Revista Colombiana de Matemáticas
Volumen 46(2012)2, páginas 185-204

The Brauer Group of K3 Covers

El grupo de Brauer de K3 cubrimientos

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Abstract. In this paper we study the injectivity of the induced morphism on the Brauer groups $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ given by the K3 cover $\pi : X \to Y$ of the Enriques surface Y.

Key words and phrases. Brauer group, K3 surface, Hochschild–Serre spectral sequence.

2010 Mathematics Subject Classification. 14J28, 16K50, 20J062.

Resumen. En este artículo estudiamos la inyectividad del morfismo inducido sobre los grupos de Brauer $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ dado por el K3 cubrimiento $\pi : X \to Y$ de la superficie de Enriques Y.

Palabras y frases clave. Grupo de Brauer, superficie K3, sucesión espectral de Hochschild–Serre.

1. Introduction

Let Y be an Enriques surface and $\pi : X \to Y$ its K3 cover with the fixed point free involution τ compatible with π. Since the Brauer group $\mathrm{Br}(Y)$ is $\mathbb{Z}/2\mathbb{Z}$, it is natural to ask about the triviality of the morphism $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$. This question was first mentioned by Harari and Skorobogatov in [3] and later answered by Beauville in [2] where he proved that the morphism is trivial if and only if the period map $\wp(Y, \varphi)$ belongs to one of the hypersurfaces H_λ for some $\lambda \in \Lambda^-$ with $\lambda^2 \equiv 2 \bmod 4$ and where H_λ is the hypersurface of Ω (this is the domain given by the equations $\omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0, \omega \cdot \lambda \neq 0$ for all $\lambda \in \Lambda^-$ with $\lambda^2 = -2$) defined by the equation $\lambda \cdot \omega = 0$. We give some group cohomology conditions for the morphism π^* to be injective. Besides, we also establish the type of the Néron Severi group of the K3 cover X of Picard number 11 such that the morphism $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is injective.

2. Basic Facts about Enriques Surfaces

We briefly recall some fundamental facts about Enriques and K3 surfaces.

Definition 1. A K3 surface is a compact complex surface X with trivial canonical bundle, i.e. $\omega_X \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.

Definition 2. An Enriques surface is a compact complex surface X with $\omega_X^2 \cong \mathcal{O}_X$, $\omega_X \neq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

The second cohomoloy of a K3 surface $H^2(X, \mathbb{Z})$ endowed with the cup-product is an even unimodular lattice of rank 22 and signature $(3, 19)$, i.e.,

$$H^2(X, \mathbb{Z}) \cong E_8^{\oplus 2} \oplus U^{\oplus 3}$$

where E_8, U are the root and hyperbolic lattices respectively.

Let Y be a smooth Enriques surface, $\pi : X \to Y$ its K3 cover and $\tau : X \to X$ the corresponding fixed point free involution such that $X/\tau \cong Y$. Thus we obtain the following lemma

Lemma 3. $0 \to \langle \omega_Y \rangle \to \operatorname{Pic}(Y) \to \operatorname{Pic}(X)^\tau \to 0$ *is an exact sequence.*

Proof. Let $\mathcal{L}$ be a sheaf with $\pi^*(\mathcal{L}) = \mathcal{O}_X$. Then $\mathcal{L} \otimes (\mathcal{O}_Y \oplus \omega_Y) = \pi_*\big(\pi^*(\mathcal{L})\big) = \pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus \omega_Y$. Therefore $\mathcal{L}$ is either $\mathcal{O}_Y$ or ω_Y. On the other hand, if $\lambda_\tau : \mathcal{M} \to \tau^*(\mathcal{M})$ is an isomorphism for some line bundle $\mathcal{M} \in \operatorname{Pic}(X)$, then, since $\mathcal{M}$ is simple (because it is a line bundle), $\tau^*\lambda_\tau \circ \lambda_\tau = c.\mathrm{id}$ for some $c \in \mathbb{C}$. Thus, we can replace λ_τ by $\frac{1}{\sqrt{c}}\lambda_\tau$ to obtain a linearization on $\mathcal{M}$ (see Definition 7 below). Hence, there exists a line bundle $\mathcal{L}$ on Y such that $\pi^*\mathcal{L} = \mathcal{M}$. ☑

Lemma 4.

i) If X is a K3 surface, then $H_1(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{tors} = 0$ (see [1, Prop. 3.3]).

ii) If Y is an Enriques surface, then $H_1(Y, \mathbb{Z}) = H^2(Y\mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$.

Lemma 5. *If Y is an Enriques surface, then* $\operatorname{Br}'(Y) = H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. By Serre duality and Lemma 4(i), it follows that $0 = b_1(Y) = b_3(Y)$ and $H^3(Y, \mathbb{Z})_{tors} = H^2(Y, \mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$ (see [1, page 15]). Since $p_g(Y) = 0$, the exponential sequence induces the following exact sequence

$$0 \to H^2(Y, \mathcal{O}_Y^*) \to H^3(Y, \mathbb{Z}) \to H^3(Y, \mathcal{O}_X).$$

Then, from the vanishing of $H^3(Y, \mathcal{O}_X)$, we conclude the isomorphism $\operatorname{Br}'(Y) = H^3(Y, \mathbb{Z})$ and from the vanishing $b_3(Y) = 0$, we deduce that $H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. ☑

3. The Kernel of $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$

We will study the kernel of the map $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ induced by the universal cover, $\pi : X \to Y$, of the Enriques surface Y. In a particular case we will be able to describe the non trivial element of $\mathrm{Br}'(Y)$ as a Brauer–Severi variety over Y. For the basic facts about group cohomology we refer to [11]. In order to describe $\ker(\pi^*)$, we use the Hochschild–Serre spectral sequence (see [5, Theorem 14.9])

$$E_2^{p,q} := H^p\big(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathcal{O}_X^*)\big) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y^*). \tag{1}$$

and the following theorem (cf. [11, Theorem 6.2.2]). First, we recall that for a cyclic group G of order m with a generator τ, the norm in $\mathbb{Z}G$ is the element $N = 1 + \tau + \cdots + \tau^{m-1}$.

Theorem 6. *If A is a G-module with G a cyclic group generated by τ, then*

$$H^n(G, A) = \begin{cases} A^G, & \text{if } \quad n = 0; \\ \big\{a \in A : Na = 0\big\}/(\tau - 1)A, & \text{if } \quad n \quad \text{is odd;} \\ A^G/NA, & \text{otherwise.} \end{cases}$$

The last theorem can be used to compute $E_2^{n,0}$ for all n. First, since the action of $\langle\tau\rangle = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^* = H^0(X, \mathcal{O}_X^*)$ is trivial, one has

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0 \tag{2}$$

for all even integers $n \neq 0$. On the other hand, if n is an odd integer and $a \in \mathbb{C}^*$ with $N(a) = 1$, it follows from the definition of the norm map that $1 = a\tau(a) = a^2$. Thus

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}. \tag{3}$$

Since $E_2^{2,0} = 0$, also $E_\infty^{2,0} = 0$ and the following exact sequence follows:

$$0 \to E_\infty^{1,1} \to H^2(Y, \mathcal{O}_Y^*) \to H^2(X, \mathcal{O}_X^*)^\tau. \tag{4}$$

Let us recall now a few facts about linearization for finite group actions. Let Z be a smooth projective variety with an action by a finite group G. Let $\sigma : G \times Z \to Z$ be the action on Z, $\mu : G \times G \to G$ be the multiplication map of G and $p_2 : G \times Z \to Z$, $p_{23} : G \times G \times Z \to G \times Z$ be the projections.

