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# THE MIDDLE GRAPH OF A HYPERGRAPH

by

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**Introduction.** This paragraph is meant to present some definitions that are necessary to follow the further notes, our graph theoretic terminology being fairly standard [2], [3], as well the matroid terminology [5].

The characterization of a middle graph of a graph is given by Akiyama, Hamada and Yoshimura [1]. Some other properties of the middle graphs are presented in [4]. In a similar way, we introduce the middle graph of a hypergraph and we give a characterization of this graph. With any middle graph, we associate a matroid and we prove that it is graphic.

Let $X = \{x_1,\ldots,x_n\}$ be a finite set and let $\mathcal{E} = \{E_i : i \in I\}$ be a family of subsets of $X$. The pair $H = (X,\mathcal{E})$ is called a *hypergraph on* $X$, of *order* $n$, if $E_i \neq \emptyset, i \in I$, and $\bigcup_{i\in I} E_i = X$. It will also be denoted as a pair $H = (V(H), E(H))$, where $V(H) = X$ is the set of *vertices*, and $E(H) = \mathcal{E}$ is the set of *edges*.

A hypergraph is *simple*, if the edges $E_i (i \in I)$ are all distinct, and *multiple*, otherwise. If $|E_i| \leqslant 2$, for all $i \in I$, then a multiple hypergraph is a *multigraph* without isolated vertices, and if $|E_i| = 2$, for all $i \in I$, a simple hypergraph is a *graph* without isolated vertices.

We define the *middle graph* $M(H)$ *of the hypergraph* $H = (X,\underline{\mathcal{E}})$, as an intersection graph $\underline{\Omega}(F)$, where

$$F = X' \cup \mathcal{E} \text{ and } X' = \left\{\{x_1\},\ldots,\{x_n\}\right\}.$$

A graph is called a *middle graph*, if it is isomorphic to the middle graph $M(H)$ of a hypergraph $H$.

If $H$ is a hypergraph and $x \in V(H)$, then we denote by $N(x)$ and $\overline{N}(x)$ the *open* and the *closed neighbourhood* of the vertex $x$, in the hypergraph $H$, respectively, i.e., $x' \in N(x)$ if and only if $x \neq x'$ and there exists an edge $E$ of $H$, such that $\{x',x\} \subset E$, and $\overline{N}(x) = N(x) \cup \{x\}$.

Let $G$ be a graph. The set $\{C_i : i = 1,\ldots,m\}$ of the cliques of $G$ is called a *C-cover* of $G$, if

$$\bigcup_{i=1}^{m} V(C_i) = V(G) \text{ and } \bigcup_{i=1}^{m} E(C_i) = E(G).$$

If in the graph $G$ there exists a stable set $S$, such that the collection $\{G[\overline{N}(x)] : x \in S\}$ is a C-cover of $G$, then the set $S$ is called *C-stable*. Here, $G[A]$ denotes the subgraph of $G$, induced by $A \subset V(G)$.

A *matroid* $M$ is a pair $(Q,\underline{B})$, where $Q$ is a nonempty finite set and $\underline{B}$ is a nonempty collection of subsets of $Q$, called *bases*, satisfying the following properties:

(B1) no basis properly contains another basis,

(B2) if $B_1$ and $B_2$ are bases and if $b$ is any element of $B_1$, then there exists an element $b'$ of $B_2$, such that $(B_1\setminus\{b\}) \cup \{b'\}$ is also a basis.

**The main results**. In this section, we shall present our main results.

**THEOREM 1**. *A graph $G$ is a middle graph if and only if there exists a maximal stable set $S = \{x_1,\ldots,x_k\} \subset V(G)$, such that the collection $\{G[\overline{N}(x_i)] : i = 1,\ldots,k\}$ is a C-cover of $G$.*

**Proof**. Let us assume that $G$ is the middle graph of the hypergraph $H$. We consider the set $S = \left\{\{x_1\},\ldots,\{x_n\}\right\}$ and the collection $\left\{G[\bar{N}(\{x_i\})]: i = 1,\dots,n\right\}$. From the definition of the middle graph of the hypergraph $H$, the set $S$ is stable and maximal. Moreover, any two elements of $N(\{x_i\})$ have a nonempty intersection. Therefore, $G[N(\{x_i\})]$ is a clique of $G$, for all $i = 1,\dots,n$. Obviously, $G[\bar{N}(\{x_i\})]$ is also a clique of $G$, and the collection $\left\{G[\bar{N}(\{x_i\})]: i = 1,\dots,n\right\}$ is a $C$-cover of $G$.

Now, assume that the collection $\{G[\bar{N}(x_i)]: i = 1,\dots,k\}$ is a $C$-cover of $G$, and $S = \{x_1,\dots,x_k\}$ is a maximal stable set of $G$. A hypergraph $H$, whose middle graph is isomorphic to $G$, may be obtained in the following way. Let $V(H) = S$ and $V(G)\setminus S = \{e_1,\dots,e_m\}$. The family of edges of our hypergraph is $\{E_i : i = 1,\dots,m\}$, where $E_i = \{x_j : x_j \in S$ and $e_i \in \bar{N}(x_j)$, for $j = 1,\dots,k\}$. It is easy to see that $M(H) \simeq G$, and the proof is complete. ▲

Let $G$ be a graph and let $\underline{B}(G)$ be the collection

$$\{B : B \subset V(G) \text{ and } B \text{ is a } C\text{-stable set of } G\}.$$

For example, if $G = K_n, V(K_n) = \{x_1,\dots,x_n\}$, then, $\underline{B}(G) = \left\{\{x_i\}: i = 1,\dots,n\right\}$. If $G = K_{1,n}, V(G) = \{y, x_1,\dots,x_n\}$, then $\underline{B}(G) = \left\{\{x_1,\dots,x_n\}\right\}, n \geqslant 2$. If $G = P_n, V(P_n) = \{x_1,\dots,x_n\}$, $n > 4$, then $\underline{B}(G) = \emptyset$.

