Existence of global bounded weak solutions to systems of Keyfitz-Kranzer type with a source

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Abstract

In this thesis we prove the existence of a weak solution for two Cauchy problems associated with 2×2 systems of Keyfitz-Kranzer type with certain source terms and bounded measurable initial data. One is a symmetric system of Keyfitz-Kranzer type and the other is the Aw-Rascle model for traffic flow which is a non-symmetric system of Keyfitz-Kranzer type. For it, in both cases, we obtain an estimate in $L^1(\mathbb{R})$ related to one of the Riemann invariants, then using this estimate we apply the compensated compactness method to prove the pointwise convergence of the viscosity solutions. For the above problems in the absence of source terms we obtain as a particular case the existence of weak entropy solution. Finally we study a $(n+1) \times (n+1)$ non-symmetric system of Keyfitz-Kranzer type with a particular source term.

Keywords: Systems of Keyfitz-Kranzer type, Aw-Rascle model, source terms, weak solution, weak entropy solution.

Resumen

En esta tesis probamos la existencia de solución débil para dos problemas de Cauchy asociados con sistemas 2×2 de tipo Keyfitz-Kranzer con ciertos términos fuente y valores iniciales acotados y medibles. Uno es un sistema de tipo Keyfitz-Kranzer simétrico y el otro es el modelo Aw-Rascle para flujo de tráfico el cual es un sistema de tipo Keyfitz-kranzer no simétrico. Para esto, en ambos casos, obtenemos una $L^1(\mathbb{R})$ estimativa relacionada con uno de los invariantes de Riemann, entonces usando esta estimativa aplicamos el método de la compacidad compensada para probar la convergencia puntual de las soluciones viscosas. Para los problemas anteriores en ausencia de términos fuente obtenemos como un caso particular la existencia de solución débil entrópica. Finalmente estudiamos un sistema $(n+1) \times (n+1)$ de tipo keyfitz-kranzer no simétrico con un término fuente particular.

Palabras clave: Sistemas de tipo Keyfitz-Kranzer, modelo Aw-Rascle, términos fuente, solución débil, solución débil entrópica.

Contents

List of symbols

C^2	space of functions with continuous derivates of orders ≤ 2
C_0^∞	space of infinitely differentiable functions with compact support
L^∞	space of measurable functions which are bounded almost everywhere
L^1	space of measurable functions which are Lebesgue integrable
L^1_{loc}	space of functions which are locally integrable
$meas\,\{\}$	measure of the set $\{\}$
$\mathcal{M}$	space of Radon measures
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$TV(u)$	total variation of u
$\overline{u^\epsilon}$	weak-star limit
$W^{m,p}$	space of all functions in L^p such that all distribution derivatives upto order m are also in L^p
$W_0^{m,p}$	closure of C_0^∞ in $W^{m,p}$
H^{-1}	dual space of $H_0^1 = W_0^{1,2}$
$z(\cdot, t)$	function $z(x,t)$, where t is fixed
∇	the gradient operador

1 Introduction

A $n \times n$ system of Keyfitz-Kranzer type is a $n \times n$ system of partial differential equations of the following form

$$(u_i)_t + \big(u_i \phi(u_1, \ldots, u_n)\big)_x = 0, \quad i = 1, \ldots, n. \tag{0.1}$$

This type of system was first introduced for two equations by Barbara L. Keyfitz and Herbert C. Kranzer in [10] as a model of an elastic string in the plane and was almost one of the first examples of nonstrictly hyperbolic systems. Systems of the form (0.1) appear in areas as elasticity theory ([10]) and magnetohydrodynamics ([8]).

The symmetric system of Keyfitz-Kranzer type where the function ϕ is of the form

$$\phi(u_1, \ldots, u_n) = \phi(r), \qquad r = \sum_{i=1}^{n} u_i^2\,,$$

has been studied by different authors, see for example [9, 10, 21, 4, 6, 14, 17]. When the symmetric function ϕ is given by

$$\phi(u_1, \ldots, u_n) = \phi(r), \qquad r = \sum_{i=1}^{n} |u_i|,$$

the system (0.1) is the system of multicomponent chromatography studied in [8].

An example of a non-symmetric system of Keyfitz-Kranzer type is the known system of two equations proposed by A. Aw and M. Rascle in [2] for traffic flow

$$\begin{cases} \rho_t + \big(\rho v\big)_x = 0 \\ (\rho u)_t + \big(\rho u v\big)_x = 0, \end{cases} \tag{0.2}$$

here ρ and $v = u - p(\rho)$ denote, respectively, the density and the velocity of cars on the roadway and $p(\rho)$ is a smooth strictly increasing function. In [30] Zhang independently proposed the same model. Aw and Rascle in their paper [2] studied the Riemann problem (i.e., a Cauchy problem with piecewise constant initial data) for the system (0.2).

In [7], the authors show the existence of a weak entropy solution of the Cauchy problem for system (0.2) with initial data taking values in a domain $\mathcal{D}_V$, provided that the initial

Riemann invariants be in $BV(\mathbb{R})$, the Bounded Variation Space function, and take values in $\mathcal{D}_V$, $p(\rho)$ satisfies the same conditions given in [2] together with the following

$$p'(0) = 0, \quad p'(\rho) > 0 \text{ for } \rho > 0 \quad \text{and} \quad \big|p(\rho_1) - p(\rho_2)\big| \leq L|\rho_1 - \rho_2|,$$

for some constant L.

An improved version of the model (0.2) includes a relaxation term (a source term) in the second equation, in [7] also the existence of a weak entropy solution is proved for the Cauchy problem associated with this improved version and initial data in $\mathcal{D}_V$. Their result is valid for another source term introduced in [27] by Siebel and Mauser to the second equation in the Aw-Rascle model, as well as for the specific choice of the relaxation term as given in [25] under other extra conditions. In [3] the Aw-Rascle model is extended to include a source term that models a highway entry. The Zhang version of the Aw-Rascle model with relaxation is studied in [12] and [13]. Features due to inhomogeneities of the roadway such as entries, exists and changes in the traffic speed are introduced in the models for traffic flow as different choices of source terms.

The objective of this thesis is to establish the existence of weak solutions for two Cauchy problems associated with 2×2 systems of Keyfitz-Kranzer type with a source term and in both cases with bounded and measurable initial data. The first problem considered is relative to a symmetric system of Keyfitz-Kranzer type, problem for which $r = u^{2n} + v^{2n}$ where n is any fixed positive integer, and the second to a non-symmetric system of Keyfitz-Kranzer type, namely the so-called Aw-Rascle model (0.2).

We illustrate the content of the chapters of this thesis. For more details the reader is referred to the introduction of each chapter.
In chapter 2, we consider the Cauchy problem for a 2×2 symmetric system of the form (0.1) with a source term, where $\phi(u, v) = \phi(r)$, $r = u^{2n} + v^{2n}$, n is a fixed positive integer, and the initial data is bounded and measurable. We prove the existence of weak solution for this problem, by using classical viscosity, an estimate in $L^1(\mathbb{R})$ of one of the Riemann invariants and the div-curl lemma, but avoiding the use of Young measures.
Starting from Chapter 3 we begin to consider a non-symmetric system of Keyfitz-Kranzer type, more specifically we deal with the Cauchy problem for the Aw-Rascle model of traffic flow ([2]) with a source. With the same conditions on $p(\rho)$ given by Aw and Rascle in [2] we establish the existence of weak solution.
Under new conditions on the function $p(\rho)$, in chapter 4, we given a result on the existence of weak solution for the Cauchy problem associated with the Aw-Rascle model. To this end we introduce a technique due to Lu ([17]).
In chapter 5, we state a theorem on the existence of weak solution for a $(n + 1) \times (n + 1)$ non-symmetric system of Keyfitz-Kranzer type with source term.

2 A symmetric system of Keyfitz-Kranzer type

This chapter is devoted to the study of the Cauchy problem for a 2×2 symmetric system of Keyfitz-Kranzer type. Section 2.1 deals the case in which the system has source terms, namely

$$\begin{cases} u_t + \big(u\phi(r)\big)_x + g_1(u,v) = 0 \\ v_t + \big(v\phi(r)\big)_x + g_2(u,v) = 0, \end{cases} \tag{2-1}$$

with initial data

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big), \tag{2-2}$$

where $g_i(u,v)$, $i = 1, 2$ are locally Lipschitz continuous functions, $u_0(x), v_0(x) \in L^\infty(\mathbb{R})$, ϕ is a nonlinear smooth function and r is given by

$$r = u^{2n} + v^{2n}, \tag{2-3}$$

for any n fixed positive integer.

In subsection 2.1.1 we get a-priori estimates for the solutions of the diffusion system associated with the system (2-1),

$$\begin{cases} u^\epsilon_t + \big(u^\epsilon\phi(r^\epsilon)\big)_x + g_1(u^\epsilon,v^\epsilon) = \epsilon u^\epsilon_{xx} \\ v^\epsilon_t + \big(v^\epsilon\phi(r^\epsilon)\big)_x + g_2(u^\epsilon,v^\epsilon) = \epsilon v^\epsilon_{xx}\,. \end{cases} \tag{2-4}$$

In subsection 2.1.2, a slight adaptation of an argument due to Bereux and Sainsaulieu allows us to show the positivity of u^ϵ provided that the initial data $u_0(x)$ is positive. By adding some reasonable conditions, in subsection 2.1.3, we get a bound on the total variation (with respect to x) in one of the Riemann invariants. In subsection 2.1.4 we introduce two entropy-entropy flux pairs for the homogeneous system related to (2-1), these pairs will be applied in subsection 2.1.6. In subsection 2.1.5 we prove some results on compactness in $H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$, which will then be used in subsections 2.1.6 to 2.1.8 where we focus on the convergence of some subsequences of the sequences $\{r^\epsilon\}$, $\{u^\epsilon\}$ and $\{v^\epsilon\}$, for this we use the div-cur lemma but without involving Young measures. In the last subsection of this section, finally we present a result on the existence of a weak solution. The Cauchy problem without source term is discussed in section 2.2, for such problem we establish the existence of entropy weak solution in subsection 2.2.1.

The scheme given here to prove existence of weak solutions will be used in the next chapters in order to study the Cauchy problem for some non-symmetric system of Keyfitz-Kranzer type.

For the system (2-1) we have that the Jacobian matrix of the flux functions

$$dF_{(u,v)} = \begin{pmatrix} \phi(r) + 2nu^{2n}\phi'(r) & 2nuv^{2n-1}\phi'(r) \\ 2nvu^{2n-1}\phi'(r) & \phi(r) + 2nv^{2n}\phi'(r) \end{pmatrix}$$

has the two real eigenvalues

$$\lambda_1 = \phi(r) + 2nr\phi'(r), \qquad \lambda_2 = \phi(r),$$

with corresponding right eigenvectors

$$r_{\lambda_1} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad r_{\lambda_2} = \begin{pmatrix} -v^{2n-1} \\ u^{2n-1} \end{pmatrix},$$

the functions

$$z(u,v) = \frac{v}{u}, \qquad w(u,v) = \phi(r), \tag{2-5}$$

are Riemann invariants for system (2-1), associated respectively with λ_1 and λ_2.

2.1 The Cauchy problem for the symmetric system with a source

We shall give a proof of the global existence of bounded weak solution for the Cauchy problem (2-1)-(2-2), using the vanishing viscosity method with the help of the theory of compensated compactness. So, we study the convergence of the viscosity solutions $(u^\epsilon(x,t), v^\epsilon(x,t))$ for the parabolic systems (2-4) as $\epsilon \to 0$. Hereafter we write the functions u, v with an index u^ϵ, v^ϵ only when it avoids confusion with the system (2-1).

2.1.1 Existence of viscosity solutions

We initially prove the existence of a sequence $(u^\epsilon(x,t), v^\epsilon(x,t))$, solutions of problem (2-4)-(2-2) on $\mathbb{R} \times [0,T]$, uniformly bounded with respect to ϵ.

Lemma 2.1.1. *If $g_1(u,v), g_2(u,v)$ satisfy the inequality*

$$u^{2n-1}g_1 + v^{2n-1}g_2 \geq c_1 r + c_2, \tag{2-6}$$

where c_i $i = 1, 2$ are real constants. Then for any $\epsilon > 0$ and any $T > 0$, the following a-priori bounds hold for the Cauchy problem (2-4)-(2-2)

$$|u^\epsilon(x,t)| \leq M(T), \qquad |v^\epsilon(x,t)| \leq M(T), \qquad (x,t) \in \mathbb{R} \times [0,T], \tag{2-7}$$

for a positive constant $M(T)$ independent of ϵ.

Proof. We multiply the first and second equations of system (2-4) respectively by $2nu^{2n-1}$ and $2nv^{2n-1}$, and adding the results, we obtain

$$r_t + \lambda_1 r_x + 2nu^{2n-1}g_1 + 2nv^{2n-1}g_2 = \epsilon r_{xx} - 2n(2n-1)\epsilon\Big(u^{2n-2}u_x^2 + v^{2n-2}v_x^2\Big). \tag{2-8}$$

Using the inequality (2-6), we have from (2-8) the following inequality

$$r_t + \lambda_1 r_x + 2nc_1 r + 2nc_2 \leq r_{xx}. \tag{2-9}$$

Applying the maximum principle to (2-9) we get the estimate $r^\epsilon \leq N(T)$, where $N(T)$ is a positive constant, being independent of ϵ. Then for u^ϵ and v^ϵ we have the a-priori bounds in (2-7), which implies the existence of viscosity solutions for the Cauchy problem (2-4)-(2-2). □

2.1.2 A positive lower bound of $u^\epsilon(x,t)$

In this subsection we present a lemma that provides conditions on the initial data $u_0(x)$ and on the source term g_1 to get the positivity of $u^\epsilon(x,t)$. We give below a simpler direct proof of this fact, proof that we adapt from Bereux and Sainsaulieu ([22],[15]). This result will be required in subsection 2.1.3.

Lemma 2.1.2. *Suppose the condition of lemma 2.1.1 holds. If the initial data $u_0(x)$ is such that $u_0(x) \geq c_3 > 0$ for a constant c_3 and $g_1(u,v) = uh(u,v)$ for a continuous function $h(u,v)$, then we have*

$$u^\epsilon(x,t) \geq c(t,\epsilon,c_3) > 0, \tag{2-10}$$

where $c(t,\epsilon,c_3)$ could tend to 0 as $t \to +\infty$ or $\epsilon \to 0$.

Proof. Algebraic manipulations on the system (2-4) give the equalities

$$\begin{aligned}(\ln u)_t + \phi(r)_x + \phi(r)(\ln u)_x + h(u,v) &= \epsilon\frac{1}{u}u_{xx} \\ &= \epsilon(\ln u)_{xx} + \epsilon\big((\ln u)_x\big)^2.\end{aligned}$$

We set $\nu = -\ln u$ and deduce from the above equalities that

$$\begin{aligned}\nu_t - \epsilon\nu_{xx} &= -\epsilon(\nu_x)^2 + \phi(r)_x + \phi(r)\nu_x + h(u,v)\\ &= -\epsilon\left(\nu_x + \frac{\phi(r)}{2\epsilon}\right)^2 + \frac{\phi(r))^2}{4\epsilon} + \phi(r)_x + h(u,v)\\ &\le \frac{\phi(r))^2}{\epsilon} + \phi(r)_x + h(u,v).\end{aligned}$$

Thus,

$$\nu(x,t) \le \nu_0(x) * k_\epsilon(x,t) + \int_0^t \left(\tfrac{1}{\epsilon}\big(\phi(r)\big)^2 + \phi(r)_x + h(u,v)\right) *_x k_\epsilon(x,t-s)ds\,,$$

where $\nu_0(x) = -\ln u_0(x)$ and here

$$k_\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\exp\left(-\frac{x^2}{4\epsilon t}\right), \tag{2-11}$$

denotes the heat kerner for $\nu_t - \epsilon\nu_{xx}$. Hence

$$\begin{aligned}\nu(x,t) &\le \nu_0^\epsilon(x) * k_\epsilon(x,t) + \frac{N_1}{\epsilon}t + \int_0^t \phi(r) *_x \big(k_\epsilon(x,s)\big)_x ds\\ &\le -\ln c_3 + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\,,\end{aligned}$$

because $u_0(x) \ge c_3 > 0$. Whence

$$u(x,t) \ge c_3 \exp -\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) \ge c(t,\epsilon,c_3) > 0.$$

This proves (2-10). □

2.1.3 Estimates for $z(x,t)$ and $z_x(\cdot,t)$

Let z be the Riemann invariant given in (2-5), from now on, $z(x,t) = z(u^\epsilon, v^\epsilon)$ and $z_x(\cdot,t)$ denotes the function $z_x(x,t)$ where t is fixed. In this section, we prove bounds for $z(x,t)$ in $L^\infty(\mathbb{R}\times[0,T])$ and for $z_x(\cdot,t)$ in $L^1(\mathbb{R})$. By using the same argument given in [16], we have the following lemma, also as in [26], the key is that the total variation of the Riemann invariant $z(\cdot,t)$ is decreasing in t.

