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AN ALGEBRAIC FORMULATION OF ERGODIC PROBLEMS

by

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ABSTRACT

We show how the conventional realization of the statistical mechanics of quantum and classical systems can be adapted to obtain a realization in terms of C^*-algebras. The latter provides a unified algebraic method for the analysis of ergodic problems. In particular, we discuss symmetry groups, ergodic states, and the Kolmogorov entropy in the algebraic formalism.

RESUMEN

Mostramos cómo las realizaciones convencionales de la mecánica estadística de los sistemas cuánticos y clásicos pueden ser adaptados para obtener una realización en términos de C^*-álgebras. La última da un método unificado y algebraico para el análisis de los problemas ergódicos. En particular, discutimos grupos de simetría, estados ergódicos, y la entropía de Kolmogorov dentro del formalismo algebraico.

§1. *Introduction* . We assume that the reader is acquainted with the elementary facts about C^*-algebras and Von Neumann algebras [1] . We discuss some of these here in order to establish notational conventions. A Banach * - algebra $\mathfrak{A}$ is a C^*-algebra if it satisfies $\| T T^* \| = \| T \|^2$ for each $T \in \mathfrak{A}$. A C^*-algebra $\mathfrak{A}$ is a von Neumann algebra if there exists a Banach space X such that the dual of X is $\mathfrak{A}$. The obvious example of a von Neumann algebra is $\mathcal{B}(\mathcal{H})$, the set of bounded operators on a Hilbert space $\mathcal{H}$. There are many others.

A state on a C^*-algebra $\mathfrak{A}$ is a positive, continuous, linear functional ρ of norm 1. We denote the set of states by E. We say that $\rho \in E$ is pure if it cannot be written in the form $\lambda \rho_1 + (1-\lambda)\rho_2$ where $0 < \lambda < 1$, $\rho_1, \rho_2 \in E$, $\rho_1 \neq \rho_2$.

Let π be a function from $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ which is a morphism of *-algebras, where $\mathcal{H}$ is some Hilbert space. The continuity of such a map is automatic; in fact, $\| \pi(T) \| \leq \| T \|$ for all T in $\mathfrak{A}$. Such a π is called a representation of $\mathfrak{A}$. It is cyclic if there exists a vector Ω in $\mathcal{H}$ such that $\{\pi(T)\Omega : T \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. In this case, Ω is called a cyclic vector. A subspace M in $\mathcal{H}$ (a closed linear manifold) is invariant for the representation π if $\{\pi(T)\Phi : T \in \mathfrak{A}, \Phi \in M\}$ is contained in M. π is irreducible if its only invariant subspaces are (0) and $\mathcal{H}$.

The GNS construction [2] states that for every $\rho \in E$ there is a unique Hilbert space $\mathcal{H}_\rho$ (up to isometries) and a cyclic representation π_ρ of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H}_\rho)$ with cyclic vector Ω_ρ such that

$$\rho(T) = (\pi_\rho(T)\Omega_\rho, \Omega_\rho), \quad T \in \mathfrak{A} \tag{1}$$

Moreover, ρ is pure if, and only if, π_ρ is irreducible.

§ 2. *Realization of classical systems in terms of C^*-algebras.* An advantage of the C^*-algebra realization of statistical mechanics is that it provides a unified treatement of classical and quantum systems which in turn allows an easy comparison between the two. As we will see, the C^*-algebra for a quantum system almost never is abelian, whereas a classical system has an abelian C^*-algebra.

For classical systems [3], let (q,p) be a system of coordinates for $\mathbb{R}^{2n}$ (or, more generally, a C^∞-manifold). Let $H(q,p)$ be a C^∞-function. The equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{2}$$

are Hamilton's equations and define a one-parameter group $\{V_t\}$ of diffeomorphisms of $\mathbb{R}^{2n}$. Since

$$\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0 \tag{3}$$

the diffeomorphisms conserve the measure $dqdp$. The same function H is a first integral of (2), since $\frac{dH}{dt} = 0$. Thus $H(q,p) = E$ defines a C^∞-manifold X. We define a measure on X by

$$d\lambda = \frac{dv}{|\, grad\, H\,|}$$

where dv is the volume element in X induced by the Lebesgue measure of $\mathbb{R}^{2n}$. λ is called the Liouville measure. Since $\lambda(V_t\Delta) = \lambda(\Delta)$ for each Borel set Δ in X, λ is invariant with respect to the diffeomorphisms V_t.

