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On the existence of uniform bundles

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To Carlos Fabio Zárate, in memoriam

ABSTRACT. Sufficient conditions for the existence of uniform bundles are given. The data are provided by a uniform structure and a family of sections. This supersedes previous results of the author by deleting the restriction of the uniformity being presented via a family of pseudometrics.

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RESUMEN. Se dan condiciones sufficientes para la existencia de campos uniformes, contando con datos suministrados por una estructura uniforme y una familia de secciones. Se obvia la restricción sobre la presentación de la uniformidad por medio de una familia de pseudométricas en un resultado anterior del autor.

For some time, conditions have been studied which secure the existence of bundles (fields) of Banach spaces in terms of the sections defining the bundle. Fell postulated the continuity of the map $t \longmapsto \|\sigma(t)\|$ for each section σ; latter it turned out that the upper semicontinuity of these maps was a sufficient condition for the existence of these fields. In recent years it has been possible to decompose other classes of algebras and groups that do not quite fit in the context laid down by Fell. In the general framework introduced by Dauns and Hofmann [1], there are sufficient conditions for the existence of uniform bundles depending on various rather intricate technical concepts. We now replace them by a simple condition —in terms of the general entourages— that is an immediate generalization of the semicontinuity condition that was postulated for bundles of Banach spaces. In a previous paper a similar setting in terms of families of pseudometrics was studied [7] that is is now generalized to the setting of families of entourages.

The condition is stated in Theorem 3 and is readily applicable to Banach modules, automorphisms and derivations of C^*–algebras.

1. Definition. Let $p\colon G \longrightarrow T$ be a surjective function. A *Uniformity* for p is a filter $\mathcal{U}$ on $G \vee G = \{(u,v)\colon p(u) = p(v)\}$ such that the filter generate by $\mathcal{U}$ on $G \times G$ is a uniform structure for G.

A *selection* for p is a function $\sigma\colon Q \longrightarrow G$, with $Q \subset T$, such that $p\sigma$ is the identity map of Q. If $Q = T$, σ is a *global selection*.

If T is a topological space and Q is open, σ is a *local selection*. When both G and T are topological spaces, a continuous selection is called *a section*. A set Σ of selections is *full* if for each $u \in G$ there exists $\sigma \in \Sigma$ such that $\sigma(p(u)) = u$.

If $U \in \mathcal{U}$ and σ is a selection for p, $U(\sigma)$ denotes the set $\{u \in G\colon (\sigma(p(u)), u) \in U\}$ and is called the *U–tube around σ*.

The following definition of uniform bundle deviates somewhat from established terminology but is more compatible with recent trends in this theory, mainly in applications related to Banach modules.

2 . Definition. Let G and T be topological spaces, $p\colon G \longrightarrow T$ be a surjective function, and $\mathcal{U}$ a uniformity for p such that for every $u \in G$ and every $U \in \mathcal{U}$ there is a local section σ such that $u \in U(\sigma)$. Then (G,p,T) is called a *uniform bundle*, provided that the sets $U(\sigma)$, where U runs through a suitable fundamental system of entourages $\mathfrak{S}$ and σ through the local sections for p, form a basis for the topology of G. Hence, for each $u \in G$, a fundamental system of neighborhoods consists of the U–tubes containing u, with $U \in \mathcal{U}$, around local sections for p.

The space T is called the *base space* of the bundle. For each $t \in T$, $p^{-1}(t)$ is the *fiber* above t. The space G is the *fiber space*. A uniform bundle is *full* if the set of all global sections is full.

3. Theorem. *Let T be a topological space and $p\colon G \longrightarrow T$ be a surjective function. Denote by Σ a set of local selections for p and by $\mathfrak{S}$ a fundamental system of entourages of a uniformity for p, closed under inversion (i.e., $U^{-1} \in \mathfrak{S}$, if $U \in \mathfrak{S}$), and such that for every $U \in \mathfrak{S}$ and $(u,v) \in U$ there exist $V, W \in \mathfrak{S}$ for which $(u,v) \in V$ and $VW \subset U$.*

We make the following assumptions:

(a) *For every $u \in G$ and every $U \in \mathfrak{S}$ there exists $\alpha \in \Sigma$ such that $u \in U(\alpha)$.*

(b) *For every $U \in \mathfrak{S}$ and every $(\alpha,\beta) \in \Sigma \times \Sigma$, the set $\{s \in T\colon (\alpha(s),\beta(s)) \in U\}$ is open in T.*

Then G can be equipped with a topology $\mathcal{T}$ such that:

(1) *The topology $\mathcal{T}$ has a basis consisting of the sets of the form $U(\alpha_Q)$*

where $U \in \mathfrak{S}$ and α_Q is the restriction to an open set $Q \subset \text{dom}\,\alpha$ of an $\alpha \in \Sigma$.

