Algebro-geometric characterizations of commuting differential operators in semi-graded rings

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TITLE

ALGEBRO-GEOMETRIC CHARACTERIZATIONS OF COMMUTING DIFFERENTIAL OPERATORS IN SEMI-GRADED RINGS

TÍTULO

CARACTERIZACIONES ALGEBRO-GEOMÉTRICAS DE OPERADORES DIFERENCIALES CONMUTATIVOS EN ANILLOS SEMI-GRADUADOS

ABSTRACT: In this thesis, we study algebro-geometric characterizations of commuting differential operators in families of semi-graded rings. First, we present some ring-theoretical notions of semi-graded rings that are necessary throughout the thesis. We include a non-exhaustive list of noncommutative rings that are particular examples of these rings. Second, to motivate the study of commuting differential operators beloging to noncommutative algebras, and hence to develop a possible Burchnall-Chaundy ($\mathscr{BC}$) theory for them, we review algebraic and matrix results appearing in the literature on the theory of these operators in some families of semi-graded rings. Third, we introduce the notion of pseudo-multidegree function as a generalization of pseudo-degree function, and hence we establish a criterion to determine whether the centralizer of an element has finite dimension over a noncommutative ring having PBW basis. In this way, we formulate a $\mathscr{BC}$ theorem for rings having pseudo-multidegree functions. We illustrate our results with families of algebras appearing in ring theory and noncommutative geometry. Fourth, we develop a first approach to the $\mathscr{BC}$ theory for quadratic algebras having PBW bases defined by Golovashkin and Maksimov. We prove combinatorial properties on products of elements in these algebras, and then consider the notions of Sylvester matrix and resultant for quadratic algebras with the purpose of exploring common right factors. Then, by using the concept of determinant polynomial, we formulate the version of $\mathscr{BC}$ theory for these algebras. We present illustrative examples of the assertions about these algebras. Finally, we establish some bridging ideas with the aim of extending results on centralizers for graded rings to the setting of semi-graded rings.

RESUMEN: En esta tesis, estudiamos caracterizaciones álgebro-geométricas de operadores diferenciales conmutativos en familias de anillos semi-graduados. Primero, presentamos algunas nociones de la teoría de anillos de anillos semi-graduados que son necesarias a lo largo de la tesis. Incluimos una lista no exhaustiva de anillos no conmutativos que son ejemplos particulares de estos anillos. Segundo, para motivar el estudio de operadores diferenciales conmutativos pertenecientes a álgebras no conmutativas, y así desarrollar una posible teoría Burchnall-Chaundy ($\mathscr{BC}$) para ellos, consideramos resultados algebraicos y matriciales presentes en la literatura sobre la teoría de estos operadores en algunas familias de anillos semi-graduados. Tercero, introducimos la noción de función pseudo-multigrado como una generalización de función pseudo-grado, y así establecemos un criterio para determinar si el centralizador de un elemento tiene dimensión finita sobre un anillo no conmutativo con base PBW. De esta manera, formulamos un teorema ($\mathscr{BC}$) para anillos que tienen funciones pseudo-multigrado. Ilustramos nuestros resultados con familias de álgebras presentes en la teoría de anillos y la geometría no conmutativa. Cuarto, desarrollamos un primer acercamiento a la teoría ($\mathscr{BC}$) para las álgebras cuadráticas con base PBW definidas por Golovashkin y Maksimov. Demostramos propiedades combinatoriales sobre productos de elementos en estas álgebras, y luego consideramos las nociones de matriz de Sylvester y resultante para álgebras cuadráticas con el fin de explorar factores comunes a derecha. Después, utilizando el concepto de determinante polinomial, formulamos la versión de la teoría ($\mathscr{BC}$) para estas álgebras. Presentamos ejemplos ilustrativos de las afirmaciones sobre estas álgebras. Finalmente, formulamos algunas ideas con el propósito de extender resultados sobre centralizadores para anillos graduados al contexto de los anillos semi-graduados.

KEYWORDS: Semi-graded ring, quantum algebra, Ore extension, PBW basis, valuation, Sylvester matrix, resultant, determinant polynomial, centralizer, Gelfand-Kirillov dimension.

PALABRAS CLAVE: Anillo semi-graduado, álgebra cuántica, extensión de Ore, base PBW, valuación, matriz de Sylvester, resultante, polinomio determinante, centralizador, dimensión de Gelfand-Kirillov.

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To my parents, Alice Torres and Adan Niño

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CONTENTS

Introduction

J. L. Burchnall and T. W. Chaundy wrote a series of papers [BC23, BC28, BC31] where they were interested in to find an algebraic curve that vanishes on two commuting differential operators. The ring of differential operators considered by them is given by the skew polynomial ring or Ore extension (introduced by Ore [Ore31, Ore33]) of derivation type $C^{\infty}(\mathbb{R},\mathbb{C})[D;\mathrm{id},\delta]$, where δ is the ordinary derivation. They found that given $P = \sum_{i=0}^{n} p_i D^i$ and $Q = \sum_{i=0}^{m} q_i D^i$ two differential operators such that $PQ - QP = 0$, where p_i, q_i are complex valued functions, there exists a complex algebraic curve $\mathscr{BC}$ determinated by some polynomial $F(x, y)$ such that $F(P,Q) = 0$. The points on this curve have coordinates which are exactly the eigenvalues associated with the operators P and Q. Since then, in the literature this theory is known as *Burchnall-Chaundy $\mathscr{BC}$ theory*.

$\mathscr{BC}$ theory was forgotten until in the 1970s Krichever in several papers discovered that these results can be useful in the study of integrable systems (e.g. [Kri76, Kri77a, Kri77b, Kri78b, Kri78a, Kri79, KN80]). As a matter of fact, in [Kri77a], Krichever established relationships between commuting differential operators and Riemann surfaces. Briefly, the idea was to relate integrable non-linear differential equations with their solutions to properties of algebraic curves and algebraic methods (Krichever allows matrix-valued coefficients, but restricts himself to the study of commuting pairs L and T both of whose leading coefficients must be constant nonsingular diagonal matrices). Note that this idea was also considered by Veselov [Ves79, Ves91], and Wilson [Wil79, Wil80]. A detailed treatment about the relationship between integrable systems and algebraic curves can be found in Mulase [Mul94], Mumford [Mum77], and Previato [PW92, Pre96]. Precisely, Mumford's paper contains an algebro-geometric construction of commuting operators and of solutions to the Toda Lattice equation and Korteweg deVries equation.

From an algebraic point of view, Burchnall and Chaundy's results can be translated into studying the algebraic dependence of a pair of commuting elements in a appropriate algebra. One of the key results in this regard has been to identify a well characterization of centralizers as finite-dimensional modules over a suitable ring of polynomials. A complete discussion of this line of research until the 1950s can be found in Flanders [Fla55]. Some years later, Amitsur [Ami58] obtained results that contributed to the solution of this problem in a more general setting. Following Amitsur's line of research, Carlson and Goodearl [CG80, Goo83] investigated the algebraic and analytic structures of families of differential operator rings.

In 1994, de Jeu et al. [dJSS09] formulated in a conjecture if this kind of research can be carried out in algebras such as the q-deformed Heisenberg algebra generated over a field $\Bbbk$

by elements A and B subject to the relation $AB - qBA = 1$. Some advances have been made to identify the key property that implies a description of centralizers belonging to this kind of algebras. For instance, Hellstrom and Silvestrov defined and studied the property $l - BDHC$ [HS07], and subsequently Richter defined the property $D(l)$ [Ric16]. It is very important to mention that both properties are defined in terms of *graded* algebras, or *pseudo-graded* algebras. These properties describe the growth of the centralizer of an element in terms to the graduation is bounded. In this way, another important approximation to this topic is to study the behavior of centralizers in terms of the Gelfand Kirillov dimension such as Bell and Small [BS04, Bel09] showed. A remarkable fact is that the $\mathscr{BC}$ curve can be found using the notion of *resultant* and *polynomial determinants* of the differential operators P and Q. This technique has been used in different algebras (e.g. Larsson [Lar14], Richter [Ric14b, RS09], Silvestrov and DeJeu [SSdJ80] and Previato et al. [PSZ23]).

On the other hand, Lezama and Latorre [LL17] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type that are not $\mathbb{N}$-graded (of course not in a trivial way). In that paper, they considered some notions of noncommutative algebraic geometry in the setting of semi-graded rings. Ring-theoretical, algebraic and geometrical properties of these rings have been investigated by some mathematicians (e.g. [AT24, Cha22, Faj18, FGL$^+$20, Gal15, Lez20, Lez21, Su7b] and references therein).

Having in mind the historical advances of the theory, we can address the following three problems:

(1) Investigate necessary and sufficient conditions for the well description of centralizers in noncommutative algebras of polynomial type.

(2) Study the behavior of the resultant technique for constructing annihilating curves in other noncommutative algebras such as quadratic algebras.

(3) Describe how is the growth of centralizers in terms of some graduations or semi-graduations.

In this thesis, we consider algebraic techniques that contribute to the mentioned problems. Chapter 1 presents the algebraic structures of our interest: *semi-graded rings*. We recall families of examples and some ring-theoretical notions of these objects that are necessary in Chapters 2 and 3. Precisely, in Chapter 2 we review some of the most important results on the theory of these operators in some families of semi-graded rings. This is a framework of techniques until today, which includes algebraic and matrix methods. Finally, Chapter 3 contains the original results of the thesis. We introduce the notion of *pseudo-multidegree function* as a generalization of pseudo-degree function, and hence we establish a criterion to determine whether the centralizer of an element has finite dimension over a noncommutative ring having PBW basis. We prove several results for this kind of rings, and extend corresponding results in the literature, so we formulate a $\mathscr{BC}$ theory for rings having pseudo-multidegree functions. Next, we formulate a first approach to the $\mathscr{BC}$ theory for quadratic algebras having PBW bases defined by Golovashkin and Maksimov. With this purpose, we present combinatorial properties on products of elements in these algebras, and consider the notions of Sylvester matrix and resultant for quadratic algebras with the aim of determining common right factors of polynomials. Then, by using the concept of determinant polynomial, we formulate the version of $\mathscr{BC}$ theory for these algebras.

Of course, we illustrate our results with different families of noncommutative algebras. Finally, we establish some bridging ideas with the aim of extending results on centralizers for graded rings to the setting of semi-graded rings.

Notation and some terminology

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers including zero
$\mathbb{Z}$	The ring of integer numbers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
R	Associative ring (not necessarily commutative) with identity
R^*	The non-zero elements of the ring R
K	Commutative ring with identity
D	Division ring
$\Bbbk, \mathbb{K}$	Field
$\mathrm{char}(\Bbbk)$	Characteristic of the field $\Bbbk$
$R[x]$	Commutative polynomial ring in the indeterminate x over R
$R[[x]]$	Ring of formal power series in the indeterminate x over R
$Z(R)$	The center of R
$N(R)$	The set of nilpotent elements of R
$C(r;R)$	Centralizer of the element r in the ring R
$M_{r\times c}(R)$	The ring of matrices of size $r \times c$ with entries in R
$\{n\}_q$	$\sum_{i=1}^{n} q^{k-1}, \quad \{0\}_q = 0, \quad q \in \Bbbk^* \quad$ (if $q = 1$, then $\{n\}_q = n$; if $q \neq 1$, then $\{n\}_q = \frac{q^n-1}{q-1}$)
$\{n\}_q!$	$\prod_{i=1}^{n} \{i\}_q, \quad \{0\}_q! = 1, \quad q \in \Bbbk^*$
$\binom{n}{i}_q$	$\frac{\{n\}_q!}{\{k\}_q!\{n-k\}_q!}, \quad i = 0, \dots, n, \quad q \in \Bbbk^*$

Throughout this thesis, the term ring means an associative (not necessarily commutative) ring with identity, and the term module means a left unital module.

Statement of contributions

The chapter three in this thesis corresponds to the following papers containing original results.

- **Niño, A.**, Reyes, A. On centralizers and pseudo-multidegree functions for non-commutative rings having PBW bases. *Journal of Algebra and Its Applications* (2023). Available online at https://www.worldscientific.com/doi/10.1142/S0219498825501099
- **Niño, A.**, Ramírez, M. C., Reyes, A. A first approach to the Burchnall-Chaundy theory for quadratic algebras having PBW bases (2024) [Manuscript submitted for publication]. Available online at https://arxiv.org/abs/2401.10023
- **Niño, A.**, Reyes, A. Some remarks on centralizers in semi-graded rings. Preprint.

CHAPTER 1

SEMI-GRADED RINGS

In this chapter, we present the algebraic structures of interest in this thesis: the *semi-graded rings*. We recall families of examples and some ring-theoretical notions of these objects that are necessary in Chapters 2 and 3.

More exactly, Section 1.1 contains definitions and some key properties of semi-graded rings, finitely semi-graded rings and modules over these rings. Next, in Section 1.2 we present a list (not exhaustive) of noncommutative algebraic structures that are particular examples of semi-graded rings. Our aim in this section is to show explicitly the generality of these rings in the setting of ring theory, noncommutative algebraic geometry and noncommutative differential geometry.

1.1 PRELIMINARIES AND KEY PROPERTIES

Lezama and Latorre [LL17] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type that are not $\mathbb{N}$-graded (not in a trivial way). In that paper, they considered some notions of noncommutative algebraic geometry in the setting of semi-graded rings such as the Hilbert series, Hilbert polynomial and Gelfand-Kirillov dimension. As a matter of fact, in that paper, they extended the notion of noncommutative projective scheme to the context of semi-graded rings and generalized the well-known Serre-Artin-Zhang-Verevkin theorem (see also [Lez21, CR23]).

Next, we recall briefly some definitions and results about semi-graded rings which are key in the following chapters.

DEFINITION 1.1 ([LL17, DEFINITION 2.1]). Let R be a ring. R is said to be *semi-graded* (SG) if there exists a collection $\{R_n\}_{n\in\mathbb{Z}}$ of subgroups of the additive group R^+ such that the following conditions hold:

(i) $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

(ii) For every $m, n \in \mathbb{Z}$, $R_m R_n \subseteq \bigoplus_{k\leq m+n} R_k$.

(iii) $1 \in R_0$.

The collection $\{R_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* R, and we say that the elements of R_n are *homogeneous of degree* n.

We say that R is *positively semi-graded* if $R_n = 0$, for every $n < 0$. If R and S are semi-graded rings and $f : R \to S$ is a ring homomorphism, then we say that f is *homogeneous* if $f(R_n) \subseteq S_n$, for every $n \in \mathbb{Z}$.

Definitions 1.2 and 1.3 recall the notion of finitely semi-graded ring and finitely semi-graded algebra, respectively.

Definition 1.2 ([LL17, Definition 2.4]). A ring R is called *finitely semi-graded* (FSG) if it satisfies the following conditions:

(i) R is SG.

(ii) There exist finitely many elements $x_1, \dots, x_n \in R$ such that the subring generated by R_0 and $x_1, \dots, x_n$ coincides with R.

(iii) For every $n \geq 0$, R_n is a free R_0-module of finite dimension.

Definition 1.3 ([LG19, Definition 10]). A $\Bbbk$-algebra R is said to be *finitely semi-graded* (FSG) if the following conditions hold:

(i) R is an FSG ring with semi-graduation given by $R = \bigoplus_{n\geq 0} R_n$.

(ii) For every $m, n \geq 1$, $R_m R_n \subseteq R_1 \oplus \cdots \oplus R_{m+n}$.

(iii) R is connected, i.e., $R_0 = \Bbbk$.

(iv) R is generated in degree 1.

From Definition 1.3, it is straightforward to see that if R is a FSG $\Bbbk$-algebra, then $R_+ := \bigoplus_{n\geq 1} R_n$ is a maximal ideal of R.

Notice that graded rings are SG. Finitely graded $\Bbbk$-algebras, PBW extensions [BG88], 3-dimensional skew polynomial rings [BS90], bi-quadratic algebras on three generators with PBW bases [Bav23], down-up algebras [Ben99b, BR98], diffusion algebras [IPR01] and skew PBW extensions [GL11] are examples of FSG rings. Definitions of these families of algebras and others are considered in Section 1.2.

Semi-graded rings and finitely semi-graded rings have been studied recently in the literature. For instance, Lezama et al. [Lez21, LG19] computed the set of point modules of finitely semi-graded rings. By considering the parametrization of the point modules for the quantum affine n-space, Lezama obtained the set of point modules for some important examples of non $\mathbb{N}$-graded quantum algebras [Lez20, Theorem 5.3].

Next, we present some results about modules in the setting of semi-graded rings.

DEFINITION 1.4 ([LL17, DEFINITION 2.1]). Let R be an SG ring and let M be an R-module. We say that M is *semi-graded* (SG) if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of the additive group M^+ such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $m, n \in \mathbb{Z}$, $R_m M_n \subseteq \bigoplus_{k\le m+n} M_k$.

The collection $\{M_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* M, and we say that the elements of M_n are *homogeneous of degree* n.

M is said to be *positively semi-graded* if $M_n = 0$, for every $n < 0$. Let $f : M \to N$ be a homomorphism of R-modules, where M and N are semi-graded R-modules. We say that f is *homogeneous* if $f(M_n) \subseteq N_n$, for every $n \in \mathbb{Z}$.

DEFINITION 1.5 ([LL17, DEFINITION 2.3]). Let R be an SG ring, M an SG R-module, and N a submodule of M. We say that N is a *semi-graded* (SG) *submodule of* M if $N = \bigoplus_{n\in\mathbb{Z}} N_n$, where $N_n = M_n \cap N$. In this case, N is an SG R-module.

PROPOSITION 1.1 ([LL17, PROPOSITION 2.6]). *If R is an* SG *ring, M is an* SG *R-module, and N is a submodule of M, then the following conditions are equivalent:*

(1) *N is a semi-graded submodule of M.*

(2) *For every $z \in N$, the homogeneous components of z belong to N.*

(3) *M/N is an* SG *R-module with semi-graduation given by*

$$(M/N)_n = (M_n + N)/N, \ n \in \mathbb{Z}.$$

REMARK 1. Let R be an SG ring and M be an SG R-module. Then:

(i) If N is an SG submodule of M, then the canonical map $M \to M/N$ is a homogeneous homomorphism.

(ii) If $\{M_i\}_{i\in I}$ is a family of SG submodules of M, then $\bigcap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are SG submodules of M.

DEFINITION 1.6. If R is a positively SG ring, for $t \in \mathbb{N}$ we define $R_{\ge t}$ as the intersection of all two-sided ideals that are SG submodules containing $\bigoplus_{k\ge t} R_k$.

Different properties of modules over families of semi-graded rings have been investigated by some people [Art15, Cha22, CR23, CR24, GL17, HR23, LR20a, NRR20, Rey19].

1.2 SOME FAMILIES OF EXAMPLES

Semi-graded rings extend several kinds of noncommutative rings of polynomial type such as Ore extensions [Ore31, Ore33], families of differential operators generalizing Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras [Bav92, BG88, Smi91], algebras appearing in mathematical physics [IPR01, RS22, Zhe91], down-up algebras [Ben99b, BR98, KMP99], ambiskew polynomial rings [Jor00, JW96], 3-dimensional skew polynomial rings [BS90, Red99, RS22, Ros95], bi-quadratic algebras on 3-generators in the sense of Bavula [Bav23], PBW extensions [BG88], skew PBW extensions [GL11], and others. Ring-theoretical, algebraic and geometrical properties of semi-graded rings have been investigated in the literature (e.g., [Art15, CR22, Rey19, RS20, SCR22, SRS23, TRS20a], and references therein).

In this section, we present families of noncommutative rings that are particular examples of semi-graded rings with the aim of showing the generality of these objects. For the completeness of the thesis, we include detailed references for every family of rings.

1.2.1 Skew polynomial rings

Skew polynomial rings (also known as Ore extensions) were introduced by Ore [Ore31, Ore33] (Noether and Schmeidler [NS20] were interested in some kind of differential operator rings). Briefly, for σ an endomorphism of a ring R, a σ-derivation on R is any additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$ (strictly speaking, this is the definition of *left σ-derivation*, but we will not need the concept of *right* σ-derivation, which is any additive map $\delta : R \to R$ satisfying the rule $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$). Notice that if σ is the identity map on R, then σ-derivations are just ordinary derivations. The condition $\delta(1) = 0$ it follows from the skew product rule. Any element r of R such that $\sigma(r) = r$ and $\delta(r) = 0$ is called a *constant*.

Definition 1.7 ([Ore31, Ore33], [GJ04, p. 34]). Let R be a ring, σ a ring endomorphism of R and δ a σ-derivation on R. We will write $R[x;\sigma,\delta]$ provided

(i) $R[x;\sigma,\delta]$ containing R as a subring;

(ii) x is not an element of R;

(iii) $R[x;\sigma,\delta]$ is a free left R-module with basis $\{1, x, x^2, \dots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$, for all $r \in R$.

Such a ring $R[x;\sigma,\delta]$ is called a *skew polynomial ring over R*, or an *Ore extension of R*. If σ is an injective map of R, then we call it an *Ore extension of injective type*, while if σ is the identity of R, then we write $R[x;\delta]$ and call it a *differential operator ring*. On the other hand, if δ is the zero map, then we write $R[x;\sigma]$ which is known as a *skew polynomial ring of endomorphism type*. Iterated skew polynomial rings are defined in the natural way. In the literature, we can find a lot of papers concerning ring-theoretical and module properties of Ore extensions. Some general details about these objects can be found in Brown and Goodearl [BG02], Fajardo et al. [FLP$^+$24], Goodearl and Warfield [GJ04], and McConnell and Robson [MR01], and references therein.

Ore extensions are one of the most important techniques to define noncommutative algebras. Next, we illustrate this situation with Weyl algebras, some of its deformations, the q-Heisenberg algebra, and the quantum matrix algebra.

About the family of Weyl algebras $A_n(\Bbbk)$, in the literature it is common to find characterizations of these algebras as rings of differential operators. An excellent treatment about Weyl algebras is presented by Coutinho [Cou95]. Briefly, the nth *Weyl algebra* $A_n(\Bbbk)$ over $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1,\dots,x_n,y_1,\dots,y_n$ where

$$\begin{aligned}
&x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, && 1\leq i<j\leq n,\\
&y_jx_i = x_iy_j+\delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker's delta}, && 1\leq i,j\leq n.
\end{aligned}$$

From the relations defining the Weyl algebras, it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but skew polynomial rings with non-trivial derivations.

Following Goodearl and Warfield [GJ04, p. 36], for an element $q\in\Bbbk^*$, $A_1^q(\Bbbk)$ denotes the $\Bbbk$-algebra presented by two generators x and y and the relation $xy-qyx=1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk)=A_1(\Bbbk)=\Bbbk[y][x;d/dy]$, when $q=1$. If $q\neq 1$, then $A_1^q(\Bbbk)=\Bbbk[y][x;\sigma,\delta]$, where σ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y))=f(qy)$, and δ is the *q-difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y)) = \frac{f(qy)-f(y)}{qy-y} = \frac{\alpha(f)-f}{\alpha(y)-y},$$

as it can be seen in [GJ04, Exercise 2N], so this algebra is not a skew polynomial ring of automorphism type.

A generalization of $A_1^q(\Bbbk)$ is given by the *additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$. For elements $q_1,\dots,q_n\in\Bbbk^*$, this algebra is generated by the indeterminates $x_1,\dots,x_n$ and $y_1,\dots,y_n$ satisfying the relations $x_jx_i=x_ix_j$, $y_jy_i=y_iy_j$, for every $1\leq i,j\leq n$, $y_ix_j=x_jy_i$, for all $i\neq j$, and $y_ix_i=q_ix_iy_i+1$, for $1\leq i\leq n$. It is clear from these definitions that these algebras are not skew polynomial rings of automorphism type.

Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [GZ95] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second Weyl algebra. By definition, the *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the indeterminates $x_1,x_2,\partial_1,\partial_2$, with relations (depending on parameters $a,b\in\Bbbk$)

$$\begin{aligned}
&x_1x_2 = x_2x_1+ax_1^2, && \partial_2\partial_1=\partial_1\partial_2+b\partial_2^2\\
&\partial_1x_1 = 1+x_1\partial_1+ax_1\partial_2, && \partial_1x_2=-ax_1\partial_1-abx_1\partial_2+x_2\partial_1+bx_2\partial_2\\
&\partial_2x_1 = x_1\partial_2, && \partial_2x_2 = 1-bx_1\partial_2+x_2\partial_2.
\end{aligned}$$

Note that if $a=b=0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$.

By definition, for $q\in\Bbbk^*$, the *q-Heisenberg algebra* $\mathbf{H}_n(q)$ is the $\Bbbk$-algebra generated over $\Bbbk$ by the indeterminates x_i,y_i,z_i, for $1\leq i\leq n$, subject to the relations

$$\begin{aligned}
x_ix_j = x_jx_i,\quad y_iy_j=y_jy_i,\quad z_jz_i=z_iz_j, &\qquad 1\leq i<j\leq n,\\
x_iz_i-qz_ix_i = z_iy_i-qy_iz_i = x_iy_i-q^{-1}y_ix_i+z_i=0, &\qquad 1\leq i\leq n,\\
x_iy_j=y_jx_i,\quad x_iz_j=z_jx_i,\quad y_iz_j=z_jy_i, &\qquad i\neq j.
\end{aligned}$$

It is easy to see that $\mathbf{H}_n(q)$ can be expressed as an iterated skew polynomial ring.

Given any $q \in \Bbbk^*$, the corresponding *quantized coordinate ring* of the ring of matrices of size 2×2 with entries in $\Bbbk$, denoted by $M_2(\Bbbk)$, is the $\Bbbk$-algebra $O_q(M_2(\Bbbk))$ presented by four generators x_{11}, x_{12}, x_{21}, and x_{22} and the six relations given by

$$\begin{aligned} x_{11}x_{12} &= qx_{12}x_{11}, \quad x_{12}x_{22} = qx_{22}x_{12}, \\ x_{11}x_{21} &= qx_{21}x_{11}, \quad x_{21}x_{22} = qx_{22}x_{21}, \\ x_{12}x_{21} &= x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q-q^{-1})x_{12}x_{21}. \end{aligned}$$

This algebra, also known as the *coordinate ring of quantum* 2×2 matrices over $\Bbbk$, or the 2×2 *quantum matrix algebra* over $\Bbbk$, can be expressed as the iterated skew polynomial ring $\Bbbk[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\delta_{22}]$ [GJ04, Exercise 2V].

Jordan [Jor95] introduced a certain class of iterated Ore extensions called *ambiskew polynomial rings*. These structures have been studied by Jordan et al. [Jor00, JW96] at various levels of generality that contain different examples of noncommutative algebras. Next, we recall briefly its definition.

Consider a commutative $\Bbbk$-algebra B, a $\Bbbk$-automorphism of B, and elements $c \in B$ and $p \in \Bbbk^*$. Let S be the Ore extension $B[x;\sigma^{-1}]$ and extend σ to S by setting $\sigma(x) = px$. By [Coh85, p. 41], there is a σ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B,\sigma,c,p)$ is the Ore extension $S[y;\sigma,\delta]$, whence the following relations hold:

$$yx - pxy = c, \quad \text{and, for all } b \in B, \quad xb = \sigma^{-1}(b)x \quad \text{and} \quad yb = \sigma(b)y. \tag{1.1}$$

Equivalently, R can be presented as $R = B[y;\sigma][x;\sigma^{-1},\delta']$ with $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$, so that $xy - p^{-1}yx = -p^{-1}c$. If we consider the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, then we can see that the definition involves twists from both sides using σ; this is the reason for the name of the objects.

1.2.2 Universal enveloping algebras and PBW extensions

If $\mathfrak{g}$ is a finite dimensional Lie algebra over a commutative ring K with basis $\{x_1,\dots,x_n\}$, then by the *Poincaré-Birkhoff-Witt theorem*, the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the algebra generated by $x_1,\dots,x_n$ subject to the relations $x_ir - rx_i = 0 \in K$, for every element $r \in K$, and $x_ix_j - x_jx_i = [x_i,x_j] \in \mathfrak{g}$, where $[x_i,x_j] \subseteq K + Kx_1 + \dots + Kx_n$, for all $1 \le i,j \le n$. As is well-known, in general these algebras are not skew polynomial rings even including non-zero trivial derivations. Some enveloping algebras can be expressed as skew polynomial rings; however, in these rings the derivations are non-trivial. Let us see an example.

Following [GJ04, p. 40], the standard basis for the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is labelled $\{e,f,h\}$, where $[e,f] = h$, $[h,e] = 2e$, and $[h,f] = -2f$. In this way, the enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra presented by three generators e,f,h and three relations $ef - fe = h$, $he - eh = 2e$, and $hf - fh = -2f$. If R is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by e and h, then $R = \Bbbk[e][h;\delta_1] = \Bbbk[h][e;\sigma_1]$, where $\Bbbk[e]$ and $\Bbbk[h]$ are commutative polynomial rings, δ_1 denotes the derivation

$2e(d/de)$ on $\Bbbk[e]$, and σ_1 is the $\Bbbk$-algebra automorphism of $\Bbbk[h]$ with $\sigma_1(h) = h-2$. Thus, $U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h;\delta_1][f;\sigma_2,\delta_2] = \Bbbk[h][e;\sigma_1][f;\sigma_2,\delta_2]$, where $\sigma_2(e) = e$, $\sigma_2(h) = h+2$, $\delta_2(e) = -h$, and $\delta_2(h) = 0$ [GJ04, Exercise 2S]. Other examples of universal enveloping algebras known as *parafermionic* and *parabosonic algebras* are considered in Section 1.2.7.

Notice that universal enveloping algebras above are PBW extensions over K in the sense of Bell and Goodearl [BG88] (these authors presented another examples of enveloping rings related to enveloping universal algebras). In Remark 6 (iv), we will say some words about these extensions.

1.2.3 3-DIMENSIONAL SKEW POLYNOMIAL ALGEBRAS

Another kind of noncommutative rings which includes the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, the Dispin algebra $U(\mathfrak{osp}(1,2))$ and the Woronowicz's algebra $W_\nu(\mathfrak{sl}_2(\Bbbk))$ [Wor87], is the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith [BS90] and are important in noncommutative algebraic geometry and noncommutative differential geometry (e.g., [Red96, Red99, RS22, Ros95], and references therein). Next, we recall its definition and classification.

Definition 1.8 ([Ros95, Definition C4.3]). *A 3-dimensional skew polynomial algebra A* is a $\Bbbk$-algebra generated by the indeterminates x, y, z restricted to relations $yz-\alpha zy = \lambda$, $zx-\beta xz = \mu$, and $xy-\gamma yx = \nu$, such that

(i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$;

(ii) standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

Proposition 1.2 ([Ros95, Theorem C.4.3.1]). *If A is a 3-dimensional skew polynomial algebra, then A is one of the following algebras:*

(1) *if $|\{\alpha, \beta, \gamma\}| = 3$, then A is given by the relations $yz-\alpha zy = 0$, $zx-\beta xz = 0$, $xy-\gamma yx = 0$.*

(2) *if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then A is one of the following algebras:*

   (i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$

   (ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$

   (iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$

   (iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$

   (v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$

   (vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0,$

   where a, b are any elements of $\Bbbk$. All non-zero values of b give isomorphic algebras.

(3) *If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then A is one of the following algebras:*

   (i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$

   (ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case, b is an arbitrary element of $\Bbbk$. Again, any non-zero values of b give isomorphic algebras.

(4) *If $\alpha = \beta = \gamma \neq 1$, then A is the algebra defined by the relations $yz - \alpha zy = a_1x + b_1$, $zx - \alpha xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. If $a_i = 0$ $(i = 1,2,3)$, then all non-zero values of b_i give isomorphic algebras.*

(5) *If $\alpha = \beta = \gamma = 1$, then A is isomorphic to one of the following algebras:*

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \Bbbk$ are arbitrary, and all non-zero values of b generate isomorphic algebras.

1.2.4 Bi-quadratic algebras on 3 generators with PBW bases

Related with algebras generated by three indeterminates, recently Bavula [Bav23] defined the *skew bi-quadratic algebras* with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

For a ring R and a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements $m_{ij} \in R$ $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in R. The set of all such matrices is denoted by $L_n(R)$.

If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an n-tuple of commuting endomorphisms of R, $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tuple of σ-endomorphisms of R (that is, δ_i is a σ_i-derivation of R for $i = 1, \ldots, n$), $Q = (q_{ij}) \in L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, $1 \leq j < i \leq n$ and $k = 1, \ldots, n$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \ldots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring R and elements $x_1, \ldots, x_n$ subject to the defining relations

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \quad \text{for } i = 1, \ldots, n, \text{ and every } r \in R, \tag{1.2}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{1.3}$$

In the particular case when $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for $i = 1, \ldots, n$, the ring A is called the *bi-quadratic algebra* (*BQA*) and is denoted by $A = R[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$. A has *PBW basis* if $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The following result classifies (up to isomorphism) the bi-quadratic algebras on three generators of Lie type, i.e., when $q_1 = q_2 = q_3 = 1$.

Proposition 1.3 ([Bav23, Theorem 1.4]). *Let A be an algebra of Lie type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the algebra A is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

(1) *$P_3 = \Bbbk[x_1, x_2, x_3]$, a polynomial in three indeterminates.*

(2) *$U(\mathfrak{sl}_2(\Bbbk))$, the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$.*

(3) *$U(\mathfrak{H}_3))$, the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{H}_3$.*

(4) *$U(\mathcal{N})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = z, [x, z] = 0, [y, z] = 1$, and the algebra $U(\mathcal{N})/\langle c-1\rangle$ is a tensor product $A_1 \otimes \Bbbk[x']$ of its subalgebras, the Weyl algebra $A_1(\Bbbk) = \Bbbk\{y, z\}/\langle [y, z] = 1\rangle$ and the polynomial algebra $[x']$ where $x' = x + \frac{1}{2}z^2$.*

(5) *$U(\mathfrak{n}_2 \times \Bbbk z) \cong \Bbbk\{x, y, z\}/\langle [x, y] = y\rangle$, and z is a central element.*

(6) *$U(\mathcal{M})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = y, [x, z] = 1, [y, z] = 0\rangle$ and the algebra $U(\mathcal{M})/\langle c-1\rangle$ is a skew polynomial algebra $A_1(\Bbbk)[y; \sigma]$ where $A_1(\Bbbk) = \Bbbk\{x, z\}/\langle [x, z] = 1\rangle$ is the Weyl algebra and σ is an automorphism of $A_1(\Bbbk)$ given by the rule $\sigma(x+1)$ and $\sigma(z) = z$.*

PROPOSITION 1.4 ([BAV23, THEOREM 2.1]). *Up to isomorphism, there are only five bi-quadratic algebras on two generators:*

(1) *The polynomial algebra $\Bbbk[x_1, x_2]$,*

(2) *The Weyl algebra $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1x_2 - x_2x_1 = 1\rangle$,*

(3) *The universal enveloping algebra of the Lie algebra $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$, $U(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 = x_1\rangle$,*

(4) *The quantum plane $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 = qx_1x_2\rangle$, where $q \in \Bbbk \setminus \{0, 1\}$, and*

(5) *The quantum Weyl algebra $A_1(q) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2 = 1\rangle$, where $q \in \Bbbk \setminus \{0, 1\}$.*

1.2.5 DIFFUSION ALGEBRAS

Diffusion algebras were introduced formally by Isaev et al. [IPR01] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice. However, its origin can be found in Krebs and Sandow [KS97].

DEFINITION 1.9. ([IPR01, p. 5817]) The *diffusion algebras type 1* are affine algebras $\mathscr{D}$ that are generated by n indeterminates $D_1, \ldots, D_n$ over $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_n}^{k_n}$ with $k_j \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and there exist elements $x_1, \ldots, x_n \in \Bbbk$ such that for all $1 \le i < j \le n$, there exist $\lambda_{ij} \in \Bbbk^*$ such that

$$\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j. \tag{1.4}$$

Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t = 0$, for all $t = 1, \ldots, n$, is a *multiparameter quantum affine $n-$space*.

Fajardo et al. [FGL+20] studied ring-theoretical properties of a graded version of these algebras.

Definition 1.10. ([FGL+20, Section 2.4]) The *diffusion algebras type 2* are affine algebras $\mathscr{D}$ generated by $2n$ variables $\{D_1,\dots,D_n,x_1,\dots,x_n\}$ over a field $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1}B_{\alpha_2}^{k_2}\cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i}\in\{D_1,\dots,D_n,x_1,\dots,x_n\}$, for all $i\le 2n$, $k_j\in\mathbb{N}$, and $\alpha_1>\alpha_2>\cdots>\alpha_n$, such that for all $1\le i<j\le n$, there exist elements $\lambda_{ij}\in\Bbbk^*$ satisfying the relations

$$\lambda_{ij}D_iD_j-\lambda_{ji}D_jD_i=x_jD_i-x_iD_j. \tag{1.5}$$

Different physical applications of algebras type 1 and 2 have been studied in the literature. From the point of view of ring-theoretical, homological and computational properties, several thesis and papers have been published (e.g., [FGL+20, HHR20, Hin05, Levrn, Twa02]). For instance, notice that a diffusion algebra type 1 generated by n indeterminates has Gelfand-Kirillov dimension n since because of the PBW basis, the vector subspace consisting of elements of total degree at most l is isomorphic to that of a commutative polynomial ring in n indeterminates. Similarly, diffusion algebras type 2 have Gelfand-Kirillov dimension $2n$.

Remark 2. About the above definitions of diffusion algebras, we have the following facts:

(i) Isaev et al. [IPR01] and Pyatov and Twarok [PT02] defined diffusion algebras type 1 by taking $\Bbbk=\mathbb{C}$. Nevertheless, for the results obtained in this thesis we can take any field not necessarily $\mathbb{C}$.

(ii) Following Krebs and Sandow [KS97], the relations (1.4) are consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta}D_\alpha D_\beta=D_\gamma X_\delta-X_\gamma D_\delta,\ \text{ for all }\ \gamma,\delta=0,1,\dots,n-1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta}\in\Bbbk$, and D_i's and X_j's are operators of a particular vector space such that not necessarily $[D_i,X_j]=0$ holds [KS97, p. 3168].

(iii) Hinchcliffe in his PhD thesis [Hin05, Definition 2.1.1] considered the following notation for diffusion algebras. Let R be the algebra generated by n indeterminates $x_1,x_2,\dots,x_n$ over $\mathbb{C}$ subject to relations $a_{ij}x_ix_j-b_{ij}x_jx_i=r_jx_i-r_ix_j$, whenever $i<j$, for some parameters $a_{ij}\in\mathbb{C}\setminus\{0\}$, for all $i<j$ and $b_{ij},r_i\in\mathbb{C}$, for all $i<j$. He defined the *standard monomials* to be those of the form $x_n^{i_n}x_{n-1}^{i_{n-1}}\cdots x_2^{i_2}x_1^{i_1}$. R is called a *diffusion algebra* if it admits a *PBW basis of these standard monomials.* In other words, R is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for R. If all the elements $q_{ij}:=\frac{b_{ij}}{a_{ij}}$'s are non-zero, then the diffusion algebras have a PBW basis in any order of the indeterminates [Hin05, Remark 2.1.6].

Diffusion algebras of n generators (also called *n-diffusion algebras*) are constructed in such a way that the subalgebras of three generators are also diffusion algebras. As we can see in Proposition 1.5, diffusion algebras type 1 of three generators can be classified into 4 families, A,B,C, and D, and these in turn are divided into classes as shown below (notice that this classification reflects the number of coefficients x_s, $s\in\{i,j,k\}$, being zero in comparison with the expression (1.4)).

Proposition 1.5 ([PT02, p. 3270]). *If $\mathscr{D}$ is a diffusion algebra type 1 generated by the indeterminates D_i,D_j and D_k with $i<j<k$, and $\Lambda\in\Bbbk$, then $\mathscr{D}$ belongs to some of the following classes*

of diffusion algebras:

(1) The case of A_I:

$$
\begin{aligned}
gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j,\\
gD_iD_k - gD_kD_i &= x_kD_i - x_iD_k,\\
gD_jD_k - gD_kD_j &= x_kD_j - x_jD_k,
\end{aligned}
$$

where $g \neq 0$.

(2) The case of A_{II}:

$$
\begin{aligned}
g_{ij}D_iD_j &= x_jD_i - x_iD_j,\\
g_{ik}D_iD_k &= x_kD_i - x_iD_k,\\
g_{jk}D_jD_k &= x_kD_j - x_jD_k,
\end{aligned}
$$

where $g_{st} := g_s - g_t$ *with* $g_s \neq g_t$, *for all* $s < t$, *and* $s, t \in \{i, j, k\}$.

(3) The case of $B^{(1)}$:

$$
\begin{aligned}
g_jD_iD_j - (g_j - \Lambda)D_jD_i &= -x_iD_j,\\
gD_iD_k - (g - \Lambda)D_kD_i &= x_kD_i - x_iD_k,\\
g_jD_jD_k - (g_j - \Lambda)D_kD_j &= x_kD_j,
\end{aligned}
$$

where $g, g_j \neq 0$.

(4) The case of $B^{(2)}$:

$$
\begin{aligned}
g_{ij}D_iD_j &= -x_iD_j,\\
g_{ik}D_iD_k - \lambda_{ki}D_kD_i &= x_kD_i - x_iD_k,\\
g_{jk}D_jD_k &= x_kD_j,
\end{aligned}
$$

where $g_{ij}, g_{ik}, g_{jk} \neq 0$.

(5) The case of $B^{(3)}$:

$$
\begin{aligned}
gD_iD_j - (g - \Lambda)D_jD_i &= x_jD_i - x_iD_j,\\
g_kD_iD_k &= -x_iD_k,\\
(g_k - \Lambda)D_jD_k &= -x_jD_k,
\end{aligned}
$$

where $g \neq 0$ *and* $g_k \neq 0, \Lambda$.

(6) The case of $B^{(4)}$:

$$
\begin{aligned}
(g_i - \Lambda)D_iD_j &= x_jD_i,\\
g_iD_iD_k &= x_kD_i,\\
gD_jD_k - (g - \Lambda)D_kD_j &= x_kD_j - x_jD_k,
\end{aligned}
$$

where $g \neq 0$ *and* $g_i \neq 0, \Lambda$.

(7) The case of $C^{(1)}$:

$$\begin{aligned} g_j D_i D_j - (g_j - \Lambda) D_j D_i &= -x_i D_j, \\ g_k D_i D_k - (g_k - \Lambda) D_k D_i &= -x_i D_k, \\ g_{jk} D_j D_k - g_{kj} D_k D_j &= 0, \end{aligned}$$

where $g_j, g_k, g_{j,k} \neq 0$.

(8) The case of $C^{(2)}$:

$$\begin{aligned} g_{ij} D_i D_j - g_{ji} D_j D_i &= -x_i D_j, \\ g_{ik} D_i D_k - g_{ki} D_k D_i &= -x_i D_k, \\ D_j D_k &= 0, \end{aligned}$$

where $g_{ij}, g_{ik} \neq 0$.

(9) The case of D: *With* $q_{st} := \frac{g_{ts}}{g_{st}}$, *where* $s, t \in \{i, j, k\}$ (*recall that* $g_{st} \neq 0$, *for* $s < t$), *we have*

$$\begin{aligned} D_i D_j - q_{ji} D_j D_i &= 0, \\ D_i D_k - q_{ki} D_k D_i &= 0, \\ D_j D_k - q_{kj} D_k D_j &= 0. \end{aligned}$$

About the relationship between diffusion algebras and skew polynomial rings, if we consider the notation in Remark 2 (3), then a 3-diffusion algebra generated by the indeterminates x_1, x_2, x_3 is a skew polynomial ring over its 2-diffusion subalgebra generated by x_2 and x_3 [Hin05, Lemma 2.2.1], where it is easy to see that a 2-diffusion algebra is a skew polynomial ring over the polynomial subalgebra generated by x_2. In general an n-diffusion algebra (generated by the indeterminates $x_1, \dots, x_n$) is a skew polynomial ring over its $(n-1)$ diffusion subalgebra generated by $x_2, \dots, x_n$ [Hin05, Remark 2.2.2].

