^{\alpha}\neq x^{\beta}$, we will write $x^{\alpha}\prec x^{\beta}$. $x^{\beta}\preceq x^{\alpha}$ means $x^{\alpha}\succeq x^{\beta}$. Each element $f\in A\setminus\{0\}$ can be represented in a unique way as $f=c_1x^{\alpha_1}+\cdots+c_tx^{\alpha_t}$, with $c_i\in R\setminus\{0\}, 1\le i\le t$, and $x^{\alpha_1}\succ\cdots\succ x^{\alpha_t}$. We say that x^{α_1} is the *leading monomial* of f and we write $\mathrm{lm}(f):=x^{\alpha_1}$; c_1 is the *leading coefficient* of f, $\mathrm{lc}(f):=c_1$, and $c_1x^{\alpha_1}$ is the *leading term* of f denoted by $\mathrm{lt}(f):=c_1x^{\alpha_1}$. If $f=0$, we define $\mathrm{lm}(0):=0, \mathrm{lc}(0):=0, \mathrm{lt}(0):=0$.

We say that $\preceq$ is a *monomial order* (also called *admissible order*) on $\mathrm{Mon}(A)$ if the following conditions hold:

(a) For every $x^{\alpha},x^{\beta},x^{\gamma},x^{\lambda}\in\mathrm{Mon}(A)$, the relation $x^{\alpha}\preceq x^{\beta}$ implies that $\mathrm{lm}(x^{\gamma}x^{\alpha}x^{\lambda})\preceq \mathrm{lm}(x^{\gamma}x^{\beta}x^{\lambda})$.

(b) $1\preceq x^{\alpha}$ for every $x^{\alpha}\in\mathrm{Mon}(A)$.

(c) $\preceq$ is degree compatible, i.e., $|\alpha|\le|\beta|$ implies $x^{\alpha}\preceq x^{\beta}$.

The condition (iv)(c) of the previous definition is needed in the proof that every monomial order on $\mathrm{Mon}(A)$ is a well order. Thus, there are not infinite decreasing chains in $\mathrm{Mon}(A)$ [GL11, Proposition 12]. Nevertheless, this hypothesis is not really needed to get a well ordering if a more elaborated argument, based upon Dickson's Lemma, is developed (e.g., [BW93, Theorem 4.6.2]).

From Definition 1.1 it follows that skew PBW extensions are not $\mathbb{N}$-graded rings in a non-trivial sense. With this in mind, Proposition 1.4 allows to define a subfamily of these extensions, the *graded skew PBW extensions* (Definition 1.4) introduced by Suárez in his PhD Thesis [Sufrm[o]–7b] (see also [Sufrm[o]–7a]). Before presenting its definition, we recall the following facts:

- If $R=\bigoplus_{p\in\mathbb{N}} R_p$ and $S=\bigoplus_{p\in\mathbb{N}} S_p$ are $\mathbb{N}$-graded rings, then a map $\varphi:R\to S$ is called *graded* if $\varphi(R_p)\subseteq S_p$, for each $p\in\mathbb{N}$. For $m\in\mathbb{N}$, $R(m):=\bigoplus_{p\in\mathbb{N}} R(m)_p$, where $R(m)_p:=R_{p+m}$.

- Suppose that $\sigma:R\to R$ is a graded algebra automorphism and $\delta:R(-1)\to R$ is a graded σ-derivation (i.e., a degree $+1$ graded σ-derivation δ of R). Let $B:=R[x;\sigma,\delta]$ be the associated *graded Ore extension* of R, that is, $B=\bigoplus_{p\ge 0} Rx^p$ as an R-module, and for $r\in R$, $xr=\sigma(r)x+\delta(r)$. If we consider x to have degree 1 in B, then under this grading B is a connected graded algebra generated in degree 1 (for more details, see [CS08, Pha12]).

PROPOSITION 1.4 ([Sufrm[o]–7a, Proposition 2.7(ii)]). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If the conditions*

(1) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$, and*

(2) *$x_j x_i - d_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in Definition 1.1* (iv) *and $d_{i,j} \in R_0$,*

hold, then A is an $\mathbb{N}$-graded algebra with graduation given by $A = \bigoplus_{p\geq 0} A_p$, where for $p \geq 0$, A_p is the $\mathbb{k}$-space generated by the set

$$\left\{ r_t x^{\alpha} \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^{\alpha} \in \mathrm{Mon}(A) \right\}.$$

DEFINITION 1.4 ([Sufrm[o]–7a, Definition 2.6]). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If A satisfies both conditions established in Proposition 1.4, then we say that A is a *graded skew PBW extension over R*.

Some properties of graded skew PBW extensions can be found in [SAR21, SCR21, SRS23]. Note that the family of graded iterated Ore extensions of injective type is strictly contained in the class of graded skew PBW extensions. For example, homogenized enveloping algebras and diffusion algebras are graded skew PBW extensions over a field but these are not iterated Ore extensions of the field. Examples of graded skew PBW extensions can be found in [GS20, Sufrm[o]–7a, SCR21].

1.2 Some families of examples

Skew PBW extensions generalize several kinds of noncommutative rings of polynomial type such as Ore extensions [Ore31, Ore33], families of differential operators generalizing Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras [Bav92, BG88, Smi91], algebras appearing in mathematical physics [IPR01, RS22, Zhe91], down-up algebras [Ben99, BR98, KMP99], ambiskew polynomial rings [Jor00, JW96], 3-dimensional skew polynomial rings [BS90, Red99, RS22, Ros95], bi-quadratic algebras on three generators with PBW bases recently characterized by Bavula [Bav23], PBW extensions [BG88], and others (e.g., [FGL+20, NR17]).

In this section, we present families of noncommutative rings that are particular examples of skew PBW extensions with the aim of showing the generality of these objects, and the scope of the results presented in the next chapters. For the completeness of the thesis, we include detailed references for every family of rings.

1.2.1 Skew polynomial rings and ambiskew polynomial rings

Skew polynomial rings (also known as Ore extensions) were introduced by Ore [Ore31, Ore33] (Noether and Schmeidler [NS20] were interested in some kind of differential operator rings). Briefly, for σ an endomorphism of a ring R, a σ-derivation on R is any additive map $\delta : R \to R$

such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$ (strictly speaking, this is the definition of *left σ-derivation*, but we will not need the concept of *right* σ-derivation, which is any additive map $\delta : R \to R$ satisfying the rule $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$). Notice that if σ is the identity map on R, then σ-derivations are just ordinary derivations. The condition $\delta(1) = 0$ it follows from the skew product rule.

DEFINITION 1.5 ([ORE31, ORE33], [GJ04, P. 34]). Let R be a ring, σ a ring endomorphism of R and δ a σ-derivation on R. We will write $R[x;\sigma,\delta]$ provided

(i) $R[x;\sigma,\delta]$ containing R as a subring;

(ii) x is not an element of R;

(iii) $R[x;\sigma,\delta]$ is a free left R-module with basis $\{1, x, x^2, \dots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$, for all $r \in R$.

Such a ring $R[x;\sigma,\delta]$ is called a *skew polynomial ring over R*, or an *Ore extension of R*. If σ is an injective map of R, then we call it an *Ore extension of injective type*, while if σ is the identity of R, then we write $R[x;\delta]$ and call it a *differential operator ring*. On the other hand, if δ is the zero map, then we write $R[x;\sigma]$ which is known as a *skew polynomial ring of endomorphism type*. Iterated skew polynomial rings are defined in the natural way. In the literature, we can find a lot of papers concerning ring-theoretical, homological and geometrical properties of Ore extensions. Some general details about these objects can be found in Brown and Goodearl [BG02], Fajardo et al. [FLP+24], Goodearl and Warfield [GJ04], and McConnell and Robson [MR01], and references therein.

REMARK 2. Skew PBW extensions of endomorphism type are more general than iterated Ore extensions of endomorphism type. Let us illustrate the situation with two and three indeterminates.

For the iterated Ore extension of endomorphism type $R[x;\sigma_x][y;\sigma_y]$, if $r \in R$ then we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. Now, if we have $\sigma(R)\langle x, y\rangle$ a skew PBW extension of endomorphism type over R, then for any $r \in R$, Definition 1.1 establishes that $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R. From these relations it is clear which one of them is more general.

If we have the iterated Ore extension $R[x;\sigma_x][y;\sigma_y][z;\sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. For the skew PBW extension of endomorphism type $\sigma(R)\langle x, y, z\rangle$, $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3, r_3', r_3''$ of R. As the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

Ore extensions are one of the most important techniques to define noncommutative algebras. Next, we illustrate this situation with Weyl algebras, some of its deformations, the q-Heisenberg algebra, and the quantum matrix algebra.

About the family of Weyl algebras $A_n(\Bbbk)$, in the literature it is common to find characterizations of these algebras as rings of differential operators. Surely, the most beautiful and excellent treatment about Weyl algebras is presented by Coutinho [Cou95]. Briefly, the nth *Weyl algebra* $A_n(\Bbbk)$ over the field $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1,\dots,x_n,\partial_1,\dots,\partial_n$ where

$$x_i x_j = x_j x_i,\quad \partial_i\partial_j = \partial_j\partial_i,\quad \partial_i x_j = x_j\partial_i + \delta_{ij},\quad 1\le i,j\le n,$$

where δ_{ij} is the Kronecker's delta. From the relations defining the Weyl algebras, it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but skew polynomial rings with non-trivial derivations.

Following Goodearl and Warfield [GJ04, p. 36], for an element $q\in\Bbbk^*$, $A_1^q(\Bbbk)$ denotes the $\Bbbk$-algebra presented by two generators x and y and the relation $xy-qyx=1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk)=A_1(\Bbbk)=\Bbbk[y][x;d/dy]$, when $q=1$. If $q\neq 1$, then $A_1^q(\Bbbk)=\Bbbk[y][x;\sigma,\delta]$, where σ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y))=f(qy)$, and δ is the *q-difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y))=\frac{f(qy)-f(y)}{qy-y}=\frac{\alpha(f)-f}{\alpha(y)-y},$$

as it can be seen in [GJ04, Exercise 2N], so this algebra is not a skew polynomial ring of automorphism type.

A generalization of $A_1^q(\Bbbk)$ is given by the *additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$. For non-zero elements $q_1,\dots,q_n\in\Bbbk$, this algebra is generated by the indeterminates $x_1,\dots,x_n$ and $y_1,\dots,y_n$ satisfying the relations $x_jx_i=x_ix_j$, $y_jy_i=y_iy_j$, for every $1\le i,j\le n$, $y_ix_j=x_jy_i$, for all $i\neq j$, and $y_ix_i=q_ix_iy_i+1$, for $1\le i\le n$. It is clear from these definitions that these algebras are not skew polynomial rings of automorphism type.

Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [GZ95] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second Weyl algebra. By definition, the *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the indeterminates $x_1,x_2,\partial_1,\partial_2$, with relations (depending on parameters $a,b\in\Bbbk$)

$$\begin{aligned}
&x_1x_2=x_2x_1+ax_1^2, && \partial_2\partial_1=\partial_1\partial_2+b\partial_2^2,\\
&\partial_1x_1=1+x_1\partial_1+ax_1\partial_2, && \partial_1x_2=-ax_1\partial_1-abx_1\partial_2+x_2\partial_1+bx_2\partial_2,\\
&\partial_2x_1=x_1\partial_2, && \partial_2x_2=1-bx_1\partial_2+x_2\partial_2.
\end{aligned}$$

Note that if $a=b=0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$.

By definition, for an element $q\in\Bbbk^*$, the *q-Heisenberg algebra* $\mathbf{H}_n(q)$ is the $\Bbbk$-algebra generated over $\Bbbk$ by the indeterminates x_i,y_i,z_i, for $1\le i\le n$, subject to the relations

$$\begin{aligned}
x_ix_j=x_jx_i,\quad y_iy_j=y_jy_i,\quad z_jz_i=z_iz_j,&\qquad 1\le i<j\le n,\\
x_iz_i-qz_ix_i=z_iy_i-qy_iz_i=x_iy_i-q^{-1}y_ix_i+z_i=0,&\qquad 1\le i\le n,\\
x_iy_j=y_jx_i,\quad x_iz_j=z_jx_i,\quad y_iz_j=z_jy_i,&\qquad i\neq j.
\end{aligned}$$

It can be seen that $\mathbf{H}_n(q)$ can be expressed as an iterated skew polynomial ring.

Given any $q\in\Bbbk^*$, the corresponding *quantized coordinate ring* of the ring of matrices of size 2×2 with entries in $\Bbbk$, denoted by $M_2(\Bbbk)$, is the $\Bbbk$-algebra $O_q(M_2(\Bbbk))$ presented by four generators x_{11}, x_{12}, x_{21}, and x_{22} and the six relations

$$\begin{aligned}
&x_{11}x_{12} = qx_{12}x_{11}, \quad x_{12}x_{22} = qx_{22}x_{12},\\
&x_{11}x_{21} = qx_{21}x_{11}, \quad x_{21}x_{22} = qx_{22}x_{21},\\
&x_{12}x_{21} = x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q-q^{-1})x_{12}x_{21}.
\end{aligned}$$

This algebra, also known as the *coordinate ring of quantum* 2×2 matrices over $\Bbbk$, or the 2×2 *quantum matrix algebra* over $\Bbbk$, can be expressed as the iterated skew polynomial ring given by $\Bbbk[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\delta_{22}]$ [GJ04, Exercise 2V].

Jordan [Jor95] introduced a certain class of iterated Ore extensions called *ambiskew polynomial rings*. These structures have been studied by Jordan et al. [Jor00, JW96] at various levels of generality that contain different examples of noncommutative algebras. Next, we recall briefly its definition.

Consider a commutative $\Bbbk$-algebra B, a $\Bbbk$-automorphism of B, and elements $c\in B$ and $p\in\Bbbk^*$. Let S be the Ore extension $B[x;\sigma^{-1}]$ and extend σ to S by setting $\sigma(x) = px$. By [Coh85, p. 41], there is a σ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B,\sigma,c,p)$ is the Ore extension $S[y;\sigma,\delta]$, whence the following relations hold:

$$yx - pxy = c, \quad \text{and, for all } b\in B, \quad xb = \sigma^{-1}(b)x \quad \text{and} \quad yb = \sigma(b)y. \tag{1.4}$$

Equivalently, R can be presented as $R = B[y;\sigma][x;\sigma^{-1},\delta']$ with $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$, so that $xy - p^{-1}yx = -p^{-1}c$. If we consider the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, then we can see that the definition involves twists from both sides using σ; this is the reason for the name of the objects. From [RS17a, Theorem 1.14], ambiskew polynomial rings are skew PBW extensions over B, that is, $R(B,\sigma,c,p)\cong\sigma(B)\langle y,x\rangle$.

1.2.2 Universal enveloping algebras and PBW extensions

If $\mathfrak{g}$ is a finite dimensional Lie algebra over a commutative ring K with basis $\{x_1,\dots,x_n\}$, then by the *Poincaré-Birkhoff-Witt theorem*, the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the algebra generated by $x_1,\dots,x_n$ subject to the relations $x_ir - rx_i = 0\in K$, for every element $r\in K$, and $x_ix_j - x_jx_i = [x_i,x_j]\in\mathfrak{g}$, where $[x_i,x_j]\subseteq K + Kx_1 + \ldots + Kx_n$, for all $1\le i,j\le n$. As is well-known, in general these algebras are not skew polynomial rings even including non-zero trivial derivations. Some enveloping algebras can be expressed as skew polynomial rings; however, in these rings the derivations are non-trivial. Let us see an example.

Following [GJ04, p. 40], the standard basis for the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is labelled $\{e,f,h\}$, where $[e,f] = h$, $[h,e] = 2e$, and $[h,f] = -2f$. In this way, the enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra presented by three generators e,f,h and three relations $ef - fe = h$, $he - eh = 2e$, and $hf - fh = -2f$. If R is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by e and h, then $R = \Bbbk[e][h;\delta_1] = \Bbbk[h][e;\sigma_1]$, where $\Bbbk[e]$ and $\Bbbk[h]$ are commutative polynomial rings, δ_1 denotes the derivation $2e(d/de)$ on $\Bbbk[e]$, and σ_1 is the $\Bbbk$-algebra automorphism of $\Bbbk[h]$ with $\sigma_1(h) = h-2$. Thus, $U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h;\delta_1][f;\sigma_2,\delta_2] = \Bbbk[h][e;\sigma_1][f;\sigma_2,\delta_2]$, where $\sigma_2(e) = e$, $\sigma_2(h) = h+2$, $\delta_2(e) = -h$, and $\delta_2(h) = 0$ [GJ04, Exercise 2S]. Other examples of universal enveloping algebras known as *parafermionic* and *parabosonic algebras* are presented in Section 1.2.7.

PBW extensions introduced by Bell and Goodearl [BG88] (Section 1.2.2) are particular examples of skew PBW extensions. More exactly, the first objects satisfy the relation $x_i r = r x_i + \delta_i(r)$, for every $i = 1, \ldots, n$, and each $r \in R$, and the elements d_{ij} in Definition 1.1 (iv) are equal to the identity of R. As examples of PBW extensions, we mention the following: the enveloping algebra of a finite-dimensional Lie algebra; any differential operator ring $R[\theta_1, \ldots, \theta_1; \delta_1, \ldots, \delta_n]$ formed from commuting derivations $\delta_1, \ldots, \delta_n$; differential operators introduced by Rinehart [Rin63]; twisted or smash product differential operator rings, and others. For more details about the generality of PBW extensions, see [BG88, p. 27].

1.2.3 3-DIMENSIONAL SKEW POLYNOMIAL ALGEBRAS

Another kind of noncommutative rings which includes the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, the Dispin algebra $U(\mathfrak{osp}(1,2))$ and the Woronowicz's algebra $W_\nu(\mathfrak{sl}_2(\Bbbk))$ [Wor87], is the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith [BS90] and are very important in noncommutative algebraic geometry and noncommutative differential geometry (e.g., [Red96, Red99, RS22, RS17c, Ros95] and references therein). Next, we recall its definition and classification.

DEFINITION 1.6 ([BS90], [ROS95, DEFINITION C4.3]). *A 3-dimensional skew polynomial algebra A* is a $\Bbbk$-algebra generated by the indeterminates x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that

(i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$;

(ii) standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

It is clear that 3-dimensional skew polynomial algebras are skew PBW extensions with three indeterminates over the field $\Bbbk$.

PROPOSITION 1.5 ([ROS95, THEOREM C.4.3.1]). *If A is a 3-dimensional skew polynomial algebra, then A is one of the following algebras:*

(1) *if $|\{\alpha, \beta, \gamma\}| = 3$, then A is given by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$.*

(2) *if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then A is one of the following algebras:*

   (i) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$; (*if $\beta = -1$, then we get the Dispin algebra*).

   (ii) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$;

   (iii) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$;

   (iv) $yz - zy = 0$, $zx - \beta xz = b$, $xy - yx = 0$;

   (v) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$;

   (vi) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$,

   where a, b are any elements of $\Bbbk$. All non-zero values of b give isomorphic algebras.

(3) *If* $|\{\alpha,\beta,\gamma\}| = 2$ *and* $\beta \neq \alpha = \gamma \neq 1$*, then* A *is one of the following algebras:*

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$

(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case, b *is an arbitrary element of* $\Bbbk$*. Again, any non-zero values of* b *give isomorphic algebras.*

(4) *If* $\alpha = \beta = \gamma \neq 1$*, then* A *is the algebra defined by the relations* $yz - \alpha zy = a_1 x + b_1$*,* $zx - \alpha xz = a_2 y + b_2$*,* $xy - \alpha yx = a_3 z + b_3$*. If* $a_i = 0$ $(i = 1,2,3)$*, then all non-zero values of* b_i *give isomorphic algebras.*

(5) *If* $\alpha = \beta = \gamma = 1$*, then* A *is isomorphic to one of the following algebras:*

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \Bbbk$ *are arbitrary, and all non-zero values of* b *generate isomorphic algebras.*

1.2.4 Bi-quadratic algebras on 3 generators with PBW bases

Related with algebras generated by three indeterminates, recently Bavula [Bav23] defined the *skew bi-quadratic algebras* with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

For a ring R and a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements $m_{ij} \in R$ $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in R. The set of all such matrices is denoted by $L_n(R)$.

If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an n-tuple of commuting endomorphisms of R, $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tuple of σ-endomorphisms of R (that is, δ_i is a σ_i-derivation of R for $i = 1, \ldots, n$), $Q = (q_{ij}) \in L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, $1 \leq j < i \leq n$ and $k = 1, \ldots, n$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \ldots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring R and elements $x_1, \ldots, x_n$ subject to the defining relations

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \quad \text{for } i = 1, \ldots, n, \text{ and every } r \in R, \tag{1.5}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{1.6}$$

In the particular case when $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for $i = 1, \ldots, n$, the ring A is called the *bi-quadratic algebra* (*BQA*) and is denoted by $A = R[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$. A has *PBW basis* if $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^{\alpha}$ where $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The following result classifies (up to isomorphism) the bi-quadratic algebras on three generators of Lie type, i.e., when $q_1 = q_2 = q_3 = 1$.

PROPOSITION 1.6 ([BAV23, THEOREM 1.4]). *Let A be an algebra of Lie type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the algebra A is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

(1) $P_3 = \Bbbk[x_1, x_2, x_3]$, *a polynomial in three indeterminates.*

(2) $U(\mathfrak{sl}_2(\Bbbk))$, *the universal enveloping algebra of the Lie algebra* $\mathfrak{sl}_2(\Bbbk)$.

(3) $U(\mathfrak{H}_3))$, *the universal enveloping algebra of the Heisenberg Lie algebra* $\mathfrak{H}_3$.

(4) $U(\mathscr{N})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = z, [x, z] = 0, [y, z] = 1$, *and the algebra* $U(\mathscr{N})/\langle c-1\rangle$ *is a tensor product* $A_1 \otimes \Bbbk[x']$ *of its subalgebras, the Weyl algebra* $A_1(\Bbbk) = \Bbbk\{y, z\}/\langle [y, z] = 1\rangle$ *and the polynomial algebra* $[x']$ *where* $x' = x + \frac{1}{2}z^2$.

(5) $U(\mathfrak{n}_2 \times \Bbbk z) \cong \Bbbk\{x, y, z\}/\langle [x, y] = y\rangle$, *and z is a central element.*

(6) $U(\mathscr{M})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = y, [x, z] = 1, [y, z] = 0\rangle$ *and the algebra* $U(\mathscr{M})/\langle c-1\rangle$ *is a skew polynomial algebra* $A_1(\Bbbk)[y; \sigma]$ *where* $A_1(\Bbbk) = \Bbbk\{x, z\}/\langle [x, z] = 1\rangle$ *is the Weyl algebra and σ is an automorphism of* $A_1(\Bbbk)$ *given by the rule* $\sigma(x+1)$ *and* $\sigma(z) = z$.

PROPOSITION 1.7 ([BAV23, THEOREM 2.1]). *Up to isomorphism, there are only five bi-quadratic algebras on two generators:*

(1) *The polynomial algebra* $\Bbbk[x_1, x_2]$,

(2) *The Weyl algebra* $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1x_2 - x_2x_1 = 1\rangle$,

(3) *The universal enveloping algebra of the Lie algebra* $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$, $U(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 = x_1\rangle$,

(4) *The quantum plane* $\mathscr{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 = qx_1x_2\rangle$, *where* $q \in \Bbbk \setminus \{0, 1\}$, *and*

(5) *The quantum Weyl algebra* $A_1(q) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2 = 1\rangle$, *where* $q \in \Bbbk \setminus \{0, 1\}$.

1.2.5 DIFFUSION ALGEBRAS

Diffusion algebras were introduced formally by Isaev et al. [IPR01] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice. However, its origin can be found in Krebs and Sandow [KS97].

DEFINITION 1.7. ([IPR01, p. 5817]) The *diffusion algebras type 1* are affine algebras $\mathscr{D}$ that are generated by n indeterminates $D_1, \dots, D_n$ over $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\cdots D_{\alpha_n}^{k_n}$ with $k_j \in \mathbb{N}$ and $\alpha_1 \succ \alpha_2 \succ \cdots \succ \alpha_n$, and there exist elements $x_1, \dots, x_n \in \Bbbk$ such that for all $1 \le i < j \le n$, there exist $\lambda_{ij} \in \Bbbk^*$ such that

$$\lambda_{ij}D_iD_j - \lambda_{ji}D_jD_i = x_jD_i - x_iD_j. \tag{1.7}$$

Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t = 0$, for all $t = 1, \dots, n$, is a *multiparameter quantum affine n−space.*

Fajardo et al. [FGL+20] studied ring-theoretical properties of a graded version of these algebras.

DEFINITION 1.8. ([FGL+20, Section 2.4]) The *diffusion algebras type 2* are affine algebras $\mathscr{D}$ generated by $2n$ variables $\{D_1,\dots,D_n,x_1,\dots,x_n\}$ over a field $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1}B_{\alpha_2}^{k_2}\cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i}\in\{D_1,\dots,D_n,x_1,\dots,x_n\}$, for all $i\leq 2n$, $k_j\in\mathbb{N}$, and $\alpha_1\succ\alpha_2\succ\cdots\succ\alpha_n$, such that for all $1\leq i<j\leq n$, there exist elements $\lambda_{ij}\in\Bbbk^*$ satisfying the relations

$$\lambda_{ij}D_iD_j-\lambda_{ji}D_jD_i=x_jD_i-x_iD_j. \tag{1.8}$$

Different physical applications of algebras type 1 and 2 have been studied in the literature. From the point of view of ring-theoretical, homological and computational properties, thesis and papers have been published (e.g., [FGL+20, Hin05, Lev05, RS22, Twa02]). For instance, notice that a diffusion algebra type 1 generated by n indeterminates has Gelfand-Kirillov dimension n since because of the PBW basis, the vector subspace consisting of elements of total degree at most l is isomorphic to that of a commutative polynomial ring in n indeterminates. Similarly, diffusion algebras type 2 have Gelfand-Kirillov dimension $2n$.

REMARK 3. About the above definitions of diffusion algebras, we have the following facts:

(i) Isaev et al. [IPR01] and Pyatov and Twarok [PT02] defined diffusion algebras type 1 by taking $\Bbbk=\mathbb{C}$. Nevertheless, for the results obtained in this thesis we can take any field or ring under certain assumptions.

(ii) Following Krebs and Sandow [KS97], the relations (1.7) are consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta}D_\alpha D_\beta=D_\gamma X_\delta-X_\gamma D_\delta,\ \text{ for all }\ \gamma,\delta=0,1,\dots,n-1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta}\in\Bbbk$, and D_i's and X_j's are operators of a particular vector space such that not necessarily $[D_i,X_j]=0$ holds [KS97, p. 3168].

(iii) Hinchcliffe in his PhD thesis [Hin05, Definition 2.1.1] considered the following notation for diffusion algebras. Let R be the algebra generated by n indeterminates $x_1,x_2,\dots,x_n$ over $\mathbb{C}$ subject to relations $a_{ij}x_ix_j-b_{ij}x_jx_i=r_jx_i-r_ix_j$, whenever $i<j$, for some parameters $a_{ij}\in\mathbb{C}\setminus\{0\}$, for all $i<j$ and $b_{ij},r_i\in\mathbb{C}$, for all $i<j$. He defined the *standard monomials* to be those of the form $x_n^{i_n}x_{n-1}^{i_{n-1}}\cdots x_2^{i_2}x_1^{i_1}$. R is called a *diffusion algebra* if it admits a *PBW basis of these standard monomials*. In other words, R is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for R. If all the elements $q_{ij}:=\frac{b_{ij}}{a_{ij}}$'s are non-zero, then the diffusion algebras have a PBW basis in any order of the indeterminates [Hin05, Remark 2.1.6].

Diffusion algebras of n generators (also called *n-diffusion algebras*) are constructed in such a way that the subalgebras of three generators are also diffusion algebras. As we can see in Proposition 1.8, diffusion algebras type 1 of three generators can be classified into 4 families, A,B,C, and D, and these in turn are divided into classes as shown below (notice that this classification reflects the number of coefficients $x_s, s\in\{i,j,k\}$, being zero in comparison with the expression (1.7)).

PROPOSITION 1.8 ([PT02, P. 3270]). *If $\mathscr{D}$ is a diffusion algebra type 1 generated by the indeterminates D_i, D_j and D_k with $i < j < k$, and $\Lambda \in \Bbbk$, then $\mathscr{D}$ belongs to some of the following classes of diffusion algebras:*

(1) The case of A_I:

$$\begin{aligned} gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j, \\ gD_iD_k - gD_kD_i &= x_kD_i - x_iD_k, \\ gD_jD_k - gD_kD_j &= x_kD_j - x_jD_k, \end{aligned}$$

where $g \neq 0$.

(2) The case of A_{II}:

$$\begin{aligned} g_{ij}D_iD_j &= x_jD_i - x_iD_j, \\ g_{ik}D_iD_k &= x_kD_i - x_iD_k, \\ g_{jk}D_jD_k &= x_kD_j - x_jD_k, \end{aligned}$$

where $g_{st} := g_s - g_t$ *with* $g_s \neq g_t$, *for all* $s < t$, *and* $s, t \in \{i, j, k\}$.

(3) The case of $B^{(1)}$:

$$\begin{aligned} g_jD_iD_j - (g_j - \Lambda)D_jD_i &= -x_iD_j, \\ gD_iD_k - (g - \Lambda)D_kD_i &= x_kD_i - x_iD_k, \\ g_jD_jD_k - (g_j - \Lambda)D_kD_j &= x_kD_j, \end{aligned}$$

where $g, g_j \neq 0$.

(4) The case of $B^{(2)}$:

$$\begin{aligned} g_{ij}D_iD_j &= -x_iD_j, \\ g_{ik}D_iD_k - \lambda_{ki}D_kD_i &= x_kD_i - x_iD_k, \\ g_{jk}D_jD_k &= x_kD_j, \end{aligned}$$

where $g_{ij}, g_{ik}, g_{jk} \neq 0$.

(5) The case of $B^{(3)}$.

$$\begin{aligned} gD_iD_j - (g - \Lambda)D_jD_i &= x_jD_i - x_iD_j, \\ g_kD_iD_k &= -x_iD_k, \\ (g_k - \Lambda)D_jD_k &= -x_jD_k, \end{aligned}$$

where $g \neq 0$ *and* $g_k \neq 0, \Lambda$.

(6) The case of $B^{(4)}$:

$$\begin{aligned}(g_i-\Lambda)D_iD_j &= x_jD_i,\\ g_iD_iD_k &= x_kD_i,\\ gD_jD_k-(g-\Lambda)D_kD_j &= x_kD_j-x_jD_k,\end{aligned}$$

where $g \neq 0$ and $g_i \neq 0, \Lambda$.

(7) The case of $C^{(1)}$:

$$\begin{aligned}g_jD_iD_j-(g_j-\Lambda)D_jD_i &= -x_iD_j,\\ g_kD_iD_k-(g_k-\Lambda)D_kD_i &= -x_iD_k,\\ g_{jk}D_jD_k-g_{kj}D_kD_j &= 0,\end{aligned}$$

where $g_j, g_k, g_{j,k} \neq 0$.

(8) The case of $C^{(2)}$:

$$\begin{aligned}g_{ij}D_iD_j-g_{ji}D_jD_i &= -x_iD_j,\\ g_{ik}D_iD_k-g_{ki}D_kD_i &= -x_iD_k,\\ D_jD_k &= 0,\end{aligned}$$

where $g_{ij}, g_{ik} \neq 0$.

(9) The case of D: *With $q_{st} := \frac{g_{ts}}{g_{st}}$, where $s,t \in \{i,j,k\}$ (recall that $g_{st} \neq 0$, for $s<t$), we have*

$$\begin{aligned}D_iD_j-q_{ji}D_jD_i &= 0,\\ D_iD_k-q_{ki}D_kD_i &= 0,\\ D_jD_k-q_{kj}D_kD_j &= 0.\end{aligned}$$

About the relationship between diffusion algebras and skew polynomial rings, if we consider the notation in Remark 3(iii), then a 3-diffusion algebra generated by the indeterminates x_1, x_2, x_3 is a skew polynomial ring over its 2-diffusion subalgebra generated by x_2 and x_3 [Hin05, Lemma 2.2.1], where it is easy to see that a 2-diffusion algebra is a skew polynomial ring over the polynomial subalgebra generated by x_2. In general an n-diffusion algebra (generated by the indeterminates $x_1,\dots,x_n$) is a skew polynomial ring over its $(n-1)$-diffusion subalgebra generated by $x_2,\dots,x_n$ [Hin05, Remark 2.2.2].

Since a diffusion algebra on $n \geq 2$ generators is left Noetherian if and only if $q_{ij} \neq 0$, for all $i<j$ [Hin05, Proposition 2.2.5], where q_{ij} is given in Remark 3 (3), then every Noetherian 2-diffusion algebra is isomorphic to one of the following three types of algebra [Hin05, Proposition 3.3.1]:

- *The quantum affine plane*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - qx_2x_1 = 0$, for some $q \in \mathbb{C} \setminus \{0\}$ (allowing the possibility $q = 1$) (Proposition 1.7(4))

- *The quantized Weyl algebra*, i.e., the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - qx_2x_1 = 1$, for some $q \in \mathbb{C} \setminus \{0,1\}$ (Proposition 1.7(5)).

- *The universal enveloping algebra of the 2-d soluble Lie algebra*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - x_2x_1 = x_1$ (Proposition 1.7(3)).

Related to Proposition 1.8, Hinchcliffe [Hin05] proved the following result about classification of diffusion algebras assuming certain conditions on the coefficients of commutation of the indeterminates.

Proposition 1.9 ([Hin05, Proposition 3.1.4]). *If $q_{ij} \notin \{0,1\}$, for all i, j, then a diffusion algebra R is isomorphic either to multiparameter quantum affine n-space or to the $\mathbb{C}$-algebra generated by the indeterminates $x_1, x_2, x_3, \ldots, x_n$ subject to relations*

$$\begin{aligned}
x_1x_2 - q_{12}x_2x_1 &= 1, \ \text{ where } \ q_{12} \neq 1,\\
x_1x_i - q_{1i}x_ix_1 &= 0, \ \text{ where } \ q_{1i} \neq 1,\\
x_2x_i - q_{2i}^{-1}x_ix_2 &= 0,\\
x_ix_j - q_{ij}x_jx_i &= 0, \ \text{ for all } 3 \leq i < j.
\end{aligned}$$

1.2.6 Generalized Weyl algebras and down-up algebras

Other algebraic structures that illustrate the results obtained in this thesis are some kinds of *generalized Weyl algebras* and *down-up algebras*. We briefly present the definitions and some relations between these algebras (see [Jor95, Jor00, JW96] for a detailed description).

Given an automorphism σ and a central element a of a ring R, Bavula [Bav92] defined the *generalized Weyl algebra* $R(\sigma, a)$ as the ring extension of R generated by the indeterminates X^- and X^+ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings. Generalized Weyl algebras have been extensively studied in the literature by various authors (see [Bav23, Jor00], and references therein).

On the other hand, the *down-up algebras* $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, were defined by Benkart and Roby [Ben99, BR98] as generalizations of algebras generated by a pair of operators, precisely, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered set P. More exactly, consider a partially ordered set $(P, \prec)$ and let $\mathbb{C}P$ be the complex vector space with basis P. If for an element p of P, the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite, then we can define the "down" operator d and the "up" operator u in $\mathrm{End}_{\mathbb{C}}\, \mathbb{C}P$ as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define d and u). For any $\alpha, \beta, \gamma \in \mathbb{C}$, the *down-up algebra* is the $\mathbb{C}$-algebra generated by d and u subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2d + \gamma u$. A partially ordered set P is called (q, r)-*differential* if there exist $q, r \in \mathbb{C}$ such that the down and up operators for P satisfy both relations, and $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$. From [BR98], we know that for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha, \beta, \gamma) \simeq A(\alpha, \beta, \lambda\gamma)$. This means that when $\gamma \neq 0$, no problem if we assume $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [Ben99, Section 1].

Remarkable examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and some of its deformations introduced by Witten [Wit90] and Woronowicz [Wor87]. Related to the theoretical properties of these algebras, Kirkman et al., [KMP99] proved that a down-up algebra $A(\alpha, \beta, \gamma)$ is Noetherian if and only if β is non-zero. As a matter of fact, they showed that $A(\alpha, \beta, \gamma)$ is a generalized Weyl algebra and that $A(\alpha, \beta, \gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

Following [Ben99, p. 32], if $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbb{C}$ with basis $x, y, [x, y]$ such that $[x,[x, y]] = \gamma x$ and $[[x, y], y] = \gamma y$, then in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ these relations are given by $x^2y - 2xyx + yx^2 = \gamma x$ and $xy^2 - 2yxy + y^2x = \gamma y$. Notice that $U(\mathfrak{g})$ is a homomorphic algebra of the down-up algebra $A(2,-1,\gamma)$ via the mapping $\phi : A(2,-1,\gamma) \to U(\mathfrak{g})$, $d \mapsto x, u \mapsto y$, and the mapping $\psi : \mathfrak{g} \to A(2,-1,\gamma)$, $x \mapsto d, y \mapsto u$, $[x, y] \mapsto du - ud$, extends by the universal property of $U(\mathfrak{g})$ to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2,-1,\gamma)$ which is the inverse of ψ. Hence, $U(\mathfrak{g})$ is isomorphic to $A(2,-1,\gamma)$. It is straightforward to see that $U(\mathfrak{sl}_2(\mathbb{C})) \cong A(2,-1,-2)$. Also, for the Heisenberg Lie algebra $\mathfrak{h}$ with basis x, y, z where $[x, y] = z$ and $[z, x] = [z, y] = 0$, $U(\mathfrak{h}) \cong A(2,-1,0)$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [Wit90, Wit91] introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1, \dots, \xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates x, y, z subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. From [Ben99, Section 2], we know that a Witten's deformation algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.9}$$

is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha, \beta, \gamma)$ with not both α and β equal to 0 is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy (1.9).

Since algebras $W(\underline{\xi})$ are filtered, Le Bruyn [LB94, LB95] studied the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by the indeterminates x, y, z over a field $\Bbbk$ subject to the relations given by $zx - axz = x$, $zy - ayz = y$, and $yx - cxy = bz^2 + z$. In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three defining relations is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1 + ac)$, $\beta = -ac^{-1}$ and $\gamma = -c^{-1}$. Notice that if $c = b = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}$, $\beta = 0$, and $\gamma = -a^{-1}$. As one can check, conformal $\mathfrak{sl}_2$ algebras are not Ore extensions.

Kulkarni [Kul99, Theorem 3.0.3] showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2(\Bbbk)$ algebra or to an iterated Ore extension. More exactly, if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to one of the following algebras: (i) a conformal $\mathfrak{sl}_2$ algebra with generators x, y, z and relations given above or (ii) an iterated Ore extension whose generators satisfy

- $xz - zx = x$, $zy - yz = \zeta y$, $yx - \eta xy = 0$, or
- $xw = \theta wx$, $wy = \kappa yw$, $yx = \lambda xy$, for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \Bbbk$.

Notice that iterated Ore extensions above are defined in the following way: (i) the Witten deformation algebra is isomorphic to $\Bbbk[z][y,\sigma_1][x,\sigma_2]$ where σ_1 is the automorphism of $\Bbbk[z]$ defined as $\sigma_1(z) = z - \zeta$, with $zy - yz = \zeta y$; σ_2 is the automorphism of $\Bbbk[z][y,\sigma_1]$ defined as $\sigma_2(y) = \eta^{-1} y$, $\sigma_2(z) = z+1$, which satisfies $xz - zx = x$ and $yx - \eta xy = 0$. (ii) The Witten deformation algebra is isomorphic to $\Bbbk[w][y,\sigma_1][x,\sigma_2]$ where σ_1 is the automorphism of $\Bbbk[w]$ defined as $\sigma_1(w) = \kappa^{-1} w$ with $wy = \kappa yw$, and σ_2 is the automorphism of $\Bbbk[w][y,\sigma_1]$ defined as $\sigma_2(w) = \theta w$, $\sigma_2(y) = \lambda^{-1} y$ such that $wy = \kappa yw$ and $yx = \lambda xy$.

1.2.7 Other families of quantum algebras

In this section, we recall some examples of noncommutative rings known in the literature as *quantum algebras* or *quantized algebras*.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \dots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The *homogenized enveloping algebra of* $\mathfrak{g}$ is $\mathscr{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R \rangle$, where $T(\mathfrak{g} \oplus \Bbbk z)$ denotes the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\}$ and $\{x \otimes y - y \otimes x - [x,y] \otimes z \mid x, y \in \mathfrak{g}\}$.

From [GJ04, p. 41], for q an element of $\Bbbk$ with $q \neq \pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of q is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators E, F, K, K^{-1} and the relations $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$, $KE = q^2 EK$, and $KF = q^{-2} FK$. From [GJ04, Exercise 2T], we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1}; \sigma_1][F; \sigma_2, \delta_2]$, so that this algebra is not of automorphism type.

The *Lie-deformed Heisenberg* is the free $\mathbb{C}$-algebra defined by the commutation relations

$$q_j(1 + i\lambda_{jk}) p_k - p_k(1 - i\lambda_{jk}) q_j = i\hbar \delta_{jk},$$
$$[q_j, q_k] = [p_j, p_k] = 0, \quad j, k = 1, 2, 3,$$

where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k \delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$, then one recovers the usual Heisenberg algebra.

With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [Hay90] considered the A_q^- algebra by using the free algebra $\mathbf{U}$. Following Berger [Ber92, Example 2.7.7], this $\Bbbk$-algebra $\mathbf{U}$ is generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n$, and $\psi_1^*, \dots, \psi_n^*$, subject to the relations

$$\psi_j \psi_i - \psi_i \psi_j = \psi_j^* \psi_i^* - \psi_i^* \psi_j^* = \omega_j \omega_i - \omega_i \omega_j = \psi_j^* \psi_i - \psi_i \psi_j^* = 0, \qquad 1 \leq i < j \leq n,$$
$$\omega_j \psi_i - q^{-\delta_{ij}} \psi_i \omega_j = \psi_j^* \omega_i - q^{-\delta_{ij}} \omega_i \psi_j^* = 0, \qquad 1 \leq i, j \leq n,$$
$$\psi_i^* \psi_i - q^2 \psi_i \psi_i^* = -q^2 \omega_i^2, \qquad 1 \leq i \leq n.$$

Following Yamane [Yam89], if $q \in \mathbb{C}$ with $q^8 \neq 1$, the complex algebra A generated by the indeterminates $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ subject to the relations

$$
\begin{array}{lll}
e_{13}e_{12} = q^{-2}e_{12}e_{13}, & f_{13}f_{12} = q^{-2}f_{12}f_{13}, & \\
e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, & f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, & \\
e_{23}e_{13} = q^{-2}e_{13}e_{23}, & f_{23}f_{13} = q^{-2}f_{13}f_{23}, & \\
e_{12}f_{12} = f_{12}e_{12} + \dfrac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & e_{12}k_1 = q^{-2}k_1e_{12}, & k_1f_{12} = q^{-2}f_{12}k_1, \\
e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, & e_{12}k_2 = qk_2e_{12}, & k_2f_{12} = qf_{12}k_2, \\
e_{12}f_{23} = f_{23}e_{12}, & e_{13}k_1 = q^{-1}k_1e_{13}, & k_1f_{13} = q^{-1}f_{13}k_1, \\
e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & e_{13}k_2 = q^{-1}k_2e_{13}, & k_2f_{13} = q^{-1}f_{13}k_2, \\
e_{13}f_{13} = f_{13}e_{13} - \dfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & e_{23}k_1 = qk_1e_{23}, & k_1f_{23} = qf_{23}k_1, \\
e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & e_{23}k_2 = q^{-2}k_2e_{23}, & k_2f_{23} = q^{-2}f_{23}k_2, \\
e_{23}f_{12} = f_{12}e_{23}, & e_{12}l_1 = q^{2}l_1e_{12}, & l_1f_{12} = q^{2}f_{12}l_1, \\
e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & e_{12}l_2 = q^{-1}l_2e_{12}, & l_2f_{12} = q^{-1}f_{12}l_2, \\
e_{23}f_{23} = f_{23}e_{23} + \dfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & e_{13}l_1 = ql_1e_{13}, & l_1f_{13} = qf_{13}l_1, \\
e_{13}l_2 = ql_2e_{13}, & l_2f_{13} = qf_{13}l_2, & e_{23}l_1 = q^{-1}l_1e_{23}, \\
l_1f_{23} = q^{-1}f_{23}l_1, & e_{23}l_2 = q^{2}l_2e_{23}, & l_2f_{23} = q^{2}f_{23}l_2, \\
l_1k_1 = k_1l_1, & l_2k_1 = k_1l_2, & k_2k_1 = k_1k_2, \\
l_1k_2 = k_2l_1, & l_2k_2 = k_2l_2, & l_2l_1 = l_1l_2,
\end{array}
$$

is very important in the definition of the *quantized enveloping algebra of* $\mathfrak{sl}_3(\mathbb{C})$.