(2) *Each $\alpha \in \Sigma$ is a section.*

(3) *(G, p, T) is a uniform bundle.*

Proof.

(1) We first show that the collection of all sets $U(\alpha_Q)$ with the specifications given in (1) is a basis for a topology $\mathcal{T}$ in G. Given two such sets $U_1(\alpha_P)$ and $U_2(\beta_Q)$ and $u \in U_1(\alpha_P) \cap U_2(\beta_Q)$,there exist V_1, V_2, W_1, W_2 such that $u \in V_1(\alpha_P) \cap V_2(\beta_Q)$ and $W_1V_1 \subset U_1$ and $W_2V_2 \subset U_2$. Iterating the argument, we find M_1, M_2, N_1, $N_2 \in \mathfrak{S}$ such that $u \in M_1(\alpha_P) \cap M_2(\beta_Q)$, $N_1M_1 \subset V_1$, $N_2M_2 \subset V_2$. Now pick $W \in \mathfrak{S}$ such that $W \subset N_1^{-1} \cap N_2^{-1} \cap W_1 \cap W_2$ and $\xi \in \Sigma$ such that $u \in W(\xi)$; then

$$S = P \cap Q \cap \{s \in T \colon (\alpha(s), \xi(s)) \in V_1\} \cap \{s \in T \colon (\beta(s), \xi(s)) \in V_2\}$$

is a neighborhood of $t = p(u)$ in the space T and $W(\xi_S) \subset U_1(\alpha_P)$; indeed, from $(\xi(p(v)), v) \in W$ and $p(v) \in S$ it follows that $(\alpha(p(v)), v) \in WV_1 \subset W_1V_1 \subset U_1$, and thus $v \in U_1(\alpha_P)$. The inclusion $W(\xi_S) \subset U_2(\beta_Q)$ is obtained in the same manner.

(2) Let $\alpha \in \Sigma$ and $t \in T$. A fundamental neighborhood of $\alpha(t)$ in G is of the form $U(\beta_Q)$, where $\beta \in \Sigma$, Q is open in T, $U \in \mathfrak{S}$ and $\alpha(t) \in U(\beta_Q)$. By assumption (b), the set $\alpha^{-1}U(\beta_Q) = \{s \in Q \colon (\alpha(s), \beta(s)) \in U\}$ is open in T. Hence, α is a section.

(3) Now let $u \in G$, $U \in \mathfrak{S}$ and σ a be local section for p (not necessarily in Σ) such that $u \in U(\sigma)$. In order to prove that (G, p, T) is a uniform field, we must exhibit $W \in \mathfrak{S}$ and $\alpha \in \Sigma$ such that $u \in W(\alpha)$ and $W(\alpha_P) \subset U(\sigma)$ for some neighborhood P of $p(u)$ in T. Let V, $W \in \mathfrak{S}$ be such that $WV \subset U$, and $u \in V(\sigma)$ and M, $N \in \mathfrak{S}$, such that $NM \subset V$ and $u \in M(\sigma)$. Choose $W' \in \mathfrak{S}$ with $W' \subset N^{-1} \cap W$ and $\alpha \in \Sigma$ such that $u \in W'(\alpha)$. Since $(\sigma(p(u)), u) \in M$ and $(\alpha(p(u)), u) \in W'$ then $V^{-1}(\alpha)$ is an open neighborhood of $\sigma(p(u))$ in G; it follows by continuity of σ that $P = \sigma^{-1}(V^{-1}(\alpha))$ is an open neighborhood of $p(u)$ in the space T; then $v \in W'(\alpha_P)$ implies $p(v) \in P$; hence $(\sigma(p(v)), \alpha(p(v))) \in V$, but since $(\alpha(p(v)), v) \in W' \subset W$ it follows that $(\sigma(p(v)), v) \in WV \subset U$. Thus $v \in U(\sigma)$, that is, $W'(\alpha_P) \subset U(\sigma)$. □

