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COMMENTS ON THE CANONICAL FORMALISM OF TIME-DEPENDENT HIGHER ORDER LAGRANGIANS

by

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Abstract. We examine two types of canonical formalisms of time dependent Lagrangians of higher order in the framework of almost tangent geometry. We examine also a relation of both formalisms with the help of the constraint theory.

§1. **Introduction.** During these last years a good number of articles devoted to the study of Particle Mechanics and Field Theories involving higher order derivatives have been made (see for example [DLR] and the references therein). All these studies may be treated from the Differential Geometry point of view. In fact the geometrical study of higher order theories would be based on the theory of fibered manifolds and Ehresmann's jet theory: if (E,p,N) is a fibered manifold, or fibre bundle, that is, if $p:E \to N$ is a surjective submersion and if J^kE is the k-jet prolongation of (E,p,N) then a Lagrangian L depending on n-independent variables x^a, m-functions $y^A(x^a)$ and on all derivatives of the y's with respect to the x' up to order k (L is said Lagrangian of order k) is a real function $L:J^kE \to \mathbb{R}$. Using the geometrical formalism underlying the theory of fibre bundles, it has been possible to show that there exists an equivalence between variational problems generated by functionals depending on regular Lagrangians of order 1 and "modified" variational problems given by functionals depending on the Poincaré-Cartan invariant form. This equivalence is characterized by an injective/surjective relation between the sets of extremals of the respective problems. Furthermore, it has been possible to show that the above equivalence remains valid for the cases $k = 1$ and $\dim N > 1$ and $k > 1$ and $\dim N = 1$, (see Dedecker [D1],[D2], Goldschmidt & Sternberg [DS]). The case $k > 1$ and $\dim N > 1$ is more complex since the Poincaré-Cartan generalized form is not uniquely determined as in the two situations above (see Garcia & Muñoz [GM] and Horák & Kolár [HK]).

The choice of a local expression for the Poincaré-Cartan generalized form may determine or not a Hamiltonian formalism of higher order. If it is the case this formalism is developed on the jet bundle $J^{2k-1}E$ (for a Lagrandian of order k, $L:J^kE \to \mathbb{R}$) since the momenta are a set of functions whose dependence on higher derivatives ranges from 1 to $2k-1$. This possibility will be called *direct*. However the Hamiltonization of higher order Lagrangians many also be studied in an *indirect* form. In this case we use the well-known fact that J^kE is canonically embedded in $J^1(J^{k-1}E)$. We can therefore develop the Hamiltonian theory on the dual manifold $J^{1*}(J^{k-1}M)$ (see for example Aldaya & de Azcárraga [AA1],[AA2] and Rodrigues [R]).

In this note we will consider only the case $k > 1$, finite, $N = \mathbb{R}$ and $E = \mathbb{R}\times M$, where M is a manifold of dimension m. Thus J^kE may be identified with the bundle $\mathbb{R}\times T^kM$, where T^kM is the tangent bundle of order k of M. As in such situation there is a unique Poincaré-Cartan form, we propose to express it in terms of the almost tangent geometry (see [DLR]). Also, we will study a possible relation between the direct and indirect canonical formalisms. More clearly, suppose that $L:J^kE \to \mathbb{R}$ is a regular Lagrangian of order k (see section 3). Then the canonical formalism is given by a real function H defined on $J^{2k-1}E$, called Hamiltonian of order k. On the other hand we may pull-back L to $J^1(J^{2k-2}E)$, using the canonical projection $\rho_k^{2k-1}: J^{2k-1}E \to J^kE$ and the canonical embedding $\phi: J^{2k-1}E \to J^1(J^{2k-2}E)$ (see section 5). This will give a function for which the Hessian matrix is not (trivially) of maximal rank. We apply the constraint procedure to obtain locally a Hamiltonian $h^{1,2k-2}$ on $J^{1*}(J^{2k-2}E)$ which is of order k. Therefore we may say that, at the local level, direct canonical formalisms may be developed in the framework of the standard formulation of constraint theory.

§2. Preliminaries. Throughout the text all functions, manifolds, etc. are smooth (C^∞-class). Let $(\mathbb{R}\times M, p, \mathbb{R})$ be a (trivial) fibred manifold. By $\sec(\mathbb{R}\times M)$ we denote the set of all sections of $(\mathbb{R}\times M, p, \mathbb{R})$, that is, the maps $s: \mathbb{R} \to \mathbb{R}\times M$ such that $p\circ s = I_d$, (a local section s is defined on a domain U of $\mathbb{R}$ such that $p\circ s = I_d$ along U). Locally $(\mathbb{R}\times M, p, \mathbb{R})$ is characterized by coordinates (t, y^A), $1 \leqslant A \leqslant m = \dim M$. The manifold of all k-jets of sections $s \in \mathrm{Sec}(\mathbb{R}\times M)$, denoted by $J^k(\mathbb{R}\times M)$ is locally given by coordinates of type

$$(t, y^A, y_1^A, \ldots, y_k^A). \tag{2.1}$$

If $s \in \mathrm{Sec}(\mathbb{R}\times M)$ and $\tilde{s}^k(t)$ is the corresponding k-jet of s at $t \in \mathbb{R}$ then we have

$$y^A = s^A(t), \quad y_i^A = \frac{1}{i!}\frac{d^i}{dt^i} s^A(t), \quad 1 \leqslant i \leqslant k.$$