The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al. [JLM95] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators $A_j,\ B_k$, where the following relations are satisfied:

$$
\begin{aligned}
A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j &= i\delta_{jk},\\
[A_j, B_k] &= 0 \ \ (j \neq k),\\
[A_j, A_k] = [B_j, B_k] &= 0,
\end{aligned}
$$

and,

$$
\begin{aligned}
A_j^{+}(1 + i\lambda_{jk})B_k^{+} - B_k^{+}(1 - i\lambda_{jk})A_j^{+} &= i\delta_{jk},\\
[A_j^{+}, B_k^{+}] &= 0 \ (j \neq k),\\
[A_j^{+}, A_k^{+}] = [B_j^{+}, B_k^{+}] &= 0,
\end{aligned}
\tag{1.10}
$$

with $A_j \neq A_j^{+},\ B_k \neq B_k^{+}\ (j,k = 1,2,3)$. If the operators $A_j,\ B_k$ are in the form $A_j = f_j(N_j + 1)a_j,\ B_k = a_k^{+}f_k(N_k+1)$, where $a_j,\ a_j^{+}$ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator $(N_j = a_j^{+}a_j,\ \langle N_j \mid n_j\rangle = \langle n_j | n_j\rangle)$, and the structure functions $f_j(N_j+1)$ complex, then it is showed that A_j and B_k are given by

$$A_j = \sqrt{\frac{i}{1+i\lambda_j}}\Big(\frac{[(1-i\lambda_j)/(1+i\lambda_j)]^{N_j+1}-1}{(1-i\lambda_j)/(1+i\lambda_j)-1}\frac{1}{N_j+1}\Big)^{\frac{1}{2}}a_j,$$
$$B_k = \sqrt{\frac{i}{1+i\lambda_k}}a_k^+\Big(\frac{[(1-i\lambda_k)/(1+i\lambda_k)]^{N_k+1}-1}{(1-i\lambda_k)/(1+i\lambda_k)-1}\frac{1}{N_k+1}\Big)^{\frac{1}{2}}.$$

Following Havliček et al. [HKP00, p. 79], the $\mathbb{C}$-algebra $U_q'(\mathfrak{so}_3)$ is generated by the indeterminates I_1, I_2, and I_3, subject to the relations given by

$$I_2I_1 - qI_1I_2 = -q^{\frac{1}{2}}I_3, \quad I_3I_1 - q^{-1}I_1I_3 = q^{-\frac{1}{2}}I_2, \quad I_3I_2 - qI_2I_3 = -q^{\frac{1}{2}}I_1,$$

where q is a non-zero element of $\mathbb{C}$. It is straightforward to show that $U_q'(\mathfrak{so}_3)$ cannot be expressed as an iterated Ore extension. By using [RS17a, Theorem 1.14], it can be shown that $U_q'(\mathfrak{so}_3)$ is a skew PBW extension over $\Bbbk$, i.e., $U_q'(\mathfrak{so}_3) \cong \sigma(\Bbbk)\langle I_1, I_2, I_3\rangle$.

Zhedanov [Zhe91, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the $\mathbb{R}$-algebra generated by three operators K_0, K_1, and K_2, that satisfy the commutation relations

$$[K_0, K_1]_\omega = K_2, \quad [K_2, K_0]_\omega = BK_0 + C_1K_1 + D_1, \quad \text{and} \quad [K_1, K_2]_\omega = BK_1 + C_0K_0 + D_0,$$

where B, C_0, C_1, D_0, and D_1 are elements of $\mathbb{R}$ that represent the structure constants of the algebra, and the q-commutator $[-,-]_\omega$ is given by $[\square, \triangle]_\omega := e^\omega\square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$\begin{aligned} e^\omega K_0K_1 - e^{-\omega}K_1K_0 &= K_2, \\ e^\omega K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1, \\ e^\omega K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0. \end{aligned}$$

According to these relations that define the algebra, it is clear that AW(3) cannot be expressed as an iterated Ore extension. Using techniques such as those presented in [RS17a, Theorem 1.14], it can be shown that AW(3) is a skew PBW extension of endomorphism type, that is, AW(3) $\cong \sigma(\mathbb{R})\langle K_0, K_1, K_2\rangle$.

With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [Gre53] and Greenberg and Messiah [GM65] introduced by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the completeness of the thesis, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [KD09]. Let $[\square, \triangle] := \square\triangle - \triangle\square$ and $\{\square, \triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space V_F freely generated by the elements f_i^+, f_j^-, with $i, j = 1, \dots, n$.

If $T(V_F)$ is the tensor algebra of V_F and I_F is the two-sided ideal I_F generated by the elements $[[f_i^{\xi}, f_j^{\eta}], f_k^{\varepsilon}] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^{\xi} + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^{\eta}$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j, k = 1, \dots, n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ (n parafermions) is the quotient algebra of $T(V_F)$ with the ideal I_F, that is,

$$P_F^{(n)} = \frac{T(V_F)}{\langle [[f_i^{\xi}, f_j^{\eta}], f_k^{\varepsilon}] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^{\xi} + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \dots, n\rangle}.$$

It is well-known (e.g., [KD09, Section 18.2]) that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$.

Similarly, if V_B denotes the $\Bbbk$-vector space freely generated by the elements b_i^+, b_j^-, $i, j = 1, \dots, n$, $T(V_B)$ is the tensor algebra of V_B, and I_B is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon-\eta)\delta_{jk}b_i^{\xi} - (\varepsilon-\xi)\delta_{ik}b_j^{\eta}$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j = 1, \dots, n$, then the *parabosonic algebra* $P_B^{(n)}$ in $2n$ generators (n parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$P_B^{(n)} = \frac{T(V_B)}{\langle [\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon-\eta)\delta_{jk}b_i^{\xi} - (\varepsilon-\xi)\delta_{ik}b_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j = 1, \dots, n\rangle}.$$

The parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0,n)$, that is, $P_B^{(n)} \cong U(B(0,n))$. For more details about *parafermionic* and *parabosonic algebras*, see [KD09, Proposition 18.2], and references therein.

EXAMPLE 1.1. The Jordan plane $\mathscr{J}$ introduced by Jordan [Jor01] is the free $\Bbbk$-algebra generated by the indeterminates x, y subject to the relation $yx = xy + x^2$. If we guarantee the PBW basis condition, then it is easy to see that $\mathscr{J} \cong \sigma(\Bbbk[x])\langle y\rangle$. Also, homogenized enveloping algebras (Section 1.2.7), and some classes of diffusion algebras (Section 1.2.5) are graded skew PBW extensions. If we assume the condition of the PBW basis, then graded Clifford algebras defined by Le Bruyn [LB95] are also examples of graded skew PBW extensions. Let us see the details.

Following Cassidy and Vancliff [CV10], let $\Bbbk$ be an algebraically closed field such that $\text{char}(\Bbbk) \neq 2$ and let $M_1, \dots, M_n \in \mathbb{M}_n(\Bbbk)$ be symmetric matrices of order $n \times n$ with entries in $\Bbbk$. A *graded Clifford algebra* $\mathscr{A}$ is a $\Bbbk$-algebra on degree-one generators $x_1, \dots, x_n$ and on degree-two generators $y_1, \dots, y_n$ with defining relations given by

(i) $x_ix_j + x_jx_i = \sum_{k=1}^{n}(M_k)_{ij}y_k$ for all $i, j = 1, \dots, n$;

(ii) y_k central for all $k = 1, \dots, n$.

Note that the commutative polynomial ring $R = \Bbbk[y_1, \dots, y_n]$ is an $\mathbb{N}$-graded algebra where $R_0 = \Bbbk$, $R_1 = \{0\}$, $y_1, \dots, y_n \in R_2$, and $R_i = \{0\}$, for $i \geq 3$. If we suppose that the set $\{x_1^{a_1}\cdots x_n^{a_n} \mid a_i \in \mathbb{N}, i = 1, \dots, n\}$ is a left PBW R-basis for $\mathscr{A}$, then the graded Clifford algebra $\mathscr{A}$ is a graded skew PBW extension over the connected algebra R, that is, $\mathscr{A} \cong \sigma(R)\langle x_1, \dots, x_n\rangle$. Indeed, from the relations (i) and (ii) above, it is clear that $\sigma_i = \mathrm{id}_R$, $\delta_i = 0$, $d_{i,j} = -1 \in R_0$, for $1 \leq i, j \leq n$, and $\sum_{k=1}^{n} (M_k)_{ij} y_k \in R_2$, where $d_{i,j}$ is given as in Definition 1.1 (iv). In this way, $\mathscr{A}$ is a bijective skew PBW extension that satisfies both conditions of Proposition 1.4.

CHAPTER 2

HOMOGENIZED SKEW PBW EXTENSIONS

This chapter presents the first original results on skew PBW extensions obtained in the thesis, which were published in [SRS23].

The chapter is organized as follows. In Section 2.1, we recall some elementary definitions and properties of ring theory that are needed throughout the chapter. Section 2.2 contains the definition of filtration on skew PBW extensions over positively filtered algebras, examples and some properties of these noncommutative rings (Theorem 2.1 and Propositions 2.2, 2.3, 2.4, and 2.5). Next, Section 2.3 presents properties of σ-filtered skew PBW extensions over finitely presented algebras (Propositions 2.6, 2.7, and 2.9). We prove that the homogenization of a σ-filtered skew PBW extension over a finitely presented algebra R is a graded skew PBW extension over the homogenization of R (Theorem 2.8). In Section 2.4, for A a σ-filtered skew PBW extension over a ring R, we establish different homological properties for A, its associated graded ring $G(A)$, and the homogenization $H(A)$ of A (Theorems 2.10 and 2.13, and Proposition 2.12). Finally, Section 2.5 presents some ideas for a possible future work concerning the ideas developed here and topics of interest noncommutative algebraic geometry.

2.1 PRELIMINARIES

An algebra R is called $\mathbb{Z}$-*graded* if there exists a family of subspaces $\{R_p\}_{p\in\mathbb{Z}}$ of R such that $R = \bigoplus_{p\in\mathbb{Z}} R_p$ and $R_pR_q \subseteq R_{p+q}$, for all $p, q \in \mathbb{Z}$. A graded algebra R is called *positively graded* (or $\mathbb{N}$-*graded*) if $R_p = 0$, for all $p < 0$. An $\mathbb{N}$-graded algebra R is called *connected* if $R_0 = \Bbbk$. A non-zero element $r \in R_p$ is called a *homogeneous element* of R of degree p. A homogeneous element r of a graded algebra R is said to be *regular* if it is neither a left nor a right zero divisor. For R and S two connected graded algebras, if there exists a central element $s \in S_1$ such that $R \cong S/\langle s\rangle$, then S is called a *central extension* of R. If further s is regular in S, then S is called a *central regular extension* of R. If R is a $\mathbb{Z}$-graded algebra, $R(l) := \bigoplus_{p\in\mathbb{Z}} R(l)_p$, where $R(l)_p = R_{p+l}$, for $l \in \mathbb{Z}$.

An algebra R is said to be *finitely graded* if the following conditions hold:

- R is $\mathbb{N}$-graded,
- R is connected,
- R is *finitely generated* as $\Bbbk$-algebra, i.e., there are finite elements $t_1, \dots, t_m \in R$ such that the set $\{t_{i_1} t_{i_2} \cdots t_{i_p} \mid 1 \leq i_j \leq m,\ p \geq 1\} \cup \{1\}$ spans R as a $\Bbbk$-space.

A *filtration* $\mathscr{F}$ on an algebra R is a collection of vector spaces $\{\mathscr{F}_p(R)\}_{p\in\mathbb{Z}}$ such that $\mathscr{F}_p(R) \subseteq \mathscr{F}_{p+1}(R)$, $\mathscr{F}_p(R)\cdot\mathscr{F}_q(R) \subseteq \mathscr{F}_{p+q}(R)$, for every $p, q \in \mathbb{Z}$, and $\bigcup_{p\in\mathbb{Z}} \mathscr{F}_p(R) = R$. The filtration $\mathscr{F}$ is said to be *finite* if each $\mathscr{F}_p(R)$ is a finite dimensional subspace. The filtration is *positive* if $\mathscr{F}_{-1}(R) = 0$. In this case, we say that R is *positively* filtered ($\mathbb{N}$-filtered). If $0 \neq r \in \mathscr{F}_p(R) \setminus \mathscr{F}_{p-1}(R)$, then r is said to have *degree* p, and write $\deg(r) = p$. A positive filtration is said to be *connected* if $\mathscr{F}_0(R) = \Bbbk$; in this case, we say that R is *connected filtered*. The associated graded algebra of R is given by $G_{\mathscr{F}}(R) := \bigoplus_{p\geq 0} \mathscr{F}_p(R)/\mathscr{F}_{p-1}(R)$. Notice that $G_{\mathscr{F}}(R)$ is connected if the filtration $\mathscr{F}$ is connected. We simply write $G(R)$ if no confusion arises.

The associated *Rees algebra* to R is defined as $\mathrm{Rees}_{\mathscr{F}}(R) := \bigoplus_{p\geq 0} \mathscr{F}_p(R)z^p$. The filtration $\{F_p(R)\}_{p\in\mathbb{Z}}$ is *left* (*right*) *Zariskian* and R is called a *left* (*right*) *Zariski ring* if $F_{-1}(R) \subseteq \mathrm{Rad}(F_0(R))$ (where $\mathrm{Rad}(F_0(R))$ is the Jacobson radical of $F_0(R)$), and the associated Rees ring $\mathrm{Rees}_{\mathscr{F}}(R)$ is left (right) Noetherian. Of course, if R is graded, then $R = G(R)$. In this case, we write R_p for the vector space spanned by homogeneous elements of degree p. If R is a filtered algebra with filtration $\{\mathscr{F}_p(R)\}_{p\in\mathbb{Z}}$ and M is an R-module, then we say that M is *filtered* if there exists a family $\{\mathscr{F}_p(M)\}_{p\in\mathbb{Z}}$ of subspaces of M such that $\mathscr{F}_p(M) \subseteq \mathscr{F}_{p+1}(M)$, $\mathscr{F}_p(R)\cdot\mathscr{F}_q(M) \subseteq \mathscr{F}_{p+q}(M)$, and $\bigcup_{p\in\mathbb{Z}} \mathscr{F}_p(M) = M$. If $m \in M_p \setminus M_{p-1}$, then m is said to have *degree* p. For further details about filtered and Rees rings, see Li and Van Oystaeyen [LvO96].

For R a connected graded algebra, its *global homological dimension* $\mathrm{gld}(R)$ is the projective dimension of the trivial R-module $\Bbbk = R/R_+$, where R_+ is the augmentation ideal generated by all degree one elements. If V is a generating set for R and V^n is the set of elements of degree n, then the *Gelfand-Kirillov dimension* of R is defined as $\mathrm{GKdim}(R) := \overline{\lim}_{n\to\infty}\log_n(\dim V^n)$.

The *free associative algebra* L in m generators $t_1, \dots, t_m$, denoted by $L := \Bbbk\{t_1, \dots, t_m\}$, is the $\Bbbk$-algebra whose underlying $\Bbbk$-vector space is the set of all words in the indeterminates t_i, that is, expressions of the form $t_{i_1} t_{i_2} \dots t_{i_p}$, for some $p \geq 1$, where $1 \leq i_j \leq m$, for all j. The *degree* (deg) of a word $t_{i_1} t_{i_2} \dots t_{i_p}$ is p, and the degree of an element $f \in L$ is the maximum of the degrees of the words in f. We include among the words a symbol 1, which we think of as the empty word, and which has degree 0. The product of two words is concatenation, and this operation is extended linearly to define an associative product on all elements. Notice that L is positively graded with graduation given by $L := \bigoplus_{p\geq 0} L_p$, where $L_0 = \Bbbk$, and L_p is spanned by all words of degree p in the alphabet $\{t_1, \dots, t_m\}$, for $p > 0$. L is connected and therefore augmented, where the augmentation of L is given by the natural projection $\varepsilon : \Bbbk\{t_1, \dots, t_n\} \to L_0 = \Bbbk$ and the augmentation ideal is given by $L_+ := \bigoplus_{p>0} L_p$. L is connected filtered with the standard filtration $\{\mathscr{F}_q(L)\}_{q\in\mathbb{N}}$, where $\mathscr{F}_q(L) = \bigoplus_{p\leq q} L_p$.

An algebra R is *finitely presented* if it is a quotient $\Bbbk\{t_1, \dots, t_m\}/I$ where I is a finitely generated two-sided ideal of $\Bbbk\{t_1, \dots, t_m\}$, say $I = \langle r_1, \dots, r_s\rangle$. $\Bbbk\{t_1, \dots, t_m\}/I$ is said to be a *presentation* of

R with generators $t_1,\dots,t_m$ and relations $r_1,\dots,r_s$. Throughout this section, we assume that $\{r_1,\dots,r_s\}$ is a minimal set of relations for R, the generators t_i all have degree 1, and none of the relations r_i are linear. Notice that if the relations $r_1,\dots,r_s$ are all homogeneous, then R is called a *connected graded algebra*. Now, by a *deformation* of a connected graded algebra R we mean an algebra

$$U = \Bbbk\{t_1,\dots,t_m\}/\langle r_1+l_1,\dots,r_s+l_s\rangle, \tag{2.1}$$

where $l_1,\dots,l_s$ are (not necessarily homogenous) elements of $\Bbbk\{t_1,\dots,t_m\}$ such that $\deg(l_i) < \deg(r_i)$, for all i. There is a standard filtration on U induced by the standard filtration on $\Bbbk\{t_1,\dots,t_m\}$. If $g = \sum_{k=0}^{p} g_k \in \Bbbk\{t_1,\dots,t_m\}$, where each non-zero g_k is a homogeneous polynomial of degree k, and $\deg(g_1) < \deg(g_2) < \cdots < \deg(g_p)$, then g_p is said to be the *leading homogeneous polynomial* of g, which is denoted by $\mathrm{lh}(g)$. The *homogenization* $\widehat{g}$ of g is given by $\widehat{g} = \sum_{k=0}^{p} g_k z^{p-k}$, where z is a new central indeterminate. Let $R = \Bbbk\{t_1,\dots,t_m\}/\langle f_1,\dots,f_s\rangle$ be a finitely presented algebra. Since R is not necessarily graded, if we homogenize every polynomial $f_i \in R$, we obtain a graded algebra known as the *homogenization of* R.

In the setting of noncommutative rings having PBW bases, Cassidy and Shelton [CS07, Theorem 1.3] proved that a deformation U of the graded algebra R is a PBW deformation if and only if the homogenization of U is a regular central extension. Other properties of central extensions and homogenization have been used by several authors to study certain classes of algebras (e.g. [CS07, CSW18, Gad16, SL16, WZ13]).

Of interest for us in this section, we recall the following definition.

DEFINITION 2.1. ([Gad16, Definition 2.1]). Let $U = \Bbbk\{t_1,\dots,t_m\}/\langle f_1,\dots,f_s\rangle$ be an algebra, where $f_1,\dots,f_s \in L = \Bbbk\{t_1,\dots,t_m\}$. The graded algebra $H(U) = L[z]/\langle \widehat{f_1},\dots,\widehat{f_s}\rangle$ is called the *homogenization* of U.

In other words, the homogenization $H(R)$ of $R = L/\langle f_1,\dots,f_s\rangle$ is the algebra with $n+1$ generators $t_1,\dots,t_n$, and z, subject to the homogenized relations $\widehat{f_k}$ as well as the additional relations $zt_i - t_iz$, for $1 \le i \le n$.

Notice that if $U := \Bbbk\{t_1,\dots,t_m\}/\langle f_1,\dots,f_s\rangle$ is an algebra and we consider

$$R := \Bbbk\{t_1,\dots,t_m\}/\langle \mathrm{lh}(f_1),\dots,\mathrm{lh(f_s)}\rangle,$$

then we have a natural graded surjective homomorphism $\phi : R \to G(U)$.

When ϕ is an isomorphism, we say that U is a *Poincaré-Birkhoff-Witt* (*PBW*) *deformation* of $G(U)$ [Gad16, Definition 2.6]. If R is a connected graded algebra, a deformation U of R as in (2.1) is said to be a *PBW deformation* if $G(U)$ is isomorphic to R.

DEFINITION 2.2. ([AS87, p. 171]). A connected graded algebra R is said to be *Artin-Schelter regular* of dimension d if the following conditions hold:

(i) R has finite global dimension d;

(ii) R has finite Gelfand-Kirillov dimension;

(iii) (Gorenstein's condition) $\operatorname{Ext}_R^i(\Bbbk, R) = 0$ if $i \neq d$, and $\operatorname{Ext}_R^d(\Bbbk, R) \cong \Bbbk$.

Let $I \subseteq \sum_{n\geq 2} L_n$ be a finitely generated homogeneous ideal of $\Bbbk\{t_1,\dots,t_m\}$ and let $R = \Bbbk\{t_1,\dots,t_m\}/I$ be a connected graded algebra generated in degree 1. Suppose that $\sigma : R \to R$ is a graded algebra automorphism and $\delta : R(-1) \to R$ is a graded σ-derivation (i.e., a degree $+1$ graded σ-derivation δ of R). As we saw in Section 1.1, let $B := R[x;\sigma,\delta]$ be the associated *graded Ore extension* of R, that is, $B = \bigoplus_{p\geq 0} Rx^p$ as an R-module, and for $r \in R$, $xr = \sigma(r)x + \delta(r)$. If we consider x to have degree 1 in B, then under this grading B is a connected graded algebra generated in degree 1.

2.2 σ-FILTERED SKEW PBW EXTENSIONS

If R is an arbitrary algebra, then it is clear that R is a filtered algebra with filtration given by $\mathscr{F}_p(R) = R$, for all $p \in \mathbb{Z}$. In this case, we say that R has the *trivial filtration*. R has the *trivial positive filtration* if $\mathscr{F}_p(R) = R$, for all $p \geq 0$ and $\mathscr{F}_{-1}(R) = 0$. If $l \geq 0$, then R is connected filtered with filtration given by

$$\mathscr{F}_p(R) = \begin{cases} 0, & \text{if } p = -1; \\ \Bbbk, & \text{if } 0 \leq p \leq l; \\ R, & \text{if } p > l. \end{cases}$$

In this case, we say that R is an *l-trivial connected filtered algebra*. If $l = 0$, then we say that R is a *trivial connected filtered algebra*. We assume that $\Bbbk$ has trivial connected filtration.

DEFINITION 2.3. Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over a positively filtered algebra $R = \bigcup_{p\in\mathbb{N}} \mathscr{F}_p(R)$.

(i) For $X = x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \in \operatorname{Mon}(A)$ and $c \in R \setminus \{0\}$, $\operatorname{tdeg}(cX) := \deg(c) + \deg(X)$.

(ii) Let $f = c_1X_1 + \cdots + c_dX_d \in A \setminus \{0\}$, $\operatorname{tdeg}(f) := \max\{\operatorname{tdeg}(c_iX_i)\}_{i=1}^d$.

(iii) Let $\sigma : R \to R$ be an endomorphism of algebras. If $\sigma(\mathscr{F}_p(R)) \subseteq \mathscr{F}_p(R)$, then we say that σ is a *filtered endomorphism*.

(iv) Let $\delta : R \to R$ be a σ-derivation. If $\delta(\mathscr{F}_p(R)) \subseteq \mathscr{F}_{p+1}(R)$, we say that δ is a *filtered σ-derivation*, and if $\delta(\mathscr{F}_p(R)) \subseteq \mathscr{F}_{p+m}(R)$, then we say that δ is an m-filtered σ-derivation, for $m > 1$.

(v) We say that A *preserves* tdeg if for each x_jx_i as in (1.2),

$$\operatorname{tdeg}(x_jx_i) = \operatorname{tdeg}(c_{i,j}x_ix_j + r_{0_{j,i}} + r_{1_{j,i}}x_1 + \cdots + r_{n_{j,i}}x_n) = 2.$$

The following theorem, one of the most important results of the chapter, provides a general filtration to the skew PBW extensions.

THEOREM 2.1. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension over a positively filtered algebra R such that the following conditions hold:*

(1) *σ_i and δ_i are filtered, for $1 \leq i \leq n$;*

(2) *A preserves* tdeg*,*

then $\{\mathscr{F}_p(A)\}_{p\in\mathbb{N}}$ is a filtration on A, where

$$\mathscr{F}_p(A) := \left\{f \in A \mid \operatorname{tdeg}(f) \leq p\right\} \cup \{0\}. \tag{2.2}$$

Moreover, A is a filtered R-module with the same filtration.

Proof. Notice that for each $f \in A \setminus \{0\}$, $\operatorname{tdeg}(f) \geq 0$. By definition, $0 \in \mathscr{F}_p(A)$. Let $f, g \in \mathscr{F}_p(A)$ with $f \neq 0$ and $g \neq 0$. By Remark 1(iv), f and g have a unique representation as $f = c_1X_1 + \cdots + c_dX_d$ and $g = r_1Y_1 + \cdots + r_eY_e$, with $c_i, r_j \in R \setminus \{0\}$ and $X_i, Y_j \in \operatorname{Mon}(A)$ for $1 \leq i \leq d$ and $1 \leq j \leq e$. In this way, $\operatorname{tdeg}(c_iX_i) \leq p$ and $\operatorname{tdeg}(r_jY_j) \leq p$, for $1 \leq i \leq d$, $1 \leq j \leq e$. Thus $\operatorname{tdeg}(f + g) = \operatorname{tdeg}(c_1X_1 + \cdots + c_dX_d + r_1Y_1 + \cdots + r_eY_e) = \max\{\operatorname{tdeg}(c_iX_i), \operatorname{tdeg}(r_jY_j) \mid 1 \leq i \leq d, 1 \leq j \leq e\} \leq p$, and so $f + g \in \mathscr{F}_p(A)$. Now, if $k \in \Bbbk$, then $\operatorname{tdeg}(kf) = \operatorname{tdeg}((kc_1)X_1 + \cdots + (kc_d)X_d) \leq p$, whence $\mathscr{F}_p(A)$ is a subspace of A, for each $p \in \mathbb{N}$. It is clear that $\bigcup\limits_{p\in\mathbb{N}} \mathscr{F}_p(A) = A$. If $0 \neq f = c_1X_1 + \cdots + c_dX_d \in \mathscr{F}_p(A)$, then $\operatorname{tdeg}(c_iX_i) \leq p < p + 1$, for $1 \leq i \leq d$, and $\mathscr{F}_p(A) \subseteq \mathscr{F}_{p+1}(A)$. Let $h \in \mathscr{F}_p(A) \cdot \mathscr{F}_q(A)$. Without loss of generality we assume that $h = h_ph_q$ with $h_p \in \mathscr{F}_p(A)$ and $h_q \in \mathscr{F}_q(A)$. Let $h_p = a_1X_1 + \cdots + a_mX_m$, $h_q = b_1Y_1 + \cdots + b_tY_t$. Then $\operatorname{tdeg}(a_iX_i) \leq p$ and $\operatorname{tdeg}(b_jY_j) \leq q$, for $1 \leq i \leq m$, $1 \leq j \leq t$. Hence

$$h = (a_1X_1 + \cdots + a_mX_m)(b_1Y_1 + \cdots + b_tY_t) = \sum_{k=1}^{m+t}\Big(\sum_{i+j=k} a_iX_ib_jY_j\Big), \tag{2.3}$$

and so $\operatorname{tdeg}(h) = \max\{\operatorname{tdeg}(a_iX_ib_jY_j) \mid 1 \leq i \leq m, 1 \leq j \leq t\}$, but obtaining the unique representation of $a_iX_ib_jY_j$ once the commutation rules have been made taking into account (iii) and (iv) in the Definition 1.1.

Let $X_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $Y_j = x_1^{\beta_1} \cdots x_n^{\beta_n}$, with $\alpha_i, \beta_i \in \mathbb{N}$. By [RS18a, Remark 2.7],

$$\begin{aligned}
a_iX_ib_jY_j &= a_i(x_1^{\alpha_1} \cdots x_n^{\alpha_n} b_j)x_1^{\beta_1} \cdots x_n^{\beta_n}\\
&= a_ix_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}\Big(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j}\delta_n(\sigma_n^{j-1}(b_j))x_n^{j-1}\Big)x_1^{\beta_1} \cdots x_n^{\beta_n}\\
&+ a_ix_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}}\Big(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1} - j}\delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(b_j)))x_{n-1}^{j-1}\Big)x_n^{\alpha_n}x_1^{\beta_1} \cdots x_n^{\beta_n}\\
&+ a_ix_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}}\Big(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2} - j}\delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(b_j))))x_{n-2}^{j-1}\Big)x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}x_1^{\beta_1} \cdots x_n^{\beta_n} +\\
&\cdots + a_ix_1^{\alpha_1}\Big(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j}\delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(b_j))))))x_2^{j-1}\Big)x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}x_1^{\beta_1} \cdots x_n^{\beta_n}\\
&+ a_i\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(b_j))))x_1^{\alpha_1} \cdots x_n^{\alpha_n}x_1^{\beta_1} \cdots x_n^{\beta_n}, \qquad \sigma_j^0 := \mathrm{id}_R \text{ for } 1 \leq j \leq n.
\end{aligned} \tag{2.4}$$

Notice that as σ_i and δ_i are filtered, then $\sigma_i^k(\mathscr{F}_p(R)) \subseteq \sigma_i^{k-1}(\mathscr{F}_p(R)) \subseteq \cdots \subseteq \sigma_i(\mathscr{F}_p(R))$ and $\delta_i^k(\mathscr{F}_p(R)) \subseteq \mathscr{F}_{p+k}(R))$, and thus σ_i^k is filtered and δ_i^k is k-filtered. Furthermore, as A preserves tdeg, then for each of the summands in (2.4), $\operatorname{tdeg} \leq p + q$ (once the commutation rules have been made taking into account (iii) and (iv) in the Definition 1.1). In this way, $\operatorname{tdeg}(h) \leq p + q$, and so $h \in \mathscr{F}_{p+q}$, whence $\{\mathscr{F}_p(A)\}_{p\in\mathbb{N}}$ is a filtration on A.

On the other hand, let $g \in \mathscr{F}_p(R)\mathscr{F}_q(A)$. Then $g = rf$, for some $r \in \mathscr{F}_p(R)$ and $f = r_1X_1 + \cdots + r_tX_t \in \mathscr{F}_q(A)$. So, $\operatorname{tdeg}(r_iX_i) = \deg(r_i) + \deg(X_i) \le q$, for $1 \le i \le t$. Thus $\operatorname{tdeg}(r(r_iX_i)) = \operatorname{tdeg}((rr_i)X_i)) = \deg(rr_i) + \deg(X_i) \le p + \deg(r_i) + \deg(X_i) \le p + q$, which implies that $\operatorname{tdeg}(rf) \le p + q$, and therefore $rf = g \in \mathscr{F}_{p+q}(A)$. This shows that A is a filtered R-module. $\square$

Theorem 2.1 suggests the following definition.

DEFINITION 2.4. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a positively filtered algebra R. We say that A is a *σ-filtered skew PBW extension over R* if A satisfies the conditions (1) and (2) in Theorem 2.1.

In this case, it is understood that A has the filtration $\{\mathscr{F}_p(A)\}_{p\in\mathbb{N}}$, where $\mathscr{F}_p(A)$ is as in (2.2).

PROPOSITION 2.2. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a σ-filtered skew PBW extension over R.*

(1) *If $\{\mathscr{F}_p(R)\}_{p\in\mathbb{N}}$ is a positive filtration on R, then $\mathscr{F}_p(R)$ is a subspace of $\mathscr{F}_p(A)$.*

(2) *If the filtration on R is finite, then the filtration of A is finite.*

Proof. Let $\{\mathscr{F}_p(R)\}_{p\in\mathbb{N}}$ be a connected filtration on R and $\{\mathscr{F}_p(A)\}_{p\in\mathbb{N}}$ the filtration on A.

(1) If $0 \ne r \in \mathscr{F}_p(R)$, then $\deg(r) \le p$. By Definition 1.1(i), we have $R \subseteq A$, and $r = rx_1^0 \cdots x_n^0$ is the unique representation of r. This means that $\operatorname{tdeg}(r) = \deg(r) \le p$, and so $r \in \mathscr{F}_p(A)$.

(2) Let $\mathscr{B}_k^{(R)}$ be a finite basis for $\mathscr{F}_k(R)$. By Definition 1.1(ii), we have that A is a left free R-module with basis $\operatorname{Mon}(A) = \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, whence $\mathscr{B}_p = \left(\bigcup_{k=0}^{p} \mathscr{B}_k^{(R)}\right) \cup \{X \in \operatorname{Mon}(A) \mid \deg(X) \le p\}$ is a finite basis for $\mathscr{F}_p(A)$.

$\square$

PROPOSITION 2.3. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a σ-filtered skew PBW extension over R, then R is connected filtered if and only if A is connected filtered.*

Proof. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle = \bigcup_{p\in\mathbb{N}} \mathscr{F}_p(A)$ be a σ-filtered skew PBW extension over a connected filtered algebra R. If $0 \ne h \in \mathscr{F}_0(A)$, then h has a unique representation as $h = a_1X_1 + \cdots + a_tX_t$, whence $\operatorname{tdeg}(a_iX_i) = \deg(a_i) + \deg(X_i) = 0$, for $1 \le i \le t$. Thus, $\deg(a_i) = 0 = \deg(X_i)$, $X_i = 1$, for $1 \le i \le t$, and so $h \in R$ with $\deg(h) = 0$, that is, $h \in \mathscr{F}_0(R) = \Bbbk$ since R is connected filtered. This proves that $\mathscr{F}_0(A) = \Bbbk$, i.e., A is connected filtered.

For the converse, let $0 \ne r \in \mathscr{F}_0(R)$. By Proposition 2.2(1), $\mathscr{F}_0(R) \subseteq \mathscr{F}_0(A) = \Bbbk$, whence $r \in \Bbbk$, that is, $\mathscr{F}_0(R) = \Bbbk$. $\square$

From Proposition 1.3 we know that skew PBW extensions are filtered rings. Notice that if R has the trivial positive filtration, then $\deg(r) = 0$, for all $r \in R$. Thus, $\deg(f) = \operatorname{tdeg}(f)$ ($\operatorname{tdeg}(f)$ as in Definition 2.3(iv) and $\deg(f)$ as in (1.3)), i.e., the filtration (2.2) coincides with the filtration (1.3). More precisely,

Proposition 2.4. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R with filtration as in (1.3), then A is a σ-filtered skew PBW extension if and only if R has the trivial positive filtration.*

Proof. Suppose that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is σ-filtered with the filtration given in (1.3). Then R is positively filtered and $\mathscr{F}'_0(A) = R$. By (2.2), $\mathscr{F}'_0(A) = \{f \in A \mid \operatorname{tdeg}(f) \le 0\} \cup \{0\} = R$. Let $r \in R = \mathscr{F}'_0(A)$. Then $\operatorname{tdeg}(r) = \deg(r) = 0 = \min\{p \in \mathbb{N} \mid r \in \mathscr{F}_p(R)\}$, i.e., $r \in \mathscr{F}_0(R)$. Therefore, $\mathscr{F}_0(R) = R$ and so $\mathscr{F}_p(R) = R$ for $p \ge 0$.

For the converse, assume that R has the trivial positive filtration, i.e., $\mathscr{F}_p(R) = R$ for all $p \ge 0$. Notice that $\sigma_i(\mathscr{F}_p(R)) = \sigma_i(R) \subseteq R$ and $\delta_i(\mathscr{F}_p(R)) = \delta_i(R) \subseteq R = \mathscr{F}_{p+1}(R)$. Thus σ_i and δ_i are filtered. Now, as $x_j x_i = c_{i,j} x_i x_j + r_{0_{j,i}} + r_{1_{j,i}} x_1 + \cdots + r_{n_{j,i}} x_n$ with $c_{i,j}, r_{0_{j,i}}, r_{1_{j,i}}, \dots, r_{n_{j,i}} \in R = \mathscr{F}_0(R)$, then $\deg(c_{i,j}) = \deg(r_{0_{j,i}}) = \deg(r_{1_{j,i}}) = \cdots = \deg(r_{n_{j,i}}) = 0$, so, $\operatorname{tdeg}(c_{i,j} x_i x_j + r_{0_{j,i}} + r_{1_{j,i}} x_1 + \cdots + r_{n_{j,i}} x_n) = 2$, that is, A preserves tdeg. Thus A is a σ-filtered skew PBW extension. $\square$

Remark 4. From Proposition 2.4 it follows that every skew PBW extension A over a ring R is σ-filtered with the filtration given by Lezama and Reyes (Proposition 1.3). Of course, graded skew PBW extensions are trivially σ-filtered skew PBW extensions. For some examples of skew PBW extensions, filtrations (2.2) and (1.3) coincide. More exactly, if we consider a skew PBW extension A over R where σ_i is the identity map on R and $\delta_i = 0$, for each $1 \le i \le n$, where σ_i and δ_i are as in Proposition 1.1, then the skew PBW extensions are trivially σ-filtered skew PBW extensions. For instance, if $\mathfrak{g}$ is a finite dimensional Lie algebra over $\Bbbk$, then its universal enveloping algebra $U(\mathfrak{g})$ satisfies these conditions.

Example 2.1. (i) The Weyl algebra $\Bbbk[t_1, \dots, t_n][x_1, \partial/\partial t_1] \cdots [x_n, \partial/\partial t_n] = A_n(\Bbbk)$ (c.f. Section 1.2.1) is a skew PBW extension over the commutative polynomial ring $\Bbbk[t_1, \dots, t_n]$, where $x_i t_j = t_j x_i + \delta_{ij}$, $x_i x_j - x_j x_i = 0$, and $\delta_{ij} = 0$ for $i \ne j$ and $\delta_{ii} = 1$, $1 \le i, j \le n$. The endomorphisms and derivations of Proposition 1.1 are σ_i, the identity map of R, and $\delta_i = \delta_{ij}$, respectively. If R is endowed with the standard filtration, then $A_n(\Bbbk)$ is a σ-filtered skew PBW extension.

(ii) Let $\Bbbk$ be a field of characteristic zero. It is well known that $A_n(\Bbbk) \cong U(\mathfrak{g})/\langle 1 - y\rangle$, where $U(\mathfrak{g})$ is the universal enveloping algebra of the $(2n+1)$-dimensional Heisenberg Lie algebra with basis given by the set $\{t_1, \dots, t_n, x_1, \dots, x_n, y\}$ over $\Bbbk$. Li and Van Oystaeyen [LO92, Example (i)] proved that $U(\mathfrak{g})$ is the Rees algebra of $A_n(\Bbbk)$ with respect to the filtration $\mathscr{F}''$ on $A_n(\Bbbk)$, where $\mathscr{F}''_p(A_n(\Bbbk)) = \left\{\sum_{|\alpha| \le p} f_\alpha(t_1, \dots, t_n) x^\alpha\right\}$, $f_\alpha(t_1, \dots, t_n) \in \Bbbk[t_1, \dots, t_n]$ and $x^\alpha \in \operatorname{Mon}(A_n(\Bbbk))$. Notice that $\mathscr{F}''$ coincides with the filtration given in (1.3). Thus, by Proposition 2.4, $A_n(\Bbbk) \cong U(\mathfrak{g})/\langle 1 - y\rangle$ is σ-filtered with the filtration $\mathscr{F}''$.

(iii) Let $R = \Bbbk\{t_1, t_2\}/(t_2 t_1 - t_1 t_2 - t_1^2)$ be the Jordan plane and $A = \sigma(R)\langle x_1\rangle$ be the skew PBW extension over R, where $x_1 t_1 = t_1 x_1$ and $x_1 t_2 = t_2 x_1 + 2 t_1 x_1$. According to Proposition 1.1 we have $\sigma_1(t_1) = t_1$ and $\sigma_1(t_2) = 2t_1 + t_2$ and $\delta_1 = 0$. Notice that A is a σ-filtered skew PBW extension over the Jordan plane R, when R is endowed with the standard filtration.

(iv) Let $R = \Bbbk[t]$ be the polynomial ring and $A = \sigma(R)\langle x_1\rangle$ a skew PBW extension over R, where $x_1 t = c_1 t x_1 + c_2 x_1 + c_3 t^2 + c_4 t + c_5$, for $c_1, c_2, c_3, c_4, c_5 \in \Bbbk$ and $c_1 \ne 0$. According to Proposition 1.1, $\sigma_1(t) = c_1 t + c_2$ and $\delta_1(t) = c_3 t^2 + c_4 t + c_5$. Therefore A is a σ-filtered skew

PBW extension over $\Bbbk[t]$, when $\Bbbk[t]$ is endowed with the standard filtration. Notice that since $c_1 \neq 0$ then σ_1 is an automorphism of $\Bbbk[t]$.

REMARK 5. Let A be the free algebra generated by the indeterminates x, y subject to the relation $yx = xy + x^3$, that is, $\Bbbk\{x, y\}/\langle yx - xy - x^3\rangle$. As one can check (following the ideas presented in [AL15]), A is a skew PBW extension over $\Bbbk[x]$. By Proposition 1.1, $yx = \sigma(x)y + \delta(x)$, whence σ is the identity map of $\Bbbk[x]$ and $\delta(x) = x^3$. Notice that the standard filtration of $\Bbbk[x]$ is connected, σ is filtered but δ is not filtered. In this way, A is not σ-filtered.

Let R be a graded algebra. A graded R-module M is *free-graded* on the basis $\{e_j \mid j \in J\}$ if M is free as a left R-module on the basis $\{e_j\}$, and also every e_j is homogeneous, say of degree $d(j)$. A filtered module $M = \bigcup_{p\in\mathbb{N}} \mathscr{F}_p(M)$ over a filtered algebra $R = \bigcup_{p\in\mathbb{N}} \mathscr{F}_p(R)$ is *free-filtered* with filtered basis $\{e_j \mid j \in J\}$ if M is a free R-module with basis $\{e_j \mid j \in J\}$, and $\mathscr{F}_p(M) = \bigoplus_j \mathscr{F}_{p-p(j)}(R)e_j$, where $p(j)$ is the degree of e_j.

PROPOSITION 2.5. *If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a σ-filtered skew PBW extension, then A is free-filtered with filtered basis* $\mathrm{Mon}(A)$. *Moreover, $G(A)$ is free-graded over $G(R)$.*

Proof. By Definition 1.1(ii), we have that A is a free R-module with basis given by $\mathrm{Mon}(A) = \{x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$. If the degree of x^α is denoted by $\deg(x^\alpha) := |\alpha|$, the idea is to show that

$$\mathscr{F}_p(A) = \bigoplus_{\alpha\in\mathbb{N}^n} \mathscr{F}_{p-|\alpha|}(R)x^\alpha. \tag{2.5}$$

Let $0 \neq f \in \mathscr{F}_p(A)$. By Remark 1(iv), f has a unique representation given by $f = c_1X_1 + \cdots + c_dX_d$, with $c_i \in R \setminus \{0\}$ and $X_i := x^{\alpha^i} = x_1^{\alpha_1^i}\cdots x_n^{\alpha_n^i} \in \mathrm{Mon}(A)$, where $\mathrm{tdeg}(c_iX_i) = \mathrm{tdeg}(c_ix^{\alpha^i}) = \deg(c_i) + \deg(X_i) = \deg(c_i) + |\alpha^i| \le p$, for $1 \le i \le d$. Hence, $\deg(c_i) \le p - |\alpha^i|$, i.e., $c_i \in \mathscr{F}_{p-|\alpha^i|}(R)$, for $1 \le i \le d$, and so $f \in \mathscr{F}_{p-|\alpha^1|}(R)x^{\alpha^1} + \cdots + \mathscr{F}_{p-|\alpha^d|}(R)x^{\alpha^d}$. From the uniqueness of the representation of f, it follows that $f \in \bigoplus_{\alpha\in\mathbb{N}^n} \mathscr{F}_{p-|\alpha|}(R)x^\alpha$.

