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On some Formulae for Ramanujan's tau Function

Sobre algunas fórmulas para la función tau de Ramanujan

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Abstract. Some formulae of Niebur and Lanphier are derived in an elementary manner from previously known formulae. A new congruence formula for $\tau(p)$ modulo p is derived as a consequence. We use this congruence to numerically investigate the order of $\tau(p)$ modulo p.

Key words and phrases. Ramanujan's tau formulae, Congruences.

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Resumen. Obtenemos algunas fórmulas de Niebur y Lanphier de manera elemental a partir de formulas conocidas. Deducimos una nueva fórmula para la congruencia $\tau(p)$ modulo p. Utilizamos esa fórmula para estudiar numéricamente el orden multiplicativo de $\tau(p)$ modulo p.

Palabras y frases clave. Fórmulas para la función tau de Ramanujan, congruencias.

1. Introduction

For a positive integer n we denote by $\sigma_k(n)$ the sum of all k-th powers of the positive divisors of n and let $\sigma(n)$ denote $\sigma_1(n)$. Ramanujan's tau function is denoted by $\tau(n)$.

We consider convolution sums of the form

$$S_a(r,n) = \sum_{k=1}^{n-1} k^r \sigma_a(k)\ \sigma_a(n-k)$$

where $r \geq 0$ is a non-negative integer and $a > 0$ is a positive integer.

By changing k by $n-k$ in the summation, we get an elementary property of these sums (see also [20]):

$$S_a(r,n) = \sum_{j=0}^{r} n^{r-j}\binom{r}{j}(-1)^j S_a(j,n). \tag{1}$$

It turns out that by putting together (1) with some classical formulae given below (see Section 2), we can prove (see Section 3) some formulae of Niebur and Lanphier for Ramanujan's tau function.

The values of some of the convolution sums modulo a prime number p are computed in Section 4. Some computations concerning properties of the order $o_\tau(p)$ of $\tau(p)$ modulo p were done which improve some known results. It turns out that $o_\tau(p)$ does not seem to be uniformly distributed in $1,\dots,p-1$. More precisely, we have $o_\tau(p) > \frac{p-1}{12}$ for about 92% of primes in a range of length roughly about 800 000. This seems to be a surprising property of the tau function. Indeed (see Section 5), this property is a consequence of the *random behavior* of the function $f(p) = \tau(p)$ modulo p as Jean-Pierre Serre kindly explained it to me. In other words, any such random function $f(p)$ should have the same behavior as $\tau(p)$ and viceversa.

2. The Known Classical Formulae

Lemma 1. *Let $n > 0$ be a positive integer. Then*

$$12S_1(0,n) = 5\sigma_3(n) - (6n-1)\sigma(n) \tag{2}$$

$$n^2(n-1)\sigma(n) = 18n^2S_1(0,n) - 60S_1(2,n) \tag{3}$$

$$n^3(n-1)\sigma(n) = 48n^2S_1(2,n) - 72S_1(3,n) \tag{4}$$

$$120S_3(0,n) = \sigma_7(n) - \sigma_3(n) \tag{5}$$

$$756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 252\cdot 691S_5(0,n) \tag{6}$$

$$\tau(n) = n^2\sigma_7(n) - 540\big(nS_3(1,n) - S_3(2,n)\big) \tag{7}$$

$$\tau(n) = 15n^4\sigma_3(n) - 14n^5\sigma(n) - 840\big(n^2S_1(2,n) - 2nS_1(3,n) + S_1(4,n)\big) \tag{8}$$

Proof. Formula (2) comes from Glaisher [5], later was proved by Ramanujan [14], and appears also in [4, p. 300]. It is Formula (3.10) in [8] where the complete history of the formula is described. Touchard [20] proved Formulae (3) and (4). He used some results of Van der Pol [21]. Glaisher [6] first considered Formula (5). It appears also in [1, p. 140, exercise 9] and in Lahiri's paper [9, Formula (9.1), p. 199]. It is Formula (3.27) in [8] where the complete history of the formula is described. Formula (6) appears in Lehmer's paper [11, Formula (9), p. 683] and also in [1, p. 140, exercise 10]. Formula (7) appears in [21], corrected in [15]; see also [10, Theorem 1, Formula (i)]. Formula (8) appears in [11, Formula (10), p. 683]. ☑