Definition 7. A G-linearization of a coherent sheaf F is an isomorphism $\lambda : \sigma^* F \xrightarrow{\sim} p_2^* F$ of $\mathcal{O}_{G\times Z}$-modules that satisfies the cocycle condition $(\mu \times \mathrm{id}_Z)^*\lambda = p_{23}^*\lambda \circ (\sigma \times \mathrm{id}_G)^*\lambda$.

In the particular case that G is a finite group, the last definition can be reformulated as follows: A G-linearization of F is given by isomorphisms $\lambda_g : F \xrightarrow{\sim} g^*F$ for all $g \in G$ satisfying $\lambda_1 = \mathrm{id}_F$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. If (F, λ) and (F', λ') are two G-linearised sheaves, then $\mathrm{Hom}(F, F')$ becomes a G-representation defined by the right action $g.f = (\lambda'_g)^{-1} \circ g^* f \circ \lambda_g$ for $f : F \to F'$.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal cover map. We proceed to define the *relative norm homomorphis* $N_{X/Y}$. Let U_i be an open covering of Y such that $\widehat{U}_i := \pi^{-1}(U_i)$ consists of two copies of U_i. Take $f = (f_0, f_1) \in \mathcal{O}^*(\widehat{U}_i)$ and define the *sheaf relative norm map* by $f_0 f_1$. Thus, the relative norm homorphism induced in the Picard groups can be defined as follows: take a 1-cocycle $\big\{\widehat{\varphi}_i = (\varphi_0^i, \varphi_1^i)\big\}_i$ over X that represents a line bundle $\mathcal{L}$, and define our desired morphism by $N_{X/Y}\big(\{(\varphi_0^i, \varphi_1^i)\}_i\big) = \big\{\varphi_0^i \cdot \varphi_1^i\big\}_i$. This is also the cocycle defining the line bundle $\det(\pi_*(\mathcal{L}))$. Hence, we obtain $N_{X/Y}(-) = \det(\pi_*(-))$ and one can show the following lemma whose proof can be found in [2].

Lemma 8. *The kernel of* $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is*

$$\big(\ker N_{X/Y}\big)/\big((1-\tau)\,\mathrm{Pic}(X)\big).$$

Definition 9. Let X be a surface and $\mathcal{P}$ a $\mathbb{P}^1$-bundle on X. We say that $\mathcal{P}$ comes from a vector bundle if there exists a vector bundle E on X such that $\mathcal{P} \cong \mathbb{P}(E)$.

Lemma 10. *Let* Y *be an Enriques surface and* $\pi : X \to Y$ *its universal cover map. Let* $\mathcal{L}$ *be a line bundle satisfying* $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, $N_{X/Y}(\mathcal{L}) = 0$, *and such that* $[\mathcal{L}]$ *is nontrivial in* $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$. *Then* $\mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ *descends to a projective bundle that does not come from a vector bundle.*

Proof**.** Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle with $N_{X/Y}(\mathcal{L}) = 0$ representing a nontrivial element in

$$\begin{aligned} E_2^{1,1} &= H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) \\ &= \frac{\{L \in \mathrm{Pic}(X) : \tau^*L \otimes L = \mathcal{O}_X\}}{\{\tau^*M \otimes M^\vee : M \in \mathrm{Pic}(X)\}}. \end{aligned}$$

We proceed to give a G-linearization on $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$:

$$\lambda_\tau : \mathbb{P}\big(\tau^*(\mathcal{O}_X \oplus \mathcal{L})\big) \longrightarrow \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}).$$

Since $N_{X/Y}(\mathcal{L}) = 0$ we can find a G-linearised isomorphism $i : \mathcal{L} \otimes \tau^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$ where we consider $\mathcal{O}_X$ endowed with the canonical G-linearization. We define λ_τ as the composition of morphisms

$$\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \to \mathbb{P}\big(\tau^*\mathcal{L} \oplus (\mathcal{L} \otimes \tau^*\mathcal{L})\big) \to \mathbb{P}(\tau^*\mathcal{L} \oplus \mathcal{O}_X) \to \mathbb{P}(\mathcal{O}_X \oplus \tau^*\mathcal{L})$$

$$[a:b] \mapsto [a\tau^*b : b\tau^*b] \mapsto \big[a\tau^*b : i(b\tau^*b)\big] \mapsto \big[i(b\tau^*b) : a\tau^*b\big]$$

where a and b are sections of $\mathcal{O}_X$ and $\mathcal{L}$ respectively. Note that $\mathbb{P}(\mathcal{O}_X \oplus \tau^*\mathcal{L}) = \mathbb{P}(\tau^*\mathcal{O}_X \oplus \tau^*\mathcal{L})$ because we consider the canonical linearization on $\mathcal{O}_X$, i.e. $\tau^*\mathcal{O}_X = \mathcal{O}_X$. Since i is a G-linearised isomorphism, it commutes with τ and from this we can check that $\lambda_\tau^2 = \mathrm{id}$ as follows:

$$\begin{aligned}
\lambda_\tau^2\big([a:b]\big) &= \lambda_\tau\big(\big[i(b\tau^*b) : a\tau^*b\big]\big)\\
&= \big[i\big((a\tau^*b)\tau^*(a\tau^*b)\big) : i(b\tau^*b)\tau^*(a\tau^*b)\big]\\
&= \big[a\tau^*a.i(b\tau^*b) : i(b\tau^*b)\tau^*(a\tau^*b)\big]\\
&= \big[a\tau^*a : \tau^*(a\tau^*b)\big]\\
&= \big[a\tau^*a : b\tau^*a\big]\\
&= [a:b].
\end{aligned}$$

Hence, the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ descends to a projective bundle $\mathcal{P}$ over Y. Now, we show that $\mathcal{P}$ does not come from a vector bundle on Y. Suppose $\mathcal{P} = \mathbb{P}(E)$ for some vector bundle E over Y and so $\mathbb{P}\big(\pi^*(E)\big) = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$. Thus, it follows that $\pi^*(E) = M \otimes (\mathcal{O}_X \oplus \mathcal{L})$, for some $M \in \mathrm{Pic}(X)$. By taking determinants on both sides of this isomorphism we get $\det\big(\pi^*(E)\big) = M^{\otimes 2} \otimes \mathcal{L}$.