For the another inclusion, let $f \in \bigoplus_{\alpha\in\mathbb{N}^n} \mathscr{F}_{p-|\alpha|}(R)x^\alpha$. Then f has a unique representation as $f = f_{\alpha^1} + \cdots + f_{\alpha^t}$, where $f_{\alpha^j} \in \mathscr{F}_{p-|\alpha^j|}(R)x^{\alpha^j}$, for $1 \le j \le t$. Thus, $f_{\alpha^j} = r_jx^{\alpha^j}$, with $r_j \in \mathscr{F}_{p-|\alpha^j|}$ and $x^{\alpha^j} \in \mathrm{Mon}(A)$. In this way, $\deg(r_j) \le p - |\alpha^j|$, and therefore $\mathrm{tdeg}(f_{\alpha^j}) = \deg(r_j) + |\alpha^j| \le p$, whence $\mathrm{tdeg}(f) \le p$, that is, $f \in \mathscr{F}_p(A)$. This means that A is free-filtered with filtered basis $\mathrm{Mon}(A)$. Finally, since A free-filtered, by [MR01, Proposition 7.6.15], we obtain that $G(A)$ is free-graded over $G(R)$ on the graded basis $\overline{\mathrm{Mon}(A)}$. □

2.3 HOMOGENIZATION OF σ-FILTERED SKEW PBW EXTENSIONS

PROPOSITION 2.6. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over an algebra R.*

(1) *If R is finitely generated as algebra, then A is finitely generated as algebra.*

(2) *If R is finitely presented, then A is finitely presented.*

Proof. (1) If R is finitely generated as algebra, then there exists a finite set of elements $t_1,\dots,t_s \in R$ such that the set $\left\{t_{i_1}t_{i_2}\cdots t_{i_m} \mid 1 \le i_j \le s, m \ge 1\right\} \cup \{1\}$ spans R as a $\Bbbk$-space. By Definition 1.1(ii), $\mathrm{Mon}(A) = \{x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n\}$ is an R-basis for A. There exists a finite set of elements $t_1,\dots,t_s,x_1,\dots,x_n \in A$ such that $\{t_{i_1}t_{i_2}\cdots t_{i_m}x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid 1 \le i_j \le s, m \ge 1, \alpha_1,\dots,\alpha_n \in \mathbb{N}\}$ spans A as a $\Bbbk$-space.

(2) If R is finitely presented, then $R = \Bbbk\{t_1,\dots,t_m\}/I$, where

$$I = \langle r_1,\dots,r_s\rangle, \tag{2.6}$$

is a two-sided ideal of $\Bbbk\{t_1,\dots,t_m\}$ generated by a finite set $r_1,\dots,r_s$ of polynomials in $\Bbbk\{t_1,\dots,t_m\}$. In this way,

$$\begin{aligned} &A = \Bbbk\{t_1,\dots,t_m,x_1,\dots,x_n\}/J, \text{ where} \\ &J = \langle r_1,\dots,r_s,\ f_{ik},\ g_{ji} \mid 1 \le i,j \le n,\ 1 \le k \le m\rangle, \end{aligned} \tag{2.7}$$

is the two-sided ideal of $\Bbbk\{t_1,\dots,t_m,x_1,\dots,x_n\}$ generated by a finite set of polynomials $r_1,\dots,r_s,\ f_{ik},\ g_{ji}$ with $r_1,\dots,r_s$ as in (2.6), that is,

$$f_{ik} := x_it_k - \sigma_i(t_k)x_i - \delta_i(t_k), \tag{2.8}$$

where σ_i and δ_i are as in Proposition 1.1, i.e.,

$$g_{ji} := x_jx_i - c_{i,j}x_ix_j - (r_{0_{j,i}} + r_{1_{j,i}}x_1 + \cdots + r_{n_{j,i}}x_n) \tag{2.9}$$

as in (1.2). Therefore, A is finitely presented.

□

PROPOSITION 2.7. *If A is a skew PBW extension over a finitely presented algebra R such that σ_i, δ_i are filtered, and A preserves* tdeg, *then A is a connected σ-filtered algebra, and the filtration of A is finite.*

Proof. Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over a finitely presented algebra $R = \Bbbk\{t_1,\dots,t_m\}/\langle r_1,\dots,r_s\rangle$. As $L = \Bbbk\{t_1,\dots,t_m\}$ is connected filtered, R inherits a connected filtration $\{\mathscr{F}_q(R)\}_{q\in\mathbb{N}}$, from the standard filtration on the free algebra $\Bbbk\{t_1,\dots,t_m\}$. Since σ_i and δ_i are filtered and A preserves tdeg, then A is σ-filtered. Now, as R is connected filtered, Proposition 2.3 implies that A is connected filtered. Notice that L_p is a finite dimensional subspace of L, for all $p \in \mathbb{N}$. In this way, $\mathscr{F}_q(L)$ is also a finite dimensional subspace, and so $\mathscr{F}_q(R)$ is a finite dimensional subspace of R, for all $q \in \mathbb{N}$, i.e., the filtration of R is finite. Proposition 2.2(2) guarantees that the filtration of A is finite. □

THEOREM 2.8. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a bijective skew PBW extension over a finitely presented algebra R such that σ_i, δ_i are filtered and A preserves* tdeg, *then $H(A)$ is a graded skew PBW extension over $H(R)$.*

Proof. Let $R = \Bbbk\{t_1,\dots,t_m\}/\langle r_1,\dots,r_s\rangle = L/\langle r_1,\dots,r_s\rangle$, and consider $H(R) = L[z]/\langle\widehat{r}_1,\dots,\widehat{r}_s\rangle = \Bbbk\{t_1,\dots,t_m,z\}/\langle\widehat{r}_1,\dots,\widehat{r}_s,\ t_kz - zt_k \mid 1 \le k \le m\rangle$ be the homogenization of R. By Proposition 2.6(2) and its proof, A is finitely presented with presentation given by

$$A = \frac{\Bbbk\{t_1,\dots,t_m,x_1,\dots,x_n\}}{\langle r_1,\dots,r_s,\ f_{ik},\ g_{ji} \mid 1\le i,j\le n,\ 1\le k\le m\rangle}, \tag{2.10}$$

where $f_{ik} = x_i t_k - \sigma_i(t_k)x_i - \delta_i(t_k)$, $g_{ji} = x_j x_i - c_{i,j}x_i x_j - (r_{0_{j,i}} + r_{1_{j,i}}x_1 + \cdots + r_{n_{j,i}}x_n)$, with $c_{i,j}, r_{0_{j,i}}, r_{1_{j,i}},\dots, r_{n_{j,i}} \in R$. Let $L_{tx} := \Bbbk\{t_1,\dots,t_m,x_1,\dots,x_n\}$. Then

$$H(A) = L_{tx}[z]/\langle \widehat{r}_1,\dots,\widehat{r}_s,\ \widehat{f_{ik}},\ \widehat{g_{ji}} \mid 1\le i,j\le n,\ 1\le k\le m\rangle \tag{2.11}$$

$$= \frac{\Bbbk\{z,t_1,\dots,t_m,x_1,\dots,x_n\}}{\langle \widehat{r}_1,\dots,\widehat{r}_s,\ \widehat{f_{ik}},\ \widehat{g_{ji}}, t_k z - z t_k, x_i z - z x_i \mid 1\le i,j\le n,\ 1\le k\le m\rangle}. \tag{2.12}$$

By Proposition 2.7, A is a connected σ-filtered algebra, whence σ_i and δ_i are filtered and A preserves tdeg. Hence,

$$\mathrm{tdeg}(f_{ik}) = \mathrm{tdeg}(x_i t_k - \sigma_i(t_k)x_i - \delta_i(t_k)) = 2,$$
$$\mathrm{tdeg}(g_{ji}) = \mathrm{tdeg}(x_j x_i - c_{i,j}x_i x_j - (r_{0_{j,i}} + r_{1_{j,i}}x_1 + \cdots + r_{n_{j,i}}x_n)) = 2.$$

Notice that

$$\widehat{f_{ik}} = x_i t_k - (\sigma_i(t_k)z^{1-\deg(\sigma_i(t_k))})x_i - \delta_i(t_k)z^{2-\deg(\delta_i(t_k))}, \tag{2.13}$$

$$\widehat{g_{ji}} = x_j x_i - c_{i,j}x_i x_j - r_{0_{j,i}}z^{2-\deg(r_{0_{j,i}})} - (r_{1_{j,i}}z^{1-\deg(r_{1_{j,i}})})x_1 - \cdots - (r_{n_{j,i}}z^{1-\deg(r_{n_{j,i}})})x_n, \tag{2.14}$$

where $c_{i,j} \in \Bbbk\setminus\{0\}$, $r_{0_{j,i}}z^{2-\deg(r_{0_{j,i}})},\ r_{1_{j,i}}z^{1-\deg(r_{1_{j,i}})},\dots, r_{n_{j,i}}z^{1-\deg(r_{n_{j,i}})} \in H(R)$, for $1\le i,j\le n$.

From (2.13) and (2.14), in $H(A)$ we have the relations

$$x_i t_k = (\sigma_i(t_k)z^{1-\deg(\sigma_i(t_k))})x_i + \delta_i(t_k)z^{2-\deg(\delta_i(t_k))}, \tag{2.15}$$

and

$$x_j x_i - c_{i,j}x_i x_j = r_{0_{j,i}}z^{2-\deg(r_{0_{j,i}})} + (r_{1_{j,i}}z^{1-\deg(r_{1_{j,i}})})x_1 + \cdots + (r_{n_{j,i}}z^{1-\deg(r_{n_{j,i}})})x_n, \tag{2.16}$$

where $c_{i,j} \in \Bbbk\setminus\{0\}$, $r_{0_{j,i}}z^{2-\deg(r_{0_{j,i}})},\ r_{1_{j,i}}z^{1-\deg(r_{1_{j,i}})},\dots, r_{n_{j,i}}z^{1-\deg(r_{n_{j,i}})} \in H(R)$, for $1\le i,j\le n$.

Relations given in (2.15) and (2.16) correspond to Definition 1.1(iii) and (iv), respectively, applied to $H(R)$ and $H(A)$. Notice that $H(R)\subseteq H(A)$ and $H(A)$ is an $H(R)$-free module with basis

$$\mathrm{Mon}(A) := \mathrm{Mon}\{x_1,\dots,x_n\} := \{x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\},$$

whence $H(A)$ is a skew PBW extension over $H(R)$ in the variables $x_1,\dots,x_n$. Thus $H(A)\cong \sigma(H(R))\langle x_1,\dots,x_n\rangle$. Notice that $H(R)$ is a connected graded algebra, i.e., $H(R) = \bigoplus_{p\ge 0} H(R)_p$. From (2.15) and applying Proposition 1.1 to $H(A)$, it follows that $\widehat{\sigma_i}(t_k) = (\sigma_i(t_k)z^{1-\deg(\sigma_i(t_k))})$;

$\widehat{\sigma_i}(z) = z$; $\widehat{\delta}_i(t_k) = \delta_i(t_k)z^{2-\deg(\delta_i(t_k))}$ and $\widehat{\delta}_i(z) = 0$, for $1 \leq i \leq n$. Thus, $\widehat{\sigma_i} : H(R) \to H(R)$ is a graded ring homomorphism and $\widehat{\delta}_i : H(R)(-1) \to H(R)$ is a graded $\widehat{\sigma_i}$-derivation for all $1 \leq i \leq n$. From (2.16) we have $c_{i,j} \in \Bbbk \setminus \{0\}$, $r_{0_{j,i}} z^{2-\deg(r_{0_{j,i}})} \in H(R)_2$, $r_{1_{j,i}} z^{1-\deg(r_{1_{j,i}})}, \dots, r_{n_{j,i}} z^{1-\deg(r_{n_{j,i}})} \in H(R)_1$, for $1 \leq i, j \leq n$. Thus, $x_j x_i - c_{i,j} x_i x_j \in H(R)_2 + H(R)_1 x_1 + \cdots + H(R)_1 x_n$, and $c_{i,j} \in H(R)_0 = \Bbbk$. As σ_i is bijective then $\widehat{\sigma_i}$ is bijective and as $c_{i,j}$ is invertible, then $H(A)$ is bijective. Therefore, $H(A) = \sigma(H(R))\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension over the $\mathbb{N}$-graded algebra $H(R)$ that satisfies both conditions formulated in Proposition 1.4, and so $H(A)$ is a graded skew PBW extension over $H(R)$. □

Remark 6. Consider A as in Theorem 2.8. By (2.12), we know that

$$H(A) = \frac{\Bbbk\{z, t_1, \dots, t_m, x_1, \dots, x_n\}}{\langle \widehat{r}_1, \dots, \widehat{r}_s, \widehat{f_{ik}}, \widehat{g_{ji}}, t_k z - z t_k, x_i z - z x_i \mid 1 \leq i, j \leq n, 1 \leq k \leq m\rangle}.$$

Let $f \in H(A)$. By Theorem 2.8, $H(A) = \sigma(H(R))\langle x_1, \dots, x_n\rangle$ is a graded skew PBW extension over $H(R)$. Remark 1(iv) shows that f has a unique representation as $f = c_1 X_1 + \cdots + c_d X_d$, with $c_i \in H(R) \setminus \{0\}$ and $X_i \in \mathrm{Mon}(A)$. As $H(A)$ is graded, $c_i X_i$ is homogeneous in $H(A)$, let us say of degree $d(i)$, and c_i is homogeneous in $H(R)$. Then $c_i X_i = k_i z^{\beta_i} \overline{t_{i_1} t_{i_2} \dots t_{i_p}} X_i$, $0 \leq i_j \leq m$, where $\overline{t_{i_1} t_{i_2} \dots t_{i_p}} \in R = \Bbbk\{t_1, \dots, t_m\}/\langle r_1, \dots, r_s\rangle$, $\overline{t_0} = x_0 = z^0 = 1$, $k_i \in \Bbbk$ and $\beta_i + p + \deg(X_i) = d(i)$. Therefore f has a unique representation as $f = k_1 z^{\beta_1} \overline{t_{1_1} t_{1_2} \dots t_{1_p}} X_1 + \cdots + k_d z^{\beta_d} \overline{t_{d_1} t_{d_2} \dots t_{d_p}} X_d$.

Let A be a bijective σ-filtered skew PBW extension over a finitely presented algebra R. Let us fix the notation: $G(A)$ and $\mathrm{Rees}(A)$ denote the associated graded algebra and the associated Rees algebra of A, respectively, regarding the filtration given in Theorem 2.1. One can always recover A and $G(A)$ from $H(A)$, via $A \cong H(A)/\langle z-1\rangle$ and $G(A) \cong H(A)/\langle z\rangle$, respectively.

Proposition 2.9. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective σ-filtered skew PBW extension over a finitely presented algebra R, then the following assertions hold:*

(1) $A \cong H(A)/\langle z-1\rangle$.

(2) $H(A)/\langle z\rangle \cong G(A)$.

Proof. (1) It is clear.

(2) From (2.7) and (2.10),

$$\begin{aligned} A &= \Bbbk\{t_1, \dots, t_m, x_1, \dots, x_n\}/J \\ &= \Bbbk\{t_1, \dots, t_m, x_1, \dots, x_n\}/\langle r_1, \dots, r_s, \ f_{ik}, \ g_{ji} \mid 1 \leq i, j \leq n, \ 1 \leq k \leq m\rangle, \end{aligned}$$

i.e., A is an algebra defined by generators and relations. Then there is a standard connected filtration $\{\mathscr{F}_p^*(A)\}_{p\in\mathbb{N}}$ on A wherein $\mathscr{F}_p^*(A) = (\mathscr{F}_p(L_{tx}) + J)/J$, i.e., $\mathscr{F}_p^*(A)$ is the span of all words in the variables $t_1, \dots, t_m, x_1, \dots, x_n$ of degree at most p. Notice that for σ-filtered skew PBW extensions of a finitely presented algebra R, $\mathscr{F}_p^*(A)$ coincides with $\mathscr{F}_p(A)$ as in (2.2), for all $p \geq 0$. Therefore, $H(A)/\langle z\rangle \cong G(A)$.

□

EXAMPLE 2.2. The Weyl algebra $A_n(\Bbbk)$ in Example 2.1 is the free associative algebra with generators $t_1,\dots,t_n$, $x_1,\dots,x_n$ modulo the relations $t_jt_i = t_it_j$, $x_jx_i = x_ix_j$, and $x_it_j = t_jx_i + \delta_{ij}$, where δ_{ij} is the Kronecker delta, $1 \le i, j \le n$. Adding another generator h that commutes with $t_1,\dots,t_n$, $x_1,\dots,x_n$ and replacing δ_{ij} with $\delta_{ij}h^2$ in the above relations yields the n homogenized Weyl algebra $H(A_n(\Bbbk))$. Since $A_n(\Bbbk)$ is a bijective skew PBW extension over a finitely presented algebra $R = \Bbbk[t_1,\dots,t_n]$, such that σ_i, δ_i are filtered and $A_n(\Bbbk)$ preserves tdeg, by Theorem 2.8 we have that $H(A_n(\Bbbk))$ is a graded skew PBW extension over $H(R) = \Bbbk[t_1,\dots,t_n,h]$. By Proposition 2.9, $A_n(\Bbbk) \cong H(A_n(\Bbbk))/\langle h-1\rangle$ and $H(A_n(\Bbbk))/\langle h\rangle \cong G(A_n(\Bbbk))$, which is just a commutative polynomial ring in 2n variables, where $G(A_n(\Bbbk))$ is the associated graded algebra of $A_n(\Bbbk)$ with respect to the filtration $\mathscr{F}$ as in (2.2).

EXAMPLE 2.3. Redman [Red99] studied the geometry of the homogenizations of two classes of 3-dimensional skew polynomial algebras (Section 1.2.3). Following the terminology used by Bell and Smith [BS90], the algebras Type I and Type II (these objects are called like this because these are two classes of three dimensional skew polynomial rings that have finite dimensional simple modules of arbitrarily large dimensions) are defined as

$$\text{Type I}\begin{cases} g_1 = yz - \alpha zy, \\ g_2 = zx - \beta xz - ay - b, \\ g_3 = xy - \alpha yx \end{cases} \quad \text{and} \quad \text{Type II}\begin{cases} g_1 = yz - \alpha zy - x - b_1, \\ g_2 = zx - \alpha xz - y - b_2, \\ g_3 = xy - \alpha yx - z - b_3 \end{cases}$$

where $a, b, b_1, b_2, b_3, \alpha, \beta \in \mathbb{C}$ with such that $\alpha\beta \neq 0$.

The homogenization $H(\mathscr{A})$ of both types of algebras with respect to a central variable t is given by $\mathbb{C}\langle x, y, z, t\rangle$ with defining relations

$$\begin{cases} yz - \alpha zy = 0, \\ zx - \beta xz = ayt + bt^2, \\ xy - \alpha yx = 0, \end{cases} \quad \text{or} \quad \begin{cases} yz - \alpha zy = xt + b_1t^2, \\ zx - \alpha xz = yt + b_2t^2, \\ xy - \alpha yx = zt + b_3t^2, \end{cases}$$

and $xt - tx = yt - ty = zt - tz = 0$. From [BS90, Proposition 2.1.1], the standard monomials $\{x^iy^jz^kt^l \mid i, j, k, l \ge 0\}$ form a $\mathbb{C}$-basis for the algebra D with the degree and dictionary ordering being $x > y > z > t$, whence t is a non-zero divisor.

By Proposition 2.9, $\mathscr{A} \cong H(\mathscr{A})/\langle t-1\rangle$ and $H(\mathscr{A})/\langle t\rangle \cong G(\mathscr{A})$, where $G(\mathscr{A})$ is the associated graded algebra of $\mathscr{A}$ with respect to the filtration $\mathscr{F}$ as in (2.2). Thus, $H(\mathscr{A})$ is a central extension of the algebra $H(\mathscr{A})/\langle t\rangle$, and therefore $H(\mathscr{A})$ is a central extension of $G(\mathscr{A})$. These facts were used by [Red99] to prove that the quotient algebra $H(\mathscr{A})/\langle t\rangle$ is a 3-dimensional Artin-Schelter regular algebra ([Red99, Lemma 1.1]), $H(\mathscr{A})$ is 4-dimensional Artin-Schelter regular, graded Noetherian domain, and Cohen Macaulay with Hilbert series $(1-t)^{-4}$ [Red99, Proposition 1.2]. He also described the noncommutative projective geometry of these objects, and compute the finite dimensional simple modules for the homogenization of Type I algebras in the case that α is not a primitive root of unity. In this case, all finite dimensional simple modules are quotients of line modules that are homogenizations of Verma modules. From Theorem 2.8, we know that $H(\mathscr{A})$ is a graded skew PBW extension over $\mathbb{C}[t]$. In Theorem 2.10 below, we generalize some of these properties for σ-filtered skew PBW extensions over a ring R such that $H(R)$ is

Auslander-regular.

EXAMPLE 2.4. Following Le Bruyn and Smith [LBSdB96], we write $\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ and define a vector space isomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to h,$$

and we transfer the Lie bracket on $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{g}$ giving $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. The homogenization $H(U(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$ with respect to a central variable t is $\mathbb{C}\langle e,f,h,t\rangle$ with defining equations

$$ef - fe = ht, \quad he - eh = 2et, \quad hf - fh = -2ft \quad et - te = ft - tf = ht - th = 0.$$

By Proposition 2.9, $U(\mathfrak{g}) \cong H(U(\mathfrak{g}))/\langle t-1\rangle$ and $H(U(\mathfrak{g}))/\langle t\rangle \cong G(U(\mathfrak{g}))$, where $G(U(\mathfrak{g}))$ is the associated graded algebra of $U(\mathfrak{g})$ with respect to the filtration $\mathscr{F}$ as in (2.2). Thus, $H(U(\mathfrak{g}))$ is a central extension of $H(U(\mathfrak{g}))/\langle t\rangle \cong G(U(\mathfrak{g})) \cong \mathbb{C}[e,f,h] = S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. These facts were used by Le Bruyn and Smith [LBS93] to deduce that $H(U(\mathfrak{g}))$ has Hilbert series $(1-t)^{-4}$, it is a positively graded Noetherian domain, Auslander-regular of dimension 4, satisfies the Cohen-Macaulay property, and its center is $\mathbb{C}[\Omega, t]$, where $\Omega = h^2 + 2ef + 2fe$ is the Casimir element. Recall that $U(\mathfrak{g})$ is a σ-filtered skew PBW extension (Example 2.1), and by Theorem 2.8, $H(U(\mathfrak{g}))$ is a graded skew PBW extension over $\mathbb{C}[t]$. Below, using Theorem 2.10, we obtain some of the above properties for σ-filtered skew PBW extensions over R such that $H(R)$ is Auslander-regular.

REMARK 7. A graded algebra R is said to be *generated in degree one if R_1 generates R as an algebra*. Let A be a σ-filtered skew PBW extension over a commutative polynomial ring $R = \Bbbk[t_1,\dots,t_m]$. Notice that if we use the filtration given in (1.3), then $H(A)$ and $G(A)$ are not generated in degree one, while if we use the standard filtration for R, then $H(A)$ and $G(A)$ are generated in degree one. For the study of certain properties in an algebra such as Artin-Schelter regular, strongly Noetherian, Auslander regular, Cohen-Macaulay, Koszul and the Jacobson radical, some authors impose the condition that the algebra be generated in degree one (e.g. [GSZ19, Theorem 3.3] or [ZZ09, Theorem 0.1]). For the example presented by Greenfeld et al. [GSZ19, Section 6], it was considered that in Ore extensions of endomorphism type $R[x,\sigma]$, $\deg(r) = 0$, for all non-zero $r \in R$ and $\deg(rx^j) = j$, for all natural number $j > 0$. This fact was used to study the properties graded nilpotent (a graded algebra is *graded nilpotent* if the algebra generated by any set of homogeneous elements of the same degree is nilpotent), graded locally nilpotent (a graded algebra is *graded locally nilpotent* if the algebra generated by any finite set of homogeneous elements of the same degree is nilpotent), the Jacobson radical, and to ask some related questions in the Ore extension $R[x;\sigma]$.

2.4 OTHER HOMOLOGICAL PROPERTIES

It is well-known that a graded algebra R is right (left) Noetherian if and only if it is graded right (left) Noetherian, which means that every graded right (left) ideal is finitely generated

[Lev92, Proposition 1.4]. Let M be an R-module. The *grade number* of M is $j_R(M) := \min\{p \mid \operatorname{Ext}_R^p(M,R) \neq 0\}$ or ∞ if no such p exists. Notice that $j_R(0) = \infty$. When R is Noetherian, $j_R(M) \leq \operatorname{pd}_R(M)$ (where $\operatorname{pd}_R(M)$ denotes the projective dimension of M), and if furthermore $\operatorname{injdim}(R) = q < \infty$, we have $j_R(M) \leq q$, for all non-zero finitely generated R-module M (see [Lev92] for further details).

DEFINITION 2.5. ([Lev92, Definition 2.1]). Let R be a Noetherian ring.

(i) An R-module M satisfies the *Auslander-condition* if for all $p \geq 0$, $j_R(N) \geq p$, for every R-submodule N of $\operatorname{Ext}_R^p(M,R)$.

(ii) R is called *Auslander-Gorenstein* of dimension q if $\operatorname{injdim}(R) = q < \infty$, and every left or right finitely generated R-module satisfies the Auslander-condition.

(iii) R is said to be *Auslander-regular* of dimension q if $\operatorname{gld}(R) = q < \infty$, and every left or right finitely generated R-module satisfies the Auslander-condition.

THEOREM 2.10. *If A is a σ-filtered skew PBW extension over a ring R such that $H(R)$ is Auslander-regular, then the following assertions hold:*

(1) *$H(A)$ is graded Noetherian.*

(2) *$H(A)$ is a domain.*

(3) *A is Noetherian.*

(4) *A is a PBW deformation of $G(A)$.*

(5) *$G(A)$ is Noetherian.*

(6) $\operatorname{Rees}(A) \cong H(A)$.

(7) *A is Zariski.*

(8) *$H(A)$ is Artin-Schelter regular.*

(9) *$G(A)$ is Artin-Schelter regular.*

Proof. From Theorem 2.8, we know that $H(A)$ is a graded bijective skew PBW extension over $H(R)$.

(1) Since $H(R)$ is Noetherian and graded, by [Lev92, Proposition 1.4] $H(R)$ is graded Noetherian. From [SLR17, Proposition 2.7], we obtain that $H(A)$ is graded Noetherian.

(2) By [LV17, Theorem 2.9], we know that $H(A)$ is Auslander-regular. Since $H(R)$ is connected graded, by [Sufrm[o]–7a, Remark 2.10] we get that $H(A)$ is connected graded, and thus the assertion follows from [Lev92, Theorem 4.8].

(3) Part (1) above shows that $H(A)$ is Noetherian, whence $A \cong H(A)/\langle z-1\rangle$ is Noetherian.

(4) Notice that A is a deformation of $G(A)$. By (2), z is regular in $H(A)$, and since $G(A) \cong H(A)/\langle z\rangle$, then $H(A)$ is a central regular extension of $G(A)$. By [CS07, Theorem 1.3], A is a PBW deformation of $G(A)$.

(5) Since $H(A)$ is connected graded and $z \in H(A)_1$ is central regular, then by [Lev92, Proposition 3.5] we have that $H(A)$ is Noetherian if and only if $H(A)/\langle z\rangle \cong G(A)$ is Noetherian.

(6) It follows from (4) and [WZ13, Proposition 2.6].

(7) From (1) and (6), Rees(A) is Noetherian. As A is connected filtered, then A is Zariski.

(8) Since $H(R)$ is finitely presented connected Auslander-regular and $H(A)$ is a graded skew PBW extension over $H(R)$, then by [SLR17, Proposition 3.5 (iii)] we have that $H(A)$ is Artin-Schelter regular.

(9) As $H(A)$ is a connected graded, by (2) $H(A)$ is a domain, and by (8) $H(A)$ is Artin-Schelter regular. From [RZ12, Corollary 1.2], $G(A) \cong H(A)/\langle z\rangle$ is Artin-Schelter regular.

□

Example 2.1 showed that the Weyl algebra $A_n(\Bbbk)$ is a σ-filtered skew PBW extension over $R = \Bbbk[t_1,\dots,t_n]$, and by Example 2.2, $H(R) = \Bbbk[t_1,\dots,t_n,z]$, which is Auslander-regular. Hence, Theorem 2.10 guarantees that $A_n(\Bbbk)$ is Noetherian, Zariski, and a PBW deformation of $G(A_n(\Bbbk))$, $H(A_n(\Bbbk))$ is a domain graded Noetherian and Artin-Schelter regular, $G(A_n(\Bbbk))$ is Noetherian and Artin-Schelter regular, and Rees$(A_n(\Bbbk)) \cong H(A_n(\Bbbk))$.

From Proposition 2.9 and Theorem 2.10, we immediately get the following result.

Corollary 2.11. *If A is a σ-filtered skew PBW extension over a ring $R = \Bbbk\langle t_1,\dots,t_m\rangle/\langle r_1,\dots,r_s\rangle$ such that σ_i is graded, then $G(A)$ is a graded skew PBW extension over $G(R)$ in n variables $y_1,\dots,y_n$ given by*

$$\begin{aligned} y_i t_k &= \sigma_i(t_k)y_i + r_i, \\ y_j y_i &= c_{i,j} y_i y_j + r_{0_{j,i}} + r_{1_{j,i}} y_1 + \cdots + r_{n_{j,i}} y_n, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} r_i &= \begin{cases} \delta_i(t_k), & \text{if } \deg(\delta_i(t_k)) = 2, \\ 0, & \text{otherwise,} \end{cases} \\ r_{0_{j,i}} &= \begin{cases} r_{0_{j,i}}, & \text{if } \deg(r_{0_{j,i}}) = 2, \\ 0, & \text{otherwise,} \end{cases} \\ r_{l_{j,i}} &= \begin{cases} r_{l_{j,i}}, & \text{if } \deg(r_{l_{j,i}}) = 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{2.18}$$

for $1 \le l \le n$, with $c_{i,j}, r_{0_{j,i}}, r_{1_{j,i}}, \dots, r_{n_{j,i}}$, $1 \le i,j \le n$, the constants that define A as in (1.2), and $t_k \in G(R)$ is the coset of t_k.

Proposition 2.12. *PBW deformations of Artin-Schelter regular algebras of dimension two are σ-filtered skew PBW extensions.*

Proof. By [Gad16, Corollary 2.13], PBW deformations of Artin-Schelter regular algebras of dimension two are isomorphic to one of the following algebras: $\Bbbk\{x,y\}/\langle xy-qyx\rangle$, $\Bbbk\{x,y\}/\langle xy-qyx+1\rangle$, $\Bbbk\{x,y\}/\langle yx-xy+y\rangle$, $\Bbbk\{x,y\}/\langle yx-xy+y^2\rangle$, $\Bbbk\{x,y\}/\langle yx-xy+y^2+1\rangle$, where $q\in\Bbbk\setminus\{0\}$. Notice that the first three algebras are skew PBW extensions over $\Bbbk$ and the last two are skew PBW extensions over $\Bbbk[y]$. As one can check, every algebra satisfies the conditions (1) and (2) established in Theorem 2.1. □

EXAMPLE 2.5. (1) Andruskiewitsch et al. [ADP22] studied the Hopf algebra $\mathscr{D}$ which was called *the double of the Jordan plane*. The authors considered the field $\Bbbk$ to be characteristic zero and algebraically closed. Following [ADP22, Definition 2.1], the Hopf algebra $\mathscr{D}$ is presented by generators $u, v, \zeta, g^{\pm1}, x, y$ and relations $g^{\pm1}g^{\pm1}=1$, $\zeta g=g\zeta$, $gx=xg$, $gy=yg+xg$, $\zeta y=y\zeta+y$, $\zeta x=x\zeta+x$, $ug=gu$, $vg=gv+gu$, $v\zeta=\zeta v+v$, $u\zeta=\zeta u+u$, $yx=xy-\frac{1}{2}x^2$, $vu=uv-\frac{1}{2}u^2$, $ux=xu$, $vx=xv+(1-g)+xu$, $uy=yu+(1-g)$, $vy=yv-g\zeta+yu$. According to Andruskiewitsch et al. [ADP22, Lemma 4.1], the algebra $\mathscr{D}$ can be described as the iterated Ore extension

$$\mathscr{D}\cong\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1][\zeta;\sigma_2,\delta_2][v;\sigma_3,\delta_3],$$

with σ_1 the identity automorphism of $\Bbbk[g^{\pm1},x,u]$, and δ_1 is the σ_1-derivation of $\Bbbk[g^{\pm1},x,u]$ given by $\delta_1(x)=-\frac{1}{2}x^2$, $\delta_1(u)=g-1$, and $\delta_1(g)=-xg$; σ_2 is the identity automorphism of $\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1]$, and δ_2 is the σ_2-derivation of $\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1]$ defined by $\delta_2(x)=x$, $\delta_2(u)=-u$, $\delta_2(g)=0$, and $\delta_2(y)=y$. Finally, σ_3 and δ_3 are the automorphism and the σ_3-derivation of $\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1][\zeta;\sigma_2,\delta_2]$, respectively, given by $\sigma_3(x)=x$, $\sigma_3(u)=u$, $\sigma_3(g)=g$, $\sigma_3(y)=y$ and $\sigma_3(\zeta)=\zeta+1$, $\delta_3(x)=1-g+xu$, $\delta_3(u)=-\frac{1}{2}u^2$, $\delta_3(g)=gu$, $\delta_3(y)=yu-g\zeta$ and $\delta_3(\zeta)=0$.

Notice that $\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1][\zeta;\sigma_2,\delta_2][v;\sigma_3,\delta_3]$ satisfies the four conditions established in [FGL+20, Example 2.2 of Part I], which means that $\mathscr{D}$ is a bijective skew PBW extension over $\Bbbk[g^{\pm1},x,u]$, that is,

$$\mathscr{D}\cong\Bbbk[g^{\pm1},x,u][y;\sigma_1,\delta_1][\zeta;\sigma_2,\delta_2]\cong\sigma(\Bbbk[g^{\pm1},x,u])\langle y,\zeta,v\rangle.$$

It is clear that σ_i and δ_i restricted to $\Bbbk[g^{\pm1},x,u]$ are the endomorphism and derivation as in Proposition 1.1, whence $\mathscr{D}$ satisfies the conditions of Theorem 2.1, and therefore $\mathscr{D}$ is a σ-filtered skew PBW extension.

(2) Semi-graded rings were defined by Lezama and Latorre [LL17] in the following way: a ring R is called *semi-graded* (SG) if there exists a collection $\{R_p\}_{p\in\mathbb{N}}$ of subgroups R_p of the additive group R^+ such that the following conditions hold:

- $R=\bigoplus_{p\in\mathbb{N}}R_p$;
- For every $p,q\in\mathbb{N}$, $R_pR_q\subseteq R_0\oplus R_1\oplus\cdots\oplus R_{p+q}$;
- $1\in R_0$.

Notice that R has a standard $\mathbb{N}$-filtration given by $\mathscr{F}_p(R):=R_0\oplus\cdots\oplus R_p$ [LL17, Proposition 2.6], and $\mathbb{N}$-graded rings and skew PBW extensions are examples of semi-graded rings ([LL17, Proposition 2.7]). In the case of a skew PBW extension A over a ring R, they assumed

$A_0 = R$, i.e., R has the trivial positive filtration. Notice that under these conditions, skew PBW extensions over an algebra R with the standard $\mathbb{N}$-filtration are σ-filtered. In this way, if R does not have the trivial positive filtration, then A is not generally σ-filtered, as can be seen in Remark 5.

Recently, Lezama [Lez21, Definition 4.3] introduced the notion of *semi-graded Artin-Schelter regular algebra*, and proved under certain assumptions that skew PBW extensions are semi-graded Artin-Schelter regular [Lez21, Theorem 4.14]. With this purpose, he showed that A is a connected semi-graded algebra with semi-graduation $A_0 = \Bbbk$, and A_p is the $\Bbbk$-subspace generated by $R_q x^{\alpha}$ such that $q + |\alpha| = p$, for $p \geq 1$. In this regard, notice that A with the standard $\mathbb{N}$-filtration given by this semi-graduation is σ-filtered [LL17, Proposition 2.6].

(3) Zhang and Zhang [ZZ08] defined double Ore extensions as a generalization of Ore extensions. If R is an algebra, and B is another algebra containing R as a subring, then B is a *right double Ore extension* of R if the following conditions hold:

- B is generated by R and two new variables x_1 and x_2.
- The variables x_1 and x_2 satisfy the relation
$$x_2x_1 = p_{12}x_1x_2 + p_{11}x_1^2 + \tau_1x_1 + \tau_2x_2 + \tau_0,$$
where $p_{12}, p_{11} \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in R$.
- As a left R-module, $B = \sum_{\alpha_1,\alpha_2 \geq 0} Rx_1^{\alpha_1}x_2^{\alpha_2}$ and it is a left free R-module with basis the set $\{x_1^{\alpha_1}x_2^{\alpha_2} \mid \alpha_1 \geq 0, \alpha_2 \geq 0\}$.
- $x_1R + x_2R \subseteq Rx_1 + Rx_2 + R$.

Left double Ore extensions are defined similarly. B is a *double Ore extension* if it is left and right double Ore extension of R with the same generating set $\{x_1, x_2\}$ [ZZ09, Definition 1.3]. B is a *graded right* (*left*) *double Ore extension* if all relations of B are homogeneous with assignment $\deg(x_1) = \deg(x_2) = 1$. They studied the property of being Artin-Schelter for these extensions [ZZ08, Theorem 3.3].

Later, Zhang and Zhang [ZZ09] constructed 26 families of Artin-Schelter regular algebras of global dimension four using double Ore extensions. Briefly, to prove that a connected graded double Ore extension of an Artin-Schelter regular algebra is Artin-Schelter regular, Zhang and Zhang needed to pass the Artin-Schelter regularity from the trimmed double extension $R_P[x_1, x_2; \sigma]$ to $R_P[x_1, x_2; \sigma, \delta, \tau]$ (details about the notation used for double Ore extensions can be seen in Zhang and Zhang [ZZ08]). For this purpose they defined a new grading and with this a filtration: let $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ be a graded (or ungraded) double extension of R with $d_1 = \deg(x_1)$ and $d_2 = \deg(x_2)$ (or $\deg(x_1) = \deg(x_2) = 0$), the new defined graduation is $\deg'(x_1) = d_1 + 1$ and $\deg'(x_2) = d_2 + 1$ and $\deg'(r) = \deg(r)$ for all $r \in R$. Using this grading they defined a filtration of A by $\mathscr{F}_p(A) = \{\sum r_{n_1,n_2}x_1^{n_1}x_2^{n_2} \in A \mid \deg'(r_{n_1,n_2}) + n_1\deg'(x_1) + n_2\deg'(x_2) \leq m\}$. $\mathscr{F} = \{\mathscr{F}_p(A)\}_{p \in \mathbb{Z}}$ is an $\mathbb{N}$-filtration such that the associated graded ring $G_{\mathscr{F}}(A)$ is isomorphic to $R_P[x_1, x_2; \sigma]$; there is a central element t of degree 1 such that $\text{Rees}_{\mathscr{F}}(A)/(t) = R_P[x_1, x_2; \sigma]$ as graded rings and $\text{Rees}_{\mathscr{F}}(A)/(t-1) =$

$R_P[x_1,x_2;\sigma,\delta,\tau]$ as ungraded rings; also, if A is connected graded, then so are $G_{\mathscr{F}}(A)$ and $\text{Rees}_{\mathscr{F}}(A)$, where $\text{Rees}_{\mathscr{F}}(A)$ is the Rees ring associated to this filtration [ZZ08, Lemma 3.4].

Related with this work, Gómez and Suárez showed that for $R = \bigoplus_{m\geq 0} R_m$ an $\mathbb{N}$-graded algebra and $A = R_P[x_1,x_2;\sigma,\delta,\tau]$ be a graded right double Ore extension of R, if $P = \{p_{12}, 0\}$, $p_{12} \neq 0$ and $\sigma := \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}$, where σ_{11}, σ_{22} are automorphism of R, then A is a graded skew PBW extension over R [GS20, Theorem 3.5]. As one can check, the previous filtration on A coincides with the filtration given in (2.2), and so $A = R_P[x_1,x_2;\sigma,\delta,\tau]$ is a σ-filtered skew PBW extension.

For the last theorem of the chapter, recall that the *enveloping algebra* of an algebra R is the tensor product $R^e = R\otimes R^{op}$, where R^{op} is the opposite algebra of R. If M is an R-bimodule, and ν, $\mu : R \to R$ are two automorphisms, then the skew R-bimodule ${}^{\nu}M^{\mu}$ is equal to M as a vector space with $a\cdot m\cdot b := \nu(a)\cdot m\cdot \mu(b)$. When ν is the identity, we omit it. M is a left R^e-module with product given by $(a\otimes b)\cdot m = a\cdot m\cdot b = \nu(a)\cdot m\cdot \mu(b)$. In particular, for R and R^e, we have the structure of left R^e-module given by $(a\otimes b)\cdot x = \nu(a)x\mu(b)$, $(a\otimes b)\cdot(x\otimes y) = a\cdot(x\otimes y)\cdot b = \nu(a)\cdot(x\otimes y)\cdot\mu(b) = \nu(a)x\otimes y\mu(b)$.

An algebra R is said to be *skew Calabi-Yau* of dimension d if it has a finite resolution by finitely generated projective bimodules, and there exists an algebra automorphism ν of R such that

$$\text{Ext}^i_{R^e}(R,R^e) \cong \begin{cases} 0, & i\neq d, \\ R^{\nu}, & i = d. \end{cases}$$

as R^e -modules. If ν is the identity, then R is said to be *Calabi-Yau*. Enveloping algebras and skew Calabi-Yau algebras related to skew PBW extensions were studied in [RS17b].

THEOREM 2.13. *If A is a σ-filtered skew PBW extension over a ring R such that $H(R)$ is Auslander-regular, then A is skew Calabi-Yau.*

Proof. From Theorem 2.10(4), (5) and (9), we know that A is a PBW deformation of a Noetherian Artin-Schelter regular algebra $G(A)$. The assertion follows from [Gad16, Proposition 2.15]. □

EXAMPLE 2.6. Let $R = \Bbbk[t_1,\dots,t_m]$, $m\geq 0$. Since $H(R) = R[z]$ is Auslander-regular and a commutative polynomial ring in $m+1$ variables over $\Bbbk$, then the examples of σ-filtered skew PBW extensions over R presented in [FGL+20, GL11, LR14, Sufrm[o]–7a] are skew Calabi-Yau. In particular, every one of the algebras presented in the proof of Proposition 2.12 is skew Calabi-Yau.

2.5 FUTURE WORK

Since Redman [Red99] and Chirvasitu et al. [CSW18] studied the noncommutative geometry of the homogenization of two classes of three dimensional skew polynomial algebras, and of the homogenization of the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$, respectively, keeping in mind that these algebras are particular examples of skew PBW extensions (Examples 2.3 and 2.4), we can think of establishing several properties of noncommutative geometry of the homogenization of skew PBW extensions. It is a natural task to investigate if the treatment developed by Redman

[Red99] and Chirvasitu et al. [CSW18] can be extended to the more general setting of these extensions.

CHAPTER 3

SUBALGEBRA ANALOGUES OF GRÖBNER BASES FOR IDEALS BASES

In this chapter, we present a first approach toward a theory of SAGBI bases for skew PBW extensions over $\Bbbk$-algebras.

With this aim and to motivate the study of SAGBI bases, in Section 3.1 we consider some key facts about these bases in the commutative setting of the polynomial ring $\Bbbk[x_1, \dots, x_n]$. We follow the treatment founded in Kapur and Madlener [KM89], and Robbiano et al. [RS90, KR05]. We also consider the problem of bases under composition initiated by Hong [Hon98] in the case of Gröbner bases, and developed by Nordbeck in his PhD Thesis [Nor01a] (see also [Nor02]) for SAGBI bases of subalgebras of $\Bbbk[x_1, \dots, x_n]$. This review of the commutative case will allow us to illustrate some of the main difficulties that arise when computing SAGBI bases (see Remarks 8 and 13, and Problem 3.1).

Next, Section 3.2 contains the theory of SAGBI bases for free algebras developed by Nordbeck [Nor01a] (see also [Nor98]). Since this theory is very important for what we want to do on these bases for skew PBW extensions in Section 3.4, we reconstruct in detail the treatment performed by him. As in the commutative case, we review the problem of SAGBI bases under composition presented in [Nor01b, Nor02].

In Section 3.3, we consider the approach to SAGBI bases developed by Khan et al. [KKB19], which follows Nordbeck's ideas described in Section 3.2, for the class of G-algebras introduced by Apel [Ape88]. Our interest in these algebras is due to their similarities with skew PBW extensions [LR14, Remark 3.1(ii)].

Having in mind the developments and results formulated in Sections 3.1, 3.2, and 3.3, in Section 3.4 we present our proposal for a theory of SAGBI bases for skew PBW extensions over $\Bbbk$-algebras. We present the notion of *reduction* which is necessary in the characterization of SAGBI bases, and then establish an algorithm to find the SAGBI normal form of an element. Then, we define what a SAGBI basis is, and formulate a criterion to determine when a subset of a skew PBW extension over a field is a SAGBI basis. In addition, we establish an algorithm to find a SAGBI basis from a subset contained in a subalgebra of a skew PBW extension. We illustrate our results with examples concerning algebras appearing in Lie theory, and noncommutative algebraic geometry. We also investigate the problem of polynomial composition for `SAGBI` bases of subalgebras of skew PBW extensions. Our important results of this chapter are formulated in `Algorithm` 2, Theorem 3.20, Propositions 3.22 and 3.25, and Theorem 3.26.

Finally, Section 3.5 presents some ideas for a future work.

