Revista Colombiana de Matemáticas
Vol. XVIII (1984), págs. 99 - 108

LOCALIZATION IN BUNDLES OF METRIC SPACES

by

Gilma de VILLAMARIN

and

Januario VARELA

§1. Introduction. The essentials of the method of *topological localization*, was presented in [1] for the first time, and a few years later in Hofmann's survey article [2]. If $p:G \to T$ is a surjection, the following data are given:

a) A uniformity on G

b) A topology on T

c) A family Σ of selections for p.

One seeks to establish the continuity of each $\alpha \in \Sigma$, for an appropriate topology on G; but this in general can not be secured, unless G is modified in a drastic manner. The process was described in terms of the entourages of the given uniformity and the neighborhood filters of the base space T. A family of modified stalks is obtained and their disjoint union provides a new space $\hat{G}$ over T.

Recently, K.H. Hofmann [3], [4], gave a very elegant presentation of this localization process in terms of directed colimits, valid for bundles of Banach spaces. A feature of this presentation worth to mention is the giving of the data in the form of a presheaf.

The purpose of this paper is to give a metric version of Hofmann's localization method, generalizing the Banach bundle situation. The construction provides a universal arrow from the given presheaf to the functor that assigns to each bundle of metric spaces the presheaf of its bounded local section.

§2. Directed colimits.

2.1. Let X, Y be metric spaces and $f:X \to Y$. The map f is said to be *contractive* if $d(f(a),f(b)) \leqslant d(a,b)$ for every pair of elements $a,b \in X$. Denote by $\mathfrak{M}$ the category of metric spaces and contractive maps.

2.2. Consider a directed system in $\mathfrak{M}$, $(X_\alpha)_{\alpha \in A}$, $(\rho_{\beta\alpha})_{\beta \geqslant \alpha}$ where A is a directed set. In particular, each $\rho_{\beta\alpha}:X_\alpha \to X_\beta$ is a contrative map such that

i) $\rho_{\alpha\alpha} = id_{X_\alpha}$

ii) if $\gamma \geqslant \beta \geqslant \alpha$, then $\rho_{\gamma\alpha} = \rho_{\gamma\beta}\rho_{\beta\alpha}$.

Let X be the disjoint union of the family of metric spaces $(X_\alpha)_{\alpha \in A}$. Define on X the relation $\sim$ by:

$u \sim v$ if and only if for every $\varepsilon > 0$, if $u \in X_\alpha$ and $v \in X_\beta$, there exists $\gamma \in A$ such that $\gamma \geqslant \alpha$, $\gamma \geqslant \beta$ and

$d(\rho_{\gamma\alpha}(u), \rho_{\gamma\beta}(v)) < \varepsilon$.

It follows, by a straighforward verification that $\sim$ is an equivalence relation [6].

2.3. Consider the quotient Z of X by the equivalence relation $\sim$, $Z = X/\sim$. Define on $Z\times Z$,

$$d(\bar{u}_\alpha, \bar{v}_\beta) = \inf d(\rho_{\gamma\alpha}(u_\alpha), \rho_{\gamma\beta}(v_\beta)),$$

where the infimum is taken over all $\gamma \in A$ such that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$, and $\bar{u}_\alpha, \bar{v}_\beta$ are the equivalence classes, module the equivalence relation $\sim$, of $u_\alpha, v_\beta \in X$. It can be shown that d is a well defined map and that it defines a metric on Z [6]. Moreover, the canonical map $\tau_\alpha : X_\alpha \to Z$ is contractive.

2.4. The metric space Z and the maps $(\tau_\alpha)_{\alpha \in A}$ define an inductive cone for the directed system of metric spaces. This cone turns out to be the directed colimit of the system. In fact, given another inductive cone, $Y, \sigma_\alpha : X_\alpha \to Y$, where α runs through A, define $\phi : Z \to Y$ by $\phi(\bar{u}_\alpha) = \sigma_\alpha(u_\alpha)$. One can easily see that ϕ is a well defined contractive map satisfying the universal property for Z [6]. Hence we have the following statement.

2.5. *Every directed system of metric spaces and contractive maps has a directed colimit.*

2.6. REMARK. The above construction remains valid even if the metric is allowed to take the value ∞.

§3. Bundles of metric spaces.