Lemma 2.1.3. *If in addition to the assumption of lemma 2.1.2, $z_0(x) = z(x,0) \in L^\infty(\mathbb{R})$, $z_0'(x) \in L^1(\mathbb{R})$ and there exist a function $G(s)$ satisfying*

$$G\left(\tfrac{v}{u}\right) = \frac{ug_2 - vg_1}{u^2}, \qquad G\left(\tfrac{v}{u}\right) \ge c_4\frac{v}{u} + c_5, \qquad G'(s) \ge 0, \tag{2-12}$$

where c_4, c_5 *are constants, then* $\left(\frac{v^\epsilon}{u^\epsilon}\right)(x,t) \in L^\infty(\mathbb{R}\times[0,T])$, $\left(\frac{v^\epsilon}{u^\epsilon}\right)_x(\cdot,t) \in L^1(\mathbb{R})$, *moreover*

$$TV\Big(\left(\tfrac{v^\epsilon}{u^\epsilon}\right)(\cdot,t)\Big) = \int_{-\infty}^{+\infty}\Big|\left(\tfrac{v^\epsilon}{u^\epsilon}\right)_x(x,t)\Big|\,dx \le \int_{-\infty}^{+\infty}\Big|\left(\tfrac{v_0}{u_0}\right)'(x)\Big|\,dx = TV\Big(\left(\tfrac{v_0}{u_0}\right)(x)\Big), \tag{2-13}$$

where TV *is the total variation.*

Proof. Multiplying the first and second equations of system (2-4) by $-\frac{v}{u^2}$ and $\frac{1}{u}$, respectively, then adding the results, we have

$$\left(\frac{v}{u}\right)_t + \phi(r)\left(\frac{v}{u}\right)_x + \frac{ug_2 - vg_1}{u^2} = \epsilon\left(\frac{v}{u}\right)_{xx} + 2\epsilon\frac{u_x}{u}\left(\frac{v}{u}\right)_x, \tag{2-14}$$

using (2-12) we obtain

$$\left(\frac{v}{u}\right)_t + \phi(r)\left(\frac{v}{u}\right)_x + c_4\left(\frac{v}{u}\right) + c_5 \le \epsilon\left(\frac{v}{u}\right)_{xx} + 2\epsilon\frac{u_x}{u}\left(\frac{v}{u}\right)_x. \tag{2-15}$$

Appliying the maximum principle to (2-15), we thus find that $\left(\frac{v^\epsilon}{u^\epsilon}\right)(x,t) \in L^\infty(\mathbb{R}\times[0,T])$. Now we differentiate (2-14) with respect to x and then we do $\theta = \left(\frac{v}{u}\right)_x$ to get

$$\theta_t + \big(\phi(r)\theta\big)_x = \epsilon\theta_{xx} + \big(2\epsilon u^{-1}u_x\theta\big)_x - \left(\frac{ug_2 - vg_1}{u^2}\right)_x,$$

multiplying this equation by the sequence of smooth functions $g'(\theta,\alpha)$, where α is a parameter, we obtain

$$\begin{aligned} g(\theta,\alpha)_t + \big(\phi(r)g(\theta,\alpha)\big)_x + \phi(r)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) &= \epsilon g(\theta,\alpha)_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 \\ + \big(2\epsilon u^{-1}u_x g(\theta,\alpha)\big)_x + \big(2\epsilon u^{-1}u_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) &- \left(\frac{ug_2 - vg_1}{u^2}\right)_x g'(\theta,\alpha). \end{aligned} \tag{2-16}$$

If we choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha) \ge 0$, $g'(\theta,\alpha) \to \text{sign}\theta$ and $g(\theta,\alpha) \to |\theta|$ as $\alpha \to 0$, we have from (2-16)

$$|\theta|_t + \big(\phi(r)|\theta|\big)_x = \epsilon|\theta|_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon u^{-1}u_x|\theta|\big)_x - \text{sign}\theta\left(\frac{ug_2 - vg_1}{u^2}\right)_x,$$

from this inequality and using (2-12) it follows that

$$\begin{aligned} |\theta|_t + \big(\phi(r)|\theta|\big)_x &\le \epsilon|\theta|_{xx} + \big(2\epsilon u^{-1}u_x|\theta|\big)_x - G'\left(\tfrac{v}{u}\right)|\theta| \\ &\le \epsilon|\theta|_{xx} + \big(2\epsilon u^{-1}u_x|\theta|\big)_x. \end{aligned} \tag{2-17}$$

Integrating (2-17) in $\mathbb{R}\times[0,t]$, we obtain (2-13). □

Remark 2.1.4. *There are functions* g_1, g_2 *and* $G(s)$, *satisfying the conditions given in lemmas 2.1.1 and 2.1.3. Consider*

$$g_1(u,v) = au, \quad g_2(u,v) = bv, \quad G(s) = (b-a)s,$$

where a, b *are constants such that* $b \ge a$.

$$g_1(u,v) = u^3, \quad g_2(u,v) = vu^2, \quad G(s) = 0.$$

2.1.4 Two pairs of entropy-entropy flux

A function $\eta = \eta(u, v)$ is called an entropy for the homogeneous system associated to (2-1), with entropy flux $q = q(u, v)$ if

$$\nabla q(u,v) = \nabla \eta(u,v) \begin{pmatrix} \phi(r) + 2nu^{2n}\phi'(r) & 2nuv^{2n-1}\phi'(r) \\ 2nvu^{2n-1}\phi'(r) & \phi(r) + 2nv^{2n}\phi'(r) \end{pmatrix}.$$

A pair of functions η, q satisfying the above equation is called an entropy-entropy flux pair and we denote it (η, q).

Two entropy-entropy flux pairs of homogeneous system associated to (2-1) are given by

$$\big(\eta(u,v), q(u,v)\big) = \left(r, \int^{r} \big(\phi(s) + 2ns\phi'(s)\big)\, ds\right), \tag{2-18}$$

$$\big(\eta(u,v), q(u,v)\big) = \left(\int^{r} \big(\phi(s) + 2ns\phi'(s)\big)\, ds, \int^{r} \big(\phi(s) + 2ns\phi'(s)\big)^2\, ds\right), \tag{2-19}$$

for r in (2-3). By means of these pairs, we shall obtain the pointwise convergence of a subsequence of $\{r^\epsilon(x,t)\}$.

2.1.5 H^{-1}_{loc} compactness

Throughout this subsection we establish the results related to compactness in H^{-1}_{loc} that allow us to apply the div-curl lemma in subsections 2.1.6 to 2.1.8 in order to prove for each of the sequences $\{r^\epsilon\}$, $\{u^\epsilon\}$ and $\{v^\epsilon\}$ the pointwise convergence of a subsequence. The first two lemmas given here refer to the two pairs of entropy-entropy flux (2-18) and (2-19). The results of the lemmas 2.1.7, 2.1.8 y 2.1.10 are possible thanks to the estimate (2-13) obtained in lemma 2.1.3 of the previous section.

Lemma 2.1.5. *We assume the same conditions given in the lemma 2.1.1. Then*

$$r^\epsilon_t + \left(\int^{r^\epsilon} \big(\phi(s) + 2ns\phi'(s)\big)\, ds\right)_x \tag{2-20}$$

is compact in $H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. The equation (2-8) may be written as

$$r_t + \left(\int^{r} \big(\phi(s) + 2ns\phi'(s)\big)\, ds\right)_x = \epsilon r_{xx} - 2n(2n-1)\epsilon\Big(u^{2n-2}u_x^2 + v^{2n-2}v_x^2\Big) - 2n\Big(u^{2n-1}g_1 - v^{2n-1}g_2\Big). \tag{2-21}$$

From (2-21) it follows that

$$\epsilon\big(u^\epsilon\big)^{2n-2}\big(u^\epsilon_x\big)^2 \text{ and } \epsilon\big(v^\epsilon\big)^{2n-2}\big(v^\epsilon_x\big)^2 \text{ are bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{2-22}$$

Also the last term in the right-hand side of (2-21) is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, thus $-2n(2n-1)\epsilon\big(u^{2n-2}u_x^2+v^{2n-2}v_x^2\big)-2n\big(u^{2n-1}g_1-v^{2n-1}g_2\big)$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$ (the space of Radon measures). The bounds in (2-22), imply that the term ϵr_{xx} is $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ compact, by using Cauchy-Schwarz inequality. The left-hand side of (2-21) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Hence by Murat's lemma ([19],[20]), (2-20) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

Lemma 2.1.6. *Under the assumptions of lemma 2.1.1, it follows that*

$$\left(\int^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)\,ds\right)_t+\left(\int^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)^2\,ds\right)_x \tag{2-23}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Multiplying the equation (2-8) by $\phi(r)+2nr\phi'(r)$, we obtain

$$\begin{aligned}\left(\int^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)\,ds\right)_t+\left(\int^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)^2\,ds\right)_x &= \epsilon r_{xx}\big(\phi(r)+2nr\phi'(r)\big)\\ -2n(2n-1)\epsilon\Big(u^{2n-2}u_x^2+v^{2n-2}v_x^2\Big)\big(\phi(r)+2nr\phi'(r)\big)&\\ -2n\Big(u^{2n-1}g_1-v^{2n-1}g_2\Big)\big(\phi(r)+2nr\phi'(r)\big).&\end{aligned} \tag{2-24}$$

The left-hand side of the above equation is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Making use of (2-22), we get that the second term on the right-hand side of (2-24) is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, and as the last term is also bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, then these are bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. It follows from the estimates (2-22) and from the Cauchy-Schwarz inequality that $\epsilon r_{xx}\big(\phi(r)+2nr\phi'(r)\big)$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. We can apply Murat's lemma and thus conclude that (1.23) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

Lemma 2.1.7. *Suppose the conditions of lemma 2.1.3 holds. Then*

$$\Big(\big(u^\epsilon\big)^{2n}\Big)_t+\left(\frac{(u^\epsilon)^{2n}}{r^\epsilon}\int^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)\,ds\right)_x \tag{2-25}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Introducing the function $\varphi(x,t)$ defined by

$$\varphi=1+\left(\frac{v}{u}\right)^{2n}, \tag{2-26}$$

we can write the first equation of system (2-4) as

$$u_t + u_x\Big(\phi\big(u^{2n}\varphi\big) + 2nu^{2n}\varphi\phi'\big(u^{2n}\varphi\big)\Big) = \epsilon u_{xx} - u^{2n+1}\varphi_x\phi'\big(u^{2n}\varphi\big) - g_1. \tag{2-27}$$

Multiplying both sides of (2-27) by $2nu^{2n-1}$, we obtain

$$\Big(u^{2n}\Big)_t + \Big(u^{2n}\Big)_x\Big(\phi\big(u^{2n}\varphi\big) + 2nu^{2n}\varphi\phi'\big(u^{2n}\varphi\big)\Big) = 2\epsilon nu^{2n-1}u_{xx} - 2n\Big(u^{2n}\Big)^2\varphi_x\phi'\big(u^{2n}\varphi\big) - 2nu^{2n-1}g_1, \tag{2-28}$$

this equation is equivalent to

$$\Big(u^{2n}\Big)_t + \left(\frac{u^{2n}}{r}\int^r\big(\phi(s) + 2ns\phi'(s)\big)\,ds\right)_x = \epsilon\Big(u^{2n}\Big)_{xx} - 2\epsilon n(2n-1)u^{2n-2}u_x^2 - 2n\Big(u^{2n}\Big)^2\varphi_x\phi'\big(u^{2n}\varphi\big) - 2nu^{2n-1}g_1, \tag{2-29}$$

where we have used that

$$\int^{u^{2n}}\big(\phi(s\varphi) + 2ns\varphi\phi'(s\varphi)\big)\,ds = \frac{u^{2n}}{r}\int^r\big(\phi(s) + 2ns\phi'(s)\big)\,ds\,.$$

By lemma 2.1.3 $\varphi(\cdot,t)_x$ is bounded in $L^1(\mathbb{R})$, this implies that the third term on the right-hand side of (2-29) is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Also the term $2nu^{2n-1}g_1$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. It follows from (2-29) that

$$\epsilon\big(u^\epsilon\big)^{2n-2}\big(u_x^\epsilon\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+), \tag{2-30}$$

since $\varphi(\cdot,t)_x$ is bounded in $L^1(\mathbb{R})$. Thus these last three terms are bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. Using the estimate (2-30), we can prove that the first term on the right is $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ compact. In addition, the left-hand side of (2-29) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. So, we are in a position to apply Murat's lemma to see that (2-25) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

Lemma 2.1.8. *If conditions in lemma 2.1.3 are satisfied, then*

$$u_t^\epsilon + \Big(u^\epsilon\phi\big(r^\epsilon\big)\Big)_x \tag{2-31}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Multiply (2-27) by $2u$ to obtain

$$\Big(u^2\Big)_t + \left(\int^{u^2}\big(\phi(s^n\varphi) + 2ns^n\varphi\phi'(s^n\varphi)\big)\,ds\right)_x = \epsilon\Big(u^2\Big)_{xx} - 2\epsilon u_x^2 - 2u^{2n+2}\varphi_x\phi'\big(u^{2n}\varphi\big) - 2ug_1. \tag{2-32}$$

The equation (2-32) shows that

$$\epsilon\big(u_x^\epsilon\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+), \tag{2-33}$$

indeed, here one makes use of the two estimates given in lemma 2.1.3.
According to first equation of system (2-4), we have

$$u_t + \big(u\phi(r)\big)_x = \epsilon u_{xx} - g_1, \tag{2-34}$$

the term g_1 is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$, since it is locally integrable. One can show that ϵu_{xx} is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, to this end we use Cauchy-Schwarz inequality together with the estimate (2-33). Hence, in view of the Murat's lemma, we conclude that (2-31) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

As a corollary of lemma 2.1.8 one can prove the following result.

Corollary 2.1.9. *With the assumptions given in lemma 2.1.3, it follows that*

$$u_t^\epsilon + \left(u^\epsilon\phi\big(r^\epsilon\big) + \frac{v^\epsilon}{u^\epsilon}\right)_x \tag{2-35}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Due to the compactness of (2-31) in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, it is sufficient to show that $\big(\frac{v^\epsilon}{u^\epsilon}\big)_x$ is also compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, which is true since $\big(\frac{v^\epsilon}{u^\epsilon}\big)_x$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, and in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, this concludes the proof. □

Lemma 2.1.10. *Assuming the hypotheses as in lemma 2.1.3, then*

$$v_t^\epsilon + \Big(v^\epsilon\phi\big(r^\epsilon\big)\Big)_x \tag{2-36}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. We begin by multiplying the first equation in system (2-4) by $\big(\frac{v}{u}\big)^2$ and (2-14) by $2u\big(\frac{v}{u}\big)$, adding the above results, we obtain

$$\left(u\left(\frac{v}{u}\right)^2\right)_t + \left(u\left(\frac{v}{u}\right)^2\phi\right)_x + 2\frac{v}{u}g_2 - \left(\frac{v}{u}\right)^2 g_1 = \epsilon\left(u\left(\frac{v}{u}\right)^2\right)_{xx} - 2\epsilon u\left(\frac{v}{u}\right)_x^2, \tag{2-37}$$

from (2-37) it would follow that

$$\epsilon u^\epsilon\left(\frac{v^\epsilon}{u^\epsilon}\right)_x^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{2-38}$$

The second equation of system (2-4) may written in the form

$$v_t + \big(v\phi(r)\big)_x = \epsilon\left(u\frac{v}{u}\right)_{xx} = \epsilon\left(u_x\left(\frac{v}{u}\right) + u\left(\frac{v}{u}\right)_x\right)_x - g_2. \tag{2-39}$$

The term g_2 is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, then it is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. We now claim that

$$\epsilon\left(u_x\left(\frac{v}{u}\right)\right)_x \text{ and } \epsilon\left(u\left(\frac{v}{u}\right)_x\right)_x \text{ are compact in } H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+),$$

to see this, we use respectively the estimates (2-33) and (2-38), together with the Cauchy-Schwarz inequality. Hence by Murat's lemma, the sequence $v^\epsilon_t + \big(v^\epsilon\phi(r^\epsilon)\big)_x$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

A consequence of the previous lemma is the next corollary.

Corollary 2.1.11. *Suppose the conditions of lemma 2.1.3. Then we have*

$$v^\epsilon_t + \left(v^\epsilon\phi(r^\epsilon) + \left(\frac{v^\epsilon}{u^\epsilon}\right)^2\right)_x \tag{2-40}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Being $\left(\left(\frac{v^\epsilon}{u^\epsilon}\right)^2\right)_x$ bounded in both $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, we deduce that $\left(\left(\frac{v^\epsilon}{u^\epsilon}\right)^2\right)_x$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. So by lemma 2.1.10 the conclusion in this corollary holds. □

We will omit subscripts on subsequences and let any subsequence of the sequence $\{u^\epsilon\}$ be denoted by $\{u^\epsilon\}$. In particular, whenever speaking of convergence of $\{u^\epsilon\}$, we really mean convergence of some subsequence.

2.1.6 Pointwise convergence of $\{r^\epsilon(x,t)\}$

In order to show that the sequence $\{r^\epsilon(x,t)\}$ converges pointwise, we use only the two pairs of entropy-entropy flux (2-18) and (2-19) together with the div-curl lemma.

Lemma 2.1.12. *When the assumptions of lemma 1.1.1 are satisfied, the function* $\phi(r)\in C^2(\mathbb{R}^+)$ *and*

$$meas\{r : (2n+1)\phi'(r) + 2nr\phi''(r) = 0\} = 0, \tag{2-41}$$

then there exists a subsequence of $\{r^\epsilon(x,t)\}$ *which converges pointwisely.*

Proof. Due to lemmas 2.1.5 and 2.1.6, the div-curl lemma can be applied to (2-18) and (2-19), this yields to the equation

$$\overline{r^\epsilon \int_k^{r^\epsilon} f'^2(s)\,ds - f^2(r^\epsilon)} = \overline{r^\epsilon}\,\overline{\int_k^{r^\epsilon} f'^2(s)\,ds} \;-\; \overline{f(r^\epsilon)}^2, \tag{2-42}$$

where k is real constant and $f(r^\epsilon) = \int^{r^\epsilon} \big(\phi(s) + 2ns\phi'(s)\big)\,ds$. Here the overline denotes the weak-star limit (i.e. $\overline{r^\epsilon} = w^\star - \lim r^\epsilon$).
Let $\overline{r^\epsilon} = r$, we first notice that

$$\begin{aligned}\overline{r^\epsilon \int_k^{r^\epsilon} f'^2(s)\,ds - f^2(r^\epsilon)} \;&=\; \overline{r^\epsilon \int_r^{r^\epsilon} f'^2(s)\,ds - \big(f(r^\epsilon) - f(r)\big)^2} \;+\; \overline{r^\epsilon} \int_k^r f'^2(s)\,ds \\ &\quad + \Big(\overline{f(r^\epsilon)} - f(r)\Big)^2 - \overline{f(r^\epsilon)}^2,\end{aligned} \tag{2-43}$$

and

$$\overline{r^\epsilon}\,\overline{\int_k^{r^\epsilon} f'^2(s)\,ds} \;=\; \overline{r^\epsilon}\,\overline{\int_r^{r^\epsilon} f'^2(s)\,ds} \;+\; \overline{r^\epsilon}\int_k^r f'^2(s)\,ds\,, \tag{2-44}$$

using the above equalities in (2-42), we obtain

$$\overline{(r^\epsilon - r)\int_r^{r^\epsilon} f'^2(s)\,ds - \big(f(r^\epsilon) - f(r)\big)^2} \;+\; \Big(\overline{f(r^\epsilon)} - f(r)\Big)^2 = 0. \tag{2-45}$$

But on the other hand, for by Cauchy-Schwarz inequality

$$\big(f(r^\epsilon) - f(r)\big)^2 = \left(\int_r^{r^\epsilon} f'(s)\,ds\right)^2 \le (r^\epsilon - r)\int_r^{r^\epsilon} f'^2(s)\,ds.$$

Therefore, the weak-star limit of $(r^\epsilon - r)\int_r^{r^\epsilon} f'^2(s)\,ds - \big(f(r^\epsilon) - f(r)\big)^2$ is nonnegative. Since both terms in the left-hand of equation (2-42) are nonnegative, we deduce

$$\overline{(r^\epsilon - r)\int_r^{r^\epsilon} f'^2(s)\,ds - \big(f(r^\epsilon) - f(r)\big)^2} \;= 0. \tag{2-46}$$

From (2-46) we get the conclusion in this lemma (see the argument used in the proof of theorem 3.1.1 in [15]). □

2.1.7 Pointwise convergence of $\{u^\epsilon(x,t)\}$

On the basis of the lemmas 2.1.7 and 2.1.8, we can now establish the following result.