3.1 Commutative setting

Gordan [Gor00] introduced the idea of Gröbner bases in 1900 while Gröbner bases for commutative polynomial rings over a field $\Bbbk$ were defined and developed by Buchberger in his PhD Thesis [Buc65] under the direction of Gröbner. In this commutative setting, Shannon and Sweedler [SS88] solved the membership problem in subalgebras transforming it into a problem of ideals. Almost at the same time, Kapur and Madlener [KM89] presented a procedure to compute a canonical basis for a finitely presented $\Bbbk$-subalgebra. In that paper they move away a bit from what was worked by Shannon and Sweedler [SS88], generalizing the theory of the Buchberger's algorithm using the *term rewriting method*. Nevertheless, this direct method has a disadvantage because for some orders on indeterminates and terms, the completion algorithm may not terminate and thus generate an infinite canonical basis, something that does not happen in the approach of Shannon and Sweedler. Later, Robbiano and Sweedler [RS90] studied the analogue of Gröbner bases for subalgebras of commutative polynomial rings, and used for the first time the term `SAGBI` (*Subalgebras Analogue to Gröbner Basis for Ideals*). They showed explicitly that the `SAGBI` theory is not simply a formal translation of Buchberger's theory from ideals to subalgebras. As a matter of fact, the theory of Gröbner bases of ideals of a subalgebra in a polynomial ring was developed by Miller [Mil98], while Lezama and Marín [LM09] used `SAGBI` bases to determine the equality of subalgebras based on the the ideas presented by Kreuzer and Robbiano [KR05].

In this section we present briefly the treatment on `SAGBI` bases of the commutative polynomial ring $\Bbbk[x_1,\dots,x_n]$ following [KR05], where they consolidated the theory of the `SAGBI` bases, gave a criterion for `SAGBI` bases and the algorithm for their construction, and proved that finite `SAGBI` bases exist for some classes of subalgebras.

3.1.1 Basic definitions

For the set of products of powers of the indeterminates $x_1,\dots,x_n$ of $\Bbbk[x_1,\dots,x_n]$, which is denoted by $\mathrm{Mon}(\Bbbk[x_1,\dots,x_n])$, and whose elements are called *monomials* (at least for the moment), recall that a *monomial ordering* $\prec$ on $\Bbbk[x_1,\dots,x_n]$ is a relation on $\mathbb{Z}_{\geq 0}^n$, or equivalently, a relation on the set of monomials x^{α}, $\alpha\in\mathbb{Z}_{\geq 0}^n$, satisfying:

(i) $\prec$ is a total (or linear) ordering on $\mathbb{Z}_{\geq 0}^n$.

(ii) If $\alpha\prec\beta$ and $\gamma\in\mathbb{Z}_{\geq 0}^n$, then $\alpha+\gamma\prec\beta+\gamma$.

(iii) $\prec$ is a well-ordering on $\mathbb{Z}_{\geq 0}^n$. This means that every nonempty subset of $\mathbb{Z}_{\geq 0}^n$ has a smallest element under $\prec$. In other words, if $C \subset \mathbb{Z}^n_{\geq 0}$ is nonempty, then there is $\alpha \in C$ such that $\alpha \prec \beta$ for every $\beta \neq \alpha$ in C.

Given a monomial ordering $\prec$ on $\Bbbk[x_1,\dots,x_n]$, $\alpha \prec \beta$ can be written as $\beta \succ \alpha$. We say that $\alpha \preceq \beta$ when either $\alpha \prec \beta$ or $\alpha = \beta$. More details about monomial orders can be found in excellent texts such as [AL94, BW93, CLO15, KR05][1].

Notice that an order relation $\prec$ on $\mathbb{Z}^n_{\geq 0}$ is a well-ordering if and only if every strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$, $\alpha(1) \succ \alpha(2) \succ \alpha(3) \succ$ eventually terminates [CLO15, Lemma 2].

Let us see some classical examples of monomial orderings (for more details, see [CLO15, Chapter 2, Section 2]).

EXAMPLES 3.1.

(i) (`Lexicographic Order`) Let $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say $\alpha > \beta$ if the leftmost non-zero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n_{\geq 0}$ is positive. We will write $x^\alpha \succ_{\texttt{lex}} x^\beta$ if $\alpha \succ_{\texttt{lex}} \beta$.

The indeterminates $x_1,\dots,x_n$ are ordered in the usual way by the `lex` ordering

$$(1,0,\dots,0) \succ_{\texttt{lex}} (0,1,0,\dots,0) \succ_{\texttt{lex}} \cdots \succ_{\texttt{lex}} (0,\dots,0,1),$$

so $x_1 \succ_{\texttt{lex}} x_2 \succ_{\texttt{lex}} \cdots \succ_{\texttt{lex}} x_n$.

If we work with polynomials in two or three indeterminates, we will write x, y, z rather than x_1, x_2, x_3.

(ii) (`Degree Lex Order`) Let $\alpha, \beta \in \mathbb{Z}^n_{n\geq 0}$. We say $\alpha \succ_{\texttt{deglex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and } \alpha \succ_{\text{lex}} \beta.$$

We will write $x^\alpha \succ_{\texttt{deglex}} x^\beta$ if $\alpha \succ_{\texttt{deglex}} \beta$.

(iii) (`Degree Reverse Lex Order`) Let $\alpha, \beta \in \mathbb{Z}^n_{n\geq 0}$. We say $\alpha \succ_{\texttt{degrevlex}} \beta$ if

$$\begin{aligned} &|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \quad \text{or} \\ &|\alpha| = |\beta| \text{ and the rightmost non}-\text{zero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is negative.} \end{aligned}$$

We will write $x^\alpha \succ_{\texttt{degrevlex}} x^\beta$ if $\alpha \succ_{\texttt{degrevlex}} \beta$.

Given a non-zero polynomial $f = \sum_{i=1}^{m} c_i x_1^{\alpha_{1_i}} \cdots x_n^{\alpha_{i_n}} \in \Bbbk[x_1,\dots,x_n]$, the set $\left\{x_1^{\alpha_{1_i}} \cdots x_n^{\alpha_{i_n}} \mid c_i \neq 0\right\}$ is called the *support* of f and is denoted by $\mathrm{Supp}(f)$. Throughout this section, for an ordering $\prec$ on the set $\mathbb{Z}^n_{\geq 0}$, which we can identify with the set of products of powers of indeterminates of $\Bbbk[x_1,\dots,x_n]$, $\mathrm{Mon}(\Bbbk[x_1,\dots,x_n])$, the leading monomial of f, denoted by $\mathrm{lm}(f)$, is the largest

[1]We have to be a little careful because in some texts and papers considered in this chapter, *monomial orderings* are called *term orderings*, and vice versa. We will clarify the meaning of *monomial* and *term* where appropriate.

element (with respect to $\prec$), $\mathrm{lc}(f)$ denotes the leading coefficient of f, and the leading term of f under $\prec$ is given by $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lm}(f)$. For $F \subseteq \Bbbk[x_1, \ldots, x_n]$, $\mathrm{lt}(F) := \{\mathrm{lt}(f) \mid f \in F\}$.

Throughout this section, S will denote a non-zero finitely generated $\Bbbk$-subalgebra of the polynomial ring $\Bbbk[x_1, \ldots, x_n]$. Thus, there are polynomials $f_1, \ldots, f_s \in \Bbbk[x_1, \ldots, x_n]$ such that $S = \Bbbk[f_1, \ldots, f_s]$. Equivalently, the ring S is the image of the $\Bbbk$-algebra homomorphism $\varphi : \Bbbk[y_1, \ldots, y_s] \to \Bbbk[x_1, \ldots, x_n]$ defined by $\varphi(y_i) = f_i$.

Definition 3.1 ([KR05, Definition 6.6.2]). A set $F \subseteq S$ is called a `SAGBI` *basis of* S if

$$\Bbbk\left[\mathrm{lt}(f) \mid f \in S \setminus \{0\}\right] = \Bbbk\left[\mathrm{lt}(F)\right].$$

Notice that every `SAGBI` basis of S is a system of $\Bbbk$-algebra generators of S [KR05, Proposition 6.6.3].

Remark 8 shows that there exist subalgebras of $\Bbbk[x_1, \ldots, x_n]$ which have no finite `SAGBI` basis at all (more examples that illustrates this situation have been recently presented by Kuroda [Kur23]).

Remark 8 ([KR05, Example 6.6.7]). Let $S = \Bbbk[f_1, f_2, f_3] \subseteq \Bbbk[x_1, x_2]$ where $f_1 = x_1 + x_2$, $f_2 = x_1x_2$, and $f_3 = x_1x_2^2$. Then S has no finite `SAGBI` basis, no matter which term ordering $\prec$ we use.

From [KR05, Proposition 6.6.11], we know that for non-zero elements $f_1, \ldots, f_s \in \Bbbk[x_1, \ldots, x_n]$, if $S = \Bbbk\left[f_1, \ldots, f_s\right]$ and the leading terms $\mathrm{lt}\left(f_1\right), \ldots, \mathrm{lt}\left(f_s\right)$ are algebraically independent, then $\left\{f_1, \ldots, f_s\right\}$ is a `SAGBI` basis of S.

3.1.2 Reduction

With the aim of computing `SAGBI` bases (Propositions 3.1 and 3.2 below), we recall the following reduction process.

Reduction 3.1 ([KR05, Definition 6.6.16]). Let F be a non-empty subset of $\Bbbk[x_1, \ldots, x_n]$.

(i) Consider $h_1 \in \Bbbk[x_1, \ldots, x_n]$, and suppose there exist a constant $k \in \Bbbk$, polynomials $f_1, \ldots, f_s$ belonging to F, and a monomial $t \in \Bbbk\left[y_1, \ldots, y_s\right]$ such that the polynomial $h_2 = h_1 - kt\left(f_1, \ldots, f_s\right)$ satisfies $t\left(\mathrm{lt}\left(f_1\right), \ldots, \mathrm{lt}\left(f_s\right)\right) \notin \mathrm{Supp}\left(h_2\right)$. Then we say that h_1 *subalgebra reduces to* h_2 *in one step* and we write $h_1 \xrightarrow{F}_{\mathrm{ss}} h_2$. The passage from h_1 to h_2 is called *a subalgebra reduction step.*

(ii) The transitive closure of the relations $\xrightarrow{F}_{\mathrm{ss}}$ is called *the subalgebra rewrite relation defined by* F and is denoted by $\xrightarrow{F}_{\mathrm{s}}$.

(iii) An element $h_1 \in \Bbbk[x_1, \ldots, x_n]$ with the property that there exists no subalgebra reduction step $h_1 \xrightarrow{F}_{\mathrm{ss}} h_2$ for which $h_2 \neq h_1$ is called *irreducible with respect to* $\xrightarrow{F}_{\mathrm{s}}$.

(iv) The equivalence relation defined by $\xrightarrow{F}_{\mathrm{s}}$ will be denoted by $\xleftrightarrow{F}_{\mathrm{s}}$.

For $f_1,\dots,f_s \in \mathbb{k}[x_1,\dots,x_n]$ and $\mathscr{F} = (f_1,\dots,f_s)$, the ideal

$$\{f \in \mathbb{k}[y_1,\dots,y_s] \mid f(f_1,\dots,f_s) = 0\}$$

is called the *ideal of algebraic relations of* $\mathscr{F}$ and is denoted by $\mathrm{Rel}(\mathscr{F})$ or $\mathrm{Rel}(f_1,\dots,f_s)$ [KR05, Definition 6.6.20].

3.1.3 SAGBI BASES CHARACTERIZATION

PROPOSITION 3.1 ([KR05, THEOREM 6.6.25]). *Let* $F \subseteq \mathbb{k}[x_1,\dots,x_n] \setminus \{0\}$ *and* $S = \mathbb{k}\langle F\rangle_{\mathbb{k}[x_1,\dots,x_n]}$ *denotes the subalgebra of* $\mathbb{k}[x_1,\dots,x_n]$ generated by F *in* $\mathbb{k}[x_1,\dots,x_n]$*. Then the following conditions are equivalent:*

(A_1) *For each* $f \in S \setminus \{0\}$*, there are* $f_1,\dots,f_s \in F$ *and* $h \in \mathbb{k}[y_1,\dots,y_s]$ *with* $f = h(f_1,\dots,f_s)$ *and* $\mathrm{lt}(f) \geq \mathrm{lt}(t(f_1,\dots,f_s))$ *for all* $t \in \mathrm{Supp}(h)$.

(A_2) *For each* $f \in S\setminus\{0\}$*, there are* $f_1,\dots,f_s \in F$ *and* $h \in \mathbb{k}[y_1,\dots,y_s]$ *with* $f = h(f_1,\dots,f_s)$ *and* $\mathrm{lt}(f) = \max\{\mathrm{lt}(t(f_1,\dots,f_s)) \mid t \in \mathrm{Supp}(h)\}$.

(B_1) *The set* F *is a* SAGBI *basis of* S*. By definition, this means that we have* $\mathbb{k}[\mathrm{lt}(S)] = \mathbb{k}[\mathrm{lt}(F)]$.

(B_2) *The monoid* $\{\mathrm{lt}(f) \mid f \in S\setminus\{0\}\}$ *is generated by* $\{\mathrm{lt}(g) \mid g \in F\}$.

(C_1) *For an element* $f \in \mathbb{k}[x_1,\dots,x_n]$*, we have* $f \xrightarrow{F} 0$ *if and only if* $f \in S$.

(C_2) *If* $f \in S$ *is irreducible with respect to* $\xrightarrow{F}$ *then we have* $f = 0$.

(C_3) *For every element* $f_1 \in \mathbb{k}[x_1,\dots,x_n]$*, there is a unique element* $f_2 \in \mathbb{k}[x_1,\dots,x_n]$ *for which* $f_1 \xrightarrow{F} f_2$ *and* f_2 *is irreducible with respect to* $\xrightarrow{F}{}_{S}$.

DEFINITION 3.2 ([KR05, DEFINITION 6.6.26]). Let $G = \{g_1,\dots,g_s\} \subseteq \mathbb{k}[x_1,\dots,x_n] \setminus \{0\}$ be monic polynomials, and let $b := t_1 - t_2$ be a pure binomial in $\mathrm{Rel}(\mathrm{lt}(G))$, where $t_1, t_2 \in \mathrm{Mon}(\mathbb{k}[x_1,\dots,x_n])$. Then the polynomial $b(g_1,\dots,g_s) \in \mathbb{k}[x_1,\dots,x_n]$ is called the *T-polynomial of* b.

"T-polynomials will be for the computation of SAGBI bases what S-polynomials are for the computation of Gröbner bases. The letter "T" reminds us of the "toric origin" of these polynomials and also of the fact that the pair $(\log(t_1), \log(t_2))$ was called a *tête-a-tête* in the pioneering paper [RS90]." [KR05, p. 494].

Proposition 3.2 is known as *The* SAGBI *Basis Criterion.*

PROPOSITION 3.2 ([KR05, PROPOSITION 6.6.28]). *Let* F *be a non-zero set of monic polynomials of* $\mathbb{k}[x_1,\dots,x_n]$*, and* $S := \mathbb{k}\langle F\rangle_{\mathbb{k}[x_1,\dots,x_n]}$*. Then the following conditions are equivalent:*

(1) *The set* F *is a* SAGBI *basis of* S.

(2) *For every tuple* $\mathscr{F} = (f_1,\dots,f_s)$ *of elements of* F*, there exists a set* B *of pure binomials in* $\mathbb{k}[y_1,\dots,y_s]$ *which generates the ideal* $\mathrm{Rel}(\mathrm{lt}(\mathscr{F}))$ *and which satisfies* $b(f_1,\dots,f_s) \xrightarrow{F} 0$ *for all* $b \in B$.

Proposition 3.3 is an important result for the SAGBI basis procedure formulated in Proposition 3.4. Next, we establish some notation and preliminary definitions.

Given further indeterminates $y_1,\dots,y_m$, let $L := \mathbb{k}\left[y_1,\dots,y_m,y_1^{-1},\dots,y_m^{-1}\right]$ be the Laurent polynomial ring in the indeterminates $y_1,\dots,y_m$ over $\mathbb{k}$. An element of the form $y_1^{i_1}y_2^{i_2}\cdots y_m^{i_m} \in L$ with $i_1,\dots,i_m \in \mathbb{Z}$ is called an *extended monomial*. The monoid of all extended monomials is denoted by $\mathbb{E}^m$. For an extended monomial $t \in \mathbb{E}^m$, there exists a unique minimal number $\tau(t) \in \mathbb{N}$ such that $t\cdot\left(y_1\cdots y_m\right)^{\tau(t)} \in \mathbb{k}\left[y_1,\dots,y_m\right]$ [KR05, Section 6.1.A].

Recall that a *toric ideal* associated to the matrix $A = \left(a_{ij}\right) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ is $I(A) = \ker(\varphi)$ in $\mathbb{k}[x_1,\dots,x_n]$ where $\varphi : \mathbb{k}[x_1,\dots,x_n] \longrightarrow L$ is a $\mathbb{k}$-algebra homomorphism given by $\varphi(x_i) = t_i$ with $t_i = y_1^{a_{1i}}y_2^{a_{2i}}\cdots y_m^{a_{mi}}$ for $i = 1,\dots,n$.

Proposition 3.3 ([KR05, Proposition 6.1.3]). *Let $t_1,\dots,t_n \in \mathbb{E}^m$, let $I \subseteq \mathbb{k}[x_1,\dots,x_n]$ be the toric ideal associated to $(t_1,\dots,t_n)$, and let $J \subseteq \mathbb{k}\left[x_1,\dots,x_n,y_1,\dots,y_m\right]$ be the binomial ideal generated by $\left\{\pi^{\tau(t_1)}(x_1-t_1),\dots,\pi^{\tau(t_n)}(x_n-t_n)\right\}$ where $\pi = y_1\cdots y_m$.*

(1) *$I = \langle J:\pi^\infty\rangle \cap \mathbb{k}[x_1,\dots,x_n]$, where $J:\pi^\infty$ is the saturation of I with respect to the element π, defined as $I:\pi^\infty = \bigcup_{i\in\mathbb{N}} I:\pi^i$, with $I:\pi^i := \left\{f \in \mathbb{k}[x_1,\dots,x_n] \mid \pi^i f \in I\right\}$.*

(2) *Let z be a new indeterminate, and let G be a Gröbner basis of the ideal $J + (\pi z - 1)$ with respect to an elimination ordering for $\left\{y_1,\dots,y_m,z\right\}$. Then the toric ideal I is generated by $G \cap \mathbb{k}[x_1,\dots,x_n]$.*

(3) *The toric ideal I is generated by pure binomials.*

3.1.4 SAGBI bases Algorithm

Proposition 3.4 ([KR05, Theorem 6.6.29]). *Consider a non-zero set of monic polynomials $F = \left\{f_1,\dots,f_s\right\} \subseteq \mathbb{k}[x_1,\dots,x_n]$, and let $\mathcal{F} := \left(f_1,\dots,f_s\right)$, $S := \mathbb{k}[f_1,\dots,f_s]$, and the following sequence of instructions:*

(1) *Let $s' = s$, let $H = F$, and let $\mathcal{H} = \mathcal{F}$.*

(2) *Using Proposition 3.3, compute a set B of pure binomials which generates the ideal $\mathrm{Rel}\,(\mathrm{lt}(\mathcal{H}))$ in $\mathbb{k}\left[y_1,\dots,y_{s'}\right]$.*

(3) *If $B = \emptyset$, return the tuple $\mathcal{H}$ and stop. Otherwise, let $B' = \emptyset$.*

(4) *For every $b \in B$, reduce the polynomial $b\left(f_1,\dots,f_{s'}\right)$ via $\xrightarrow{H}_{\mathrm{s}}$ until an irreducible element $b' \in \mathbb{k}[x_1,\dots,x_n]$ is found, and if $b' \neq 0$, adjoin the element $\mathrm{lc}\left(b'\right)^{-1}b'$ to the set B'.*

(5) *If $B' = \emptyset$ then return the tuple $\mathcal{H}$ and stop. Otherwise, let $t = \#B'$, increase s' by t, and append the elements $f_{s'-t+1},\dots,f_{s'}$ of B' to H and $\mathcal{H}$.*

(6) *Using Proposition 3.3, compute a set B of pure binomials which generate the ideal $\mathrm{Rel}\,(\mathrm{lt}(\mathcal{H}))$ in $\mathbb{k}\left[y_1,\dots,y_{s'}\right]$. Replace B by its subset consisting of those elements which involve at least one of the indeterminates $y_{s'-t+1},\dots,y_{s'}$. Then continue with step (3.4).*

Then this is an enumerating procedure. The set of all elements contained in H at some point is a `SAGBI` *basis of S. The procedure stops if and only if S has a finite* `SAGBI` *basis. In this case, it returns a tuple $\mathscr{H}$ of polynomials which form a* `SAGBI` *basis of S.*

The next example presents one case where a subset not is a `SAGBI` basis of a subalgebra, but we can find it.

Example 3.1 ([KR05, Example 6.6.30]). Consider the $\Bbbk$-algebra $A = \Bbbk[x_1, x_2]$ and let $F := \{x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4\}$ be a subset of A with the order `deglex` with $x_1 \succ x_2$.

For the element $f_1 = \left(x_1^2 - x_2^2\right)\left(x_1^2x_2^2 - x_2^4\right) - \left(x_1^2x_2\right)\left(x_1^4x_2\right) = -2x_1^2x_2^4 + x_2^6$, it follows that $f_1 \in S = \Bbbk\langle F\rangle_A$.

Notice that $t\left(y_1, y_2, y_3, y_4\right) = y_4$, $t\left(\mathrm{lt}(f_1), \dots, \mathrm{lt}(f_4)\right) = x_1^2x_2^4 \notin \mathrm{Supp}\left(f_2\right)$, whence $f_1 \xrightarrow{F}_+ f_2$. Thus, F is not a `SAGBI` basis of S [KR05, Theorem 6.6.25].

To find a `SAGBI` basis of S we use the algorithm formulated in Proposition 3.4.

(i) Let $s' = 4, H = \left\{x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4\right\}$, and $\mathscr{H} = (x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4)$.

(ii) We compute a Gröbner basis for the ideal

$$\tilde{J} = \left\langle y_1 - x_1^2x_2, y_2 - x_1^2, y_3 - x_1^2x_2^2, y_4 - x_1^2x_2^4, x_1x_2z - 1\right\rangle$$

considering the order `lex` with $x_1 \succ x_2 \succ z \succ y_1 \succ y_2 \succ y_3 \succ y_4$. Using the package `SINGULAR` we obtain the basis given by $\mathscr{B} = \left\{y_1^2 - y_2y_3, y_2y_4 - y_3^2\right\}$. Proposition 3.3 implies that $\mathscr{B}$ generates the ideal of relationships $\mathrm{Rel}\left(\mathrm{LT}(\mathscr{H})\right)$ in $\Bbbk\left[y_1, \dots, y_4\right]$

(iii) Let $B' = \emptyset$.

(iv) For $b_1 = y_2y_4 - y_3^2$ we find

$$b_1\left(x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4\right) = x_1^2x_2^6 - x_2^8.$$

From the process described in Reduction 3.1 we know that b_1 is irreducible with respect to F, and so $b_1 = b_1'$.

For $b_2 = y_1^2 - y_2y_3$, we compute

$$b_2\left(x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4\right) = 2x_1^2x_2^4 - x_2^6,$$

and by the reduction process, $b_2 \xrightarrow{H}_S -x_2^6$. Thus, $x_2^6 = b_2'$.

We obtain $B' = \{x_1^2x_2^6 - x_2^8, x_2^6\}$.

(v) Let $t = 2, s' = 6, H = \left\{x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4, x_2^6, x_1^2x_2^6 - x_2^8\right\}$, and $\mathscr{H} = (x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4, x_2^6, x_1^2x_2^6 - x_2^8)$.

(vi) We compute a Gröbner basis for the ideal

$$\tilde{J} = \left\langle y_1 - x_1^2x_2, y_2 - x_1^2, y_3 - x_1^2x_2^2, y_4 - x_1^2x_2^4, y_5 - x_1^2x_2^6, y_6 - x_2^6x_1x_2z - 1\right\rangle$$

with the order `lex` and $x_1 > x_2 > z > y_1 > y_2 > y_3 > y_4 > y_5 > y_6$. `SINGULAR` gives us the basis

$$G = \{y_1^2 - y_2y_3, y_2y_4 - y_3^2, y_2y_6 - y_5, y_3y_5 - y_4^2, y_2y_5 - y_3y_4\}.$$

We choose the elements with the indeterminates y_5 and y_6, and so

$$B = \{y_2y_6 - y_5, y_3y_5 - y_4^2, y_2y_5 - y_3y_4\}.$$

(vii) Let $B' = \emptyset$.

(viii) For the element $b_1 = y_2y_6 - y_5$, we have

$$b_1\left(x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4, x_2^6, x_1^2x_2^6 - x_2^8\right) = 0.$$

For $b_2 = y_3y_5 - y_4^2$,

$$b_2(x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4, x_2^6, x_1^2x_2^6 - x_2^8) = -x_1^4x_2^8 + x_1^2x_2^8 - x_2^{10},$$

and by the reduction process, $b_2 \xrightarrow{H}_S 0$.

Finally, for $b_3 = y_2y_5 - y_3y_4$ we compute

$$b_3(x_1^2x_2, x_1^2 - x_2^2, x_1^2x_2^2 - x_2^4, x_1^2x_2^4, x_2^6, x_1^2x_2^6 - x_2^8) = -x_1^4x_2^6 + x_1^2x_2^8 + x_1^2x_2^6 - x_2^8,$$

and applying the reduction process it follows that $b_3 \xrightarrow{H}_S 0$.

In this way, $B' = \emptyset$, and therefore, $H \cup \{x_1^2x_2^6 - x_2^8, x_2^6\}$ is a finite `SAGBI` basis for S.

The next proposition gives us the *normal form* of a polynomial which is necessary to define a reduced `SAGBI` basis.

PROPOSITION 3.5 ([LM09, PROPOSITION 3.2]). *For every $f \in \Bbbk[x_1,\dots,x_n]$ there exists a unique polynomial $f_F \in \Bbbk[x_1,\dots,x_n]$ with the properties that $f - f_F \in S$ and* $\operatorname{Supp}(f_F) \cap \Bbbk[\operatorname{lt}(F)] = \emptyset$.

The element f_F is called *the normal form of f with respect to S*, and it is denoted by $\mathtt{NF}_S(f)$.

PROPOSITION 3.6 ([LM09, PROPOSITION 3.3]). *Let $f, f_1, f_2 \in \Bbbk[x_1,\dots,x_n]$. Then*

(1) $\mathtt{NF}_S(\mathtt{NF}_S(f)) = \mathtt{NF}_S(f)$.

(2) $\mathtt{NF}_S(f_1 - f_2) = \mathtt{NF}_S(f_1) - \mathtt{NF}_S(f_2)$.

(3) $\mathtt{NF}_S(f_1f_2) = \mathtt{NF}_S(\mathtt{NF}_S(f_1)\mathtt{NF}_S(f_2))$.

Recall that Kreuzer and Robbiano [KR05, p. 501] called F a *reduced* `SAGBI` *basis of S* if the set $\{\operatorname{lt}(f_1),\dots,\operatorname{lt}(f_s)\}$ is the minimal monomial system of algebra generators of $\Bbbk[\operatorname{lt}(F)]$ and $\operatorname{Supp}(f_i - \operatorname{lt}(ff_i)) \cap \Bbbk[\operatorname{lt}(F)] = \emptyset$, for $i = 1,\dots,s$. This reduced `SAGBI` basis of S is unique [LM09, Proposition 3.4]. The next example shows the process for computing a reduced `SAGBI` basis.

EXAMPLE 3.2 ([LM09, EXAMPLE 3.2]). Let $F = \{x_2x_3 - x_3^2, x_1^4 - x_2^2x_3^2\}$ be a SAGBI basis of $S = \mathbb{Q}[F]$ with the order deglex with $x_1 > x_2 > x_3$. Then:

$$\begin{aligned}
\mathtt{NF}_S\left(x_2x_3 - x_3^2\right) &= \mathtt{NF}_S\left(x_3\left(x_2 - x_3\right)\right)\\
&= \mathtt{NF}_S(\mathtt{NF}_S\left(x_3\right)\mathtt{NF}_S\left(x_2 - x_3\right))\\
&= \mathtt{NF}_S\left(x_3\left(x_2 - x_3\right)\right)\\
&= \mathtt{NF}_S\left(x_2x_3\right) - \mathtt{NF}_S\left(x_3^2\right)\\
&= x_2x_3 - x_3^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathtt{NF}_S\left(x_1^4 - x_2^2x_3^2\right) &= \mathtt{NF}_S\left(\left(x_1^2 - x_2x_3\right)\left(x_1^2 + x_2x_3\right)\right)\\
&= \mathtt{NF}_S\left(\mathtt{NF}_S\left(x_1^2 - x_2x_3\right)\mathtt{NF}_S\left(x_1^2 + x_2x_3\right)\right)\\
&= \mathtt{NF}_S\left\{(\mathtt{NF}_S\left(x_1^2\right) - \mathtt{NF}_S(x_2x_3))\left(\mathtt{NF}_S\left(x_1^2\right) + \mathtt{NF}_S\left(x_2x_3\right)\right)\right\}\\
&= \mathtt{NF}_S\left(\left(x_1^2 - x_3^2\right)\left(x_1^2 + x_3^2\right)\right)\\
&= \mathtt{NF}_S\left(x_1^4 - x_3^4\right)\\
&= \mathtt{NF}_S\left(x_1^4\right) - \mathtt{NF}_S\left(x_3^4\right)\\
&= x_1^4 - x_3^4.
\end{aligned}$$

In this way, a reduced SAGBI basis of S is given by $\{x_2x_3 - x_3^2, x_1^4 - x_3^4\}$.

From [LM09, Corollary 3.2], we know that two subalgebras S and T are equal if and only if S and T have the same reduced SAGBI bases. Example 3.3 illustrates this situation.

EXAMPLE 3.3 ([LM09, EXAMPLE 3.3]). Determine whether or not the following two subalgebras are equal. Let $F_1 = \{x_2x_3 - x_3^2, x_1^4 - x_2^2x_3^2\}$ be a SAGBI basis of the $\mathbb{Q}$-subalgebra $S_1 = \mathbb{Q}[F_1]$ and $F_2 = \{x_1^2 - 1, x_2^2 - 1, x_3^2 - 1\}$ a SAGBI basis of the $\mathbb{Q}$-subalgebra $S_2 = \mathbb{Q}[F_2]$. Consider the monomial order degrevlex.

The reduced SAGBI basis of the $\mathbb{Q}$-subalgebra S_1 is $\{x_2x_3 - x_3^2, x_1^4 - x_3^4\}$ and the reduced SAGBI basis of the $\mathbb{Q}$-subalgebra S_2 is $\{x_1^2 - 1, x_2^2 - 1, x_3^2 - 1\}$. Since the reduced SAGBI basis of S_1 is different from the reduced SAGBI basis of S_2, the two subalgebras S_1 and S_2 are different.

3.1.5 SAGBI BASES UNDER COMPOSITION

In this section, we consider the question about polynomial composition of SAGBI bases investigated by Nordbeck [Nor01a] (see also [Nor02]). We start by recalling the origin of the question and the corresponding answer made by Hong [Hon98] in the setting of Gröbner bases.

3.1.5.1 MOTIVATION: GRÖBNER BASES UNDER COMPOSITION

Polynomial composition is the operation of replacing the indeterminates in a polynomial with other polynomials. Hong asked in his paper: *When does composition commute with Gröbner ba-*

sis computation? [Hon98, p. 643] More exactly, let F be a finite set of polynomials in $\Bbbk[x_1,\dots,x_n]$, and let G be a Gröbner basis of the ideal generated by F under some monomial ordering $\prec$ on $\Bbbk[x_1,\dots,x_n]$[2]. For $\Theta = \{\theta_1,\dots,\theta_n\}$ a list of n polynomials in $\Bbbk[x_1,\dots,x_n]$, let F^* be the set obtained from F by replacing x_i by θ_i, and likewise let G^* be the set obtained from G by replacing x_i by θ_i. The question is if G^* is also a Gröbner basis of F^* (under the same monomial ordering $\prec$), or equivalently, when does Gröbner basis computation commute with composition? (see Definition 3.4). Hong proved that this is the case if and only if the composition is "compatible" with the monomial ordering and the nondivisibility (see Definition 3.5 and Proposition 3.7). Gutiérrez and San Miguel [GM98] studied composition of Gröbner bases for reduced Gröbner bases, while Liu and Wang [LW07] investigated homogeneous Gröbner bases under composition.

Throughout this section, Θ denotes a list $\{\theta_1,\dots,\theta_n\}$ of n non-zero polynomials in $\Bbbk[x_1,\dots,x_n]$, $\mathrm{lt}(\Theta) := \{\mathrm{lt}(\theta_1),\dots,\mathrm{lt}(\theta_n)\}$, $\mathrm{lm}(\Theta) := \{\mathrm{lm}(\theta_1),\dots,\mathrm{lm}(\theta_n)\}$, and F means a finite set of polynomials in $\Bbbk[x_1,\dots,x_n]$.

Definition 3.3 ([Hon98, Definition 3.1]). The *composition of h by* Θ, written as $h \circ \Theta$, is the polynomial obtained from h by replacing each x_i in it with Θ_i. Likewise, $H \circ \Theta$ is the set $\{h \circ \Theta \mid h \in H\}$.

One might consider the possibility of defining composition as the "function composition", namely, for all $(k_1,\dots,k_n) \in \Bbbk^n$, $(h \circ \Theta)(k_1,\dots,k_n) = h(\theta_1(k_1,\dots,k_n),\dots,\theta_n(k_1,\dots,k_n))$. However, this is not suitable since $h \circ \Theta$ is *not* uniquely determined when $\Bbbk$ is a finite field.

Definition 3.4 ([Hon98, Definition 3.2]). *Composition by* Θ *commutes with Gröbner basis computation if and only if the following formula is true for* Θ: if G is a Gröbner basis for the ideal generated by F, then $G \circ \Theta$ is a Gröbner basis for the ideal generated by $F \circ \Theta$.

With the aim of answering the question formulated above, Hong introduced the following notions of compatibility.

Definition 3.5. Let $\Theta = \{\theta_1,\dots,\theta_n\}$ and $\prec$ be a monomial ordering on $\Bbbk[x_1,\dots,x_n]$.

(i) ([Hon98, Definition 3.3]) *Composition by* Θ *is compatible with the monomial ordering* $\prec$ if and only if for all monomials x^α and x^β belonging to $\mathrm{Mon}(\Bbbk[x_1,\dots,x_n])$, we have

$$x^\alpha \prec x^\beta \quad \text{implies that} \quad x^\alpha \circ \mathrm{lt}(\Theta) \prec x^\beta \circ \mathrm{lt}(\Theta).$$

(ii) ([Hon98, Definition 3.4]) *Composition by* Θ *is compatible with nondivisibility* if and only if for all monomials x^α and x^β belonging to $\mathrm{Mon}(\Bbbk[x_1,\dots,x_n])$, we have

$$x^\alpha \nmid x^\beta \quad \text{implies that} \quad x^\alpha \circ \mathrm{lt}(\Theta) \nmid x^\beta \circ \mathrm{lt}(\Theta),$$

where $\mid$ denotes the classical divisibility relation over monomials (that is, sum on $\mathbb{Z}^n_{\geq 0}$).

The main result in Hong's paper [Hon98] is formulated in Proposition 3.7.

Proposition 3.7 ([Hon98, Theorem 3.1]). *The following are equivalent:*

[2] Hong called *term* what here we means *monomial*.

(1) *Composition by* Θ *commutes with Gröbner basis computation.*

(2) *Composition by* Θ *is*

(a) *compatible with monomial ordering* $\prec$ *and*

(b) *compatible with nondivisibility.*

3.1.5.2 SAGBI BASES UNDER COMPOSITION

Following Nordbeck [Nor02, Section 2] (see also [Nor01a]), by a *term* t in $\Bbbk[x_1,\dots,x_n]$ he means an element of the form $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ with $\alpha_1,\dots,\alpha_n \in \mathbb{N}$. The set of all terms of $\Bbbk[x_1,\dots,x_n]$ is denoted by T. Of course, $1 = x_1^0\cdots x_n^0$. If we compare with the notation established in Section 3.1.1, then $T = \operatorname{Mon}(\Bbbk[x_1,\dots,x_n])$. Note that this is the terminology used by Hong [Hon98].

If H is a subset of $\Bbbk[x_1,\dots,x_n]$ (not necessarily finite), then the subalgebra of $\Bbbk[x_1,\dots,x_n]$ generated by H is usually denoted as $\Bbbk[H]$. The elements of $\Bbbk[H]$ are precisely the polynomials in the set of formal intederminates H, viewed as elements of $\Bbbk[x_1,\dots,x_n]$. Of course, the elements of $\Bbbk \subset \Bbbk[H]$ correspond to the constant polynomials. Nordbeck [Nor02, Section 2] reserved the word *monomial* for the "terms in H" (in contrast to the terminology used in Section 3.1.1 for $\Bbbk[x_1,\dots,x_n]$). In this way, *by a monomial he means a finite product of elements from* H (or some other subset of $\Bbbk[x_1,\dots,x_n]$ that we are working with). Such a product will usually be written $m(H)$. The "empty" monomial 1 is considered. Of course, these monomials are not (in general) terms viewed as elements of $\Bbbk[x_1,\dots,x_n]$.

We assume that T is given a *term ordering* $\prec$ in the sense of monomial orderings considered in Section 3.1.1. For a non-zero polynomial $f \in \Bbbk[x_1,\dots,x_n]$, Nordbeck denoted its *leading term* as $\widehat{f} \in T$. He called the coefficient of $\widehat{f}$ the *leading coefficient* of f. For a subset $F \subset \Bbbk[x_1,\dots,x_n]$, $\widehat{F} := \{\widehat{f} \mid f \in F\}$.

Definition 3.6 ([Nor02, Definition 1]). Let S be a subalgebra of $\Bbbk[x_1,\dots,x_n]$. A subset $H \subset S$ is called a SAGBI *basis for* S if for every non-zero $s \in S$, there exists a monomial $m(H) \in H$ such that $\widehat{s} = \widehat{m(H)}$.

Since term orderings are preserved by multiplication,

$$m(H) = \prod_i h_i \ (h_i \in H) \text{ implies } \widehat{m(H)} = \prod_i \widehat{h_i},$$

that is,

$$\widehat{m(H)} = m(\widehat{H}). \tag{3.1}$$

This means that an equivalent formulation of Definition 3.6 is that H is a SAGBI basis if the leading term of every non-zero element in S can be written as a product of leading terms of elements in H [Nor02, Remark 1]. It can be seen that if H is a SAGBI basis for S, then H generates S, that is, $S = \Bbbk[H]$. We say that H is a SAGBI basis meaning that H is a SAGBI basis for $\Bbbk[H]$.

Definition 3.7 ([Nor02, Definition 2]). Two monomials $m(H), m'(H) \in H$ form a *critical pair* $(m(H), m'(H))$ of H if $\widehat{m(H)} = \widehat{m'(H)}$. If $c \in \Bbbk^*$ is such that $m(H)$ and $cm'(H)$ have the same leading coefficient, then we define the T*-polynomial* of $(m(H), m'(H))$ as $T(m(H), m'(H)) =$

$m(H) - cm'(H)$ (c.f. Definition 3.2). The idea with the constant c is that the leading words cancel in $T(m(H), m'(H))$, whence $\widehat{T(m(H), m'(H))} \prec \widehat{m(H)} = \widehat{m'(H)}$.

Proposition 3.8 characterizes SAGBI basis by using the notion of T-polynomial (c.f. Proposition 3.1(B1) and (C3)).

PROPOSITION 3.8 ([NOR02, THEOREM 1]). *A subset $H \subset \Bbbk[x_1,\ldots,x_n]$ is a* SAGBI *basis if and only if the T-polynomial of every critical pair $(m(H), m'(H))$ of H either is equal to zero, or can be written as*

$$T(m(H), m'(H)) = \sum_{i=1}^{t} c_i m_i(H), \quad \widehat{m_i(H)} \prec \widehat{m(H)} = \widehat{m'(H)}, \quad \text{for every } i, \tag{3.2}$$

where the m_i are monomials belonging to H and the $c_i \in \Bbbk$.

Note that every subset $H \subset \Bbbk[x_1,\ldots,x_n]$ consisting only of terms (or terms times coefficients) is a SAGBI basis.

Now, let us define the process of composition of polynomials (c.f. Definition 3.3).

DEFINITION 3.8 ([NOR02, DEFINITION 3]). Let $\Theta = \{\theta_1,\ldots,\theta_n\}$ be a subset of $\Bbbk[x_1,\ldots,x_n]$, and let $f \in \Bbbk[x_1,\ldots,x_n]$. The *composition of f by Θ*, written $f \circ \Theta$, is the polynomial obtained from f by replacing each ocurrence of the x_i with θ_i. For a subset $F \subset \Bbbk[x_1,\ldots,x_n]$, $F \circ \Theta := \{f \circ \Theta \mid f \in F\}$.

It is assumed that $\theta \notin \Bbbk$ for all i. This is used to guarantee that $t \neq 1$ implies $t \circ \widehat{\Theta} \neq 1$, for every $t \in T$.

REMARK 9 ([NOR02, REMARK 4]). Considering the notation in Definition 3.8, it seems natural to write monomials $m(H)$ above as $m \circ H$. Nevertheless, Nordbeck retained the notation $m(H)$ with the aim of distinguishing "compositions" by H from compositions by Θ. As a matter of fact, the two forms of compositions are associative in the sense that

$$m(H) \circ \Theta = m(H \circ \Theta). \tag{3.3}$$

Notice that the notion $m(H \circ \Theta)$ makes sense due to the natural correspondence between the sets $H = \{h_1,\ldots\}$ and $H \circ \Theta = \{h_1 \circ \Theta, h_2 \circ \Theta,\ldots\}$. Note also that for elements $f, g \in \Bbbk[x_1,\ldots,x_n]$,

$$(fg) \circ \Theta = (f \circ \Theta)(g \circ \Theta), \tag{3.4}$$

$$(f+g) \circ \Theta = f \circ \Theta + g \circ \Theta. \tag{3.5}$$

Having in mind that the term order is preserved by multiplication, in a similar way to expression (3.1), we have the equality

$$\widehat{t \circ \Theta} = t \circ \widehat{\Theta}. \tag{3.6}$$

Definition 3.9 is the analogue of Definition 3.4 but now in the setting of SAGBI bases.

DEFINITION 3.9 ([NOR02, DEFINITION 4]). *Composition by Θ commutes with* SAGBI *bases computation* if for every SAGBI basis H, also $H \circ \Theta$ is a SAGBI basis (under the same ordering as

H).

REMARK 10 ([NOR02, REMARK 5]). "In Hong's paper [Hon98], the counterpart of Definition 3.9 (see Definition 3.4) requires that if G is a Gröbner basis for the ideal generated by a set of polynomials F, then $G \circ \Theta$ is a Gröbner basis for the ideal generated by $F \circ \Theta$. A direct translation to the subalgebra language would of course be: if H is a SAGBI basis for the subalgebra generated by a set of polynomials F, then $H \circ \Theta$ is a SAGBI basis for the subalgebra generated by $F \circ \Theta$. That this implies the statement in Definition 3.9 is clear (take $F = H$). The two formulations are equivalent since it is easy to prove (using (3.4) and (3.5) above) that $\Bbbk[H] = \Bbbk[F]$ implies $\Bbbk[H \circ \Theta] = \Bbbk[F \circ \Theta]$."

Nordbeck decided under which conditions on Θ, composition by Θ commutes with SAGBI bases computation (Proposition 3.9).

DEFINITION 3.10. Let $\prec$ be a term ordering on $\Bbbk[x_1, \ldots, x_n]$ and $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a subset of $\Bbbk[x_1, \ldots, x_n]$.

(i) ([Nor02, Definition 5]) Composition by Θ is *compatible with the given ordering* if for all terms $x^\alpha, x^\beta \in T$, we have

$$x^\alpha \prec x^\beta \quad \text{implies} \quad x^\alpha \circ \widehat{\Theta} \prec x^\beta \circ \widehat{\Theta}. \tag{3.7}$$

For an element $f \in \Bbbk[x_1, \ldots, x_n]$ written as a linear combination of terms in decreasing order $f = \sum_{i=1}^{s} c_i t_i$, $t_1 \succ \cdots \succ t_s$, if composition by Θ is compatible with the ordering, then $t_1 \circ \widehat{\Theta} \succ \cdots \succ t_s \circ \widehat{\Theta}$, so expressions (3.5) and (3.6) guarantee that

$$\widehat{f \circ \Theta} = \widehat{f} \circ \widehat{\Theta}. \tag{3.8}$$

(ii) ([Nor02, Definition 6]) Composition by Θ is *compatible with nonequality* if for all terms $x^\alpha, x^\beta \in T$, we have

$$x^\alpha \neq x^\beta \quad \text{implies} \quad x^\alpha \circ \widehat{\Theta} \neq x^\beta \circ \widehat{\Theta}. \tag{3.9}$$

Definition 3.10 contains the two sufficient conditions needed for commutation of SAGBI bases computation. However, since term orderings are total, the second condition follows from the first one [Nor02, Lemma 1].