Sometimes we will employ the notation $y^A = y_o^A$. The factor $(\frac{1}{i!})$ appears only for technical reasons. We may adopt the following coordinate system for $J^k(\mathbb{R}, M)$,

$$(t, q^A, q_1^A, \ldots, q_k^A) \tag{2.2}$$

where

$$q^A = q_o^A = s^A(t) \; ; \quad q_i^A = \frac{d^i}{dt^i} s^A(t), \quad 1 \leq i \leq k,$$

and we have

$$q_i^A = i! y_i^A, \quad 0 \leq i \leq k, \quad 1 \leq A \leq m.$$

Tha manifold $J^k(\mathbb{R}\times M)$ can be fibred along $\mathbb{R}$, M and $J^i(\mathbb{R}\times M)$, $i < k$, with projections $\alpha^k : J^k(\mathbb{R}\times M) \to \mathbb{R}$, $\beta^k : J^k(\mathbb{R}\times M) \to M$ and $\rho_i^k : J^k(\mathbb{R}\times M) \to J^i(\mathbb{R}\times M)$ defined by

$$\alpha^k(\tilde{s}(t)) = t; \quad \beta^k(\tilde{s}^k(t)) = s(t) \quad \text{and} \quad \rho_i^k(\tilde{s}^k(t)) = \tilde{s}^i(t)$$

we identify $J^o(\mathbb{R}\times M)$ with $\mathbb{R}\times M$.

The mapping $\tilde{s}^k : t \to \tilde{s}^k(t)$ is a section of $(J^k(\mathbb{R}\times M), \alpha^k, \mathbb{R})$ and is called *the* k*-jet prolongation of* s.

As $(\mathbb{R}\times M), p, \mathbb{R})$ is a trivial bundle we may identify maps from $\mathbb{R}$ to M with sections of $(\mathbb{R}\times M, p, \mathbb{R})$ as well as their k-jets. Thus we set $J^k(\mathbb{R}\times M) = J^k(\mathbb{R}, M)$. Also, $J^k(\mathbb{R}, M)$ may be identified with $\mathbb{R}\times T^k M$, where $T^k M = J_o^k(\mathbb{R}, M)$ is the k-jet manifold of all maps $s : \mathbb{R} \to M$ with source at the origin $t = 0$. $T^k M$ is the *tangent bundle of order* k.

We recall the following important result (see Goldschmidt & Sternberg [GS] or de León & Rodrigues [DLR]): *On* $J^k(\mathbb{R}, M)$ *there is defined a unique one-form* θ_k *with values in* $T(J^{k-1}(\mathbb{R}, M))$ (T means tangent) *such that for every* $s \in Sec(\mathbb{R}\times M)$ *and all vector fields* X *on* $T\mathbb{R}$ (respectively *any vertical vector field* Z *on* $J^k(\mathbb{R}, M)$) *one has*

$$\theta_k(T\tilde{s}^k(X)) = 0; \quad \theta_k(Z) = T\rho_{k-1}^k(Z).$$

Therefore, if u *is a section of* $(J^k(\mathbb{R}, M), \alpha^k, \mathbb{R})$ *then there is* $s \in Sec(\mathbb{R}\times M)$ *such that* $u = \tilde{s}^k$ *is the* k*-jet prolongation of* s *if and only if* θ_k *vanishes along the image* u(R) (*or if we want* $u^*\theta_k \equiv 0$). *In such a case we say that* u *is* "holonomic".

The components of θ_k are real one-forms, locally expressed by

$$\theta^A = dq^A - q_1^A dt \; ; \quad \theta_1^A = dq_1^A - q_2^A dt ; \ldots ; \theta_{k-1}^A = dq_{k-1}^A - q_k^A dt,$$

(if we adopt the coordinate y's, then we consider the one-forms

$$\bar\theta^A = dy^A - y_1^A dt \; ; \quad \bar\theta^A{}_1 = dy_1^A - 2y_2^A dt ; \ldots ; \bar\theta_{k-1}^A = dy_{k-1}^A - k y_k^A dt,$$

and

$$\bar\theta_i^A = \frac{1}{i!}\,\theta_i^A, \qquad 0 \leqslant i \leqslant k-1). \tag{2.3}$$

So, a section $u : \mathbb{R} \to J^k(\mathbb{R},M)$ is holonomic if and only if $u^*\theta_i^A = 0$, for all $i \in \{0,1,\ldots,k-1\}$. If $u^*\theta_i^A = 0$, $0 \leqslant i \leqslant r < k-1$, then u is said r-*holonomic*.