4. Corollary. *Let $p \colon G \longrightarrow T$ be a surjection, Σ a set of local selections for p and $d_i \colon G \vee G \longrightarrow \mathbb{R}_+$, $i \in I$, a family of functions such that their restriction to $G_t \times G_t$ is a pseudometric for each $t \in T$. Let $\mathfrak{U}$ a uniformity for p with fundamental system of entourages of the form $\{(u, v) \in G \vee G \colon d_i(u, v) < \delta\}$. Assume that the set $\{s \in T \colon d_i(\alpha(s), \beta(s)) < \delta\}$ is open in T for every $i \in I$, $\delta > 0$ and $(\alpha, \beta) \in \Sigma \times \Sigma$. Then condition (b) of Theorem 3 is fulfilled. In the sectional representation theory of Banach modules, automorphisms and derivations of C^*–algebras, this is precisely the case [4, 5].*

5. Corollary. *Let T be a topological space, $p\colon G \longrightarrow T$ be a surjection, Σ a full set of local selections for p and $d_i : G \vee G \longrightarrow \mathbb{R}_+$, $i \in I$, a family of functions such that their restriction to $G_t \times G_t$ is a pseudometric for each $t \in T$. Let $\mathfrak{U}$ a uniformity for p with fundamental system of entourages of the form $\{(u,v) \in G \vee G\colon d_i(u,v) < \delta\}$. For each $t \in T$, define an equivalent relation R_t on Σ by $\sigma R_t \tau$ if and only if for every $i \in I$, $\limsup_{s\to t} d_i(\sigma(s),\tau(s)) = 0$. Let $\widehat{G}$ be the disjoint union of the family $\{\Sigma/R_t\colon t \in T\}$ and $\widehat{p} : \widehat{G} \longrightarrow T$ be the obvious proyection. For each $t \in T$ the fiber (with respect to $\widehat{p}$) is equipped with a family of pseudometrics, indexed by the same index set I, given by $\widehat{d}_i([\sigma]_t,[\tau]_t) = \limsup_{s\to t} d_i(\sigma(s),\tau(s))$. This definition does not depend on the representatives σ or τ taken on $\widehat{G}_t$. The map $t \longmapsto \widehat{d}_i([\sigma]_t,[\tau]_t) = \limsup_{s\to t} d_i(\sigma(s),\tau(s))$ is the upper envelope of $t \longmapsto d_i(\sigma(t),\tau(t))$. Its upper semicontinuty secures that condition (b) of Theorem 3 holds.*

This setup was discussed in detail in [8] as a localization process that produces stalks in a canonical way via inductive limits of directed systems defined through the neighborhood filter of each $t \in T$ (under the assumption that the uniformity defined by the family $(d_i)_{i\in I}$ of pseudometrics is Hausdorff). One of the main features of the localization procedure is that one can recover the set Σ in an isometric way in the set of sections of the uniform bundle $(\widehat{G},\widehat{p},T)$. It is worth mentioning that if the initial family of pseudometrics reduces to a single metric d, then $\widehat{d}$ is also a metric.

The following question arises naturally: What can we conclude if we do not assume in theorem 3 that for every $U \in \mathfrak{S}$ and $(u,v) \in U$ there exist $V, W \in \mathfrak{S}$ for which $(u,v) \in V$ and $WV \subset U$?

We can still construct a topology on G, provided that Σ is assumed to be full, but we can not describe T in terms of a basis. We may assert —if the rest of the set up in Theorem 3 remains unchanged— that

(1) For each $u \in G$, a fundamental system of neighborhoods consists of the sets of the form $U(\alpha_Q)$, where $U \in \mathfrak{S}$, and α_Q is the restriction to a neighborhood $Q \subset \operatorname{dom}\alpha$ of $t = p(u)$, of an $\alpha \in \Sigma$ such that $\alpha(p(u)) = u$.
(2) Each $\alpha \in \Sigma$ is a local section.
(3) (G,p,T) is a uniform field in the original Dauns–Hofmann sense [1].