The following is the most important result presented by Nordbeck [Nor02].

PROPOSITION 3.9 ([NOR02, THEOREM 2]). *Composition by Θ commutes with* SAGBI *bases computation if and only if the composition is compatible with the term ordering $\prec$ on $\Bbbk[x_1, \ldots, x_n]$.*

Proof. The sufficiency of the compatibility condition requires of the following preliminary result: if composition by Θ is compatible with the ordering, and $(m(H), m'(H))$ is a critical pair of $H \circ \Theta$, then $(m(H), m'(H))$ is also a critical pair of H [Nor02, Lemma 2].

We follow the ideas presented in [Nor02, Proposition 1]. Consider an arbitrary SAGBI basis H. We have to show that $H \circ \Theta$ is also a SAGBI basis. With this aim, let $(m(H), m'(H))$ be a critical pair of $H \circ \Theta$. By the previous result, $(m(H), m'(H))$ is also a critical pair of H. Proposition 3.8

guarantees the expression

$$m(H) - cm'(H) = \sum_i c_i m_i(H) \quad \text{(or zero)}, \quad \widehat{m_i(H)} \prec \widehat{m(H)} = \widehat{m'(H)}, \quad \text{for all } i. \tag{3.10}$$

If we compose with the T-polynomial by Θ, in expression (3.10), (3.3) and (3.4) and (3.5) imply that

$$m(H\circ\Theta) - cm'(H\circ\Theta) = \sum_i c_i m_i(H\circ\Theta) \quad \text{(or zero)}. \tag{3.11}$$

Now, if we compose the inequality in (3.10) by $\widehat{\Theta}$, by (3.6) and (3.7) we have

$$\widehat{m_i(H\circ\Theta)} \prec \widehat{m(H\circ\Theta)} = \widehat{m'(H\circ\Theta)}, \quad \text{for all } i. \tag{3.12}$$

Notice that the leading words in the left-hand side of expression (3.11) cancel, so the constant c must be the same as in the definition of the T-polynomial of $(m(H), m'(H))$ with respect to $H\circ\Theta$. Therefore, expressions (3.11) and (3.12) are a representation in the sense of Proposition 3.8, and having in mind that the critical pair $(m(H), m'(H))$ of $H\circ\Theta$ was arbitrary, it follows that $H\circ\Theta$ is a `SAGBI` basis.

On the other hand, the proof of necessity of the compatibility condition requires of the two following results:

- ([Nor02, Lemma 3]) Let $u, v \in T$ be two terms with $u \neq v$ but $u\circ\widehat{\theta} = v\circ\widehat{\Theta}$. Then for every $w \succ u$, $H = \{u - w, v\}$ is a `SAGBI` basis.
- ([Nor02, Proposition 2]) If composition by Θ commutes with `SAGBI` bases computation, then composition by Θ is compatible with nonequality.

With these two preliminary results, let us rewrite the proof of the Proposition presented by Nordbeck [Nor02, Theorem 2 and Proposition 3].

Assume that composition by Θ commutes with `SAGBI` bases computation. If $u, v \in T$ are two terms with $u \succ v$, then we have to show that $u\circ\widehat{\Theta} \succ v\circ\widehat{\Theta}$. Since in particular $u \neq v$, we know from [Nor02, Proposition 2] that we cannot have $u\circ\widehat{\Theta} = v\circ\widehat{\Theta}$, so we need only exclude the case $u\circ\widehat{\Theta} \prec v\circ\widehat{\Theta}$.

We first claim that $H = \{u - v, v\}$ is a `SAGBI` basis. In fact, we saw above that every subset $H \subset \Bbbk[x_1, \ldots, x_n]$ consisting only of terms (or terms times coefficients) is a `SAGBI` basis, whence $H' = \{u, v\}$ is a `SAGBI` basis so, having [Nor02, Remark 1], our claim follows since $\Bbbk[H] = \Bbbk[H']$ and $\widehat{H} = \widehat{H'}$. We conclude that $H\circ\Theta = \{u\circ\Theta - v\circ\Theta, v\circ\Theta\}$ must be also a `SAGBI` basis.

Assume now that $u\circ\widehat{\Theta} \prec v\circ\widehat{\Theta}$. We then have $\widehat{H\circ\Theta} = \{v\circ\widehat{\Theta}\}$, and $u\circ\Theta = (u\circ\Theta - v\circ\Theta) + v\circ\Theta \in \Bbbk[H\circ\Theta]$. But since $u\circ\widehat{\Theta} \prec v\circ\widehat{\Theta}$, $u\circ\widehat{\Theta} \neq 1$ cannot be written as a power of $v\circ\widehat{\Theta}$, so $H\circ\Theta$ cannot be a `SAGBI` basis. Thus our assumption that $u\circ\widehat{\Theta} \prec v\circ\widehat{\Theta}$ was false, so composition by Θ is compatible with the ordering. □

REMARK 11 ([NOR02, REMARK 6]). Nordbeck used the assumption that $\theta_i \in \Bbbk$, for all i. Hong [Hon98] used instead the fact that in the definition of a Gröbner basis G, it is common to assume that zero polynomial is not allowed as a member of G. Notice that if do not allow zero in SAGBI bases, then if $\theta_i = c \in \Bbbk$ for some i, by using that $H = \{x_i - c\}$ is obviously a SAGBI basis, we assert that $H \circ \Theta = \{x_i \circ \Theta - c\} = \{\Theta_i - c\} = \{0\}$, which contradicts the commutation of SAGBI bases computation. This fact shows that it would have worked equally well to assume that SAGBI bases may not contain the zero polynomial.

In Sections 3.2.5 and 3.4.4 we will consider the problem of composition in the setting of free associative algebras and skew PBW extensions, respectively.

3.2 Free algebras

In his PhD Thesis [Nor01a], (see [Nor98, Nor02]), Nordbeck also studied SAGBI bases for subalgebras of the free associative algebra over a field $\Bbbk$. In this section, we recall his key ideas.

3.2.1 Basic definitions

As in Section 2.1, let $X = \{x_1, \dots, x_n\}$ be a finite alphabet, and $\Bbbk\{x_1, \dots, x_n\}$ the free associative algebra over $\Bbbk$. The symbol W means the set of all *words* in X including the empty word 1 (i.e. $W = X^*$, the free monoid generated by X). For H a subset of $\Bbbk\{x_1, \dots, x_n\}$, the subalgebra S of $\Bbbk\{x_1, \dots, x_n\}$ generated by H is denoted by $\Bbbk\{H\}$. In other words, *the elements of S are the polynomials in the set of formal and noncommutative indeterminates H viewed as elements of* $\Bbbk\{x_1, \dots, x_n\}$. It is assumed that the coefficient of such monomials is the identity 1. Notice that with this description, for Nordbeck the monomials are not general words appearing in $\Bbbk\{x_1, \dots, x_n\}$. As it is clear, the elements of $\Bbbk \subset S$ correspond to the constant polynomials.

Example 3.4 ([Nor98, Example 1]). Consider the subset $H := \{h_1 = yx^2 + 1,\ h_2 = 2xy - y,\ h_3 = yx\}$ of the free algebra $\Bbbk\{x, y\}$. An example of a monomial $m(H) \in \Bbbk\{X\}$ is given by $h_1 h_2 = (yx^2 + 1)(2xy - y) = 2yx^3y - yx^2y + 2xy - y$.

For Nordbeck [Nor98], an *admissible order* $\prec$ on a set W is a well-order preserving multiplication, that is, $f \prec g$ implies $hfk \prec hgk$ for all $f, g, h, k \in W$, such that the smallest word is the unity 1. It follows that every infinite sequence $u_1 \succ u_2 \succ \cdots \succ$ in W stabilizes. He used the order `deglex` (Definition 3.1(ii)), so we can, if terms with identical words are collected together using the operations over $\Bbbk$, with every non-zero element $f \in \Bbbk\{x_1, \dots, x_n\}$ associate its *leading word* $\mathrm{lw}(f)$, that is, the word in f that is larger (relative the order $\prec$) than every other word occurring in f. The *leading term* $\mathrm{lt}(f)$ of f is the leading word times its coefficient. For instance, in Example 3.4, using `deglex`, $\mathrm{lw}(m(H)) = yx^3y$ and $\mathrm{lt}(m(H)) = 2yx^3y$. As expected, for $H \subseteq \Bbbk\{X\}$, $\mathrm{lw}(H) := \{\mathrm{lw}(h) \mid h \in H\}$.

Definition 3.11 ([Nor98, Definition 1]). Let $H \subset \Bbbk\{x_1, \dots, x_n\}$, and let $\sum_{i=1}^{t} k_i m_i(H)$, $k_i \in \Bbbk$, $m_i \in \Bbbk\{H\}$, be a sum of monomials. The *height* of the sum is $\max\{\mathrm{lw}(m_i(H)) \mid 1 \le i \le t\}$, where the maximum is taken relative the order in $\Bbbk\{x_1, \dots, x_n\}$. The *breadth* of the sum is the number of i 's such that $\mathrm{lw}(m_i(H))$ is equal to the height.

As one can see, the leading word of $f = \sum_{i=1}^{t} k_i m_i(H)$ can be smaller than the height of $\sum_{i=1}^{t} k_i m_i(H)$. As a matter fact, this happens if and only if some words larger than $\mathrm{lw}(f)$ cancel in the sum, and the breadth of the sum is then necessarily at least two.

Next, we present the definition of `SAGBI` basis in the case of free associative algebras (c.f. Definition 3.1).

DEFINITION 3.12 ([NOR98, DEFINITION 2]). Let S be a subalgebra of $\Bbbk\{x_1,\ldots,x_n\}$. A subset $H \subset S$ is called a `SAGBI` basis for S if for every $f \in S$, $f \neq 0$, there exists a monomial $m \in \Bbbk\{H\}$ such that $\mathrm{lw}(f) = \mathrm{lw}(m(H))$.

By recalling that orders are preserved after multiplication, if $m(H) = h_{i_1} h_{i_2} \cdots h_{i_t}$ with $h_{ij} \in H$, then $\mathrm{lw}(m(H)) = \mathrm{lw}(h_{i_1})\mathrm{lw}(h_{i_2})\cdots\mathrm{lw}(h_{i_t})$. This means that an equivalent formulation of Definition 3.12 is that H is a `SAGBI` basis if the leading word of every non-zero element in S can be written as a product of leading words of elements in H. Notice that every subalgebra is a `SAGBI` basis for itself, so every subalgebra has a `SAGBI` basis. Also, if H consists only of words (or terms), then H is a `SAGBI` basis for the subalgebra S generated by H [Nor98, Example 2].

An important fact is that the `SAGBI` property depends on which order we consider, as the following example illustrates (c.f. Remark 8).

EXAMPLE 3.5 ([NOR98, EXAMPLE 3]). Let $H := \{h_1 = yz,\ h_2 = zy,\ h_3 = x - y\} \in \Bbbk\{x, y, z\}$, S the subalgebra generated by H, and consider the order `deglex` with $x \succ y \succ z$. Then $\mathrm{lw}(H) = \{yz, zy, x\}$, and it is easy to check (by using Proposition 3.13 below) that H is a `SAGBI` basis.

On the other hand, if we let $x \prec y \prec z$, then $\mathrm{lw}(H) = \{yz, zy, y\}$. However, $p(H) = h_1h_3 - h_3h_2 = yzx - xzy \in S$ and $\mathrm{lw}(p(H)) = yzx$ cannot be written as a product of words in $\mathrm{lw}(H)$. This means that H is not a `SAGBI` basis for S.

Next, we present the process of reduction in `SAGBI` bases (c.f. Section 3.1.2).

3.2.2 REDUCTION

REDUCTION 3.2 ([NOR98, P. 141]). The reduction of $f \in \Bbbk\{x_1,\ldots,x_n\}$ over a subset $H \subset \Bbbk\{x_1,\ldots,x_n\}$ is performed as follows:

(1) $f_0 = f$.

(2) If $f_i = 0$, or if there is no monomial $m \in \Bbbk\{H\}$ with $\mathrm{lw}(f_i) = \mathrm{lw}(m(H))$, then terminate. In case of termination this f_i will be referred as the result of the reduction.

(3) Find a monomial $m_i \in \Bbbk\{H\}$ and $k_i \in \Bbbk$ such that $\mathrm{lt}(f_i) = \mathrm{lt}(k_i m_i(H))$. (This is possible since we have not terminated in step (2)). Now let $s_{i+1} = s_i - k_i m_i(H)$.

(4) Go to step (2) $(i + 1 \mapsto i)$.

When step (3) has been performed, the leading word of s_{i+1} is strictly smaller than the leading word of s_i (by the choice of m_i and k_i). Now, since the order is well-founded, the reduction always terminates after a finite number of steps.

Notice that if H is a finite set, then it is a constructive matter to determine whether a given word is a product of elements in $\mathrm{lw}(H)$, and hence the reduction is algorithmic. This is also the case if H is infinite but sorted by e.g. the length of the leading words.

If the result (i.e., the last f_i) of a reduction of f over H is denoted $\overline{f^H}$, and if the reduction terminated after t iterations of step (3) above, then

$$f = \sum_{i=0}^{t-1} k_i m_i(H) + \overline{f^H}. \tag{3.13}$$

If $t = 0$ the right hand side of (3.13) is of course just $\overline{f^H}$. If step (3) is performed at least once we have, then

$$\mathrm{lw}(f) = \mathrm{lw}(m_0(H)) > \mathrm{lw}(m_1(H)) > \cdots > \mathrm{lw}(m_{t-1}(H)), \tag{3.14}$$

so the sum in the right hand side of (3.13) is clearly of breadth one and height equal to $\mathrm{lw}(f)$. Nordbeck used these facts several times (in particular when $\overline{f^H} = 0$).

Notice that there are several different possibilities to choose the m_i's in step (3), so the result of the reduction depends (in general) on how we choose these monomials.

It is interesting to consider the case when $\overline{s^H} = 0$ above. Nordbeck [Nor98, p. 142] said that f *reduces to zero weakly over* H if there exists one reduction (i. e., one choice of the m_i's) with $\overline{s^H} = 0$, and that s *reduces to zero strongly over* H if every reduction (every choice) yields $\overline{s^H} = 0$. As he asserted, in most cases it does not matter which formulation we use, and we will then simply say that s *reduces to zero over* H. By definition, $f = 0$ reduces to zero.

The following result is an application of `SAGBI` bases for the *Subalgebra Membership Problem*, that is, the decision whether an element $f \in \Bbbk\{x_1, \dots, x_n\}$ is in a given subalgebra (c.f. Proposition 3.1(C1)).

Proposition 3.10 ([Nor98, Proposition 1]). *Let $H \subset \Bbbk\{x_1, \dots, x_n\}$ be a* `SAGBI` *basis for the subalgebra S, and let $f \in \Bbbk\{x_1, \dots, x_n\}$. Then $f \in S$ if and only if f reduces to zero over H.*

As expected, if H is a `SAGBI` basis for the subalgebra S, then H generates S [Nor98, Corollary 1]. We will simply say that $H \subset \Bbbk\{x_1, \dots, x_n\}$ is a `SAGBI` basis meaning that H is a `SAGBI` basis for the subalgebra of $\Bbbk\{x_1, \dots, x_n\}$ generated by H.

Remark 12. Notice that an arbitrary element of $\Bbbk\{x_1, \dots, x_n\}$ has not in general a unique result of reduction over a `SAGBI` basis H. The uniqueness can be obtained by modifying step (2) above. Instead of terminating when the leading term no longer can be written as a product of $\mathrm{lw}(H)$, we move this term to some kind of remainder (generalizing $\overline{f^H}$) and continue with the other terms.

3.2.3 `SAGBI` basis criterion

We start with the following proposition that gives a first method to test the `SAGBI` basis property (c.f. Proposition 3.2).

Proposition 3.11 ([Nor98, Proposition 2]). *$H \subset S$ is a* `SAGBI` *basis for the subalgebra S if and only if every $f \in S$ reduces to zero over H.*

Definition 3.13 and Proposition 3.12 allow us to reduce the number of elements we need to consider for the SAGBI basis test. The definition of a T-polynomial is as expected from Definitions 3.2 and 3.7.

DEFINITION 3.13 ([Nor98, DEFINITION 3]). Let H be a subset of $\Bbbk\{x_1,\ldots,x_n\}$. A critical pair $(m(H), m'(H))$ of H is a pair of monomials $m(H), m'(H) \in \Bbbk\{x_1,\ldots,x_n\}$ with $\mathrm{lw}(m(H)) = \mathrm{lw}(m'(H))$. If $k \in \Bbbk^*$ is such that $\mathrm{lt}(m(H)) = \mathrm{lt}(km'(H))$ we define the *T-polynomial* of $(m(H), m'(H))$ as $T(m(H), m'(H)) = m(H) - km'(H)$.

As in Definition 3.7, the idea of the constant is that the leading words cancel in $T(m(H), m'(H))$, whence it follows that $\mathrm{lw}(T(m(H), m'(H))) \prec \mathrm{lw}(m(H)) = \mathrm{lw}(m'(H))$. Note that $T(m(H), m'(H)) = k'T(m(H), m'(H))$, for some $k' \in \Bbbk$, and that $T(m(H), m'(H))$ reduces to zero weakly (resp. strongly) if and only if $T(m(H), m'(H))$ does.

Proposition 3.12 is the analogue of Propositions 3.1(B1) and (C3), and 3.8.

PROPOSITION 3.12 ([Nor98, PROPOSITION 3]). *H is a* SAGBI *basis if and only if the T-polynomials of all critical pairs of H reduce to zero over H.*

The following result establishes sufficient conditions for a set to be a SAGBI basis.

PROPOSITION 3.13 ([Nor98, PROPOSITION 4]). *Let $H \subset \Bbbk\{x_1,\ldots,x_n\}$ be such that $\mathrm{lw}(h_i) \neq \mathrm{lw}(h_j)$ if $h_i \neq h_j$, $h_i, h_j \in H$. If either no word in $\mathrm{lw}(H)$ is a prefix (proper left factor) of some other word in $\mathrm{lw}(H)$, or no word in $\mathrm{lw}(H)$ is a suffix (proper right factor) of some other, then H is a* SAGBI *basis.*

With the aim of reducing the number of critical pairs to be considered for the SAGBI basis test, Nordbeck formulated another version of Proposition 3.12. Briefly, the idea is that in the proof of the sufficiency of that result, it was only used that every $T(m(H), m'(H))$ either was zero, or could be written as a sum of monomials of height less than $\mathrm{lw}(m(H)) = \mathrm{lw}(m'(H))$. Nordbeck said that, sometimes without mentioning the corresponding $m(H)$ and $m'(H)$, that such a T-polynomial *admits a low representation*. Since this property of the $T(m(H), m'(H))$'s is clear if H is a SAGBI basis (by reduction), then H is a SAGBI basis if and only if every T-polynomial of H admits a low representation [Nor98, Proposition 5].

PROPOSITION 3.14 ([Nor98, PROPOSITION 6]). *Let $(m(H), m'(H))$ be a critical pair of H, and assume that there are factorizations $m(H) = m_1(H)m_2(H)$, $m'(H) = m'_1(H)m'_2(H)$, where $m_1(H)$, $m_2(H)$, $m'_1(H)$, $m'_2(H)$ are monomials in $\Bbbk\{H\}$ with $\mathrm{lw}(m_i(H)) = \mathrm{lw}(m'_i(H))$, $i = 1,2$. If we have that $T(m_i(H), m'_i(H))$ admits a low representation for $i = 1,2$, then also $T(m(H), m'(H))$ admits a low representation.*

Of course, by induction we have that Proposition 3.14 holds for every number of factors, i.e., that for any factorizations $m = m_1\cdots m_t$ and $m' = m'_1\cdots m'_t$ with $\mathrm{lw}(m_i(H)) = \mathrm{lw}(m'_i(H))$ for all i, then the only critical pairs necessary for the SAGBI test are those which can not be factored in the sense above. Notice that this is the case only for the critical pairs of the form (h, h'), $h, h' \in H$ with $\mathrm{lw}(h) = \mathrm{lw}(h')$, and pairs (m, m'), $m = h_{i_1}\cdots h_{i_s}$, $m' = h'_{i_1}\cdots h'_{i_t}$, s or (and) $t > 1$, where the leading words of the factors overlap. Nordbeck called a critical pair with leading words of this form an *overlapping pair*. With the aim of finding all such pairs we can proceed in the following way.

IDEA 3.1 ([NOR98, P. 143]). "An overlapping pair must begin with two elements of H where one of the leading words is a prefix of the other. If these words are u_1 and $u_2 = u_1 v$ we get the *overlap* v. We must then find all possibilities to continue on v. This could be with a leading word equal to v (in which case we have obtained a critical pair), with a prefix of v, or finally with a word of which v is a prefix. In the two latter cases we get new overlaps that might be continued. If we get an overlap we have obtained before (starting with u_1, u_2 or a previous pair) we need of course not continue with this; we have already examined how it can be be continued. As mentioned before, we get critical pairs whenever a leading word fits on an overlap."

"If H is a finite set, then the process above is algorithmic. This rests on the fact that there only can be a finite number of different overlaps shorter than a given length. However, there may be an infinite number of overlapping pairs. This is the case if (and only if) the pair of leading words of some overlapping pair contains a segment beginning and ending with the same overlap. We will call such a segment a *loop*."

The following is one of the main aspects in SAGBI theory.

PROBLEM 3.1 ([NOR98, PROBLEM, P. 144]). For a finite set H, is the SAGBI basis test in general algorithmic?

Below, we will some answers to this question (c.f. Propositions 3.3 and 3.4).

PROPOSITION 3.15 ([NOR98, THEOREM 1]). *A subset $H \subseteq \Bbbk\{x_1, \ldots, x_n\}$ is a* SAGBI *basis if and only if the T-polynomials of all the necessary critical pairs describes above reduce to zero over H.*

3.2.4 SAGBI BASIS CONSTRUCTION

Nordbeck [Nor98] presented the `Algorithm 1` to compute SAGBI bases, where we know that H_∞ is a SAGBI basis for the subalgebra S generated by H [Nor98, Proposition 12]. As Nordbeck stated, "In general the algorithm will not stop, and we have seen in Section 3.2.3 that Step (2) may not be algorithms even for a finite set H. We conclude that the construction algorithm is mostly of theoretical value" [Nor98, p. 146].

Algorithm 1:

INPUT : A subset H of $\Bbbk\{x_1, \ldots, x_n\}$
OUTPUT: A SAGBI basis for the subalgebra S generated by H
INITIALIZATION: $H_0 = H$;

Use the methods in Section 3.2.3 to find the set M_i of all necessary critical pairs (m, m') of H_i.

$H_{i+1} = H_i \bigcup \{\overline{T(m(H), m'(H))} \mid (m(H), m'(H)) \in M_i, \overline{T(m(H), m'(H))} \neq 0\}$. Here $\overline{T(m(H), m'(H))}$ denotes a result of reduction over H_i.

If $H_{i+1} \neq H_i$ then go to Step (2) $(i + 1 \mapsto i)$.

$H_\infty = \bigcup H_i$

The following remark shows the importance of homogeneous elements in the computation of SAGBI bases.

REMARK 13 ([NOR98, P. 146]). If H is a set of homogeneous elements, then the situation is a bit more promising. It is then clear that all monomials $m(H)$ are homogeneous, and thus also all T-polynomials. Since reduction over H involves only homogeneous elements, every T-polynomial will reduce to a homogeneous element (or zero). It follows that H_1 must be homogeneous, and by induction every H_i (and H_∞).

If, moreover, H has only a finite number of elements of degree (word length) less than or equal to $d \in \mathbb{N}$ (and we can find all these elements), then we can algorithmically obtain a "partial" SAGBI basis $H_{(d)} = \{h \in H_\infty \mid \deg(h) \leq d\}$ as follows.

Performing Step (2) and (3) above we need only consider critical pairs of degree $\leq d$, because critical pairs of degree $> d$ can only reduce to new elements of degree $> d$ (or to zero). It is clear that the results $\overline{T(m(H), m'(H))} \neq 0$ of reduction over H_i all have leading words not lying in $\mathrm{lw}(H_i)$ (since the reduction algorithm terminated). Since there are only a finite number of different words of length $\leq d$, we must sooner or later get $\overline{T(m(H), m'(H))} = 0$ for all T-polynomials of degree $\leq d$ in Step (3), and the current H_i is then our requested $H_{(d)}$.

To perform the reduction of an arbitrary element of $\Bbbk\{x_1, \ldots, x_n\}$ over some subset, we need of course in each step of the reduction only know this subset up to a certain degree. We conclude that if H consists of homogeneous elements (with the finiteness condition above), then the *Subalgebra Membership Problem* (Proposition 3.10) is algorithmic in the sense of Problem 3.1.

3.2.5 SAGBI BASES UNDER COMPOSITION

In this section, we study the problem of polynomial composition of SAGBI bases in free associative algebras solved also by Nordbeck [Nor01a] (see also [Nor02, Section 5]). Note that in [Nor01b] he investigated this problem for noncommutative Gröbner bases.

Nordbeck [Nor02, Section 5] gave a sufficient and necessary condition on a set Θ of polynomials of $\Bbbk\{x_1, \ldots, x_n\}$ to assure that the set $F \circ \Theta$ of composed polynomials is a SAGBI basis whenever F is. He proved [Nor02, Theorem 4] that exactly the same results of the commutative setting (Proposition 3.9) hold in the noncommutative case, that is, composition by Θ commutes with noncommutative SAGBI bases computation if and only if the composition is compatible with the ordering. Just as we did in the commutative case, we will separate the proof into sufficiency and necessity of the condition of compatibility with the ordering.

PROPOSITION 3.16 ([NOR02, THEOREM 4]). *Composition by Θ commutes with noncommutative* SAGBI *bases computation if and only if the composition is compatible with the ordering.*

Sketch of the proof. With respect to the sufficiency of the compatibility condition, Nordbeck stated that "The reader can check that the proofs of [Nor02, Lemma 2] and [Nor02, Proposition 1] (see the beginning of the proof of Proposition 3.9) still work when H and the monomials are noncommutative" [Nor02, p. 75]. We have verified the details and will present them in the setting of SAGBI bases of skew PBW extensions in Section 3.4.4 (more exactly, Theorem 3.26).

With respect to the necessity of this condition, Nordbeck proved the following preliminary results:

- ([Nor02, Lemma 3]) Let $u, v \in T$ be two words with $u \neq v$ but $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$. Then, for every

$w \prec u$, the set $H = \{u - w, v\}$ is a `SAGBI` basis.

- ([Nor02, Proposition 2]) If composition by Θ commutes with `SAGBI` bases computation, then composition by Θ is compatible with nonequality.

Since that we will build our proposal in Section 3.4.4 following Nordbeck's ideas, we will leave the detailed proofs of these results for that section.

3.3 G-ALGEBRAS

G-algebras were introduced by Apel [Ape88] (c.f. Gómez-Torrecillas and Lobillo [GTL00] and Li [Li02]) without requiring the vanishing of the non–degeneracy conditions to guarantee the existence of a PBW basis as can be seen in Kandri-Rody and Weispfenning [KRW90] and Levandovskyy's PhD thesis [Lev05, Chapter 1, Section 2]. Some examples of G-algebras are quasi–commutative polynomial rings, universal enveloping algebras of finite dimensional Lie algebras, some iterated Ore extensions, many quantum groups, some nonstandard quantum deformations, Weyl algebras and most of various flavors of quantizations of Weyl algebras (like additive and multiplicative analogues of Weyl algebras), many important operator algebras, Witten's algebra, some of diffusion algebras and many more. Several algorithmic properties of these algebras have been investigated in [BHL17, LH17], and references therein.

Despite the similarities between the definitions of G-algebra (Definition 3.14) and skew PBW extension (Definition 1.1), Lezama and Reyes [LR14, Remark 3.1(ii)] showed that there are no inclusions between the classes of all G-algebras of a field $\Bbbk$ and skew PBW extensions over this field (of course, these two families share several examples of non-commutative algebras like those mentioned above). In this way, our purpose in this section is to recall briefly the theory presented by Khan et al. [KKB19] of `SAGBI` bases of G-algebras, and hence motivate our proposal of `SAGBI` bases theory for skew PBW extensions in Section 3.4.

Definition 3.14 ([Ape88], [Lev05, Section 6.4.33]). Let $\Bbbk\{x_1, \dots, x_n\}$ be the free associative $\Bbbk$-algebra generated by the indeterminates $x_1, \dots, x_n$ over $\Bbbk$. Let $c_{ij} \in \Bbbk^*$ and d_{ij} denote the standard polynomials in $\Bbbk\{x_1, \dots, x_n\}$, where $1 \le i < j \le n$. Let

$$A := \frac{\Bbbk\{x_1, \dots, x_n\}}{\langle x_j x_i = c_{ij} \cdot x_i x_j + d_{ij},\ 1 \le i < j \le n\rangle}.$$

A is said to be a G-*algebra* if the following conditions hold:

(i) There exists a *monomial ordering* (in the sense of Section 3.1) $\prec$ on the set

$$\mathrm{Mon}(A) := \left\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_k \ge 0, \text{for every } k\right\},$$

such that $\mathrm{lm}(d_{ij}) \prec x_i x_j$.

(ii) For all $1 \le i < j < k \le n$, the polynomial

$$c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk}$$

reduces to 0 with respect to the relations of A.

As a $\mathbb{k}$-algebra, A has a PBW basis if the set $\mathrm{Mon}(A)$ is a $\mathbb{k}$-basis of A.

DEFINITION 3.15 ([KKB19, DEFINITION 2]). Let A be a G-algebra in n indeterminates.

(i) As we know, $\mathrm{Mon}(A)$ forms a $\mathbb{k}$-basis of A, so any non-zero element f in A can be uniquely written as $f = c_\alpha x^\alpha + g$ with $c_\alpha \in \mathbb{k}^*$ and x^α a monomial (for any non-zero term $c_\beta x^\beta$ of g, $x^\beta \prec x^\alpha$). The monomial $x^\alpha \in \mathrm{Mon}(A)$ represents the leading monomial of f, denoted by $\mathrm{lm}(f)$. The element $c_\alpha \in \mathbb{k} \setminus \{0\}$ represents the leading coefficient of f, denoted by $\mathrm{lc}(f)$, and $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lm}(f)$.

(ii) Let $F \subseteq A$. The notation $\mathbb{k}\langle F\rangle_A$ means the subalgebra S of A generated by F. It is the polynomials set in the F-indeterminates in A.

(iii) For $F \subseteq A$, $m(F)$ denotes a monomial in terms of elements of F, we call it F-*monomial*. For $m(F) = f_{i1} f_{i2} \cdots f_{it}, f_{ij} \in F$, we define

$$\overline{\mathrm{lm}}\, m(F) = \mathrm{lm}(\mathrm{lm}(f_{i1})\mathrm{lm}(f_{i2}) \cdots \mathrm{lm}(f_{it})),$$

and

$$\overline{\mathrm{lt}}\, m(F) = \mathrm{lt}(\mathrm{lt}(f_{i1})\mathrm{lt}(f_{i2}) \cdots \mathrm{lt}(f_{it})).$$

Next, we recall the process of reduction together with the `SAGBI` *normal form* in G-algebras presented by Khan et al. [KKB19] (c.f. Proposition 3.5).

REDUCTION 3.3 ([KKB19, DEFINITION 3]). Let F and s be a subset and a polynomial in a G-algebra, respectively. If there exists an F-monomial $m(F)$ and $k \in \mathbb{k}$ satisfying $\overline{\mathrm{lt}}(km(F)) = \mathrm{lt}(s)$, then we say that

$$s_0 = s - km(F) \tag{3.15}$$

is a *one-step s-reduction of s with respect to F*. Otherwise, the s-reduction of s with respect to F is s itself.

If we apply the one-step s-reduction process iteratively, we can achieve a special form of s with respect to H (which cannot be s-reduced further with respect to F), called `SAGBI` *normal form*, and write it as, $s_0 := \mathtt{SNF}(s|F)$.

Khan et al. [KKB19] presented an algorithm which computes the output of `SNF`. It is important to say that for different choices of $km(F)$ in this algorithm, the output of `SNF` may also be different [KKB19, Remark 2].

EXAMPLE 3.6 ([KKB19], EXAMPLE 2). Let $A = \mathbb{Q}\langle e, f, h \mid fe = ef - h, he = eh + 2e, hf = fh - 2f\rangle$ (Section 1.2.2). Let S be a subalgebra of A generated by $F = \{q_1, q_2, q_3\} = \{e^2, f, fh + f\}$ and $g = e^2 fh + eh + f$, associated with `degrevlex` ordering. Using the Algorithm 1 presented by Khan et al. [KKB19] to compute $\mathtt{SNF}(g|F)$ we find the following results (first and second possible choice, respectively):

Turn	f_i	F_{f_i}	**Choose**	f_{i+1}
$i=0$	g	$\{q_1q_3, q_3q_1\}$	q_1q_3	$-e^2f+eh+f$
$i=1$	h_1	$\{q_1q_2, q_2q_1\}$	q_1q_2	$eh+f$
$i=2$	f_2	$\emptyset$		$\mathtt{SNF}(g\|F)=eh+f$

Turn	f_i	F_{f_i}	**Choose**	f_{i+1}
$i=0$	g	$\{q_1q_3, q_3q_1\}$	q_3q_1	$-5e^2f+2eh^2+13eh+10e+f$
$i=1$	f_1	$\{q_1q_2, q_2q_1\}$	q_1q_2	$2eh^2+13eh+10e+f$
$i=2$	f_2	$\emptyset$		$\mathtt{SNF}(g\|F)=2eh^2+13eh+10e+f$

Let S be a subalgebra of a G-algebra A and $F\subseteq S$. Khan et al. [KKB19] asks when SAGBI normal form $s_0=0$. If there is at least one choice of F-monomials such that $s_0=0$, then they said that s *reduces weakly over* F, and *reduces strongly* if all possible choices give $s_0=0$.

The next definition is the analogue of Definitions 3.1 and 3.12.

Definition 3.16 ([KKB19, Definition 4]). Let S be a subalgebra of a G-algebra A. A subset $F\subseteq S$ is called a SAGBI *basis for* S if for every non-zero element $s\in S$ there exists an F-monomial $m(F)\in \Bbbk\langle F\rangle_A$ such that $\mathrm{lm}(s)=\overline{\mathrm{lm}}(m(F))$.

The next result establishes that $s\in\Bbbk\langle F\rangle_A$ reduces strongly to $s_0=0$ if F is a SAGBI basis of S (c.f. Propositions 3.1(C1) and 3.10).

Proposition 3.17 ([KKB19, Proposition 2]). *Let S be a subalgebra of A and $F\subseteq S$. If F is a* SAGBI *basis of S, then the following conditions hold:*

(1) *For each $s\in A$, $s\in S$ if and only if* $\mathtt{SNF}(s|F)=0$.

(2) *F generates the subalgebra S, i.e., $S=\Bbbk\langle F\rangle_A$.*

For the computation of SAGBI bases in a G-algebra, Khan et al. proposed an algorithm and explored some necessary elements for its construction. Next, we state the notion of critical pair (c.f. Definitions 3.2 and 3.7).

Definition 3.17 ([KKB19, Definition 5]). Let $F\subseteq A$ and $m(F)$ and $m'(F)$ be F-monomials. The pair $(m(F),m'(F))$ is a *critical pair of* F if $\overline{\mathrm{lm}}(m(F))=\overline{\mathrm{lm}}(m'(F))$. The T-polynomial of critical pair is defined as $T(m(F),m'(F))=m(F)-km'(F)$, where $k\in\Bbbk$ satisfies $\overline{\mathrm{lt}}(m(F))=\overline{\mathrm{lt}}(m'(F))$.

Definition 3.18 is the analogous of Definition 3.11.

Definition 3.18 ([KKB19, Definition 6]). Let F be a set of polynomials in A and $S=\Bbbk\langle F\rangle_A$ be a subalgebra in A. We consider $P\in S$ with the representation $P=\sum_{i=1}^{t}k_im_i(F)$. Then the height of P with respect to this representation is defined as $\mathrm{ht}(P)=\max_{i=1}^{t}\{\overline{\mathrm{lm}}(m_i(F))\}$, where the maximum is taken with respect to term ordering in A (the height is defined for a specific representation of elements of A, not for the elements itself).

The next result is the most important theorem appearing in Khan et al. [KKB19]. This is the analogous to Proposition 3.12).

PROPOSITION 3.18 ([KKB19, THEOREM 1]). *Assume that F generates S as a subalgebra in A. Then F is a* `SAGBI` *basis of S if every T-polynomial of every critical pair of F gives zero* `SAGBI` *normal form.*

REMARK 14 ([KKB19, REMARK 4]). The necessary critical pairs used in `SAGBI` basis testing are those critical pairs $(m(F), m'(F))$ which cannot be factor as $m(F) = m_1(F)\cdots m_t(F)$, $m'(F) = m'_1(F)\cdots m'_t(F)$ with $\overline{\text{lm}}(m_i(F)) = \overline{\text{lm}}(m'_i(F))$, for all i. The T-polynomial induced by a necessary critical pair is called the necessary T-polynomial. Since G-algebras are finite factorization domains [BHL17, Theorem 1.3], therefore for any critical pair given by $(m(F), m'(F))$ (possibly not a necessary critical pair), the F-monomials $m(F)$ and $m'(F)$ have finite irreducible factors. The necessary critical pairs are formed by these irreducible factors, therefore the zero `SAGBI` normal form of T-polynomials induced by necessary critical pairs implies the `SAGBI` normal form of T-polynomial of a critical pair $(m(F), m'(F))$ are zero [Nor98, Proposition 6].

REMARK 15. To date, the polynomial composition problem for G-algebras has not been studied, neither in Gröbner bases nor in `SAGBI` bases. Of course, these pending tasks can be considered as near future work.

3.4 SKEW PBW EXTENSIONS

Gallego in her PhD Thesis [Gal15] (and related papers with Lezama [GL11, Gal16a]) developed the Gröbner basis theory for skew PBW extensions (see also Fajardo et al. [Faj18, Faj19, Faj22, FLP+24]), and then studied several homological properties of projective modules over these extensions [GL17, Gal16b]. Nevertheless, the problem of `SAGBI` bases for these objects has never been considered before. This fact motivates us to present a first approach toward a theory of `SAGBI` bases for skew PBW extensions over $\Bbbk$-algebras. We are interested in finding sufficient or necessary conditions to guarantee the existence of these bases (Section 3.4.3), and in the topic of bases under composition (Section 3.4.4).

We start by presenting some terminology used in this section.

3.4.1 BASIC DEFINITIONS

DEFINITION 3.19. Throughout this section, let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a $\Bbbk$-algebra R (that is, $\sigma_i(k) = k$ and $\delta_i(k) = 0$, for every $k \in \Bbbk$, and $1 \le i \le n$, as in Proposition 1.1), and F a finite set of non-zero elements of A. Let $\preceq$ be a monomial ordering on A in the sense of Definition 1.3(iv).

(i) The notation $\Bbbk\langle F\rangle_A$ means *the subalgebra of A generated by F*. The elements of $\Bbbk\langle F\rangle_A$ are precisely the polynomials in the set of formal intederminates F, viewed as elements of A. The elements of $\Bbbk \subset \Bbbk\langle F\rangle_A$ correspond to the constant polynomials.

(ii) *By an F-monomial we mean a finite product of elements from F* that will usually be written as $m(F)$ (the "empty" monomial 1 is considered). When we speak of the leading monomial, leading coefficient and leading term of an element in $\Bbbk\langle F\rangle_A$, we will always mean the leading monomial, leading coefficient and leading term, respectively, of the element viewed as an element of A, relative to the monomial ordering $\preceq$ in A.

(iii) $\operatorname{lm}(F) := \{\operatorname{lm}(f) \mid f \in F\}$. Since orders are preserved under multiplication, if $m(F) = f_{i_1} f_{i_2} \cdots f_{i_t}$, with $f_{i_j} \in F$, then

$$\operatorname{lt}(m(F)) = \operatorname{lt}\left(\operatorname{lt}\left(f_{i_1}\right)\operatorname{lt}\left(f_{i_2}\right)\cdots\operatorname{lt}\left(f_{i_t}\right)\right) \quad \text{and} \quad \operatorname{lm}(m(F)) = \operatorname{lm}\left(\operatorname{lm}\left(f_{i_1}\right)\operatorname{lm}\left(f_{i_2}\right)\cdots\operatorname{lm}\left(f_{i_t}\right)\right).$$

EXAMPLE 3.7. Consider the diffusion algebra (Definition 1.8) $\sigma(\mathbb{Q}[x_1, x_2, x_3])\langle D_1, D_2, D_3\rangle$, and let $F := \{f_1 := x_1x_2D_1D_2 + x_3D_1D_3, f_2 := D_2^2D_3^2\}$ be a subset of the algebra with the monomial ordering `deglex`, and $D_1 \succ D_2 \succ D_3$. An example of an F-monomial $m(F) \in \Bbbk\langle F\rangle_A$ is given by

$$\begin{aligned}
m(F) &= f_1f_2 = (x_1x_2D_1D_2 + x_3D_1D_3)D_2^2D_3^2\\
&= x_1x_2D_1D_2^3D_3^2 + x_3D_1(D_3D_2)(D_2D_3^2)\\
&= x_1x_2D_1D_2^3D_3^2 + x_3D_1(D_2D_3 + x_3D_2 - x_2D_3)(D_2D_3^2)\\
&= x_1x_2D_1D_2^3D_3^2 + x_3D_1D_2(D_3D_2)D_3^2 + x_3^2D_1D_2^2D_3^2 - x_2x_3D_1(D_3D_2)D_3^2\\
&= x_1x_2D_1D_2^3D_3^2 + x_3D_1D_2(D_2D_3 + x_3D_2 - x_2D_3)D_3^2 + x_3^2D_1D_2^2D_3^2\\
&\quad - x_2x_3D_1(D_2D_3 + x_3D_2 - x_2D_3)D_3^2\\
&= x_1x_2D_1D_2^3D_3^2 + x_2^2x_3D_1D_3^3 - x_2x_3^2D_1D_2D_3^2\\
&\quad - 2x_2x_3D_1D_2D_3^3 + 2x_3^2D_1D_2^2D_3^2 + x_3D_1D_2^2D_3^3.
\end{aligned}$$

EXAMPLE 3.8. For the algebra $U'(\mathfrak{so}_3)$ defined in Example 1.2.7, consider the subset $F = \{f_1 := I_1I_2 + I_3, f_2 := I_1I_3 + I_2\}$ of $U_q'(\mathfrak{so}_3)$ with the monomial ordering `deglex`, and $I_1 \succ I_2 \succ I_3$. An F-monomial $m(F) \in \Bbbk\langle F\rangle_A$ is the following:

$$\begin{aligned}
m(F) &= f_1f_2\\
&= (I_1I_2 + I_3)(I_1I_3 + I_2)\\
&= I_1(I_2I_1)I_3 + I_1I_2^2 + (I_3I_1)I_3 + I_3I_2\\
&= I_1(qI_1I_2 - q^{1/2}I_3)I_3 + I_1I_2^2 + I_3(q^{-1}I_1I_3 + q^{-1/2}I_2) + qI_2I_3 - q^{-1/2}I_1\\
&= qI_1^2I_2I_3 - q^{-1/2}I_1I_3^2 + I_1I_2^2 + q^{-1}(q^{-1}I_1I_3 + q^{-1/2}I_2)I_3\\
&\quad + q^{-1/2}(qI_2I_3 - q^{1/2}I_1) + qI_2I_3 - q^{-1/2}I_1\\
&= qI_1^2I_2I_3 + I_1I_2^2 + (q^{-2} - q^{-1/2})I_1I_3^2 + (q^{-3/2} + q^{1/2})I_2I_3\\
&\quad + qI_1I_3 - (1 + q^{-1/2})I_1
\end{aligned}$$

Notice that $\operatorname{lm}(f_1)\operatorname{lm}(f_2) = (I_1I_2)(I_1I_3) = I_1(qI_1I_2 - q^{1/2}I_3)I_3 = qI_1^2I_2I_3 - q^{1/2}I_1I_3^2$, and $\operatorname{lm}\left(\operatorname{lm}\left(f_1\right)\operatorname{lm}\left(f_2\right)\right) = qI_1^2I_2I_3 = \operatorname{lm}(m(F))$.