Let us now take k = 2. We recall some basic results on almost tangent geometry and we indicate our book [DLR] for further details. Let M be a m-dimensional manifold. Then the tangent bundle T^2M of order 2 is endowed with a canonical endomorphism $J_1 : T(T^2M) \to T(T^2M)$, called *almost tangent structure of order 2 and type 1*, locally defined by

$$J_1 = \frac{\partial}{\partial y_1^A} \otimes dy^A + \frac{\partial}{\partial y_2^A} \otimes dy_1^A.$$

If we set $J_2 = (J_1)^2$ then J_2 is an endomorphism on $T(T^2M)$. Locally

$$J_2 = \frac{\partial}{\partial y_2^A} \otimes dy^A$$

Also, it is possible to show that on T^2M there are defined vector fields C_1 and C_2, locally characterized by

$$C_1 = y_1^A \frac{\partial}{\partial y_1^A} + 2y_2^A \frac{\partial}{\partial y_2^A}; \quad C_2 = y_1^A \frac{\partial}{\partial y_2^A}.$$

We call C_1 by *Liouville vector field* (of order 2). These results can be extended for any finite k, that is, the tangent bundle of order k, T^kM, is also endowed with an almost tangent structure $J_1:T(T^kM) \to T(T^kM)$ and we have a family of endomorphisms $J_r = (J_1)^r$, $2 \leqslant r \leqslant k$ as well as canonical vector fields $C_1, C_2, \ldots, C_k$, where $C_r = J_{r-1}(C_1)$.

The autonomous (or time-independent) Lagrangian formalism of higher order as well as its canonical (or Hamiltonian) formalism can be deduced in an intrinsical way with the help of the above structures and an appropriate exterior calculus on higher order tangent bundles. In our book [DLR] it is shown that in these formalisms the form

$$w_L = -dd_{J_1}L + \frac{1}{2}d_T dd_{J_2}L - \frac{1}{3!}d_T^2 dd_{J_3}L + \ldots + (-1)^k \frac{1}{k!} d_T^{k-1} dd_{J_k}L$$

takes a fundamental role in the theory. Here $L:T^kM \to \mathbb{R}$ is the Lagrangian of order k, d_{J_S} and d_T are appropriate operators (see below). The intrinsical form of the $2k^{th}$-order Lagrange equations is

$$i_\xi w_L = dE_L$$

where $\xi:T^kM \to T(T^kM)$ is a semispray (see below) and E_L is the energy associated to L, defined by

$$E_L = C_1L - \frac{1}{2} d_T(C_2L) + \frac{1}{3!} d_T^2(C_3L) + \ldots + (-1)^{k-1} \frac{1}{k!} d_T^{k-1}(C_kL) - L.$$

Let us return to the case k = 2. We denote by $\bar{J}_1$ (respectively $\bar{J}_2$) the induced endomorphisms on $J^2(\mathbb{R},M)$ given by

$$\bar{J}_1 = J_1 - C_1 \otimes dt \; ; \qquad \bar{J}_2 = J_2 - C_2 \otimes dt.$$

Thus, locally, one has

$$\bar{J}_1 = \frac{\partial}{\partial y_1^A} \otimes dy^A + \frac{\partial}{\partial y_2^A} \otimes dy_1^A - \left(y_1^A \frac{\partial}{\partial y_1^A} + 2y_2^A \frac{\partial}{\partial y_2^A}\right) \otimes dt$$

$$\bar{J}_2 = \frac{\partial}{\partial y_1^A} \otimes dy^A \times y_1^A \frac{\partial}{\partial y_2^A} \otimes dt,$$

and we have the following equalities:

$$\bar{J}_1(\partial/\partial t) = -C_1 \ ; \quad \bar{J}_1^*(dt) = dt\circ\bar{J}_1 = 0$$

$$\bar{J}_2(\partial/\partial t) = -C_2 \ ; \quad \bar{J}_2^*(dt) = dt\circ\bar{J}_2 = 0 \ ,$$

$$\bar{J}_1\left(\frac{\partial}{\partial y_i^A}\right) = \begin{cases} \dfrac{\partial}{\partial y_{i+1}^A}, & 0 \leqslant i \leqslant 1 \\ 0 & i = 2 \ ; \end{cases} \qquad \bar{J}_1^*(dy_i^A) = \begin{cases} \bar{\theta}_{i-1}^A, & 1 \leqslant i \leqslant 2 \\ 0 & i = 0 \ ; \end{cases}$$

$$\bar{J}_2\left(\frac{\partial}{\partial y_i^A}\right) = \begin{cases} \dfrac{\partial}{\partial y_{i+2}^A} & i = 0 \\ 0 & 1 \leqslant i \leqslant 2 \ ; \end{cases} \qquad \bar{J}_2^*(dy_i^A) = \begin{cases} \bar{\theta}_{i-2}^A, & i = 2 \\ 0 \ , & 0 \leqslant i \leqslant 1 \end{cases}$$

Therefore $\bar{J}_1^*(dy_1^A) = \bar{\theta}^A = \bar{J}_2^*(dy_2^A)$, $1 \leqslant i \leqslant 2$.