3.1. DEFINITION. Let $p : G \to T$ be a surjective function

A *metric for* p is a map $d: G \times G \to [0,\infty]$ such that its restriction to each fiber $G_t = \{u \in G : p(u) = t\}$ is a metric in G_t and $d(u,v) = \infty$ if $p(u) \neq p(v)$.

We refer to [4],[5] for each definitions of *selection, section* and *local section.*

A set M of selections is called *bounded* if $d(\alpha,\beta) = \sup\{d(\alpha(t), \beta(t)) : t \in \mathrm{dom}\,\alpha \cap \mathrm{dom}\beta\}$ is finite for every $\alpha,\beta \in M$, in this case $(\alpha,\beta) \to d(\alpha,\beta)$ is a metric on M. Nevertheless it is convenient for us to allow the value ∞ for d.

By definition, a *bundle of metric spaces* is a bundle of uniform spaces in the sense given in [5], such that the family of pseudometrics reduces to the metric d. In particular, the tubes around local sections are a basis for the topology of G and the map $t \mapsto d(\alpha(t),\beta(t)): U \to \mathbb{R}$ is upper semicontinuous whenever α,β are local sections over U.

3.2. Let (E,p,T) and (F,q,T) be bundles of metric spaces, Σ a presheaf of local sections in the field (E,p,T) and Σ' a presheaf of local sections in the field (F,q,T), such that for every open set U of T the domain of each section in $\Sigma(U)$ or in $\Sigma'(U)$ is U.

Consider a morphism of presheaves $\phi: \Sigma \to \Sigma'$, then for every open subset U of T and $\alpha,\beta \in \Sigma(U)$, $d(\phi_U(\alpha),\phi_U(\beta)) \leqslant d(\alpha,\beta)$.

Assume that for every $t \in T$, every $x \in E_t = \overset{-1}{p}(t)$ and every open neighbordhood V of t, there exists $\alpha \in \Sigma(W)$ such $\alpha(t) = x$, with $W \in V$. That is, assume that Σ is full.

Define $f: E \to F$ by $f(x) = \phi_U(\alpha)(t)$ if $p(x) = t$, $\alpha(t) = x$ and $U = \mathrm{dom}\,\alpha$ is an open neighborhood of t.

This is a well defined map: suppose $\beta \in \Sigma(V)$ is such that $\beta(t) = x$. Then there exists an open neighborhood $W \subset V \cap U$ of t such that

$$d(\phi_W(\alpha_W)(t),\phi_W(\beta_W)(t)) \leqslant d(\phi_W(\alpha_W),\phi_W(\beta_W)) \leqslant d(\alpha_W,\beta_W) < \varepsilon$$

Thus $\phi_U(\alpha)(t) = \phi_W(\alpha_W)(t) = \phi_W(\beta_W)(t)) = \phi_V(\beta)(t)$.

The map f is contractive fiberwise; in fact, take $x,y \in E$ with $p(x) = p(y) = t$. Let $\alpha,\beta \in \Sigma(U)$ be such that $\alpha(t) = x$ and $\beta(t) = y$, where U is open in T and $t \in U$. Given $\varepsilon > 0$, there exists $V \subset U$ open and containing t such that

$$d(f(x),f(y)) = d(\phi_V(\alpha_V)(t),\phi_V(\beta_V)(t)) \leqslant d(\phi_V(\alpha_V),\phi_V(\beta_V))$$

$$\leqslant d(\alpha_V,\beta_V) < d(\alpha(t),\beta(t)) + \varepsilon = d(x,y) + \varepsilon.$$

Hence $d(f(x),f(y)) \leqslant d(x,y)$.

3.3. LEMA. *Let* (E,p,T) *and* (F,q,T) *be bundles of metric spaces,* Σ *a presheaf of local sections in* (E,p,T) *and* Σ' *a presheaf of local sections in* (F,q,T). *Assume that* Σ *is full. If* ϕ *is a morphism between the presheaves* Σ *and* Σ', *let* $f:E \to F$ *be defined as described above in terms of* ϕ, *then*

a) *For every open subset* U *of* T *and every* $\alpha \in \Sigma(U)$,

$f\mathcal{T}_\varepsilon(\alpha_U) \subset \mathcal{T}_\varepsilon(f\alpha_U) = \mathcal{T}_\varepsilon(\phi_U(\alpha_U))$.

b) f *is continuous.*

c) $d(f\sigma,f\tau) \leqslant d(\sigma,\tau)$ *for every pair of local sections* $\sigma,\tau \in \Sigma_p(U)$.