In particular, this implies that M is not invariant. Indeed, if M is an invariant line bundle, $\mathcal{L} = \det\big(\pi^*(E)\big) \otimes (M^\vee)^{\otimes 2}$ is an invariant bundle. Hence $\mathcal{L} \cong \mathcal{O}_X$ because $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, a contradiction. Since $M^{\otimes 2} \otimes \mathcal{L}$ is invariant and $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, one has

$$M^{\otimes 2} \otimes \mathcal{L} = \tau^*(M^{\otimes 2} \otimes \mathcal{L}) = \tau^*M^{\otimes 2} \otimes \mathcal{L}^\vee$$

and so, $\tau^*M^{\otimes 2} = M^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}$. Hence, from the torsion freeness of $\mathrm{Pic}(X)$ we obtain $\tau^*M = M \otimes \mathcal{L}$, i.e., $\mathcal{L} = \tau^*M \otimes M^\vee$, but this contradicts the assumption that $\mathcal{L}$ defines a non trivial element in $E_2^{1,1}$. ☑

Lemma 11. *Let $\pi : X \to Y$ be the universal cover of an Enriques surface Y with $\rho(X) = 10$. Then $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is a nontrivial homomorphism.*

Proof. We show that $\rho(X) = 10$ implies $\mathrm{Pic}(X)^\tau = \mathrm{Pic}(X)$, i.e., all the line bundles on X are invariant. Since $\rho(X) = 10$, $\mathrm{Pic}(X)^\tau \subseteq \mathrm{Pic}(X)$ is a sublattice of finite index. Thus, if $\mathcal{L}$ is a line bundle, there exists a positive integer r with $\mathcal{L}^{\otimes r} \in \mathrm{Pic}(X)^\tau$, i.e.,

$$\tau^*\mathcal{L}^{\otimes r} = \mathcal{L}^{\otimes r}.$$

Hence

$$(\tau^*\mathcal{L} \otimes \mathcal{L}^\vee)^{\otimes r} = \mathcal{O}_X.$$

Since $\mathrm{Pic}(X)$ is torsion free, we obtain

$$\tau^*\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}_X,$$

i.e., $\mathcal{L}$ is an invariant line bundle. Thus, the group $H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$ vanishes and the lemma holds. ☑

Example 12. In this example we show the existence of a K3 surface X with $\rho(X) = 10$ that covers an Enriques surface. First, we find a K3 surface with Picard number 10. Let us define $\Lambda := E_8 \oplus E_8 \oplus U \oplus U \oplus U$ and an involution ρ of L by

$$\rho : \Lambda \to \Lambda, (e_1, e_2, h_1, h_2, h_3) \mapsto (e_2, e_1, -h_1, h_3, h_2).$$

Note that this involution is the universal action (cf. [1, Ch. VIII, Lemma 19.1]), i.e. whenever $\pi : X \to Y$ is the universal covering of an Enriques surface Y with $\tau : X \to X$ the covering involution, then there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. The ρ-invariant sublattice of Λ is

$$\Lambda^+ = \{x \in \Lambda : \rho(x) = x\} = \{(e, e, 0, h, h) : e \in E_8, h \in U\},$$

which is isometric to $E_8(2) \oplus U(2)$, where the isometry is given as follows

$$\rho^+ : \Lambda^+ \to E_8(2) \oplus U(2), \qquad (e, e, 0, h, h) \mapsto (e, h).$$

Hence, $E_8(2) \oplus U(2) \hookrightarrow E_8^{\oplus 2} \oplus U^{\oplus 3}$ is a primitive embedding. Since this lattice has Picard number 10 and signature (1,9), by [6, Cor. 2.9] we can find an algebraic K3 surface X with $\mathrm{NS}(X) = E_8(2) \oplus U(2)$. Now, we show that X has a fixed point free involution. The isometry ρ^+ also yields an isomorphism

$$\big(\Lambda^+\big)^\vee / \Lambda^+ \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

It means that Λ^+ is a 2-elementary lattice with $l(A_{\Lambda^+}) = 10$. This gives us an involution

$$\tau^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

which is the identity on Λ^+ and acts like multiplication by (-1) on $T_X = (\Lambda^+)^\perp = \big(\mathrm{NS}(X)\big)^\perp$ where the orthogonal complement is taken in $H^2(X, \mathbb{Z})$. Since τ^* is the identity on Λ^+ (=NS(X) through the isometry ρ^+), it is effective and so it maps a Kähler class to a Kähler class. By the global Torelli Theorem for K3 surfaces, there exists a unique involution $\tau : X \to X$ which induces τ^* on $H^2(X, \mathbb{Z})$. Then it follows from [8, Thm. 4.2.2], that the set of fixed points X^τ is empty. It means that the involution τ is fixed point free, hence X/τ is an Enriques surface.

Now, we introduce the following spectral sequence

$$E_{2,\mathbb{Z}}^{p,q} := H^p\big(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathbb{Z})\big) \Rightarrow H^{p+q}(Y, \mathbb{Z}) \tag{5}$$

associated to the covering map $\pi : X \to Y$ of an Enriques surface Y and we compute some terms of this. Since X is a K3 surface, the vanishing $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$ implies

$$E_{2,\mathbb{Z}}^{n,1} = E_{2,\mathbb{Z}}^{n,3} = 0 \tag{6}$$

for all integers n. Now, we compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for all integers n. First, we note that the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $\mathbb{Z}$. Since the term $E_{2,\mathbb{Z}}^{0,0}$ corresponds to the invariant elements of $\mathbb{Z}$ under the action of $\mathbb{Z}/2\mathbb{Z}$ we obtain that $E_{2,\mathbb{Z}}^{0,0} = \mathbb{Z}$. Now, let us compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for odd n. Since the action is trivial, we deduce that

$$0 = N(m) = \tau^*(m) + m = 2m.$$

Then it follows that $m = 0$ and hence by Theorem 6 that $E_{2,\mathbb{Z}}^{n,0} = 0$. On the other hand, if n is an even number we can see that $E_{2,\mathbb{Z}}^{n,0} = \mathbb{Z}/2\mathbb{Z}$. Summarizing,

$$E_{2,\mathbb{Z}}^{n,0} = \begin{cases} \mathbb{Z}, & \text{if} \quad n = 0; \\ 0, & \text{if} \quad n \quad \text{is odd}; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if} \quad n \quad \text{is even}, n \neq 0. \end{cases} \tag{7}$$

From (6) and (7) we deduce

$$E_{\infty,\mathbb{Z}}^{0,3} = E_{\infty,\mathbb{Z}}^{2,1} = E_{\infty,\mathbb{Z}}^{3,0} = 0$$

and this implies

$$E_{\infty,\mathbb{Z}}^{1,2} = \mathbb{Z}/2\mathbb{Z}. \tag{8}$$

The homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ induces a homomorphism $C : E_2^{1,1} \to E_{2,\mathbb{Z}}^{1,2}$ which can be easily described using Theorem 6 as

$$C : \frac{\{L \in \mathrm{Pic}(X) : \tau^* L \otimes L \cong \mathcal{O}_X\}}{\{\tau^* M \otimes M^\vee : M \in \mathrm{Pic}(X)\}} \to \frac{\{\ell \in H^2(X,\mathbb{Z}) : \tau^*\ell + \ell = 0\}}{\{\tau^* m - m : m \in H^2(X,\mathbb{Z})\}}, \tag{9}$$

sending $[L]$ to $[c_1(L)]$.