Since we are considering statistical mechanics, it is natural to consider the case where n is large (say 10^{23}). In this case we can only give the probability that the classical systems starts out in a certain point of X when $t = 0$.

Let μ be a probability measure defined on the Borel sets of X ; we assume that μ is absolutely continuous and regular [4] with respect to λ. In this case we also have $\mu(V_t \Delta) = \mu(\Delta)$. Thus we can talk about a set of probability measures invariant with respect to the group $\{V_t\}$. Each μ determines a state of the classical system.

We now convert these ideas to the C^*-algebra formalism. If X is not compact, let $\tilde{X}$ be the one-point compactification of X [5]. If X is compact, let $\tilde{X} = X$. (X usually is compact, an important fact for many arguments in classical statistical mechanics.) Let $\mathfrak{A}$ be the C^*-algebra of all continuous functions f from X into $\mathbb{C}$. The states of $\mathfrak{A}$ are in one-to-one correspondence with the probability measures by means of the formula

$$\rho(f) = \int f \, d\mu_\rho \ . \qquad (5)$$

(This is just Riesz' theorem.) In particular, for the GNS construction, $\mathcal{H}_\rho = \mathcal{L}^2(\mu_\rho)$, $\pi_\rho : f \to T_f \in \mathcal{B}(\mathcal{L}^2(\mu_\rho))$ where $(T_f\Phi)(\xi) = f(\xi)\,\Phi(\xi)$, and Ω_ρ is the function identically equal to 1.

§ 3. *Realization of quantum systems in terms of C^*-algebras.* The conventional realization of non-relativistic quantum mechanics can be found in many elementary texts [6]. One begins with the notion of a state of a quantum system and observables for the system. A state is a normalized vector Ψ in a separable Hilbert space $\mathcal{H}$. The observables are self-adjoint operators A defined on dense domains of $\mathcal{H}$. The expectation value of A for the state Ψ is defined by $\langle A;\Psi\rangle = (A\Psi, \Psi)$. The expectation values are possible results of experiments performed on the quantum system and represent all that we can know about the system. One notes that Ψ is not determined uniquely, since $e^{i\alpha}\Psi$ for real α determines the same expectation value. One solution to this problem is to identify the state with the scalar multiples of Ψ, which is called a ray, and is a one-dimensional subspace. The projection W on this subspace is defined by $W\Phi = (\Phi, \Psi)\Psi$. Thus we can identify the state with a one-dimensional projection. Observe that

$$W > 0 \ , \quad Tr(W) = 1 \ , \quad W^2 \leq W \tag{6}$$

Here an operator is positive, written $A > 0$, if $(A\Phi, \Phi) > 0$ for each Φ in the domain of A ; $W^2 \leq W$ says that $W - W^2$ is positive or equal to 0 ; and

$$Tr(A) = \sum_{n=1}^{\infty} (A\Phi_n, \Phi_n) \tag{7}$$

(where $\{\Phi_n\}$ is a complete orthonormal set), assuming that the sum is well-defined. If we assume that (6) defines what we mean by state, we have the statistical point of view. A one-dimensional projection is then a pure state $(W^2 = W)$, whereas a general W that satisfies (6) with $W^2 < W$ is a mixed state.

Thus we have the conventional realization of non-relativistic quantum mechanics : the states are self-adjoint operators W which satisfy (6) ; the observables are self-adjoint operators. W is usually called the density matrix.