Since a diffusion algebra on $n \geq 2$ generators is left Noetherian if and only if $q_{ij} \neq 0$, for all $i < j$ [Hin05, Proposition 2.2.5], where q_{ij} is given in Remark 2 (3), then every Noetherian 2-diffusion algebra is isomorphic to one of the following three types of algebra [Hin05, Proposition 3.3.1]:

- *The quantum affine plane*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1 x_2 - q x_2 x_1 = 0$, for some $q \in \mathbb{C} \setminus \{0\}$ (allowing the possibility $q = 1$) (Proposition 1.4(4)).
- *The quantized Weyl algebra*, i.e., the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1 x_2 - q x_2 x_1 = 1$, for some $q \in \mathbb{C} \setminus \{0, 1\}$ (Proposition 1.4(5)).
- *The universal enveloping algebra of the 2-d soluble Lie algebra*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1 x_2 - x_2 x_1 = x_1$ (Proposition 1.4(3)).

Related to Proposition 1.5, Hinchcliffe [Hin05] proved the following result about classification of diffusion algebras assuming certain conditions on the coefficients of commutation of the indeterminates.

Proposition 1.6 ([Hin05, Proposition 3.1.4]). *If $q_{ij} \notin \{0,1\}$, for all i, j, then a diffusion algebra R is isomorphic either to multiparameter quantum affine n-space or to the $\mathbb{C}$-algebra generated by the indeterminates $x_1, x_2, x_3, \dots, x_n$ subject to relations*

$$
\begin{aligned}
x_1x_2 - q_{12}x_2x_1 &= 1, \text{ where } q_{12} \neq 1,\\
x_1x_i - q_{1i}x_ix_1 &= 0, \text{ where } q_{1i} \neq 1,\\
x_2x_i - q_{2i}^{-1}x_ix_2 &= 0,\\
x_ix_j - q_{ij}x_jx_i &= 0, \text{ for all } 3 \leq i < j.
\end{aligned}
$$

1.2.6 Generalized Weyl algebras, hyperbolic rings and down-up algebras

Other algebraic structures that illustrate the results obtained in this thesis are the *generalized Weyl algebras* and *down-up algebras*. We briefly present the definitions and some relations between these algebras (see [Jor95, Jor00, JW96] for a detailed description).

Given an automorphism σ and a central element a of a ring R, Bavula [Bav92] defined the *generalized Weyl algebra* $R(\sigma, a)$ as the ring extension of R generated by the indeterminates X^- and X^+ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings. Generalized Weyl algebras have been extensively studied in the literature by various authors (see [Bav92, Jor00], and references therein).

Definition 1.11 ([Ros95, Definition 3.1.0]). Let K be a commutative ring, ν an automorphism of K, u a fixed element of K. With this data, we relate the ring $K\langle \nu, u\rangle$ generated by the ring K and by the indeterminates x, y subject to the following relations:

$$
\begin{aligned}
xa = \nu(a)x,\ \ ya = \nu^{-1}(a)y, &\text{ for any } a \in K,\\
xy - yx = u, &\text{ for some } u \in K.
\end{aligned}
$$

Remark 3 ([Ros95], 3.1.3). The defining ring $K\langle \nu, u\rangle$, the relations show that the element $\zeta = xy$ commutes with every element of the ring K. In other words, the ring $K[\zeta]$ is commutative. This fact suggest to consider $K\langle \nu, u\rangle$ not as a K-ring, but as an $K[\zeta]$-ring.

Define the extensions θ and θ' of the automorphisms ν and ν^{-1} respectively onto $K[\zeta]$, setting $\theta(\zeta) = \zeta + \nu(u)$ and $\nu'(\zeta) = \zeta - u$. We have:

$$\theta \circ \theta'(\zeta) = \theta(\zeta - u) = (\zeta + \nu(u)) - \nu(u) = \zeta,$$

and

$$\theta' \circ \theta(\zeta) = \theta'(\zeta + \nu(u)) = (\zeta - u) + u = \zeta$$

In consequence, $\theta' = \theta^{-1}$. Now the relations defining the ring $K\langle \nu, u\rangle$ can be rewritten in the following way:

$$xb = \theta(b)x \text{ and } yb = \theta^{-1}(b)y \text{ for all } b \in K[\zeta];$$

$$xy = \zeta, \; yx = \theta^{-1}(\zeta).$$

Definition 1.12 ([Ros95], 3.1.4). Let θ be an automorphism of a commutative ring K; and let ζ be an element of K. Denote by $K\{\theta,\zeta\}$ the K-ring generated by the indeterminates x, y with the relations

$$xa = \theta(a)x \text{ and } ya = \theta^{-1}(a)y \text{ for any } a \in K;$$

$$xy = \zeta \;\; yx = \theta^{-1}(\zeta).$$

The ring $K\{\theta,\zeta\}$ is called *hyperbolic ring.*

On the other hand, the *down-up algebras* $A(\alpha,\beta,\gamma)$, where $\alpha,\beta,\gamma \in \mathbb{C}$, were defined by Benkart and Roby [Ben99b, BR98] as generalizations of algebras generated by a pair of operators, precisely, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered set P. More exactly, consider a partially ordered set $(P,\prec)$ and let $\mathbb{C}P$ be the complex vector space with basis P. If for an element p of P, the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite, then we can define the "down" operator d and the "up" operator u in $\operatorname{End}_{\mathbb{C}} \mathbb{C}P$ as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define d and u). For any $\alpha,\beta,\gamma \in \mathbb{C}$, the *down-up algebra* is the $\mathbb{C}$-algebra generated by d and u subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2 d + \gamma u$. A partially ordered set P is called (q,r)-*differential* if there exist $q, r \in \mathbb{C}$ such that the down and up operators for P satisfy both relations, and $\alpha = q(q+1), \beta = -q^3$, and $\gamma = r$. From [BR98], we know that for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha,\beta,\gamma) \simeq A(\alpha,\beta,\lambda\gamma)$. This means that when $\gamma \neq 0$, no problem if we assume $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [Ben99b, Section 1].

Remarkable examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and some of its deformations introduced by Witten [Wit90] and Woronowicz [Wor87]. Related to the theoretical properties of these algebras, Kirkman et al. [KMP99] proved that a down-up algebra $A(\alpha,\beta,\gamma)$ is Noetherian if and only if β is non-zero. As a matter of fact, they showed that $A(\alpha,\beta,\gamma)$ is a generalized Weyl algebra and that $A(\alpha,\beta,\gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

Following [Ben99b, p. 32], if $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbb{C}$ with basis $x, y, [x, y]$ such that $[x,[x,y]] = \gamma x$ and $[[x,y],y] = \gamma y$, then in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ these relations are given by $x^2y - 2xyx + yx^2 = \gamma x$ and $xy^2 - 2yxy + y^2x = \gamma y$. Notice that $U(\mathfrak{g})$ is a homomorphic algebra of the down-up algebra $A(2,-1,\gamma)$ via the mapping $\phi : A(2,-1,\gamma) \to U(\mathfrak{g})$, $d \mapsto x, u \mapsto y$, and the mapping $\psi : \mathfrak{g} \to A(2,-1,\gamma)$, $x \mapsto d, y \mapsto u$, $[x,y] \mapsto du - ud$, extends by the universal property of $U(\mathfrak{g})$ to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2,-1,\gamma)$ which is the inverse of ψ. Hence, $U(\mathfrak{g})$ is isomorphic to $A(2,-1,\gamma)$.

It is straightforward to see that $U(\mathfrak{sl}_2(\mathbb{C})) \cong A(2,-1,-2)$. Also, for the Heisenberg Lie algebra $\mathfrak{h}$ with basis x, y, z where $[x,y] = z$ and $[z,x] = [z,y] = 0$, $U(\mathfrak{h}) \cong A(2,-1,0)$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [Wit90, Wit91] introduced a 7-parameter deformation of the universal enveloping algebra

$U(\mathfrak{sl}_2(\Bbbk))$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1, \dots, \xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates x, y, z subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. From [Ben99b, Section 2], we know that a Witten's deformation algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.6}$$

is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha, \beta, \gamma)$ with not both α and β equal to 0 is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy (1.6).

Since algebras $W(\underline{\xi})$ are filtered, Le Bruyn [LB94, LB95] studied the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by the indeterminates x, y, z over a field $\Bbbk$ subject to the relations given by $zx - axz = x$, $zy - ayz = y$, and $yx - cxy = bz^2 + z$. In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three defining relations is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1+ac)$, $\beta = -ac^{-1}$ and $\gamma = -c^{-1}$. Notice that if $c = b = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}$, $\beta = 0$, and $\gamma = -a^{-1}$. As one can check, conformal $\mathfrak{sl}_2$ algebras are not Ore extensions.

Kulkarni [Kul99] showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2(\Bbbk)$ algebra or to an iterated Ore extension. More exactly, following [Kul99, Theorem 3.0.3] if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to one of the following algebras: (i) a conformal $\mathfrak{sl}_2$ algebra with generators x, y, z and relations given above or (ii) an iterated Ore extension whose generators satisfy

- $xz - zx = x$, $zy - yz = \zeta y$, $yx - \eta xy = 0$, or
- $xw = \theta wx$, $wy = \kappa yw$, $yx = \lambda xy$, for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \Bbbk$.

Notice that iterated Ore extensions above are defined in the following way: (i) the Witten deformation algebra is isomorphic to $\Bbbk[z][y, \sigma_1][x, \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[z]$ defined as $\sigma_1(z) = z - \zeta$, with $zy - yz = \zeta y$; σ_2 is the automorphism of $\Bbbk[z][y, \sigma_1]$ defined as $\sigma_2(y) = \eta^{-1} y$, $\sigma_2(z) = z + 1$, which satisfies $xz - zx = x$ and $yx - \eta xy = 0$. (ii) The Witten deformation algebra is isomorphic to $\Bbbk[w][y, \sigma_1][x, \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[w]$ defined as $\sigma_1(w) = \kappa^{-1} w$ with $wy = \kappa yw$, and σ_2 is the automorphism of $\Bbbk[w][y, \sigma_1]$ defined as $\sigma_2(w) = \theta w$, $\sigma_2(y) = \lambda^{-1} y$ such that $wy = \kappa yw$ and $yx = \lambda xy$.

1.2.7 Other families of quantum algebras

In this section, we recall the definitions of some examples of noncommutative rings known in the literature as *quantum algebras* or *quantized algebras*.

The term *"quantum group"* was independently popularized by Drinfel'd [Dri88] and Jimbo [Jim85] around 1985. They used it to build solutions to the quantum Yang-Baxter equations.

These "groups" represent certain special Hopf algebras which are *deformations* of the universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac–Moody algebra. Intuitively, a deformation is a family of algebras that depends "nicely" on a parameter q such that we get back the initial structure for some special value of q. For example, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Choose a generic parameter q. Then, for each q, we have a Hopf algebra $U_q(\mathfrak{g})$, called the *quantum group* or the *quantized universal enveloping algebra*, whose structure tends to that of $U(\mathfrak{g})$ as q approaches 1, it is same as the Hopf algebra $U(\mathfrak{g})$ [Jan96].

We describe briefly this type of associative algebras introduced by Drinfel'd and Jimbo. Following [YGO15], let $A = (a_{ij})$ be an integral symmetrizable $n\times n$ Cartan matrix, so that $a_{ii} = 2$ and $a_{ij} \le 0$, for $i \ne j$, and there exists a diagonal matrix D with diagonal entries d_i non-zero integers such that the product DA is symmetric. Let $0 \ne q \in \Bbbk$ so that $q^{4d_i} \ne 1$, for each i. Then the *quantum group* $U_q(A)$ is the $\Bbbk$-algebra generated by $4n$ elements, $E_i, K_i^{\pm 1}, F_i$, for $1 \le i, j \le n$, subject to the following set of relations:

$$K = \{K_iK_j - K_jK_i, K_iK_i^{-1} - 1, K_i^{-1}K_i - 1, \tag{1.7}$$

$$E_jK_i^{\pm 1} - q^{\pm d_ia_{ij}}K_i^{\pm 1}E_j, K_i^{\pm 1}F_j - q^{\pm d_ia_{ij}}F_jq^{\pm d_ia_{ij}}\}, \tag{1.8}$$

$$T = \left\{E_iF_j - F_jE_i - \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\right\}, \tag{1.9}$$

$$S^+ = \left\{\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t E_i^{1-a_{ij}-\mu}E_jE_i^\mu : i \ne j, t = q^{2d_i}\right\}, \tag{1.10}$$

$$S^- = \left\{\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t F_i^{1-a_{ij}-\mu}F_jF_i^\mu : i \ne j, t = q^{2d_i}\right\}, \tag{1.11}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^{n} \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}}, \text{ for } m > n > 0, \\ \\ 1, \text{ for } n = 0 \text{ or } n = m. \end{cases}$$

One of the basic properties of these algebras is that they have a triangular decomposition, i.e., $U_q(A) \cong U_q^+(A)\otimes U_q^0(A)\otimes U_q^-(A)$, where $U_q^+(A)$ (resp., $U_q^-(A)$) is the subalgebra of $U_q(A)$ generated by E_i (resp., F_i), and $U_q^0(A)$ is the subalgebra of $U_q(A)$ generated by $K_i^{\pm 1}$ [Jan96, Chapter 4].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \dots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The *homogenized enveloping algebra of* $\mathfrak{g}$ is $\mathscr{A}(\mathfrak{g}) := T(\mathfrak{g}\oplus \Bbbk z)/\langle R\rangle$, where $T(\mathfrak{g}\oplus\Bbbk z)$ denotes the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z\otimes x - x\otimes z \mid x \in \mathfrak{g}\}$ and $\{x\otimes y - y\otimes x - [x, y]\otimes z \mid x, y \in \mathfrak{g}\}$.

From [GJ04, p. 41], for q an element of $\Bbbk$ with $q \ne \pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of q is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators E, F, K, K^{-1} and the relations $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$. From [GJ04, Exercise 2T], we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1};\sigma_1][F;\sigma_2,\delta_2]$ [GJ04, Exercise 2T], so that this algebra is not of automorphism type.

Following Yamane [Yam89], if $q \in \mathbb{C}$ with $q^8 \neq 1$, the complex algebra A generated by the indeterminates $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ subject to the relations

$$
\begin{array}{lll}
e_{13}e_{12} = q^{-2}e_{12}e_{13}, & f_{13}f_{12} = q^{-2}f_{12}f_{13}, & \\
e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, & f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, & \\
e_{23}e_{13} = q^{-2}e_{13}e_{23}, & f_{23}f_{13} = q^{-2}f_{13}f_{23}, & \\
e_{12}f_{12} = f_{12}e_{12} + \dfrac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & e_{12}k_1 = q^{-2}k_1e_{12}, & k_1f_{12} = q^{-2}f_{12}k_1, \\
e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, & e_{12}k_2 = qk_2e_{12}, & k_2f_{12} = qf_{12}k_2, \\
e_{12}f_{23} = f_{23}e_{12}, & e_{13}k_1 = q^{-1}k_1e_{13}, & k_1f_{13} = q^{-1}f_{13}k_1, \\
e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & e_{13}k_2 = q^{-1}k_2e_{13}, & k_2f_{13} = q^{-1}f_{13}k_2, \\
e_{13}f_{13} = f_{13}e_{13} - \dfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & e_{23}k_1 = qk_1e_{23}, & k_1f_{23} = qf_{23}k_1, \\
e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & e_{23}k_2 = q^{-2}k_2e_{23}, & k_2f_{23} = q^{-2}f_{23}k_2, \\
e_{23}f_{12} = f_{12}e_{23}, & e_{12}l_1 = q^{2}l_1e_{12}, & l_1f_{12} = q^{2}f_{12}l_1, \\
e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & e_{12}l_2 = q^{-1}l_2e_{12}, & l_2f_{12} = q^{-1}f_{12}l_2, \\
e_{23}f_{23} = f_{23}e_{23} + \dfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & e_{13}l_1 = ql_1e_{13}, & l_1f_{13} = qf_{13}l_1, \\
e_{13}l_2 = ql_2e_{13}, & l_2f_{13} = qf_{13}l_2, & e_{23}l_1 = q^{-1}l_1e_{23}, \\
l_1f_{23} = q^{-1}f_{23}l_1, & e_{23}l_2 = q^{2}l_2e_{23}, & l_2f_{23} = q^{2}f_{23}l_2, \\
l_1k_1 = k_1l_1, & l_2k_1 = k_1l_2, & k_2k_1 = k_1k_2, \\
l_1k_2 = k_2l_1, & l_2k_2 = k_2l_2, & l_2l_1 = l_1l_2,
\end{array}
$$

is very important in the definition of the *quantized enveloping algebra of* $\mathfrak{sl}_3(\mathbb{C})$.

The *Lie-deformed Heisenberg* is the free $\mathbb{C}$-algebra defined by the commutation relations

$$
\begin{aligned}
q_j(1 + i\lambda_{jk})p_k - p_k(1 - i\lambda_{jk})q_j &= i\hbar\delta_{jk}, \\
[q_j, q_k] = [p_j, p_k] &= 0, \quad j, k = 1, 2, 3,
\end{aligned}
$$

where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$, then one recovers the usual Heisenberg algebra.

With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [Hay90] considered the A_q^- algebra by using the free algebra $\mathbf{U}$. Following Berger [Ber92, Example 2.7.7], this $\Bbbk$-algebra $\mathbf{U}$ is generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n$, and $\psi_1^*, \dots, \psi_n^*$, subject to the relations

$$
\begin{aligned}
\psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* = 0, &\qquad 1 \le i < j \le n, \\
\omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* = 0, &\qquad 1 \le i, j \le n, \\
\psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, &\qquad 1 \le i \le n.
\end{aligned}
$$

The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al. [JLM95] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators A_j, B_k, where the following relations are satisfied:

$$
\begin{aligned}
A_j(1+i\lambda_{jk})B_k - B_k(1-i\lambda_{jk})A_j &= i\delta_{jk},\\
[A_j, B_k] &= 0 \ \ (j \neq k),\\
[A_j, A_k] &= [B_j, B_k] = 0,
\end{aligned}
$$

and,

$$
\begin{aligned}
A_j^{+}(1+i\lambda_{jk})B_k^{+} - B_k^{+}(1-i\lambda_{jk})A_j^{+} &= i\delta_{jk},\\
[A_j^{+}, B_k^{+}] &= 0 \ (j \neq k),\\
[A_j^{+}, A_k^{+}] &= [B_j^{+}, B_k^{+}] = 0,
\end{aligned}
\tag{1.12}
$$

with $A_j \neq A_j^{+}$, $B_k \neq B_k^{+}$ $(j,k = 1,2,3)$. If the operators A_j, B_k are in the form $A_j = f_j(N_j+1)a_j$, $B_k = a_k^{+} f_k(N_k+1)$, where a_j, a_j^{+} are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator ($N_j = a_j^{+}a_j$, $\langle N_j \mid n_j\rangle = \langle n_j | n_j\rangle$), and the structure functions $f_j(N_j+1)$ complex, then it is showed that A_j and B_k are given by

$$
\begin{aligned}
A_j &= \sqrt{\frac{i}{1+i\lambda_j}}\left(\frac{[(1-i\lambda_j)/(1+i\lambda_j)]^{N_j+1}-1}{(1-i\lambda_j)/(1+i\lambda_j)-1}\,\frac{1}{N_j+1}\right)^{\frac{1}{2}} a_j,\\
B_k &= \sqrt{\frac{i}{1+i\lambda_k}}\,a_k^{+}\left(\frac{[(1-i\lambda_k)/(1+i\lambda_k)]^{N_k+1}-1}{(1-i\lambda_k)/(1+i\lambda_k)-1}\,\frac{1}{N_k+1}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Following Havliček et al. [HKP00, p. 79], the $\mathbb{C}$-algebra $U_q'(\mathfrak{so}_3)$ is generated by the indeterminates I_1, I_2, and I_3, subject to the relations given by

$$
I_2I_1 - qI_1I_2 = -q^{\frac{1}{2}}I_3, \quad I_3I_1 - q^{-1}I_1I_3 = q^{-\frac{1}{2}}I_2, \quad I_3I_2 - qI_2I_3 = -q^{\frac{1}{2}}I_1,
$$

where q is a non-zero element of $\mathbb{C}$. It is straightforward to show that $U_q'(\mathfrak{so}_3)$ cannot be expressed as an iterated Ore extension.

Zhedanov [Zhe91, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the algebra generated by three operators K_0, K_1, and K_2, that satisfy the commutation relations

$$
[K_0, K_1]_\omega = K_2, [K_2, K_0]_\omega = BK_0 + C_1K_1 + D_1, \text{ and } [K_1, K_2]_\omega = BK_1 + C_0K_0 + D_0,
$$

where B, C_0, C_1, D_0, and D_1 are the structure constants of the algebra, which Zhedanov assumes are real, and the q-commutator $[-,-]_\omega$ is given by $[\square, \triangle]_\omega := e^{\omega}\square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$
\begin{aligned}
e^{\omega}K_0K_1 - e^{-\omega}K_1K_0 &= K_2,\\
e^{\omega}K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1,\\
e^{\omega}K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0.
\end{aligned}
$$

According to these relations that define the algebra, it is clear that AW(3) cannot be expressed as an iterated Ore extension.

With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [Gre53] and Greenberg and Messiah [GM65] introduced by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the completeness of the thesis, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [KD09]. Let $[\square, \triangle] := \square\triangle - \triangle\square$ and $\{\square, \triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space V_F freely generated by the elements f_i^+, f_j^-, with $i, j = 1, \dots, n$. If $T(V_F)$ is the tensor algebra of V_F and I_F is the two-sided ideal I_F generated by the elements $[[f_i^{\xi}, f_j^{\eta}], f_k^{\varepsilon}] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^{\xi} + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^{\eta}$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j, k = 1, \dots, n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ (n parafermions) is the quotient algebra of $T(V_F)$ with the ideal I_F, that is,

$$
P_F^{(n)} = \frac{T(V_F)}{\langle [[f_i^{\xi}, f_j^{\eta}], f_k^{\varepsilon}] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^{\xi} + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \dots, n\rangle}.
$$

It is well-known (e.g., [KD09, Section 18.2]) that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$.

Similarly, if V_B denotes the $\Bbbk$-vector space freely generated by the elements b_i^+, b_j^-, $i, j = 1, \dots, n$, $T(V_B)$ is the tensor algebra of V_B, and I_B is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon-\eta)\delta_{jk}b_i^{\xi} - (\varepsilon-\xi)\delta_{ik}b_j^{\eta}$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j = 1, \dots, n$, then the *parabosonic algebra* $P_B^{(n)}$ in $2n$ generators (n parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$
P_B^{(n)} = \frac{T(V_B)}{\langle [\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon-\eta)\delta_{jk}b_i^{\xi} - (\varepsilon-\xi)\delta_{ik}b_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j = 1, \dots, n\rangle}.
$$

It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$, that is, $P_B^{(n)} \cong U(B(0, n))$. For more details about *parafermionic* and *parabosonic algebras*, see [KD09, Proposition 18.2], and references therein.

1.2.8 Ore polynomials of higher order generated by homogeneous quadratic relations

For a ring R, as we saw in Section 1.2.1, the *Ore extensions* introduced by Ore [Ore31, Ore33] consist of the uniquely representable elements $r_0 + r_1x + \cdots + r_kx^k$, $k = k(r) = 0, 1, 2, \ldots, r_i \in R$, with the commutation relation $xr = \sigma(r)x + \delta(r)$, where σ is an endomorphism of R and δ is a σ-derivation of R. Different generalizations, called *skew Ore polynomials*, have been introduced and studied by Cohn [Coh61, Coh85], Dumas [Dum91], and Smits [Smi68], considering the commutation relation $xr = \Psi_1(r)x + \Psi_2(r)x^2 + \cdots$, where the Ψ's are endomorphisms of R. Nevertheless, there are cases of quadratic algebras such as Clifford algebras, Weyl-Heisenberg algebras, and Sklyanin algebras, in which this commutation relation is not sufficient to define the noncommutative structure of the algebras since a free non-zero term Ψ_0 is required (e.g., Ostrovskii and Samoilenko [OS89]). Precisely, *skew Ore polynomials of higher order* with commutation relation with this free term, that is, $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_n(r)x^n + \cdots$, were studied by Maksimov [Mak00], where, for every $r, s \in R$, the free term Ψ_0 satisfies the relation

$$\Psi_0(rs) = \Psi_0(r)s + \Psi_1(r)\Psi_0(s) + \Psi_2(r)\Psi_0^2(s) + \cdots,$$

or the equivalent operator equation $\Psi_0 r = \Psi_0(r) + \Psi_1(r)\Psi_0 + \Psi_2(r)\Psi_0^2$, where r is considered as the operator of left multiplication by r on R. Notice that one may consider Ψ_0 as a singular differentiation operator with respect to $\Psi_1, \Psi_2, \ldots$, but where Ψ_1 need not be an endomorphism of R.

Later, Golovashkin and Maksimov [GM05] investigated the representation of algebras $Q(a, b, c)$ over a field $\Bbbk$ of characteristic zero defined by a *quadratic relation* in two generators x, y given by

$$yx = ax^2 + bxy + cy^2, \tag{1.13}$$

as an algebra of Ore polynomials of higher degree with commutation relation (1.13) with a, b, c belong to $\Bbbk$. As one can check, the algebra generated by the relation is represented in the form of an algebra of skew Ore polynomials of higher order if the elements $\{x^m y^n\}$ form a linear basis of the algebra. Hence, this algebra can be defined by a system of linear mappings $\Psi_0, \Psi_1, \Psi_2, \ldots$ of the algebra of polynomials $\Bbbk[x]$ into itself such that for an arbitrary element $p(x) \in \Bbbk[x]$, $yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_k(p(x))y^k$, $k = k(p(x))$, $k = 0, 1, 2, \ldots$ If this representation exists, then one can obtain the relations between the operators $\Psi_0, \Psi_1, \Psi_2, \ldots$ They found conditions for such an algebra $Q(a, b, c)$ to be expressed as a skew polynomial with generator y over the polynomial ring $\Bbbk[x]$ (cf. [GM98]), and proved that these conditions are equivalent to the existence of a PBW basis, i.e., basis of the form $\{x^m y^n\}$. Notice that this kind of algebras have been previously studied in the literature where its Poincaré series was calculated by Ufnarovskii [Ufn90].

Next, we recall briefly some of the results presented in [GM05] about PBW bases of these algebras which are useful in Chapter 3.

First of all, Golovashkin and Maksimov [GM05, Section 1] distinguished three types of algebras that can be occur from relation (1.13):

(i) Algebras in which the monomials $\{x^m y^n \mid m,n \in \mathbb{N}\}$ form a PBW basis.

(ii) Algebras in which the monomials $\{x^m y^n \mid m,n \in \mathbb{N}\}$ are linearly dependent (for instance, the algebra determined by the relation $yx = x^2 + xy + y^2$).

(iii) Algebras in which the monomials $\{x^m y^n \mid m,n \in \mathbb{N}\}$, are linearly independent, but do not form a PBW basis (for instance, the algebra subjected to the relation $yx = x^2 - xy + y^2$).

Case (i) is of interest since in this situation the quadratic algebra is determined by the structural constants that arise when expanding the products $(x^k y^r)(x^l y^s)$ in terms of the basis $\{x^m y^n\}$. Nevertheless, it is more useful to use special linear mappings of the ring of polynomials $\Bbbk[x]$ rather than structural constraints. Let us see the details.

If the monomials $\{x^m y^n\}$ form a basis, then for every power $x^n \in \Bbbk[x]$, yx^n has a unique expression given by

$$yx^n = \Psi_{0,n}(x) + \Psi_{1,n}(x)y + \cdots + \Psi_{m(n),n}(x)y^{m(n)}, \tag{1.14}$$

where $\Psi_{k,n}(x)$, for each k, are polynomials from $\Bbbk[x]$. Precisely, for $k = 0,1,\dots$, it can be defined a linear mapping $\Psi_k : \Bbbk[x] \to \Bbbk[x]$ given by $\Psi_k(x^n) = \Psi_{k,n}(x)$. If we define $x^0 = y^0 = 1_{\Bbbk}$, then $yx^0 = y \cdot 1 = 1 \cdot y + 0 \cdot y^2 + \cdots$. By (1.14),

$$\Psi_0(1) = 0, \quad \Psi_1(1) = 1, \quad \Psi_k(1) = 0, \quad k = 2,3,\dots \tag{1.15}$$

which shows that for every element $p(x) \in \Bbbk[x]$, there is a unique expresion given by

$$yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_{m(p(x))}(p(x))y^{m(p(x))}. \tag{1.16}$$

Having in mind that $y^n(p(x)) = y(y^{n-1}p(x)) = \cdots = y(y(\cdots y(yp(x))))$, it follows that the values of the operators Ψ_k, $k = 0,1,\dots$, uniquely determine the algebra of skew polynomials generated by (1.13) in the case that $\{x^m y^n \mid m,n \in \mathbb{N}\}$.

Remark 4. A general algebra of skew polynomials with indeterminate x over R is also determined by certain linear operators $\Psi_k : R \to R$ such that for each $r \in R$, there is a unique representation $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_{m(r)}x^{m(r)}$. Of course, if R is a ring with one generator, then the algebra has a PBW basis. If $m_0 = 1$, then we obtain the classical Ore extensions [Ore33]. It it important to say that the associativity and uniqueness of the representation of the product xr guarantee conditions on the set of operators $\{\Psi_k\}_{k\in\mathbb{N}}$ which are necessary and sufficient to determinate the algebra of skew polynomials (for more details, see [Mak00, Mak05]).

Example 1.1 contains examples of skew polynomials with PBW basis over the ring $\Bbbk[x]$ generated by the quadratic homogeneous relation (1.13).

Example 1.1 ([GM05, Section 1.3]). The following two cases arise in the study of operators in functional analysis [OS89, VK95]:

(i) $a = 0$ and $c \neq 0$. Here, expression (1.13) turns out to be $yx = bxy + cy^2$. If $x_1 := x$, $y_1 := cy$, then we obtain $y_1x_1 = bx_1y_1 + y_1^2$.

(ii) $a \neq 0$ and $c = 0$. Again, if $x_1 := x$, $y_1 := cy$, then (1.13) is equivalent to $y_1x_1 = bx_1y_1 + x_1^2$.

Denote by T the operator of multiplying a polynomial $f(x)$, that is, $Tf(x) = xf(x)$. Let D be the ordinary operator of differentiation, that is, $Dx^n = nx^{n-1}$, and D_q be the operator of q-differentiation given by $D_q f(x) = \frac{f(x)-f(qx)}{x-qx}$, for every $q \in \Bbbk$. Of course, for $q = 1$, $D_1 = D$, while for $q = 0$, the operator $D_0 = \overline{D}$ is the operator of difference quotient given by $\overline{D}f(x) = \frac{f(x)-f(0)}{x}$. If one consider the operator of integration J, $Jx^n = \frac{1}{n+1}x^{n+1}$, the operator of q-integration J_q defined by $J_q x^n = \frac{1-q}{1-q^{n+1}}x^{n+1}$, the Dirac operator V_0 given by $V_0 f(x) = f(0)$, and the identity operator I, then the following relations hold at the basis $\{x^n \mid n \in \mathbb{N}\}$:

$$\overline{D}T = I, \quad T\overline{D} = I - V_0, \quad D_q J_q = I, \quad J_q D_q = I - V_0, D_q X - qXD_q = I,$$

and

$$\overline{D}D_q = qD_q\overline{D} + \overline{D}^2, \text{ and } TJ_q = qJ_qT + J_q^2,$$

which are precisely the Cases (i) and (ii) considered in Example 1.1. If $q = 1$, we obtain the Weyl relation $DT - TD = I$, and the equivalent relations $\overline{D}D - D\overline{D} = \overline{D}^2$ and $TJ - JT = J^2$.

It is straightforward to see that the sets of operators $\{T^m D_q^n\}$ and $\{D_q^m T^n\}$, for $m, n \in \mathbb{N}$, are PBW bases of the algebra generated by the operators T and D_q. Similarly, the sets $\{T^m J_q^n\}$ and $\{J_q T^n\}$ ($\{\overline{D}^m D^n\}$ and $\{D^m \overline{D}^n\}$), where $m, n \in \mathbb{N}$, are PBW bases of the algebras generated by the operators T and J_q (D and $\overline{D}$). It follows that for both Cases (i) and (ii) in Example 1.1, the sets $\left\{a^m b^n \mid m, n \in \mathbb{N}\right\}$ and $\{b^m a^n \mid m, n \in \mathbb{N}\}$ are bases of the algebra (1.13).

Notice that if $a = 0$ in (1.13), then the quadratic algebra is given by $yx = bxy + cy^2$. From [GM05, Expression (11)], we know that this is an algebra of skew polynomials determined by the operators $\Psi_0 = 0, \Psi_1(x^n) = b^n x^n, \Psi_2(x^n) = \Psi_1 c D_b(x^n), \Psi_3(x^n) = \Psi_1 c^2 D_b^2(x^n)$, that is, $\Psi_k = \Psi_1 c^{k-1} D_b^{k-1}$, for $k = 1, 2, \ldots$

Example 1.2 ([GM05, Section 1.5]). When two of the three coefficients a, b, and c are equal to zero, the resulting algebras are given by three types: (i) $yx = ax^2$, (ii) $yx = cy^2$, and (iii) $yx = bxy$. The set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a PBW basis for algebras (iii) and (iv), while the algebra (v) is an Ore extension.

Examples 1.1 and 1.2 guarantee that if $ab = 0$, then the quadratic algebra defined by (1.13) is an algebra of skew polynomials over $\Bbbk[x]$ and has a PBW basis $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [GM05, Proposition 1].

A useful tool in the study of the PBW bases for quadratic algebras defined by (1.13) is the matrix given by

$$M := \begin{pmatrix} b & a \\ -c & 1 \end{pmatrix},$$

which is called the *companion* matrix for (1.13) [GM05, Section 2.2]. If the lower-right elements of the matrices M^l does not vanish for all $l \in \mathbb{N}$, then $Q(a, b, c)$ has a PBW basis of the form $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [GM05, Proposition 4]. Equivalently, $Q(a, b, c)$ is the ring of skew polynomials over $\Bbbk[x]$ determined by the sequence of operators Ψ_k, $k = 0, 1, \ldots$, satisfying the infinite relations

$$\begin{aligned}
\Psi_0 X &= aX^2 + bX\Psi_0 + c\Psi_0^{(2)},\\
\Psi_1 &= \qquad\quad bX\Psi_1 + c\Psi_1^{(2)},\\
&\vdots\\
\Psi_k X &= \qquad\quad bX\Psi_k + c\Psi_k^{(2)},
\end{aligned}$$

where

$$\Psi_0^{(2)} = \Psi_0^2, \qquad \Psi_k^{(2)} = \Psi_0\Psi_k + \Psi_1\Psi_{k-1} + \cdots + \Psi_k\Psi_0,$$

see [GM05, Lemma 1]. General relations between $\Psi_j^{(k)}$ were formulated in [Mak00, Mak05, Smi68].

If $b + ac \neq 0$ a necessary and sufficient condition for the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ form a basis of $Q(a,b,c)$ is precisely that the lower-right elements of the matrices M^l does not vanish for all $l \in \mathbb{N}$. If $a = c = 1$ and $b = -1$, then the elements $\{x^m y^n \mid m, n \in \mathbb{N}\}$ are linearly independent but do not form a PBW basis of $Q(a,b,c)$ [GM05, Propositions 5 and 10]. Notice that when two of the three coefficients $a, b,$ and c are equal to zero, the resulting algebras are given by three types: (i) $yx = ax^2$ (ii) $yx = cy^2$ and (iii) $yx = bxy$. The set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a PBW basis for algebras (i) and (ii), while the algebra (iii) is an Ore extension [GM05, Section 1.5].

From the facts above, it follows that every polynomial $f(x,y) \in Q(a,b,c)$ has a unique representation as a finite sum of the monomials of the PBW basis in the form $f(x,y) = \sum_{i,j} a_{i,j} x^i y^j$, with $a_{i,j} \in \Bbbk$. As usual, the *degree* of every monomial $x^i y^j$ is defined to be $i + j$, and the *degree* of $f(x,y)$ is the greatest degree of the monomials that appear in the expansion of $f(x,y)$. This degree is known as the *total degree* of $f(x,y)$. As in the commutative polynomial ring $\Bbbk[x,y]$, we also have the expression $f(x,y) = \sum_{i=0}^{n} p_i(x) y^i$, where the $p_i(x)$'s are elements of the commutative polynomial ring $\Bbbk[x]$. We call this representation of $f(x,y)$ the *the normal form* of $f(x,y)$, and we will refer to n as the degree of f with respect to the indeterminate y.

1.2.9 Skew Poincaré-Birkhoff-Witt extensions

Skew Poincaré-Birkhoff-Witt extensions were defined by Gallego and Lezama [GL11] with the aim of generalizing Poincaré-Birkhoff-Witt extensions introduced by Bell and Goodearl [BG88] (Section 1.2.2) and Ore extensions of injective type defined by Ore [Ore31, Ore33] (Section 1.2.1). Over the years, several authors have shown that skew PBW extensions also generalize families of noncommutative algebras such as those mentioned in the previous sections. The importance of skew PBW extensions is that they do not assume that the coefficients commute with the variables, and the coefficients do not necessarily belong to fields. As a matter of fact, skew PBW extensions contain well-known groups of algebras such as some types of G-algebras in the sense of Apel [Ape98], Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, families of differential operator rings, and many other algebras of interest in noncommutative algebraic geometry and noncommutative differential geometry. Ring, theoretical and geometrical properties of skew PBW extensions have been presented in several works [AT24, Art15, Gal15, HHR20, HR22, JR18, LG19, LR20b, NR17, NR20, NRR20, Rey13b, Rey14, RR21, RS16a, RS16b, RS17a, RS17b, RS19, Su7b, SLR15, SRS23, Ven20].

Definition 1.13 ([GL11, Definition 1]). Let R and A be rings. We say that A is a *skew PBW extension* over R (also called a *σ-PBW extension of R*) if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist elements $x_1, \dots, x_n \in A \setminus R$ such that A is a left free R-module with basis given by the set $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $1 \le i \le n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \le i, j \le n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n,$$

i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{k=1}^{n} r_k^{(i,j)} x_k.$$

From now on, we use freely the notation $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ to denote a skew PBW extension A over a ring R in the indeterminates $x_1, \dots, x_n$. R will be called the *ring of coefficients* of the extension A.

Remark 5 ([GL11, Remark 2]). (i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ in Definition 1.13 are unique.

(ii) If $r = 0$, it follows that $c_{i,0} = 0$. In fact, from $0 = x_i 0 = c_{i,0} + r_i$, with $r_i \in R$, we obtain $c_{i,0} = 0 = r_i$ for all i.

(iii) In Definition 1.13 (iv), $d_{i,i} = 1$. This follows from $x_i^2 - d_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - d_{i,i} = 0 = s_i$.

(iv) Let $i < j$. From Definition 1.13 it follows that there exist elements $d_{j,i}, d_{i,j} \in R$ such that $x_i x_j - d_{j,i} x_j x_i \in R + R x_1 + \cdots + R x_n$ and $x_j x_i - d_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$, and hence $1 = d_{j,i} d_{i,j}$, that is, for each $1 \le i < j \le n$, $d_{i,j}$ has a left inverse and $d_{j,i}$ has a right inverse.

(v) Every element $f \in A \setminus \{0\}$ has a unique representation as $f = \sum_{i=0}^{t} r_i X_i$, with $r_i \in R \setminus \{0\}$ and $X_i \in \mathrm{Mon}(A)$ for $0 \le i \le t$ with $X_0 = 1$. When necessary, we use the notation $f = \sum_{i=0}^{t} r_i Y_i$.

Proposition 1.7 ([GL11, Proposition 3]). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R. For each $1 \le i \le n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$.*

We use the notation $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$ for the families of injective endomorphisms and σ_i-derivations, respectively, established in Proposition 1.7. For a skew

PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ over R, we say that the pair (Σ, Δ) is a *system of endomorphisms and* Σ*-derivations* of R with respect to A. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha := \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions.

The next definition presents some particular examples of skew PBW extensions.

Definition 1.14 ([GL11, Definition 4], [LAR15, Definition 2.3 (ii)]). Consider a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ over R.

(a) A is called *quasi-commutative* if the conditions (iii) - (iv) in Definition (1.13) are replaced by the following:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

(b) A is *bijective* if σ_i is bijective, for every $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) If σ_i is the identity map of R for each $i = 1, \dots, n$, then we say that A is a skew PBW extension of *derivation type*. Similarly, if δ_i is zero, for every i, then A is called a skew PBW extension of *endomorphism type*.

(d) A is said to be *semi-commutative* if it is quasi-commutative and $x_i r = r x_i$, for each i and every $r \in R$.

Definition 1.15 ([GL11, Definition 6]). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over R.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) If f is an element as in Remark 5(v), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

Proposition 1.8 ([GL11], Theorem 7). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension over R if and only if the following conditions hold:*

(1) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, so is r_α.*

(2) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R \setminus \{0\}$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

In $\mathrm{Mon}(A)$, we define the total order

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but there exists } i \text{ with } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases} \tag{1.17}$$

If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A \setminus \{0\}$ can be represented in a unique way as $f = r_1x^{\alpha_1} + \cdots + r_tx^{\alpha_t}$, with $r_i \in R \setminus \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leading monomial* of f and we write $\mathrm{lm}(f) := x^{\alpha_1}$; r_1 is the *leading coefficient* of f, $\mathrm{lc}(f) := r_1$; and $r_1x^{\alpha_1}$ is the *leading term* of f, $\mathrm{lt}(f) := r_1x^{\alpha_1}$. It is clear that $\succeq$ is a *monomial order* in the sense of [GL11], i.e., the following conditions hold

(i) For every $x^\alpha, x^\beta, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$

$$x^\alpha \succeq x^\beta \Longrightarrow \mathrm{lm}(x^\gamma x^\alpha x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\beta x^\lambda);$$

(ii) $x^\alpha \succeq 1$, for every $x^\alpha \in \mathrm{Mon}(A)$; and

(iii) $\succeq$ is degree compatible, i.e., $|\alpha| \geq |\beta| \Longrightarrow x^\alpha \succeq x^\beta$.

Example 1.3. (i) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a PBW extension over R. A is a positively SG ring with graduation $A = \bigoplus_{n\in\mathbb{N}} A_n$, where

$$A_n := {}_R\langle x^\alpha \in \mathrm{Mon}(A) \mid \deg(x^\alpha) = n\rangle,$$

i.e., A_n are the set of homogeneous polynomials of degree n.

Remark 6. (i) From Definition 1.13 (iv), it is clear that skew PBW extensions are more general than iterated skew polynomial rings. For example, universal enveloping algebras of finite dimensional Lie algebras and some 3-dimensional skew polynomial algebras in the sense of Bell and Smith [BS90] cannot be expressed as iterated skew polynomial rings but are skew PBW extensions. For quasi-commutative skew PBW extensions, these are isomorphic to iterated Ore extensions of endomorphism type [LR14, Theorem 2.3].

(ii) Skew PBW extensions of endomorphism type are more general than iterated Ore extensions of endomorphism type. Let us illustrate the situation with two and three indeterminates.

For the iterated Ore extension of endomorphism type $R[x;\sigma_x][y;\sigma_y]$, if $r \in R$ then we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. Now, if we have $\sigma(R)\langle x, y\rangle$ a skew PBW extension of endomorphism type over R, then for any $r \in R$, Definition 1.13 establishes that $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R. From these relations it is clear which one of them is more general.