To finish this section we remark that on $J^2(\mathbb{R},M)$ there is defined also an appropriate exterior calculus induced by $\bar{J}_1$; an interior product $i_{\bar{J}_1}$ on r-forms

$$i_{\bar{J}_1}\omega(X_1,\ldots,X_r) = \sum_{i=1}^{r}\omega(X_1,\ldots,\bar{J}_1 X_i,\ldots,X_r)$$

and a differentiation $d_{\bar{J}_1}$ defined by $d_{\bar{J}_1} = i_{\bar{J}_1}d - di_{\bar{J}_1}$. Also we say that a vector field ξ on $J^2(\mathbb{R},M)$ is a *semispray* of type r, $1 \leqslant r \leqslant 2$ if and only if

$$\bar{J}_r(\xi) = 0 \quad \text{and} \quad J_r(\xi) = C_r, \quad 1 \leqslant r \leqslant 2.$$

Thus, in local coordinates we have

$$\xi = \frac{\partial}{\partial t} + y_1^A\frac{\partial}{\partial y^A} + 2y_2^A\frac{\partial}{\partial y_1^A} + \xi_2^A\frac{\partial}{\partial y_2^A}$$

were ξ_2^A are functions of the variables (t,y^A,y_1^A,y_2^A).

§3. The direct canonical formalism. Let us set $E = \mathbb{R}\times M$ and use the identification of $J^k(\mathbb{R},M)$ with J^kE. Let $L^k : J^kE \to \mathbb{R}$ be a Lagrangian of order k. Then we denote by

$$\Omega_{L^k} = L^k dt + f_{A/1}\bar{\theta}^A + \ldots + f_{A/k}\bar{\theta}^A_{k-1} \tag{3.1}$$

the Poincaré-Cartan form on $J^{2k-1}E$, where the f's are functions defined by

$$f_{A/j} = \sum_{i=o}^{k-j} (-1)^i \frac{1}{(i+1)!} d_T^i \left(\frac{\partial L^k}{\partial y^A_{(j+i)}}\right), \qquad 1 \leq j \leq k. \tag{3.2}$$

In (3.2), d_T is the operator which maps each function g on J^rE on a function d_Tg on $J^{r+1}E$, locally expressed by

$$d_Tg(t,y^A,\ldots,y^A_{r+1}) = \frac{\partial g}{\partial t} + \sum_{i=o}^{r} (i+1)y^A_{i+1} \frac{\partial g}{\partial y^A_i} .$$

Clearly,

$$f_{A/j} = f_{A/j}\ (t,y^A,y^A_1,\ldots,y^A_{2k-j}), \quad 1 \leq j \leq k.$$

We recall that (3.2) along (2k-1)-jet prolongations takes the form

$$f_{A/j} = \frac{\partial L^k}{\partial q^A_1} - \frac{d}{dt}\left(\frac{\partial L^k}{\partial q^A_2}\right) + \ldots + (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} \left(\frac{\partial L^k}{\partial q^A_k}\right)$$

We say that a section $s \in Sec(E)$ is an *extremal* for $L^k : J^kE \to \mathbb{R}$ if

$$(\tilde{s}^{2k-1})^*(i_X\Theta_k) = 0 \tag{3.3}$$

where X is a α^{2k-1}-vertical vector field on $J^{2k-1}E$ and

$$\Theta_k = d\Omega_{L^k}. \tag{3.4}$$

A direct calculation on local coordinates shows that (3.3) takes the form

$$\frac{L^k}{q^A} - \frac{d}{dt}(\frac{L^k}{q_1^A}) + \ldots + (-1)^k \frac{d^k}{dt^k}(\frac{L^k}{q_k^A}) = 0 \qquad (3.4)'$$

which are the generalized Lagrange equations of order 2k.

We may extend the above notion of extremal to sections which are not necessarily jet prolongations: a section u of $J^{2k-1}E$ is an extremal of Ω_{L^k} if

$$u^*(i_X \Theta_k) = 0 \qquad (3.5)$$

for every α^{2k-1}-vertical vector field X on $J^{2k-1}E$. In local coordinates we see that (3.5) takes the form

$$\frac{\partial L^k}{\partial y^A} - d_T f_{A/1} = 0 \qquad (3.5)'$$

where $f_{A/1}$ is defined by (3.2) and so (3.5)' is

$$\frac{\partial L^k}{\partial y^A} - d_T (\frac{\partial L^k}{\partial y_1^A}) + \frac{1}{2} d_T^2 (\frac{\partial L^k}{\partial y_2^A}) + \ldots (-1)^k \frac{1}{k!} d_T^k (\frac{\partial L^k}{\partial y_k^A}). \qquad (3.5)''$$

A main problem in the calculus of variations is to know if an extremal u^* of Ω_{L^k} is holonomic, i.e., if there is a section s of E such that $u = \tilde{s}^{2k-1}$. It can be shown that if L^k is regular then u is holonomic and the section s such that $\tilde{s}^{2k-1} = u$ is an extremal for L^k. We say that L^k is *regular* if the Hessian matrix $(\partial^2 L^k / \partial y_k^A\ y_k^B)$ is of maximal rank.