Lemma 2.1.13. *Assume that in addition to the hypotheses of the lemmas 2.1.3 and 2.1.12, $\phi(r)$ is strictly increasing or decreasing for positive r. Then there is a subsequence of $\{u^\epsilon(x,t)\}$ which converges pointwisely.*

Proof. Applying again the div-curl lemma to the functions (2-25) and (2-31), we get

$$\overline{(u^\epsilon)^{2n+1}\left(\frac{1}{r^\epsilon}\int_0^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)\,ds-\phi(r^\epsilon)\right)} = \overline{u^\epsilon}\;\overline{\frac{(u^\epsilon)^{2n}}{r^\epsilon}\int_0^{r^\epsilon}\big(\phi(s)+2ns\phi'(s)\big)\,ds} - \overline{(u^\epsilon)^{2n}}\;\overline{u^\epsilon\phi(r^\epsilon)}\,. \tag{2-47}$$

By lemma 1.1.12 we may extract a subsequence of $\{r^\epsilon(x,t)\}$ (still denoted $\{r^\epsilon(x,t)\}$) which converges pointwisely, let $r^\epsilon(x,t)\to R(x,t)$ (strong). Using this fact in (2-47) it follows that

$$\left(\frac{1}{R}\int_0^{R}\big(\phi(s)+2ns\phi'(s)\big)\,ds-\phi(R)\right)\left(\overline{(u^\epsilon)^{2n+1}}-\overline{u^\epsilon}\;\overline{(u^\epsilon)^{2n}}\right)=0. \tag{2-48}$$

By the condition on ϕ we conclude that $\frac{1}{r}\int_0^r\big(\phi(s)+2ns\phi'(s)\big)\,ds-\phi(r)=0$ only on $r=0$, according to (2-48) we obtain that

$$\overline{(u^\epsilon)^{2n+1}}-\overline{u^\epsilon}\;\overline{(u^\epsilon)^{2n}}=0, \tag{2-49}$$

equation from which we get the pointwise convergence of $\{u^\epsilon\}$ in the region $r>0$. □

2.1.8 Pointwise convergence of $\{v^\epsilon(x,t)\}$

Lemma 2.1.14. *If the conditions of lemma 2.1.13 are fulfilled, then there is a subsequence of $\{v^\epsilon\}$ such that it converges pointwise.*

Proof. It now follows from the div-curl lemma applied to the functions (2-35) and (2-40) that

$$\overline{u^\epsilon}\;\overline{u^\epsilon\frac{v^\epsilon}{u^\epsilon}\phi(r^\epsilon)+\left(\frac{v^\epsilon}{u^\epsilon}\right)^2}-\overline{u^\epsilon\phi(r^\epsilon)+\frac{v^\epsilon}{u^\epsilon}}\;\overline{u^\epsilon\frac{v^\epsilon}{u^\epsilon}}=0. \tag{2-50}$$

Combining (2-50) and the strong convergence of the sequences $\{\phi(r^\epsilon)\}$ and $\{u^\epsilon\}$ we find that

$$\overline{u^\epsilon}\left(\overline{\left(\frac{v^\epsilon}{u^\epsilon}\right)^2}-\overline{\left(\frac{v^\epsilon}{u^\epsilon}\right)}^{\,2}\right)=0, \tag{2-51}$$

which implies the pointwise convergence of $\left\{\frac{v^\epsilon}{u^\epsilon}\right\}$ on the region $u>0$, and therefore the convergence of $\{v^\epsilon\}$. □

2.1.9 Existence of weak solution

In this last subsection, we prove our main result, which is the existence of weak solution for the Cauchy problem (2-1)-(2-2).

Theorem 2.1.15. *Suppose that $g_1(u,v), g_2(u,v)$ satisfy the inequality $u^{2n-1}g_1 + v^{2n-1}g_2 \geq c_1 r + c_2$, $u_0(x) \geq c_3 > 0$, $g_1(u,v) = uh(u,v)$ for a continuous function $h(u,v)$, $\big(\frac{v_0}{u_0}\big)(x) \in L^\infty(\mathbb{R})$, $\big(\frac{v_0}{u_0}\big)'(x) \in L^1(\mathbb{R})$, there exist a function $G(s)$ satisfying $G\big(\frac{v}{u}\big) = \frac{ug_2 - vg_1}{u^2}$, $G\big(\frac{v}{u}\big) \geq c_4\frac{v}{u} + c_5$, $G'(s) \geq 0$, where c_i $i = 1, \dots, 5$ are real constants, $\phi(r) \in C^2(\mathbb{R}^+)$, $meas\{r : (2n+1)\phi'(r) + 2nr\phi''(r)\} = 0$ and $\phi(r)$ is strictly increasing or decreasing for positive r. Then there exist a subsequence of (u^ϵ, v^ϵ) which converges pointwisely and the limit is a weak solution of the Cauchy problem (2-1)-(2-2).*

Proof. To prove this result, we consider the sequence of viscosity solutions (u^ϵ, v^ϵ) of the approximated system (2-4). Let us consider $\varphi, \psi \in C_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$. By multiplying the first equation of the system (2-4) by φ, the second by ψ, adding the resulting equations and integrating by parts in $\mathbb{R} \times [0,\infty)$, we obtain that u^ϵ and v^ϵ satisfy the weak formulation of the Cauchy problem (2-4)-(2-2),

$$\int_{\mathbb{R}}\int_0^{+\infty} \big(u^\epsilon\varphi_t + u^\epsilon\phi(r^\epsilon)\varphi_x - g_1(u^\epsilon,v^\epsilon)\varphi\big)\,dt\,dx + \int_{\mathbb{R}} u_0\varphi(x,0)\,dx$$
$$+ \int_{\mathbb{R}}\int_0^{+\infty} \big(v^\epsilon\psi_t + v^\epsilon\phi(r^\epsilon)\psi_x - g_2(u^\epsilon,v^\epsilon)\psi\big)\,dt\,dx + \int_{\mathbb{R}} v_0\psi(x,0)\,dx =$$
$$-\epsilon\int_{\mathbb{R}}\int_0^{+\infty} \big(u^\epsilon\varphi_{xx} + v^\epsilon\psi_{xx}\big)\,dt\,dx\,. \quad (2\text{-}52)$$

By lemmas 2.1.12, 2.1.13 and 2.1.14, we can find a subsequence of (u^ϵ, v^ϵ) (no relabeled), which converges pointwise, a. e. $(x,t) \in \mathbb{R} \times [0,T]$, to (u,v) and it is such that $r^\epsilon \to u^2 + v^2$, a. e. $(x,t) \in \mathbb{R} \times [0,T]$. Since ϕ and g_i $i = 1,2$ are continuous, $\phi(r^\epsilon) \to \phi(u^2+v^2)$, $g_i(u^\epsilon, v^\epsilon) \to g_i(u,v)$ $i = 1,2$, a. e. $(x,t) \in \mathbb{R} \times [0,T]$.
Note that by (2-7),

$$\left|\epsilon\int_{\mathbb{R}}\int_0^{+\infty} u^\epsilon\varphi_{xx}\,dt\,dx\right| \leq \epsilon||\varphi_{xx}||_{L^\infty}\int_{\text{supp}(\varphi)} |u^\epsilon|\,dt\,dx$$
$$\leq \epsilon N\Big(meas\big(\text{supp}(\varphi)\big)\Big),$$

thus we obtain

$$\lim_{\epsilon\to 0}\epsilon\int_{\mathbb{R}}\int_0^{+\infty} u^\epsilon\varphi_{xx}\,dt\,dx = 0. \quad (2\text{-}53)$$

Using the above argument we also have

$$\lim_{\epsilon\to 0}\epsilon\int_{\mathbb{R}}\int_0^{+\infty} v^\epsilon\psi_{xx}\,dt\,dx = 0. \quad (2\text{-}54)$$

We want to pass to the limit the weak formulation (2-52) to complete the proof. From (2-53) and (2-54), it follows immediately that the integral on the right-hand side of (2-52) converges to 0 as $\epsilon \to 0$. Due to the convergence almost everywhere, we can apply the Lebesgue dominated convergence theorem to (2-52) to obtain that (u,v) is a weak solution of the Cauchy problem (2-4)-(2-2). □

2.2 The Cauchy problem for the symmetric homogeneous system

In this section we turn our attention on the Cauchy problem for the homogeneous system associated to (2-1) with initial data (2-2). In this case where there are no source terms (i.e. $g_i = 0$ $i = 1, 2$), the system (2-1) reduces to

$$\begin{cases} u_t + \big(u\phi(r)\big)_x = 0 \\ v_t + \big(v\phi(r)\big)_x = 0, \end{cases} \tag{2-55}$$

which is a 2×2 hyperbolic system of conservation laws.

2.2.1 Existence of weak entropy solution

Lax [11] proves that if the solution u^ϵ of the Cauchy problem for the parabolic system ($\epsilon > 0$)

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \quad u(x,t) \in \mathbb{R}^n,$$

satisfies certain a-priori estimates and converges almost everywhere to a limit u, then the function u must satisfy the following inequality

$$\int_{\mathbb{R}} \int_0^{+\infty} \big(\eta(u)\varphi_t + q(u)\varphi_x\big)\, dt\, dx \geq 0, \tag{2-56}$$

for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ and any entropy-entropy flux pair $(\eta, q) \in C^2$, with η convex.

For a system of two strictly hyperbolic conservation laws, the rigourous proof of this derivation has been given by Di Perna [1] (see also Tartar [29], Rascle [23]).

As a consequence of this fact we may now prove the next result.

Corollary 2.2.1. *Let $\phi(r) \in C^2(\mathbb{R}^+)$ such that meas$\{r : (2n+1)\phi'(r) + 2nr\phi''(r)\} = 0$ and $\phi(r)$ is strictly increasing or decreasing for positive r. Assume that $u_0(x) \geq c > 0$, $\big(\frac{v_0}{u_0}\big)(x) \in L^\infty(\mathbb{R})$ and that $\big(\frac{v_0}{u_0}\big)'(x) \in L^1(\mathbb{R})$, where c is a constant. Then the Cauchy problem (2-55)-(2-2) has a weak entropy solution.*

Proof. Applying the previous theorem to $g_1 = g_2 = 0$ it follows that there exist a subsequence (u^ϵ, v^ϵ) of solutions for (2-4) satisfying the a-priori estimates (2-7) and converging almost everywhere, which ensures that the entropy inequality (2-56) is fulfilled. □

3 Existence of weak solution for the Aw-Rascle traffic flow model: A first result

The purpose of this chapter, divided into two sections, is to study the Cauchy problem for the Aw-Rascle model of traffic flow, which is a non-symmetric system of Keyfitz-Kranzer type. Under the same conditions on $p(\rho)$ given by Aw and Rascle in [2], in the first section we prove the existence of a weak solution for the Aw-Rascle model with a source term and in the second section we obtain the existence of weak entropy solution for the homogeneous model, i.e. the Aw-Rascle model. In subsection 2.1.1 we get a-priori estimates for the viscosity solutions of the diffusion system (3-7). An estimate for the total variation of the Riemann invariant $z(\cdot,t)$ given in (3-3) is obtained in subsection 2.1.2. The div-curl lemma allows us to prove in subsection 2.1.4 the pointwise convergence of a subsequence of viscosity solutions, for this we use the results on compactness in H^{-1}_{loc} given in subsection 2.1.3.

We shall rewrite the system (0.2) in a more convenient form by introducing the variable $m = \rho u$ as

$$\begin{cases} \rho_t + \big(m - \rho p(\rho)\big)_x = 0 \\ m_t + \left(\frac{m^2}{\rho} - m p(\rho)\right)_x = 0. \end{cases} \tag{3-1}$$

Let F be the mapping defined by

$$F\begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} m - \rho p(\rho) \\ \frac{m^2}{\rho} - m p(\rho) \end{pmatrix},$$

the Jacobian matrix of F is

$$dF_{(\rho,m)} = \begin{pmatrix} -p(\rho) - \rho p'(\rho) & 1 \\ -\frac{m^2}{\rho^2} - m p'(\rho) & \frac{2m}{\rho} - p(\rho) \end{pmatrix},$$

in other words, $dF_{(\rho,m)}$ is the Jacobian matrix of the flux functions in (3-1).

Thus the eigenvalues of system (3-1) are given by

$$\lambda_1 = \frac{m}{\rho} - p(\rho)\,, \qquad \lambda_2 = \frac{m}{\rho} - p(\rho) - \rho p'(\rho)\,, \tag{3-2}$$

The vectors

$$r_{\lambda_1} = \begin{pmatrix} 1 \\ \frac{m}{\rho} + \rho p'(\rho) \end{pmatrix}, \qquad r_{\lambda_2} = \begin{pmatrix} 1 \\ \frac{m}{\rho} \end{pmatrix},$$

are respectively right eigenvectors of the system (3-1) corresponding to eigenvalues λ_1 and λ_2.

The functions $z = z(\rho, m)$ and $w = w(\rho, m)$ given below are Riemann invariants for system (3-1), associated respectively with λ_1 and λ_2

$$z(\rho, m) = \frac{m}{\rho}, \qquad w(\rho, m) = \frac{m}{\rho} - p(\rho). \tag{3-3}$$

3.1 The Cauchy problem for the Aw-Rascle model with a source

We consider the Cauchy problem for the following non-symmetric system of Keyfitz-Kranzer type or Aw-Rascle model with a source

$$\begin{cases} \rho_t + \big(\rho(u - p(\rho))\big)_x + g_1(\rho, \rho u) = 0 \\ (\rho u)_t + \big(\rho u(u - p(\rho))\big)_x + g_2(\rho, \rho u) = 0, \end{cases} \tag{3-4}$$

with data initial

$$\big(\rho(x,0), u(x,0)\big) = \big(\rho_0(x), u_0(x)\big), \quad \rho_0(x) \geq 0, \tag{3-5}$$

where $p(\rho)$ is a smooth strictly increasing function, $g_i(\rho, \rho u)$, $i = 1, 2$ are locally Lipschitz continuous functions and $\rho_0(x), u_0(x) \in L^\infty(\mathbb{R})$. For this problem, we get two results on the existence of bounded entropy solution.

Notice that if we make $m = \rho u$ in the system (3-4), this system becomes

$$\begin{cases} \rho_t + \big(m - \rho p(\rho)\big)_x + g_1(\rho, m) = 0 \\ m_t + \Big(\frac{m^2}{\rho} - m p(\rho)\Big)_x + g_2(\rho, m) = 0. \end{cases} \tag{3-6}$$

3.1.1 Existence of viscosity solutions

In order to obtain bounded weak solution to the Cauchy problem (3-6)-(3-5), we now consider the following Cauchy problem for the diffusion system associated with the system (3-6)

$$\begin{cases} \rho_t^\epsilon + \big(m^\epsilon - \rho^\epsilon p(\rho^\epsilon)\big)_x + g_1(\rho^\epsilon, m^\epsilon) = \epsilon\rho_{xx}^\epsilon \\ m_t^\epsilon + \Big(\frac{(m^\epsilon)^2}{\rho^\epsilon} - m^\epsilon p(\rho^\epsilon)\Big)_x + g_2(\rho^\epsilon, m^\epsilon) = \epsilon m_{xx}^\epsilon, \end{cases} \tag{3-7}$$

with data initial

$$\big(\rho^\epsilon(x,0), m^\epsilon(x,0)\big) = \big(\rho_0^\epsilon(x), m_0^\epsilon(x)\big) = \big(\rho_0(x)+\epsilon, \rho_0(x)u_0(x)\big), \tag{3-8}$$

where $\rho_0(x), u_0(x)$ are given by (3-5). In the following, we indicate the functions ρ, m with an index $\rho^\epsilon, m^\epsilon$, when it avoids ambiguities with the system (3-6), and we omit them in general.

To prove the existence of viscosity solutions for the Cauchy problem (3-7)-(3-8) only need to establish a-priori estimates for the solutions $(\rho^\epsilon(x,t), m^\epsilon(x,t))$. Using the Riemann invariants (3-3) and the maximum principle, we obtain a-priori estimates that we summarize in the following two lemmas.