Theorem 13 (Schwarzenberger, [10])**.** *Let X be a projective surface. A topological complex vector bundle admits a holomorphic structure if and only if its first Chern class belongs to the Neron–Severi group of the surface.*

Lemma 14. *Let Y be an Enriques surface. Then every topological vector bundle on Y has a holomorphic structure.*

Proof. Let E be a $\mathcal{C}_X$–bundle on Y. Since Y is an Enriques surface then $\mathrm{NS}(Y) \cong H^2(Y,\mathbb{Z})$. Hence $c_1(E) \in \mathrm{NS}(Y)$ and by Theorem 13, E has a holomorphic structure. ☑

Lemma 15. *The homomorphism C is injective.*

Proof. Let $[L]$ be the class of a line bundle L such that $\tau^*L \otimes L = \mathcal{O}_X$. Suppose that $C(L) = 0$. Thus, there exists a topological line bundle M such that $L = M^\vee \otimes \tau^* M$ and so

$$-c_1(M) + c_1(\tau^* M) = c_1(M^\vee \otimes \tau^* M) = c_1(L) \in \mathrm{NS}(X). \tag{10}$$

On the other hand, since the topological rank 2 vector bundle $\tau^* M \oplus M$ has a linearization (i.e. the trivial linearization), there exists a topological vector bundle E on Y such that $\pi^* E = \tau^* M \oplus M$. By Lemma 14, E has a holomorphic structure and induces one on $\tau^* M \oplus M$. Thus, by Theorem 13,

$$c_1(\tau^* M \oplus M) \in \mathrm{NS}(X). \tag{11}$$

Therefore, by (10) and (11), $2c_1(\tau^* M) = (c_1(\tau^* M) - c_1(M)) + c_1(\tau^* M \otimes M) \in \mathrm{NS}(X)$. Since X is a K3 surface, $c_1 : \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$ is injective and so

$$\frac{H^2(X, \mathbb{Z})}{\mathrm{NS}(X)} \hookrightarrow H^2(X, \mathcal{O}_X).$$

Thus $c_1(\tau^* M) \in \mathrm{NS}(X)$ because $2c_1(\tau^* M) \in \mathrm{NS}(X)$ and $H^2(X, \mathcal{O}_X)$ is torsion free, and so we conclude $[L] = 0$ in $E_2^{1,1}$. ☑

In Example 12 we have introduced the involution ρ on the K3 lattice $\Lambda := (E_8)^{\oplus 2} \oplus U^{\oplus 3}$ and also defined the invariant lattice Λ^+. We define similarly the ρ-anti-invariant sublattice of Λ by

$$\Lambda^- := \{\ell \in \Lambda : \rho(\ell) = -\ell\}.$$

Given $\ell = (x, y, z_1, z_2, z_3) \in \Lambda$, we get $\rho(\ell) = -\ell$ if and only if

$$\ell = (x, -x, z_1, z_2, -z_2).$$

Let $m = (m_1, m_1, n_1, n_2, n_3) \in \Lambda$, then

$$\rho(m) - m = \big(m_2 - m_1, -(m_2 - m_1), -2n_1, n_3 - n_2, -(n_3 - n_2)\big).$$

this yields that

$$\ell = (x, -x, z, y, -y) \in \Lambda^-$$

can be written as $\rho(m) - m$ for some $m \in \Lambda$ if and only if $z = -2n$ for some $n \in U$.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Consider the spectral sequence $E_{2,\mathbb{Z}}^{1,2}$ associated to this (see (5)). Let $\ell \in H^2(X, \mathbb{Z})$ such that $\tau^* \ell = -\ell$. Thus, $2\ell = \ell - \tau^* \ell$, i.e. $[2\ell] = 0$ in $E_{2,\mathbb{Z}}^{1,2} =$

$H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X,\mathbb{Z})\big)$. Therefore, any element in $E_{2,\mathbb{Z}}^{1,2} = H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X,\mathbb{Z})\big)$ is 2-torsion.

By definition, $E_{3,\mathbb{Z}}^{1,2} = \ker\left(d_2^{1,2} : E_{2,\mathbb{Z}}^{1,2} \to E_{2,\mathbb{Z}}^{3,1}\right)$. Thus

$$E_{3,\mathbb{Z}}^{1,2} = E_{2,\mathbb{Z}}^{1,2}$$

because $E_{2,\mathbb{Z}}^{3,1} = H^3\big(\mathbb{Z}/2\mathbb{Z}, H^1(X,\mathbb{Z})\big) = 0$. Since

$$\mathbb{Z}/2\mathbb{Z} = E_{\infty,\mathbb{Z}}^{1,2} = \ker\left(d_3^{1,2} : E_{3,\mathbb{Z}}^{1,2} \to E_{3,\mathbb{Z}}^{4,0}\right),$$

we have only the following two options:

(i) $E_{2,\mathbb{Z}}^{1,2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} \neq 0$,

(ii) $E_{2,\mathbb{Z}}^{1,2} = \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} = 0$.

Now, we show that (ii) can not occur.

Lemma 16. *Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Then the $d_3^{1,2}$ of the spectral sequence $E_{2,\mathbb{Z}}^{p,q}$ associated to the morphism $\pi : X \to Y$ is not* 0*.*

Proof. First, we compute the term $E_{\infty,\mathbb{Z}}^{0,4}$. Since

$$E_{\infty,\mathbb{Z}}^{1,3} = E_{\infty,\mathbb{Z}}^{3,1} = 0,$$

$E_{2,\mathbb{Z}}^{4,0} = \mathbb{Z}/2\mathbb{Z}$ and $E_{2,\mathbb{Z}}^{2,2}$ is a torsion group, one finds

$$E_{\infty,\mathbb{Z}}^{0,4} = \mathbb{Z}.$$

Suppose that $d_3^{1,2} = 0$. Since X is a K3 surface,

$$E_{2,\mathbb{Z}}^{0,3} = H^0\big(\mathbb{Z}/2\mathbb{Z}, H^3(X,\mathbb{Z})\big) = 0 \tag{12}$$

$$E_{2,\mathbb{Z}}^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, H^1(X,\mathbb{Z})\big) = 0. \tag{13}$$