3. Proofs of Niebur and Lanphier Formulae

The main result of Niebur's paper [12] is the formula:

$$\tau(n) = n^4\sigma(n) - 24\big(35S_1(4,n) - 52nS_1(3,n) + 18n^2S_1(2,n)\big). \tag{9}$$

Proof. Let Δ be the difference of the right hand sides of (9) and Formula (8) of Lemma 1. Then

$$\Delta = (n^4 + 14n^5)\sigma(n) - 15n^4\sigma_3(n) + 408n^2S_1(2,n) - 432nS_1(3,n).$$

By introducing the two special cases of Formula (1):

$$S_1(1,n) = \frac{1}{2}nS_1(0,n) \tag{10}$$

and

$$S_1(3,n) = \frac{1}{2}\big(n^3S_1(0,n) - 3n^2S_1(1,n) + 3nS_1(2,n)\big), \tag{11}$$

Δ becomes

$$\Delta = 108n^4S_1(0,n) - 240n^2S_1(2,n) + (n^4 + 14n^5)\sigma(n) - 15n^4\sigma_3(n).$$

Applying now Touchard's Formula (3) of Lemma 1 we get

$$\Delta = 3n^4\big(12S_1(0,n) + (6n-1)\sigma(n) - 5\sigma_3(n)\big).$$

Thus, by Formula (2) of Lemma 1 we get

$$\Delta = 0;$$

this proves Niebur's Formula (9). ☑

Lanphier (see [10, the formula after Theorem 4]) obtained the following formula as a consequence of his tau formulae:

$$2S_1(3,n) - 3nS_1(2,n) + n^2S_1(1,n) = 0. \tag{12}$$

Proof. Call δ the left hand member of (12). Applying Formula (11) above we get

$$\delta = n^2\big(nS_1(0,n) - 2S_1(1,n)\big);$$

thus $\delta = 0$ by Formula (10) above. This proves (12). ☑

Now, we prove Lanphier's [10, Theorem 1, Formula (iv) (equivalent to Formula (iii))].

$$\tau(n) = -\frac{1}{2}\sigma_7(n) + \frac{3}{2}n^2\sigma_3(n) + \frac{360}{n}S_3(3,n). \tag{13}$$

Proof. Observe that a special case of Formula (1) is

$$S_3(3,n) = \frac{1}{2}\big(n^3 S_3(0,n) - 3n^2 S_3(1,n) + 3n S_3(2,n)\big). \tag{14}$$

Let Δ_1 be the difference of the right hand sides of Formula (7) of Lemma 1 and (13). Then, from (14) we get

$$\Delta_1 = -\frac{3}{2}n^2\big(-\sigma_7(n) + \sigma_3(n) + 120 S_3(0,n)\big);$$

thus, $\Delta_1 = 0$ from Formula (4) of Lemma 1. ☑

Finally, we prove Lanphier's [10, Theorem 3].

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{2\cdot 691}{3n} S_5(1,n). \tag{15}$$

Proof. Observe that a special case of Formula (1) is

$$S_5(1,n) = \frac{1}{2} n S_5(0,n). \tag{16}$$

Let Δ_2 be the difference of the right hand sides of Formula (6) of Lemma 1 and (15). We have

$$\Delta_2 = \frac{691}{3}\frac{-2S_5(1,n) + nS_5(0,n)}{n};$$

then from (16) we get $\Delta_2 = 0$. This finishes the proof of Niebur and Lanphier results. ☑

4. Some Congruences Modulo a Prime

Proposition 1 below follows immediately from Formula (1) and from Lemma 1, Formulae (2), (3), (4) and (8).