If we have the iterated Ore extension $R[x;\sigma_x][y;\sigma_y][z;\sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. For the skew PBW extension of endomorphism type $\sigma(R)\langle x, y, z\rangle$, $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3, r_3', r_3''$ of R. As the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

(iii) Ambiskew polynomial rings (Section 1.2.1) are skew PBW extensions over B, that is, $R(B,\sigma,c,p) \cong \sigma(B)\langle y,x\rangle$.

(iv) PBW extensions introduced by Bell and Goodearl [BG88] (Section 1.2.2) are particular examples of skew PBW extensions. More exactly, the first objects satisfy the relation $x_i r = r x_i + \delta_i(r)$, for every $i = 1,\dots,n$, and each $r \in R$, and the elements d_{ij} in Definition 1.13 (iv) are equal to the identity of R. As examples of PBW extensions, we mention the following: the enveloping algebra of a finite-dimensional Lie algebra; any differential operator ring $R[\theta_1,\dots,\theta_1;\delta_1,\dots,\delta_n]$ formed from commuting derivations $\delta_1,\dots,\delta_n$; differential operators introduced by Rinehart; twisted or smash product differential operator rings, and others (for more details, see [BG88, p. 27]).

(v) 3-dimensional skew polynomial algebras (Section 1.2.3) and bi-quadratic algebras on three generators with PBW bases (Section 1.2.4) are skew PBW extensions.

(vi) The *Jordan algebra* introduced by Jordan [Jor01] is the free $\Bbbk$-algebra $\mathscr{J}$ defined by $\mathscr{J} := \Bbbk\{x,y\}/\langle yx - xy - y^2\rangle$. It is immediate to see that this algebra is not a skew polynomial ring of automorphism type but an easy computation shows that $\mathscr{J} \cong \sigma(\Bbbk[y])\langle x\rangle$.

(vii) The algebra $U_q'(\mathfrak{so}_3)$ is a skew PBW extension over $\Bbbk$, i.e., $U_q'(\mathfrak{so}_3) \cong \sigma(\Bbbk)\langle I_1,I_2,I_3\rangle$ [FGL+20, Example 1.3.3].

(viii) Using techniques such as those presented in [FGL+20, Theorem 1.3.1], it can be shown that AW(3) is a skew PBW extension of endomorphism type, that is, AW(3) $\cong \sigma(\mathbb{R})\langle K_0,K_1,K_2\rangle$.

Proposition 1.9 ([LR14, Proposition 4.1]). *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over R. If R is a domain, then A is a domain.*

From Definition 1.13 it follows that skew PBW extensions are not $\mathbb{N}$-graded rings in a non-trivial sense. With this in mind, Proposition 1.10 allows to define a subfamily of these extensions, the *graded skew PBW extensions* (Definition 1.16) introduced by Suárez in his PhD Thesis [Su7b] (see also [Su7a]). Before presenting its definition, we recall the following facts:

- If $R = \bigoplus_{p\in\mathbb{N}} R_p$ and $S = \bigoplus_{p\in\mathbb{N}} S_p$ are $\mathbb{N}$-graded rings, then a map $\varphi : R \to S$ is called *graded* if $\varphi(R_p) \subseteq S_p$, for each $p \in \mathbb{N}$. For $m \in \mathbb{N}$, $R(m) := \bigoplus_{p\in\mathbb{N}} R(m)_p$, where $R(m)_p := R_{p+m}$.

- Suppose that $\sigma : R \to R$ is a graded algebra automorphism and $\delta : R(-1) \to R$ is a graded σ-derivation (i.e., a degree $+1$ graded σ-derivation δ of R). Let $B := R[x;\sigma,\delta]$ be the associated *graded Ore extension* of R, that is, $B = \bigoplus_{p\geq 0} Rx^p$ as an R-module, and for $r \in R$, $xr = \sigma(r)x + \delta(r)$. If we consider x to have degree 1 in B, then under this grading B is a connected graded algebra generated in degree 1 (for more details, see [CS08, Pha12]).

Proposition 1.10 ([Su7a, Proposition 2.7(ii)]). *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If the following conditions hold:*

(1) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$, and*

(2) $x_j x_i - d_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, *as in Definition 1.13* (iv) *and* $d_{i,j} \in R_0$,

then A is an $\mathbb{N}$-graded algebra with graduation given by $A = \bigoplus_{p \geq 0} A_p$, *where for* $p \geq 0$, A_p *is the* $\Bbbk$*-space generated by the set*

$$\left\{ r_t x^{\alpha} \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^{\alpha} \in \mathrm{Mon}(A) \right\}.$$

Definition 1.16 ([Su7a, Definition 2.6]). Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m \geq 0} R_m$. If A satisfies both conditions established in Proposition 1.10, then we say that A is a *graded skew PBW extension over R*.

Some properties of graded skew PBW extensions can be found in [SAR21, SCR21, SR17, SRS23, SLR17]. Next, we recall some examples of these objects.

Example 1.4. The Jordan plane, homogenized enveloping algebras (Section 1.2.7), and some classes of diffusion algebras (Section 1.2.5, [Su7a, Examples 2.9]) are graded skew PBW extensions. If we assume the condition of PBW basis, then graded Clifford algebras defined by Le Bruyn [LB95] are also examples of graded skew PBW extensions. Let us see the details.

Following Cassidy and Vancliff [CV10], let $\Bbbk$ be an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$ and let $M_1, \ldots, M_n \in \mathbb{M}_n(\Bbbk)$ be symmetric matrices of order $n \times n$ with entries in $\Bbbk$. A *graded Clifford algebra* $\mathscr{A}$ is a $\Bbbk$-algebra on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by

(i) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$;

(ii) y_k central for all $k = 1, \ldots, n$.

Notice that the commutative polynomial ring $R = \Bbbk[y_1, \ldots, y_n]$ is an $\mathbb{N}$-graded algebra where $R_0 = \Bbbk, R_1 = \{0\}$, $y_1, \ldots, y_n \in R_2$, and $R_i = \{0\}$, for $i \geq 3$. If we suppose that the set $\left\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{N}, i = 1, \ldots, n\right\}$ is a left PBW R-basis for $\mathscr{A}$, then the graded Clifford algebra $\mathscr{A}$ is a graded skew PBW extension over the connected algebra R, that is, $\mathscr{A} \cong \sigma(R)\langle x_1, \ldots, x_n\rangle$. Indeed, from the relations (i) and (ii) above, it is clear that $\sigma_i = \mathrm{id}_R$, $\delta_i = 0$, $d_{i,j} = -1 \in R_0$, for $1 \leq i, j \leq n$, and $\sum_{k=1}^{n} (M_k)_{ij} y_k \in R_2$, where $d_{i,j}$ is given as in Definition 1.13 (iv). In this way, $\mathscr{A}$ is a bijective skew PBW extension that satisfies both conditions of Proposition 1.10.

CHAPTER 2

On the Burchnall-Chaundy theory in families of semi-graded rings

With the aim of motivating the study of commuting differential operators beloging to noncommutative algebras, and hence to develop a possible $\mathscr{BC}$ theory for them, in this chapter we review some of the most important results on the theory of these operators in some families of semi-graded rings presented in Chapter 2. This is a framework of techniques until today, which includes algebraic and matrix methods. As expected, our list of results is not exhaustive but we suggest possible references for more details about the subject.

In Section 2.1 we start by recalling (Section 2.1.1) some results due to Amitsur [Ami58], Carlson and Goodearl [CG80], and Goodearl [Goo83] on fields of constants, centralizers and annihilating polynomials in the setting of differential operator rings. Next, Section 2.1.2 contains key facts about $\mathscr{BC}$ theory for q-difference operators and q-deformed Heisenberg algebras developed by Silvestrov et al. [dJSS09, LS03]. Section 2.1.3 presents the notion of *Bounded Dimension Homogeneous Centralisers* (BDHC) introduced by Hellstrom and Silvestrov [HS07], which is a well-known property of many q-deformed Heisenberg algebras, and generalized Weyl structures by using the definition of ergodipotency of an endomorphism that defines the basic commutation relations in these algebras. Section 2.1.4 presents the key results about commuting operators of some families of Ore extensions in one indeterminate appearing in Richter [Ric14b]. Finally, in Section 2.1.5 we consider the approach to $\mathscr{BC}$ theory formulated by Reyes and Suárez [RS18] for skew PBW extensions.

Section 2.2 treats the construction of the annihilating curves formulated by $\mathscr{BC}$ theory. With this aim, in Section 2.2.1 we recall some important facts about the resultant theory in the commutative polynomial ring $\mathbb{k}[x]$. This is due to the importance of this theory for the computation of $\mathscr{BC}$ *curves* that annihilate pairs of commutative elements belonging to families of Ore extensions (Section 2.2.2). All these facts will be of our interest in Section 3.2 but now in the noncommutative context of quadratic algebras having PBW bases (Section 1.2.8).

Section 2.3 contains different results on centralizers of families of noncommutative algebras. We start in Section 2.3.1 with two well-known and very important results on centralizers in free associative algebras: Cohn's centralizer theorem and Bergman's centralizer theorem (Propositions 2.40 and 2.41, respectively). Section 2.3.2 concerns Makar-Limanov's treatment [ML06] on centralizers of quantum space algebras. Next, Section 2.3.3 concerns centralizers of single elements in certain Ore extensions with a non-invertible endomorphism of the ring of polynomials in one intederminate over a field, following the ideas presented by Richter and Silvestrov [RS14]. Later, Section 2.3.4 contains Richter's ideas [Ric16] on centralizers in certain algebras with valuations. Section 2.3.5 consider an affine domain of Gelfand-Kirillov dimension two over an algebraically closed field, for which Bell and Small [BS04] characterized the centralizers of any non-scalar element of this domain. By last, in Section and 2.3.6 we present some results about centralizers of some examples of skew PBW extensions.

Finally, in Section 3.4 we formulate some ideas for a future work.

2.1 Commuting operators in families of noncommutative rings

2.1.1 Differential operator rings

Amitsur [Ami58] obtained results which contributed to this problem: he considered $f = a_0 D^n + \cdots + a_n$, $a_0 \neq 0$ a differential operator degree n with constant coefficients of a field K and he characterizes the centralizer of f as the ring $F[D] = \left\{\sum_{i=0}^{m} c_i D^i \mid c_i \in K\right\}$. This result motivates further authors to continue the work on commuting differential operators throught an *algebraical perspective*. In this sense, there are numerous authors who developed some generalizations of Amitsur's results to certain families of noncommutative rings of polynomial type of great interest in noncommutative algebra, more especifically, the theory of semi-graded rings.

As we said in the Introduction, from an algebraic point of view, a related problem to the considered by Burchnall and Chaundy is to classify all the commutative subalgebras of commuting operators that are algebraically dependent. The first work in this line of thinking was Amitsur's paper [Ami58]. Our purpose in this section is to recall Amitsur's work and some algebraic generalizations of his results to the more general context of differential operator rings.

Amitsur [Ami58] considered a field $\Bbbk$ of characteristic zero with a derivation $D : \Bbbk \to \Bbbk$, $a \mapsto a'$, and the field F of constants of $\Bbbk$, that is, the elements of $\Bbbk$ such that $a' = 0$ (c.f. Section 1.2.1). Let $\Bbbk[D]$ be the ring of all *formal differential polynomials* $p(D) = p_n D^n + p_{n-1} D^{n-1} + \cdots + p_0$, $p_i \in \Bbbk$, with multiplication defined in $\Bbbk[D]$ by the relation $Da = aD + a'$, for every $a \in \Bbbk$. $\Bbbk[D]$ can be considered as *the ring of linear differential calculus on* $\Bbbk$. If $f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0$, $n \geq 1$, $a_n \neq 0$, is a polynomial of degree n in $\Bbbk[D]$, $C[f]$ denotes the centralizer of $f(D)$ in $\Bbbk[D]$, that is, $C[f] = \left\{g(D) \in \Bbbk[D] \mid gf = fg\right\}$. As we can see, $C[f]$ is indeed a subring of $\Bbbk[D]$ and it contains the ring $F[f]$ of all polynomials in $f(D)$ with constant coefficients. With this notation, the most important results of Amitsur's paper are the following:

Proposition 2.1 ([Ami58, Theorem 1]). (1) *$C[f]$ is a free $F[f]$-module of dimension t, where t is a divisor of n, the degree of $f(D)$.*

(2) *$C[f]$ is a commutative ring.*

Proposition 2.2 ([Ami58, Corollary 2]). *If $h \in C[f]$ then h is algebraic over $F(f)$ and its degree is a divisor of t, that is, there exists a polynomial $H(h, f) = 0$ with constant coefficients and* *where degree in h is a divisor of t.*

Carlson and Goodearl [CG80] considered the algebraic and analytic structure of the commutant $\mathscr{C}(L)$ of a regular ordinary differential operator L with C^∞ matrix-valued coefficients. They proved that $\mathscr{C}(L)$ is a free module of rank at most k^2n^2 over the polynomial ring of $\mathbb{C}[L]$, where k is the size of the matrices in the coefficients of L and n is the order of L. This result was proved in a completely algebraic setting, namely for differential operators whose coefficients are matrices over a commutative differential ring in which systems of homogeneous linear differential equations have finite-dimensional solution sets. In the C^∞ case, the algebraic structure of $\mathscr{C}(L)$ was obtained from an embedding of $\mathscr{C}(L)$ into the ring $nk \times nk$ matrices over a polynomial ring $\mathbb{C}[\lambda]$, and the image of $\mathscr{C}(L)$ in this matrix was explicitly determined. For $\mathscr{C}_m(L)$, the set of those operators in $\mathscr{C}(L)$ with rank at most m, Carlson and Goodearl proved that the dimension of $\mathscr{C}(L)$ is an upper semicontinuous function of L. Their paper finishes by finding some first integrals for the commutation equation $LT - TL = 0$.

With the aim of giving more details about Carlson and Goodearl's paper, we recall some definitions and preliminary results.

Definition 2.1 ([CG80, p. 340]). An (ordinary) *differential ring* is a ring R equipped with a derivation $' : R \to R$ satisfying the product rule $(xy)' = x'y + xy'$, for all $x, y \in R$. If R is also an algebra over the rational field $\mathbb{Q}$, then R is called a *differential* $\mathbb{Q}$-algebra. Given a differential ring R, we form the *formal linear differential operator ring* $R[D]$ over R as follows. Additively, $R[D]$ is the Abelian group of all polynomials over R in an indeterminate D. Multiplication in $R[D]$ is defined so that multiplication of elements of R is not changed, and so that $Dr = rD + r'$, for all $r \in R$. Any non-zero operator $T \in R[D]$ can be uniquely written in the form $t_0 + t_1D + \cdots + t_nD^n$, where the $t_i \in R$ and $t_n \neq 0$. The integer n is called the *order* of T, denoted $\mathrm{ord}(T)$. If T has positive order and the leading coefficient t_n is invertible in R, then T is called a *regular* differential operator.

Definition 2.2 ([CG80, p. 341]). The *commutant* of a differential operator $L \in R[D]$ is the set $\mathscr{C}(L)$ of all operators in $R[D]$ which commute with L, and we note that $\mathscr{C}(L)$ is a subring of $R[D]$. If $F = \{r \in R \mid r' = 0\}$, which is a subring of R, then the ring $F[L]$ of all polynomials in L with coefficients from F is always a subring of $\mathscr{C}(L)$. As a result, $\mathscr{C}(L)$ can be viewed as a module over $F[L]$, and we shall derive sufficient conditions under which $\mathscr{C}(L)$ must be a free $F[L]$-module of finite rank.

Some of the most important results proved by Carlson and Goodearl [CG80] are the following:

Proposition 2.3 ([CG80, Theorem 1.2]). *Let R be a commutative differential ring, and assume that the subring $F = \{r \in R \mid r' = 0\}$ is a field of characteristic zero. For all $a \in R$, assume that the set $\{r \in R \mid r' = ar\}$ either equals $\{0\}$ or contains an invertible element. Let $L \in R[D]$ be regular of order n. Let X be the set of those $i \in \{0, 1, \ldots, n-1\}$ for which $\mathscr{C}(L)$ contains an operator of order congruent to i modulo n. Set $T_0 = 1$, and for non-zero $i \in X$, choose $T_i \in \mathscr{C}(L)$ such that $\mathrm{ord}(T_i) \equiv i \pmod n$ and T_i has minimal order for this property. Then $\mathscr{C}(L)$ is a free $F[L]$-module with basis $\{T_i \mid i \in X\}$. Moreover, the rank of $\mathscr{C}(L)$ as a free $F[L]$-module is a divisor of n.*

Proposition 2.4 ([CG80, Theorem 1.6]). *Let I be an open interval of the real line, set $R = C^\infty(I)$, and let $L \in R[D]$ be a regular differential operator of order n. Then $\mathscr{C}(L)$ is a commutative* *integral domain, and $\mathscr{C}(L)$ is a free $\mathbb{C}[L]$-module of finite rank which is a divisor of n. Furthermore, $\mathscr{C}(L) = \mathbb{C}[L]$ if and only if all non-zero operators in $\mathscr{C}(L)$ have order divisible by n.*

Example 2.1 ([CG80, p. 345]). Let I be an open interval of the real line, set $R = C^{\infty}(I)$, and let $q \in R$. According to Proposition 2.4, $\mathscr{C}(D^2+q)$ is a free $\mathbb{C}[D^2+q]$-module of rank either 1 or 2, and both possibilities can occur. For a trivial rank 2 example, set $q = 0$. Then $\mathscr{C}(D^2) = \mathbb{C}[D]$, whence $\mathscr{C}(D^2)$ is a free $\mathbb{C}[D^2]$-module of rank 2, with basis $\{1, D\}$.

A non-trivial example is the following. Assume that $I \neq \mathbb{R}$, the set of real numbers, and choose a real number $m \notin I$. The function q defined by the rule $q(x) = -2(x-m)^{-2}$ belongs to R. If $T = D^3 - 3(x-m)^{-2}D + 3(x-m)^{-3}$, then $T(D^2+q) = (D^2+q)T$, so $\mathscr{C}(D^2+q)$ must have rank 2 over $\mathbb{C}[D^2+q]$. From Proposition 2.3 it follows that $\{1, T\}$ is a basis for $\mathscr{C}(D^2+q)$ over $\mathbb{C}[D^2+q]$. For a rank 1 example, define $q \in R$ by the rule $q(x) = x$. Carlson and Goodearl [CG80, p. 346] proved that any non-zero $T \in \mathscr{C}(D^2+q)$ must have even order, so $\mathscr{C}(D^2+q) = \mathbb{C}[D^2+q]$.

As we said above, Carlson and Goodearl also considered the commuting operators to regular differential operators L in $M_k(R)[D]$, where R is a commutative differential ring with the derivation extended from R to $M_k(R)$ in the natural way. Under certain assumptions, they proved that $\mathscr{C}(L)$ is a free module of finite rank over a polynomial ring over L; nevertheless, the bounds for the rank of this free module are not quite as good as those obtained in the previous results. Let us see the details.

Definition 2.3 ([CG80, p. 347]). Let $L \in M_k(R)[D]$. For all $m = 0, 1, 2, \ldots$, we use $\mathscr{C}_m(L)$ to denote the set of those operators in $\mathscr{C}(L)$ of order at most m.

Proposition 2.5 ([CG80, Proposition 2.1]). *Let R be a commutative differential ring, and assume that the subring $F = \{r \in R \mid r' = 0\}$ is a field. If $L \in M_k(R)[D]$ is regular of order n, then $\mathscr{C}(L)$ is a free $F[L]$-module of rank at most*

$$\lim_{m\to\infty} \sup (n/m) \dim_F(\mathscr{C}_m(L)).$$

Proposition 2.6 ([CG80, Theorem 2.2]). *Let R be a commutative differential ring and assume that the subring $F = \{r \in R \mid r' = 0\}$ is a field. For all $A \in M_m(R)$, assume that the set $\{v \in R^m \mid v' = Av\}$ is a vector space of dimension at most m over F. If $L \in M_k(R)[D]$ is regular of order n, then $\mathscr{C}(L)$ is a free $F[L]$-module of rank at most k^2n^2.*

Under the hypotheses formulated in Proposition 2.6, if $T \in \mathscr{C}(L)$ then there exist polynomials $P_0, P_1, \ldots, P_{t-1} \in F[L]$ such that $P_0 + P_1T + P_2T^2 + \cdots + P_{t-1}T^{t-1} + T^t = 0$ [CG80, Corollary 2.3].

Proposition 2.7 ([CG80, Theorem 2.4]). *Let R be a commutative differential ring, and assume that the subring $F = \{r \in R \mid r' = 0\}$ is a field of characteristic zero. For all $A \in M_m(R)$, assume that the set $\{v \in R^m \mid v' = Av\}$ is a vector space of dimension at most m over F. Let $L \in M_k(R)[D]$ be regular of order n, and assume that the leading coefficient of L lies in the center of $M_k(R)$. Then $\mathscr{C}(L)$ is a free $F[L]$-module of rank at most k^2n.*

Proposition 2.8 ([CG80, Theorem 2.5]). *Let R be a commutative differential ring, and assume that the subring $F = \{r \in R \mid r' = 0\}$ is a field of characteristic zero. For all $a \in R$, assume that the set $\{r \in R \mid r' = ar\}$ is a vector space of dimension at most 1 over F. Let $L \in M_k(R)[D]$ be regular* *of order n, and assume that the leading coefficient of L is a diagonal matrix*

$$\begin{bmatrix} a_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & a_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{k-1,k-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{k,k} \end{bmatrix}$$

such that $a_{ii} - a_{jj}$ is a non-zero divisor in R, for all $i \neq j$. Then $\mathscr{C}(L)$ is a free $F[L]$-module of rank at most kn.

EXAMPLES 2.1 ([CG80, P. 351]). (i) Let I be an open interval of the real line, $R := C^{\infty}(I)$ and $k = 2$. Consider the operators D^2 in $M_2(R)[D]$. We can see that $\mathscr{C}(D^2) = M_2(\mathbb{C})[D]$. In this way, the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix},$$

form a basis for $\mathscr{C}(D^2)$ over $\mathbb{C}[D^2]$. Hence, $\mathscr{C}(D^2)$ is a free $\mathbb{C}[D^2]$-module of rank 8. Of course, in this examples the bound $k^2 n$ in Proposition 2.7 is attained.

(ii) Consider the operator $L = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in $M_2(R)[D]$. Then $\mathscr{C}(L)$ consists of those operators $T \in M_2(R)[D]$ such that every coefficient of T is a diagonal matrix with constant entries. Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix},$$

is a basis for $\mathscr{C}(L)$ over $\mathbb{C}[L]$, whence $\mathscr{C}(L)$ is a free $\mathbb{C}[L]$-module of rank 4. Note that in this example the bound kn in Proposition 2.8 is attained.

(iii) If L is the operator given by $L = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} D^2 + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$ in $M_2(R)[D]$, where q is the function defined by the rule $q(x) = x$, then $\mathscr{C}(L)$ consists of those operators in $M_2(R)[D]$ which can be written in the form $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$, where $S \in \mathscr{C}(D^2)$ and $T \in \mathscr{C}(D^2 + q)$ (in $R[D]$). Of course, $\{1, D\}$ is a basis for $\mathscr{C}(D^2)$ over $\mathbb{C}[D^2]$, and as we saw above, $\mathscr{C}(D^2 + q) = \mathbb{C}[D^2 + q]$. Therefore,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

is a basis for $\mathscr{C}(L)$ over $\mathbb{C}[L]$, i.e., $\mathscr{C}(L)$ is a free $\mathbb{C}[L]$-module of rank 3. Note that this example shows that, in general, the rank of $\mathscr{C}(L)$ as a free $\mathbb{C}[L]$-module need not be a divisor of $k^2 n^2$, unlike the situation for $k = 1$ (Proposition 2.3).

Another result proved by Carlson and Goodearl [CG80] concerns the commutativity of $\mathscr{C}(L)$. Of course, in general, this is not the case (take the case of the operator D in $M_k(R)[D]$, where $\mathscr{C}(D) = M_k(\mathbb{C})[D]$). Nevertheless, there is one class of operators L for which $\mathscr{C}(L)$ is always commutative. Let us see this situation.

PROPOSITION 2.9 ([CG80, THEOREM 2.6]). *Let R be a commutative differential $\mathbb{Q}$-algebra. For all $a \in R$, assume that the set $\{r \in R \mid r' = ar\}$ contains no non-zero nilpotent elements. Let $L \in M_k(R)[D]$ be regular, and assume that all the coefficients of L are lower triangular matrices. Assume also that the leading coefficient of L is a lower triangular matrix*

$$\begin{bmatrix} a_{1,1} & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,k-1} & 0 \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & a_{k,k} \end{bmatrix}$$

such that $a_{ii} - a_{jj}$ is a non-zero divisor in R, for all $i \neq j$. Then $\mathscr{C}(L)$ is a commutative ring with no non-zero nilpotent elements.

Three years after Carlson and Goodearl's work, Goodearl [Goo83] extended once more again the results of Amitsur to a more general kind of noncommutative rings.

For the next assertions, Goodearl consider the differential operator ring $R[D]$ over R seen above as $R[\theta;\delta]$. In this way, $R[\theta;\delta]$ is the abelian group of all polynomials over R in an indeterminate θ. Multiplication in $R[\theta;\delta]$ is given by the multiplication in R and the obvious rules for powers of θ together with the rule $\theta r = r\theta + \delta(r)$, for all $r \in R$. Another notation and terminology established in Definition 2.1 is easily expressed for $R[\theta;\delta]$.

PROPOSITION 2.10 ([GOO83, THEOREM 1.11]). *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta;\delta]$. Let $a \in S$ be an operators of positive order n. Then $C_S(a)$ is a free $F[a]$-module of finite rank, and that rank is at most n^2.*

PROPOSITION 2.11 ([GOO83, THEOREM 1.2]). *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta;\delta]$. Let $a \in S$ be an operator of positive order n, such that n is invertible in F and the leading coefficient of a is invertible in R. Let X be the set of those $i \in \{0,1,\dots,n-1\}$ for which the centralizer $C_S(a)$ of a in S contains an operator of order congruent to i modulo n. For each $i \in X$, choose $b_i \in C_S(a)$ such that $\operatorname{ord}(b_i) \equiv i \pmod{n}$ and b_i has minimal order for this property. [In particular, $0 \in X$, and we may choose $b_0 = 1$.] Then $C_S(a)$ is a free $F[a]$-module with basis $\{b_i \mid i \in X\}$. Moreover, the rank of $C_S(a)$ as a free $F[a]$-module is a divisor of n.*

PROPOSITION 2.12 ([GOO83, THEOREM 1.11]). *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta;\delta]$. Let $a \in S$ be an operators of positive order n. Then $C_S(a)$ is a free $F[a]$-module of finite rank, and that rank is at most n^2.*

Propositions 2.12 and 2.11 guarantee the following result:

COROLLARY 2.13 ([GOO83, THEOREM 1.13]). *Let R be a semiprime commuting differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta;\delta]$. Let $a \in S$ be an operator of positive order whose leading coefficient is invertible in R. Given any operator $b \in S$ which* *commutes with a, there exist polynomials $p_0(a), p_1(a), \dots, p_{t-1}(a)$ in $F[a]$ such that*

$$p_0(t) + p_1(a)b + \cdots + p_{t-1}(a)b^{t-1} + b_t = 0.$$

In particular, there is a non-constant polynomial $q(x, y)$ over F in two commuting indeterminates x, y such that $q(a, b) = 0$.

2.1.2 q-DIFFERENCE OPERATORS AND q-DEFORMED HEISENBERG ALGEBRAS

In this section, we recall some key facts about $\mathscr{BC}$ theory for q-difference operators and q-deformed Heisenberg algebras. We follow the ideas and the treatment developed by Silvestrov et al. [LS03, dJSS09, SSdJ80].

In several papers, Jackson [Jac08, Jac09a, Jac09b, Jac10a, Jac10b], considered a detailed study of the q-difference operator

$$(D_q\phi)(x) = \frac{\phi(x) - \phi(qx)}{(1-q)x}, \ \ q \neq 1,$$

which is known as *Euler-Jackson, Jackson q-difference operator*, or simply the q-derivative. As we can see, this operator may be applied to any function not containing $x = 0$ in the domain of definition, and the limit as q approaches to 1 is the ordinary derivative, i.e., $\lim_{q\to 1}(D_q\phi)(x) = \frac{d\phi}{dx}(x)$, if ϕ is differentiable at x. In this sense, the q-differential operator is considered as a generalization of the usual derivative. Notice that the D_q-constants or multiplicatively q-periodic functions are solutions of the functional equation

$$m(qx) = m(x) \quad \text{or} \quad D_q m(x) = 0. \tag{2.1}$$

As expected, this operator satisfies some properties of the classical derivative (see Exton [Ext83] for more details). For example, the formulas for the q-difference of a sum of functions and of a product by a constant are given by

$$\begin{aligned} D_q(u(x) + v(x)) &= D_q u(x) + D_q v(x),\\ D_q(cu(x)) &= cD_q u(x), \end{aligned}$$

so the operator D_q is linear when it acts on a linear space of functions. The formula for the q-difference of a product and a quotient of functions are the following:

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + D_q f(x)g(x), \tag{2.2}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}, \tag{2.3}$$

(notice that the classical Leibniz rule for the derivative of the product is recovered when q tends to 1), while the chain rule is given by

$$
\begin{aligned}
D_q(f\circ g)(x) &= \frac{f(g(x)) - f(g(qx))}{(1-q)x}\\
&= \frac{f(g(x)) - f\left(\frac{g(qx)}{g(x)}g(x)\right)}{\left(1-\frac{g(qx)}{g(x)}\right)g(x)}\frac{\left(1-\frac{g(qx)}{g(x)}\right)g(x)}{(1-q)x}\\
&= \left(D_{\frac{g(qx)}{g(x)}}\right)(g(x))D_q(g)(x).
\end{aligned}
$$

Of course, this rule holds for all $x \neq 0$ such that $g(x)\neq 0$ and $g(qx)\neq g(x)$. As a matter of fact, the Leibniz rule for action of powers of the q-derivative operator on a product of functions is given by

$$
D_q^n(fg)(x) = \sum_{i=0}^{n}\binom{n}{i}_q D_q^k(f)(xq^{n-k})D_q^{n-k}(g)(x). \tag{2.4}
$$

If we consider the multiplicative q-shift operator $T_q : f(x)\to f(qx)$, then the Leibniz rules (2.2) and (2.4) can be written in the following way:

$$
D_q(f(x)g(x)) = T_q f(x)D_q g(x) + D_q f(x)g(x) \tag{2.5}
$$

$$
D_q^n(fg)(x) = \sum_{i=0}^{n}\binom{n}{i}_q T_q^{n-k}D_q^k(f)(x)D_q^{n-k}(g)(x). \tag{2.6}
$$

By the definition of the q-derivative, it follows that the action of D_q on the functions x^s is given by the q-analogue of the usual rule $D_q(x^s) = \{s\}_q x^{s-1}$.

From the discussion above, it follows that the linear q-difference operators are sums of the form

$$
P = \sum_{j=0}^{n} p_j D_q^j,
$$

where the coefficients p_i are functions which are assumed to be polynomials in the indeterminate x [LS03, p. 98]. All these facts motivate the definition of the q−deformed Heisenberg algebra.

Definition 2.4 ([LS03, p. 98]). The *q-deformed Heisenberg algebra for $q\in\mathbb{C}\setminus\{0\}$* is the $\mathbb{C}$-algebra $H_q(\mathbb{C})$ with identity element I and generators A and B satisfying the q-deformed Heisenberg canonical commutation relation $AB - qBA = I$.

The q-deformed Heisenberg algebra is important for q-difference operators since that Jackson q-difference operator D_q and the operator of multiplication $M_x : f(x)\to xf(x)$ satisfy the q-deformed Heisenbserg canonical commutation relation

$$
D_qM_x - qM_x = I.
$$

As a matter of fact,

$$\begin{aligned}(D_qM_x - qM_xD_q)(f)(x) &= \frac{xf(x) - qxf(x)}{x(1-q)} - \frac{qxf(x) - qxf(qx)}{x(1-q)} \\ &= \frac{xf(x) - qxf(x)}{x(1-q)} \\ &= f(x) \\ &= (If)(x).\end{aligned}$$

By using the relation $AB - qBA = I$, we can prove by induction that

$$\begin{aligned}B^nA^n &= q^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \left(BA - \left(\sum_{k=0}^{j-1} q^k \right) I \right) \\ &= q^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} (BA - \{j\}_q I).\end{aligned}$$

This relation guarantees that in the q-deformed Heisenberg algebra H_q, all monomials and linear combinations of monomials of the form B^nA^n commute with each other [LS03, Lemma 1].

The next result is a generalization of the $\mathscr{BC}$ theorem to q-deformed Heisenberg algebras.

Proposition 2.14 ([HS00]). *If $P, Q \in H_q$ commute, that is, $PQ = QP$, then there exists a non-zero polynomial F in two commutative indeterminates with coefficients from the center of H_q such that $F(P,Q) = 0$ in H_q.*

About the center of H_q, if q is not a root of unity or if $q = 1$, then the center of H_q is trivial, that is, only consists from the elements of the form λI, $\lambda \in \mathbb{C}$. In this case, to any pair of commuting elements in H_q, one can associate an algebraic curve in $\mathbb{C}^2$ given by the corresponding polynomial with complex coefficients due to Proposition 2.14. On the other hand, when q is a root of unity but not 1, the center of H_q is the subalgebra generated by A^p and B^p, where p is the smallest positive integer such that $q^p = 1$. The coefficients in the polynomial from Proposition 2.14 are some elements of this commutative subalgebra (they could be the elements of the form λI, and in this case we again would get an ordinary algebraic curve; if they are not, then we obtain an algebraic curve with coefficients, which are not scalars but polynomials in A^p and B^p).

In Section 2.2 we will present some examples to find the curve annihilating the two commuting operators in the context of q-difference operators.

2.1.3 Ergodipotent maps

Hellstrom and Silvestrov [HS07] considered $\mathbb{Z}$-graded algebras having *Bounded Dimension Homogeneous Centralisers* (BDHC), which is a well-known property of many q-deformed Heisenberg algebras [HS00, Theorem 5.13]. They also considered algebras with a *generalized Weyl structure* (GWS), which is a generalized form of the class of Generalized Weyl Algebras (GWAs).

GWS algebras are not necessarily $\mathbb{Z}$-graded but there is a gradation for them so that the BDHC condition can be a consequence of ergodipotency of an endomorphism that defines the basic commutation relations in these algebras.

Definition 2.5 ([HS07, Definition 2.1]). Let an algebra $\mathscr{A}$ and a family $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ of subspaces of $\mathscr{A}$ given. Then $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ is said to be a *pseudogradation* of $\mathscr{A}$ if

(i) $\mathscr{A}_m \cdot \mathscr{A}_n \subseteq \mathscr{A}_{m+n}$ for all $m, n \in \mathbb{Z}$.

(ii) $\mathscr{A} = \sum_{n\in\mathbb{Z}} \mathscr{A}_n$.

For $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ to be a *gradation*, it is an addition required that this sum is direct, i.e., $\mathscr{A} = \bigoplus_{n\in\mathbb{Z}} \mathscr{A}_n$. Define

$$\begin{aligned} \operatorname{Cen}(p) &= \left\{q \in \mathscr{A} \mid pq = qp\right\}, \\ \operatorname{Cen}(\mathscr{A}) &= \bigcap_{p\in\mathscr{A}} \operatorname{Cen}(p), \\ \operatorname{Cen}(n,p) &= \left\{q \in \mathscr{A}_n \mid pq = qp\right\}, \end{aligned}$$

for all $n \in \mathbb{Z}$ and $p \in \mathscr{A}$. The set $\operatorname{Cen}(p)$ (a subalgebra of $\mathscr{A}$) is called the *centraliser* of p, whereas $\operatorname{Cen}(\mathscr{A})$ is called the center of $\mathscr{A}$. The set $\operatorname{Cen}(n,p)$ is called the n-homogeneous centraliser of p. The algebra $\mathscr{A}$ is said to have *l-bounded-dimension homogeneous centralisers*, or l-BDHC for short, if $l \in \mathbb{N}$ is such that for all $n \in \mathbb{Z}$, non-zero $m \in \mathbb{Z}$, and non-zero $p \in \mathscr{A}_m$ it holds that $\dim \operatorname{Cen}(n,p) \leq l$.

Hellstrom and Silvestrov [HS07, p. 19] asserted that the reason for excepting $m = 0$ in the definition of BDHC is that if $p \in \mathscr{A}_0$, then any $p^k \in \mathscr{A}_0$ as well and thus $\dim \operatorname{Cen}(0,p) = \infty$ in many cases of practical interest.

For $\mathscr{A}$ a graded algebra, denote by $\pi_n : \mathscr{A} \to \mathscr{A}_n$ the projection of $\mathscr{A}$ onto $\mathscr{A}_n$ given by the direct sum $\mathscr{A} = \bigoplus_{n\in\mathbb{Z}} \mathscr{A}_n$. Define $\overline{\chi}, \underline{\chi} : \mathscr{A} \setminus \{0\} \to \mathbb{Z}$ through

$$\begin{aligned} \overline{\chi}(p) &= \max\{n \in \mathbb{Z} \mid \pi_n(p) \neq 0\}, \\ \underline{\chi}(p) &= \min\{n \in \mathbb{Z} \mid \pi_n(p) \neq 0\}, \end{aligned}$$

and let $\overline{\chi}(0) = -\infty$ but $\underline{\chi}(0) = \infty$. A $p \in \mathscr{A}$ is then homogeneous non-zero if and only if $\overline{\chi}(p) = \underline{\chi}(p)$. Set $\overline{\pi}(p) = \pi_{\overline{\chi}(p)}(p)$ and $\underline{\pi}(p) = \pi_{\underline{\chi}(p)}(p)$ for all non-zero $p \in \mathscr{A}$, and set $\overline{\pi}(0) = \underline{\pi}(0) = 0$.

Proposition 2.15 ([HS07, Lemma 2.2]). *The functions $\overline{\chi}$ and $\underline{\chi}$ are additive, i.e.,*

$$\begin{aligned} \overline{\chi}(pq) &= \overline{\chi}(p) + \overline{\chi}(q), \\ \underline{\chi}(pq) &= \underline{\chi}(p) + \underline{\chi}(q), \end{aligned}$$

for all $p, q \in \mathscr{A} \setminus \{0\}$ if and only if $\mathscr{A}$ has no zero divisors.

The $\overline{\chi}$ and $\underline{\chi}$ relations with respect to sums are rather that

$$\overline{\chi}(p+q) \leq \max\{\overline{\chi}(p), \overline{\chi}(q)\},$$
$$\underline{\chi}(p+q) \geq \min\{\underline{\chi}(p), \underline{\chi}(q)\}.$$

These inequalities imply

$$\begin{aligned} \overline{\chi}(p+q) &= \overline{\chi}(p) \quad \text{if } \overline{\chi}(p) > \overline{\chi}(q) \\ \overline{\chi}(p+q) &= \underline{\chi}(p) \quad \text{if } \underline{\chi}(p) < \underline{\chi}(q). \end{aligned}$$

Proposition 2.16. (1) [HS07, Lemma 2.4] *Let $p \in \mathscr{A}$ be arbitrary such that $m = \overline{\chi}(p) > 0$. If $\mathscr{A}$ has l-BDHC and there are no zero divisors in $\mathscr{A}$ then $\overline{\pi}(p)$ is invertible or $\overline{\chi}(q) \geq 0$ for all $q \in \mathrm{Cen}(p) \setminus \{0\}$.*

(2) [HS07, Lemma 2.5] *Assume $\mathscr{A}$ has l-BDHC and that there are no zero divisors in $\mathscr{A}$. If $p \in \mathscr{A} \setminus \mathscr{A}_0$ has $\overline{\chi}(p) = m > 0$ and $\overline{\pi}(p)$ is not invertible in $\mathscr{A}$ then there exist a finite $\Bbbk[p]$-module basis $\{q_1, \dots, q_k\}$ for $\mathrm{Cen}(p)$, the elements of which satisfy*

$$\overline{\chi}\left(\sum_{i=1}^{k} p_i q_i\right) = \max_{1\leq i\leq k}(\overline{\chi}(p_i) + \overline{\chi}(q_i)), \tag{2.7}$$

for all $p_1, \dots, p_k \in \Bbbk[p]$. Furthermore the number of elements k in this basis is at most lm.

Proposition 2.17 ([HS07, Theorem 2.6])**.** *Assume $\mathscr{A}$ has l-BDHC and that there are no zero divisors in $\mathscr{A}$. If $p \in \mathscr{A} \setminus \mathscr{A}_0$ and $q \in A$ such that $pq = qp$, $\overline{\chi}(p) > 0$, and $\overline{\pi}(p)$ is not invertible in $\mathscr{A}$ then there exists a non-zero polynomial P in two commuting indeterminates and with coefficients from $\Bbbk$ such that $P(p,q) = 0$.*

The next proposition is the most important result obtained by Hellstrom and Silvestrov [HS07].

Proposition 2.18 ([HS07, Theorem 2.8]). *Assume $\mathscr{A}$ has 1-BDHC and that there are no zero divisors in $\mathscr{A}$. If $p \in A \setminus \mathscr{A}_0$ has $\overline{\chi}(p) = m > 0$ and $\overline{\pi}(p)$ is not invertible in $\mathscr{S}$ then:*

(1) *The $\Bbbk[p]$-module $\mathrm{Cen}(p)$ has a finite basis $\{q_g\}_{g\in G}$, where G is a subgroup of $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $\overline{\chi}(q_g) \in g$ for all $g \in G$.*

(2) *The cardinality of the basis $\{q_g\}_{g\in G}$ of $\mathrm{Cen}(p)$ divides m.*

(3) *$\mathrm{Cen}(p)$ is a commutative subalgebra of $\mathscr{A}$.*

Example 2.2 ([HS07, Example 2.9]). There exist 2-BDHC algebras that does not satisfy claims of Proposition 2.18. Take the family of algebras

$$\mathscr{A} = \mathbb{R}\{A, B, I\}/\langle AB - (q_0 + q_1 I)BA = 1,\ AI = -IA,\ BI = -IB,\ I^2 = -1\rangle,$$

for some $q_0, q_1 \in \mathbb{R}$, which is $\mathbb{Z}$-graded with $A \in \mathscr{A}_{-1}, B \in \mathscr{A}_1$, and $I \in \mathscr{A}_0$. As a matter of fact, $\mathscr{A}$ is not 1-BDHC.

With the aim of illustrating the results about BDHC condition, let us recall the notion of Generalized Weyl Algebras introduced by Bavula [Bav92].

Let a unital algebra $\mathscr{R}$, some pairwise commuting algebra automorphisms $\sigma_1, \dots, \sigma_n : \mathscr{R} \to \mathscr{R}$, and an equal number of elements $C_1, \dots, C_n \in \mathrm{Cen}(\mathscr{R})$ be given. The *Generalized Weyl Algebra* (or GWA for short) $\mathscr{W}$ defined by these data is what one gets by adjoining to $\mathscr{R}$ another $2n$ generators $\{A_i, B_i\}_{i=1}^{n}$ that are subject to the relations

$$\begin{aligned}
A_i r &= \sigma_i(r) A_i, & r B_i &= B_i \sigma_i(r), \quad \text{for all } r \in \mathscr{R}\\
B_i A_i &= C_i, & A_i B_i &= \sigma_i(C_i),\\
A_i A_j &= A_j A_i, & B_i B_j &= B_j B_i, \quad \text{and} \quad A_i B_j = B_j A_i \quad \text{if } i \neq j,
\end{aligned}$$

for all $i, j = 1, \dots, n$. Rosenberg [Ros95] called these objects *hyperbolic* algebras or rings (Section 1.2.6).