**THEOREM 2.** *Suppose that* $\underline{B}(G) \neq \emptyset$. *Then, the pair* $M_G = (V(G), \underline{B}(G))$ *is a matroid.*

**Proof.** Let $G$ be a middle graph. We must to prove the properties (B1) and (B2). Clearly, (B1) is trivial. To prove (B2), we let $B_1, B_2 \in \underline{B}(G)$ and $b \in B_1$. If $b \in B_1 \cap B_2$, then we put $b' = b$, and (B2) is true. Suppose that $b \in B_1 \setminus B_2$. Obviously, $B_2 \setminus B_1$ is not empty. Since $B_1$ is $C$-stable, we have $N(b) \cap (B_2 \setminus B_1) \neq \emptyset$, for every $b \in B_1 \setminus B_2$. Moreover, $|N(b) \cap (B_2 \setminus B_1)| = 1$. If it were not so, the induced subgraph $G[\bar{N}(b)]$ would not be a clique, and $B_1 \notin B(G)$, in contradiction with the assumption. Let $N(b) \cap (B_2 \setminus B_1) = \{b'\}$. In a similar way, we obtain $N(b') \cap (B_1 \setminus B_2) = \{b\}$, for $b' \in B_2 \setminus B_1$. Hence, there exists a bijection $\delta : (B_1 \setminus B_2) \to (B_2 \setminus B_1)$, such that $(B_1 \setminus \{b\}) \cup \{f(b)\}$ is $C$-stable, i.e., it is an element of $\underline{B}(G)$. Thus, $M_G = (V(G), \underline{B}(G))$ is a matroid. ▲

From the above and from the properties of the matroids, it is easy to verify the facts described in the next theorem.

**THEOREM 3.** *If $G$ is a middle graph and $M_G$ is its matroid, then:*

*(a) The rank $r(M_G)$ of $M_G$ is equal to the stability number $\underline{\alpha}(G)$ of $G$.*

*(b) If $S$ is a stable set and $|S| = \underline{\alpha}(G)$, then $S \in \underline{B}(G)$.*

*(c) The hypergraph $H$ is uniquely determined up to an isomorphism by its middle graph $M(H)$.* ▲

It is a reasonable question to ask whether a given matroid $M_G$ is the circuit matroid of some multigraph. In other words, whether there exists a multigraph $G'$, such that $M_G$ is isomorphic to the circuit matroid corresponding to $G'$. The answer to this question is obtained in the next theorem. Moreover, we give the construction of a such multigraph.

Suppose that we are given the middle graph $G = M(H)$ of a hypergraph $H$ and the matroid $M_G = (V(G), \underline{B}(G))$, with the rank function $r$, and let $A = \bigcup_{B \in \underline{B}(G)} B$. Obviously, $A \subset V(G)$ and $A$ does not contain the loops of $M_G$. Note that the set $A$ contains only those elements of $G$, which correspond to the vertices and loops of $H$ (if it were not so, the collection $\underline{B}(G)$ would not satisfy (B2)). These facts imply that the matroid $M_G$ does not have a circuit of size greater than two. We define, on the set $A$, a relation $R$, in the following way:

$$xRy \text{ if and only if } r(\{x,y\}) = 1. \quad (1)$$

Note that $x$ and $y$ form a pair of parallel elements of $M_G$. The above considerations give the following

**LEMMA.** *The relation $R$, defined above, is an equivalence relation on the set $A$. The matroid $M_G$ does not contain circuits of size greater than two.* ▲

**THEOREM 4.** *Suppose that we are given a matroid* $M_G = (V(G), \underset{=}{B}(G))$ *where* $G$ *is a middlegraph. Then, there exists a connected multigraph* $G'$, *such that* $M_G$ *is isomorphic to the circuit matroid corresponding to* $G'$.

**Proof.** Let $A = \bigcup_{B \in \underset{=}{B}(G)} B$ and let $R$ be the relation defined by (1). Let us denote by $A \setminus R = \{A_1, \ldots, A_k\}$ the factor set of $A$, with respect to $R$. Now, with every set $A_i$, let us associate a multigraph $G_i$, with two vertices and $|A_i|$ parallel edges, joining these vertices. Let $H_1$ be a multigraph with one vertex and $|V(G) \setminus A|$ loops. By the above and by lemma, it is easy to see that the circuit matroid of the multigraph $G' = (\bigoplus_{i=1}^{k} G_i) \oplus H_1$, where the operation "$\oplus$" is a direct sum operation, i.e., it is a multigraph obtained by the coalescence of a vertex of $G_1$ with a vertex of $G_2$ and then of a vertex of $G_1 \oplus G_2$ with a vertex of $G_3$ and so on, satisfies the required isomorphism. Note that the size of the collection $\underset{=}{B}(G)$ is equal to $\prod_{i=1}^{k} |A_i|$. ▲

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