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ON A PROBLEM OF SAMUEL

by

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RESUMEN. Se demuestra que si A es un dominio euclideano con respecto a una función que vale 1 en los primos de A, y además A es una k-álgebra finitamente generada sobre un cuerpo $k \subsetneq A$ que contiene todos los invertibles de A, entonces k es algebráicamente cerrado y $A = k[x]$, el anillo de polinomios en x con coeficientes en k. Por otra parte, el cuerpo de fracciones de A debe tener genus 0.

Interest in euclidean rings was revived with the appearance of an excellent and interesting paper of Samuel [3]. Since then more than 20 papers have appeared on this topic. In this paper we consider the following problem of Samuel. Let A be a unique factorization domain. Then every nonzero element a of A is of the form $a = u\Pi_1^{e_1}\dots\Pi_r^{e_r}$, where u is a unit of A, the Π_i are primes of A and the $e_i \geq 0$ are integers for $1 \leq i \leq r$. Set $\phi(a) =$

$e_1 + \ldots + e_r$. Under what conditions is A euclidean with respect to ϕ?

Before considering this question, we give some examples of domains which are euclidean with respect to a function of the type ϕ.

1. $A = k[x]$, the polynomial ring with coefficients in an algebraically closed field k.

2. A semilocal principal ideal domain (see Prop. 5 in [3]).

3. A principal ideal domain A such that $A^* \to (\frac{A}{Aa})^*$ is surjective for all a in A, where A^* is the set of all units of A.

4. If A is euclidean for a function θ then localizing A at all primes Π such that $\theta(\Pi) \geq 2$, we find that the localized ring is euclidean for a function of the type ϕ.

Now we consider the following two general cases.

CASE 1. A *contains a field* k. In this case we suppose that A is a finitely-generated k-algebra. This also includes the case when characteristic of A is not 0. Since A is euclidean we find that the transcendental degree of K over k is 0 or 1, i.e. either A is a field or K is an algebraic function field in one variable over k. Thus $A = \bigcap_{P \notin S} v_P$, where S is a finite set of primes of K and v_P is the valuation ring of K at the prime P.

CASE 2. *A does not contain a field.* Thus the characteristic of A is 0 and $\mathbb{Z} \subset A$, we now assume that A is a finitely-generated $\mathbb{Z}$-algebra. Since A is euclidean, we find that K, the quotient field of A, is a number field. Thus $A = \bigcap_{P \notin S} v_P$, where S is a finite set of primes of K containing all the archimedean primes A and v_P is the valuation ring of K at the prime P.

In view of these examples we may assume that A is contained in all but a finite number of valuation rings of K, where K is the field of fractions of A.

Next we state, without proof, a theorem of Queen and Weinberger (pag.68 in [2]). Let $A = \bigcap_{P \notin S} v_P$ *be a principal ideal domain,* $\#(S) \geqslant 2$, *such that its quotient field* K *is a global field. We also assume a certain generalized* Riemann *hypothesis if* K *is a number field. Then* A *is euclidean and the smallest algorithm* θ *on* A *is given by*

$$\theta(x) = \sum_{P \notin S} \mathrm{ord}_P(x) n_P, \quad x \neq 0$$

where $n_P = 1$ *if* $A^* \to (\frac{A}{P})^*$ *is surjective, and* $n_P = 2$ *otherwise.*

In view of this we find that if a subring A of a global field K is euclidean for a function of the type ϕ such that $\phi(\Pi) = 1$ for all primes Π of A, then A is a localization at a large number of primes of K, i.e. S is infinite. We also need the following.

THEOREM [Cunnea, 1]. *Let* K *be an algebraic function field in one variable over an algebraically closed field* k. *Let* A *be a subring of* K *such that* $k \subset A$, K *is a field of fractions of* A *and* A *is contained in all but a finite number of valuation rings of* K. *Then* A *is a unique factorization domain if and only if genus of* K *is* 0.

Using this result we prove the following theorem.

MAIN THEOREM. *Let* A *be a domain which is not a field and such that* $k = \{0\} \cup \{\text{units of } A\}$ *is a field. Let* K *be the quotient field of* A. *Suppose now that* A *is a finitely-generated* k*-algebra which is euclidean for a function* ϕ *such that* $\phi(\Pi) = 1$ *for all primes* Π *of* A. *Then* k *is algebraically closed and the genus of* K *is* 0. *Moreover,* $A = k[x]$, *the polynomial ring in* x *with coefficients in* k.

Proof. Since A is euclidean and K is the field of fractions of A, we find that the transcendental degree of K over k is less than or equal to 1. Now trans.deg.(K/k) = 0 implies that A is integral over k and thus a field, a contradiction to our hypothesis. Thus trans.deg.(K/k) = 1. Choose x in A such that x is transcendental over k. Ley f(x) be an irreducible polynomial in k[x] and let

$$f(x) = u\Pi_1^{e_1}\cdots\Pi_r^{e_r}$$

be its prime decomposition in A, where u is a unit of A and the Π_i are primes of A, $1 \leqslant i \leqslant r$. Now $k = \{0\} \cup \{\text{units of } A\}$ and $\phi(\Pi_1) = 1$ imply that

$$\left[\frac{A}{\Pi_1 A}:k\right] = 1,$$

i.e. $$\left[\frac{k[x]}{(f(x))}:k\right] = 1,$$

i.e. degree of $f(x) = 1$.

Thus we see that k is algebraically closed. It now follows that K is an algebraic function field over an algebraically closed field k. Since A is euclidean, using the result of Cunnea we find that the genus of K is 0 and thus $K = k(x)$, a rational field. Since $k[x] \subsetneq A \subsetneq k(x)$ and all the units of A are in k, we find that $A = k[x]$ and whence the result.

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[3] Samuel P., *About Euclidean Ring*, J.Algebra, 19 (1971) 282-301.

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