Lemma 3.1.1. *We assume that $p(0) = 0$, $\lim_{\rho\to 0}\rho p'(\rho) = 0$, the function $\rho p(\rho)$ is strictly convex for positive ρ (i.e. $2p'(\rho)+\rho p''(\rho) > 0$ for $\rho > 0$) , $g_1(\rho, m) = \rho h(\rho, m)$ for a continuous function $h(\rho, m)$ and that $g_1(\rho, m), g_2(\rho, m)$ satisfy the inequalities*

$$z_\rho g_1 + z_m g_2 \geq c_1 z + c_2, \qquad w_\rho g_1 + w_m g_2 \leq c_3 w + c_4, \tag{3-9}$$

where c_i $i = 1, \ldots, 4$ are real constants and the functions $z(\rho, m), w(\rho, m)$ are the Riemann invariants given in (3-3). Then for any $\epsilon > 0$ and any $T > 0$, the following a-priori bounds hold for the Cauchy problem (3-7)-(3-8)

$$0 \leq \rho^\epsilon(x,t) \leq M(T), \qquad |m^\epsilon(x,t)| \leq M(T), \qquad (x,t) \in \mathbb{R}\times[0,T] \tag{3-10}$$

for a positive constant $M(T)$ independent of ϵ.

Proof. Multiplying the firts equation of the system (3-7) by z_ρ and the second equation of this system by z_m, adding these results we obtain

$$z_t + \lambda_1 z_x = \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x - (z_\rho g_1 + z_m g_2). \tag{3-11}$$

Using the Riemann invariant w one proceeds analogously to obtain the equality

$$w_t + \lambda_2 w_x = \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x + \frac{\epsilon}{\rho}\big(2p'(\rho) + \rho p''(\rho)\big)\rho_x^2 - (w_\rho g_1 + w_m g_2). \tag{3-12}$$

Inserting the inequalities (3-9) and the assumption on $\rho p(\rho)$ in the equalities (3-11)-(3-12), we obtain

$$z_t + \lambda_1 z_x + c_1 z + c_2 \leq \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x \tag{3-13}$$

and

$$w_t + \lambda_2 w_x + c_3 w + c_4 \geq \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x. \tag{3-14}$$

Applying the maximum principle to (3-13) and (3-14), we have that $z(\rho^\epsilon, m^\epsilon) \leq N_1(T)$ and $w(\rho^\epsilon, m^\epsilon) \geq N_2(T)$. Using the first equation in (3-7) and also that $\rho_0(x) \geq 0$, we get $\rho^\epsilon \geq 0$. It follows from the invariant region theory ([5],[28]) that the region

$$\Sigma = \big\{(\rho, m) : z(\rho, m) \leq N_1(T),\ w(\rho, m) \geq N_2(T),\ \rho \geq 0\big\}$$

is a bounded invariant region for two suitable constants $N_1(T), N_2(T)$. This region is depicted in *figure 3.1*. Thus we obtain the estimates in (3-10) for a suitable positive constant $M(T)$, which is independent of ϵ. □

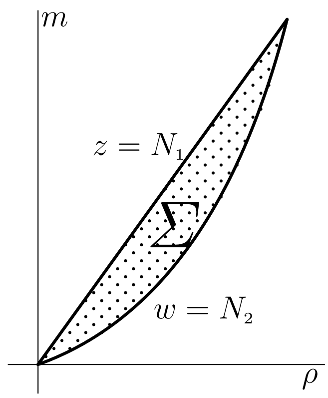

figure 3.1

Since system (3-7) is singular when $\rho = 0$, it is important to prove that a vacuum cannot occur for the diffusion system. Using the same argument as in the proof of lemma 2.1.2 in chapter one, for ρ we can obtain a-priori positive bounds dependent on ϵ.

Lemma 3.1.2. *We assume the hypotheses of lemma 3.1.1. Then for any $\epsilon > 0$, the problem (3-7)-(3-8) satisfies the a-priori positivity bound*

$$\rho^\epsilon(x, t) \geq c(t, \epsilon) > 0. \tag{3-15}$$

Proof. We rewrite the first equation in the system (3-7) as

$$\begin{aligned}(\ln\rho)_t + \big(u - p(\rho)\big)_x + \big(u - p(\rho)\big)(\ln\rho)_x + h(\rho, \rho u) &= \epsilon\frac{1}{\rho}\rho_{xx} \\ &= \epsilon(\ln\rho)_{xx} + \epsilon\big((\ln\rho)_x\big)^2.\end{aligned}$$

We set $\nu = -\ln\rho$, then we deduce that

$$\begin{aligned}\nu_t - \epsilon\nu_{xx} &= -\epsilon(\nu_x)^2 + \big(u-p(\rho)\big)(\ln\rho)_x + \big(u-p(\rho)\big)_x + h(\rho,\rho u)\\ &= -\epsilon\left(\nu_x + \frac{u-p(\rho)}{2\epsilon}\right)^2 + \frac{\big(u-p(\rho)\big)^2}{4\epsilon} + \big(u-p(\rho)\big)_x + h(\rho,\rho u)\\ &\le \frac{\big(u-p(\rho)\big)^2}{\epsilon} + \big(u-p(\rho)\big)_x + h(\rho,\rho u).\end{aligned}$$

Thus,

$$\nu(x,t) \le \nu_0^\epsilon(x) * k_\epsilon(x,t) + \int_0^t \Big(\tfrac{1}{\epsilon}\big(u-p(\rho)\big)^2 + \big(u-p(\rho)\big)_x + h(\rho,\rho u)\Big) *_x k_\epsilon(x,t-s)ds\,,$$

where $\nu_0^\epsilon(x) = -\ln\rho_0^\epsilon(x)$ and $k_\epsilon(x,t)$ is given by (2-11). Hence

$$\begin{aligned}\nu(x,t) &\le \nu_0^\epsilon(x) * k_\epsilon(x,t) + \frac{N_1}{\epsilon}t + \int_0^t \big(u-p(\rho)\big) *_x \big(k_\epsilon(x,s)\big)_x ds\\ &\le -\ln\epsilon + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\,,\end{aligned}$$

since $\rho_0^\epsilon(x) \ge \epsilon > 0$. Therefore

$$\rho(x,t) \ge \epsilon\exp -\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) \ge c(t,\epsilon) > 0.$$

and thus we obtain the bounds (3-15). □

3.1.2 An $L^1(\mathbb{R})$ estimate of $z_x(\cdot,t)$

A result is given here regarding the total variation of $z_x(\cdot,t)$, which is necessary in proofs of our lemmas concerning H^{-1}_{loc} compactness.

Lemma 3.1.3. *Let z be the Riemann invariant given in (3-3). If the total variation of $z_0(x) = z(x,0)$ $\big(TV\big(z_0(x)\big)\big)$ is bounded and there exist a function $G(s)$ satisfying*

$$G\Big(\tfrac{m}{\rho}\Big) = z_\rho g_1 + z_m g_2, \qquad G'(s) \ge 0, \tag{3-16}$$

then $z_x(\cdot,t)$ is bounded in $L^1(\mathbb{R})$, moreover

$$TV\big(z(\cdot,t)\big) = \int_{-\infty}^{+\infty} \big|z_x(x,t)\big|\,dx \le \int_{-\infty}^{+\infty} \big|z_{0x}(x)\big|\,dx = TV\big(z_0(x)\big). \tag{3-17}$$

Proof. We differentiate (3-11) with respect to x and then we do $\theta = z_x$ to get

$$\theta_t + \big(\lambda_1\theta\big)_x = \epsilon\theta_{xx} + \big(2\epsilon\rho^{-1}\rho_x\theta\big)_x - \big(z_\rho g_1 + z_m g_2\big)_x,$$

multiplying this equation by the sequence of smooth functions $g'(\theta,\alpha)$, where α is a parameter, we obtain

$$\begin{aligned} g(\theta,\alpha)_t + \big(\lambda_1 g(\theta,\alpha)\big)_x + \lambda_{1x}\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) &= \epsilon g(\theta,\alpha)_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 \\ &+ \big(2\epsilon\rho^{-1}\rho_x g(\theta,\alpha)\big)_x + \big(2\epsilon\rho^{-1}\rho_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) \\ &- \big(z_\rho g_1 + z_m g_2\big)_x g'(\theta,\alpha). \qquad (3\text{-}18) \end{aligned}$$

If we choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha) \geq 0$, $g'(\theta,\alpha) \to \text{sign}\theta$ and $g(\theta,\alpha) \to |\theta|$ as $\alpha \to 0$, we have from (3-18)

$$|\theta|_t + \big(\lambda_1|\theta|\big)_x = \epsilon|\theta|_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x - \text{sign}\theta\big(z_\rho g_1 + z_m g_2\big)_x,$$

and from this equality

$$|\theta|_t + \big(\lambda_1|\theta|\big)_x \leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x - \text{sign}\theta\big(z_\rho g_1 + z_m g_2\big)_x,$$

from this inequality and using (3-16) it follows that

$$\begin{aligned} |\theta|_t + \big(\lambda_1|\theta|\big)_x &\leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x - G'(z)|\theta| \\ &\leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x. \qquad (3\text{-}19) \end{aligned}$$

Integrating (3-19) in $\mathbb{R} \times [0,t]$, we obtain (3-17). □

Remark 3.1.4. *There are functions g_1, g_2 and $G(s)$, which satisfy the conditions in lemmas 3.1.1 and 3.1.3. As an example we have*

$$g_1(\rho,m) = a\rho, \quad g_2(\rho,m) = bm, \quad G(s) = (b-a)s,$$

where a, b are constants such that $b \geq a > 0$.
Another example

$$g_1(\rho,m) = \rho^2, \quad g_2(\rho,m) = m\rho, \quad G(s) = 0.$$

3.1.3 One family of entropy-entropy flux pairs

We begin by giving definition of a pair of convex entropy-entropy flux for the system (3-1). We say that η, q is a convex entropy-entropy flux pair if η is convex and

$$\nabla q(\rho,m) = \nabla\eta(\rho,m)\begin{pmatrix} -p(\rho) - \rho p'(\rho) & 1 \\ -\frac{m^2}{\rho^2} - mp'(\rho) & \frac{2m}{\rho} - p(\rho) \end{pmatrix}.$$

The system (3-1) has one family of entropy-entropy flux pairs, defined for arbitrary (smooth) functions F (see [24])

$$\big(\eta(\rho,m),q(\rho,m)\big)=\big(\rho F(z),\rho\phi(\rho,m)F(z)\big), \tag{3-20}$$

where $\phi(\rho,m)=\frac{m}{\rho}-p(\rho)$.

The entropy η in (3-20) is convex with respect to ρ and m if and only if the function F is convex with respect to the Riemann invariant z. Indeed, we have

$$\eta(\rho,m)=\rho F(z)=\rho F\left(\frac{m}{\rho}\right),\qquad \rho>0,$$

so that the eigenvalues of the Hessian matrix of η with respect to ρ and m are 0 and $\frac{1}{\rho^3}\big(\rho^2+m^2\big)F''\big(\frac{m}{\rho}\big)$. Therefore they are nonnegative if and only if F is convex.
We can demonstrate the lemma 2.1.7 by using these pairs of entropy-entropy flux.

3.1.4 H^{-1}_{loc} compactness

Lemma 3.1.5. *With the notation and assumptions of the lemmas 3.1.1 and 3.1.3, when $g(\rho)$ is an arbitrary smooth function then*

$$g(\rho^\epsilon)_t+\left(\int^{\rho^\epsilon} g'(s)f'(s)\,ds+g(\rho^\epsilon)u^\epsilon\right)_x \tag{3-21}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s)=-sp(s)$.

Proof. If we multiply each member of the first equation in (3-7) by $g'(\rho)$, we have

$$g(\rho)_t-g'(\rho)\big(\rho p(\rho)-\rho u\big)_x=\epsilon g(\rho)_{xx}-\epsilon g''(\rho)\rho_x^2-g'(\rho)g_1,$$

which is equivalent to

$$g(\rho)_t+\left(\int^{\rho} g'(s)f'(s)\,ds+g(\rho)u\right)_x=\epsilon g(\rho)_{xx}-\epsilon g''(\rho)\rho_x^2-g'(\rho)g_1+\big(g(\rho)-\rho g'(\rho)\big)u_x. \tag{3-22}$$

From this equation, choosing a strictly convex function $g(\rho)$ and using the L^1 estimate (3-17), we obtain

$$\epsilon(\rho_x^\epsilon)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{3-23}$$

This estimate shows that $\epsilon g''(\rho)\rho_x^2$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and also together with the Cauchy-Schwarz inequality allows to establish that $\epsilon g(\rho)_{xx}$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Moreover the two last terms in the right-hand side of (3-22) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Hence $-\epsilon g''(\rho)\rho_x^2 - g'(\rho)g_1 + \big(g(\rho)-\rho g'(\rho)\big)u_x$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$ (the space of Radon measures). The left-hand side of equation (3-22) is bounded in $W^{-1,\infty}_{loc}$. Using Murat's lemma ([20]) we get that (3-21) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

The next corollary is an immediate consequence of our previous lemma.

Corollary 3.1.6. *Suppose the conditions of the lemmas 2.1.1 and 2.1.3 hold and that $f(s)$ is as in lemma 3.1.5. Then*

$$\rho_t^\epsilon + \big(\rho^\epsilon u^\epsilon - \rho^\epsilon p(\rho^\epsilon)\big)_x \tag{3-24}$$

and

$$f(\rho^\epsilon)_t + \left(\int^{\rho^\epsilon} f'^2(s)\,ds + f(\rho^\epsilon)u^\epsilon\right)_x \tag{3-25}$$

are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. To prove this, we choose the particular functions $g(\rho)=\rho$ and $g(\rho)=f(\rho)$ in lemma 3.1.5. □

Lemma 3.1.7. *Under the same hypotheses as in the lemmas 3.1.1 and 3.1.3, it follows that*

$$(\rho^\epsilon u^\epsilon)_t + \Big(\rho^\epsilon u^\epsilon\big(u^\epsilon - p(\rho^\epsilon)\big)\Big)_x \tag{3-26}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Let $F(\cdot)$ be any convex function of the Riemann invariant z, due to what was exposed in subsection 2.1.3, it follows that $\eta(\rho,m)=\rho F(z)$ is a convex entropy of the system (3-1) with associated entropy flux $q(\rho,m)=\big(m-\rho p(\rho)\big)F(z)$. We multiply the first equation of the system (3-7) by η_ρ and the second by η_m. Adding up, we obtain

$$\eta_t + q_x = \epsilon\eta_{xx} - \epsilon F''(u)\rho u_x^2 + (ug_1 - g_2)F'(u) - g_1F(u) \tag{3-27}$$

One can choose a strictly convex function $F(u)$ in the equation (3-27) to obtain that

$$\epsilon\rho^\epsilon\big(u_x^\epsilon\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{3-28}$$

From the second equation in the system (3-7) we have

$$(\rho^\epsilon u^\epsilon)_t + \Big(\rho^\epsilon u^\epsilon\big(u^\epsilon - p(\rho^\epsilon)\big)\Big)_x = \epsilon(\rho^\epsilon u_x^\epsilon + u^\epsilon\rho_x^\epsilon)_x - g_2(\rho^\epsilon,\rho^\epsilon u^\epsilon). \tag{3-29}$$

Using the estimates (3-23) and (3-28) together with the Cauchy-Schwarz inequality, one can easily show that $\epsilon(\rho^\epsilon u_x^\epsilon + u^\epsilon\rho_x^\epsilon)_x$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and as $g_2(\rho^\epsilon,\rho^\epsilon u^\epsilon)$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and hence in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. We remark that (3-26) is bounded in $W^{-1,\infty}_{loc}$. Then applying Murat's lemma ([20]) we end the proof. □

3.1.5 Pointwise convergence of $\{\rho^\epsilon(x,t)\}$ and $\{u^\epsilon(x,t)\}$

The results given in the preceding subsection allows us to use the div-curl lemma of the compensated compactness theory to obtain the pointwise convergence of viscosity solutions.

Lemma 3.1.8. *Assuming the same notation and hypotheses as in the lemmas 3.1.1 and 3.1.3, we have that there exist a subsequence* $\{\rho^\epsilon(x,t)\}$ *and a subsequence* $\{u^\epsilon(x,t)\}$ *which converge pointwisely.*

Proof. We apply the div-curl lemma to the functions given by (3-24) and (3-25) to get

$$\overline{\rho^\epsilon \int_k^{\rho^\epsilon} f'^2(s)\,ds - f^2(\rho^\epsilon)} = \overline{\rho^\epsilon}\;\overline{\int_k^{\rho^\epsilon} f'^2(s)\,ds} \;-\; \overline{f(\rho^\epsilon)}^2 + \overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)u^\epsilon} \;-\; \overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon u^\epsilon}\,, \tag{3-30}$$

where k is real constant and for ρ^ϵ we denote by $\overline{\rho^\epsilon}$ its weak-star limit (i.e. $\overline{\rho^\epsilon} = w^\star - \lim \rho^\epsilon$). Let $\overline{\rho^\epsilon} = \rho$, so one has

$$\begin{aligned}\overline{\rho^\epsilon \int_k^{\rho^\epsilon} f'^2(s)\,ds - f^2(\rho^\epsilon)} \;=\; & \overline{\rho^\epsilon \int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon) - f(\rho)\big)^2} \\ & + \Big(\overline{f(\rho^\epsilon)} - f(\rho)\Big)^2 + \overline{\rho^\epsilon} \int_k^{\rho} f'^2(s)\,ds \;-\; \overline{f(\rho^\epsilon)}^2,\end{aligned} \tag{3-31}$$

and

$$\begin{aligned}\overline{\rho^\epsilon}\;\overline{\int_k^{\rho^\epsilon} f'^2(s)\,ds} \;-\; \overline{f(\rho^\epsilon)}^2 + \overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)u^\epsilon} \;-\; \overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon u^\epsilon} \;=\; & \overline{\rho^\epsilon}\;\overline{\int_\rho^{\rho^\epsilon} f'^2(s)\,ds} \\ & + \overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)u^\epsilon} \;-\; \overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon u^\epsilon} \;+\; \overline{\rho^\epsilon} \int_k^{\rho} f'^2(s)\,ds \;-\; \overline{f(\rho^\epsilon)}^2,\end{aligned}$$

using these two equalities in (3-30) gives

$$\overline{(\rho^\epsilon - \rho)\int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon) - f(\rho)\big)^2} \;+\; \Big(\overline{f(\rho^\epsilon)} - f(\rho)\Big)^2 = \overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)u^\epsilon} \;-\; \overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon u^\epsilon}\,. \tag{3-32}$$

It follows from the div-curl lemma applied to the functions (3-24) and (3-26) that

$$\overline{\rho^\epsilon}\;\overline{\rho^\epsilon (u^\epsilon)^2 + f(\rho^\epsilon)u^\epsilon} \;-\; \overline{\rho^\epsilon u^\epsilon}\;\overline{\rho^\epsilon u^\epsilon + f(\rho^\epsilon)} = 0,$$

which permits us to write

$$\overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)u^\epsilon} \;-\; \overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon u^\epsilon} = \overline{\rho^\epsilon u^\epsilon}^2 \;-\; \overline{\rho^\epsilon}\;\overline{\rho^\epsilon (u^\epsilon)^2}\,. \tag{3-33}$$

From the equations (3-32)-(3-33) we see that

$$\overline{(\rho^\epsilon - \rho)\int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon) - f(\rho)\big)^2} \;+\; \Big(\overline{f(\rho^\epsilon)} - f(\rho)\Big)^2 = \overline{\rho^\epsilon u^\epsilon}^2 - \overline{\rho^\epsilon}\;\overline{\rho^\epsilon (u^\epsilon)^2}\,. \tag{3-34}$$

Since both terms in the left-hand side of (3-34) are nonnegative and the right-hand side is nonpositive, then we must have that

$$\overline{(\rho^\epsilon - \rho)\int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon) - f(\rho)\big)^2} + \Big(\overline{f(\rho^\epsilon)} - f(\rho)\Big)^2 = 0, \tag{3-35}$$

and

$$\overline{\rho^\epsilon u^\epsilon}^2 - \overline{\rho^\epsilon}\;\overline{\rho^\epsilon (u^\epsilon)^2} = 0\,. \tag{3-36}$$

The last two equalities allow us to prove, respectively, the pointwise convergence of $\{\rho^\epsilon\}$ and $\{u^\epsilon\}$ in the region of $\rho > 0$. □

3.1.6 Existence of weak solution

We are now ready to give a result on the existence of weak solution. It is contained in the following theorem.