Proposition 1. *Let p be a prime number, then*

a) $S_1(0,p) = \frac{1}{12}(p-1)(5p-6)(p+1)$

b) $S_1(1,p) = \frac{1}{24}(p-1)(5p-6)(p+1)p$

c) $S_1(2,p) = \frac{1}{24}(p-1)(3p-4)(p+1)p^2$

d) $S_1(3,p) = \frac{1}{24}(p-1)(2p-3)(p+1)p^3$

e) $S_1(4,p) = \frac{1}{840}\big((50p^2 - 134p + 85)(p+1)p^4 - \tau(p)\big)$

f) $S_1(5,p) = \frac{1}{336}\big((15p^2 - 43p + 29)(p+1)p^4 - \tau(p)\big)p.$

Observe that Ramanujan's Formula a) was rediscovered by Chowla [3]. Thus we obtain.

Corollary 1. *Let p be a prime number, then*

i) $\dfrac{S_1(r,p)}{p^r} \equiv \frac{1}{2r+2} \pmod p, \quad \text{for} \quad 0 \le r \le 3$

ii) $\tau(p) \equiv -840 S_1(4,p) \pmod{p^4}$

iii) $\tau(p) \equiv -336 \frac{S_1(5,p)}{p} \pmod{p^4}$

Lehmer (see [11, p. 683]) used Formula (8) to compute $\tau(n)$ with a computer. Let p be a prime number. Proceeding as before it is easy to see that

$$\tau(p) = p^4(p+1)(15p^2 - 29p + 15) - 840 L_4(p) \tag{17}$$

is indeed Formula (8) evaluated at $n = p$, where

$$L_4(p) = \sum_{k=1}^{p-1} k^2(n-k)^2 \sigma(k)\sigma(p-k).$$

The reduction modulo p^4 of $\tau(p)$ computed with Formula (17), or equivalently, with Formula e) of Proposition 1, is (ii) of Corollary 1.

For

$$T_4(p) = \sum_{k=1}^{(p-1)/2} k^4 \sigma(k)\sigma(p-k)$$

we have

$$S_1(4,p) \equiv 2T_4(p) \pmod p.$$

From (ii) of Corollary 1 we get

Corollary 2.

$$\tau(p) \equiv -1680 T_4(p) \pmod p. \tag{18}$$

This fact was used in some computer computations.

Very little is known about $\tau(p)$ modulo p. Some nice theoretical comments appear in Serre's paper [16, p. 12-13] (see [17] for the English version).

Recently, Papanikolas [13] obtained a new formula involving a certain finite field hypergeometric function ${}_3F_2$, namely,

$$\tau(p) \equiv -1 - \frac{1}{2} \sum_{k=2}^{p-1} \left(\frac{1-k}{p} \right) \left({}_3F_2(p) p^2 \right)^5 \pmod p, \tag{19}$$

that holds for all odd prime numbers p. He states that ${}_3F_2(p)$ may take some time to compute when p is large. But, perhaps, the bottleneck with (19) and also with (18) is with the length of the summation.

Consider the following facts:

a) $\tau(p) \equiv 0 \pmod p$ for $p \in \{2, 3, 5, 7, 2411\}$, provided $p < 10^7$

b) $\tau(p) \equiv 1 \pmod p$ for $p \in \{11, 23, 691\}$, provided $p \leq 314\,747$

c) $\tau(p) \equiv -1 \pmod p$ for $p \in \{5807\}$, provided $p \leq 16091$.

For a) see [7], for b) see sequence $A000594$ in [18], for c) see [19, p. 12].

Observe that either $\tau(p) = 0 \pmod p$ or $\tau(p)$ has an order, say $o_\tau(p)$, in the multiplicative group of nonzero elements of $\mathbb{Z}/p\mathbb{Z}$.