An immediate example of GWA is the q-deformed Heisenberg-Weyl algebra $H_q(\mathbb{C})$ considered in Section 2.1.2 (Definition 2.1.2). This follows from the following data: $\mathscr{R} = \mathbb{C}[C_1]$, $\sigma_1(C_1) = qC_1 + 1$, $A = A_1$, $B = B_1$, and $AB = \sigma_1(C_1) = qC_1 + 1 = qBA + 1$.

Let $\mathscr{A}$ be an associative unital $\Bbbk$-algebra. Let $\mathscr{R} \subseteq \mathscr{A}$ be a unital subalgebra of $\mathscr{A}$, and let an algebra endomorphism $\sigma : \mathscr{R} \to \mathscr{R}$ be given. Then the algebra $\mathscr{A}$ is said to have a *generalized Weyl structure* (GWS) hung on the *spine* $\mathscr{R}$ if there are elements $A, B \in \mathscr{A} \setminus \mathscr{R}$ such that $\mathscr{A}$ is generated by $\mathscr{R} \cup \{A, B\}$ and

$$Ar = \sigma(r)A, \qquad \text{for all } r \in R, \tag{2.8}$$

$$rB = B\sigma(r), \qquad \text{for all } r \in R, \tag{2.9}$$

$$AB \in \mathscr{R}, \tag{2.10}$$

$$\sigma^n(AB)r = r\sigma^n(AB), \qquad \text{for all } r \in \mathscr{R} \text{ and } n \in \mathbb{N}. \tag{2.11}$$

The distinguished elements A and B are called the *Weylian generators* of the algebra.

Example 2.3 ([HS07, Example 3.3]). A GWS algebra that is not a GWA is given by

$$\mathscr{D}(d) = \Bbbk\{A, B\} / \langle AB = (BA)^d \rangle, \quad d \geq 2.$$

The spine $\mathscr{R}$ in this case is the unital subalgebra generated by BA, and as it happens $\mathscr{R} \cong \Bbbk[x]$ as an algebra, σ is the endomorphism sending BA to $(BA)^d$. Since the endomorphism $\sigma : \Bbbk[x] \to \Bbbk[x]$ which satisfies $\sigma(x) = x^d$ is not surjecive, it follows that this is not a GWA.

Another example is the algebra

$$\Bbbk\left\{A, B, E\right\} / \left\langle AE = A, EB = B, AB - qBA = E^2 + \frac{q^2}{4}(BA)^2 \right\rangle,$$

where $q \in \Bbbk$ is arbitrary and $\mathrm{char}(\Bbbk) \neq 2$. The subalgebra generated by $C = E + \frac{q}{2}BA$ again turns out to be isomorphic to $\Bbbk[x]$, and works as a spine with σ defined by $\sigma(C) = \frac{q}{2}C^2 + 1$.

Next, $\mathscr{A}$ denotes an arbitrary unital $\Bbbk$-algebra with generalized Weyl structure hung on spine $\mathscr{R}$.

Definition 2.6. [HS07, Definition 3.4] For any algebra $\mathscr{A}$ with a generalized Weyl structure, let $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ be the family of subspaces defined by

$$\mathscr{A}_n = \sum_{\substack{k,l\in\mathbb{N}\\ k-l=n}} B^k \mathscr{R} A^l, \tag{2.12}$$

where $\mathscr{R}$ is the spine and A and B are the Weylian generators as above. Notice that $A \in \mathscr{A}_{-1}$, $B \in \mathscr{A}_1$, and $\mathscr{R} \subseteq \mathscr{A}_0$.

As one can check, the family $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ is a pseudogradation of $\mathscr{A}$. In fact, $\mathscr{A}_n = B^n \cdot \mathscr{A}_0$ and $\mathscr{A}_{-n} = \mathscr{A}_0 \cdot A^n$, for all $n \in \mathbb{N}$ [HS07, Lemma 3.6].

Proposition 2.19. (1) [HS07, Lemma 3.8] *Unless A and B are both zero divisors, there exists a linear map $\hat{\sigma} : \mathscr{A}_0 \to \mathscr{A}_0$ which satisfies*

$$\hat{\sigma}(B^k \gamma A^k) = \begin{cases} \sigma(y) & \text{if } k = 0, \\ B^{k-1}\sigma^{k-1}(AB)\gamma A^{k-1} & \text{if } k > 0, \end{cases} \tag{2.13}$$

for all $k \geq 0$ and $\gamma \in \mathscr{R}$.

(2) [HS07, Lemma 3.9] *If there is a linear map $\hat{\sigma} : \mathscr{A}_0 \to \mathscr{A}_0$ which satisfies (2.13) for all $k \geq 0$ and $\gamma \in \mathscr{R}$, then this map is an algebra endomorphism on $\mathscr{A}_0$ which additionally satisfies $\hat{\sigma}(BA) = AB$,*

$$A\gamma = \hat{\sigma}(\gamma)A, \tag{2.14}$$

$$\gamma B = B\hat{\sigma}(\gamma), \tag{2.15}$$

for all $\gamma \in \mathscr{A}_0$.

Proposition 2.20 ([HS07, Theorem 3.10]). *Let $\mathscr{A}$ be an algebra with a generalized Weyl structure such that $\{\mathscr{A}_n\}_{n\in\mathbb{Z}}$ is a gradation. Then $\mathscr{A}$ has no zero divisors if and only if all of the following conditions hold:*

(1) *$\mathscr{A}_0$ has no zero divisors.*

(2) *$AB \neq 0$.*

(3) *The map $J : \mathscr{A}_0 \to \mathscr{A}_0$ defined by $J(\gamma) = B\gamma A$ is injective.*

(4) *$\hat{\sigma} : \mathscr{A}_0 \to \mathscr{A}_0$ is injective.*

Definition 2.7 ([HS07, Definition 3.12]). Let $\mathscr{B}$ be a unital $\Bbbk$-algebra and $\sigma : \mathscr{B} \to \mathscr{B}$ a map. Then σ is said to be *ergodipotent* if the fact that $\sigma^n(\alpha) = \alpha$ for some $n > 0$ and $\alpha \in \mathscr{B}$ implies that $\alpha \in \Bbbk$ (i.e., such an element α must be a scalar multiple of the multiplicative identity in $\mathscr{B}$).

An example of an ergodipotent endomorphism σ of $\mathscr{B} = \Bbbk[x]$ is that which has $\sigma(x) = x^2$, since this implies $\deg(\sigma(\alpha)) = 2\deg(\alpha)$, for all $\alpha \in \mathscr{B}$. Notice that a solution to $\alpha = \sigma^n(\alpha)$ must have $\deg(\alpha) = \deg(\sigma^n(\alpha)) = 2^n\deg(\alpha)$, and hence $\deg(\alpha) = 0$.

Proposition 2.21 ([HS07, Theorem 3.13]). *If $\mathscr{A}$ has no zero divisors, $\mathscr{A}_0$ is commutative, and $\hat{\sigma}$ is ergodipotent on the field of fractions of $\mathscr{A}_0$ then $\mathscr{A}$ has 1-*BDHC.

REMARK 7. The 2-BDHC algebra $\mathscr{A}$ of Example 2.2 has a GWS for which the spine $\mathscr{R}$ is the subalgebra generated by BA and I. Besides, σ satisfies $\sigma(BA) = qBA + 1$ and $\sigma(I) = -I$, which uniquely defines it since $\mathscr{R} \cong \mathbb{C}[x]$ and σ is an $\mathbb{R}$-algebra endomorphism. It turns out $\mathscr{A}_0 = \mathscr{R}$ and $\mathscr{A}$ has no zero divisors if $q \neq 0$, but $\mathscr{A}$ fails the ergodipotency condition of Proposition 2.21.

Proposition 2.22 is the version of $\mathscr{BC}$ theory for the algebras $\mathscr{D}(d)$.

2.1.3.1 POWER ENDOMORPHISM ALGEBRAS

Hellström and Silvestrov [HS07, Section 4] studied an example of an algebra which fits the GWS definition but not the more restrictive definition of a GWA. They considered the case where the initial ring $\mathscr{R}$ is the polynomial ring $\Bbbk[C]$ in one indeterminate C over some arbitrary field $\Bbbk$ and $\sigma(C) = C^d$ with $AB = \sigma(C)$ and $AB = C$.

DEFINITION 2.8 ([HS07, DEFINITION 4.1]). Given a field $\Bbbk$ consider the free associative algebra $\Bbbk\{a, b, c\}$. Choose an integer $d > 1$ and let $\mathscr{J}(d)$ be the two-sided ideal in this free algebra which is generated by $\mathsf{ba} - \mathsf{c}, \mathsf{ab} - \mathsf{c}^d, \mathsf{cb} - \mathsf{bc}^d$, and $\mathsf{ac} - \mathsf{c}^d\mathsf{a}$. Define $\mathscr{D}(d)$ to be the quotient $\Bbbk\langle \mathsf{a}, \mathsf{b}, \mathsf{c}\rangle / \mathscr{J}(d)$, and denote $\mathsf{a} + \mathscr{J}(d), \mathsf{b} + \mathscr{J}(d)$, and $\mathsf{c} + \mathscr{J}(d)$ by A, B, and C, respectively. These defining relations imply that

$$BA = C, \quad AB = C^d, \quad CB = BC^d, \quad AC = C^d A. \tag{2.16}$$

As one can check, for a polynomial $P \in \Bbbk[x]$,

$$P(C) \cdot B = B \cdot P(C^d) \quad \text{and} \quad A \cdot P(C) = P(C^d) \cdot A, \tag{2.17}$$

which means that the unital subalgebra $\mathscr{R}$ of $\mathscr{D}(d)$ that is generated by C fits the conditions for being the GWS spine of $\mathscr{D}(d)$. The defining endomorphism $\sigma : \mathscr{R} \to \mathscr{R}$ acts as $\sigma(C) = C^d$, or more generally, $\sigma(P(C)) = P(C^d)$.

Let us resume some of the key properties of the algebras $\mathscr{D}(d)$.

- Consider the $\mathbb{Z}$-gradation $\{F_n\}_{n\in\mathbb{Z}}$ such that $\mathsf{a} \in F_{-1}$, $\mathsf{b} \in \mathsf{F}_1$, and $\mathsf{c} \in F_0$ of the free algebra $\Bbbk\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$. Define $\mathscr{D}_n(d) = \{\alpha + \mathscr{J}(d) \mid \alpha \in F_n\}$, for $n \in \mathbb{Z}$. Since $\mathsf{ba} - \mathsf{c}, \mathsf{ab} - \mathsf{c}^d \in F_0$, $\mathsf{cb} - \mathsf{bc}^d \in F_1$, and $\mathsf{ac} - \mathsf{c}^d a \in F_{-1}$, the ideal $\mathscr{J}(d)$ is homogeneous with respect to the gradation $\{F_n\}_{n\in\mathbb{Z}}$, and consequently $\{\mathscr{D}_n(d)\}_{n\in\mathbb{Z}}$ is a gradation of $\mathscr{D}(d)$. Notice that this gradation coincides with the pseudogradation in Definition 2.6 for GWS algebras since the generating elements A, B, and C have the same degrees in both cases.
- A clear difference to the q-deformed Heisenberg algebras $H_q(\mathbb{C})$ is that the set $\{B^k A^l\}_{k,l\in\mathbb{N}}$ does not span the algebra $\mathscr{D}(d)$. In this way, it is necessary to seek some other set of short products that span $\mathscr{D}(d)$. From [HS07, Theorem 4.2] we know that the set

$$\left\{B^k C^m A^l \mid k, l, m \in \mathbb{N} \text{ and } k = 0, l = 0, \text{ or } d \text{ does not divide } m\right\}, \tag{2.18}$$

  is a $\Bbbk$-vector space basis of the algebra $\mathscr{D}(d)$.

Notice that if $d > 1$ and $\mathscr{A}$ is either of the two algebras

$$\Bbbk\{x, y\}/\langle yx = xy^d\rangle \quad \text{and} \quad \Bbbk\{x, y\}/\langle yx = x^d y\rangle,$$

then it is straightforward to see that $\mathscr{A}$ can be embedded into $\mathscr{D}(d)$.

The following proposition establishes the $\mathscr{BC}$ theory for the algebras $\mathscr{D}(d)$.

Proposition 2.22 ([HS07, Theorem 4.14]). *There are no zero divisors in the algebra $\mathscr{D}(d)$ and this algebra has 1-*BDHC. *The only invertible elements in $\mathscr{D}(d)$ are the non-zero elements of $\Bbbk$. If $\alpha, \beta \in \mathscr{D}(d)$ satisfy $\alpha\beta = \beta\alpha$ then there exists a non-zero polynomial $P \in \Bbbk[x, y]$ such that $P(\alpha, \beta) = 0$, that is, two commuting elements of $\mathscr{D}(d)$ are algebraically dependent.*

Example 2.4 ([HS07, Corollary 4.15]). Let $d > 1$ and $\mathscr{A}$ be either of the two algebras

$$\Bbbk\{x, y\}/\langle yx = xy^d\rangle \quad \text{and} \quad \Bbbk\{x, y\}/\langle yx = x^d y\rangle.$$

Then $\mathscr{A}$ can be embedded into $\mathscr{D}(d)$, it has no zero divisors, the only invertible elements are the non-zero scalars, and if $\alpha, \beta \in \mathscr{A}$ are such that $\alpha\beta = \beta\alpha$, then there exists some non-zero $P \in \Bbbk[x, y]$ such that $P(\alpha, \beta) = 0$.

Proposition 2.23 shows several properties of centers of elements of $\mathscr{D}(d)$.

Proposition 2.23 ([HS07, Corollary 4.16]). *If $\alpha \in \mathscr{D}(d)$ has $\overline{\chi}(\alpha) = m > 0$, then:*

(1) *The $\Bbbk[\alpha]$-module* $\mathrm{Cen}(\alpha)$ *has a finite basis $\{\beta_g\}_{g\in G}$, where G is a subgroup of $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $\overline{\chi}(\beta_g) \in g$ for all $g \in G$.*

(2) *The cardinality of the basis $\{\beta_g\}_{g\in G}$ of* $\mathrm{Cen}(\alpha)$ *divides m.*

(3) $\mathrm{Cen}(\alpha)$ *is a commutative subalgebra of $\mathscr{D}(d)$.*

2.1.4 Some families of Ore extensions

Richter [Ric14b] proved the following facts about Ore extensions. Recall that for an injective endomorphism σ of a ring R and δ a σ-derivation on R, any element r of R such that $\sigma(r) = r$ and $\delta(r) = 0$ is called a *constant* (Section 1.2.1).

Proposition 2.24 ([Ric14b, Theorem 3.3]). *Let R be an integral domain, σ an injective endomorphism of R and δ a σ-derivation on R. Suppose that the ring of constants, F, is a field. Let a be an element of $S := R[x; \sigma, \delta]$ of degree n and assume that if b and c are two elements in $C_S(a)$ such that* $\deg(b) = \deg(c) = m$, *then $b_m = \alpha c_m$, where b_m and c_m are the leading coefficients of b and c, respectively, and α is some constant. Then $C_S(a)$ is a free $F[a]$-module of rank at most n.*

Proposition 2.25 ([Ric14b, Proposition 3.1]). *Let $R = \Bbbk[y]$ and let σ be an endomorphism of R such that $\sigma(k) = k$, for all $k \in \Bbbk$ and $\sigma(y) = p(y)$, where $p(y)$ is a polynomial of degree in y greater than 1. Let δ be a σ-derivation such that $\delta(a) = 0$ for all $a \in \Bbbk$. Form the Ore extension $S = R[x; \sigma, \delta]$. We note that its ring of constants is $\Bbbk$. Let $f \notin \Bbbk$ be an element of $R[x; \sigma, \delta]$. Assume that b, c are elements of S such that* $\deg(b) = \deg(c) = m$ *(here the degree is taken with respect to*

x) and b, c both belong to $C_S(f)$. Then $b_m = \alpha c_m$, where b_m, c_m are the leading coefficients of b and c, respectively, and α is some constant.

From Proposition 2.25 it follows that $C_S(f)$ is a free $\Bbbk[a]$-module of finite rank [Ric14b, Proposition 3.2].

Proposition 2.26 ([Ric14b, Theorem 3.4]). *Let σ be an endomorphism of $\Bbbk[y]$ such that $\sigma(y) = p(y)$, where $\deg(p(y)) \geq 1$, and let δ be a σ-derivation. Suppose that $\sigma(k) = k$ and $\delta(k) = 0$ for all $a \in \Bbbk$. Let p, q be two commuting elements of $\Bbbk[y][x;\sigma,\delta]$. Then there is a non-zero polynomial $f(s,t) \in \Bbbk[s,t]$ such that $f(a,b) = 0$.*

These results were generalized by Reyes and Suárez [RS18] to the context of skew PBW extensions, as we will see in the next section.

2.1.5 Skew Poincaré-Birkhoff-Witt extensions

In this section, we recall an extension of $\mathscr{BC}$ theory to skew PBW extensions presented by Reyes and Suárez [RS18]. Recall that F is the subring of constants of R, and $F[P]$ is the ring of polynomials in P with constant coefficients, that is, $F[P] = \left\{\sum_{j=0}^{m} b_j P^j \mid b_j \in F\right\}$.

Proposition 2.27 ([RS18, Theorem 4.1]). *Let R be an integral domain and let A be a skew PBW extension of R. Suppose that the ring of constants F is a field. Let $f \in A$ with $\exp(f) = \alpha$, and assume that if $g, h \in C_A(f)$ with $|\exp(g)| = |\exp(h)|$, then $\mathrm{lc}(g) = p\mathrm{lc}(h)$, where p is some constant. Then $C_A(f)$ is a free $F[f]$-module of rank at most $|\alpha|$.*

Proposition 2.28 establishes sufficient conditions to use Theorem 2.27.

Proposition 2.28 ([RS18, Proposition 4.2]). *Let A be a skew PBW extension over $\Bbbk$. If $f \in A \setminus \Bbbk$, $g, h \in C_A(f)$ with $\exp(g) = \exp(h) = \beta$, and $\mathrm{lc}(g)$, $\frac{1}{\mathrm{lc}(h)}$ are constants, then $\mathrm{lc}(g) = p\ \mathrm{lc}(h)$, where p is a constant.*

The following result is the version of $\mathscr{BC}$ theory for these extensions.

Proposition 2.29 ([RS18, Theorem 4.3]). *Let P and Q be two commuting elements of A, where A is a skew PBW extension of a field $\Bbbk$. Then there is a polynomial $f(s,t)$ with coefficients in F satisfying the condition $f(P,Q) = 0$.*

Examples 2.2 ([RS18, Examples 4.4]). Theorem 2.29 can be applied to several skew PBW extensions. More precisely, if A is a skew PBW extension of a field $\Bbbk$ where the coefficients commute with the indeterminates, that is, $x_i r = r x_i$, for every $r \in R$ and each $i = 1, \ldots, n$, then Theorem 2.29 can be illustrated. Some examples of these extensions are the following: PBW extensions defined by Bell and Goodearl (which include the classical commutative polynomial rings, universal enveloping algebra of a Lie algebra, and others); some operator algebras (for example, the algebra of linear partial differential operators, the algebra of linear partial shift operators, the algebra of linear partial difference operators, the algebra of linear partial q-dilation operators, and the algebra of linear partial q-differential operators); the class of diffusion algebras; Weyl algebras; additive analogue of the Weyl algebra; multiplicative analogue of the Weyl algebra; some quantum Weyl algebras as $A_2(J_{a,b})$; the quantum algebra $U'(\mathfrak{so}(3,\Bbbk))$; the family of 3-dimensional skew polynomial algebras; Dispin algebra $U(osp(1,2))$; Woronowicz algebra $W_\nu(\mathfrak{sl}(2,\Bbbk))$; the complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$; q-Heisenberg algebra $\mathbf{H}_n(q)$; the Hayashi algebra $W_q(J)$, and several algebras of quantum physics (for instance, Weyl algebras, additive and multiplicative analogue of the Weyl algebra, quantum Weyl algebras, q-Heisenberg algebra, and others).

2.2 Resultant theory and construction of Burchnall-Chaundy curves

Our purpose in this section is to recall some important facts about the resultant theory in the commutative and noncommutative setting. We also review some applications of resultant theory for the computation of $\mathscr{BC}$ curves that annihilate pairs of commutative elements belonging to some families of Ore extensions. All these facts will be of our interest in Section 3.2 but now in the noncommutative context of quadratic algebras having PBW bases (Section 1.2.8).

2.2.1 Resultant theory in the commutative polynomial ring $\Bbbk[x]$

The concept of *resultant* can be introduced by asking when two polynomials in the commutative polynomial ring $\Bbbk[x]$ have a common factor. Two important applications of resultant theory are elimination theory, and the proofs of *Extension theorem* and *Bezout's theorem*. We follow the excellent treatment presented by Cox et al. [CLO15, Chapter 3].

Consider the question of whether two polynomials $f(x), g(x) \in \Bbbk[x]$ have a *common factor*, that is, there exists a polynomial $h(x) \in \Bbbk[x]$ of degree greater than zero that divides $f(x)$ and $g(x)$. As is well-known, one approach to solve this question would be to compute the greatest common divisor of $f(x)$ and $g(x)$ using the Euclidean Algorithm (see [CLO15, Chapter 1] for a detailed description) which requires divisions in $\Bbbk$. The idea is, precisely, to determine whether a common factor exists without doing any divisions in $\Bbbk$. Proposition 2.30 gives us a first answer.

Proposition 2.30 ([CLO15, Lemma 1, p. 161]). *Let $f(x), g(x)$ be polynomials of degrees $l > 0$ and $m > 0$, respectively. Then $f(x)$ and $g(x)$ have a common factor in $\Bbbk[x]$ if and only if there are polynomials $c(x), d(x)$ such that:*

(1) *$c(x)$ and $d(x)$ are not both zero.*

(2) *$c(x)$ has degree at most $m-1$ and $d(x)$ has degree at most $l-1$.*

(3) *$c(x)f(x) + d(x)g(x) = 0$.*

Proof. Suppose that $f(x)$ and $g(x)$ have a common factor $h(x) \in \Bbbk[x]$. This means that $f(x) = h(x)f_1(x)$ and $g(x) = h(x)g_1(x)$, where $f_1(x), g_1(x) \in \Bbbk[x]$. By degree arguments, $f_1(x)$ and $g_1(x)$ have degree at most $l-1$ and $m-1$, respectively. Since $g_1(x)f(x)+(-f_1(x))g(x) = g_1(x)h(x)f_1(x) - f_1(x)h(x)g_1(x) = 0$, whence $c(x) := g_1(x)$ and $d(x) := -f_1(x)$ satisfy the required properties.

Conversely, suppose that the polynomials $c(x)$ and $d(x)$ have the above three properties. By (1), it is possible to assume that $d(x)$ is non-zero. Notice that if $f(x)$ and $g(x)$ have no common factor, their greatest common divisor is 1, so we can find polynomials $\widehat{c(x)}, \widehat{d(x)} \in \Bbbk[x]$ such that

$\widehat{c(x)}f(x)+\widehat{d(x)}g(x)=1$ [CLO15, Proposition 6, p. 41]. By using that $d(x)q(x)=-c(x)p(x)$, it follows that

$$\begin{aligned}
d(x) &= d(x)(\widehat{c(x)}f(x)+\widehat{d(x)}g(x)) \\
&= \widehat{c(x)}d(x)f(x)+\widehat{d(x)}d(x)g(x) \\
&= \widehat{c(x)}d(x)f(x)+\widehat{d(x)}(-c(x)f(x)) \\
&= (\widehat{c(x)}g(x)-\widehat{d(x)}c(x))f(x).
\end{aligned}$$

Having in mind that B is non-zero, the above expression shows that $d(x)$ has degree at least l, which contradicts (3). Therefore, $c(x)$ and $d(x)$ must have a common factor of positive degree. □

As expected, we still need to decide whether the polynomials $c(x)$ and $d(x)$ exist. *Linear Algebra* is very helpful to answer this question. Let us see the details since these will be of interest in Section 3.2.

Let us start by expressing the equality $c(x)f(x)+d(x)g(x)=0$ as a system of linear equations. With this aim, consider

$$\begin{aligned}
c(x) &= u_0x^{m-1}+\cdots+u_{m-1}, \\
d(x) &= v_0x^{l-1}+\cdots+v_{l-1}
\end{aligned}$$

where the $l+m$ coefficients $u_0,\ldots,u_{m-1},v_0,\ldots v_{l-1}$ are unknown, and the objective is to find explicitly its values such that the equation

$$c(x)f(x)+d(x)g(x)=0, \tag{2.19}$$

holds.

Let us write

$$\begin{aligned}
f(x) &= c_0x^l+\cdots+c_l, \quad c_0\neq 0, \\
g(x) &= d_0x^m+\cdots+d_m, \quad d_0\neq 0,
\end{aligned}$$

where the elements $c_i,d_i\in\Bbbk$. Next, we substitute the expressions for $f(x),g(x),c(x)$, and $d(x)$ into (2.19), compare the corresponding coefficients of powers in the indeterminate x, whence we obtain the following system of linear equations with unknowns u_i,v_i and coefficients c_i,d_i belonging to $\Bbbk$:

$$\begin{aligned}
c_0u_0+d_0v_0 &= 0 \quad \text{coefficient of } x^{l+m-1} \\
c_1u_0+c_0u_1+d_1v_0+d_0v_1 &= 0 \quad \text{coefficient of } x^{l+m-2} \\
&\vdots \\
c_lu_{m-1}+d_mv_{l-1} &= 0 \quad \text{coefficient of } x^0.
\end{aligned} \tag{2.20}$$

As we can see, there are $l+m$ linear equations and $l+m$ unknowns, so there is a non-zero solution if and only if the coefficient matrix has zero determinant. This fact motivates the definition of *Sylvester matrix*.

DEFINITION 2.9 ([CLO15, DEFINITION 2]). Consider two non-zero polynomials $f(x), g(x) \in \Bbbk[x]$ of degree l, m, respectively, expressed as

$$
\begin{aligned}
f(x) &= c_0 x^l + \cdots + c_l, \quad c_0 \neq 0,\\
g(x) &= d_0 x^m + \cdots + d_m, \quad d_0 \neq 0.
\end{aligned}
$$

If $l, m > 0$, then the *Sylvester matrix* of f and g with respect to x, denoted $\mathrm{Syl}(f(x), g(x))$, is the coefficient matrix of the system of equations given in (2.20). Equivalently, $\mathrm{Syl}(f(x), g(x))$ is the $(l+m) \times (l+m)$ matrix

$$
\mathrm{Syl}(f(x), g(x)) = \begin{bmatrix}
c_0 & & & & d_0 & & & \\
c_1 & c_0 & & & d_1 & d_0 & & \\
c_2 & c_1 & \ddots & & d_2 & d_1 & \ddots & \\
\vdots & & \ddots & c_0 & \vdots & & \ddots & d_0\\
 & \vdots & & c_1 & & & & d_1\\
c_l & & & & d_m & & & \\
 & c_l & & \vdots & & d_m & & \vdots\\
 & & \ddots & & & & \ddots & \\
 & & & c_l & & & & d_m
\end{bmatrix},
$$

where the empty spaces are filled by zeros. The *resultant* of $f(x)$ and $g(x)$ with respect to x, denoted $\mathrm{Res}(f(x), g(x))$, is the determinant of the Sylvester matrix. In this way,

$$
\mathrm{Res}\big(f(x), g(x)\big) = \big|\mathrm{Syl}\big(f(x), g(x)\big)\big|.
$$

REMARK 8. When f, g do not both have positive degree, let

$$
\begin{aligned}
&\mathrm{Res}(c_0, g(x)) = c_0^m, \quad \text{when } c_0 \in \Bbbk \setminus \{0\},\ m > 0,\\
&\mathrm{Res}(c_0, g(x)) = d_0^l, \quad \text{when } c_0 \in \Bbbk \setminus \{0\},\ l > 0,\\
&\mathrm{Res}(c_0, g(x)) = 1, \quad \text{when } c_0, d_0 \in \Bbbk \setminus \{0\},
\end{aligned}
$$

Proposition 2.31 establishes a relationship between resultants and common factors. Recall that a polynomial is called an *integer polynomial* provided that all of its coefficient are integers.

PROPOSITION 2.31 ([CLO15, PROPOSITION 3]). *Given non-zero $f(x), g(x) \in \Bbbk[x]$, the resultant $\mathrm{Res}\big(f(x), g(x)\big) \in \Bbbk$ is an integer polynomial in the coefficients of $f(x)$ and $g(x)$. Furthermore, $f(x)$ and $g(x)$ have a common factor in $\Bbbk[x]$ if and only if $\mathrm{Res}\big(f(x), g(x)\big) = 0$.*

Note that when $f(x)$ or $g(x)$ is a non-zero constant, $\mathrm{Res}(f(x), g(x)) \neq 0$ by Remark 8, and f and g cannot have a common factor since by definition common factors have positive degree.

PROPOSITION 2.32 ([CLO15, PROPOSITION 5]). *Given non-zero polynomials $f(x), g(x) \in \Bbbk[x]$,* *there are polynomials* $c(x), d(x) \in \Bbbk[x]$ *such that*

$$c(x)f(x) + d(x)g(x) = \operatorname{Res}\big(f(x), g(x)\big).$$

Furthermore, if at least one $f(x), g(x)$ *has positive degree, then the coefficients of* $c(x)$ *and* $d(x)$ *are integer polynomials in the coefficients of* f *and* $g(x)$.

Proof. As we saw above, the definition of resultant is based on the equation $c(x)f(x)+d(x)g(x) = 0$. However, we will see below that the same methods apply to the equation

$$\widehat{c(x)}f(x) + \widehat{d(x)}g(x) = 1, \tag{2.21}$$

which is more convenient in the proof of the proposition.

First of all, notice that if $\operatorname{Res}(f(x), g(x)) = 0$, then the proposition holds since we can take $f(x) = g(x) = 0$. In the case $f(x) = c_0 \in \Bbbk$ and $m = \deg(g(x)) > 0$, then Remark 8 implies that

$$\operatorname{Res}\big(f(x), g(x)\big) = c_0^m = c_0^{m-1} \cdot f(x) + 0 \cdot g(x).$$

Of course, the case $l = \deg(f(x)) > 0$ and $g(x) = d_0$ is analogous.

From this reasoning, we may assume that $f(x), g(x)$ have positive degree and satisfy $\operatorname{Res}\big(f(x), g(x)\big) \neq 0$. Consider

$$\begin{aligned}
f(x) &= c_0x^l + \cdots + c_l, \quad c_0 \neq 0,\\
g(x) &= d_0x^m + \cdots + d_m, \quad d_0 \neq 0,\\
\widehat{c(x)} &= u_0x^{m-1} + \cdots + u_{m-1},\\
\widehat{d(x)} &= v_0x^{l-1} + \cdots + v_{l-1},
\end{aligned}$$

where $u_0, \ldots, u_{m-1}, v_0, v_{l-1}$ are unknowns in the field $\Bbbk$. Note that expression (2.21) is true if and only if substituting these expressions into (2.21) gives an equality of polynomials. If we compare the coefficients of powers of x, we get that (2.21) is equivalent to the following system of linear equations with unknowns u_i, v_i and coefficients c_i, d_i in $\Bbbk$:

$$\begin{aligned}
c_0u_0 + d_0v_0 &= 0 \quad \text{coefficient of } x^{l+m-1}\\
c_1u_0 + c_0u_1 + d_1v_0 + d_0v_1 &= 0 \quad \text{coefficient of } x^{l+m-2}\\
&\vdots\\
c_lu_{m-1} + d_mv_{l-1} &= 1 \quad \text{coefficient of } x^0.
\end{aligned} \tag{2.22}$$

Of course, these equations are the same as (2.20) except for the 1 on the right-hand side of the last equation. This means that the coefficient matrix is the *Sylvester matrix* of $f(x)$ and $g(x)$, and then $\operatorname{Res}(f(x), g(x)) \neq 0$, that is, (2.22) has a unique solution in $\Bbbk$.

At this point, we will use the well-known Cramer's rule of Linear Algebra. As an illustration, for the first unknown u_0 we have

$$u_0 = \frac{1}{\operatorname{Res}\big(f(x), g(x)\big)} \det \begin{bmatrix} 0 & & & & d_0 & & \\ 0 & c_0 & & & \vdots & \ddots & \\ \vdots & \vdots & \ddots & & \vdots & & d_0 \\ 0 & c_l & & c_0 & d_m & & \vdots \\ \vdots & & \ddots & \vdots & & \ddots & \vdots \\ 1 & & & c_l & & & d_m \end{bmatrix}.$$

Having in mind that a determinant is an integer polynomial in its entries, then

$$u_0 = \frac{\text{an integer polynomial in } c_i, d_i}{\operatorname{Res}\big(f(x), g(x)\big)}.$$

All of the u_i's and v_i's can be written in a similar way. Now, by using that $\widehat{c(x)} = u_0 x^{m-1} + \cdots + u_{m-1}$, we can pull out the common denominator $\operatorname{Res}(f(x), g(x))$ and so

$$\widehat{c(x)} = \frac{1}{\operatorname{Res}\big(f(x), g(x)\big)} c(x),$$

where $c(x) \in \Bbbk[x]$ and the coefficients of $c(x)$ are integer polynomials in c_i, d_i. Analogously,

$$\widehat{d(x)} = \frac{1}{\operatorname{Res}\big(f(x), g(x)\big)} d(x),$$

where $d(x)$ has the same properties as $c(x)$. Finally, since $\widehat{c(x)}$ and $\widehat{d(x)}$ satisfy $\widehat{c(x)} f(x) + \widehat{d(x)} g(x) = 1$, we can multiply adequately to obtain

$$c(x) f(x) + g(x) d(x) = \operatorname{Res}\big(f(x), g(x)\big).$$

As we saw, the polynomials $c(x)$ and $d(x)$ have the required kind of coefficients, and so the assertion follows. □

REMARK 9. Let us see the relation between the Resultant and the greatest common divisor. Given $f, g \in \Bbbk[x]$, $\operatorname{Res}\big(f(x), g(x)\big) \neq 0$ means that f and g have no common factor, and so their greatest common divisor is 1. From [CLO15, Proposition 6, p. 41], we know that there exist polynomials $\widehat{c(x)}$ and $\widehat{d(x)}$ belonging to $\Bbbk[x]$ such that $\widehat{c(x)} f(x) + \widehat{d(x)} g(x) = 1$. Having in mind that the expressions above for $\widehat{c(x)}$ and $\widehat{d(x)}$ are explicitly, the coefficients of $\widehat{c(x)}$ and $\widehat{d(x)}$ have a denominator given by the resultant. Therefore, if we clear these denominators then we get that $c(x) f(x) + d(x) g(x) = \operatorname{Res}\big(f(x), g(x)\big)$.

EXAMPLE 2.5 ([CLO15, P. 166]). Let $f(x, y) = xy - 1$ and $g(x, y) = x^2 + y^2 - 4$. The idea is to consider these polynomials in the indeterminate x with coefficients in $\Bbbk[y]$. After some computations, we can see that $\operatorname{Res}\big(f(x), g(x)\big) = y^4 - 4y^2 + 1 \neq 0$, and so their greatest common divisor is 1, and

$$-\left(\frac{y}{y^4-4y^2+1}x+\frac{1}{y^4-4y^2+1}\right)f(x,y)+\frac{y^2}{y^4-4y^2+1}g(x,y)=1,$$

it is an equation in $\Bbbk(y)[x]$, so the coefficients are *rational functions* in the intederminate y. As expected, if we want to work in $[x,y]$, we have to clear denominators to obtain the expression

$$-(yx+1)f(x,y)+y^2g(x,y)=y^4-4y^2+1.$$

This fact means that the Resultant can be regarded as a "denominator-free" version of the greatest common divisor.

2.2.2 Burchnall-Chaundy curves in some families of Ore extensions

Now, we review the key ideas of resultant theory for the computation of $\mathscr{BC}$ curves that annihilate pairs of commutative elements belonging to some families of Ore extensions. We follow the treatment presented by Richter in his PhD thesis [Ric14a] and Richter and Silvestrov [RS12].

We start by recalling the notion of *determinant polynomial* (see [Loo82, Mis93] for more details).

Definition 2.10 ([Li98, Definition 2.3]). Let M be an $r\times c$ matrix over R. Assume that $r\le c$. The *determinant polynomial of* M is

$$|M|=\sum_{i=0}^{c-r}\det(M_i)x^i, \tag{2.23}$$

where M_i is the $r\times r$ matrix whose first $(r-1)$ columns are the first $(r-1)$ columns of M and whose last column is the $(c-i)$th column of M, for $i=0,1,\dots,c-r$.

From this definition, it follows that we encounter determinants whose last columns contain polynomial belonging to $R[x]\setminus R$. When expanding such a determinant, we place the products of entries in R on the left-hand side of the entry in $R[x]\setminus R$, whence we can express determinant polynomials by determinants [Li98, p. 133], [RS12, Section 2.1].

For the matrix M defined in (2.23) given by

$$\begin{bmatrix} m_{11} & \cdots & m_{1(r-1)} & m_{1r} & \cdots & m_{1,c}\\ m_{21} & \cdots & m_{2(r-1)} & m_{2r} & \cdots & m_{2c}\\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots\\ m_{r1} & \cdots & m_{r(r-1)} & m_{rr} & \cdots & m_{rc}\end{bmatrix}$$

and, for $i=1,\dots,4$, let H_i denote the polynomial

$$H_i=m_{i1}x^{c-1}+\cdots+m_{ir}x^{c-r}+\cdots+m_{ic}.$$

From [RS12, Proposition 1], we get that

$$|M| = \det \begin{bmatrix} m_{11} & \cdots & m_{1(r-1)} & H_1 \\ m_{21} & \cdots & m_{2(r-1)} & H_2 \\ \vdots & \cdots & \vdots & \vdots \\ m_{r1} & \cdots & m_{r(r-1)} & H_r \end{bmatrix}.$$

Let $\mathscr{F} := \{f_1, f_2, \dots, f_r\}$ be a set of polynomials in $R[x]$ and let d be the maximum of the degrees of the f_i's. Suppose that $d \geq r$. The *matrix associated to* $\mathscr{F}$, denoted by mat($\mathscr{F}$), is the $r \times (d+1)$ matrix whose entry in the ith row and jth column is the coefficient of x^{d+1-j} in f_i, where $1 \leq i \leq r$ and $1 \leq j \leq d+1$. The determinant polynomial of $\mathscr{F}$ is defined to be $|\mathrm{mat}(\mathscr{F})|$, which is denoted by $|\mathscr{F}|$.

PROPOSITION 2.33 ([RS12, PROPOSITION 2]). *Let $\mathscr{F} := \{f_1, f_2, \dots, f_r\}$ be a set of polynomials in $R[x]$ of maximum degree d. Then*

$$|\mathscr{F}| = \det \begin{bmatrix} m_{1d} & \dots & m_{1(d-r+1)} & f_1 \\ m_{2d} & \dots & m_{2(d-r+1)} & f_2 \\ \vdots & \dots & \vdots & \vdots \\ m_{rd} & \dots & m_{r(d-r+1)} & f_r \end{bmatrix},$$

where m_{ij} is the coefficient of the monomial x^j in the polynomial f_i.

Next, we recall the notion of resultant for elements belonging to $R[x]$.

DEFINITION 2.11 ([RS12, SECTION 2.2]). Let $p(x)$ and $q(x)$ be two elements of $R[x]$ of degree m and n, respectively. Their resultant, denoted $\mathrm{Res}(p(x), q(x))$, is defined as the determinant polynomial of the family of polynomials

$$\{f(x), xp(x), \dots, x^{n-1}p(x), q(x), xq(x), \dots, x^{m-1}q(x)\}.$$

Notice that this will be a determinant of size $m+n$.

PROPOSITION 2.34 ([RS12, PROPOSITION 3]). *For all elements $p(x)$ and $q(x)$ in $R[x]$, there exist elements $s(x), t(x) \in R[x]$ such that*

$$\mathrm{Res}\big(p(x), q(x)\big) = s(x)p(x) + t(x)q(x).$$

Proof. From the results above, we know that for some elements $m_{ij} \in R$,

$$\mathrm{Res}\big(p(x),q(x)\big)=\det\begin{bmatrix} m_{n1} & \cdots & m_{n(n+m-1)} & p(x)\\ \vdots & \cdots & \vdots & \vdots\\ m_{21} & \cdots & m_{2(2+m-1)} & x^{n-2}p(x)\\ m_{11} & \cdots & m_{1(n+m-1)} & x^{n-1}p(x)\\ m_{n+m} & m_{n+m,n+m-1} & & q(x)\\ \vdots & \cdots & \vdots & \vdots\\ m_{n+1} & \cdots & m_{n+1,n+m-1} & x^{m-1}q(x)\end{bmatrix}.$$

By using the definition of determinant, the assertion follows. □

PROPOSITION 2.35 ([RS12, THEOREM 1]). *If $p(x)$ and $q(x)$ are commuting elements of $R[x]$, then*

$$f(s,t)=\mathrm{Res}\big(p(x)-s,q(x)-t\big)$$

is a polynomial in two commuting variables such that $f\big(p(x),q(x)\big)=0$.

Proof. Consider the commutative polynomial ring $R[x][s,t]$. Let $p(x)$ and $q(x)$ be commuting elements of $R[x]$, so they can be considered as elements in $R[x][s,t]$. Then $\mathrm{Res}(p(x)-s,q(x)-t)=S(x)(p(x)-s)+T(x)(q(x)-t)$, for some polynomials in $R[x][s,t]$. Since $\mathrm{Res}(p(x)-s,q(x)-t)$ is an elements of $R[s,t]$, it is clear that if we replace s by $p(x)$ and t by $q(x)$, then the resultant becomes zero, which concludes the proof. □

Let us see some examples that illustrate the results formulated in this section. All these are taken from [RS12, Sections 3-6].

EXAMPLE 2.6. Consider the quantum plane $\mathbb{C}[y][x;\sigma]$, where $\sigma(y)=qy$, for some $q\in\mathbb{C}^*$. Let $p(x)=yx$ and $q(x)=qy^2x^2$. Since $p(x)q(x)=q(x)p(x)$, compute $\mathrm{Res}(p(x)-s,q(x)-t)$ in the following way:

$$\mathrm{Res}(p(x)-s,q(x)-t)=\det\begin{bmatrix}0 & y & -s\\ qy & -s & 0\\ qy^2 & 0 & -t\end{bmatrix}=(t-s^2)qy^2.$$

EXAMPLE 2.7 (HEISENBERG ALGEBRA). Let $p(x)$ and $q(x)$ be commuting elements in the Heisenberg algebra $\mathbb{C}[y][D]$, of degree m and n, respectively. Consider the expression $p(x)=\sum_j a_jD^j$ and $Q=\sum_i b_iD^i$, where the a_i and b_i are polynomials over $\mathbb{C}$ in one indeterminate, which we denote by y. Let $p=yD$ and $Q=y^2D^2$. Then

$$\mathrm{Res}(p-s,q-t)=\det\begin{bmatrix}0 & y & yD-s\\ y & 1-s & yD^2+(1-s)D\\ y^2 & 0 & y^2D^2-t\end{bmatrix},$$

and expanding this determinant, we get

$$\begin{aligned}\operatorname{Res}(p-s,q-t) &= y^3(yD^2+(1-s)D) - y^2(y^2D^2-t) - y^2(1-s)(yD-s)\\ &= y^4D^2 + y^3D - sy^3D - y^4D^2 + y^2t + sy^2 - y^3D - y^2s^2 + sy^3D\\ &= (t+s-s^2)y^2.\end{aligned}$$

Note that $q+p-p^2=0$.