The regularity of L^k is related with the form Θ_k. This can be seen using the form Ω_{L^k} in (3.1). Let us take k = 2. Then the intrinsical expression of Ω_{L^k} is

$$\Omega_{L^k} = d_{\bar{J}_1} L^2 - \frac{1}{2} d_T d_{\bar{J}_2} L^2 + L^2 dt$$

$$= d_{J_1} L^2 - \frac{1}{2} d_T d_{J_2} L^2 - [C_1(L^2) - \frac{1}{2} d_T(C_2(L^2)) - L^2] dt.$$

Therefore

$$\Theta_2 = d\Omega_{L^2} \tag{3.6}$$

and from a straightforward calculation one obtains

$$\underbrace{\Theta_2 \wedge \dots \wedge \Theta_2}_{2m} = \pm\det(\partial^2 L^2/\partial y_2^A\ y_2^B)\, dy^1 \wedge dy^2 \wedge \dots \wedge dy_3^m$$

(see [DLR], p.93/94 for the autonomous situation).

Suppose that $L^k : J^k E \to \mathbb{R}$ is regular. Then we may define the *Legendre transformation* in the following manner ([K]): let (t, y^A) be a coordinate system for $E = \mathbb{R} \times M$ and $(t, y^A, y_1^A, \dots, y_k^A)$ the corresponding induced coordinates on $J^k E$. Then take into account the regularity of L^k and the equility in (3.2) for $j = k$. We see that $(t, y^A, y_1^A, \dots, y_{k-1}^A, f_{a/k})$ is a coordinate system on $J^k E$. Now, consider the induced coordinate system $(t, y^A, y_1^A, \dots, y_{k+1}^A)$ on $J^{k+1}E$ and take (2.2) for $j = k-1$. The function $f_{A/k-1}$ is defined on $J^{k+1}E$ and a direct calculation shows that

$$\frac{\partial f_{A/k-1}}{\partial y_{k+1}^A} = \frac{\partial f_{A/k-1}}{\partial y_{k+1}^B} = \frac{\partial^2 L^k}{\partial y_k^A \partial y_k^B}\ .$$

As before, the regularity of L^k gives a coordinate system $(t, y^A, y_1^A, \dots, y_{k-1}^A, f_{A/k}, f_{A/k-1})$ on $J^{k+1}E$. If we proceed in an analogous way one obtains a mapping $(t, y^A, y_1^A, \dots, y_{2k-1}^A) \to (t, y^A, y_1^A, \dots, y_{k-1}^A, f_{A/k}, f_{A/k-1}, \dots, f_{A/1})$ which is called the *Legendre transformation*, (which is assumed throughout the text as at least a local diffeomorphism). For these new coordinates, when $k = 2$ for example, we have

$$\Omega_{L^2} = f_{A/1}\, dy^A + \tfrac{1}{2} f_{A/2}\, dy_1^A - H^2\, dt\ , \tag{3.7}$$

where

$$H^2 = f_{A/1}\, y_1^A + \tfrac{1}{2} f_{A/2}\, y_2^A - L^k(t, y^A, y_1^A, y_2^A).$$

DEFINITION. Let $L^k : J^kE \to \mathbb{R}$ be a regular Lagrangian. The function $H^k : J^{2k-1}E \to \mathbb{R}$ locally characterized by

$$H^k = \sum_{j=1}^{k} f_{A/j}\ y_j^A - L^k(t, y^A, \ldots, y_k^A)$$

is called *Hamiltonian of order* k.

For these new coordinates the computation of (3.7) gives the *canonical form* of the Lagrange equations of 4^{th} order:

$$\frac{\partial H^2}{\partial y^A} = -d_T f_{A/1} \ ; \quad \frac{\partial H^2}{\partial f_{A/1}} = d_T y^A$$

$$\frac{\partial H^2}{\partial y_1^A} = -d_T f_{A/2} \ ; \quad \frac{\partial H^2}{\partial f_{A/2}} = d_T y_1^A ; \quad \frac{\partial H^2}{\partial t} = \frac{\partial L^2}{\partial t} \tag{3.8}$$

Equations (3.8) are also known as *Hamilton equations*.