After some straightforward computations with Maple using Formula (18), we got the following results that extend some of the above facts:

Let $b = 882\,389$ and let $2 \leq p \leq b$ be a prime number. Set $o_\tau(p) = 0$ if $\tau(p) \equiv 0 \pmod p$. Otherwise, set $o_\tau(p) =$ the order of $\tau(p)$ modulo p in the cyclic multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ of nonzero elements of $\mathbb{Z}/p\mathbb{Z}$. Then,

a) $o_\tau(p) < 12$ if and only if

$$\begin{aligned} p \in \{&2, 3, 5, 7, 11, 13, 19, 23, 29, 37, 67, 151, 331, 353, 659, 691, 2069, \\ &2411, 5807, 10891, 19501, 58831, 131617, 148921, 184843\}. \end{aligned}$$

More precisely, $o_\tau(p) = 0$ for $p \in \{2, 3, 5, 7, 2411\}$, $o_\tau(p) = 1$ for $p \in \{11, 23, 691\}$, $o_\tau(p) = 2$ for $p \in \{5807\}$, $o_\tau(p) = 3$ for $p \in \{19, 151, 148291\}$, $o_\tau(p) = 4$ for $p \in \{13, 37, 131617\}$, $o_\tau(p) = 6$ for $p \in \{19501\}$, $o_\tau(p) = 7$ for $p \in \{29, 659\}$, $o_\tau(p) = 9$ for $p \in \{10891, 184843\}$, $o_\tau(p) = 10$ for $p \in \{331, 58831\}$, and $o_\tau(p) = 11$ for $p \in \{67, 353, 2069\}$.

b) Let $2 \leq p \leq b$ be a fixed prime number. Let $\pi(p)$ denote the number of prime numbers $q \leq p$. Let $r(p)$ be the quantity of prime numbers $q \leq p$ such that

$$o_\tau(q) < \frac{q-1}{12};$$

then the quotient $q_p = 100 \cdot \frac{r(p)}{\pi(p)}$ oscillates, but for $p \geq 718\,187$ has the decimal form

$$q_p = 7.6 * . \tag{20}$$

In this range, for about 92.4% of primes q one has

$$o_\tau(q) \geq \frac{q-1}{12}.$$

c) Moreover, for practically all the range considered, i.e., for

$$21391 \leq p \leq b$$

one has

$$q_p < 8.0.$$

So for about 92% of such p's the order of $\tau(p)$ modulo p in $(\mathbb{Z}/p\mathbb{Z})^*$ is equal to or exceeds

$$\frac{p-1}{12}. \tag{21}$$

The computations took some time. About six days of idle time in an eighth processor Linux machine running command line, cmaple 11, (for a 9- niced process, running in background). For example to treat all the 100 primes between 491731 and 493133 the computer took about 16 minutes, while for an interval of 100 primes between 830503 and 831799, the computer took about 41 minutes.

So (20) and (21) show, for these values of p at least, that the orders of $\tau(p)$ modulo p are not uniformly (and neither randomly) distributed between 1 and $p-1$.

5. More Computations. Artin's Constant and Randomness

Several further computations of the same kind were done. Essentially all based on some suggestions of Serre. First of all, some functions that should behave randomly were tested in place of τ in the same computations undertaken in Section 4. The behavior of the (analogue) densities $d_p = q_p/100$ for all of them was about the same. That is, there were small oscillations for small primes and stabilization close to some constant for relatively large primes. For the constant function $f(p) = 2$, we obtained $d_p = 0.8625151883$ for $p = 99\,991\,667$ the latest prime considered for.

Some examples of the "random" functions that we tested (in a more extended range than in Section 4 as these functions were easier to compute) are the following:

a) For $f(p) = \sigma\left(\frac{p-1}{2}\right)$ modulo p, where $\sigma(n)$ is the sum of all the positive divisors of n, we obtained $d_p = 0.7714250998$ for $p = 99\,991\,667$.

b) For $f(p) = \phi(p-1)\phi(p+1)$ modulo p, where $\phi(n)$ is the euler function, i.e., the number of coprime integers $0 < m < n$, we obtained $d_p = 0.7680263843$ for $p = 99\,991\,667$.