EXAMPLE 2.8. Consider the Ore extension $\mathbb{C}[y][x;\sigma,\delta]$, where $\sigma(y)=qy$, for some $q\in\mathbb{C}^*$ and $\delta(y)=1$. Set $p(x)=q(x)=(yx)^2=qy^2x^2+yx$. Then

$$\operatorname{Res}(p-s,q-t)=\det\begin{bmatrix}0 & qy^2 & y & -s\\ q^3y^2 & (2q+q^2)y & 1-s & 0\\ 0 & qy^2 & y & -t\\ q^3y^2 & (2q+q^2)y & 1-t & 0\end{bmatrix},$$

and so

$$\operatorname{Res}(p-s,q-t)=q^4y^4t^2-2q^4y^4st+q^4y^4s^2.$$

EXAMPLE 2.9. Once more again, consider the Ore extension $\mathbb{C}[y][x;\sigma,\delta]$, where $\sigma(y)=qy$, for some $q\in\mathbb{C}^*$ and $\delta(y)=1$. Let $p(x)=(y^2x)^2+y^2x=q^2y^4x^2+(y^3+qy^3+y^2)x$ and $q(x)=y^2x$. Then

$$\operatorname{Res}(p-s,q-t)=\det\begin{bmatrix}y^4 & qy^3+y^3+y^2 & -s\\ 0 & y^2 & -t\\ q^2y^2 & qy+y-t & 0\end{bmatrix}=-q^2y^4t^2-q^2y^4t+q^2y^4s.$$

PROPOSITION 2.36 ([RS12, THEOREM 2]). *Let $p(x),q(x)$ be commuting elements in some Ore extension $R[x;\sigma,\delta]$ of degrees m and n. Suppose the highest coefficients a_m and b_n both belong to the kernel of σ. Then* $\operatorname{Res}(p(x)-s,q(x)-t)=0$.

From [RS12, Section 4, Theorem 3] we know the importance of the injectivity of σ. Additionally, if R is a domain, then $\operatorname{Res}(p(x)-s,q(x)-t)$ is non-zero.

Another important step done by Richter in this paper is that he not only considered a commutative ring R as the coefficient ring but he studied the case when $R=S[y]$.

PROPOSITION 2.37 ([RS12, THEOREM 6]). *Let $R=k[y]$ for some field k. Let σ be any k-endomorphism of R and assume that δ is identically zero. If $P=\sum_{i=0}^m a_i(yx)^i$ and $Q=\sum_{j=0}^n b_j(yx)^j$, then P and Q commute and*

$$\operatorname{Res}(P-s,Q-t)=G(s,t)\prod_{j=0}^{n-1}\prod_{i=0}^{j+m-1}\sigma^i(y),$$

where $G(s,t)$ does not contain any non-zero power of y.

This theorem consider elements of the form $P=\sum_{i=0}^m a_i(yx)^i$, those elements are precisely the elements in the subalgebra of homogeneous elements, that we denoted previously as R_0. So, the results exposed in this part generalize the theorems obtained for Heisenberg algebras. In fact, this results give an explicit construction of the annihilating curve, in complement with the results obtained in [Ric14b], which guarantees the existence of the annihilating curve.

Next, we recall the key ingredients of another construction following the procedure described by Larsson and Silvestrov [LS03, p. 100] and [SSdJ80, Section 23.3] in the setting of q-deformed Heisenberg algebras studied in Section 2.1.2.

The row-scheme is a first stepping-stone:

$$D^k(P-sI) = \sum_{i=0}^{n+k} \theta_{i,k} D^i - sD^k, \quad k = 0, 1, \dots, m-1 \tag{2.24}$$

$$D^k(Q-tI) = \sum_{i=0}^{m+k} \omega_{i,k} D^i - tD^k, \quad k = 0, 1, \dots, n-1, \tag{2.25}$$

where $\theta_{i,k}$ and $\omega_{i,k}$ are certain functions built from the coefficients of P and Q, respectively, whose exact form is calculated by moving D^k through to the right of the coefficients, by using Leibniz's rule. In the process, the coefficients of the powers of D on the right hand side in (2.24) and (2.25) build up the rows of a matrix exactly as written. In other words, as the first row we take the coefficients in $\sum\limits_{i=0}^{n} \theta_{i,0} D^i - sD^0$, and as the second row - the coefficients in $\sum_{i=0}^{n+1} \theta_{i,1} D^i - sD$, continuing this until $k = m-1$. Now, as the mth row, we take the coefficients in $\sum\limits_{i=0}^{m} \omega_{i,0} D^i - tD^0$, and as the $(m+1)$th row we take the coefficients in $\sum_{i=0}^{m+1} \omega_{i,1} D^i - tD$ and so on. Therefore, we obtain a $(m+n)\times(m+n)$-matrix using (2.24) and (2.25). The determinant of this matrix yields a bivariate polynomial $f(s,t)$ in the indeterminates s and t over the complex numbers $\mathbb{C}$, which is called *the $\mathscr{BC}$ polynomial corresponding to the curve $\mathscr{BC}$*, defining an algebraic curve $f(s,t) = 0$, and annihilating P and Q when we replace $s = P$ and $t = Q$. This correspondence between commuting differential operators and algebraic curves has also been discretized to classical differential operators (e.g., [Mum77, Kri78a, vMM79]).

Example 2.10 ([LS03, Example 2.1]). Consider the q-deformed Heisenberg algebra $H_q(\mathbb{C})$ in Definition 2.4, and let $P = M_x^3 D_q^3$ and $Q = M_x^2 D_q^2$. The following formulas hold:

$$\begin{aligned}
D_q^0(P-sI) &= -sI + M_x^3 D_q^3,\\
D_q(P-sI) &= -sD_q + \{3\}_q M_x^2 D_q^3 + q^3 M_x^3 D_q^4,\\
D_q^0(Q-tI) &= -tI + M_x^2 D_q^2,\\
D_q(Q-tI) &= -tD_q + \{2\}_q M_x D_q^2 + q^2 M_x^2 D_q^3,\\
D_q^2(Q-tI) &= -tD_q^2 + \{2\}_q D_q^2 + (q\{2\}_q + q^2\{2\}_q) M_x D_q^3 + q^4 M_x^2 D_q^4\\
&= (\{2\}_q - tI) D_q^2 + q\{2\}_q^2 M_x D_q^3 + q^4 M_x^2 D_q^4.
\end{aligned}$$

The coefficients in front of the powers of D_q in these equalities can be placed in an operator matrix with the determinant

$$\begin{vmatrix} -s & 0 & 0 & M_x^3 & 0 \\ 0 & -s & 0 & \{3\}_q M_x^2 & q^3 M_x^3 \\ -t & 0 & M_x^2 & 0 & 0 \\ 0 & -t & \{2\}_q M_x & q^2 M_x^2 & 0 \\ 0 & 0 & \{2\}_q - t & q\{2\}_q^2 M_x & q^4 M_x^2 \end{vmatrix} = q^3(q^3 s^2 + q(2q+1)st + \{2\}_q t^2 - t^3) M_x^6,$$

which generates the curve $F(s,t) = q^3 s^2 + q(2q+1)st + \{2\}_q t^2 - t^3 = 0$. One can check that if $s = P$ and $t = Q$, then $F(P,Q) = 0$.

Example 2.11 ([LS03, Example 2.3]). Consider the polynomials $P = M_x^2 D_q^2 + M_x^3 D_q^3$ and $Q = M_x D_q + M_x^2 D_q^2$ in $H_q(\mathbb{C})$. The determinant method gives

$$\begin{vmatrix} -s & 0 & M_x^2 & M_x^3 & 0 \\ 0 & -s & \{2\}_q M_x & (\{3\}_q + q^2) M_x^2 & q^3 M_x^3 \\ -t & M_x & M_x^2 & 0 & 0 \\ 0 & 1-t & (\{2\}_q + q) M_x & q^2 M_x^2 & 0 \\ 0 & 0 & 2\{2\}_q - t & q(\{2\}_q^2 + q) M_x & q^4 M_x^2 \end{vmatrix},$$

which gives an algebraic curve when we expand the determinant as follows

$$\begin{aligned} F(s,t) = &-t^3 + (q^3 - 3q^2 + q + 3)t^2 + q(5q - 2q^2 + 1)st && (2.26) \\ &-(q-2)(q^2 - q - 1)t + q^3 s^2 - q^2(q-1)(q-2)s = 0. && (2.27) \end{aligned}$$

Again, one can check that if $s = P$ and $t = Q$, then $F(P,Q) = 0$.

2.3 Centralizers in some quantum algebras

As we said in the Introduction, the question about the centralizers in some noncommutative rings contributes to the $\mathscr{BC}$ problem, and it seems that the first work on centralizers was presented by Schur [Sch04], where he considered the $\mathbb{C}$-algebra R consisting of ordinary differential operators over complex-valued functions which are infinitely differentiable, and proved that if p is an element of degree at least one in R, then $C(p;R)$ is a commutative $\mathbb{C}$-algebra. Our purpose in this section is to present some important results (not exhaustive) about this topic.

2.3.1 Associative algebras

In this section, we recall briefly two well-known results on centralizers in free associative algebras: Cohn's centralizer theorem and Bergman's centralizer theorem. We follow the ideas developed by Sharifi in his PhD thesis [Sha13].

Throughout this section, the symbol X denotes a set of noncommutative indeterminates. Let X^* denote the free monoid generated by X. An element of X (resp. X^*) is called a *letter* (resp. *word*) and X is said to be an alphabet. As usual, let $\Bbbk\{\{X\}\}$ and $\Bbbk\{X\}$ denote the $\Bbbk$-algebra of formal series and polynomials in X, respectively. In this way, an element of $\Bbbk\{\{X\}\}$ can be written as $f = \sum_{w\in X^*} a_w w$, where $a_w \in \Bbbk$ is the coefficient of the word w in f. The *length* $|w|$ of $w \in X^*$ is the number of letters appearing in w. We can define the valuation

$$\begin{aligned} \nu : \Bbbk\{\{X\}\} &\to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \\ 0 &\mapsto \infty, \\ f = \sum_{w\in X^*} a_w w \neq 0 &\mapsto \min\{|w| \mid a_w \neq 0\}. \end{aligned}$$

Of course, if w is constant, then $\nu(w) = 0$, and $\nu(ab) = \nu(a) + \nu(b)$, for every $a, b \in \Bbbk\{\{X\}\}$.

The Levi's lemma establishes that for elements w_1, w_2, w_3, and w_4 are elements of X^* be non-zero with $|w_2| \geq |w_4|$, if $w_1 w_2 = w_3 w_4$, then $w_2 = w w_4$, for some $w \in X^*$ ([Sha13, Lemma 2.3.1]). The extension of this lemma to $\Bbbk\{\{X\}\}$ states that for non-zero elements $a, b, c, d \in \Bbbk\{\{X\}\}$, if $\nu(a) \geq \nu(c)$ and $ab = cd$, then $a = cq$, for some $q \in \Bbbk\{\{X\}\}$ [Lot02, Lemma 9.12]. A consequence of this result is formulated in the following proposition. $C\left(f; \Bbbk\{\{X\}\}\right)$ denotes the centralizer of the element f in $\Bbbk\{\{X\}\}$.

Proposition 2.38 ([Sha13, Corollary 2.3.3]). *Let $f \in \Bbbk\langle\langle X\rangle\rangle$. Then $g \in C\left(f; \Bbbk\langle\langle X\rangle\rangle\right)$ if and only if f, g are not free, that is, $h(f,g) = 0$ for some non-zero $h \in \Bbbk\langle\langle x_1, x_2\rangle\rangle$.*

Proof. Note that if $fg = gf$, then $h(f,g) = 0$ for the element $h = x_1x_2 - x_2x_1$.

Conversely, consider $h \in \Bbbk\langle\langle x_1, x_2\rangle\rangle$ such that $h(x_1, x_2) = 0$. Let $n := \nu(fg - gf)$. We proceed by induction on n. Notice that the constant term of $fg - gf$ is zero, whence $n = 0$ if and only if $fg = gf$. Without loss of generality, we assume that the constant terms of f and g are zero since if $f = r + f_1$ and $g = s + g_1$, where r, s are the constant terms of f and g, respectively, then $fg - gf = f_1g_1 - g_1f_1$. This means that we may consider $\nu(f) \geq \nu(g) \geq 1$. Let $t = x_1 m + x_2 l + k$, for some $m, l \in \Bbbk\langle\langle x, y\rangle\rangle$ and $k \in \Bbbk$. By using that $0 = h(f,g) = fm(f,g) + gl(f,g) + k$ and the constant terms of f, g are zero, necessarily $k = 0$. Hence, $fm(f,g) = -gl(f,g) = 0$. By [Lot02, Lemma 9.12], there exists an element $q \in \Bbbk\langle\langle x_1, x_2\rangle\rangle$ with $f = gq$, and so $fg - gf = g(qg - gq)$ and $\nu(qg - gq) < \nu(fg - gf) = n$. Finally, since $f = gq$ and g are not free, then g, q are not free. By induction, $gq = qg$, and hence $fg = gf$. $\square$

Before stating the Cohn's centralizer theorem (Proposition 2.40), we need one more preliminary result.

Proposition 2.39 ([Sha13, Lemma 2.3.4]). *Suppose that the constant term of an element $f \in \Bbbk\{\{X\}\}$ is zero and $g, h \in C\left(f; \Bbbk\{\{X\}\}\right) \setminus \{0\}$. If $\nu(h) \geq \nu(g)$, then $h = gp$, for some $p \in C\left(f; \Bbbk\{\{X\}\}\right)$.*

Proposition 2.40 (Cohn's centralizer theorem, [Lot02, Theorem 9.1.1]). *If $f \in \Bbbk\{\{X\}\}$ is not a constant element, then $C\left(f; \Bbbk\{\{X\}\}\right) \cong \Bbbk[[x]]$, where $\Bbbk[[x]]$ is the algebra of formal power series in the indeterminate x over $\Bbbk$.*

Proof. Consider $C := C\left(f; \Bbbk\{\{X\}\}\right)$. As above, note that if $a_0 \in \Bbbk$ is the constant term of f, then $C = C(f - k; \Bbbk\{\{X\}\})$, so we may suppose that the constant term of f is zero. Let $C' := \{c \in C \mid \nu(c) > 0\}$. Since $\nu(f) > 0$, $C' \neq \emptyset$, and so there exists $g \in C'$ such that $\nu(g)$ is minimal. With the aim of showing that $\Bbbk[[g]] \cong \Bbbk[[f]]$, suppose that $\sum_{i\geq m} k_i g^i = 0$, with $k_i \in \Bbbk$, $k_m \neq 0$. Then $\infty = \nu\left(\sum_{i\geq m} k_i g^i\right) = \nu(g^m) = m\nu(g)$, a contradiction. Now, let us see that $C = \Bbbk[[g]]$. If $c \in C$ is a constant, then it is clear that $c \in \Bbbk[[g]]$, so we suppose that $c \notin \Bbbk$.

The aim is to prove that there exist elements $k_i \in \Bbbk$ such that

$$\nu\left(c - \sum_{i=0}^{n} k_i g^i\right) \geq (n+1)\nu(g), \tag{2.28}$$

since our result in our hands, then we are done due to the fact that $\nu\left(c - \sum_{i\geq 0} k_i g^i\right) = \infty$, whence $c = \sum_{i\geq 0} k_i g^i \in \Bbbk[[g]]$. We proceed by induction on n. Let k_0 be the constant term of c. Since $c - k_0 \in C'$, $\nu(c-k_0) \geq \nu(g)$, due to the minimality of g, which shows that (2.28) holds for $n = 0$. Suppose that there exist elements $k_0, \ldots, k_n \in \Bbbk$ such that $\nu(c - \sum_{i=0}^{n} k_i g^i) \geq (n+1)\nu(g)$. By using that $(n+1)\nu(g) = \nu(g^{n+1})$, Proposition 2.39 guarantees that

$$c - \sum_{i=0}^{n} k_i g^i = g^{n+1} d,$$

for some element $d \in C$. Again, if $d \in \Bbbk$, then $c \in \Bbbk[g] \subset \Bbbk[[g]]$, and we are done. If $d \notin \Bbbk$, then let k_{n+1} be the constant term of d. It follows that $d - k_{n+1} \in C'$, and so $\nu(d - k_{n+1}) \geq \nu(g)$, by the minimality of g. Proposition 2.39 implies that $d - k_{n+1} = gd'$, for some element $d' \in C$. Therefore,

$$c - \sum_{i=0}^{n} k_i g^i = g^{n+1} d = g^{n+1}(gd' + k_{n+1}) = g^{n+2} d' + k_{n+1} g^{n+1},$$

whence $c - \sum_{i=0}^{n+1} k_i g^i = g^{n+2} d'$, and so

$$\nu(c - \sum_{i=0}^{n+1} k_i g^i) = \nu(g^{n+2} d') = \nu(b^{n+2}) + \nu(d') = (n+2)\nu(g) + \nu(d') \geq (n+2)\nu(g),$$

which concludes the proof. $\square$

Having in mind that $\Bbbk\{X\} \subseteq \Bbbk\{\{X\}\}$, from Cohn's centralizer theorem it follows that for a non-constant element $f \in \Bbbk\{X\}$, $C\big(f; \Bbbk\{X\}\big)$ is commutative. Bergman [Ber69] proved that if $f \in \Bbbk\{X\}$ is not constant, then $C\big(f; \Bbbk\{X\}\big)$ is integrally closed, and used this fact to prove his famous theorem which states that a similar result to Proposition 2.40 holds for centralizers in $\Bbbk\{X\}$.

PROPOSITION 2.41 (BERGMAN'S CENTRALIZER THEOREM, [BER69], [LOT02, THEOREM 9.5.1]). *If $f \in \Bbbk\{X\}$ is not constant, then $C\big(f; \Bbbk\{X\}\big) \cong \Bbbk[x]$.*

2.3.2 Quantum plane

Dixmier [Dix68] found that the centralizers of elements of the first Weyl algebra have some unexpected properties, such as sometimes a centralizer is not integrally closed, or that there are cases when the field of fractions of a centralizer is not a purely transcendental field. Motivated by these facts, Makar-Limanov discussed what happens if the Weyl algebra is replaced by the quantum plane algebra or a quantum space algebra of any dimension. He found that though the centralizers (of non-central elements) are not necessarily integrally closed, and the fields of fractions of centralizers of non-constants are always purely transcendental fields of dimension one for a "general position" situation.

Let us recall the key facts in Makar-Limanov's paper.

For the first Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\{p,q\}/\langle pq-qp-1\rangle$ (Section 1.2.1), Dixmier [Dix68, p. 412] presented the following example of centralizers of elements of this algebra. If $u := p^3+q^2-a$ where $a \in \mathbb{C}$, $v := \frac{1}{2}p$, $U := u^2+4v$, $V := u^3+3(uv+vu)$, then $V^2-U^3 = a$. From this, it is straightforward to see that U and V commute, and so when $a \neq 0$, the centralizer $C(U)$ of U is isomorphic to the ring of regular functions on an elliptic curve, and its field of fractions is not isomorphic to the field of rational functions $\mathbb{C}(z)$. Notice that when $a = 0$, the field of fractions of $C(U)$ is isomorphic to $\mathbb{C}(z)$, but $C(U)$ is not integrally closed. As a matter of fact, if $t := pq$ and $h := pt(t-1)^{-1}(t-2)$, then both $h^2, h^3 \in A_1(\mathbb{C})$ but $h \in D_1(\mathbb{C})\setminus A_1(\mathbb{C})$, where $D_1(\mathbb{C})$ is the skew-field of fractions of $A_1(\mathbb{C})$. A similar example was found by Bergman [Ber69]. Burchnall and Chaundy noticed that $P = p^2-2q^{-2}$ and $Q = p^3-3q^{-2}+3q^{-3}$ commute without being polynomials of an element of the form $p+f(q)$ (since $P = q^{-2}t(t-3)$ and $Q = q^{-3}t(t-2)(t-4)$, they are the square and cube of $h = q^{-1}t(t-1)^{-1}(t-2)$ correspondingly), and in fact, also gave an example of a centralizer isomorphic to the ring of regular functions on an elliptic curve but the elements involved are not in D_1. As Makar-Limanov states, "The effect discovered by Dixmier is somewhat somewhat surprising because $A_1(\mathbb{C})$ may be looked at as a deformation of the polynomial algebra $\mathbb{C}[x,y]$, and in $\mathbb{C}[x,y]$ any maximal subalgebra of transcedence degree one is isomorphic to a polynomial ring $\mathbb{C}[z]$ [Zak71]. It is well-known that the centralizer of a non-scalar element of $A_1(\mathbb{C})$ is a maximal subalgebra of A_1 of transcendence degree one (Amitsur [Ami58] attributed this result to Flanders [Fla55] who attributed it to Schur [Sch04]). Goodearl [Goo83] investigated properties of centralizers of $A_1(\mathbb{C})$ and generalizations.

Artamonov and Cohn [AC99] and Bell and Small [BS04] showed that the centralizers of non-scalar elements of the quantum plane $\mathbb{C}_q[x,y]$ when q is not a root of the unity are commutative algebras of transcendence degree one. As a matter of fact, from [AC99] we know that this result holds for the field of fractions $\mathbb{C}_q(x,y)$ of $\mathbb{C}_q[x,y]$. In the case of the q-deformed Heisenberg algebra $H_q[x,y]$, which is a subalgebra of $\mathbb{C}_q(x,y)$, Hellström and Silvestrov [HS00] independently proved that any two commuting elements of this algebra are algebraically dependent (some examples of centralizers of $H_q[x,y]$ are provided by Larsson and Silvestrov [LS03]).

Makar-Limanov [ML06] verified that, unlike the Weyl algebra setting, these centralizers are always subalgebras of a polynomial ring in one indeterminate. He extended this result to quantum planes in n indeterminates, that is, algebras with generators $x_1,\ldots,x_n$ subject to the relations $x_jx_i = q_{ij}x_ix_j$, with $q_{ij} \in \mathbb{C}^*$. This result follows from [BS04, Theorem 1.1] which states that a centralizer of a non-scalar element of $\mathbb{C}_q[x,y]$ has transcendence degree one, and [Ber69, Proposition 6.1] which guarantees that any subalgebra of $\mathbb{C}_q[x,y]$ has a non-trivial mapping into $\mathbb{C}[z]$.

Remark 10 ([ML06, p. 413]). The following example shows that the centralizers of non-central elements of $\mathbb{C}_q[x,y]$, when q is a root of the unity, could not be subalgebras of a polynomial ring in two indeterminates.

Let $q=-1$. The center of $\mathbb{C}_{-1}[x,y]$ is $\mathbb{C}[x^2,y^2]$. If $z := x^3 + xyt^3$ with $t = x^2y^2+1$, then it can be seen that the centralizer of z is given by $C(z) = \mathbb{C}[x^2,y^2,z]$, and that $z^2 = x^6-(t-1)t^6$. Nevertheless, $\mathbb{C}[u,v][\sqrt{u^3-(t-1)t^6}]$ where $u=x^2$, $v=y^2$, and $t=uv+1$, cannot be embedded into a polynomial ring with two indeterminates.

Remark 11 ([ML06, p. 413]). Suppose that q is not a root of the unity. There are examples of centralizers not being integrally closed. Take $h = x(1-y)(1-qy)^{-1}(1-q^2y)$ in the skew-field $\mathbb{C}_q(x,y)$ of fractions of $\mathbb{C}_q[x,y]$. Then $h^2, h^3 \in \mathbb{C}_q[x,y]$ and $C(h^2) = \mathbb{C}[h^2,h^3]$. In a similar way, if $h = x(1-y)^n(1-qy)^{-1}(1-q^2y)^{-1}\cdots(1-q^ny)^{-1}(1-q^{n+1}y)^n$, then $h^i \notin \mathbb{C}_q[x,y]$ for $i<n+1$, $h^{n+1}\in\mathbb{C}_q[x,y]$, and $C(h^{n+1}) = \mathbb{C}[h^{n+1},\dots,h^{2n+1}]$. These facts show that the centralizers of elements of $\mathbb{C}_q[x,y]$ are neither integrally closed nor is there a bound on the size of a set of generators of a centralizer.

2.3.3 Some families of Ore extensions

Richter and Silvestrov [RS14] considered centralizers of single elements in certain Ore extensions with a non-invertible endomorphism of the ring of polynomials in one intederminate over a field. They proved that these centralizers are commutative and finitely generated as algebras, and that for certain classes of elements their centralizers are singly generated as an algebra. Some of the results are similar to those obtained by Bell and Small [BS04] but are logically independent since the algebras considered by Richter and Silvestrov have infinite Gelfand-Kirillov dimension. Let us recall some details presented in [RS14].

Their article is concerned with centralizers of elements in Ore extensions of the form $\Bbbk[y][x;\sigma,\delta]$, where σ is a $\Bbbk$-algebra endomorphism such that $\deg(\sigma(y))>1$ and δ is a $\Bbbk$-linear derivation. The following are some of the most important results of the paper.

Proposition 2.42 ([RS14, Theorem 3.1]). *Let P be any element of $S=\Bbbk[y][x;\sigma,\delta]$ that is not constant. Then the following assertions hold:*

(1) *The centralizer of P in S, denoted by $C_S(P)$, is a free $\Bbbk[P]$-module of rank at most $n=\deg(P)$.*

(2) *$C_S(P)$ is commutative.*

(3) *Let D be any subset of S. Then $C_S(A)$ is equal to either $\Bbbk[y][x;\sigma,\delta]$, $\Bbbk$ or $C_S(P)$, where P is a non-constant element in S*

From [RS14, Theorem 4.1], we know that if P is an element of S that is not a constant, then . As a matter of fact, if $A\subseteq\Bbbk[y][x;\sigma,\delta]$, [RS14, Proposition 4.2].

For instance, if $\sigma(y)=y^s$, $\delta(y)=0$, and $P=y^ix^j$, where $i+j>0$, then $C_S(P)$ is singly generated.

Next, we review the characterizations of centralizers presented by Tumwesigye et al. [TRS20b] for the Ore extension of the algebra of functions with finite support on a countable set.

Let $X := [n] = \{1, 2, 3, \dots, n\}$ be a finite set and let $R := \{f : X \to \mathbb{R}\}$ denote the algebra of real-valued functions on X with respect to the usual pointwise operations, that is, pointwise addition, scalar multiplication and pointwise multiplication. By writing $f_n := f(n)$, we can identify R with $\mathbb{R}^n$. $\mathbb{R}^n$ is equipped with the usual operations of pointwise addition, scalar multiplication and pointwise multiplication. Let $\sigma : X \to X$ be a bijection such that R is invariant under σ, $\tilde{\sigma} : R \to R$ be the automorphism induced by σ, that is, $\tilde{\sigma}(f) = f \circ \sigma^{-1}$ for every $f \in R$, and consider the Ore extension $R[x; \tilde{\sigma}, \delta]$.

Definition 2.12 ([TRS20b, Definition 19.3]). For any non-zero $n \in \mathbb{Z}$, set

$$\begin{aligned} \operatorname{Sep}^n(X) &:= \{x \in X \mid x \neq \sigma^n(x)\}, \quad \text{and} \\ \operatorname{Per}^n(X) &:= \{x \in X \mid x \neq \sigma^n(x)\}. \end{aligned}$$

Notice that $\tilde{\sigma}_n(h)(x) \neq h(x)$ if and only if $\sigma_n(x) \neq x$, for every $x \in X$ and each $h \in R$.

Proposition 2.43 ([TRS20b, Theorem 19.3]). *The centralizer $C(R)$ of R in the Ore extension $R[x; \tilde{\sigma}]$ is given by*

$$C(R) = \left\{ \sum_{n \in \mathbb{Z}} f_n x^n \text{ such that } f_n = 0 \text{ on } \operatorname{Sep}^n(X) \right\},$$

where $\operatorname{Sep}^k(X)$ is as given in Definition 2.12.

Now, suppose $\tilde{\sigma} \neq \mathrm{id}$ is of order $j \in \mathbb{Z}^+$, that is, $\tilde{\sigma}^j = \mathrm{id}$ but $\tilde{\sigma}^k \neq \mathrm{id}$, for all $k < j$. In the following proposition we have a necessary condition for an element to belong to the centralizer of R.

Proposition 2.44 ([TRS20b, Theorem 19.4]). *If an element of degree m, $\sum_{k=0}^{m} f_k x^k \in R[x; \tilde{\sigma}, \delta]$ belongs to the centralizer of R, then $f_m = 0$ on $\operatorname{Sep}^m(X)$.*

Tumwesigye et al. presented some examples that show that conditions satisfied by all elements in the centralizer of R are actually quite complicated for establishing, for example, in the case when $n = 2$ [TRS20b, Example 19.4.1]. Also they gave a description of the center of our Ore extension algebra.

Proposition 2.45 ([TRS20b, Theorem 19.5]). *The center of the Ore extension algebra $R[x, \tilde{\sigma}, 0]$ is given by*

$$Z(R[x, \tilde{\sigma}, 0]) = \left\{ \sum_{k=0}^{m} f_k x^k \mid \text{where } f_k = 0 \text{ on } \operatorname{Sep}_k(X) \text{ and } \tilde{\sigma}(f_k) = f_k \right\}.$$

Proof. Let $f = \sum_{k=0}^{m} f_k x^k$ be an element in $Z(R[x, \tilde{\sigma}, 0])$. Then $f \in C(R)$, that is $f_k(x) = 0$ for every $x \in \operatorname{Sep}_k(x)$. Since the Ore extension $R[x, \tilde{\sigma}, 0]$ is associative, it is enough to derive conditions under which $xf = fx$. We have

$$fx = \left(\sum_{k=0}^{m} f_k x^k\right) x = \sum_{k=0}^{m} f_k x^{k+1},$$

and

$$xf = x\left(\sum_{k=0}^{m} f_k x^k\right) = \sum_{k=0}^{m} x f_k x^k = \sum_{k=0}^{m} \tilde{\sigma}(f_k) x^{k+1},$$

which implies the assertion. □

Now, we mention some results in the infinite dimensional case. Let J be a countable subset of $\mathbb{R}$ and let R be the set of functions $f : J \to J$ such that $f(i) = 0$, for all except finitely many $i \in J$. In this case, R is a commutative algebra non-unital with respect to the usual pointwise operations of addition, scalar multiplication and multiplication. For $i \in J$, let $e_i \in R$ denote the characteristic function of i, that is,

$$e_i(j) = \chi_{\{i\}}(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, every $f \in R$ can be written in the form $f = \sum_{i \in J} f_i e_i$, where $f_i = 0$ for all except finitely many $i \in J$. Let $\sigma : J \to J$ be a bijection and let $\tilde{\sigma} : R \to R$ be the automorphism of R induced by σ, that is, $\tilde{\sigma} = f \circ \sigma^{-1}$ for every $f \in R$. It is possible to consider the non-unital Ore extension $R[x;\tilde{\sigma},\delta]$ as

$$R[x,\tilde{\sigma},\delta] := \left\{ \sum_{k=0}^{m} f_k x^k \text{ where } f_k \in R \right\},$$

with addition and scalar multiplication given by the usual pointwise operations and multiplication determined by the relation $(fx)g = \tilde{\sigma}(g) fx + \delta(g)$, where δ is a $\tilde{\sigma}$-derivation on R. Consider the skew ring of formal power series over R, $R[x;\tilde{\sigma}]$, that is, the set

$$R[x;\tilde{\sigma}] = \left\{ \sum_{n=0}^{\infty} \text{ such that } f_n \in R \right\},$$

with pointwise addition and multiplication determined by the relations $xf = \tilde{\sigma}(f)x$. Equivalently, if $f = \sum_{n=0}^{\infty} f_n x^n$ and $g = \sum_{n=0}^{\infty} g_n x^n$ are elements of $R[x;\tilde{\sigma}]$, then

$$f + g = \sum_{n=0}^{\infty} (f_n + g_n) x^n$$

and

$$fg = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k \tilde{\sigma}^k (g_{n-k}) \right) x^n.$$

Once more again, we recall the theorems that characterize the centralizer of R in this skew polynomial ring $R[x, \tilde{\sigma}, \delta]$ for both cases $\delta = 0$ and $\delta \neq 0$.

Proposition 2.46 ([TRS20b, Theorem 19.7]). *The centralizer $C(R)$, of R in the skew power series ring $R[x; \tilde{\sigma}]$ is given by*

$$C(R) = \left\{ \sum_{n \in \mathbb{Z}} f_n x^n \text{such that } f_n = 0 \text{ on } \operatorname{Sep}^n(X) \right\},$$

where $\operatorname{Sep}^k(X)$ *is as given in Definition 2.12.*

Now, suppose $\tilde{\sigma} \neq \text{id}$ is of order $j \in \mathbb{Z}_{>0}$, that is, $\tilde{\sigma}^j = \text{id}$ but $\tilde{\sigma}^k \neq \text{id}$ for all $k < j$

Proposition 2.47 ([TRS20b, Theorem 19.8]). *Let $\tilde{\sigma} : R \to R$ be an automorphism on R. If an element of order m, $\sum_{k=0}^{m} f_k x^k \in R[x, \tilde{\sigma}, \delta]$ belongs to the centralizer of R, then $f_m = 0$ on* $\operatorname{Sep}^m(X)$.

These results can be generalized to the context of skew power series rings. As before, we let $X = [n] = \{1, 2, \ldots, n\}$ be a finite set, and let $R = \{f : X \to \mathbb{R}\}$ denote the unital algebra of real-valued functions on X with respect to the usual pointwise operations. Let $\sigma : X \to X$ be a bijection such that R is invariant under σ (that is, σ is a permutation on X), and let $\tilde{\sigma} : A \to A$ be the automorphism induced by σ, that is, $\sigma\left(\widetilde{f}\right) = f \circ \sigma^{-1}$, for every $f \in A$.

Consider the skew ring of formal power series over A, $A[x; \tilde{\sigma}]$; that is the set

$$\left\{ \sum_{n=0}^{\infty} f_n x^n \mid f_n \in A \right\}$$

with pointwise addition and multiplication determined by the relations $x \cdot f = \tilde{\sigma}(f) \cdot x$, that is, if $f = \sum_{n=0}^{\infty} f_n x^n$ and $g = \sum_{n=0}^{\infty} g_n x^n$ are elements of $A[x; \tilde{\sigma}]$, then

$$f + g = \sum_{n=0}^{\infty} (f_n + g_n) x^n,$$

and

$$fg = \left(\sum_{n=0}^{\infty} f_n x^n \right) \left(\sum_{n=0}^{\infty} g_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k \tilde{\sigma}^k g_{n-k} \right) x^n.$$

Proposition 2.48 ([TRS20b, Theorem 19.10]). *The centralizer $C(A)$ of A in the skew power series ring $A[x; \tilde{\sigma}]$ is given by*

$$C(A) = \left\{ \sum_{n \in \mathbb{Z}} f_n x^n \;\middle|\; f_n = 0 \textit{ on } Sep_n(X) \right\},$$

where $Sep_k(X)$ is as given in Definition 2.12.

Now, the fact that $\tilde{\sigma}$ is an automorphism of A motivates to the consideration of the skew-Laurent ring $A[x, x^{-1}; \tilde{\sigma}]$.

Definition 2.13 ([TRS20b, Definition 19.4]). Let R be a ring and σ an automorphism of R. By a skew-Laurent ring $R[x, x^{-1}; \sigma]$ we mean that

(i) $R[x, x^{-1}; \sigma]$ is a ring, containing R as a subring,

(ii) x is an invertible element of $R[x,x^{-1};\sigma]$,

(iii) $R[x,x^{-1};\sigma]$ is a free left R-module with basis $\{1,x,x^{-1},x^2,x^{-2},\ldots\}$,

(iv) $xr=\sigma(r)x$, (and $x^{-1}r=\sigma^{-1}(r)x^{-1}$) for all $r\in R$.

Consider the skew-Laurent ring $A[x,x^{-1};\tilde{\sigma}]$, that is the set

$$\left\{\sum_{n\in\mathbb{Z}} f_n x^n \mid f_n\in A \ \text{ and } \ f_n=0 \ \text{ for all except finitely many } \ n\right\},$$

with pointwise addition and multiplication determined by the relations

$$xf=\tilde{\sigma}(f)x \ \text{ and } \ x^{-1}f=\tilde{\sigma}^{-1}(f)\,x^{-1}.$$

Finally, we mention the following theorem about the centralizer of A in this algebra.

Proposition 2.49 ([TRS20b, Theorem 19.12]). *The centralizer of A in the skew-Laurent extension $A[x,x^{-1};\tilde{\sigma}]$ is given by*

$$C(A)=\left\{\sum_{n\in\mathbb{Z}} f_n x^n \mid f_n=0 \ \textit{ on } \ \mathrm{Sep}^n(X)\right\}.$$

Proof. Let $f=\sum_{n=0}^{\infty} f_n x^n\in A[x;\tilde{\sigma}]$ be an element that belongs to $C(A)$. Then $fg=gf$ should hold for every $g\in A$. Now,

$$gf=g\left(\sum_{n=0}^{\infty} f_n x^n\right)=\sum_{n=0}^{\infty} g f_n x^n.$$

On the other hand,

$$fg=\left(\sum_{n=0}^{\infty} f_n x^n\right)g=\sum_{n=0}^{\infty} f_n x^n g=\sum_{n=0}^{\infty} f_n\left(x^n g\right)=\sum_{n=0}^{\infty} f_n\tilde{\sigma}^n(g)x^n.$$

Therefore, $gf=fg$ if and only if

$$gf_n=f_n\tilde{\sigma}^n(g), \quad \text{for all } n\in\mathbb{N}.$$

Since A is commutative, then the above equation holds on $Per_n(X)$. Therefore,

$$C(A)=\left\{\sum_{n\in\mathbb{Z}} f_n x^n \;\middle|\; f_n=0 \text{ on } \mathrm{Sep}_n(X)\right\}.$$

This fact concludes the proof. □

2.3.4 Pseudo-degree functions

Motivated by Hellström and Silvestrov's paper [HS07], Richter [Ric16] studied centralizers in certain algebras with valuations, and showed that the centralizer of an element in these algebras is a free module over a certain ring (in some cases, the centralizer is commutative).

Let us see some key facts of Richter's paper.

Definition 2.14 ([Ric16, Definition 3]). Let S be a $\Bbbk$-algebra. A function $\chi : S \to \mathbb{Z} \cup \{-\infty\}$ is called a *pseudo-degree function* if it satisfies the following conditions:

(i) $\chi(a) = -\infty$ if and only if $a = 0$.

(ii) $\chi(ab) = \chi(a) + \chi(b)$, for all $a, b \in S$.

(iii) $\chi(a+b) \leq \max\big(\chi(a), \chi(b)\big)$.

(iv) $\chi(a+b) = \chi(a)$ if $\chi(b) < \chi(a)$.

Remark 12. Richter said that his definition of pseudo-degree function is essentially a special case of the concept of a *valuation* [Ric16, p. 68]. Definition 2.14 presents some differences with respect to another definitions of valuation (cf. [Coh95, Tig15]).

Following [Tig15, p. 2], a *valuation* v on a division ring D is a function $v : D \longrightarrow \Gamma \cup \{\infty\}$, where Γ is a totally ordered additive Abelian group, and ∞ is a symbol such that $\gamma < \infty$ and $\gamma + \infty = \infty + \infty = \infty$, for all $\gamma \in \Gamma$, subject to the following conditions:

(i) $v(a) = \infty$ if and only if $a = 0$;

(ii) $v(a+b) \geq \min(v(a), v(b))$, for all $a, b \in D$;

(iii) $v(ab) = v(a) + v(b)$, for all $a, b \in D$.

Sometimes, the second condition is replaced by $v(a-b) = \min(v(a), v(b))$. On the other hand, Richter defined a pseudo-degree function by replacing again this second condition by asking that $v(a+b) \leq v(a) + v(b)$, and $v(a+b) = v(a)$ if $v(b) \leq v(a)$.

The next definition presents a condition that replaces the notion l-BDHC introduced by Hellstrom and Silvestrov [HS07].

Definition 2.15 ([Ric16, Definition 4]). Let S be a $\Bbbk$-algebra with a pseudo-degree function χ, and let l be a positive integer. A subalgebra $B \subset S$ is said to satisfy *condition* $D(l)$ if $\chi(b) \geq 0$ for all non-zero $b \in B$ and if, whenever we have $l+1$ elements $b_1, \ldots, b_{l+1} \in B$ all mapped to the same integer by χ, there exist elements $\alpha_1, \ldots, \alpha_{l+1} \in \Bbbk$, not all zero, such that $\chi\left(\sum_{i=1}^{l+1} \alpha_i b_i\right) < \chi(b_1)$.

The requirement that $\alpha_1, \ldots, \alpha_{l+1}$ are mapped to the same integer by χ excludes the possibility that they are equal to 0. If S is a $\Bbbk$-algebra, $a \in S$ is such that $C_S(a)$ satisfies condition $D(l)$ for some l, and b is an invertible element, then $\chi(b^{-1}) = -\chi(b)$. This means that all invertible elements of $C_S(a)$ must be mapped to zero by χ (in particular, the non-zero scalars are all mapped to zero by χ).

The following result is a version of $\mathscr{BC}$ theory in terms of centralizers following Amitsur's ideas [Ami58].

PROPOSITION 2.50 ([RIC16, COROLLARY 1]). *Let S be a $\Bbbk$-algebra with a pseudo-degree function χ. Let $a \in S$ be such that the centralizer of a in S, denoted by $C_S(a)$, satisfies condition $D(l)$ for some $l > 0$. If b is any element in $C_S(a)$, then there exists a non-zero polynomial $P(s,t) \in \Bbbk[s,t]$ such that $P(a,b) = 0$.*

Richter proved Proposition 2.51 which ensures that some centralizers are commutative when the condition $D(1)$ is satisfied.

PROPOSITION 2.51 ([RIC16, THEOREM 7]). *Let S be a $\Bbbk$-algebra with a pseudo-degree function χ. If $a \in S$ satisfies $\chi(a) = m > 0$ and $C_S(a)$ satisfies condition $D(1)$, then:*

(1) *$C_S(a)$ has a finite basis as $\Bbbk[a]$-module, the cardinality of which divides m.*

(2) *$C_S(a)$ is a commutative algebra.*

These results can be applied in certain situations that are not covered by the theory developed by Hellstrom and Silvestrov [HS07] as the following result shows.

PROPOSITION 2.52 ([RIC16, PROPOSITION 1]). *Set $R = \Bbbk[y]$, let σ be an endomorphism of R such that $s = \deg_y(\sigma(y)) > 1$ and let δ be a σ-derivation. Consider the Ore extension $S = R[x;\sigma,\delta]$. If $a \in S \setminus \Bbbk$, then $C_S(a)$ is a free $\Bbbk[a]$-module of finite rank and a commutative subalgebra of S.*

These results can be applied in certain situations that are not covered by the theory developed by Hellstrom and Silvestrov [HS07] as the following example shows.

PROPOSITION 2.53 ([RIC16, PROPOSITION 1]). *Set $R = \Bbbk[y]$, let σ be an endomorphism of R such that $s = \deg_y(\sigma(y)) > 1$ and let δ be a σ-derivation. Consider the Ore extension $S = R[x;\sigma,\delta]$. If $a \in S \setminus \Bbbk$, then $C_S(a)$ is a free $\Bbbk[a]$-module of finite rank and commutative subalgebra of S.*

2.3.5 DOMAINS OF FINITE GK DIMENSION

A different approach to study centralizers consists of the Gelfand-Kirillov dimension of a $\Bbbk$-algebra which was introduced by Gelfand and Kirillov [IMG66]. Briefly, given a field $\Bbbk$ and an affine $\Bbbk$-algebra A, the *Gelfand-Kirillov dimension of A*, denoted by GKdim(A) is given by

$$\text{GKdim}(A) := \limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log n},$$

where V is a finite-dimensional subspace of A that generates A as an $\Bbbk$-algebra. As one can check, this definition is independent of choice of V. If A is not affine, then its Gelfand-Kirillov dimension is defined to be the supremum of the Gelfand-Kirillov dimensions of all affine subalgebras of A. For more details about this dimension, see the excellent treatment developed by Krause and Lenagan [KL00].