Theorem 3.1.9. *We assume that $p(0) = 0$, $\lim_{\rho\to 0} \rho p'(\rho) = 0$, the function $\rho p(\rho)$ is strictly convex for positive ρ, $g_1(\rho, m) = \rho h(\rho, m)$ for a continuous function $h(\rho, m)$ and that $g_1(\rho, m)$, $g_2(\rho, m)$ satisfy the inequalities $z_\rho g_1 + z_m g_2 \geq c_1 z + c_2$, $w_\rho g_1 + w_m g_2 \leq c_3 w + c_4$, where c_i $i = 1, \dots, 4$ are real constants and the functions $z(\rho, m), w(\rho, m)$ are the Riemann invariants given in (3-4). If the total variation of $z_0(x) = z(x, 0)$ is bounded and there exist a function $G(s)$ satisfying $G\left(\frac{m}{\rho}\right) = z_\rho g_1 + z_m g_2$, $G'(s) \geq 0$, then the Cauchy problem (3-4)-(3-5) has a weak solution.*

Proof. Multiplying the firts equation of the system (3-7) by φ and the second equation of this system by ψ, where $\varphi, \psi \in C_0^\infty\big(\mathbb{R} \times [0, \infty)\big)$, adding these results and integrating over $\mathbb{R} \times [0, \infty)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{+\infty} \Big(\rho^\epsilon \varphi_t + \rho^\epsilon\big(u^\epsilon - p(\rho^\epsilon)\big)\varphi_x - g_1(\rho^\epsilon, \rho^\epsilon u^\epsilon)\varphi\Big)\,dt\,dx + \int_{\mathbb{R}} \rho_0 \varphi(x, 0)\,dx \\
&\quad + \int_{\mathbb{R}}\int_0^{+\infty} \Big(\rho^\epsilon u^\epsilon \psi_t + \rho^\epsilon u^\epsilon\big(u^\epsilon - p(\rho^\epsilon)\big)\psi_x - g_2(\rho^\epsilon, \rho^\epsilon u^\epsilon)\psi\Big)\,dt\,dx + \int_{\mathbb{R}} \rho_0 u_0 \psi(x, 0)\,dx = \\
&\qquad\qquad - \epsilon \int_{\mathbb{R}}\int_0^{+\infty} (\rho^\epsilon \varphi_{xx} + \rho^\epsilon u^\epsilon \psi_{xx})\,dt\,dx.
\end{aligned} \tag{3-37}$$

The argument used in (2-53) yield

$$\lim_{\epsilon\to 0} \epsilon \int_{\mathbb{R}}\int_0^{+\infty} (\rho^\epsilon \varphi_{xx} + \rho^\epsilon u^\epsilon \psi_{xx})\,dt\,dx = 0. \tag{3-38}$$

From the equality (3-37), and using (3-38) and lemma 3.1.8, we obtain the conclusion in this theorem. □

3.2 The Cauchy problem for the Aw-Rascle model

We have now the following corollary concerning the Cauchy problem for the Aw-Rascle traffic flow model

$$\begin{cases} \rho_t + \big(\rho(u - p(\rho))\big)_x = 0 \\ (\rho u)_t + \big(\rho u(u - p(\rho))\big)_x = 0, \end{cases} \tag{3-39}$$

with bounded measurable initial data

$$\big(\rho(x,0), u(x,0)\big) = \big(\rho_0(x), u_0(x)\big), \quad \rho_0(x) \geq 0, \tag{3-40}$$

where $p(\rho)$ is a smooth strictly increasing function.

3.2.1 Existence of weak entropy solution

In this subsection we consider the existence of a weak entropy solution.

Corollary 3.2.1. *We assume that* $p(0) = 0$, $\lim_{\rho \to 0} \rho p'(\rho) = 0$, *the function* $\rho p(\rho)$ *is strictly convex for positive* ρ, *let* z *be the Riemann invariant given in (3-4). If the total variation of* $z_0(x) = z(x,0)$ *is bounded, then the Cauchy problem (3-39)-(3-40) has a weak entropy solution.*

Proof. The result follows immediately from theorem 3.1.9 by taking $g_1(\rho, \rho u) = g_2(\rho, \rho u) = 0$. □

4 Existence of weak solution for the Aw-Rascle traffic flow model: A second result

In this chapter we again study the Cauchy problem associated to the Aw-Rascle model with a source but modifying the conditions on $p(\rho)$ given in [2] and under which the results on existence of weak solution (Theorem 3.1.9) and weak entropy solution (Corollary 3.2.1) were established in the previous chapter.

With the conditions on $p(\rho)$ assumed in this chapter an argument following the ideas of Bereux and Sainsaulieu is not valid to prove the positivity of ρ^ϵ, which shows that the vacuum state is not present in the diffusion system (3-7). The technique used here for a positivity proof is an adaptation of a technique due to Lu, which he first introduced in [17] to study the isentropic gas dynamics system for general pressure function. For this reason in Section 4.1, we introduce a new flux approximation and also an approximation of the source term in the first equation of the Aw-Rascle system with a source (3-6) by adding a small perturbation. We can now obtain in subsection 4.1.1 the estimate $\rho^{\epsilon,\delta} \geq \delta > 0$, in the same subsection we can also obtain a priori bounds dependent on ϵ and δ. Borrowing the scheme given in section 3.1, we get a second result on the existence of weak solution for the Cauchy problem (3-4)-(3-5). In section 4.2, we prove the existence of weak entropy solution to the Cauchy problem (3-39)-(3-40).

4.1 The Cauchy problem for the Aw-Rascle model with a source

To establish our second result, we first consider the following approximate system

$$\begin{cases} \rho_t + \left(\frac{(\rho-\delta)m}{\rho} - (\rho-\delta)p(\rho)\right)_x + \frac{\rho-\delta}{\rho} g_1(\rho,m) = 0 \\ m_t + \left(\frac{(\rho-\delta)m^2}{\rho^2} - \frac{(\rho-\delta)m}{\rho}p(\rho)\right)_x + g_2(\rho,m) = 0, \end{cases} \tag{4-1}$$

where $\delta > 0$ is a small perturbation constant.

The matrix

$$dF_{\delta(\rho,m)} = \begin{pmatrix} \frac{\delta m}{\rho^2} - p(\rho) - (\rho-\delta)p'(\rho) & \frac{\rho-\delta}{\rho} \\ \frac{(2\delta-\rho)m^2}{\rho^3} - \frac{\delta m}{\rho^2}p(\rho) - \frac{(\rho-\delta)m}{\rho}p'(\rho) & \frac{2(\rho-\delta)m}{\rho^2} - \frac{\rho-\delta}{\rho}p(\rho) \end{pmatrix},$$

is the Jacobian matrix of the flux functions in (4-1). The matrix $dF_{\delta(\rho,m)}$ has eigenvalues

$$\lambda_{\delta 1} = \frac{\rho-\delta}{\rho}\left(\frac{m}{\rho} - p(\rho)\right), \qquad \lambda_{\delta 2} = \frac{m}{\rho} - p(\rho) - (\rho-\delta)p'(\rho)\,, \tag{4-2}$$

with corresponding Riemann invariants

$$z_\delta(\rho,m) = \frac{m}{\rho} = z(\rho,m), \qquad w_\delta(\rho,m) = \frac{\rho-\delta}{\rho}\left(\frac{m}{\rho} - p(\rho)\right) = \frac{\rho-\delta}{\rho}w(\rho,m), \tag{4-3}$$

where $z(\rho,m)$ and $w(\rho,m)$ are given by (3-3).

4.1.1 Existence of viscosity solutions

To (4-1) there is associated the diffusive system

$$\begin{cases} \rho^\epsilon_t + \left(\frac{(\rho^\epsilon-\delta)m^\epsilon}{\rho^\epsilon} - (\rho^\epsilon-\delta)p(\rho^\epsilon)\right)_x + \frac{\rho^\epsilon-\delta}{\rho^\epsilon}g_1(\rho^\epsilon, m^\epsilon) = \epsilon\rho^\epsilon_{xx} \\ m^\epsilon_t + \left(\frac{(\rho^\epsilon-\delta)(m^\epsilon)^2}{(\rho^\epsilon)^2} - \frac{(\rho^\epsilon-\delta)m^\epsilon}{\rho^\epsilon}p(\rho^\epsilon)\right)_x + g_2(\rho^\epsilon, m^\epsilon) = \epsilon m^\epsilon_{xx}. \end{cases} \tag{4-4}$$

We now consider the Cauchy problem (4-4) with initial data (3-5). We denote the functions ρ, m with the indexes $\rho^{\epsilon,\delta}, m^{\epsilon,\delta}$, only when it avoids ambiguities.
One can deduce the existence of solutions to the diffusive system under certain conditions on the function p, the source terms g_i, $i = 1,2$, on $\rho_0(x)$ and on the Riemann invariant z given by (3-4). For it we derive L^∞ bounds for solutions.

Lemma 4.1.1. *Let p satisfy $\lim_{\rho\to 0}\rho p(\rho) = 0$, $2p'(\rho)+\rho p''(\rho) > 0$ for $\rho > 0$, and assume that $\rho_0(x) \geq c_0 > 0$ and $w_0(x) = w(x,0) \geq c_1 > 0$ for two constants c_0, c_1, $g_1(\rho,m) = \rho h(\rho,m)$ for a continuous function $h(\rho,m)$ and that*

$$\frac{\rho-\delta}{\rho}z_{\delta\rho}g_1 + z_{\delta m}g_2 \geq c_2 z_\delta + c_3, \qquad \frac{\rho-\delta}{\rho}w_{\delta\rho}g_1 + w_{\delta m}g_2 \leq c_4 w_\delta + c_5, \tag{4-5}$$

where c_i $i = 2,\ldots,5$ are real constants and the functions $z_\delta(\rho,m), w_\delta(\rho,m)$ are the Riemann invariants given in (4-3). Then for any $\epsilon > 0$ and any $T > 0$, we have the a-priori bounds for the Cauchy problem (4-4)-(3-5)

$$\delta \leq \rho^{\epsilon,\delta} \leq M(T), \qquad \left|m^{\epsilon,\delta}\right| \leq M(T), \qquad (x,t) \in \mathbb{R}\times[0,T] \tag{4-6}$$

for a positive constant $M(T)$ independent of ϵ and δ.

Proof. We multiply the first and second equations of system (4-4) respectively by $z_{\delta\rho}$ and $z_{\delta m}$ and adding the results, we obtain

$$z_{\delta t} + \lambda_{\delta 1} z_{\delta x} + \frac{\rho-\delta}{\rho} g_1 z_{\delta\rho} + g_2 z_{\delta m} = \epsilon z_{\delta xx} + \frac{2\epsilon}{\rho}\rho_x z_{\delta x}. \tag{4-7}$$

Proceed similarly with the Riemman invariant w_δ, we find

$$w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{\rho-\delta}{\rho} g_1 w_{\delta\rho} + g_2 w_{\delta m} = \epsilon w_{\delta xx} - \epsilon\Big(w_{\delta\rho\rho}\rho_x^2 + 2w_{\delta\rho m}\rho_x m_x + w_{\delta mm} m_x^2\Big). \tag{4-8}$$

Algebraic manipulations on the equation (4-8) yields

$$\begin{aligned} w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{2\epsilon}{\rho}\Big(\frac{\delta}{\rho-\delta} - 1\Big)\rho_x w_{\delta x} - \frac{2\epsilon\delta^2}{\rho^2(\rho-\delta)^2}\rho_x^2 w_\delta + \frac{\rho-\delta}{\rho} g_1 w_{\delta\rho} \\ + g_2 w_{\delta m} = \epsilon w_{\delta xx} + \frac{\epsilon(\rho-\delta)}{\rho^2}\big(2p'(\rho) + \rho p''(\rho)\big)\rho_x^2. \end{aligned} \tag{4-9}$$

Using the inequalities (4-5) in the equalities (4-8)-(4-9) together with the assumption of strict convexity for $\rho p(\rho)$, we get the following inequalities

$$z_{\delta t} + \lambda_{\delta 1} z_{\delta x} + c_2 z_\delta + c_3 \le \epsilon z_{\delta xx} + \frac{2\epsilon}{\rho}\rho_x z_{\delta x}, \tag{4-10}$$

and

$$w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{2\epsilon}{\rho}\Big(\frac{\delta}{\rho-\delta} - 1\Big)\rho_x w_{\delta x} - \frac{2\epsilon\delta^2}{\rho^2(\rho-\delta)^2}\rho_x^2 w_\delta + c_4 w_\delta + c_5 \ge \epsilon w_{\delta xx}. \tag{4-11}$$

By applying the maximum principle to the inequality (4-10) we get the estimate $z(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) = z_\delta(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \le N(T)$. Since $\rho_0(x) \ge c_0 > 0$ and $w_0(x) \ge c_1 > 0$, this means that $w_{\delta 0}(x) = w_\delta(x,0) \ge \frac{1}{2}c_1$ for small δ, again applying the maximum principle to (4-11) we obtain the estimate $w_\delta(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \ge \frac{1}{2}c_1$. We have from (4-3) the equality

$$w\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big) = \frac{\rho^{\epsilon,\delta}}{\rho^{\epsilon,\delta} - \delta} w_\delta\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big),$$

this readily leads to the bounded $w\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big) \ge \frac{1}{2}c_1 > 0$. Using the first equation in (4-4), we get $\rho^{\epsilon,\delta} \ge \delta$. The region

$$\varSigma = \Big\{(\rho,m) : z(\rho,m) \le N,\, w(\rho,m) \ge \tfrac{1}{2}c_1,\, \rho \ge \delta\Big\}$$

is a bounded invariant region (see *figure 4.1*) for a suitable constant N. Then for $\rho^{\epsilon,\delta}, m^{\epsilon,\delta}$ we have the bounds

$$\delta \le \rho^{\epsilon,\delta} \le M(T), \qquad \Big|m^{\epsilon,\delta}\Big| \le M(T),$$

for a suitable constant $M(T)$, which is independent of ϵ and δ. □

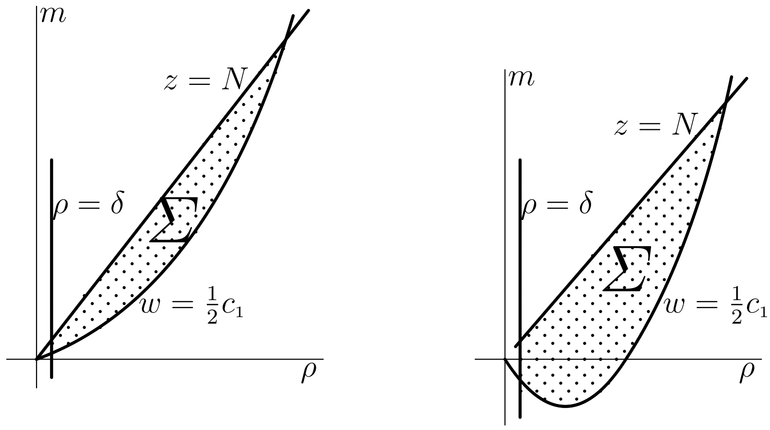

figure 4.1

Remark 4.1.2. *It may appear at the first glance that the hypothesis on $w_0(x)$ is unnecessary for the validity of lemma 4.1.1. An example however will show that the lemma is not generally true if omit this hypothesis. If one does not assume this condition, then for $p(\rho) = -\frac{1}{a}\rho^{-a}$, $0 < a < 1$ the region Σ is unbounded, this region is depicted in figure 4.2.*
Indeed, the condition $w_0(x) = w(x,0) \geq c_1 > 0$ in lemma 4.1.1 is necessary to ensures that the invariant region Σ is bounded when $p(\rho) \leq 0$.