We have yet to extend the notion of expectation value to the statistical case. This we do as follows : the spectral theorem [7] says that to each self-adjoint operator A there corresponds a unique spectral measure E_A defined on the σ-algebra of Borel sets in $\mathbb{R}$. Moreover

$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda \ , \quad I = \int_{-\infty}^{\infty} dE_\lambda \tag{8}$$

where $E_\lambda = E_A(\,]-\infty, \lambda\,])$. The integrals in (8) have the following strong convergence interpretation :

$$(I\Phi, \Phi) = \int_{-\infty}^{\infty} d(E_\lambda \Phi, \Phi) \tag{9a}$$

$$(A\Phi, \Phi) = \int_{-\infty}^{\infty} \lambda\, d(E_\lambda \Phi, \Phi) \tag{9b}$$

where Φ is in the domain of A. (Here $\mu_\Phi(\Delta) = (E_A(\Delta)\Phi, \Phi)$ is an ordinary measure.)

Let W be a state, A an observable. We define the trace of WA by

$$Tr(WA) = \int_{-\infty}^{\infty} \lambda\, d\,Tr(WE_\lambda) \tag{10}$$

Here $Tr(WE_\lambda)$ is well-defined since we can use the formula (7) . For a pure state, i.e. a one-dimensional projection W onto the subspace generated by Ψ, we have from (10) that

$$Tr(WA) = (A\,\Psi, \Psi) = \langle A\,;\Psi\rangle \tag{11}$$

Thus it is natural to define

$$\rho(A) = Tr(WA) \tag{12}$$

as the expectation value of A for the state W. This is in fact the definition used in quantum statistical mechanics. (Unfortunately from the mathematician's point of view, most elementary texts in quantum mechanics do not consider the problem of defining $Tr(WA)$ well. [6 d] is an exception.)

According to (10), we have

$$\rho(\alpha A + \beta B) = \alpha\,\rho(A) + \beta\,\rho(B) \tag{13}$$

for any pair of observables A , B and any pair α, β of complex numbers. In addition,

$$\rho(A) \geq 0 \qquad \text{if} \quad A \geq 0 \tag{14}$$

Thus ρ is a linear, positive functional on the observables. We now want to show how this ρ can be redefined so that it is a state on a certain C^*-algebra.

Let $\mathcal{O} = \{ E_A(\Delta) \}$ where A is an observable and Δ is a Borel set in $\mathbb{R}$. All possible experimental experimental information about the system is contained in $\mathcal{O}$. Let $\mathcal{A}$ be the von Neumann algebra generated by $\mathcal{O}$, i.e. the double commutator of $\mathcal{O}$. $\mathcal{A}$ contains all the bounded self-adjoint operators of $\mathcal{B}(\mathcal{H})$ which are observables, a set which we denote by $\mathcal{A}_b$. The restriction of ρ to $\mathcal{A}_b$ (which we continue to denote by ρ) is a positive linear functional on $\mathcal{A}_b$. We extend ρ to $\mathcal{A}$ by defining

$$\rho(T) = \rho(T_1) + i\rho(T_2) \tag{15}$$

where T_1, T_2 are self-adjoint operators defined by

$$T_1 = \frac{T + T^*}{2} \quad , \quad T_2 = \frac{T - T^*}{2i} \tag{16}$$

We then have that

$$\sup \{ |\rho(T)| : T \in \mathcal{A} , \| T \| = 1 \} = \rho(I) = 1 \tag{17}$$

i.e. $\|\rho\| = 1$. Thus ρ is a positive, continuous, linear functional of norm 1 on $\mathcal{A}$. Reciprocally, a positive, continuous, linear functional of norm 1 on $\mathcal{A}$ has the form

$$\rho(T) = Tr(WT), \quad T \in \mathcal{A}_b \tag{18}$$

which can then be extended to all observables. This means that we can identify states on $\mathfrak{A}$ with the states of the quantum system. The conclusion is that we do not lose any information if we use the realization of non-relativistic quantum mechanics in terms of C^*-algebras.