§4. The indirect canonical formalism. Let us consider the jet bundle $J^1(J^{k-1}E) = \mathbb{R}\times T(T^{k-1}M)$, which may be locally characterized by the following coordinates

$$(t, y^A, y_1^A, \ldots, y_{k-1}^A \ ; \quad z_{o,1}^A, z_{1,1}^A, \ldots, z_{k-1,1}^A).$$

Let us forget for a moment that $J^{k-1}E$ is a (k-1) jet bundle, that is, we think $J^{k-1}E$ as a differentiable manifold of dimension km. Denote by $L^{1,k-1}$ a Lagrangian defined on $J^1(J^{k-1}E)$. We may also think that $L^{1,k-1}$ is a Lagrangian of order one and as we know from the order-one situation, the Lagrange equations have the form

$$\frac{\partial L^{1,k-1}}{\partial u^A} - d_T\left(\frac{\partial L^{1,k-1}}{\partial v^A}\right) = 0 \tag{4.1}$$

where the u is the position and the v the velocity. So, returning to the situation where $J^{k-1}E$ is the jet bundle, we may take $z^A_{o,1}$, $z^A_{1,1}$ etc. as the "velocities" and (4.1) takes the form

$$\frac{\partial L^{1,k-1}}{\partial y^A} - d_T\left(\frac{\partial L^{1,k-1}}{\partial z^A_{o,1}}\right) = 0 \;\ldots\; \frac{\partial L^{1,k-1}}{\partial y^A_{k-1}} - d_T\left(\frac{\partial L^{1,k-1}}{\partial z^A_{k-1,1}}\right) = 0 \qquad (4.2)$$

REMARK. If $q:t \to q(t) \in M$ is a curve then the following relations hold

$$q^A_j(t) = j!y^A_j; \quad q^A_{j,1}(t) = j!z^A_{j,1}; \quad q^A_{j,2}(t) = j!2!z^A_{j,2},$$

$0 \leqslant j \leqslant k-1$. Also, $d_T(\partial L^{1,k-1}/\partial z^A_{j,1})$, $0 \leqslant j \leqslant k-1$, is a function on $J^2(J^{k-1}E)$ and it is not hard to see that

$$d_T g = z^A_{o,1}\frac{\partial g}{\partial y^A} + \ldots + z^A_{k-1,1}\frac{\partial g}{\partial y^A_{k-1}} +$$

$$+ 2\left[z^A_{o,2}\frac{\partial g}{\partial z^A_{o,1}} + \ldots + z^A_{k-1,2}\frac{\partial g}{\partial z^A_{k-1,2}}\right]$$

where $g = (\partial L^{1,k-1}/\partial z^A_{j,2})$ and $(t, y^A, \ldots, y^A_{k-1}; z^A_{o-1}, \ldots, z^A_{k-1,1}; z^A_{o,2}, \ldots, z^A_{k-1,2})$ are coordinates for $J^2(J^{k-1}E)$.

The Lagrangian $L^{1,k-1}:J^1(J^{k-1}E) \to \mathbb{R}$ will be regular if the Hessian matrix

$$\left(\frac{\partial^2 L^{1,k-1}}{\partial z^A_{i,1}\partial z^B_{j,1}}\right), \qquad 0 \leqslant i,j \leqslant k-1$$

is of maximal rank. If this is the case we may define a Legendre transformation $Leg:J^1(J^{k-1}E) \to J^{1*}(J^{k-1}E)$ in a similar way as the standard situation and we define the corresponding Hamiltonian $H^{1,k-1}:J^{1*}(J^{k-1}E) \to \mathbb{R}$, *or order 1*, by

$$H^{1,k-1} = \sum_{j=o}^{k-1} p^{j,1}_A\, z^A_{j,1} - L^{1,2}. \qquad (4.3)$$

where the functions p are the momenta defined by

$$p_A^{j,1} = \frac{1}{j!}\frac{\partial L^{1,k-1}}{\partial z_{j,1}^A}, \qquad 0 \leq j \leq k-1 \tag{4.4}$$

We leave to the reader the deduction of the corresponding Hamilton equations.

REMARK. We may derive the above equations directly from the corresponding Poincaré-Cartan form $\Omega_L{}^{t,k-1} = L^{1,k-1}dt + (\partial L^{1,k-1}/\partial y_{j-1}^A)\Theta_{j-1}^A$, where $\theta_{j-1}^A = dy_{j-1}^A - z_{j,1}^4 dt$ $1 \leq j \leq k-1$.

§5. Relating both formalisms. In the present section we will work with regular Lagrangians of 2^{nd} order, $L^2 : J^2E \to \mathbb{R}$. All results may be extended for any finite k, but we take k = 2 only to simplify the notations. Also, we will work with indirect formalisms on $J^1(J^2E)$ instead of $J^1(J^1E)$. This is due to the fact that the corresponding Hamiltonian of order 2, $H^2 : J^3E \to \mathbb{R}$ is defined on the jet bundle J^3E and J^3E may be embeded in $J^1(J^2E)$ as we will see below.