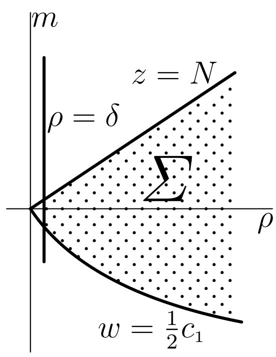

figure 4.2

Using exactly the same method as in the proof of lemma 3.1.2, for $\rho^{\epsilon,\delta}$ we can also obtain a-priori positive lower bounds dependent on ϵ and δ.

Lemma 4.1.3. *With assumptions given in lemma 4.1.1, the following a-priori bounds hold for the problem (4-5)-(3-5)*

$$\rho^{\epsilon,\delta}(x,t) \geq c(t,\epsilon,\delta) > \delta, \tag{4-12}$$

where $c(t,\epsilon,\delta)$ could tend to δ as $t \to +\infty$ or $\epsilon \to 0$.

Proof. We obtain as in the proof of lemma 3.1.2,

$$\nu(x,t) \le \nu_0^\epsilon(x) * k_\epsilon(x,t) + \frac{N_1}{\epsilon}t + \int_0^t \big(u - p(\rho)\big) *_x \big(k_\epsilon(x,t-s)\big)_x\, ds,$$

where $\nu = -\ln(\rho - \delta)$, $\nu_0^\epsilon(x) = -\ln\big(\rho_0(x) - \delta\big)$ and $k_\epsilon(x,t)$ is given by (2-11).
It follows from the above inequality that

$$\nu(x,t) \le -\ln\delta + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\,,$$

since $\rho_0(x) \ge 2\delta$. Therefore

$$\rho(x,t) \ge \delta \exp -\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) + \delta \ge c(t,\epsilon,\delta) > \delta > 0,$$

and thus we obtain the bounds (4-12). □

4.1.2 An L^1 estimate of $z_{\delta x}(\cdot,t)$

Lemma 4.1.4. *Let z be the Riemann invariant given in (3-3). If the total variation of $z_0(x) = z(x,0)$ is bounded and there exist a function $G(s)$ satisfying*

$$G\Big(\tfrac{m}{\rho}\Big) = \frac{\rho-\delta}{\rho} z_\rho g_1 + z_m g_2, \qquad G'(s) \ge 0, \tag{4-13}$$

then $z_{\delta x}(\cdot,t)$ is bounded in $L^1(\mathbb{R})$, moreover

$$TV\big(z_\delta(\cdot,t)\big) = \int_{-\infty}^{+\infty} \big|z_{\delta x}(x,t)\big|\, dx \le \int_{-\infty}^{+\infty} \big|z_{0x}(x)\big|\, dx = TV\big(z_0(x)\big). \tag{4-14}$$

Proof. Following the same kind of calculation as in the proof of the lemma 3.1.3, from the equality (4-7) one readily checks the next inequality

$$|\theta|_t + \big(\lambda_\delta|\theta|\big)_x + \operatorname{sign}\theta \left(\frac{\rho-\delta}{\rho} g_1 z_{\delta\rho} + g_2 z_{\delta m}\right)_x \le \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x,$$

where $\theta = z_{\delta x}$. By using the assumption (4-13) in the above inequality, we obtain

$$\begin{aligned} |\theta|_t + \big(\lambda_\delta|\theta|\big)_x &\le |\theta|_t + \big(\lambda_\delta|\theta|\big)_x + G'(z_\delta)|\theta| \\ &\le \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x\,, \end{aligned} \tag{4-15}$$

and integrate it over $\mathbb{R} \times [0,t]$, we conclude the result of the lemma. □

Remark 4.1.5. *There are functions g_1, g_2 and $G(s)$ satisfying the assumptions of the lemmas 4.1.1 and 4.1.4, such as*

$$g_1(\rho,m) = a\rho^2, \quad g_2(\rho,m) = a\rho m, \quad G(s) = a\delta s,$$

or

$$g_1(\rho,m) = 0, \quad g_2(\rho,m) = am, \quad G(s) = as,$$

where $a > 0$ is a constant.

4.1.3 One family of entropy-entropy flux pairs

A pair (η, q) is called an entropy-entropy flux pair of system (4-1) if it satisfies the system

$$\nabla q(\rho,m) = \nabla \eta(\rho,m) \begin{pmatrix} \frac{\delta m}{\rho^2} - p(\rho) - (\rho-\delta)p'(\rho) & \frac{\rho-\delta}{\rho} \\ \frac{(2\delta-\rho)m^2}{\rho^3} - \frac{\delta m}{\rho^2}p(\rho) - \frac{(\rho-\delta)m}{\rho}p'(\rho) & \frac{2(\rho-\delta)m}{\rho^2} - \frac{\rho-\delta}{\rho}p(\rho) \end{pmatrix}.$$

Furthermore, $\eta(\rho, m)$ is called a convex entropy if $\eta(\rho, m)$ is convex.

Let $F(\cdot)$ be any convex function of the Riemman invariant z given in (3-3), the pair

$$\big(\eta(\rho,m), q(\rho,m)\big) = \big(\rho F(z), \rho\phi_\delta(\rho,m)F(z)\big), \tag{4-16}$$

where $\phi_\delta(\rho,m) = \frac{\rho-\delta}{\rho}\big(\frac{m}{\rho} - p(\rho)\big)$, is a pair convex entropy-entropy flux for the system (4-1), this pair will be used later in subsection 3.1.3.

4.1.4 H^{-1}_{loc} compactness

Lemma 4.1.6. *We assume the same conditions given in the lemmas 4.1.1 and 4.1.4. Let $g(\rho)$ be an arbitrary smooth function, then*

$$g\big(\rho^{\epsilon,\delta}\big)_t + \left(\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds + g\big(\rho^{\epsilon,\delta}\big)u^{\epsilon,\delta}\right)_x \tag{4-17}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s) = -sp(s)$.

Proof. Multiplying the first equation of system (4-4) by $g'(\rho)$, we obtain

$$\begin{aligned} g(\rho)_t - g'(\rho)\big(\rho p(\rho) - \rho u\big)_x - \delta g'(\rho)u_x + \delta g'(\rho)p(\rho)_x = \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2 \\ - (\rho-\delta)g'(\rho)h(\rho,\rho u), \end{aligned}$$

this equation is equivalent to

$$\begin{aligned} g(\rho)_t + \left(\int^{\rho} g'(s)f'(s)\,ds + g(\rho)u\right)_x &= \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2 \\ - \delta\left(\int^{\rho} g'(s)p'(s)ds\right)_x &+ \big(g(\rho) - \rho g'(\rho) + \delta g'(\rho)\big)u_x - (\rho-\delta)g'(\rho)h(\rho,\rho u). \end{aligned} \tag{4-18}$$

By using the L^1 estimate (4-14) and taking a strictly convex function $g(\rho)$ into (4-18), we see that

$$\epsilon\big(\rho_x^{\epsilon,\delta}\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+), \tag{4-19}$$

and from here that $-\epsilon g''(\rho)\rho_x^2$ is also bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Since the two last terms in the right-hand side of (4-18) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, we have that $-\epsilon g''(\rho)\rho_x^2+\big(g(\rho)-\rho g'(\rho)+\delta g'(\rho)\big)u_x-(\rho-\delta)g'(\rho)h(\rho,\rho u)$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. On the other hand, again the estimate (4-19) together with the Cauchy-Schwarz inequality allows to establish that $\epsilon g(\rho)_{xx}$ and $-\delta\big(\int^\rho g'(s)f'(s)ds\big)_x$ are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Finally, the $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ compactness of (4-17) follows from Murat's lemma since (4-17) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. This completes the proof. □

As a special case of lemma 4.1.6 we have the next corollary.

Corollary 4.1.7. *Assuming the hypotheses as in the lemmas 4.1.1 and 4.1.4. If $g(\rho)$ is an arbitrary smooth function then*

$$\left(\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_t+\left(\int^{\rho^{\epsilon,\delta}} g'(s)f'^2(s)\,ds+u^{\epsilon,\delta}\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_x, \tag{4-20}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s)=-sp(s)$.

Lemma 4.1.8. *We assume that $g(\rho)$ is a smooth function, strictly increasing such that $\lim_{\rho\to 0} g(\rho)p(\rho)=0$, so that $\int^\rho g'(s)f'(s)\,ds$ is a regular function at $\rho=0$, where $f(s)=-sp(s)$, and together with the hypotheses of the lemmas 4.1.1 and 4.1.4, we have*

$$\left(g\big(\rho^{\epsilon,\delta}\big)u^{\epsilon,\delta}\right)_t+\left(g\big(\rho^{\epsilon,\delta}\big)\big(u^{\epsilon,\delta}\big)^2+u^{\epsilon,\delta}\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_x \tag{4-21}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Multiplying the firts equation of the system (4-4) by η_ρ and the second equation by η_m, where η is the convex entropy given in (4-16). Adding up, we obtain

$$\eta_t+q_x=\epsilon\eta_{xx}-\epsilon F''(u)\rho u_x^2-\left(\frac{\rho-\delta}{\rho}\eta_\rho g_1+\eta_m g_2\right).$$

We can choose a strictly convex function $F(u)$ in the above equation to obtain that

$$\epsilon\rho^{\epsilon,\delta}\big(u_x^{\epsilon,\delta}\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{4-22}$$

In order to show the compactness of (4-21), we now multiply the equations (4-7) by $g(\rho)$ and (4-18) by z_δ, then adding the results and using that $z_\delta=u$ together with (4-14), we get

$$\begin{aligned}\big(g(\rho)u\big)_t&+\left(g(\rho)u^2+u\int^\rho g'(s)f'(s)\,ds\right)_x=\epsilon\big(g(\rho)u\big)_{xx}+\frac{2\epsilon}{\rho}\big(g(\rho)-\rho g'(\rho)\big)\rho_x u_x\\&-\epsilon g''(\rho)\rho_x^2u-\delta\left(u\int^\rho g'(s)p'(s)\,ds\right)_x+\big(g(\rho)-(\rho-\delta)g'(\rho)\big)u_xu-(\rho-\delta)g'(\rho)hu\\&-G(u)g(\rho)+\left(g(\rho)u-\lambda_{\delta 1}g(\rho)+\int^\rho g'(s)f'(s)\,ds+\delta\int^\rho g'(s)p'(s)\,ds\right)z_{\delta x}.\end{aligned} \tag{4-23}$$

The terms $\epsilon\big(g(\rho)u\big)_{xx}$ and $-\delta\big(u\int^{\rho} g'(s)p'(s)\,ds\big)_x$ in the above equation are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^{+})$, the other terms on the right-hand side of (4-23) are bounded in $L^{1}_{loc}(\mathbb{R}\times\mathbb{R}^{+})$. The left-hand side of (4-23) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^{+})$. The Murat's lemma implies the H^{-1}_{loc} compactness of (4-21). □

4.1.5 Pointwise convergence of $\{\rho^{\epsilon,\delta}(x,t)\}$ and $\{u^{\epsilon,\delta}(x,t)\}$

To show that the sequence of solutions to the system (4-4) has a subsequence which converges strongly again use the div-curl lemma for one appropriate function g satisfying the conditions in lemma 4.1.8.

Lemma 4.1.9. *When the hypotheses in the lemmas 4.1.1 and 4.1.4 are satisfied, then a subsequence of* $\{\rho^{\epsilon,\delta}\}$ *and a subsequence of* $\{u^{\epsilon,\delta}\}$ *converge pointwisely.*

Proof. Let $g(\rho)$ be a function as in lemma 4.1.7, but such that it is nonnegative strictly increasing. We use the div-curl lemma to the functions (4-17), (4-21) and then making $g(\rho^{\epsilon,\delta}) = \mu^{\epsilon,\delta}$ and $\int^{\rho^{\epsilon,\delta}} g'(s)f'(s\,)ds = F\big(\mu^{\epsilon,\delta}\big)$, this yields

$$\Big(\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\Big)^2 - \overline{\mu^{\epsilon,\delta}}\;\overline{\mu^{\epsilon,\delta}\big(u^{\epsilon,\delta}\big)^2} = \overline{\mu^{\epsilon,\delta}}\;\overline{\mu^{\epsilon,\delta}F\big(\mu^{\epsilon,\delta}\big)} - \overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\;\overline{F\big(\mu^{\epsilon,\delta}\big)}\,. \tag{4-24}$$

We have that (4-20) can be written as

$$F\big(\mu^{\epsilon,\delta}\big)_t + \left(\int^{\mu^{\epsilon,\delta}} F'^2(s)\,ds + u^{\epsilon,\delta}F\big(\mu^{\epsilon,\delta}\big)\right)_x, \tag{4-25}$$

since $\int^{\rho^{\epsilon,\delta}} g'(s)f'^2(s)\,ds = \int^{\mu^{\epsilon,\delta}} F'^2(s)\,ds$.
By the div-curl lemma, for the functions (4-17) and (4-25), it follows that

$$\overline{\mu^{\epsilon,\delta}\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - F^2\big(\mu^{\epsilon,\delta}\big)} = \overline{\mu^{\epsilon,\delta}}\;\overline{\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} + \overline{\mu^{\epsilon,\delta}}\;\overline{u^{\epsilon,\delta}F\big(\mu^{\epsilon,\delta}\big)} - \overline{F\big(\mu^{\epsilon,\delta}\big)}^2 - \overline{F\big(\mu^{\epsilon,\delta}\big)}\;\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\,, \tag{4-26}$$

where k is real constant. Let $\overline{\mu^{\epsilon,\delta}} = \mu$, so from (4-26) we obtain that

$$\overline{\big(\mu^{\epsilon,\delta}-\mu\big)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \Big(F\big(\mu^{\epsilon,\delta}\big) - F(\mu)\Big)^2} + \Big(\overline{F\big(\mu^{\epsilon,\delta}\big)} - F(\mu)\Big)^2 = \overline{\mu^{\epsilon,\delta}}\;\overline{u^{\epsilon,\delta}F\big(\mu^{\epsilon,\delta}\big)} - \overline{F\big(\mu^{\epsilon,\delta}\big)}\;\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\,, \tag{4-27}$$

where we have used that

$$\overline{\mu^{\epsilon,\delta}\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - F^2\big(\mu^{\epsilon,\delta}\big)} = \overline{\mu^{\epsilon,\delta}\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \Big(F\big(\mu^{\epsilon,\delta}\big) - F(\mu)\Big)^2} + \Big(\overline{F\big(\mu^{\epsilon,\delta}\big)} - F(\mu)\Big)^2 + \overline{\mu^{\epsilon,\delta}}\int_k^{\mu} F'^2(s)\,ds - \overline{F\big(\mu^{\epsilon,\delta}\big)}^2$$

and

$$\overline{\mu^{\epsilon,\delta}}\ \overline{\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} = \overline{\mu^{\epsilon,\delta}}\ \overline{\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} + \overline{\mu^{\epsilon,\delta}}\ \overline{\int_k^{\mu} F'^2(s)\,ds}\,.$$

Since the right-hand side of both equations (4-24) and (4-27) are equal, we have

$$\Big(\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\Big)^2 - \overline{\mu^{\epsilon,\delta}}\ \overline{\mu^{\epsilon,\delta}\left(u^{\epsilon,\delta}\right)^2} = \overline{\left(\mu^{\epsilon,\delta}-\mu\right)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \Big(F\left(\mu^{\epsilon,\delta}\right)-F(\mu)\Big)^2} + \Big(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\Big)^2. \quad (4\text{-}28)$$

As the left-hand side of the above equation is nonpositive and the right-hand side is nonnegative, then both sides of this equation must be zero, i.e.,

$$\overline{\left(\mu^{\epsilon,\delta}-\mu\right)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \Big(F\left(\mu^{\epsilon,\delta}\right)-F(\mu)\Big)^2} + \Big(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\Big)^2 = 0, \quad (4\text{-}29)$$

and

$$\Big(\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\Big)^2 - \overline{\mu^{\epsilon,\delta}}\ \overline{\mu^{\epsilon,\delta}\left(u^{\epsilon,\delta}\right)^2} = 0. \quad (4\text{-}30)$$

The equality (4-29) allow us to prove the pointwise convergence of $\left\{\mu^{\epsilon,\delta}\right\}$ and so we get the convergence of $\left\{\rho^{\epsilon,\delta}\right\}$ since $g(\rho)$ is a strictly increasing function. From (4-30) we get the pointwise convergence of $\left\{u^{\epsilon,\delta}\right\}$ in the region of $\rho > 0$. □

4.1.6 Existence of weak solution

Using the lemma 4.1.9, we reach the following result, with the aid of the Lebesgue dominated convergence theorem.