More generally, we have the following theorem which applies in particular to the localization processes studied by Hofmann [2]:

6. Theorem. *Let $p\colon G \longrightarrow T$ be a surjective function, Σ a full set of local selections and $\mathfrak{S}$ a fundamental system of a uniformity for p such that $U^{-1} \in \mathfrak{S}$ for each $U \in \mathfrak{S}$.*
Assume that the set T is endowed with a topology satisfying the following condition:

(a) *For every $U \in \mathfrak{S}$ there exists $V \in \mathfrak{S}$, ($V \subset U$) such that for every $(\alpha,\beta) \in \Sigma \times \Sigma$ the set $\{s \in T\colon (\alpha(s),\beta(s)) \in U\}$ is a neighborhood of*

$\{s \in T\colon (\alpha(s), \beta(s)) \in V\}$.

Then, there exists a topology on G which has the properties (1), (2) and (3) on page 98.

Proof. We must show that the filter $\mathcal{V}(u)$ of the sets of the form $U(\alpha_Q)$ as described in (1), page 98, is the neighborhood filter of u for a topology on G. To this effect we show that for each set $U(\alpha_Q)$, with $Q \subset \operatorname{dom} \alpha$ open in T, there exists $D \in \mathcal{V}(u)$ such that for every $v \in D$, $U(\alpha_Q) \in \mathcal{V}(v)$.

Pick $W \in \mathfrak{S}$ such that $WW \subset U$. By condition (a) there exists $V \in \mathfrak{S}$ ($V \subset W$) such that for every $(\zeta, \eta) \in \Sigma \times \Sigma$ the set $\{s \in T\colon (\zeta(s), \eta(s)) \in W\}$ is a neighborhood of $\{s \in T\colon (\zeta(s), \eta(s)) \in V\}$. Take $D = V(\alpha_Q)$, $v \in D$ and $\beta \in \Sigma$ such that $\beta(p(v)) = v$. We claim that $V(\beta_{P \cap Q}) \subset U(\alpha_Q)$, where $P = \{s \in T\colon (\alpha(s), \beta(s)) \in W\}$. Indeed, if $w \in V(\beta_{P\cap Q})$ then $(\beta(p(w)), w) \in V$; but also $(\alpha(p(w)), \beta(p(w))) \in W$; hence $(\alpha(p(w), w) \in VW \subset WW \subset U$, and thus $w \in U(\alpha)$.

To see that the intersection of two elements of $\mathcal{V}(u)$ belongs to $\mathcal{V}(u)$, consider neighborhoods P, Q of $t = p(u)$ in the space T and tubes $U_1(\alpha_P)$, $U_2(\alpha_Q)$ with $\alpha(t) = u = \beta(t)$. Set $U = U_1 \cap U_2$, choose $W \in \mathfrak{S}$ such that $WW \subset U$, and define $S = \{s \in P \cap Q\colon (\alpha(s), \beta(s)) \in W\}$. By condition (a), the set S is a neighborhood of $t = p(u)$ in the space T. Also, $W(\beta_S) \subset U_1(\alpha_P) \cap U_2(\beta_q)$. In fact, take $v \in W(\beta_S)$; then $(\beta(p(v), v) \in W$ and $p(v) \in S$, hence $(\alpha(p(v)), v) \in WW$ and so $v \in U(\alpha_S) \subset U_1(\alpha_Q)$. Hence, for each $u \in G$, $\mathcal{V}(u)$ is a neighborhood filter of u for a topology on G.

(2) Now let $\alpha \in \Sigma$, $t \in \operatorname{dom} \alpha$ and $U \in \mathcal{U}$. By hypothesis (a), for every $\beta \in \Sigma$ such that $\alpha(t) = \beta(t)$ and every neighborhood $Q \subset \operatorname{dom} \beta$ of $t \in T$, the set $\alpha^{-1}U(\beta_Q) = \{s \in Q \cap \operatorname{dom} \alpha\colon (\alpha(s), \beta(s)) \in U)$ is a neighborhood of t. Hence, α is continuous.