Proof. Parts a) and c) follow from the contractivity of f established above.

b) Let $x \in E$, $t = p(x)$ and σ a local section in (F,q,T) such that $f(x) \in \mathcal{T}_\varepsilon(\sigma)$. Take $\alpha \in \Sigma(U)$ such that $\alpha(t) = x$, then $W = \{s \in U \cap \mathrm{dom}\sigma\colon d(\phi_U(\alpha(s)),\sigma(s)) < \delta\}$, with $d(f(\alpha(t),\sigma(t)) < \delta < \varepsilon$, is an open neighbordhood of $t = p(x) = q(f(x))$. Now,

$\mathcal{T}_{\varepsilon-\delta}(\sigma_W(\alpha_W)) \subset \mathcal{T}_{\varepsilon}(\sigma)$; in fact, if $y \in \mathcal{T}_{\varepsilon-\delta}(\sigma_W(\alpha_W))$, then $s = q(y) \in W$ and

$$d(y,\sigma(s)) \leqslant d(y,\phi_W(\alpha_W(s))) + d(\phi_W(\alpha_W(s)),\sigma(s)) < \varepsilon-\delta + \delta = \varepsilon.$$

By part (a) of this lemma $f\mathcal{T}_{\varepsilon-\delta}(\alpha_W) \subset \mathcal{T}_{\varepsilon-\delta}(\sigma_W(\alpha_W))$. Thus f is continuous at x.

3.4. Let (E,p,T) and (F,q,T) be bundles of metric spaces, a continuous map $h:E \to F$ is called a *morphism of bundles of metric spaces if* h is fiber preserving (i.e. $qh = p$) and h is contractive.

To a bundle of metric spaces (E,p,T) we can associate a sheaf of metric spaces Σ_p such that for each open subset U of T, $\Sigma_p(U)$ is the space of all local sections whose domain is U, and to each morphism $h:E \to F$ of bundles of metric spaces we can associate a sheaf morphism $\Sigma_p(h) = \theta$ such that if $\alpha \in \Sigma_p(U)$, $\theta\alpha = h\alpha \in \Sigma_q(U)$.

3.5. THEOREM. *Let* T *be a topological space,* A *the set of all open subsets* U *of* T *and* $(\Sigma(U))$, $U \in A$, (ρ_{VU}), $V \subset U$ *a presheaf of metric spaces. Then there exists a bundle of metric spaces* $(\hat{G},\hat{p},T)$ *and maps* $\phi_U:\alpha \to \hat{\alpha}$, $\Sigma(U) \to \Sigma_{\hat{p}}(U)$, *where* $\Sigma_{\hat{p}}(U)$ *are the local section for* p *over* U, *compatible with restriction such that for every open subset* U *of* T *and every pair* $\alpha,\beta \in \Sigma(U)$, $d(\hat{\alpha},\hat{\beta}) \leqslant d(\alpha,\beta)$.

Proof. As in the second paragraph, $\mathfrak{M}$ denotes the category of metric spaces and contractive maps. For each $t \in T$, denote by $V(t)$ the directed set of all open neighborhoods of t in the space T and $(\Sigma(U))$, $U \in V(t)$, (ρ_{VU}), $V \subseteq U$, the directed system determined by the given presheaf. Call $\hat{G}_t$ its directed colimit. It was shown before that $\hat{G}_t$ is endowed with a metric $\hat{d}_t$.

Let $\hat{G}$ be the disjoint union of the family $\{\hat{G}_t : t \in T\}$. Define $p:\hat{G} \to T$ by $\hat{p}(\hat{u}) = t$ if $u \in \hat{G}_t$ and a metric $\hat{d}$ for $\hat{p}$ by $\hat{d}(\hat{u},\hat{v}) = \hat{d}_t(\hat{u},\hat{v})$ if $\hat{p}(\hat{u}) = \hat{p}(\hat{v}) = t$ and $\hat{d}(\hat{u},\hat{v}) = \infty$ if $\hat{p}(\hat{u}) \neq \hat{p}(\hat{v})$.