Theorem 4.1.10. *Let p satisfy $\lim_{\rho\to 0}\rho p(\rho) = 0$, $2p'(\rho) + \rho p''(\rho) > 0$ for $\rho > 0$, and assume that $\rho_0(x) \geq c_0 > 0$ and $w_0(x) = w(x,0) \geq c_1 > 0$ for two constants c_0, c_1, $g_1(\rho,m) = \rho h(\rho,m)$ for a continuous function $h(\rho,m)$ and that $\frac{\rho-\delta}{\rho} z_{\delta\rho} g_1 + z_{\delta m} g_2 \geq c_2 z_\delta + c_3$, $\frac{\rho-\delta}{\rho} w_{\delta\rho} g_1 + w_{\delta m} g_2 \leq c_4 w_\delta + c_5$, where c_i $i = 2,\ldots,5$ are real constants and the functions $z_\delta(\rho,m), w_\delta(\rho,m)$ are the Riemann invariants given in (4-3). Let z be the Riemann invariant given in (3-3). If the total variation of $z_0(x) = z(x,0)$ is bounded and there exist a function $G(s)$ satisfying $G\Big(\frac{m}{\rho}\Big) = \frac{\rho-\delta}{\rho} z_\rho g_1 + z_m g_2$, $G'(s) \geq 0$, then we have that the Cauchy problem (3-4)-(3-5) has a weak solution.*

Proof. We write the approximate system (4-4) in the weak form

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{+\infty}\Big(\rho^{\epsilon,\delta}\varphi_t+\big(\rho^{\epsilon,\delta}-\delta\big)\Big(u^{\epsilon,\delta}-p\big(\rho^{\epsilon,\delta}\big)\Big)\varphi_x-\big(\rho^{\epsilon,\delta}-\delta\big)h\big(\rho^{\epsilon,\delta},\rho^{\epsilon,\delta}u^{\epsilon,\delta}\big)\varphi\Big)\,dt\,dx\\
&\quad+\int_{\mathbb{R}}\int_0^{+\infty}\Big(\rho^{\epsilon,\delta}u^{\epsilon,\delta}\psi_t+\big(\rho^{\epsilon,\delta}-\delta\big)u^{\epsilon,\delta}\Big(u^{\epsilon,\delta}-p\big(\rho^{\epsilon,\delta}\big)\Big)\psi_x-g_2\big(\rho^{\epsilon,\delta},\rho^{\epsilon,\delta}u^{\epsilon,\delta}\big)\psi\Big)\,dt\,dx\\
&\quad+\int_{\mathbb{R}}\rho_0\varphi(x,0)\,dx+\int_{\mathbb{R}}\rho_0u_0\psi(x,0)\,dx=-\epsilon\int_{\mathbb{R}}\int_0^{+\infty}\big(\rho^{\epsilon,\delta}\varphi_{xx}+\rho^{\epsilon,\delta}u^{\epsilon,\delta}\psi_{xx}\big)\,dt\,dx. \quad (4\text{-}31)
\end{aligned}$$

for all functions $\varphi,\psi\in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$.
The same argument as in (2-53) shows that the term on the right-hand side of (4-31) goes to zero as ϵ,δ go to zero and the pointwise convergence of $\{\rho^{\epsilon,\delta}(x,t)\}$ and $\{u^{\epsilon,\delta}(x,t)\}$, ensures the existence of a weak solution of (3-4)-(3-5). □

4.2 The Cauchy problem for the Aw-Rascle model

The next subsection deals with the existence of weak entropy solution for the Cauchy problem (3-39)-(3-40).

4.2.1 Existence of weak entropy solution

The following result on the existence of entropy solution is a consequence of theorem 4.1.10 established for the Aw-Rascle traffic flow model with a source.

Corollary 4.2.1. *Let z and w be the Riemann invariants given in (3-3), let p satisfy $\lim_{\rho\to 0}\rho p(\rho)=0$, $2p'(\rho)+\rho p''(\rho)>0$ for $\rho>0$, and assume that $\rho_0(x)\ge c_0>0$ and $w_0(x)=w(x,0)\ge c_1>0$ for two constants c_0,c_1. If the total variation of $z_0(x)=z(x,0)$ is bounded, then the Cauchy problem (3-39)-(3-40) has a weak entropy solution in the sense of Lax.*

Proof. The assertion follows easily from the theorem 4.1.10 if we set $g_1(\rho,\rho u)=g_2(\rho,\rho u)=0$. □

5 A $(n+1)\times(n+1)$ non-symmetric system of Keyfitz-Kranzer type

Following the ideas of chapters 2 and 3, in this chapter we shall be concerned with a $(n+1)\times(n+1)$ non-symmetric system of Keyfitz-Kranzer type with specific source terms. Such a system is

$$\begin{cases} \rho_t + \Big(\rho\big(\psi(\rho,u_1,\ldots,u_n) - p(\rho)\big)\Big)_x = 0 \\ (\rho u_i)_t + \Big(\rho u_i\big(\psi(\rho,u_1,\ldots,u_n) - p(\rho)\big)\Big)_x + \rho h_i(u_i) = 0, \quad i = 1,\ldots,n, \end{cases} \tag{5-1}$$

In section 4.1, we will consider the Cauchy problem for the system (5-1) with bounded measurable initial data

$$\big(\rho(x,0), u_1(x,0),\ldots,u_n(x,0)\big) = \big(\rho_0(x), u_{10}(x),\ldots,u_{n0}(x)\big), \quad \rho_0(x) \geq 0. \tag{5-2}$$

In subsection 4.1.1, by using the maximum principle and a positivity proof given by Bereux and Sainsaulieu, we obtain a priori estimates for the viscosity solutions of the diffusion system (5-3) with initial data (5-4). In subsection 4.1.2, under suitable conditions on the initial data $u_{i0}(x)$ and the functions $h_i(u_i)$ is possible to get L^1 estimates para $u^{\epsilon}_{i\,x}(\cdot,t)$. Motivated by the convex entropies in (3-20) (see subseccion 2.1.3, chapter 2), in subsection 4.1.3, we give n convex entropy-entropy flux pairs for the homogeneous system associated to (5-1) rewritten by introducing the variables $m_i = \rho u_i$. The strong convergence of (a subsequence of) viscosity solutions is proved in subsection 4.1.5. In subsection 4.2.1, we state a corollary on the existence of weak entropy solution for the system (5-1) without source terms and initial data (5-2).

5.1 The Cauchy problem for the $(n+1)\times(n+1)$ system with a source term

Let us show the existence of a weak solution for the Cauchy problem (5-1)-(5-2), for it we use classical viscosity. Hence, we consider here the system (5-1) with viscosity terms; namely

$$\begin{cases} \rho^\epsilon_t + \Big(\rho^\epsilon\big(\psi(u^\epsilon_1,\dots,u^\epsilon_n) - p(\rho^\epsilon)\big)\Big)_x = \epsilon\rho^\epsilon_{xx} \\ (\rho^\epsilon u^\epsilon_i)_t + \Big(\rho^\epsilon u^\epsilon_i\big(\psi(u^\epsilon_1,\dots,u^\epsilon_n) - p(\rho^\epsilon)\big)\Big)_x + \rho^\epsilon h_i(u^\epsilon_i) = \epsilon(\rho^\epsilon u^\epsilon_i)_{xx}, \quad i=1,\dots,n, \end{cases} \tag{5-3}$$

with initial data

$$\big(\rho^\epsilon(x,0), u^\epsilon_1(x,0),\dots,u^\epsilon_n(x,0)\big) = \big(\rho_0(x)+\epsilon, u_{10}(x),\dots,u_{n0}(x)\big), \tag{5-4}$$

where $\rho_0(x), u_{10}(x),\dots,u_{n0}(x)$ are given by (5-2).

5.1.1 Existence of viscosity solutions

According to the theory of parabolic systems, to prove the existence of the viscosity solutions for the Cauchy problem (5-3)-(5-4), we only need to derive an a-priori L^∞ estimate for the solutions $\rho^\epsilon(x,t), u^\epsilon(x,t)$.

Lemma 5.1.1. *We assume that $p(0)=0, \lim_{\rho\to 0}\rho p'(\rho)=0, \lim_{\rho\to+\infty}p(\rho)=-\infty$, $p(\rho)\le 0$ and the function $\rho p(\rho)$ is strictly concave for positive ρ (i.e. $2p'(\rho)+\rho p''(\rho)<0$ for $\rho>0$); let $\psi(u_1,\dots,u_n)\in C^2(\mathbb{R}^n)$ be a nonlinear function, nonnegative and covex. If $g_i(\rho,u_i)=\rho h_i(u_i)$, $i=1,\dots,n$ are locally Lipschitz continuous functions and each function $h_i(u_i)$ satisfies the inequalities*

$$c_{i1}u_i + c_{i2} \le h_i(u_i) \le c_{i3}u_i + c_{i4}\,, \tag{5-5}$$

where c_{ij}, $i=1,\dots,n$, $j=1,\dots,4$ are constants, then for any $\epsilon>0$ the viscosity solution $(\rho^\epsilon(x,t), u^\epsilon_1(x,t),\dots,u^\epsilon_n(x,t))$ of the Cauchy problem (5-3)-(5-4) exists and satisfies

$$\rho^\epsilon \le M(T), \qquad |u^\epsilon_i|\le M(T), \quad i=1,\dots,n, \qquad (x,t)\in\mathbb{R}\times[0,T], \tag{5-6}$$

where $M(T)$ is a positive constant, not dependent on ϵ, for arbitrary fixed $T>0$.

Proof. For each fixed i we proceed as follows. We substitute the first equation of system (5-3) in the second equation, finding

$$u_{it} + \big(\psi - p(\rho)\big)u_{ix} + h_i(u_i) = \epsilon u_{ixx} + 2\epsilon\frac{\rho_x}{\rho}u_{ix}. \tag{5-7}$$

Using the inequalities (5-5) in the above equation together with the maximum principle we get the a-priori estimate of u^ϵ_i given in (5-6).

We multiply the first equation of system (5-3) by $p'(\rho)$ and the equation (5-7) by ψ_{u_i} to obtain

$$p(\rho)_t + \big(\psi - p(\rho)\big)p(\rho)_x + \rho p'(\rho)\big(\psi - p(\rho)\big)_x = \epsilon p(\rho)_{xx} - \epsilon p''(\rho)\rho_x^2, \tag{5-8}$$

and

$$\psi_{u_i}u_{it} + \big(\psi - p(\rho)\big)\psi_{u_i}u_{ix} + \psi_{u_i}h_i(u_i) = \epsilon\psi_{u_i}u_{ixx} + 2\epsilon\frac{\rho_x}{\rho}\psi_{u_i}u_{ix}. \tag{5-9}$$

From this last equation, we deduce

$$\psi_t + \big(\psi - p(\rho)\big)\psi_x + \sum_{i=1}^{n}\psi_{u_i}h_i(u_i) = \epsilon\psi_{xx} - \epsilon\sum_{i,j=1}^{n}\psi_{u_iu_j}u_{ix}u_{jx} + 2\epsilon\frac{\rho_x}{\rho}\psi_x. \tag{5-10}$$

Subtracting the members of (5-8) from those of (5-10), we have

$$\begin{aligned}\big(\psi - p(\rho)\big)_t + \big(\psi - p(\rho) - \rho p'(\rho)\big)\big(\psi - p(\rho)\big)_x + \sum_{i=1}^{n}\psi_{u_i}h_i(u_i) = \epsilon\big(\psi - p(\rho)\big)_{xx}\\ + 2\epsilon\frac{\rho_x}{\rho}\big(\psi - p(\rho)\big)_x - \epsilon\sum_{i,j=1}^{n}\psi_{u_iu_j}u_{ix}u_{jx} + \frac{\epsilon}{\rho}\big(2p'(\rho) + \rho p''(\rho)\big)\rho_x^2. \end{aligned}\tag{5-11}$$

From (5-11) and by the assumptions on the functions p and ψ one finds that

$$\big(\psi - p(\rho)\big)_t + \big(\psi - p(\rho) - \rho p'(\rho)\big)\big(\psi - p(\rho)\big)_x - c \le \epsilon\big(\psi - p(\rho)\big)_{xx} + 2\epsilon\frac{\rho_x}{\rho}\big(\psi - p(\rho)\big)_x, \tag{5-12}$$

where c is a positive constant. Appliying the maximum principle to this inequality we get the estimate $\psi - p(\rho) \le N(T)$ and therefore $p(\rho) \ge -N(T)$, estimate from which we obtain that $\rho^\epsilon \le M(T)$. □

Next, we can show that ρ^ϵ has a positive lower bound.

Lemma 5.1.2. *With the hypotheses of lemma 5.1.1, we assume (5-4). Then, the following a-priori bounds hold for the diffusion system (5-3)*

$$\rho^\epsilon(x,t) \ge c(t,\epsilon) > 0, \tag{5-13}$$

where $c(t,\epsilon)$ could tend to 0 as $t \to +\infty$ or $\epsilon \to 0$.

Proof. Let us consider the first equation in (5-3), we rewrite this equation as

$$\nu_t - \epsilon\nu_{xx} = -\epsilon(\nu_x)^2 + -\big(\psi - p(\rho)\big)\nu_x + \big(\psi - p(\rho)\big)_x, \tag{5-14}$$

where $\nu = -\ln\rho$.
Let $k_\epsilon(x,t)$ be the function given by (2-19), then a solution of (5-14) with initial data $\nu_0^\epsilon(x) = -\ln\big(\rho_0(x) + \epsilon\big)$ satisfies the following integral equation

$$\begin{aligned}\nu(x,t) &= \nu_0^\epsilon(x) * k_\epsilon(x,t)\\ &+ \int_0^t\left(-\epsilon\Big(\nu_x + \tfrac{1}{2\epsilon}\big(\psi - p(\rho)\big)\Big)^2 + \tfrac{1}{4\epsilon}\big(\psi - p(\rho)\big)^2 + \big(\psi - p(\rho)\big)_x\right) *_x k_\epsilon(x,t-s)ds\,,\end{aligned}$$

from which we get

$$\begin{aligned}\nu(x,t) &\le \nu_0^\epsilon(x) * k_\epsilon(x,t) + \int_0^t \left(\tfrac{1}{4\epsilon}\big(\psi - p(\rho)\big)^2 + \big(\psi - p(\rho)\big)_x \right) *_x k_\epsilon(x,t-s)ds \\ &= \nu_0^\epsilon(x) * k_\epsilon(x,t) + \int_0^t \tfrac{1}{4\epsilon}\big(\psi - p(\rho)\big)^2 *_x k_\epsilon(x,t-s)ds \\ &\qquad + \int_0^t \big(\psi - p(\rho)\big) *_x \big(k_\epsilon(x,t-s)\big)_x ds \\ &\le -\ln\epsilon + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}.\end{aligned}$$

Thus

$$\rho(x,t) \ge \epsilon \exp - \left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}} \right) \ge c(t,\epsilon) > 0.$$

□

5.1.2 A L^1 estimate of $u^\epsilon_{i\,x}(\cdot,t)$

Lemma 5.1.3. *If the total variation of $u_{i0}(x) = u_i(x,0)$ is bounded and the functions $h_i(u_i)$ are such that $h_i'(u_i) \ge 0$, then $u_{ix}(\cdot,t)$, $i = 1,\dots,n$ is bounded in $L^1(\mathbb{R})$, moreover we have*

$$TV\big(u_i(x,t)\big) = \int_{-\infty}^{+\infty} \big|u_{ix}(x,t)\big|\,dx \le \int_{-\infty}^{+\infty} \big|u_{i0x}(x)\big|\,dx = TV\big(u_{i0}(x)\big), \quad i = 1,\dots,n \tag{5-15}$$

Proof. Differentiating equation (5-7) with respect to x, we have

$$u_{itx} + \Big(\big(\psi - p(\rho)\big)u_{ix}\Big)_x + h_{ix}(u_i) = \epsilon u_{ixxx} + \big(2\epsilon\rho^{-1}\rho_x u_{ix}\big)_x.$$

Let $\theta = u_{ix}$, then from the above equation we obtain

$$\theta_t + \Big(\big(\psi - p(\rho)\big)\theta\Big)_x + h'(u_i)\theta = \epsilon\theta_{xx} + \big(2\epsilon\rho^{-1}\rho_x\theta\big)_x, \tag{5-16}$$

multiplying (5-16) by the sequence of smooth functions $g'(\theta,\alpha)$, where α is a parameter, we see that

$$\begin{aligned} g(\theta,\alpha)_t &+ \Big(\big(\psi - p(\rho)\big)g(\theta,\alpha)\Big)_x + \big(\psi - p(\rho)\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) + h_i'(u_i)g'(\theta,\alpha)\theta \\ &= \epsilon g(\theta,\alpha)_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon\rho^{-1}\rho_x g(\theta,\alpha)\big)_x + \big(2\epsilon\rho^{-1}\rho_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big).\end{aligned}$$

we choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha) \ge 0$, $g'(\theta,\alpha) \to \mathrm{sign}\theta$ and $g(\theta,\alpha) \to |\theta|$ as $\alpha \to 0$, so that

$$|\theta|_t + \Big(\big(\psi - p(\rho)\big)|\theta|\Big)_x + h_i'(u_i)|\theta| = \epsilon|\theta|_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x,$$

this equation yields

$$|\theta|_t + \Big(\big(\psi - p(\rho)\big)|\theta|\Big)_x \le \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x. \tag{5-17}$$

Integrating (5-17) in $\mathbb{R}\times[0,t]$, we obtain (5-15). □

5.1.3 n entropy-entropy flux pairs

In this subsection, we given n entropy-entropy flux pairs (η_i, q_i) for the system (5-18). These pairs will be used later to prove two results on H^{-1}_{loc} compactness with respect to the viscosity solutions $\big(\rho^\epsilon(x,t),, u_1^\epsilon(x,t),\ldots,u_n^\epsilon(x,t)\big)$ of the Cauchy problem (5-3)-(5-4).

Introducing the variables $m_i=\rho u_i$, we rewrite the homogeneous system associated to (5-1) as

$$\begin{cases} \rho_t + \Big(\rho\Big(\psi\big(\rho,\frac{m_1}{\rho},\ldots,\frac{m_n}{\rho}\big)-p(\rho)\Big)\Big)_x = 0 \\ m_{it} + \Big(\rho\Big(\psi\big(\rho,\frac{m_1}{\rho},\ldots,\frac{m_n}{\rho}\big)-p(\rho)\Big)\Big)_x = 0, \quad i=1,\ldots,n. \end{cases} \tag{5-18}$$

The Jacobian matrix of the flux functions in (5-18) is given by

$$dF_{(\rho,m_1,\ldots,m_2)} = \begin{pmatrix} \psi-p(\rho)+\rho(\psi_\rho-p'(\rho)) & \cdots & \rho\psi_{m_i} & \cdots & \rho\psi_{m_n} \\ \vdots & & \vdots & & \vdots \\ m_i(\psi_\rho-p'(\rho)) & \cdots & \psi-p(\rho)+m_i\psi_{m_i} & \cdots & m_i\psi_{m_n} \\ \vdots & & \vdots & & \vdots \\ m_n(\psi_\rho-p'(\rho)) & \cdots & m_n\psi m_i & \cdots & \psi-p(\rho)+m_n\psi_{m_n} \end{pmatrix}$$

An entropy $\eta=\eta(\rho,m_1,\cdots,m_n)$ for (5-18) and its associated entropy flux $q=q(\rho,m_1,\cdots,m_n)$ are functions satisfying

$$\nabla q = \nabla\eta dF_{(\rho,m_1,\ldots,m_2)}. \tag{5-19}$$

A pair (η,q) satisfying (5-19) is called a pair of entropy-entropy flux of system (5-18). If η is convex then (η,q) is a covex entropy-entropy flux pair.