(3) Let $U \in \mathfrak{S}$ and σ be a local section for p (not necessarily in Σ) with $\sigma(p(u)) = u$. To prove (3) we must find $V \in \mathfrak{S}$ and $\alpha \in \Sigma$ such that $\alpha(p(u)) = u$ and $V(\alpha_P) \subset U(\sigma)$ for some neighborhood $P \subset \operatorname{dom} \alpha$ of $p(u)$ in T. Take an entourage $V \in \mathfrak{S}$ such that $VV \subset U$ and choose any $\alpha \in \Sigma$ with $\alpha(p(u)) = u$. Since σ is continuous, $P = \sigma^{-1}(V^{-1}(\alpha))$ is a neighborhood of $p(u)$ in the space T; then $v \in V(\alpha_P)$ implies $p(v) \in P$; hence $(\alpha(p(v), \sigma(p(v))) \in V^{-1}$; but $(\alpha(p(v), v) \in V$, and therefore $(\sigma(p(v)), v) \in VV \subset U$; thus $v \in U(\sigma)$, that is, $V(\alpha_P) \subset U(\sigma)$, as desired. □

7. Remark. Observe that if for every $U \in \mathfrak{S}$ and every $(\alpha, \beta) \in \Sigma \times \Sigma$ the set $\{s \in T\colon (\alpha(s), \beta(s)) \in U\}$ is open in T, then condition a) of Theorem 6 is trivially satisfied by taking $V = U$. This takes care of the claims (1), (2), (3) on page 98.

8. Remark. Using Hofmann's notation in the so called method of topological localization [2], section 6, it can be easily verified that condition (a) of theorem 6 holds for that particular setup. Indeed, for every entourage $U \in \mathcal{U}$ and every

$(\tilde{\sigma}, \tilde{\tau}) \in \tilde{\Sigma} \times \tilde{\Sigma}$, the set $\{c \in C : (\tilde{\sigma}(c), \tilde{\tau}(c)) \in U_1^C\}$ is a neighborhood of the set $\{c \in C : (\tilde{\sigma}(c), \tilde{\tau}(c)) \in U^C\}$.

9. Example. Let T be a topological space, Y a Hausdorff uniform space with a saturated family of pseudometrics $(d_i)_{i\in I}$, $p\colon T \times Y \longrightarrow T$ the canonical proyection onto T, and $\mathfrak{S}$ a non empty set of subsets of T.

Define a family of pseudometrics for p as follows. For each $S \in \mathfrak{S}$ and $i \in I$, let

$$d_S^i : (T \times Y) \times (T \times Y) \longrightarrow \mathbb{R}$$

be such that for $u = (t, x)$, $v = (s, y)$,

$$d_S^i(u, v) = \begin{cases} d_i(x, y) & \text{if } p(u) = t = s = p(v) \in S, \\ 0 & \text{if } p(u) = t = s = p(v) \notin S, \\ \infty & \text{if } p(u) = t \neq s = p(v). \end{cases}$$

We can now define pseudometrics between selections (denoted also by d_S^i) by

$$d_S^i(\alpha, \beta) = \sup_{t\in S} d_S^i(\alpha(t), \beta(t)).$$

Note that the selections are, in this example, in a biyective correspondance with Y^T. This family of pseudometrics determines, as i ranges through I and S through $\mathfrak{S}$, the uniformity of the uniform convergence on the sets of $\mathfrak{S}$. Let Σ be the presheaf defined by $\Sigma(U) = \{\alpha_U : \alpha \in \mathcal{H}\}$, $\mathcal{H}$ being the set of all global selections for p and α_U the restriction of α to U. Since for α, $\beta \in \mathcal{H}$ we cannot in general secure the upper semicontinuity of $t \longmapsto d_S^i(\alpha(t), \beta(t))$, the triple $(T \times Y, p, T)$ is not a uniform bundle, but through localization [8] we can construct a uniform bundle $((\widehat{T \times Y}), \widehat{p}, \widehat{T})$ and a presheaf $\widehat{\Sigma}$ of local sections for $\widehat{p}$ such that $d_S^i(\widehat{\alpha}, \widehat{\beta}) = d_S^i(\alpha, \beta)$. Nevertheless, there is an instance, the uniformity of the pointwise convergence, in which we do have upper semicontinuity of the pseudometrics (provided that T is a T_1 space). If that is the case, we automatically have a uniforme bundle, with no additional hypothesis on the data.

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