By definition of the terms of the spectral sequence,

$$E_{3,\mathbb{Z}}^{4,0} = \frac{E_{2,\mathbb{Z}}^{4,0}}{\operatorname{im}\left(d_2^{2,1} : E_{2,\mathbb{Z}}^{2,1} \to E_{2,\mathbb{Z}}^{4,0}\right)}$$

and by (13), $E_{3,\mathbb{Z}}^{4,0} = E_{2,\mathbb{Z}}^{4,0}$. Since $d_3^{1,2} = 0$,

$$E_{4,\mathbb{Z}}^{4,0} = \frac{E_{3,\mathbb{Z}}^{4,0}}{\operatorname{im}\left(d_3^{1,2} : E_{3,\mathbb{Z}}^{1,2} \to E_{3,\mathbb{Z}}^{4,0}\right)} = E_{3,\mathbb{Z}}^{4,0},$$

and finally by (12)

$$E_{\infty,\mathbb{Z}}^{4,0} = E_{5,\mathbb{Z}}^{4,0} = \frac{E_{4,\mathbb{Z}}^{4,0}}{\operatorname{im}\left(d_4^{0,3} : E_{4,\mathbb{Z}}^{0,3} \to E_{4,\mathbb{Z}}^{4,0}\right)} = E_{4,\mathbb{Z}}^{4,0}.$$

Hence we conclude $E_{\infty,\mathbb{Z}}^{4,0} = E_{2,\mathbb{Z}}^{4,0} = \mathbb{Z}/2\mathbb{Z}$, a contradiction. ☑

4. More about the Morphism $\mathbf{Br}'(Y) \to \mathbf{Br}'(X)$

We recall the following two results due to Beauville:

Proposition 17. ([2, Cor. 5.6 and its proof]) *Let* $\lambda = (\alpha, \alpha', \beta) \in H^2(X, \mathbb{Z})$ *such that* $\alpha, \alpha' \in E_8 \oplus U$ *and* $\beta \in U$ *and* ε *the class of* $e+f$ *in* $U_2 := U/2U$ *where* $\{e, f\}$ *is the basis of the hyperbolic lattice* U*. Then the following conditions are equivalent:*

i) $\pi_*\lambda = 0$ *and* $\lambda \notin (1-\tau^*)\big(H^2(X,\mathbb{Z})\big)$*;*

ii) $\tau^*\lambda = -\lambda$ *and* $\lambda^2 \equiv 2 \bmod 4$*.*

iii) the class $\overline{\beta} = \varepsilon$ *and* $\alpha' = -\alpha$*.*

Corollary 18. $\pi : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is trivial if and only if there exists a line bundle* L *on* X *with* $\tau^*L = L^\vee$ *and* $c_1(L)^2 \equiv 2 \bmod 4$*.*

Now, we quickly recall a kind of divisors in the period domain Ω of $E_8(2) \oplus U(2)$-polarized marked K3 surfaces. If we fix the unique primitive embedding of $E_8(2) \oplus U(2)$ in the K3 lattice Λ, then Ω is by definition

$$\Omega := \Big\{[\omega] \in \mathbb{P}\Big(\big(E_8(2) \oplus U(2)\big)_{\mathbb{C}}^{\perp}\Big) : \omega^2 = 0,\ \omega\overline{\omega} > 0\Big\}.$$

Let $S \subset \Lambda$ be a primitive sublattice of rank 11 containing the lattice $E_8(2) \oplus U(2)$. Then the subset

$$\Omega(S) := \Big\{[\omega] \in \mathbb{P}\big(S_{\mathbb{C}}^{\perp}\big) : \omega^2 = 0,\ \omega\overline{\omega} > 0\Big\}$$

is called the Heegner divisor of type S in Ω.

Proposition 19. ([9, Proposition. 3.1]) *If* X *corresponds to a very general point of* $\Omega(S)$*, i.e. in the complement of a union of countably many proper closed analytic subset of* $\Omega(S)$*, then we have* $\mathrm{NS}(X) = S$*.*

***Remark* 20.** Ohashi proved in [9, Theorem. 3.4], that for a lattice $S = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$ with $N \equiv 0 \bmod 4$, there exists a K3 surface X with an Enriques quotient and such that $\mathrm{NS}(X) = S$.

Example 21. Now, we will show the existence of a K3 surface X covering an Enriques surface Y with $\rho(X) = 11$ and $E_2^{1,1} = 0$ which from (4) implies that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. Let $\alpha \in \Lambda$, defined by (see [7])

$$\alpha = \Bigg(\sum_{i\,\mathrm{odd}} a_i e_i, -\sum_{i\,\mathrm{odd}} a_i e_i, 0, f_1 - f_2, -f_1 + f_2 \Bigg),$$

where the $a_i's$ are integers. This is a primitive element, $\alpha = \beta - \rho(\beta)$ where

$$\beta = (a_1 e_1 + a_3 e_3, -a_5 e_5 - a_7 e_7, 0, f_1, f_2)$$

and

$$\alpha^2 = -4 \sum_{i\,\mathrm{odd}} a_i^2 = -4m.$$

Thus, $E_8(2) \oplus U(2) \oplus \alpha\mathbb{Z} \hookrightarrow E_8^{\oplus 2} \oplus U^{\oplus 3}$ is a primitive embedding (note that $E_8(2) \oplus U(2)$ diagonally embeds in $E_8^{\oplus 2} \oplus U^{\oplus 3}$). Note that by the Lagrange's four-square Theorem ([4, Proposition 17.7.1]), m can take any positive integer value. By Proposition 19 and Remark 20, there exists a K3 surface X with an Enriques quotient Y and such that

$$\mathrm{NS}(X) = E_8(2) \oplus U(2) \oplus \alpha\mathbb{Z}$$

and by [1, Lemma 19.1] there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. Now, we take a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \phi^{-1}(\alpha)$. Then,

$$\begin{aligned} \alpha &= -\rho(\alpha) \\ &= -\rho\big(\phi\big(\phi^{-1}(\alpha)\big)\big) \\ &= -\phi\big(\tau^*\big(\phi^{-1}(\alpha)\big)\big) \\ &= -\phi\big(\tau^*\big(c_1(\mathcal{L})\big)\big) \\ &= -\phi\big(c_1(\tau^*\mathcal{L})\big) \\ &= \phi\big(c_1\big(\tau^*\mathcal{L}^\vee\big)\big). \end{aligned}$$

Then, from the injectivity of ϕ, it follows that

$$c_1(\tau^*\mathcal{L} \otimes \mathcal{L}) = 0,$$

and since X is a K3 surface we deduce

$$\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X,$$

i.e. $[\mathcal{L}] \in E_2^{1,1}$. Now, since $\alpha = \beta - \rho(\beta)$ and $E_2^{1,1} \subseteq E_{2,\mathbb{Z}}^{1,2}$ (Lemma 15), then $[\mathcal{L}] = 0$ in $E_2^{1,1}$.