An affine domain of Gelfand-Kirillov dimension zero is precisely a division ring that is finite-dimensional over its center. In the case of an affine domain of Gelfand-Kirillov dimension one over a field $\Bbbk$, this is precisely a finite module over its center, and thus polynomial identity. Besides, if $\Bbbk$ is algebraically closed, then Tsen's theorem guarantees that this domain is commutative. However, domains of Gelfand-Kirillov dimension two are not well understood. For example, Small conjectured that such domains must either be primitive or polynomial identity. In fact, in the case that such a domain is a finitely generated $\mathbb{N}$-graded algebra with the property that the homogeneous elements of any given degree form a finite-dimensional vector space, Artin and Stafford [AS95] proved that this conjecture holds.

For an affine domain of Gelfand-Kirillov dimension two over an algebraically closed field, Bell and Small [BS04] showed that the centralizer of any non-scalar element of this domain is a commutative domain of Gelfand-Kirillov dimension one whenever the domain is not polynomial identity. In fact, they also characterized centralizers of elements in a finitely graded Goldie domain of Gelfand-Kirillov dimension two over an algebraically closed field, and proved that the centralizer of a non-scalar element is an affine commutative domain of Gelfand-Kirillov dimension one.

Proposition 2.54 ([BS04, Theorem 2.2]). *Let A be a non-polynomial identity affine domain of Gelfand-Kirillov dimension two over an algebraically closed field $\Bbbk$. Then the centralizer of a non-scalar element of A is a commutative domain of Gelfand-Kirillov dimension one.*

Recall that if $A = \bigoplus_{n=0}^{\infty} A_n$ is a finitely generated $\mathbb{N}$-graded $\Bbbk$-algebra, then A is said to be *finitely graded* if $\dim_{\Bbbk}(A_n) < \infty$ for all $n \geq 0$. Given an algebra A and a right A-module M, the Krull dimension of M is denoted by $\mathcal{K}(M)$.

Proposition 2.55 ([BS04, Theorem 3.5]). *Let A be a finitely graded non-polynomial identity domain of Gelfand-Kirillov dimension two over a field $\Bbbk$. Then any subfield of $Q(A)$ has transcendence degree at most one over $\Bbbk$.*

Proof. We follow the arguments presented by Bell and Small. By [AS95, Theorem 0.1], A has a graded quotient ring $Q_{\rm gr}(A) \cong D[x, x^{-1}; \sigma]$ for some division ring D which is a finite module over its center Z, some automorphism σ of D, and with Z a finitely generated extension of $\Bbbk$ of transcendence degree one. Let $\mathbb{K}$ be a subfield of $Q_{\rm gr}(A)$ that is a purely transcendental extension of $\Bbbk' := Z(Q_{\rm gr}(A)) \subseteq D$. Note that $\mathbb{K} \otimes_{\Bbbk'} Q_{\rm gr}(A) \cong (D \otimes_{\Bbbk'} \mathbb{K})[x, x^{-1}; \sigma]$, and that $D \otimes_{\Bbbk'} \mathbb{K}$ is a prime Noetherian algebra. Since it is polynomial identity and not simple, then it is not primitive. Now, by using that $Q_{\rm gr}(A) \otimes_{\Bbbk'} \mathbb{K} \cong (D \otimes_{\Bbbk'} \mathbb{K})[x, x^{-1}; \sigma]$ is simple [BS04, Proposition 3.4], it follows that $Q_{\rm gr}(A)$ is a central simple $\Bbbk'$-algebra and $\mathbb{K}$ is a simple $\Bbbk'$-algebra. From [BS04, Lemma 3.2], $\mathcal{K}(Q_{\rm gr}(A) \otimes_{\Bbbk'} \mathbb{K}) = \mathcal{K}(D \otimes_{\Bbbk'} \mathbb{K})$. Now, having in mind that D is a finite Z-module,

$$\mathcal{K}(D \otimes_{\Bbbk'} \mathbb{K}) = \mathcal{K}(Z \otimes_{\Bbbk'} \mathbb{K}) = \min\{\text{trdeg}_{\Bbbk'}(Z), \text{trdeg}_{\Bbbk'}(\mathbb{K})\},$$

whence

$$\mathcal{K}(Q_{\rm gr}(A) \otimes_{\Bbbk'} \mathbb{K}) = \min\{\text{trdeg}_{\Bbbk'}(Z), \text{trdeg}_{\Bbbk'}(\mathbb{K})\}. \tag{2.29}$$

Notice that we also have

$$\text{trdeg}_{\Bbbk'}(\mathbb{K}) = \mathcal{K}(\mathbb{K} \otimes_{\Bbbk'} \mathbb{K}).$$

From [MR01, Corollary 6.5.3], $\mathcal{K}(\mathbb{K} \otimes_{\Bbbk'} \mathbb{K}) \leq \mathcal{K}(Q(A) \otimes_{\Bbbk'} \mathbb{K})$, and by [MR01, Lemma 6.5.3.ii],

$$\mathcal{K}(Q(A) \otimes_{\Bbbk'} \mathbb{K}) \leq \mathcal{K}(Q_{gr}(A) \otimes_{\Bbbk'} \mathbb{K}) = \min\left(\text{trdeg}_{\Bbbk'}(\mathbb{K}), \text{trdeg}_{\Bbbk'}(Z)\right).$$

Since Z has transcendence degree at most 1 over $\Bbbk'$, then $\operatorname{trdeg}_{\Bbbk'}(K) \leq 1$. Finally, by using that $\Bbbk'$ is a finite extension of $\Bbbk$, the desired result follows. □

PROPOSITION 2.56 ([BS04, THEOREM 3.6]). *Let A be a finitely graded non-polynomial identity domain of Gelfand-Kirillov dimension two over an algebraically closed field $\Bbbk$. Then the centralizer of a non-scalar element $a \in Q(A)$ is a finitely generated field extension of $\Bbbk$ of transcendence degree one.*

Proof. Again, we follow the arguments presented by Bell and Small. Notice that $C(a)$ is a division algebra over $\Bbbk$, and that for any $b \in C(a)$, a and b commute and hence $\Bbbk(a,b)$ is a field. By Proposition 2.55, $\Bbbk(a,b)$ has transcendence degree at most 1 over $\Bbbk$, so b is algebraic over $\Bbbk(a)$, as $\Bbbk(a)$ has transcendence degree 1 over $\Bbbk$. Thus, $C(a)$ is a division algebra that is algebraic over $\Bbbk(a)$. If Z is the center of $C(a)$, then $Z \supseteq \Bbbk(a)$, and so Z has transcendence degree at least 1 over $\Bbbk$. On the other hand, Z is a subfield of $Q(A)$, and thus has transcendence degree at most 1 over $\Bbbk$. By [BS04, Lemma 3.3], Z is a finitely generated field extension of $\Bbbk$. Let $\mathbb{K}$ denote a maximal subfield of $C(a)$. Then $\mathbb{K}$ has transcendence degree 1 over $\Bbbk$ and is a finitely generated extension of $\Bbbk$ by the same lemma. In this way, $\mathbb{K}$ is a finitely generated algebraic extension of Z, and so $[\mathbb{K} : Z] < \infty$. Therefore, $C(a)$ is finite-dimensional over Z [Jac64, p. 165], and in fact, $[C(a) : Z] = [K : Z]^2$. Since Z is a finitely generated extension of transcendence degree 1 of an algebraically closed field, Z is necessarily a C_1 field. By Tsen's theorem, $C(a)$ is commutative, and so is a field of transcendence degree 1 over $\Bbbk$. □

PROPOSITION 2.57 ([BS04, PROPOSITION 3.7]). *Let B be a commutative graded domain of Gelfand-Kirillov dimension 1 over an algebraically closed field $\Bbbk$. Then there exist positive integers $m_1, \ldots, m_l$ and there exists a homogeneous element $t \in Q_{\rm gr}(B)$ such that $B = \Bbbk[t^{m_1}, \ldots, t^{m_l}]$.*

PROPOSITION 2.58 ([BS04, THEOREM 3.8]). *Let A be a finitely graded Goldie $\Bbbk$-algebra with the property that for any homogeneous element of positive degree, $a \in A$, there exist positive integers $m_1, \ldots, m_l$ and a homogeneous element $t \in Q_{\rm gr}(A)$ of positive degree d such that*

$$C(a) = \Bbbk[t^{m_1}, \ldots, t^{m_l}].$$

Then the centralizer of any element of A is an affine $\Bbbk$-algebra.

Proof. Consider $a \in A$. Write $a = a_0 + a_1 + \cdots + a_n$, where a_i is homogeneous of degree i and $a_n \neq 0$. Without loss of generality, $n > 0$. Notice that if $b = b_0 + \cdots + b_p \in C(a)$ with b_i homogeneous of degree i, then $b_p \in C(a_n)$. There exist a homogeneous element $t \in Q_{\rm gr}(A)$ of degree d and positive integers $m_1, \ldots, m_l$ such that $C(a_n) = \Bbbk[t^{m_1}, \ldots, t^{m_l}]$. Hence, if $b = b_0 + \cdots + b_p \in C(a)$, then $b_p = kt^m$, with $k \in \Bbbk$ and $m \in \mathbb{N}m_1 + \cdots + \mathbb{N}m_l$. Let

$$\mathscr{S} := \left\{\deg(b) \mid b \in C(a),\ b \neq 0\right\}.$$

Let d' be the greatest common divisor of $\mathscr{S}$. Then there exists an integer N such that all integers larger than N can be expressed as a *nonnegative* integer linear combination of elements of $\mathscr{S}$. Choose elements $r_1, \ldots, r_l \in C(a)$ such that

$$\left\{\deg(r_1), \ldots, \deg(r_l)\right\} = \mathscr{S} \cap \{0, 1, \ldots, N\}.$$

By using that the leading homogeneous part of r_i is in $C(a_n)$, we can multiply by appropriate scalars so that the leading homogeneous part of r_i is t^{m_i} for some positive integer m_i. The idea is to show that $C(a) = \Bbbk[r_1,\dots,r_l]$. Suppose this is not the case. Choose $b = b_0 + \cdots + b_p \in C(a) \setminus \Bbbk[r_1,\dots,r_l]$ with p_i minimal. Again, we may assume that $b_p = t^{\frac{p}{d}}$. By assumption, $\frac{p}{d} = i_1 m_1 + \cdots i_l m_l$ for some nonnegative integers $i_1,\dots,i_l$. Observe that both b and $b' := r_1^{i_1} \cdots r_l i^l$ have degree p and both have the same homogeneous part of degree p, namely $t^{\frac{p}{d}}$. Hence, $b - b' \in C(a)$ has degree at most $p-1$. By using the minimality of $\deg(b)$, we can see that $b - b' \in \Bbbk[r_1,\dots,r_l]$, which contradicts the fact that $b \notin \Bbbk[r_1,\dots,r_l]$. □

Corollary 2.59 ([BS04, Corollary 3.9]). *Let A be a finitely graded non-polynomial identity Goldie domain of Gelfand-Kirillov dimension two over an algebraically closed field $\Bbbk$. Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand-Kirillov dimension one.*

Bell and Small [BS04, Conjecture 3.10] conjectured that for A an affine Noetherian non-polynomial identity domain of Gelfand-Kirillov dimension two over an algebraically closed field, the centralizer of a non-scalar element is an affine domain.

Corollary 2.60 ([BS04, Corollary 3.11]). *Let A be an affine domain of Gelfand-Kirillov dimension two with a non-polynomial identity domain for an associated graded ring. Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand-Kirillov dimension one.*

Remark 13. (i) Five years later, Bell [Bel09] studied centralizers in domains. He extended a result established in [BS04] by showing that if A is a finitely generated domain of finite Gelfand-Kirillov dimension and $a \in A$ is not algebraic over the extended center of A, then the centralizer of a has Gelfand-Kirillov dimension at most one less than $\mathrm{GK}(A)$. If A is a finitely generated Noetherian domain of GK dimension 3 over the complex numbers, Bell proved that the centralizer of an element $a \in A$ that is not algebraic over the extended center of A satisfies a polynomial identity.

(ii) In his PhD thesis [Sha13], Sharifi proved that over an algebraically closed field of characteristic zero, the centralizer of a non-constant element in the second Weyl algebra has Gelfand-Kirillov dimension one, two or three. The centralizers of GK dimension one or two are commutative and those of GK dimension three contain a finitely generated subalgebra which does not satisfy a polynomial identity. He also showed that for every $n = 1, 2, 3$, there exists a centralizer of GK dimension n, and gave explicit forms of centralizers for some elements of the second Weyl algebra.

Since algebras such as the first Weyl algebra, quantum planes and finitely generated graded algebras of GK dimension two can be viewed as subalgebras of some skew Laurent polynomial algebra over a field, Sharifi proved that for σ an automorphism of $\Bbbk$ and the fixed field of σ is algebraically closed, then the centralizer of a non-constant element of a subalgebra of $\Bbbk[x^{\pm 1}; \sigma]$ is commutative and a free module of finite rank over some polynomial algebra in one indeterminate.

Proposition 2.61 ([Sha13, Proposition 3.2.4]). *Let $A = A_2(k)$, $a \in A \setminus k$ and $C := C(a; A)$. Then $\mathrm{GKdim}(C) \in \{1, 2, 3\}$. If $\mathrm{GKdim}(C) \in \{1, 2\}$, then C is commutative and if $\mathrm{GKdim}(C) = 3$, then*

C is not locally PI.

2.3.6 Skew PBW extensions

In this part, we recall the description of centralizers of elements of skew PBW extensions following the treatment presented by Tumwesigye et al. [TRS20a]. They provided an explicit description in the quasi-commutative case and state a necessary condition in the general case. Additionally, they considered the skew PBW extension over the algebra of functions with finite support on a countable set, describing the centralizer of the extension and the center of the skew PBW extension. Note that Venegas in his PhD Thesis [Ven20] (see also [LV20a, LV20b]) computed the center of different families of skew PBW extensions.

Proposition 2.62 ([TRS20a, Theorem 20.1]). *Let R be a commutative ring and suppose that for all $1 \leq i \leq n$, $\delta_i = 0$. Then the centralizer $C(R)$ of R in the skew PBW extension $\sigma(R)\langle x_1, \dots, x_n\rangle$ is given by*

$$C(R) = \left\{\sum_{\alpha} f_\alpha x^\alpha \mid (\forall r \in R),\ \ (\sigma^\alpha(r) - r) f_\alpha = 0\right\}. \tag{2.30}$$

Proof. An element $f = \sum_\alpha f_\alpha x^\alpha \in \sigma(R)\langle x_1, \dots, x_n\rangle$ belongs to $C(R)$ if and only if for every $r \in R$, $rf = fr$. We see that:

$$\begin{aligned} rf &= r\sum_\alpha f_\alpha x^\alpha \\ &= \sum_\alpha r f_\alpha x^\alpha. \end{aligned}$$

On the other hand, if $\delta_i = 0$ for $1 \leq i \leq n$, then for every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and every $r \in R$, we have

$$x^\alpha r = \sigma^\alpha(r) x^\alpha.$$

Therefore,

$$\begin{aligned} fr &= \sum_\alpha f_\alpha x^\alpha r \\ &= \sum_\alpha f_\alpha x^\alpha \sigma^\alpha(r) \\ &= \sum_\alpha f_\alpha \sigma^\alpha(r) x^\alpha. \end{aligned}$$

Since R is commutative, it follows that $rf = fr$ if and only if $(\sigma^\alpha(r) - r)f_\alpha = 0$. In this way, we can conclude that

$$C(R) = \left\{\sum_\alpha f_\alpha x^\alpha \,\middle|\, \forall r \in R, (\sigma^\alpha(r) - r) f_\alpha = 0\right\}.$$

□

Using similar arguments, it can be seen the following results:

Proposition 2.63 ([TRS20a, Theorem 20.2]). *Let R be a commutative ring. If an element $\sum_\alpha f_\alpha x^\alpha \in \sigma(R)\langle x_1,\dots,x_n\rangle$ belongs to the centralizer $C(R)$, then $(\sigma^\alpha(r)-r)f_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$.*

Corollary 2.64 ([TRS20a, Corollary 20.1]). *Let R be a commutative ring. If for every $\alpha \in \mathbb{N}^n$ there exists $r \in R$ such that $(\sigma^\alpha(r)-r)$ is a regular element, then $C(R) = R$.*

Proof. Suppose $f = \sum_\alpha f_\alpha x^\alpha \in \sigma(A)\langle x_1,\dots,x_n\rangle$ is a non-constant element of degree α which belongs to the centralizer of R. Then $fr = rf$ for every $r \in R$.

$$\begin{aligned} rf &= r\sum_\alpha f_\alpha x^\alpha \\ &= \sum_\alpha r f_\alpha x^\alpha. \end{aligned}$$

On the other hand, by (1.8), for every $x^\alpha \in \mathrm{Mon}(A)$ and every $r \in A$, we have

$$x^\alpha r = \sigma^\alpha(r) + p_{\alpha,r},$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Therefore,

$$\begin{aligned} fr &= \sum_\alpha f_\alpha x^\alpha r \\ &= \sum_\alpha f_\alpha \left(x^\alpha \sigma^\alpha(r) + p_{\alpha,r}\right). \end{aligned}$$

By using the commutativity of R, we get

$$r f_\alpha = \sigma^\alpha(r) f_\alpha,$$

or equivalently, $(\sigma^\alpha(r)-r)f_\alpha = 0$. Since $\sigma^\alpha(r)-r$ is a regular element, then we have $f_\alpha = 0$ for all α, which is a contradiction. □

Let $\mathbb{P} = \bigcup_{k=0}^{2N} I_k$ be a partition of $\mathbb{R}$, where $I_k = (t_k, t_{k+1})$, for $k = 0,1,\dots,N$ with $t_0 = -\infty$ and $t_{N+1} = \infty$ and $I_{N+k} = \{t_k\}$, $k, 1, \dots, N$ and let A be the algebra of functions which are constant on the intervals I_k, $k = 0,1\dots,2N$. Then A is the algebra of piecewise constant functions $h : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1,\dots,t_N$. Let $\Omega = \{0,1,\dots,2N\}$ be a finite set and let $\mathbb{R}^\Omega$ denote the algebra of all functions $f : \Omega \to \mathbb{R}$.

Proposition 2.65 ([TRS20a, Proposition 20.2]). *The algebra A is isomorphic to the algebra $\mathbb{R}^\Omega$.*

Now, let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that A is invariant under σ (and σ^{-1}). Let $\tau : \Omega \to \Omega$ be a bijection such that $\tau(\omega) = \theta$ if and only if $\sigma(I_\omega) = I_\theta$. Suppose $\tilde{\sigma} : A \to A$ is the automorphism induced by σ and $\tilde{\tau} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is the automorphism induced by τ, that is, for every $h \in A$ and every $f \in \mathbb{R}^\Omega$,

$$\tilde{\sigma}(h) = h \circ \sigma^{-1} \;\text{ and }\; \tilde{\tau}(f) = f \circ \tau^{-1}$$

Definition 2.16 ([TRS20a, Definition 20.4]). For $n \in \mathbb{Z}$ set,

$$\begin{aligned}
\mathrm{Sep}_A^n(\mathbb{R}) &:= \left\{x \in \mathbb{R} \mid (\exists h \in A),\ \ h(x) \neq \tilde{\sigma}^n(h)(x)\right\},\\
\mathrm{Sep}_A^n(\Omega) &:= \left\{x \in \Omega \mid (\exists f \in \mathbb{R}^{\Omega}),\ \ f(w) \neq \tilde{\tau}^n(f)(w)\right\},\\
\mathrm{Sep}_A^n(\Omega) &:= \left\{w \in \Omega \mid (\exists f \in \mathbb{R}^{\Omega}), \tau^n(w) \neq w\right\}.
\end{aligned}$$

2.3.6.1 Algebra of Functions on a Finite Set

Let $\Omega = \{0, 1, 2, \dots, 2N\}$ be a finite set and let $\mathbb{R}^{\Omega} = \{f : \Omega \to \mathbb{R}\}$ denote the algebra of real-valued functions on Ω with respect to the usual pointwise operations. By writing $f_k := f(k)$, $\mathbb{R}^{\Omega}$ can be identified with $\mathbb{R}^{2N+1}$ where $\mathbb{R}^{2N+1}$ is equipped with the usual operations of pointwise addition, scalar multiplication and multiplication defined by

$$xy = (x_1 y_1, x_2 y_2, \dots, x_n y^n),$$

for every $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. Now, for $1 \leq i \leq n$, let $\tau_i : \Omega \to \Omega$ be a bijection such that $\mathbb{R}^{\Omega}$ is invariant under τ_i and τ_i^{-1} (that is, both τ_i and τ_i^{-1} are permutations on Ω). For $1 \leq i \leq n$, let $\tilde{\tau}_i : A \to A$ be the automorphism induced by τ_i, that is,

$$\tilde{\tau}_i(f) = f \circ \tau_i^{-1} \quad \text{for every } f \in \mathbb{R}^{\Omega},$$

and let δ_i, $1 \leq i \leq n$, be a $\tilde{\tau}_i$-derivation. Consider the skew-PBW extension $\tau(\tilde{R}_{\Omega})\langle x_1, \dots, x_n\rangle$.

Definition 2.17 ([TRS20a, Definition 20.5]). For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, define

(i) $\mathrm{Sep}^{\alpha}(\Omega) := \{w \in \Omega \mid \tau^{\alpha}(w) \neq w\}$, and

(ii) $\mathrm{Per}^{\alpha}(\Omega) := \{w \in \Omega \mid \tau^{\alpha}(w) = w\}$.

With this setting, we mention the following results.

Proposition 2.66. (1) [TRS20a, Theorem 20.3] *Suppose that for $1 \leq i \leq n$, $\delta_i = 0$. Then the centralizer $C(\mathbb{R}^{\Omega})$, of $\mathbb{R}^{\Omega}$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega})\langle x_1, \dots, x_n\rangle$ is given by*

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{\sum_{\alpha} f_{\alpha} x^{\alpha} \mid f_{\alpha} = 0 \ \ on \ \ \mathrm{Sep}^{\alpha}(\Omega)\right\}.$$

(2) [TRS20a, Theorem 20.4] *If an element of degree m, $\sum_{k=0}^{m} f_k x^k \in \mathbb{R}^{\Omega}[x, \tilde{\tau}, \delta]$ belongs to the centralizer of $\mathbb{R}^{\Omega}$, then $f_m = 0$ on $\mathrm{Sep}^m(\Omega)$.*

Again, they give a neccesary condition for an element to being in the centralizer of the coefficient ring.

Proposition 2.67 ([TRS20a, Theorem 20.5]). *If an element $\sum_{\alpha} f_{\alpha} x^{\alpha} \in \tilde{\tau}(\mathbb{R}^{\Omega})\langle x_1, \dots, x_n\rangle$ belongs to the centralizer of $\mathbb{R}^{\Omega}$, then $f_{\alpha} = 0$ on $\mathrm{Sep}^{\alpha}(\Omega)$.*

The following are examples of how this theory can be used.

Example 2.12 ([TRS20a, Example 20.4.2]). Consider the quasi-commutative skew PBW extension $A = \tilde{\tau}\left(\mathbb{R}^{\Omega}\right)\langle x_1, x_2\rangle$ with the following conditions:

(i) The automorphisms $\tilde{\tau}_1, \tilde{\tau}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows: $\tilde{\tau}_1 = id$, $\tilde{\tau}_2(e_1) = e_2$, $\tilde{\tau}_2(e_2) = e_1$, where e_1, e_2 are the standard basis vectors in $\mathbb{R}^2$.

(ii) $x_2 x_1 = (c_1, c_2) x_1 x_2$ where $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq 0 \neq c_2$.

From Theorem 2.66, the centralizer of $\mathbb{R}^\Omega$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^\Omega\langle x_1, x_2\rangle)$ is given by

$$
\begin{aligned}
C(\mathbb{R}^\Omega) &= \left\{\sum_\alpha f_\alpha x^\alpha \mid f_\alpha = 0 \ \text{ on } \ \operatorname{Sep}^\alpha(\Omega)\right\} \\
&= \left\{\sum_{j,k} f_{j,2k} x_1^j x_2^{2k}\right\}.
\end{aligned}
$$

2.3.6.2 PBW Extensions for the Algebra of Piecewise Constant Functions

The algebra A of piecewise constant functions $h : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1, t_2, \ldots, t_N$ was introduced, and in fact this algebra is isomorphic to $\mathbb{R}^\Omega$, the algebra of all functions $f : \Omega \to \mathbb{R}$ indexed by $\Omega = \{0, 1, \ldots, 2N\}$. The previous description of the centralizer of $\mathbb{R}^\Omega$ in the skew PBW extension $\tilde{\tau}(R^\Omega)\langle x_1, x_2, \ldots, x_n\rangle$ can be translated in terms of the centralizer of the coefficient algebra A in the skew PBW extension $\sigma(\tilde{A})\langle x_1, x_2, \ldots, x_n\rangle$ in terms of the isomorphism between this algebras, $\mu : \mathbb{R}^\Omega \to A$, and $\operatorname{Sep}^\alpha(\Omega)$. We start with the following definition.

Definition 2.18 ([TRS20a, Definition 20.6]). For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, define

(i) $\operatorname{Sep}_A^\alpha(R) := \{x \in R : \exists h \in A \text{ such that } \tilde{\sigma}^\alpha(h)(x) \neq h(x)\}$.

(ii) $\operatorname{Per}_A^\alpha(R) := \{x \in R : \tilde{\sigma}^\alpha(h)(x) = h(x)\}$.

Using methods similar to the proof of Proposition 2.66, it can be shown that the centralizer of A in the quasi-commutative skew PBW extension $\tilde{\sigma}(A)\langle x_1, x_2, \ldots, x_n\rangle$ is given by the following:

Proposition 2.68. (1) [TRS20a, Proposition 20.4] *Suppose that for* $1 \leq i \leq n$, $\delta_i = 0$. *Then the centralizer* $C(A)$ *of* A *in the skew PBW extension* $\tilde{\sigma}(A)\langle x_1, x_2, \ldots, x_n\rangle$ *is given by*

$$
C(A) = \left\{\sum_\alpha h_\alpha x^\alpha \;\middle|\; h_\alpha = 0 \textit{ on } \operatorname{Sep}_\alpha^A(R)\right\}.
$$

(2) [TRS20a, Theorem 20.7] *The centralizer* $C(A)$ *of* A *in the quasi-commutative* $\tilde{\sigma}$*-PBW extension* $\tilde{\sigma}(A)\langle x_1, x_2, \ldots, x_n\rangle$ *is given by:*

$$
C(A) = \left\{\sum_\alpha h_\alpha x^\alpha \;\middle|\; \mu^{-1}(h_\alpha) = 0 \textit{ on } Sep^\alpha(\Omega)\right\},
$$

where μ *is given by*

$$
\mu(f)(x) = f(\omega) \quad \textit{if } x \in I_\omega, \quad \omega = 0, 1, \ldots, 2N. \tag{2.31}
$$

Example 2.13. Consider the quasi-commutative skew PBW extension $A = \tilde{\tau}(\mathbb{R}^\Omega)\langle x_1, x_2\rangle$ with the following conditions:

(i) The automorphisms $\tilde{\tau}_1, \tilde{\tau}_2 : A \to A$ are defined as follows:

$$\begin{aligned}\tilde{\tau}_1 &= \mathrm{id},\\ \tilde{\tau}_2(e_1) &= e_2,\\ \tilde{\tau}_2(e_2) &= e_1,\end{aligned}$$

where e_1 and e_2 are the standard basis vectors in $\mathbb{R}^2$.

(ii) $x_2x_1 = (c_1, c_2)x_1x_2 \iff x_1x_2 = \left(\frac{1}{c_1}, \frac{1}{c_2}\right)x_2x_1$ where $c_1, c_2 \in R$ with $c_1 \neq 0$ and $c_2 \neq 0$.

This corresponds to the algebra A of piecewise constant functions with one fixed jump point t_1, with $\mathbb{R}$ partitioned into intervals $I_0 = (-\infty, t_1)$, $I_1 = (t_1, \infty)$, and $I_3 = \{t_1\}$. Invariance of A under any bijection $\sigma : R \to R$ implies that $\sigma(t_1) = t_1$. From the definition of the automorphisms $\tilde{\tau}_1$ and $\tilde{\tau}_2$, we see that the corresponding bijections σ_1 and $\sigma_2 : \mathbb{R} \to \mathbb{R}$ behave as follows:

- $\sigma_1(t_1) = \sigma_2(t_1) = t_1$.
- $\sigma_1(I_0) = I_0$ and hence $\sigma_1(I_1) = I_1$.
- $\sigma_2(I_0) = I_1$, which implies $\sigma_2(I_1) = I_0$.

From Theorem 2.68, it follows that for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$,

$$\gamma^{-1}(\mathrm{Sep}^{\alpha}(\Omega)) = \mathrm{Sep}^{\alpha}_{A}(R) = \begin{cases} I_0 \cup I_1, & \text{if } \alpha_2 \text{ is odd,} \\ \varnothing, & \text{if } \alpha_2 \text{ is even.} \end{cases}$$

Therefore, the centralizer of A in the skew PBW extension $\tilde{\sigma}(A)\langle x_1, x_2\rangle$ is given by

$$C(A) = \left\{ \sum_{j,k} h_{j,k} x_j^1 x_k^2 \,\middle|\, h_{j,2k+1} = 0 \text{ on } I_0 \cup I_1 \right\}. \tag{2.32}$$

CHAPTER 3

BURCHNALL-CHAUNDY THEORY IN SEMI-GRADED RINGS

This chapter contains the original results of the thesis.

In Section 3.1, we introduce the notion of *pseudo-multidegree function* as a generalization of pseudo-degree function in the sense of Richter [Ric16] (Section 2.3.4), and hence we establish a criterion to determine whether the centralizer of an element has finite dimension over a noncommutative ring having PBW basis. We prove Theorems 3.2 and 3.4, and Corollary 3.3, which are the important results of the section, extending the corresponding results obtained by Richter. Precisely, Corollary 3.3 is the formulation of $\mathscr{BC}$ theory for rings having pseudo-multidegree functions. Finally, Section 3.1.3 presents the illustration of our results with families of algebras appearing in ring theory and noncommutative geometry.

Next, Section 3.2 contains a first approach to the $\mathscr{BC}$ theory for quadratic algebras having PBW bases defined by Golovashkin and Maksimov [GM05] (Section 1.2.8). With this purpose, in Section 3.2.1, Propositions 3.5, 3.6, 3.7, 3.8, and 3.10, we present combinatorial properties on products of elements in these algebras. Next, in Section 3.2.2 we consider the notions of Sylvester matrix and resultant for quadratic algebras with the aim of exploring common right factors of polynomials. In Section 3.2.3, by using the concept of determinant polynomial, we formulate the version of $\mathscr{BC}$ theory for these algebras (Theorem 3.16). Section 3.2.4 contains illustrative examples of the results formulated in the previous sections.

Section 3.3 contains some ideas with the aim of extending some of the results on centralizers formulated by Bell et al. [BS04, BR16] for graded rings to the setting of semi-graded rings.

In Section 3.4 we present some ideas for a future work.

3.1 PSEUDO-MULTIDEGREE FUNCTIONS ON SEMI-GRADED RINGS

As we said in the Introduction, a very important approach to the study of centralizers with the aim of formulating a Burchnall and Chaundy's result was presented by Silvestrov and Hellstrom [HS07] and Richter [Ric16]. They identified some sufficient conditions that ensure that a version of the $\mathscr{BC}$ theory exists in certain types of algebras over a field. In the first paper, they worked on the condition l-BDHC (Section 2.1.3), which is stated for graded algebras over a field $\Bbbk$. This condition says that for any non-zero homogeneous element a, the dimension of every *homogeneous centralizer* $\text{Cen}(a,n)$ is less or equal than l. This is crucial for a version of $\mathscr{BC}$ theory for some types of generalized Weyl algebras. Later, Richter [Ric16] generalized certain results of [HS07], and investigated centralizers in skew polynomial rings with valuations (more exactly, *pseudo-degree functions*), and proved that the centralizer of an element in these algebras is a free module over a certain ring. Under additional conditions, one can assert that the centralizer is commutative [Ric16, Theorem 7]. Richter exemplified his results with skew polynomial rings of the form $R[y][x;\sigma,\delta]$ with $\deg(\sigma(y)) > 1$ [Ric16, Proposition 1].

Motivated by Richter's treatment developed in [Ric16] and presented in Section 2.3.4, and in order to extend his results to other more general algebras (having PBW bases) than the skew polynomial rings mentioned (such as those considered in Proposition 2.53), the purpose of this section is to investigate a more general version of the concept of pseudo-degree function which we call *pseudo-multidegree function* (see Definition 3.1). In this way, we contribute to the study of $\mathscr{BC}$ theory of several noncommutative algebras more general than skew polynomial rings from an algebraic point of view.

3.1.1 Preliminaries

Throughout Section 3.1, we consider $\mathbb{Z}^n$ as an ordered additive group with the usual sum and the well-known `degree-lexicographic order` $\preceq$. Let $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$ be in $\mathbb{Z}^n$. We say that $\alpha \prec \beta$ if the leftmost non-zero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. Additionally, $\mathbb{Z}^n \cup \{-\infty\}$ will be considered as a group in such way that $m + (-\infty) = -\infty$ and $-\infty \preceq a$, for all $a \in \mathbb{Z}^n$.

Definition 3.1. Let S be a $\Bbbk$-algebra. A function $\chi : S \to \mathbb{Z}^n \cup \{-\infty\}$ is called a *pseudo-multidegree function* if it satisfies the following conditions:

(i) $\chi(a) = -\infty$ if and only if $a = 0$;

(ii) $\chi(ab) = \chi(a) + \chi(b)$ for all $a, b \in S$;

(iii) $\chi(a+b) \preceq \max(\chi(a), \chi(b))$;

(iv) $\chi(a+b) = \chi(a)$ if $\chi(b) \prec \chi(a)$.

Inductively, it can be seen that Definition 3.1 (iii) applies for any finite sum, that is,

$$\chi\left(\sum_{i=1}^{k} a_i\right) \preceq \max_{i\leq k}(\chi(a_i)).$$

Richter [Ric16] not only introduced the notion of pseudo-degree function but he also gave sufficient conditions on a $\Bbbk$-algebra that allow one to decide when the centralizer of an element a of the algebra has finite rank over $\Bbbk[a]$. This condition is exactly the *condition* $D(l)$, and our adaptation is the following:

DEFINITION 3.2. Let S be a $\Bbbk$-algebra with a pseudo-multidegree function χ, and l be a positive integer. A subalgebra $B \subseteq A$ is said to satisfy *condition* $MD(l)$ if $\chi(b) \succeq \mathbf{0} := (0,\ldots,0)$ for all non-zero $b \in B$ and if, whenever we have $l+1$ elements $b_1, b_2, \ldots, b_{l+1} \in B$, such that $\chi(b_1) = \chi(b_2) = \cdots = \chi(b_{l+1})$, then there exist $\alpha_1, \ldots, \alpha_{l+1} \in \Bbbk$, not all zero, such that $\chi\left(\sum_{i=1}^{l+1} \alpha_i b_i\right) \prec \chi(b_1)$.

An important fact in the proof of our principal result, Theorem 3.2, is that we need an upper bound for the amount of possible values of the pseudo-multidegree function on the elements in the centralizer of an element modulo an adequate equivalence relation. With this purpose, we introduce the following condition.

DEFINITION 3.3. Let S be a $\Bbbk$-algebra with a pseudo-multidegree function χ, $a \in S$ and m be a positive integer. A subalgebra $B \subseteq S$ is said to satisfy condition $NC(m, a)$ if there exists a subset of m elements $\{b_1, \ldots, b_m\} \subseteq B$ such that for every $b \in B$, there exist $c \in \mathbb{Z}$ and $1 \leq j \leq m$ such that $\chi(b) - \chi(b_j) = c \cdot \chi(a)$.

REMARK 14. When the pseudo-multidegree function of an algebra S takes values on $\mathbb{Z}$, the function is a pseudo-degree function in the sense of Richter (Definition 2.14). In this case, the condition $MD(l)$ is exactly the condition $D(l)$ which is satisfied for some cases of Ore extensions [Ric16, Example 1], and the condition $NC(\chi(a), a)$ also holds since $\mathbb{Z}$ has a finite number of classes of equivalence modulo $m = \chi(a)$.

The following lemma will be useful in the proof the Theorem 3.2.

LEMMA 3.1. *Let S be a $\Bbbk$-algebra and $a \in S$ such that χ maps all non-zero scalars to $\mathbf{0}$. If $a, b \in S$ are such that $\chi(b) \prec \chi(a)$, then $\chi(a+b) = \chi(a)$.*

Proof. By Definition 3.1 (ii), we know that $\chi(a+b) \preceq \max(\chi(a), \chi(b))$, but since $\chi(b) \prec \chi(a)$, it follows that $\chi(a+b) \preceq \chi(a)$. On the other hand, the relations $\chi(b) \prec \chi(a)$ and $\chi(-b) = \chi(-1)\chi(b) = \mathbf{0} + \chi(b) = \chi(b)$ guarantee that

$$\chi(a) = \chi(a+b-b) \preceq \max(\chi(a+b), \chi(-b)) = \max(\chi(a+b), \chi(b)) = \chi(a+b).$$

□

3.1.2 BURCHNALL-CHAUNDY THEORY

Theorem 3.2 is one of the most important results of the thesis.

THEOREM 3.2. *Let S be a $\Bbbk$-algebra with a pseudo-multidegree function χ. Let a be an element of S such that $\chi(a) \succ \mathbf{0}$. If $C_S(a)$ satisfies condition $MD(l)$ for some $l \in \mathbb{N}$, and there exists a positive integer m such that $C_S(a)$ satisfies condition $NC(m, a)$, then $C_S(a)$ is a free $\Bbbk[a]$-module of finite rank.*

Proof. The proof is divided into four steps.

(1) We define a sequence $\mathscr{H}$ of elements $b_1, b_2, \ldots$ in the following way: $b_1 = 1$, and $b_{k+1} \in C_S(a)$ is an element that does not belong to the vector space generated by linear combinations of elements from $\{b_1, \ldots, b_k\}$ with coefficients in $\Bbbk[a]$, and it is minimal with

respect to the values of the pseudo-multidegree function χ. A first observation is that this sequence is well defined since the value of χ in the elements of $C_S[a]$ is always greater than $\mathbf{0}$ ($MD(l)$ condition). The idea is to prove that this sequence is, in fact, finite. If it is true, that means that $C_S(a)$ is a free $\Bbbk[a]$-module of finite rank.

(2) We claim that the following formula holds for any choice of $c_1,\dots,c_k \in \Bbbk[a]$:

$$\chi\left(\sum_{i=1}^{k} c_i b_i\right) = \max_{i\leq k}\big(\chi(c_i)+\chi(b_i)\big). \tag{3.1}$$

First, notice that according to Definition 3.1 (iii), we have

$$\chi\left(\sum_{i=1}^{k} c_i b_i\right) \preceq \max_{i\leq k}\big(\chi(c_i b_i)\big) = \max_{i\leq k}\big(\chi(c_i)+\chi(b_i)\big).$$

In this way, our aim is to prove the inequality

$$\max_{i\leq k}\big(\chi(c_i)+\chi(b_i)\big) \preceq \chi\left(\sum_{i=1}^{k} c_i b_i\right). \tag{3.2}$$

Since we suppose that $C_S(a)$ satisfies condition $MD(l)$, we know that $\chi\left(\sum_{i=1}^{k} c_i b_i\right) \succeq 0$. This means that the inequality (3.2) holds when $\max_{i\leq k}\big(\chi(c_i)+\chi(b_i)\big) = -\infty$ or $\max_{i\leq k}(\chi(c_i)+\chi(b_i)) = \mathbf{0}$.

Now, let $\mathbf{v} = \max_{i\leq k}\big(\chi(c_i)+\chi(b_i)\big)$. We will prove that if the inequality above holds when the value of the left side of (3.2) is less than $\mathbf{v}$, then it holds for $\mathbf{v}$. We can assume that $\chi(c_k)+\chi(b_k)=\mathbf{v}$. This is due that if $\chi(c_j b_j) \prec \mathbf{v}$, then Lemma 3.1 guarantees the equality

$$\chi\left(\sum_{i=1}^{k} c_i b_i\right) = \chi\left(\sum_{i=1, i\neq j}^{k} c_i b_i\right).$$

There are two cases to consider: $c_k \in \Bbbk$ or $c_k \in \Bbbk[a] \setminus \Bbbk$. First, if $c_k \in \Bbbk$, then $\chi(c_k)=\mathbf{0}$, so $\chi(b_k) = \mathbf{v}-\chi(c_k) = \mathbf{v}-\mathbf{0} = \mathbf{v}$, but this implies that $\chi\left(\sum_{i=1}^{k} c_i b_i\right)$ has to be greater or equal than $\mathbf{v}$; otherwise, b_i wouldn't be minimal over the values of χ, a contradiction to the definition of the sequence. Now, if $c_k \in \Bbbk[a] \setminus \Bbbk$, we have $\chi(c_k) \prec \mathbf{v}$. Since $c_i \in \Bbbk[a]$, it can be written in the form $c_i = av_i + r_i$, where $v_i \in \Bbbk[a]$ and $r_i \in \Bbbk$, for every $1 \leq i \leq k$. In this way,

$$\chi\left(\sum_{i=1}^{k} c_i b_i\right) = \chi\left(\sum_{i=1}^{k} (av_i + r_i) b_i\right) = \chi\left(\sum_{i=1}^{k} av_i b_i + \sum_{i=1}^{k} r_i b_i\right). \tag{3.3}$$

However, as $r_i \in k$ and $v_i \in \Bbbk[a]$,

$$\chi\left(\sum_{i=1}^{k} r_i b_i\right) \prec \chi\left(\sum_{i=1}^{k} av_i b_i\right).$$

By Lemma 3.1 and Definition 3.3, we conclude that

$$\begin{aligned}\chi\left(\sum_{i=1}^{k} c_i b_i\right) &= \chi\left(a\sum_{i=1}^{k} v_i b_i\right)\\ &= \chi(a)+\chi\left(\sum_{i=1}^{k} v_i b_i\right)\\ &\preceq \chi(a)+\max_{i\le k}\big(\chi(v_i)+\chi(b_i)\big),\end{aligned}$$

where the last inequality holds by our inductive assumption.

Now, since that

$$\chi(a)+\max_{i\le k}\big(\chi(v_i)+\chi(b_i)\big)=\max_{i\le k}\big(\chi(c_i)+\chi(b_i)\big),$$

it follows

$$\chi\left(\sum_{i=1}^{k} c_i b_i\right)\preceq \max_{i\le k}\big(\chi(c_i)+\chi(b_i)\big),$$

as we wanted to prove.

(3) We claim that if $\chi(b_i)=\chi(b_{i+1})=\cdots=\chi(b_{i+k})$ for some i, then $k\le l$. For this, suppose that there exist elements $b_i, b_{i+1},\dots,b_{i+l+1}$ such that their images by χ are the same. Then, using the hypothesis that $C_S(a)$ satisfies $MD(l)$ condition, we know that there exist a sequence of $l+1$ elements of $\Bbbk\subseteq\Bbbk[a]$, $\{c_k\}_{k=1}^{l+1}$, such that

$$\chi\left(\sum_{k=1}^{l+1} c_k b_{i+k}\right)\prec\chi(b_i),$$

but this contradicts expression (3.1).

(4) The sequence must be finite. In this step, let us consider the following relation $\mathscr{R}$ on $\mathbb{Z}^n$: let $\mathbf{v}\in\mathbb{Z}^n$ be a fixed element. For $\mathbf{a},\mathbf{b}\in\mathbb{Z}^n$, $\mathbf{a}\mathscr{R}\mathbf{b}$ if $\mathbf{a}-\mathbf{b}=k\mathbf{v}$, for some element $k\in\mathbb{Z}$. It is straightforward to see that $\mathscr{R}$ is an equivalence relation on $\mathbb{Z}^n$.