As we mentioned before, the obvious example of a C^*-algebra is $\mathcal{B}(\mathcal{H})$, which is certainly a possibility for $\mathfrak{A}$ in our case. But in general $\mathfrak{A} \neq \mathcal{B}(\mathcal{H})$. In Nature there are superselection rules which restrict $\mathfrak{A}$. For example, one usually considers the proton and the neutron as two orthogonal states of the same particle, the nucleon, but no one has ever seen a linear combination of these states. Thus the associated observable, the isospin of the nucleon, commutes with all the operators in $\mathcal{O}$, i.e. the single commutator of $\mathcal{O}$ is larger than $\mathbb{C} I$, which tells us that $\mathfrak{A}$, the double commutator of $\mathcal{O}$, cannot be equal to $\mathcal{B}(\mathcal{H})$.

As for the GNS construction, we can give π_ρ explicitly if ρ is pure. If $\rho(T) = Tr(WT)$ for all $T \in \mathfrak{A}_h$, where $W\Phi = (\Phi, \Psi)\Psi$ with $\|\Psi\| = 1$, then

$$\rho(T) = (T\psi, \psi), \quad T \in \mathfrak{A} \tag{19}$$

In this case $\mathcal{H}_\rho = \mathcal{H}$, $\pi_\rho(T) = T$, and $\Omega_\rho = \Psi$ satisfy the conditions of the theorem. The advantage of the GNS construction only exists when we want to consider mixed states. If ρ is a mixed state, the GNS construction states that there is a projection P on $\mathcal{B}(\mathcal{H}_\rho)$ that has the form $P\Phi = (\Phi, \Omega_\rho)\Omega_\rho$ so that

$$Tr(WT) = (\pi_\rho(T)\Omega_\rho, \Omega_\rho), \qquad T \in \mathfrak{A} \tag{20}$$

with $\pi_\rho(W) = P$. Of course, $\mathcal{H}_\rho \neq \mathcal{H}$ in this case. The proof of the GNS construction shows that the reduction of W to a simple one-dimensional projection is made at the cost of a reduction of $\mathcal{H}$.

§ 4. *Symmetries and ergodic states.* As an example of the use of the C^*-algebra formalism, let us consider the notion of symmetries. Beginning with pure states in the conventional realization of quantum mechanics, we may define a symmetry as a one-to-one correspondence α between rays which conserve the transition probabilities. If we identify each ray with a one-dimensional projection P_Φ on the subspace generated by a normalized vector Φ in the ray, the transition probability to go from P_Ψ to P_Φ is

$$p_\Phi(P_\Psi) = |(\Phi , \Psi)|^2 = < P_\Psi ; \Phi > \tag{21}$$

From this we see that two possible ways to realize such an α are $P_\Psi \xrightarrow{\alpha} P_{U\Psi}$ where $(U\Phi, U\Psi) = (\Phi, \Psi)$, or $(U\Phi, U\Psi) = \overline{(\Phi, \Psi)}$. Wigner's theorem assures us that these are the only possibilities [8]. If we translate this result into the C^*-algebra realization of quantum mechanics, we see that a symmetry α is an automorphism or antiautomorphism of the C^*-algebra $\mathfrak{A}$ (either $\alpha(S\,T) = \alpha(S)\,\alpha(T)$ or $\alpha(ST) = \alpha(T)\,\alpha(S)$, while α is linear and $*$-preserving in both cases).

We now restrict our attention to symmetries where $(U\Phi, U\Psi) = (\Phi, \Psi)$, i.e. unitary symmetries. The corresponding α is then an automorphism of $\mathfrak{A}$, i.e. an element of $Aut(\mathfrak{A})$. The latter is a group, which allows us to make the connection with symmetry groups. The action of a symmetry group $\mathcal{G}$ on the quantum system is determined by a group morphism $\alpha : \mathcal{G} \to Aut(\mathfrak{A})$; α is called the action of $\mathcal{G}$ on $\mathfrak{A}$, and $\alpha_g T$ will denote the image of T under a particular automorphism α_g. There have been proposed many definitions of continuity for α in the case that $\mathcal{G}$ is a topological group. The most physical definition and the easiest to apply is that $g \to \alpha_g T$ is a continuous function for all $T \in \mathfrak{A}$ [9].