Let us recall here that a Lagrangian $L^1 : J^1E \to \mathbb{R}$ of order one is degenerate if the Hessian matrix of L^1 with respect to the velocities in not of maximal rank. We suppose that the set of points where the non-maximality of the matrix occurs define a manifold S embedded in the evolution space J^1E. Also, for simplicity, we may suppose that, if the rank of $(\partial^2 L/\partial v^A\ v^B)$ is n = m-r, where m = dim M, then the dynamical variables are supposed to be labelled in such a way that the regular part of the matrix lie among the first n-velocities. The degeneracy assumption on L implies that only n-momenta $p^B = \partial L^1/\partial v^B$, $1 \leq B \leq n$, can be defined as independent variables. Thus, in the dual evolution space $J^{1*}E$, there exists r-*constraints* relations of type

$$f^i(t, q^A, p^A) = 0, \qquad 1 \leq i \leq r,$$

defining also a manifold embedded in $J^{1*}E$, with codimension r. We define a Legendre transformation (supposed diffeomorphism) between these manifolds and a Hamiltonian H^1 which may be extended to $J^{1*}E$ by incorporating multipliers, that is,

$$h^1 = H^1 + \psi_i f^i(t, q^A, p^A),$$

(for more details about the constraint canonical formalism see Dirac [DI] or Sudharsan and Mukunda [SM]). We will apply this method to give a relation between the above canonical formalisms. To do this, let us first recall that J^kE may be embedded in $J^1(J^{k-1}E)$ by a mappijg ϕ such that in local terms one has

$$t(\phi(u(s))) = t(u(s)); \quad y^A(\phi(u(s))) = y^A(u(s));$$

$$z^A_{o,1}(\phi(u(s))) = y^A_1(u(s)) \ldots z^A_{k-2,1}(\phi(u(s))) = y^A_{k-1}(u(s)); \qquad (5.1)$$

$$z^A_{k,1,1}(\phi(u(s))) = y^A_k(u(s))$$

where $u \in Sec(J^kE)$ is a section. Also, the dual mapping ϕ^* allow us to inject functions on J^kE into functions on $J^1(J^{k-1}E)$.

In what follows we will consider only regular Lagrangians of order 2 (or 3), but all results are valid for any finite k. Let us take first a Lagrangian $L^3 : J^3E \to \mathbb{R}$ of order 3 and let us use the same symbol for the induced Lagrangian obtained from the action of the embedding ϕ. Then the incorporation of the constraint relations (5.1) in the Lagrangian

$$L^{1,2} = L^3 - \lambda^1_A(y^A_1 - z^A_{o,1}) + \frac{1}{2}\lambda^2_A(y^A_2 - z^A_{1,1}), \qquad (5.2)$$

where the λ's are the Lagrange multipliers permit us to show that the Lagrange equations (3.4)', for k = 3, can be derived from the Lagrange equations (4.2) (for a proof see [DLR] p. 190/191 or Aldaya and de Azcárraga [AA2], p.2546).

PROPOSITION 1. *Let* $H^3:J^5E \to \mathbb{R}$ *be the Hamiltonian counterpart of the regular Lagrangian* $L^3:J^3E \to \mathbb{R}$ *and* $H^{1,2}:J^{1*}(J^2E) \to \mathbb{R}$ *the Hamiltonian counterpart of the regular Lagrangian* $L^{1,2}:J^1(J^2E) \to \mathbb{R}$. *If we consider the relation (5.2) then the momenta* $f_{A/j}$ *given by (3.2) assume the form* $p_A^{j-1,1}$ *given by (4.4),* $1 < j < 3$, *under the action of* ϕ.

Proof. Let $v \in J^3E$ and consider the embedding $\phi:J^3E \to J^1(J^2E)$ and the Legendre transformation $\mathrm{Leg}:J^1(J^2E) \to J^{1*}(J^2E)$. Then

$$y^A(v) = (\phi\circ\mathrm{Leg})^* y^A(v);\quad y_1^A(v) = (\phi\circ\mathrm{Leg})^* p_A^{o,1}(v);\ldots;y_3^A(v)$$
$$= (\phi\circ\mathrm{Leg})^* p_A^{2,1}(v),$$

where the $p_A^{j-1,1}$, $1 \leqslant j \leqslant 3$, are the momenta defined by (4.4). Recalling that L^3 is supposed to be a function of type $L^3 = L^3(t,y^A,z^A_{o,1},z^A_{1,1},z^A_{2,1})$ we have, after the action of ϕ, the using (5.2), that

$$p_A^{o,1} = \frac{\partial L^{1,2}}{\partial z^A_{o,1}} = \frac{\partial L^3}{\partial z^A_{o,1}} + \lambda_A^1 \;;\; p_A^{1,1} = \frac{\partial L^{1,2}}{\partial z^A_{1,1}} = \frac{\partial L^3}{\partial z^A_{1,1}} - \frac{1}{2}\lambda_A^2$$

$$p_A^{2,1} = \frac{1}{2}\frac{\partial L^{1,2}}{\partial z^A_{2,1}} = \frac{1}{2}\frac{\partial L^3}{\partial z^A_{2,1}}. \tag{5.3}$$