Now, we show that for any line bundle $\mathcal{M}$ such that $\tau^*\mathcal{M} \otimes \mathcal{M} = \mathcal{O}_X$, there exists an integer n such that $\mathcal{M} = \mathcal{L}^{\otimes n}$. By construction of the above primitive embedding, we have that the action of τ^* on $E_8(2) \oplus U(2)$ is the identity. Thus, if $\mathcal{M}$ is a line bundle, it can be written as $\mathcal{M} = \mathcal{L}^{\otimes n} \otimes \mathcal{F}$ for some invariant line bundle $\mathcal{F}$. Hence

$$\mathcal{O}_X = \tau^*\mathcal{M} \otimes \mathcal{M} = \tau^*\mathcal{L}^{\otimes n} \otimes \tau^*\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{F} = \mathcal{F}^{\otimes 2}.$$

Hence $\mathcal{F} = \mathcal{O}_X$ because $\operatorname{Pic}(X)$ is torsion free and thus $\mathcal{M} = \mathcal{L}^{\otimes n}$. Thus, we have shown that $E_2^{1,1} = 0$.

Example 22. Let E_1, E_2 be elliptic curves over k (a field of characteristic 0) which are not isogeneous over $\overline{k}$ and such that their points of order 2 are defined over k. For $i = 1, 2$, let D_i be a principal homogeneous space of E_i whose class in $H^1\big(Gal(\overline{k}/k), E_i\big)$ has order 2. The antipodal involution $P \mapsto -P$ defines an involution on D_1 and on D_2, and defines a Kummer surface X by considering the minimal desingularization of the quotient of $D_1 \times D_2$ by the simultaneous antipodal involution. Since X is a Kummer surface, it covers an Enriques surface Y. Harari and Skorobogatov were able to prove that for this example the morphism $\pi^* : \mathrm{Br}'\left(\overline{Y}\right) \to \mathrm{Br}'\left(\overline{X}\right)$ is injective (see [3, Corollary 2.8]) where $\overline{X}$ and $\overline{Y}$ are the surfaces over $\overline{k}$ obtained from X and Y respectively by extending the ground field from k to $\overline{k}$. We also know from Corollary 4.4 in [6] that $\rho\big(\overline{X}\big) \geq 17$ because X is a Kummer surface.

Let $\pi : X \to Y$ be the universal covering map of an Enriques surface Y and let τ be the fixed point free involution of X associated to π. We proceed to study how τ acts on $H^2\big(X, \mathcal{O}_X^*\big)$ and on $H^3\big(X, \mathcal{O}_X^*\big)$.

Lemma 23. *Let X be a K3 surface with a fixed point free involution τ. The involution τ acts on $H^2\big(X, \mathcal{O}_X^*\big)$ as $\tau^*\alpha = \alpha^{-1}$.*

Proof. The involution τ acts on $H^2(X, \mathcal{O}_X)$ as $-\operatorname{id}$. Indeed, since $H^2\big(X, \mathcal{O}_X\big)$ is one dimensional then the action τ on this is $\pm\operatorname{id}$. If θ is a $2-$form and $\tau^*\theta = \theta$, the form descends to a $2-$form on $Y := X/\tau$. This is a contradiction because for any Enriques surface, $h^{0,2}(Y) = 0$. From the exponential sequence we get

$$\begin{array}{ccc} H^2\big(X, \mathcal{O}_X\big) & \twoheadrightarrow & H^2\big(X, \mathcal{O}_X^*\big) \\ \downarrow{\scriptstyle -\operatorname{id}} & & \downarrow{\scriptstyle \tau^*} \\ H^2\big(X, \mathcal{O}_X\big) & \twoheadrightarrow & H^2\big(X, \mathcal{O}_X^*\big) \end{array}$$

Hence for every $\alpha \in H^2\big(X, \mathcal{O}_X^*\big)$, $\tau^*\alpha = \alpha^{-1}$. ☑

Lemma 24. *Let X be a K3 surface. Any element in the Brauer group* $\mathrm{Br}'(X)$ *is* $2-$*divisible.*

Proof. From the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

we get

$$0 \to \mathrm{Br}'(X)_2 \to \mathrm{Br}'(X) \to \mathrm{Br}'(X) \to 0$$

because $H^3(X, \mathbb{Z}/2\mathbb{Z}) = 0$. ☑

***Remark* 25.** Let $\rho := \rho(X)$ denote the Picard number of a surface X. Let X be a K3 surface with an involution τ that has no fixed points. For any invariant line bundle L under τ, there is a line bundle M on the Enriques surface $Y := X/\tau$ such that $\pi^*M = L$. This is no longer true for Brauer classes. Indeed, by Lemma 23, the invariant elements of $\mathrm{Br}'(X)$ under τ consist of all the 2-torsion elements of $\mathrm{Br}'(X)$. Since X is a K3 surface, $\mathrm{Br}'(X)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Hence, since $\rho \leq 20$, there exists an element $\alpha \in \mathrm{Br}'(X)$ such $\tau^*\alpha = \alpha$ which is not in the image $\mathrm{im}\big(\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)\big)$. In conclusion, you may have picked α that happens to be in the image, but since $22 - \rho \geq 2$, there is always another one.

Now, let us compute some elements of the spectral sequence $E_2^{p,q}$ introduced in (1), associated to the universal covering map $\pi : X \to Y$ of an Enriques surface Y. First, we know from the exponential sequence that

$$H^3(Y, \mathcal{O}_Y^*) \cong H^4(Y, \mathbb{Z}) = \mathbb{Z}. \tag{14}$$

***Remark* 26.** By Theorem 6,

$$E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = \frac{\big\{L \in \mathrm{Pic}(X) : \tau^*L \otimes L^\vee = \mathcal{O}_X\big\}}{\big\{\tau^*M \otimes M : M \in \mathrm{Pic}(X)\big\}}$$

and

$$E_2^{1,2} = H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathcal{O}_X^*)\big) = \frac{\big\{\alpha \in H^2\big(X, \mathcal{O}_X^*\big) : \tau^*(\alpha).\alpha = 1\big\}}{\big\{\tau^*(\beta).\beta^{-1} : \beta \in H^2\big(X, \mathcal{O}_X^*\big)\big\}}.$$

By Lemmas 23 and 24, $E_2^{1,2} = 0$. Now, if $L \in \mathrm{Pic}(X)$ with $\tau^*L \otimes L^\vee = \mathcal{O}_X$. Then $[L^{\otimes 2}] = [\tau^*(L) \otimes L]$, i.e. $[L]$ is a 2-torsion element in $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$.