To see how the C^*-algebra formalism aids us now in the discussion of symmetries, we say $\rho \in E$ is $\mathcal{G}$-invariant if $\rho(\alpha_g T) = \rho(T)$ for all $g \in \mathcal{G}$, $T \in \mathfrak{A}$. If $\mathcal{H}_\rho$, π_ρ, Ω_ρ are the Hilbert space, cyclic representation, and cyclic vector, respectively, obtained from the GNS construction for the $\mathcal{G}$-invariant state ρ, there is a unique unitary representation $U_\rho(g)$ of $\mathcal{G}$ in $\mathcal{B}(\mathcal{H}_\rho)$ such that

$$U_\rho(g)\, \pi_\rho(T)\, U_\rho(g)^{-1} = \pi_\rho(\alpha_g T), \tag{22a}$$

$$U_\rho(g)\, \Omega_\rho = \Omega_\rho . \tag{22b}$$

Moreover $U_\rho(g)$ is a strongly-continuous unitary representation of $\mathcal{G}$ if $g \to \alpha_g T$ is continuous for all $T \in \mathfrak{A}$. This result is known as Segal's theorem [9, 10], and shows how unitary representations come almost for free in the C^*-algebra formalism.

A particularly important symmetry group for a quantum system is its dynamical group. This is the one-parameter group of unitary operators $\{V_t\}$, where $V_t = exp(-iHt)$; H being the energy operator or Hamiltonian. The action on $\mathfrak{A}$ is $\alpha_t = V_t^{-1} T\, V_t$ where we have used the so-called Heisenberg picture for the time development. Since W constant in the Heisenberg picture, $\rho(\alpha_t T) = \rho(T)$ for every T in $\mathfrak{A}$. Thus every state is an invariant state for $\{V_t\}$.

We now turn to classical systems. If α is the action of a symmetry group $\mathcal{G}$ such that $g \to \alpha_g f$ is continuous, then α'_g defined by $\alpha'_g(\rho)(f) = \rho(\alpha_g(f))$ is a homeomorphism of the space $\tilde{X}$. The invariance of ρ is equivalent to the invariance of μ_ρ with respect to these homeomorphisms. Again a particularly important symmetry group is $\{V_t\}$, the dynamical group of the classical system, and every state is invariant with respect to this group.

The study of invariant measures with respect to the action of a certain symmetry group represents part of ergodic theory. In classical ergodic theory one considers transformations τ_g generated by the action of the group on a measure space $(X, \mathfrak{M})$. If we have a probability measure μ defined on $\mathfrak{M}$, a bijection τ_g is an automorphism of the probability space $(X, \mathfrak{M}, \mu)$ if τ_g conserves measure :

$$\mu(\tau_g(\Delta)) = \mu(\Delta) \quad , \quad \Delta \in \mathfrak{M} \tag{23}$$

μ is ergodic with respect to $\mathcal{G}$ if $\tau_g(\Delta) = \Delta$ for each $g \in \mathcal{G}$ implies that either $\mu(\Delta) = 0$ or $\mu(\Delta) = 1$.

For a classical system, a state ρ is invariant if, and only if, each α'_g is an automorphism of the probability space $(\tilde{X}$, Borel sets, $\mu_\rho)$, and ρ is invariant and pure if, and only if, μ_ρ is ergodic with respect to $\mathcal{G}$. From these results we can obtain the logical extension of ergodicity to quantum systems : if $\mathfrak{A}$ is the C^*-algebra of a quantum system, ρ is an ergodic state with respect to $\mathcal{G}$ if ρ is pure and invariant [10].