First, note that for every class of equivalence defined by $\mathscr{R}$, there are at most l elements in the sequence such that their images belong to this class. For this, let us suppose that the elements $b_i,\dots,b_{i+l}$ in the sequence S are such that their images are in the same equivalence class, i.e., $\chi(b_i)\mathscr{R}\chi(b_{i+1})\mathscr{R}\cdots\mathscr{R}\chi(b_{i+l})$. Then, we can multiply each b_i by some power of a to make the χ values the same. Without loss of generality, we can assume that $\chi(b_i)\preceq\chi(b_{i+1})\preceq\cdots\preceq\chi(b_{i+l})$, with $\chi(b_{i+l})$ the maximum value in this list. In this way, by taking a^{k_j}, where k_j is such that $\chi(b_{i+l})-\chi(b_{i+j})=k_j\chi(a)$, we have

$$\chi\left(\sum_{j=0}^{l}(a^{k_j})b_{i+j}\right)=\chi(b_{i+l}).$$

Since $C_S(a)$ satisfies the $D(l)$ condition, we know that there exist some elements $\gamma_0,\dots,\gamma_l\in$

$\Bbbk[a]$ such that
$$\chi\left(\sum_{j=0}^{l}(\gamma_j a^{k_j})b_{i+j}\right) \prec \chi(b_{i+l}),$$
which is a contradiction with (3.1). The second observation is that there are finite equivalence classes occurring in the values of the sequence $\mathscr{H}$. For this, we will use the fact that $C_S(a)$ satisfies the condition $NC(m,a)$. Since $\mathscr{H} \subset C_S(a)$, it is not posible that there are more than m classes of equivalence appearing in S. Moreover, this allows us to conclude that the rank of $C(a)$ is at most lm.

□

As expected, an immediate consequence of Theorem 3.2 is a version of $\mathscr{BC}$ theory for those algebras and centralizers that satisfy the conditions described above.

Corollary 3.3. *Let S be a $\Bbbk$-algebra with a pseudo-multidegree function χ. Let $a \in S$ with $\chi(a) \succ \mathbf{0}$. If $C_S(a)$ satisfies condition $MD(l)$ for some $l \in \mathbb{N}$, and there exists a positive integer m such that $C_S(a)$ satisfies condition $NC(m,a)$, then for any element $b \in S$ such that $ab = ba$, there exists a polynomial $F(s,t) \in \Bbbk[s,t]$ such that $F(a,b) = 0$.*

Proof. Note that $b^n \in C_S(a)$ for every $n \in \mathbb{Z}^+$. Let k be the rank of $C_S(a)$ over $\Bbbk[a]$ (which is finite by Theorem 3.2). Then $\{b^i\}_{i=1}^{k+1}$ is a linearly dependent set over $C_S(a)$, so there exists a sequence of elements $\phi_i(a) \in \Bbbk[a]$, $1 \leq i \leq k+1$, such that
$$\sum_{i=1}^{k+1}\phi_i(a)b^i = 0.$$
The element $F(s,t) = \sum_{i=1}^{k+1}\phi_i(s)t^i \in \Bbbk[s,t]$ is the desired polynomial. □

We finish this section with a theorem which gives criteria to establish when the condition $NC(m,a)$ is not only sufficient but also necessary.

Theorem 3.4. *Let S be a $\Bbbk$-algebra with a pseudo-multidegree function χ. Consider $a \in S$. If $C_S(a)$ is a module of finite rank m over $\Bbbk[a]$ such that there exists a basis $\{b_1, b_2, \ldots, b_m\}$ with the property that condition* (iii) *of Definition 3.1 holds, with equality for any combination with coefficients in $\Bbbk[a]$, then $C_S(a)$ satisfies condition $NC(m,a)$.*

Proof. Let $b_1,\ldots,b_m \in C_S(a)$ such that $C_S(a) = \Bbbk[a]b_1 \bigoplus \Bbbk[a]b_2 \bigoplus \cdots \bigoplus \Bbbk[a]b_m$. If $b \in C_S(a)$, then
$$b = \sum_{i=1}^{m} p_i(a)b_i, \quad p_i(a) \in \Bbbk[a],$$

whence,

$$\begin{aligned}\chi(b) &= \chi\left(\sum_{i=1}^{m} p_i(a)b_i,\right)\\ &= \max_{1\leq i\leq m} \chi(p_i(a)b_i)\\ &= \max_{1\leq i\leq m} \left(\chi(p_i(a))+\chi(b_i)\right)\\ &= \chi\left(p_j(a)\right)+\chi\left(b_j\right).\end{aligned}$$

In this way, $\chi(b)-\chi(b_j)=\chi(p_j(a))=k_j\chi(a)$, which concludes the proof. □

3.1.3 Examples

Next, we present different examples that illustrate the results in Section 3.1.2. Another examples can be taken from Section 1.2.

Example 3.1. Let I be an open interval of the real line $\mathbb{R}$, $R=C^{\infty}(I)$, and consider L a differential operator. In this case, the pseudo-multidegree function is given by

$$\begin{aligned}\chi : R &\longrightarrow \mathbb{Z}\\ L=\sum_{j=0}^{n} p_j(t)D^t &\longmapsto \chi(L)=n,\end{aligned}$$

where n is the order of the differential operator L. From Proposition 2.4 we know that $C_R(L)$ is a $\mathbb{C}[L]$-module of finite rank, which is a divisor of n. In this case, we can assert that the condition $NC(n,L)$ holds since $\mathbb{Z}$ has finite classes modulo n.

Example 3.2. Consider once more again the skew polynomial rings described in Proposition 2.53. Set $R=\Bbbk[y]$, let σ be an endomorphism of R such that $s=\deg_y(\sigma(y))>1$, δ a σ-derivation, and form the Ore extension $S=R[x;\sigma,\delta]$. We know that if $a\in S\setminus\Bbbk$, then $C_S(a)$ is a free $\Bbbk[a]$-module of finite rank. Richter and Silvestrov [RS14] presented examples of some centralizers that satisfy the conditions established in Theorems 3.2 and 3.4, and Corollary 3.3. For example, $C_S(x^ny^n)=\Bbbk[x^ny^n]$ and $C_S(y^nx^n)=\Bbbk[y^nx^n]$. In this case, the pseudo-multidegree function is given by

$$\begin{aligned}\chi : S &\longrightarrow \mathbb{Z}^2\\ p(x,y)=\sum_{i,j} c_{i,j}x^iy^i &\longmapsto (m,n),\end{aligned}$$

where x^my^n is the highest monomial that appears in the expansion of $p(x,y)$ considering the `degree-lexicographic order`. Notice that this function is well-defined because the set $\{x^iy^j \mid i,j\in\mathbb{Z}\}$ is a basis for S as a $\Bbbk$-module. As a matter of fact, for each $q(x,y)\in\Bbbk[x^ny^n]$ we have $\chi(q(x,y))=(kn,kn)$, for some $k\in\mathbb{Z}$, and so it is clear that $C_S(x^ny^n)$ satisfies both conditions.

Example 3.3. Let us consider the $\Bbbk$-algebra S generated by two indeterminates subject to the relation

$$yx = q_1xy + q_2x + q_3y + q_4, \quad q_1, q_2, q_3, q_4 \in \Bbbk.$$

Suppose $q_2 = q_3 = 0$. Let us describe the function χ on the centralizer of any element of the form $a = x^r y^r \in S$. We can use [LV20a, Lemma 2.1] with the aim of studying the leading terms of the elements of $C_S(a)$. Let $b = \sum b_{ij}x^i y^j \in C_S(a)$ and suppose that $\chi(b) = (m, n)$. Then

$$\begin{aligned} ab &= (x^r y^r)(b_{m,n}x^m y^n + p(x, y)), \text{ where } \chi(p(x, y)) \preceq (m, n) \\ &= q_1^{rn} x^{rn} y^m + p_1(x, y). \end{aligned}$$

On the other hand,

$$\begin{aligned} ba &= (b_{m,n}x^n y^n + p(x, y))(x^r y^r), \text{ where } \chi(p(x, y)) \preceq \chi(x^{n+r} y^{m+r}) \\ &= q_1^{rm} x^{r+n} y^{m+r} + p_2(x, y), \end{aligned}$$

whence $rm = rn$, which implies that $m = n$. Therefore, for any element $b \in C_S(x^r y^r)$, we have that $\chi(b) = (n, n)$ for some $n \in \mathbb{N}$. Notice that if $n = qr + d$, then $\chi(b) = (n, n) = q\chi(a) + (d, d)$, and d can have r possible values, so the condition $NC(r, a)$ is satisfied. Finally, the condition $MD(1)$ is also satisfied since the coefficients of the polynomials belong to $\Bbbk$. Therefore, a version of $\mathscr{BC}$ theory holds for every pair of elements $a = x^r y^r$ and $b \in C_S(a)$.

Example 3.4. In Section 1.2.8, we saw that Golovashkin and Maksimov [GM05] defined the *quadratic algebras* $Q(a, b, c)$ in two generators x, y given by

$$yx = ax^2 + bxy + cy^2, \quad a, b, c \in \Bbbk, \tag{3.4}$$

and that they found conditions for such an algebra $Q(a, b, c)$ to be expressed as a skew polynomial with generator y over the polynomial ring $\Bbbk[x]$ (cf. [GM98]), that is, the existence of a *Poincaré-Birkhoff-Witt* basis of the form $\{x^m y^n\}$.

Precisely, if the set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ forms a PBW basis for $Q(a, b, c)$, then we can define the pseudo-multidegree function given by

$$\begin{aligned} \chi : Q(a, b, c) &\to \mathbb{Z}^2 \\ p(x, y) = \sum c_{i,j} x^i y^j &\to (m, n), \end{aligned}$$

where $x^m y^n$ is the highest monomial that appears in the expansion of $p(x, y)$ considering the `degree lexicographic order`. Let us see an example.

If $a, b \neq 0$, then for $b \in \Bbbk$ and $k \geq 1$, we write

$$[k]_b := \sum_{i=0}^{k-1} b^i \quad \text{and} \quad [k]_b! := \prod_{i=1}^{k} [i]_b.$$

Notice that if $b \neq 1$, then $[k]_b = \frac{b^k - 1}{b - 1}$ and $[k]_1 = k$. From [Ben99a, Proposition 1] or [Cha22, Section 3.2.1.3], we know that in $Q(a,b,0)$ (i. e., $c=0$) the following commutation rules hold:

$$\begin{aligned} yx^k &= b^k x^k y + a[k]_b x^{k+1},\\ y^k x &= \sum_{r=0}^{k} \frac{[k]_b!}{[k-r]_b!} b^{k-r} a^r x^{r+1} y^{k-r}. \end{aligned}$$

If $b=1$ and $\mathrm{char}(\Bbbk)=n$, then for the element $p(x,y)=x^n y^n$, the equality $C_{Q(a,b,0)}(p(x,y)) = Q(a,b,0)$ holds. In this case, it is clear that $C_{Q(a,b,0)}(p(x,y))$ cannot be a module of finite rank over $\Bbbk[x^n y^n]$.

Consider the case $c=0$ and b is not a root of the unity. If we take the element $v=xy$, it can be seen that if $w \in C_{Q(a,b,0)}(v)$, with $w = d_{n,m} x^n y^m + \sum_{i,j} d_{i,j} x^i y^j$ and $\chi(w) = (m,n)$ then by using that $vw = wv$ we have

$$\begin{aligned} vw &= (xy)\left(d_{n,m} x^n y^m + \sum_{i,j} d_{i,j} x^i y^j\right)\\ &= d_{n,m}(xy)(x^n y^m) + \text{lower monomial terms}\\ &= d_{n,m} x(yx^n) y^m + \text{lower monomial terms}\\ &= d_{n,m}(xy)(b^n x^n y + a[n]_b x^{n+1}) y^m + \text{lower monomial terms}\\ &= d_{n,m} b^n x^{n+1} y^{m+1} + d_{n,m} a[n]_b x^{n+2} y^m + \text{lower monomial terms} \end{aligned}$$

and

$$\begin{aligned} wv &= \left(d_{n,m} x^n y^m + \sum_{i,j} d_{i,j} x^i y^j\right)(xy)\\ &= (d_{n,m} x^n y^m)(xy) + \text{lower monomial terms}\\ &= d_{n,m} x^n (y^m x) y + \text{lower monomial terms}\\ &= d_{n,m} x^n \left(\sum_{h=0}^{m} \frac{[m]_b}{[m-h]_b} b^{m-h} a^h x^{h+n+1} y^{m-h+1}\right) y + \text{lower monomial terms}\\ &= d_{n,m} b^m x^{n+1} y^{m+1} + d_{n,m}[m]_b b^{m-1} a x^{n+2} y^m + \text{lower monomial terms} \end{aligned}$$

Comparing the coefficients of the leading terms of wv and vw, we conclude that $b^n = b^m$, so $n=m$, whence $\chi(w) = (n,n) = n(1,1) = \chi(xy)$, so we can infer that the set $\{1\}$ proves that $C_S(xy)$ satisfies the condition $NC(1,xy)$. On the other hand, it is clear that $C_S(xy)$ satisfies condition $MD(1)$ since the coefficients of the polynomials are elements of the field $\Bbbk$.

Example 3.5. In Section 2.3.6, we saw that for the skew PBW extensions, Reyes and Suárez [RS18] made a first approach to a $\mathscr{BC}$ theory for these objects. A possible pseudo-multidegree function on a polynomial f is given by $\chi(f) = \exp(\mathrm{lm}(f))$. In this way, Proposition 2.27 says that the centralizer of an element belonging to a skew PBW extension satisfies condition $NC(1,f)$. As it is clear, if $R=\Bbbk$ then the skew PBW extension satisfies condition $MD(1)$.

Of course, there are examples of algebras where their centers and centralizers do not satisfy the conditions imposed above. Let us see two illustrative examples.

EXAMPLE 3.6. From Section 2.3.2 we know that a well-known deformation of the classical ring of polynomials in two variables $\mathbb{C}[x,y]$ is the *Manin's plane* or *quantum plane* $S=\mathbb{C}_q[x,y]$, i.e., the $\mathbb{C}$-algebra generated by x and y subject to the relation $yx=qxy$ with $q\in\mathbb{C}^*$. If $q=-1$, then the centralizer of the element $z=x^3+xyt^3$, with $t=x^2y^2+1$, is $C_S(z)=\mathbb{C}[x^2,y^2,z]$. In this case, we consider the following pseudo-multidegree function

$$\begin{aligned}\chi: S &\longrightarrow \mathbb{Z}^2\\ p(x,y)=\sum_{i,j}c_{i,j}x^iy^i &\longmapsto (m,n),\end{aligned}$$

where x^my^n is the highest monomial that appears in the expansion of $p(x,y)$ considering the `degree lexicographic monomial ordering`. Notice that $\chi(z)=(7,7)$ and for every element $p(x,y)$ in $\mathbb{C}[x^2,y^2,z]$, we have that $\chi(p(x,y))=(7r+2s,7r+2t)$, with $r,t,s\in\mathbb{Z}$. Hence, $C_S(z)$ satisfies the condition $MD(1)$ but it does not satisfy our condition $NC(m,z)$ for every $m\geq 0$ (c.f. Remark 10).

EXAMPLE 3.7. By definition, the *Jordan plane* (Remark 6 (vi)) is the $\mathbb{k}$-algebra defined by $\mathscr{J}=\mathbb{k}\{x,y\}/\langle yx-xy-x^2\rangle$. According to [LV20a, Example 1.11], if $\mathrm{char}(\mathbb{k})=p>0$, then x^py^p is in the center of $\mathscr{J}$. In this way, $C_S(x^py^p)$ does not satisfy the condition $MD(l)$.

EXAMPLE 3.8. Following Jakobsen [JZ97, p. 458], consider the quantum group (in the sense of Section 1.2.7) known as *the coordinate algebra of a quantum matrix space $M_q(A)$*, which is by definition the associative algebra $\mathscr{A}_n$ generated by elements $Z_{i,j}$, $i,j=1,2,\dots,n$, subject to the relations

$$\begin{aligned}
Z_{i,j}Z_{i,k}&=qZ_{i,k}Z_{i,j}, \ \text{ if } j<k,\\
Z_{i,j}Z_{k,j}&=qZ_{k,j}Z_{i,j} \ \text{ if } i<k,\\
Z_{i,j}Z_{s,t}&=qZ_{s,t}Z_{i,j}, \ \text{ if } i<s, t<j,\\
Z_{i,j}Z_{s,t}&=qZ_{s,t}Z_{i,j}+(q-q^{-1})Z_{i,t}Z_{s,j}, \ \text{ if } i<s, j<t.
\end{aligned}$$

The ring $\mathscr{A}_n$ has a PBW basis where each element can be written as linear combination of finite elements from the set

$$\left\{Z^A \mid A\in M_n(\mathbb{Z}^+)\right\}, \quad Z^A=\prod_{i,j=1}^n Z_{i,j}^{a_{ij}},$$

where the factors are arranged in `degree lexicographic order` in $I(n)=\left\{(i,j)\mid i,j=1,\dots,n\right\}$. In the case when $n=2$, we can formulate examples where our theory can be applied. In this case, our pseudo multidegree function χ is the following

$$\begin{aligned}\chi:\mathscr{A}_n &\longrightarrow \mathbb{Z}^4,\\ f=\sum_{j=1}^n c_jZ_{11}^{a_{j,1,1}}Z_{12}^{a_{j,1,2}}Z_{21}^{a_{j,2,1}}Z_{22}^{a_{j,2,2}} &\longmapsto (m,n,v,w),\end{aligned}$$

where (m,n,v,w) is again, the highest monomial, ordered by `degree-lexicographic order`. According to [JZ97], if q is a $m-th$ root of unity, the element Z_{ij}^m is central for all $i,j=1,2$. For example, $C_{\mathscr{A}_n}(Z_{12})=\mathscr{A}_n$, whence the condition $NC(m,Z_{12})$ is not satisfied for any $m\in\mathbb{Z}^+$, as expected. On the other hand, condition $MD(1)$ is satisfied since the coefficients belong to the field $\Bbbk$.

3.2 Burchnall-Chaundy theory for quadratic algebras having PBW bases

As we saw in Chapter 2, several authors have formulated different algebraic approaches toward $\mathscr{BC}$ theory for families of noncommutative rings related with Ore extensions (e.g., Amitsur [Ami58], Flanders [Fla55], Carlson and Goodearl [CG80, Goo83], Hellstrom and Silvestrov [HS00, HS07], Bell and Small [Bel09, BS04], Richter [Ric16], Reyes and Suárez [RS18], and references therein). For instance, with the aim of constructing the $\mathscr{BC}$ curve in the setting of some families Ore extensions, the notions of *determinant polynomial, subresultant* and *resultant* have been investigated by Li in his PhD thesis [Li96] (see also [Li98]), Richter in his PhD thesis [Ric14a], Richter and Silvestrov [RS12], and Larsson [Lar14]. It is important to note that the concepts of resultant and determinant polynomials are intimately related since both aim to encode the common factors of pairs (right factors in the noncommutative case) of polynomials. For instance, the theory of the commutative subresultant has been of interest since the work of Collins, Brown and Traub [BT71, Col67, Loo82, Mis93], which seeks to establish algorithms that solve the problem of finding greatest common divisors in rings of polynomial type, while the concept of multivariate resultant serves as a criterion for determining the existence of common factors in a pair of homogeneous polynomials [Cay48, CLO15]. For example, Chardin [Cha91] studied a subresultant theory for linear ordinary differential operators, while Carra'-Ferro [CF94, CF97], McCallum [MW18], Rueda and Sendra [RS10], and Zhang, Yuan and Gao [ZYG14], established different results on differential resultants. Several details, from classical ones to those obtained relatively recently, about resultant theory and their importance in physical research were presented by Morozov and Shakirov [MS10].

Motivated by the algebraic approach to the $\mathscr{BC}$ theory for noncommutative algebras such as described above, our aim in this section is to present a first approach to the $\mathscr{BC}$ theory for skew Ore polynomials of higher order generated by homogenous quadratic relations considered in Section 1.2.8. Since some of its ring-theoretical, homological, geometrical and combinatorial properties have been investigated (e.g., [Cha22, CR23, FGL+20, NRR20, NR20, Ros00] and references therein), this paper can be considered as a contribution to the study of algebraic characterizations of these objects.

The organization of the section is as follows. In Section 3.2.1, Propositions 3.5, 3.6, 3.7, 3.8, and 3.10 present combinatorial properties on products of elements in these algebras. Next, in Section 3.2.2 we consider the notions of Sylvester matrix and resultant for quadratic algebras with the aim of determining whether this object characterize common right factors of polynomials. In Section 3.2.3, by using the concept of determinant polynomial, we formulate the version of $\mathscr{BC}$ theory for these algebras (Theorem 3.16). Finally, Section 3.2.4 contains illustrative examples of the results formulated in the previous sections.

3.2.1 Products of homogeneous elements

We consider some combinatorial properties of products of elements for several values of a, b and c in the defining relation (1.13). It is possible that these are found in the literature; however, we could not find them explicitly somewhere, so we provide their corresponding proofs.

Let $Q(a, b, c)$ be a quadratic algebra having a PBW basis of the form $\{x^m y^n \mid m, n \in \mathbb{N}\}$. We divide our treatment in the Cases 3.2.1.1, 3.2.1.2 and 3.2.1.3.

3.2.1.1 Case $Q(a, -1, c)$

As we mentioned before, Golovashkin and Maksimov proved that if $b = -1$ and $ac = 1$, then the set $\{x^n y^m \mid n, m \in \mathbb{N}\}$ is not a PBW basis, so we consider the case $ac \neq 1$. From [CR24, Proposition 3.8], we know that the set $\{x^n y^m \mid n, m \in \mathbb{N}\}$ is a PBW basis for $Q(a, -1, c)$, and by [CR24, Lemma 3.9], the following commuting rules hold in this algebra:

(1) If k is even, then $yx^k = x^k y$ and $y^k x = xy^k$.

(2) If k is odd, then $yx^k = ax^{k+1} - x^k y + cx^{k-1} y^2$ and $y^k x = ax^2 y^{k-1} - xy^k + cy^{k+1}$.

We can go further and prove the facts established in Propositions 3.5, 3.6 and 3.7.

Proposition 3.5. *In $Q(a, -1, c)$, the following formulas hold:*

(1) $y^n x^m = x^m y^n$, *if m is even or n is even.*

(2) $y^n x^m = ax^{m+1} y^{n-1} - x^m y^n + cx^{m-1} y^{n+1}$, *if m and n are odd.*

Proof. (1) If n is even, then y^n commutes with x, whence y^n belongs to the center of $Q(a, -1, c)$. In particular, $y^n x^m = x^m y^n$. The argument for x^m is analogous.

(2) In this case, we have

$$\begin{aligned} y^n x^m = y^n x^{m-1} x = x^{m-1} y^n x &= x^{m-1} \left(ax^2 y^{n-1} - xy^n + cy^{n+1} \right) \\ &= ax^{m+1} y^{n-1} - x^m y^n + cx^{m-1} y^{n+1}. \end{aligned}$$

□

In the following propositions, the symbol $\lfloor \square \rfloor$ denote the greatest integer less than or equal to $\square$.

Proposition 3.6. *Suppose that n is an odd number.*

(1) *If m is an even number, then*

$$y^n\left(\sum_{j=0}^{m} e_j x^{m-j} y^j\right) = \sum_{j=0}^{m}(-1)^j e_j x^{m-j} y^{n+j} + \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}\left(ax^{m-2j}y^{n+2j} + cx^{m-(2j+2)}y^{n+(2j+2)}\right).$$

(2) *If m is an odd number, then we have*

$$y^n\left(\sum_{j=0}^{m} e_j x^{m-j} y^j\right) = \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{2j}(ax^{m-2j+1}y^{2j+n-1} - x^{m-2j}y^{2j+n} + cx^{m-(2j+1)}y^{2j+n+1}) + \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{(2j+1)}x^{m-(2j+1)}y^{n+(2j+1)}.$$

Proof. (1) Suppose that m is an even number. Then:

$$\begin{aligned}
y^n\left(\sum_{j=0}^{m} e_j x^{m-j} y^j\right) &= y^n\left(\sum_{j=0}^{\frac{m}{2}} e_{2j}x^{m-2j}y^{2j}\right) + y^n\left(\sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}x^{m-(2j+1)}y^{2j+1}\right)\\
&= \sum_{j=0}^{\frac{m}{2}} e_{2j}x^{m-2j}y^{n+2j} + \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}y^n x^{m-(2j+1)}y^{2j+1}\\
&= \sum_{j=0}^{\frac{m}{2}} e_{2j}x^{m-2j}y^{n+2j}\\
&\quad + \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}\left(ax^{m-2j}y^{n-1} - x^{m-(2j+1)}y^n + cx^{m-(2j+2)}y^{n+1}\right)y^{2j+1}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
y^n\left(\sum_{j=0}^{m} e_j x^{m-j} y^j\right) &= \sum_{j=0}^{\frac{m}{2}} e_{2j}x^{m-2j}y^{n+2j} - \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}x^{m-(2j+1)}y^{n+(2j+1)}\\
&\quad + \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}\left(ax^{m-2j}y^{n+2j} + cx^{m-(2j+2)}y^{n+(2j+2)}\right)\\
&= \sum_{j=0}^{m}(-1)^j e_j x^{m-j}y^{n+j}\\
&\quad + \sum_{j=0}^{\frac{m}{2}-1} e_{(2j+1)}\left(ax^{m-2j}y^{n+2j} + cx^{m-(2j+2)}y^{n+(2j+2)}\right).
\end{aligned}$$

(2) If m is odd, it follows that

$$
\begin{aligned}
y^n\left(\sum_{j=0}^{m} e_j x^{m-j} y^j\right) &= y^n\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{2j} x^{m-2j} y^{2j}\right) + y^n\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{(2j+1)} x^{m-(2j+1)} y^{2j+1}\right)\\
&= \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{2j} y^n x^{m-2j} y^{2j} + \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{(2j+1)} x^{m-(2j+1)} y^{(n+(2j+1))}\\
&= \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{2j}(a x^{m-2j+1} y^{2j+n-1} - x^{m-2j} y^{2j+n}\\
&\quad + c x^{m-(2j+1)} y^{2j+n+1}) + \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} e_{(2j+1)} x^{m-(2j+1)} y^{n+(2j+1)}.
\end{aligned}
$$

□

The following formulas can be used for computing the product of two homogeneous polynomials.

Proposition 3.7. *Let $f(x,y), g(x,y) \in Q(a,-1,c)$ be two homogeneous polynomials given by*

$$
f(x,y) = \sum_{i=0}^{m} e_i x^{m-i} y^i, \text{ and } g(x,y) = \sum_{j=0}^{n} l_j x^{n-j} y^j. \tag{3.5}
$$

If

$$
p = \begin{cases} \lfloor\frac{m}{2}\rfloor, & \text{if } m \text{ is odd,}\\ \frac{m}{2}, & \text{if } m \text{ is even,}\end{cases} \quad \text{and} \quad q = \begin{cases} \lfloor\frac{m}{2}\rfloor, & \text{if } m \text{ is odd,}\\ \frac{m}{2}-1, & \text{if } m \text{ is even,}\end{cases}
$$

then we have the following equalities:

(1) *If n is an even number, then*

$$
\begin{aligned}
f(x,y)g(x,y) &= \sum_{i=0}^{p}\sum_{j=0}^{\frac{n}{2}} e_{2i} l_{2j} x^{(m+n)-(2i+2j)} y^{2i+2j}\\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\frac{n}{2}-1} e_{2i+1} l_{2j+1} (a x^{(m+n)-(2i+2j+1)} y^{(2i+2j+1)}\\
&\quad + x^{(m+n)-(2i+2j+2)} y^{(2i+2j+2)} + c x^{(m+n)-(2i+2j+3)} y^{(2i+2j+3)}).
\end{aligned}
$$

(2) *On the other hand, if n is an odd number, then we have*

$$
\begin{aligned}
f(x,y)g(x,y) &= \sum_{i=0}^{p}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i} l_{2j} x^{(m+n)-(2i+2j)} y^{(2i+2j)}\\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i+1} l_{2j+1} x^{(m+n)-(2i+2j+2)} y^{(2i+2j+2)}.
\end{aligned}
$$

Proof. (1) Suppose that n is even. We get

$$
\begin{aligned}
f(x,y)g(x,y) &= \left(\sum_{i=0}^{m} e_i x^{m-i} y^i\right)\left(\sum_{j=0}^{n} l_j x^{n-j} y^j\right) = \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} y^i x^{n-j} y^j \\
&= \sum_{i=0}^{p}\sum_{j=0}^{\frac{n}{2}} e_{2i} l_{2j} x^{m-2i} y^{2i} x^{n-2j} y^{2j} \\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\frac{n}{2}-1} e_{2i+1} l_{2j+1} x^{m-(2i+1)} y^{2i+1} x^{n-(2j+1)} y^{2j+1} \\
&= \sum_{i=0}^{p}\sum_{j=0}^{\frac{n}{2}} e_{2i} l_{2j} x^{(m+n)-(2i+2j)} y^{2i+2j} \\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\frac{n}{2}-1} e_{2i+1} l_{2j+1} x^{m-(2i+1)} (a x^{n-2j} y^{2i} \\
&\quad - x^{n-(2j+1)} y^{2i+1} + c x^{n-(2j+2)} y^{2i+2}) y^{2j+1} \\
&= \sum_{i=0}^{p}\sum_{j=0}^{\frac{n}{2}} e_{2i} l_{2j} x^{(m+n)-(2i+2j)} y^{2i+2j} \\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\frac{n}{2}-1} e_{2i+1} l_{2j+1} (a x^{(m+n)-(2i+2j+1)} y^{(2i+2j+1)} \\
&\quad + x^{(m+n)-(2i+2j+2)} y^{(2i+2j+2)} + c x^{(m+n)-(2i+2j+3)} y^{(2i+2j+3)}).
\end{aligned}
$$

(2) Let n be an odd number. Then

$$
\begin{aligned}
f(x,y)g(x,y) &= \left(\sum_{i=0}^{m} e_i x^{m-i} y^i\right)\left(\sum_{j=0}^{n} l_j x^{n-j} y^j\right) = \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} y^i x^{n-j} y^j \\
&= \sum_{i=0}^{p}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i} l_{2j} x^{m-2i} y^{2i} x^{n-2j} y^{2j} \\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i+1} l_{2j+1} x^{m-(2i+1)} y^{2i+1} x^{n-(2j+1)} y^{2j+1} \\
&= \sum_{i=0}^{p}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i} l_{2j} x^{(m+n)-(2i+2j)} y^{(2i+2j)} \\
&\quad + \sum_{i=0}^{q}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} e_{2i+1} l_{2j+1} x^{(m+n)-(2i+2j+2)} y^{(2i+2j+2)}.
\end{aligned}
$$

□

3.2.1.2 CASE $Q(a,0,0)$

It is clear that this algebra has a PBW basis of the form $\{x^m y^n \mid m,n \in \mathbb{N}\}$, and from [CR24], we know that the relation $y^n x^k = a^n x^{n+k}$ holds for all $n \geq 0$ and $k \geq 1$.

PROPOSITION 3.8. *If $f(x,y), g(x,y) \in Q(a,0,0)$ are two homogeneous elements given by the expressions*

$$f(x,y) = \sum_{i=0}^{m} e_i x^{m-i} y^i \quad \text{and} \quad g(x,y) = \sum_{j=0}^{n} l_j x^{n-j} y^j, \tag{3.6}$$

then

$$f(x,y)g(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left(e_i l_j a^i\right) x^{m+n-j} y^j.$$

Proof. Since $y^n x^k = a^n x^{n+k}$, for all $n \geq 0$ and $k \geq 1$, it follows that

$$\begin{aligned}
f(x,y)g(x,y) &= \left(\sum_{i=0}^{m} e_i x^{m-i} y^i\right)\left(\sum_{j=0}^{n} l_j x^{n-j} y^j\right) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} y^i x^{n-j} y^j \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} \left(a^i x^{n-j+i}\right) y^j \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \left(e_i l_j a^i\right) x^{m+n-j} y^j.
\end{aligned}$$

□

COROLLARY 3.9. *For the element $g(x,y)$ given by the expression (3.6), we have*

$$y^n g(x,y) = \sum_{j=0} l_j a^n x^{2n-j} y^j.$$

Proof. The assertion follows from the equalities

$$y^n g(x,y) = y^n \sum_{j=0}^{n} l_j x^{n-j} y^j = \sum_{j=0}^{n} l_j y^n x^{n-j} y^j = \sum_{j=0}^{n} l_j a^n x^{2n-j} y^j.$$

□

3.2.1.3 CASE $Q(0,b,0)$

The algebra $Q(0,b,0)$ is known in the literature as the *Manin's plane* or the *quantum plane*. Algebraic descriptions of the centralizer of elements belonging to this algebra can be found in Artamanov and Cohn's paper [AC99]. It is straightforward to see that this algebra satisfies the relation $y^n x^k = b^{kn} x^k y^n$ holds for all $n,k \geq 0$. Proposition 3.10 extends this relationship by considering the product of two polynomials belonging to this algebra.

PROPOSITION 3.10. *If $f(x,y), g(x,y) \in Q(0,b,0)$ are two homogeneous elements given by*

$$f(x,y) = \sum_{i=0}^{m} e_i x^{m-i} y^i \text{ and } g(x,y) = \sum_{j=0}^{n} l_j x^{n-j} y^j,$$

then

$$f(x,y)g(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left(e_i l_j b^{in-ij}\right) x^{(m+n)-(i+j)} y^{i+j}.$$

Proof. Since $y^n x^k = b^{kn} x^k y^n$, for all $n, k \geq 0$,

$$\begin{aligned}
f(x,y)g(x,y) &= \left(\sum_{i=0}^{m} e_i x^{m-i} y^i\right)\left(\sum_{j=0}^{n} l_j x^{n-j} y^j\right)\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} y^i x^{n-j} y^j\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} e_i l_j x^{m-i} b^{(i)(n-j)} x^{n-j} y^i y^j\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \left(e_i l_j b^{in-ij}\right) x^{(m+n)-(i+j)} y^{i+j}.
\end{aligned}$$

□

COROLLARY 3.11. *The following rule of commutation holds for any value of n, m and i.*

$$x^{m-i} y^i \left(\sum_{j=0}^{n} e_j x^{n-j} y^j\right) = \sum_{j=0}^{n} e_j b^{i(n-j)} x^{m+n-(i+j)} y^{i+j}.$$

In the next section we introduce the notions of Sylvester matrix and resultant for quadratic algebras for the algebras considered above.

3.2.2 SYLVESTER MATRIX AND RESULTANTS

In this section, we follow the ideas used in the commutative case to define the Sylvester matrix associated with a pair homogeneous polynomials in several variables [MS10]. We also explore whether this notion determines the existence of common right factors in the context of the algebras of our interest.

3.2.2.1 SYLVESTER MATRIX FOR $Q(a,b,c)$

From the properties obtained in Section 3.2.1, we can see that the product of two homogeneous polynomials is a homogeneous polynomial. In this way, if we define the sets

$$H_n = \{p(x,y) \in Q(a,b,c) \mid p(x,y) \text{ is homogeneous of total degree } n\},$$

then it is straightforward to see that $H_nH_m \subseteq H_{n+m}$. Now, since $Q(a,b,c) = \bigoplus_{n\in\mathbb{N}} H_n$, it is clear that $\{H_n\}_{n\in\mathbb{N}}$ is a graduation for $Q(a,b,c)$.

Related with this, an important fact to formulate a definition of resultant appears when we consider the homomorphism of left $\Bbbk$-modules

$$\begin{aligned} \phi : H_{n-1} \times H_{m-1} &\longrightarrow H_{m+n-1} \\ (c(x,y), d(x,y)) &\longmapsto c(x,y)f(x,y) + d(x,y)g(x,y), \end{aligned} \tag{3.7}$$

which is well defined due to the fact that the set $\{x^i y^j\}_{i,j\geq 0}$ is a PBW basis and the graduation above.

Given two polynomials $f(x,y)$ and $g(x,y)$ in the quadratic algebra $Q(a,b,c)$, the question about the existence of two homogeneous polynomials $c(x,y)$ and $d(x,y)$ such that

$$c(x,y)f(x,y) + d(x,y)g(x,y) = 0, \tag{3.8}$$

can be formulated on the characterization of the kernel of the homomorphism ϕ in expression (3.7). Having in mind that the set $\{x^n y^m \mid n,m \in \mathbb{N}\}$ is a PBW basis for $Q(a,b,c)$, it follows that $\{x^i y^j \mid i+j=n\}$ is a basis for the $\Bbbk$-module H_n. Hence, we can think about defining the matrix that represents the homomorphism ϕ, and for this we will consider a fixed monomial order on the PBW basis: the lexicographic monomial ordering. In this way, the following is the ordered set that we consider for the PBW basis

$$\mathscr{B} = \left\{(x^n,0),(x^{n-1}y,0),(x^{n-2}y^2,0),\dots,(y^n,0),(0,x^m),(0,x^{m-1}y),\dots,(0,y^m)\right\} \tag{3.9}$$

Now, we define the *Sylvester matrix* for homogeneous elements in quadratic algebras.

DEFINITION 3.4. Let $f(x,y)$ and $g(x,y)$ be two homogeneous polynomials in $Q(a,b,c)$ with degree m and n, respectively. The *Sylvester matrix* of $f(x,y)$ and $g(x,y)$ is the matrix that represents the homomorphism (3.7) in the basis given by the set (3.9). This matrix will be denoted by $\mathrm{Syl}_{Q(a,b,c)}(f,g)$ and has size $(m+n)\times(m+n)$. The determinant of this matrix will be called *the resultant* of $f(x,y)$ and $g(x,y)$, and it will be denoted by $\mathrm{Res}_{Q(a,b,c)}$.

The Sylvester matrix for the algebras of our interest has one of the forms which are presented below. Note that all the formulas deduced in Section 3.2.1 are used to construct these matrices. Let us describe how the Sylvester matrix is constructed: at position i,j (ith row and jth column), the entry corresponds to the coefficient of the monomial $x^{m+n-i}y^{i-1}$ of the polynomial $x^{n-j}y^{j-1}f$, for the first n columns, and in the last columns, the coefficient of the same monomial but of the polynomial $x^{m-k}y^{k-1}g$. Since quadratic algebras we are considering have PBW basis, the function $\gamma_{i,j} : Q(a,b,c) \to \Bbbk,\ p(x,y) \to c_{i,j}$, where $c_{i,j}$ is the coefficient of the monomial $x^i y^j$ in the expansion of $p(x,y)$ in terms of the basis $\{x^i y^j \mid i,j \geq 0\}$, is well defined.

With this notation, we can describe the Sylvester matrix $\mathrm{Syl}_{Q(a,b,c)}(f,g)$ as

$$\begin{bmatrix} \gamma_{m+n-1,0}(x^{n-1}f) & \dots & \gamma_{m+n-1,0}(y^{n-1}f) & \gamma_{m+n-1,0}(x^{m-1}g) & \dots & \gamma_{m+n-1,0}(y^{m-1}g) \\ \gamma_{m+n-2,1}(x^{n-1}f) & \dots & \gamma_{m+n-2,1}(y^{n-1}f) & \gamma_{m+n-2,1}(x^{m-1}g) & \dots & \gamma_{m+n-2,1}(y^{m-1}g) \\ \gamma_{m+n-3,2}(x^{n-1}f) & \dots & \gamma_{m+n-3,2}(y^{n-1}f) & \gamma_{m+n-3,2}(x^{m-1}g) & \dots & \gamma_{m+n-3,2}(y^{m-1}g) \\ \vdots & \vdots & & \vdots & & \vdots \\ \gamma_{n+1,m-2}(x^{n-1}f) & \dots & \gamma_{n+1,m-2}(y^{n-1}f) & \gamma_{n+1,m-2}(x^{m-1}g) & \dots & \gamma_{n+1,m-2}(y^{m-1}g) \\ \gamma_{n,m-1}(x^{n-1}f) & \dots & \gamma_{n,m-1}(y^{n-1}f) & \gamma_{n,m-1}(x^{m-1}g) & \dots & \gamma_{n,m-1}(y^{m-1}g) \\ \vdots & \vdots & & \vdots & & \vdots \\ \gamma_{0,m+n-1}(x^{n-1}f) & \dots & \gamma_{n,m+n-1}(y^{n-1}f) & \gamma_{n,m+n-1}(x^{m-1}g) & \dots & \gamma_{n,m+n-1}(y^{m-1}g) \end{bmatrix}.$$

Let us see some illustrative examples of Definition 3.4.

3.2.2.2 Case $Q(a,-1,c)$

In this algebra, the Sylvester matrix depends on the parity or oddness of the degrees of the elements $f(x,y) = \sum_{i=0}^{m} e_i x^{m-i} y^i$ and $g(x,y) = \sum_{j=0}^{n} l_j x^{n-j} y^j$. The corresponding matrices are shown in (3.10), (3.11), (3.12), and (3.13).