EXAMPLE 3.9. Let $U(\mathfrak{osp}(1,2))$ be the algebra generated by the indeterminates x, y, z subject to the relations $yz - zy = z$, $zx + xz = y$, and $xy - yx = x$. $U(\mathfrak{osp}(1,2))$ corresponds to the universal enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1,2)$ (see Sections 1.2.2 and 1.2.3). From [LR14, p. 1215], $U(\mathfrak{osp}(1,2)) \cong \sigma(\Bbbk)\langle x, y, z\rangle$. Consider the subset $F = \{f_1 = x^2y, f_2 = xy + z\}$ of $U(\mathfrak{osp}(1,2))$ with the monomial ordering `deglex`, and $x \succ y \succ z$. An example of an F-monomial $m(F) \in \Bbbk\langle F\rangle_A$

can be

$$\begin{aligned}
m(F) = f_2 f_1 &= (xy+z)x^2y \\
&= x(yx)xy + (zx)xy \\
&= x(xy-x)xy + (-xz+y)xy \\
&= x^2(yx)y - x^3y - x(zx)y + (yx)y \\
&= x^2(xy-x)y - x^3y - x(-xz+y)y + (xy-x)y \\
&= x^3y^2 - 2x^3y + x^2(zy) - xy \\
&= x^3y^2 - 2x^3y + x^2(yz-z) - xy \\
&= x^3y^2 - 2x^3y + x^2yz - x^2z - xy.
\end{aligned}$$

Note that $\operatorname{lm}(f_1)\operatorname{lm}(f_2) = (xy)(x^2y) = x(xy-x)xy = x^2(yx)y - x^3y = x^2(xy-x) - x^3y = x^3y^2 - 2x^3y$, and $\operatorname{lm}(m(F)) = \operatorname{lm}\left(\operatorname{lm}\left(f_1\right)\operatorname{lm}\left(f_2\right)\right) = x^3y^2$.

Example 3.10. In Example 3.7, $\operatorname{lm}(f_1)\operatorname{lm}(f_2) = (D_1D_2)(D_2^2D_3^2) = D_1D_2^3D_3^2$, and $\operatorname{lm}(m(F))$ is equal to $\operatorname{lm}\left(\operatorname{lm}\left(f_1\right)\operatorname{lm}\left(f_2\right)\right) = D_1D_2^3D_3^2$.

3.4.2 Reduction and Algorithm for the SAGBI normal form

Next, we present the notion of *reduction* which is necessary in the characterization of `SAGBI` bases (c.f. Reductions 3.1, 3.2 and 3.3).

Reduction 3.4. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a $\Bbbk$-algebra R, F a finite set of non-zero elements of A, and s, s_0 elements of A. We say that s_0 is a *one-step s-reduction of s* with respect to F if there exist an F-monomial $m(F)$ and $k \in \Bbbk$ such that the following two conditions hold:

(i) $k\operatorname{lt}(m(F)) = \operatorname{lt}(s)$, and

(ii) $s_0 = s - km(F)$.

If the first condition of Reduction 3.4 fails, then the s-reduction of s with respect to F is s. When we apply the one-step s-reduction process iteratively, we obtain a special form of s with respect to F (which cannot be s-reduced further with respect to F), called `SAGBI` normal form (c.f. Proposition 3.5). In this case, we write $s_0 := \mathtt{SNF}(s|F)$ (c.f. Proposition 3.5 and Reduction 3.3). `Algorithm` 2 allows us to compute the `SNF`.

Algorithm 2:

INPUT :A fixed monomial ordering $\succeq$ on $\operatorname{Mon}(A)$, $F \subseteq A$ and $s \in A$.
OUTPUT: $h \in A$, the `SAGBI` normal form
INITIALIZATION: $s_0 := s$; $F_{s_0} := \{km(F) \mid k \in \Bbbk \text{ and } k\operatorname{lt}(m(F)) = \operatorname{lt}(s_0)\}$;

while $s_0 \neq 0$ and $F_{s_0} \neq \emptyset$ **do**
  choose $km(F) \in F_{s_0}$; $s_0 := s_0 - km(F)$; $F_{s_0} := \{km(F) \mid k \in \Bbbk \text{ and } k\operatorname{lt}(m(F)) = \operatorname{lt}(s_0)\}$;
return s_0;

Notice that for different choices of $km(F)$, the output of SNF may also be different. Examples 3.11, 3.15 and 3.13 illustrate this situation.

EXAMPLE 3.11. Consider the Jordan plane (Example 1.1). Let $\Bbbk\langle F\rangle_A$ be the subalgebra generated by the set $F := \{q_1, q_2, q_3\}$, where $q_1 := x^2, q_2 := y, q_3 := xy + y$, and $g = x^3y + y \in \mathcal{J}$. For the computation of $\mathtt{SNF}(g|F)$, we use `Algorithm` 2.

Let $g = x^3y + y := s_0$. If $F_{s_0} = \{q_1q_3, q_3q_1\}$, then we have two possibilities to choose $km(F) \in F_{s_0}$. Let us see them:

(i) If $km(F) := q_1q_3$, then $s_0 := -x^2y + y$ and $F_{s_0} := \{-q_1q_2, -q_2q_1\}$. Now, we take $km(F) = -q_2q_1$, which implies $s_0 := y \in F$, whence $\mathtt{SNF}(g|F) = 0$.

(ii) If $km(F) := q_3q_1$, then $s_0 := -2x^4 - x^2y - 2x^3 + y$ and $F_{s_0} := \{-2(q_1)^2\}$, and so $km(F) = -2(q_1)^2$, whence $s_0 := -x^2 - 2x^3 + y$, which implies that $F_{s_0} := \{-q_1q_2, -q_2q_1\}$. In this way, $km(F) = -q_1q_2$, and we obtain $s_0 := -2x^3 + y$ and $F_{s_0} = \emptyset$. Thus, $\mathtt{SNF}(g|F) = -2x^3 + y$.

EXAMPLE 3.12. Consider the *Askey-Wilson algebra* $AW(3)$ (Section 1.2.7) with the monomial ordering `deglex` and $K_0 \succ K_1 \succ K_2$. Let $\Bbbk\langle F\rangle_A$ be the subalgebra generated by the set $F := \{q_1, q_2, q_3\}$, where $q_1 := K_0K_1 + K_2,\ q_2 := K_0,\ q_3 := K_0K_2$, and $g = K_0^2K_1K_2 + K_2 \in AW(3)$. Let us find $\mathtt{SNF}(g|F)$ by using `Algorithm` 2.

Let $s_0 := g = K_0^2K_1K_2 + K_2$. If $F_{s_0} = \{e^{-2\omega}q_1q_3, q_3q_1\}$, then we have two options to choose $km(F) \in F_{s_0}$:

(i) If

$$\begin{aligned} km(F) &:= e^{-2\omega}q_1q_3 \\ &= e^{-2\omega}(e^{2\omega}K_0^2K_1K_2 + \left(e^{-2\omega} - e^{\omega}\right)K_0K_2^2 + e^{-\omega}BK_0K_2 + e^{-\omega}C_1K_1K_2 + e^{-\omega}D_1K_2), \end{aligned}$$

then

$$s_0 := \left(e^{-\omega} - e^{-4\omega}\right)K_0K_2^2 - e^{-3\omega}BK_0K_2 - e^{-3\omega}C_1K_1K_2 + \left(1 - e^{-3w}D_1\right)K_2,$$

and $F_{s_0} = \emptyset$. Thus,

$$\mathtt{SNF}(g|F) = \left(e^{-\omega} - e^{-4\omega}\right)K_0K_2^2 - e^{-3\omega}BK_0K_2 - e^{-3\omega}C_1K_1K_2 + \left(1 - e^{-3w}D_1\right)K_2.$$

(ii) If

$$\begin{aligned} km(F) &:= q_3q_1 \\ &= K_0^2K_1K_2 - e^{-\omega}C_0K_0^3 + e^{-\omega}C_1K_0K_1^2 + K_0K_2^2 - e^{-\omega}D_0K_0^2 + e^{-\omega}D_1K_0K_1, \end{aligned}$$

then

$$s_0 := e^{-\omega}C_0K_0^3 - e^{-\omega}C_1K_0K_1^2 - K_0K_2^2 + e^{-\omega}D_0K_0^2 - e^{-\omega}D_1K_0K_1 + K_2,$$

and $F_{s_0} := \{e^{-\omega}C_0q_2^3\}$. In this way, $km(F) = e^{-\omega}C_0q_2^3$, whence

$$s_0 := -e^{-\omega}C_1K_0K_1^2 - K_0K_2^2 + e^{-\omega}D_0K_0^2 - e^{-\omega}D_1K_0K_1 + K_2,$$

which implies that $F_{s_0} = \emptyset$. We conclude that

$$\texttt{SNF}(g|F) = -e^{-\omega}C_1K_0K_1^2 - K_0K_2^2 + e^{-\omega}D_0K_0^2 - e^{-\omega}D_1K_0K_1 + K_2.$$

EXAMPLE 3.13. Consider the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ (Section 1.2.7) with the monomial ordering $\texttt{deglex}$, and $E \succ F \succ K \succ K^{-1}$. Let $\Bbbk\langle \mathscr{F}\rangle_A$ be the subalgebra generated by the set $\mathscr{F} := \{q_1, q_2, q_3\}$, with $q_1 := EK - FK^{-1}$, $q_2 := EF$, $q_3 := FK^{-1}$, and $g = E^2FK + EFK^2 \in A$. With the aim of finding $\texttt{SNF}(g|\mathscr{F})$, let $g = K_0^2K_1K_2 + K_2 := s_0$. If $F_{s_0} = \{q_1q_2, q_2q_1\}$, then:

(i) If $km(\mathscr{F}) := q_1q_2 = E^2FK - EF^2K^{-1} + \frac{F-FK^{-2}}{q-q^{-1}}$, then $s_0 := EF^2K^{-1} + EFK^2 - \frac{F-FK^{-2}}{q-q^{-1}}$ and $F_{s_0} = \{q_2q_3, q_3q_2\}$. Now, if $km(\mathscr{F}) = q_2q_3 = EF^2K^{-1}$, we get $s_0 := EFK^2 - \frac{F-FK^{-2}}{q-q^{-1}}$, which implies $F_{s_0} = \emptyset$. Thus, $\texttt{SNF}(g|\mathscr{F}) = EFK^2 - \frac{F-FK^{-2}}{q-q^{-1}}$. On the other hand, if $km(\mathscr{F}) = q_3q_2 = EF^2K^{-1} - \frac{F-FK^{-2}}{q-q^{-1}}$, it follows that $s_0 := EFK^2$ which implies that $F_{s_0} = \emptyset$. Thus, $\texttt{SNF}(g|\mathscr{F}) = EFK^2$.

(ii) Let $km(\mathscr{F}) := q_2q_1 = E^2FK - EF^2K^{-1} - \frac{EK^2-E}{q-q^{-1}}$. We have $s_0 := EF^2K^{-1} + EFK^2 + \frac{EK^2-E}{q-q^{-1}}$ and $F_{s_0} = \{q_2q_3, q_3q_2\}$. If $km(\mathscr{F}) = q_2q_3 = EF^2K^{-1}$, we get $s_0 := EFK^2 + \frac{EK^2-E}{q-q^{-1}}$, whence $F_{s_0} = \emptyset$. This implies that $\texttt{SNF}(g|\mathscr{F}) = EFK^2 - \frac{EK^2-E}{q-q^{-1}}$. Now, considering $km(\mathscr{F}) = q_3q_2 = EF^2K^{-1} - \frac{F-FK^{-2}}{q-q^{-1}}$, it follows that $s_0 := EFK^2 + \frac{EK^2-E}{q-q^{-1}} + \frac{F-FK^{-2}}{q-q^{-1}}$, and so $F_{s_0} = \emptyset$. Hence, $\texttt{SNF}(g|\mathscr{F}) = EFK^2 + \frac{EK^2-E}{q-q^{-1}} + \frac{F-FK^{-2}}{q-q^{-1}}$.

If S is a subalgebra of a skew PBW extension A and F is a subset of S, then our interest lies in the case when $\texttt{SAGBI}$ normal form is zero. If there is at least one choice of F-monomials such that $s_0 = 0$, then we say s *reduces weakly over F*, and *reduces strongly* if all possible choices give $s_0 = 0$. This is the same as the case of G-algebras.

EXAMPLE 3.14. The reduction process done in Example 3.11 shows us that $g = x^3y + y \in \mathscr{J}$ reduces weakly over F.

EXAMPLE 3.15. Consider the 3-dimensional skew polynomial algebra generated by the indeterminates x, y, z restricted to relations $yz - zy = z$, $zx + xz = 0$, and $xy - yx = x$ (Proposition 1.5(2)(v)). Let $\Bbbk\langle F\rangle_A$ be the subalgebra generated by the set $F := \{q_1, q_2, q_3, q_4\}$, where $q_1 := x^2, q_2 := z, q_3 = xy + z, q_4 = x^3$ with the monomial ordering $\texttt{deglex}$, $x \succ y \succ z$, and consider the element $s = x^3y + z \in A$. Let $s = x^3y + z := s_0$. If $F_{s_0} = \{q_1q_3, q_3q_1\}$, then we have two options to choose $km(F) \in F_{s_0}$:

(i) If $km(F) := q_1q_3$, then $s_0 := -x^2z + z$ and $F_{s_0} := \{-q_1q_2 = -q_2q_1\}$. Now, we take $km(F) = -q_2q_1$, whence $s_0 := z \in F$, whence $\texttt{SNF}(s|F) = 0$.

(ii) If $km(F) := q_3q_1$, then $s_0 := 2x^3 + x^2z + z$ and $F_{s_0} = \{2q_4\}$. By taking $km(F) = 2q_4$, we get $s_0 := x^2z + z$ and $F_{s_0} := \{q_1q_2 = q_2q_1\}$. If $km(F) = q_2q_1$, then $s_0 := z \in F$ and $\texttt{SNF}(s|F) = 0$.

These facts show that s reduces strongly over F.

The next definition is completely analogous to Definitions 3.1, 3.12, and 3.16.

DEFINITION 3.20. Let S be a subalgebra of a skew PBW extension A. A subset $F \subseteq S$ is called a SAGBI *basis* for S if for all non-zero element $s \in S$, there exists an F-monomial $m(F)$ in $\Bbbk\langle F\rangle_A$ such that $\mathrm{lm}(s) = \mathrm{lm}(m(F))$.

As occurs in the setting of free algebras, since monomial orderings are compatible with the multiplication, an equivalent formulation of the definition is that F is a SAGBI basis if the leading term of every non-zero element in S can be written as a product of leading terms of elements in F.

EXAMPLE 3.16. Consider A as in the Example 3.15 and let $\Bbbk\langle F\rangle_A$ be the subalgebra generated by the set $F = \{q_1 = zy, q_2 = -zy + x, q_3 = -y + x\}$. If we use deglex with $z > y > x$ we have $\mathrm{lm}(F) = \{zy, y\}$. But $f = q_1q_3 - q_3q_2 = 2zyx - z^2 + y^2 + yx - x^2 - z \in S$, and $\mathrm{lm}(f) = zyx$ cannot be written as a product of terms in $\mathrm{lm}(F)$. Thus, F is not a SAGBI basis.

The following result establishes that when F is a SAGBI basis of S, then $s \in \Bbbk\langle F\rangle_A$ reduces strongly to $s_0 = 0$. This is the corresponding version of Propositions 3.1(C1), 3.10, and 3.17.

PROPOSITION 3.19. *Let S be subalgebra of a skew PBW extension A and $F \subseteq S$. If F is a* SAGBI *basis of S, then the following assertions hold:*

(1) *For each $s \in A$, $s \in S$ if and only* $\mathtt{SNF}(s|F) = 0$.

(2) *F generates the subalgebra S, i.e., $S = \Bbbk\langle F\rangle_A$.*

Proof. (1) If $\mathtt{SNF}(s|F) = 0$, then $s = \sum_{i=1}^{l} k_i m_i(F)$, where $k_i \in \Bbbk$, and hence $s \in S$.

Conversely, suppose that $s \in S$ and $\mathtt{SNF}(s|F) \neq 0$. This means that it cannot be reduced further, i.e., $\mathrm{lm}(\mathtt{SNF}(s|F)) \neq \mathrm{lm}(m(F))$, for any F -monomial $m(F)$, and this contradicts that F is a SAGBI basis.

(2) It follows from Proposition 3.19(1). More exactly, $s \in S$ if and only if $\mathtt{SNF}(s|F) = 0$, that is, $s = \sum_{i=1}^{l} k_i m_i(F)$, with $k_i \in \Bbbk$. Therefore $s \in \Bbbk\langle F\rangle_A$, which shows that $S = \Bbbk\langle F\rangle_A$.

□

3.4.3 SAGBI BASES CRITERION

In this section, we present an algorithm to calculate SAGBI basis in skew PBW extensions. Once more again, we assume that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a $\Bbbk$-algebra R.

As in the previous sections, below we present the definition of critical pair (c.f. Definitions 3.2, 3.7, and 3.17).

DEFINITION 3.21. Let F be a subset of a skew PBW extension A, and $m_1(F)$, $m_2(F)$ two F-monomials. The pair given by $(m_1(F), m_2(F))$ is called a *critical pair* of F if $\mathrm{lm}(m_1(F)) = \mathrm{lm}(m_2(F))$. The *$T$-polynomial of a critical pair* $(m_1(F), m_2(F))$ is defined as $T(m_1(F), m_2(F)) = m_1(F) - km_2(F)$, where $k \in \Bbbk$ is such that $\mathrm{lt}(m_1(F)) = k\,\mathrm{lt}(m_2(F))$.

The following definition is precisely our adaptation of Definition 3.11.

DEFINITION 3.22. Let F be a subset of a skew PBW extension A, and $\Bbbk\langle F\rangle_A$ the subalgebra generated by F in A. Let $P = \sum_{i=1}^{t} k_i m_i(F) \in \Bbbk\langle F\rangle_A$. The *height* of P is defined as $\mathrm{ht}(P) = \max_{i=1}^{t}\{\mathrm{lm}(m_i(F))\}$, where the maximum is taken with respect to one monomial ordering $\preceq$ in A.

The height is defined for a specific representation of elements of A, not for the elements itself.

The following result is one of the most important of the chapter. This is the analogous version of Propositions 3.8 and 3.18 in the case of free algebras and G-algebras, respectively.

THEOREM 3.20. *Suppose that F generates S as a subalgebra in the skew PBW extension A. Then F is a* `SAGBI` *basis of S if an only if every T-polynomial of every critical pair of F gives zero* `SAGBI` *normal form.*

Proof. If H is a `SAGBI` basis of S then every T-polynomial is an element of $S = \Bbbk\langle F\rangle_A$ and its `SAGBI` normal form is equal to zero by part (1) of Proposition 3.19.

Conversely, suppose given $0 \neq s \in S$. It is sufficient to prove that it has a representation $s = \sum_{p=1}^{t} k_p m_p(F)$, where $k_p \in \Bbbk$ and $m_p(F) \in \Bbbk\langle F\rangle_A$ with $\mathrm{lm}(s) = \mathrm{ht}\left(\sum_{p=1}^{t} k_p m_p(F)\right) = \mathrm{lm}(m_i(F))$. Let $s \in S$ given by $s = \sum_{p=1}^{t} k_p m_p(F)$ with smallest possible height X among all possible representations of s in S, that is, $X = \max_{p=1}^{t}\left\{\mathrm{lm}\left(m_p(F)\right)\right\}$. It is clear that $X \succeq \mathrm{lm}(s)$.

Suppose $X \succ \mathrm{lm}(s)$ i.e., cancellation of terms occur then there exist at least two F-monomials such that their leading monomial is equal to X. Assume we have only two F-monomials $m_i(F), m_j(F)$ in the representation $s = \sum_{p=1}^{t} k_p m_p(F)$ such that $\mathrm{lm}(m_i(F)) = \mathrm{lm}\left(m_j(F)\right) = X$. If $T\left(m_i(F), m_j(F)\right) = m_i(F) - k m_j(F)$, we can write

$$s = \sum_{p=1}^{t} k_p m_p(F) \tag{3.16}$$

$$= k_i\left(m_i(F) - k m_j(F)\right) + \left(k_j + k_i k\right) m_j(F) + \sum_{p=1, p\neq i,j}^{t} k_p m_p(F) \tag{3.17}$$

$$= k_i T\left(m_i(F), m_j(F)\right) + \left(k_j + k_i k\right) m_j(F) + \sum_{p=1, p\neq i,j}^{t} k_p m_p(F). \tag{3.18}$$

Since $T\left(m_i(F), m_j(F)\right)$ has a zero `SAGBI` normal form, then this T-polynomial is either zero or can be written as sum of F-monomials of height $\mathrm{lm}\left(T\left(m_i(F), m_j(F)\right)\right)$ which is less than X. If $k_j + k_i k$ is equal to zero, then the right-hand side of expression (3.18) is a representation of s with height less than X, which contradicts our initial assumption that we have chosen a representation of s with smallest possible height. Otherwise, the height is preserved, but on the right-hand side of expression (3.18), we have only one F-monomial $m_j(F)$ such that $\mathrm{lm}\left(m_j(F)\right) = X$, which is a contradiction as at least two F-monomials of such type must exist in the representation of s. $\square$

Remark 16. From Theorem 3.20, it follows that every subset $H \subset A$ consisting only of terms or monomials is a `SAGBI` basis since every T-polynomial is clearly equal to zero.

As we can see the elements to consider for the test of the basis `SAGBI` are many and sometimes unlimited. For the case of free associative algebras over the arbitrary field $\Bbbk$, Nordbeck in [Nor98] gives special conditions on the critical pairs to be taken into account for the construction of a SAGBI basis, reducing the number of such pairs, following we show two examples of skew PBW extensions where we show that this construction is not valid in our object of study. From the above, a new question arises: What conditions should we impose so that in the construction of a `SAGBI` basis the critical pairs to be taken into account are reduced and thus be able to build a finite algorithm?

The following two examples show that the assertion [Nor98, Proposition 6] for free algebras does not hold for skew PBW extensions.

Example 3.17. Consider the Dispin algebra $U(\mathfrak{osp}(1,2))$ (Proposition 1.5 (2)(i)), the set F given by $F = \{xy, yz, xz, z^2, xy + xz, yz + z^2\}$ and $S = \Bbbk\langle F\rangle_A$. If we use the monomial ordering `deglex` with $x \succ y \succ z$, then we get the critical pair (m, m') of F, where

$$\begin{aligned} m &= (xy + xz)yz = xy^2z + xyz^2 - xz^2, \quad \text{and} \\ m' &= xy(yz + z^2) = xy^2z + xyz^2. \end{aligned}$$

Notice that $m = m_1m_2$ and $m' = m'_1m'_2$, with $\mathrm{lm}(m_i(F)) = \mathrm{lm}(m'_i(F))$, $i = 1,2$. It follows that $T(m_1(F), m'_1(F)) = xz$ is a monomial of S with height less than $\mathrm{lm}(m_1(F)) = xy$, and $T(m_2(F), m'_2(F)) = z^2$ is a monomial of S with height less than $\mathrm{lm}(m_2(F)) = yz$. Nevertheless, the T-polynomial $T(m(F), m'(F)) = -xz^2$ is not a sum of monomials of S, which shows that [Nor98, Proposition 6] does not hold in the setting of skew PBW extensions.

Example 3.18. Consider the *Sklyanin algebra* $\mathscr{S} = \Bbbk\{x, y, z\}/\langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2\rangle$, where $a, b, c \in \Bbbk$ ([Rog16, Example 1.14]). If $c = 0$ and $a, b \neq 0$ then we obtain the defining relations $yx = -\frac{b}{a}xy$; $zx = -\frac{a}{b}xz$ and $zy = -\frac{b}{a}yz$. It can be seen by using [RS17a, Theorem 1.14] that $\mathscr{S} \cong \sigma(\Bbbk)\langle x, y, z\rangle$. Let the set $F \subseteq \mathscr{S}$ given by

$$F = \{yz + z^2, yz + z, z^2 - z\}$$

and $S = \Bbbk\langle F\rangle_{\mathscr{S}}$. If we use the monomial ordering `deglex` with $x \succ y \succ z$, then let (m, m') be a critical pair of F, with

$$m = (yz + z^2)(yz + z^2) = -\frac{b}{a}y^2z^2 + yz^3 + \frac{b^2}{a^2}yz^3 + z^4 \tag{3.19}$$

$$m' = (yz + z)(yz + z) = -\frac{b}{a}y^2z^2 + yz^2 - \frac{b}{a}yz^2 + z^2. \tag{3.20}$$

Note that $m = m_1m_2$ and $m' = m'_1m'_2$, where $\mathrm{lm}(m_i(F)) = \mathrm{lm}(m'_i(F))$, for $i = 1,2$, and $T(m_1(F), m'_1(F)) = z^2 - z$ is a monomial of S with height less than $\mathrm{lm}(m_1(F)) = yz$; similarly, $T(m_2(F), m'_2(F)) = z^2 - z$ is a monomial of S with height less than $\mathrm{lm}(m_2(F)) = yz$. However,

$T(m(F), m'(F)) = \left(1 + \dfrac{b^2}{a^2}\right) yz^3 + z^4 + \left(\dfrac{b}{a} - 1\right) yz^2 - z^2$ is not a sum of monomials of S. Once more again, [Nor98, Proposition 6] does not hold.

3.4.4 SAGBI BASES UNDER COMPOSITION

In this section, we present the original results about SAGBI bases under composition in the setting of skew PBW extensions. We will consider similar definitions to those corresponding presented in Sections 3.1.5, 3.2.5, and the notation used in Section 3.4.1. More exactly:

(1) Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension over R and F a finite set of non-zero elements of A.

(2) $\Bbbk\langle F\rangle_A$ means *the subalgebra of A generated by F*, that is, the polynomial set in the F-variables in A.

(3) $m(F)$ denotes a monomial in terms of the elements of F, and we call it an *F-monomial.*

(4) When we speak of the leading monomial, leading coefficient and leading term of an element in $\Bbbk\langle F\rangle_A$, we will always mean the leading monomial, leading coefficient and leading term, respectively, of the element viewed as an element of A, relative to the fix monomial ordering in $\mathrm{Mon}(A)$. $\mathrm{lm}(F) := \{\mathrm{lm}(f) \mid f \in F\}$.

(5) Let S be a subalgebra of A. A subset $H \subset S$ is called a SAGBI basis for S if for every non-zero $s \in S$, there exists a monomial m such that $\widehat{s} = \widehat{m(H)}$.

(6) Since monomial orderings are preserved by multiplication,

$$m(H) = \prod_i h_i \ (h_i \in H) \text{ implies } \widehat{m(H)} = \prod_i \widehat{h_i}$$

that is,

$$\widehat{m(H)} = m(\widehat{H}). \tag{3.21}$$

This means that an equivalent formulation of Definition 3.6 is that H is a SAGBI basis if the leading term of every non-zero element in S can be written as a product of leading terms of elements in H [Nor02, Remark 1]. It can be seen that if H is a SAGBI basis for S, then H generates S, that is, $S = \Bbbk\langle H\rangle_A$. We say that H is a SAGBI basis meaning that H is a SAGBI basis for $\Bbbk\langle H\rangle_A$.

(7) We will say that two monomials m, m' form a *critical pair* (m, m') of H if $\widehat{m(H)} = \widehat{m'(H)}$. If $c \in \Bbbk$ is such that $m(H)$ and $cm'(H)$ have the same leading coefficient, then we define the *T-polynomial* of (m, m') as $T(m, m') = m(H) - cm'(H)$. [Nor02, Definition 2]. The idea with the constant c is that the leading terms cancel in $T(m, m')$, whence $\widehat{T(m, m')} < \widehat{m(H)} = \widehat{m'(H)}$.

(8) Every subset $H \subset A$ consisting only of terms (or terms times coefficients) is a SAGBI basis.

(9) Let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a subset of A, and let $f \in A$. The *composition* of f by Θ, written $f \circ \Theta$, is the polynomial obtained from f by replacing each ocurrence of the x_i with θ_i. For a subset $F \subset A$, $F \circ \Theta := \left\{ f \circ \Theta \mid f \in F \right\}$. Once more again, it is assumed that $\theta_i \notin \Bbbk$ for all i. This is used to guarantee that $X \neq 1$ implies $X \circ \widehat{\Theta} \neq 1$, for every $X \in \mathrm{Mon}(A)$. It is straightforward to see that two forms of compositions are associative in the following sense:

$$m(H) \circ \Theta = m(H \circ \Theta). \tag{3.22}$$

(10) Notice that the notion $m(H \circ \Theta)$ makes sense due to the natural correspondence between the sets $H = \{h_1, h_2, \dots, \}$ and $H \circ \Theta = \{h_1 \circ \Theta, h_2 \circ \Theta, \dots\}$. Note also that for elements $f, g \in A$,

$$(fg) \circ \Theta = f \circ \Theta g \circ \Theta, \tag{3.23}$$

$$(f+g) \circ \Theta = f \circ \Theta + g \circ \Theta \tag{3.24}$$

Having in mind that the order is preserved by multiplication, in a similar way to (3.1),

$$\widehat{X \circ \Theta} = X \circ \widehat{\Theta}, \quad \text{for every} \quad X \in \mathrm{Mon}(A). \tag{3.25}$$

(11) Let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a subset of A. Composition by Θ is *compatible with the monomial ordering* $\prec$ if for all monomials $X_i, X_j \in \mathrm{Mon}(A)$, we have

$$X_i \prec X_j \quad \text{implies} \quad X_i \circ \widehat{\Theta} \prec X_j \circ \widehat{\Theta}. \tag{3.26}$$

For an element $f \in A$ written as a linear combination of monomials in decreasing order $f = \sum_{i=1}^{t} r_i X_i$, $X_1 \succ \cdots \succ X_t$ (Remark 1 (iv)), if composition by Θ is compatible with the monomial ordering $\prec$, then $X_1 \circ \widehat{\Theta} \succ \cdots \succ X_s \circ \widehat{\Theta}$, so expressions (3.5) and (3.6) guarantee that

$$\widehat{f \circ \Theta} = \widehat{f} \circ \widehat{\Theta}. \tag{3.27}$$

Composition by Θ is *compatible with nonequality* if for all monomials $X_i, X_j \in \mathrm{Mon}(A)$, we have

$$X_i \neq X_j \quad \text{implies} \quad X_i \circ \widehat{\Theta} \neq X_j \circ \widehat{\Theta}. \tag{3.28}$$

Since monomial orderings are total, if composition by Θ is compatible with the monomial ordering $\prec$, then composition by Θ is compatible with nonequality [Nor02, Lemma 1].

Lemma 3.21 and Proposition 3.22 are the analogues of [Nor02, Lemma 2 and Proposition 1], respectively. We need these two results to prove the sufficiency of the compatibility with the ordering in Theorem 3.26.

Lemma 3.21. *Suppose that the composition by Θ is compatible with the ordering $\prec$. If (m, m') is a critical pair of $H \circ \Theta$, then (m, m') is also a critical pair of H.*

Proof. Suppose that (m, m') is a critical pair of $H \circ \Theta$. By (3.22), $m(H \circ \Theta) = m(H) \circ \Theta$ and $m'(H \circ \Theta) = m'(H) \circ \Theta$, whence $\mathrm{lt}(m(H \circ \Theta)) = \mathrm{lt}(m'(H \circ \Theta))$ and so $\mathrm{lt}(m(H)) \circ \mathrm{lt}(H) = \mathrm{lt}(m'(H)) \circ \mathrm{lt}(\Theta)$

due to (3.25). Now, by assumption the composition is compatible with the ordering, and hence with the nonequality, which implies that $\mathrm{lt}(m(H)) = \mathrm{lt}(m'(H))$, that is, (m, m') is a critical pair of H. □

The following result illustrates the necessity of the compatibility with the ordering.

PROPOSITION 3.22. *If composition by Θ is compatible with the ordering $\prec$, then composition by Θ commutes with noncommutative* `SAGBI` *bases computation.*

Proof. Consider an arbitrary `SAGBI` basis H. We have to show that $H \circ \Theta$ is also a `SAGBI` basis. Consider an arbitrary critical pair of $H \circ \Theta$. By Lemma 3.21 we know that (m, m') is also a critical pair of H. Theorem 3.20 guarantees the expression

$$m(H) - cm'(H) = \sum_i c_i m_i(H) \quad \text{(or zero)}, \quad \widehat{m_i(H)} \prec \widehat{m(H)} = \widehat{m'(H)}, \quad \text{for all } i. \tag{3.29}$$

If we compose the T-polynomial by Θ, then expressions (3.22), (3.23), and (3.24) guarantee

$$m(H \circ \Theta) - cm'(H \circ \Theta) = \sum_i c_i m_i(H \circ \Theta) \quad \text{(or zero)}, \tag{3.30}$$

Now, if we compose the inequality in expression (3.29) by $\widehat{\Theta}$, by (3.25) and (3.26) we get

$$\widehat{m_i(H \circ \Theta)} \prec \widehat{m(H \circ \Theta)} = \widehat{m'(H \circ \Theta)}, \quad \text{for all } i. \tag{3.31}$$

Notice that the leading terms in the left-hand side of expression (3.30) cancel, so the constant c must be the same as in the definition of the T-polynomial of (m, m') with respect to $H \circ \Theta$. Therefore, expressions (3.30) and (3.31) are a representation in the sense of Theorem 3.20, and having in mind that the critical pair (m, m') of $H \circ \Theta$ was arbitrary, it follows that $H \circ \Theta$ is a `SAGBI` basis. □

Now, the proof of the necessity of the compatibility with the ordering in Theorem 3.26 requires of Lemma 3.23 and Proposition 3.24 which are the analogue versions of [Nor02, Lemma 3 and Proposition 2], respectively.

LEMMA 3.23. *If u, v are two monomials with $u \neq v$ but $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, then for every $w \prec u$, $H = \{u - w, v\}$ is a* `SAGBI` *basis.*

Proof. It is clear that both u and v must be different from 1. If this is not the case, say $v = 1$, then $u \neq 1$, and having in mind that the elements θ_i's are nonconstant, $v \circ \widehat{\Theta} = u \circ \widehat{\Theta} \neq 1$, a contradiction.

The idea is to show that H has no non-trivial critical pairs, that is, if $\widehat{m(H)} = \widehat{m'(H)}$ then $m = m'$, because in this situation H is a `SAGBI` basis, and hence every T-polynomial must be identically equal to zero. Let us see the proof by contradiction.

Let (m, m') be a non-trivial arbitrary critical pair of H. Notice that $\widehat{H} = \{u, v\}$, and so $\widehat{m(H)} = m(\widehat{H}) = u^k v^l$ and $\widehat{m'(H)} = u^s v^t$, and $u^k v^l = u^s v^t$. Since the critical pair is non-trivial, then $k \neq s$

and $l \neq t$, which means that $k > s$ and $l < t$ or vice versa. In this way, it follows that $u^i = v^j$, with $i, j \geq 0$. If we compose this equality by $\widehat{\Theta}$, expression (3.23) implies that $(u \circ \widehat{\Theta})^i = (v \circ \widehat{\Theta})^j$. By using that $u \circ \widehat{\Theta} = v \circ \widehat{\Theta} \neq 1$, it follows that $i = j$. Finally, the equality $u^i = v^j$ implies $u = v$, a contradiction, whence the critical pair (m, m') is trivial. We conclude that H is a `SAGBI` basis. □

PROPOSITION 3.24. *If composition by* Θ *commutes with noncommutative* `SAGBI` *bases computation, then composition by* Θ *is compatible with nonequality.*

Proof. Let Θ be commuting with `SAGBI` bases computation. Once more again, we proceed by contradiction.

Suppose that there exist two different monomials $u, v \in \text{Mon}(A)$ but $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$. As we saw in the proof of Lemma 3.23, $u, v \neq 1$. Since every subset $H \subset A$ consisting only of monomials (or terms) is a `SAGBI` bases, $H = \{u, v\}$ is a `SAGBI` basis, and so $H \circ \Theta = \{u \circ \Theta, v \circ \Theta\}$ also is. Hence, if $f = u \circ \Theta - v \circ \Theta \in \Bbbk\langle H \circ \Theta \rangle_A$ is not equal to 1 or zero, then the assertion follows. Since $\widehat{f} \prec u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, $\widehat{f}$ cannot be written as a product from $\widehat{H \circ \Theta} = \{u \circ \widehat{\Theta}, v \circ \widehat{\Theta}\}$, which means that $H \circ \Theta$ cannot be a `SAGBI` basis [Nor02, Remark 1].

Let $u' = ux_i$ and $v' = vx_i$ for some indeterminate x_i in the skew PBW extension of A. Note that $u' \circ \Theta = (u \circ \Theta)\theta_i$ and

$$u' \neq v' \quad \text{and} \quad \widehat{u' \circ \Theta} = \widehat{v' \circ \Theta}. \tag{3.32}$$

If $f = u \circ \Theta - v \circ \Theta = 1$, then we consider $H' = \{u', v'\}$ which is a `SAGBI` basis by Remark 16. Now,

$$f' = u' \circ \Theta - v' \circ \Theta = (u \circ \Theta)\theta_i - (v \circ \Theta)\theta_i = f\theta_i = \theta_i \in \Bbbk\langle H' \circ \Theta \rangle_A,$$

and, once more again, it follows that $H = \{u' \circ \Theta, v' \circ \Theta\}$ cannot be a `SAGBI` basis (note that $\widehat{\theta}_i \prec \widehat{u' \circ \Theta} = \widehat{v' \circ \Theta}$).

We only need to consider the case $f = u \circ \Theta - v \circ \Theta = 0$ (e.g., if $\Theta = \widehat{\Theta}$). Let $H' = \{u' + x_i, v'\}$. By expression (3.32) and $x_i \prec u' = ux_i$, it follows that H' is a `SAGBI` basis by Lemma 3.23. As above, we obtain a contradiction from $f' = u' \circ \Theta - v' \circ \Theta = \theta_i \in \Bbbk\langle H' \circ \Theta \rangle_A$. □

PROPOSITION 3.25. *If composition by* Θ *commutes with noncommutative* `SAGBI` *bases computation, then composition by* Θ *is compatible with the ordering* $\prec$.

Proof. Suppose that composition by Θ commutes with `SAGBI` bases computation. Let $u, v \in \text{Mon}(A)$ two monomials with $u \succ v$. We want to show that $u \circ \widehat{\Theta} \succ v \circ \widehat{\Theta}$. Due to $u \neq v$, Proposition 3.24 shows that we cannot have $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, which means that we exclude the case $u \circ \widehat{\Theta} \prec v \circ \widehat{\Theta}$.

By Remark 16 we know that $H' = \{u, v\}$ is a `SAGBI` basis, and due to (3.21), $\Bbbk\langle H \rangle_A = \Bbbk\langle H' \rangle_A$, and $\widehat{H} = \widehat{H}'$, we get that $H = \{u - v, v\}$ is a `SAGBI` basis. Thus, $H \circ \Theta = \{u \circ \Theta - v \circ \Theta, v \circ \Theta\}$ must be also a `SAGBI` basis.

Consider $u \circ \widehat{\Theta} \prec v \circ \widehat{\Theta}$. It follows that $\widehat{H \circ \Theta} = \{v \circ \widehat{\Theta}\}$ and $u \circ \Theta = (u \circ \Theta - v \circ \Theta) + v \circ \Theta \in \Bbbk\langle H \circ \Theta \rangle_A$. However, as we saw in the proof of Proposition 3.24, $u \circ \widehat{\Theta} \prec v \circ \widehat{\Theta}$, and so $u \circ \widehat{\Theta} \neq 1$ cannot be expressed as a power of $v \circ \widehat{\Theta}$ which means that $H \circ \Theta$ is not a `SAGBI` basis. In other words, the assumption $u \circ \widehat{\Theta} \prec v \circ \widehat{\Theta}$ is false, and hence the composition by Θ is compatible with the ordering $\prec$. □

Finally, we present the most important result of this section that follows from Propositions 3.22 and 3.25.

THEOREM 3.26. *Composition by* Θ *commutes with noncommutative* `SAGBI` *bases computation if and only if the composition is compatible with the ordering* $\prec$.

3.5 FUTURE WORK

As we said in Section 3.4, Gallego developed the theory of Gröbner bases of skew PBW extensions. However, the problem of Gröbner bases under composition of these extensions has not been investigated, so a first natural task is to investigate this problem. In fact, recently Kanwal and Khan [KK23] considered the question of `SAGBI`-Gröbner bases under composition. As expected, this paper is of our interest in the near future.

Related with the topic of Gröbner bases, another interesting questions concern the notions of *Universal Gröbner basis* and the *Gröbner Walk method*. In the commutative setting, a set F which is a Gröbner basis for an ideal I of the commutative polynomial ring $\Bbbk[x_1,\dots,x_n]$ with respect to every monomial ordering is called a *universal Gröbner basis*. From [AL94, Exercise 1.8.6(d)] we know that every ideal of $\Bbbk[x_1,\dots,x_n]$ has a basis of this kind. Since some approaches to this notion of basis in the noncommmutative case have been developed by Weispfenning [Wei89], we consider interesting to ask the possibility of these bases in the setting of skew PBW extensions. With respect to the Gröbner Walk method, this is a basis conversion method proposed by Collart, Kalkbrener, and Mall [CKM97] with the aim of converting a given Gröbner basis, respect to a fixed monomial order, of a polynomial ideal I of the commutative polynomial ring $\Bbbk[x_1,\dots,x_n]$ to a Gröbner basis of I with respect to another monomial order. Since some approaches have been realized to Gröbner Walk method in the noncommutative setting (for instance, Evans in his PhD Thesis [Eva05] who presented an algorithm for a noncommutative Gröbner Walk in the case of conversion between two harmonious monomial orderings), it is important to us to investigate the possibility of this procedure in the setting of Gröbner bases of skew PBW extensions.

Last, but not least, regarding the computational development of `SAGBI` bases, this has been strongly driven in the commutative case as we can see in the literature (e.g., [HMAR13, KR05, GP08] and references therein), and recently in Robbiano and Bigatti's paper [RB22]. Now, since Fajardo in his PhD Thesis [Faj18], and related papers [Faj19, Faj22, FLP+24] and [FGL+20, Appendices C, D, E], developed the library `SPBWE.lib` in `Maple` with the aim of making homological and constructive computations of Gröbner bases of skew PBW extensions, it is natural to consider the problem of developing a computational approach to the `SAGBI` basis theory for skew PBW extensions following the ideas presented in Section 3.4.

CHAPTER 4

`Involutive` BASES IN SKEW PBW EXTENSIONS

In this chapter, we present a first approach toward a theory of `Involutive` bases for skew PBW extensions.

We start in Section 4.1 with a motivation about the study of `Involutive` bases. We mention general facts of the theory in comparison with the theory of Gröbner bases, and present some examples that illustrate the differences between both kinds of bases.

Next, Section 4.2 presents an illustrative example for the computation of `Involutive` bases in the noncommutative setting, and more exactly, the Weyl algebras. We also recall the most important preliminaries on `Involutive` divisions and total orders following the ideas developed by Gerdt and Blinkov [GB98a, GB98b], Hausdorff et al. [HSS02], and Saito et al. [SST00].

In Section 4.3, we consider the *polynomial algebras of solvable type* defined by Seiler [Sei10] with some technical details about its theory of Gröbner and `Involutive` bases. This will be useful in the next section due to the similarities between these algebras and skew PBW extensions.

Section 4.4 contains the treatment on `Involutive` bases for skew PBW extensions. We explore the relationships between Gerdt and Blinkov's ideas [GB98a, GB98b], Seiler's theory [Sei10] and the theory of Gröbner bases presented by Gallego and Lezama [Gal15, Gal16a, GL11]. We start in Section 4.4.1 with definitions and preliminaries on `Involutive` bases in the setting of these extensions. Then, Section 4.4.2 presents some results with the aim of proving the existence of a finite `Involutive` basis for a left ideal of a skew PBW extension (Theorem 4.16).

Finally, Section 4.5 contains some ideas for a possible research on `Involutive` bases.

4.1 Motivation

Janet-Riquier theory of linear systems of partial differential equations is the origin of `Involutive` *bases* (see Janet [Jan20, Jan24, Jan29], Riquier [MR90, Riq10], Thomas [Tho37], and Tresse [Tre94] for a detailed history of the topic). Briefly, the key idea in the theory of `Involutive` bases developed by Janet is to assign to every generator in a basis a subset of all indeterminates: its *multiplicative indeterminates*. This way of assigning is called an *involutive division* and it is a restriction of the usual divisibility relation on terms (also called monomials by some authors, as we saw in different sections of Chapter 3). If one only allow to multiply each generator by polynomials in its multiplicative indeterminates, then the involutive standard representation is unique, and yields to interesting combinatorial properties not shared by classical Gröbner bases in the sense considered in Chapter 3 (that is, works such as Buchberger [Buc65], Adams and Loustaunau [AL94], Becker and Weispfenning [BW93], and Cox et al., [CLO15]). Note that, in comparison with the theory of Gröbner bases, where it is well-known that given a Gröbner basis G the remainder obtained from dividing a polynomial with respect to G will always be the same no matter how the division is carried out, in the case of `Involutive` bases the difference is that there is only one way for the division to be performed, which guarantees that unique remainders are also obtained in a unique way.