If we derive (5.2) with respect to y_1^A and y_2^A, one has

$$\frac{\partial L^{1,2}}{\partial y_1^A} = -\lambda_A^1 = d_T\left(\frac{\partial L^{1,2}}{\partial z^A_{1,2}}\right)$$

$$\frac{\partial L^{1,2}}{\partial y_2^A} = \frac{1}{2}\lambda_A^2 = \frac{1}{2}\, d_T\left(\frac{\partial L^{1,2}}{\partial z^A_{2,1}}\right) \tag{5.4}$$

where the equalities on the right of (5.4) are given by (4.2). Now, from (5.2) one has

$$\frac{\partial L^{1,2}}{\partial z^A_{1,1}} = \frac{\partial L^3}{\partial z^A_{1,1}} - \frac{1}{2}\lambda^2_A \ ; \quad \frac{\partial L^{1,2}}{\partial z^A_{2,1}} = \frac{\partial L^3}{\partial z^A_{2,1}} \tag{5.5}$$

Therefore,

$$-\lambda^1_A = d_T\left(\frac{\partial L^3}{\partial z^A_{1,1}} - \frac{1}{2}\lambda^2_A\right) = d_T\left(\frac{\partial L^3}{\partial z^A_{1,1}} - \frac{1}{2}\ d_T\left(\frac{\partial L^{1,2}}{\partial z^A_{2,1}}\right)\right)$$

$$= d_T\left(\frac{\partial L^3}{\partial z^A_{1,1}}\right) - \frac{1}{2}\ d^2_T\left(\frac{\partial L^3}{\partial z^A_{2,1}}\right) \qquad \text{(see (5.3))}$$

and (5.3) becomes

$$p^{o,1}_A = \frac{\partial L^3}{\partial z^A_{o,1}} - d_T\left(\frac{\partial L^3}{\partial z^A_{1,1}}\right) + \frac{1}{2}d^2_T\left(\frac{\partial L^3}{\partial z^A_{2,1}}\right) ;$$

$$p^{1,1}_A = \frac{\partial L^3}{\partial z^A_{1,1}} - \frac{1}{2}d_T\ \frac{\partial L^3}{\partial z^A_{2,2}} \quad ; \tag{5.6}$$

$$p^{2,1}_A = \frac{1}{2}\ \frac{\partial L^3}{\partial z^A_{2,1}} \ ;$$

which are, respectively, the momenta $f_{A/1}$, $f_{A/2}$ and $f_{A/3}$ expressed after the action of the embedding ϕ. ▲

PROPOSITION 2. *Let* $L^2:J^2E \to \mathbb{R}$ *be a regular Lagrangian of order 2. Let* $\rho^3_2:J^3E \to J^2E$ *be the canonical projection and put* $L^3 = (\rho^3_2)^*L^2$ *as well as for the induced Lagrangian on* $J^1(J^2E)$ *by* ϕ. *Then the Lagrangian* $L^{1,2}:J^1(J^2E) \to \mathbb{R}$ *is independent of the variables* $z^A_{2,1}$.

Proof. It is obvious, since $L^3 = (\rho^3_2)^* \circ L^2(t,y^A,z^A_{o,1}, z^A_{1,1})$ is not a function of the variables $z^A_{2,1}$. Therefore the Lagrangian $L^{1,2}:J^1(J^2E) \to \mathbb{R}$ related with such L^3, initially given by

$$L^{1,2} = L^3 - \lambda^2_A(y^A_1 - z^A_{o,1}) + \lambda^2_A(y^A_2 - z^A_{1,2})$$

is in fact of type $L^{1,2} = L^3 - \lambda_A^1(y_1^A - z_{o,1}^A)$, since $\lambda_A^2 = d_T(\frac{\partial L^{1,2}}{\partial z_{2,1}^A}) = 0.$ Δ

Obviously, rank $\frac{\partial L^{1,2}}{\partial z_{j-1,1}^A} = 2m$, $1 \leqslant j \leqslant 3$ and we may define a Legendre transformation on the submanifold of $J^1(J^2E)$ of codimension m by $(t, y^A, z_{o,1}^A, z_{1,1}^A) \to (t, y^A, p_A^{o,1}, p_A^{1,1})$, where now

$$p_A^{o,1} = \frac{\partial L^3}{\partial z_{o,1}^A} - d_T(\frac{\partial L^3}{\partial z_{1,1}^A}); \qquad p_A^{1,1} = \frac{\partial L^3}{\partial z_{1,1}^A}$$

The Hamiltonian $H^{1,2}$ defined by $H^{1,2} = p_A^{o,1} z_{o,1}^A + p_A^{1,1} z_{1,1}^A - L^{1,2}$ (which is of 2^{nd} order) may be extended to a neighborhood in $J^{1*}(J^2E)$ by the Hamiltonian

$$h^{1,2} = H^{1,2} + \beta_A^1(y_1^A - p_A^{o,1}).$$

Therefore:

If $L^2 : J^2E \to \mathbb{R}$ is a regular Lagrangian of 2^{nd} order, then it is locally possible to construct a Hamiltonian on a neighborhood in $J^{1*}(J^2E)$ which is also of 2^{nd} order.

To conclude this note, we remark that the above procedure may be extended to non-regular Lagrangians of order 2, 3, etc. Even in such situations we may obtain a Hamiltonian counterpart on $J^{1*}(J^2E)$, $J^{1*}(J^3E)$, etc. of order 2, 3, etc.

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