The next function was tested for two of the properties in the second step below:

c) For $f(p) = q_2(p) = \frac{2^{p-1}-1}{p}$ modulo p, the "2-Fermat quotient", we obtained the approximations (to the densities d_1, d_2) $d_{1p} = 0.4998536691$ and $d_{2p} = 0.3741391367$ for $p = 254\,269\,523$. Where d_1 is the density of the primes p with $f(p)$ a square in $(\mathbb{Z}/p\mathbb{Z})^*$ and d_2 is the density of the primes p with $f(p)$ a generator of $(\mathbb{Z}/p\mathbb{Z})^*$.

In a second (more important) step and following Serre suggestions, we considered more directly the random behavior of the tau function by trying the computations listed below. They confirmed experimentally, or in other words, gave computational evidence, of the randomness of the function $\tau(p)$ modulo p. In particular e) below explains the density 0.92 discovered in Section 4.

More precisely the following was tested:

a) The density d of primes p with $\tau(p)$ a square in $(\mathbb{Z}/p\mathbb{Z})^*$ should be equal to $\frac{1}{2}$. In fact we got the approximation $d_p = 0.4997695853$ for $p = 815\,671$.

b) The density d_1, d_3, d_5, d_{15} of primes $p \equiv 1 \pmod{15}$ with $\tau(p)$ of order $1, 3, 5$ or 15 in $(\mathbb{Z}/p\mathbb{Z})^*$ modulo the 15-th powers, should be equal to $\frac{1}{15}$, $\frac{2}{15}$, $\frac{4}{15}$, $\frac{8}{15}$ respectively. We got the approximations $d_{1p} = 0.06423327896$, $d_{3p} = 0.1320350734$, $d_{5p} = 0.2635603589$, and $d_{15p} = 0.5400693312$ for $p = 2\,412\,391$.

c) The density d of primes p with $\tau(p)$ primitive, i.e., of maximal order $p-1$ in $(\mathbb{Z}/p\mathbb{Z})^*$, should equal Artin's constant

$$A = 0.3739558136\cdots$$

We got the approximation $d_p = 0.3756374808$ for $p = 815\,671$.

d) The quotient $d_p = \frac{r(p)}{u(p)}$ should be close to 1, for large p, where $r(p)$ is the number of primes $q \leq p$ such that $\tau(p)$ modulo p is primitive, and $u(p)$ is the sum:

$$u(p) = \sum_{q \leq p,\ q \text{ prime}} \frac{\phi(q-1)}{\phi(q)}$$

where ϕ is the euler function. It is well known (see [2]) that $e_p = u(p)/\pi(p)$ converges to the Artin's constant A. We got $d_p = 1.004598290$ and $e_p = 0.3741287045$ for $p = 723\,467$.

e) The sum S_{12} of the densities A_m for $m \leq 12$ is close to 0.92, (more precisely $S_{12} = 0.92273\cdots$) where A_m is the density of the primes p such that the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by a "generic" fixed integer is of given index m. So that $A_1 = A$, and e.g., $A_3 = \frac{8}{45}A$.

We got the following approximations A_{pm} to A_m for $p = 787\,217$: $A_{p1} = 0.3758095238$, $A_{p2} = .2813809524$, $A_{p3} = 0.06569841270$, $A_{p4} =$

0.06842857143, $A_{p5} = 0.01890476190$, $A_{p6} = 0.04919047619$, $A_{p7} = 0.008777777778$, $A_{p8} = 0.01800000000$, $A_{p9} = 0.007761904762$, $A_{p10} = 0.01449206349$, $A_{p11} = 0.002904761905$, $A_{p12} = 0.01228571429$. Thus the sum $A_{p1} + \cdots + A_{p12} = 0.9236349206$.

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