§ 5. *The Kolmogorov entropy.* Let us analyze the classical system one more time. We note that the basic object is the functional

$$r(T_f) = (T_f \Omega_\rho, \Omega_\rho) \tag{24}$$

For each projection P in $\pi_\rho(\mathfrak{A})$ there is a measurable set Δ (unique, modulo sets of measure zero) such that

$$(P\Phi)(\xi) = C_\Delta(\xi)\ \Phi(\xi) \tag{25}$$

where C_Δ is the characteristic function of Δ. If we put $r(P) = \mu_\rho(\Delta)$, we see that

$$r(P) = \mu_\rho(\Delta) = \int C_\Delta(\xi)\, d\mu_\rho(\xi)$$

$$= \int (C_\Delta(\xi)\, \Omega_\rho(\xi))\, \Omega_\rho(\xi)\, d\mu_\rho(\xi) \qquad (26)$$

$$= (P\, \Omega_\rho ,\, \Omega_\rho)$$

and we return to (24) for projections. This suggests that in the non-abelian case we may use

$$r(P) = (P\, \Omega_\rho ,\ \Omega_\rho) \qquad (27)$$

as a substitute for a probability measure. Further justification is contained in the following properties :

a) $0 \leq r(P) \leq 1$

b) $r(0) = 0$, $r(I) = 1$

c) $r(\Sigma P_n) = \Sigma\, r(P_n)$ for each countable set $\{ P_n \}$ of mutually orthogonal projections.

Even more justification of the use of r as a probability measure comes from non-commutative integration theory. I. E. Segal [11] has developed the concept of an integral which corresponds to a linear functional s on a C^*-algebra in the case that s satisfies a), b), c), and

d) $s(U P U^{-1}) = s(P)$

for each projection P and each unitary operator U. Our r does not satisfy d) ; instead it satisfies

d') $r(U_\rho(g)\, P\, U_\rho(g)^{-1}) = r(P)$.

The problem here is that the unitary group of $\pi_\rho(\mathfrak{A})$ is not necessarily contained in $\{U_\rho(g) : g \in \mathcal{G}\}$. But Segal also introduced the following concept : our r is absolutely continuous with respect to an s which satisfies a), b), c), d), if there is a positive operator T which is integrable with respect to s and satisfies $r(P) = s(TP)$. The necessary and sufficient condition is that $r(P) = 0$ for all the projections P for which $s(P) = 0$. It's possible that such an s could be determined by τ in our case. The connection between our τ and Segal's non-commutative integration theory is still an unsolved problem, but in any case this connection does not appear to be necessary for our study of ergodic problems. It would seem that we can discuss these problems completely within the algebraic context.

As an example, we define the Kolmogorov entropy algebraically. In 1958 Kolmogorov introduced a new concept of entropy which has proven useful in the analysis of ergodic problems in classical mechanics [12]. We want to give an algebraic formulation of this entropy which, of course, will include quantum systems. First we review the classical definition.

Let $(X, \mathfrak{M})$ be a measure space and μ a probability measure defined on the σ-algebra $\mathfrak{M}$. A partition of X is a set $\alpha = \{\Delta_1, \ldots, \Delta_n\} \subset \mathfrak{M}$ such that

a) $\mu[X \setminus (\bigcup_{i=1}^{n} \Delta_i)] = 0$;

b) $\mu(\Delta_i \cap \Delta_j) = 0$, $i \neq j$.

We define

$$h(\alpha) = -\sum_{i=1}^{n} \mu(\Delta_i) \log_2(\mu(\Delta_i)) \tag{28}$$

as the Kolmogorov entropy of the partition. (To make contact with information theory it is customary to use logarithms to the base 2, but it is not necessary.)

Now the algebraic definition is a direct extension. r serves as the substitute for the probability measure. A partition of $\mathcal{H}_\rho$ is a set $\alpha = \{ M_1, \ldots, M_n \}$ such that $\mathcal{H}_\rho = \bigoplus_{i=1}^{n} M_i$. The extension of (28) is

$$h(\alpha) = -\sum_{i=1}^{n} r(P_i) \; \log_2 (r(P_i)) \tag{29}$$

where the P_i are the projections on the M_i.

§ 6. *Conclusions.* We have seen that it is possible to give an algebraic formulation of quantum and classical statistical mechanics. The C^*-algebra realization provides us with a powerful tool, especially for attacking the ergodic problem of quantum statistical mechanics. In a subsequent paper [13], we will analyze the Kolmogorov entropy in much more detail and see which classical results can be extended to the quantum domain.