It is important to say that the Janet-Riquier theory in its original form lacks only the concept of reduction to a normal form; in other case, it contains all the ingredients of Gröbner bases. Several relations between both kinds of bases have been established by Wu [Wu91], and then by Blinkov et al., [GB98a, GB98b, ZB93] who introduced a special form of non-reduced Gröbner bases for polynomial ideals. In this way, `Involutive` *bases are a special family of non-reduced Gröbner bases with additional combinatorial properties.*

Due to these combinatorial properties appearing in the theory of `Involutive` bases it is possible to define Stanley decompositions (c.f. Seiler [Sei01]). This kind of decompositions were introduced with the aim of carrying Hilbert function computations (see Stanley [Sta78] for more details) and it has been showed that these are very useful in applications like invariant theory (e.g., Sturmfels and White [SW91], Gatermann [Gat00]) or the computation of syzygy resolutions (Seiler [Sei01]). With respect to Gröbner bases, Gerdt et al., [GBY01] proved that `Involutive` bases are very efficient and represent an useful alternative to the Buchberger algorithm for the computation of Gröbner bases.

In the noncommutative setting, it is interesting to determine whether `Involutive` bases can also be defined for ideals in rings of polynomial type. Some approaches have been realized in this line of thinking. For example, the definition and the construction of `Involutive` bases for order terms was realized by Apel [Ape95, Ape98] in the case of algebras of solvable type; Gerdt [Ger99] studied the case of linear differential operators; Seiler [Sei01] considered examples of solvable algebras; Hausdorff et al., [HSS02] studied the problem of defining these bases for Weyl algebras; Evans in his PhD Thesis [Eva05] investigated `Involutive` bases in the setting of free algebras; and more recently, Seiler [Sei09] has studied these bases for a rather general class of polynomial algebras including non-commutative algebras like linear differential and difference operators or universal enveloping algebras of finite dimensional Lie algebras.

As expected, motivated by all works mentioned above, it is interesting to investigate the possibility of defining `Involutive` bases of skew PBW extensions.

With the aim of motivating and illustrate some key concepts of interest for us in the theory of involutive bases for noncommutative rings, next we present some notions used in the development of the theory.

4.2 WEYL ALGEBRAS: AN ILLUSTRATIVE EXAMPLE

4.2.1 Involutive DIVISIONS

The notion of an involutive division was introduced for the commutative polynomial ring $\Bbbk[x_1,\ldots,x_n]$ and leads to a restriction of the divisibility relation on power products. If we want to extend this concept, then we need to establish the theory on the set of n-tuples with non-negative integer entries. When we say "division" we are using addition and substraction.

An element of the Abelian monoid $(\mathbb{N}^n,+)$ is called a *multi index*. For two multi indices $\mu=(\mu^1,\ldots,\mu^n)$ and $\nu=(\nu^1,\ldots,\nu^n)\in\mathbb{N}^n$, we say that $\mu \mid \nu$ (μ *divides* ν) if $\mu^i\leq\nu^i$, for all i. The *cone* of μ with respect to a subset $N\subseteq\{1,\ldots,n\}$ is the set $C_N(\mu):=\mu+\mathbb{N}^n_N\subset\mathbb{N}^n$, where $\mathbb{N}^n_N=\left\{\nu\in\mathbb{N}^n \mid \nu^i=0, \text{ for } i\notin N\right\}$.

DEFINITION 4.1 ([HSS02, P. 183]). An Involutive *division* L on $\mathbb{N}^n$ is given by prescribing for each finite subset $\mathcal{N}\subset\mathbb{N}^n$ and for each multi index $\mu\in\mathcal{N}$, a set $N_{L,\mathcal{N}}(\mu)$ of multiplicative indices such that the following holds:

(i) For all $\mu,\nu\in\mathcal{N}$ with $C_{L,\mathcal{N}}(\mu)\cap C_{L,\mathcal{N}}(\nu)\neq\emptyset$ either $C_{L,\mathcal{N}}(\mu)\subseteq C_{L,\mathcal{N}}(\nu)$ or $C_{L,\mathcal{N}}(\nu)\subseteq C_{L,\mathcal{N}}(\mu)$.

(ii) If $\mathcal{M}\subset\mathcal{N}$, then for all $\mu\in\mathcal{M}$, $N_{L,\mathcal{N}}(\mu)\subseteq N_{L,\mathcal{M}}(\mu)$.

$C_{L,\mathcal{N}}(\mu)$ is called the *involutive cone* of μ with respect to L and $\mathcal{N}$. We denote the complement of $N_{L,\mathcal{N}}(\mu)$ in $\{1,\ldots,n\}$, the *non-multiplicative indices* of μ, by $\overline{N}_{L,\mathcal{N}}(\mu)$. For $\mu\in\mathcal{N}$ and $\nu\in\mathbb{N}^n$, we write $\mu|_{L,\mathcal{N}}\ \nu$ (μ *involutively divides* or is an *involutive divisor* of ν), if and only if $\nu\in C_{L,\mathcal{N}}(\mu)$.

Next, we recall the definitions of two involutive divisions that are most frequently found (see [HSS02] or [Sei09] for more details).

(i) *Janet division J*: Let $\mathcal{N}$ be a finite subset of $\mathbb{N}^n$ and $\mu\in\mathcal{N}$.

- $1\in N_{J,\mathcal{N}}(\mu)$ if and only if $\mu^1\geq\nu^1$, for all $\nu\in\mathcal{N}$.
- For $i=2,\ldots,n$ we have $i\in N_{J,\mathcal{N}}(\mu)$ if and only if $\mu^i\geq\nu^i$, for all those $\nu\in\mathcal{N}$ with $\nu^l=\mu^l$, for $l=1,\ldots,i-1$.

(ii) *Pommaret division P*: Let $\mu\in\mathbb{N}^n$ be an arbitrary multi index and let k be the position of the right-most non-zero entry. Then i is a multiplicative index if and only if $i\geq k$. For $\mu=(0,\ldots,0)$, all indices are multiplicative.

EXAMPLE 4.1. (i) [Eva05, p. 4] Consider the Janet *Involutive* Basis $H=\{xy-z, yz+2x+z, 2x^2+xz+z^2, 2x^2z+xz^2+z^3\}$ with multiplicative indeterminates given in the next table:

Polynomial	Janet Multiplicative Indeterminates
$xy - z$	$\{x, y\}$
$yz + 2x + z$	$\{x, y, z\}$
$2x^2 + xz + z^2$	$\{x\}$
$2x^2z + xz^2 + z^3$	$\{x, z\}$

Any polynomial may only be *involutively divisible* by at most one member of any *Involutive* basis. Note that $G = \{xy - z, yz + 2x + z, 2x^2 + xz + z^2\}$ is a Gröbner basis for the same ideal generated by H.

(ii) [Eva05, Example 4.1.16] Let $U = \{x^5y^2z, x^4yz^2, x^2y^2z, xyz^3, xz^3, y^2z, z\}$ be a set of monomials over the polynomial ring $\mathbb{Q}[x, y, z]$, with $x \succ y \succ z$. According the Janet and Pommaret divisions, the following table presents the multiplicative indeterminates for the elements of U:

Monomial	Janet	Pommaret
x^5y^2z	$\{x, y\}$	$\{x\}$
x^4yz^2	$\{x, y\}$	$\{x\}$
x^2y^2z	$\{y\}$	$\{x\}$
xyz^3	$\{x, y, z\}$	$\{x\}$
xz^3	$\{x, z\}$	$\{x\}$
y^2z	$\{y\}$	$\{x, y\}$
z	$\{x\}$	$\{x, y, z\}$

(iii) [Eva05, Example 4.1.18] Consider the polynomial ring $\mathbb{Q}[x, y]$, the monomial ordering `deglex`, and the Gröbner basis $G = \{g_1, g_2, g_3\} = \{x^2 - 2xy + 3, 2xy + y^2 + 5, \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\}$. Let $p = x^2y + y^3 + 8y \in \mathbb{Q}[x, y]$. It can be seen that p reduces to 0 (i.e., p belongs to the ideal generated by G) by g_1 and then by g_2, or by g_2 twice, and then by g_3. A Pommaret `Involutive` basis for G is given by the set $P = \{g_1, g_2, g_3, g_4\} = \{x^2 - 2xy + 3, 2xy + y^2 + 5, \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y, -5xy^2 - 5x + 6y\}$, with the following assignation of multiplicative indeterminates:

Polynomial	Pommaret Multiplicative Indeterminates
$x^2 - 2xy + 3$	$\{x\}$
$2xy + y^2 + 5$	$\{x\}$
$\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y$	$\{x, y\}$
$-5xy^2 - 5x + 6y$	$\{x\}$

As an illustration, consider the polynomial $p = x^2y + y^3 + 8y$. It can be seen that p is reduced to 0 (i.e., p belongs to the ideal generated by G) by g_1 and then by g_2, or by g_2 (twice) and then by g_3. Nevertheless, p is only involutively divisible by one element of the `Involutive` basis P (g_2) starting the following unique involutive reduction path for p:

$$x^2y + y^3 + 8y \xrightarrow{g_2} -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y \xrightarrow{g_4} y^3 - 2x + \frac{37}{5} \xrightarrow{g_3} 0.$$

The multiplicative indices in the Pommaret division are determined independently of a set $\mathcal{N}$; these are known as *globally defined*. For both divisions above multi indices are ordered tuples, as the entries are considered one after the other. In this way we may introduce variants of them by applying a fixed permutation to any multi index before computing its multiplicative indices.

Definition 4.2 ([HSS02, p. 183]). Let L be an involutive division on $\mathbb{N}^n$ and $\mathcal{N} \subset \mathbb{N}^n$ a finite set.

(i) The *involutive span* of $\mathcal{N}$ is the union of the involutive cones of its elements

$$\langle \mathcal{N} \rangle_L = \bigcup_{\mu \in \mathcal{N}} C_{L,\mathcal{N}}(\mu).$$

(ii) $\mathcal{N}$ is *involutively autoreduced* (with respect to the division L), if $C_{L,\mathcal{N}}(\mu) \bigcap C_{L,\mathcal{N}}(\nu) = \emptyset$, for all $\mu \neq \nu$ with $\mu, \nu \in \mathcal{N}$.

(iii) $\mathcal{N}$ is called *weakly* (L)-*involutive*, if $\langle \mathcal{N} \rangle_L = \langle \mathcal{N} \rangle$, where $\langle \mathcal{N} \rangle$ denotes the ordinary span of $\mathcal{N}$, i.e., the monoid ideal $\langle \mathcal{N} \rangle = \mathcal{N} + \mathbb{N}^n$.

(iv) A finite subset $\widehat{\mathcal{N}} \subset \mathcal{N}$ is called a *weak involutive basis* of $\langle \mathcal{N} \rangle$ if $\langle \widehat{\mathcal{N}} \rangle = \langle \mathcal{N} \rangle$. If $\widehat{\mathcal{N}}$ contains $\mathcal{N}$, it is a *weak* (L)-*completion* of $\mathcal{N}$. If the set $\widehat{\mathcal{N}}$ is furthermore autoreduced, it is a (*strong*) *involutive basis* of $\langle \mathcal{N} \rangle$ or a (*strong*) *completion* of $\mathcal{N}$, respectively.

(v) An involutive division L is called *Noetherian* if every finite set $\mathcal{N} \subset \mathbb{N}^n$ of multi indices has a completion.

We know that if $\mathcal{N}$ is a weakly involutive set, there exists a subset $\mathcal{N}' \subset \mathcal{N}$ such that $\mathcal{N}'$ is a strong involutive basis of $\langle \mathcal{N} \rangle$ [HSS02, Lemma 2.1].

With the aim of obtaining an efficient tool to find involutive completions of a given set, it is necessary that the involutive division L possess two additional properties introduced by Gerdt and Blinkov [GB98a]. Let us see the details.

Definition 4.3 ([HSS02, p. 184]). Let L be an involutive division on $\mathbb{N}^n$.

(i) L is *continuous* if for all finite subsets $\mathcal{N} \subset \mathbb{N}^n$ the following condition is satisfied: for each finite sequence $(\mu_i)_{i=1,\dots,r}$ of elements from $\mathcal{N}$ obeying that for all i the sum $\mu_i + 1_j \in C_{L,\mathcal{N}}(\mu_{i+1})$ for some non-multiplicative index $j \in \overline{N}_{L,\mathcal{N}}(\mu_i)$, the inequality $\mu_k \neq \mu_l$ holds, for all $k \neq l$ (the multi index 1_i has 1 in the ith place and all other entries are 0).

(ii) L is *constructive* if it satisfies the following condition: for all finite subsets $\mathcal{N} \subset \mathbb{N}^n$ and for each multi index $\mu \in \mathcal{N}$ and $i \in \overline{N}_{L,\mathcal{N}}(\mu)$ with

  (a) $\mu + 1_i \notin \langle \mathcal{N} \rangle_L$,

  (b) $\mu + 1_i$ is minimal in the sense that each of its proper divisors lies in the involutive span of $\mathcal{N}$, i.e., if there exists $\nu \in \mathcal{N}$ and $j \in \overline{N}_{L,\mathcal{N}}(\nu)$ such that $\nu + 1_j \mid \mu + 1_i$ and $\nu + 1_j \neq \mu + 1_i$, then $\nu + 1_j \in \langle \mathcal{N} \rangle_L$.

  $\mathcal{N}$ cannot be enlarged by a multiplicative multiple to include $\mu + 1$, i.e., there does not exist $\rho \in \langle \mathcal{N} \rangle_L$ such that $\mu + 1_i \in \langle \mathcal{N} \cup \{\rho\} \rangle_L$.

It was shown by Gerdt and Blinkov [GB98a] that for a continuous division L a set $\mathcal{N} \subset \mathbb{N}^n$ is weakly L-involutive, if it is *locally L-involutive*, i.e., if for every $\mu \in \mathcal{N}$ and $i \in \overline{N}_{L,\mathcal{N}}(\mu)$ there exists some $\nu \in \mathcal{N}$ with $\mu + 1_i \in C_{L,\mathcal{N}}(\nu)$. By using this result, a simple strategy for completing a set $\mathcal{N}$ is established by Gerdt and Blinkov [GB98a] and Calmet et al., [CHS01]: if $i \in \overline{N}_{L,\mathcal{N}}(\mu)$ is a non-multiplicative index for $\mu \in \mathcal{N}$, we check whether $\mu + 1_i \in \langle \mathcal{N} \rangle_L$; in other case, $\mu + 1_i$ is added to $\mathcal{N}$. The algorithm which illustrates this procedure can be found in Hausdorff et al., [HSS02, p. 185]. From Gerdt and Blinkov [GB98a] we know that if the division L is continuous and constructive, this algorithm terminates with an L-completion for any finite input set $\mathcal{N}$ possessing an L-completion. Janet and Pommaret involutive divisions are all continuous [Eva05, Proposition 4.3.2].

In Section 1.2.1 and Example 2.1(i), we saw that for $n \in \mathbb{N}$, the n-dimensional Weyl algebra $A_n(\mathbb{k})$ is the free associative $\mathbb{k}$-algebra with generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ modulo the relations

$$x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i x_j = x_j \partial_i + \delta_{ij}, \quad 1 \le i, j \le n,$$

where δ_{ij} is the Kronecker's delta. In Example 2.2, we also saw that adding another generator h that commutes with the indeterminates $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and replacing δ_{ij} with $\delta_{ij} h^2$ in the above relations we obtain the *n-dimensional homogenized Weyl algebra $H(A_n(\mathbb{k}))$*.

Since any element $f \in A_n(\mathbb{k})$ can be written in a unique *normally form*

$$f = \sum_{\mu, \nu \in \mathbb{N}^n} a_{\mu\nu} x^\mu \partial^\nu, \tag{4.1}$$

where only finitely many $a_{\mu\nu} \in \mathbb{k}$ do not vanish, the normally ordered monomials $\mathbb{T}^n := \{x^\mu \partial^\nu = x_1^{\mu_1} \cdots x_n^{\mu_n} \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \mid \mu, \nu \in \mathbb{N}^n\}$ form a $\mathbb{k}$-vector space basis of $A_n(\mathbb{k})$. Similarly, $\mathbb{T}_n^{(h)} := \{h^\lambda x^\mu \partial^\nu \mid \lambda \in \mathbb{N}, \mu, \nu \in \mathbb{N}^n\}$ forms a basis of $H(A_n(\mathbb{k}))$.

Definition 4.4 ([HSS02, p. 186]). Let $\preceq$ be a total order on $\mathbb{T}_n$. We call $\preceq$ a *multiplicative monomial order* on $\mathbb{T}_n$ if the following conditions hold:

(i) $1 \preceq x_i \partial_i$, for all $1 \le i \le n$, and

(ii) $x^\alpha \partial^\beta \preceq x^a \partial^b$ implies $x^{\alpha+\mu} \partial^{\beta+\nu} \preceq x^{a+\mu} \partial^{b+\nu}$, for all $\mu, \nu \in \mathbb{N}^n$.

If in addition $1 \preceq t$ holds for all $t \in \mathbb{T}_n$, then we call $\preceq$ a *term order* on $\mathbb{T}_n$; otherwise, we refer to $\preceq$ as a *non-term order*. Notice the differences with respect to *monomial orderings* considered in Definition 1.3(iv) and Section 3.1.1.

Replacing the first of the above conditions with $h^2 \preceq x_i \partial_i$ and adapting the second one condition, we obtain the corresponding definitions for the terms $\mathbb{T}_n^{(h)}$ in the homogenized Weyl algebra.

Example 4.2 ([HSS02, p. 186]). Let $(\xi, \zeta) \in \mathbb{R}^{2n}$ be a *weight vector*, that is, $\xi + \zeta \in \mathbb{R}^n$ is non-negative and $\preceq$ an arbitrary term order. It is defined a multiplicative monomial order $\preceq_{(\xi,\zeta)}$ by setting $x^\alpha \partial^\beta \prec_{(\xi,\zeta)} x^a \partial^b$ if either $\alpha \cdot \xi + \beta \cdot \zeta < a \cdot \xi + b \cdot \zeta$, or $\alpha \cdot \xi + \beta \cdot \zeta = a \cdot \xi + b \cdot \zeta$ and $x^\alpha \partial^\beta \prec x^a \partial^b$. This yields a term order if and only if (ξ, ζ) is non-negative.

Once one fix a multiplicative monomial order $\preceq$ on $\mathbb{T}_n$, we define the *leading term, leading exponent* and *leading coefficient* of $f \in A_n(\Bbbk) \setminus \{0\}$ by selecting in the normally ordered form (4.1) the $\preceq$-maximal term ocurring in f. This yields applications $\mathrm{lt}_{\preceq} : A_n(\Bbbk) \setminus \{0\} \to \mathbb{T}_n$, $\mathrm{le}_{\preceq} : A_n(\Bbbk) \setminus \{0\} \to \mathbb{N}^{2n}$, and $\mathrm{lc}_{\preceq} : A_n(\Bbbk) \setminus \{0\} \to \Bbbk$. In a similar way we have maps $\mathrm{lt}_{\preceq}^{(h)}$, $\mathrm{le}_{\preceq}^{(h)}$ and $\mathrm{lc}_{\preceq}^{(h)}$ for $H(A_n(\Bbbk))$.

The next definition contains the notion of Gröbner basis of a left ideal in $A_n(\Bbbk)$ with respect to a term order. It is well-known that using suitable analogues of the commutative syzygy-polynomials and the commutative normal form algorithm, a Gröbner basis can be computed with Buchberger's algorithm. As a matter of fact, in the Weyl algebra the notion of a Gröbner basis can be extended to multiplicative monomial orders.

Definition 4.5 ([HSS02, p. 187]). Let I be a left ideal in $A_n(\Bbbk)$ and $\preceq$ a multiplicative monomial ordering on $\mathbb{T}_n$. A finite subset $G \subset A_n(\Bbbk)$ with $0 \notin G$ is a *Gröbner basis* of I if the following conditions hold:

(i) $A_n(\Bbbk) \cdot G = I$, and

(ii) $\langle \mathrm{le}(g) \mid g \in G \rangle = \{\mathrm{le}(f) \mid f \in I\} = \mathrm{le}(I)$.

A Gröbner basis G is *reduced* if for any two distinct $f, g \in G$ the following condition holds: for all monomials $t \in \mathbb{T}_n$ occurring in the normally ordered form (4.1) of f with non-zero coefficient, $\mathrm{le}(g)$ does not divide $\mathrm{le}(t)$.

Saito et al., [SST00] showed how to compute effectively Gröbner bases in $A_n(\Bbbk)$ with respect to non-term orders $\preceq$. Briefly, the idea is to derive from $\preceq$ a term order $\preceq_h$ on $\mathbb{T}_n^{(h)}$ that respects the total degree:

$$\begin{aligned} h^{\lambda} x^{a} \partial^{\beta} \preceq_h h^{l} x^{a} \partial^{b} \Leftrightarrow \lambda + |\alpha| + |\beta| &< l + |a| + |b| \quad \text{or both} \\ \lambda + |\alpha| + |\beta| &= l + |a| + |b| \quad \text{and } x^{\alpha}\partial^{\beta} \preceq x^{a}\partial^{b}, \end{aligned}$$

where $|\mu| = \sum_{i=1}^{n} \mu^i$, for a multi index $\mu = (\mu^1, \dots, \mu^n) \in \mathbb{N}^n$.

The following results establishes the relation between homogenization and dehomogenization of Gröbner bases.

Proposition 4.1 ([SST00, Theorem 1.2.4]). *Let $F \subset A_n(\Bbbk)$ be a finite generating set of some left ideal $I \subseteq A_n(\Bbbk)$, $\preceq$ a multiplicative monomial order on $\mathbb{T}_n$, and $F^{(h)} \subset H(A_n(\Bbbk))$ the set obtained by homogenizing the elements in F. Then applying Buchberger's algorithm to $F^{(h)}$ and the induced term order $\preceq_h$ on $\mathbb{T}_n^{(h)}$ yields a set $\widehat{G} \subset H(A_n(\Bbbk))$ whose dehomogenization G is a Gröbner basis of I with respect to $\preceq$.*

4.2.2 Weyl algebras

With the aim of introducing `Involutive` bases for left ideals in the Weyl algebra, Hausdorff et al., [HSS02] considered as starting point the choice of some order $\preceq$ on $\mathbb{T}_n$, either a term order or a multiplicative monomial order, for selecting the leading term of an element. Following

Hausdorff et al., [HSS02], "the situation for the more general multiplicative monomial orders is considerably more difficult, as for them normal form algorithms do not necessarily terminate". One example of multiplicative monomial orders which are not order terms is given by means of weight vectors of the form $(-\varepsilon, \varepsilon) \in \mathbb{R}^{2n}$ which appears from the action of the algebraic torus $(\Bbbk^*)^n$ on the Weyl algebra (see Saito et al., [SST00] or Oaku et al., [OTW00]). With the aim of solving this problem, for Gröbner bases the situation of non-terminating normal form algorithms is given by computing in the homogenized Weyl algebra: any multiplicative monomial order on this algebra can be lifted to a term order on the homogenized Weyl algebra. This technique goes back to Lazard [Laz83] in the commutative case, and in the Weyl algebra by Castro-Jiménez and Narváez-Maccaro [CJNM97]. Having in mind that involutive bases do not only depend on an order but also on an involutive division, the problem is to lift this division to the homogenized Weyl algebra; Hausdorff et al., [HSS02] shown that this is possible.

Definition 4.6 ([HSS02, p. 188]). Let $F = \{f_1, \ldots, f_r\} \subset A_n(\Bbbk)$ be a finite set. With respect to a given involutive division L on $\mathbb{N}^{2n}$, the *involutive span* of F is defined as

$$\langle F \rangle_L = \sum_{i=1}^{r} \Bbbk[(\iota \circ N_{L,\mathrm{le}(f)} \circ \mathrm{le})(f_i)] \cdot f_i. \tag{4.2}$$

By $\langle F \rangle$ we denote the left ideal $A_n(\Bbbk) \cdot F$ in $A_n(\Bbbk)$. The set F is a *weak involutive basis* of the left ideal $I \subseteq A_n(\Bbbk)$ if $\langle F \rangle_L = \langle F \rangle = I$.

Saito et al., [SST00] presented details on the notions of reducibility and normal form in the Weyl algebra.

A finite set $F \subset \mathbb{W}_n$ is said to be *autoreduced* if every $f \in F$ is in normal form modulo $F \setminus \{f\}$. It is *involutively autoreduced* if no $f \in F$ contains a term $t = x^\mu \partial^\nu$ such that $\mathrm{le}(f')|_{L,\mathrm{le}(f)} \mathrm{le}(t)$, for some $f' \in F$.

Definition 4.7 ([HSS02, p. 188]). Let $F = \{f_1, \ldots, f_r\}$ be a weak involutive basis of the left ideal I. If there do not exist two indices $i \neq j$ such that $\mathrm{le}(f_i)|_{L,\mathrm{le}(F)} \mathrm{le}(f_j)$, then F is called a (*strong*) *involutive basis*.

It is well-known that the sum in (4.2) is direct if and only if F is a strong involutive basis. In this way, any such basis induces a Stanley decomposition of the given ideal into a direct sum of free modules over subalgebras of the Weyl algebra. If one think this results in the context of commutative polynomials, this property allows for a trivial determination of the Hilbert function, see Stanley [Sta78] for more details.

The next result is one of the most important about involutive basis for Weyl algebras; this present its relation with Gröbner bases.

Proposition 4.2 ([HSS02, Theorem 4.1]). *Let $I \subseteq A_n(\Bbbk)$ be a left ideal. The finite set $F = \{f_1, \ldots, f_r\} \subset I$ is an involutive basis of I for the multiplicative monomial order $\preceq$ if and only if every polynomial $f \in I$ possesses a unique involutive standard representations $f = \sum_{i=1}^{r} g_i f_i$ where $g_i \in \Bbbk[(\iota \circ N_{L,\mathrm{le}(f)} \circ \mathrm{le})(f_i)]$ and $\mathrm{lt}(g_i f_i) \preceq \mathrm{lt}(f)$. Any involutive basis F of I is also a Gröbner basis of I (for the order $\preceq$).*

A consequence of Proposition 4.2 is that the involutive standard representation is unique in contrast to the ordinary one with respect to an arbitrary Gröbner basis; this is due to the fact that the sum in (4.2) is direct for involutive bases.

The following result establishes that involutive bases with respect to term orders are characterized by the fact that the leading terms involutively generate the leading ideal. Recall that this is a characteristic property of Gröbner bases.

Proposition 4.3 ([HSS02, Theorem 4.2]). *Let $I \subseteq A_n(\Bbbk)$ be a left ideal, $\preceq$ a term order on $\mathbb{T}_n$, and $F \subset I$ a finite set. If $\operatorname{le}(F)$ is a (weak) involutive basis of $\operatorname{le}(I)$ with respect to the division L, then F is a (weak) involutive basis of I. If F is a strong involutive basis, the converse is true, too.*

A natural question is how we can explicitly compute an involutive basis for a given involutive division and a given order. For a term order, the simplest approach is the following one: we first compute a Gröbner basis and then we complete the leading terms of its elements to an involutive basis of the leading ideal; Proposition 4.3 yields the desired result. This algorithm was proposed by Sturmfels and White [SW91] for the determination of Stanley decompositions in the case of commutative polynomials. About this question, Hausdorff et al., [HSS02] presents an answer which includes an algorithm to perform the computations (this algorithm can be seen in [HSS02, Figure 3]).

Proposition 4.4. *Let L be a continuous and constructive involutive division, $\preceq$ a term order on $\mathbb{T}^n$ and $F \subset A_n(\Bbbk)$ a finite set such that the monoid ideal $\operatorname{le}(\langle F\rangle)$ possesses an involutive basis with respect to L. Then there is an algorithm which terminates with an involutive basis of $\langle F\rangle$.*

Remark 17 ([HSS02, p. 190]). Involutive bases of left ideals in $A_n(\Bbbk)$ have beautiful combinatorial properties as it is shown by the following fact: consider a left ideal $I \subset A_n(\Bbbk)$ with a strong involutive basis $F = \{f_1, \ldots, f_r\}$ with respect to a degree compatible term order $\preceq$. $A_n(\Bbbk)$ is endowed with the Bernstein filtration $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots$, where the $\Bbbk$-vector spaces $\mathscr{F}_k$ are spanned by the terms $x^\alpha\partial^\beta$ with $\alpha + \beta \leq k$, respectively (see Coutinho [Cou95] for a detailed description of this filtration). If we denote by σ_k, $k \in \mathbb{N}$, the canonical projection of $\mathscr{F}_k$ onto $\mathscr{F}_k/\mathscr{F}_{k-1}$, then the set $\{\sigma_{|\operatorname{le}(f_1)|}(f_1), \ldots, \sigma_{|\operatorname{le}(f_r)|}(f_r)\}$ is a strong involutive basis of the homogeneous graded ideal $\operatorname{Gr}^{\mathscr{F}}(I) = \bigoplus_{k \geq 1} \frac{I \cap \mathscr{F}_k}{\mathscr{F}_{k-1}}$ of the associated graded algebra of $\mathbb{W}_n$ with respect to $\mathscr{F}$ (this algebra is precisely a commutative polynomial ring in $2n$ indeterminates). If one consider the Hilbert series of I as that of $\operatorname{Gr}^{\mathscr{F}}(I)$, we know from Stanley [Sta78] that

$$\mathscr{H}_I(\lambda) = \sum_{i=1}^{r} \frac{\lambda^{|\operatorname{le}(f_i)|}}{(1-\lambda)^{|N_{L,\operatorname{le}(f)}(f_i)|}}.$$

This amazing fact means that we can compute the Hilbert function and polynomial from $\mathscr{H}_I$.

4.3 Polynomial algebras of solvable type

Following Seiler [Sei10, p. 63], *polynomial algebras of solvable type* "comprises many classical algebras which are important for applications like rings of linear differential or difference operators or universal enveloping algebras of Lie algebras". Let us recall its definition.

Let $\mathscr{P} = \mathscr{R}[x_1, \ldots, x_n]$ be a polynomial ring over a ring $\mathscr{R}$. Seiler equipped the $\mathscr{R}$-module $\mathscr{P}$ with alternative noncommutative multiplications with the aim of allowing that the indeterminates x_i's do not commute and that they operate on the coefficients. He denoted the usual multiplication either by a dot $\cdot$ or by no symbol at all, and alternative multiplications $* : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$ by $f * g$. We follow his ideas presented in [Sei10, Section 3.2].

Analogously to Definition 1.3, with respect to a *monomial ordering* $\prec$ (Seiler called a *term order*), for each polynomial $f \in \mathscr{P}$, its *leading monomial* is $\mathrm{lt}_\prec f = x^\mu$ with leading exponent $\mathrm{le}_\prec f = \mu$; the coefficient $r \in \mathscr{R}$ of x^μ in f is the *leading coefficient* $\mathrm{lc}_\prec f$ and the product $r x^\mu$ is the *leading term* $\mathrm{lt}_\prec f$.

Definition 4.8 ([Sei10, Definition 3.2.1]). The triple $(\mathscr{P} = \mathscr{R}[x_1, \ldots, x_n], *, \prec)$ is a *polynomial algebra of solvable type* over the coefficient ring $\mathscr{R}$ for the monomial order $\prec$, if multiplication $* : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$ satisfies the following three axioms:

(i) $(\mathscr{P}, *)$ is a ring with neutral element 1.

(ii) For all $r \in \mathscr{R}, f \in \mathscr{P}$, $r * f = r f$.

(iii) For all $\mu, \nu \in \mathbb{N}$, $r \in \mathscr{R} \setminus \{0\}$, $\mathrm{le}_\prec(x^\mu * x^\nu) = \mu + \nu$ and $\mathrm{le}_\prec(x^\mu * r) = \mu$.

Condition (i) guarantees that the arithmetics in $(\mathscr{P}, *)$ satisfies the usual associative and distributive laws, while condition (ii) implies that $(\mathscr{P}, *)$ is a left $\mathscr{R}$-module. Finally, condition (iii) ensures the compatibility of the new multiplication $*$ and the monomial order $\prec$: $*$ is an *order respecting multiplication*. This property implies the existence of injective maps $\rho_\mu : \mathscr{R} \to \mathscr{R}$, and maps $h_\mu : \mathscr{R} \to \mathscr{P}$ with $\mathrm{le}_\prec(h_\mu(r)) \prec \mu$ for all $r \in \mathscr{R}$, coefficients $r_{\mu\nu} \in \mathscr{R} \setminus \{0\}$ and polynomials $h_{\mu\nu} \in \mathscr{P}$ with $\mathrm{le}_\prec(h_{\mu\nu}) \prec \mu + \nu$ such that

$$
\begin{aligned}
x^\mu * r &= \rho_\mu(r) x^\mu + h_\mu(r), \\
x^\mu * x^\nu &= r_{\mu\nu} x^{\mu+\nu} + h_{\mu\nu}.
\end{aligned}
$$

Seiler [Sei10, Definition 3.2.1] introduced the polynomial algebras of solvable type. If we compare its [Sei10, Proposition 3.2.3] with Propositions 1.1 and 1.2, then we can see the similarities between skew PBW extensions and polynomial algebras of solvable type. Examples of these algebras are the commutative polynomial rings, rings of linear differential operators, Ore extensions, recurrence and difference operators, universal enveloping algebras of finite-dimensional Lie algebras, and PBW extensions [BG88]. The definition of Gröbner basis for this algebras was introduced in [Sei10, Definition 3.3.12].

Definition 4.9 ([Sei10, Definition 3.3.12]). Let $(\mathscr{P}, *, \prec)$ be a polynomial algebra of solvable type over a (skew) field $\Bbbk$ and $\mathscr{I} \subseteq \mathscr{P}$ a left ideal. A finite set $\mathscr{G} \subset \mathscr{P}$ is a *Gröbner basis* of the ideal $\mathscr{I}$ (for the term order $\prec$) if $\langle \mathrm{le}_\prec \mathscr{G} \rangle = \mathrm{le}_\prec \mathscr{I}$.

Note that if $(\mathscr{P}, *, \prec)$ is a polynomial algebra of solvable type over a (skew) field $\Bbbk$, then $\mathscr{P}$ is a left Noetherian ring and every left ideal $\mathscr{I} \subseteq \mathscr{P}$ possesses a Gröbner basis with respect to $\prec$.

Next, we proceed to recall the definition of `Involutive` basis for left ideals in polynomial algebras of solvable type introduced by Seiler [Sei10, Section 3.4].

DEFINITION 4.10 ([SEI10, DEFINITION 3.4.1]). Let $(\mathscr{P}, *, \prec)$ be a polynomial algebra of solvable type over a coefficient field $\Bbbk$ and $\mathscr{I} \subset \mathscr{P}$ a non-zero left ideal. A finite set $\mathscr{H} \subset \mathscr{I}$ is a *weak involutive basis* of $\mathscr{I}$ for an involutive division L on $\mathbb{N}^n$, if $\mathrm{le}_\prec \mathscr{H}$ is a weak involutive basis of the monoid ideal $\mathrm{le}_\prec \mathscr{I}$. The set $\mathscr{H}$ is a (*strong*) *involutive basis* of $\mathscr{I}$ if $\mathrm{le}_\prec \mathscr{H}$ is a strong involutive basis of $\mathrm{le}_\prec \mathscr{I}$ and two distinct elements of $\mathscr{H}$ never possess the same leading exponents.

As Seiler stated, Definition 4.10 represents a natural extension of Definition 4.9 of a Gröbner basis in $\mathscr{P}$. This implies that any weak involutive basis is a Gröbner basis. Any finite set $\mathscr{F} \subset \mathscr{P}$ is said to be (*weakly*) *involutive* if it is a (weak) involutive basis of the ideal $\langle \mathscr{F} \rangle$ generated by it.

DEFINITION 4.11 ([SEI10, DEFINITION 3.4.2]). Let $\mathscr{F} \subset \mathscr{P} \setminus \{0\}$ be a finite set of polynomials and L and involutive division on $\mathbb{N}^n$. We assign to each element $f \in \mathscr{F}$ a set of *multiplicative variables*

$$X_{L,\mathscr{F},\prec}(f) = \left\{ x^i \mid i \in N_{L,\mathrm{le}_\prec \mathscr{F}}(\mathrm{le}_\prec f) \right\}. \tag{4.3}$$

The *involutive span* of $\mathscr{F}$ is then the set

$$\langle \mathscr{F} \rangle_{L,\prec} = \sum_{f \in \mathscr{F}} \Bbbk[X_{L,\mathscr{F},\prec}(f)] * f \subseteq \langle \mathscr{F} \rangle. \tag{4.4}$$

Note that the involutive span of a set $\mathscr{F}$ depends on both the involutive division L and the term order $\prec$.

As is well-known, a key property of Gröbner bases is the existence of standard representations for any ideal element. The following result establishes that for (weak) involutive bases a similar characterization exists and in the case of strong bases we obtain unique representations.

PROPOSITION 4.5 ([SEI10, THEOREM 3.4.4]). *Let $\mathscr{I} \subseteq \mathscr{P}$ be a non-zero ideal, $\mathscr{H} \subset \mathscr{I} \setminus \{0\}$ a finite set and L an involutive division on $\mathbb{N}_0^n$. Then the following statements are equivalent:*

(1) *The set $\mathscr{H} \subset \mathscr{I}$ is a weak involutive basis of $\mathscr{I}$ with respect to L and $\prec$.*

(2) *Every polynomial $f \in \mathscr{I}$ can be written in the form*

$$f = \sum_{h \in \mathscr{H}} P_h * h \tag{4.5}$$

with coefficients $P_h \in \Bbbk[X_{L,\mathscr{H},\prec}(h)]$ satisfying $\mathrm{lt}_\prec(P_h * h) \preceq \mathrm{lt}_\prec f$ for all polynomials $h \in \mathscr{H}$ such that $P_h \neq 0$.

$\mathscr{H}$ is a strong involutive basis if and only if the representation (4.5) is unique.

COROLLARY 4.6 ([SEI10, COROLLARY 3.4.5]). *Let $\mathscr{H}$ be a weak involutive of the left ideal $\mathscr{I} \subseteq \mathscr{P}$. Then $\langle \mathscr{H} \rangle_{L,\prec} = \mathscr{I}$. If $\mathscr{H}$ is even a strong involutive basis of $\mathscr{I}$, then $\mathscr{I}$ considered as a $\Bbbk$-linear space possesses a direct sum decomposition $\mathscr{I} = \bigoplus_{h \in \mathscr{H}} \Bbbk[X_{L,\mathscr{H},\prec}(h)] * h$.*

There are cases where a finite set $\mathscr{F}$ with $\langle \mathscr{F} \rangle_L = \mathscr{I}$ is not a weak involutive basis of the ideal $\mathscr{I}$. For example, the ideal $\mathscr{I}$ in the commutative polynomial ring $\Bbbk[x, y]$ generated by the polynomials $f_1 = y^2$ and $f_2 = y^2 + x^2$, with $x_1 = x$ and $x_2 = y$, satisfies that for $\mathscr{F} = \{f_1, f_2\}$,

$\langle \mathscr{F} \rangle_{J,\prec} = \mathscr{I}$, as with respect to the Janet division all variables are multiplicative for each generator. Nevertheless, $\mathrm{le}_{\prec}\mathscr{F} = \{[0,2]\}$ does not generate $\mathrm{le}_{\prec}\mathscr{I}$, as it is clear that $[2,0] \in \mathrm{le}_{\prec}\mathscr{I} \setminus \langle \{[0,2]\} \rangle$ [Sei10, Example 3.4.6].

PROPOSITION 4.7 ([SEI10, PROPOSITION 3.4.7]). *Let $\mathscr{I} \subseteq \mathscr{P}$ be an ideal and $\mathscr{H} \subset \mathscr{I}$ a weak involutive basis of it for the involutive division L. Then there exists a subset $\mathscr{H}' \subseteq \mathscr{H}$ which is a strong involutive basis of $\mathscr{I}$.*

Seiler [Sei10, Chapter 3] defined the notion of `Involutive` basis but he did not consider the question of the existence of such bases. He proved in [Sei10, Section 3.3] that ideals of polynomial algebras of solvable type over fields have Gröbner bases but for Involutive bases the situation is more complicated. In [Sei10, Example 3.1.16], we can see that an ideal not possessing a finite Pommaret basis, so we cannot expect that an arbitrary polynomial ideal has for every involutive division a finite involutive basis. With this in mind, in [Sei10, Section 4.1] Seiler introduced a special class of involutive divisions, the *constructive divisions*, which is related to an algorithm for computing involutive basses. The idea is that if such a division is Noetherian, then the algorithm will always terminate with an involutive basis and this provides us with a proof of the existence of such bases for many divisions (including the Janet division). As Seiler said, “Unfortunately, both the definition of constructive divisions and the termination proof are highly technical and not very intuitive” [Sei10, p. 105].

We start with the notion of *local* involution that requires only a finite number of checks in contrast with any involution.

DEFINITION 4.12 ([SEI10, DEFINITION 4.1.1]). A finite set $\mathscr{B} \subset \mathbb{N}^n$ of multi indices is *locally involutive* for the involutive division L, if $\nu + 1_j \in \langle \mathscr{B} \rangle_L$ for every non-multiplicative index $j \in \overline{N}_{L,\mathscr{B}}(\nu)$ of every multi index $\nu \in \mathscr{B}$.

It is clear that while involution implies local involution but the converse does not hold as the following division due to Gerdt and Blinkov shows.

EXAMPLE 4.3 ([GB98A, EXAMPLE 4.8]). Consider in $\mathbb{N}^3$ the involutive division L defined in the following way: with the exception of four multi indices all elements of $\mathbb{N}^3$ do not have any multiplicative indices. The four exceptions are $N_L([0,0,0]) = \{1,2,3\}$, $N_L([1,0,0]) = \{1,3\}$, $N_L([0,1,0]) = \{1,2\}$, and $N_L([0,0,1]) = \{2,3\}$. It can be seen that this assignment defines a global involutive division.

Take the set $\mathscr{B} = \{[1,0,0],[0,1,0],[0,0,1]\}$. Since the multi index $[1,1,1] \in \langle \mathscr{B} \rangle$ is not contained in the involutive span $\langle \mathscr{B} \rangle_L$, then $\mathscr{B}$ is not involutive. Nevertheless, it is locally involutive, as the three multi indices $[1,1,0]$, $[0,1,1]$ and $[1,0,1]$ obtained by taking the elements of $\mathscr{B}$ and adding their respective non-multiplicative index are contained in $\langle \mathscr{B} \rangle_L$.

Next, we recall the definition of *continuous* division (Definition 4.3(i)) that is not only sufficient condition for the equivalence of local involution and involution but also useful for proving other properties.

DEFINITION 4.13 ([SEI10, DEFINITION 4.1.3]). Let L be an involutive division and $\mathscr{B} \subset \mathbb{N}^n$ a finite set. Let furthermore $(\nu^{(1)}, \dots, \nu^{(t)})$ be a finite sequence of elements of $\mathscr{B}$ where every multi index $\nu^{(k)}$ with $k < t$ has a non-multiplicative index $j_k \in \overline{N}_{L,\mathscr{B}}(\nu^{(k)})$ such that $\nu^{(k+1)} \mid_{L,\mathscr{B}} \nu^{(k)} + 1_{j_k}$.

The division L is *continuous*, if any such sequence consists only of distinct elements, i.e. if $\nu^{(k)} \neq \nu^{(l)}$ for all $k \neq l$.

The division of Example 4.3 is not continuous. As expected, for continuous divisions, local involution implies involution. More exactly, if the involutive division L is continuous, then any locally involutive set $\mathscr{B} \subset \mathbb{N}^n$ is weakly involutive, and it can be proved that the Janet and the Pommaret divisions are continuous [Sei10, Proposition 4.1.4 and Lemma 4.1.5].

DEFINITION 4.14 ([SEI10, DEFINITION 4.1.7]). Let $\mathscr{B} \subset \mathbb{N}^n$ be a finite set of multi indices and choose a multi index $\nu \in \mathscr{B}$ and a non-multiplicative index $j \in \overline{N}_{L,\mathscr{B}}(\nu)$ such that

(i) $\nu + 1_j \notin \langle \mathscr{B} \rangle_L$.

(ii) If there exists a multi index $\mu \in \mathscr{B}$ and $k \in \overline{N}_{L,\mathscr{B}}(\nu)$ such that $\mu + 1_k \mid \nu + 1_j$ but $\mu + 1_k \neq \nu + 1_j$, then $\mu + 1_k \in \langle \mathscr{B} \rangle_L$.

The continuous division L is *constructive* (Definition 4.3(ii)) if for any such set $\mathscr{B}$ and any such multi index $\nu + 1_j$ no multi index $\rho \in \langle \mathscr{B} \rangle_L$ with $\nu + 1_j \in \mathscr{C}_{L,\mathscr{B} \cup \{\rho\}}(\rho)$ exists.

Note that any globally defined division is constructive, so Pommaret division also is [Sei10, Lema 4.1.8]. In fact, Janet division is also constructive.