Let $\tau_{tU}:\Sigma(U) \to \hat{G}_t$ be the colimit map. Given $\alpha \in \Sigma(U)$, define $\hat{\alpha}:U \to \hat{G}$ by $\hat{\alpha}(t) = \tau_{tU}(\alpha)$, with U and open neighborhood of t. Clearly $\hat{p}\hat{\alpha} = id_U$. Let $\hat{\Sigma}(U) = \{\hat{\alpha}:\alpha \in \Sigma(U)\}$ and $\phi_U:\Sigma(U) \to \hat{\Sigma}(U)$ be defined by $\phi_U(\alpha) = \hat{\alpha}$. It is apparent that $\hat{G}_t = \{\hat{\alpha}(t) \mid \hat{\alpha} \in \hat{\Sigma}(U)\}$.

We show now that $t \to \hat{d}(\hat{\alpha}(t),\hat{\beta}(t)):U \to \mathbb{R}$ is upper semicontinuous. Given $\varepsilon > 0$, let $t \in T$ such that $\hat{d}(\hat{\alpha}(t),\hat{\beta}(t)) < \varepsilon$. By the definition of $\hat{d}_t$ as an infimum, there exists an open neighborhood $W \subset U$ of t in T such that

$$\hat{d}_t(\hat{\alpha}(t),\hat{\beta}(t)) \leqslant d(\alpha_W,\beta_W) < \varepsilon,$$

but W is also an open neighborhood of any $s \in W$, hence $\hat{d}_s(\hat{\alpha}(s),\hat{\beta}(s)) < \varepsilon$. Then $W \subset \{t \in U: \hat{d}_t(\hat{\alpha}(t),\hat{\beta}(t)) < \varepsilon\}$; that is $\{t \in U : \hat{d}_t(\hat{\alpha}(t),\hat{\beta}(t)) < \varepsilon\}$ is an open subset of T. This proves the asserted upper semicontinuity.

By Theorem 4 [5], we conclude that $(\hat{G},\hat{p},T)$ is a bundle of metric spaces and that each $\hat{\alpha} \in \hat{\Sigma}(U)$ is a local section.

Let U be an open subset of T, $\alpha,\beta \in \Sigma(U)$ and $\varepsilon > 0$, then there exists $t \in T$ such that $d(\hat{\alpha},\hat{\beta}) - \varepsilon < d(\hat{\alpha}(t),\hat{\beta}(t)) \leqslant d(\alpha,\beta)$. Thus $d(\hat{\alpha},\hat{\beta}) \leqslant d(\alpha,\beta)$.

3.6. THEOREM. *Under the same hypothesis of the preceding theorem, assume that there is a bundle of metric spaces* $(\bar{G},\bar{p},T)$ *and contractive maps* $\psi_U:\Sigma(U) \to \bar{\Sigma}(U)$, $\alpha \to \bar{\alpha}$ *compatible with restriction maps, where* $\bar{\Sigma}(U)$ *are the local section for* $\bar{p}$ *over* U. *Then there exists a unique continuous map* $h:\hat{G} \to \bar{G}$ *such* *that*

a) h *is fiber preserving and contractive.*

b) $h\,\hat{\alpha} = \bar{\alpha}$ *for every* $\alpha \in \Sigma(U)$.

Proof. Consider again the directed system $(\Sigma(U),\rho_{VU})$, where $V \subset U$ runs through the open neighborhoods of $t \in T$. Let $\bar{G}_t$ the fiber above t in the bundle $(\bar{G},\bar{p},T)$. For $V \in V(t)$ define $\sigma_{tV}:\Sigma(V) \to \bar{G}_t$ by $\sigma_{tV}(\alpha) = \bar{\alpha}(t)$. Clearly $\sigma_{tW}\rho_{WV} = \sigma_{tV}$ when $W \subset V$. On the other hand, $d(\sigma_{tV}\alpha_V,\sigma_{tV}\beta_V) = d(\bar{\alpha}(t),\bar{\beta}(t)) \leqslant d(\bar{\alpha},\bar{\beta}) \leqslant d(\alpha,\beta)$. Hence α_{tV} is contractive and consequently we have an inductive cone for the directed system.