Example 3.9. Let $f(x,y) = x^2 + y^2$ and $g(x,y) = xy$ be elements of $Q(a,-1,c)$. Then $c(x,y) = c_1x + c_2y$ and $d(x,y) = d_1x + d_2y$, whence $cf = c_1x^3 + c_2x^2y + c_1xy^2 + c_2y^3$ and $dg = (d_1 + d_2a)x^2y - d_2xy^2 + cd_2y^3$, and the Sylvester matrix of f and g is given by

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & c \end{bmatrix}.$$

Example 3.10. Let $f(x,y) = x^3 + y^3$, $g(x,y) = x^2y + xy^2 \in Q(a,-1,c)$. Then $c(x,y) = c_1x^2 + c_2xy + c_3y^2$ and $d(x,y) = d_1x^2 + d_2xy + d_3y^2$. In this way,

$$\begin{aligned} cf &= (c_1 + ac_2)x^5 - c_2x^4y + (c_3 - c)x^3y^2 + c_1x^2y^3 + c_2xy^4 + c_3y^5, \text{ and} \\ dg &= d_1x^4y + (d_1 + d_2 + ad_2)x^3y^2 + (d_3 - d_2)x^2y^3 + (d_3 + d_2c)xy^4. \end{aligned}$$

The Sylvester resultant matrix of f and g is given by

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g) = \begin{bmatrix} 1 & a & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -c & 1 & 1 & 1 & a \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & c & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

(1) If m is even and n is even, then the matrix has the following form:

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix}
e_0 & 0 & 0 & \dots & 0 & l_0 & 0 & \dots & 0\\
e_1 & e_0+ae_1 & 0 & \dots & 0 & l_1 & l_0+al_1 & \dots & 0\\
e_2 & -e_1 & e_0 & \dots & 0 & l_2 & -l_1 & \dots & 0\\
e_3 & ce_1+e_2+ae_3 & e_1 & \dots & 0 & l_3 & cl_1+l_2+al_3 & \dots & 0\\
e_4 & -e_3 & e_4 & \dots & 0 & l_4 & -l_3 & \dots & 0\\
e_5 & ce_3+e_4+ae_5 & e_5 & \dots & 0 & l_5 & cl_3+l_4+al_5 & \dots & 0\\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots\\
e_{m-2} & -e_{m-3} & e_{m-4} & \dots & 0 & l_{n-3} & cl_{n-5}+l_{n-4}+al_{n-3} & \dots & 0\\
e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & e_{m-3} & \dots & 0 & l_{n-2} & -l_{n-3} & \dots & 0\\
e_m & -e_{m-1} & e_{m-2} & \dots & 0 & l_{n-1} & cl_{n-3}+l_{n-2}+al_{n-1} & \dots & 0\\
0 & ce_{m-1}+e_m & e_{m-1} & \dots & e_0 & l_n & -l_{n-1} & \dots & l_0\\
0 & 0 & e_m & \dots & e_1 & 0 & cl_{n-1}+l_n & \dots & l_n\\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & e_{m-2} & 0 & 0 & \dots & l_{n-2}\\
0 & 0 & 0 & \dots & e_{m-1} & 0 & 0 & \dots & l_{n-1}\\
0 & 0 & 0 & \dots & e_m & 0 & 0 & \dots & l_n
\end{bmatrix}. \tag{3.10}$$

(2) If m is even and n is odd, then the matrix has the following form:

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix}
e_0 & 0 & 0 & \dots & 0 & l_0 & al_0 & \dots & 0\\
e_1 & e_0+ae_1 & 0 & \dots & 0 & l_1 & -l_0 & \dots & 0\\
e_2 & -e_1 & e_0 & \dots & 0 & l_2 & cl_0+l_1+al_2 & \dots & 0\\
e_3 & ce_1+e_2+ae_3 & e_1 & \dots & 0 & l_3 & -l_2 & \dots & 0\\
e_4 & -e_3 & e_4 & \dots & 0 & l_4 & cl_2+l_3+al_4 & \dots & 0\\
e_5 & ce_3+e_4+ae_5 & e_5 & \dots & 0 & l_5 & -l_4 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots\\
e_{m-2} & -e_{m-3} & e_{m-4} & \dots & 0 & l_{n-3} & cl_{n-5}+l_{n-4}+al_{n-3} & \dots & 0\\
e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & e_{m-3} & \dots & 0 & l_{n-2} & -l_{n-3} & \dots & 0\\
e_m & -e_{m-1} & e_{m-2} & \dots & 0 & l_{n-1} & cl_{n-3}+l_{n-2}+al_{n-1} & \dots & 0\\
0 & ce_{m-1}+e_m & e_{m-1} & \dots & e_0 & l_n & -l_{n-1} & \dots & l_0\\
0 & 0 & e_m & \dots & e_1 & 0 & cl_{n-1}+l_n & \dots & l_n\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & e_{m-2} & 0 & 0 & \dots & l_{n-2}\\
0 & 0 & 0 & \dots & e_{m-1} & 0 & 0 & \dots & l_{n-1}\\
0 & 0 & 0 & \dots & e_m & 0 & 0 & \dots & l_n
\end{bmatrix}. \tag{3.11}$$

(3) If m is odd and n is even, then the matrix has the following form:

$$\operatorname{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix}
e_0 & ae_0 & 0 & 0 & \dots & 0 & l_0 & 0 & \dots & 0\\
e_1 & -e_0 & 0 & 0 & \dots & 0 & l_1 & l_0+al_1 & \dots & 0\\
e_2 & ce_0+e_1+ae_2 & e_0 & ae_0 & \dots & 0 & l_2 & -l_1 & \dots & 0\\
e_3 & -e_2 & e_1 & -e_0 & \dots & 0 & l_3 & cl_1+l_2+al_3 & \dots & 0\\
e_4 & ce_2+e_3+ae_4 & e_2 & ce_0+e_1+ae_2 & \dots & 0 & l_4 & -l_3 & \dots & 0\\
e_5 & -e_4 & e_3 & -e_2 & \dots & 0 & l_5 & cl_3+l_4+al_5 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\
e_{m-2} & -e_{m-3} & e_{m-4} & -ce_{m-5} & \dots & ae_0 & l_{n-3} & cl_{m-5}+l_{m-4}+al_{m-3} & \dots & 0\\
e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & e_{m-3} & ce_{m-5}+e_{m-4}+ae_{m-3} & \dots & -e_0 & l_{n-2} & -l_{n-3} & \dots & l_0\\
e_m & -e_{m-1} & e_{m-2} & -e_{m-3} & \dots & ce_0+e_1+ae_2 & l_{n-1} & cl_{n-3}+l_{n-2}+al_{n-1} & \dots & l_1\\
0 & ce_{m-1}+e_m & e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & \dots & -e_2 & l_n & -l_{n-1} & \dots & l_2\\
0 & 0 & e_m & -e_{m-1} & \dots & ce_2+e_3+ae_4 & 0 & cl_{n-1}+l_n & \dots & l_3\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\
0 & 0 & 0 & 0 & \dots & ce_{m-3}+e_{m-2}+ae_{m-1} & 0 & 0 & \dots & l_{n-2}\\
0 & 0 & 0 & 0 & \dots & -e_{m-1} & 0 & 0 & \dots & l_{n-1}\\
0 & 0 & 0 & 0 & \dots & ce_{m-1}+e_m & 0 & 0 & \dots & l_n
\end{bmatrix}. \tag{3.12}$$

(4) If m is odd and n is odd, then the matrix has the following form:

$$\operatorname{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix}
e_0 & ae_0 & 0 & 0 & \dots & 0 & l_0 & al_0 & \dots & 0\\
e_1 & -e_0 & 0 & 0 & \dots & 0 & l_1 & -l_0 & \dots & 0\\
e_2 & ce_0+e_1+ae_2 & e_0 & ae_0 & \dots & 0 & l_2 & cl_0+l_1+al_2 & \dots & 0\\
e_3 & -e_2 & e_1 & -e_0 & \dots & 0 & l_3 & -l_2 & \dots & 0\\
e_4 & ce_2+e_3+ae_4 & e_2 & ce_0+e_1+ae_2 & \dots & 0 & l_4 & cl_2+l_3+al_4 & \dots & 0\\
e_5 & -e_4 & e_3 & -e_2 & \dots & 0 & l_5 & -l_4 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\
e_{m-2} & -e_{m-3} & e_{m-4} & -ce_{m-5} & \dots & ae_0 & l_{n-3} & -l_{m-3} & \dots & 0\\
e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & e_{m-3} & ce_{m-5}+e_{m-4}+ae_{m-3} & \dots & -e_0 & l_{n-2} & cl_{n-3}+l_{n-2}+al_{n-1} & \dots & l_0\\
e_m & -e_{m-1} & e_{m-2} & -e_{m-3} & \dots & ce_0+e_1+ae_2 & l_{n-1} & -l_{n-1} & \dots & l_1\\
0 & ce_{m-1}+e_m & e_{m-1} & ce_{m-3}+e_{m-2}+ae_{m-1} & \dots & -e_2 & l_n & cl_{n-1}+l_m & \dots & l_2\\
0 & 0 & e_m & -e_{m-1} & \dots & ce_2+e_3+ae_4 & 0 & 0 & \dots & l_3\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\
0 & 0 & 0 & 0 & \dots & ce_{m-3}+e_{m-2}+ae_{m-1} & 0 & 0 & \dots & l_{n-2}\\
0 & 0 & 0 & 0 & \dots & -e_{m-1} & 0 & 0 & \dots & l_{n-1}\\
0 & 0 & 0 & 0 & \dots & ce_{m-1}+e_m & 0 & 0 & \dots & l_n
\end{bmatrix}. \tag{3.13}$$

3.2.2.3 CASE $Q(a,0,0)$

The Sylvester matrix $\mathrm{Syl}_{Q(a,0,0)}(f,g)$ is given by

$$\begin{bmatrix}
e_0 & ae_0 & a^2e_0 & \dots & a^{n-1}e_0 & l_0 & al_0 & a^2l_0 & \dots & a^{m-1}l_0 \\
e_1 & ae_1 & a^2e_1 & & a^{n-1}e_1 & l_1 & al_1 & a^2l_1 & & a^{m-1}l_1 \\
e_2 & ae_2 & a^2e_2 & & a^{n-1}e_2 & l_2 & al_2 & a^2l_2 & & a^{m-1}l_2 \\
e_3 & ae_3 & a^2e_3 & & a^{n-1}e_3 & l_3 & al_3 & a^2l_3 & \dots & a^{m-1}l_3 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{m-2} & ae_{m-2} & a^2e_{m-2} & & a^{n-1}e_{m-2} & l_{n-3} & al_{n-3} & a^2l_{n-3} & & a^{m-1}l_{n-3} \\
e_{m-1} & ae_{m-1} & a^2e_{m-1} & & a^{n-1}e_{m-1} & l_{n-2} & al_{n-2} & a^2l_{n-2} & & a^{m-1}l_{n-2} \\
e_m & 0 & 0 & & 0 & l_{n-1} & al_{n-1} & a^2l_{n-1} & & a^{m-1}l_{n-1} \\
0 & e_m & 0 & & 0 & l_n & 0 & 0 & & 0 \\
0 & 0 & e_m & & 0 & 0 & l_n & 0 & & 0 \\
0 & 0 & 0 & & 0 & 0 & 0 & l_n & & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \dots & e_m & 0 & 0 & 0 & \dots & l_n
\end{bmatrix}.$$

3.2.2.4 CASE $Q(0,b,0)$

The Sylvester matrix $\mathrm{Syl}_{Q(0,b,0)}(f,g)$ is given by

$$\begin{bmatrix}
e_0 & 0 & 0 & \cdots & l_0 & \cdots & 0 & 0 \\
e_1 & b^m e_0 & 0 & & l_1 & & 0 & 0 \\
e_2 & b^{m-1}e_1 & b^{2m}e_0 & & l_2 & & 0 & 0 \\
e_3 & b^{m-2}e_2 & b^{2(m-1)}e_1 & & l_3 & & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
e_{m-2} & b^3e_{m-3} & b^8e_{m-4} & & l_{n-1} & & 0 & 0 \\
e_{m-1} & b^2e_{m-2} & b^{15}e_{m-3} & & l_n & & 0 & 0 \\
e_m & be_{m-1} & b^4e_{m-2} & & 0 & & l_0b^{n(m-2)} & 0 \\
0 & e_m & b^2e_{m-1} & & 0 & & l_1b^{(n-1)(m-2)} & l_0b^{(n)(m-1)} \\
0 & 0 & e_m & & 0 & & l_2b^{(n-2)(m-2)} & l_1b^{(n-1)(m-1)} \\
0 & 0 & 0 & & 0 & & l_3b^{(n-3)(m-2)} & l_2b^{(n-2)(m-1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & & 0 & & l_{n-1}b^{2(m-2)} & l_{n-2}b^{2(m-1)} \\
0 & 0 & 0 & & 0 & & l_nb^{(m-2)} & l_{n-1}b^{(m-1)} \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & l_n
\end{bmatrix}. \tag{3.14}$$

3.2.2.5 RESULTANTS AND RIGHT COMMON FACTORS

It is well-known that the concept of *resultant* can be introduced by asking when two polynomials in the commutative polynomial ring $\Bbbk[x]$ have a common factor. Two important applications of resultant theory are elimination theory, and the proofs of *Extension theorem* and *Bezout's theorem* (see Cox et al. [CLO15, Chapter 3] for more details). In the commutative and some non-commutative cases, it has been proven that the existence of common factors is equivalent to the existence of polynomials which satisfy expression (3.8) [AC99, Eri08, MS10, RS10, ZYG14]. This implies that the Sylvester matrix and the notion of resultant encodes the existence of common factors (right factors in the noncommutative case) of a pair of polynomials and homogeneous polynomials in commutative multivariate algebras. Next, we explore whether the resultant is a complete criterion to decide the existence of common right factor for a pair of homogeneous polynomials in the quadratic algebra defined by the relation (1.13).

EXAMPLES 3.1. (i) Let $f = x^2+(1-b)xy-y^2 = (x-y)(x+y)$ and $g = x^2+(1+b)xy+y^2 = (x+y)^2$ be polynomials in $Q(0,b,0)$. Then $x+y$ is a common right factor of f and g, and according to expression (3.14), its Sylvester matrix is given by

$$\mathrm{Syl}_{Q(0,b,0)}(f,g) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1-b & b & 1+b & b \\ -1 & (1-b)b & 1 & (1+b)b \\ 0 & -b & 0 & b \end{bmatrix},$$

which implies that $\mathrm{Res}_{Q(0,b,0)}(f,g) = -4b^3+4b^2 = -4(b^2)(b-1)$. In this way, f and g have a common right factor but $\mathrm{Res}_{Q(0,b,0)}(f,g) = 0$.

(ii) Let us take $f(x,y) = e_0x^2+e_2y^2$, $g(x,y) = l_0x^2+l_2y^2$ be elements in $Q(a,0,0)$ with e_0, e_2, l_0 and l_2 being non-zero elements such that $e_0l_2 \neq e_0l_2$. Then

$$\mathrm{Syl}_{Q(a,0,0)}(f,g) = \begin{bmatrix} e_0 & ae_0 & l_0 & al_0 \\ 0 & 0 & 0 & 0 \\ e_2 & 0 & l_2 & 0 \\ 0 & ae_2 & 0 & al_2 \end{bmatrix},$$

whence $\mathrm{Res}_{Q(a,0,0)} = 0$. However, these polynomials do not have a common right factor. If there exists a polynomial $p(x,y) = b_0x+b_1y$ such that

$$\begin{aligned} e_0x^2+e_2y^2 &= (a_0x+a_1y)(b_0x+b_1y) \\ l_0x^2+l_2y^2 &= (c_0x+c_1y)(b_0x+b_1y), \end{aligned}$$

we can see that the system has a solution if and only if $e_0l_2 = e_0l_2$, a contradiction.

(iii) Consider the polynomials $f(x,y) = (1+a)x^2+(1+c)y^2 = (x+y)^2$ and $g(x,y) = (1-a)x^2+2xy+(1-c)y^2 = (x-y)(x+y)$ in the quadratic algebra $Q(a,-1,c)$. Then

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g) = \begin{bmatrix} 1+a & 0 & 1-a & 0 \\ 0 & 1+a & 2 & 1+a \\ 1+c & 0 & 1-c & -2 \\ 0 & 1+c & 0 & 1+c \end{bmatrix}.$$

This fact implies that $\mathrm{Res}_{Q(a,-1,c)}(f,g) = -4(a+ac+c+1) \neq 0$. Again, the fact that f and g

have common right factor does not imply that $\mathrm{Res}_{Q(a,-1,c)}(f,g)=0$. On the other hand, if we take the following polynomials $f(x,y)=x^2$ and $g(x,y)=xy$, it can be seen that

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix}1&0&0&0\\0&1&1&a\\0&0&0&-1\\0&0&0&c\end{bmatrix},$$

which implies that $\mathrm{Res}_{Q(a,-1,c)}=0$. However f and g cannot have a common right factor: all possible factorizations of x^2 are given by $x^2=(kx)(k^{-1}x)$ and $x^2=\frac{k}{k+aw}(x-cy)(x-cy)$, while the unique possible factorizations of xy are $xy=(kx)(k^{-1}y)$, so the assertion follows.

Theorem 3.12. *Let $f(x,y), g(x,y)$ be two polynomials in $Q(a,0,0)$ having the same total degree n. Then $\mathrm{Res}_{Q(a,0,0)}(f,g)=0$.*

Proof. Let f,g be two polynomials in $Q(a,0,0)$ such that $f(x,y)=\sum\limits_{i=0}^{n} e_i x^{n-i}y^i$ and $g(x,y)=\sum\limits_{j=0}^{n} l_j x^{n-j}y^j$. The Sylvester matrix of f and g, $\mathrm{Syl}_{Q(a,0,0)}(f,g)$, is a square matrix $2n\times 2n$ of the form

$$\begin{bmatrix}
e_0 & ae_0 & a^2e_0 & \dots & a^{n-1}e_0 & l_0 & al_0 & a^2l_0 & \dots & a^{n-1}l_0\\
e_1 & ae_1 & a^2e_1 & & a^{n-1}e_1 & l_1 & al_1 & a^2l_1 & & a^{n-1}l_1\\
e_2 & ae_2 & a^2e_2 & & a^{n-1}e_2 & l_2 & al_2 & a^2l_2 & & a^{n-1}l_2\\
e_3 & ae_3 & a^2e_3 & & a^{n-1}e_3 & l_3 & al_3 & a^2l_3 & \dots & a^{n-1}l_3\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots\\
e_{n-2} & ae_{n-2} & a^2e_{n-2} & & a^{n-1}e_{n-2} & l_{n-2} & al_{n-2} & a^2l_{n-2} & & a^{n-1}l_{n-2}\\
e_{n-1} & ae_{n-1} & a^2e_{n-1} & & a^{n-1}e_{n-1} & l_{n-1} & al_{n-1} & a^2l_{n-1} & & a^{n-1}l_{n-1}\\
e_n & 0 & 0 & & 0 & l_n & 0 & 0 & & 0\\
0 & e_n & 0 & & 0 & 0 & l_n & 0 & & 0\\
0 & 0 & e_n & & 0 & 0 & 0 & l_n & & 0\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \dots & e_n & 0 & 0 & 0 & \dots & l_n
\end{bmatrix}.$$

It is easy to see that some columns can be reduced to zero. For example, the second column has the coefficients of the polynomial $f(x,y)$ multiplied by a (except for the last coefficient), as it also happens with the column $n+2$ where we have the coefficients of $g(x,y)$ multiplied by a. Thus, the matrix can be reduced by operations between columns, in this case, multiplying the first column and column $n+1$ by a and subtracting with columns 2 and $n+2$, respectively. With this operation, we obtain the following equivalent matrix:

$$\begin{bmatrix} e_0 & 0 & a^2e_0 & \dots & a^{n-1}e_0 & l_0 & 0 & a^2l_0 & \dots & a^{n-1}l_0 \\ e_1 & 0 & a^2e_1 & & a^{n-1}e_1 & l_1 & 0 & a^2l_1 & & a^{n-1}l_1 \\ e_2 & 0 & a^2e_2 & & a^{n-1}e_2 & l_2 & 0 & a^2l_2 & & a^{n-1}l_2 \\ e_3 & 0 & a^2e_3 & & a^{n-1}e_3 & l_3 & 0 & a^2l_3 & \dots & a^{n-1}l_3 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{n-2} & 0 & a^2e_{n-2} & & a^{n-1}e_{n-2} & l_{n-2} & 0 & a^2l_{n-2} & & a^{n-1}l_{n-2} \\ e_{n-1} & 0 & a^2e_{n-1} & & a^{n-1}e_{n-1} & l_{n-1} & 0 & a^2l_{n-1} & & a^{n-1}l_{n-1} \\ e_n & 0 & 0 & & 0 & l_n & 0 & 0 & & 0 \\ 0 & e_n & 0 & & 0 & 0 & l_n & 0 & & 0 \\ 0 & 0 & e_n & & 0 & 0 & 0 & l_n & & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & e_n & 0 & 0 & 0 & \dots & l_n \end{bmatrix}.$$

Now, subtracting the columns 2 and $n+2$ multiplied by e_n/l_n, we can reduce the second column to zero. Thus, after some operations, the Sylvester matrix is equivalent to a matrix that has a column with zero entries, so that its determinant is zero, i.e. $\mathrm{Res}_{Q(a,0,0)}(f,g)=0$. □

REMARK 15. We compare the form of the Sylvester matrix for the same values of f and g seeing how different can be according to the values of the parameters a, b and c. Let us take:

$$\begin{aligned} f(x,y) &= e_0x^3+e_1x^2y+e_2xy^2+e_3y^3 \\ g(x,y) &= l_0x^2+l_1xy+l_2y^2. \end{aligned}$$

In this way, the matrix has the following form in every possibility:

$$\mathrm{Syl}_{Q(a,0,0)}(f,g)=\begin{bmatrix} e_0 & ae_0 & l_0 & al_0 & a^2l_0 \\ e_1 & ae_1 & l_1 & al_1 & a^2l_1 \\ e_2 & ae_2 & l_2 & 0 & a^2l_2 \\ e_3 & 0 & 0 & l_2 & 0 \\ 0 & e_3 & 0 & 0 & l_2 \end{bmatrix},$$

$$\mathrm{Syl}_{Q(0,b,0)}(f,g)=\begin{bmatrix} e_0 & 0 & l_0 & 0 & 0 \\ e_1 & b^3e_0 & l_1 & b^2l_0 & 0 \\ e_2 & b^2e_1 & l_2 & bl_1 & b^4l_0 \\ e_3 & be_2 & 0 & l_2 & b^2l_1 \\ 0 & e_3 & 0 & 0 & l_2 \end{bmatrix},$$

$$\mathrm{Syl}_{Q(a,-1,c)}(f,g)=\begin{bmatrix} e_0 & e_0a & l_0 & 0 & 0 \\ e_1 & -e_0 & l_1 & l_0+al_1 & 0 \\ e_2 & e_0c+e_1+e_2a & l_2 & -l_1 & l_0 \\ e_3 & -e_2 & 0 & b+l_2 & l_1 \\ 0 & e_2+e_3 & 0 & 0 & l_2 \end{bmatrix}.$$

3.2.3 Burchnall-Chaundy theory

Following [Lar14, Section 3], we generalize the notion of resultant to the context of quadratic algebras. The key fact of our treatment is that $Q(a,b,c)$ is an $\Bbbk$-algebra finitely generated over $\Bbbk[x]$ because the set $\{x^i y^j \mid i,j \geq 0\}$ is a PBW basis. The following definition is analog to the concept of determinant polynomial found in [Lar14, section 3], which is a matrix concept that is, in fact, independent of the noncommutative structure of algebra.

Definition 3.5 ([Li96, Definition 1.3.1]; [Mis93, p. 241]). Let $M \in M_{r\times c}(\Bbbk[x])$. Then we define the *determinant polynomial of* M, denoted by $|M|$, in the following way:

$$|M| = \sum_{i=0}^{c-r} \det(M_i)y^i,$$

where M_i is the square matrix that satisfies the following properties:

(i) The first $r-1$ columns of M_i are the same that the first $r-1$ columns of M.

(ii) The last column of M_i is the $(c-i)$th column of M.

An important remark is that the following proposition remains valid for $Q(a,b,c)$. This is due to the fact that the calculations involved in the determinant will be done under certain assumptions about the order in which the multiplications are done.

Proposition 3.13 ([Ric14b, Proposition B.2.1]). *Let $M \in M_{r\times c}(\Bbbk[x])$ with determinant polynomial $|M|$. For $i = 1,\ldots,r$, if H_i is the polynomial given by*

$$H_i = m_{i1}y^{c-1} + \cdots + m_{ir}y^{c-r} + \cdots + m_{ic},$$

then

$$|M| = \det\begin{bmatrix} m_{1,d} & \ldots & m_{1,d-r+1} & H_1 \\ m_{2,d} & \ldots & m_{2,d-r+1} & H_2 \\ \vdots & \ldots & \vdots & \vdots \\ m_{r,d} & \ldots & m_{r,d-r+1} & H_r \end{bmatrix}.$$

In the expansion of this determinant, the elements not involving y will be always multiplied from the left. This will be denoted as $\texttt{mult}(ay^i, x) := xay^i$. In this way, there is no any problem with the expansion of this determinant.

Having in mind [Lar14, Definition 3.1], we introduce the concept of determinant polynomial associated to a sequence of polynomials in $Q(a,b,c)$.

Definition 3.6 ([Ric14b, Section B.2.1]). Let $G := (g_1, g_2, \ldots, g_r)$ be a sequence of polynomials in $Q(a,b,c)$, and let d be the maximum degree of the polynomials. Assume $d \geq r$. We define the *matrix of size* $r \times (d+1)$, denoted by $M(G)$, whose entry in the ith row and jth column is the coefficient of the monomial y^{d+1-j} in g_j. The determinant polynomial of G is $|M(G)|$ and it is denoted by $|G|$.

The next proposition describes an easy way to calculate the determinant polynomial of a sequence of polynomials in $Q(a,b,c)$.

PROPOSITION 3.14 ([RIC14B, PROPOSITION B.2.2]). *Let $G := (g_1, g_2, \ldots, g_r)$ be a sequence of elements of $Q(a,b,c)$ of maximum degree d with respect to y. Then*

$$|G| = \det \begin{bmatrix} a_{1,d} & \cdots & a_{1,d-r+1} & g_1 \\ a_{2,d} & \cdots & a_{2,d-r+1} & g_2 \\ \vdots & \cdots & \vdots & \vdots \\ a_{r,d} & \cdots & a_{r,d-r+1} & g_r \end{bmatrix}.$$

where $a_{i,j}$ is the coefficient of y^j in $g_j(x,y)$ written in its normal form.

Proof. This proposition is formulated within the same assumptions as in [Li98, p. 133]. The assertion follows from Proposition 3.13 since the matrix entries correspond to the coefficients of the given polynomials. □

The following definition is motivated by Richter and Silvestrov [RS12, Section 2.2].

DEFINITION 3.7. Let f and g be two elements of $Q(a,b,c)$ of degree n and m, respectively. The *resultant of f and g*, denoted $\mathtt{Res}(f,g)$, is the determinant polynomial of the sequence

$$f, yf, y^2 f, \ldots, y^{m-1} f, g, yg, y^2 g, \ldots, y^{n-1} g.$$

An important consequence that can be visualized is the following property.

PROPOSITION 3.15. *For f and g polynomials in $Q(a,b,c)$, $\mathtt{Res}(f,g) = F_1(x) f + F_2(x) g$.*

Proof.

$$\mathtt{Res}(f,g) = \det \begin{bmatrix} a_{1,d} & \cdots & a_{1,d-r+1} & f \\ a_{2,d} & \cdots & a_{2,d-r+1} & yf \\ \vdots & \cdots & \vdots & \vdots \\ a_{r,d} & \cdots & a_{r,d-r+1} & y^{n-1} f \\ b_{1,d} & \cdots & b_{1,d-r+1} & g \\ b_{2,d} & \cdots & b_{2,d-r+1} & yg \\ \vdots & \cdots & \vdots & \vdots \\ b_{r,d} & \cdots & b_{r,d-r+1} & y^{m-1} g \end{bmatrix}.$$

The expansion of this determinant along the last column shows the result has the expression that we are describing. □

The most important application for our interest is the following result, which establish a version of $\mathscr{BC}$ theory for the cases of $Q(a,b,c)$ that we are considering.

THEOREM 3.16. *Let f and g be two polynomials such that $fg = gf$. Then there exists a polynomial $F(s,t)$ such that $F(f,g) = 0$.*

Proof. Let s, t be two new indeterminates such that they commute with all the elements in $Q(a,b,c)$. Let us establish that the s and t have degree zero, then according to Proposition 3.15, the following equality holds for some polynomials:

$$F(s,t) = \texttt{Res}(f-s, g-t) = F_1(x)(f-s) + F_2(x)(g-t)$$

Since $fg = gf$, it is not ambiguous the evaluation of f and g in $F(s,t)$, and then, the fact that $F(f,g) = 0$. □

As we have mentioned previously, our approach is fully inspired by exploring the application of the ideas exposed by Li [Li98] in our context. It is worth noting that Li develops a theory not only of resultants but more generally about subresultants. In fact, the concept of subresultant generalizes what we have described regarding the notion of a resultant, as we can see in the following definition.

DEFINITION 3.8. Let $f(x,y)$ and $g(x,y)$ be two polynomials in $Q(a,b,c)$ of degree n and m respectively, with respect to the variable y. Suppose $n \geq m$, the lth *subresultant* of f and g is

$$\texttt{sRes}_l(f,g) := |y^{m-l-1}f, \ldots, yf, f, y^{n-l-1}g, \ldots, yg, g|$$

The 0th subresultant $\texttt{sRes}_0(f,g)$ is in fact $\texttt{Res}(f,g)$.

3.2.4 EXAMPLES

We are going to present some illustrative examples of the results obtained in the previous sections.

EXAMPLE 3.11. Let us take the polynomials $f = x^2y^2, g = (x^3+x)y^2 \in Q(a,0,0)$. It can be seen that $fg = gf$, whence

$$\begin{aligned}
F(s,t) &= \texttt{Res}(f-s, g-t)\\
&= \det\begin{bmatrix} -s & 0 & x^2 & x^2y^2 - s \\ 0 & -s & ax^3 & y(x^2y^2-s) \\ -t & 0 & x^3+x & (x^3+x)y^2 - t \\ 0 & -t & a(x^4+x^2) & y\left((x^3+x)y^2-t\right) \end{bmatrix}\\
&= \det\begin{bmatrix} -s & 0 & x^2 & (x^2y^2 - s) \\ 0 & -s & ax^3 & ax^3y^2 - sy \\ -t & 0 & x^3+x & (x^3+x)y^2 - t \\ 0 & -t & a(x^4+x^2) & a(x^4+x^2)y^2 - ty \end{bmatrix}\\
&= 0.
\end{aligned}$$

EXAMPLE 3.12. For the polynomials $f = y^2, g = (x^2+1)y^2+1$ in $Q(a,0,0)$, we have $fg = gf =$, and

$$\begin{aligned}
F(s,t) &= \mathtt{Res}(f-s, g-t)\\
&= \det\begin{bmatrix} -s & 0 & 1 & y^2-s\\ 0 & -s & 0 & y^3-ys\\ 1-t & 0 & x+1 & (x+1)y^2+(1-t)\\ 0 & (1-t) & ax^2 & y^3+ax^2y^2+(1-t)y \end{bmatrix}\\
&= s^2y^3+2sy^3+t^2y^3-2sty^3-2ty^3+s^2xy^3+sxy^3-stxy^3+y^3\\
&= (s^2+2s+t^2-2st-2t+s^2x+sx-stx+1)y^3.
\end{aligned}$$

The previous examples give us some ideas about the behavior of the resultant in $Q(a,0,0)$. Next, we give sufficient conditions on $f(x,y)$ and $g(x,y)$ for $\mathtt{Res}(f,g)=0$. First, we mention a case where the resultant is non-zero.

PROPOSITION 3.17. *For any pair of commuting elements of degree 1 in* $Q(a,0,0)$, $F(s,t) = Res(f-s, g-t) \neq 0$.

Proof. Let $f = p_0 + p_1 y$ and $g = q_0 + q_1 y$ be two commuting elements in $Q(a,0,0)$. Then

$$\begin{aligned}
F(s,t) = \mathtt{Res}(f-s,g-t) = \det\begin{bmatrix} p_0-s & p_0-s+p_1y\\ q_0-t & q_0-t+q_1y\end{bmatrix}\\
&= (p_0-s)(q_0-t+q_1y)-(q_0-t)(p_0-s+p_1y)\\
&= (p_1t-q_1s+p_0q_1-q_0p_1)y.
\end{aligned}$$

Since y is a non-zero divisor, if $F(s,t) := p_1t-q_1s+p_0q_1-q_0p_1$, then

$$\begin{aligned}
F(f,g) &= p_1(q_0+q_1y)-q_1(p_0+p_1y)+p_0q_1-q_0p_1\\
&= p_1q_0+p_1q_1y-q_1p_0-q_1p_1y+p_0q_1-q_0p_1 = 0.
\end{aligned}$$

□

REMARK 16. In the proof of Proposition 3.17 we do not use the noncommutative structure of the algebra; it only uses that polynomials in the indeterminate x are commutative which holds in every algebra of our interest in the paper.

LEMMA 3.18. *Let* $f(x,y) = \sum_{i=0}^{n} p_i(x)y^i$ *be a polynomial of degree* m *with respect to the indeterminate* y *in* $Q(a,0,0)$. *If* x *is a factor of the polynomial* $p_n(x)$, *then* $y^m f(x,y)$ *is a polynomial of degree less than* $m+n$ *with respect to* y.

Proof. Since x is a factor of $p_n(x)$, there exists a nonzero polynomial $r(x)$ such that $p_n(x) = xr(x)$.

Then

$$
\begin{aligned}
y^m f(x,y) = y^m\left(\sum_{i=0}^{n} p_i(x)y^i\right) &= \sum_{i=0}^{n} y^m p_i(x)y^i = y^m p_n(x)y^n + \sum_{i=0}^{n-1} y^m p_i(x)y^i\\
&= y^m x r(x) y^n + \sum_{i=0}^{n-1} y^m p_i(x)y^i = a^m x^{m+1} r(x)y^n + \sum_{i=0}^{n-1} y^m p_i(x)y^i,
\end{aligned}
$$

since the polynomial $\sum_{i=0}^{n-1} y^m p_i(x)y^i$ has degree at most $m+(n-1)$ with respect to y. It follows that $y^m f(x,y)$ has degree at most $m+(n-1)$. □

THEOREM 3.19. *Let $f(x,y) = \sum_{i=0}^{n} p_i(x)y^i$ and $g(x,y) = \sum_{j=0}^{m} q_j(x)y^j$ be two polynomials of $Q(a,0,0)$ with $m > 1$ presented in their normal form. If x is a factor of $p_n(x)$ and $q_m(x)$, then* $\texttt{Res}(f,g) = 0$.

Proof. Let $f(x,y) = \sum_{i=0}^{n} p_i(x)y^i$ and $\sum_{j=0}^{m} q_j(x)y^j$ be two polynomials presented in their normal form. Since we assume the degree of g greater than 1, this case is not mentioned in Proposition 3.17. Now, from the definition, we can compute $\texttt{Res}(f,g)$ as the determinant polynomial of the matrix. In the following matrix, the expression $\pi_i(p(x,y))$ denotes the coefficient of the monomial y^i of $p(x,y)$ written in its normal form.

$$
\begin{bmatrix}
p_0 & p_1 & \cdots & p_n & 0 & 0 & \cdots & 0\\
\pi_0(yf) & \pi_1(yf) & \cdots & \pi_n(yf) & \pi_{n+1}(yf) & 0 & \cdots & 0\\
\pi_0(y^2f) & \pi_1(y^2f) & \cdots & \pi_n(y^2f) & \pi_{n+1}(y^2f) & \pi_{n+2}(y^2f) & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\pi_0(y^{m-1}f) & \pi_1(y^{m-1}f) & \cdots & \pi_n(y^{m-1}f) & \pi_{n+1}(y^{m-1}f) & \pi_{n+2}(y^{m-1}f) & \cdots & \pi_{n+m-1}(y^{m-1}f)\\
q_0 & q_1 & \cdots & q_n & q_{n+1} & 0 & \cdots & 0\\
\pi_0(yg) & \pi_1(yg) & \cdots & \pi_n(yg) & \pi_{n+1}(yg) & 0 & \cdots & 0\\
\pi_0(y^2g) & \pi_1(y^2g) & \cdots & \pi_n(y^2g) & \pi_{n+1}(y^2g) & \pi_{n+2}(y^2g) & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\pi_0(y^{n-1}g) & \pi_1(y^{n-1}g) & \cdots & \pi_n(y^{n-1}g) & \pi_{n+1}(y^{n-1}g) & \pi_{n+2}(y^{n-1}g) & \cdots & \pi_{n+m-1}(y^{n-1}g)
\end{bmatrix}.
$$

By using Lemma 3.18, we conclude that the entries along the last column are zero. This fact implies that the resultant is zero. □

EXAMPLE 3.13. Consider the polynomials $f = x^2y^2 + xy, g = xy \in Q(0,b,0)$. Then

$$
\begin{aligned}
fg &= \left(x^2y^2 + xy\right)\left(xy\right) = b^2x^3y^3 + bx^2y^2, \quad \text{and}\\
gf &= \left(xy\right)\left(x^2y^2 + xy\right) = b^2x^3y^3 + bx^2y^2.
\end{aligned}
$$

Hence,

$$
F(s,t) = \texttt{Res}(f-s, g-t) = \det\begin{bmatrix} -s & x & x^2y^2 + xy - s\\ -t & x & xy - t\\ 0 & -t & bxy^2 - ty\end{bmatrix} = \left(-bs + bt + t^2\right)x^2y^2.
$$

Since $x^2y^2 \neq 0$, it is easy to see that $F(s,t) = t^2 - bs + bt$, and so

$$\begin{aligned} F(f,g) &= (xy)^2 - b(x^2y^2 + xy) + b(xy) \\ &= bx^2y^2 - bx^2y^2 - bxy + bxy = 0. \end{aligned}$$

EXAMPLE 3.14. Let $f = y^2, g = x^2y^2 + x^2y \in Q(a,-1,c)$. It follows that

$$\begin{aligned} fg &= \left(y^2\right)\left(x^2y^2 + x^2y\right) = x^2y^4 + x^2y^3, \quad \text{and} \\ gf &= \left(x^2y^2 + x^2y\right)\left(y^2\right) = x^2y^4 + x^2y^3, \end{aligned}$$

whence

$$\begin{aligned} F(s,t) &= \mathtt{Res}(f-s, g-t) = \det \begin{bmatrix} -s & 0 & 0 & y^2 - s \\ 0 & -s & 0 & y^3 - sy \\ -t & x^2 & x^2 & x^2y^2 + x^2y - t \\ 0 & -t & x^2 & x^2y^3 + x^2y^2 - ty \end{bmatrix} \\ &= s^2x^4y^3 - sx^4y^3 - stx^2y^3 + stx^2y^2. \end{aligned}$$

After some computations, $F(s,t) = x^2ys - (y-1)t - x^2y$. It can be seen that

$$F(f,g) = x^2y^3 - (y-1)(x^2y^2 + x^2y) - x^2y = 0.$$

This example allows us to illustrate the lth subresultant of f and g with $l = 1$:

$$F_1(s,t) = \mathtt{sRes}_1(f-s, g-t) = \det \begin{bmatrix} -s & y^2 - s \\ -t & x^2y^2 + x^2y - t \end{bmatrix} = ty - x^2(y+1)s.$$

By replacing f and g, we get $F_1(f,g) = (x^2y^2)y - x^2(y+1)y^2 = 0$.

Examples 3.12 and 3.14 show an interesting behavior with respect to the coefficients of the curve. The same behavior can be seen in [Lar14, Example 5.3] in the sense that the curve is not defined strictly on the coefficient field $\Bbbk$. A natural question would be to determine in which cases the curve must have coefficients in $\Bbbk$.

3.3 CENTRALIZERS IN SEMI-GRADED RINGS

In this short section, we consider some ideas with the aim of extending some of the results presented by Bell et al. [BS04, BR16] for graded rings to the setting of semi-graded rings. We focus on the key results in this process.

Recall that if $A = \bigoplus_{n=0}^{\infty} A_n$ is a finitely generated $\mathbb{N}$-graded $\Bbbk$-algebra, then A is said to be *finitely graded* if $\dim_{\Bbbk}(A_n) < \infty$ for all $n \geq 0$.

(1) Let A be a $\mathbb{Z}$-graded ring with at least one homogeneous, regular element in positive degree.

Then A is a *graded division ring* if every homogeneous element is invertible. Năstăsescu and Van Oystaeyen [NvO82, A. I.4.3] showed that this graded division ring A has the form of a skew Laurent polynomial ring $A = D[z^{\pm 1}; \sigma]$, where σ is an automorphism of a division ring D and z is an element of positive degree.

(2) By using (1), Artin and Stafford [AS95, Theorem 0.1] showed that if A is a finitely graded Goldie domain of Gelfand-Kirillov dimension two, then A has a graded quotient ring $Q_{\rm gr}(A) \cong D[x^{\pm 1}; \sigma]$ for some division ring D which is a finite module over its center and some automorphism σ of D.

(3) The subfields of $Q(D[x^{\pm 1}; \sigma])$ are finitely generated [RSW79, Theorem 3].

(4) From (2) and (3), we get that for A a finitely graded Goldie domain of Gelfand-Kirillov dimension two, the subfields of $Q(A)$ are finitely generated [BS04, Lemma 3.3].

(5) Following the ideas presented by McConnell and Robson [MR01, Corollary 6.6.7 and Theorem 6.6.10], Bell and Small [BS04, Lemma 3.2] proved that for A a prime Noetherian polynomial identity ring that is not primitive, and σ an automorphism of A such that $A[x^{\pm 1}; \sigma]$ is simple, then
$$\mathscr{K}(A) = \mathscr{K}(A[x^{\pm 1}; \sigma]),$$
where $\mathscr{K}(-)$ denotes Krull dimension.

(6) Let A be a finitely graded non-polynomial identity domain of Gelfand-Kirillov dimension two over a field $\Bbbk$. Then $Q_{\rm gr}(A)$ is a simple $\Bbbk$-algebra and $Z(Q_{\rm gr}(A))$ is a finite extension of F consisting of homogeneous elements of degree zero [BS04, Proposition 3.4].

(7) Let A be a finitely graded non-polynomial identity domain of Gelfand-Kirillov dimension 2 over a field $\Bbbk$. Then any subfield of $Q(A)$ has transcendence degree at most 1 over $\Bbbk$.

(8) As we saw in Proposition 2.56, Bell and Smith's main result [BS04, Theorem 3.6] says that for A a finitely graded non-polynomial identity domain of Gelfand-Kirillov dimension two over an algebraically closed field $\Bbbk$, the centralizer of a non-scalar element $a \in Q(A)$ is a finitely generated field extension of $\Bbbk$ of transcendence degree one.

Thinking about a version of Proposition 2.56 in the semi-graded case, we consider useful to study the papers on the Gelfand-Kirillov dimension. For example, Reyes [Rey13a] computed the Gelfand-Kirillov dimension of skew PBW extensions (Section 1.2.9), while Bell and Zhang [BZ17] extended this dimension to algebras over commutative domains, and having in mind the examples of semi-graded rings considered in Chapter 1, this could be a good idea. In fact, Lezama and Venegas [LV20c] investigated the Gelfand–Kirillov transcendence degree to algebras over the ring of integers. We believe that all these works are relevant for our interests.

As seen above, it is also necessary to characterize the semi-graded ring of fractions of a $\mathbb{Z}$-semi-graded Ore domain R, and study its relations with its center (see the proof of Proposition 2.56). For R a right Ore domain and $Q(R)$ its total ring of fractions, Lezama and Venegas [LV20b] proved that if R is finitely generated and ${\rm GKdim}(R) < {\rm GKdim}(Z(R)) + 1$, then $Z(Q(R)) \cong Q(Z(R))$. As they showed, different examples of skew PBW extensions illustrate this result (see [Ven20] for more details). Related to this, recall that for a right Noetherian ring R, Goldie's theorem asserts that the ring of fractions of R is a matrix ring over a division ring, so for several of the examples in Chapter 1 this result can be applied. On the other hand, it can be useful the well-known fact that the injective envelope of a $\mathbb{N}$-graded ring R is its graded ring of fractions. If we consider this possibility, then the characterization of these envelopes in the semi-grade case is an immediate task.

As of today, we have a paper with some advances in this direction, so we trust that for the day of the dissertation we can present the desired results.

3.4 Future work

We are interested in the interactions between geometric, analytic and algebraic interpretations of $\mathscr{BC}$ theory in the sense of Mumford [Mum77, vMM79], Larsson [Lar14], and Previato [PW92, Pre96]. For example, Larsson investigated the problem of commuting operators from a geometric perspective by considering the notion of *Picard-Vessiot ring*. In the same way, and more recently, Previato et al. [PSZ23] defined the *matrix differential resultant* and use it to compute the $\mathscr{BC}$ curve $\mathscr{BC}$ of a pair of commuting matrix ordinary differential operators (MODOs). They proved that this resultant provides the necessary and sufficient condition for the spectral problem to have a solution, and then by considering the Picard–Vessiot theory (c.f. [Lar14]), they described explicitly isomorphisms between commutative rings of (MODOs) and a finite product of rings of irreducible algebraic curves. It is interesting to investigate the matrix differential resultant and its consequences for the quadratic algebras considered in this paper.

Since our first version of homogeneous determinants considered in Section 3.2.2 is not a criterion for determining the existence of common right factors, we consider interesting to explore some other technique that can improve this first approach. On the other hand, Rosengren [Ros00] proved a *binomial formula* for the quadratic algebras studied in Section 3.2, so we consider interesting to continue investigating a $\mathscr{BC}$ theory for these algebras that is as general as possible. Also, the research on $\mathscr{BC}$ theory for noncommutative algebras defined by two indeterminates can be continued for *PBW deformations of Artin-Schelter regular algebras* investigated by Gaddis in his PhD thesis [Gad13, Gad16] and other families of semi-graded rings.

On centralizers, we consider important to characterize centralizers of differential operators in the sense of $\mathscr{BC}$ bundles following the ideas presented by Makar-Limanov and Previato [MLP23, Pre98]. As we can see in the literature, this is an active area of research of the noncommutative geometry.

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