With all notions above, Seiler [Sei10, Section 4.2] presented an algorithm for determining involutive bases of left ideals in a polynomial algebra of solvable type $(\mathscr{P}, *, \prec)$. It is important to say that if we assume that the division is constructive, then a very simple algorithm exists due to basic ideas of which go back to Janet [Jan20, Jan24, Jan29]. Seiler [Sei10, p. 114] an easy way to compute an involutive basis for an ideal $\mathscr{I}$ in a polynomial algebra $(\mathscr{P}, *, \prec)$ of solvable type follows from [Sei10, Remark 3.4.12]. First, we determine a Gröbner basis $\mathscr{G}$ of $\mathscr{I}$ and by using [Sei10, Algorithm 4.1] we compute an involutive completion of the monomial set $\mathrm{le}_\prec \mathscr{G}$. It is important to highlight that this idea is similar to the method proposed by Sturmfels and White [SW91] for the construction of Stanley decompositions that corresponds to the computation of an involutive basis. Nevertheless, he extended this `Algorithm` to a direct completion algorithm for polynomial ideals, by considering subalgorithms for two important tasks: *involutive normal forms* and *involutive head autoreductions*.

4.4 SKEW PBW EXTENSIONS

Having in mind the ideas about `Involutive` bases for noncommutative algebras presented in the previous sections, we explore the relationships between Gerdt and Blinkov's ideas [GB98a, GB98b], Seiler's theory [Sei10] and the theory of Gröbner bases presented by Gallego and Lezama [Gal15, Gal16a, GL11], with the aim of presenting a theory of `Involutive` bases theory for skew PBW extensions over fields.

4.4.1 Involutive BASES

Let A be a skew PBW extension over a field $\Bbbk$ and consider the notation in Definition 1.3. For every finite set of polynomials $F \subset A \setminus \{0\}$. We say that x_i is a multiplicative variable of a polynomial $f \in F$ if $i \in N_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(f)))$. Also, we define $L(v, B) = \{\mu \in \mathbb{N}^n \mid \forall j \notin N_{L,B}(v) : \mu_j = 0\}$ for any finite subset $B \in \mathbb{N}^n$ (see [Sei10, Definition 3.1.1])

Now, for a finite subset of $\mathrm{Mon}(A)$, let $\exp(F) = \{\exp(x^\alpha) \mid x^\alpha \in F\}$, and for $x^\alpha \in \mathrm{Mon}(A)$, we define the set

$$\mathrm{Mon}(N_{L,B}(\alpha)) = \left\{x^\beta \mid \beta_i = 0 \text{ if } i \notin N_{L,B}(\alpha)\right\}.$$

For every $\mathrm{Mon}(N_{L,B}(\alpha))$, let

$$\mathbf{X}_{x^\alpha} := \left\{ g = \sum_{i=1}^{t} a_i X_i \in A \mid X_i \in \mathrm{Mon}(N_{L,B}(\alpha)) \right\}.$$

Motivated by the ideas presented above, we consider the following definition.

Definition 4.15. Let $A = \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $\Bbbk$ and $I \subseteq A$ a non-zero left ideal. A finite set $F \subset I$ is a *weak involutive basis of* I for an involutive division L on $\mathbb{N}^n$, if $\exp(\mathrm{lm}(F))$ is a weak involutive basis of the monoid ideal $\exp(\mathrm{lm}(I))$. The set F is a (*strong*) *involutive basis of* I if $\exp(\mathrm{lm}(F))$ is a strong involutive basis of $\exp(\mathrm{lm}(I))$ and for two distinct elements f_1, f_2 of F, $\exp(\mathrm{lm}(f_1)) \neq \exp(\mathrm{lm}(f_2))$

Definition 4.16. Let $A = \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $\Bbbk$ and $F = \{f_1, \dots, f_r\} \subset A$ a finite set. Fix an involutive division L on $\mathbb{N}^n$ and let $B := \exp(\mathrm{lm}(F))$. *The involutive span of* F is defined as

$$\langle F\rangle_L = \sum_{i=1}^{r} \mathbf{X}_{\mathrm{lm}(f_i)} \cdot f_i \subseteq \langle F\rangle,$$

where $\langle F\rangle$ denotes the left ideal of A generated by F.

Next, we consider an involution division by restricting the divisibility relation $\mid$ on $\mathrm{Mon}(A)$ considered in [FGL+20, Definition 13.1.3]: we say that $x^\alpha \mid_{L,F} x^\beta$ if and only if $\alpha \mid_{L,\exp(F)} \beta$ for monomials x^α and x^β belonging to a finite set $F \subseteq \mathrm{Mon}(A)$. As expected, an involutive division satisfies the criterion of divisibility formulated by Gallego and Lezama [GL11, Definition 13].

Proposition 4.8. *Let $F \subset \mathrm{Mon}(A)$ be a finite set. For any monomials x^α and x^β in F, if $x^\alpha \mid_{L,F} x^\beta$, then $x^\alpha \mid x^\beta$.*

Proof. The assertion follows directly from the corresponding definitions. □

By using Proposition 4.8 we define the involutive reduction process in a similar way to its corresponding for Gröbner bases [GL11, Definition 18].

Definition 4.17. Let F be a finite set of non-zero elements of a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, L an involutive division, and $f, h \in A$. We say that f *involutively reduces* to h by F w.r.t. L in one step, denoted by $f \xrightarrow{L,F} h$, if there exist elements $f_1, \dots, f_t \in F$ and $r_1, \dots, r_t \in R$

such that
$$\operatorname{lm}\left(f_i\right) \mid_{L,\operatorname{lm}(F)} \operatorname{lm}(f), \quad \text{for } 1 \le i \le t. \tag{4.6}$$

f reduces to h by F w.r.t. L, denoted by $f \xrightarrow{L,F}_+ h$, if there exist elements $h_1, \dots, h_{t-1} \in A$ such that
$$f \xrightarrow{L,F} h_1 \xrightarrow{L,F} h_2 \xrightarrow{L,F} \cdots \xrightarrow{L,F} h_{t-1} \xrightarrow{L,F} h.$$

f is *involutively reduced* w.r.t. F and L, if $f = 0$ or there is no reduction of f by F and L. Otherwise, we say that f is *involutively reducible* w.r.t. F and L. If f is involutively reduced to h w.r.t. F and L, then we say that h is an *involutive remainder* for f w.r.t. F and L.

REMARK 18. Proposition 4.8 gives us an equivalent condition to expression (4.6): if $\operatorname{lm}(f_i) \mid_{L,\operatorname{lm}(F)} \operatorname{lm}(f)$, for $1 \le i \le t$, then there exists $\alpha_i \in L(\exp(\operatorname{lm}(f_i)), \exp(F))$ with $\exp(\operatorname{lm}(f)) = \exp(\operatorname{lm}(f_i)) + \alpha_i$. From [FGL+20, Proposition 13.1.4], there exists $x^{\alpha_i} \in \operatorname{Mon}(A)$ tal que $\operatorname{lm}(f) = \operatorname{lm}\left(x^{\alpha_i} \operatorname{lm}(f_i)\right)$. Note that the monomials x^{α_i} appearing in the expression of f due to [GL11, Definition 18(iii)] depend on the involutive division we are considering.

In the next examples, let $\mathscr{N}_i := N_{L,\exp(\operatorname{lm}(F))}(\exp(\operatorname{lm}(f_i)))$ for every $f_i \in F$ and a fixed involutive division L.

EXAMPLE 4.4. Consider the Dispin algebra $U(\mathfrak{osp}(1,2))$ and the set F given by
$$F = \{f_1 = x^2 y, f_2 = xy + z, f_3 = x + y, f_4 = x + z\}.$$

If we use `deglex` with $x \succ y \succ z$ and the Janet divison J, the reduction of $f = xy$ w.r.t F and J is as follows:

Step 1: (i) $\mathscr{N}_1 = \{1,2,3\}$ and $C_{J,\mathscr{N}_1}(\exp(\operatorname{lm}(f_1))) = \{(2+n_1, 1+n_2, n_3) \mid n_i \in \mathbb{N}\}$ since $\exp(\operatorname{lm}(f)) \notin C_{J,\mathscr{N}_1}(\exp(\operatorname{lm}(f_1)))$. We get $\operatorname{lm}(f_1) \nmid_{J,\operatorname{lm}(F)} \operatorname{lm}(f)$

(ii) $\mathscr{N}_2 = \{2,3\}$ and $C_{J,\mathscr{N}_2}(\exp(\operatorname{lm}(f_2))) = \{(1, 1+n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}$ due to $\exp(\operatorname{lm}(f)) \in C_{J,\mathscr{N}_2}(\exp(\operatorname{lm}(f_2)))$. In this way, $\operatorname{lm}(f_2) |_{J,\operatorname{lm}(F)} \operatorname{lm}(f)$.

(iii) $\mathscr{N}_3 = \{3\}$ and $C_{J,\mathscr{N}_3}(\exp(\operatorname{lm}(f_3))) = \{(1,0,n) \mid n \in \mathbb{N}\}$. Then $\operatorname{lm}(f_3) \nmid_{J,\operatorname{lm}(F)} \operatorname{lm}(f)$

(iv) $\mathscr{N}_4 = \mathscr{N}_3$ and so $\operatorname{lm}(f_4) \nmid_{J,\operatorname{lm}(F)} \operatorname{lm}(f)$

Step 2: $\alpha_2 = (0,0,0)$ and $1 = \operatorname{lc}(f) = r_2 \sigma^{\alpha_2}(1) d_{\alpha_2, \exp(\operatorname{lm}(f_2))} = r_2$.

Step 3: $h = xy - (xy + z) = -z$.

Step 4: If we reduce involutively to h w.r.t F and J, then we get $\operatorname{lm}(h) \notin \mathscr{N}_i$ for $i = 1,2,3,4$, so it is reduced involutively. We say that h is an *involutive remainder* for f.

The next example illustrates that if we consider Pommaret division instead Janet division, then we get surprising results due to the remainders will depend on elements of the field $\Bbbk$. Let us see the details.

EXAMPLE 4.5. Consider :

Step 1: (i) $\mathscr{N}_1 = \{2,3\}$ and $C_{P,\mathscr{N}_1}(\exp(\operatorname{lm}(f_1))) = \{(2, 1+n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}$ since $\exp(\operatorname{lm}(f)) \notin C_{P,\mathscr{N}_1}(\exp(\operatorname{lm}(f_1)))$ then $\operatorname{lm}(f_1) \nmid_{P,\operatorname{lm}(F)} \operatorname{lm}(f)$

(ii $\mathcal{N}_2 = \{2,3\}$ and $C_{P,\mathcal{N}_2}(\exp(\operatorname{lm}(f_2))) = \{(1, 1+n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}$ since $\exp(\operatorname{lm}(f)) \in C_{P,\mathcal{N}_2}(\exp(\operatorname{lm}(f_2)))$ then $\operatorname{lm}(f_2)|_{P,\operatorname{lm}(F)} \operatorname{lm}(f)$

(iii) $\mathcal{N}_3 = \{1,2,3\}$ and $C_{P,\mathcal{N}_3}(\exp(\operatorname{lm}(f_3))) = \{(1+n_1, n_2, n_3) \mid n_1, n_2, n_3 \in \mathbb{N}\}$ since $\exp(\operatorname{lm}(f)) \in C_{P,\mathcal{N}_3}(\exp(\operatorname{lm}(f_3)))$ then $\operatorname{lm}(f_3)\,|_{P,\operatorname{lm}(F)} \operatorname{lm}(f)$

(iv) $\mathcal{N}_4 = \mathcal{N}_3$ then $\operatorname{lm}(f_4)\,|_{P,\operatorname{lm}(F)} \operatorname{lm}(f)$

Step 2: $\alpha_2 = (0,0,0)$, $\alpha_3 = (0,1,0) = \alpha_4$ then $1 = \operatorname{lc}(f) = r_2\sigma^{\alpha_2}(1)d_{\alpha_2,\exp(\operatorname{lm}(f_2))} + r_3\sigma^{\alpha_3}(1)d_{\alpha_3,\exp(\operatorname{lm}(f_3))} + r_4\sigma^{\alpha_4}(1)d_{\alpha_4,\exp(\operatorname{lm}(f_4))} = r_2 + r_3 + r_4$ then $r_2 = 1 - r_3 - r_4$.

Step 3:

$$\begin{aligned}
h =& xy - [(1-r_3-r_4)(xy+z) + r_3y(x+y) + r_4y(x+z)]\\
=& xy - xy - z + r_3xy + r_3z + +r_4xy + r_4z - r_3yx - r_3y^2 - r_4yx\\
=& r_3xy + r_4xy - r_3(xy-x) - r_3y^2 - r_4(xy-x) - r_4yz + z(r_3+r_4-1)\\
=& -r_3y^2 - r_4yz + (r_3+r_4)x + (r_3+r_4-1)z
\end{aligned}$$

Step 4: Again, if we reduce involutively h w.r.t F and P, then $\operatorname{lm}(h) \notin \mathcal{N}_i$ for $i = 1,2,3,4$, so it is reduced involutively. h is an involutive remainder for f.

Theorem 4.9 presents the relation between involutive reduction and the process of reduction in the setting of Gröbner bases of skew PBW extensions formulated in [GL11, Theorem 21].

THEOREM 4.9. *Let $F = \{f_1, \ldots, f_t\}$ be a finite set of nonzero polynomials of A and $f \in A$. Then the involutive reduction process produces polynomials $q_1, \ldots, q_t, h \in A$, with h reduced involutively w.r.t. F and an involutive division L, such that $f \xrightarrow{L,F} +h$ and $f = q_1f_1 + \cdots + q_tf_t + h$ with $q_i \in \mathbf{X}_{\operatorname{lm}(f_i)}$.*

Proof. If f is involutively reduced with respect to F and L, then $h = f, q_1 = \cdots = q_t = 0$ and $q_i \in \mathbf{X}_{\operatorname{lm}(f_i)}$. If f is not involutively reduced, then we make the first involutive reduction, $f \xrightarrow{L,F} h_1$, where $f = \sum_{j\in J_1} r_{j1}x^{\alpha_j}f_j + h_1$, with $J_1 := \left\{j \mid \operatorname{lm}\left(f_j\right)|_{L,\operatorname{lm}(F)} \operatorname{lm}(f)\right\}$ and $r_{j1} \in \Bbbk$. If h_1 is involutively reduced with respect to F, then the cycle WHILE ends and we have that $q_j = r_{j1}x^{\alpha_j}$ for $j \in J_1$ and $q_j = 0$ for $j \notin J_1$. Moreover, by Remark 18 $\alpha_j \in L(\exp\left(\operatorname{lm}\left(f_j\right)\right), \exp(F))$, i.e. $q_j \in \mathbf{X}_{\operatorname{lm}(f_j)}$ for $j \in J_1$. If h_1 is not reduced, we make the second involutive reduction with respect to F and L, $h_1 \xrightarrow{L,F} h_2$, with $h_1 = \sum_{j\in J_2} r_{j2}x^{\alpha_j}f_j + h_2$, $J_2 := \left\{j \mid \operatorname{lm}\left(f_j\right)|_{L,\operatorname{lm}(F)} \operatorname{lm}(h_1)\right\}$ and $r_{j2} \in \Bbbk$. We get

$$f = \sum_{j\in J_1} r_{j1}x^{\alpha_j}f_j + \sum_{j\in J_2} r_{j2}x^{\alpha_j}f_j + h_2.$$

If h_2 is involutively reduced with respect to F the procedure ends and we get that $q_j = q_j$ for $j \notin J_2$ and $q_j = q_j + r_{j2}x^{\alpha_j}$ for $j \in J_2$, and again we have $q_j \in \mathbf{X}_{\operatorname{lm}(f_j)}$. We can continue this way and the algorithm ends by the Proposition 4.8 and [FGL+20, Theorem 13.2.6]. □

PROPOSITION 4.10. *Let $I \neq 0$ be a left ideal of A. Let F be a nonempty finite subset of nonzero polynomials of I and L a involutive division on $\mathbb{N}_0^n$. If each element $0 \neq f \in I$ is involutively reducible w.r.t. F and L then F is a weak involutive basis of I.*

Proof. Let us show that $\langle \exp(\mathrm{lm}(F))\rangle_L = \exp(\mathrm{lm}(I))$. Since $F \subseteq I$, $\exp(\mathrm{lm}(F)) \subseteq \exp(\mathrm{lm}(I))$, so if $\mu \in C_{L,\exp(\mathrm{lm}(F))}(v)$ for some $v \in \exp(\mathrm{lm}(F))$, then $\mu \in C(v)$ for some $v \in \exp(\mathrm{lm}(I))$, whence $\langle \exp(\mathrm{lm}(F))\rangle_L \subseteq \exp(\mathrm{lm}(I))$.

Take $0 \neq f \in I$ an arbitrary element. By assummption, f is involutively reduced w.r.t F and L, so Remark18 guarantees that $\mathrm{lm}(f) = \mathrm{lm}\left(x^{\alpha_i}\mathrm{lm}\left(f_i\right)\right)$, and hence $\exp(\mathrm{lm}(f)) = \alpha_i + \exp\left(\mathrm{lm}\left(f_i\right)\right)$. Since $f_i \in F$, $\exp\left(\mathrm{lm}\left(f_i\right)\right) \in \exp(\mathrm{lm}(F))$ and $\alpha_i \in L\left(\exp\left(\mathrm{lm}\left(f_i\right)\right), \exp(\mathrm{lm}(F))\right)$ due to Theorem 4.9, and thus $\exp(\mathrm{lm}(f)) \in C_{L,\exp(\mathrm{lm}(F))}(f_i)$. In this way, $\exp(\mathrm{lm}(f))$ is an arbitrary element of $\exp(\mathrm{lm}(I))$, so $\langle \exp(\mathrm{lm}(F))\rangle_L \supseteq \exp(\mathrm{lm}(I))$, and therefore F is a weak involutive basis of I. □

The characterization of a weak (strong) involutive basis can be given through how the elements of the ideal I are represented. For skew PBW extensions this representation is due to Theorem 4.9.

THEOREM 4.11. *Let I be a non-zero ideal of a skew PBW extension A, $F = \left\{f_1, \ldots, f_r\right\} \subset I \setminus \{0\}$ a finite set and L an involutive division on $\mathbb{N}^n$. Then the following two statements are equivalent:*

(1) *F is a weak involutive basis of I.*

(2) *For any polynomial $f \in A$, $f \in I$ if and only if $f \xrightarrow{F,L}_+ 0$.*

Proof.
- (1) ⇒ (2) Let F be a weak involutive basis of I and $f \in I$. By Defintion 4.15, $\exp(\mathrm{lm}(f)) \in C_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(f_1))$ for some $f_1 \in F$, which implies that $\exp(\mathrm{lm}(f_1)) \mid_{L,\exp(\mathrm{lm}(F))} \exp(\mathrm{lm}(f))$, that is, $\mathrm{lm}(f_1) \mid_{L,\mathrm{lm}(F)} \mathrm{lm}(f)$. Due to Definition 4.17, there exists $h_1 \in A$ such that $f \xrightarrow{F,L} h_1$ with $\mathrm{lm}(f) \succ \mathrm{lm}(h_1)$ and $h_1 = f - c_1 x^{\alpha_1} f_1$, and so $h_1 \in I$. If $h_1 = 0$, then the assertion follows. In other case, we apply the same reasoning to h_1. Since $\mathrm{Mon}(A)$ is well-ordered, then $f \xrightarrow{F,L}_+ 0$.

  Now, if $f \xrightarrow{F,L}_+ 0$, then by Theorem 4.9 there exist elements $f_1, \ldots, f_t \in F$ and $q_1, \ldots, q_t \in A$ such that $f = q_1 f_1 + \cdots q_t f_t$, i.e., $f \in I$.
- (2) ⇒ (1) It follows from Proposition 4.10.

□

From Theorem 4.11, a representation for the elements of I in terms of the elements of the weak involutive basis is given as follows: $F = \{f_1, \ldots, f_t\}$ is a weak involutive basis of I if and only if

$$f = \sum_{i=1}^{t} q_i f_i \quad \text{with} \quad q_i \in \mathbf{X}_{\mathrm{lm}(f_i)}, \quad \text{for all } f \in I. \tag{4.7}$$

We call this representation the *standard involutive representation of f* since it is unique.

EXAMPLE 4.6. Consider the commutative polynomial ring $\Bbbk[x, y, z]$. The set $F = \{f_1 = x^2 + y^2, f_2 = x^2 z - 1, f_3 = y^2 z + 1\}$ is a weak `Involutive` basis w.r.t Janet division and the ideal I generated by $\left\{x^2 z - 1, y^2 z + 1\right\}$ [Sei10, Example 4.6.2]. For the element $f = f_2 - z f_1 = -y^2 z - 1$, it is clear that $f \xrightarrow{F,J}_+ 0$, whence we obtain a standard involutive representation for 0 given by $0 = z f_1 - f_2 - f_3$. This shows that the representation (4.7) is not unique.

COROLLARY 4.12. *F is a strong involutive basis if and only if the representation (4.7) is unique.*

Proof. Suppose that F is a strong `Involutive` basis, and take a standard involutive representation for an element $f \in I$ as in (4.7). By definition, there exists a unique element $f' \in F$ such that $\operatorname{lt}(qf') = \operatorname{lt}(f)$ which means that $\operatorname{lt}(q)$ is also unique. If we apply the same reasoning to $f - (\operatorname{lt} q)f'$, inductively we get that the representation (4.7) is also unique.

On the other hand, suppose that F is a weak but not strong `Involutive` basis of I. Then there exist elements $f_1, f_2 \in F$ with $C_{L,\exp(\operatorname{lm}(F))}(\exp(\operatorname{lm}(f_2)) \subseteq C_{L,\exp(\operatorname{lm}(F))}(\exp(\operatorname{lm}(f_1))$, whence $\operatorname{lm}(f_2) = \operatorname{lm}(ax^{\alpha_1} f_1)$ for some $a \in \Bbbk$ and $x^{\alpha_1} \in \operatorname{Mon}(A)$. Consider the element $f_2 - ax^{\alpha_1} f_1 \in I$. Note that if this polynomial is annihilated, then it is a non-trivial standard involutive representation of the zero element. In other case, there exists a standard involutive representation $f_2 - ax^{\alpha_1} f_1 = \sum_{i=1}^{t} q_i f_i$ with $q_i \in \mathbf{X}_{\operatorname{lm}(f_i)}$. By taking $q_i' := q_i$ for all $f_i \neq f_1, f_2$ and $q_1' = q_1 + ax^{\alpha_1}$, $q_2' = q_2 - 1$, we obtain a non-trivial standard involutive representation of $0 = \sum_{i=1}^{t} q_i' f_i$. By last, notice that the existence of a non-trivial standard involutive representation of 0 implies that the representation (4.7) is not unique. $\square$

As usual, it is easier to show when F is not a weak `Involutive` basis. The following result gives us a way to see this fact using Definition 4.16.

COROLLARY 4.13. *Let F be a weak involutive basis of the left ideal I. Then $\langle F\rangle_L = I$.*

Proof. Proposition 4.8 guarantees that if $f \xrightarrow{F,L}_+ 0$ for $f \in I$, then $f \longrightarrow_+ 0$ in the sense of [GL11, Definition 18]. Hence, F is a Gröbner basis for I [GL11, Theorem 24], and so $\langle F\rangle_L = I$. $\square$

EXAMPLE 4.7. Consider the 3-dimensional skew polynomial algebra A generated by the indeterminates x, y, z over the field of rational numbers $\mathbb{Q}$ subject to the relations $yx - q_2xy = x$, $zx - q_1xz = z$ and $zy = yz$. From Section 1.2.3 we know that $A \cong \sigma(\mathbb{Q})\langle x, y, z\rangle$. If $q_1 = \frac{3}{4}$ and $q_2 = \frac{2}{3}$, for the left ideal $I = \langle y^2z + 3xz, x^2z - yz\rangle$ with the monomial ordering `deglex` and $x \succ y \succ z$, the Buchberger's algorithm [FGL+20, Section 13.4] shows that $F = \{xz, yz\}$ is a Gröbner basis for I. The aim is to determine if F is a Pommaret weak involutive basis of I. Since $\mathcal{N}_1 = \{3\} = \mathcal{N}_2$ and $\mathbf{X}_{\operatorname{lm}(f_1)} = \{a_n z^n + \cdots + a_1 z + a_0 \mid a_i \in \Bbbk\} = \mathbf{X}_{\operatorname{lm}(f_2)}$, it follows that $\langle F\rangle_P = \mathbf{X}_{\operatorname{lm}(f_1)} \cdot (xz + yz)$ and $y^2z + 3xz \notin \langle F\rangle_P$, which implies that $\langle F\rangle_P \neq I$. Hence, F is not a weak `Involutive` Pommaret basis of I.

EXAMPLE 4.8. In the 2-*Heisenberg algebra* $\mathbf{H}_2(2) = \sigma(\mathbb{Q})\langle x_1, x_2, y_1, y_2, z_1, z_2\rangle$ (Section 1.2.1), consider the monomial ordering `deglex` with $x_1 \succ x_2 \succ y_1 \succ y_2 \succ z_1 \succ z_2$. Let $f_1 := x_1x_2y_1y_2$, $f_2 := x_2y_1$, and $f_3 := x_1z_2$. By using Buchberger's algorithm, a Gröbner basis for the left ideal $I = \langle f_1, f_2, f_3\rangle$ is given by the set $F = \{x_1x_2y_1y_2, x_2y_1, x_1z_2, x_1y_1y_2^2, x_1y_1y_2\}$ [FGL+20, Example 13.4.10]. Let us check if F is a Janet or Pommaret weak involutive basis of I.

(i) Since $\exp(\operatorname{lm}(F)) = \{(1,1,1,1,0,0), (0,1,1,0,0,0), (1,0,0,0,0,1), (1,0,1,2,0,0), (1,0,1,1,0,0)\}$, then $\mathcal{N}_1 = \{1,2,3,4,5,6\}$, $\mathcal{N}_2 = \{2,3,4,5,6\}$, $\mathcal{N}_3 = \{1,4,5,6\}$, $\mathcal{N}_4 = \{1,3,4,5,6\}$ and $\mathcal{N}_5 =$

$\{1,3,5,6\}$, whence

$$\begin{aligned}
\mathrm{Mon}(\mathcal{N}_1) &= \mathrm{Mon}(A),\\
\mathrm{Mon}(\mathcal{N}_2) &= \{x_2^{\alpha_1} y_1^{\alpha_2} y_2^{\alpha_3} z_1^{\alpha_4} z_2^{\alpha_5} \mid \alpha_i \in \mathbb{N}\},\\
\mathrm{Mon}(\mathcal{N}_3) &= \{x_1^{\beta_1} y_2^{\beta_2} z_1^{\beta_3} z_2^{\beta_4} \mid \beta_i \in \mathbb{N}\},\\
\mathrm{Mon}(\mathcal{N}_4) &= \{x_1^{\gamma_1} y_1^{\gamma_2} y_2^{\gamma_3} z_1^{\gamma_4} z_2^{\gamma_5} \mid \gamma_i \in \mathbb{N}\},\\
\mathrm{Mon}(\mathcal{N}_5) &= \{x_1^{\lambda_1} y_1^{\lambda_2} z_1^{\lambda_3} z_2^{\lambda_4} \mid \lambda_i \in \mathbb{N}\}.
\end{aligned}$$

In this way,

$$\begin{aligned}
\langle F\rangle_J = \mathbf{X}_{\mathrm{lm}(f_1)} \cdot (x_1x_2y_1y_2) + \mathbf{X}_{\mathrm{lm}(f_2)} \cdot (x_2y_1) + \mathbf{X}_{\mathrm{lm}(f_3)} \cdot (x_1z_2)\\
+ \mathbf{X}_{\mathrm{lm}(f_4)} \cdot (x_1y_1y_2^2) + \mathbf{X}_{\mathrm{lm}(f_5)} \cdot (x_1y_1y_2).
\end{aligned}$$

Since $g(x) = x_1(x_2y_1) \in I$ but $g(x) \notin \langle F\rangle_J$, whence $\langle F\rangle_J \neq I$. This means that F is not a weak `Involutive` Janet basis of I.

(ii) Note that $\mathcal{N}_1 = \{4,5,6\} = \mathcal{N}_4 = \mathcal{N}_5$, $\mathcal{N}_2 = \{3,4,5,6\}$ and $\mathcal{N}_3 = \{6\}$, which implies that

$$\begin{aligned}
\mathrm{Mon}(\mathcal{N}_1) &= \{y_2^{\alpha_1} z_1^{\alpha_2} z_2^{\alpha_3} \mid \alpha_i \in \mathbb{N}\} = \mathrm{Mon}(\mathcal{N}_4) = \mathrm{Mon}(\mathcal{N}_5),\\
\mathrm{Mon}(\mathcal{N}_2) &= \{y_1^{\beta_1} y_2^{\beta_2} z_1^{\beta_3} z_2^{\beta_4} \mid \beta_i \in \mathbb{N}\},\\
\mathrm{Mon}(\mathcal{N}_3) &= \{z_2^{\gamma_1} \mid \gamma_1 \in \mathbb{N}\},
\end{aligned}$$

and hence

$$\begin{aligned}
\langle F\rangle_P = \mathbf{X}_{\mathrm{lm}(f_1)} \cdot (x_1x_2y_1y_2) + \mathbf{X}_{\mathrm{lm}(f_2)} \cdot (x_2y_1) + \mathbf{X}_{\mathrm{lm}(f_3)} \cdot (x_1z_2) + \mathbf{X}_{\mathrm{lm}(f_4)} \cdot (x_1y_1y_2^2)\\
+ \mathbf{X}_{\mathrm{lm}(f_5)} \cdot (x_1y_1y_2).
\end{aligned}$$

By using that $g(x) = x_1^2x_2y_1y_2 \in I$ and $g(x) \notin \langle F\rangle_P$, we obtain $\langle F\rangle_P \neq I$, and hence F is not also a weak `Involutive` Pommaret basis of I.

The involutive reduction process given at the beginning of this chapter allows us to relate in a more direct way the theory of Gröbner bases developed in [Gal15] with the theory of `Involutive` bases of skew PBW extensions. Next, we present how this reduction process allows us to introduce the notion of *involutively reduced set* and *involutively head reduced set.*

DEFINITION 4.18. The set F is *involutively autoreduced* if no polynomial $f = \sum_{i=1}^{t} r_iX_i \in F$ contains a monomial X_i such that another polynomial $f' \in F \setminus \{f\}$ exists with $\mathrm{lm}(f') \mid_{L,\mathrm{lm}(F)} X_i$. F is *involutively head autoreduced* if no element $f \in F$ satisfies $\mathrm{lm}(f') \mid_{L,\mathrm{lm}(F)} \mathrm{lm}(f)$ for $f' \in F \setminus \{f\}$.

Let us note that the definition of a strong involutive basis immediately implies that it is involutively head autoreduced.

DEFINITION 4.19. Let I be a non-zero ideal of A and L an involutive division. An `Involutive` basis F of I with respect to L is *minimal* if $\exp(\mathrm{lm}(F))$ is the minimal involutive basis of the monoid ideal $\exp(\mathrm{lm}(I))$ for the division L (see [Sei10, Definition 3.1.20]).

PROPOSITION 4.14. *Let I be a non-zero ideal of A and L an involutive division. Then I possesses at most one monic, involutively autoreduced, minimal involutive basis for the division L.*

Proof. Suppose that there exist F_1 and F_2 minimal involutive bases, involutively reduced and monic, different from I with respect to L. By definition of minimal involutive basis, $\operatorname{lm}(F_1) = \operatorname{lm}(F_2)$, and since $F_1 \neq F_2$ there exist $f_1 \in F_1$ and $f_2 \in F_2$ such that $\operatorname{lm}(f_1) = \operatorname{lm}(f_2)$ but $f_1 \neq f_2$. Let $f := f_1 - f_2 \in I$. Note that $f \in \langle F_1 \rangle_L$ and $\operatorname{lm}(f)$ is a monomial in the unique representation of f_1 or f_2 (Remark 1 (iv)). However, this implies that F_1 of F_2 is not involutively autoreduced, a contradiction. □

4.4.2 EXISTENCE AND CORRECTNESS

With the aim of formulating an algorithm to compute an `Involutive` basis of a left ideal of a skew PBW extension, we consider the algorithm [FGL+20, Division algorithm in A] to Gröbner bases theory together Seiler's ideas [Sei10, Section 4.2]. As expected, we consider an involutive division instead a general algorithm division. Algorithm 3 gives us an involutive standard representation according with Theorem 4.9.

Algorithm 3: Involutive Standard Representation

INPUT :A fixed monomial ordering $\succeq$ on $\operatorname{Mon}(A)$, a involutive divison L,
$F = \{f_1, \ldots, f_t\} \subset A$ and $f \in A$.
OUTPUT: $q_1, \ldots, q_t, h \in A$ with $f = q_1 f_1 + \cdots + q_t f_t + h$, h reduced w.r.t. F and $q_i \in \mathbf{X}_{\operatorname{lm}(f_i)}$
INITIALIZATION: $q_1 := 0, q_2 := 0, \ldots, q_t := 0, h := f$;

while $h \neq 0$ *and there exists* j *such that* $\operatorname{lm}(f_j) \mid_{L,\operatorname{lm}(F)} \operatorname{lm}(h)$ **do**
  Calculate $J := \{ j \mid \operatorname{lm}(f_j) \mid_{L,\operatorname{lm}(F)} \operatorname{lm}(h) \}$

for $j \in J$ **do**
  Calculate $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(\operatorname{lm}(f_j)) = \exp(\operatorname{lm}(h))$

if *the equation* $\operatorname{lc}(h) = \sum_{j \in J} r_j \sigma^{\alpha_j}(\operatorname{lc}(f_j))\, d_{\alpha_j, \exp(\operatorname{lm}(f_j))}$ *is soluble, where* $d_{\alpha_j, \exp(\operatorname{lm}(f_j))}$ *are defined as in Proposition 1.2* **then**
  Calculate one solution $(r_j)_{j \in J}$
  $h := h - \sum_{j \in J} r_j x^{\alpha_j} f_j$

for $j \in J$ **do**
  $q_j := q_j + r_j x^{\alpha_j}$

else
  Stop

We can obtain a simple implementation of `Algorithm` 3 for computing a principal involutive autoreduction (c.f. [Sei10, Algorithm 4.2]).

`Algorithm` 4 terminates since in every iteration of the bucle **while** a polynomial $h \in H$ is eliminated or replaced by other polynomial $\overline{h}$ with $\operatorname{lm}(\overline{h})$ less than $\operatorname{lm}(h)$ (recall that each monomial ordering on $\operatorname{Mon}(A)$ is a well-order, see Definition 1.3(iv), so the bucle finishes).

`Algorithm` 5 follow the corresponding formulated by Seiler [Sei10, $\texttt{MonomialComplete}_{L,\prec}$].

Algorithm 4: Principal Involutive Autoreduction

INPUT :A fixed monomial ordering $\succeq$ on $\mathrm{Mon}(A)$, a involutive divison L and $F \subset A$ finite

OUTPUT:an involutively head autoreduced set H with $\langle H\rangle = \langle F\rangle$

INITIALIZATION: $H := F$;

while *exist* $h \in H, f \in H\backslash\{h\} : \mathrm{lm}(f) \mid_{L,\mathrm{lm}(H)} \mathrm{lm}(h)$ **do**

  choose such a pair (h, f)

  Calculate $\mu := \exp(\mathrm{lm}(h)) - \exp(\mathrm{lm}(f))$; $c := \mathrm{lc}(h)/\mathrm{lc}(x^\mu f)$

  $H := H\backslash\{h\}$; $\overline{h} := h - cx^\mu f$

if $\overline{h} \neq 0$ **then**

  $H := H \cup \{\overline{h}\}$

return H

Algorithm 5:

INPUT :A fixed monomial ordering $\succeq$ on $\mathrm{Mon}(A)$, a involutive divison L and $F \subset A$ finite

OUTPUT:an involutively completion set $\overline{F}$

INITIALIZATION: $\overline{F} := F$; $S :=$
$\left\{x_i f \mid f \in \overline{F}, x_i \text{ is not a multiplicative variable of } \mathrm{lm}(f), \exp(\mathrm{lm}(x_i f)) \notin \left\langle \exp(\mathrm{lm}(\overline{F}))\right\rangle_L\right\}$

while $S \neq \emptyset$ **do**

  choose $s \in S$ such that S does not contain a proper involutive divisor of $\mathrm{lm}(s)$;

  $\overline{F} := \overline{F} \cup \{s\}$

return S

`Algorithms` 3, 4 and 5 generate `Algorithm` 6 to compute `Involutive` bases of left ideals of skew PBW extensions.

With the aim of obtaining the *correctness* of `Algorithm` 6 (Theorem 4.16), we need Definition 4.20 and Proposition 4.15.

DEFINITION 4.20. A finite set $F \subset A$ is called involutive up to the multi index $\lambda \in \mathbb{N}_0^n$ (*partially involutive*) for the involutive division L, if for every $f \in F$ and every monomial $x^\alpha \in \mathrm{Mon}(A)$ such that $\exp(\mathrm{lm}(x^\alpha f)) \prec \lambda$, the remainder involutive of $x^\alpha f$ with respect to F and L vanishes, i.e., we have $x^\alpha f \in \langle F\rangle_L$. The set F is *locally involutive* up to the multi index $\lambda \in \mathbb{N}_0^n$ for the division L, if $x_i f \in \langle F\rangle_L$ for any non-multiplicative variable of any polynomial $f \in F$ (such that $\exp(\mathrm{lm}(x_i f)) \prec \lambda$).

PROPOSITION 4.15. *Given a continuous involutive division L, any finite involutively head autoreduced set and locally involutive $F \subset A$ is involutive.*

Proof. Consider a "*complete*" local involutive division, that is, without some restriction for some multi index λ. Of course, the proof remanis valid for any *partial* involutive division.

The idea is to show that for every monomial $x^\alpha \in \mathrm{Mon}(A)$ and each $f_1 \in F$, there exists $h \in F$ such that $\exp(\mathrm{lm}(x^\alpha f_1)) \in C_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(h)))$. As it is clear, if x^α consists of only multiplicative indeterminates for f_1, taking $h := f_1$ the assertion follows.

Algorithm 6: Involutive basis in A

INPUT :A fixed monomial ordering $\succeq$ on $\mathrm{Mon}(A)$, a involutive division L and $F \subset A$ finite
OUTPUT : an involutive basis $\mathscr{F}$ of $I = \langle F\rangle$ with respect to L
INITIALIZATION: $\mathscr{F} := F$; $S := \emptyset$

repeat
$\mathscr{F} := \mathscr{H}$, with $\mathscr{H}$ an involutively head autoreduced set of $\mathscr{F} \cup S$
$\mathscr{F} := \overline{\mathscr{F}}$, with $\overline{\mathscr{F}}$ an involutively completion set of $\mathscr{F}$
$S := \{h \mid x_i f \xrightarrow{L,F} +h, f \in \mathscr{F}, x_i$ is not a multiplicative variable of $\mathrm{lm}(f)\}\setminus\{0\}$
until $S = \emptyset$;
return $\mathscr{F}$

On the contrary, let $i_1 \in \overline{N}_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(f_1)))$ with $\alpha_{i_1} > 0$. By assumption, F is locally involutive, so for every $f \in F$ there exists $p_f^{(1)} \in \mathbf{X}_{\mathrm{lm}(f)}$ such that $x_{i_1} f_1 = \sum_{f\in F} p_f^{(1)} f$. Note that if we suppose that the leading monomial of the right hand of this expression is given by $\mathrm{lm}(p_{f_2}^{(1)} f_2)$, then $\exp(\mathrm{lm}(x^\alpha f_1)) = \alpha + \exp(\mathrm{lm}(f_2)) + \exp(\mathrm{lm}(p_{f_2}^{(1)} - 1_{i_1})$, due to that if $p_{f_2}^{(1)} \in \mathbf{X}_{\mathrm{lm}(f_2)}$ then $\mathrm{lm}(p_{f_2}^{(1)})$ only posses multiplicative variables of f_2. If we take $h := f_2$ and $x^{\alpha - 1_{i_1}}$ only posses multiplicative indeterminates of f_2, then we are done.

Again, on the contrary, let $i_2 \in \overline{N}_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(f_2)))$ with $(\alpha_{i_1} - 1_{i_1})_{i_2} > 0$. By applying once more again the local involution of F, we get that for every $f \in F$ there exists $p_f^{(2)} \in \mathbf{X}_{\mathrm{lm}(f)}$ with $x_{i_2} f_2 = \sum_{f\in F} p_f^{(2)} f$. Suppose that the leading monomial of the right hand of the expression above is $\mathrm{lm}(p_{f_3}^{(2)} f_3)$, and let $\beta = \exp(\mathrm{lm}(x_{i_1} f)) - \exp(\mathrm{lm}(f_2))$, whence $\exp(\mathrm{lm}(x^\alpha f_1)) = \exp(\mathrm{lm}(x^{\alpha+\beta-1_{i_1}-1_{i_2}} f_3))$. Note that for $h := f_3$, if $x^{\alpha+\beta-1_{i_1}-1_{i_2}}$ only have multiplicative indeterminates of f_3, then the proof concludes. If this is not the case, we iterate the process by choosing a non-multiplicative index i_3 and decompose $x_{i_3} f_3$ (by using the local involutiveness of F) into multiplicative products, and so we obtain a polynomial f_4. By repeating this procedure we obtain a sequence $v^{(1)} = \exp(\mathrm{lm}(f_1)), v^{(2)} = \exp(\mathrm{lm}(f_2)), \dots$ with $v^{(k)} \in \exp(\mathrm{lm}(F))$, and for each $v^{(k)}$ there exists an index $i_k \in \overline{N}_{L,\exp(\mathrm{lm}(F))}(v^{(k)})$ such that $v^{(k+1)} \mid_{L,\exp(\mathrm{lm}(F))} v^{(k)} + 1_{i_k}$. Since the division is continuous, this sequence cannot be infinite and this process must terminate. In this way, we assert the existence of a polynomial $h \in F$ satisfying $\exp(\mathrm{lm}(x^\alpha f_1)) \in C_{L,\exp(\mathrm{lm}(F))}(\exp(\mathrm{lm}(h)))$ for every monomial $x^\alpha \in \mathrm{Mon}(A)$ and each $f_1 \in F$. Therefore, we have the possibility of consider linear combinations by using these products (and coefficients in the field $\Bbbk$) in such a way that the leading monomials cannot be reduced since F is head involutively reduced, whence the leading exponent of every polynomial in $\langle F\rangle$ belongs to the monomial ideal $\langle \exp(\mathrm{lm}(F))\rangle_L$, that is, F is involutive. □

THEOREM 4.16. *Let L be a constructive Noetherian division and A a skew PBW extension over a field $\Bbbk$. Then for any finite input set $F \subset A$,* `Algorithm` 6 *terminates with an* `Involutive` *basis $\mathscr{F}$ of $I = \langle F\rangle$.*

Proof. The correctness of the `Algorithm` 6 follows from Proposition 4.15. Let us show that it terminates. Since L is Noetherian, the set $\exp(\mathrm{lm}(\mathscr{F}))$ in the second step of the bucle **repeat** posses a finite involutive completation that has an end due to [Sei10, Proposition 4.2.1]. Then, the set itself is an `Involutive` basis, which means that for every polynomial $g \in S$, we get $\exp(\text{lm}(g)) \notin \langle \exp(\text{lm}(\mathscr{F})) \rangle$. Thus, all these monoidal ideals form a strict ascendant sequence that cannot be infinite [Sei10, Lemma A.1.2]. This concludes the proof. □

EXAMPLE 4.9. Let A, I and F as in Example 4.7. If we use the Janet division J which is constructive Noetherian and apply the Algorithm 4, then we get that F is an involutively head autoreduced set. Now, if we apply the Algorithm 5, then F is an involutively completion of F. Finally, the Algorithm 6 guarantees that F is an `Involutive` Janet basis of I.

REMARK 19. It is important to note that if the division is non-Noetherian, then `Algorithm` 6 could not terminate, and hence we obtain an infinite `Involutive` basis [Sei10, Example 4.2.9].

4.5 FUTURE WORK

As we said in Section 4.3, Seiler [Sei10] formulated a detailed theory of `Involutive bases` of its polynomial algebras of solvable type. From the theoretic point of view [Sei10, Chapter 3] and its algorithmic implementation [Sei10, Section 4.2], he extends the theory to non-term orders [Sei10, Section 4.5] (c.f. Hausdorff et al. [HSS02]) and used it to characterize several homological properties of commutative polynomial rings [Sei10, Chapter 6]. Having these facts in mind, by considering the approach to `Involutive` bases presented in Section 4.4, our interest is to implement computationally these ideas in an analogous way as was made for Gröbner bases (see [FGL+20, Appendices C, D and E]). Of course, a possible extension of our results to the setting of skew PBW extensions over *Gröbner-soluble* rings will also be considered. By last, the question on the importance of non-term orders with the technique of homogenization and dehomogenization remains open.

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