By the universal property of $\hat{G}_t$ there exists an unique contractive map $\theta_t:\hat{G}_t \to \bar{G}_t$ such that $\theta_t\tau_{tV} = \sigma_{tV}$, for every open neighborhood V of t. Therefore, for every $t \in V$ and every $\alpha \in \Sigma(V)$, $\theta_t(\hat{\alpha}(t)) = \bar{\alpha}(t)$.

By means of the family $\{\theta_t : t \in T\}$ we can define $\theta_U:\hat{\Sigma}(U) \to \bar{\Sigma}(U)$ such that $\theta_U(\hat{\alpha})(t) = \theta_t(\hat{\alpha}(t)) = \bar{\alpha}(t)$. Since $d(\bar{\alpha}(t),\bar{\beta}(t)) = d(\theta_t\hat{\alpha}(t),\theta_t\hat{\beta}(t)) \leqslant \hat{d}_t(\hat{\alpha}(t),\hat{\beta}(t))$ we have $d(\bar{\alpha},\bar{\beta}) = d(\theta_U\hat{\alpha},\theta_U\hat{\beta}) \leqslant d(\hat{\alpha},\hat{\beta})$ for every $\alpha,\beta \in \Sigma(U)$. By Lemma 3.3 it follows that the map $h:\hat{G} \to \bar{G}$ defined by $h(\hat{x}) = h(\hat{\alpha}(t)) = \theta_t(\hat{\alpha}(t)) = \bar{\alpha}(t)$, is continuous.

Uniqueness of h is obvious.

3.7. EXAMPLE. Let T be a topological space and $\Sigma(U)$ be the (bounded) upper semicontinuous functions defined in U. As U runs through the open sets of T, $\Sigma(U)$ defines a sheaf of metric spaces by taking $d(f,g) = \sup\{d(f(t),g(t)) : t \in T\}$ and the obvious restrictions maps.

By Theorem 3.5 there exists a bundle of metric spaces $(\hat{\mathbb{R}},\hat{p},T)$ and contractive maps $\phi_U:f \to \hat{f}:\Sigma(U) \to \hat{\Sigma}(U)$ compatible with restrictions such that for every pair $f,g \in \Sigma(U)$, $d(\hat{f},\hat{g}) = d(f,g)$. On the other hand if (E,p,T) is any bundle of metric spaces and σ, τ are (bounded) local section for p over $U \subset T$, then $t \mapsto d(\sigma(t),\tau(t)):U \to \mathbb{R}$ is upper semicontinuous and hence it determines an element of $\hat{\Sigma}(U)$, call it $\hat{d}(\sigma,\tau)$. The bundle $(\hat{\mathbb{R}},\hat{p},T)$ can thus be considered as the object "*real numbers*" in the category of bundles of metric spaces and contractive maps.

* *

REFERENCES

[1] Dauns, J. and Hofmann, K.H., *Representation of rings by sections*, Mem. Amer. Math. Soc. 83 (1968).

[2] Hofmann, K.H., *Representation of algebras by continuous sections*, Bull. Aner. Math. Soc. 78 (1972), 291-373.

[3] ——————, "Sheaves and Bundles of Banach Spaces", Tulane University preprint. 1975.

[4] Hofmann, K.H. and Keimel, K., *Sheaf Theoretical Concepts in Analysis, Bundles and Sheaves of Banach Spaces*, Lecture Notes in Mathematics 753, Springer Verlag, 1979.

[5] Varela, J., *Existence of uniform bundles*, Revista Colombiana de Matemáticas, Vol. XVIII, Nº 1-2 (1984), 1-8.

[6] Villamarín G.de, "Campos de Espacios Métricos", Master's Thesis, Departamento de Matemáticas, Universidad Nacional de Colombia, 1983.

* *

Departamento de Matemáticas y Estadística.
Universidad Nacional de Colombia
Bogotá, D.E., Colombia.

(Recibido en marzo de 1984, versión revisada en Diciembre de 1984).