*Revista Colombiana de Matemáticas*
*Volumen X (1976), págs. 121-124*

# EIGENVALUES OF NONSINGULAR MATRICES AND COMBINATORIAL APPLICATIONS

*by*

*Osvaldo* MARRERO

ABSTRACT

The purpose of this article is to present a result about eigenvalues of nonsingular matrices and to observe that this result implies a theorem of this author on combinatorial designs as well as other combinatorial results. The material presented herein lends itself well for use as an illustration of some nontrivial applications in a first course in Linear Algebra; these applications may be mentioned right after the concepts of eigenvalue and eigenvector have been defined.

**Throughout the sequel, $J$ will denote the matrix having all its entries equal to $+1$, and $I$ will denote the identity matrix. Subscripts will be used whenever it is necessary or convenient to emphasize the order of a matrix ; thus, $A_{m,n}$ will be an $m$ by $n$ matrix, and $A_m$ will be a square matrix of order $m$. The transpose of the matrix $A$ will be $A^T$. The scalar $\mu$ is an eigenvalue of the matrix $A_v$ with corresponding (nonzero) eigenvector $(a_1, a_2, \ldots, a_v)^T$ if $A(a_1, a_2, \ldots, a_v)^T = \mu(a_1, a_2, \ldots, a_v)^T$.**

Let $X = \{x_1, x_2, \ldots, x_v\}$, and let $X_1, X_2, \ldots, X_v$ be subsets of $X$. The subsets $X_1, X_2, \ldots, X_v$ are said to form a *$(v, k, \lambda)$- design* if each $X_j (1 \leq j \leq v)$ has $k$ elements ; each two distinct $X_i, X_j$ $(1 \leq i, j \leq v)$ intersect in $\lambda$ elements ; and $0 \leq \lambda < k < v-1$.

The preceding combinatorial design is completely determined by its *incidence matrix* ; this is the $(0,1)$- matrix $A = [a_{ij}]$ defined by taking $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ if $x_j \notin X_i$.

Let $0 \leq \lambda < k < v-1$. Then a $(0,1)$ - matrix $A_v$ is the incidence matrix of a $(v, k, \lambda)$ - design if and only if $AA^T = (k - \lambda) I + \lambda J$. More information about $(v, k, \lambda)$- designs is available, for example, in [1] and [3] .

*LEMMA. Let A be a v by v nonsingular matrix, and suppose that k is an eigenvalue of A with corresponding eigenvector $(1, 1, \ldots, 1)^T$. Then $A(a_1, a_2, \ldots, a_v)^T = \mu (1, 1, \ldots, 1)^T$ for some scalar $\mu$ if and only if $a_1 = a_2 = \ldots = a_v = \mu k^{-1}$.*

*Proof.* First, it is observed that $k \neq 0$, for $det\ A$ is the product of the $v$ (not necessarily distinct) eigenvalues of $A$, and $det\ A \neq 0$ since $A$ is assumed nonsingular.

Now, using the hypotheses that $k$ is an eigenvalue of $A$ with corresponding eigenvector $(1, 1, \ldots, 1)^T$, and that $A$ is nonsingular, one sees that $A(a_1, a_2, \ldots, a_v)^T = \mu (1, 1, \ldots, 1)^T$ for some scalar $\mu$ if and only if $A[(a_1, a_2, \ldots, a_v)^T - \mu k^{-1} (1, 1, \ldots, 1)^T] = 0$, which holds if and only if $(a_1, a_2, \ldots, a_v)^T = (\mu k^{-1}, \mu k^{-1}, \ldots, \mu k^{-1})^T$

The preceding Lemma yields the following more complete version of the Theorem in [2]:

*COROLLARY 1. Suppose the subsets $X_1, X_2, \ldots, X_v$ of a set $X = \{x_1, x_2, \ldots, x_v\}$ form a $(v, k, \lambda)$- design. Then, except for the empty set and X itself, X contains no subset Y that intersects each $X_j$ $(1 \leq j \leq v)$ in the same number $\lambda_1$ of elements.*

*Proof.* Let $A$ be the incidence matrix of the given $(v, k, \lambda)$ - design; then $A$

is a $v$ by $v$ nonsingular matrix (for a proof of this, the reader is referred to the first 4 sentences in the proof of Theorem 2.1 on p. 103 of [3] ) and $k$ is an eigenvalue of $A$ with corresponding eigenvector $(1,1,\ldots,1)^T$ ; that is ,

$$A(1,1,\ldots,1)^T = k\,(1,1,\ldots,1)^T .$$

If there exists a subset $Y$ of $X$ intersecting each $X_j$ in the same number $\lambda_1$ of elements, then there is a vector $(a_1,a_2,\ldots,a_v)^T$ (defined by taking $a_j = 1$ if $x_j \in Y$ and $a_j = 0$ if $x_j \notin Y$ for $j = 1,2,\ldots, v)$ such that

$$A(a_1,a_2,\ldots,a_v)^T = \lambda_1(1,1,\ldots,1)^T .$$

Now, as a consequence of the Lemma above, it follows that $a_1 = a_2 = \ldots = a_v$ ; therefore each $a_j = 0$, or each $a_j = 1$ ; that is, $Y$ is the empty set, or $Y = X$ .

The following three combinatorial results, all of which are mentioned in [2], are also simple consequences of the preceding Lemma (as well as of Corollary 1).

COROLLARY 2. (Theorem in [2] ). *Suppose the subsets* $X_1,X_2,\ldots,X_v$ *of a set* $X = \{x_1,x_2,\ldots,x_v\}$ *form a* $(v,k,\lambda)$*-design. Then there does not exist another subset* $X_{v+1}$ *of* $X$ *such that* $X_{v+1}$ *has* $k_1$ *elements and* $X_{v+1}$ *intersects each* $X_j (1 \le j \le v)$ *in* $\lambda_1$ *elements, where* $0 < k_1 < v$ *and* $0 < \lambda_1 < k$ .

COROLLARY 3. *Suppose the subsets* $X_1,X_2,\ldots,X_v$ *of a set* $X = \{x_1,x_2,\ldots,x_v\}$ *form a* $(v,k,\lambda)$*- design. Then there does not exist another subset* $X_{v+1}$ *of* $X$ *such that* $X_{v+1}$ *has* $k$ *elements and* $X_{v+1}$ *intersects each* $X_j$ $(1 \le j \le v)$ *in* $\lambda$ *elements.*

COROLLARY 4. *Suppose the subsets* $X_1,X_2,\ldots,X_v$ *of a set* $X = \{x_1,x_2,\ldots,x_v\}$ *form a* $(v,k,\lambda)$*- design. Then there does not exist another subset* $X_{v+1}$ *of* $X$ *such that* $X_{v+1}$ *has* $k_1$ *elements and* $X_{v+1}$ *intersects each* $X_j$ $(1 \le j \le v)$ *in* $\lambda$ *elements, where* $0 < k_1 < v$ .

*The author wishes to express his gratitude to Professor Andrew M. Gleason of Harvard University for his comments on an earlier version of this paper.*

*References*

1. M. Hall, Jr., *"Combinatorial Theory"*, Blaisdell, Waltham, Mass., 1967.
2. O. Marrero, *A property of (v,k, λ)- designs*, Israel Jour. Math. 12(1972), 277-278.
3. H. J. Ryser, *"Combinatorial Mathematics"*, Wiley, New York, N.Y., 1963.



*Department of Mathematics*
*Francis Marion College*
*Florence, South Carolina 29501, U.S.A.*



*(Recibido en mayo de 1976).*