References

1. The following books are fundamental works on C^*-algebras and von Neumann algebras :

   a) J. Dixmier. *Les Algèbres d'Opérateurs dans l'Espace Hilbertien (Algèbres de von Neumann)* (2nd edition). Gauthier-Villars, Paris, 1969.

   b) J. Dixmier. *Les C^*-Algèbres et Leurs Représentations.* Gauthier-Villars, Paris, 1964.

   c) S. Sakai. *C^*-Algebras and W^*-Algebras.* Springer-Verlag, New York, 1971.

   Also see the more general works of Naĭmark and Rudin indicated below.

2. The letters G, N, and S in "GNS construction" stand for Gel'fand, Naĭmark, and Segal :

   a) I. Gel'fand, M. A. Naĭmark. *On the Embedding of Normed Rings into the*

Ring of Operators in Hilbert Space. Matem. Šb., 12, 197-213 (1943).

b) *I. Segal. Postulates for General Quantum Mechanics.* Ann. Math., 48,930 948 (1947).

The proof of the GNS construction is also contained in the books of Sakai and Dixmier.

3. Two books that we can recommend for their discussion of both quantum and classical systems are :

a) *G. Mackey . Mathematical Foundations of Quantum Mechanics.* W. A.Benjamin, New York, 1968.

b) *J. Jauch. Foundations of Quantum Mechanics.* Addison Wesley, Reading, Mass., 1968.

4. Two basic references on measure theory are :

a) *W. Rudin. Real and Complex Analysis.* McGraw-Hill, New York, 1966.

b) *P. Halmos. Measure Theory.* Van Nostrand, Princeton, N. J., 1950.

5. *J. Kelley General Topology.* Van Nostrand, Princeton, N. J., 1955.

6. One can find a presentation of the conventional realization of non-relativistic quantum mechanics including statistics in :

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b) *E. Merzbacher. Quantum Mechanics* (2nd edition). Wiley, New York, 1970.

c) *L. I. Schiff. Quantum Mechanics* (3rd edition). McGraw-Hill, New York, 1968.

d) *S. M. Moore. La Mecánica Cuántica,* Tomo 2 : La Teoría no Relativista, Segunda Parte. Universidad Pedagógica Nacional, Departamento de Física (in preparation).

7. The following books discuss the spectral theorem :

a) *M. A. Naĭmark. Normed Rings.* Noordhoff, New York, 1964.

b) *W. Rudin. Functional Analysis.* McGraw-Hill, New York, N.Y., 1972 .

It is interesting that both authors prove the spectral theorem using C^*-algebras.

8. For various proofs of Wigner's theorem, see :

a) V. Bargmann. JMP 5 862 (1964).

b) G. Emch. Helv. Phys. Acta 36 739 (1963).

c) G. Emch, C. Piron. JMP 4 469 (1963).

9. Several books contain more details about symmetries in the context of C^*-algebras. We can mention the books of Dixmier and Sakai again, as well as :

a) G. Emch. Algebraic Methods in Statistical Mechanics and Quantum Field Theory. Wiley, New York, 1972.

b) D. Ruelle. Statistical Mechanics - Rigorous Results. W. A. Benjamin, New York, 1972.

10. Segal's theorem and the definition of ergodic state for C^*-algebras appeared for the first time in :

I. E. Segal. A Class of Operator Algebras which are Determined by Groups. Duke Math. J., 18, 221-265 (1951).

11. *I. E. Segal. A Non-Commutative Extension of Abstract Integration.* Ann. Math., 57, 401-457 (1953).

12. *A. N. Kolmogorov. A New Metric Invariant of Transitive Dynamic Systems and Automorphisms of Lebesgue Fields.* Doklady Akad. Nauk., 119, 861-864 (1958).

13. *S. M. Moore. An Algebraic Study of the Kolmogorov Entropy* (in preparation).

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