Let F_i be a convex function, and

$$\eta_i(\rho,m_1,\ldots,m_n) = \rho F\left(\frac{m_i}{\rho}\right), \tag{5-20}$$

$$q_i(\rho,m_1,\ldots,m_n) = \rho F\left(\frac{m_i}{\rho}\right)\left(\psi\left(\frac{m_1}{\rho},\ldots,\frac{m_n}{\rho}\right)-p(\rho)\right), \tag{5-21}$$

$i=1,\ldots,n$, then the pairs (η_i, m_i) are covex entropy-entropy flux pairs. Indeed

$$\begin{aligned}\nabla\eta_i dF = \Big(&\big(\psi-p(\rho)\big)\eta_{i\rho} + \big(\rho\eta_{i\rho}+m_i\eta_{im_i}\big)\big(\psi_\rho-p'(\rho)\big),\ldots,\big(\rho\eta_{i\rho} \\ &+ m_i\eta_{im_i}\big)\psi_{m_i} + \big(\psi-p(\rho)\big)\eta_{im_i},\ldots,(\rho\eta_{i\rho}+m_i\eta_{im_i})\psi_{m_n}\Big),\end{aligned}$$

since $\rho\eta_{i\rho}+m_i\eta_{im_i}=\eta_i$ it follows that (5-20) and (5-21) satisfy (5-19).

5.1.4 H^{-1}_{loc} compactness

Lemma 5.1.4. *Under the conditions given in the lemmas 5.1.1 and 5.1.3 and if $g(\rho)$ is an arbitrary smooth function, it follows that*

$$g(\rho^\epsilon)_t + \left(\int^{\rho^\epsilon} g'(s)f'(s)\,ds + g(\rho^\epsilon)\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big)\right)_x \tag{5-22}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s) = -sp(s)$.

Proof. Multiplying each member of the first equation in (5-3) by $g'(\rho)$, we obtain

$$g(\rho)_t + g'(\rho)f'(\rho)\rho_x + g'(\rho)\psi\rho_x + g'(\rho)\rho\psi_x = \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2,$$

which is equivalent to

$$g(\rho)_t + \left(\int^{\rho} g'(s)f'(s)\,ds + g(\rho)\psi\right)_x = \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2 + \big(g(\rho)-\rho g'(\rho)\big)\psi_x. \tag{5-23}$$

From (5-23) and by choosing a strictly convex function $g(\rho)$ together with the estimate (5-15), we obtain

$$\epsilon(\rho_x^\epsilon)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+), \tag{5-24}$$

so that $\epsilon g''(\rho)\rho_x^2$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The two last terms in the right-hand side of (5-23) are bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$ since these are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Also it follows from (5-24) and the Cauchy-Schwarz inequality that $\epsilon g(\rho)_{xx}$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The sum on the left of the (5-23) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Applying Murat's lemma we have that (5-22) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

A consequence of the previous lemma is the following result.

Corollary 5.1.5. *Let $f(s)$ be the function given in lemma 5.1.4. We assume the hypotheses of the lemmas 5.1.1 and 5.1.3. Then*

$$\rho_t^\epsilon + \Big(\rho^\epsilon\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big) - \rho^\epsilon p(\rho^\epsilon)\Big)_x, \tag{5-25}$$

$$f(\rho^\epsilon)_t + \left(\int^{\rho^\epsilon} f'^2(s)\,ds + f(\rho^\epsilon)\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big)\right)_x \tag{5-26}$$

$$\rho_t^\epsilon + \Big(\rho^\epsilon\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big) - \rho^\epsilon p(\rho^\epsilon) + u_i^\epsilon\Big)_x, \quad i = 1,\dots,n, \tag{5-27}$$

are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. By choosing $g(\rho) = \rho$ in the previous lemma, then (5-22) becomes (5-25), which shows that (2-22) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.
(5-26) is H^{-1}_{loc} compact. In fact, the H^{-1}_{loc} compactness of (5-22) remains valid if we replace $g(\rho)$ with $f(\rho)$.
As the functions $u^\epsilon_{i\,x}$ are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, then they are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Using this fact and the H^{-1}_{loc} compactness of (5-25) the proof is completed. □

Lemma 5.1.6. *With the hypotheses of the lemmas 5.1.1 and 5.1.3, it follows that for each $i = 1,\ldots,n$ we have*

$$(\rho^\epsilon u^\epsilon_i)_t + \Big(\rho^\epsilon u^\epsilon_i\big(\psi\big(u^\epsilon_1,\ldots,u^\epsilon_n\big) - p(\rho^\epsilon)\big)\Big)_x \tag{5-28}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. If (η_i, q_i) is an convex entropy-entropy flux pair given by (5-20)-(5-21), then multiplying by $\nabla\eta_i$ the system (5-18) gives

$$\eta_{it} + q_{ix} = \epsilon\eta_{ixx} - \epsilon\rho F''(u_i)u_{ix}^2 - \rho h_i(u_i)F'(u_i). \tag{5-29}$$

We shall take a strictly convex function $F(u)$ in the equation (5-29), so we obtain that

$$\epsilon\rho^\epsilon\big(u^\epsilon_{i\,x}\big)^2,\quad i = 1,\ldots,n,\ \text{ are bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{5-30}$$

According to the system (5-3) we can write

$$(\rho^\epsilon u^\epsilon_i)_t + \Big(\rho^\epsilon u^\epsilon_i\big(\psi\big(u^\epsilon_1,\ldots,u^\epsilon_n\big) - p(\rho^\epsilon)\big)\Big)_x = \epsilon(\rho^\epsilon_x u^\epsilon_i + \rho^\epsilon u^\epsilon_{i\,x})_x - \rho^\epsilon h_i(u^\epsilon_i). \tag{5-31}$$

The estimates (5-24) and (5-30) allow us to show respectively that $\epsilon(\rho^\epsilon_x u^\epsilon_i)_x$ and $\epsilon(\rho^\epsilon u^\epsilon_{i\,x})_x$ are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, with the aid of the Cauchy-Schwarz inequality. The term $\rho h_i(u_i)$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$ since it is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. (5-28) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Using now Murat's lemma we see that (5-28) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

The preceding lemma admits the next corollary.

Corollary 5.1.7. *The assumptions in the lemmas 5.1.1 and 5.1.3 imply that*

$$(\rho^\epsilon u^\epsilon_i)_t + \Big(\rho^\epsilon u^\epsilon_i\big(\psi\big(u^\epsilon_1,\ldots,u^\epsilon_n\big) - p(\rho^\epsilon)\big) + (u^\epsilon_i)^2\Big)_x \tag{5-32}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. The bounding of the functions $\big((u^\epsilon_i)^2\big)_x$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ shows that these are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, it together with the compactness given in the lemma 5.1.6 prove the corollary. □

Lemma 5.1.8. *If the conditions of the lemmas 5.1.1 and 5.1.3 are satisfied, then*

$$\Big(\rho^\epsilon\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big)\Big)_t+\Big(\rho^\epsilon\psi^2\big(u_1^\epsilon,\dots,u_n^\epsilon\big)-\rho^\epsilon p(\rho^\epsilon)\psi\big(u_1^\epsilon,\dots,u_n^\epsilon\big)\Big)_x \tag{5-33}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Using (5-10) and the first equation of the system (5-3), it is possible to write

$$(\rho\psi)_t+\big(\rho\psi^2-\rho p(\rho)\psi\big)_x=\epsilon(\rho\psi)_{xx}-\epsilon\rho\sum_{i,j=1}^{n}\psi_{u_iu_j}u_{ix}u_{jx}-\rho\sum_{i=1}^{n}\psi_{u_i}h_i(u_i). \tag{5-34}$$

By the Cauchy-Schwarz inequality, we deduce from (5-24) and (5-30) that $\epsilon(\rho\psi)_{xx}$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The other two terms on the right-hand side of (5-34) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and hence in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. $(\rho\psi)_t+\big(\rho\psi^2-\rho p(\rho)\psi\big)_x$ is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and by Murat's lemma is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

5.1.5 Pointwise convergence of $\{\rho^\epsilon(x,t)\}$ and $\{u_i^\epsilon(x,t)\}$

Lemma 5.1.9. *We assume the hypotheses of the lemmas 5.1.1 and 5.1.3. Then (a subsequence of)* $\{\rho^\epsilon(x,t)\}$ *and (a subsequence of)* $\{u_i^\epsilon(x,t)\}$, $i=1,\dots,n$ *converge pointwisely.*

Proof. From div-curl lemma applied to the functions given by (5-25) and (5-26) it follows that

$$\overline{\rho^\epsilon\int_k^{\rho^\epsilon}f'^2(s)\,ds-f^2(\rho^\epsilon)}=\overline{\rho^\epsilon}\;\overline{\int_k^{\rho^\epsilon}f'^2(s)\,ds}\;-\;\overline{f(\rho^\epsilon)}^2\;+\;\overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)\psi(u^\epsilon)}$$
$$-\;\overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon\psi(u^\epsilon)}\,, \tag{5-35}$$

where k is real constant and the overbar denotes the weak-star limit of the indicated quantities after extraction of appropriate subsequences, if necessary.
Let $\overline{\rho^\epsilon}=\rho$, we have

$$\overline{\rho^\epsilon}\;\overline{\int_k^{\rho^\epsilon}f'^2(s)\,ds}\;=\;\overline{\rho^\epsilon}\;\overline{\int_\rho^{\rho^\epsilon}f'^2(s)\,ds}\;+\;\overline{\rho^\epsilon}\int_k^{\rho}f'^2(s)\,ds\,, \tag{5-36}$$

then using (3-31) and (5-36) in (5-35) one obtains

$$\overline{(\rho^\epsilon-\rho)\int_\rho^{\rho^\epsilon}f'^2(s)\,ds-\big(f(\rho^\epsilon)-f(\rho)\big)^2}\;+\;\Big(\overline{f(\rho^\epsilon)}-f(\rho)\Big)^2=\overline{\rho^\epsilon}\;\overline{f(\rho^\epsilon)\psi(u^\epsilon)}$$
$$-\;\overline{f(\rho^\epsilon)}\;\overline{\rho^\epsilon\psi(u^\epsilon)}\,. \tag{5-37}$$

Now, we apply the div-curl lemma to the functions given by (5-25) and (5-33), this yields

$$\overline{\rho^\epsilon}\;\overline{\rho^\epsilon\psi^2(u^\epsilon)+f(\rho^\epsilon)\psi(u^\epsilon)}\;-\;\overline{\rho^\epsilon\psi(u^\epsilon)}\;\overline{\rho^\epsilon\psi(u^\epsilon)+f(\rho^\epsilon)}=0,$$

which is equivalent to

$$\overline{\rho^\epsilon}\ \overline{f(\rho^\epsilon)\psi(u^\epsilon)} - \overline{f(\rho^\epsilon)}\ \overline{\rho^\epsilon\psi(u^\epsilon)} = \overline{\rho^\epsilon\psi(u^\epsilon)}^2 - \overline{\rho^\epsilon}\ \overline{\rho^\epsilon\psi^2(u^\epsilon)}\,. \tag{5-38}$$

From the equations (5-37)-(5-38) we get that

$$\overline{(\rho^\epsilon-\rho)\int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon)-f(\rho)\big)^2} + \Big(\overline{f(\rho^\epsilon)}-f(\rho)\Big)^2 = \overline{\rho^\epsilon\psi(u^\epsilon)}^2 - \overline{\rho^\epsilon}\ \overline{\rho^\epsilon\psi^2(u^\epsilon)}\,. \tag{5-39}$$

Since the right-hand side of the equation (5-39) is nonpositive and as the left-hand side of this equation is nonnegative, we deduce

$$\overline{(\rho^\epsilon-\rho)\int_\rho^{\rho^\epsilon} f'^2(s)\,ds - \big(f(\rho^\epsilon)-f(\rho)\big)^2} + \Big(\overline{f(\rho^\epsilon)}-f(\rho)\Big)^2 = 0, \tag{5-40}$$

and

$$\overline{\rho^\epsilon\psi(u^\epsilon)}^2 - \overline{\rho^\epsilon}\ \overline{\rho^\epsilon\psi^2(u^\epsilon)} = 0\,. \tag{5-41}$$

The equality (5-40) allow us get the pointwise convergence of $\{\rho^\epsilon\}$ and from the equality (5-41) we get the pointwise convergence of $\{\psi(u^\epsilon)\}$ in the region of $\rho > 0$.
We use the div-curl lemma to the functions (5-27) and (5-32), one obtains

$$\overline{\rho^\epsilon}\ \overline{\rho^\epsilon u_i^\epsilon\big(\psi(u^\epsilon)-p(\rho^\epsilon)\big)+(u_i^\epsilon)^2} - \overline{\rho^\epsilon u_i^\epsilon}\ \overline{\rho^\epsilon\big(\psi(u^\epsilon)-p(\rho^\epsilon)\big)+u_i^\epsilon} = 0, \tag{5-42}$$

from this equation and due to the pointwise convergence of $\{\rho^\epsilon\}$ and $\{\psi(u^\epsilon)\}$, we have

$$\rho\Big(\overline{(u_i^\epsilon)^2} - \overline{u_i^\epsilon}\Big) = 0, \tag{5-43}$$

which implies the pointwise convergence of $\{u_i^\epsilon\}$ in the region of $\rho > 0$. □

5.1.6 Existence of weak solution

Theorem 5.1.10. *We assume that* $p(0)=0$, $\lim_{\rho\to 0}\rho p'(\rho)=0$, $\lim_{\rho\to+\infty}p(\rho)=-\infty$, $p(\rho)\le 0$, $2p'(\rho)+\rho p''(\rho)<0$ *for* $\rho>0$*; let* $\psi(u_1,\ldots,u_n)\in C^2(\mathbb{R}^n)$ *be a nonlinear function, nonnegative and covex;* $g_i(\rho,u_i)=\rho h_i(u_i)$, $i=1,\ldots,n$ *are locally Lipschitz continuous functions and each function* $h_i(u_i)$ *satisfies the inequalities* $c_{i_1}u_i+c_{i_2}\le h_i(u_i)\le c_{i_3}u_i+c_{i_4}$, *where* c_{ij}, $i=1,\ldots,n$, $j=1,\ldots,4$ *are constants. If the total variation of* $u_{i0}(x)=u_i(x,0)$ *is bounded and the functions* $h_i(u_i)$ *are such that* $h_i'(u_i)\ge 0$, *then the Cauchy problem (5-1)-(5-2) has a weak solution.*

Proof. To prove the existence of a weak solution to problem (5-1)-(5-2), we recall the weak formulation of (5-3)-(5-4)

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{\infty}\Big(\rho^\epsilon\phi_t+\rho^\epsilon\big(\psi(u_1^\epsilon,\dots,u_n^\epsilon)-p(\rho^\epsilon)\big)\phi_x\Big)\,dt\,dx+\int_{\mathbb{R}}\big(\rho_0(x)+\epsilon\big)\phi(x,0)\,dx\\&\quad+\sum_{i=1}^n\Bigg(\int_{\mathbb{R}}\int_0^{\infty}\Big(\rho^\epsilon u_1^\epsilon\varphi_{it}+\rho^\epsilon u_i^\epsilon\big(\psi(u_1^\epsilon,\dots,u_n^\epsilon)-p(\rho^\epsilon)\big)\varphi_{ix}-\rho^\epsilon h_i(u_i^\epsilon)\varphi_i\Big)\,dt\,dx\\&\quad+\int_{\mathbb{R}}u_{i0}(x)\varphi_i(x,0)\,dx\Bigg)=-\epsilon\int_{\mathbb{R}}\int_0^{\infty}\rho^\epsilon\phi_{xx}\,dt\,dx-\epsilon\sum_{i=1}^n\int_{\mathbb{R}}\int_0^{\infty}\rho^\epsilon u_i^\epsilon\varphi_{ixx}\,dt\,dx\,,\quad(5\text{-}44)\end{aligned}$$

for all test functions $\phi,\varphi_i\in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$, $i=1,\dots,n$.
The same argument as in (2-53) shows that

$$\lim_{\epsilon\to0}\epsilon\int_{\mathbb{R}}\int_0^{\infty}\rho^\epsilon\phi_{xx}\,dt\,dx=0,\qquad\lim_{\epsilon\to0}\int_{\mathbb{R}}\int_0^{\infty}\rho^\epsilon u_i^\epsilon\varphi_{ixx}\,dt\,dx=0,\quad i=1,\dots,n.\qquad(5\text{-}45)$$

By using lemma 5.1.9 together with (5-45) and the Lebesgue dominated convergence theorem, we pass to the limit the weak formulation (5-44) to get the existence of a weak solution. □

5.2 The Cauchy problem for the homogeneous system

For the system without this source term, i.e. the homogeneous system

$$\begin{cases}\rho_t+\Big(\rho\big(\psi(\rho,u_1,\dots,u_n)-p(\rho)\big)\Big)_x=0\\(\rho u_i)_t+\Big(\rho u_i\big(\psi(\rho,u_1,\dots,u_n)-p(\rho)\big)\Big)_x=0,\quad i=1,\dots,n,\end{cases}\qquad(5\text{-}46)$$

with initial data (5-2), we obtain as a consequence of theorem 5.1.10 the existence of weak entropy solution.

5.2.1 Existence of weak entropy solution

As a corollary of this theorem we can prove the result announced at the beginning of the chapter.

Corollary 5.2.1. *Suppose that p satisfies $p(0)=0,\lim_{\rho\to0}\rho p'(\rho)=0,\lim_{\rho\to+\infty}p(\rho)=-\infty$, $p(\rho)\le0$, $2p'(\rho)+\rho p''(\rho)<0$ for $\rho>0$; let $\psi(u_1,\dots,u_n)\in C^2(\mathbb{R}^n)$ be a nonlinear function, nonnegative and covex. If the total variation of $u_{i0}(x)=u_i(x,0)$, $i=1,\dots,n$ is bounded, then the Cauchy problem (5-46)-(5-2) has weak entropy solution.*

Proof. We reach the conclusion if we set $h_i=0$, $i=1,\dots,n$ in theorem 5.1.10.

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