Thus $E_2^{1,2}=0$, $E_2^{3,0}=\mathbb{Z}/2\mathbb{Z}$ (cf. (3)) and $E_2^{2,1}$ is a torsion group (by (26)). In conclusion, we get from the (14) which says that $E^3=\mathbb{Z}$, that

$$E_\infty^{0,3}=\mathbb{Z}, \tag{15}$$

$$E_\infty^{1,2}=E_\infty^{2,1}=E_\infty^{3,0}=0. \tag{16}$$

The action τ on $H^3\big(X,\mathcal{O}_X^*\big)=H^4(X,\mathbb{Z})=\mathbb{Z}$ is $\pm\,\mathrm{id}$. If $\tau^*=-\,\mathrm{id}$, then $E_2^{0,3}=H^0\big(\mathbb{Z}/2\mathbb{Z},H^3\big(X,\mathcal{O}_X^*\big)\big)=H^3\big(X,\mathcal{O}_X^*\big)^\tau=0$, but this contradicts the fact $E_\infty^{0,3}=\mathbb{Z}$. Thus, we have shown the following lemma. (Note that this lemma trivially follows only from the fact that $H^3\big(X,\mathcal{O}_X^*\big)=H^4(X,\mathbb{Z})=\mathbb{Z}$ and the action on the last cohomology group is id, but the computations above are needed).

Lemma 27. *Let X be a K3 surface with a fixed point free involution τ. Then the action of τ on $H^3\big(X,\mathcal{O}_X^*\big)$ is trivial.*

***Remark* 28.** Let L be a line bundle such that $\tau^*L\otimes L=\mathcal{O}_X$. Thus, $L^{\otimes 2}=L\otimes(\tau^*L)^\vee$, i.e. $[L]\otimes[L]=[L^{\otimes 2}]=0$ in $E_2^{1,1}=H^1(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X))$. Since

$$E_2^{0,2}=H^0\big(\mathbb{Z}/2\mathbb{Z},H^2\big(X,\mathcal{O}_X^*\big)\big)=H^2\big(X,\mathcal{O}_X^*\big)^\tau,$$

by Lemma 23, $E_2^{0,2}=\mathrm{Br}'(X)_2$. Indeed, if $\alpha\in\mathrm{Br}'(X)$ with $\tau^*\alpha=\alpha$, then by Lemma 23, $\alpha=\tau^*\alpha=\alpha^{-1}$, i.e. α is a $2-$torsion element of $\mathrm{Br}'(X)$. On the other hand, if $\alpha\in\mathrm{Br}'(X)_2$, then by Lemma 23, $\alpha=\alpha^{-1}=\tau^*\alpha$. Finally, by Remark 25, $E_2^{0,2}=\mathrm{Br}'(X)_2=(\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Since any element in $E_2^{1,1}$ is a 2-torsion element, we have only the following four cases:

i) $E_2^{1,1}=0$ or

ii) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}=\mathrm{id}$, $\quad$ i.e. $\quad E_\infty^{1,1}=0$ or

iii) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}=0$, $\quad$ i.e. $\quad E_\infty^{1,1}=\mathbb{Z}/2\mathbb{Z}$ or

iv) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}\neq 0$, $\quad$ i.e. $\quad 0\to\mathbb{Z}/2\mathbb{Z}\to E_2^{1,1}\overset{d_2^{1,1}}{\to}E_2^{3,0}\to 0$.

Lemma 29. *Let Y be an Enriques surface, $\pi:X\to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1}=H^1\big(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X)\big)=0$. Then $E_2^{2,1}=H^2\big(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X)\big)=(\mathbb{Z}/2\mathbb{Z})^{20-\rho}$.*

***Proof*.** Since $E_2^{1,1}=0$,

$$E_3^{3,0}=\frac{E_2^{3,0}}{\mathrm{im}\left(d_2^{1,1}:E_2^{1,1}\to E_2^{3,0}\right)}=E_2^{3,0}=\mathbb{Z}/2\mathbb{Z}$$

and by (16)

$$0 = E_\infty^{3,0} = E_4^{3,0} = \frac{E_3^{3,0}}{\operatorname{im}\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)}.$$

Thus $d_3^{0,2}$ is surjective. Since $E_2^{1,1} = 0$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right), \tag{17}$$

and since $E_3^{3,0} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ and all elements in $E_2^{0,2}$ are 2-torsion,

$$E_3^{0,2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{18}$$

By (16)

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)},$$

and thus the morphism $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is surjective. Hence, by (17) and the fact that any element in $E_2^{0,2}$ is a 2−torsion element (cf. Remark 28),

$$E_2^{0,2} = E_3^{0,2} \times E_2^{2,1}.$$

From $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 28) and (18), $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. ☑

Lemma 30. *Let Y be an Enriques surface, $\pi : X \to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = 0$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. Since $E_2^{1,1} \neq 0$ and $E_\infty^{1,1} = 0$, $\operatorname{im}\left(d_2^{1,1}\right) = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ (cf. (3)). Thus

$$E_3^{3,0} = \frac{E_2^{3,0}}{\operatorname{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = 0. \tag{19}$$

By Remark 26, any element in $E_2^{2,1}$ is 2−torsion. Then there is an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By (16),

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)}$$

and thus $\operatorname{im}\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^m$. Hence

$$\ker\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

because $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Since $E_\infty^{0,2} = \mathbb{Z}/2\mathbb{Z}$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : \ker\left(d_2^{0,2}\right) \to E_3^{3,0}\right)$$

and from (19)

$$\mathbb{Z}/2\mathbb{Z} = \ker\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

and so $m = 21 - \rho$. ☑

Lemma 31. *Let X be a K3 surface that covers an Enriques surface Y and such that its spectral sequence satisfies $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. By assumptions $d_2^{1,1}$ is trivial and so

$$E_3^{3,0} = \frac{E_2^{3,0}}{\operatorname{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$$

and by definition

$$E_4^{3,0} = \frac{E_3^{3,0}}{\operatorname{im}\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)}. \tag{20}$$

On the other hand,

$$0 = E_\infty^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)$$

because $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Hence $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0} = \mathbb{Z}/2\mathbb{Z}$ is injective and this and (20) imply the following equivalence:

$$E_3^{0,2} = \mathbb{Z}/2\mathbb{Z} \quad \text{if and only if} \quad E_\infty^{3,0} = E_4^{3,0} = 0. \tag{21}$$

By (16), $E_\infty^{3,0} = 0$. Thus, the equivalence (21) implies $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Since by Remark 26, any element in $E_2^{2,1}$ is a 2-torsion element, there exists an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By (16),

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)},$$

and thus

$$\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right) = (\mathbb{Z}/2\mathbb{Z})^m,$$

i.e. the map $d_2^{0,2}$ is surjective. Since $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 28), $E_3^{0,2} = \ker\left(d_2^{0,2}\right)$, it yields from the surjectivity of $d_2^{0,2}$ that

$$E_3^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}.$$

Thus, $m = 21 - \rho$ because $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Hence

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}.$$ ☑

Lemma 32. *Let Y be an Enriques surface and $\pi : X \to Y$ the universal covering map of Y such that $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^2$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Moreover $\rho(X) \geq 12$.*

Proof. Since $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2$ and $E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$, the map $d_2^{1,1} \neq 0$. Hence $E_\infty^{1,1} = E_3^{1,1} = \ker\left(d_2^{1,1}\right)$ is nontrivial, and it must be $\mathbb{Z}/2\mathbb{Z}$. By definition,

$$E_3^{3,0} = \frac{E_2^{3,0}}{\operatorname{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = 0 \tag{22}$$

and by (16)

$$E_\infty^{2,1} = E_3^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)} = 0. \tag{23}$$

Since $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$, then

$$0 = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right).$$

Thus, by (22), $E_3^{0,2} = 0$. By definition,

$$E_3^{0,2} = \ker\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)$$

and then $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is injective. Hence, by (23),

$$E_2^{2,1} = E_2^{0,2}.$$

Since

$$E_2^{0,2} = \operatorname{Br}'(X)_2 = (\mathbb{Z}/2\mathbb{Z})^{22-\rho},$$

one finds

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}.$$

Since $E_2^{2,1}$ is a quotient of $\operatorname{Pic}(X)^\tau$ and thus of $\operatorname{Pic}(Y) = \mathbb{Z}^{10} \times \mathbb{Z}/2\mathbb{Z}$, one finds $22 - \rho \leq 10$ (note that $\mathbb{Z}/2\mathbb{Z} \subset \operatorname{Pic}(Y)$ goes to zero in $E_2^{2,1}$). Thus $\rho \geq 12$ ☑

In conclusion, by lemmas 29, 30, 31, 32 and the statement before Lemma 29, we only have the following four cases:

i) $E_2^{1,1} = 0, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$ or

ii) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_\infty^{1,1} = 0, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

iii) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

iv) $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2, \quad E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Note that in the cases ii) and iii) we have that $\rho \geq 11$.

Theorem 33. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2\big(\mathbb{Z}/2\mathbb{Z}\operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Then the morphism $\pi^* : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is trivial, where $Y := X/\langle\tau\rangle$.*

***Proof*.** Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$, we are in case iv). Hence $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. By (4), the morphism $\pi : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is trivial. ☑

Theorem 34. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2\big(\mathbb{Z}/2\mathbb{Z}\operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. Then the morphism $\pi^* : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is nontrivial, where $Y := X/\langle\tau\rangle$.*

***Proof*.** Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$, we are in case i). Hence $E_\infty^{1,1} = 0$. By (4), the morphism $\pi^* : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is injective. ☑

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. We know that if X is as in the first case above, then $\rho(X) \geq 10$, and if X is one of the cases ii) or iii), then $\rho(X) \geq 11$ and if X is as in the case iv), then $\rho(X) \geq 12$. Thus, if $\rho(X) = 10$, the K3 surface X satisfies the conditions of the first case and we obtain $E_2^{1,1} = 0$. Hence, by (4), the morphism $\pi^* : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is injective. This is another proof of the same result obtained before out Lemma 11.

Proposition 35. *Let X be a K3 cover of an Enriques surface Y such that $\rho(X) = 11$ and $\operatorname{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle$, where $N \geq 2$. Then $\pi^* : \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$ is injective if and only if N is an even number.*

***Proof*.** Note that $\operatorname{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle = \pi^* \operatorname{NS}(Y) \oplus \langle -2N\rangle$ (because, as in Example 12, $\Lambda^+ \cong U(2) \oplus E_8(2)$ and this is diagonally embedded in the K3 lattice), i.e. τ^* acts trivially on $U(2) \oplus E_8(2)$. Now, we show that τ acts as $-\operatorname{id}$ on $\langle -2N\rangle$. Let $L \in \operatorname{NS}(X)$ denote the generator of $\langle -2N\rangle$, i.e. $c_1^2(L) = -2N$. Thus,

$$\tau^* L = I \otimes L^{\otimes k} \tag{24}$$

for some integer k and invariant line bundle I and since τ is an involution:

$$\begin{aligned} L = \tau^*\tau^* L &= \tau^* I \otimes \tau^* L^{\otimes k} \\ &= I \otimes \tau^* L^{\otimes k} \\ &= I \otimes \big(I \otimes L^{\otimes k}\big)^{\otimes k} \\ &= I^{\otimes(k+1)} \otimes L^{\otimes k^2}. \end{aligned}$$

Hence

$$L^{\otimes(k^2-1)} \otimes I^{\otimes(k+1)} = \mathcal{O}_X \tag{25}$$

and we find that $L^{\otimes(k^2-1)}$ is an invariant line bundle. Thus,

$$\mathcal{O}_X = L^{\otimes(-k^2+1)} \otimes \tau^* L^{\otimes(k^2-1)} = \left(L^\vee \otimes \tau^* L\right)^{\otimes(k^2-1)}$$

and if $k \neq 1, -1$, then $\mathcal{O}_X = L^\vee \otimes \tau^* L$ (because $\mathrm{Pic}(X)$ is a free torsion group), i.e. L is an invariant line bundle which contradicts our assumption about L. If $k = 1$, then from (25) we get $I = \mathcal{O}_X$ and then by (24), L is an invariant bundle which contradicts our assumption on L. Thus $k = -1$ and from (25), $I = \mathcal{O}_X$ and from (24) we obtain $\tau^* L \otimes L = \mathcal{O}_X$, i.e. τ acts as $-\mathrm{id}$ in $\langle -2N \rangle$.

Now, we show that if M is a line bundle such that $\tau^* M \otimes M = \mathcal{O}_X$, then $M = L^{\otimes m}$ for some integer m. Indeed, if $M = L^{\otimes m} \otimes F$ where F is an invariant line bundle, then

$$\mathcal{O}_X = \tau^* M \otimes M = \tau^* L^{\otimes m} \otimes \tau^* F \otimes L^{\otimes m} \otimes F = F^{\otimes 2}.$$

Hence $F = \mathcal{O}_X$ because $\mathrm{Pic}(X)$ is torsion free and thus $M = L^{\otimes m}$.

Suppose that N is an even number and that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial. By Corollary 18, there exists a line bundle $M = L^{\otimes m}$ for some integer m such that $c_1(M)^2 \equiv 2 \bmod 4$. Thus $-2m^2 N \equiv 2 \bmod 4$, which implies that $m^2 N$ is an odd number and thus N is an odd number, a contradiction. On the other hand, let us suppose that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. By Corollary 18, $c_1^2(L) \not\equiv 2 \bmod 4$. Hence, $(1-N) \not\equiv 0 \bmod 2$ and thus N is an even number. ☑

Acknowledgements. I am grateful to my advisor, Daniel Huybrechts, for all his help and encouragement and the Max-Planck Institute for Mathematics in Bonn for all its support.

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(Recibido en mayo de 2012. Aceptado en octubre de 2012)

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